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DOCTORAL THESIS

# STUDIES IN ENDOGENOUS MACROECONOMIC DYNAMICS

A DISSERTATION SUBMITTED TO THE DOCTORAL PROGRAM IN ECONOMICS AND MANAGEMENT IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DOCTORAL DEGREE (PH.D) IN ECONOMICS AND MANAGEMENT

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# Abstract

This thesis focuses on endogenous approaches to studying macroeconomic dynamics and evaluates this tradition from different perspectives. It traces the origins and development of non-linear, endogenous theories of business cycles from its early beginnings up to its present frontiers. It argues that these theories emerged out of an attempt to reconcile the then existing corpus of (essentially static) economic theory with empirically observed persistent fluctuations. It offers a re-reading of Harrod's book- 'The Trade Cycle' and demonstrates the accelerator in his theory to be non-linear and consequently claims that Harrod's text contains essential elements that constitute an endogenous theory.

On the mathematical front, it examines the role of existence and uniqueness theorems (in particular, the Poincaré–Bendixson Theorem) in planar endogenous models of economic dynamics. Their underpinnings, their use and influence on the mathematical models of aggregate macroeconomic fluctuations are critically evaluated. In this context, it considers Goodwin (1951)'s nonlinear model of business cycles and shows how existence and uniqueness of limit cycles can be established even for the case of an asymmetric, nonlinear, accelerator with only one nonlinearity. This is achieved using a result by de Figueiredo (1960). It argues that an excessive reliance on proving 'existence' and 'uniqueness' hampered the enlargement of scope in these nonlinear, endogenous theories. It outlines the non-constructive aspects of these theorems and discusses the issue of computability for limit cycles in these planar models.

Furthermore, some methodological issues related to computational economic dynamics are analyzed. From an algorithmic point of view, it contends that there are inherent undecidabilities associated with many important properties of these dynamic models. These include characterizing attractors, determining their number, the domains of attraction and the possibility of exhibiting chaos, all of which have been important for dynamic economic theories. It makes a case for resorting to *algorithmic economic dynamics* in the future in order to overcome some of the limitations faced by the endogenous tradition, which exclusively relies on dynamical systems theory for modelling. A broad outline of what such an approach would look like is illustrated.

# Contents

# Chapter 1

# Introduction

Macroeconomic dynamics has been an active area of research for almost over a century now. Changes registered over time in aggregate economic measures such as GDP, investment, employment, inflation and wealth trigger enormous public interest and lies at the heart of contemporary political debates and policy decisions. Economists over the years have attempted to understand the empirical regularities of these aggregate variables and have formulated various theories to explain their nature and dynamics. This involves unearthing the nature of relationships between different economic variables and theoretically framing the observed dynamics of the aggregate economic system. Among the different topics under the umbrella of macroeconomic dynamics, growth and business cycles (along with monetary dynamics, unemployment and inflation dynamics) are among the most important areas that have been the subjects of extensive theoretical and empirical investigations.

Theoretical developments over the last century in explaining economic growth and business cycles have progressed in various directions. The different schools of thought that engage with these issues vary in terms of the causal factors and operational mechanisms that they attribute to growth and fluctuations. One useful way to classify these different views can be to make a distinction between those that subscribe to the view that sources of growth and fluctuations are from 'within' the economic system and those that hold such dynamics to be coming from 'outside'. Consequently, the theories can be grouped as either 'endogenous or 'exogenous' respectively. Theoretical developments in these two areas have evolved in an interesting manner over the years: Growth theory started out as being predominantly an exogenous theory and later become endogenous over the years. For Business Cycle Theory, the opposite holds true.

In this thesis, the focus will be largely on cycle theory. However, there are important implications for growth theory as well. The exogenous view of economic fluctuations is often thought to have originated from the work of Frisch (1933), who made a distinction between 'impulse' and 'propagation' mechanisms[1]. The argument here

[1]This is also referred as the *Frisch-Slutsky* methodology. However, as we point out later, Simon

is that the impulse mechanisms are to be distinguished from the propagation mechanisms that are associated with the structure of the economic system. The impulses are exogenous to the system and they take the form of stochastic shocks or random disturbances to one of the variables (for example, supply, productivity or demand shocks). The economic system is assumed to be stable and would always return to a state of equilibrium, if not for the disturbances that displace the system from its equilibrium. In other words, the system is endowed with self-regulating capabilities that bring it back to equilibrium. The metaphor that captures this worldview is that of a 'rocking horse', which remains in a state of equilibrium unless it is disturbed. In the event of being disturbed, the system fluctuates (or rocks) because of the manner in which these shocks propagate. However, these oscillations gradually die down due to the structure of the system. This view has been popular for almost two decades now, in the form of Real Business Cycle theory (RBC) put forward by Kydland and Prescott (1982).

A brief sketch of the exogenous viewpoint, in its RBC rendition, would be the following: Business cycles are viewed as fluctuations in output in a steady-state growth path of one or other neoclassical growth model. The baseline version of the model considers a representative agent endowed with rational expectations, who maximizes his or her utility and the representative firm maximizes its profit. The equilibrium is characterized by the paths of evolution of different variables (such as output, consumption and prices) at the steady state. In the presence of stochastic shocks, the equilibrium (known as the Dynamic Stochastic General Equilibrium or DSGE) paths of evolution for quantities and prices are characterized as stochastic processes. Here, studying business cycles translates to understanding how positive or negative stochastic shocks to a variable, for example technology[2], translates to output fluctuations. This view has been quite influential in the empirical and policy front as well in the form of impulse-response investigations[3].

Endogenous theories, on the other hand, hold that *persistent* cyclical tendencies are in-built in capitalistic economies. The endogenous tradition in cycle theory can probably be traced to Karl Marx or even earlier. The structure and the relationship between different economic variables are such that they make the system prone to sustained fluctuations, even if they are insulated from the disturbances. While exogenous shocks may very well have an impact, in the endogenous view, this only plays a subordinate role. It is not central to explaining the persistent cyclical tendencies of the economic system. There are many different strands under this broad view. Some focus on highly aggregated systems (for example, those in the Keynesian tradition) and disequilibrium fluctuations, while others look at systems with optimizing agents, competitive markets and equilibrium fluctuations. The unifying theme is that

Kuznets deserves to be placed along with them.

[2] Variables are assumed to be auto-correlated processes, which are non-explosive.

[3] For instance, the Nobel memorial prize for Economics for 2011 was awarded to two of its proponents, Thomas Sargent and Christopher Sims.

the source of these fluctuations are from 'within' the system, which does not have any self-regulating mechanisms to bring itself back to a stable equilibrium or continue to evolve without cycles.

In the past, economic growth and cycles have been mostly analyzed separately with the exception of scholars like Schumpeter, Lundberg and Goodwin (1967). However, a puritan endogenous theorist would hold that trend and the cycle are inseparable. There are also differing views within the endogenous approach regarding whether which of the two phenomena - cycles or growth - should be given a central role. Some view that there is no economic growth that is possible without a cycle, while others hold that cyclical tendencies are merely the form that growth takes. This thesis will mainly focus on endogenous cycle theories. In its development in the mathematical mode, the presence of nonlinear relationships between different variables proved to be a crucial ingredient in explaining persistent endogenous cycles. Consequently, mathematical tools from the theory of nonlinear oscillations (in the formative years) and nonlinear dynamical systems theory (in the later years) were utilized for building theoretical models. These theories have often concentrated on explaining plausible long and short-term properties of the system and their capacity to oscillate by resorting to the application of different existence theorems. The progress in this tradition also paralleled important mathematical developments in nonlinear dynamics and this cross fertilization led to interesting hybrids. Similarly, developments in the world of computing had profound implications in studying nonlinear models via simulations, since explicit solutions are often hard to obtain.

Among the vast literature in this area of endogenous economic dynamics, attention will be paid to theories that are explicitly mathematical. Additionally, the focus will be on the 'real' theories of endogenous cycle, hence the monetary theories of endogenous fluctuations will only feature occasionally in this analysis. This choice is due to reasons of space and the fact that most mathematical models of endogenous fluctuations in the past have concentrated on the 'real' side and monetary factors have not been at the center stage. The unfortunate consequence of this would be that some important endogenous (monetary) theories, for e.g., monetary models of fluctuations underpinned by Swedish, Austrian schools of thought, price and output fluctuations with optimizing agents in equilibrium models with money, theories of endogenous (financial) instability (Hyman Minsky) will not be discussed at length. However, this should not be mistaken as merely an attempt to trace the use of one or other kind of mathematics in macroeconomics. Rather, it is the interplay between endogenous economic theories and their mathematical characterizations that are of interest here. It should be stated explicitly that this thesis neither claims to propose a new model nor provide an alternative theory of endogenous fluctuations in output, prices or employment. Instead, it closely examines a specific exciting field of research in economics, with the aim of understanding its origins, evolution, strengths and shortcomings. This is important for few reasons: First, it helps us to be aware of the scope, limitations and the appropriateness of the (mathematical) tools that we utilize in our economic theories. Second, it is only by understanding the limits built into a particular mode of reasoning can we better qualify our theoretical views and predictions about the economic system. This knowledge will, hopefully, also guide us to transcend these limitations and point us to newer tools, paradigms and explorations.

After a century-long efforts in understanding, theorizing and modeling endogenous fluctuations, it may be useful to take stock of these developments - the various turns that have been taken by this tradition at different times - and critically analyze its merit. In doing so, some of the following questions need to be addressed: How did the mathematical theories of endogenous dynamics emerge and develop over the years? What were the crucial ingredients in their mathematical structure that facilitated the demonstration of endogenous dynamics? What are the constraints posed by the mathematical tools utilized by this tradition? What are the probable explanations for the (seeming) stagnation in the theoretical developments of this tradition? How to transcend these limitations and what new future directions should the endogenous economic theorist embark upon? The two different view points, (i.e, exogenous and endogenous) may appear to be merely different methodological or philosophical positions, which may be of interest only within academia and not relevant outside its walls. However, this is not the case. Adopting one or other view has tremendous consequences and often leads to diametrically opposing positions, especially in terms of macroeconomic policy. While undertaking such a critical evaluation, it is also important to be constructive. This would mean not just identifying the limitations to further development in this tradition, both in terms of our economic understanding and the tools we employ but also to propose possible and feasible alternative paths that go beyond the limitations. This hopefully will lead to the development of new theories, which incorporate and present new economic insights, and yet all the while remain within the scope of an endogenous theorist. This is the aim of this thesis.

One of the recurring themes in this thesis is the insistence on constructive and algorithmic procedures to study endogenous economic dynamics. This is along the lines of Computable Economics, a field pioneered by Velupillai (2000), who since then[4] has made a sustained plea for constructive approaches in economic theory. By computation, we do not merely refer to the use of computational tools in economics, but also to the theory of computation (Computability theory/recursion theory) and mathematical analysis of algorithms as formal objects. This algorithmic approach involves constructing economic models, which are themselves algorithms, to encapsulate patterns from the economic data. These algorithms are simulated and also studied as formal objects in their own right. The aim would be to understand their evolution, long-term properties, transition paths, complexity and so on. This will pay greater attention to economic *processes* and allow us to search and develop methodological solu-

[4] A recent companion with important contributions in this field can be found in Velupillai et al. (2011).

tions rather than merely be restricted to proving existence and uniqueness. This means venturing beyond the exclusive reliance on dynamical systems theory for understanding economic dynamics. The conventional approach to economic dynamics works on unrestricted domains and does not pay careful attention to the natural data types associated with the economic phenomena. Most models assume the data domain to be that of real numbers, while economic quantities are often, at best, rational numbers. The results and theorems in this standard approach often do not carry over to dynamics defined on rational numbers. The correspondence between these two domains is unclear and its implications of using one vis-à-vis the other are not well understood. This complicates relating the theories meaningfully to the available data. Algorithmic approaches overcome this weakness since formal algorithms themselves are defined over rational numbers. The epistemological limits to formal reasoning are made explicit in this approach since there are algorithmically undecidable problems.

With those introductory remarks, we can now look at the structure of this thesis. It is composed of five chapters that address different aspects that concern endogenous macrodynamics. The first chapter traces the origins and early development of nonlinear, endogenous theories of business cycle. It argues that these theories came out of an attempt to reconcile the then existing corpus of (essentially static) economic theory with empirically observed fluctuations. It outlines the extraordinary set of events that eventually led to the birth of a full fledged endogenous tradition. Its development in different directions over the years is traced all the way up to the current frontiers.

The second chapter takes up a key contribution by Roy Harrod, one to which the nonlinear, endogenous theories of the Keyneisan tradition owes much to. It is often argued that Harrod's attempt to combine the accelerator and the multiplier resulted in an unstable system, which relied on exogenous factors to explain the turning points. An attempt is made to re-read Harrod in order to find the true nature of the accelerator that underlies his literary exposition. Following a clue in Ichimura (1955b), this chapter presents arguments in support of the view that a nonlinear accelerator underpins Harrod's theory and consequently, claims that there were essential elements in Harrod (1936) to constitute an endogenous theory.

In chapter 3, the role of existence theorems, in particular, the Poincaré–Bendixson theorem and Levinson-Smith theorem, in the planar models of nonlinear, endogenous mathematical theories of the business cycle (NETBC, henceforth) is analyzed. This chapter investigates the mathematical and economic underpinnings of these theorems, the features which enable them to demonstrate persistent fluctuations and the different ways in which they were employed. Their influence on the mathematization of various theories of aggregate macroeconomic fluctuations in this tradition is critically evaluated. It argues that an excessive reliance on proving 'existence' hampered the enlargement of the scope of these nonlinear theories. The trade-offs involved in attempting to strait-jacket economic theories to existing mathematical results are pointed out. We outline the non-constructive aspects of these theorems and discuss the issue of computability of limit cycles for these planar models.

Chapter 4 examines the theorems which establish the uniqueness of periodic orbits in the planar models of endogenous business cycles. It identifies and catalogues the different uniqueness theorems that are used in this tradition. By focusing on Goodwin (1951)'s nonlinear model of business cycles, it shows how existence and uniqueness of limit cycles can be established even for the case of an asymmetric nonlinear accelerator with only one nonlinearity (discussed in Goodwin (1950)). This is achieved using a result by de Figueiredo (1960), which in turn benefited from important insights from Le Corbeiller on Goodwin's one-sided oscillator. The relationship between this result and the sufficiency conditions used by Sasakura (1996) are discussed. The chapter opens up ways to move beyond the 'existence-uniqueness' mode of theorizing that has restricted the development of NETBC. It argues in favour of resorting to algorithmic methods to model economic dynamics and presents a broad outline of the proposed approach. This can be a way to transcend the exclusive reliance on dynamical systems theory and the limitations that come with it.

In chapter 5, we analyze some methodological issues in computational approaches to studying economic dynamics in general. Computational models of economic dynamics are evaluated from the perspective of 'decidability'. Tools from computable analysis are utilized to study the dynamics of these economic models. From an algorithmic point of view, it contends that there are inherent undecidabilities associated with many properties of these systems. These properties include characterizing attractors, determining their number, domains of attraction and the possibility of exhibiting chaos - all of which have been important for dynamic economic theories. It is shown that some of these properties are however decidable for planar models of economic dynamics- an important class of endogenous models. In particular, stability assumption seems to play an important role in ensuring computability. It also presents a brief evaluation of the use of numerical procedures in simulating continuous time models. Finally, it attempts to make a case for moving towards the use of algorithmic methods to study endogenous economic dynamics.

# Chapter 2

# Origins and Early Development of NETBC[1]

## 2.1 Prologue

> "He [Oppenheimer] studied me with his remarkable blue eyes and asked, 'What is new and firm in Physics?' The '. . . and firm' impressed me."
>
> **Oppenheimer: Portrait of an Enigma** by Jeremy Bernstein, Ivy Publishers, Chicago, 2005

We begin with a puzzle: Wicksell observes a 20-year deflation and constructs an *unstable model of inflation for stabilization purposes*. Why? The same fact, observed and recorded in their writings, led Fisher and Schumpeter to emphasize other aspects of the behaviour of economic institutions, agents and the economic system's *evolutionary dynamics*. Fisher developed the link between appreciation and interest via *expectations*; Schumpeter, on the other hand, that between *deflation* and *innovation* to justify the tendency for a capitalist system to undergo benign fluctuations.

A young macroeconomist facing, say an ageing Walras, at the turn of the century that took the 19th into the 20th, and confronted with the kind of question Bernstein was posed by Oppenheimer, may have had difficulties identifying the unstable cumulative process, the Fisher equation and Schumpeterian evolutionary dynamics as being part of the '. . . and firm' description of the subject; although she may have recognized them as 'new'. After all, *even the subject did not exist at that time*.

In March 1952, during a lecture in Stockholm, Eli Heckscher (1952) recalled, on *14 April 1898*, Wicksell 'somewhat unexpectedly revealed before the [Stockholm Economic] Society what was perhaps his greatest theoretical achievement, his theory of the connection between interest rate and money value' (*ibid*, p. 119). *Thus was born modern macroeconomics*.

[1] An extended version of this chapter, co-authored with Vela Velupillai, was published as Ragupathy and Velupillai (2012b). It includes suggested future directions to go beyond dynamical systems theory.

Macroeconomics is a word coined in 1939 by the Swedish economist Erik Lindahl[2], himself Wicksell's distinguished pupil in the theory of public finance and taxation. The word had been in use, in academic circles in Sweden and Norway, from the early 1930s after Ragnar Frisch and Michal Kalecki had popularised the term macrodynamics in discussions about the problems of the trade cycle. But it was Lindahl who explicitly contrasted the word macroeconomics with microeconomics, in the senses in which we use them in modern economic theoretical discourse; and he did so in his famous book **Studies in the Theory of Money and Capital** (Lindahl (1939)).

It is, proverbially, a new name for an old subject. However, it was Wicksell – and, to a lesser extent, Fisher - not Keynes nor Hayek, who first stamped it with modernism in an unmistakable way – the modernism we associate with providing microfoundations for aggregate variables and behaviour. This he provided for the twin horns of macroeconomics – the real and the monetary sides; for the former on the basis of Austrian capital theory, which he almost single-handedly and rigorously re-wrote and redid for Menger, Böhm-Bawerk and von Wieser; for the latter, on the basis of a wholly new approach to monetary theory by devising an innovative thought-experiment - *gedankenexperiment* - which obviated the need for a reliance on the quantity theory of money to explain inflation. This thought-experiment constructed a pure credit economy in which monetary transactions were conducted in an imaginary giro system.

The crucial event that spurred him to these conceptual innovations was the 20-year deflation – not recession – experienced, without exception, by all the advanced industrial nations, from the mid-1870s to the mid-1890s. He was – as Fisher was - deeply concerned that this deflation meant an unwarranted redistribution of wealth and income between lenders and borrowers. The theoretical discussion on bimetallism, and its policy ramification, had reached its summit.

The only conceptual tool that was available for *policy purposes* was the quantity theory of money. A reliance on this would have meant a further deepening of the deflationary process and an exacerbation of the unjust income and wealth distributions. He had to devise an alternative vision of the monetary mechanism in such a way that it would yield policy perspectives and tools that would stabilize the price level, whilst preserving consistency with the microeconomics of relative prices in a situation of deflationary dynamics. Thus was born the Wicksellian analogue of the Malthusian mechanism: the discrepancy between the money rate of interest, determined by banking policy, and the natural rate of profit resulting from the capital structure of the production system.

Independently, and motivated by the same events and concerns, Irving Fisher had suggested an alternative mechanism for the interpretation and resolution of the same problem. In a sense, modern macroeconomics is an uncoordinated amalgam of Fisher's expectational mechanism and Wicksell's capital theoretic underpinnings on *Clower's monetary macroeconomic thought-experiments*.

[2]See, however, Velupillai (2009a), for reasonably complete details on the issue of the origins of the word Macroeconomics. In passing it should be stressed that the origins of the word attributed to Jacob Marschak in **The Economist**'s article on *The Other Worldy Philosophers*, on 16 July 2009, is incorrect.

In this chapter our implicit working hypothesis is that the dynamics of Keynesian macroeconomics in Harrod (1936), the sequence analysis of the Swedes, most explicitly formulated in Lundberg (1937) that which has come to be called the 'time-to-build' approach to business cycle theory, but originally in mathematical form encapsulated in the early work by Tinbergen (1931) and Kalecki, and the 'cobweb' tradition, most elegantly broached, in a mathematical mode, by Leontief (1934), were the first successes in the drive *to integrate cycle theory, intrinsically, to macroeconomic theory*, as this subject itself emerged in a definable form in the 1930s. That these theories and their mathematical formulations have been subverted at the frontiers does not mean they have disappeared from the active research agenda of many scholars, working in a variety of traditions that cannot be encapsulated within any kind of equilibrium orthodoxy. Moreover, we would like to assert, quite categorically, that we adhere to the *methodology* of nonlinear, endogenous mathematical modelling of macroeconomic fluctuations. It is our definite belief – going beyond 'opinion' - that *epistemologically*, too, this approach is superior to the dominant linear stochastic approach to modelling macroeconomic fluctuation. In parallel work we have demonstrated, formally, this claim, from the point of view of the epistemology of computation.

In the next section we outline, in a very concise form, the early – essentially confined to the early years of the 1930s – attempts and discussions on the need to integrate cyclical phenomena with economic theory, especially, though not exclusively, equilibrium economic theory. In section 3 we attempt to describe the kinds of ways intrinsically nonlinear macroeconomic theories were mathematised nonlinearly. The next section traces the second stage, when consolidation of both the macroeconomic theory of the business cycle, and its mathematical formalisation, outlined in the chapter came to maturity in the Golden quarter century of Keynesian Macroeconomics, i.e., 1947-1972 – then declined, rose - and, in recent years, seems to have fallen again.

## 2.2 Integrating Cyclical Phenomena with Economic Theory

> "Eine Krisentheorie kann nie die Untersuchung eines abgesonderten Theiles der socialwirtschaftliches Phänomene sein, sondern sie ist, wenn sie nicht ein diletantisches Unding sein soll, immer das letzte oder vorletzte Capitel eines geschriebenen oder ungeschriebenen socialwirtschaftlichen Systems, die reife Frucht der Erkenntnis sämmtlicher socialwirtschaftlichen Vorgänge und ihres wechselwirkenden Zusammenhanges. Daraus geht ein Doppeltes hervor. Erstens, dass jedem wissenschaftlichen System eine andere Krisentheorie entspricht; und zweitens, dass je weniger reif und vollendet das zugehörige wissenschaftliche System ist, desto hypothetischer, gewagter, sogar abenteuerlicher die darauf gebaute Krisentheorie geraten kann. Es is wie mit den volksthümlichen Auffassungen und Erklärungen vom Wesen der Krankheiten, die nicht auf eine solide Anatomie und Phys-

iologie des menschliehen Organismus aufgebaut sind."

Eugen von Böhm-Bawerk: *Review* of *Die Wirtschaftskrisen. Geshichte der nationalökonomischen Krisentheorien*, **Zeitschrift für Volkswirtschaft Sozialpolitik und Verwaltung**, Vol. vii, p. 132[3]

What began as an exercise in attempting a reconciliation between 'theoretical economics' and the phenomenon displayed as 'business cycles', in 1898, became, by the 1930s, the attempt to graft business cycle phenomena to equilibrium theory. Three interrelated, simultaneous, phenomena emerged from the attempt to synthesise traditional static, equilibrium, economic theory with *dynamic method*: business cycle theory, monetary macroeconomic theory (as outlined in an ultra-brief mode in the previous section) and the theory of economic policy (for long also referred to as stabilization policy). Two diametrically opposing *visions* – in the strict Schumpeterian sense[4] – of this attempted synthesis were enunciated by two of the giants of 20th century economics: Simon Kuznets and Friedrich von Hayek, both early Nobel Laureates (in 1971 & 1974, respectively). Kuznets, in a fundamental paper[5], outlining the nature of the synthesis that was being attempted so as to incorporate, in particular, business cycle phenomena that were considered naturally 'dynamic', within the fold of the then orthodox equilibrium economic theory, came out with the radical conclusion:

> "*What [should] be discarded is the notion of a stable or slowly varying equilibrium and the equational system of solving economic problems*. What is substituted for it is a general recognition of the importance of the time element

[3]A free translation by Velupillai would be as follows (where *socialwirtschaftliches* is rendered *economic*, although, perhaps, a direct translation of the word may suggest *social economy*, which is a more a 19th century word/phrase):

> A theory of crisis can never be based on the analysis of one separate aspect of the economy alone. Unless it is to be an amateurish absurdity, it is always the last or last but one chapter of a written or unwritten system of economics, the ripe fruit of the insight obtained from the totality of the economic processes and their interaction. Two implications follow from this. First, that each scientific system requires its own crisis theory, and second, that the less mature and complete the corresponding scientific system is, the more hypothetical, daring, even preposterous the crisis theory built on it will be. This is similar to the popular understanding and explanation of illnesses, which are not based on a solid anatomy and physiology of the human organism.

[4]See p. 41, ff., (Schumpeter (1954)).

[5]In which he also pointed out that Böhm-Bawerk (see also the opening quote in this section), as early as 1898, had taken up this topic (Kuznets (1930), p. 384):

> "The organic relation between business-cycle theory and theoretical economics was stated by Böhm-Bawerk as early as 1898 (in a book review in the *Zeitschrift für Volkwirtschaft Sozialpolitik und Verwaltung*, Vol, vii, p.132)."

It is interesting to recall, as pointed out in section 1, that it was in 1898 that Wicksell's similar concern for the 'organic relation' between *Monetary Theory and Theoretical Economics* – which was, at that time, not specifically identified with 'equilibrium economics' – was first expressed in the international literature (Knut Wicksell, (*1898*, [1936]), **Interest and Prices**, translated by Richard F. Kahn, Macmillan, London).

– a recognition which permits the utilization of the generalized experience of various special investigations in a more complex and a more realistic general theory of economic change. The equilibrium theory, in the limited meaning in which it is retained, will also be enriched, since the general theory of economic change will point out many more important economic factors than have heretofore been included in the equational systems of the mathematical school. *If we are to develop any effective general theory of economic change and any complete theory of economic behaviour, the practice of treating change as a deviation from an imaginary picture of a rigid equilibrium system must be abandoned.*"

Kuznets (1930), p. 415; italics added.

Hayek, on the other hand, suggested that[6]:

"[T]he thesis of Löwe (which remains .... the basis of my own work) that the incorporation of cyclical phenomena into the system of economic equilibrium theory, with which they are in apparent contradiction, remains the crucial *problem of Trade Cycle theory*. .....

..

By 'equilibrium theory' we here primarily understand the modern theory of the general interdependence of all economic quantities, which has been most perfectly expressed by the Lausanne School of theoretical economics."

Hayek (1933), p.33 & p.42; italics added.

It should be noted that for Kuznets[7] it was equilibrium theory that faced the problem of incorporating business cycle phenomena into its framework; the opposite is the case for Hayek. Somewhere in between there was Johan Åkerman, perhaps best characterised as the lone *Schumpeterian*[8] voice, in an otherwise *Wicksellian Sweden*, whose methodological views were refreshingly original in that he also brought into consideration issues of the roles played by deductive and inductive processes of reasoning in equilibrium theory and cycle theory[9]. A representative view of his stance on the problem of integrating the phenomenon of the business cycle with equilibrium theory, on

[6]Quoted in a fractured way, out of context, and inaccurately by Lucas (1981), p.215.

[7]Incidentally, the almost 'universal' reference to Frisch (1933) as the macrodynamic origins of what is now referred to as the Frisch-Slutsky methodology is seriously unfair to Kuznets (1929), who also pointed out that the classic Slutsky work was even referred to by Mitchell (1927); p. 478.

[8]Long before Schumpeterian evolutionary economics, where the cycle was an intrinsic manifestation of the dynamic growth process, was a codified chapter in macroeconomic theory.

[9]It should be recalled that business cycle theory was referred to as *konjunkturtheorie*, as in German, and was differentiated from crisis theory by the use of the word *krisen* for the latter phenomenon. Johan Åkerman's doctoral dissertation, (Åkerman (1928)), is an important document in the history of mathematical business cycle theories, not least because Ragnar Frisch was the official examiner. It is the only document, to the best of our knowledge, by any Swedish economist in the interwar period, where there is an explicit acknowledgement to S.D. Wicksell, the statistician son of the great Knut Wicksell:

which he wrote systematically during the decade late 1920s and the whole of the 1930s, may be gleaned from his superbly pedagogical article in the **Ekonomisk Tidskrift** of 1932 (Åkerman, J. (1932)), where also copious references to his previous writings on the subject is made available. It is clear, even with only rudimentary mathematical mastery of nonlinear dynamics, he was advocating an endogenous, nonlinear, deterministic approach to the modelling of business cycle phenomena, although he did not neglect seasonal factors and, to some extent, both exogenous shocks and psychological factors (although critical of Pigou's stance on this factor in the latter's **Industrial Fluctuations** (Pigou (1927)) also played a role in his desiderata for a formal theory of the cycle within economic theory.

Hicks, in 1933, as, indeed, Kaldor at that time[10], was 'minimising [his] differences from Hayek' (Hicks (1982), p. 28) and went so far as to claim (*ibid*, p. 29; italics added):

> "The great advances that have been made in recent years in our understanding of the Trade Cycle have consisted chiefly of the successful application of economic theory (and especially monetary theory) to the problem of fluctuations. ... The development in our knowledge of the Cycle was thus, from one point of view, a purely theoretical development. It took the form of the construction of a theory of Money that finds a place inside general economic theory rather than outside it.
>
> The object of the present chapter is to make a small contribution to this theoretical development by enquiring into the place that is to be occupied in the new theory of Money and of the Cycle by the central notion of pure economics: *the concept of equilibrium*."

That this 'new theory of Money' was untenable, both from the point of view of a seamless integration with economic theory *and* as *a foundation for a cycle theory within equilibrium economic theory*, was the message of the two classics by Myrdal (1931), and Sraffa (1932), but it took Hicks more than a quarter of a century to acknowledge the twin messages of the great Swede and the Cambridge Italian maestro!

With the benefit of melancholy – at least from our point of view – hindsight, we now know that the Hayekian vision, in the form of old wine in new bottles, prevailed and is the dominant current approach; the enlightened and challenging vision of a

> "Under min studietid vid universitetet i Lund har professor Emil Sommarin och professor S.D. Wicksell visat ett livligt intresse för min undersökning och givit mig många värdefulla råd och anvisningar, för vilket jag härmed får uttrycka min stora tacksamhet."

[I express my immense gratitude to Professor Emil Sommarin and to Professor S.D. Wicksell for the lively interest they have shown for my investigations and for having given me valuable advice and instruction during my stuies at Lund (KVV's translation of the original Swedish)]. During Velupillai's early years as a doctoral student at the University of Lund, Johan Åkerman was an occasional auditor at special advanced seminars in the department of economics. He was, by then, almost totally deaf and was always accompanied by his wife, who helped him interpret any talk.

[10]Kaldor was the joint translator (together with H.M. Croome) of Hayek's classic *Monetary Theory and the Trade Cycle*, Hayek (1933).

dynamic theory free of viewing change as simply 'a deviation from an imaginary picture of a rigid equilibrium system', now survives only in the underworlds of modern day reincarnations of Karl Marx, Silvio Gesell or Major Douglas[11]. Our adherence to this underworld is uncompromisingly complete. It is based on exactly the reasons for which Kuznets advocated the abandonment of equilibrium economics and its formalisations.

Formalisation of *dynamic method*[12] that could encapsulate proper disequilibria, the existence of multiple equilibria and even lack of any equilibria to which the system may or may not tend, or around which fluctuations may or may not recur – whether as small deviations or large and sustained departures, was the sought after criterion such that it was possible to incorporate it coherently with the the formal systems of general equilibrium equations of the real economy of orthodox theory. Hence, *dynamic method*, formalised as ordinary differential, difference or mixed difference-differential equations, and, very occasionally, also as differential *inequalities* were to be made an adjunct of, or an integral part of, the systems of equilibrium equations, for which, then, solutions would be sought in a similar manner to traditional methods (*whatever they may have been*). The first, tentative, steps – methodologically – were simple additions of time subscripts to standard variables and a claim that the consistent equilibrium formulation and solutions to this new system of equations was an answer to the puzzle of synthesising 'change' or 'dynamics' and 'static equilibrium'.

In this chapter, we concentrate on those macroeconomic business cycle theories that tried to encapsulate *dynamic method* in terms of *nonlinear* differential, difference and mixed difference-differential equations such that the solutions – *the attractors* in the language of dynamical systems theory – had the potential to display *multiple*, *unstable*, *endogenously generated*, *equilibria*, where the trajectories in any relevant *basin of attraction* would be consistent with well defined economic disequilibria. This is the standard approach of the nonlinear, endogenous, business cycle theories, when appropriately formalised. However, we shall also suggest that theories of the business cycle, for example that associated with *Swedish Sequence Analysis*, may not be consistent with generalised nonlinear dynamical systems modelling. This is because a literal, purist, interpretation of *Swedish Sequence Analysis* suggests that they were seeking to model economic dynamics of a kind that was not associated with any equilibrium. We suggest that this interpretation is not consistent with modelling in terms of any kind of dynamical system and one has to seek, at least in the first instance, a formalism for dynamics that cannot be associated with any kind of differential, difference or mixed

[11] Paraphrasing Keynes (Keynes (1936), p. 32; italics added):

> "The great puzzle of Effective Demand .... could only live on furtively, below the surface, *in the underworlds of Karl Marx, Silvio Gesell or Major Douglas*."

[12] We have in mind the idea of formalising an intuitive concept in a precisely defined scientific context. This is similar to the way Alan Turing, and others, formalised the intuitive notion of *calculability* with the precise notion of *computability*. The intuitive notion of continuity is still to find a definitive formalisation, despite claims to the contrary by Bourbaki, and others.

difference-differential equation system.

This observation is, in our opinion, dual to Samuelson's important remark on the existence of dynamical systems that cannot be associated with any (useful) maximum principle (Samuelson (1971)) and he gave the homely example of the (nonlinear) multiplier-accelerator model of the business cycle to illustrate the point.

These two principles of modelling – nonmaximum and nonequilibrium economic dynamics – will form the touchstone for the structure and content of the entire work, and their crucial roles will emerge only as the whole tapestry is completed. This part of the story is but one aspect of the final tapestry envisaged.

Finally, the problem setting itself *should* be provided by a background narrative, at the outset, of *two parallel stories*: one, an outline of the business cycle theories that provided the foundations for nonlinear, endogenous, dynamic modelling; two, a concise outline of the parallel development of nonlinear dynamics, but extending backwards to Poincaré, and coming down the years till the dawn of the era of dynamical systems theory – i.e., from Poincaré and the elder Birkhoff, via van der Pol and the Andronov school, and ending with Cartwright-Littlewood, Levinson and the Lefschetz school. This is an outline of a 70-year history[13], as the backdrop for the kind of mathematical formalisms used in the dynamic method of the theories of nonlinear, endogenous, business cycle theories. The interaction between the formal dynamics invoked by the macroeconomist and that being developed by the mathematician did have some felicitous outcomes, and we will highlight some of them. But this chapter will, inevitably, be crippled by leaving out the parallel development in the mathematics of nonlinear dynamics, as itself emerged from nonlinear oscillations theory to become dynamical systems theory. This latter story will also form a part of the completed tapestry.

From the strictly macroeconomic point of view the following fourteen classics will provide the textual foundations on which we will outline the emergence of nonlinear, endogenous, business cycle theories (all of them produced during the 1930s): Tinbergen (1931), Kalecki (1939), Fisher (1933), Hayek (1931), Hawtrey (1931), Myrdal (1931), Frisch (1933), Hicks (1982), Leontief (1934), Keynes (1936), Harrod (1936), Lundberg (1937), Lindahl (1939) and Schumpeter (1939). It is not without significance that eleven of these classics emanated on 'this side' of the Atlantic and three were by the members of the 'Stockholm School'.

Connoiseurs of the history of business cycle theories may wonder at the absence of many classics – in particular the three *League of Nations* commissioned studies by Haberler (1937a) and Tinbergen (1939a,b). To them our answer is that this is *not* a study of the origins and development of *business cycle theories*; it is, instead, a study

[13]Our 'model' here is the excellent expository historical narrative by David Aubin and Amy Dahan Dalmedico (Aubin and Dalmedico (2002)). The essays by Anosov, Arnold, Il'yashenko, Shil'nikov and Sinai in (Bolibruch et al. (2005)) were also important for the way we structure our own story. Finally, we are also deeply influenced by the 'insider's accounts' given in a series of papers by Mary Cartwright, spanning almost forty years of the history of how nonlinear oscillations theory became, first, topological dynamics and, eventually, dynamical systems theory (Cartwright (1952), Cartwright (1964) & Cartwright (1974)).

of the way a mathematical mode was introduced to study the nonlinear, endogenous, vision of business cycle theory.

From the point of view of the differential, difference and mixed difference-differential equations that were canonical in the formalisation of the dynamics of the emerging nonlinear, endogenous, business cycle theories, the following played crucial roles[14]:

The van der Pol equation:

$$\ddot{x} - k(1 - x^2)\dot{x} + x = 0 \tag{2.1}$$

Equations of the Liénard type:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{2.2}$$

studied in the Liénard Plane:

$$\dot{x} = y - F(x), \dot{y} = -g(x) \tag{2.3}$$

The generalized, forced, van der Pol equation:

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = p(t) \tag{2.4}$$

The Rayleigh equation:

$$\ddot{x} + \eta \left( -\dot{x} + \frac{\dot{x}^3}{3} \right) + x = 0, \{0 < \eta < \infty\} \tag{2.5}$$

The Logistic Map:

$$x_{n+1} = \lambda x_n (1 - x_n) \tag{2.6}$$

The difference-differential equation:

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} y^{(\nu)} (x + \mu) = 0 \tag{2.7}$$

The second-order difference equation:

$$\begin{gathered} y_{n+1} = F(y_n, y_{n-1}) \quad \forall n = 0, 1, 2, .... \\ \text{where } F : \mathbb{R}^2 \to \mathbb{R} \text{ \& given initial conditions } y_0, y_{-1} \in \mathbb{R} \end{gathered} \tag{2.8}$$

The first five encapsulated the business cycle theories of Fisher, Keynes, Harrod, Schumpeter and Hawtrey; the sixth, models of the 'cobweb' type, as in Leontief; the seventh, in various specialised forms, the business cycle theories of Tinbergen, Kalecki

[14]In all of the cases, when used in macrodynamic models of the business cycle, $x$ and $y$ signified either aggregate output, income or sectoral (for example agricultural in cobweb models) output values. The nonlinearities encapsulated in the functions $f(x), g(x)$ and $F(x)$ represented the nonlinear accelerator or delayed adjustment of an independent variable.

and Frisch; the last one, with variously specified functional forms for $F$, encapsulated variations on the dynamics of *Swedish Sequence Analysis* (although we do not fully subscribe to this interpretation of their 'dynamic method'), on the one hand, and the Hicks version of a Keynes-Harrod model of the trade cycle.

## 2.3 Excitement at Birth: 1928 – 1957

> "van der Pol believes[15] that even periodic business cycles show a certain analogy to the relaxation oscillation of a physical system. The essential condition for such oscillations is negative damping for small deviations and a rather rapidly increasing positive damping for large deviations from the equilibrium position. The psychological response of certain groups of people to changing business conditions shows doubtless some analogy to the behaviour of mechanical systems capable of relaxations oscillations."
>
> von Karman (1940), p.624.

How reliable are 'analogies' in devising fruitful models in economics in general and in economic dynamics in particular? Is it sufficient to rely on analogies at a phenomenological level to justify mathematical modelling of a particular variety and then to seek behavioural and other basic hypotheses to justify that particular kind of formalization? Arguably, no field of formal economic analysis has been subject to serious and systematic 'analogical thinking' that has led to mathematical formalizations of one sort or another in more fruitful ways than business cycle theory.

From time to time, distinguished mathematicians, physicists, biologists and other natural scientists make important forays into economics, make fundamental contributions that changes the face of the subject in profound ways, and they themselves return to their own, original disciplines, whilst the economists and economics continue to reap the results of such beneficial influences for years on end. von Neumann, Wald, Mandelbrot, Smale, Gale and a few others come immediately to mind as outstanding examples of such remarkable individuals. There are, of course, less obvious successes and, equally, also less edifying examples of such attempts. The early 30s was a fertile time for this kind of activity and economic theory was at the dawn of becoming almost swamped by a wave of mathematizations that was to change its character beyond recognition forever. Two outstanding natural scientists - one an applied mathematician, in the sense in which the phrase was commonly used a few decades

[15] For example:

> "Returning to a general consideration of relaxation oscillations many more instances of these oscillations can be cited .. . Even the periodic reoccurrence of economical crises and epidemics may possibly follow similar laws."
>
> van der Pol (1934) p. 1081.

ago, another a classic polymath - suggested a particular formalization for the modelling of the macroeconomic phenomenon of business cycles: Philippe Le Corbeiller and J.B.S. Haldane. The former advocated the formalization of business cycles as relaxation phenomena in a non-linear dynamical system; the latter advocated the use of integral equations to formalize similar phenomena. Their individual advocacies reflected the particular concerns that had, at that point in time, occupied their fertile minds: maintained oscillations in electrical and mechanical units in the case of Le Corbeiller and evolutionary biological phenomena in the case of Haldane. We try to tell the circumstances that led to Le Corbeiller's innovative suggestion being taken up by an economist who, subsequently, pioneered the non-linear approach to business cycle modelling. However, we do not mention Haldane's name in these contexts frivolously! The same economist, in a later 'incarnation', was directly and personally influenced by Haldane to further the non-linear cause in macrodynamic modelling in even more dramatic ways. That, too, forms a lining in this story - but only as a kind of icing on the cake. Perhaps the implicit message in the way the story will be constructed and narrated is that fertile cross-disciplinary harvests require timely seedings in receptive soil to be nurtured by men and women of imaginative, tenacious and audacious temperament. This is because harvests take time to mature and blossom.

One important theme here is to tell the story of mathematical business cycle theories as adventures in non-linear dynamics. Thus, it will not be a complete story - of the past, the present or possible future - of mathematical business cycle theories; only the part that embraced and was fertilised and enriched by being modelled as non-linear dynamical systems.

In this section, a succinct description of the way nonlinear dynamics was introduced into formal business cycle theory is given. There is a discussion of the way a purely economically motivated hypothesis was fruitfully formalised as a *characteristic* underpinning a special case of Liénard's equation. The serendipitous way Goodwin and Le Corbeiller came to meet and collaborate is also described.

> "[E]conomists will be led, as natural scientists have been led, to seek in nonlinearities as explanation of the maintenance of oscillation. Advice to this effect, given by Professor Le Corbeiller is one of the earliest issues of this journal, has gone largely unheeded"
>
> Goodwin (1951), p. 1.

The thirty years in consideration was a period of flourishing and fertile research in the mathematical modelling of business cycles. Our choice of precisely these initial and terminal years are motivated by *ex-post* considerations. To the best of our knowledge, it was in 1928 that the idea of interpreting economic cycles as being generated by a *non-linear dynamical system* capable of *relaxation oscillations* was first hypothesized:

> "The present writer would like to point out that the applicability of the principle of relaxation-oscillations to economic cycles was first emphasized by

> him in 1928 [at the May 7, 1928, Meeting of the Batavian Society of Logic Empirical Philosophy] in a discussion following a paper read by Messers. Van der Pol and J. van der Mark on 'The Heartbeat Considered as a Relaxation-Oscillation, and an Electrical Model of the Heart."
>
> Hamburger (1934), p. 112[16]

The terminal year is defined as the dawn after the twilight characterised by the classic by Hugh Hudson (1957) which summarised, in elegant prose and classic diagrammatic exegesis, the nonlinear, endogenous, business cycle theories that had became, for that time, the standard approach.

We will outline the idea that invoking *non-linear* models capable of *relaxation oscillations* to encapsulate economic data had to rely on reasonably reliable empirical evidence of a particular kind, historically and theoretically substantiated:

- evidence of the *persistence of fluctuations*;
- of asymmetric cycles (in the sense of time series of aggregate variable displaying *significantly non 'sinusoidal' behaviour*);
- of *multiple equilibria*;
- of, at least, *local instability of equilibria*;
- of significant intrinsic non-linearities in economic relationships or behaviour in variables defining macroeconomic fluctuations.

The five desiderata, *persistence*, *asymmetry*, *multiple equilibria*, *instability* and *non-linearity* as criteria for a model of macroeconomic fluctuations implied, in turn, an endogenous cycle. The key *economic hypotheses* underpinning these ideas (multiple equilibria, instability and non-linear behavioural relations) and the *stylized facts* (persistent and asymmetrical fluctuations) underlined departures from orthodox visions of the workings of the economic system in advanced industrial economies. Thus the instability hypothesis meant that deviations from equilibria did not call forth automatic self-adjusting mechanisms of the metaphorical world of the *invisible hands*. The hypothesis of multiple equilibria implied, in conjunction with the loss of self-adjustment capabilities, that economies could, for endogenous or exogenous reasons, end up in undesirable basins of attraction, out of which the system could not, of its own accord, extricate itself and, hence, signalled an active role for policy. That, in turn, called forth

[16] Velupillai's discovery of Hamburger's work is as follows: Concisely summarised, it was the late Professor Sukhamoy Chakravarty who, during a personal conversation in Cambridge in 1982, referred Velupillai to Hamburger's claims to priority in this area. Some of this information was summarised, after he passed it on to her, in the doctoral dissertation of his brilliant student, Serena Sordi.

a theory of macroeconomic policy to be developed within the same context[17]. Instability, multiple equilibria and a theory of policy within a framework of growth and business cycles in an advanced industrial monetary economy were themes broached by, and models for them were crafted by, four pioneering economists: Wicksell, Lindahl, Keynes and Harrod. None of them, however, fashioned an explicit mathematical model. We conjecture that none had the theoretical technology to construct meaningful unstable, multiple equilibria, models mathematically. Their deep economic insights, expressed in every one of their cases in exceptionally elegant prose[18], left no doubt as to the necessity of non-linear tools to encapsulate their fertile ideas. It was left to their students and near contemporaries - in the chronological order in which their works came to be published, Erik Lundberg, Nicholas Kaldor, Richard Goodwin and John Hicks - to realise that aim.

Several authors, in the period considered, appealed to one or more of the above desiderata. However, to the best of our knowledge, only these four invoked the whole set as defining criteria for a model of macroeconomic fluctuations. Of these four, the first and the last, Erik Lundberg and John Hicks, framed their models in terms of piecewise linear relations; the second, Nicholas Kaldor, described his economic model graphically and set out the defining economic relationships algebraically in non-linear functional forms without, however, deriving the final, crucial, non-linear equation which would encapsulate the dynamics and show the nature of its underlying relaxation oscillation behaviour. This significant task, for the Kaldor economic model, was first accomplished by Takuma Yasui only in 1952-3 and it was shown, in a masterly pedagogical piece of analysis, that the Kaldor (1940) non-linear *Model of The Trade Cycle* implied a formalism in terms of the *van der Pol equation*. Only Richard Goodwin developed a formal mathematical macrodynamic model, explicitly satisfying every one of the criteria listed above, and derived the final, formal, equation - as it happened it was the *Rayleigh form for maintained oscillations* - in one fell swoop, so to speak.

These four supreme macroeconomic theorists did not invoke these desiderata arbitrarily or in an *atheoretical* vacuum. The intrinsic structure of the theoretical foundations on which each, in their own distinctive way, erected their respective business cycle models implied non-linear mathematical equations encapsulating, naturally, the five desiderata. It was not as if a non-linear equation was chosen, *a priori*, and, then, economic assumptions were tailored to fit the chosen equation; it was, instead, quite the other way about and according to the noblest Ockhamian traditions of model building and theorising. Indeed, it was precisely because these outstanding theorists went about the construction of their theoretical model of the business cycle in this traditional, noble, way that non-linear macroeconomic modelling of business cycles had many false starts, several still-born episodes and even unfortunate and unfounded dismissals, at least in the period under consideration. None of them, except Goodwin, ever managed to master the mathematical sophistication required for the understand-

[17]The choice between a van der Pol formalism and a Rayleigh formalism for non-linear business cycle theory had, as its economic backdrop, a precise stance on policy.

[18]In Swedish of impeccable clarity and admirable directness, in the case of Wicksell and Lindahl.

ing of the full formalism of non-linear dynamics. That Goodwin became a master - at least of some aspects of this fascinating area - was almost wholly due to the personal tutoring he received from Philippe Le Corbeiller.

Lundberg, Kaldor, Goodwin and Hicks had, each of them independently, constructed non-linear business cycle models of innovative and imaginative structure and each had their own sources of theoretical inspiration. Lundberg built on Wicksell and the contemporary work of his Swedish macroeconomic colleagues, particularly Erik Lindahl, Gunnar Myrdal and Dag Hammarskjöld; Kaldor subtly synthesised the works of Keynes, Harrod and Kalecki; Goodwin combined, with outstanding innovative imagination, elements of Schumpeter, Keynes and Harrod; Hicks, in his own, characteristic, low-key way, seemed to have relied on modified aspects of Keynesian and Harrodian elements to construct his piecewise linear model of the trade cycle[19]. In passing, it must be noted that modern studies on non-linear macrodynamics, particularly when it relates to business cycle theory, have had a tendency to pay justifiable *homage* to these pioneers - with the exception of Lundberg.

Thus, before concluding this section, four issues must be faced and resolved.

1. First of all, why did Hamburger's pioneering conjectures fail to elicit any response at all?

2. Secondly, why is Lundberg's impressive and highly original work *not* bracketed together with Kaldor, Goodwin and Hicks as one of the pioneers of non-linear business cycle modelling?

3. Thirdly, what of many other significant calls for the 'non-linearization' of macrodynamics in general and business cycle theory in particular, of this period, and why didn't any of them - some by outstanding theorists of the profession such as Paul Samuelson and Nicholas Georgescu-Roegen - lead to serious modelling exercises, satisfying the five desiderata enumerated above?

4. Only one such 'clarion call', that by Ph. Le Corbeiller, elicited any response at all, by economic theorists - why?

[19]It is interesting to recall the reflections of one of the pioneers of macroeconometric model building on the theoretical sources that inspired them:

> "The econometric models that I have constructed as practical tools for analyzing or predicting the economies of the United States, Canada, United Kingdom, and Japan have been based on combinations from the theoretical models of Marx, Kalecki, Keynes, Lange, Hicks, Kaldor, Metzler, Goodwin, and others. .... Actually most models in existence today could be decomposed into ideas first found in the models of Kalecki, Kaldor, Metzler, and Goodwin."
>
> Klein (1964) p.189.

It is interesting that Metzler's name appears in both lists. The precise role of the particular contribution by Metzler to which Klein refers, in the 'subverting' of the piecewise linear Lundberg model, is discussed above. The only surprise in the lists above is the absence of Harrod's name.

Hamburger's imaginative and original line of economic research was sadly terminated by the tragedy of the holocaust. Despite the valiant empirical case he tried to make to substantiate his claims that economic fluctuations should be modelled as the relaxation oscillations of a nonlinear differential equation, his work did not attract much - or, indeed, any - attention in the vibrant efforts that were being made, throughout the 30s, to model the business cycle. 'Emphasizing the applicability of the principle of relaxation-oscillations to model economic cycles', is one thing; to actually build a formal mathematical model of aggregate fluctuations, *ab initio* from economic principles, encapsulated in the dynamics of a nonlinear (or even a linear) system of equations capable of relaxation oscillations, is quite another thing. Hamburger pointed out (*ibid*) that his 'suggestion .. was ..corroborated by results indicated in [his] paper[s]' in Dutch and French, published, respectively, in 1930 and 1931[20]. However, the 'corroboration' is simply by way of appeal to descriptive similarities of crude statistical plots of time series pertaining to arbitrary economic variables.[21] Although it is surprising that his innovative suggestions were not taken up in serious research circles, the reasons for the failure of the modelling effort he wished to promote to take-off are equally unsurprising. Except for what may be called a tendentious preoccupation with the importance of relaxation oscillations, Hamburger provided no unifying economic theoretic modelling principle within which a theory of the business cycle could be embedded and at least a few of the desired criteria satisfied[22].

The full details of Lundberg's model of the inventory cycle cannot be discussed here[23]. All we shall do here is to report the main conclusion. Lundberg's construc-

[20] As Hamburger (1930) and Hamburger (1931). The van der Pol equation does appear in both of these papers (as equation # 7, on p.5, in the former and in footnote 7, p.6 in the latter) in the form:

$$\frac{d^2y}{dt^2} - \alpha\left(1 - y^2\right)\frac{dy}{dt} + \omega^2 y = 0 \tag{2.9}$$

Figures 1 to 3 (in both papers) show the increasing loss of (nearly) sinusoidal behaviour of the time variation of $y$ for increasing values of $\alpha$ (0.1, 1.0, 10), presumably for a given value of $\omega$ (unspecified in the papers). The equation and the simulations are supplemented by a couple of pages of a discursive discussion on the meaning of relaxation oscillations in the abstract.

[21] For example, figure 4 plotting the monthly variation in sales in so-called 'Five- and ten-cent chain stores' in the US, for the five years from 1921 to 1925, does show a remarkable consistency with a possible underlying relaxation mechanism. The hard work is to go from suggestive statistics to the underlying model and that does not seem to have exercised Hamburger's considerably fertile mind. I have devoted more space than warranted on the marginalised work of Hamburger simply because I feel his untimely demise may have deprived the economic profession of an unusual talent that may have helped speed up the introduction of nonlinear mathematical modelling to the art of business cycle theorising much sooner than happened in his absence. The only reference in the mainstream economic literature to anything by Hamburger is the one by Tinbergen in his famous *Survey* (cf. Tinbergen (1935), footnote 71, p.288).

[22] For the same reason we have not gone into details of the contributions by Marrama and Palomba to the nonlinear macrodynamic tradition. Our friend, Professor Giancarlo Gandolfo's sterling effort on this front may be referred to, for the interested reader (for example, Gandolfo (2010)).

[23] Readers wishing to get a partial idea of what is meant here could profitably read Berg (1991) and Baumol (1991).

tion was of a linear, unstable model of inventory cycles, made to generate bounded fluctuations by building in natural, economic, constraints that would act as bounds on unlimited expansion and catastrophic contractions. In effect, the formal model was in terms of a piecewise linear difference equations. Lloyd Metzler endogenised the bounds and converted the model into a completely linear system. Why did he do it? We had to wait thirty years to get a straight, candid, answer - as always with characteristic directness from Paul Samuelson:

> "In leaving Frisch's work of the 1930's on stochastic difference, differential and other functional equations, let me point out that a great man's work can, in its impact on lesser men, have bad as well as good effects. Thus, by 1940, Metzler and I as graduate students at Harvard fell into the dogma - I use the word 'dogma' in the non-perjorative sense of Crick's dogma on DNA and RNA, as a leading hypothesis - that **all economic business-cycle models** should have *damped* roots. .... [W]hat was so bad about the dogma? Well, **it slowed down our recognition of the importance of non-linear autorelaxation models of the van der Pol-Rayleigh type**, with their characteristic amplitude features lacked by linear systems."
>
> Paul Samuelson (Samuelson (1974), p.10; bold emphasis added.

Lundberg's non-linear, unstable, model of the inventory cycle was, after its unfortunate transmogrification by Metzler, forever cast into the linear mould, until recent, sporadic, revivalist attempts, with hardly a ripple in mainstream thought or practice.

In 1933, in the very first volume of *Econometrica*, Philippe Le Corbeiller had written, suggestively and challengingly:

> "Le problème des crises, et plus généralment des oscillations des prix, est assurément l'un des plus difficiles de l'Économie Politique; il ne sera sans doute pas de trop, pour approcher de sa solution, de la mise en commun de *toutes les ressources de la théorie des oscillations* et de la théorie économique. C'est pouquoi j'ai pensé pouvoir vous présenter un compte-rendu succinct d'un avance récente, que je crois importante, de la théorie des oscillations: celle apportée au problème des *systèmes autoentretenus par la découverte des oscillations de relaxation*, due à un savant hollandais, le Dr Balth. van der Pol."
>
> \- Le Corbeiller (1933), pp.328-9; italics added.

The suggestion was not one of those famed 'bolts from the blue'. First of all, by the time it came to be published, it had been in the hands of, Ragnar Frisch, the Editor of *Econometrica*, for over an year.[24] Secondly, there is ample evidence, even at

[24]Unfortunately, the University of Oslo library where, at present most of the Frisch Archives are deposited, do not allow copying of personal letters without the written permission from descendents on

those very early stages in the development of the analytic apparatus of (non-linear[25]) *relaxation oscillations*, that Le Corbeiller was deeply interested in, and committed to, an investigation of diverse phenomena in the natural and physical world that were amenable to an interpretation in terms of a non-linear formalization emphasising this aspect in its dynamics.[26] Thirdly, here we are conjecturing without hard evidence, it is more than likely that his lifelong intimacy and friendship with van der Pol had already begun in the late 20s. He may, therefore, have been aware of Hamburger's remarks on the van der Pol-van der Mark paper, via personal discussions or communications from van der Pol himself. We believe a little more research effort may close this minor gap and help present a complete picture of the background to Le Corbeiller's fascinating and suggestive paper. There is no mention of possible interpretations of economic fluctuations as relaxation oscillations in his 1931 monograph, the contents of which were given as seminars in May, 1931. Frisch had received[27] a copy of the first draft by July, 1932. Sometime in the 14-month period between these two dates, Le Corbeiller had conceived and written this pioneering paper. The source of the inspiration remains to be discovered.

To the best of our knowledge, there are only three explicit references to Le Corbeiller's call for a non-linear, relaxation oscillation, approach to the modelling of economic fluctuations: Paul Samuelson in his path-breaking monograph, *Foundations of*

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both sides of a correspondence! Many of the letters between Le Corbeiller and Frisch, particularly from the former, are in handwriting that is indecipherable without expert help. On 12 July 1932 Frisch wrote as follows to Le Corbeiller (typewritten):

> "My dear Professor Le Corbeiller,
>
> Your manuscript 'Les systremes autoentretenus....' has been referred to me as Editor of the newly established journal 'Econometrica', the journal of the Econometric Society. If this paper has not been published elsewhere and if you do not plan to have it published elsewhere, I shall be glad to accept it for publication in an early issue of 'Econometrica'. Please drop me a line about this at your earliest opportunity.
>
> Sincerely yours,
>
> Ragnar Frisch"

Le Corbeiller replied, with a handwritten note, from Paris, three days later, expressing his gratitude for the honour Frisch was bestowing upon him with the proposal to publish his piece.

[25] Lest the unwary reader think we are being facetious with the qualifying 'non-linear', we must point out that, in economics, an early attempt at applying the ideas underlying relaxation methods emphasised linearity. We shall deal with this later, in this paper.

[26] This is eminently clear in his elegant booklet of 1931 (Le Corbeiller (1931)), based on Seminars given at the *Conservatoire National des Arts et Métiers* on 6-7, May, 1931. In particular, the concluding section, sub-titled Aperçu historique et conclusion (pp.43-5), although the whole work reflects the mind of a scientist with an admirably broad vision of natural and physical phenomena. It will not come as a surprise to anyone familiar with this beautiful little exposition that this fertile mind saw the possibility of a fruitful interpretation of fluctuating economic phenomena in terms of non-linear relaxation oscillation mechanisms as the underlying cause. The significant step of identifying these mechanisms in terms of meaningful and incontrovertible economic factors had to wait another decade and a half, much due to the personal efforts of Le Corbeiller himself, albeit indirectly.

[27] Although through which channels is still a mystery.

*Economic Analysis* (Samuelson (1947)); Georgescu-Roegen in one of his contributions in the **Cowles Foundation Monograph** on *Activity Analysis of Production & Allocation* (Gorgescu-Roegen (1951)) and, finally, Richard Goodwin (1951). It was only this latter work that directly took up the challenge posed by Le Corbeiller and codified into a usable formalization, within standard macroeconomic theory, a model of the business cycle in a theoretically sound and empirically implementable way.

Paul Samuelson simply catalogued some possibilities for mathematically modelling endogenous business cycles using non-linear differential and difference equations, in a brief section of two and a half pages, in his monumentally influential book of 447 pages. Perhaps the very fact that a voice as mathematically competent as Samuelson's, expressing that a non-linear, relaxation oscillation, approach to mathematical modelling of business cycles entails 'formal difficulties of solution ... so great that very much remains to be done' (ibid, p.340), immediately after a reference to Le Corbeiller's paper, may have diverted the profession's attention away from the potential gains that may have been available with a little effort. Apart from this brief and wholly discouraging reference to Le Corbeiller, there are discursive remarks on general properties of non-linear dynamical systems, with explicit references to van der Pol's equation, without, however, any indication or attempt at encapsulating meaningful economic hypotheses in a mathematical formalism that may have resulted in such an equation.

Georgescu-Roegen opens his illuminating and interesting paper with an explicit reference to Le Corbeiller's pioneering role in emphasising the relevance of 'relaxation phenomena as a model for business cycles', (ibid, p.116). He, then, goes on:

> "However, Le Corbeiller's suggestion has found little echo among economists, and the literature shows only sporadic references to his paper. Paul A. Samuelson .., speaking of this possible approach, admits that practically nothing has been done along this line. The only economic problem which could be regarded as having something to do with relaxation is the famous cobweb problem, but this has been developed independently of any relation to the concept of relaxation'
>
> [ibid, p.116]

Georgescu-Roegen's attempt at introducing relaxation phenomena in economic dynamics took the unusual form of emphasising the discontinuity residing in them by highlighting the fact there were two time-phased regimes encapsulated in the system. He, then, interpreted all attempts at encapsulating the discontinuity within one functional equation, such as van der Pol's, as 'veiling the real meaning of relaxation, which is the discontinuity of the regime'. He went on, therefore, to consider the two regimes formalised as two separate systems of linear differential equations. There was, therefore, no scope for taking seriously the full message of Le Corbeiller's challenge and, indeed, like Samuelson's reference to it, had the unfortunate consequence of diverting the attention of the business cycle theorist away from it.

The first formal attempt at a fully developed *non-linear relaxation oscillation* mathematical model of the *The Business Cycle as a Self-Sustained Oscillation*[28] was presented by Richard Goodwin at the Cleveland Meetings of the *Econometric Society*, on 30 December, 1948, (Goodwin (1949)). The full paper was published subsequently in the same Journal as the lead article in the first issue of 1951 (Goodwin (1951)). The mathematical model of the business cycle presented in this paper was the first fully-fledged formalization of the phenomenon that satisfied all the five criteria discussed above: *persistence*, *asymmetry*, *multiple equilibria*, *instability* and *non-linearity*. Le Corbeiller's role in the development of the work that enabled Goodwin to produce this pioneering paper is evident in the footnote to the lead quote of this section (above):

> "My debt to Professor Le Corbeiller is very great, not only for the original stimulation to search for the essential nonlinearities, but also for his patient insistence, in the face of the many difficulties which turned up, that this type of analysis *must* somehow be worked out."
>
> ibid, p.2; italics in original

## 2.4 Towards Consolidation, Decline and Renewal

> "Certainly we do not want a theory of the cycle which clamps the facts into a vice; but this theory [which Frisch has called the theory of *erratic shocks*[29]] ... does not explain *enough*."
>
> John Hicks (1950) pp. 90-1; italics in the original.

In no uncertain terms, based on lucid economic and mathematical reasoning, Hicks pointed out (Hicks (1950), p. 91) that:

> "[T]he theory of damped fluctuations and erratic shocks proves unacceptable; but if we reject it, what is the alternative? There is an alternative ... "

[28] In view of the fact that Goodwin, in his own celebrated non-linear model of the business cycle, emphasised the Rayleigh rather than the van der Pol equation, it may be of interest to recall the title of the pioneering paper by Lord Rayleigh in which that system was developed: *'On Maintained Vibrations'* (Strutt (1883)). It was, perhaps, not a coincidence that, forty three years later, van der Pol's classic paper, '*On Relaxation Oscillations*', was also published in the same Journal (van der Pol (1926)). Incidentally, Marshall was Second Wrangler to Lord Rayleigh in 1865 and, for those numerologists interested in coincidences, 1883 was, of course the year Keynes and Schumpeter were born and Marx died! The non-linear business cycle theories in discussion in this paper rely, to some extent, on the economic theories of Marx, Keynes and Schumpeter. Some substantiation for this statement can be found in the first footnote in Goodwin (1951) and the last sentence of the second footnote of the same paper.

[29] Or, in Richard Day's more felicitous, if slightly less complementary, phrasing: the theory of '*ad-hoc shockeries*' (cf. Day (1992), p. 180).

The 'alternative' is, of course, the subject matter of this chapter: *non-linear theory*.

The first 'Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel'[30] was shared by Tinbergen and Frisch in 1969. The citation for Frisch stated that he was awarded the Prize 'for having developed and applied *dynamic models for the analysis of economic processes*'. Thirty five and forty two years later, we read that the 2004 Prize was to be shared by Prescott with another Norwegian, Finn Kydland, and the 2011 Prize was to be shared by Sargent and Sims. The former were awarded it 'for their contributions to *dynamic macroeconomics*: the time consistency of economic policy and *the driving forces behind business cycles*'. The latter award was for ostensibly different contributions – although by simply changing 'for their contributions to *recursive* macroeconomics', rather than '*dynamic* macroeconomics', nothing else would have changed. The metaphor of the *rocking horse* was the cementing concept that unified the mathematical methodologies underpinning Frisch's 'dynamic models for the analysis of economic processes' and the Kydland-Prescott real business cycle models of 'dynamic macroeconomics' (and the underpinning for Sargent's so-called 'recursive macroeconomics'). That much maligned metaphor was incorrectly attributed, by Frisch (cf. Frisch (1933), footnote 5, p.178) to Wicksell's famous lecture in Oslo, to the Stats$\phi$konomisk F$\phi$rening, on May 6, 1907 (cf. Wicksell (1907)). No amount of fine-toothed combing of that fine lecture will unearth any reference to a rocking (or, more appropriately, an unrocking[31] horse. Wicksell invoked the metaphor of the rocking horse in a review of an obscure and best-forgotten book titled *Goda och Dåliga Tider*[32] by a long-forgotten minor Swedish economist by the name of Karl Petander (cf. Wicksell (1918), p. 71, footnote 1)[33].

Hicks was, of course, not alone in finding the 'the theory of damped fluctuations

[30]Sometimes, misleadingly, referred to as the *Nobel Prize in Economics* and placed, incorrectly, on a par with the those awarded for Peace, Literature, Physics, Chemistry and Medicine & Physiology. Surely, it would have been more appropriate for the Bank of Sweden to follow the practice of the Mathematicians and award the equivalent of a Fields Medal – say, calling it a *Wicksell Medal* – to honour and celebrate excellence in economics!

[31]Zambelli (1992)) has shown, unambiguously and convincingly, that Frisch's 'rocking horse' does not 'rock'. It is a pity that this exceptionally careful and detailed analysis of the untenability of the numerical underpinnings of Frisch's economic assumptions, such as implausible initial conditions and unsustainable historical trajectories, have received hardly any attention in the macrodynamic profession.

[32]'Good and Bad Times'.

[33]Frisch translated only the first of the two sentences in this footnote which referred to the now famous rocking-horse metaphor. Just for the record, the full Swedish statement in this Wicksellian footnote is as follows (ibid; italics in the original):

> "Om man slår på en gunghäst med en klubba, så bli gunghästens rörelser mycket olika klubbans. Stöten är *orsaken* till rörelsen, men föremålets egna jämiktsbetingelser äro *förutsättningarna* för rörelsens *form*".

[The impulse is the *reason* for the movement, but the object's own equilibrium tendencies {structure} are the *prerequisites* for the form of the movement. (KVV's translation from the original Swedish, of the second sentence].

and erratic shocks unacceptable'; the names we have invoked in the pages of this chapter are a testimony to that fact.

But is it necessary to choose between such starkly different alternatives - between a linear stochastic theory and a non-linear deterministic theory? It was not in Hicks' nature, nor in the nature of Schumpeter, Keynes, Lindahl, Lundberg, Tinbergen, Leontief, Kalecki, Kaldor, Goodwin, Yasui, Morishima or Day, to depict possible worlds in starkly contrasting colours; their's was a world of shades and many colours and this was so even in their theories of the trade cycle. Even though Hicks opted for the alternative of theorising without reliance on *ad-hoc shockeries*, he did add the characteristic caveat (Hicks (1950), p.90):

> "It [the theory of *erratic shocks*] certainly is an interesting theory; it is quite likely that a 'stochastic' hypothesis of this sort has some part to play in the explanation of what happens. But this particular hypothesis will not do."

There was a time when the theoretical technology of computing mitigated against the use of non-linear dynamical systems to model macroeconomic fluctuations in excess of two or three dimensions. However, advances in the technology of feasible, large-scale computations and simulations of high-dimensional non-linear dynamical systems suggests new approaches to the modelling of macroeconomic fluctuations. Moreover, it is also possible, with the new developments in theory and technology at hand, to use modelling techniques and strategies that *go beyond* the traditional reliance on difference, differential and mixed difference-differential systems, whether deterministic or stochastic, whether linear or nonlinear. Indeed, even the traditional and worn dichotomy between deterministic and stochastic systems can be questioned from the point of view of newer mathematical modelling possibilities brought to the fore by concepts of incompleteness, uncomputability and undecidability[34].

The rest of this concluding section, apart from summarising very briefly the way nonlinear, endogenous, mathematical theorising of the phenomenon of aggregate fluctuations proceeded, is also a mini-manifesto of hope.

### 2.4.1 The Interregnum: 1958-1970

We have called this period an *Interregnum*. This is an era that seemed to have reached a nadir in the nonlinear, endogenous, mathematical theory of the business cycle, with the provocative and perennially falsified thought that the business cycle was 'obsolete'. A conference convened by the *Social Science Research Committee on Economic Stability*, with distinguished business cycle theorists in attendance - Erik Lundberg, Robin Matthews, Lawrence Klein, Bert Hickman, R.A. Gordon, P.J.Verdoorn and many others - with the

[34] Note that we carefully avoid mentioning the fashionable – although 'fashion' has a way of making obsolete even current 'buzz' words faster than adherents to them can imagine – notion of deterministic randomness, deterministic chaos, and so on – at least here.

main theme being: *Is the Business Cycle Obsolete* (Bronfenbrenner (1969)). The closing year of the period is significant in that it was also approximately midway between the year of Friedman's famous AEA address that ushered in the natural rate of unemployment as an essential ingredient in macroeconomic thinking and modelling and the birth of newclassical macroeconomics at the hands of Lucas (1972)[35]. Apart from sporadic contributions to business cycle theory - mostly in the linear mode - the significance of the period for the story being told here is that 1967 marked the year that Goodwin's remarkable '*A Growth Cycle*' was published, in the *Dobb Festschrift* (Goodwin (1967)) and a new impetus that was given to the worn out mantle of *IS-LM* by Hugh Rose in an influential and inspired series of contributions that integrated the non-linear Phillips curve within the fold of the dying embers of the Neoclassical Synthesis and helped revive it, at least for a few years[36]. The former introduced, into mainstream macrodynamic modelling, the famous *Lotka-Volterra equations* and with it a wholly different set of issues from non-linear dynamical systems theory - even while that theory was itself undergoing, literally, cataclysmic changes with the publication of Steve Smale's famous survey paper on *Differential Dynamical Systems* (Smale (1967)). The latter – i.e., the contributions of Hugh Rose – introduced into the toolbox of the macrodynamic student, once and forever, the powerful *Poincaré-Bendixson theorem*. In the early years of the *Interregnum*, crossing over and overlapping with the period of *Excitement at Birth*, there was a sudden burst of activity, probably inspired by the powerful contributions by Yasui (1953), in the late 1940s and early 1950s, by Japanese economists. Kurihara (1955), Ichimura (1955b) and Morishima (1958), surveyed and pushed the frontiers of non-linear Keynesian macroeconomics in interesting directions. Indeed, few realise that Morishima's doctoral dissertation was on *Non-Linear Macrodynamics*. There is also another important contribution to the main theme of this section: Hugh Hudson's little acknowledged but hugely important pedagogical effort at making non-linear trade cycle theory comprehensible to the general macroeconomic community (Hudson (1957)) – which, by and large, did an admirable job of completely bypassing it in the manner of Robertson's 'Scottish Preacher' (Robertson (1952), p.70). Finally, the re-formalisation of Kaldor's model, in formally more precise ways than in Yasui's early paper that was referred to above, was expertly attempted by Chang and Smyth (1971). This paper had a significant influence in inspiring some interesting work on non-linear Keynesian models of the business cycle and further helped in making the economist

[35]Indeed, the 'closing year' – 1970 – was the year Lucas submitted his famous paper to the **Journal of Economic Theory**, received by the Journal on 'September 4, 1970'. However, it had been subject to 'a withering rejection from the journal to which it was first submitted' (Lucas (1981), p.10), which must date its completion in draft form a little earlier. It is clear from the Introduction to Lucas (1981) that the foundations of Newclassical economics lay in the concepts – natural rate, neutrality, rational expectations, etc., – and metaphors – the island paradigm, for example – conceived by Muth, Friedman and Phelps during the previous decade.

[36]The most illuminating and comprehensive of a series of three papers by Hugh Rose (1966, 1967, 1969) was the 1967, **RES** contribution. Rose had been a pupil of Hicks; so it is entirely natural that his fundamental contributions arose from considering the neglected 'monetary chapters', XI and XII (as did Hudson, a decade earlier, Hudson (1957) – see below), in the Hicksian classic (Hicks (1950)).

more familiar with the mathematics of *planar dynamical systems*. We may add that it also imprisoned the mathematically inclined business cycle theorist within the strait-jacket of two-dimensional modelling. A large part of the story, both adventurous and monotonous, was due to the dominance of planar dynamic modelling. That it was a necessity in the early years cannot be denied; that it was a straitjacket in a later period is something to be established by argument.

But the story of this part, as befits the meaning of the word *Interregnum*[37], will be about an afterglow and a setting of the scene for a new thrust. An afterglow after the excitements of birth and early growth of a nascent discipline and the expectations of continued progress in understanding and taming the more virulent aspects of cyclical fluctuations. With hindsight, it will also be a story of the scene that was being set for the new developments in non-linear dynamical systems theory to be embraced by macroeconomic theories that were going beyond and away from Keynesian paradigms and freeing themselves from the somnambulance of the Neo-Classical Synthesis.

### 2.4.2 Hopes Betrayed: 1970-1987

The dawn of this period saw the challenge posed by Clower to the Neoclassical Synthesis, even while the capital, growth and distribution controversies were going on at another end of the macroeconomic spectrum. Meanwhile Friedman was mounting a sustained and increasingly plausible attempt at reviving Monetarism to place it as the centerpiece not just of macroeconomics traditionally conceived, but also as a basis for business cycle theories. Out of these challenging developments at the core of macroeconomic theory emerged, at first with great promise and much excitement, varieties of *Fix-Price Macroeconomics*[38]. There were two immediate fountainheads for these theories: the challenge to the Neoclassical Synthesis posed, on the one hand, by Clower from a Keynesian perspective; and, from another end, by Barro and Grossman. The former line of macroeconomics was further codified by Malinvaud's famous *Yrjö Jahnsson Lectures* (Malinvaud (1977)) and added a new impetus to non-linear modelling of economic fluctuations. New tools of non-linear dynamics, particularly René Thom's *Catastrophe Theory* and Christopher Zeeman's work at the University of Warwick in the same tradition, came to dominate that version of macroeconomic fluctuations emanating from the *French* version of *Fix-Price Macroeconomics.* Perhaps the most comprehensive study along these lines, summarised the economics and the non-linear mathematics of catastrophe theory and was used to formalize regime changes as phases in economic dynamics. They were, then, interpreted as macroeconomic fluctuations (Michael Blad's doctoral dissertation at Warwick University (Blad (1969)), out of which he was to spawn some influential articles on 'new' methodologies for modelling non-

[37]The *OED* definition, #4, is: '*A breach of continuity; an interval, pause, vacant space'.* The other three definitions are almost equally applicable, for the sense we have in mind.

[38]However, the fix-flex price divide in macroeconomics had first been broached by Hicks much earlier, in his comparison of aggregate accounting by Lindahl and Keynes, in a severely neglected masterpiece in the *Lindahl Festschrift* (Hicks (1956)).

linear, fluctuating, phenomena, Blad (1981) & Blad and Zeeman (1982), exemplify this work). At the level of graduate pedagogy, with a specific application of the mathematical methods introduced by Rose (for example in Rose (1967)) in the framework of the macroeconomics of the neoclassical synthesis, there were important contributions by, for example Benassy (1986).

The quintessential nonlinear, endogenous, business cycle contribution, which was to lead to a flurry of activity in the application of modern dynamical systems theory in a variety of Non-Newclassical macrodynamic models – New Keynesian Economics is what it transmogrified into, but that was to be in a future where the nonlinear mathematical underpinnings were diluted – was of course the classic by Grandmont (1985).

Almost all of these developments that emerged out of the ruins of the Neoclassical Synthesis were, initially, theories of *Disequilibrium Macrodynamics*[39]. The tide, however, was turning against this paradigm as the defining theme for macroeconomics and the early years of the period, particularly after the *Phelps Volume* (**Phelps et.al (1970)**), saw a revival of the equilibrium approach to macroeconomics in general reasserting itself. The 1970s saw the codification of Lucasian Macroeconomics, re-named *Newclassical Macroeconomics*, built on fusing of eight fundamental concepts in a remarkable *tour de force* of model building by Lucas:

- the *natural rate of unemployment* (from Friedman and Phelps);
- the *rational expectations hypothesis* (from Muth);
- endogenising labour supply via the *search model* (from Stigler and McCall);
- exploiting the local-global divide to formalise misperceptions in a monetary economy subject to shocks by situating the rational agent in *Phelpsian Islands*;
- reintroducing *Human capital* as an additional factor of production in aggregate production functions;
- incorporating all these elements in an *overlapping generations model* (from Samuelson);
- reinterpreting business cycles as *equilibrium phenomena* (claiming allegiance to Hayek's thesis of the early 30s)
- and utilising developments in linear filtering theory to reinterpret the rational agent as a *signal processor* (from Kalman and Wesley Clare Mitchell, as explicitly acknowledged by Lucas)

[39]To be distinguished from *current* work on *Keynesian Disequilibrium Macrodynamics*, most systematically and competently developed and pursued by Carl Chiarella, Peter Flaschel and their collaborators (cf., for example, Chiarella and Flaschel (2000), Chiarella et al. (2005),Flaschel et al. (1997) & Asada et al. (2010)).

By the end of this era of *Hopes Betrayed*, Newclassical economics *was* Macroeconomics and at least so far as business cycle theory was concerned, non-linear, disequilibrium theories of macroeconomic fluctuations had been banished to the hinterlands. Kydland and Prescott published, in 1982, their celebrated paper that defined the dominant research paradigm for business cycle theory for the whole of the period after that, *Real Business Cycle Theory* (Kydland and Prescott (1982)), Lucas and Romer 'endogenised' growth theory (Lucas (1988), Romer (1986)), Lucas gave up on his original monetary misperception theory of the business cycle, Kydland and Precott nailed the coffin that bore the remains of the fix price macrodynamic visions with their own policy nihilistic codification via Kydland and Prescott (1977), the first of Sargent's hugely successful series of Newclassical textbooks appeared (Sargent (1987)), *DSGE* modelling became the paradigm and with it six decades of adventures with non-linear dynamics in business cycle modelling came to an end- *or so it seemed*.

Mercifully - or is it better to say, fortunately - not all was lost and not all was as it seemed or appeared. There had been *momentous* - the word is chosen carefully - developments in the theory of non-linear dynamics. *Chaos* and, more generally, *sensitive dependence on initial conditions* had been *rediscovered* and the Poincaré- Birkhoff tradition in non-linear dynamical systems theory was about to explode into a frenzy of research activity, much facilitated by the new power brought into that branch of work by the availability of cheap computing resources. Lorenz, Takens, Ruelle, May, Feigenbaum, Smale, Abraham, Arnold and others had taken non-linear dynamics into new frontiers, beyond where it had been left off by the giants of the first half of the 20th century: Poincaré, above all; but also van der Pol, the Russian school fostered and nurtured by the great Andronov; the Latin American schools inspired by Peixoto and Lefshetz, in Mexico, Brazil, Argentina and Uruguay; Littlewood and Cartwright; Levinson, Minorsky and Lefshetz (now, in his US roles) and, of course, many others in Continental Western Europe. While all this was going on, two significant papers were published in core economic journals that pointed the way towards the usefulness of these new developments in non-linear dynamical systems theory for the modelling of macroeconomic fluctuations. First of all, there was the remarkably elegant and almost deceptively simple paper by David Gale (1973); and, then, building on this, a series of papers by Richard Day, beginning with (a joint work with Jess Benhabib), Benhabib and Day (1982).

The period was dominated by the emergence of the Newclassical approach to business cycle modelling; but it ended with a hope for the revival of non-linear business cycle modelling due, primarily, to external factors. The external factors were something entirely new in the adventures of non-linear mathematical business cycle modelling: the power, facility and feasibility of studying non-linear systems by simulation due to the cheap and easy availability of computers, literally at one's fingertips, and the increasingly well documented and competently prepared software for studying and simulating complex non-linear equations.

In the excitement that was brewing for the dawn of the next period all and sundry forgot that much had been written and claimed for *chaos* and its existence; but little had been done about developing a *theory of chaos*.

### 2.4.3 Adventures in Poincaré's Paradise: 1988-2003

In the early years of this period, a leading non-linear theorist remarked:

> "The chaotic attractor of *mathematical theory* began with Birkhoff in 1916. The chaotic attractor of *simulation experiment* arrived with Lorenz in 1962. .. The identification of these two objects has not yet succeeded, despite many attempts during the past twenty years. Of course, everyone (including myself) expects this to happen soon."
> Ralph Abraham (1985), p.117; italics added.

The ferment and the plethora of articles, books and manuscripts on non-linear economic dynamics describing complex behaviour paid little or no attention to the above dichotomy. This sense of careless excitement was compounded by a habitual disregard, in economic modelling, for the need to understand three interrelated issues:

- the digital computer, with floating-point precision, needs to be fed discrete dynamical systems; hence, if economic modelling has been done in continuous time, then such systems have to be discretized in a way that preserves the characteristics of its attractor (supposing there to be one for the system);
- the non-linear dynamical system, when implemented in a digital computer, takes on the characteristics of a recursive function that is iterated, or that of a Turing Machine that is initialised to implement a computation; hence, the theory of computation acts to constrain the feasible trajectories and the characteristics of the basin of attraction of the dynamical system;
- in view of the above two points, any study – theoretically or experimentally – of a non-linear dynamical system cannot be complete without a correspondence with a theory of numerical analysis and recursion theory (the theory of computation).

In describing the work on mathematical business cycle theory in the non-linear mode of this period, against the backdrop of the development in the mathematics of non-linear dynamical systems theory, the above three caveats and Ralph Abraham's cautionary note must be kept in mind.

Bifurcation theory played a crucial role in the non-linear economic models that were developed in this period. Examples are the Andronov-Hopf Bifurcations in classic Keynesian models of the business cycle and Turing bifurcations in Marxian models of distribution cycles. All kinds of macrodynamic models made this tool and concept, by the end of the period, as familiar to mathematically minded economists as the Perron-Frobenius theorem had been to linear economic model builders and economic theorists, and the Brouwer and other fix point theorems had become to general equilibrium theorists in earlier periods.

The economic workhorse, for the non-linear theorist of business cycles, turned out to be the overlapping generations model, owing a great deal to the pioneering two contributions by Gale and Day, mentioned above. This workhorse, encapsulating non-linearities in an ingenious way - exploiting, for example, the differences in attitude to risk by different generations populating the economy - served a dual purpose in what had become an intellectual battle between Newclassical visions of the economy as a self-sustaining, self-adjusting, equilibrium phenomenon and those on an obverse side challenging all or some of these characteristics. The first purpose was to demonstrate the existence of multiple equilibria and, hence, the possibility of selection via policy active measures. The second purpose was to show that even incorporating rational behaviour as the underlying disciplining criterion for a model, there was the possibility of persistency in disequilibrium configurations for long periods of time. In both of these ways, this signalled a return to the program that initiated the non-linear adventures in the mathematical modelling of business cycles, in 1928-1957. It gives some substance to that famous Robertsonian wit and wisdom:

> "Now, as I have often pointed out to my students, some of whom have been brought up in sporting circles, highbrow opinion is like the hunted hare; if you stand in the same place, or nearly the same place, it can be relied upon to come round to you in a circle."
>
> Dennis Robertson (1956), p. 81.

In some sense the way the story of this period will be told keeping this Robertsonian precept in mind; but it applies only to a part of the story. The remarkable developments in the mathematics of non-linear dynamical systems is an undoubted advance in theory. Whether, and to what extent, there was progress in the economics of business cycle analysis, outside the Newclassical framework, to match the powerful non-linear dynamical system theories remains a moot point - or a 'Robertsonian point'.

# Chapter 3

# Nonlinear Accelerator and Harrod's Trade Cycle

## 3.1 Harrod's Trade Cycle

The nonlinear cycle theories in the Keynesian tradition owe their core analytical insight to Roy Harrod in one way or another. Harrod (1936) attempted to close the Keynesian system and infuse explicit dynamic elements into it. Despite the fact that his book did not always attract favourable reviews from noted economists like Joan Robinson, Tinbergen, Harberler and Hansen, his insight turned out to be crucial[1] in the further developments of endogenous cycle theory in the hands of Hicks, Goodwin, Kaldor and others.

For the interests that concern this thesis, attention will be focused on examining Harrod's contribution to see whether his system had plausible mechanisms that could result in a perpetual, endogenous cycle. The key contribution of Harrod (1936) was to combine the forces of the (Keynes-Kahn) multiplier and the accelerator in order to have a theory of trade cycle. However, he did not call the principle of acceleration - the functional relationship between aggregate investment level and the rate of change of income, by its well known name. The accelerator mechanism was not definitely not a new concept by that time and had been widely studied. Instead, Harrod called it by a new name -'The Relation', which attracted much criticism for an undue claim

[1] Although, the recognition given to his contribution to trade cycle theory may not have been commensurable with its importance. Samuelson, in his reply to Heertje-Heemeijer's article (Heertje and Heemeijer (2002)) on the origins of the multiplier-accelerator model reiterates Harrod's role (Samuelson (2002), p.219):

> I applaud these authors for calling attention to Roy Harrod's important post-1935 contributions to the literature on the acceleration principle and macroeconomic business cycles. Harrod was a brilliant innovator who felt – properly felt – that a number of his seminal novelties failed to receive their deserved recognition.

on novelty, especially by Hansen, who was well acquainted with the work of Clark, Bickerdike and Aftalion (Samuelson (2002), p.220). Samuelson, who was a student of Hansen, developed a mathematical model of the multiplier-accelerator story. He was apparently influenced by Hansen's work, especially by his book '*Full recovery or Stagnation*', published in 1938. Samuelson employed linear difference equations (by resorting to lags) and the resulting second order system was capable of sustained fluctuations only for a very special set of parameters. The fluctuations otherwise were either damped or explosive.

Harrod's explanation about the combined effect of the multiplier and the accelerator and the upper and lower turning points were purely verbal. He did clearly perceive the importance of the constraints that are associated with the economic system - the presence of full employment and a lower limit to net investment (which cannot below the amount of depreciation). However, he did not explicitly relate them to the changing values of the accelerator co-efficient depending on the level of output, as Goodwin eventually did. Instead, he attributed the slowdown of the rate of increase of consumption (consequently, the investment) in the later stages of the boom and vice versa during the recession to a set of static and dynamic determinants. The elaborate explanations concerning the operation of these determinants seem to have obscured the potential behind the role of nonlinearities in explaining the possibility of a perpetual cycle. The main (and relevant) criticism seems to be that of Tinbergen, who interpreted Harrod's model as a linear, first order, differential equation, and argued that such a system was not capable of generating cycles.

Was it merely that Harrod's exposition fitted a linear, unstable system and that he failed to see how the bounds on either end that accounted for the turning points could have been deftly utilized for generating perpetual cycle? Or was Harrod's description of the accelerator was in fact nonlinear and therefore endowed with the power to explain sustained oscillations? This might be important not just for the reasons of doctrine history, but also for a close understanding of the nonlinear elements that underlie the non mathematical exposition of these older theories. The contention of this chapter is that Harrod's accelerator was in fact nonlinear and we attempt to substantiate this claim further.

## 3.2 The Issue of Lags:

The linear, unstable system with bounds on either side would mean that there would be a discontinuous switch between two regimes at the turning points. It would be along the lines of what Samuelson called as the 'billiard table' theory and the turning points themselves would have to be explained by exogenous factors. This, for example, is the view taken by Heertje and Heemeijer (2002), p. 211:

"What then stops us now from concluding outright that it was Harrod who

> laid the verbal basis for Samuelson's model? It is not in his under- standing of the interaction between the multiplier and accelerator that we can find shortcomings, but in the fact that Harrod himself did not see the explanatory possibilities of this interaction. The problem will be evident to any reader of his work: the movement of production between the turning points is made perfectly clear, but when it comes to explaining the turning points themselves, the analysis seems to drown in a massive swamp of exogenous variables."

This shares certain similarities with Hicks (1950), where disequilibrium fluctuations are superimposed on to exogenous growth factors. The system bounces between the ceiling and the floor and there are clearly two regimes at play. However, there are remarkable differences between Hicks and Harrod's approach too. First of all, Hicks' model was squarely in the tradition of period analysis and therefore showed explicit allegiance to lags. Harrod, however, was not a great supporter of lag theories[2].

> "All references to time-intervals in this topic are highly dangerous; it is so easy to give plausible explanations on the basis of a time-lag hypothesis; the hypotheses that may be introduced are so many and various that with their aid the facts can be made to fit almost any theory; it is extremely difficult to demonstrate that one hypothesis is more probable than the another." - p. 88, Harrod (1936)

Potential inconsistencies in letting the multiplier and the accelerator operate in the same period (in the absence of lags) and Harrod's unclear description about the length of the period involved were the two factors that attracted criticism from people like Tinbergen and Haberler (1937b). Tinbergen reviewed Harrod's book (Tinbergen, 1937) and pointed out that Harrod's system (without the lags) would not be able to produce sustained oscillations[3]. In the absence of lags, Tinbergen interpreted Harrod's system as being a *linear*, first-order, continuous time system and contended that it can only generate exponential growth and not cycles. This system, upon reaching the full-employment ceiling would break down. In his reply to Tinbergen (dated 1$^{st}$ July, 1937),

[2]This is made clear right from the preface of the book:

> "In fact, writers seeking to introduce dynamic considerations have often tended to confine themselves to mere description or to develop a theory regarding time-lags. But is not a theory of time-lags or of friction premature when the fundamental propositions relating to velocity and acceleration remain unformulated?" - p. viii, Harrod (1936)

However, his exposition of the interaction of multiplier and the accelerator does seem to introduce lags implicitly. See Haberler (1937b), p. 691, where he points out the tension between Harrod's outspoken aversion to the use of lags and the standing of his own theory without time-lags.

[3]This is along the same lines, in principle, of Frisch's criticism of Kalecki's model for its ability to generate sustained oscillations for a special set of parameters. For a linear rendition of Harrod's theory, Samuelson(1939) showed that sustained fluctuations are possible only for special value of parameters for which the system lies in the boundary between regions.

Harrod mentioned that such a rigid mathematical formulation would not do justice to his theory[4]:

> I fear that my mathematics are rather rudimentary and that *any single-handed attempt to give a rigid mathematical formulation to my theory would not be successful. . . . I am aware that my fundamental propositions do not yield a sine curve of the kind that your soul delights in. I do not think it follows that they necessarily fail to demonstrate the inevitability of the cycle.* On the look out for a certain type of equation you have, I think, done less than justice to my argument at this point.

Harrod, while admitting in his reply that he resorted to the use of lags implicitly in his argument, felt that though lags might have role to play it would not be fundamental for the explanation of an endogenous cycle. Tinbergen's criticism allegedly played an important part in the transition that Harrod made from his theory of trade cycle to his famous model of growth[5].

> I very much hope therefore that you will pause a little further to consider the significance for the cycle of my multiplier/relation propositions, and not dismiss them because the solution is not so neat as (those) you get by certain lag assumptions. My own intuition, for what it is worth, is that you will not get at the *vera causa* of the cycle by looking at lags only. I have no doubt they play some part in the whole thing, but I believe it will be found to be a relatively minor part.
> I see you convict me of bringing lags into my argument at various points. Of course I do. But I do not think that the assumption of a lag is present in the fundamental part of my argument formulated in the equations above.

Harrod seems to have been interested in providing a theory that does not resort to lags, yet having the power to explain the discontinuity between the boom and the recession. That is, discontinuity occurring at the turning points are to be accounted for using endogenous mechanisms.

## 3.3 The Nonlinear Accelerator

The crucial detail to remembered from the earlier section is that the functional relationship of the accelerator described by Harrod was assumed to be linear by Tinbergen. Goodwin, who later developed the nonlinear accelerator model (Goodwin (1951)), indicated that his inspiration came from Harrod (1936).

[4]Source: http://economia.unipv.it/harrod/edition/editionstuff/rfh.2dd.htm#pgfId=73882

[5]This issue has been discussed in detail by Jolink (1995), Velupillai (1988) and interested readers are referred to these articles.

> "I continued to believe in Harrod's primal insight but did not know how to validate it. *Both Frisch and Tinbergen, had failed to pay adequate attention to a short note in an early issue of Econometrica to the effect that to explain a self-generated cycle, it was necessary to have an unstable equilibrium with a pair of non linearities, in the outer regions of the state space, to convert the instability into global stability*"
> Goodwin (1988) p. 20.

Harrod's description of the multiplier-accelerator has been widely perceived in the literature as being a linear, unstable system. A remarkable exception was a perceptive remark by Ichimura[6], which seems to have gone unnoticed:

> "Mr. Harrod's cycle theory is really a forerunner of the Hicks-Goodwin type of nonlinear macrodynamics, though he presented it verbally. Mr. Harrod combines the multiplier and the accelerator, which latter is made nonlinear by reason of the following effects of the changes in the level of output on the acceleration co-efficient:
>
> 1. the influence of the rising rate of interest in the upswing and that of the falling rate of interest in the downswing;
> 2. the changes in the relative prices of the capital goods; and
> 3. the variations in profitability due to the law of diminishing returns and the elasticity of demand."
>    - Page 217, Ichimura (1955b)

However, Ichimura does not cite the exact arguments in Harrod's book to bolster his claim. The following sections attempt to fill this gap.

### 3.3.1 Ceiling and the floor

Harrod begins his description of the 'relation' with the observation that the activity in production of capital goods is more rapid than that of consumer goods in upswing and conversely in downswing. Based on their use, the produced capital goods can be divided loosely into two parts - to compensate for depreciation and to increment the existing capital stock (net investment). Assuming no technological growth, the amount of increase in net investment depends on the rate at which consumption is increasing. In this case, "since net investment is responsible for a large proportion of the activity of capital goods industries, **a cessation of the advance of consumption, without any decrease in its absolute amount, would entail a vast falling off in the activity of capital goods industries**(p.55, Harrod(1936))". Thus, Harrod realized that the accelerator provided a way to link changes in consumption to investment, dynamically. With the

[6] To the best of our knowledge, the only other exception was Besomi (1998). In particular, see section 7 (pp. 126-128), where Marshack's correspondence with Harrod on this issue is discussed.

multiplier, now there was a possibility of closing the system because it provided a link between investment and income (or consumption).

The entire mechanism operates within the natural constraints posed by the economic system. Increase in output, once it has set in, cannot go on forever when it is not in line with the increase in population and improvements in technology. If the increase in output that has been brought about by bringing in the unemployed into the labour force, keeping the technology constant, diminishing marginal returns for labour ought to set in. This would lead to a reduction in the level of activity. In other words, the full employment ceiling comes in to effect.

> "After any outstanding surplus capital plant is brought back into use, the activity of the capital goods trades becomes abnormally high; for a time the increase of general activity is itself above normal, the unemployed is taken up and the monetary-destabilizer has exerted all its influence or a great part of its possible influence in raising activity, any further advance must depend on increasing population or improving technique;. . .
> When the period of abnormal advance comes to an end, there must be some recession in the capital goods industries. But if the prime factors of production are incompletely mobile and cannot be readily absorbed in the consumption goods industries as they are displaced from the capital goods industries, total activity must recede."
> - Page 57, ibid

Similarly, during the downturn, the inescapable constraint that net investment cannot be negative (except in situations like war) sets the bound on the other side. These two constraints establish the natural bounds within which any advance or contraction in the capitalistic mode of production can ensue.

> "If consumption actually recedes, complete replacement of existing capital goods will not be necessary. Certain replacements and repairs are necessary from time to time to keep a centre of output in operation at all. These might be called 'overhead' replacements. But some are related to the volume of its activity. To give an example in very simple terms, suppose that the machines of a firm last ten years and ten percent of these machines are normally replaced each year when the level of output is steady. Suppose a recession in out of 20 percent; in the first two years it will not be necessary to replace machines at all; but there after replacement must revive. This necessity involves increased activity in the capital goods industry, although no increase in consumption occurs. But if total output and income thus increase, an increase of consumption is highly probable. This is the revival."
> - pp. 57-58, ibid

As the level of output changes during the phases of boom and recession, we need to understand the magnitude (and direction) of change in the net investment that results. In other words, we need to understand the value and variations in the co-efficient of the accelerator. Apart from the above mentioned bounds, whose presence can be incorporated into the accelerator co-efficient and making it nonlinear, Harrod's discussions also reveal other possible reasons to regard the accelerator as being non-linear. The rate of growth of consumption is related to the rate of growth of investment via what Harrod's calls as three dynamic determinants[7]: propensity to save, shift to profit and the amount of capital used in production. The first two dynamic determinants cause a restrictive influence on the advance of output. Higher the savings by the households out of their income, lesser will be the effect of the multiplier. A reduction in the share of income available for consumption due to shift to profits during the boom exerts a restrictive force on expansion of income. The argument about shift to profit is a bit more complex and lets analyze it in a bit more detail. Before that, we shall examine the role of interest rates during boom and the slump in his theory.

### 3.3.2 Role of interest rates:

The influence of the interest varies during the phase of expansion or contraction as it increases during the former and decreases in the latter. When it decreases, it changes the relative price of capital by making it cheaper and tends to make the productive methods more capitalistic.

> Net Investment, in addition to that required as a basis of increase of consumption, may also occur because a representative parcel of consumable goods comes to require more capital for its production. This may occur either (i) owing to a fall in the rate of interest which makes the capital relatively cheaper factor of production and so stimulates its use, or (ii) owing to a fall in the relative prices of capital goods compared to that of consumable goods, or (iii) owing to improvements of productive technique requiring a larger use of capital to make representative parcel.. (p.59 ibid)

This reinforces the expansionary power associated with the third dynamic determinant viz: the amount of capital used in production process. This effect helps overpower the restrictive force exerted by the other two determinants[8]. Furthermore, when the

[7]This is along with four static determinants which determine the level of output at any given point in time: Plasticity of prime costs, Law of diminishing marginal returns, Law of decreasing elasticity of demand and price level

[8]This made clear from the following discussion:

> Drawing again from the field of observation, we find interest rates tending to be high in the boom and low in the slump. Thus, so far as the rate of interest is concerned, net investment is discouraged in the boom. ...The stimulus to the net investment afforded by the Relation being so great, it is fortunate that we have the rate of interest to provide some counterweight. High rates in the boom and low rates in the slump do something to check the vagaries of net investment which we should expect otherwise. Furthermore, (ii) the relative prices of capital goods tend to be lower in the slump and higher in the boom

changes in the rate of interest also influences the propensity to save, the expansionary forces (due to the multiplier effect resulting from reduced saving) overcome the restrictive forces, thereby leading to the process of recovery. During the upswing, the influence of the increasing interest rate is the contrary and it works to restrict the expansionary effect discussed earlier. In all, the role of interest rate with varying influence across the cycle has a nonlinear influence on the acceleration co-efficient ( i.e, on investment resulting from changes in rate of change of consumption (output)).

### 3.3.3 Changes in Profit

Changes in profit per unit of output during the cycle is determined by the difference between price and marginal revenue or that between average and marginal cost under the assumption of profit maximization (p.78, ibid).

> In the recession the average prime cost will, in so far as it is affected by this 'overhead' factor[9], tends to rise. There is an asymmetry here which is worth noticing. In the later phases of expansion this overhead item may have grown; it does not follow that it can be reduced in recession. ...This asymmetry is due to the irreversible nature of decisions regarding the scale of capital equipment. It is worth emphasizing, since its effect corresponds with the observed facts of the cycle, viz. *a smaller rise of profit per unit for the last n units of expansion than the fall of profit per unit for the first n units of contraction* (p.80, ibid)

Harrod accounts for the shift to and away from profit during the boom and the slump through the forces exerted by the law of diminishing returns (LDR) and the law of decreasing elasticity of demand (LDE). In case of the former (LDR), an increase in output is associated with a shift to profit whenever there is an increase in the ratio between marginal costs and average prime costs. As the system nears a situation where the labour availability becomes scarce, marginal cost ought to be increasing. Although his judgment is not clear cut, Harrod claims that it is highly probable that with the operation of LDR, shift to profit occurs. More importantly, LDE provides an explanation for shift to profits, operating through rise and fall of prices. Note that the changes in profits and prices, due to changes in elasticity of demand, is not linear under the assumption of profit maximization[10]. Therefore, LDE exerts varying impacts on prices and profits during different phases of the cycle. The intuitive reasoning would be that the forces of imperfect competition set in with the increase in income more forcefully and this effect is explained by LDE[11]. The shift to profit can occur with reduction in

> in relation to general prices, and this should reinforce the fluctuations in the interest rate. (p.59 ibid)

[9] This refers to the minimum cadre required maintaining operations, which may not be easily variable

[10] See this in conjunction with the pricing rule in footnote 1, p.86. ibid

[11] Even though the nonlinear aspects are not discussed explicitly, Lokanathan (1938) summarizes this argument in a very lucid manner in his review of Harrod's book. See pp. 519 – 521.

output under imperfect competition.

From the hypothesis that the 'amount by which methods of production become more capitalistic' is relatively higher in the slump, he concludes that the stabilizing power exerted 'shift to profit' would be eventually overcome by the expansive effect triggered due to the third dynamic determinant.

> This tendency of production to become more capitalistic will offset the restrictive influence of the first two determinants. ...There is a reason to suppose that the shift to profit is intensified as the advance continues and available human material is used up. If this is so, the point will come when a given rate of increase of net investment proves no longer justified. This happens as soon as the restrictive force of first two determinants comes to exceed the expansive force of the third. (p.94, ibid)

### 3.3.4 Nonlinear accelerator and Relaxation Oscillations

Thus, we see evidence for Ichimura's remark that the accelerator is indeed nonlinear. This is due to the varying role and influence of interest rates and the imperfect competition, the latter in the form of LDE operating through the profit channel. The accelerator coefficient has to be defined as being nonlinear to capture these effects. This opens up the possibility of explaining the turning points under these assumptions mathematically, without relying on any lag theories as Harrod intended. If one does not resort to lag theories, the model has to be formulated in continuous time and the model ought to explain the asymmetry associated with rapid booms and long recessions in conformity with observed data. The presence of such an asymmetry rules out employing sine-like curves, which were the popular mode of theorizing about periodic phenomena during that period. However, the existence of non-linearities give a straight forward method of viewing this as relaxation oscillations. This is also the message of LeCorbeiller's note in *Econometrica*, advocating the view that business cycles can be viewed as relaxation phenomena.

In the case of relaxation oscillations, the system slowly builds up energy and then there is a rapid discharge of this energy. This is due to the nonlinear damping coefficient and the oscillations are self-sustained. It would be useful to consider an example (the van der Pol oscillator):

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{3.1}$$

The coefficient of $\dot{x}, \epsilon(x^2 - 1)$ is positive for small values of $x$ ($x < 1$) and thereby facilitating the increase in the value of $x$. However, for larger values of $x$, the coefficient switches sign and the damping reduces the value of $x$ as the rate of change of $x$ is negative. Thus, there is a local instability, but the system is globally stable. The switch between the increasing and decreasing branches of x is sudden and is almost instantaneous. With higher values of $\epsilon$, the switch between the two branches (increasing and decreasing)becomes increasingly rapid. In Harrod(1936), the presence of nonlinearities mentioned earlier mean that the co-efficient of accelerator changes. It may alternate its sign, but not necessarily so. This alone would have sufficed to address the criticisms posed on mathematical grounds that Harrod's theory was incapable of generating endogenous cycles without resorting to exogenous ceilings and floors. The economic system encourages net investment until the expansionary forces are counterbalanced by the stabilizing forces of imperfect competition and diminishing returns, so that the system does not explode. Similarly, the very same forces do not let the system collapse either. In the light of Harrod's letter to Tinbergen mentioned earlier, this realization would have helped Harrod explain, even on mathematical grounds, that the asymmetric trade cycle can be endogenously explained as a result of the multiplier-accelerator interaction, without lags. The key is to view them as relaxation oscillations instead of harmonic, sinusoidal oscillations (however much we like our souls to delight in such things).

This is precisely what Goodwin(1951) went on to do - by skillfully utilizing the two bounds on either side to correspond to the changes in the value of the acceleration co-efficient. Two remarks related to this context might be useful. First, if the above relaxation metaphor were to be accepted, the rapid switch in the sign of the damping co-efficient needs further explanation. What does infinitely fast transition between increasing and decreasing rate of change (of investment) mean in terms of economic quantities involved? Can this be validated by observation? Gorgescue-Rogen made an interesting observation that relaxation phenomena masked a discontinuity between upward and the downward regimes (see Gorgescu-Roegen (1951), p.116-117). His emphasis on the discontinuity between regimes led him to characterize such a scenario via two separate functions for each regime. One might not agree with the solution proposed by him, but it seems important to provide an economic rationale for this discontinuous or infinitely fast transition. Second, it should be noted that the net investment of the system does not have to reach to zero. This is elaborated in Harrod (1936)(pp.58-59). The improvements in productive techniques (innovations) have a place in this framework and this can be modeled by introducing a forcing term.Concerning the lags, the use of continuous time model does help solve the problem mathematically. However, the real issue of the length of the period, in terms of the economics, still haunts to be a thorny issue in this context.

Harrod's economic intuition, with or without mathematics, was probably right after all. Even more than he himself knew!

# Chapter 4

# Existence theorems in Planar NETBC[1]

> "The modern business cycle theories in the Keynesian paradigm have a long history. Among them the most relevant ones are of nonlinear types... From the analytical point of view these nonlinear theories, whether verbally or analytically presented, essentially owe the core of their results - that is, the *possibility of business cycles* - either implicitly or explicitly, to the *Poincaré-Bendixson .. theorem* on the *existence* of a limit cycle, except for Goodwin (1967) results, which depend on the Volterra differential equation[2] simulating the symbiosis of prey and predator, that generates infinitely many concentric trajectories"
> -Nikaido (1996), pp. 217-218 (italics added).

It would not be an exaggeration to state that the tradition of modelling, and proving the existence of, 'cycles' in *Nonlinear Endogenous Theories of the Business Cycle* (henceforth, **NETBC**) *on the plane* (the *phase-plane*), from the late 1940s and the early 1950s, was largely reliant on the use of one or other of the available formal existence theorems, such as, for example, the *Poincaré-Bendixson theorem*[3] (henceforth, *the P-B theorem*). We

[1]This chapter expands on the ideas from a paper co-authored with K. Vela Velupillai and published as Ragupathy and Velupillai (2012a). The same theme resonates in another of our papers, explicitly focusing on Hugh Hudson's contribution to nonlinear cycle theory. See: Ragupathy et al. (2013).

[2]Of course, the existence of such 'centre-type' attractors for this system, of differential equations also requires a 'proof'. Nikaido, with characteristic candour, goes on (p. 218, italics added):

> "I am myself allied in spirit with the Keynesian paradigm, and will show in this study the possibility of a long-term growth cycle with explicit consideration of both demand-side and supply-side potential based on the [Poincaré-Bendixson] theorem. .... Ignoring the monetary factors here is just for the sake of obtaining a complete growth cycle based on the [Poincaré-Bendixson] theorem."

Nikaido's transparent statement exemplifies the main theme of this chapter: an investigation into the way the application of a mathematical theorem determined the nature of the constructed economic model.

[3]To the best of our knowledge, the first time this important theorem was given a pedagogical exposition in an advanced *textbook*, aimed essentially at graduate students in economics, was in (what

have claimed (see, Velupillai (2008)) that this tradition originated in the early, pioneering, contribution of Yasui (1952, 1953)), Ichimura (1955b,a) and Morishima (1953, 1958). Perhaps the crucial role played by the P-B theorem was first *highlighted* by the important contributions of Rose (1966, 1967, 1969), Chang and Smyth (1971).

In a sense the reign of what may be called *the P-B* (and allied existence) *theorems* and the dominance of planar dynamical modelling of NETBC could be said to have lasted – and coincided with – the 'Golden Years of Keynesian Economics', approximately the quarter of a century from 1949 ((Goodwin, 1949)) to 1973[4].

However both Goodwin (1951) and Hicks (1950) were well aware of the need to prove the *existence* (and *uniqueness*[5]) of endogenously generated aggregate fluctuations in formally acceptable, mathematically rigorous, modes in their economic models of fluctuations. Although one would have expected the distinguished author of **Value & Capital** (Hicks, 1939) to have emphasised this aspect, it was, in fact, Goodwin who was more explicit (ibid, pp. 13-14, italics added):

> "It is intuitively clear that [the aggregate fluctuations] will settle down to [a limit cycle] although *proof* requires the rigorous methods developed by Poincaré. ..... Of another equation [the van der Pol equation] mathematically equivalent to ours [the Rayleigh equation], Andronov and Chaikin say:'Thus while there is no convenient *method for solving* van der Pol's equation, it is known that: (a) there is a unique periodic solution and it is stable; (b) every solution tends asymptotically to *the* periodic solution.' "

Two further points should be noted and emphasised. First of all, there is the important distinction between 'methods of solution' and 'proof of existence of solutions'. Secondly, the classic Andronov and Chaikin (1949) text did, in fact, discuss explicitly *the P-B theorem* for planar dynamical systems (*ibid*, 208-9). The former distinction was a practising *credo* Goodwin maintained in all his work on nonlinear macrodynamics. It is not surprising, therefore, that he did not pay attention to the fact that he could have applied the Poincaré-Bendixson theorem to *prove the existence* of a limit cycle in his (Rayleigh-type) model of nonlinear macrodynamics[6]. The *cognoscenti* would, of course, realise that this concern with existence proofs in NETBC was neither an independent research activity in one isolated field of economics, nor a 'flash in the pan'.

eventually turned out to be the first of four editions of) Gandolfo (1971), p.407 & p. 421).

[4]David Gale (1973) is, for us, the fountainhead of the era of NETBC beyond planar dynamics and its underpinnings in *the P-B theorem*, although it took another decade before this was recognised (with the pioneering works of Richard Day (for example, Benhabib and Day (1982)). As in all such 'approximate' historical delineations, there is slippage at both ends. Thus, Torre (1977)) is an anticipation and Schinasi (1981) and Benassy (1984) are 'hangovers'.

[5]And, indeed, stability, too.

[6]In a personal letter to Velupillai, dated 23 August 1990, Goodwin wrote (italics added):

> "As you are well aware, *I am hopeless at formalism* and I was always pleased to be told by you that what I practice, innocently, is constructive analysis (*proofs being quite beyond me*)."

Formal concern with existence proofs,using fix point theorems[7], in core areas of economics can be said to have begun with von Neumann (1928) to reach a kind of *zenith* with the Arrow-Debreu classic (Arrow and Debreu (1954))[8].

In this chapter, we examine the role of existence proofs in modelling NETBC as planar dynamical systems. As a by-product we also point out that this can be interpreted as a further attempt to straight-jacket economic theories to existing mathematical results, thereby restricting the attempts at unravelling the real nature of attractor(s) that characterize economic dynamics. Efforts seem to have been directed at adapting economic theories and assumptions to fit those for which existence results concerning the attractors were available.

Thus, the questions we pose are the following: Why did mathematically oriented macrodynamic modellers appeal to existence theorems in the context of business cycles? What were the different existence theorems that were widely used and how? How did these theorems, directly or indirectly, influence the direction in which endogenous business cycle theory proceeded? Are these proofs *constructive*, if not, how can we make them constructive? Are these mathematical objects encapsulating economic phenomena (that is, attractors like limit points and limit cycles) that are *proved to exist*, *computable*?

In order to address these questions, we investigate the role of important *existence* theorems, in particular, the Poincaré-Bendixson theorem (and the Levinson-Smith theorem), in the NETBC models on the plane. The chapter is organised in the following way: Section 2 provides a brief overview of endogenous business cycle theory. Section 3 investigates the different existence theorems and the way in which they were invoked using Kaldor's model as an example. Section 4 undertakes a detailed discussion of the use and the impact of Poincaré-Bendixson theorem in NEBCT. In section 5 we discuss the differences between classical existence proofs and that of Poincaré-Bendixson theorem and issue of computability of attractors in this context.

[7]Temple (1981), p.119, italics added) made an important observation of the utmost relevance to the general 'vision' underpinning the message in this chapter:

> "One of the most fruitful studies in topology has considered the mapping $T$ of a set of points $S$ into $S$, and the *existence* of fixed points such that $T(x) = x$. The importance of these studies is largely due to their application to ordinary and partial differential equations which can often be transformed into a functional equation $Fx = 0$ with $F = T - I$ where $Ix = x$.

[8]We do not ignore Walras' s valiant – often unjustly belittled – efforts to juxtapose the question of existence of equilibrium with devising methods to solve for its determination, in line with the dominant $19^{th}$ century tradition of algorithmic proofs. In other writings by us (Velupillai (2009a, 2011a)) these issues have been tackled in greater detail. A comprehensive discussion is forthcoming in Velupillai (2012).

## 4.1 Endogenous Business Cycle Theories

The nature and cause of aggregate fluctuations that characterize capitalistic economies has been one of the overriding themes of macroeconomic research and theorizing for more than a century now. Business cycle theory was born as a result of the attempts to incorporate the observed phenomena of cyclical fluctuations into the existing corpus of equilibrium economic theory. Different schools of thought in macroeconomics vary in terms of the sources that they attribute to these fluctuations. Their views can be classified, broadly, into two categories: Those who view that these sources are from 'within' the economic system and those whose view them as being from 'outside'. Consequently, their theories can be classified as 'endogenous' and 'exogenous' business cycles, respectively. These different approaches have differed in their choice of tools and formalisms while theorizing in a mathematical mode and these choices have in turn influenced the economic assumptions made for these models.

The endogenous view of the business cycles perceives that the aggregate economic fluctuations arise from *within* the system and holds that the fluctuations are a result of an interaction between different economic forces that operate in an economy and it is considered as an intrinsic feature of the system. This is in sharp contrast with the exogenous view, which considers that these fluctuations stem essentially due to factors that are *outside* the system, often termed as exogenous *shocks*, that perturbs the economic system that is, or has a tendency to return to, its equilibrium state. This meant that the formalization of latter was in terms of equations that necessarily had damped roots.

The initial attempts to characterize business cycles in a mathematical mode aimed at building models to demonstrate certain qualitative properties that were observed in the advanced industrial economies at that time. Broadly, these desired properties were: the persistence of economic fluctuations that made the economic system unstable; these upswings and downswings were not symmetrical; the possibility of multiple equilibria was tied together with a widely held belief that the instability is endogenous to the economic system. Characterizing economic fluctuations in a mathematical model with these desired properties meant that there was a need to go beyond linear models, since the latter are capable of exhibiting either damped oscillations, or maintained oscillations only for very specific parameter values. To this end, nonlinearity became an essential feature for these theories, in the mathematical mode. When they started out, the leading theories of business cycles were predominantly endogenous, which later became exogenous. For the endogenous theories, nonlinearity in economic relationships was a crucial element in generating persistent oscillations. Here, we focus on the mathematical apparatus of the endogenous business cycle theories and investigate the role of existence theorems in shaping the mathematization of the economic models in this tradition.

The pioneering works that spearheaded the area of mathematical Nonlinear Endogenous Business Cycle Theory are : Lundberg (1937), Kaldor (1940), Goodwin (1949, 1951) and Hicks (1950). In Kaldor's model, which builds on the works of Harrod (1936), Kalecki (1939) and Keynes (1936), a trade cycle arises due to a dynamic, nonlinear relationship, between investment and savings and, in particular, in the specification of the investment demand function, which responds to the changes in the *level* of capital stock (although a part is also played by an intrinsic nonlinearity in the aggregate savings function). Goodwin and Hicks built on Harrod's work on trade cycles which combines the Keynesian multiplier and the accelerator. Neither was influenced at all by Kalecki, although both Lundberg and Schumpeter (1912, 1939)) underpin the foundations of many aspects of Goodwin's approach to NETBC. Hicks developed a piecewise linear multiplier-accelerator model of a growing economy, with the presence of a ceiling and a floor (upper and lower bounds). The economy is constrained by these bounds and continues to oscillate within this corridor. Goodwin (1951) constructed a model using the Keynesian dynamic multiplier and a nonlinear accelerator. Kaldor presented his model in terms of graphical analysis, Hicks modeled his system in discrete time and as a piecewise linear system using difference equations, while Goodwin used difference-differential (which was reduced to a differential equation) equations. However, none of them proved the formal existence theorems concerning these persistent fluctuations. Lundberg's formal models was in the Wicksellian tradition developed by Lindahl, Myrdal and Hammarskjöld and resulted in a piecewise linear discrete model, formally similar to the Hicksian theory. Lundberg, however, resorted to numerical simulations to study the analytical properties of his essentially nonlinear difference equation model (see Velupillai (2012a) for a detailed discussion of the Hammarskjöld-Lundberg tradition, subverted by the unfortunate linearization by Metzler (1941), but revived by the 'traverse' dynamics work of Amendola and Gaffard (1998)).

## 4.2 Existence theorems in NETBC

Though the models of Hicks and Kaldor were nonlinear, they did not succeed in deriving the final nonlinear equation of the model that was necessary to demonstrate the *existence* of sustained fluctuations. It was Goodwin who succeeded in doing so. He reduced his model to a nonlinear second-order differential equation of the Rayleigh-van der Pol type that is capable of exhibiting maintained (relaxation) oscillations. However, he showed the *existence* of the limit cycle for the equation *geometrically* and did not provide an analytical proof of existence. He came very close to hinting at the use of Poincaré's methods, but did not, for reasons given above, make the final 'leap' towards a formal, analytical, proof of existence (see also the previous quote of a part of this passage)[9]:

[9] In retrospect, this remark could be considered as envisaging the entry of existence theorems in NETBC.

> "..the system oscillates with increasing violence in the central region, but as it expands into the outer regions, it enters more and more into an area of positive damping with a growing tendency to attenuation. ***It is intuitively clear that it will settle down to such a motion as will just balance the two tendencies, although proof requires the rigorous methods developed by Poincaré.*** It is interesting to note that this is how the problem of the maintenance of oscillation was originally conceived by Lord Rayleigh ... The result is that we get, instead of a stable equilibrium, a stable motion. This concept is the more general one, for a stable equilibrium point may be considered as a stable motion so small that it degenerates into a point. Perfectly general conditions for the stability of motion are complicated and difficult to formulate, but what we can say is that any curve of the general shape of $X(\dot{x})$ [or $\varphi(\dot{y})$] will give rise to a single, stable limit cycle." - Goodwin (1951), pg. 13 [Italics added]

Though Goodwin provided the intuition for the existence and hinted at the direction for proving the maintained oscillations, the final steps were accomplished by the works of the Japanese economists- Yasui, Morishima and Ichimura (roughly in that order; cf., Velupillai (2008), for a full discussion). Though proving existence theorems was not entirely new to the economic (and game) theorists by then, it was the Japanese trio who were responsible for raising the formal question of *existence* proofs into business cycle theory. Yasui (1953) was interested in formulating the Kaldor model in terms of the van der Pol equation, like the way Goodwin formulated the multiplier accelerator model using the Rayleigh equation. Yasui outlined the methods by which one can cast business cycle models via graphical discussions, especially of nonlinear systems. In this context, he posed the question of formal existence and invoked the (*Levinson-Smith*) *theorem*, thereby bringing questions of existence into the discussions on business cycles.

The subsequent use of *existence* proofs in business cycle theory was by Ichimura, who appealed to the Levinson-Smith theorem once again and discussed the questions of existence in a more detailed manner. The earliest use of Poincaré-Bendixson Theorem in business cycle theory was by Morishima (1953, 1958), where he used both *the Levinson-Smith theorem* and *P-B theorem* in his analysis. Thus, the Japanese trio, made possible the entry of *formal existence* theorems concerning planar differential equations into the business cycle theory, which later went on to crucially determine how nonlinear cycle theory evolved in two dimensions.

For the purposes of this chapter, we will focus on the existence theorems related to the following models: Goodwin (1951), Kaldor (1940), Hicks (1950), Kalecki (1935), Lundberg (1937) and their variations. It can be shown that all these models are special cases of the following canonical difference-differential equation.

## Canonical difference-differential Equation

Bellman and Cooke (1963), p. 43:

$$F[t, u(t), u(t-\omega_1) \ldots, u(t-\omega_m), u'(t), u'(t-\omega_1), \ldots, \ldots, \\ u^{(n)}(t), u^{(n)}(t-\omega_1), \ldots, u^n(t-\omega_m)] = 0 \tag{4.1}$$

This is a $n^{th}$ order difference-differential equation and it is function of $1 + (m + 1) + (n + 1)$ variables. The functions $F$ and $u$ are real functions, $\omega_i \in \mathbb{R}$ and $n \in \mathbb{Z}$.

## Richard Goodwin:

Goodwin (1951) formulated a nonlinear model of business cycle, in which he combined the nonlinear accelerator with dynamic multiplier, allowing for the presence of investment lags to account for the lag between the time in the enhanced version of the model. The presence of a nonlinearity in the investment function meant that the investment that comes forth is proportional to the change in national income around the unstable equilibrium value. However, capital accumulation becomes highly inflexible once the deviation of the national income from the equilibrium level. He reduces the model to the following equation.

$$\epsilon\dot{y}(t+\theta) + (1-\alpha)y(t+\theta) = O_A(t+\theta) + \phi[\dot{y}(t)] \tag{4.2}$$

where, $y$ is the aggregate output and $\phi$ is the nonlinear accelerator, $\theta$ is the delay parameter, which accounts for the time-to-build[10] $1-\alpha$ is the propensity to save, $O_A$ is the sum of autonomous consumption and investment outlays and $1/\epsilon$ is the adjustment parameter. Simplifying this we get,

$$\dot{y}(t+\theta) + \frac{1}{\epsilon}(1-\alpha)y(t+\theta) - \frac{1}{\epsilon}O_A(t+\theta) - \frac{1}{\epsilon}\phi[\dot{y}(t)] = 0$$

This is a first order nonlinear difference-differential equation, which is a special case of the equation (4.1) with 4, i.e, (1+1+2) variables. Goodwin then takes the Taylor series expansion of the terms with time lags $(t+\theta)$ and by retaining the leading terms of the expansion obtains the following second order, nonlinear, differential equation, a special case of the equation (4.1).

$$\epsilon\theta\ddot{y} + [\epsilon + (1-\alpha)\theta]\dot{y}(t)] - \phi[\dot{y} + (1-\alpha)y = 0 \tag{4.3}$$

## Mihał Kalecki:

Kalecki (1935) introduces an investment function

$$\frac{I}{K} = \phi\left(\frac{C_1 + A}{K}\right)$$

[10] For a detailed discussion of the Time-to-Build Tradition in Business Cycle Modelling, refer Dharmaraj and Velupillai (2011).

where $C_1, A, K, I$ are: constant part of the accumulation of the capitalist, gross accumulation equal to the production of capital goods, volume of the existing capital equipment and investment respectively. He then linearizes the function as $I = m(C_1 + A) - nk$ with $m, n$ as the linearization parameters to derive the following final equation:

$$I'(t) = \frac{m}{\theta}[I(t) - I(t-\theta)] - n[I(t-\theta) - U]$$

This can be simplified further as,

$$I'(t) - \frac{m}{\theta}I(t) + \left(\frac{m}{\theta} + n\right) I(t-\theta) + nU = 0 \tag{4.4}$$

This is a linear, first order difference-differential equation[11], a special case of the canonical equation (4.1) as a function of three variables. However instead of linearizing the investment function had he used a nonlinear investment function(see Velupillai (1997), the model would reduced to

$$\frac{K(t) - K(t-1)}{K(t-\theta)} = \phi\left(\frac{C_1 + U + \frac{1}{\theta}[K(t) - K(t-\theta)]}{K(t-\theta)}\right) - U$$

This in turn can be simplified and rewritten as

$$K(t) - K(t-1) - \phi(.)K(t-\theta) + UK(t-\theta) = 0$$

where $\left(\frac{C_1 + U + \frac{1}{\theta}[K(t) - K(t-\theta)]}{K(t-\theta)}\right) = \phi$. This, in turn, is a *nonlinear* difference equation, again a special case of (4.1), here a function of three variables.

## John Hicks:

Hicks presented his model in terms of linear relationships in discrete time. He starts with the national income identity,

$$Y_n = C_n + I_n + A_n$$

where, $A_n$is the autonomous investment , $Y_n, C_n, I_n$ are total income, consumption and induced investment respectively. By introducing an appropriate number of lags, p, in

[11] In case of Frisch, the model that he presented which was supposed to generate endogenous (non-persistent) fluctuations can be presented as the following second order linear difference-differential equation.

$$\ddot{x} = \left(\frac{s\mu}{\epsilon} - \lambda r\right)\dot{x}(t) + \left(\frac{s\mu}{\epsilon}\right)\dot{x}(t-\epsilon) + \frac{sm}{\epsilon}[x(t) - x(t-\epsilon)] \tag{4.5}$$

Here, $x$ is the amount of consumer goods produced per year. $m, \mu, s, \epsilon$ are parameters which stand for: depreciation of capital stock for every unit of consumer good produced, amount of capital stock required for production of one unit of consumer good, the desired cash balance (*encaisse désirée*) parameter for production of capital goods, respectively. However, this model, for the parameter values presented by Frisch do not generate any oscillations as shown by Zambelli (2007).

the presence of the accelerator, the above equation becomes a $P^{th}$ order, linear difference equation.

$$Y_n = A_n + \sum_{r=1}^{p} c_r Y_{n-r} + \sum_{r=1}^{p-1} v_r \left(Y_{n-r} - Y_{n-r-1}\right) + K \tag{4.6}$$

$c_r$ is the propensity to consume out of the income in period r, or the weight of period p income on the current consumption and $v_r$ is the corresponding accelerator coefficient. This relation above, together with the bounds, that is, a ceiling and a floor that are a result of natural constraints to the system in the form of limited available factors and lower limit of investment, becomes a piecewise linear (hence, nonlinear) $P^{th}$ order difference equation. This, in turn, is a particular case of (4.1).

## Erik Lundberg:

Lundberg's approach was based on the *logic of sequence analysis* and the idea of *cumulative causation*. He constructed *model sequences* in an expanding economy by varying the assumptions on the nature of investment, parameters and initial conditions and studied them for the presence of cyclical behaviour. He worked with a piecewise linear, unstable, model of inventory cycles. This model had built-in natural, economic, constraints acting as bounds that checked the system from unlimited expansion and catastrophic contractions and based on this, the model was made to generate bounded fluctuations. Among the cases he considered, for example, the expansion determined by investment in working capital and fixed capital, respectively, were formulated as second-order difference equations, in terms of the expenditure receipts. Let us consider the case of investment in fixed capital (housing). This was expressed as the following *model sequence*:

$$R_t - R_{t-1} = \left(\frac{\mu(1-\Lambda)}{\sigma}\right)(R_{t-1} - R_{t-2}) + \left(1 - (1-\Lambda)(1 - b - \frac{h-b}{\sigma})\right) R_{t-1} + C \tag{4.7}$$

$R_i$ - Expenditure receipts from the output of consumer goods at period i, C is a constant autonomous investment in consumer goods inventory, $\mu$ is the ratio of income generated from building a house and the sum of expected rent payments during a given period to cover costs, $\Lambda$ - the propensity to save, $1/\sigma$ is the proportion of total expenditure on consumer expenditure spent on rent payments for housing, $h$ is the proportion of consumer expenditure for dwellings that does not become income during the next period. Simplifying this equation and setting

$$\left(1 + \frac{\mu(1-\Lambda)}{\sigma} + \left(1 - (1-\Lambda)(1 - b - \frac{h-b}{\sigma})\right)\right) = \Theta$$

and

$$\left(\frac{\mu(1-\Lambda)}{\sigma}\right) = \Phi$$

we get the following second order *nonlinear* difference equation, (piecewise linear due to the presence of natural constraints) which is a special case of equation (4.1) which is a function of 3 variables.

$$R_t - \Theta R_{t-1} + \Phi R_{t-2} - C = 0$$

Lundberg was exploring these *model sequences*, numerically, rather than finding a general solutions for these equations. This exercise can be thought of as a numerical experimentation to identify the possibility of turning points in an expanding economy. This model was later considered by Metzler (1941). He endogenised the bounds and transformed the model into a purely *linear*, second order, difference equation, which again is a very straight forward, special case of (4.1)

### 4.2.1 Nicholas Kaldor

**Chang and Smyth:**

Let us take the case of Chang and Smyth's exposition of Kaldor's model with nonlinear income and savings functions.

$$\begin{aligned} \frac{dY}{dt} &= \alpha[I(Y,K) - S(Y,K)] \\ \frac{dK}{dt} &= I(Y,K) \end{aligned} \tag{4.8}$$

Here $Y, K, I, S$ are net income, capital stock, net investment and savings respectively. $\alpha$ is the goods market adjustment parameter. Here the savings and the investment functions are nonlinear. Let us assume, as with Chang and Smyth, $I_Y \equiv \frac{\partial I}{\partial Y} > 0$, $S_Y \equiv \frac{\partial S}{\partial Y} > 0$ and $I_K \equiv \frac{\partial I}{\partial Y} < 0$, $S_K \equiv \frac{\partial S}{\partial Y} < 0$. Differentiating $\frac{dY}{dt}$ and substituting $\frac{dK}{dt} = I(Y,K)$, given the above assumptions, we have

$$\ddot{Y} - \alpha\left(I_Y - S_Y\right)\dot{Y} - \alpha\left(I_K - S_K\right)I(Y,K) = 0$$

This is a second order, nonlinear, differential equation - a special case of (4.1).

**Takuma Yasui:**

Yasui casts Kaldor's model into a van der Pol type equation, which is expressed below.

$$\ddot{y} + \frac{1}{\sqrt{\mu\gamma s}}[s + \mu\gamma - \phi'(y)]\dot{y} + y = 0$$

Here, $y$ is deviation of the national income from its stationary or equilibrium value. $s$ is the propensity to save and $1/\gamma$ is the proportion by which the national income changes according to the difference between savings and investment. We will discuss this model in some detail below. The final form of his equation is a continuous time, second order, nonlinear differential equation, a particular case of (4.1) as well.

Why were the existence theorems used in the first place, in business cycle theory? Is it because the geometrical demonstrations were considered not rigorous enough and therefore an inevitable outcome in the quest for rigour, or was it was purely an accident ? It is possible that the age of existence proofs was dawning upon economics profession, with von Neumann's use of the min-max theorem, the application of Brouwer's fixed point theorem by Nash in game theory and the Arrow-Debreu proof of the existence of the competitive equilibrium, in turn, using Nash's approach. The use existence theorems in business cycle theory by the trio of Japanese economists, who were familiar with nonlinear mathematics and the theory of oscillations, may seem like an inevitable spillover. We will explore the way in which different existence theorems were used in NETBC and the kind of assumptions that were forced to introduce into economic models.

### 4.2.2 Kaldor's Model of Trade Cycle

Proving the existence of limit cycles in NETBC for two dimensional autonomous systems has proceeded in two ways and it is useful to distinguish between them. The first approach is to reduce the model of an economy into an autonomous second-order nonlinear differential equation of the Liénard type and appeal to one of the theorems to establish the existence of a unique limit cycle. The second method involves proving the existence of *atleast* one limit cycle for a model which is eventually reduced to a planar dynamical system, by making use of the P-B theorem. Let us begin by considering a slightly modified version of Kaldor model of trade cycles, along the lines of Yasui (1953). We choose this model because it has been studied widely and it was one of the first models in this tradition to which the existence proofs were applied. However, it is equally possible to do the same analysis with the Goodwin's nonlinear accelerator model as well.

We begin with the national income accounting identity,

$$Y = C + I \tag{4.9}$$

Here, $Y$ stands for aggregate income and $C$ and $I$ stand for consumption and investment, respectively. We now define behavioural equations for the aggregate consumption and investment functions as follows:

$$\begin{aligned} C_t &= c(Y_\tau) + \alpha \qquad (\tau < t) \\ I_t &= \phi(Y_t) - \mu K_t \end{aligned}$$

$\alpha$ for autonomous consumption and the investment function $\phi$ is assumed to be a nonlinear function. In Kaldor's model, this is an S shaped function. Let us ignore the time subscripts of the variables to keep the notations simple. Taking Taylor series expansion and retaining only the first two terms of the consumption function,

$$C = \beta Y - \gamma \dot{Y} + \alpha$$

Substituting C and I in (1), we get,

$$Y = \beta Y - \gamma \dot{Y} + \alpha + \phi(Y) - \mu K$$

Differentiating the above equation, we get

$$\dot{Y} = \beta \dot{Y} - \gamma \ddot{Y} + \phi'(Y)\dot{Y} - \mu \dot{K} \tag{4.10}$$

By viewing investment as the change in capital stock:

$$\begin{aligned}\dot{K} = I &= Y - C \\ &= Y - \beta Y + \gamma \dot{Y} - \alpha \\ &= (1-\beta)Y + \gamma \dot{Y} - \alpha \\ &= sY + \gamma \dot{Y} - \alpha\end{aligned}$$

A more illuminating way of looking at this equation, in the context of Kaldor model, is to remember that in the Investment-Savings theory of income determination, the change in income is proportional to the difference between savings and investment.[12] Substituting this in (2), and rearranging the equation, we get

$$\begin{aligned}\gamma \ddot{Y} + [1 - \beta + \mu\gamma - \phi'(Y)]\dot{Y} + \mu s Y - \mu\alpha = 0 \\ \gamma \ddot{Y} + [s + \mu\gamma - \phi'(Y)]\dot{Y} + \mu(sY - \alpha) = 0\end{aligned}$$

Let us redefine the variables Y and time(t). Write Y in terms of the deviations from the equilibrium(z),

$$z = Y - (\alpha/s)$$

and time as

$$T = \sqrt{\mu s/\gamma}t$$

we obtain a second-order, nonlinear, differential equation.

$$\ddot{z} + \frac{1}{\sqrt{\mu\gamma s}}[s + \mu\gamma - \phi'(z)]\dot{z} + z = 0 \tag{4.11}$$

Now let,

$$\frac{1}{\sqrt{\mu\gamma s}}[s + \mu\gamma - \phi'(z)] = \zeta'(z)$$

[12] We can express this as follows: $\gamma \dot{Y} = I - sY$

Substituting this in (3), we get a simpler equation,

$$\ddot{z} + \zeta'(z)\dot{z} + z = 0 \tag{4.12}$$

Here, assume that $\phi'(z) > \mu\gamma + s$ in the neighbourhood of $z = 0$ and this would imply that $\zeta(z)$ is positive or negative, depending on whether the absolute values of $z$ are small or large. Under these assumptions, the equation(3) is the unforced, van der Pol equation and this in turn is a special case of the Liénard equation

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

Note that the van der Pol equation is a special case of this equation and that this equation can be obtained by transforming the Rayleigh equation. Goodwin, for example, reduced his model to an equation of Rayleigh type and demonstrated the existence of limit cycle using the Liénard method[13]. Yasui reduced the Kaldor model to the van der Pol equation, a particular case of the Liénard equation. The idea was to reduce the model under investigation to this canonical, nonlinear equation and invoke the appropriate theorems that ensure the existence of periodic solutions.

It may be useful to clarify the role of nonlinearity in the models of NETBC. Since linear models are also capable of exhibiting oscillatory behaviour, we need to distinguish the essential differences in the nature of oscillations between these two and the additional value of placing nonlinearity at the heart of the matter. Linear systems are capable of having periodic solutions (closed paths) if and only if the characteristic equation of these systems have purely imaginary roots. If the roots are purely imaginary, the trace of the coefficient matrix of this characteristic equation vanishes and the system has a center. This means that either all paths are closed, else no path is closed. In contrast, nonlinear models are capable of having *isolated* closed paths, that is, without other closed paths lying next to them. The solution curves that are near wrap themselves around these isolated closed orbits. Closed paths of this type are called as limit cycles. Therefore, nonlinearity becomes a crucial ingredient for constructing endogenous models that are capable of persistent fluctuations.

### 4.2.3 Levinson-Smith theorem & the Kaldor Model

Now we can proceed in three different directions in establishing the existence of limit cycle(s) for the above equation. The first way is to go the old-fashioned geometric way - by transforming the equation by relevant change of variables and study it on the Liénard plane and demonstrate the existence of a limit cycle by constructing it purely geometrically. This is the method resorted by Goodwin and by Yasui. However, Yasui also discussed the applicability of formal existence proofs. The first method is discussed in the appendix. In this section, we analyze the second way, which is to

[13]Refer to the appendix for a detailed exposition of the Liénard method.

appeal to the Levinson-Smith theorem, like the way the Japanese economists – Yasui, Morishima and Ichimura did, by assuming that certain properties hold for the functions under consideration. The third way is to invoke the P-B theorem, widely used in NETBC, which will be dealt with in the next section.

Yasui succeeded in formulating the Kaldor model in terms of a generalized van der Pol type equation (or Liénard equation) and showed the presence of self-excited oscillations, graphically. It was in this exercise, for the first time, an *existence* theorem was referred to in the theory of business cycles.

> "Thus equation (2.18), in which $\varphi(y)$ and $g(y)$ are assumed to have the above stated properties, is known to be the generalized van der Pol-type equation or Liénard-type equation mentioned above. ... It has been already proved mathematically that in this case (2.18) will have a unique periodic solution"
> -Yasui (1953), pg. 233.[14]

He appealed to the Levinson-Smith theorem which guarantees the existence of a unique limit cycle under certain conditions.

**Theorem 1.** ***Levinson Smith Theorem***
*Consider a two-dimensional differential equation system*

$$\begin{aligned} \dot{x} &= y - f(x) \\ \dot{y} &= -g(x) \end{aligned}$$

*which is represented as a second-order differential equation,*

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

*The above equation has a unique periodic solution if the following conditions are satisfied.*

1. $f'$ *and* $g'$ *are* $C^1$
2. $\exists x_1 > 0$ *and* $x_2 > 0$ *such that for* $-x_1 < x < x_2 : f'(x) < 0$ *and* $> 0$ *otherwise.*
3. $xg(x) > 0 \; \forall x \neq 0$
4. $\lim\limits_{x\to\infty} F(x) = \lim\limits_{x\to\infty} G(x) = \infty$ *where* $F(x) = \int_0^x f'(s)ds$ *and* $G(x) = \int_0^x g(s)ds$
5. $G(-x_1) = G(x_2)$

[14] We are using the mimeographed, condensed, version of the original Cowles Foundation Discussion Paper: Economics No. 2065, 1953. This mimeographed version's pagination is from 219-240 and quotations refer to this pagination.

Proving existence in this case simply becomes an exercise of verifying whether the final nonlinear equation to which the model is reduced does satisfy the conditions required by the above theorem. By assuming functions assumed satisfy the above mentioned properties, the proof of existence of a limit cycle was established. Often, these conditions were too stringent or unrealistic for the economic system to satisfy, so the functional forms were assumed to satisfy these properties. In this case, $\zeta'(z)$ and $z$ are assumed to be $C^1$ implies that these functions satisfy the *Lipschitz* condition. The last condition is called the symmetry condition and if the functions $f'(x)$ and $g(x)$ to be even and odd functions respectively, this condition is automatically satisfied. This condition plays a crucial role in establishing existence of cycle in this case. Since the van der Pol equation satisfies these conditions, it was possible to prove the existence of a unique limit cycle. This was the practice, for example, in Morishima (1958), Schinasi (1981), Ichimura (1955b).

**Remark 2.** *Note here that the version of the Kaldor's model that we have analyzed assumes that the rate of change of capital stock($\dot{K} = I$) is independent of the level of capital stock and is only a function of income. This assumption is not trivial, since making investment dependent on both capital and income (as it is the case in version investigated by Chang and Smyth (1971)), then the dynamical system does not reduce to Liénard equation (Lorenz (1987), p.286). Consequently, it is not possible to apply the Levinson-Smith theorem so long as investment is dependent on capital stock. Alternatively, one can assume that the change in capital stock ($\dot{K}$) is determined by the savings function alone (which is only dependent on income and not on capital stock). For proving existence and uniqueness using theorems other than Levinson-Smith for this model, see Galeotti and Gori (1989).*

### 4.2.4 The Poincaré-Bendixson theorem

Now we analyze the use of another important existence theorem that was very widely used (not just for the theories expressed in terms of Liénard equations) in NETBC.

#### A Brief Overview of the P-B theorem

*The Poincaré-Bendixson theorem* is an important *existence* theorem that is used in the study of the qualitative behaviour of the planar dynamical systems and provides positive criterion for presence of limit cycles in the plane.The origin of this theorem dates back to Poincaré, who pioneered the field of the qualitative theory of differential equations. Instead of trying to solve differential equations in terms of explicit solutions, Poincaré developed methods to analyse the qualitative behaviour of solution curves of these differential equations. He thereby developed a geometric approach to understand the global behaviour of these equations on the plane via phase portraits. He classified different kinds of limit sets for these planar differential equations and introduced the concept of 'limit cycles'. We now know that his attempts at an exhaustive classification may not have been successful. He concluded that if the curves do not end up in one of the singular or stationary points, then they are either *closed orbits* or they wrap themselves around these closed orbits. Such limit sets are known as limit cycles.

Later, building on the contributions of Poincaré, the Swedish mathematician Bendixson (1901) proved the same theorem, with much weaker assumptions. The P-B theorem guarantees the existence of limit cycles under certain assumptions, providing the sufficient conditions for its existence on the plane. It provides a precise description of the structure of limit sets in the case of planar dynamical systems and it rules out the possibility of 'chaos' on the plane.[15]

**The P-B theorem & the Kaldor Model**

Equation (12), which is a second order equation, can be rewritten as a system of two first-order, ordinary differential equations (ODEs) in the following manner:

$$\begin{aligned} \frac{dz}{dt} &= y - \zeta(z) \\ \frac{dy}{dt} &= -z \end{aligned} \tag{4.13}$$

This system has a unique equilibrium given by (0,$\zeta(0)$). The z and y-nullcline for the above system are

$$y = \zeta(z)\ ; z = 0$$

respectively. Following Kaldor's assumption that the investment curve is S-shaped, it is clear that $y = \zeta(z)$ has a cubic characteristic and we can divide the $(z - y)$ plane in to four regions.

$$\begin{aligned} V^{+} &= \{(z, y) \mid\ y > 0, z = 0\} \\ V^{-} &= \{(z, y) \mid\ y < 0, z = 0\} \\ g^{+} &= \{(z, y) \mid\ z > 0, y = \zeta(z)\} \\ g^{-} &= \{(z, y) \mid\ z < 0, y = \zeta(z)\} \end{aligned}$$

In the case of the original van der Pol equation, $\zeta(z) = z^3 - z$. It is worth noting that the characteristic of our equation, based on Kaldor's theory also has a cubic characteristic. Therefore, the arguments in our case are analogous to the case of the original van der Pol equation and let us for now assume that $\zeta(z) = z^3 - z$. The Jacobian matrix is the following:

$$\begin{pmatrix} \zeta'(0) & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues, computed from the characteristic equation of this system are:

$$\lambda_{\pm} = \frac{1}{2}(-\zeta'(z) \pm \sqrt{\zeta'^2 - 4})$$

[15] Refer to Ciesielski (2001) for a detailed discussion on its history and its development during the last century.

Given that system has a unique equilibrium $(0, \zeta(0))$, one can analyze the above Jacobian matrix around this point in order to identify the nature of this equilibrium. If $\phi'(z) > \mu\gamma + s$ around $z = 0$, then $\zeta'(z) < 0$ and the equilibrium is a source. Consequently, no solution curve can tend to the equilibrium point over time. It is also the case that any solution curve, which starts in $V^+$, has to pass through $g^-$, $V^-$ and $g^+$ before it enters back[16] to $V^+$.

Now, define a closed, invariant region on the plane (trapping region) (call it $\Omega$) surrounding the origin, whose boundary is a Jordan curve. Given that the region $\Omega \subset \mathbb{R}^2$ is closed and invariant with an unstable equilibrium point, one can invoke the following theorem to establish the existence of a periodic orbit. In case of the van der Pol equation, the periodic solution is also a limit cycle- i.e, all the other solutions, except the equilibrium point, tend to this periodic solution.

**Theorem 3.** ***Poincaré-Bendixson Theorem:*** *Consider a nonlinear autonomous system*

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned} \tag{4.14}$$

*Let $\Omega$ be a bounded region of the phase plane together with its boundary, and assume that $\Omega$ does not contain any critical points of the above system. If $\phi$ is a path of system that lies in $\Omega$ for some $t_0$ and remains in $\Omega$ $\forall t > t_0$, then $\phi$ is either itself a closed path or it spirals toward a closed path as $t \to \infty$. Thus in either case the system has a closed path in $\Omega$.*

Here, the vector field all along the boundary of this closed and bounded (hence compact) region points inwards into $\Omega$. This would indicate that the path must spiral towards a closed orbit or it is in itself a closed orbit. The compactness of the space on which these these variables are studied is therefore crucial in ensuring the presence of a closed orbit. In our case, the assumption of compactness of the income space is introduced so as to invoke this theorem. This is even more explicit in the following treatment of the Kaldor's model by Chang and Smyth.

### The Kaldor NETBC Model as a Planar Dynamical System and the P-B theorem

There is an alternative way to formalize the Kaldor model, without reducing it in to a Liénard type equation. Instead, it is possible to define relationships between the different variables involved and characterising the direction of changes in one variable with respect to changes in the other. Firstly, *ex-ante* savings and investment are functions of aggregate income and aggregate capital stock. As mentioned earlier, Kaldor's model does not endorse the acceleration principle and it relies on the Savings-Investment theory of income determination. Accordingly, change in aggregate income (Y) is proportional to the difference aggregate savings (S) and investment (I). Investment is defined

[16] Refer to Hirsch et al. (2004), Pg. 263-64 for the proof

as the change in capital stock (K) over time. These relations define the following dynamical system.

$$\begin{aligned}\frac{dY}{dt} &= \alpha[I(Y,K) - S(Y,K)] \\ \frac{dK}{dt} &= I(Y,K)\end{aligned} \tag{4.15}$$

Here we assume nonlinear investment and savings curves as in the original Kaldor model, which twice differentiable, therefore satisfying the *Lipschitz* condition. The partial derivatives, according the assumptions of the Kaldor model can be stated as $I_Y \equiv \frac{\partial I}{\partial Y} > 0$, $S_Y \equiv \frac{\partial S}{\partial Y} > 0$ and $I_K \equiv \frac{\partial I}{\partial Y} < 0$, $S_K \equiv \frac{\partial S}{\partial Y} < 0$. This system can be studied on the Y-K plane and the nullclines are given by

$$\begin{aligned}\left.\frac{dK}{dY}\right|_{\dot{Y}=0} &= \frac{S_Y - I_Y}{I_K - S_K} \\ \left.\frac{dY}{dK}\right|_{\dot{K}=0} &= -\frac{I_Y}{I_K} > 0\end{aligned} \tag{4.16}$$

The slope of the nullcline $\frac{dK}{dY}|_{\dot{Y}=0}$ is greater, less than or equal to zero, depending on whether $S_Y$ is less, greater than or equal to $I_Y$, respectively. Further, assume that $I_K\, S_Y < S_K\, I_Y$, that is if the slope of the nullcline along which the capital stock is constant, is steeper than the slope of the nullcline along which the income is constant(when it is rising). The Jacobian of this dynamical system is given by

$$J = \begin{bmatrix} \alpha(I_Y - S_Y) & \alpha(I_K - S_K) \\ I_Y & I_K \end{bmatrix}$$

The characteristic roots can be analyzed to understand the nature of the singular point. The assumption that $I_K\, S_Y < S_K\, I_Y$ implies that there is a unique, singular point $(y^*, K^*)$ and assuming that $\alpha(I_Y - S_Y) + I_K > 0$ will ensure that this singular point is unstable node or a focus. By choosing $\Omega$ – a compact subset (or assuming that such a compact subset *exists*) in the Y-K plane (non-negative quadrant of $\mathbb{R}^2$), we are ready to invoke the P-B theorem, which guarantees the existence of a closed orbit. Further, if we choose this compact subset $\Omega$ in such a way that the closed orbits are in the interior of $\Omega$ and assume that the system is structurally stable in $\Omega$, we can establish that these closed orbits are limit cycles.

Proving the existence of limit cycle using this theorem can be synthesized as below.

1. Formulate the dynamic model of the economy as a system of differential equations, encapsulating the nonlinearities in relationships between different economic variables.

2. Reduce the model to a planar (two dimensional) dynamical system.

3. Demonstrate that the economic model thus formulated has a unique equilibrium point.

4. Examine the Jacobian matrix of this system, evaluated at the unique equilibrium point, for the possibility of the equilibrium to be a saddle point and by showing that the determinant of the Jacobian is positive, rule out this possibility. Therefore, the equilibrium can either be a node or a focus.

5. Show that the trace of the Jacobian evaluated at the unique equilibrium point is positive and consequently establish that the unique equilibrium is unstable.

6. Choose an appropriate compact region in the space on which the model is defined in such a way that the chosen compact space is invariant and includes the equilibrium in the interior.

7. Following from the assumption that the chosen set is invariant, establish that the system is structurally stable.

8. Demonstrate that in the boundary of the compact region thus chosen, the velocity vector field points to the interior.

9. Use the Poincaré-Bendixson theorem to prove the existence of at least one limit cycle.

This was also the way it was used by Rose (1967), later by Chang and Smyth (1971), which later became a standard practice in the profession and this was emulated by most other studies mentioned in the previous section. However, it should be noted that not all studies end up demonstrating all the above mentioned points in the strategy and instead simply assume that some of these requirements hold. For example, as Sasakura (1994) points out, Schinasi (1982) and Varian (1979) take that point 6 in the above list, i.e, the compact region with the vector field pointing inwards at the boundaries, *exists by assumption*. Some others assume structural stability of the system in the chosen region, *apriori*.

**Remark 4.** *The Poincaré-Bendixson theorem guarantees the existence of* ***atleast*** *one limit cycle and therefore uniqueness of this limit cycle is not automatically established. On the other hand, the Levinson-Smith theorem for the Liénard equation does guarantee uniqueness as well. In the case of the van der Pol equation, the uniqueness is established by appealing to the intermediate value theorem. Also, note that the van der Pol equation satisfies the requirements of the Levinson-Smith theorem, which can be invoked to prove existence.*

## 4.3 Existence Theorems in NETBC: A Survey

In this section, we provide a survey of the models of endogenous cycles that use the P-B theorem. Morishima (1958) built on the works of Yasui and Goodwin, attempting to find a way to combine and synthesize the models of Kaldor (Yasui's version) and Goodwin-Hicks, formulated as van der Pol and Rayleigh equations, respectively. The rate of investment was a function of the *level* of income in Kaldor's model and 'rate of change' of income (accelerator) in the model of Hicks-Goodwin. He defined the rate of investment as a linear combination of the investment functions of these two models. He then derived the final equation representing the generalization of these two models and investigated the existence of cycles *formally* by invoking the Levinson- Smith and P-B theorems.

Almost a decade later, Rose (1967) made an important contribution, where he developed a theory of employment cycles in the neoclassical tradition, with profit maximizing firms. In his model, cyclical movements in employment (real effects) arise due to changes in money wages, even in the absence of real balance effects, money illusion and pressure on interest rates. The crucial element that drives the cycles is the nonlinear relationship between employment rate and the wage inflation, that is, the nonlinear Phillips curve. He demonstrates the existence of (real) employment cycles by invoking the P-B theorem.

By exploiting this nonlinear relationship capable of generating cycles endogenously, Rose managed to influence NETBC in important ways. First, by taking endogenous cycle theory outside the Keynesian circles to a broader arena, subsequently, forging new ways forward in modeling cycles. Second, his work presented a full fledged invocation and a detailed demonstration of the way in which the Poincaré-Bendixson theorem can be applied in NETBC. By introducing the Poincaré-Bendixson theorem, he provided a strategy for establishing the existence of business cycles in more general conditions than earlier. Further, his works (especially Rose (1969)) provided attention to monetary factors in NETBC, which till then was largely focused on 'real' elements of the cycles[17]

After the contributions by the Japanese economists mentioned earlier, another important contribution to Kaldor's model was made by Chang and Smyth (1971). They undertook a study of Kaldor (1940), addressing questions regarding existence and persistence of cycles. Kaldor's model had nonlinear investment and saving functions, which change according to the direction of capital accumulation, which set the economic system into cyclical motion between stable and unstable equilibrium points. Though the questions of existence were posed in relation to this model by Yasui and Morishima, they did so by reducing the model to a nonlinear differential equations of Liénard type that are known to have stable limit cycles. It was Chang & Smyth, following the path of Rose, who reduced Kaldor's model to a planar dynamical system and showed the existence and persistence of cycles in this system using the P-B theorem.[18]

[17]The exceptions include Hicks (1950), Hudson (1957) and the contributions by Swedish economists like Lindahl, Lundberg, whose theories were intrinsically monetary. ).

[18]It is interesting to note that Rose (1967) discusses the possibility of proving the existence of cycles,

Kaldor's model was analyzed by Varian (1979) in the light of catastrophe theory that was developed mainly by Thom and Zeeman. In his interpretation of the Kaldor model, he assumes a linear savings function and a nonlinear (sigmoid) investment function, and demonstrates the presence of cyclical behaviour of the system that is locally stable and globally unstable. He introduces the idea of having different rates of change for parameters and the state variables of the functions involved, 'slow' and 'fast' variables respectively, and explains plausible situations where there is a jump in short run equilibrium between different regions of the state space(catastrophes). Depending on whether there are one or two slow variables, the resulting behaviour of the system can be either in terms of 'fold' or 'cusp' catastrophes and the latter is shown to account for the possibilities of slow and fast recoveries in one model. In this framework, Varian uses the P-B theorem to prove the existence of limit cycles.

Research on Hicks' IS-LM model in the context of business cycles also made use of the Poincaré-Bendixson theorem (see Velupillai (2008) for a more comprehensive discussion of this strand of research). Notable works on this line of research include Schinasi (1981, 1982), Benassy (1984) and this came to be called the fix-price macroeconomics approach. Schinasi (1981) in his model of short run fluctuations, combined the dynamic version of the traditional IS-LM model, augmented for the government budget constraint, with the idea of a having a nonlinear investment function. He then showed that this model can be reduced to the Liénard equation and thereby appealing to the Levinson-Smith theorem to prove the existence of a unique limit cycle. In Schinasi (1982), he works in the same framework for the intermediate run, but this time appealing to *the P-B theorem*. The existence proof for the cycles in this model, in particular, the use of P-B theorem was later refined by Sasakura (1994).

Benassy developed a non-Walrasian model, in which business cycles arise due to interaction between 'stabilizing' and 'destabilizing' effects - the former role is played by prices and the latter is due to the unstable accelerator and its resulting quantity dynamics. He works with the IS-LM framework and the traditional Phillips curve (as opposed to the nonlinear Phillips curve like Rose) and incorporates expected demand explicitly into the investment function. The model focuses long run dynamics of the short run (non-Walrasian) equilibrium and proves the *existence* of cyclical behaviour in this case, by invoking the P-B theorem.

In the case of Goodwin's model, Sasakura (1996) and Flaschel (2009) demonstrate the existence of limit cycle under more general conditions than the earlier attempts (using the Liénard method) by using the P-B theorem. The missing case – the application of P-B theorem in proving the existence of cycles in Hicks' trade cycle model, is proba-

albeit in a modified version of the Kaldor's model, adapted for the presence of growth in the system. He outlines the proof strategy in footnote no.1, p.170 for proving the existence of cycles. We don't know whether Chang and Smyth took the hint from here.

bly due to the fact that the original Hicks model was formulated in terms of difference equations and as a piecewise linear model, unlike the other two nonlinear models, which were presented (or else reduced or approximated) as differential equations[19]. However, a recent paper by Matsumoto and Szidarovszky (2010), on a modified version of Hicks' model presents in continuous time with consumption and investment time delays. They make use of the P-B theorem to prove the existence of cycles.

There are many other strands of work in NETBC which make use of the Poincaré-Bendixson theorem – for example numerous studies which focus on inventory cycles, but they do not belong to the 'origins' and, therefore, we do not discuss them here. It is evident from various studies that were mentioned in this section that the role played by the P-B theorem is both crucial and pervasive across different schools of thought within the nonlinear tradition of endogenous (business) cycles on the plane.

### 4.3.1 The Impact of the Poincaré-Bendixson Theorem

As we can observe from the studies mentioned above, this tendency to equate rigour with mathematics of a particular tradition or one kind of mathematics and more specially, proving existence in this case, went beyond the case Walrasian equilibrium to business cycle theory as well. This mode of theorizing meant that, in some sense, methods dictated the questions and hypothesis that were posed and restricted the evolution of this tradition. It also hampered the enlargement of the scope of theories since theorizing concentrated on proving existence, rather than gaining insight into the real nature of the attractors. The questions relating to dynamic methods, processes through which one can understand were ignored.

The introduction of the P-B theorem marked a subtle shift from a modeling strategy in NETBC. Earlier, the strategy was the following: First, reduce the models to some form of Liénard's equation, i.e, nonlinear differential equations like the van der Pol or the Rayleigh equations (which are known to exhibit relaxation oscillations). Then, show that the system has limit cycles, either graphically using the Liénard method as in the case of Goodwin or by appealing to an existence theorem for the uniqueness of limit cycles for the Liénard-type equation. Instead, with the introduction of this theorem, the models were formulated as dynamical systems to investigate the existence of cycles in more general situations, without restrictions on the functional forms as in the earlier case.

One of the important limitations that was posed by the use of the P-B theorem was the constraint on the dimensions of the economic model it demanded. Recall our

[19]However, Goodwin's model and Hicks' model are in some sense equivalent, though the former uses a nonlinear accelerator. This is also why the early work on NETBC by the Japanese economists considered these two models together.

earlier remark that the vector field has to constantly turning *inwards* at the boundaries of the compact, invariant space and that there is no equivalent for this theorem in higher dimensions. This has to do with the fact that the P-B theorem in turn depends on the *Jordan curve theorem on the plane*. In the case of a plane, it is easy to decide the direction of the vector field to be either *inside* or *outside* the compact region. Whereas, this is extremely difficult when the dimension is higher than two. For dimensions greater than 2, it can be shown that there is a always a vector that *exists*, that is not in the set of all tangent vectors to the simple closed curve. For example, in the case of a three-dimensional object like a Möbius strip, what is *inside* and *outside* is *undecidable*. Even on 2-dimensional plane, a formal verification of the validity of P-B theorem would necessitate checking for the satisfiability of the Jordan curve theorem. This is computationally a non-trivial task (See the discussion in Ragupathy et al. (2013), section III.) This restriction meant that the scope for theorizing was rather limited, since it was not always possible to reduce the dynamic relationships between different economic variables to two dimensions, without making strong assumptions or sacrificing some relevant economic factors which might be crucial in explaining economic fluctuations. To overcome this restriction, NETBC researchers had to either adopt new mathematical tools or to be confined to two dimensions. Eventually, researchers working in the tradition of NETBC reacted by moving on to endorse the tools of bifurcation theory, catastrophe theory and chaos theory.

It should be noted that this theorem is concerned with the qualitative global dynamics of the system. In this framework, it was not possible to meaningfully assert much regarding the short-run behaviour of the system as economists would desire otherwise. Further, since the theorem proves the existence of *atleast* one limit cycle and the fact that the knowledge of the precise number of limit cycles for the planar dynamical systems is not known in general meant that details regarding the possible kinds (number and nature) of the fluctuations were not available in this mode of theorizing.

The fact that P-B theorem was purely an existence theorem had its impact on NETBC in another important methodological influence for the years that followed. While the use of this theorem may have been decisive in the earlier period of development of NETBC, later models that were developed that were made possible solely because of the availability of this theorem. In other words, economic theory was straight-jacketed to fit the assumptions of these theorems. Once this was the established practice, NETBC moved from depending on one existence theorem to another (those in bifurcation theory, for example) which probably, in retrospect, limited the development of algorithmic methods to study business cycles in NETBC.

The use of the P-B theorem meant that the theory of NETBC was largely restricted to continuous time formalisms, i.e., in terms of differential equations. A corresponding theory of limit cycles, where economic variables exhibit nonlinear relationships between themselves, was not available for difference equations.[20] This also meant that NETBC endorsed continuous time formalisms and topological assumptions needed for the validity of P-B theorem, such as an existence of a compact, invariant space, which might not correspond with either the underlying economic intuition or the nature of the economic quantities which are being studied.[21]

### 4.3.2 A Very *Brief* Note on Bifurcation theorems

Besides the criterion provided by the above existence theorems for detecting limit cycles, it is also possible to prove the existence of business cycles using bifurcation theorems. We briefly discuss them in this section, without attempting to be comprehensive. Bifurcation refers to instances where parametrized dynamical systems undergo sudden changes in their topological structure. That is, the qualitative behaviour of their trajectories change abruptly at certain points of the parameter space. These points are called bifurcation points and some of the commonly discussed bifurcations include: Poincaré-Andronov-Hopf Bifurcation, Turing bifurcations, pitch fork bifurcation, saddle-node bifurcations, etc. Bifurcation theorems facilitate, in particular, the possibility of working with dimensions higher than two, which is a serious restriction in the case of the *P-B theorem*.

Bifurcation theorems, too, for proving the existence of limit cycles have been widely used in the Keynesian tradition of business cycle theories. Torre (1977) used bifurcation theorems to establish the presence of a stable limit cycle in what he called as the 'Complete Keynesian system', which is structurally stable. However, bifurcation theorems have also been widely used in other traditions of endogenous business cycle theory, for example, to establish competitive endogenous business cycles in overlapping generations models (see, for example, Grandmont (1985)).

Within the tradition of endogenous business cycles that this chapter has concentrated on, there has been a recent interest in applying bifurcation theorems to Kaldor, Goodwin and Hicks models. Flaschel (2009) summarizes some of the common criteria used for establishing the presence of bifurcations in the appendix. Kaldor's system has been modified to incorporate time-delays and bifurcations occur in these models due to changes in the parameters, capturing either the time delay or the intensity of the effect of the difference between savings and investment on income (see Kaddar and Alaoui (2008), Krawiec and Szydlowski (1999) and Wang and Wu (2009)). Similar investiga-

[20] There is the other case of difference-differential equations, also known as the delay differential equations, which was used by Frisch, Kalecki (in their *time-to-build* models and by Goodwin (1951). The latter reduced this to a Rayleigh equation by taking a Taylor series expansion and neglecting higher order terms (which he should not have done, see Dharmaraj and Velupillai (2011).

[21] Refer to Velupillai (2010) for arguments against the indiscriminate use of real numbers (and functions defined on real number domains) in economics, since economic quantities can, at best, be rational numbers.

tions have been carried out for modified versions of Goodwin's model (for example, in Matsumoto (2009)) and Hicks' model (in Matsumoto and Szidarovszky (2010)). By the way, all these modified models are again are special cases of the canonical equation (4.1).

## 4.4 Existence theorems and Computability

### Existence of solutions for IVP of an ODE Vs Poincaré-Bendixson theorem:

While proving existence is fairly straight forward in the case of linear system of differential equations, it is more complicated for nonlinear equations. For some nonlinear equations, there may not be any solutions for certain initial conditions and in some cases there might be many solutions for the same initial condition. For a general case, including nonlinear differential equations, there are additional conditions that need to be imposed and which guarantee (local) existence and uniqueness of solutions. At this point, it is pertinent to wonder why this general existence, uniqueness theorem for solution of ordinary differential equations, a more general theorem, valid for any finite dimension(n), was not used here for establishing the periodic solution. To understand this, we need to look more into the difference between these two theorems.

**Theorem 5.** ***Existence and Uniqueness of Solutions:*** *Consider the initial value problem*

$$X' = F(x), X(0) = X_0$$

*where, $X_0 \in \mathbb{R}^n$. Suppose that F: $\mathbb{R}^n \to \mathbb{R}^n$ is $C^1$. Then there exists a unique solution of this initial value problem. More precisely, there exists $a > 0$ and a unique solution*

$$X : (-a, a) \to \mathbb{R}^n$$

*of this differential equation satisfying the initial condition $X(0) = X_0$*

In this proof of existence, a sequence of functions are defined via Picard's iteration procedure, by successive approximation of the functions. Later, it is shown that the sequence of functions thus defined would *uniformly converge* to the solution of the given differential equation. Note that $F$ is $C^1$ implies that the function $F$ satisfies the *Lipschitz condition*, which in turns ensures the contraction to a unique fixed point.

**Definition 6.** ***Lipschitz Continuity****: Let $D \subset \mathbb{R}^n$ and F: $D \to \mathbb{R}^n$. F is called Lipschitz continuous if for any closed and bounded interval $I \subset D$ there exists a $K \in \mathbb{R}$ and $K < \infty$ with*

$$|F(x) - F(y)| \leq K|x - y|, \forall x, y \in I \tag{4.17}$$

*and K is called the Lipschitz constant.*

If the functions $f(x)$ and $g(x)$ in the Liénard equation satisfy the of the *Lipschitz condition*, then, why was the general existence proof for a system of differential equations not used and what is different in the planar case?

Firstly, the above existence theorem, known as the *Picard-Lindelöf theorem*, like the *Cauchy-Peano existence theorem*, is a part of the analytical tradition of studying differential equations. In this approach, the solution of the differential equation is obtained in terms of finding analytic functions equivalent to the differential equation. Though this quantitative approach is valid and accurate, it is quite complicated in practice and often limited in scope. These difficulties led Poincaré to develop the qualitative theory of differential equations. The idea in the qualitative, geometrical, approach is to find methods to understand the qualitative behaviour of the solutions of the differential equations. Theorems such as P-B and Levinson-Smith, are existence theorems in this latter tradition. Therefore, it needs to be clarified that the existence theorems in nonlinear, endogenous theories of business cycles were essentially invoked to establish certain properties of solutions in the spirit of the qualitative tradition.

Secondly, this qualitative approach to understanding dynamics meant that NETBC was an exercise to understand certain global properties of the economic system. The Picard-Lindelöf theorem, however, is a local existence theorem. Initial efforts in NETBC were to establish global behaviour of economic system, in this case – the presence and persistence of cyclical fluctuations, independent of the initial conditions. For planar dynamical systems, the P-B theorem provides a positive criterion for establishing this, in terms of identifying the presence of limit cycles. It is more complicated for higher dimensions and there equivalent for this theorem in higher dimensions (for $n \geq 3$).

Thirdly, the P-B theorem proves the plausibility of *atleast* one limit cycle. However, it does not say anything about the number of limit cycles that are present in the given system and their location, which might be of interest to economists. This leads directly to the unresolved Part B of Hilbert's 16th Problem (see Velupillai (2008).

### 4.4.1 Computability of the attractors on the plane

Besides proving the existence of endogenous business cycles, economists would like study the nature of these cycles, more generally, the properties of the attractors. In the case of planar dynamical systems that characterize the nonlinear models we have discussed, the P-B theorem provides a criterion for the existence of limit cycles. The question that we ask is the following: Is it possible to precisely determine these attractors (in the case of the P-B theorem) and their basins of attraction, in order to gain a deeper insight into the precise nature of the aggregate dynamics of the economy and to undertake meaningful computational experiments? In other words, are attractors of a given dynamical system on the plane *decidable* (*algorithmically*)? Alternatively, can we determine the basin of attraction in which the economy is located at a given time, from the data we have?

Before answering these questions, it would be fruitful to understand why computability of the attractors is important. Studying these attractors to which an economy tends to, and their properties, would have important bearing on theorizing on the kind of policies and control mechanisms to be adopted. As mentioned earlier, *the P-B theorem is an existence theorem, which is proved non-constructively*. This means that the proof does not offer a procedure to identify these attractors, instead, it merely states that such a mathematical object exists. *It might be the case that there exists no finite procedure to determine these attractors*.

Given that the nonlinear dynamical systems are often impossible to solve analytically, simulation experiments to study these systems have relied on numerical methods. Since the above theorem is valid only for a (planar) system of differential equations defined over real number domains, it poses challenges to compute, since not all real numbers are computable. There have been attempts to bridge this gap on two fronts – numerical analysis and computable analysis (and, more recently, also via constructive analysis).

Three questions can be posed at this point.

1. What are the conditions under which we can establish that a solution of the dynamical system and a numerical procedure used to approximate it are equivalent?

2. Can we say something about the possibility of computing the solutions of these dynamical systems from the point of view of computability, instead of numerical analysis?

3. What are the other mathematical tools to gain insight into the nature of the attractors?

To answer the first question, it is worthwhile to note that a numerical procedure is essentially a discrete dynamic object. The numerical procedures – like the Euler method and Runge-Kutta method – can themselves be viewed as dynamical systems. For the solutions approximated by these numerical procedures to be same as that of the original system, it is necessary to show the equivalence between the two and the conditions under which they are so. That is, it is essential to demonstrate that the numerical procedure employed would converge to same invariant sets of the original dynamical system and that structural properties of the original system are retained by the numerical procedure as well. Therefore, the choice of numerical procedures and the conditions under which these procedures faithfully approximate the dynamics of the system which is studied need to considered. Even in this case, it is necessary to discipline these computational investigations using the theory of computability to delineate what can and cannot be computed. These ideas are discussed in Stuart and

Humphries (1998) and Velupillai (2010), pp.127-151).

For the second, we need to evaluate the P-B theorem from a constructive and computable point of view and identify the source of the non-constructive or uncomputable content, if there are any. P-B theorem requires that the limit sets are compact subsets of Euclidean space. For example, a classic Compactness criterion for a metric space is given by the Heine-Borel Theorem.

First, a subset X of E is said to be closed if and only if every sequence $\{x_n\}$ of elements of X which converges in E has its limit in X. A subset X of a metric space $(E, d)$ is said to be bounded if there exists a number $A$ such that $d(x, y) \leq A$ for all $x, y \in X$.

**Theorem 7.** *Heine-Borel Theorem:*
*A subset X of $\mathbb{R}^n$ is compact if and only if X is closed and bounded. Alternatively, A subset X of $\mathbb{R}^n$ is compact if and only if each open cover of X has a finite subcover.*

This in turn appeals to the Bolzano-Weierstrass theorem, which states that every bounded real sequence in $\mathbb{R}^n$ has a convergent subsequence. This theorem is in turn based on the axiom of completeness for the real numbers. Therefore this kind of definition of compactness is dependent on the completeness axiom of the Cauchy sequences.

**Definition 8.** *Completeness Axiom*
*Every non-empty set of $\mathbb{R}$ which is bounded from above has a least upper bound. Every non-empty set of $\mathbb{R}$ which is bounded from below has a greatest lower bound.*

This property does not hold for the set of computable or constructive real numbers.

**Theorem 9.** *Specker's Theorem:*
*A sequence exists with an upper bound, but without a least upper bound.*

The Bolzano-Weierstrass theorem is not valid in Constructive Analysis and the *classical* version of the Heine-Borel theorem is not valid in many variants of Computable Analysis (it is however valid in Weihrauch's program with the so-called *type 2 effectivity*). This means that the attractors that are proved to exist by the nonlinear models of business cycles invoking the P-B theorem cannot be computed, unless we have a *computable (or constructive) definition of compact sets*. This is so as long as one works on the domain of real numbers as in the case of the models of NETBC. The question then is whether there is a way to make these models computable?

First of all, we need to restrict the domain of numbers on which we theorize to computable real numbers and work with only computable functions. Results from computable analysis (of the Weihrauch variety) provides some answers. Graça and Zhong (2011) conclude that the attractors and the basins of attractions are *semi-computable* if we assume that the system is stable. In their scheme, they work with so-called type-2 machines. Stability becomes a necessary condition for ensuring the computability of attractors[22]. By employing a notion of computability on closed, open and compact sets as outlined, for example, in Brattka and Weihrauch (1999) and Weihrauch (2000)), they are able to prove the following.

**Theorem 10.** *Let $x' = f(x)$ be a planar dynamical system. Assume that $f \in C^1(\mathbb{R}^2)$ and that the system is structurally stable. Let $K \subseteq \mathbb{R}^2$ be a computable compact set and let $K_{cycles}$ be the union of all hyperbolic periodic orbits of the system, is contained in K. Then, given as input ρ-names of f and K, one can compute a sequence of closed sets $\{K^n_{cycles}\}_{n\in\mathbb{N}}$ with the following properties:*

1. $K^n_{cycles} \subseteq K$ *for every* $n \in \mathbb{N}$
2. $K^{n+1}_{cycles} \subseteq K^n_{cycles}$ *for every* $n \in \mathbb{N}$
3. $\lim_{n\to\infty} K^n_{cycles} = K_{cycles}$

This means that, under the assumption of structural stability, if one can supply the $\rho$ names of $f$ and the compact set $K$ as input, there is an algorithm which can tell, in finite time, whether $f$ has a periodic orbit of the above dynamical system in the compact set $K$. Since the periodic orbits are only *semi-decidable* in this case, one may need an infinite amount of time, countably calibrated, to conclude that $K$ does not contain a periodic orbit. The same is true for the equilibrium points of the above dynamical system. However, the number of attractors of a given compact set are, in general, undecidable - even if the functional forms are analytic.
All this is part of living with the Halting Problem for Turing Machines and the problem of recursively enumerable sets that are not recursive.

Finally to answer the third question, in the affirmative, we want to mention about the possibilities offered by non-standard analysis (see Velupillai (2012b)).

## 4.5 Concluding Notes

The introduction of existence proofs, in particular, the Poincaré-Bendixson theorem, transformed the way mathematical NETBC on the plane envisioned the economics of aggregate fluctuations. In particular, it had an important methodological influence on NETBC, in terms of the mathematical formalisms that the economic theory of aggregate fluctuations embraced and also the role of existence proofs becoming a dominant way of theorizing about economic fluctuations. Whether this development was due to the absence of results in dynamical systems theory or due to the shortcoming of the theorists in terms of developing appropriate mathematical tools for the theoretical problems at hand is not clear. So a categorical judgement on the overall benefit due

[22] These themes are discussed more elaborately in the last chapter of this thesis.

to the wide acceptance of this theorem, for the development NETBC, is not easy to evaluate - one way or the other. What is undeniable, however, is its importance in the modelling of theories of nonlinear endogenous business cycles.

## 4.A Limit cycles on the Liénard plane

> "Therefore, making only assumptions acceptable to most business cycle theorists, along with two simple approximations, we have been able to arrive at a stable, cyclical motion which is self-generating and self-perpetuating. For performing the graphical integration it is convenient, letting $v = \dot{x}$, to rewrite
>
> $$\ddot{x} + X(\dot{x}) + x = 0$$
>
> as
>
> $$vdv + X(v) + x = 0$$
>
> "Thus we have an extremely simple, nonlinear, first order, differential equation, which may easily (**the Liénard method makes it truly easy** ) be integrated graphically, provided we have an empirically given $X(v)$ curve. $X(v)$ need not be expressible in any simple mathematical form, although some approximation, say by a cubic expression, does facilitate qualitative discussion of the type of system."
> Goodwin (1951), p. 14, emphasis added.

In this section, we elaborate on the 'Liénard method' and how it makes the graphical integration truly easy, as mentioned by Goodwin in the above quote. The questions that we are interested in exploring are: What is the difference between the normal phase plane and the Liénard plane? What is the Liénard graphical method? What are the advantages of using Liénard plane and how this has been used in the theory of endogenous business cycles?

### 4.A.1 Liénard plane:

The early phase of NETBC proceeded in terms of reducing a model of the economy to a canonical nonlinear equation and establishing that the economy was capable of endogenous, self-sustaining oscillations. As we have shown earlier, the two important nonlinear equations in NETBC are the van der pol equation and the Rayleigh equation, which are special cases of a more general equation, namely, the Liénard equation. These equations were then analyzed on a special phase plane (Liénard plane), different from the normal phase plane. By using a special method to construct integral curves on this Liénard plane, it was possible to demonstrate the presence of a limit cycle, graphically.

Let us consider the Liénard equation,

$$\ddot{x} + f(x)\dot{x} + x = 0 \tag{4.18}$$

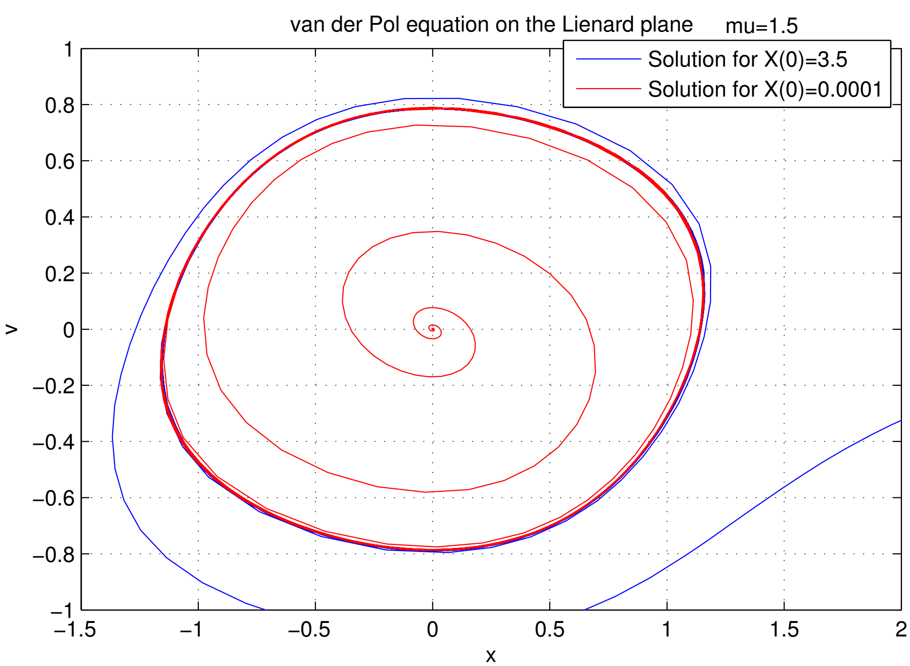
Now let us introduce a new variable $v = \dot{x} + F(x)$, where $F(x) = \int_0^x f(s)ds$
We can write the above equation, equivalently, as the following system:

$$\begin{aligned} \frac{dx}{dt} &= v - F(x) \\ \frac{dv}{dt} &= -x \end{aligned} \tag{4.19}$$

For analyzing the behaviour of this system, let us define a modified plane, Liénard plane:

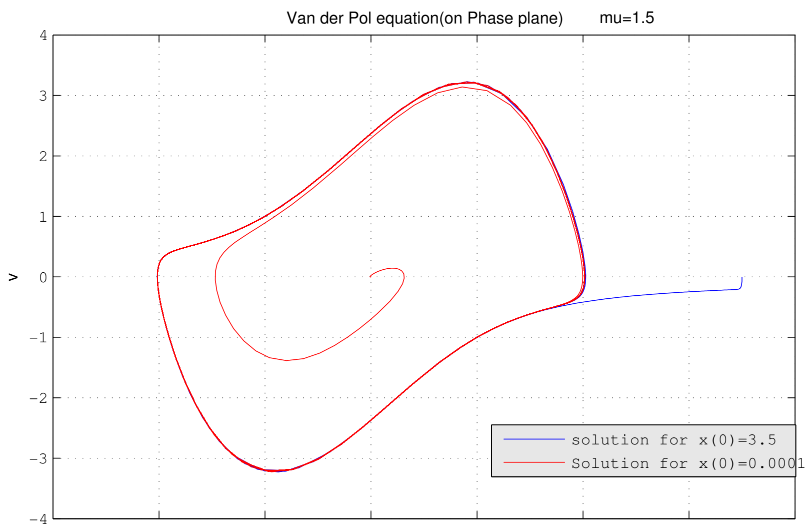
$$\begin{aligned} v &= \dot{x} + F(x) \\ x &= -\dot{v} \end{aligned}$$

Note that this is different from the ordinary phase plane$(x, y)$, where $y = \dot{x}$. In the ordinary phase plane, the velocity is counted along the vertical axis with respect to the abscissa(x axis). In the Liénard plane$(x, v)$, the velocity is counted along the vertical axis, not with respect to the regular abscissa, but the new 'curvilinear abscissa'. That is, because of the new transformation of co-ordinates that we introduced, $v = \dot{x} + F(x)$, our abscissa has been redefined. Note that the curve $F(x)$ is the characteristic of the equation. Therefore, the trajectories that are traced on the Liénard plane look different from those on the phase plane.




But, what are the advantages of this co-ordinate change, in comparison to the regular phase plane? In order to see this, let us first understand the relationship between the two planes, in terms of the trajectories. There is a one-to-one correspondence implies that the trajectories traced on the Liénard plane can be easily transformed to the ordinary phase plane through the relation,

$$\dot{x} = v - F(x)$$




If the functions $f(x)$ and $x$ are assumed to satisfy the *Lipschitz* condition, existence and uniqueness of solutions are guaranteed on the ordinary phase plane. Since, $x$ is $C^1$ and $F(x)$ is an integral of a $C^1$ function $f(x)$, $F(x)$ is also *Lipschitz*. This means that the existence and uniqueness of solutions to the ODE is applicable to the Liénard plane $(x, v)$ as well. This guarantees that the trajectories cannot cross. The one-to-one correspondence, which is continuous both ways, in turn means that the closed path in one plane has a corresponding closed path in the other plane and these qualitative attributes of their solutions do not change across the two planes.

Eliminating time from the above system of equations, we have

$$\frac{dv}{dx} = \frac{-x}{v - F(x)} \tag{4.20}$$

The above equation gives the equation for the paths of the above dynamical system and in order to analyze these paths, the Liénard plane offers some advantages over the normal phase plane, in the absence of the possibilities of numerical integration using computers. However, this geometric method is not without its advantages and this pre-digital computing era method in fact enabled one of the important discoveries in dynamical systems theory, entirely due to pure macrodynamic motivations: the *one-sided oscillator* with the *Goodwin-characteristic* (for a fairly full discussion of this 'story', see Velupillai (2008).

On examining the above equation, we can infer that the paths described by the equation become horizontal when they cross the $v$-axis. Similarly, only when they cross the curve $v = F(x)$, they become vertical. This feature makes graphical construction of the integral curves easier. Also, if we assume that $x$ and $F(x)$ are odd functions[23], then the equation of the path does not change if the $(x, v)$ are replaced by $(-x, -v)$. This implies that the curve is symmetric with respect to the origin and therefore the paths traced on one half of the plane, then those on the other half of the plane can be obtained by mere reflection by exploiting the property that they are symmetric with respect to the origin. This property of the Liénard plane, in particular, makes it easier, to analyze the paths geometrically.

### 4.A.2 Liénard's Method of Graphical Integration:

By setting the variable, $y = \dot{x}$, we can rewrite the Liénard equation as the following:

$$\frac{dy}{dx} + f(x) + \frac{x}{y} = 0 \tag{4.21}$$

If we change the co-ordinates by introducing $v = y + F(x)$, where $F(x) = \int_0^x f(s)ds$, then the differential equation becomes:

$$\frac{dv}{dx} - \frac{dF(x)}{dx} + f(x) + \frac{x}{v - F(x)} = 0 \tag{4.22}$$

$$\frac{dv}{dx} + \frac{x}{v - F(x)} = 0 \tag{4.23}$$

This can be rewritten as

$$xdx + (v - F(x))dv = 0$$

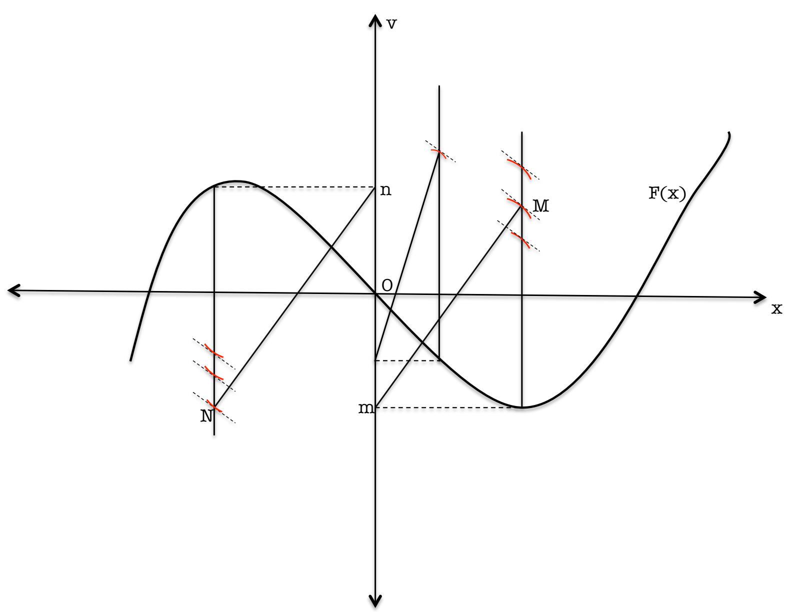
This is nothing but the equation of the normal, with respect to the curvilinear (abscissa) axis and the corresponding ordinate. It is easier to visualize it as the normal passing through $(0, F(x))$:

$$(x - X)dx + (v - V)dv = 0 \tag{4.24}$$

The procedure to construct integral curves on the plane is the following: Choose any point, say M, on the plane (x,v). In order to construct the curve, we can drop a perpendicular to the curve $F(x)$ and obtain a projection $m$ on the axis parallel to this perpendicular (in the case of the van der Pol equation, the projection would be on the $V$-axis). Keeping $m$ as the center, one can trace a small arc that passes through $M$. Similarly, one can trace a family of small arcs keeping m as the center along the perpendicular from M.The line $mM$ gives the normal and it is easy to construct a line perpendicular to $mM$ to obtain the tangent at this point. By repeating the same procedure for different points and consequently, the projections on $V$- axis, we can obtain a family of curves. Using this, we can then construct the integral curves and one of these curves in the family will be a closed path, provided the conditions that one

[23]Note that if $f(x)$ is assumed to be even, $F(x) = \int_0^x f(s)ds$ becomes an odd function.

imposes on the Liénard equation are met. This, in essence, is the method of graphical integration that was developed by Liénard. On the Liénard plane, given that we know that the paths can be horizontal and vertical when they cross the $v$ axis and the curve $F(x)$ respectively makes this procedure very intuitive and easy. Goodwin uses this method in order to geometrically demonstrate the presence of a limit cycle. Yasui uses the same technique to proceed with graphical integration for a van der Pol type equation. In addition to this geometric demonstration, he formulates his investigation along the lines of existence proofs, stating the conditions that need to be met by the functions in order to formally ensure the existence of periodic solutions. We discuss this in detail in the following section. In contrast, Goodwin's demonstration was purely by geometric construction. In his case, he showed the presence of limit cycle in a Rayleigh type equation, which has a cubic characteristic. Note that the precise shape of the characteristic does not limit this graphical integration method - which is more general and can be applied regardless of the shape of the characteristic.




**Construction of Integral curves - Liénard's Graphical Method**

### 4.A.3 Liénard's criterion:

Given the advantages listed in first section above, how can one go about establishing the presence of a unique, stable, limit cycle on the Liénard plane? To see this, we need to introduce the variable that defines the total every of the system$\lambda$ as the sum of kinetic and potential energies:

$$\lambda(x,v) = \frac{v^2}{2} + \int_0^x x ds$$

$$\lambda(x,v) = \frac{v^2}{2} + \frac{x^2}{2}$$

$$\begin{aligned}
\frac{d\lambda(x,v)}{dt} &= \frac{d}{dt}\left[\frac{v^2}{2} + \frac{x^2}{2}\right] \\
&= \frac{d}{dt}\left[\frac{1}{2}(\dot{x} + F(x))^2 + \frac{x^2}{2}\right] \\
&= \dot{x}[\ddot{x} + f(x)\dot{x} + x] + F(x)\frac{d}{dt}(\dot{x} + F(x)) \\
&= F(x)\frac{d}{dt}(\dot{x} + F(x)) \\
d\lambda(x,v) &= F(x)dv
\end{aligned} \tag{4.25}$$

The rationale behind this is given by the Liénard's criterion, which states that for a system to be in a state of sustained oscillation, the change in total energy of the system over a given cycle must be zero. For this the curvilinear integral taken along the trajectory should be zero.

$$\oint F(x)dv = 0$$

First, the condition for the presence of limit cycles can be intuitively inferred from the symmetry condition discussed above. Given that the paths are symmetric to the origin, let us first analyze the length of the two intercepts, along the ordinate $(v - axis)$, due these paths traced on the right half of the plane (call them OA and OC). If the length of these intercepts are not equal, then the paths cannot be closed, due to the fact the the paths are symmetric about the origin and the reflection on the other half of the plane would suggest that the paths will never meet if OC is greater or smaller than OA. In order to prove that there is a unique closed path, it would be sufficient to show that intercepts are equal. This is also the idea behind the proof [24] that we invoked a purely geometric criterion for showing the existence of limit cycles in the Kaldor model.

The use of Levinson-Smith theorem in NETBC, by Yasui, Morishima, Ichimura and others, was essentially along these lines - assuming that the relevant economic variables and their functional forms are odd and even, along with other requirements of this theorem, guarantees the existence of an unique, stable limit cycle. Symmetry property of the trajectories on the Liénard plane is particularly helpful in the search for closed paths.

[24] Refer Minorsky (1974) and Hirsch et al. (2004) for the detailed proof.

# Chapter 5

# Uniqueness theorems in NETBC

In this chapter, we look at the uniqueness theorems that have been employed in macrodynamics in planar models. We confine our attention to the Nonlinear, Endogenous theories of Business cycle (NETBC), in particular, to the pioneering models of Goodwin, Kaldor, Hicks and their variations. We focus on the uniqueness proofs concerning the attractors (limit cycles) in these models and examine them. Section 2 provides a survey of different uniqueness theorems that were used in this tradition. In Section 3 Goodwin's nonlinear cycle model is considered and we apply a sufficiency theorem for the nonlinear accelerator model, with just one non-linearity. We point out the connection that this theorem has with Goodwin's own contribution. Section 4 presents some ways in which one can go beyond the existence-uniqueness mode of theorizing in this tradition.

## 5.1 Uniqueness proofs in NETBC

In the planar models of NETBC, the qualitative nature of the attractors that underpin these theories is fairly obvious, viz, limit cycles[1]. Important models in this tradition are Goodwin (1951), Kaldor (1940), Hicks (1950), Lundberg (1937). These models were largely in the Keynesian tradition. Their unifying thread was the presence of nonlinearities in how different economic variables were related. For example: relationship between income, savings and investment; presence of limits to investment and growth due to natural economic constraints such as full employment. This nonlinearity played a crucial role in explaining the observed, sustained fluctuations in aggregate variables such as output and employment. Mathematically, what these economic theories sought to explain (viz, sustained fluctuations of aggregate economic variables over time) were translated to demonstrating the presence of periodic solutions at a local or global level[2]. These theories were formulated in terms of models using differential or

[1]There is also the case of 'centers', which is associated with the growth cycle model of Goodwin (1967)

[2]However, this may not be applicable in the case of Lundberg(1937).

difference equations or dynamical systems. The early models that were formulated in continuous time were mostly reduced to one or the other special case of the Liénard equation[3] – van der pol equation (in the case of Kaldor's model) or the Rayleigh equation Goodwin (1951)), which were known to possess stable periodic solutions. Later, with the development in dynamical systems theory, these models were formulated in terms of dynamical systems and the attention shifted to demonstrating the existence of sustained oscillations by means of formal existence proofs.Theorems such as Poincaré-Bendixson theorem were used to establish the necessary and sufficient conditions for the presence of limit cycles.

Compared to the use of existence proofs in NETBC, studies providing results concerning the number of possible attractors have been relatively few. The proof of existence and uniqueness was established in some of the early models models by invoking the Levinson-Smith theorem. While Poincaré- Bendixson guarantees the existence of *atleast* one limit cycle for planar dynamical systems, the Levinson and Smith theorem establishes the sufficient conditions under which a Liénard equation (a special case of a second-order differential equation) can have a unique isolated periodic solution(limit cycle). There are a couple of observations that may be relevant here. First, uniqueness theorems that are used often provide only sufficient conditions and not the necessary and sufficient conditions for the presence of a unique limit cycle. Presupposing that a cycle already exists for a given system, these theorems provide the conditions for such a cycle to be unique. Therefore, proof of existence is provided first and then these sufficiency conditions are provided for the strip in which the limit cycle exists. Secondly, it is, in some sense, relatively easy to provide sufficient conditions, compared to proving the existence of the cycle. The more general mathematical problem concerning the upper bound on the number of limit cycles for a planar polynomial vector field (as a function of the degree of the polynomial), posed as the second part of Hilbert's $16^{th}$ problem, still remains unresolved till today. Consequently, a 'complete' characterization of the nature and number of attractors, even for Liénard equation (which is a special case of the planar polynomial vector fields), is still beyond reach.

### 5.1.1 Uniqueness of limit cycle - Kaldor's Model

In the early mathematical models of NETBC, the proof of uniqueness of the limit cycle in the planar models has involved reducing the dynamic model to a generalized Liénard equation and invoking the Levinson-Smith theorem, which provides sufficient conditions for the existence and uniqueness of limit cycles. Depending on the way in which one approximates the economic assumptions into a mathematical model, the number of limit cycles can vary. Therefore, any categorical statement regarding the

[3]Liénard equation is written as

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

presence of unique limit cycle must be evaluated in the light of the approximation involved.
We provide a survey of the studies that employ uniqueness theorems in NETBC. For the purpose of this chapter, we restrict our attention only to analytical proofs for establishing uniqueness and therefore we will not focus on studies which use numerical simulations and other approximate methods. In case of Kaldor's model, the issue of the uniqueness of attractors has been taken up for the different versions of the model by Lorenz (1987), Galeotti and Gori (1989), Ichimura (1955b). Earliest application of sufficient conditions to guarantee a unique limit cycle was by Yasui(1953) who applied Levinson-Smith theorem to his version of Kaldor's model. In this case, the model was reduced to a van der Pol type equation, which is a particular case of the more general Liénard equation.

**Theorem 1 (Levinson Smith Theorem).** *[Gandolfo (2005), p.440]*
*Consider a two-dimensional differential equation system*

$$\dot{x} = y - f(x)$$
$$\dot{y} = -g(x)$$

*which is represented as a second-order differential equation,*

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

*The above equation has a unique periodic solution if the following conditions are satisfied.*

1. $f'(x)$ *and* $g(x)$ *are* $C^1$
2. $\exists x_1 > 0$ *and* $x_2 > 0$ *such that for* $-x_1 < x < x_2 : f'(x) < 0$ *and* $\geq 0$ *otherwise.*
3. $xg(x) > 0 \; \forall x \neq 0$
4. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} G(x) = \infty$ *where* $f(x) = \int_0^x f'(s)ds$ *and* $G(x) = \int_0^x g(s)ds$
5. $G(-x_1) = G(x_2)$

Ichimura (1955b) discussed the possibility of applying the above theorem to a particular case of his model, in which he attempted to synthesize the theory of Kaldor, Goodwin and Hicks. However, he perceptively noted that the symmetry condition ($G(-x_1) = G(x_2)$) may not hold for his system and therefore uniqueness of the limit cycle was not certain. Lorenz (1987) later took up the Kaldor model and explicitly addressed the question of uniqueness. He observed that Kaldor's model (Chang and Smyth's version) does not reduce to a generalized Liénard equation, and therefore Levinson and Smith theorem cannot be applied. In particular, he argued the assumption concerning symmetry of $G(x)$ may be too restrictive from an economic point of view. He contended that it is not possible to apply the above theorem unless one of the following assumptions are made: Investment is assumed to be independent of capital stock or that changes in capital stock is entirely determined by savings, which is independent of the level of capital. This is because in the Kaldor model, as long as one assumes investment function is dependent on both capital stock and income, the resulting second-order differential equation is not a Liénard equation. His main message was that it was not possible to retain all the assumptions of the original economic model, without making further simplifying assumptions to satisfy mathematical requirements of the theorems invoked.

In reply to this, Galeotti and Gori (1989) contended that it is possible to make use of other uniqueness theorems for Kaldor's trade cycle model and demonstrate the presence of unique limit cycle. By making convenient assumptions and appropriate transformations, they show that Kaldor's system is reducible to a Liénard system.

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x) \end{aligned} \tag{5.1}$$

The existence of a limit cycle is proved by using the theorem by Fillipov and they provide the sufficiency conditions for uniqueness of the limit cycle for this model through the following theorem.

**Theorem 2** (Zhang Zhi-Fen[4]). *Suppose the system (5.1) above satisfies the following conditions:*

1. *There exists* $a \geq 0$ *such that,* $F_1(z) \leq 0 \leq F_2(z)$ *for* $0 \leq z \leq a$, $F_1(z) \not\equiv F_2(z)$ *for* $0 < z \ll 1$, $F_1(z) > 0$ *for* $z > a$; $F_2'(z) \leq 0$ *for* $z \in \{z > 0 | F_2(z) < 0\}$
2. $F_1'(z)$ *is non-decreasing for* $z > a$
3. *if* $F_1(z) = F_2(u)$ *with* $a < z < u$, *then* $F_1'(z) \geq F_2'(u)$

*Then the system has at most one limit cycle, which, if exists, must be stable.*

The above theorem is utilized by applying the Fillipov transformation[5].

However, in the process of doing so, they overlook the essential message that there is a tendency among economists to force their economic intuitions to the demands of the existence-uniqueness theorems.

[4]Galeotti and Gori refer to Zhi-Fen (1986b) and Zhi-Fen (1986a). Since the second reference is difficult to access, we refer the reader to Theorem 1, section 2 of Xianwu et al. (1994) for the proof.

[5]Fillipov transformation is done as follows:

$$\begin{aligned} z_1 &= \int_0^x g(t)dt \quad \text{if } x \geq 0; F_1(z_1) = F(x) as\ x \geq 0 \\ z_2 &= \int_0^x g(t)dt \quad \text{if } x \leq 0; F_2(z_2) = F(x) as\ x \leq 0 \end{aligned}$$

$z_i$ is the integral curve of $g(x)$ and therefore instead of the trajectory of (5.1) in the right and left half-planes, due to the transformation, one deals with the integral curves $\frac{dz}{dy} = F_i(z) - y, i = 1, 2$. Uniqueness in this case is established by way of an comparison argument involving $F_1(z)$ and $F_2(z)$ for $z > 0$.

Goodwin (1951) did not'prove' existence or uniqueness of the limit cycle by invoking theorems of these kind, instead he demonstrated the presence of single stable limit cycle (for an S-shaped characteristic) by using geometric methods (Liénard integration). This was later addressed by Sasakura (1996), pp.1771-73:

> In Goodwin's (1951) model there has been something mysterious to business cycle theorists, since, in spite of the simple structure, the question: "Does the model have really a unique stable limit cycle?" could not be solved in general circumstances. In this paper I gave a correct answer to the question: 'Yes, as was expected.' The model has a unique stable limit cycle in an economically meaningful region.

His demonstrated the uniqueness of the limit cycle by using the following theorem, which provides the sufficient conditions.

**Theorem 3.** *Luo Ding-Jun Uniqueness Theorem*

*For the system*

$$\dot{x} = y - F(x)$$
$$\dot{y} = -x$$

*if $F'(x) - F(x)/x \geq 0$ (or $\leq 0$) for all $x \neq 0$, and* ***in the strip where the limit cycle exists*** *the left side of the above formula is not identically zero, then the system has at most one limit cycle. (Ye et al. (1986), 139-140)*

The above theorems comprehensively cover almost all the uniqueness results that were employed within this tradition.

## 5.2 Uniqueness of Limit cycle and Goodwin's (1951) model

There has been a fair amount of ambiguity about the number of limit cycles that are associated with the Goodwin model. On the one hand, Sasakura points out the 'academic belief' that was held about the presence of a unique limit cycle for this model. Flaschel(2009) also resonates the same belief for his version of Goodwin's model, but at the same time makes a puzzling remark[6] that there could be multiple limit cycles:

> ... **We state without proof that a [Goodwin] system such as the one we depicted earlier will not only have a closed orbit, but will in fact exhibit just one limit cycle, which will be a globally stable attractor for all other trajectories** of this dynamical system in the above depicted domain.
>
> For economic purposes, **it is, however, not necessary to have a unique and stable limit cycle under all circumstances**. Figure 3.13 shows that all points

[6] In footnote 75, Flaschel's remark about uniqueness is a bit misleading and his reference to Ye et al. (1986) is inaccurate. It is misleading because the uniqueness theorems discussed in Ye et al. (1986) deal with 'sufficiency' conditions and not 'necessary' conditions.

close to the boundary of the box as well as points close to the stationary state of the dynamics cannot lie on the limit cycle, but will be attracted by one (not necessarily the same) in the interior of the box.
- Flaschel (2009), p. 97–98

In this section, let us take a closer look at Goodwin's model in the light of the above discussion on the uniqueness of limit cycle and hope to clarify some of these ambiguities.

### 5.2.1 Goodwin's Nonlinear Model

Let us briefly sketch the business cycle model developed by Goodwin (1951). In this model, cyclical fluctuations result from the interaction of the dynamic multiplier with a nonlinear accelerator. Let $y, \alpha, \dot{k}, \beta, \epsilon$ be the income, marginal propensity to consume, change in capital stock, autonomous consumption and the lag in consumption for the changes in income, respectively. The multiplier relation can be written as,

$$y = \frac{1}{1-\alpha}(\beta + \dot{k} - \epsilon\dot{y})$$

By introducing a lag between the time at which investment outlays are made and their realization ($\theta$ - the time to build parameter), we have

$$(1-\alpha)y(t+\theta) + \epsilon\dot{y}(t+\theta) = O_A(t+\theta) + O_I(t+\theta)$$

where,

$$O_A(t+\theta) = \beta(t+\theta) + l(t+\theta)$$

and

$$O_I(t+\theta) \simeq O_D \simeq \psi(\dot{y})$$

Here $O_A$ is the sum of autonomous investment and consumption outlays and $O_I$ is the induced investment $\psi(\dot{y})$ (the nonlinear accelerator or the flexible accelerator). This investment function is assumed to be nonlinear[7]. We then have

$$(1-\alpha)y(t+\theta) + \epsilon\dot{y}(t+\theta) = O_A(t+\theta) + \psi(\dot{y}(t)) \tag{5.2}$$

This we shall call as the *Canonical Goodwin Equation*. From here, based on the kind of approximation we would like, we can have different final equations and consequently, the nature and the number of attractors can vary.

Taking the *Canonical Goodwin Equation* and approximating the equation through Taylor-series expansion for the $(t+\theta)$ terms and retaining only the first two terms of $y$ and $\dot{y}$ variable[8], we have:

$$\epsilon\theta\ddot{y} + [\epsilon + (1-\alpha)\theta]\dot{y}] - \varphi(\dot{y}) + (1-\alpha)y = O_A(t+\theta) \tag{5.3}$$

[7] Goodwin also introduces lag in investment and decision outlays - the time lag between decisions on investment and actual investment outlays. However,it is assumed that $O_I(t+\theta) \simeq O_D(t)$

[8] That is, without taking $O_A(t+\theta)$ term into account

This is a second-order nonlinear difference-differential equation. Goodwin shifts the time co-ordinate of the autonomous injections by $\theta$ units and arrives at,

$$\epsilon\theta\ddot{y} + [\epsilon + (1-\alpha)\theta]\dot{y}(t)] - \varphi(\dot{y}) + (1-\alpha)y = O_A(t) \tag{5.4}$$

The delay term here can be viewed as a periodic forcing to the differential system by the injection of autonomous investment. This equation corresponds to a Rayleigh type equation with external forcing[9] and to the best of our knowledge, there is no general result characterizing the attractors completely or establishing that this system has a unique limit cycle. As in the case of the forced van der Pol equation, results are known only for some special cases of the forced Rayleigh type equation. However, Goodwin himself went on to further approximate this equation by assuming that $O_A$ to be constant $O*$ and redefined the variable $y$ in terms of $z$, representing deviations from the equilibrium value $\frac{O*}{(1-\alpha)}$. The system is assumed to have a cubic characteristic, i.e, the nonlinear accelerator is assumed to be a S-shaped function, with two nonlinearities representing the built-in economic constraints.

$$\epsilon\theta\ddot{z} + [\epsilon + (1-\alpha)\theta]\dot{z}(t)] - \varphi(\dot{z}) + (1-\alpha)z = 0 \tag{5.5}$$

To this, he adds the requirement that the equilibrium is locally unstable and that $d\varphi(0)/d\dot{z} > \epsilon + (1-\alpha)\theta$. By defining the following variables, $x = \sqrt{\frac{1-\alpha}{\epsilon\theta}}z/\dot{z}_0$[10] and $t_1 = \sqrt{\frac{1-\alpha}{\epsilon\theta}}t$, the above equation can be reduced to a dimension-less form[11], to the following equation.

$$\ddot{x} + \chi(\dot{x}) + x = 0 \tag{5.6}$$

$$where, \chi(\dot{x}) = \frac{[\epsilon + (1-\alpha)\theta]\dot{z}(t)] - \varphi(\dot{z})}{\sqrt{(1-\alpha)\epsilon\theta}}$$

Goodwin states that

> Consequently the system oscillates with increasing violence in the central region, but as it expands into the outer regions, it enters more and more into an area of positive damping with a growing tendency to attenuation. It is intuitively clear that it will settle down to such a motion as will just balance the two tendencies, although proof requires the rigorous methods developed by Poincaré.
> ... Perfectly general conditions for the stability of motion are complicated and difficult to formulate, but what we can say is that any curve of the general shape of $X(\dot{x})$ [or $\varphi(\dot{y})$] will give rise to **a single, stable limit cycle**.

[pp.13-14, Goodwin (1951)] (emphasis added)

[9] In this case, constant and periodic
[10] $\dot{z}_0$ is any unit to measure velocity
[11] Refer Goodwin (1951), pp. 12-13

He uses the graphical integration method of Liénard, a geometric method and not a proof of existence and uniqueness, to establish the presence of a limit cycle. However, whether there will be 'single, stable limit cycle' for more complicated functional forms of nonlinear accelerator is not addressed.

We can rewrite the above system as

$$\begin{aligned} \dot{x} &= u - \Theta(x) \\ \dot{u} &= -x \end{aligned} \tag{5.7}$$

where,

$$\Theta(x) = \sigma\tau(x)$$

$$\sigma = \frac{1}{\sqrt{(1-\alpha)\epsilon\theta}}$$

and

$$\tau(x) = [\epsilon + (1-\alpha)\theta]\dot{x}(t)] - \varphi(\dot{x})$$

Does this system have a unique limit cycle? The answer to that question depends on the approximation of the nonlinear investment function. For example, Matsumoto (2009) shows that assuming $\varphi(\dot{x}) = vtan^{-1}(x)$(an odd function), the system can have single or multiple limit cycles depending on the values of $\theta$ and the local instability condition $\epsilon + (1-\alpha)\theta - v < 0$ on $\Theta(x)$.

Approximating the nonlinear accelerator $\varphi(\dot{x}) = vtan^{-1}(x)$ by $x - \frac{1}{3}x^3$ would have different result on the number of limit cycles as opposed to taking the higher order terms, by say the fifth order expression, where $\tau(x) = x - \frac{5}{3}x^3 + \frac{2}{5}x^5$, which has two limit cycles[12]. For the above Liénard form representation of the Goodwin model, the number of limit cycles for the system will depend on the degree of $\Theta$ function. Therefore, approximations play a crucial role and it is also necessary to have these approximations correspond to the actual shape of the investment function and compatible with the economic assumptions [13]. Goodwin was aware of this and notes:

> Finally, it should be noted that, while I have assumed a particular shape for $\varphi(\dot{y})$, the power of the Liénard construction is shown by the fact that an equation containing any given curve may be easily integrated. Therefore, whatever sort of investment function is found actually to hold, that type may be completely analyzed in its cyclical functioning. If we look closely into this problem, we find that what is really necessary is to take individual account of many different industries because, while one industry may still have excess capacity, another may be short of fixed capital. Therefore, the combined operation may depend as much on the points at which different

[12] For $\sigma = 0.08$. Refer Matsumoto (2009) for the details of the approximation

[13] For example, Matsumoto notes that the fifth order approximation specified above does not have asymptotic bounds that correspond to the ceiling and the floor and the same goes for the values of $\theta$ being small or large

industries fire into investment activity as on the actual shape of the X function for each industry or any conceivable aggregation of all of them.
-Goodwin (1951), pg. 17.

Instead of approximations of the investment functions, if one resorts to higher order approximations around the time lag parameter $\theta$, it would result in multiple limit cycles as well. This exercise was carried out using analogue and digital simulations by Strotz(et.al) (1953) and Dharmaraj and Velupillai (2011) respectively. The latter study uses a fifth order approximation and shows that the number of periodic solutions increase with retaining more higher order terms in the Taylor series approximation. Over all, the uniqueness of the limit cycle in Goodwin's model and its independence from initial conditions is not so clearcut.

### 5.2.2 Sasakura's proof

Sasakura (1996) starts his analysis by taking the following equation mentioned in the last section:

$$\epsilon\theta\ddot{Y} + [\epsilon + (1-\alpha)\theta]\dot{Y}(t)] - \varphi(\dot{Y}) + (1-\alpha)Y = \beta + l \tag{5.8}$$

He then replaces it with the following, mathematically equivalent system of the previous equation:

$$\begin{aligned} \dot{Y} &= (1/\epsilon)\left(I - (1-\alpha)y + \beta\right) \\ \dot{I} &= (1/\theta)\left(\varphi[\dot{y}(t)] + l - I\right) \end{aligned} \tag{5.9}$$

Here $I = \dot{K}$ and income in not in terms of deviations from its equilibrium value, but its absolute value. By restricting the domain of income and investment values to an economically meaningful region, he proved the existence of a limit cycle for this region using the Poincaré- Bendixson theorem. He considers the dimensionless form (eq. 5.6)

$$\ddot{z} + \zeta'(z)\dot{z} + z = 0 \tag{5.10}$$

Let $u = \dot{z}$ and $v = -z$

$$\dot{u} = v - \zeta(u)$$

$$\dot{v} = -u$$

and uses a theorem that provides sufficient conditions for uniqueness of the limit cycle for this system. Since this theorem does not require any assumptions regarding symmetry, the functional form of the induced investment (or the characteristic) need not be restricted to be symmetric around the origin. From the economic point of view, the meaning of this asymmetry is quite clear.

**Theorem 11** (Luo Ding-Jun Theorem)**.** *For the system*

$$\dot{x} = y - F(x)$$

$$\dot{y} = -x$$

*if $F'(x) - F(x)/x \geq 0$ (or $\leq 0$) for all $x \neq 0$, and* ***in the strip where the limit cycle exists*** *the left side of the above formula is not identically zero, then the system has at most one limit cycle. (Ye et al. (1986),p. 139-140)*

The intuitive meaning of the condition is the following: if the marginal propensity to invest due to an increase in income is less than the average propensity in the strip where the limit cycle exists, such a cycle will be unique, within that strip.

**One Sided Oscillator**

Since the approximations and the assumptions that go into defining the shape of the investment function are crucial in Goodwin's model, we shall examine alternative considerations regarding the shape of the nonlinear accelerator. The nonlinear accelerator in Goodwin (1951) is motivated by the fact that aggregate capital accumulation in an economy cannot go on unhindered forever, since the built-in constraints of the economic system come in to play at some point. In the multiplier accelerator mode, this had meant that the operation of the accelerator mechanism (changes in investment as induced by changes in income) is restricted during the upswing by having reached the desired level of capital (which is a function of income) or by hitting full employment of economic resources. In the downswing, the fact that net investment in fixed capital cannot be lower than the amount required corresponding to rate(s) of depreciation and therefore, poses a natural lower bound. These constraints have traditionally been dubbed as the ceiling and the floor of the economic system. This means that there are two economically plausible nonlinearities, or two plausible bends on the either ends of the accelerator if we wish, in the model of the economy. In Hicks' model, exogenous, autonomous growth factors are superimposed on to a disequilibrium model of fluctuations which incorporates both the ceiling and the floor. Goodwin(1951) model also had two built-in constraints on either side for the accelerator, therefore, utilizing two nonlinearities in order to explain sustained oscillations. In terms of the mathematical structure, this means that the dimensionless form (equation 5.6)of the reduced master equation has a cubic characteristic.

However, the presence of two bounds is not a necessary condition for showing persistence of cycles in this class of planar models. It was persuasively argued by Goodwin in his review of Hicks' book that economic intuition would suggest that *either* the ceiling (full employment) *or* the floor (lower limit on disinvestment, which is zero) would be sufficient to guarantee the presence of sustained oscillations[14]. In terms

[14] He notes:

of the mathematical structure, this meant that a single nonlinearity would be enough to have sustained oscillations, which until then was not thought to be possible. This was shown to be a mathematically plausible by Goodwin himself and thus was born the 'one-sided oscillator'[15].

Now, if we approximate the investment function with a piecewise linear function[16], with only one nonlinearity - either a ceiling or a floor, it is still possible to show the presence of a limit cycle. Let us consider the case where the accelerator becomes inflexible after having reached a ceiling[17]:

$$\phi(\dot{y}) = \begin{cases} \kappa\dot{y} & \text{if } \kappa\dot{y} \leq \dot{k}^U \\ \dot{k} & \text{if } \dot{y} > \kappa\dot{k}^U \end{cases}$$

Here $\kappa$ is the accelerator co-efficient and $\dot{k}^U$ is the investment level once the system reaches the ceiling. This can be appropriately modified in case one wants to shift the focus to the floor and on the downswing. Let us measure income in terms of the deviations from its equilibirum value and express

$$\tau(\dot{z}) = \begin{cases} -[\kappa - (\epsilon + (1-\alpha)\theta)]\dot{z} & \text{if } \dot{z} \leq \dot{k}^U/\kappa \\ -[\dot{k}^U - (\epsilon + (1-\alpha)\theta)]\dot{z} & \text{if } \dot{z} > \dot{k}^U/\kappa \end{cases}$$

When we substitute this and following the time translations $x = \sqrt{\frac{1-\alpha}{\epsilon\theta}}z/\dot{z}_0$ and $t_1 = \sqrt{\frac{1-\alpha}{\epsilon\theta}}t$, we arrive at the 5.5 and reduce it to the dimensionless[18] form, the characteristic of which is as follows:

$$\ddot{x} + \zeta(\dot{x}) + x = 0$$

---

Either the "ceiling" or the "floor" will suffice to check and hence perpetuate it. Thus the boom may die before hitting full employment,but then it will be checked on the downswing by the limit on disinvestment. Or again it may, indeed it ordinarily does, start up again before eliminating the excess capital as a result of autonomous outlays by business or government.
Goodwin (1950), p.319

[15] A detailed discussion of this discovery can be found in Velupillai (1998)

[16] See: Sordi (2006). However, her study focuses on the simulation aspects of the cycle more than the proof of uniqueness.

[17] Note that this ceiling, given by desired capital level given the income level, need not necessarily coincide with the full employment ceiling. This ceiling can come into effect even before the full employment ceiling is reached.

[18] The piecewise linear characteristic above can be approximated by a function to smoothen the discontinuity (Refer Le Corbeiller (1960)) and we have the following second order differential equation.

$$\ddot{z} - \rho(2 - e^{-\dot{z}})\dot{z} + z = 0 \tag{5.11}$$

Under the instability condition assumed by Goodwin, i.e, $\kappa > \epsilon + (1-\alpha)$ and for appropriate parameter values, we can establish the presence of sustained oscillations.

where,

$$\zeta(\dot{z}) = \frac{\tau(\dot{x}\dot{z_0})}{\dot{z_0}\sqrt{(1-\alpha)\epsilon\theta}}$$

The above dimensionless equation can be rewritten as:

$$\begin{aligned}\dot{x} &= -u - \zeta(x) \\ \dot{u} &= x\end{aligned} \tag{5.12}$$

The proof of existence is given by a theorem by de Figueiredo, making use of the Poincaré-Bendixson theorem. (Theorem 1,p. 274, de Figueiredo (1960) ).

**Theorem 12.** ***de Figueiredo's Existence Theorem***
*Consider the system (5.12) above and let*

1. $\zeta(0) = 0$
2. $\zeta'(0)$ *exists and is negative and provided there exists a* $y_0 > 0$ *such that*
3. $\zeta(\dot{x}) > 0$, *min*($\dot{x} \geq y_0$)
4. $2 > -min\zeta'(\dot{x}) < \zeta'(-\dot{x}), (\dot{x} \leq -y_0)$ *except for values of* $\dot{x}$ *at which* $\zeta'(\dot{x})$ *undergoes simple discontinuities.*

*Under the above conditions, the system has* ***atleast*** *one periodic solution.*

Given the above conditions for existence, the sufficient conditions for uniqueness of the periodic orbit are provided by the following theorem:

**Theorem 13.** *de Figueiredo's Uniqueness Theorem*
*Suppose the above system satisfies the above conditions for existence and therefore has a periodic solution. Let there exist a* $y_1 > 0$ *such that following conditions hold:*

1. $\zeta(y_1) = \zeta(0) = 0$
2. $\dot{x}\zeta(\dot{x}) < 0, (0 < |\dot{x}| < y_1)$
3. $\zeta(\dot{x}) > 0, (\dot{x} > y_1)$
4. $\zeta'(\dot{x}) \geq \frac{1}{\dot{x}}\,\zeta(\dot{x}), \dot{x} < 0, \dot{x} > y_1$ *except at values of* $\dot{x}$ *where* $\zeta'(\dot{x})$ *undergoes simple discontinuities. Then the system has a unique periodic solution, except for translations in t.*

A more general theorem for uniqueness is given in de Figueiredo (1970) for a generalized Liénard system.

**Theorem 14.**

$$\dot{x} = y - F(x)$$
$$\dot{y} = -g(x)$$

*Suppose the above system has a periodic solution. Let $xg(x) > 0$ for $x \neq 0$ and the following conditions hold for $g, G$ and $F$:*

1. *$f$ and $g$ are real valued functions which are $C^1$ (Lipschitz condition, which in turn, guarantees the local uniqueness of the solution of the system)*
2. *$\lim_{x \to 0}[g(x)/x]$ exists and is$\neq 0$*
3. *$G(x) \to \infty$ as $x \to \pm\infty$*
4. *$2G(x) + y^2 - F(x)y \neq 0 \; \forall (x, y) \neq (0, 0)$*

*If the inequality $2G(x)f(x) - F(x)g(x) \geq 0$ holds on the interval $x < 0, x > x_0$, where $x_0$ is a positive constant such that $xF(x) < 0$ on $0 < |x| < x_0$, $F(x) > 0$ on $x > x_0$ and $G(x_0) = G(-x_0)$, then the above system has a unique periodic orbit (except for time translations along t axis).*

**Remark 15.** *For $g(x) = x$, the above inequality reduces to $f(x) - F(x)/x \geq 0$. Note that this is the same condition that ensures the uniqueness of periodic oribit in Sasakura's use of Luo Ding-Jun's theorem*[19]*.*

If one feels that we have merely provided yet another sufficiency theorem that is applicable to one of the nonlinear models of business cycle, then may be some justification is warranted. This is not meant to be an exercise in showing that a certain sufficiency theorem can be applied, by searching in the compendium of results on sufficiency conditions for unique limit cycles. In fact, the motivation is the contrary - to show that economic intuition ought to steer the boat, as in the case of Goodwin's review of Hicks' book. The economic intuition that the accelerator was dead during the downswing motivated to Goodwin to come up with the one-sided oscillator as a mode

[19]On Saskura's theorem, Sordi (2006) says:

> ".. a good starting point is the recent contribution by Sasakura (1996), where the existence of a unique stable limit cycle in Goodwin's model (for the general case of asymmetric nonlinearity of the investment function) is rigorously proved."

Sordi, who mentions de Figueiredo's work and Goodwin's role in the discovery of the one-sided oscillator, overlooks the fact the sufficient conditions in Sasakura's theorem that she discusses are in fact the same conditions (see above) that de Figuerido obtained in his thesis, where the one-sided oscillator played a crucial role. Sasakura also mentions about the one-sided oscillator, but does not discuss similarities of the sufficient conditions. Moreover, in footnote 3 in Sasakura (1996), he says that "one of the referees suggested another easier method for proving at least the uniqueness and stability parts as follows: This is to compute the derivative of the Poincaré map and show that (2) is hyperbolic and orbitally asymptotically stable. Then uniqueness drops out easily too.". This is the way de Figuerido proves uniqueness in his 1970 paper.

of clarifying his economic intuition. This was acknowledged by de Figueiredo and Le Corbeiller. de Figueiredo provided sufficient conditions for the unique limit cycles as early as 1958. The economic interpretation of these sufficient conditions are exactly the same as the ones which we seem to have been 'rediscovered' after almost 40 years! In addition, the aim was also show that a more general, economically grounded, yet parsimonious explanation for the persistence of the cycle can be provided within the framework of Goodwin's nonlinear cycle model. If one wishes to prove uniqueness and existence and so on, it is possible, but that however was not the concern for Goodwin. He seemed to be much more interested in unearthing the nature of the cycle. It is in lines with Hick's reaction to the criticism on Value and Capital model, which he felt was out of place:

> "It may also be observed that on this interpretation, VC model is not much affected by the criticism, made against it by some mathematical economists, that the existence of an equilibrium, at positive prices, is not demonstrated. I admire the elegance of the Samuelson–Solow proof of existence; but I still do not think that for my purpose I needed it. **Existence, from my point of view, was a part of the hypothesis; I was asking, if such a system existed, how would it work?** I can understand that for those who are concerned with the defense of 'capitalism', to show the possibility of an arm's length equilibrium (an 'Invisible Hand') is a matter of importance. But that was not, and still is not, my concern."
> - Hicks (1983), pg. 374-5.

## 5.3 Beyond Existence and Uniqueness

Proving existence and uniqueness has preoccupied the theorists of NETBC, and this seems to have limited the potential that nonlinear theories of the cycle hold. This meant not having to uncover the nature of the cycles, their properties, amplitude, frequency etc, instead merely proving that there are cycles, without providing any explicit method either to find or analyze them. Goodwin's disinterest in proving existence may have been partly because he was concerned with solving or simulating them (geometrically). In contrast, the approximations and simplifications in NETBC often got trapped to the practice of reducing models to equations with known results on uniqueness.

> .. we discuss three different accounts of the original model derived from alternative assumptions...
> In each case the corresponding dynamics is written in Lienard form, **so as to apply a classical result of A. Fillipov** and a more recent theorem of the Chinese mathematician Zhang Zhi Fen.
> . . . Indeed in business-cycle Kaldor's systems, which can be driven to Lienard form, several periodic orbits are ruled out: even with an imperfect knowledge of the initial state, the limit cycle to which the economy will

> eventually tend is univocally determined.
> Galeotti and Gori (1989), pp. 137-38

Rather than exploring ways to generalize the models, for example to higher dimensions, removing first approximations and so on, the above quote clearly captures the way in which NETBC modeling activities were directed.

> Lorenz repeatedly underlines the 'ad hoc' character of these further assumptions and their lack of economic meaning ... it is very important, in our opinion, to underline the fact that the empirical relevance of systems presenting a certain number of limit cycles cannot be deduced from the 'realism' of the formal conditions, which are known to guarantee such a dynamical morphology.
> In particular, in the case discussed, if the mathematical hypotheses employed in proving existence and uniqueness of a limit cycle do not appear economically justifiable, this cannot imply that models reducible to Lienard(sic) form, which exhibit a unique periodic orbit, are necessarily 'unrealistic'.
> Galeotti and Gori (1989)), p. 137

Expecting realism of formal conditions is one thing, demanding that the economic assumptions be modified to suit these requirements is quite another. It is regrettable that this always came at the cost of resorting to ad-hoc assumptions which compromised the rich economic intuitions in these models. Galeotti and Gori assume that either savings or investment is a function of income level alone and independent of the level of capital stock. But this does not escape the criticism that was posed by Lorenz that these assumptions are economically restrictive. These simplifications are done so that the model is reducible to the general Liénard form, so that already available theorems can be readily applied. Even when moving on to higher dimensions and other generalizations, the same temptation to trail behind mathematical results is likely to prevail, for example, in the use of bifurcation theory.

This tendency to sacrifice economic theory at the altar of mathematics can actually be reversed. This does not mean giving up on mathematical models altogether. Instead, we need to develop methods and tools that are faithful to the domain of data that we deal with. For this, we have to explicitly acknowledge and remember that the domain of data in which economic quantities, particularly those relevant for cycle theory - employment, income, investment, prices- are rational numbers at best. By assuming that the domain of these quantities are real numbers, we create a wedge - one that keeps us away from meaningfully relating the results to the data we observe on the one hand, and making claims regarding existence and uniqueness which do not have any known correspondence with the economic world, once the assumptions on their domain are relaxed.

Our interest, therefore, should be directed towards models of economic dynamics that are not merely explanatory devices. For endogenous, nonlinear cycle theory, there are possible ways to move beyond the existence-uniqueness paradigm. We suggest that the mathematical business cycle theorists should, after 75 years of adventures with non-linear differential, difference and mixed differential-difference equations, move on to other formalisms and other adventures - but still remaining within the fold of the non-linear theorist. One way to move forward, we believe, is to embrace an algorithmic approach to model economic dynamics. This involves modelling and simulations with *formal algorithms*, so that *dynamic method* is divorced from exclusive reliance on dynamical systems theory.

## 5.4 Algorithmic Economic Dynamics:

In the algorithmic approach, dynamic models of the macroeconomy would be formulated as formal algorithms and these, in turn, can be analyzed. This approach can be fruitful for the following reasons:

1. The mathematics of the digital computers and economic system share a common meeting ground – rational numbers[20].

2. Respecting the nature of the domain of economic data (which is rational numbers, at best) would mean that these models can now be meaningfully related to observed data.

3. Constructing algorithmic models would aim to encapsulate patterns and stylized facts observed in economic data. It involves modeling and analyzing an economic system by constructing algorithms (equivalently, Turing machines[21].) and formally study their evolution to understand its dynamic properties.

4. It would help devote attention to the 'methods' and the 'processes' that are involved while studying economic systems. Instead of being exclusively concerned with steady states and long-run properties, adjustment processes and transition paths, structural change can be addressed and analyzed meaningfully.

5. Questions like 'does this model have an attractor, if so, is the attractor unique?' can be rephrased as: 'Can we devise algorithmic procedures that will help us identify attractors or other long term properties of the system? Can we algorithmically 'decide' whether the attractor is unique?

[20]This does not mean that digital computers can process rational numbers alone. They can handle algebraic numbers too.

[21]Note that Turing machines are themselves discrete dynamical systems in their own right (Moore, 1991), however, acting on natural or rational numbers.

6. Interesting questions related to the characterization and decidability of the attractors can be posed. These will have straightforward implications when it comes to forecasting.

7. There is natural place for complexity and indeterminacy within this framework in the form of computational irreducibility, algorithmic complexity and algorithmic undecidability, respectively.

8. Methodologically, this could be a stepping stone for establishing a constructive approach, avoiding the non-constructive aspects that are routinely used in current mathematical approaches to study economic dynamics, in general.

The above direction is geared towards developing methods and results that are specific to the discipline of economics. It might help to advance the field of endogenous economic dynamics by infusing constructive and algorithmic elements.

# Chapter 6

# Algorithmic Undecidability and Dynamic Economic Theory[1]

In the previous chapters, we discussed the role of non-linearity in endogenous models of business cycles. It was pointed out that there was an excessive focus on proving the existence of periodic attractors (limit cycles) on the plane. This can be thought of as an exercise in constructing dynamic models to illustrate a plausible story in a way that explains the observed phenomena[2]. This method of ascribing an attractor to a system involved the use of existence proofs, which provided the necessary and sufficient conditions for the existence of an equilibrium or periodic cycles. Typically, these existence proofs are non-constructive, devoid of any algorithmic content and do not offer an explicit procedure to determine the actual properties of these attractors. However, computational approaches have been utilized in the past to study such models of economic dynamics. This rises important methodological questions since the mathematics of these models are different from the mathematics that underpins digital computation.

Our task here is to examine some questions concerning the methodology of economic dynamics, in general, from the viewpoint of computation. We will be discussing the issues pertaining to computation in the context of dynamical systems and differential equations, which are widely used for modeling economic dynamics. This has implications for computational approaches to studying endogenous economic dynamics as well. We will largely focus on the methodology of computational dynamics in the context of continuous time models or flows. Most results that are shown for this restricted class can then be extended to hybrid dynamical systems as well.

[1]The inspiration for this chapter comes from various contributions by Velupillai (1999, 2011b, 2010, 2011c, 2009b) on this topic. It expands on some of the themes discussed in a paper co-authored with my colleagues in the Algorithmic Social Sciences Research Unit, See: Kao et al. (2012).

[2]In the case of cycle theory, the phenomenon in question was the persistent fluctuations in output and employment. These models described specific (nonlinear) economic relationships between variables, and due to these relationships, the aggregate system is shown to exhibit periodic cycles.

It may be useful to highlight some aspects of the traditional models of economic dynamics that are relevant for this chapter, even though they have been mentioned in the earlier chapters. First of all, almost all the theoretical models in economic dynamics are defined on real number domains and the functions are real valued functions, whereas the mathematics of digital computers deals with numbers and functions that are defined on natural or rational numbers. It is also worth noting that not all real numbers are computable – when we simulate these models on digital computers, we are in fact dealing with functions that are defined over natural, rational or algebraic numbers. Secondly, the natural data-type of the economic system are themselves rational numbers, at best. Many results that are valid for dynamics defined over real domains do not easily carry over to dynamics defined over rational numbers. The functions over these numbers also need to be equipped with computational content. It is the quest to develop formal models that respect the natural data type for economics that leads us to algorithmic dynamics. If we view the economy as a computational system and economic processes as computations, then the need for algorithmic models for studying economic dynamics is clear, and they can help discover the dynamic properties of economic systems while offering the possibility of performing computational experiments.

Computational models are often understood as those which employ numerical methods in order to compute solutions. However, by algorithmic economic models, we mean those models that are explicitly faithful to the nature of the economic data as well as the mathematics of the computer (computability theory). By algorithmic models, we refer to algorithms in the sense in which they are formally understood in Computability theory and not merely any iterative procedure. Therefore, this should not be confused with the use of numerical procedures to solve economic models [3]. The focus of the algorithmic approach is therefore on computation, the kind of numbers and processes that are involved in it. If we choose to retain continuous time models and solve them using numerical methods, we need to understand the scope and limitations of this approach. As we will discuss subsequently, an indiscriminate reliance on numerical procedures may not perhaps be justified.

In this note, we focus on providing computational meaning[4] to continuous time dynamic models in macroeconomics. By imposing a computability structure, we then ask whether their long-run dynamic properties can be answered algorithmically. Questions concerning the extent of predictability of future events and states of the economy arise naturally in this context.

We focus on the following questions related to long-run properties:

1. Is it possible to characterize the attractors of dynamic economic models algorith-

[3]For example, the use of numerical methods to solve for economic models that are formulated as differential equations. We will briefly discuss some questions related to convergence of numerical methods and discretization in economic models in the later sections.

[4]In the sense in which 'computability' was defined by Alan Turing.

mically? Are their domains of attraction computable?

2. Given knowledge of the attractors, can we 'decide' whether and when the economy would reach a given attractor?

3. Can we 'decide' the number of attractors algorithmically?

This brings us to the interface between economic theory, computability theory and dynamical systems. In order to provide a computational structure to dynamical systems, we appeal to definitions and results from computable analysis so as to keep the discussion relevant to continuous time models that are widely used in economic dynamics. That means, while talking about computation, we do not impose restrictions such as that the numbers have to be natural or rational (or computable reals) alone, as in classical computability theory. Instead, we resort to computation over real numbers and topological spaces using type-2 machines. The main contention of this chapter is that several important properties in these are algorithmically undecidable. Since we are interested in endogenous models of economic dynamics and the role of non-linearity, the emphasis will be on nonlinear dynamical systems and the algorithmic undecidabilities and unpredictabilities associated with them. We also discuss the decidable fragments of these models and the assumptions that may be necessary (or sufficient) for them to be decidable. We do not address questions related to computational complexity here.

The rest of the chapter is organized as follows: Section 2 provides an brief introduction to the notion of computability in continuous time models. Section 3 deals with algorithmic questions related to computing the attractors, and section 4 investigates algorithmic unpredictabilities in the models of economic dynamics, with a focus on complex dynamical systems.

## 6.1 Continuous-time models and Computability

Many of the models in economic dynamics, especially endogenous dynamics, are formulated in continuous time and we need to explore the status of these models from a computational standpoint. For this, we require that these models are endowed with computational content. Then, the questions that we have raised in the previous section can be framed as decision problems concerning economic dynamics. In a decision problem, we are interested in a 'yes' or 'no' decision using an effective procedure. Typically, this involves a problem that has many individual sub-problems and one looks for a general method or procedure to answer each of those problems. For example, to decide whether an arbitrary Diophantine equation is solvable is a more general problem with countably infinite sub-problems. Instead of addressing the solvability of each specific problem, one looks at whether there is a general method to decide. If there is no such method available, then each problem or a sub-class of problems might need a specific decision procedure. Some of these sub-problems might not be decidable as well.

Before proceeding, we need to formally define what we mean by a 'procedure'. This leads us to the definition of an algorithm, provided by the work of Turing, Church and others. The conventional definition of an algorithm in computability theory is over discrete mathematical objects. This can be viewed as a theory for (discrete) computation on words in some alphabet $\Sigma$[5]. According to the Church-Turing thesis, the set of intuitively computable functions are precisely those that are computable via Turing machines. Turing machines themselves are discrete dynamical systems in their own right, and the operation of a Turing machine can be viewed as the evolution of a corresponding discrete dynamical system. Since most models of the economy are formulated in terms of continuous-time dynamical systems that are often defined over (compact) metric spaces, we need a way to tackle this problem for continuous time systems.

For continuous time systems, there is a need to bridge the gap between the structure on which traditional computability theory is defined ($\mathbb{N}$, which are countable) and continuous systems (defined on sets in $\mathbb{R}^n, n = 1, 2..$ for example, which are uncountable). If we can link continuous-time dynamical systems to structures with explicit computational meaning, then decision problems concerning dynamic economic models can be addressed. Notions such as enumerable, recursively enumerable, co-recursively enumerable, semi-decidable, recursive are central to understanding the notion of computability of the functions defined over sets.

**Definition 16.** ***Recursively Enumerable Set***
*A set $S \subset \mathbb{N}$ is said to be recursively enumerable if and only if it can be accepted by a Turing machine.*

**Definition 17.** ***Recursive Set***
*A set $S \subset \mathbb{N}$ is recursive if and only if both the set $S$ and its complement $S^c$ are recursively enumerable.*

The idea of enumerability deals with listing the elements of a set, while recursive sets have the property that the problem of membership is solvable for them. There are at least two possible ways to extend the notion of computation to continuous time dynamic models. The first is to extend the theory of computability to continuous-time systems. In classical computability theory, the set $S \subset \mathbb{N}$ and we need to extend it to subsets of $\mathbb{R}$ and functions defined on these sets. The second is to simulate the continuous time model using continuous-time analog machines. The former is the subject of Computable analysis or recursive analysis, which extends computability to continuous objects. On the other hand, the analog computation of continuous time models has a fairly long and established history, even within economics. MONIAC - designed by

[5]In a digital computer, one can think of an alphabet as being composed of {0,1}. But it need not necessarily be restricted to binary alphabets and can be generalized to more expressive ones. See (Collins, 2010)

A.W.H. Phillips and popularly known as the 'Phillips machine' is one of the classic examples of this approach. See Velupillai (2011a) for a detailed account of this tradition[6].

In Computable Analysis,'representations' or naming systems provide a way to link continuous objects (such as real numbers, continuous functions) to other objects that have an explicit computational content or meaning. It is instructive to think of this as a code word that links elements in one domain to those in another. Using these representations, we can induce computability on sets, and the results of the computation can be interpreted in light of these representations. This allows to carry out accurate computations with arbitrary finite precision.

When dealing with real numbers, we require an infinite amount of information to describe an object exactly. We also need to be able to reliably approximate these infinite objects in $\mathbb{R}$ using finite information, and use them for computation or find other ways to overcome this problem. It is possible to extend the traditional notion of computability on words of an alphabet ($\Sigma \rightarrow \Sigma$) to sequences of words on an alphabet($\Sigma^\omega \rightarrow \Sigma^\omega$). Since real numbers can be represented via infinite sequences, computability can now be defined for mappings between infinite sequences. This is referred to as Type-2 computability or Type-2 effectivity (see Weihrauch (2000), Ch:2 for more details), where infinite sequences act as 'representations' for a real number. Note that Type-2 computability is still explicitly based on Turing computability and it is only as powerful. The infinite amount of computation associated while dealing with infinite inputs and outputs (sequence of decimals for example) can be finitely approximated to any desired level of precision in this framework and can be simulated using digital computers.

These representations can be extended to topological spaces as well (Weihrauch (2000), Brattka and Weihrauch (1999)), and concepts such as effective or computable topological spaces, admissible representations and names, computability over real numbers, computability over closed, open and compact sets, can be appropriately defined (Chapters 2-5, Weihrauch (2000)). This approach relies heavily on the nexus between continuity and computability of functions. Please refer to the appendix on computable analysis for a list of basic definitions in this area.

[6]One of the well known theoretical models of a universal continuous-time analogue machine is perhaps Claude Shannon's General Purpose Analog Computer (GPAC). The computational power of computable analysis and GPAC is shown to be equivalent, at least in the case of real computable functions over compact domains (Bournez et al. (2007)). However, there seems to be no explicit agreement on the class of computable functions via different models of analog machines, as opposed to the case of digital computation, where we have the Church-Turing thesis.

## 6.2 Attractors of the economic models and Computability

Having extended the notion of computability to real numbers and topological spaces via representations, we can now turn to dynamic economic models formulated in these spaces. It should be remembered that these extensions do not enable us to compute more than what is computable by a Turing machine and hence we are still within the Turing boundary. This should not be interpreted as advocating or dismissing the use of continuous time models. Rather, we are interested in exploring what properties can be declared computable in this class of economic models. In macroeconomic dynamics, given certain assumptions regarding the relationships between different economic variables, we are interested in exploring and characterizing certain long term properties of an economic system such as steady state paths, equilibrium points, limit cycles, periodic or chaotic attractors and their stability properties. If more than one attractor is possible, then we need understand the conditions under which the system tends to one or the other, i.e, their respective domains of attraction. This can be viewed as the characterization of the $\omega$ limit set associated with a given dynamic economic model. In the case of endogenous economic dynamics, the focus will be on the algorithmic characterization of equilibrium points and periodic attractors, as has been the major focus of this tradition[7].

In the context of aggregate economic dynamics, we may ask: given a representation of an economy as a (nonlinear) dynamical system, can we algorithmically characterize:

1. Long run patterns of growth (steady states, unique or multiple equilibria)
2. Features concerning the periodic attractors (of relevance to business cycle theory)
3. The domains of attraction and the stability of these attractors.

It turns out that many long-run properties associated with these models are, in general, undecidable. Therefore, exhaustive algorithmic classification of attractors is generally not possible. Since this is not a big surprise for the case of continuous time models, we focus on a class for which these properties are known to be decidable. Fortunately, the major class of economic models of endogenous dynamics – which happen to be planar systems – do seem to have important properties that are algorithmically decidable under certain stability conditions. In nonlinear models, Poincaré-Bendixson theorem

[7]This query is directly related to the kind of questions posed by Roger Penrose in his book "The Emperor's New Mind". He asks whether mathematical structures such as the Mandelbrot set and the Julia set that one encounters in complex dynamics are recursive. In other words, whether there is an algorithm that can compute the Julia set or the Mandelbrot set precisely. Note that this is different from asking whether a given point belongs to the attractor. Instead, this is about the characterization of the attractors themselves.

helps to classify the attractors on the plane. In higher dimensions, the attractors can be highly complicated and we do not have general results for classifying them as we do on the plane.

Some formal definitions concerning dynamical systems and related notions that are relevant for this chapter are provided in the appendix.

For now, let us assume that we know the qualitative properties of the possible attractors for the economic model. This is what is usually done when one invokes the Poincaré-Bendixson theorem to prove the existence of limit cycles. Computability and decidability questions about the $\omega$-limit set can be analyzed in different ways. One is to compute the attracting set - such as fixed points and limit cycles, explicitly. The second is to algorithmically decide whether a trajectory belongs to the domain of attraction of a given attractor. There is also the issue of explicitly computing the domain(s) of attraction for members of the $\omega$-limit set, but we will not discuss this here.

When we ask the above questions about attractors, we are essentially asking whether the set is recursive. By providing a computable structure, we endow the set with the property of recursive enumerability. That is, there is a rule or an algorithm or a partial recursive function to list the successive members of this set. In order for this set to be decidable, we require an algorithm that will decide whether a given element belongs to the set or not in finite time. It is intuitive that all recursive sets are recursively enumerable, but the converse is not true. The major theme of the following sections is to understand the possibility of determining the properties of an economy, which is viewed as a recursively enumerable, but not recursive set. By establishing a correspondence between the economy formulated as a dynamical system and a Turing machine, we can study the dynamic trajectories of an economy via the evolution of a Turing machine.

### 6.2.1 Computing the Attractors: Planar models

Planar, nonlinear models form an important class of models in the tradition of endogenous economic dynamics, especially, business cycle theory. What is the status of these models when it comes for algorithmic decidability and computability of attractors? Although several of their properties are undecidable, there are some decidable fragments. Graça and Zhong (2011) conclude that attractors and their basins of attractions are *semi-computable* if we assume that the system is stable. They work within the framework of computable analysis and type-2 machines. Since the attractors cannot be computed in general, we need to explore the conditions under which they become computable. Stability becomes a crucial condition for ensuring the computability of attractors. They employ the notion of computability on closed, open and compact sets as outlined earlier, following the work of Brattka and Weihrauch (1999) and Weihrauch (2000). Please refer to appendix A for detailed definitions.

**Theorem 18.** *Let $x' = f(x)$ be a planar dynamical system. Assume that $f \in C^1(\mathbb{R}^2)$ and that the system is structurally stable. Let $K \subseteq \mathbb{R}^2$ be a computable compact set and let $K_{cycles}$*

*be the union of all hyperbolic periodic orbits of the system, which is contained in K. Then, given as input ρ-names of f and K, one can compute a sequence of closed sets* $\{K^n_{cycles}\}_{n\in\mathbb{N}}$ *with the following properties:*

1. $K^n_{cycles} \subseteq K$ *for every* $n \in \mathbb{N}$
2. $K^{n+1}_{cycles} \subseteq K^n_{cycles}$ *for every* $n \in \mathbb{N}$
3. $\lim_{n\to\infty} K^n_{cycles} = K_{cycles}$

This means that, under the assumption of structural stability (together with the Lipschitz property), if one can supply the $\rho$ names of $f$ and the compact set $K$ as input, there is an algorithm which can tell, in finite time, whether $f$ has a periodic orbit of the above dynamical system in the compact set $K$. Since the periodic orbits are only *semi-decidable* in this case, one may need an infinite amount of time, countably calibrated, to conclude that $K$ does not contain a periodic orbit. The same is true for the equilibrium points of the above dynamical system. This is a necessary consequence of dealing with a recursively enumerable, but not recursive set.

### 6.2.2 Stability

The notion of stability has played an important role in the models of economic dynamics. The stability properties associated with the equilibria, limit cycles as well as structural stability are routinely discussed in relation to these models. Bifurcations occupy an important role in these models as well. Bifurcations are intimately related to the idea of stability. Bifurcations occur when there is change in the stability properties of the solutions of a (parameterized) dynamical system due to changes in parametric values. Bifurcations are essentially transitions of the phase portraits as the parameters vary, where they move from being topologically equivalent to being nonequivalent. In other words, as the parameter values pass certain critical values, similarity of the phase portraits no longer holds and the qualitative behaviour of the system undergoes a change. These critical values are called *bifurcation points*, where the stability of the equilibirium undergoes a change. For example, Hopf bifurcation has been widely used to show the possibility of endogenously generated economic fluctuations. Structural stability, on the other hand, is concerned with perturbation to the entire system (family of solutions), while other notions of stability such as Lyapunov stability are based on small perturbations to individual orbits.

We saw earlier that (structural) stability and Lipschitz property are crucial for computing the attractors on the plane. This indicates that we might have to impose some *apriori* assumptions concerning stability of the economic system in order to achieve computability. But can considering only structurally stable[8] systems be justified in eco-

[8]Or even some other notion of stability.

nomics? If we were to adopt the view held by Schumpeter or Goodwin, it may be pertinent to view the capitalist evolution as one that is throughout unstable. Therefore, the assumption of structural stability can be seen as being too stringent or even counter intuitive. Structural stability may be considered a reasonable assumption for planar models, since structurally stable systems are 'typical' on the plane. But for higher dimensional systems, this is not the case[9]. The definitions of stability associated with dynamical systems and smooth manifolds may not be appropriate for capturing the intuitive notion of stability and instability in an economy. It might be necessary to develop discipline-specific notions of stability for economics that respect the nature of economic data.

### 6.2.3 Decidability of the Number of attractors

Along with existence theorems, uniqueness theorems related to equilibrium and limit cycles have also been discussed in macroeconomic dynamics. These theorems provide sufficient conditions for uniqueness. We briefly consider the algorithmic decidability of the number of attractors for a given planar dynamical system. Since the models invoking these theorems often assume compactness, we grant these assumptions and ask whether the number of attractors for a given economy (dynamical system) is algorithmically decidable. We can consider a situation that is general enough by allowing the function to be analytic.

**Proposition 19.** *Consider a nonlinear model of an economy, formulated as a planar dynamical system $\dot{x} = \psi(x)$ on a compact subset $K \subseteq \mathbb{R}^2$. Consider the case where $\psi$ is described by an analytic function and assume that the system is structurally stable.*

1. *The possible limit sets (hyperbolic equilibria and hyperbolic limit cycles) are necessarily finite.*

2. *Given the $\rho$ names of the compact set $K \subseteq \mathbb{R}^2$ and the analytic function $\psi$, the problem of deciding the number of equilibria and limit cycles is in general undecidable.*

*Proof.* The first part follows from Dulac's theorem (Perko (2001), pg. 205-6). The second part follows from Graça and Zhong (2011), theorems 20 & 21. □

Within the framework of computable analysis, the algorithmic decidability of the number of attractors is not possible, in general. The precise number and position of limit cycles of planar polynomial vector fields is the subject of Hilbert's $16^{th}$ problem, which is yet to be resolved. When there is more than one attractor, computing their respective domains of attraction becomes important. A system that has a unique, stable equilibrium point may seem trivial from the point of view of dynamic economic theory. No matter where one starts from, the system will end up in that unique equilibirum.

[9]Without listing the theorems and their proofs, we just note that several stability properties associated with dynamical systems (continuous and discrete) are in general undecidable.

However, for a nonlinear system this is not trivial from the point of view of computing its domain of attraction. For a nonlinear system, the problem of deciding the domain of attraction is uncomputable even for hyperbolic systems with a unique equilibrium.

**Theorem 20.** *(Zhong, 2009)*
*Let $\dot{x} = \psi(x)$ be a nonlinear system with a stable hyperbolic equilibrium point that is computable. The domain of attraction of this equilibrium point $S$ is not necessarily computable even when the equilibrium is unique.*

## 6.3 Unpredictability in Models of Economic Dynamics

### 6.3.1 Decidability of Domains of Attraction

We now turn to the decidability of the domain(s) of attraction. Our interest is to obtain a better understanding of the power and limitations of dynamic economic models. When the economic system has more than one attractor [10], the knowledge regarding the limit set of the economic system is of natural interest. Let us consider a situation where the attractor(s) associated with the model of the economy can be characterized explicitly apriori. Given this information, we are interested in finding out the attractor that is currently associated with the economy. If the number of attractors is finite, then the observed data on variables in the model can be used to describe the 'current' state so as to *verify* whether the economy is in the domain of attraction of one or other attractor.

Here, we focus on the possibility of verification using a computer - digital or analogue. Formally, the question can be stated as follows: *Is it possible to algorithmically verify, in general, whether the trajectory of an economic system (through the observed data) would ever reach a pre-specified attractor of the theoretical model?* In other words, the idea is to *verify* whether the economy is in the domain of attraction of a given attractor. What would be the use of such a verification? In case the economy is headed towards one of the undesirable attractors, then there is a need to steer it away from these basins with the aid of policy. The appropriate policy instruments and targets for such an exercise in controlling the economic system is an familiar topic in the control theoretic approach to economic policy. This is important in order to meaningfully link models with available economic data and also to validate such economic models. In terms of real-life economic questions, here are some ways in which the above question can be translated: Is the economy tending to a specific steady growth rate? Is the economy already in a periodic trajectory or tending to one? Is the economy going to enter a *ponzi* regime from the current regime[11]? The idea is to see whether these type of can be

[10] This is can be of the same qualitative type, for e.g. multiple equilibria, or different types - for e.g. an equilibrium and a limit cycle

[11] This example is related to Minsky's model of financial crises

answered algorithmically.

We can pose these problems as a canonical reachability problem, since the decidability of the domains of attraction of the $\omega$-limit set is a special case of the problem of reachability. The destination of the trajectory will be one of the members of the $\omega$-limit set of the model.

**Definition 21.** ***Reachability***
*Let $(X, f)$ be a dynamical system and $X \in \mathbb{R}^{n \times n}, x_0 \in \mathbb{R}^n, Y \in \mathbb{R}^n$. The system is said to reach $Y$ from $x_0$ if there exists $t \in \mathbb{R}$ such that $\phi(t, x_0) = Y$ with $\phi(t, x)$ the trajectory of the dynamical system with an the initial condition $x_0$.*

**Definition 22.** ***Reachability Problem***
*Given a trajectory $\phi$ of the dynamical system $(X, f)$, such that $X \in \mathbb{R}^{n \times n}, x_0 \in \mathbb{R}^n$, and a point $Y \in \mathbb{R}^n$, can we decide whether $Y$ can be reached from $x_0$. Alternatively, is $Reach(X, x_0, Y)$ decidable?*

Note that the reachability problem is a 'decision' problem - i.e, a yes/no problem. It is not restricted to *point-point* reachability (i.e the destination need not always be an point), and we modify the problem appropriately to answer *point-to-set* or *set-to-set* reachability as well. It is even possible to specify a particular region of the state space and ask whether the trajectory will ever enter this region.

**Lemma 23.** *There exist polynomial dynamical systems that can simulate universal Turing machines.*

*Proof.* Refer Graça et al. (2008) □

**Lemma 24.** *There exist analytic maps and flows which can simulate universal Turing machines.*

**Remark 25.** *The main result follows from these two lemmas. Note that the above theorems are valid even in the presence of some perturbation and therefore, are error-robust. Here, each and every configuration of a Turing machine is associated with a unique point in $\mathbb{R}^n$. The next step is to show that the evolution of the TM can be related to the flow (or a map) of a dynamical system whose state space is in $\mathbb{R}^n$.*

**Lemma 26.** *The Reachability problem is decidable for strictly linear dynamical systems.*

*Proof.* Refer Hainry (2008) □

**Proposition 27.** *Consider a nonlinear model of the economy formulated as a continuous-time dynamical system (H). Given knowledge of the attractors ($\Omega$) apriori, and an observed (rational) data point $x_0$, there is no finite procedure to decide, in general, whether the trajectory of H through x will reach a given attractor $\omega_i \in \Omega$.*

*Proof.* Assume that such a rule ($\Psi$) does exist and that it can be defined explicitly. From Lemma 23, Turing machines can be simulated by polynomial dynamical systems. That is, an injective function can encode every state of the Turing machine to a point on the state space in $\mathbb{R}^n$. Since $\Psi$ is capable of deciding reachability for an arbitrary dynamical system, we can do the following: Let ($x_0 \in \mathbb{R}$) representing the current state of the Turing machine be the initial condition. Encode the attracting set ($\Omega$) set as the halting set. Since we can decide whether or not the trajectory through $x_0$ will reach $\omega_i$, we can now decide whether or not the Turing machine will halt. But this implies that the halting problem for Turing machines is decidable, which is known to be undecidable. Hence, a contradiction. □

**Reachability via shift maps**

Another way to establish this undecidability of reachability is through symbolic dynamics. In symbolic dynamics, the dynamical system can be studied using another dynamical system which is its conjugate, defined over a discrete space. We can consider a specific class of dynamical systems. The dynamics of this class of systems are defined using a shift-operator over a set of bi-infinite sequences of some finite alphabet A. Shift maps have been frequently used in models to demonstrate chaotic economic dynamics.

Let us look at some definitions:

**Definition 28.** ***Shift Map***
*Let* $\Sigma = A^{\mathbb{Z}}$

$$\sigma : \Sigma \to \Sigma : (\sigma a)_k := a_{k+1}$$

The discretized space is labelled using symbols, and the trajectory of any point over time can be viewed as a sequence of symbols denoting its position in terms of these labels. In order to explicitly relate this to the idea of computation, one needs to generalize the notion of a shift map (Moore (1991), Moore (1990)).

**Definition 29.** ***Generalized Shift Map***
*Let* $a \in \Sigma$ *and* $F : \Sigma \to \mathbb{Z}$

$$\Phi : a \to \sigma^{F(a)}(a \oplus G(a))$$

The maps $F$ and $G$ above depend on a finite number of cells in $a$, called the *domain of dependence*. $F$ maps $a$ to integers and $G$ maps $a$ to finite sequences. The generalized shift map has two distinct operations - while G governs the rules related to changing a finite number of cells in $a$, $F(a)$ is responsible for shifting the sequence to the right or left. From this one can show the correspondence between Generalized Shift map and Turing Machines, which will then link us explicitly to the world of computation. Refer Moore(1991) for details.

**Proposition 30.** *[Moore(1991)] Any generalized shift map $\Phi$ on an alphabet A can be simulated by a Turing Machine, and vice versa.*

*Proof.* Refer to the proof of Theorems 7 and 8 in Moore(1991) □

It is not difficult to find a dynamical system that embeds the evolution of a generalized shift map (Moore, 1990). Given this correspondence, the problem of reachability can be posed as follows: Given an initial condition, is there a finite procedure to verify that a trajectory through the initial condition will reach a particular point on a region A. It is equivalent to asking for a procedure to decide whether a Turing machine will halt for a given initial state. This is because we can specify a particular cell or a block of the sequence as the halting state of the Turing machine, and pose the above question. We know that the above question is undecidable, in general, due to the unsolvability of the halting problem for Turing Machines.

Note that the sets associated with partial recursive functions (Turing machines), for example their domain, range, image and its inverse, are recursively enumerable sets (Moore,1991, 214-215). Due to Rice's theorem, again, we know that any nontrivial properties[12] of these recursively enumerable sets are undecidable.

**Proposition 31.** *Moore(1991)*
*Given a point $x_0$ and a periodic attractor $\omega_p$, it is impossible to decide, in general, whether the trajectory through $x_0$ with converge to $\omega_p$.*

This is one way to establish that the reachability problem is in general undecidable. Also, note that the domain of attraction of a dynamical system is recursively enumerable, but not recursive, in general. It follows from the above results that the reachability problem is undecidable for this class of dynamical systems and therefore undecidable in general.

### 6.3.2 Complex Economic Dynamics

Complex economic dynamics have been extensively studied in the past (Day (1994), Grandmont (1985), Goodwin (1990)). This area has mainly concentrated on studying economic dynamics through the framework of nonlinear dynamical systems. Loosely speaking, the (endogenous) dynamics of the economic system are understood to be complex, if it does not converge to a fixed point (or balanced path) or periodic attractor Rosser (2009). It has been associated with the presence of chaotic attractors, aperiodic fluctuations, bifurcations, catastrophes and/or endogenous structural change. This view has concentrated on defining complexity in terms of the attractors. 'Chaos' is understood as systems having sensitive dependence on initial conditions, where even a minute error in specifying initial conditions gets exponentially amplified over time. The trajectories which are close to each other diverge exponentially fast, and predicting the future becomes extremely hard or even impossible. This is especially relevant for continuous time models which require infinite precision while specifying initial conditions.

[12] This means that the only decidable properties are true either for all Turing Machines or for none.

Dynamical systems exhibiting chaos are often studied using maps, and we will therefore focus on maps (instead of flows). A relevant decision problem in this case would be to ask whether a given model of the economy is capable of such complex (chaotic) dynamics, in general. As usual, we are interested in an algorithmic verification of this property.

**Definition 32.** ***Sensitive Dependence on Initial Conditions*** *(Robinson (1995), p.82)*
*A map $f$ on a metric space $X$ is said to have sensitive dependence on initial conditions provided there is an $r > 0$ (independent of the point) such that for each point $x \in X$ and for each $\epsilon > 0$ there is a point $y \in X$ with $d(x, y) < \epsilon$ and $k \geq 0$ such that $d(f^k(x), f^k(y)) \geq r$.*

**Definition 33.** ***Chaotic map***[13] *(Robinson(1995), p.83)*
*A map $f$ on a metric space $X$ is said to be chaotic on an invariant set $Y$ or exhibits chaos, provided (i) $f$ is transitive on $Y$ and (ii) $f$ has sensitive dependence on the initial conditions on $Y$.*

**Proposition 34.** *It is impossible to algorithmically decide, in general, whether a given model of the economy will exhibit chaos.*

*Proof.* Application of theorems 10 and 11 from Moore(1991). □

The above statement means that for a class of piece-wise linear dynamical systems (generalized shift maps), it is impossible to have a general algorithmic rule to decide whether an arbitrary model from this class has sensitive dependence on initial conditions. This naturally implies that such a general rule does not exist for nonlinear dynamical systems as a whole. But so far, we have restricted complex dynamics only to SDIC. One can question whether the complexities associated with economic dynamics are faithfully captured by the unpredictability arising from the presence of SDIC. If the modeling exercise is merely linking unpredictability of the economic system qualitatively to those arising in a certain class of dynamical system, then it is likely that this could just be a pseudo-unpredictability. In other words, it could simply be an artefact of the modeling exercise. The debate concerning the appropriate notion of dynamic complexity that would encapsulate the complexities of the economic system can be found in Rosser (2009) and Velupillai (2011c).

In contrast to the 'negative' definition[14] of dynamic complexity provided by Day-Rosser, Velupillai (2011c) points to computational universality as the relevant property that one ought to associate with a 'dynamically complex' economic system. This definition is more general than the Day-Rosser definition and is explicit in terms of the computational content. The inherent unpredictability associated with such complex

[13] An important class of chaotic maps that illustrates the idea of homoclinic orbits is the 'Horseshoe map' developed by Stephen Smale. The Smale horseshoe map, however, is computable within the framework of Computable Analysis (See:Graça et al. (2012)).

[14] It is negative because the definition is in terms of what the system *does not* do (i.e., reach one of the elementary attractors), as opposed to what it does.

systems is different from the chaotic dynamics because it is more than mere SDIC. It concerns the unpredictability in determining the long term property of the system, even with complete knowledge of initial conditions.

> For a dynamical system to be chaotic means that it exponentially amplifies ignorance of its initial condition; for it to be undecidable means that essential aspects of its long-term behaviour – such as whether a trajectory ever enters a certain region – though determined, are unpredictable even with total knowledge of initial conditions.
> Pg. 606, Bennett (1990)

Perhaps it is reasonable to doubt the relevance of 'computational universality' as a required property specifically in the context of economic dynamics. The absence of a general rule to decide some specific properties of dynamical systems may not be relevant. This is true if the characteristics that we desire for our economic models do not require the system to be computationally universal. However this is not so!

To place this discussion squarely within the interests of this thesis – endogenous macrodynamics – it is worth listing some important observations and results from Velupillai(1999). He lists the following criteria as the 'minimal' requirements of an endogenous macroeconomic model (Velupillai (1999), pg.663):

1. It must be capable of extracting patterns inherent in aggregate data, without assuming that such data have been generated by an underlying probability model;

2. It must be capable of generating multiple equilibria;

3. It must be capable of persistent (or observably) unstable equilibria;

4. It must be consistent with the fundamental individually rational principle of no arbitrage;

5. It must be capable of encapsulating rational disequilibria;

6. It must be capable of non-maximum dynamics;

**Lemma 35.** ***(Velupillai, 1999)***
*If an endogenous dynamic economic model satisfies all the above criteria simultaneously, then such a system has to be computationally universal.*

Notice that we require that the model should be capable of encapsulating all the above desired characteristics and not merely a subset of them. Also, these criteria are not arbitrary, but are common features extracted from the models in this tradition. We can ask: Is it possible to verify whether an observed trajectory is in the basin of attraction of a system that is complex (in the above mentioned sense)?

**Theorem 36.** *(Velupillai (2011c), p.556)*
*There is no effective procedure to decide whether a given observable trajectory is in the basin of attraction of a dynamical system capable of computation universality.*

The above theorem states that there is no algorithmic procedure to verify whether the system is in fact complex in the above defined sense. The preceding sections have outlined some undecidable questions associated with dynamic economic models. On the face of it, these results may appear negative or even nihilistic towards mathematical models of dynamic economic theory. But this is not the case. These results have to be understood as establishing the boundary between what can and cannot be known (algorithmically) for a particular kind of formalism. The general message is that we can say very little, algorithmically, about the long-term properties of these dynamic models in general. We may wonder whether we can resort to other ways, perhaps not algorithmic, to answer these questions. But 'to decide' is invariably connected with 'to decide by what'. If the decision has to be reached by using a 'procedure'[15], we need to find or define a universally agreed-upon notion of a procedure that is general enough. We are now back again to the definition of an algorithm, an effective procedure or a mechanism, all of which culminated in the definition of a Turing machine and the Church-Turing thesis.

We need to clarify the implications of undecidability in this context. One interpretation is that there exist no general algorithmic rules that will enable us know the long term properties of an economy (viewed as a computational model). We need to develop specific algorithms for specific models and this is a non-trivial task. For some class of models, it might not be possible at all. The second, and related interpretation is more to do with what we usually understand as (un)predictability: There is no way to know what happens after $n$-periods, before explicitly computing these n-steps of the computation. It means that we cannot unravel answers to some questions concerning the economy, apriori, before the future actually unfolds. This is true even for the world of theoretical models of the economy, even if they are entirely deterministic. The second interpretation is related to the time involved in gaining knowledge about the computation. It is not possible to take shortcuts to know these answers before the economic system (or the model) evolves through time. If we have to provide our models with meaningful computational content, one that is computationally equivalent (in the sense of Wolfram[16]) to the original system, then we have to live with the fact that there are no general methods to know the outcome of this computation before it actually takes place. Simulation is the best we can do, especially when it comes to finding the long-term properties of a system. This is similar to Wolfram's notion of 'Computational irreducibility'[17] in the context of economic dynamics. Computational irreducibility can be interpreted as computational universality in the context of economic dynamics, as

[15] If not by a procedure, how else can one decide!

[16] Although similar, Wolfram's notion of computational equivalence is in the context of discrete systems such as Cellular Automata. Although he discusses the relevance of this property for continuous systems and differential equations, he says that this is still a guess, and stops short of pronouncing a verdict (Wolfram (2002), pp.730-733). But a more detailed discussion of this topic can be found in Velupillai (2013)

[17] For a discussion on the definitions of computational irreducibility, refer Zwirn and Delahaye (2013)

argued in Velupillai (2013).

The lesson here seems to be this: Even if the underlying rules governing economic processes are simple and known, if the system is complex enough (interpreted as computationally universal or computationally irreducible), then there is an inescapable unpredictability. It is impossible to predict reliably with knowledge of the past behaviour of the system, or even with knowledge of the underlying rules and initial conditions. This does not mean that there is no place for a theory of economic dynamics. Instead, the answers about these undecidable properties can be understood only by actually simulating them. Neither does this preclude a place for educated guesses, nor does it undermine economic intuition. In fact, it establishes the presence of limitations to pure mathematical reasoning and the inherent unpredictability even within deterministic mathematical models, that we cannot bridge.

## 6.A Numerical Methods and Computability:

> For the moment note that algorithms are the main object of study in scientific computation, yet there is not a formal definition of algorithm. ...
>
> In contrast to numerical analysis, the use of Turing machines gives the computer scientists a unifying concept of algorithm, well formalized. Thus complexity theory can speak of lower bounds of all algorithms without ambiguity. ...
>
> I hope the above communicates my view of numerical analysis as an eclectic subject with weak foundations. On the other hand,its achievements over many centuries, and especially since the revolution of the computer have an undeniable greatness.
> Smale (1990), p. 212.

We mentioned earlier that the algorithmic approach to economic dynamics should not equated with the use of numerical procedures, which are often used in simulations. The important difference between the algorithms in numerical analysis and computability theory is that the algorithms in computability theory have a strong foundation in terms of Turing machines. This formal definition of algorithms was originally in the domain of natural numbers and are now extended to real numbers, still retaining Turing machines foundations as we have seen earlier in the discussion on computable analysis[18]. Algorithms in numerical analysis lack any such explicit foundation and they typically operate over real number domains. In this section, we will briefly attempt to discuss a few issues that are relevant for simulations and economic dynamics. The discussion will not address all aspects of numerical analysis, but only those which are relevant to differential equations.

### 6.A.1 Numerical Methods and Dynamical Systems

While simulating dynamic models, one often resorts to numerical methods to solve a ODEs.

**Definition 37 (Dynamical System).** *Consider the following ODE:*

$$\frac{dx}{dt} = f(x), \quad x(0) = X \in \mathbb{R}^n \tag{6.1}$$

*where $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $x(t) \in \mathbb{R}^n$ denotes a vector valued function of $t \in \mathbb{R}$. The above equation is said to define a dynamical system on a subset $E \subseteq \mathbb{R}^n$ if, for every $X \in E$, there exists a unique solution for the equation, which is defined for all $t \in [0, \infty)$ and remaining in $E$ for all $t \in [0, \infty)$.*

[18]There are other developments which seek new foundations as well. We will not discuss them here.

Numerical methods such as Euler method, Runge-Kutta methods are used to solve the Initial Value Problems (IVP) like 6.1 and these methods can themselves be viewed as discrete dynamical systems in their own right. The following questions are relevant in this context:

1. Whether the numerical method in question generates a dynamical system?
2. If it does generate a dynamical system, whether they preserve the structure of the original dynamical system?

The conditions under which the long term, global properties and structure of the original, continuous time dynamical system is preserved while using a discrete dynamical system to approximate the former need to be understood. This is important to ensure that the limit sets generated by the numerical approximation is are equivalent. This will help us avoid ending up with spurious limit sets that are introduced by discretization. This is particularly relevant while studying chaotic systems. These issues are discussed in great detail in Stuart and Humphries (1998). Lipschitz conditions on $f$ plays a crucial role for the numerical method to define a dynamical system. Similarly, the size of the time steps used in these numerical methods have important implications on the accuracy of the solutions approximated by the numerical methods. For the long term (asymptotic) properties of economic models, the accuracy of numerical methods in determining the $\omega$ limit sets becomes relevant. In general, we need to place some restrictions on the time step size in numerical procedures and on initial conditions, for them to preserve the asymptotic properties of the original system. Some numerical methods are capable of generating additional fixed points and periodic solutions, which are purely a result of the numerical discretization procedure. That is, use of discrete dynamical systems that are not equivalent to the original continuous time systems and therefore it is possible to end up with spurious solutions[19]. Even if it might not be possible to undertake a systematic study of spurious solutions while studying economic dynamics, this fact needs to be borne in mind to discriminate between different numerical procedures and choose those that do not admit such solutions.

### 6.A.2 ODEs and Computability

We need to understand the assumptions that are needed to guarantee the presence of solutions for these systems, and the conditions under which their solution paths can be simulated on digital computers using numerical procedures. Let us consider the initial value problem (IVP) for ODEs (or a system of ODEs). In IVP, given the initial condition, the task is to find a solution to the ODE. This is guaranteed under certain conditions provided by this fundamental theorem on the existence and uniqueness of

[19] Besides the equivalence of numerical methods, there are also potential pitfalls while trying to discretize a continuous system. See (Potts, 1982) for phantom solutions that might arise due to non-equivalent discretization of nonlinear equations.

solutions to IVP. Let us consider the problem of existence of solutions for an ordinary differential equation, given an initial condition.

**Theorem 38.** ***Existence and Uniqueness of Solutions:*** *Consider the initial value problem*

$$X' = F(x), X(0) = X_0$$

*where, $X_0 \in \mathbb{R}^n$. Suppose that F: $\mathbb{R}^n \to \mathbb{R}^n$ is $C^1$. Then there exists a unique solution of this initial value problem. More precisely, there exists $a > 0$ and a unique solution*

$$X : (-a, a) \to \mathbb{R}^n$$

*of this differential equation satisfying the initial condition $X(0) = X_0$*

We can now contrast this with another existence theorem available for the solutions of an IVP, which is attributed to Peano.

**Theorem 39.** ***Peano's Existence Theorem:*** *Consider the initial value problem*

$$X' = F(x, t), X(0) = X_0$$

*where, $X_0 \in \mathbb{R}^n$. Suppose that U is open in $\mathbb{R}^n \times \mathbb{R}$ and $(x_0, t_0) \in U$. Suppose $f : U \to \mathbb{R}^n$ is continuous. Then there exists some positive number $\alpha$ and a $C^1$ function $h : [t_0, t_0 + \alpha] \to \mathbb{R}^n$ which satisfies*

$$h'(t) = F(h(t), t), \;\; h(t_0) = X_0 \tag{6.2}$$

The difference between the Picard-Lindelöf theorem and Peano's theorem lies in the assumptions that they make regarding the function $F$. While $F$ is assumed to both continuous and differentiable $C^1$ in the first case, it is only continuous in Peano's theorem. It states that there exists at least one solution for the IVP, but does not guarantee that this solution is unique. On the other hand, Picard-Lindelöf theorem assumes more, namely that $F$ is (locally) Lipschitz continuous[20], and concludes that the solution is unique. Lipschitz continuity, intuitively, requires that the slope of the continuous function cannot vary more than a constant, called the Lipschitz constant.

**Definition 40.** ***Lipschitz Continuity****: Let $D \subset \mathbb{R}$ and F: $D \to \mathbb{R}$. F is called Lipschitz continuous if for any closed and bounded interval $I \subset D$ there exists a $K \in \mathbb{R}$ and $K < \infty$ with*

$$|F(x) - F(y)| \leq K|x - y|, \forall x, y \in I \tag{6.3}$$

*and K is called the Lipschitz constant.*

This provides the space for a contraction mapping to a unique solution, through successive approximations of the solutions by an iterative scheme.Lipschitz continuity has important implications for the computability of trajectories. For example, what if we decide to stick to the world of Peano and do not impose Lipschitz continuity as

[20]Note that $C^1$ implies that the function F is locally Lipschitz. (See p.387, Hirsch and Smale for proof)

a requirement for the functions in our dynamic economic models (given that Peano's theorem asserts that there exists at least one solution)? Since we are concerned about computability, our solutions are computable with the requirements of Peano's theorem alone. The following theorem by Aberth (1971) shows that computability fails for this classical existence theorem.

**Theorem 41.** *Let $y'(x) = f(x, y(x))$*
*There exists a function $f(x, y)$, uniformly continuous in the rectangle $R : |x| \leq 1$ and $|y| \leq 1$ and with $|f(x, y)| \leq 1$ for $(x, y)$ in R, such that for any interval $[a, b]$ which is a subset of $[-1, 1]$ and contains the point 0, there is no function $y(x)$ defined in the interval which satisfies the above equation and the initial condition y(0) = 0.*

A similar theorem is by Pour-El and Richards (1979) in their study of computability properties associated with Peano's existence theorem. They analyze an ODE which takes computable inputs and has a rule of evolution $f$ which is computable, and still generates an uncomputable solution.

**Theorem 42.** ***Pour-El and Richards Theorem:***
*There exists an ODE: $y'(x) = f(x, y(x))$ with $y(0) = 0$, such that f(x,y) is computable on the rectangle $[0 \leq x \leq 1, -1 \leq y \leq 1]$, but no solution of the ODE is computable on any interval $[0, \delta], \delta \geq 0$*

Note that uniqueness of the solutions is important for the computability of trajectories. While studying dynamics, simulation of ODEs on computers rely on the condition that the function has the Lipschitz property. Theorems in numerical analysis concerning convergence often rely on the existence of Lipschitz constants. If the ODE has a unique solution, then we can say something about the computability of the solution for IVPs. However, if the solution is not unique, computability of the solution is not always possible (Collins and Graça (2008)). Since global Lipschitz constants are extremely hard to find and may not even exist for most cases, local Lipschitz constants becomes relevant in deciding whether the solution can be computed over its entire life span. If the solution is locally Lipschitz, and if the Lipschitz constant(s) for each and every compact neighbourhood are computable (effective Lipschitz constant), then the trajectory is computable over its entire life span. Thus, we see a strong connection between computability of trajectories, uniqueness of the solution to IVP and Lipschitz condition.

The existence theorems for IVP stated above are both local theorems and we still have to determine the length of the interval over which the solution to the IVP exists. This is important, particularly in the case of nonlinear models, to establish the adequacy of numerical methods that are employed to simulate these models. The following theorem by Graça et al. (2009) establishes that this maximal interval of existence is uncomputable in general, even if we have $f$ and the initial condition are computable.

**Theorem 43.** *There exists a continuous computable and effectively locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that the unique solution of the problem*

$$\begin{cases} \dot{x} = f(x) \\ x(0) = 0 \end{cases}$$

*is defined on a non-computable maximal interval.*

They also show that the question of whether or not this maximal interval is bounded is also undecidable. This is a caution against exclusive reliance on numerical methods to infer global properties of the dynamic system, with inadequate attention to the computability concerns associated with the models that are being simulated.

> The non-computability of the lifespan suggests limitations concerning numerical methods for solving ODE problems, because numerical methods often assume the existence of some time interval where the solution is defined, and this assumption is crucial in error analysis. In the case where the lifespan is non-computable, one may have to settle for a numerical algorithm computing only a local solution.
> Graça et al. (2009), p.2914.

Numerical simulations of continuous time dynamical systems rely on some notion of stability of the underlying system so that computed trajectories and phase portraits are approximately close to the original. One such assumption is the *shadowing* property that concerns global stability[21]. This property requires that the whole set of orbits are stable to perturbations. If so, there is a real orbit that always *shadows* or stays close to the simulated 'pseudo-orbit'. This can be used to justify that these simulated orbits approximate the original system. However, this property is not sufficient if one is interested accurate computation of properties and their algorithmic verification with arbitrary finite precision(Hoyrup (2007)). For this, the system needs to posses a much stronger property: robustness. This, unlike shadowing, requires that each and every pseudo-orbit (not merely the whole set of orbits) stays close to the original orbit starting from the same initial condition. This is a local stability property and hence, a more restrictive property. Therefore, relying on simulations (both approximate or exact computation - in the sense of computable analysis) of economic models that lack robustness, especially for prediction, needs to be thought carefully.

[21]See results in section 6.5, Stuart and Humphries (1998)

## 6.B Dynamical Systems - Definitions

**Definition 44.** ***Dynamical System***
*A dynamical system describes the evolution of points on a state space over time. Let X is an open subset of $\mathbb{R}^n$. A dynamical system on X is a $C^1$ function*

$$\phi : \mathbb{R} \times X \to X$$

*, where $\phi_t(x) = \phi(t, x)$ and $\phi_t$ satisfies the following conditions:*

1. *$\phi_0(x) = x$ for all $x \in X$*
2. *$\phi_t \circ \phi_s(x) = \phi_{t+s}(x)$ for all $s, t \in \mathbb{R}$ (in case of discrete time systems $s, t \in \mathbb{N}$) and $x \in X$.*

*The evolution rule $\phi$ is a map (for discrete-time system) or can be written as a differential equation (for a continuous-time, $C^1$ system).*

**Definition 45.** *$\omega$* ***limit set***
*Let $\phi(t, x)$ be the flow of the above dynamical system and z be a point on this trajectory. z is called a $\omega$ limit point of the trajectory of the dynamical system if there exists a sequence $t_n \to \infty$ such that $lim_{n\to\infty}\phi(t_n, x) = z$. The $\omega$ limit set of x ,$\omega(x)$, is the set of all $\omega$-limit points $z \in X$*

**Definition 46.** ***Invariant Set***
*A set $L \subset \mathbb{R}^n$ is called an invariant set if $\phi(t, x) \in L$, for all $x \in L$ and $t \to \infty$.*

**Definition 47.** ***Attracting Set***
*The closed invariant set L is called an* ***attracting set*** *of the flow $\phi(t, x)$ if $\exists$ some neighbourhood V of L, such that, $\forall x \in V$ and $\forall t \geq 0, \phi(t, x) \in V$ and $\phi(t, x) \to L$ as $t \to \infty$*

**Definition 48.** ***Domain of Attraction*** *Domain of attraction of the attracting set L of $\phi(t, x)$ is defined as,*

$$\Theta_L = \bigcup_{t\leq 0} \phi_t(V)$$

*the union of all neighbourhoods V of the attracting set, for which $\forall x \in V$ and $\forall t \geq 0, \phi(t, x) \in V$ and $\phi(t, x) \to L$ as $t \to \infty$*

## 6.C Computable Analysis - Some definitions

We quote only the minimum definitions (from Graça and Zhong (2011)) required for the present context and for more details please refer Weirauch(2000).

**Definition 49.** 1. *A sequence $\{r_n\}$ of rational numbers is called a $\rho$-name of a real number $x$ if there are three functions $a, b$ and $c$ from $\mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,*
*$r_n = (1)^{a(n)} \frac{b(n)}{c(n)+1}$ and $|r_n - x| \frac{1}{2^n}$*

2. *A double sequence $\{r_{n,k}\}_{n,k \in \mathbb{N}}$ of rational numbers is called a $\rho$-name for a sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers if there are three computable functions $a, b, c$ from $\mathbb{N}^2 \to \mathbb{N}$ such that, for all $k, n \in N$, $r_{n,k} = (1)^{a(k,n)} \frac{b(k,n)}{c(k,n)+1}$ and*
*$|r_{n,k} - x_n| \frac{1}{2^k}$*

3. *A real number $x$ (a sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers) is called computable if it has a computable $\rho$-name, i.e. there is a Type-2 machine that computes the $\rho$-name without any input.*

**Definition 50.** ***Computable Functions***
*Let $A, B$ be sets, where $\rho$-names can be defined for elements of $A$ and $B$. A function $f : A \to B$ is computable if there is a Type-2 machine such that on any $\rho$-name of $x \in A$, the machine computes as output a $\rho$-name of $f(x) \in B$.*

**Definition 51.** 1. *An open set $E \subseteq \mathbb{R}^m$ is called **recursively enumerable** (r.e. for short) open if there are computable sequences $\{a_n\}$ and $\{r_n\}$, $a_n \in \mathbb{Q}^m$ and $r_n \in \mathbb{Q}$, such that $E = \bigcup_{n=0}^{\infty} B(a_n, r_n)$.*

2. *A closed subset $K \subseteq \mathbb{R}^m$ is called **r.e. closed** if there exist computable sequences $\{b_n\}$ and $\{s_n\}$, $b_n \in \mathbb{Q}^m$ and $s_n \in \mathbb{Q}$, such that $\{B(b_n, s_n)\}_{n \in \mathbb{N}}$ lists all rational open balls intersecting $K$.*

3. *An open set $E \subseteq \mathbb{R}$ is called **computable** (or recursive) if $E$ is r.e. open and its complement $E^c$ is r.e. closed. Similarly, a closed set $K \subseteq \mathbb{R}^m$is called computable (or recursive) if $K$ is r.e. closed and its complement $K^c$ is r.e. open.*

4. *A compact set $M \subseteq \mathbb{R}^m$is called **computable** if it is computable as a closed set and, in addition, there is a rational number $b$ such that $\| x \| \leq b \forall x \in M$.*

**Definition 52.** ***Semi-Computable Functions***
*A function $\psi : A \to O(\mathbb{R}^m)$, (where $O(\mathbb{R}^m) = \{O | O \subseteq \mathbb{R}^m$ is open in the standard topology$\}$ is called **semi-computable** if there is a Type-2 machine such that on any $\rho$-name of $x \in A$, the machine computes as output two sequences $\{a_n\}$ and $\{r_n\}$, $a_n \in \mathbb{Q}^m$ and $r_n \in \mathbb{Q}$, such that $\psi(x) = \bigcup_{n=0}^{\infty} B(a_n, r_n)$.*

**Definition 53.** ***Type-2 Machine*** *[Weihrauch(2000), p.15]*
*Let $\Sigma^*$ be the set of finite words over some arbitrary finite alphabet $\Sigma$. Similarly, let $\Sigma^\omega$ be the set of infinite sequence of symbols from some arbitrary finite alphabet $\Sigma$, which has atleast two elements. A Type-2 Machine M is a Turing machine with k input tapes together with a type specification $(Y_1, Y_2....Y_k, Y_0)$ with $Y_i \in (\Sigma^*, \Sigma^\omega)$, giving the type for each input tape and the output tape.*

# Chapter 7

# Conclusion

In this thesis, we critically evaluated and took stock of developments in the field of endogenous macroeconomic dynamics. The focus is largely on the mathematical aspects of these theories, more specifically the models of endogenous fluctuations. We traced the origins of this tradition, its rich history and the different directions in which it has developed. We distilled some methodological elements that underpin these theories and analyzed them critically. We also looked at some methodological and epistemological aspects in computational economic dynamics, in terms of the algorithmic undecidablities that are associated with these models.

We have seen that the endogenous tradition in economic dynamics has gone through various phases - from being the predominant paradigm for understanding business cycles to being an unfashionable mode of theorizing subject to intermittent resurgences. We are left with some questions concerning the future of the endogenous tradition in economic modeling. Is this modeling philosophy which attributes aggregate phenomena to endogenous factors still tenable? Can it pose a formidable challenge to the exogenous view on both theoretical and empirical fronts? Can nonlinearity still retain its value as a reliable source for characterizing endogenous fluctuations? The answer, in light of arguments presented in this thesis, is a simple Yes. It is not just an episode in the development of economic thought that got replaced by 'better' theories and we believe that it holds a lot of potential in contributing to our knowledge about the dynamics of capitalistic economies. Schumpeter's words are most relevant here:

> The essential point to grasp is that in dealing with capitalism we are dealing with *an evolutionary process*. It may seem strange that anyone can fail to see so obvious a fact which moreover was long ago emphasized by Karl Marx. Yet that fragmentary analysis which yields the bulk of our propositions about the functioning of modern capitalism persistently neglects it. ... *Capitalism, then, is by nature a form or method of economic change and not only never is but never can be stationary.* ...The fundamental impulse that sets and keeps the capitalist engine in motion comes from the new consumers?

> goods, the new methods of production or transportation, the new markets, the new forms of industrial organization that capitalist enterprise creates.
> -Schumpeter (1942), pp. 82-83 (Italics added)

Capitalistic economies are highly intricate, complex structures that continue to evolve, constantly generating new products, methods, organizational forms. Its dynamics presents the investigator with very interesting patterns. Understanding the driving forces, mechanisms and the fluctuating nature of its evolution remains a formidable challenge even today. An endogenous approach to analyzing macroeconomic dynamics is still very relevant and we believe it is a superior modelling philosophy when compared with the exogenous view to understand and theorize about these conundrums. However, these investigations should treat economic phenomena in its actual form, rather than distorting its inherent structure to suit available tools. This thesis argued that the power of endogenous approach can be better exploited if we transcend the limitations posed by its current mathematical tools. It also tried to make a case for resorting to algorithmic and constructive approaches to modeling economic dynamics, pioneered by Velupillai (2010, 2000), which we hope will invigorate this tradition with newer challenges and insights. This could be one possible way in which endogenous macroeconomic dynamics - a tradition that flourished under the venerable hands of Marx, Keynes, Schumpeter, Hicks, Goodwin, Day and a legion of other scholars -could move forward.

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