Luca Prakash Argani received his MEng in Civil Engineering and the Engineer professional habilitation at the University of Trento (Italy) where he is a member of the Mechanical and Structural Systems Group.
His Ph.D. research, involving joint work between the University of Trento and Aberystwyth University, focuses on dislocations and thermoplasticity in continuum media.
During his Ph.D. studies he was Teaching Assistant for the course of Mechanics of Solids ("Scienza delle Costruzioni") at the University of Trento.
The participation as invited speaker at the 5th International Workshop of Young Researchers on the Mechanics of Materials and Structures (YRMMS 2013), held at University of Salerno (Italy), is an award for his Ph.D. studies.

SUBJECT HEADINGS

- Green's functions, inclusions and dislocations
  - Elements of tensor algebra and analysis
  - Elements of continuum mechanics
  - Incremental deformations
  - Two-dimensional Green's functions
  - Inclusion and dislocation problems
  - Three-dimensional Green's functions
- Non-standard elastoplastic constitutive laws
  - Integration algorithms of elastoplasticity for ceramic powder compaction
  - Comparison with semi-analytical solutions

Luca Prakash Argani – DISLOCATIONS AND GREEN'S FUNCTIONS IN PRESTRESSED SOLIDS

UNIVERSITY OF TRENTO - ITALY
DEPARTMENT OF CIVIL, ENVIRONMENTAL AND MECHANICAL ENGINEERING

DOCTORAL SCHOOL IN
STRUCTURAL CIVIL AND MECHANICAL ENGINEERING
XXVI CYCLE – SCIENTIFIC SECTOR ICAR/08

Luca Prakash Argani

DISLOCATIONS AND GREEN'S FUNCTIONS IN PRESTRESSED SOLIDS

April 2014

This text provides a summary of the fundamental equations of tensor algebra and continuum mechanics.
The set of Green's functions is derived and generalized to the problem of a point load in a two dimensional incompressible incremental non-linear elastic and homogeneously prestressed medium. This problem is extended to the three-dimensional case: these new Green's functions pave the way for a boundary element formulation of three-dimensional problems of incremental elasticity.
The effect of prestress on dislocation and inclusion fields in non-linear elastic solids is analysed by extending previous results of Eshelby and Willis.
The investigation is addressed to ductile materials subject to extreme strain. When the level of prestress is high enough that the limit of ellipticity loss is approached, localized deformations are shown to be organized in shear bands lines (2D case) and in conical surfaces (3D case): dislocation activity and material failure (fracture, shatter cones) are promoted along these bands/surfaces.
Two novel integration algorithms are proposed for the implementation of non-standard elastoplastic constitutive laws, based of pressure-sensitive yield functions, for which the gradient based algorithms cannot be directly employed.
The investigation is addressed to ceramic materials through two interesting applications: the compaction of a ceramic powder and the expansion of a thick spherical shell made up of a green body.

UNIVERSITY OF TRENTO
DEPARTMENT OF CIVIL, ENVIRONMENTAL AND MECHANICAL ENGINEERING

DOCTORAL SCHOOL IN
STRUCTURAL CIVIL AND MECHANICAL ENGINEERING
XXVI CYCLE – SCIENTIFIC SECTOR ICAR/08

Luca Prakash Argani

DISLOCATIONS AND GREEN'S FUNCTIONS IN PRESTRESSED SOLIDS

Advisor:	
Prof. Davide Bigoni	Università di Trento (IT)
Board of Examiners:	
Prof. Oreste Salvatore Bursi	Università di Trento (IT)
Prof. Dionisio P. Bernal	Northeastern University Boston (US)
Prof. Michel Destrade	National University of Ireland (IE)
Dott. Andrea Giovanni Calogero	Università Milano Bicocca (IT)
Dott. Paola Falugi	Imperial College London (GB)
Head of the Doctoral School:	
Prof. Davide Bigoni	
Final Examination:	Monday 7th April, 2014

Daiano, Thursday 8th May, 2014

Ing. Luca Prakash Argani

Edition: 08/05/2014
Revision: 08/05/2014

TEXT NOTE

This document has been created with LaTeX (MiKTeX distribution) using TeXnic-Center, TeXmaker and TeXstudio editors, while the images have been created with GIMP, Inkscape, LibreCAD, and FreeCAD. *All* the above mentioned programs are multiplatform and, especially, they are *open source* and released with GNU GPL license.

CONTACTS

✎ luca.argani@ing.unitn.it – Mail address of the author
✎ luca.argani@unitn.it – Mail address of the author
☎ +39 0461 282593
☎ +39 0461 282594
☞ http://www.unitn.it/dicam/7282/computational-solid-and-structural-mechanics – Research group website, computational laboratory.
☞ http://www.unitn.it/dicam/7286/laboratory-for-physical-modeling-of-structures-and-photoelasticity – Research group website, experimental laboratory.

Quan che la Roca la fa ’l capel, laga la falze e ciapa ’l restel

Poplar adage
Daiano

To my father

Abstract

THE PRESENT PhD dissertation is divided into two Parts: Green's function and problems of the inclusions and dislocations are addressed in the first Part, while implementation of elastoplastic constitutive laws are treated in the second. These subjects can be seen as different approaches to the investigation of the plastic behaviour of materials.

In the **first Part**, infinite-body two-dimensional Green's functions are derived for the incremental deformation of an incompressible, anisotropic, prestressed body. These functions, given by Bigoni & Capuani, show the response of an infinite body to a concentrated force.

The effect of prestress on dislocation (and inclusion) fields in non-linear elastic solids is analyzed by extending previous solutions by Eshelby & Willis. Employing a plane strain constitutive model (for incompressible incremental non-linear elasticity) to describe the behaviour a broad class of (anisotropic) materials, but with a special emphasis on ductile metals (J_2-deformation theory of plasticity), it is shown that strongly localized strain patterns emerge, when a dislocation dipole is emitted by a source and the prestress level is high enough. These strain patterns may explain cascade activation of dislocation clustering along slip band directions. Several of the presented results remain valid within a three-dimensional context.

Novel infinite-body three-dimensional Green's functions are derived for the incremental deformation of an incompressible, anisotropic, prestressed body. The case of a force dipole is developed within this framework. Results are used to investigate the behaviour of a material deformed near the limit of ellipticity loss and to reveal features related to shear failure cones development in a three-dimensional solid medium.

Non-standard elastoplastic constitutive laws are treated in the **second Part** of

the present PhD dissertation, based on pressure-sensitive yield functions, such that proposed by Bigoni & Piccolroaz, which describes the inelastic deformation of ceramic powders and of a broad class of rock-like and granular materials. This yield function is not defined outside the yield locus, so that 'gradient-based' integration algorithms of elastoplasticity cannot be directly employed. Therefore, two *ad hoc* integration algorithms are proposed: an explicit scheme based on a forward Euler technique with a 'centre-of-mass' return correction and an implicit scheme based on a 'cutoff-substepping' return algorithm.

CONTENTS

List of Figures

List of Tables

Acknowledgments

MY MOST IMPORTANT ACKNOWLEDGMENTS are towards my Family, in particular my Father. I want to thank Prof. Davide Bigoni because of his courtesy and helpfulness reserved for my requests for materials, explanations, clarifications, and suggestions. The environment he created inside the Mechanics of Solids Group is something special that can not be described in few words: only a restricted number of students have been as lucky as me in finding such a good place. The credit for this belong also to the members of this group, cooperating with my advisor for the realization of this small organization, but very important within the international sphere.

I would like to thank in particular all the people that contributed for the development and the correction of this work, but also have been with and encouraged me throughout this educational path, like Prof. Gennady Mishuris (Department of Mathematics and Physics, Aberystwyth University, UK), Professors Alexander B. Movchan and Natasha Movchan (Department of Mathematical Sciences, University of Liverpool, UK), Prof. Domenico Capuani (Department of Architecture, University of Ferrara) and Dr. Andrea Piccolroaz (DICAM, University of Trento). It would be a crime to forget their courtesy, patience and helpfulness, as well as their commitment to accomodating me for short and long periods in their offices to carry on the research activity.

I am greatly indebted to I.F. Ph.D. Eng. Daniele Veber, with whom I shared and managed the *Independent Reign of the Loft*. His ever-present and continuous support and suggestions in many fields (research, civil engineering and didactics) have been extremely valuable and significantly contributed to my formation. Even the presence of Ph.D. Eng. Giulia Franceschini has been very important, even though we seldom encountered. The discussions with Ph.D. Eng. Panagiotis Gourgiotis on the subjects treated in this text have been interesting and productive, as well as his revision of this work.

Two persons that revealed to be very important for my didactic experience are Ph.D. Eng. Francesco Dal Corso, essential and irreplaceable handyman of the Group, and Prof. Massimiliano Gei.

Among many of Ph.D. students (currently, a very long list), I foremost want to thank Ph.D. Diego Misseroni, extraordinary hard worker and unique responsible of the laboratory, Ph.D. Eng. Mattia Bacca, adventure companion throughout the university, Eng. Federico Bosi, versatile and multidisciplinary assistant, Eng. Aldo Madaschi, and finally Stefano Signetti, sport and mountain enthusiast.

Daiano, Thursday 8th May, 2014 — L. P. A.

NOTATION

THE NOTATION used in this PhD dissertation corresponds to that of the literature, even though some variations may occur. Whenever such exceptions occur, the symbols used in the expressions will be immediately understandable from the context and, if necessary, clarified in an appropriate manner. The conventions used throughout the text are listed below.

- Italic greek or latin letters (a, A, α, ...) are used for mathematical variables such as numerical or scalar quantities.
- Italic bold-face lowercase greek or latin letters ($\boldsymbol{b}$, $\boldsymbol{\beta}$, ...) denote vector quantities.
- Italic bold-face uppercase greek or latin letters ($\boldsymbol{C}$, $\boldsymbol{\Sigma}$, ...) denote tensor quantities, usually of the second or third order.
- Blackboard bold letters ($\mathbb{K}$, $\mathbb{N}$, ...) are used for both fourth order tensor quantities and numeric sets (see table D.1); the distinction between these cases will be clear from the context.
- Italic greek or latin letters with italic subscripts (a_i, A_{ij}, ε_{ijk}, $\mathbb{E}_{ijkl}$, ...) denote the components of the corresponding vectors or tensors.
- Uppercase calligraphic letters are used both for spaces (for instance $\mathcal{V}$ is the vector space) and regions or domains (i.e. $\mathcal{B}$ is the domain of a body); the distinction between these cases will be clear from the context.
- A comma inserted in subscripts containing greek or latin letters (in any font style) ($\psi_{,i}$, $v_{i,j}$, $A_{ij,i}$, ...) denotes the derivative with respect to a

variable that is defined through the text, or that may be deduced from the context.

- An apostrophe inserted in superscripts of greek or latin letters (in any font style) ($\psi^{'}$, $\boldsymbol{v}^{'}$, $\boldsymbol{C}^{'}$, ...) denotes the derivative with respect to the whole argument of the function; this argument is explicit or may be deduced from the context.

- Numbers and italic latin letters in superscripts of greek or latin letters (in any font style) (ϕ^2, $\boldsymbol{B}^3$, ...) denote the exponents; nevertheless some *exceptions* may occur, that will be clarified in the text.

- A slash mark in between two numbers in subscripts of italic greek or latin letters denotes the solutions of an equation; for instance, $x_{1/2}$ corresponds to the solutions 1 and 2 of an equation in the variable x.

- When and index appears twice in the same product, summation over the repeated index (*Einstein notation*) is implied unless otherwise stated. For instance,

$$a_i b_i = \sum_{i=1}^{3} a_i b_i$$

  It is worth noting the following *exception*: in the description of incremental numeric and/or iterative procedures, the subscripts denote the specific increment numbers, so that automatically the *Einstein notation* does not hold. For instance,

$$\Delta\varepsilon = \varepsilon_{n+1} - \varepsilon_n$$

  After the description of the involved quantities, the distinction between these two cases will be clear from the context.

All the logical, algebraic, geometric symbols and operators, and, more in general, whatever not specified in this Chapter, are reported in detail in Appendix D.

Part I

GREEN'S FUNCTIONS, INCLUSIONS AND DISLOCATIONS

Chapter 1

INTRODUCTION

THE PROBLEM of the concentrated force in an infinite anisotropic elastic medium subject to prestress is presented following Bigoni & Capuani [1] within a two-dimensional context, referring to the Biot [2] material model (which embraces the cases of Mooney-Rivlin [3] and Ogden [4] materials). To obtain the solutions for the Biot material, the plane wave expansion method of the *stream function* has been exploited following Hill & Hutchinson [5]. In particular, the Green's functions [6–8] have been used for the static problem [1] and for the dynamic problem in the case of *time-harmonic* incremental loads (Bigoni *et al*. [9, 10]); the singular terms are derived in a closed form. Furthermore, the characteristic equation associated to the equilibrium equations is analyzed in order to determinate the regime classification (Bigoni [77]). The prestress and its role in the formation of the *shear bands* (localization of the deformation) are systematically investigated. The introduction of an incremental formulation based on the nominal stress is required, following directions previously explored for different problems, like the bifurcation of non-linear solids [77].

The Green's functions have been obtained for the two-dimensional problem of the concentrated force in terms of incremental displacement and mean stress. The singular solutions obtained for the infinite medium can be used to analyze the effects of a perturbation superimposed to a given homogeneous deformation of linear elastic medium [77]. Many self-equilibrated load conditions can be constructed by superposition of the concentrated load solution; the simplest

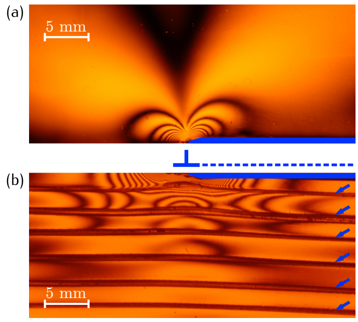


Figure 1.1: Photoelasticity (in monochromatic light) discloses the stress field (in-plane principal stress difference) around an edge dislocation (as sketched) in an isotropic material (**a**) and in an orthotropic material (**b**) (with orthotropy axes aligned parallel and orthogonal to the dislocation line). Orthotropy has been simulated by cutting parallel groves (indicated with arrows) in a photoelastic 5 mm thick) two-component resin, in which two parallel metallic (0.5 mm thick) steel laminae simulate a dislocation, when forced to slide one against the other. Compared with the isotropic case (**a**), the fields become strongly elongated along the orthotropy axis parallel to the dislocation for anisotropy (**b**).

force system is represented by a force dipole. It can be demonstrated that, when approaching the elliptic threshold, the incremental solution tends to organize along well-defined *shear bands* (Bigoni & Capuani [1]).

The classic problems of the inclusion and of the dislocation, reported in the literature only for the linear elastic compressible material (Eshelby [12–14] and Willis [15]), have been extended by Argani, Bigoni & Mishuris [16][1] to the more general case represented by Biot material; furthermore, the case of material subject to an initial state of prestress has been analyzed. These problems have been approached through the Green's function technique; it is important to highlight that the general solutions obtained are valid not only for two-dimensional cases, but also for the three-dimensional: the difference consists in the Green's functions set (2-D or 3-D) that is employed in the integral expressions for the incremental displacement and mean stress fields. In this text, the two-dimensional problem is considered, so that the Green's functions set for the concentrated force in a plane medium is employed. The solutions

[1] L.P. Argani, D. Bigoni, and G. Mishuris. "Dislocations and inclusions in prestressed metals". In: Proc. Roy. Soc. A 469 (2013).

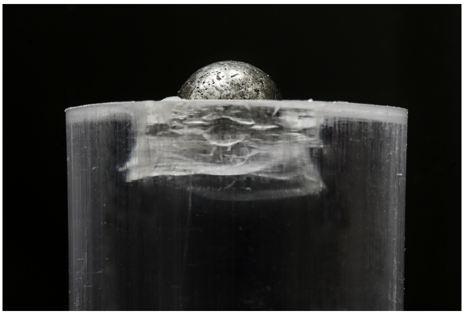


Figure 1.2: Conical fracture produced by a spherical indenter (14 mm diameter) at an indentation speed of 8 mm s^{-1} in a polycarbonate cylindrical specimen at 0 °C (experiment performed by the authors at the Lab. for Physical Modeling of Structures and Photoelasticity of the University of Trento).

are derived for the incremental displacement, mean stress and tension fields as functions of the prestress and the incremental shear moduli of the material.

Examples about a circular inclusion subject to an homothetic inelastic initial expansion and of a *straight edge dislocations* dipole have been developed; it can be evidenced the perfect correspondence with the solutions available in the literature for the simple cases of linear elastic isotropic material; observations are reported on the development of the deformation localization induced by different levels of prestress and of the dislocation dipole inclination in the case of anisotropic material. The results show the behaviour of material in severe deformation condition; in particular, simulations on metallic materials have been performed employing the J_2-deformation theory of plasticity (Hutchinson & Neale [17]), which is suitable in describing the behaviour of ductile metals: the formation of *shear bands* is observed for specific values of prestress, that can promote other dislocations with a cascade effect along the sliding directions. Note that the effect of prestress consists in an induced anisotropy, which has a strong effect on the stress field near an *edge dislocation* that can be evidenced through photoelastic experiments (Figure 1.1).

Three-dimensional Green's functions have been obtained by Argani, Bigoni, Capuani & Movchan[2] for the problem of the concentrated force in an infinite,

[2] L.P. Argani, D. Bigoni, D. Capuani, and N.V. Movchan. "Cones of localized shear strain in incompressible elasticity with prestress: Green's function and integral representations". (2014).

incompressible, anisotropic, elastic material subject to an initial prestress; consequently, similarly to the problems of the inclusion and the dislocation, an incremental formulation has been employed within the general case of anisotropic material and a subsequent introduction of the J_2-deformation theory. The solutions are expressed in terms of incremental displacements and mean stress and, in particular, are represented for simple configurations of self-equilibrated forces, for instance the force dipoles; these dipoles are assumed with a generic inclination with respect to the infinite medium. In particular, the investigation is focused on the problem of a force dipole in an axisymmetric state of stress; it is demonstrated that the material, when approaching the elliptic threshold, shows a localization of the deformation, which preserves the cylindrical symmetry, namely, the formation of *shear cones*; the formation of a failure cone has been noticed by performing experimental tests on cylindrical polycarbonate specimens with spherical indenter (Figure 1.2). One of the most relevant motivations for the research on the three-dimensional Green's functions is the generalization of the results obtained for the inclusion and the dislocation to the three-dimensional case.

Outline of Part I

Part I is organized as follows. After a brief overview on tensor algebra and analysis (provided in the **second Chapter**), elements of continuum mechanics at large strain (kinematics, forces, elasticity, equilibrium and motion equations) are described in detail within the **third Chapter**. The **fourth Chapter** summarizes the incremental problem, focusing on the constitutive equations of elasticity; non-linear elasticity constitutes an interesting tool allowing to analyze the plastic behaviour of materials within the loading branch.

The perturbative approach used by Bigoni & Capuani is introduced in the **fifth Chapter** for the investigation of material instabilities by analyzing the effects of a concentrated force in an infinite medium within a two-dimensional incremental non-linear incompressible elasticity framework.

The two-dimensional Green's functions set of Bigoni & Capuani is employed within *a new generalization* of the inclusion and the dislocation problems, described in the **sixth Chapter**. As a generalization to the constitutive framework introduced in Chapters 5 and 6, *a novel formulation* of three-dimensional Green's

Submitted

functions is derived in the **seventh Chapter**, to analyze the material instabilities of an infinite incremental non-linear elastic incompressible material.

Chapter 2

Elements of tensor algebra and analysis

A brief summary of definitions and properties of vectors and tensors is provided. The possible representations and notations of the quantities are illustrated. The aim of this Chapter is to recall the main aspects, rules and theorems of tensor algebra and calculus that will be used throughout the text.

TENSOR ALGEBRA AND ANALYSIS are largely used in the next Chapters: it is important to have an overview on these fundamentals of mathematics and to summarize the main aspects at the beginning of this text. The attention is centred on definitions and properties of vectors and second and fourth order tensors in a three-dimensional Euclidean space $\mathbb{E}^3$ and its associated vector space $\mathcal{V}$. Vector operators and the main relations between tensors are illustrated; all the properties are expressed using both index and compact notations. Finally, some important theorems are presented.

2.1 Points, vectors and tensors

2.1.1 Points and vectors

In a three-dimensional Euclidean space $\mathbb{E}^3$, a point is an element of $\mathbb{E}^3$ and a vector is an element of the associated vector space $\mathcal{V}$.

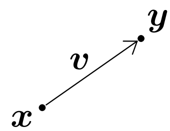


Figure 2.1: Points and vectors.

The difference between two points

$$\boldsymbol{v} = \boldsymbol{y} - \boldsymbol{x} \tag{2.1}$$

is a vector, while the sum of a point $\boldsymbol{x}$ and a vector $\boldsymbol{v}$

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{v} \tag{2.2}$$

is a point (Figure 2.1).

In the following, the *convention of sum over repeated indexes* has been adopted, unless otherwise stated.

2.1.2 Inner (or scalar) product of two vectors

The inner (or scalar) product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \qquad i = 1, 2, 3 \tag{2.3}$$

From (2.3) we have

$$|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} \tag{2.4}$$

A Cartesian coordinate system is composed by an orthonormal basis $\{ \boldsymbol{e}_i \} = \{ \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \}$ and by a point $\boldsymbol{O}$ called origin. The components of a vector $\boldsymbol{v}$ described through $\{ \boldsymbol{e}_i \}$ are

$$u_i = \boldsymbol{u} \cdot \boldsymbol{e}_i \tag{2.5}$$

2.1.3 Second-order tensors

A second-order tensor $\boldsymbol{S}$ is a linear operator that transforms the vector $\boldsymbol{u}$ in the vector $\boldsymbol{v}$ through the following relation

$$v_i = S_{ij} u_j \tag{2.6}$$

The set of all the second-order tensors $\boldsymbol{S}$ forms a vector space, called Lin, when the following operations may apply:

- sum
$$(\boldsymbol{S}+\boldsymbol{T})\boldsymbol{v} = \boldsymbol{S}\boldsymbol{v} + \boldsymbol{T}\boldsymbol{v} \tag{2.7}$$

- product with a scalar quantity
$$(\alpha \boldsymbol{S})\boldsymbol{v} = \alpha(\boldsymbol{S}\boldsymbol{v}) \qquad \alpha \in \mathbb{R} \tag{2.8}$$

- null second-order tensor $\mathbf{0}$
$$\mathbf{0}\boldsymbol{v} = \mathbf{0} \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{2.9}$$

- identity $\boldsymbol{I}$
$$\boldsymbol{I}\boldsymbol{v} = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{2.10}$$

2.1.4 Product of two second-order tensors

The product of two second-order tensors $\boldsymbol{S}$ and $\boldsymbol{T}$ is defined as

$$(\boldsymbol{S}\boldsymbol{T})\boldsymbol{v} = \boldsymbol{S}(\boldsymbol{T}\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{2.11}$$

In general, the product of second-order tensors does not commute, so that $\boldsymbol{S}\boldsymbol{T} \neq \boldsymbol{T}\boldsymbol{S}$.

2.1.5 Transposed second-order tensor

The transposed of the second-order tensor $\boldsymbol{S}$, denoted with $\boldsymbol{S}^{\mathrm{T}}$, is the unique tensor such that:

$$\boldsymbol{S}\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{S}^{\mathrm{T}}\boldsymbol{v} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V} \tag{2.12}$$

The following rules of transposition hold:

$$(\boldsymbol{S}+\boldsymbol{T})^{\mathrm{T}} = \boldsymbol{S}^{\mathrm{T}} + \boldsymbol{T}^{\mathrm{T}} \tag{2.13a}$$

$$(\boldsymbol{S}\boldsymbol{T})^{\mathrm{T}} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}} \tag{2.13b}$$

$$\left(\boldsymbol{S}^{T}\right)^{\mathrm{T}} = \boldsymbol{S} \tag{2.13c}$$

2.1.6 Symmetric and skew-symmetric second-order tensors

A second-order tensor $\boldsymbol{S}$ is symmetric if

$$\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}} \tag{2.14}$$

while it is skew-symmetric if

$$\boldsymbol{S} = -\boldsymbol{S}^{\mathrm{T}} \tag{2.15}$$

The set of all the symmetric tensors is called Sym, while the set of all the skew-symmetric tensors is called Skw. Every second-order tensor $\boldsymbol{S}$ may be expressed as

$$\boldsymbol{S} = \boldsymbol{A} + \boldsymbol{W} \tag{2.16}$$

where $\boldsymbol{A} \in \mathrm{Sym}$ and $\boldsymbol{W} \in \mathrm{Skw}$, that are defined as

$$\boldsymbol{A} = \frac{1}{2}\left(\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}}\right) \tag{2.17a}$$

$$\boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{S} - \boldsymbol{S}^{\mathrm{T}}\right) \tag{2.17b}$$

2.1.7 Tensor (or dyadic) product of two vectors

The tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ is a second-order tensor such that

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v} = (\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a} \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{2.18}$$

The Cartesian components of $\boldsymbol{a} \otimes \boldsymbol{b}$ with respect to $\{\, \boldsymbol{e}_i \,\}$ are

$$(\boldsymbol{a} \otimes \boldsymbol{b})_{ij} = a_i b_j = (\boldsymbol{e}_i \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{e}_j) \tag{2.19}$$

$\boldsymbol{a} \otimes \boldsymbol{b}$ is not symmetric since $(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}}_{ij} = b_i a_j$.

Given an orthonormal basis $\{\, \boldsymbol{e}_i \,\}$, every second-order tensor $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.20}$$

2.1.8 Trace of a second-order tensor

The trace operator assigns to every second-order tensor $\boldsymbol{S}$ the scalar $(\operatorname{tr}\boldsymbol{S})$ such that

$$\operatorname{tr}\boldsymbol{S} = S_{ii} = S_{11} + S_{22} + S_{33} \tag{2.21}$$

This operator has the following properties:

$$\operatorname{tr}\boldsymbol{S}^{\mathrm{T}} = \operatorname{tr}\boldsymbol{S} \tag{2.22a}$$

$$\operatorname{tr}(\boldsymbol{S}\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{T}\boldsymbol{S}) \tag{2.22b}$$

2.1.9 Inner (or scalar) product of two second-order tensors

The inner (or scalar) product of two second-order tensors $\boldsymbol{S}$ and $\boldsymbol{T}$ is defined as

$$\boldsymbol{S} \cdot \boldsymbol{T} = \operatorname{tr}\big(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{T}\big) = S_{ij}T_{ij} \tag{2.23}$$

and has the following properties:

$$\boldsymbol{I} \cdot \boldsymbol{S} = \operatorname{tr}\boldsymbol{S} \tag{2.24a}$$

$$\boldsymbol{R} \cdot (\boldsymbol{S}\boldsymbol{T}) = \big(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{R}\big) \cdot \boldsymbol{T} = \big(\boldsymbol{R}\boldsymbol{T}^{\mathrm{T}}\big) \cdot \boldsymbol{S} \tag{2.24b}$$

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{a} \cdot \boldsymbol{u})(\boldsymbol{b} \cdot \boldsymbol{v}) = a_i b_j u_i v_j \tag{2.24c}$$

Furthermore,

- if $\boldsymbol{S}$ is symmetric

$$\boldsymbol{S} \cdot \boldsymbol{T} = \boldsymbol{S} \cdot \boldsymbol{T}^{\mathrm{T}} = \boldsymbol{S} \cdot \frac{1}{2}\big(\boldsymbol{T} + \boldsymbol{T}^{\mathrm{T}}\big) \tag{2.25}$$

- if $\boldsymbol{W}$ is skew-symmetric

$$\boldsymbol{W} \cdot \boldsymbol{T} = -\boldsymbol{W} \cdot \boldsymbol{T}^{\mathrm{T}} = \boldsymbol{W} \cdot \frac{1}{2}\big(\boldsymbol{T} - \boldsymbol{T}^{\mathrm{T}}\big) \tag{2.26}$$

- if $\boldsymbol{S}$ is symmetric and $\boldsymbol{W}$ is skew-symmetric

$$\boldsymbol{S} \cdot \boldsymbol{W} = 0 \tag{2.27}$$

- if $\boldsymbol{T} \cdot \boldsymbol{S} = 0$ for every second-order tensor $\boldsymbol{S}$, then $\boldsymbol{T} = \boldsymbol{0}$
- if $\boldsymbol{T} \cdot \boldsymbol{S} = 0$ for every symmetric second-order tensor $\boldsymbol{S}$, then $\boldsymbol{T}$ is skew-symmetric
- if $\boldsymbol{T} \cdot \boldsymbol{W} = 0$ for every skew-symmetric second-order tensor $\boldsymbol{W}$, then $\boldsymbol{T}$ is symmetric.

2.1.10 Determinant of a second-order tensor

The determinant of a second-order tensor $\boldsymbol{S}$ is defined as the determinant of the matrix $[\boldsymbol{S}]$

$$\det \boldsymbol{S} = |\boldsymbol{S}| = \varepsilon_{ijk} S_{1i} S_{2j} S_{3k} \tag{2.28}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for even permutations of } i, j, k \\ -1 & \text{for odd permutations of } i, j, k \\ 0 & \text{if an index is repeated} \end{cases} \tag{2.29}$$

is the Ricci or Levi-Civita tensor. The determinant of a second-order tensor is independent from the orthonormal basis $\{ \boldsymbol{e}_i \}$. The inverse of $\boldsymbol{S}$ is defined as the unique tensor $\boldsymbol{S}^{-1}$ such that

$$\boldsymbol{S}\boldsymbol{S}^{-1} = \boldsymbol{S}^{-1}\boldsymbol{S} = \boldsymbol{I} \tag{2.30}$$

$\boldsymbol{S}$ is invertible if and only if $\det \boldsymbol{S} \neq 0$. The determinant and the inverse of a second-order tensor have the following properties:

$$\det(\boldsymbol{S}\boldsymbol{T}) = (\det \boldsymbol{S})(\det \boldsymbol{T}) \tag{2.31a}$$
$$\det\left(\boldsymbol{S}^{\mathrm{T}}\right) = \det \boldsymbol{S} \tag{2.31b}$$
$$\det\left(\boldsymbol{S}^{-1}\right) = (\det \boldsymbol{S})^{-1} \tag{2.31c}$$
$$(\boldsymbol{S}\boldsymbol{T})^{-1} = \boldsymbol{T}^{-1}\boldsymbol{S}^{-1} \tag{2.31d}$$
$$\left(\boldsymbol{S}^{-1}\right)^{\mathrm{T}} = \left(\boldsymbol{S}^{\mathrm{T}}\right)^{-1} \tag{2.31e}$$

2.1.11 Orthogonal second-order tensors

A second-order tensor $\boldsymbol{Q}$ is orthogonal if

$$\boldsymbol{Q}\boldsymbol{v} \cdot \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{v} \cdot \boldsymbol{u} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V} \tag{2.32}$$

Condition (2.32) is satisfied if and only if

$$\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \boldsymbol{I} \quad \Longleftrightarrow \quad \boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q}^{-1} \tag{2.33}$$

Relation (2.33) implies

$$\det\left(\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}}\right) = (\det \boldsymbol{Q})^2 = (\det \boldsymbol{I})^2 = 1 \tag{2.34}$$

so that we have

$$\det \boldsymbol{Q} = \pm 1 \tag{2.35}$$

If an orthogonal second-order tensor $\boldsymbol{Q}$ has $\det \boldsymbol{Q} = +1$, we have a *rotation*, while if it has $\det \boldsymbol{Q} = -1$ we have a *reflection*.

The set of all orthogonal tensors is denoted with Orth set of all orthogonal rotation tensors is denoted with Orth^+. The orthogonal tensors form a *multiplicative group*, namely, the product of orthogonal tensors is always an orthogonal tensor

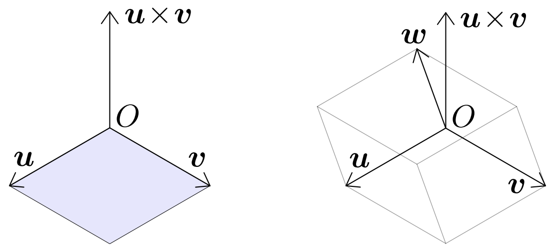
$$\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathrm{Orth} \quad \Longrightarrow \quad \boldsymbol{Q} = \boldsymbol{Q}_1\boldsymbol{Q}_2 \in \mathrm{Orth} \tag{2.36}$$

2.1.12 Positive-definite second-order tensors

A second-order tensor $\boldsymbol{S}$ is a positive-definite tensor if

$$\boldsymbol{v} \cdot \boldsymbol{S}\boldsymbol{v} > 0 \tag{2.37}$$

for every vector $\boldsymbol{v} \neq \boldsymbol{0}$. The set of all positive-definite tensors is denoted with Lin^+.



(a) *Parallelogram generated by $\boldsymbol{u}$ and $\boldsymbol{v}$.*

(b) *Parallelepiped generated by $\boldsymbol{u}$, $\boldsymbol{w}$ and $\boldsymbol{v}$.*

Figure 2.2: Graphic representation of the geometric meaning of vector product.

2.1.13 External (or vector) product of two vectors

The external (or vector) product of two vectors $\boldsymbol{u} \times \boldsymbol{v}$ is defined as

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \tag{2.38}$$

or, with indexes,

$$(\boldsymbol{u} \times \boldsymbol{v})_i = \varepsilon_{ijk} u_j v_k \tag{2.39}$$

where ε_{ijk} is the Ricci or Levi Civita tensor. The external product has the following properties:

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u} \tag{2.40a}$$

$$\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0} \tag{2.40b}$$

The modulus of the external product $\boldsymbol{u} \times \boldsymbol{v}$ represents the area of the parallelogram generated by $\boldsymbol{u}$ and $\boldsymbol{v}$ (Figure 2.2a), while the scalar $\boldsymbol{u} \cdot (\boldsymbol{w} \times \boldsymbol{v})$ represents the volume of the parallelepiped generated by $\boldsymbol{u}$, $\boldsymbol{w}$ and $\boldsymbol{v}$ (Figure 2.2b).

2.1.14 Fourth-order tensors

Throughout this text, fourth-order tensors are denoted with blackboard characters, for instance the elasticity tensor $\mathbb{E}$, and they define linear mappings, assigning to each second-order tensors $\boldsymbol{A}$ a second-order tensor $\mathbb{E}[\boldsymbol{A}]$. If we define an orthonormal basis $\boldsymbol{e}_i$, we have

$$\mathbb{E} = \mathbb{E}_{ijkl}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{2.41}$$

so that

$$(\mathbb{E}[\boldsymbol{A}])_{ij} = \mathbb{E}_{ijkl}\boldsymbol{A}_{kl} \tag{2.42}$$

The product $\mathbb{A}\mathbb{B}$ between two fourth-order tensors is defined, analogously to second-order tensors, by composition, namely, for every $\boldsymbol{C} \in \mathrm{Lin}$ we have

$$(\mathbb{A}\mathbb{B})[\boldsymbol{C}] = \mathbb{A}\big[\mathbb{B}[\boldsymbol{C}]\big] \quad \longleftrightarrow \quad (\mathbb{A}\mathbb{B})_{ijkl} = \mathbb{A}_{ijmn}\mathbb{B}_{mnkl} \tag{2.43}$$

We define the three following tensor products between second-order tensors:

$$(\boldsymbol{A} \otimes \boldsymbol{B})[\boldsymbol{C}] = (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A} \quad \longleftrightarrow \quad (\boldsymbol{A} \otimes \boldsymbol{B})_{ijkl} = \boldsymbol{A}_{ij}\boldsymbol{B}_{kl} \tag{2.44a}$$

$$(\boldsymbol{A} \boxtimes \boldsymbol{B})[\boldsymbol{C}] = \boldsymbol{A}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}} \quad \longleftrightarrow \quad (\boldsymbol{A} \boxtimes \boldsymbol{B})_{ijkl} = \boldsymbol{A}_{ik}\boldsymbol{B}_{jl} \tag{2.44b}$$

$$(\boldsymbol{A} \boxdot \boldsymbol{B})[\boldsymbol{C}] = \frac{1}{2}\left(\boldsymbol{A}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}} + \boldsymbol{A}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\right)$$

$$\longleftrightarrow \quad (\boldsymbol{A} \boxdot \boldsymbol{B})_{ijkl} = \frac{1}{2}\left(\boldsymbol{A}_{ik}\boldsymbol{B}_{jl} + \boldsymbol{A}_{il}\boldsymbol{B}_{jk}\right) \tag{2.44c}$$

From equations (2.44b) and (2.44b) it follows that

$$(\boldsymbol{A} \boxdot \boldsymbol{B})[\boldsymbol{C}] = \frac{1}{2}(\boldsymbol{A} \boxtimes \boldsymbol{B})\left[\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}\right] \tag{2.45}$$

From relations (2.44) we can define the fourth-order identity tensor:

$$\mathbb{I} = \boldsymbol{I} \boxtimes \boldsymbol{I} \tag{2.46}$$

The transpose of a fourth-order tensor is defined for every $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}$ as the unique tensor $\mathbb{E}^{\mathrm{T}}$ with the property

$$\boldsymbol{B} \cdot \mathbb{E}^{\mathrm{T}}[\boldsymbol{A}] = \boldsymbol{A} \cdot \mathbb{E}[\boldsymbol{B}] \tag{2.47}$$

from which we have these definitions

- $\mathbb{E}$ has the major (or diagonal) symmetry whenever $\mathbb{E} = \mathbb{E}^{\mathrm{T}}$;
- $\mathbb{E}$ has the left minor symmetry whenever, for every $\boldsymbol{A}$, $\mathbb{E}[\boldsymbol{A}] \in \mathrm{Sym}$;
- $\mathbb{E}$ has the right minor symmetry whenever, for every $\boldsymbol{A}$, $\mathbb{E}^{\mathrm{T}}[\boldsymbol{A}] \in \mathrm{Sym}$.

The following relations hold

$$(\boldsymbol{A} \otimes \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B} \otimes \boldsymbol{A} \tag{2.48a}$$
$$(\boldsymbol{A} \boxtimes \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} \boxtimes \boldsymbol{B}^{\mathrm{T}} \tag{2.48b}$$
$$(\boldsymbol{A} \boxdot \boldsymbol{B})^{\mathrm{T}} = \frac{1}{2}\left(\boldsymbol{A}^{\mathrm{T}} \boxtimes \boldsymbol{B}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}} \boxtimes \boldsymbol{A}^{\mathrm{T}}\right) \tag{2.48c}$$

Positive definiteness and invertibility of fourth-order tensors are defined analogously to second-order tensors. In fact, for every second-order tensor $\boldsymbol{A} \neq \boldsymbol{0}$, a fourth-order tensor $\mathbb{E}$ is positive-definite if

$$\boldsymbol{A} \cdot \mathbb{E}[\boldsymbol{A}] > 0 \quad \longleftrightarrow \quad A_{ij}\mathbb{E}_{ijkl}A_{kl} > 0 \qquad \forall \boldsymbol{A} \neq \boldsymbol{0} \tag{2.49}$$

while the inverse $\mathbb{E}^{-1}$ of $\mathbb{E}$ exists if

$$\mathbb{E}^{-1}\big[\mathbb{E}[\boldsymbol{A}]\big] = \boldsymbol{A} \qquad \forall \boldsymbol{A} \quad \longleftrightarrow \quad A_{ij}\mathbb{E}_{ijkl}A_{kl} > 0 \tag{2.50}$$

from which we have the following property

$$\mathbb{E}^{-1}\mathbb{E} = \mathbb{I} \tag{2.51}$$

We are now in a position to define the following operators, that will be used in the text

$$\mathbb{S} = \boldsymbol{I} \boxdot \boldsymbol{I} \tag{2.52a}$$
$$\mathbb{W} = \boldsymbol{I} \boxtimes \boldsymbol{I} - \boldsymbol{I} \boxdot \boldsymbol{I} \tag{2.52b}$$

where $\mathbb{S}$ is the *symmetrising operator* and $\mathbb{W}$ is the *skew-symmetrising operator*. Through these quantities, the decomposition (2.16), together with relations (2.17), can be rewritten as

$$\boldsymbol{A} = \mathbb{S}[\boldsymbol{S}] \tag{2.53a}$$
$$\boldsymbol{W} = \mathbb{W}[\boldsymbol{S}] \tag{2.53b}$$

Note that the symmetrising operator is singular, because it associates the null second-order tensor to every skew-symmetric tensor, but its restriction to symmetric tensors is invertible, and the inverse is the tensor itself.

The following relations hold

$$(\boldsymbol{A} \boxtimes \boldsymbol{B})(\boldsymbol{C} \boxtimes \boldsymbol{D}) = \boldsymbol{AC} \boxtimes \boldsymbol{BD} \tag{2.54a}$$
$$(\boldsymbol{A} \boxdot \boldsymbol{B})(\boldsymbol{C} \boxdot \boldsymbol{D}) = \boldsymbol{AC} \boxdot \boldsymbol{BD} + \boldsymbol{AD} \boxdot \boldsymbol{BC} \tag{2.54b}$$

so that we have

$$(\boldsymbol{A} \boxtimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \boxtimes \boldsymbol{B}^{-1} \tag{2.55a}$$
$$(\boldsymbol{A} \boxdot \boldsymbol{A})^{-1} = \boldsymbol{A}^{-1} \boxdot \boldsymbol{A}^{-1} \tag{2.55b}$$

where the inverse are defined in such a way that

$$(\boldsymbol{A} \boxtimes \boldsymbol{B})(\boldsymbol{A}^{-1} \boxtimes \boldsymbol{B}^{-1}) = \boldsymbol{I} \boxtimes \boldsymbol{I} \tag{2.56a}$$
$$(\boldsymbol{A} \boxdot \boldsymbol{A})(\boldsymbol{A}^{-1} \boxdot \boldsymbol{A}^{-1}) = \boldsymbol{I} \boxdot \boldsymbol{I} \tag{2.56b}$$

2.2 Cayley-Hamilton theorem and spectral theorem

A scalar ω is an eigenvalue of $\boldsymbol{S}$ if a unit vector $\boldsymbol{e}$ such that

$$\boldsymbol{S}\boldsymbol{e} = \omega \boldsymbol{e} \tag{2.57}$$

exists, where $\boldsymbol{e}$ is an eigenvector of $\boldsymbol{S}$. The spectrum $\boldsymbol{S}$ is composed by all the eigenvalues of $\boldsymbol{S}$, $\{\, \omega_1, \omega_2, \dots \}$. The set of all the eigenvectors associated to an eigenvalue is called *eigenspace*.

Theorem 2.1. *The eigenvalues of a positive-definite tensor are strictly positive.*

Theorem 2.2. *The characteristic spaces of a symmetric tensor are mutually orthogonal.*

Theorem 2.3 (Spectral theorem)**.** *Let $\boldsymbol{S}$ be a symmetric tensor. Let us consider an orthonormal basis for $\mathcal{V}$ entirely composed by eigenvectors of $\boldsymbol{S}$. Then $\boldsymbol{S}$ can be* *expressed as*

$$\boldsymbol{S} = \sum_{i=3}^{n} \omega_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{2.58}$$

where $\boldsymbol{e}_i$ *are the eigenvectors of* $\boldsymbol{S}$ *and* ω_i *are the corresponding eigenvalues. Furthermore:*

- $\boldsymbol{S}$ *has exactly three different eigenvalues if and only if the characteristic space of* $\boldsymbol{S}$ *is determined by three mutually orthogonal lines*

- $\boldsymbol{S}$ *has exactly two different eigenvalues if and only if* $\boldsymbol{S}$ *admits the representation*

  $$\boldsymbol{S} = \omega_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \omega_2(\boldsymbol{I} - \boldsymbol{e}_1 \otimes \boldsymbol{e}_1) \tag{2.59}$$

  with $|\boldsymbol{e}| = 1$ *and* $\omega_1 \neq \omega_2 = \omega_3$. *In this case* ω_1 *and* ω_2 *are two different eigenvalues and the eigenvector* $\boldsymbol{e}_1$ *is perpendicular to a plane defined by the mutually orthogonal eigenvectors* $\boldsymbol{e}_2$ *and* $\boldsymbol{e}_3$.

- $\boldsymbol{S}$ *has exactly one eigenvalues if and only if*

  $$\boldsymbol{S} = \omega \boldsymbol{I} \tag{2.60}$$

  In this case ω *is the eigenvalue and* $\mathcal{V}$ *the corresponding characteristic space.*

Theorem 2.4. $\omega \in \mathbb{R}$ *is an eigenvalue of the tensor* $\boldsymbol{S}$ *if and only if*

$$\det(\boldsymbol{S} - \omega \boldsymbol{I}) = -\omega^3 + J_1\omega^2 - J_2\omega + J_3 = 0 \tag{2.61}$$

where

$$J_1 = \operatorname{tr} \boldsymbol{S} \tag{2.62a}$$

$$J_2 = \frac{1}{2}\left[(\operatorname{tr} \boldsymbol{S})^2 - \operatorname{tr}(\boldsymbol{S}^2)\right] \tag{2.62b}$$

$$J_3 = \det \boldsymbol{S} \tag{2.62c}$$

are the principal invariants of $\boldsymbol{S}$. *An alternative form for the invariants is the following:*

$$I_1 = J_1 \tag{2.63a}$$

$$I_2 = \operatorname{tr} \boldsymbol{S}^2 \tag{2.63b}$$

$$I_3 = J_3 \tag{2.63c}$$

Theorem 2.5. *Let $\boldsymbol{S}$ and $\boldsymbol{T}$ be symmetric and assume that*

$$J_1(\boldsymbol{S}) = J_1(\boldsymbol{T}) \tag{2.64a}$$

$$J_2(\boldsymbol{S}) = J_2(\boldsymbol{T}) \tag{2.64b}$$

$$J_3(\boldsymbol{S}) = J_3(\boldsymbol{T}) \tag{2.64c}$$

Then $\boldsymbol{S}$ and $\boldsymbol{T}$ have the same spectrum.

Theorem 2.6 (Cayley-Hamilton theorem)**.** *Every tensor $\boldsymbol{S}$ satisfies its own characteristic equation*

$$\boldsymbol{S}^3 - J_1 \boldsymbol{S}^2 + J_2 \boldsymbol{S} - J_3 \boldsymbol{I} = \boldsymbol{0} \tag{2.65}$$

2.3 Gradient, divergence and curl

Let us consider functions defined in an open domain $\mathcal{B}$ of the Euclidean space. A function defined in $\mathcal{B}$ is called *scalar*, *vector*, *tensor* or *point function* depending on its values that may, respectively, be scalars, vectors, tensors or points.

Let ϕ be a scalar field, regular on $\mathcal{B}$. The Taylor series expansion of the first order of ϕ, in a generic point $\boldsymbol{x}$, is:

$$\phi(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) = \phi(\boldsymbol{x}) + \mathrm{d}\phi + o(\mathrm{d}\boldsymbol{x}) \tag{2.66}$$

where

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial \boldsymbol{x}} \cdot \mathrm{d}\boldsymbol{x} \tag{2.67}$$

is the *total differential* of ϕ. The term $o(\mathrm{d}\boldsymbol{x})$ tends quickly to zero when $\mathrm{d}\boldsymbol{x} \to \boldsymbol{0}$. The total differential, given by the difference $\phi(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) - \phi(\boldsymbol{x})$, can be written as

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x_i \tag{2.68}$$

or as

$$\mathrm{d}\phi = \nabla\phi \cdot \mathrm{d}\boldsymbol{x} \tag{2.69a}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x_i}\boldsymbol{e}_i \tag{2.69b}$$

where $\nabla\phi$ is the *gradient* of the scalar field ϕ.

Similarly, if $\boldsymbol{u}$ is a regular vector field, the tensor

$$\nabla\boldsymbol{u} = \frac{\partial u_i}{\partial x_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \nabla\boldsymbol{u} \in \mathrm{Lin} \tag{2.70}$$

is the *gradient* of $\boldsymbol{u}$ and its Cartesian components are

$$(\nabla\boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j} \tag{2.71}$$

The scalar field

$$\mathrm{div}\,\boldsymbol{u} = \frac{\partial u_i}{\partial x_i} \tag{2.72}$$

is the *divergence* of $\boldsymbol{u}$, while the vector field

$$\mathrm{curl}\,\boldsymbol{u} = \nabla \times \boldsymbol{u} \tag{2.73}$$

is called *curl* of $\boldsymbol{u}$. Finally, if $\boldsymbol{S}$ is a regular tensor field, the *divergence* is defined as

$$(\mathrm{div}\,\boldsymbol{S})_i = \frac{\partial S_{ij}}{\partial x_j} \tag{2.74}$$

according to Gurtin [18, 19] and Truesdell & Noll [20]. Another definition of divergence (see for instance Hill [21] and Ogden [22]) is

$$(\mathrm{div}\,\boldsymbol{S})_i = \frac{\partial S_{ij}}{\partial x_i} \tag{2.75}$$

Some important and useful relations involving divergence and gradient opera-

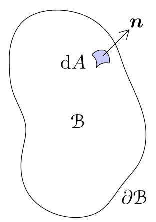


Figure 2.3: Closed region $\mathcal{B}$, boundary $\partial\mathcal{B}$, infinitesimal element $\mathrm{d}A$ of the outer surface $\partial\mathcal{B}$ and its normal vector $\boldsymbol{n}$.

tors are the following:

$$\nabla(\phi\boldsymbol{v}) = \phi(\nabla\boldsymbol{v}) + \boldsymbol{v}\otimes\nabla\phi \tag{2.76a}$$

$$\operatorname{div}(\phi\boldsymbol{u}) = \phi\operatorname{div}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\phi \tag{2.76b}$$

$$\nabla(\boldsymbol{u}\cdot\boldsymbol{v}) = (\nabla\boldsymbol{v})^T\boldsymbol{u} + (\nabla\boldsymbol{u})^T\boldsymbol{v} \tag{2.76c}$$

$$\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v}) = \boldsymbol{u}\operatorname{div}\boldsymbol{v} + (\nabla\boldsymbol{u})\boldsymbol{v} \tag{2.76d}$$

$$\operatorname{div}(\boldsymbol{S}^T\boldsymbol{u}) = \boldsymbol{S}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\operatorname{div}\boldsymbol{S} \tag{2.76e}$$

$$\operatorname{div}(\phi\boldsymbol{S}) = \phi\operatorname{div}\boldsymbol{S} + \boldsymbol{S}\nabla\phi \tag{2.76f}$$

2.4 Divergence theorem

Let $\mathcal{B}$ be a region of the Euclidean space $\mathbb{E}^3$ occupied by a three-dimensional continuous medium with piecewise regular boundary $\partial\mathcal{B}$ (Figure 2.3).

Theorem 2.7 (Divergence theorem)**.**
Let $\phi\colon \mathcal{B}\to\mathbb{R}$, $\boldsymbol{v}\colon \mathcal{B}\to\mathcal{V}$ *and* $\boldsymbol{S}\colon \mathcal{B}\to\mathrm{Lin}$ *be three regular fields. Then:*

$$\int_{\partial\mathcal{B}} \phi\boldsymbol{n}\,\mathrm{d}A = \int_{\partial\mathcal{B}} \nabla\phi\,\mathrm{d}V \tag{2.77a}$$

$$\int_{\partial\mathcal{B}} \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}A = \int_{\partial\mathcal{B}} \operatorname{div}\boldsymbol{v}\,\mathrm{d}V \tag{2.77b}$$

$$\int_{\partial\mathcal{B}} \boldsymbol{S}\boldsymbol{n}\,\mathrm{d}A = \int_{\partial\mathcal{B}} \operatorname{div}\boldsymbol{S}\,\mathrm{d}V \tag{2.77c}$$

where $\boldsymbol{n}$ *is the outward normal unit vector with respect to* $\partial\mathcal{B}$.

Chapter 3

ELEMENTS OF CONTINUUM MECHANICS

In this Chapter the following concepts are explained: deformations, forces, strains, stress, elasticity, hyperelasticity, elastic energy, and also equilibrium and motion equations. A detailed explanation of the rules describing the material response due to a change of the reference system is provided.

CONTINUUM MECHANICS is a science focused on the description of motions, deformations and stresses that are generated inside a solid body subject to external actions. Many are the treated arguments, and each of them is fundamental; in the present Chapter, the main concepts given in Gurtin [18] and Holzapfel [23] are recalled. The first part of this Chapter consists in kinematics, in the introduction of the concept of deformation, and in the description of motions. The concepts of forces and stresses are defined, then equations of equilibrium, and of mass and momentum conservation are formulated. Rules governing the change of the reference system are defined in detail to clarify the concept of spatial and material fields. The Chapter ends with the introduction of the constitutive equations for elastic bodies and the formulation of boundary problems.

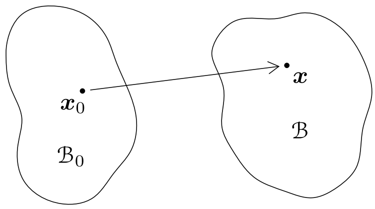


Figure 3.1: Reference configuration and current configuration of a continuum.

3.1 Kinematics

A deformable body $\mathcal{B}$ is a set of material points occupying a regular region of the three-dimensional Euclidean space $\mathbb{E}^3$. The position occupied by the continuum $\mathcal{B}$ at every time is called *configuration*. Let $\mathcal{B}_0$ be the configuration at time $t = 0$, usually characterized by the absence of external loads.

To describe every quantity depending on the position, we can use two variables:

- the position $\boldsymbol{x}$ in the current configuration $\mathcal{B}$ at any given time; the motion is defined through a *spatial* (or *Eulerian*) *description*;

- the position $\boldsymbol{x}_0$ in the reference configuration $\mathcal{B}_0$; the motion is defined through a *material* (or *Lagrangian*) *description*.

3.1.1 Deformation

The deformation is described by a function transforming the material points $\boldsymbol{x}_0$ into the spatial points $\boldsymbol{x}$,

$$\begin{aligned} \boldsymbol{\phi}&\colon \mathcal{B}_0 \to \mathcal{B} \\ \boldsymbol{x} &= \boldsymbol{\phi}(\boldsymbol{x}_0) \end{aligned} \tag{3.1}$$

Assuming that during the transformation interpenetration and tearing are avoided, $\boldsymbol{\phi}$ must be C^2 class continuous bijective application. Therefore, the inverse function $\boldsymbol{\phi}^{-1}$ exists and takes every point of the current configuration back to the corresponding point of the reference configuration

$$\boldsymbol{\phi}^{-1} \colon \mathcal{B} \to \mathcal{B}_0$$
$$\boldsymbol{x}_0 = \boldsymbol{\phi}^{-1}(\boldsymbol{x}) \tag{3.2}$$

Deformation gradient

Let $\boldsymbol{F}$ be the deformation gradient

$$\boldsymbol{F} = \nabla \boldsymbol{\phi} \qquad \boldsymbol{F} \in \mathrm{Lin}^+ \tag{3.3}$$

Let $\mathrm{d}\boldsymbol{x}_0$ be an element with infinitesimal length in $\mathcal{B}_0$, and $\mathrm{d}\boldsymbol{x}$ its relative image in $\mathcal{B}$. Since

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0) \tag{3.4a}$$
$$\boldsymbol{x} + \mathrm{d}\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0 + \mathrm{d}\boldsymbol{x}_0) \tag{3.4b}$$

we obtain:

$$\mathrm{d}\boldsymbol{x} = (\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) - \boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0 + \mathrm{d}\boldsymbol{x}_0) - \boldsymbol{\phi}(\boldsymbol{x}_0) = (\nabla \boldsymbol{\phi})\, \mathrm{d}\boldsymbol{x}_0 = \boldsymbol{F}\, \mathrm{d}\boldsymbol{x}_0 \tag{3.5}$$

$\boldsymbol{F}$ describes the variations of the linear elements during the motion. Furthermore, $\boldsymbol{F}$ is a so-called *two-point* tensor, since it relates the geometric elements of the reference configuration to the corresponding elements of the current configuration. An important property of the deformation gradient is

$$J = \det(\boldsymbol{F}) > 0 \tag{3.6}$$

Relation (3.6) can be demonstrated ab absurdo. If $J = 0$, one of the eigenvalues of $\boldsymbol{F}$, for instance λ_0, should be equal to zero, so that

$$\boldsymbol{F}\, \mathrm{d}\boldsymbol{x}_0 = \lambda_0\, \mathrm{d}\boldsymbol{x}_0 = \boldsymbol{0} \quad \Longrightarrow \quad \mathrm{d}\boldsymbol{x} = \boldsymbol{0} \tag{3.7}$$

where $\mathrm{d}\boldsymbol{x}_0$ is the eigenvector associated to λ_0. Hence, if $J = 0$, a linear material element should collapse in a spatial element with null length. To avoid these physically impossible behaviours, we assume $J \neq 0$. Since $J = \det(\boldsymbol{F})$ is a continuous function, and taking into account that for $\mathcal{B} \equiv \mathcal{B}_0$ (namely, $\boldsymbol{x} \equiv \boldsymbol{x}_0$) we have $\boldsymbol{F} = \boldsymbol{I}$ and $J = 1$, we have proved the property (3.6).

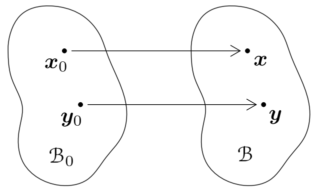


Figure 3.2: Rigid translation.

Displacement field

The vector field

$$\boldsymbol{u}(\boldsymbol{x}_0) = \boldsymbol{\phi}(\boldsymbol{x}_0) - \boldsymbol{x}_0 \tag{3.8}$$

represents the displacement field of a material point. The displacement field can also be expressed as a function of the current position

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{\phi}^{-1}(\boldsymbol{x}_0) \tag{3.9}$$

The two above descriptions are equivalent and can be interchanged, but are conceptually different due to the different meaning assigned to the deformation $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0)$

$$\boldsymbol{u}(\boldsymbol{x}_0) = \boldsymbol{u}\big(\boldsymbol{\phi}^{-1}(\boldsymbol{x})\big) = \boldsymbol{u}(\boldsymbol{x}) \tag{3.10}$$

The material gradient of the displacement field is

$$\boldsymbol{H} = \nabla \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_0} - \frac{\partial \boldsymbol{x}_0}{\partial \boldsymbol{x}_0} = \boldsymbol{F} - \boldsymbol{I} \tag{3.11}$$

so that the deformation gradient can be rewritten as

$$\boldsymbol{F} = \boldsymbol{H} + \boldsymbol{I} = \nabla \boldsymbol{u} + \boldsymbol{I} \tag{3.12}$$

Homogeneous deformations

In general, $\boldsymbol{F}$ is a function of $\boldsymbol{x}_0$. If $\boldsymbol{F}$ is independent from $\boldsymbol{x}_0$, the deformation is defined as homogeneous.

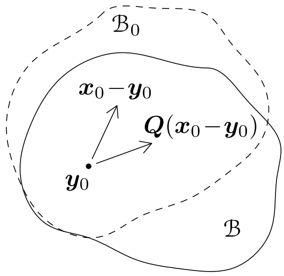


Figure 3.3: Rigid rotation.

Rigid deformations. Rigid deformations constitute a particular class of homogeneous deformations. For rigid deformations, the distance among the points in the reference configuration remains constant for all the other configurations.

In a rigid translation, every point moves with the same distance, direction, and time. Therefore $\boldsymbol{u}$ is not a function of the position of the material point, but is constant within the whole body.

The rigid translation of the body is described by the deformation

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0) = \boldsymbol{x}_0 + \boldsymbol{y} \tag{3.13}$$

so that

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{x}_0}{\partial \boldsymbol{x}_0} + \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_0} = \boldsymbol{I} + \boldsymbol{0} = \boldsymbol{I} = \text{const} \tag{3.14}$$

In a rigid rotation, the whole body rotates around a fixed point $\boldsymbol{y}_0$. In this case, the deformation is described by

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0) = \boldsymbol{y}_0 + \boldsymbol{Q}(\boldsymbol{x}_0 - \boldsymbol{y}_0) \qquad \boldsymbol{Q} \in \mathrm{Orth}^+ \tag{3.15}$$

so that

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}_0} = \boldsymbol{0} + \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{0}) = \boldsymbol{Q} = \text{const} \tag{3.16}$$

Uniform deformations. An other class of homogeneous deformations is given by the uniform deformations, described by

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0) = \alpha \boldsymbol{x}_0 \tag{3.17}$$

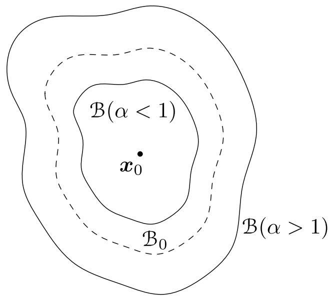


Figure 3.4: Uniform deformation.

where α is a scalar. If $\alpha > 1$ we have an expansion, while if $\alpha < 1$ we have a contraction. The deformation gradient of a uniform deformation is

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}_0} = \alpha \boldsymbol{I} = \text{const} \tag{3.18}$$

Polar decomposition

If $\boldsymbol{F} \in \mathrm{Lin}^+$, then it can be expressed through the polar decomposition

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R} \tag{3.19}$$

where $\boldsymbol{R}$ is an orthogonal tensor, called rotation tensor, while $\boldsymbol{U}$ and $\boldsymbol{V}$ are called, respectively, *right stretch tensor* and *left stretch tensor*, which are defined as

$$\boldsymbol{U} = \sqrt{\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}} \qquad \boldsymbol{U} \in \mathrm{Sym}^+ \tag{3.20a}$$

$$\boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}} \qquad \boldsymbol{V} \in \mathrm{Sym}^+ \tag{3.20b}$$

The decomposition $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ describes the transformation composed by a pure deformation motion of $\mathrm{d}\boldsymbol{x}_0$, represented by the material tensor $\boldsymbol{U}$, followed by a pure rotation $\boldsymbol{R}$. Therefore, the tensor $\boldsymbol{R}$ transforms the material vector $\boldsymbol{U}\,\mathrm{d}\boldsymbol{x}_0$ into the spatial vector $\mathrm{d}\boldsymbol{x}$.

In the same way, the decomposition $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$ describes the transformation composed by a rotation of the material vector $\mathrm{d}\boldsymbol{x}_0$, represented by the rotation tensor $\boldsymbol{R}$, followed by a pure deformation motion of the spatial vector $\boldsymbol{R}\,\mathrm{d}\boldsymbol{x}_0$, applied by the spatial tensor $\boldsymbol{V}$.

Hence, the tensor $\boldsymbol{R}$ is a *two-point* tensor, like $\boldsymbol{F}$, since it works in between the material and the current configurations. Being $\boldsymbol{U}$ and $\boldsymbol{V}$ definite positive, all their eigenvalues are positive.

Let λ_i $(i = 1, 2, 3)$ be the positive eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{u}_i$ $(i = 1, 2, 3)$ the corresponding normalized eigenvectors; in the relative principal reference system we have

$$\boldsymbol{U} = \lambda_1 \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \lambda_2 \boldsymbol{u}_2 \otimes \boldsymbol{u}_2 + \lambda_3 \boldsymbol{u}_3 \otimes \boldsymbol{u}_3 \tag{3.21}$$

and from (3.19) it follows that

$$\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}\left(\sum_{i=1}^{3} \lambda_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i\right)\boldsymbol{R}^{\mathrm{T}} = \sum_{i=1}^{3} \lambda_i (\boldsymbol{R}\boldsymbol{u}_i \otimes \boldsymbol{R}\boldsymbol{u}_i) \tag{3.22}$$

We can conclude that $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same eigenvalues, while the eigenvectors of $\boldsymbol{V}$ are those of $\boldsymbol{U}$, but rotated through $\boldsymbol{R}$ (or, inversely, through $\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$).

We introduce now the following tensors

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \boldsymbol{U}^2 = \sum_{1=1}^{3} \lambda_i^2 \boldsymbol{u}_i \otimes \boldsymbol{u}_i \qquad \boldsymbol{U} \in \mathrm{Sym}^+ \tag{3.23}$$

and

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{V}^2 = \sum_{1=1}^{3} \lambda_i^2 (\boldsymbol{R}\boldsymbol{u}_i \otimes \boldsymbol{R}\boldsymbol{u}_i) \qquad \boldsymbol{V} \in \mathrm{Sym}^+ \tag{3.24}$$

that are called, respectively, *right* and *left Cauchy-Green tensors*. Their eigenvectors coincide with those of $\boldsymbol{U}$ and $\boldsymbol{V}$, while their eigenvalues are the squared of the eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$.

Deformation measures

Transformation of a line element. In order to measure the elongation of a line element $\mathrm{d}\boldsymbol{x}_0$, we use the linear dilatation coeffiecient ε_n

$$\varepsilon_n = \frac{|\mathrm{d}\boldsymbol{x}| - |\mathrm{d}\boldsymbol{x}_0|}{|\mathrm{d}\boldsymbol{x}_0|} = \frac{|\boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0|}{|\mathrm{d}\boldsymbol{x}_0|} - 1 \tag{3.25}$$

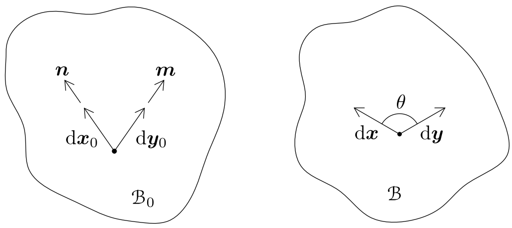


Figure 3.5: Angular variations.

ε_n can also be rewritten as

$$\varepsilon_n = \frac{\sqrt{\boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0 \cdot \boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0}}{|\mathrm{d}\boldsymbol{x}_0|} - 1 = \frac{\sqrt{\mathrm{d}\boldsymbol{x}_0 \cdot \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0}}{|\mathrm{d}\boldsymbol{x}_0|} - 1 = \sqrt{\boldsymbol{n}\cdot\boldsymbol{C}\boldsymbol{n}} - 1 \tag{3.26}$$

where

$$\boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{x}_0}{|\mathrm{d}\boldsymbol{x}_0|} \tag{3.27}$$

represents the direction of the material vector $\mathrm{d}\boldsymbol{x}_0$.

Angular transformations. Let $\boldsymbol{n}$ and $\boldsymbol{m}$ be the directions of the material vectors $\mathrm{d}\boldsymbol{x}_0$ and $\mathrm{d}\boldsymbol{y}_0$, defined as

$$\boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{x}_0}{|\mathrm{d}\boldsymbol{x}_0|} \tag{3.28a}$$

$$\boldsymbol{m} = \frac{\mathrm{d}\boldsymbol{y}_0}{|\mathrm{d}\boldsymbol{y}_0|} \tag{3.28b}$$

and θ be the angle between the corresponding spatial vectors $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{y}$. If

$$\gamma_{\mathrm{mn}} = \frac{\pi}{2} - \theta \tag{3.29}$$

is the complementary angle of θ, the angular variation is described by

$$\sin(\gamma_{\mathrm{mn}}) = \cos(\theta) = \frac{\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{y}}{|\mathrm{d}\boldsymbol{x}||\mathrm{d}\boldsymbol{y}|} = \frac{\boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0 \cdot \boldsymbol{F}\,\mathrm{d}\boldsymbol{y}_0}{|\mathrm{d}\boldsymbol{x}_0||\mathrm{d}\boldsymbol{y}_0|} \tag{3.30}$$

or, equivalently, by

$$\sin(\gamma_{\mathrm{mn}}) = \frac{\mathrm{d}\boldsymbol{x}_0 \cdot \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\,\mathrm{d}\boldsymbol{y}_0}{\sqrt{\mathrm{d}\boldsymbol{x}_0 \cdot \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0}\sqrt{\mathrm{d}\boldsymbol{y}_0 \cdot \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\,\mathrm{d}\boldsymbol{y}_0}} = \frac{\boldsymbol{n} \cdot \boldsymbol{C}\boldsymbol{m}}{\sqrt{\boldsymbol{n} \cdot \boldsymbol{C}\boldsymbol{n}}\sqrt{\boldsymbol{m} \cdot \boldsymbol{C}\boldsymbol{m}}} \tag{3.31}$$

Volume element trasformation. In Section 2.1 we have seen that the volume determined by three material vectors is

$$\mathrm{d}V_0 = \mathrm{d}\boldsymbol{x}_0 \cdot (\mathrm{d}\boldsymbol{y}_0 \times \mathrm{d}\boldsymbol{z}_0) \tag{3.32}$$

in the reference configuration, while it is equal to

$$\mathrm{d}V = \mathrm{d}\boldsymbol{x} \cdot (\mathrm{d}\boldsymbol{y} \times \mathrm{d}\boldsymbol{z}) = \boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0 \cdot (\boldsymbol{F}\,\mathrm{d}\boldsymbol{y}_0 \times \boldsymbol{F}\,\mathrm{d}\boldsymbol{z}_0) \tag{3.33}$$

after the deformation. In the two- or three-dimensional Euclidean space, the following theorem of linear algebra holds:

$$\boldsymbol{A}\boldsymbol{a} \times \boldsymbol{A}\boldsymbol{b} = (\det \boldsymbol{A})\boldsymbol{A}^{-T}(\boldsymbol{a} \times \boldsymbol{b}) \qquad \forall \boldsymbol{A} \in \mathrm{Lin}^+, \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R} \tag{3.34}$$

with which equation (3.33) can be rewritten as

$$\begin{aligned} \mathrm{d}V &= \boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0 \cdot \big(\det(\boldsymbol{F})\boldsymbol{F}^{-\mathrm{T}}(\mathrm{d}\boldsymbol{y}_0 \times \mathrm{d}\boldsymbol{z}_0)\big) = J\,\mathrm{d}\boldsymbol{x}_0 \cdot \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}^{-\mathrm{T}}(\mathrm{d}\boldsymbol{y}_0 \times \mathrm{d}\boldsymbol{z}_0) = \\ &= J\,\mathrm{d}V_0 \end{aligned} \tag{3.35}$$

Consequently, in a constant volume deformation ($\mathrm{d}V = \mathrm{d}V_0$)

$$J = \det(\boldsymbol{F}) = 1 \tag{3.36}$$

The volumetric expansion can be evaluated as

$$\Theta = \frac{\mathrm{d}V - \mathrm{d}V_0}{\mathrm{d}V_0} = \det(\boldsymbol{F}) - 1 \tag{3.37}$$

Transformation of an oriented surface element. A surface element $\mathrm{d}\boldsymbol{A}$ is an elementary surface $\mathrm{d}A$ with a specific orientation given by the unit vector $\boldsymbol{n}$ normal to the surface. As described in Section 2.1, two material vectors determine the surface element

$$\mathrm{d}\boldsymbol{A}_0 = \boldsymbol{n}_0\,\mathrm{d}A_0 = \mathrm{d}\boldsymbol{x}_0 \times \mathrm{d}\boldsymbol{y}_0 \tag{3.38}$$

where $\boldsymbol{n}_0$ is the normal vector of $\mathrm{d}\boldsymbol{A}_0$. After the deformation, $\mathrm{d}\boldsymbol{A}_0$ transforms into

$$\mathrm{d}\boldsymbol{A} = \boldsymbol{n}\,\mathrm{d}A = \mathrm{d}\boldsymbol{x} \times \mathrm{d}\boldsymbol{y} = \boldsymbol{F}\,\mathrm{d}\boldsymbol{x}_0 \times \boldsymbol{F}\,\mathrm{d}\boldsymbol{y}_0 \tag{3.39}$$

On application of the theorem (3.34) to (3.39) yields

$$\boldsymbol{n}\,\mathrm{d}A = \det(\boldsymbol{F})\boldsymbol{F}^{-\mathrm{T}}(\mathrm{d}\boldsymbol{x}_0 \times \mathrm{d}\boldsymbol{y}_0) = J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0\,\mathrm{d}A_0 \tag{3.40}$$

which is known as *Nanson rule for the area transformation*. From equation (3.40) we obtain

$$\mathrm{d}A = J\left|\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0\right|\mathrm{d}A_0 \tag{3.41}$$

while the relation between $\boldsymbol{n}$ and $\boldsymbol{n}_0$ is given by

$$\boldsymbol{n} = \frac{\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0}{|\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0|} \tag{3.42a}$$

$$\boldsymbol{n}_0 = \frac{\boldsymbol{F}^{\mathrm{T}}\boldsymbol{n}}{|\boldsymbol{F}^{\mathrm{T}}\boldsymbol{n}|} \tag{3.42b}$$

Green-Lagrange tensor. If we consider finite deformations, the infinite deformation measures can be obtained using $\boldsymbol{F}$ and its related tensors $\boldsymbol{C}$, $\boldsymbol{B}$, $\boldsymbol{U}$, and $\boldsymbol{V}$. However, these measures are non null in the undeformed configuration ($\mathcal{B} \equiv \mathcal{B}_0$). For this reason we define the *Green-Lagrange strain tensor*

$$\boldsymbol{G} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \qquad \boldsymbol{G} \in \mathrm{Sym}^+ \tag{3.43}$$

such that it is zero in the undeformed configuration ($\boldsymbol{G}(\boldsymbol{x}_0) = \boldsymbol{0}$).

Small deformations

Within the small deformation theory, $\nabla\boldsymbol{u}$ is an infinitesimal quantity, so that the product $\nabla\boldsymbol{u}^{\mathrm{T}}\nabla\boldsymbol{u}$ can be neglected. Using equation (3.12), the Cauchy-Green deformation tensor $\boldsymbol{C}$ becomes

$$\boldsymbol{C} = \boldsymbol{I} + \nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}} + \nabla\boldsymbol{u}^{\mathrm{T}}\nabla\boldsymbol{u} \simeq \boldsymbol{I} + \nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}} \tag{3.44}$$

Therefore, the Green-Lagrange deformation tensor $\boldsymbol{G}$ has the form

$$\boldsymbol{G} = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}\right) \tag{3.45}$$

that corresponds to the deformation tensor within the infinitesimal elasticity.

3.1.2 Motions

The motion of a material body is a collection of configurations depending on a unique parameter, the time t. The position of the point $\boldsymbol{x}_0$ at time t is

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0, t) \tag{3.46}$$

while the trajectory described by the material point is

$$\mathcal{T} = \{\, (\boldsymbol{x}, t) \mid \boldsymbol{x} \in \mathcal{B}(t), t \in \mathbb{R} \,\} \tag{3.47}$$

The motion description in relation (3.46) is called *Lagrangian* (or *material*); this description is focused on the material point, once the initial configuration is known.

Since $\boldsymbol{\phi}$ is a regular bijective function, the inverse function $\boldsymbol{\phi}^{-1}$ exists, such that

$$\boldsymbol{x}_0 = \boldsymbol{\phi}^{-1}(\boldsymbol{x}, t) \tag{3.48}$$

The motion description given by relation (3.48) is called *Eulerian* (or *spatial*). In this case, we focus on a point in the space. Usually, in the continuum mechanics the Lagrangian description is adopted, while in fluid mechanics the Eulerian description is preferred.

Velocity and acceleration fields

The velocity and acceleration fields are obtained, respectively, through the first and second order derivatives of the motion $\boldsymbol{\phi}$ with respect to time t, with $\boldsymbol{x}_0$ fixed; the material description of the velocity and acceleration fields is given by

$$\dot{\boldsymbol{x}}(\boldsymbol{x}_0, t) = \frac{\partial \boldsymbol{\phi}(\boldsymbol{x}_0, t)}{\partial t} \tag{3.49a}$$

$$\ddot{\boldsymbol{x}}(\boldsymbol{x}_0, t) = \frac{\partial^2 \boldsymbol{\phi}(\boldsymbol{x}_0, t)}{\partial t^2} \tag{3.49b}$$

where $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ represent the increment, with respect to time, of the position and the velocity of the material point $\boldsymbol{x}_0$ at time t. The spatial description of the velocity and acceleration fields is

$$\boldsymbol{v}(\boldsymbol{x}, t) = \dot{\boldsymbol{x}}(\boldsymbol{\phi}^{-1}(\boldsymbol{x}, t), t) = \dot{\boldsymbol{x}}(\boldsymbol{x}_0, t) \tag{3.50a}$$

$$\boldsymbol{a}(\boldsymbol{x}, t) = \ddot{\boldsymbol{x}}(\boldsymbol{\phi}^{-1}(\boldsymbol{x}, t), t) = \ddot{\boldsymbol{x}}(\boldsymbol{x}_0, t) \tag{3.50b}$$

where $\boldsymbol{v}$ and $\boldsymbol{a}$ represent, respectively, the increment, with respect to time, of the position and the velocity at point $\boldsymbol{x}$. Furthermore, it is important to note that

$$\boldsymbol{a}(\boldsymbol{x}, t) =\neq \frac{\partial \boldsymbol{v}}{\partial t} \tag{3.51}$$

3.1.3 Material and spatial derivatives

Let $f(\boldsymbol{x}_0, t)$ be a regular material field and $g(\boldsymbol{x}, t)$ a regular spatial field.

Material time derivative of a material field

The material time derivative of f, with a symbolic notation $\mathrm{D}f/\,\mathrm{D}t$ or $\dot{f}$, is the derivative of f with respect to time t with $\boldsymbol{x}_0$ fixed, namely

$$\dot{f}(\boldsymbol{x}_0, t) = \frac{\mathrm{D}f(\boldsymbol{x}_0, t)}{\mathrm{D}t} = \left.\frac{\partial f(\boldsymbol{x}_0, t)}{\partial t}\right|_{\boldsymbol{x}_0} \tag{3.52}$$

$\dot{f}$ represents the increment of f with time, evaluated by a mobile observer following the linear path of $\boldsymbol{x}_0$.

Spatial time derivative of a spatial field

The spatial time derivative of g, with a symbolic notation $\partial g/\partial t$, is the derivative of g with respect to time t with $\boldsymbol{x}$ fixed, namely

$$\frac{\partial g(\boldsymbol{x},t)}{\partial t} = \left.\frac{\partial g(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{x}} \tag{3.53}$$

$\dot{g}$ represents the increment of g with time, evaluated by a mobile observer in $\boldsymbol{x}$.

Material gradient of a material field

The material gradient of f, $\operatorname{Grad} f$, is the derivative of f with respect to $\boldsymbol{x}_0$ with t fixed, namely

$$\operatorname{Grad} f(\boldsymbol{x}_0,t) = \nabla_{\boldsymbol{x}_0} f(\boldsymbol{x}_0,t) = \left.\frac{\partial f(\boldsymbol{x}_0,t)}{\partial \boldsymbol{x}_0}\right|_{t} \tag{3.54}$$

Spatial gradient of a spatial field

The spatial gradient of g, $\operatorname{grad} g$, is the derivative of g with respect to $\boldsymbol{x}$ with t fixed, namely

$$\operatorname{grad} g(\boldsymbol{x}_0,t) = \nabla_{\boldsymbol{x}} g(\boldsymbol{x}_0,t) = \left.\frac{\partial g(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\right|_{t} \tag{3.55}$$

Material time derivative of a spatial field

To obtain the material time derivative of g we must first express g as a function of $\boldsymbol{x}_0$, in order to obtain the material field

$$g(\boldsymbol{x},t) = g\big(\boldsymbol{\phi}(\boldsymbol{x}_0,t),t\big) \tag{3.56}$$

then we calculate the material time derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} g\big(\boldsymbol{\phi}(\boldsymbol{x}_0,t),t\big) = \left.\frac{\partial g\big(\boldsymbol{\phi}(\boldsymbol{x}_0,t),t\big)}{\partial t}\right|_{\boldsymbol{x}_0} \tag{3.57}$$

and finally, we take the obtained result back to the spatial description

$$\frac{\mathrm{D}}{\mathrm{D}t} g\big(\boldsymbol{\phi}(\boldsymbol{x}_0, t), t\big) = \left.\frac{\partial g\big(\boldsymbol{\phi}(\boldsymbol{x}_0, t), t\big)}{\partial t}\right|_{\boldsymbol{x}_0 = \boldsymbol{\phi}^{-1}(\boldsymbol{x}, t)} \tag{3.58}$$

On application of the chain rule yields

$$\begin{aligned} \frac{\mathrm{D}g(\boldsymbol{x}, t)}{\mathrm{D}t} &= \left.\frac{\partial g(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x}} + \left.\frac{\partial g(\boldsymbol{x}, t)}{\partial t}\right|_{t} \cdot \left.\frac{\partial \boldsymbol{\phi}(\boldsymbol{x}_0, t)}{\partial t}\right|_{\boldsymbol{x}_0 = \boldsymbol{\phi}^{-1}(\boldsymbol{x}, t)} = \\ &= \left.\frac{\partial g(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \operatorname{grad}\big(g(\boldsymbol{x}, t)\big) \end{aligned} \tag{3.59}$$

where the first term represents the spatial time derivative of g, while the second term is called *convective derivative* of g and describes the position changes of $\boldsymbol{x}$.

3.1.4 Deformation increment

Since $\boldsymbol{F} = \operatorname{Grad} \boldsymbol{\phi}$, we have

$$\dot{\boldsymbol{F}} = \frac{\mathrm{D}\boldsymbol{F}}{\mathrm{D}t} = \operatorname{Grad} \dot{\boldsymbol{x}}(\boldsymbol{x}_0, t) = \operatorname{Grad} \boldsymbol{v}\big(\boldsymbol{\phi}(\boldsymbol{x}_0, t), t\big) \tag{3.60}$$

If we apply the chain rule, we obtain

$$\dot{\boldsymbol{F}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_0} = \boldsymbol{L}\boldsymbol{F} \tag{3.61}$$

where

$$\boldsymbol{L} = \operatorname{grad} \boldsymbol{v} \tag{3.62}$$

The tensor $\boldsymbol{L}$ can be decomposed in the form

$$\boldsymbol{L}(\boldsymbol{x}, t) = \boldsymbol{D}(\boldsymbol{x}, t) + \boldsymbol{W}(\boldsymbol{x}, t) \tag{3.63}$$

where $\boldsymbol{D}$ is the symmetric part of $\boldsymbol{L}$ and is called *deformation velocity tensor*

$$\boldsymbol{D} = \frac{1}{2}\big(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\big) = \boldsymbol{D}^{\mathrm{T}} \qquad \boldsymbol{D} \in \mathrm{Sym} \tag{3.64}$$

while $\boldsymbol{W}$ is the skew-symmetric part of $\boldsymbol{L}$, called *spin or rotation tensor*

$$\boldsymbol{W} = \frac{1}{2}\big(\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}}\big) = -\boldsymbol{W}^{\mathrm{T}} \qquad \boldsymbol{W} \in \mathrm{Skw} \tag{3.65}$$

The tensor $\boldsymbol{D}$ is associated to a pure deformation, and $\boldsymbol{W}$ to a rotation. If $\boldsymbol{D} = \boldsymbol{0}$, we have a rigid deformation.

3.1.5 Volume increments

Since $\mathrm{d}V = J\,\mathrm{d}V_0$, to determine the variation of $\mathrm{d}V$ as a function of time during the motion, we must calculate the derivative of J with respect to time. In particular, we have

$$J = \det(\boldsymbol{F}) = \det(\boldsymbol{R}\boldsymbol{U}) = \det(\boldsymbol{R})\det(\boldsymbol{U}) = \det(\boldsymbol{U}) = \lambda_1\lambda_2\lambda_3 \tag{3.66}$$

where λ_i $(i = 1, 2, 3)$ are the principal stretches (namely, the eigenvalues of $\boldsymbol{U}$). It follows that

$$\dot{J} = (\det \boldsymbol{F})^{\cdot} = \dot{\lambda}_1\lambda_2\lambda_3 + \lambda_1\dot{\lambda}_2\lambda_3 + \lambda_1\lambda_2\dot{\lambda}_3 = J\left(\frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{\lambda}_3}{\lambda_3}\right) \tag{3.67}$$

In the principal reference system, the inverse and the derivative with respect to time of $\boldsymbol{U}$ are

$$\boldsymbol{U}^{-1} = \sum_{1=1}^{3} \frac{1}{\lambda_i}\boldsymbol{u}_i \otimes \boldsymbol{u}_i \tag{3.68}$$

and

$$\dot{\boldsymbol{U}} = \sum_{1=1}^{3} \dot{\lambda}_i\boldsymbol{u}_i \otimes \boldsymbol{u}_i + \sum_{1=1}^{3} \lambda_i\dot{\boldsymbol{u}}_i \otimes \boldsymbol{u}_i + \sum_{1=1}^{3} \lambda_i\boldsymbol{u}_i \otimes \dot{\boldsymbol{u}}_i \tag{3.69}$$

respectively. Consequently,

$$\dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} = \mathrm{tr}\big(\dot{\boldsymbol{U}}^{\mathrm{T}}\boldsymbol{U}^{-1}\big) = \left(\frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{\lambda}_3}{\lambda_3}\right) \tag{3.70}$$

since $\boldsymbol{u}_i \cdot \boldsymbol{u}_i = 1$, so that

$$(\boldsymbol{u}_i \cdot \boldsymbol{u}_i)^{\cdot} = 2(\dot{\boldsymbol{u}}_i \cdot \boldsymbol{u}_i) = 0 \qquad \Longrightarrow \qquad \dot{\boldsymbol{u}}_i \perp \boldsymbol{u}_i \tag{3.71}$$

Taking into account that

$$\dot{\boldsymbol{U}} = \dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{F} + \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{F}} \tag{3.72}$$

and

$$\boldsymbol{U}^{-1} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{F}^{-\mathrm{T}} \tag{3.73}$$

and using equation (3.61), the equation (3.70) becomes

$$\dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} = \operatorname{tr}\big(\dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R}\big) + \operatorname{tr}\big(\boldsymbol{R}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{R}\big) \tag{3.74}$$

Furthermore, since $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$, we observe that

$$\big(\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}\big)^{\cdot} = \dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}} = 0 \tag{3.75}$$

hence

$$\operatorname{tr}\big(\dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R}\big) + \operatorname{tr}\big(\boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}}\big) = 2\operatorname{tr}\big(\dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R}\big) = 0 \tag{3.76}$$

We can conclude that

$$\dot{J} = J\operatorname{tr}\big(\dot{\boldsymbol{U}}^{\mathrm{T}}\boldsymbol{U}^{-1}\big) = J(\operatorname{tr}\boldsymbol{L}) = J\operatorname{div}\boldsymbol{v} \tag{3.77}$$

from which we deduce that in a constant volume deformation we have

$$\operatorname{div}\boldsymbol{v} = 0 \tag{3.78}$$

3.2 Forces

In continuum mechanics we consider the motions of deforming bodies and we assume that the forces acting on the bodies are due to external effects independent from the observer. These forces are called *applied forces* and are acting on the current configuration $\mathcal{B}$. We can distinguish three types of forces:

- *contact forces* among separated parts of a body;
- *surface forces* of the external environment on the boundary of a body;
- *volume forces* of the external environment on the interior points of a body.

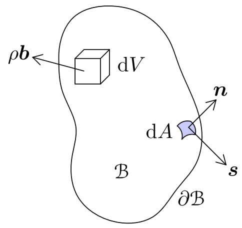


Figure 3.6: Forces acting on the current configuration.

The *contact force per surface unit* $\boldsymbol{s}(\boldsymbol{x}, \partial\mathcal{B}, t)$ is a vector field defined on every oriented surface piecewise regular on $\mathcal{B}$. To determine the contact force between two separated parts $\mathcal{B}$ and $\mathcal{P}$ we integrate $\boldsymbol{s}$ on the contact surface

$$\mathcal{L} = \mathcal{B} \cap \mathcal{P} \tag{3.79}$$

Consequently

$$\int_{\partial\mathcal{L}} \boldsymbol{s}(\boldsymbol{x}, \partial\mathcal{B}, t) \,\mathrm{d}A \tag{3.80}$$

is the force acting on $\mathcal{B}$ by $\mathcal{P}$ at time t.

The *volume force per volume unit* $\boldsymbol{b}$ is a vector field defined on $\mathcal{B}$, such that the forces of the external environment on $\mathcal{B}$, that are not due to contact, are

$$\int_{\mathcal{B}} \rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t) \,\mathrm{d}V \tag{3.81}$$

3.2.1 Cauchy stress theorem

The *fundamental postulate of Cauchy* asserts the existence of the surface tension $\boldsymbol{s}$ depending on the position of the normal $\boldsymbol{n}$ to the surface we are referring to

$$\boldsymbol{s}(\boldsymbol{x}, \partial\mathcal{B}, t) = \boldsymbol{s}(\boldsymbol{x}, \boldsymbol{n}, t) \tag{3.82}$$

The *Cauchy stress theorem* asserts that the tension vector $\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{n}, t)$ linearly depends on $\boldsymbol{n}$, so that a second order tensor field $\boldsymbol{\sigma}$, independent from $\boldsymbol{n}$, exists such that

$$\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{n}, t) = \boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n} \qquad \forall \boldsymbol{x} \in \mathcal{B} \tag{3.83}$$

$\boldsymbol{\sigma}(\boldsymbol{x}, t)$ is called *Cauchy tensor* and is a symmetric tensor. Note that $\boldsymbol{\sigma} \in \mathrm{Sym}$ due to balance of angular momentum.

3.2.2 First Piola-Kirchhoff tensor

The Cauchy tensor $\boldsymbol{\sigma}$ is defined in the spatial configuration $\mathcal{B}$. Since $\mathcal{B}$ is often unknown, in many cases its is useful to work with a tensor referring to the reference configuration.

The contact force $\mathrm{d}\boldsymbol{f}$ acting on the surface element $\mathrm{d}\boldsymbol{A}$ is defined as

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{s}(\boldsymbol{x}, \boldsymbol{n}, t)\,\mathrm{d}A = \boldsymbol{s}_0(\boldsymbol{x}_0, \boldsymbol{n}_0, t)\,\mathrm{d}A_0 \tag{3.84}$$

where $\boldsymbol{s}_0$ is the *nominal tension vector*, representing the surface force on the current configuration, but acting on the reference configuration $\mathcal{B}_0$ and is a function of $\boldsymbol{x}_0$ and $\boldsymbol{n}_0$. Therefore, $\boldsymbol{s}_0$ has not a physical meaning, has the same direction of $\boldsymbol{s}$, but $|\boldsymbol{s}| \neq |\boldsymbol{s}_0|$.

Since $\boldsymbol{s}_0 = \boldsymbol{S}\boldsymbol{n}_0$, where $\boldsymbol{S}$ is the *first Piola-Kirchhoff tensor*, equations (3.83) and (3.84) yield

$$\boldsymbol{\sigma}\boldsymbol{n}\,\mathrm{d}A = \boldsymbol{S}\boldsymbol{n}_0\,\mathrm{d}A_0 \tag{3.85}$$

If we use the Nanson rule (3.40), the equation (3.85) becomes

$$\boldsymbol{S}\boldsymbol{n}_0\,\mathrm{d}A_0 = J\boldsymbol{\sigma}\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0\,\mathrm{d}A_0 \tag{3.86}$$

so that

$$\boldsymbol{S} = J\boldsymbol{\sigma}\boldsymbol{F}^{-\mathrm{T}} \tag{3.87}$$

Hence, $\boldsymbol{S}$ is a *two-point* tensor like $\boldsymbol{F}$. Usually, $\boldsymbol{S}^{\mathrm{T}} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma} \neq \boldsymbol{S}$, so that $\boldsymbol{S}$ is not a symmetric tensor. The tensor $\boldsymbol{S}$ becomes symmetric in the particular cases in which $\boldsymbol{F}$ is symmetric.

3.2.3 Other known tensors

Kirchhoff tensor

The Kirchhoff tensor $\boldsymbol{K}$ yields a spatial measure of the stress. The Kirchhoff tensor is different from the Cauchy tensor by the volume ratio J, and is defined as

$$\boldsymbol{K} = J\boldsymbol{\sigma}\,, \qquad \boldsymbol{K} \in \mathrm{Sym} \tag{3.88}$$

Consequently, if the material satisfies the incompressibility constraint ($J = 1$) we obtain

$$\boldsymbol{K} = \boldsymbol{\sigma} \tag{3.89}$$

Second Piola Kirchhoff tensor

The second Piola Kirchhoff tensor $\boldsymbol{T}^{(2)}$ yields a material measure of the stress and is defined as

$$\boldsymbol{T}^{(2)} = \boldsymbol{F}^{-1}\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{F}^{-\mathrm{T}} \qquad \boldsymbol{T}^{(2)} \in \mathrm{Sym} \tag{3.90}$$

From equations (3.84) and (3.90) we obtain

$$\boldsymbol{T}^{(2)}\,\mathrm{d}A_0 = \boldsymbol{F}^{-1}\,\mathrm{d}\boldsymbol{f} \tag{3.91}$$

Biot tensor

The Biot tensor $\boldsymbol{T}^{(1)}$ yields a material measure of the stress and is defined as

$$\boldsymbol{T}^{(1)} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{R} = \boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{R} \tag{3.92}$$

A substitution of the polar decomposition, $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, into equation (3.92), yields

$$\boldsymbol{T}^{(1)} = \boldsymbol{P}\boldsymbol{U}^{-1} \tag{3.93}$$

Hence, from equations (3.84) and (3.92) we obtain

$$\boldsymbol{T}^{(1)}\boldsymbol{R}^{\mathrm{T}}\,\mathrm{d}\boldsymbol{A} = J\boldsymbol{F}^{-1}\,\mathrm{d}\boldsymbol{f} \tag{3.94}$$

Nominal tensor

The nominal tensor $\boldsymbol{t}$ is defined as

$$\boldsymbol{t} = \boldsymbol{S}^{\mathrm{T}} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma} \tag{3.95}$$

so that $\boldsymbol{t}$ is a *two-point* tensor like $\boldsymbol{S}$ and, in general, is not symmetric.

3.3 Equilibrium equations

3.3.1 Mass conservation

The mass conservation law asserts that during a motion, the mass m of a body is a quantity that does not change. The mass density at the points $\boldsymbol{x}$ and $\boldsymbol{x}_0$ is defined by the limits

$$\rho(\boldsymbol{x}) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \tag{3.96}$$

and

$$\rho_0(\boldsymbol{x}_0) = \lim_{\Delta V_0 \to 0} \frac{\Delta m}{\Delta V_0} \tag{3.97}$$

Let $\mathcal{P}_0$ be one part of $\mathcal{B}_0$ and $\mathcal{P}$ its relative image in $\mathcal{B}$; the mass conservation implies

$$m(\mathcal{P}_t) = \int_{\mathcal{P}_t} \rho \, \mathrm{d}V = \int_{\mathcal{P}_0} \rho_0 \, \mathrm{d}V_0 \tag{3.98}$$

and therefore, because $\mathrm{d}V = J \, \mathrm{d}V_0$,

$$\int_{\mathcal{P}_t} \rho \, \mathrm{d}V = \int_{\mathcal{P}_t} \frac{\rho_0}{J} \, \mathrm{d}V \qquad \forall \mathcal{P}_t \in \mathcal{B}_t \tag{3.99}$$

Since the two integrands in (3.99) must coincide in every point, we have

$$J\rho = \rho_0 \tag{3.100}$$

and taking into account that the density ρ_0 is constant with respect to time in $\mathcal{P}_0$, the mass conservation equation (3.100) becomes

$$\dot{J}\rho + J\dot{\rho} = 0 \tag{3.101}$$

or, equivalently, using equation (3.33),

$$\rho \, \mathrm{div}(\boldsymbol{v}) + \dot{\rho} = 0 \tag{3.102}$$

Note that in an isochoric motion we have $\dot{\rho} = 0$ since $\mathrm{div}(\boldsymbol{v}) = 0$.

3.3.2 Momentum and angular momentum

Definitions

The *momentum* $\boldsymbol{l}(t)$ is defined as

$$\boldsymbol{l}(t) = \int_{\mathcal{B}} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)\,\mathrm{d}V \tag{3.103}$$

The *angular momentum* $\boldsymbol{a}(t)$ referred to the fixed point $\boldsymbol{y}$ is defined as

$$\boldsymbol{a}(\boldsymbol{y}, t) = \int_{\mathcal{B}} \rho(\boldsymbol{x}, t)\boldsymbol{r} \times \boldsymbol{v}(\boldsymbol{x}, t)\,\mathrm{d}V \tag{3.104}$$

where

$$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x}_0, t) - \boldsymbol{y} \tag{3.105}$$

Properties

The momentum and the angular momentum have the following properties:

$$\dot{\boldsymbol{l}}(t) = \int_{\mathcal{B}} \rho(\boldsymbol{x}, t)\dot{\boldsymbol{v}}(\boldsymbol{x}, t)\,\mathrm{d}V \tag{3.106a}$$

$$\dot{\boldsymbol{a}}(\boldsymbol{y}, t) = \int_{\mathcal{B}} \boldsymbol{r} \times \rho(\boldsymbol{x}, t)\dot{\boldsymbol{v}}(\boldsymbol{x}, t)\,\mathrm{d}V \tag{3.106b}$$

To demonstrate equations (3.106a) and (3.106b) we must observe that

$$\int_{\mathcal{B}} \phi_s \rho\,\mathrm{d}V = \int_{\mathcal{B}_0} \phi_m \rho J\,\mathrm{d}V_0 = \int_{\mathcal{B}_0} \phi_m \rho_0\,\mathrm{d}V_0 \tag{3.107}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{B}} \phi_s \rho\,\mathrm{d}V = \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{B}_0} \phi_m \rho J\,\mathrm{d}V_0 = \int_{\mathcal{B}_0} \dot{\phi}_m \rho_0\,\mathrm{d}V_0 = \int_{\mathcal{B}_t} \dot{\phi}_s \rho\,\mathrm{d}V \tag{3.108}$$

where $\phi_s = \phi(\boldsymbol{x}, t)$ and $\phi_m = \phi\big(\boldsymbol{\phi}(\boldsymbol{x}_0, t), t\big)$ are generic scalar, vector, or tensor functions.

Hence, the relation (3.106a) has been obtained using equation (3.108), while (3.106b) has been obtained using equation (3.108) taking into account that

$$(\boldsymbol{r} \times \boldsymbol{v})^{\cdot} = \dot{\boldsymbol{r}} \times \boldsymbol{v} + \boldsymbol{r} \times \dot{\boldsymbol{v}} = \boldsymbol{r} \times \dot{\boldsymbol{v}} \tag{3.109}$$

since $\dot{\boldsymbol{r}} = \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{v} = 0$.

3.3.3 Centre of mass

The centre of mass of $\mathcal{B}_t$ is defined as

$$\boldsymbol{\alpha}(t) - \mathbf{0} = \frac{1}{m(\mathcal{B}_t)} \int_{\mathcal{B}_t} \rho \boldsymbol{r} \, \mathrm{d}V \tag{3.110}$$

Furthermore, the following property holds

$$\dot{\boldsymbol{\alpha}}(t) = \frac{1}{m(\mathcal{B}_t)} \int_{\mathcal{B}_t} \dot{\boldsymbol{r}} \rho \, \mathrm{d}V \tag{3.111}$$

which can be demonstrated using equation (3.108).

A comparison between equations (3.111) and (3.106a) shows that the momentum of a body $\mathcal{B}$ is equivalent to that of a particle of mass $m(\mathcal{B})$ and located in the centre of mass of $\mathcal{B}$

$$m(\mathcal{B}) \dot{\boldsymbol{\alpha}}(t) = \boldsymbol{l}(\mathcal{B}, t) \tag{3.112}$$

3.3.4 Momentum equilibrium equations

The resultant force applied on the current configuration $\mathcal{B}$ is

$$\boldsymbol{f}(\mathcal{B}, t) = \int_{\mathcal{B}} \rho(\boldsymbol{x}, t) \boldsymbol{b}(\boldsymbol{x}, t) \, \mathrm{d}V + \int_{\partial \mathcal{B}} \boldsymbol{\sigma}(\boldsymbol{x}, t) \boldsymbol{n} \, \mathrm{d}A \tag{3.113}$$

and the resultant moment of the applied forces applied in a point $\boldsymbol{y} \in \mathbb{E}^3$ is

$$\boldsymbol{m}(\mathcal{B}, t) = \int_{\mathcal{B}} \rho(\boldsymbol{x}, t) \boldsymbol{r}(\boldsymbol{x}, t) \times \boldsymbol{b}(\boldsymbol{x}, t) \, \mathrm{d}V + \int_{\partial \mathcal{B}} \boldsymbol{r}(\boldsymbol{x}, t) \times \boldsymbol{\sigma}(\boldsymbol{x}, t) \boldsymbol{n} \, \mathrm{d}A \tag{3.114}$$

where $\boldsymbol{r}(\boldsymbol{x}, t) = \boldsymbol{x} - \mathbf{0}$. It is important to note that we are considering only the non-polar continua (continua in absence of distributed torques).

The motion of a continuum is governed by the *momentum equilibrium equation laws*. The first law asserts that the momentum $\boldsymbol{l}(t)$ evolves through time if a non null resultant force $\boldsymbol{f}(t)$ is applied, thus

$$\frac{\mathrm{D}\boldsymbol{l}(t)}{\mathrm{D}t} = \boldsymbol{f}(t) \tag{3.115}$$

The second law consists in the momentum equilibrium

$$\frac{\mathrm{D}\boldsymbol{a}(\mathbf{0},t)}{\mathrm{D}t} = \boldsymbol{m}(\mathbf{0},t) \tag{3.116}$$

where $\boldsymbol{a}(\mathbf{0},t)$ and $\boldsymbol{m}(\mathbf{0},t)$ depend on the point $\mathbf{0}$.

In equations (3.115) and (3.116), the existence of an inertial observer (inertial reference system) is implicitly assumed, and the equilibrium laws are referred to this system. The momentum equilibrium equations (3.115) and (3.116) in terms of integrals on $\mathcal{B}$ and $\partial\mathcal{B}$ become

$$\int_{\mathcal{B}} \rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t)\,\mathrm{d}V + \int_{\partial\mathcal{B}} \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}\,\mathrm{d}A = \int_{\mathcal{B}} \rho(\boldsymbol{x},t)\dot{\boldsymbol{v}}(\boldsymbol{x},t)\,\mathrm{d}V \tag{3.117}$$

and

$$\int_{\partial\mathcal{B}} \boldsymbol{r}(\boldsymbol{x},t)\times\boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}\,\mathrm{d}A + \int_{\mathcal{B}} \rho(\boldsymbol{x},t)\boldsymbol{r}(\boldsymbol{x},t)\times\boldsymbol{b}(\boldsymbol{x},t)\,\mathrm{d}V =$$
$$= \int_{\mathcal{B}} \rho(\boldsymbol{x},t)\boldsymbol{r}(\boldsymbol{x},t)\times\dot{\boldsymbol{v}}(\boldsymbol{x},t)\,\mathrm{d}V \tag{3.118}$$

If we denote the *resultant of the body forces* (which includes the inertial force $-\rho\dot{\boldsymbol{v}}$) with

$$\boldsymbol{b}_* = \boldsymbol{b} - \dot{\boldsymbol{v}} \tag{3.119}$$

and if we define

$$\boldsymbol{f}_*(\mathcal{B},t) = \int_{\mathcal{B}} \rho(\boldsymbol{x},t)\boldsymbol{b}_*(\boldsymbol{x},t)\,\mathrm{d}V + \int_{\partial\mathcal{B}} \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}\,\mathrm{d}A \tag{3.120}$$

and

$$\boldsymbol{m}_*(\mathcal{B}, t) = \int_{\mathcal{B}} \rho(\boldsymbol{x}, t)\boldsymbol{r}(\boldsymbol{x}, t) \times \boldsymbol{b}_*(\boldsymbol{x}, t)\, \mathrm{d}V + \int_{\partial\mathcal{B}} \boldsymbol{r}(\boldsymbol{x}, t) \times \boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n}\, \mathrm{d}A \quad (3.121)$$

then, the momentum equilibrium laws (3.115) and (3.116) assume the form

$$\boldsymbol{f}_*(\mathcal{B}, t) = 0 \quad (3.122a)$$

$$\boldsymbol{m}_*(\mathcal{B}, t) = 0 \quad (3.122b)$$

The momentum equilibrium equations (3.115) and (3.116) can be rewritten in terms of integrals on $\mathcal{B}_0$ and $\partial\mathcal{B}_0$ as

$$\int_{\mathcal{B}_0} \rho_0(\boldsymbol{x}_0, t)\boldsymbol{b}(\boldsymbol{x}_0, t)\, \mathrm{d}V_0 + \int_{\partial\mathcal{B}_0} \boldsymbol{S}(\boldsymbol{x}_0, t)\boldsymbol{n}_0\, \mathrm{d}A_0 = \\ = \int_{\mathcal{B}_0} \rho_0(\boldsymbol{x}_0, t)\frac{\mathrm{D}^2}{\mathrm{D}t^2}\boldsymbol{\phi}(\boldsymbol{x}_0, t)\, \mathrm{d}V \quad (3.123)$$

and

$$\int_{\partial\mathcal{B}_0} \boldsymbol{r}(\boldsymbol{x}_0, t) \times \boldsymbol{S}(\boldsymbol{x}_0, t)\boldsymbol{n}_0\, \mathrm{d}A_0 + \int_{\mathcal{B}} \rho_0(\boldsymbol{x}_0, t)\boldsymbol{r}(\boldsymbol{x}_0, t) \times \boldsymbol{b}(\boldsymbol{x}_0, t)\, \mathrm{d}V_0 = \\ = \int_{\mathcal{B}_0} \rho_0(\boldsymbol{x}_0)\boldsymbol{r}(\boldsymbol{x}_0, t) \times \frac{\mathrm{D}^2}{\mathrm{D}t^2}\boldsymbol{\phi}(\boldsymbol{x}_0, t)\, \mathrm{d}V_0 \quad (3.124)$$

where

$$\boldsymbol{r}(\boldsymbol{x}, t)_0 = \boldsymbol{\phi}(\boldsymbol{x}_0, t) - \boldsymbol{0} \quad (3.125)$$

3.3.5 Mechanical energy equilibrium

The external mechanical power, or increment of the external mechanical work W_{ext}, is defined as the power produced by the set of the forces $(\boldsymbol{s}, \boldsymbol{b})$ on the region $\mathcal{B}$ at time t

$$W_{\text{ext}}(t) = \int_{\partial\mathcal{B}} \boldsymbol{s} \cdot \boldsymbol{v}\, \mathrm{d}A + \int_{\mathcal{B}} \rho\boldsymbol{b} \cdot \boldsymbol{v}\, \mathrm{d}V \quad (3.126)$$

The kinetic energy K of a continuum occupying the region $\mathcal{B}$ at time t t is defined as

$$K(t) = \frac{1}{2} \int_{\mathcal{B}} \rho \boldsymbol{v}^2 \, \mathrm{d}V \tag{3.127}$$

If we neglect other forms of energy, for instance thermal, nuclear, electric, magnetic or chemical energy, then the mechanical energy equilibrium in the spatial description becomes

$$W_{\text{ext}}(t) = W_{\text{int}}(t) + \frac{\mathrm{D}}{\mathrm{D}t} K(t) \tag{3.128}$$

where W_{int} is the stress power or increment of the internal mechanical work, and W_{int} describes the response of the region $\mathcal{B}$ at time t given by the stress field. Now we want to determine W_{int} in a way such that the mechanical energy equilibrium is satisfied. Recalling that $\boldsymbol{s} = \boldsymbol{\sigma n}$, the first integral in equation (3.126) can be rewritten as

$$\int_{\partial \mathcal{B}} \boldsymbol{\sigma n} \cdot \boldsymbol{v} \, \mathrm{d}A = \int_{\partial \mathcal{B}} \boldsymbol{\sigma v} \cdot \boldsymbol{n} \, \mathrm{d}A \tag{3.129}$$

and, on application of the divergence theorem (2.77) yields

$$\int_{\partial \mathcal{B}} \boldsymbol{s} \cdot \boldsymbol{v} \, \mathrm{d}A = \int_{\mathcal{B}} \operatorname{div}(\boldsymbol{\sigma v}) \, \mathrm{d}V \tag{3.130}$$

where

$$\operatorname{div}(\boldsymbol{\sigma v}) = (\sigma_{ij} v_j)_{,i} = \sigma_{ij,i} v_j + \sigma_{ij} v_{j,i} = (\operatorname{div} \boldsymbol{\sigma}) \cdot \boldsymbol{v} + \boldsymbol{\sigma} \cdot \boldsymbol{D} \tag{3.131}$$

Consequently

$$\int_{\partial \mathcal{B}} \boldsymbol{s} \cdot \boldsymbol{v} \, \mathrm{d}A = \int_{\mathcal{B}} \left[(\rho \dot{\boldsymbol{v}} - \boldsymbol{b}) \cdot \boldsymbol{v} + \boldsymbol{\sigma} \cdot \boldsymbol{D} \right] \mathrm{d}V \tag{3.132}$$

and

$$\int_{\partial \mathcal{B}} \boldsymbol{s} \cdot \boldsymbol{v} \, \mathrm{d}A + \int_{\mathcal{B}} \boldsymbol{b} \cdot \boldsymbol{v} \, \mathrm{d}V = \int_{\mathcal{B}} \boldsymbol{\sigma} \cdot \boldsymbol{D} \, \mathrm{d}V + \frac{1}{2} \int_{\mathcal{B}} \rho \boldsymbol{v}^2 \, \mathrm{d}V \tag{3.133}$$

which represents the power theorem in terms of spatial coordinates.

A comparison between equations (3.128) and (3.133) yields

$$W_{\text{int}}(t) = \int_{\mathcal{B}} \boldsymbol{\sigma} \cdot \boldsymbol{D} \, \mathrm{d}V \tag{3.134}$$

Now we want to determine the mechanical energy equilibrium equation as a function of the Lagrangian (namely, reference) coordinates.

The external mechanical power W_{ext} and the kinetic energy K in the material configuration are expressed as

$$W_{\text{ext}}(t) = \int_{\partial\mathcal{B}_0} \boldsymbol{S}\boldsymbol{n}_0 \cdot \dot{\boldsymbol{x}} \, \mathrm{d}A_0 + \int_{\mathcal{B}_0} \boldsymbol{b}_0 \cdot \dot{\boldsymbol{x}} \, \mathrm{d}V_0 \tag{3.135}$$

and

$$K(t) = \frac{1}{2} \int_{\mathcal{B}_0} \rho_0 \dot{\boldsymbol{x}}^2 \, \mathrm{d}V_0 \tag{3.136}$$

respectively. The stress power in terms of material coordinates becomes

$$W_{\text{int}}(t) = \int_{\mathcal{B}_0} \boldsymbol{\sigma} \cdot \boldsymbol{D} J \, \mathrm{d}V_0 \tag{3.137}$$

and noting that

$$\boldsymbol{\sigma} \cdot \boldsymbol{D} = \boldsymbol{\sigma} \cdot (\boldsymbol{L} - \boldsymbol{W}) = \boldsymbol{\sigma} \cdot \boldsymbol{L} - \boldsymbol{\sigma} \cdot \boldsymbol{W} = \boldsymbol{\sigma} \cdot \boldsymbol{L} \tag{3.138}$$

since the spin tensor $\boldsymbol{W}$ corresponds to the skew-symmetric part of $\boldsymbol{L}$ and the Cauchy tensor is symmetric (therefore $\boldsymbol{\sigma} \cdot \boldsymbol{W} = 0$), equation (3.137) can be rewritten as

$$W_{\text{int}}(t) = \int_{\mathcal{B}_0} \boldsymbol{\sigma} \cdot \boldsymbol{L} J \, \mathrm{d}V_0 \tag{3.139}$$

If we introduce the first Piola-Kirchhoff tensor, we have

$$W_{\text{int}}(t) = \int_{\mathcal{B}_0} \boldsymbol{S}\boldsymbol{F}^T \cdot \boldsymbol{L} \, \mathrm{d}V_0 = \int_{\mathcal{B}_0} \boldsymbol{S} \cdot \dot{\boldsymbol{F}} \, \mathrm{d}V_0 \tag{3.140}$$

A comparison between equations (3.137) and (3.140) yields

$$\boldsymbol{S} \cdot \dot{\boldsymbol{F}} = J\boldsymbol{\sigma} \cdot \boldsymbol{D} \tag{3.141}$$

from which we conclude that the mechanical energy equilibrium is satisfied using the Kirchhoff tensor $\boldsymbol{K}$ ($\boldsymbol{K} = J\boldsymbol{\sigma}$) and $\boldsymbol{D}$, or the tensors $\boldsymbol{S}$ and $\dot{\boldsymbol{F}}$ to measure the power per volume unit in the material configuration.

If a stress measure $\boldsymbol{H}$ and a strain measure $\boldsymbol{Z}$ are such that $\boldsymbol{H} \cdot \dot{\boldsymbol{Z}} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}}$, they are called *work-conjugated functions*. An infinite number of stress and strain measures of this type exists; if we define a strain tensor

$$\boldsymbol{E}^{(m)} = \frac{1}{m}(\boldsymbol{U}^{(m)} - \boldsymbol{I}) \qquad m \in \mathbb{N} \tag{3.142}$$

it is always possible to find a tensor $\boldsymbol{T}^{(m)}$ such that

$$\boldsymbol{T}^{(m)} \cdot \dot{\boldsymbol{E}}^{(m)} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}} \tag{3.143}$$

Note that two important work-conjugated measures of stress and strain are the Biot tensor $\boldsymbol{T}^{(1)}$ and the right stretch tensor $\boldsymbol{U}$.

3.4 Motion equations

3.4.1 Governing equations in the spatial configuration

On application of the divergence theorem (2.77), the linear momentum equilibrium equation (3.117) yields

$$\int_{\mathcal{B}} (\rho \boldsymbol{b} + \operatorname{div} \boldsymbol{\sigma} - \rho \dot{\boldsymbol{v}}) \, \mathrm{d}V = \boldsymbol{0} \tag{3.144}$$

and since (3.144) holds for an arbitrary body $\mathcal{B}$, we have

$$\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \dot{\boldsymbol{v}} \tag{3.145}$$

or, if we introduce the total force $\boldsymbol{b}_* = \boldsymbol{b} - \dot{\boldsymbol{v}}$:

$$\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b}_* = \boldsymbol{0} \tag{3.146}$$

Furthermore, for a non-polar continuum the momentum balance (3.118) is satisfied only if equation (3.145) holds and if the Cauchy tensor is symmetric

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \tag{3.147}$$

Since in equations (3.145) and (3.147) the independent variables are $\boldsymbol{x}$ and t, these two equations yield a spatial description of the motion of the body. In terms of spatial coordinates, we have

- motion equations

$$\operatorname{div}(\boldsymbol{\sigma}) + \boldsymbol{b} = \rho\dot{\boldsymbol{v}} \tag{3.148a}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \tag{3.148b}$$

- mass conservation equations

$$\rho_0 = J\rho \tag{3.149}$$

- constitutive equations

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\boldsymbol{B}) \tag{3.150}$$

3.4.2 Governing equations in the reference configuration

On application of the divergence theorem (2.77), the linear momentum equilibrium equation (3.123) in terms of material coordinates yields:

$$\int_{\mathcal{B}_0} \left(\rho_0 \boldsymbol{b}_0 + \operatorname{div} \boldsymbol{S} - \rho_0 \frac{\mathrm{D}^2 \boldsymbol{\phi}}{\mathrm{D}t^2} \right) \mathrm{d}V = \mathbf{0} \tag{3.151}$$

where $\boldsymbol{b}_0 = \boldsymbol{b}(\boldsymbol{x}_0, t)$. Since equation (3.151) holds for any arbitrary material configuration $\mathcal{B}_0$, we have

$$\operatorname{div} \boldsymbol{S} + \rho \boldsymbol{b}_0 = \rho_0 \frac{\mathrm{D}^2 \boldsymbol{\phi}}{\mathrm{D}t^2} \tag{3.152}$$

which holds for every $\boldsymbol{x}_0 \in \mathcal{B}_0$. Furthermore, using a particular form of the divergence theorem involving the vector product, the momentum balance (3.124) becomes:

$$\boldsymbol{S}\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}\boldsymbol{S}^{\mathrm{T}} \tag{3.153}$$

Consequently, in terms of reference coordinates , the motion of the continuum body $\mathcal{B}$ is governed by the following relations:

- motion equations

$$\operatorname{Div} \boldsymbol{S} + \boldsymbol{b}_0 = \rho_0 \frac{\mathrm{D}^2 \boldsymbol{\phi}}{\mathrm{D}t^2} \tag{3.154a}$$

$$\boldsymbol{S}\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}\boldsymbol{S}^{\mathrm{T}} \tag{3.154b}$$

- constitutive equations

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F}), \tag{3.155}$$

- kinematic condition

$$\boldsymbol{F} = \operatorname{Grad} \boldsymbol{x} \tag{3.156}$$

The mass conservation equation is omitted because in the reference configuration $\rho_0 = \text{const}$.

3.5 Invariance of the material response

3.5.1 Change in observer

Analyzing the motion of the body $\mathcal{B}$, two different observers, O and O^+, in general record two different motions, respectively denoted with $\boldsymbol{\phi}(\boldsymbol{x}_0, t)$ and $\boldsymbol{\phi}^+(\boldsymbol{x}_0^+, t^+)$. These motions are called *equivalent motions* if the mutual distances between the points and the time intervals among the events are constant during the observation, so that

$$|\boldsymbol{x} - \boldsymbol{x}_0| = \left|\boldsymbol{x}^+ - \boldsymbol{x}_0^+\right| \tag{3.157a}$$

$$t - t_0 = t^+ - t_0^+ \tag{3.157b}$$

The requirement (3.157a) is satisfied by the spatial reference system

$$\boldsymbol{x}^+ - \boldsymbol{x}_0^+ = \boldsymbol{Q}(t)(\boldsymbol{x} - \boldsymbol{x}_0) \tag{3.158}$$

where $\boldsymbol{Q}$ is a rotation tensor.

Consequently, equations (3.157) can be rewritten in the form

$$\boldsymbol{x}^+ = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x} \tag{3.159a}$$

$$t^+ = t + \alpha \tag{3.159b}$$

where

$$\boldsymbol{c}(t) = \boldsymbol{x}_0^+ - \boldsymbol{Q}(t)\boldsymbol{x}_0 \tag{3.160}$$

and

$$\alpha = t_0^+ - t_0 \tag{3.161}$$

Therefore, at every time t the deformation $\boldsymbol{x}^+(\boldsymbol{x}_0, t)$ can be seen as a specific rigid body motion imposed on $\boldsymbol{x}(\boldsymbol{x}_0, t)$.

The relations (3.159) represent the analytic formulation of the *change in observer* and are referred to an *Euclidean transformation*. A spatial tensor field of order n ($\boldsymbol{q}_1 \otimes \boldsymbol{q}_2 \otimes ... \otimes \boldsymbol{q}_n$) is called *objective* if in the change in observer it transforms through the relation

$$(\boldsymbol{q}_1 \otimes \boldsymbol{q}_2 \otimes ... \otimes \boldsymbol{q}_n)^+ = \boldsymbol{Q}\boldsymbol{q}_1 \otimes \boldsymbol{Q}\boldsymbol{q}_2 \otimes ... \otimes \boldsymbol{Q}\boldsymbol{q}_n \tag{3.162}$$

In particular:

- if $n = 1$, we have a spatial vector field $\boldsymbol{q}$, for which the equation (3.162) becomes
$$\boldsymbol{q}^+(\boldsymbol{x}^+, t^+) = \boldsymbol{Q}(t)\boldsymbol{q}(\boldsymbol{x}, t) \tag{3.163}$$

- if $n = 2$, we have a second-order tensor field that can be defined as $\boldsymbol{A}(\boldsymbol{x}, t) = \boldsymbol{q}_1(\boldsymbol{x}, t) \otimes \boldsymbol{q}_2(\boldsymbol{x}, t)$, for which the equation (3.162) becomes
$$\boldsymbol{A}(\boldsymbol{x}^+, t^+) = \boldsymbol{Q}(t)\boldsymbol{A}(\boldsymbol{x}, t)\boldsymbol{Q}(t)^{\mathrm{T}} \tag{3.164}$$
in which the following property has been used
$$\boldsymbol{Q}\boldsymbol{q}_1 \otimes \boldsymbol{Q}\boldsymbol{q}_2 = \boldsymbol{Q}(\boldsymbol{q}_1 \otimes \boldsymbol{q}_2)\boldsymbol{Q}^{\mathrm{T}} \tag{3.165}$$

It is obvious that a scalar field is not affected by a change in observer, so that

$$\psi^+(\boldsymbol{x}^+, t^+) = \psi(\boldsymbol{x}, t) \tag{3.166}$$

Material fields do not always vary after a change in observer; for instance, they remain unchanged after an imposed rigid body motion. Consequently,

$$\boldsymbol{q}_0^+ = \boldsymbol{q}_0 \tag{3.167}$$

and

$$\boldsymbol{A}_0^+ = \boldsymbol{A}_0 \tag{3.168}$$

where $\boldsymbol{q}_0$ and $\boldsymbol{A}_0$ are, respectively, a vector and a tensor field.

We illustrate now the transformation laws for some known field, and their objectivity is analyzed. We start considering by the deformation gradient $\boldsymbol{F}$ at point $\boldsymbol{x} \in \mathcal{B}$ and at its relative point $\boldsymbol{x}^+ \in \mathcal{B}^+$

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_0} \tag{3.169a}$$

$$\boldsymbol{F}^+ = \frac{\partial \boldsymbol{x}^+}{\partial \boldsymbol{x}_0^+} \tag{3.169b}$$

The differentiation of equation (3.159) with respect to $\boldsymbol{x}_0$ yields

$$\boldsymbol{F}^+ = \frac{\partial \boldsymbol{x}^+}{\partial \boldsymbol{x}_0^+} = \boldsymbol{Q}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_0} = \boldsymbol{Q}\boldsymbol{F} \tag{3.170}$$

representing the transformation law of $\boldsymbol{F}$. Note that $\boldsymbol{F}$ is objective even if the requirement (3.164) is not satisfied, because, being a *two-point* tensor, one of its indexes describes the material coordinates, which are independent from the change in observer; consequently, $\boldsymbol{F}$ transforms as a vector.

Also the first Piola-Kirchhoff tensor $\boldsymbol{S}$ transforms into

$$\boldsymbol{S}^+ = \boldsymbol{Q}\boldsymbol{S} \tag{3.171}$$

so that also $\boldsymbol{S}$ is an objective *two-point* tensor.

The polar decomposition for the observer O^+ at point $\boldsymbol{x} \in \mathcal{B}^+$ is

$$\boldsymbol{F}^+ = \boldsymbol{R}^+\boldsymbol{U}^+ = \boldsymbol{V}^+\boldsymbol{R}^+ \tag{3.172}$$

and, on application of equation (3.170), we obtain

$$\boldsymbol{Q}\boldsymbol{R}\boldsymbol{U} = \boldsymbol{R}^+\boldsymbol{U}^+ \tag{3.173a}$$

$$\boldsymbol{Q}\boldsymbol{V}\boldsymbol{R} = \boldsymbol{V}^+\boldsymbol{R}^+ \tag{3.173b}$$

Since the tensor $\boldsymbol{QR}$ is orthogonal, we have

$$\boldsymbol{R}^+ = \boldsymbol{QR} \tag{3.174a}$$
$$\boldsymbol{U}^+ = \boldsymbol{Q} \tag{3.174b}$$
$$\boldsymbol{V}^+ = \boldsymbol{QRQ}^{\mathrm{T}} \tag{3.174c}$$

Therefore, being $\boldsymbol{R}$ a *two-point* tensor, $\boldsymbol{U}$ a material tensor, and $\boldsymbol{V}$ a spatial tensor, they all are objective. Consequently, the Cauchy-Green tensors $\boldsymbol{C}$ and $\boldsymbol{B}$ are objective too, since

$$\boldsymbol{C}^+ = \boldsymbol{C} \tag{3.175a}$$
$$\boldsymbol{B}^+ = \boldsymbol{QBQ}^{\mathrm{T}} \tag{3.175b}$$

An example of a non-objective tensor is given by the velocity gradient $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, a spatial tensor that transforms as follows

$$\boldsymbol{L}^+ = \boldsymbol{QLQ}^{\mathrm{T}} + \dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}} \tag{3.176}$$

where $\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}$ represents the rotation of the reference system of the observer O around the reference system of O^+. However, since $\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}$, we have

$$\boldsymbol{D}^+ = \boldsymbol{QDQ}^{\mathrm{T}} \tag{3.177a}$$
$$\boldsymbol{W}^+ = \dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{QWQ}^{\mathrm{T}} \tag{3.177b}$$

thus, the tensor $\boldsymbol{D}$ is objective.

3.5.2 Invariance in the change in observer

The material invariance principle asserts that the material properties must remain unchanged when changing the observer. In other words, O and O^+ must record the same stress under the same actions. It is important to define a constitutive law that remains the same for every observer. Consequently, the constitutive equations must be defined using parameters that have the same transformation rule in change in observer.

Let $\boldsymbol{s} = \boldsymbol{\sigma n}$ be the Cauchy tension vector, measured at time t by the first observer. We should expect that

$$\boldsymbol{s}^{+}(\boldsymbol{n}^{+}) = \boldsymbol{Q}\boldsymbol{s}(\boldsymbol{n}) \tag{3.178}$$

Since $\boldsymbol{s} = \boldsymbol{\sigma}\boldsymbol{n}$ and $\boldsymbol{s}^{+} = \boldsymbol{\sigma}^{+}\boldsymbol{n}^{+}$, we have

$$\boldsymbol{\sigma}^{+}\boldsymbol{n}^{+} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{n} \tag{3.179}$$

and using $\boldsymbol{n}^{+} = \boldsymbol{Q}\boldsymbol{n}$, we obtain

$$\boldsymbol{\sigma}^{+}\boldsymbol{Q}\boldsymbol{n} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{n} \tag{3.180}$$

Consequently, the tensor $\boldsymbol{\sigma}$ is an objective tensor, since

$$\boldsymbol{\sigma}^{+} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^{\mathrm{T}} \tag{3.181}$$

3.5.3 Objective derivatives

The aim of this Section is to understand the transformation of the material derivatives with respect to time when there is a change in observer. Let us consider a spatial vector $\boldsymbol{q}$ and a spatial tensor $\boldsymbol{A}$ transforming into

$$\boldsymbol{A}^{+} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{\mathrm{T}} \tag{3.182a}$$

$$\boldsymbol{q}^{+} = \boldsymbol{Q}\boldsymbol{q} \tag{3.182b}$$

Their relative material derivatives do not follow these rules, in fact

$$\dot{\boldsymbol{q}}^{+} = \boldsymbol{Q}\dot{\boldsymbol{q}} + \dot{\boldsymbol{Q}}\boldsymbol{q} \tag{3.183a}$$

$$\dot{\boldsymbol{A}}^{+} = \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^{\mathrm{T}} \tag{3.183b}$$

Consequently, $\dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{A}}$ are not objective fields, so that they are not suitable in formulating constitutive equations in a differential form. If $\dot{\boldsymbol{\sigma}}$ were employed in the incremental constitutive equations, a rigid rotation would generate a stress increment. For this reason we introduce the *objective derivatives*, which are essentially modified material time derivatives.

The equation (3.177b) becomes $\dot{\boldsymbol{Q}} = \boldsymbol{W}^{+}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{W}$, and a substitution into (3.183) leads to

$$\overset{\triangledown}{\boldsymbol{q}}^{+} = \boldsymbol{Q}\overset{\triangledown}{\boldsymbol{q}} \tag{3.184a}$$

$$\overset{\triangledown}{\boldsymbol{A}}^{+} = \boldsymbol{Q}\overset{\triangledown}{\boldsymbol{A}}\boldsymbol{Q}^{\mathrm{T}} \tag{3.184b}$$

where

$$\overset{\triangledown}{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \boldsymbol{W}\boldsymbol{q} \tag{3.185a}$$

$$\overset{\triangledown}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{W}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{W} \tag{3.185b}$$

are the definitions of the *Jaumann derivative* of a vector and a tensor, respectively. Many other derivatives can be defined, for instance the *Oldroyd derivative*

$$\overset{\circ}{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \boldsymbol{L}\boldsymbol{q} \tag{3.186a}$$

$$\overset{\circ}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{L}^{\mathrm{T}} \tag{3.186b}$$

or the *Cotter-Rivlin derivative*

$$\overset{\diamond}{\boldsymbol{q}} = \dot{\boldsymbol{q}} + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{q} \tag{3.187a}$$

$$\overset{\diamond}{\boldsymbol{A}} = \dot{\boldsymbol{A}} + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{L} \tag{3.187b}$$

The objective derivatives are usually employed to define the derivatives of the Eulerian stress that are suitable in the formulation of constitutive equations. Note that if the Lagrangian fields remain unchanged in the change in observer, their corresponding material derivatives behave in the same way, so that

$$\dot{\boldsymbol{q}}_0^{+} = \dot{\boldsymbol{q}}_0 \tag{3.188a}$$

$$\dot{\boldsymbol{A}}_0^{+} = \dot{\boldsymbol{A}}_0 \tag{3.188b}$$

3.6 Elastic bodies

A material is elastic if its relative stress state depends only on the final deformation and not on the deformation path. The constitutive class of an elastic material is defined as

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \hat{\boldsymbol{\sigma}}\big(\boldsymbol{F}(\boldsymbol{x}_0,t),\boldsymbol{x}_0\big) \tag{3.189}$$

or simply

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\boldsymbol{F}) \tag{3.190}$$

where $\boldsymbol{F}$ measures the local changes of the distance, while $\hat{\boldsymbol{\sigma}}$ is called response function. Since the material response is independent from the observer, it must be

$$\boldsymbol{\sigma}^+ = \hat{\boldsymbol{\sigma}}(\boldsymbol{F}^+) \tag{3.191}$$

and, introducing the transformation law for $\boldsymbol{F}$ and $\boldsymbol{\sigma}$,

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{QF}) = \boldsymbol{Q}\hat{\boldsymbol{\sigma}}(\boldsymbol{F})\boldsymbol{Q}^{\mathrm{T}} \qquad \boldsymbol{Q} \in \mathrm{Orth}^+ \tag{3.192}$$

which represents the necessary and sufficient condition to have a constitutive law satisfying the material invariance principle.

We want to determine the alternative forms for the constitutive equations. Since $\boldsymbol{R}$ is a particular rotation, as $\boldsymbol{Q}$ ans $\boldsymbol{F} = \boldsymbol{RU}$, the relation (3.190) can be rewritten as

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\boldsymbol{RU}) = \boldsymbol{R}\hat{\boldsymbol{\sigma}}(\boldsymbol{U})\boldsymbol{R}^{\mathrm{T}} \tag{3.193}$$

so that, substituting $\boldsymbol{R} = \boldsymbol{FU}^{-1}$,

$$\boldsymbol{\sigma} = \boldsymbol{FU}^{-1}\hat{\boldsymbol{\sigma}}(\boldsymbol{U})\boldsymbol{U}^{-\mathrm{T}}\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}\breve{\boldsymbol{\sigma}}(\boldsymbol{U})\boldsymbol{F}^{\mathrm{T}} \tag{3.194}$$

where $\breve{\boldsymbol{\sigma}}(\boldsymbol{U}) = \boldsymbol{U}^{-1}\hat{\boldsymbol{\sigma}}(\boldsymbol{U})\boldsymbol{U}^{-\mathrm{T}}$ represents the constitutive equation in terms of $\boldsymbol{U}$. In the same manner we can determine a form of the constitutive equation as a function of the deformation tensor $\boldsymbol{C}$,

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{F}) = \boldsymbol{R}\tilde{\boldsymbol{\sigma}}(\boldsymbol{C})\boldsymbol{R}^{\mathrm{T}} \tag{3.195}$$

or

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{F}) = \boldsymbol{F}\bar{\boldsymbol{\sigma}}(\boldsymbol{C})\boldsymbol{F}^{\mathrm{T}} \tag{3.196}$$

Note that also the differential forms of the constitutive equations (3.194), (3.195), and (3.196) are independent from the observer.

3.6.1 Isotropic materials

Let us consider a body $\mathcal{B}_0$. In a first case we apply to $\mathcal{B}_0$ a deformation generated by the gradient $\boldsymbol{F}$, while in a second case the undeformed body $\mathcal{B}_0$ is rotated by $\boldsymbol{Q}$ ($\boldsymbol{Q} \in \mathrm{Orth}^+$) and then subject to the same deformation of the first case, so that the total deformation gradient becomes $\boldsymbol{FQ}$. A material is isotropic if the material response is the same for every rotation $\boldsymbol{Q}$, namely

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{F}) = \hat{\boldsymbol{\sigma}}(\boldsymbol{FQ}) \qquad \forall \boldsymbol{Q} \in \mathrm{Orth}^+ \tag{3.197}$$

If we consider finite deformation, a material is not isotropic in all the configurations, but only initially, since the body loses the isotropy due to the deformation.

The response of an isotropic material should be also independent from the observer. Since the rotation tensors $\boldsymbol{Q}$ used in equation (3.192) and in (3.197) are different, we choose $\boldsymbol{Q}$ in (3.197) equal to $\boldsymbol{Q}^{\mathrm{T}}$ in (3.192) and, in this way, a substitution of (3.192) into (3.197), allows us to write the response invariance in the form

$$\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^{\mathrm{T}} = \hat{\boldsymbol{\sigma}}(\boldsymbol{QFQ}^{\mathrm{T}}) \tag{3.198}$$

Constitutive equations for an isotropic material

If a material is isotropic, its constitutive equation in terms of Cauchy stresses $\boldsymbol{\sigma}$ and of left Cauchy-Green tensor $\boldsymbol{B} = \boldsymbol{FF}^{\mathrm{T}}$ is

$$\boldsymbol{\sigma} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_2 \boldsymbol{B}^{-1} \tag{3.199}$$

where β_0, β_1, and β_2 are functions of the three invariants of $\boldsymbol{B}$. Furthermore, if the elastic Cauchy material is incompressible and isotropic, relation (3.199) becomes (Truesdell & Noll [20]):

$$\boldsymbol{\sigma} = -q\boldsymbol{I} + \beta_0 \boldsymbol{B} + \beta_1 \boldsymbol{B}^{-1} \tag{3.200}$$

since $I_3 = \det(\boldsymbol{B}) = 1$ for an incompressible material.

The parameter q in equation (3.200) can be determined on application of the Cayley-Hamilton theorem (2.65). In this way, we note that q is connected to the hydrostatic pressure $\hat{p} = \operatorname{tr} \boldsymbol{\sigma}/3$ through the relation

$$q = -\hat{p} + \beta_0 I_1 + \frac{1}{2}\beta_1 \left(I_1^2 - I_2\right) \tag{3.201}$$

where $I_1 = \operatorname{tr} \boldsymbol{B}$ and $I_2 = \operatorname{tr} \boldsymbol{B}^2$, while β_0 and β_1 are generic functions of the two invariants of $\boldsymbol{B}$

$$\beta_0 = \beta_0(I_1, I_2) \tag{3.202a}$$

$$\beta_1 = \beta_1(I_1, I_2) \tag{3.202b}$$

Two particular cases of equation (3.200) are the *Mooney-Rivlin material* [3], for which

$$\boldsymbol{\sigma} = -q\boldsymbol{I} + \alpha\boldsymbol{B} + \beta\boldsymbol{B}^{-1} \tag{3.203}$$

where α and β are constants, and the *neo-Hookean material*, for which

$$\boldsymbol{\sigma} = -q\boldsymbol{I} + \alpha\boldsymbol{B} \tag{3.204}$$

where α is constant.

The constitutive equation (3.200) implies the coaxiality between tensors $\boldsymbol{B}$ and $\boldsymbol{\sigma}$, so that they share (at least) a principal reference system where

$$\operatorname{diag} \boldsymbol{B} = \left(\lambda_1^2, \lambda_2^2, \lambda_3^2\right) \tag{3.205a}$$

$$\operatorname{diag} \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \tag{3.205b}$$

in which $\lambda_i > 0$ (for $i = 1, 2, 3$) are the principal stretches satisfying the incompressibility constraint

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{3.206}$$

and σ_i are the principal stresses. Consequently, the functions β_0 and β_1 can be determined by expressing equation (3.200) in the principal Eulerian reference system as follows

$$\beta_0 = \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{(\sigma_1 - \sigma_3)\lambda_1^2}{\lambda_1^2 - \lambda_3^2} - \frac{(\sigma_2 - \sigma_3)\lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right] \tag{3.207a}$$

$$\beta_1 = \frac{1}{\lambda_1^2 - \lambda_2^2}\left(\frac{\sigma_1 - \sigma_3}{\lambda_1^2 - \lambda_3^2} - \frac{\sigma_2 - \sigma_3}{\lambda_2^2 - \lambda_3^2}\right) \tag{3.207b}$$

Equations (3.207) can be expressed in an alternative manner on application of any permutation of the indexes 1, 2, and 3.

Newtonian fluids

A particular case of equation (3.200) is the constitutive equation describing an incompressible Newtonian fluid

$$\boldsymbol{\sigma} = \hat{p}\boldsymbol{I} + 2\mu\boldsymbol{D} \tag{3.208}$$

where the Cauchy stress is connected to the Eulerian deformation increment through the viscous parameter μ and the fluid pressure $\hat{p} = \operatorname{tr}\boldsymbol{\sigma}/3$ because $\operatorname{tr}\boldsymbol{D} = v_{i,i} = 0$. Therefore, in the case of plane strain we have

$$D_{i3} = D_{3i} = 0 \tag{3.209}$$

for $i = 1, 2, 3$, while the non null components of $\boldsymbol{\sigma}$ are

$$\sigma_{i3} = 0 \qquad i = 1, 2 \tag{3.210a}$$

$$\sigma_{33} = \frac{\sigma_1 + \sigma_2}{2} \tag{3.210b}$$

$$\sigma_{11} - \sigma_{22} = 2\mu\left(D_{11} - D_{22}\right) \tag{3.210c}$$

$$\sigma_{12} = \mu D_{12} \tag{3.210d}$$

3.6.2 Hyperelastic materials

An elastic material, for which exists an elastic potential $W = W(\boldsymbol{F})$ such that

$$\boldsymbol{S}^{\mathrm{T}} = \frac{\partial W}{\partial \boldsymbol{F}}[\boldsymbol{F}] \tag{3.211}$$

is called *hyperelastic material* (Ogden [4]). If W is an isotropic function, we have the following relation

$$\frac{\partial W}{\partial \boldsymbol{F}} = \frac{\partial W}{\partial \boldsymbol{U}} \tag{3.212}$$

and the Biot stresses $\boldsymbol{T}^{(1)}$ and $\boldsymbol{U}$ are coaxial, so that

$$\boldsymbol{T}^{(1)} = \frac{\partial W}{\partial \boldsymbol{U}} = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)} \tag{3.213}$$

where $\boldsymbol{u}^{(i)}$, for $i = 1, 2, 3$, are the unit vectors aligned to the principal Lagrangian directions.

On application of the definition of the Biot stress

$$\boldsymbol{T}^{(1)} = \boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{R} \tag{3.214}$$

and taking into account that

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \tag{3.215a}$$

$$\boldsymbol{U} = \sum_{j=1}^{3} \lambda_j \boldsymbol{u}^{(j)} \otimes \boldsymbol{u}^{(j)} \tag{3.215b}$$

we obtain

$$\boldsymbol{K} = \boldsymbol{R}\sum_{j=1}^{3} \lambda_j \boldsymbol{u}^{(j)} \otimes \boldsymbol{u}^{(j)} \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)} \boldsymbol{R}^{\mathrm{T}} \tag{3.216}$$

so that

$$\boldsymbol{R}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{R} = \sum_{i=1}^{3} \lambda_i \frac{\partial W}{\partial \lambda_i} \boldsymbol{u}^{(i)} \otimes \boldsymbol{u}^{(i)} \tag{3.217}$$

therefore, the principal components of $\boldsymbol{K}$ are

$$k_i = \lambda_i \frac{\partial W}{\partial \lambda_i} \tag{3.218}$$

in which the indexes $i = 1, 2, 3$ are not summed. For an incompressible material we introduce the constraint $J = \lambda_1\lambda_2\lambda_3 = 1$ and the principal Cauchy stresses σ_i are given by

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - \hat{p} \tag{3.219}$$

an expression in which the indexes $i = 1, 2, 3$ are not summed, and $\hat{p} = \mathrm{tr}\,\boldsymbol{\sigma}/3$ is the hydrostatic pressure; therefore, eliminating $\hat{p}$, we obtain

$$\sigma_i - \sigma_j = \lambda_i \frac{\partial W}{\partial \lambda_i} - \lambda_j \frac{\partial W}{\partial \lambda_j} \tag{3.220}$$

where $i \neq j$ and the indexes $i, j = 1, 2, 3$ are not summed. If the material is incompressible, only two of its principal stretches λ_i are independent, because, to satisfy the constraint $J = 1$, it must be $\lambda_3 = 1/(\lambda_1\lambda_2)$. Consequently, the elastic potential can be expressed as

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}) \tag{3.221}$$

and, on application of the chain rule, we obtain

$$\frac{\partial \hat{W}}{\partial \lambda_i} = \frac{\partial W}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial \lambda_i} + \frac{\partial W}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial \lambda_i} + \frac{\partial W}{\partial \lambda_3}\frac{\partial \lambda_3}{\partial \lambda_i} \tag{3.222}$$

for $i = 1, 2, 3$, so that equation (3.219) rewrites as

$$\sigma_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \hat{p} = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} + \lambda_3 \frac{\partial W}{\partial \lambda_3} - \hat{p} \tag{3.223a}$$

$$\sigma_2 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - \hat{p} = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2} + \lambda_3 \frac{\partial W}{\partial \lambda_3} - \hat{p} \tag{3.223b}$$

$$\sigma_3 = \lambda_3 \frac{\partial W}{\partial \lambda_3} - \hat{p} \tag{3.223c}$$

Furthermore, in the case of plane strain, we have

$$\lambda_1 = \lambda \tag{3.224a}$$

$$\lambda_2 = \frac{1}{\lambda} \tag{3.224b}$$

$$\lambda_3 = 1 \tag{3.224c}$$

and the elastic potential assumes the form

$$\breve{W}(\lambda) = \hat{W}\left(\lambda, \frac{1}{\lambda}\right) \tag{3.225}$$

On a second application of the chain rule, we can verify that

$$\frac{\partial \hat{W}}{\partial \lambda} = \frac{\partial \hat{W}}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial \lambda} + \frac{\partial \hat{W}}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial \lambda} \tag{3.226}$$

so that

$$\lambda \frac{\partial \hat{W}}{\partial \lambda} = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} - \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2} \tag{3.227}$$

and, consequently, equation (3.220) can be rewritten as

$$\sigma_1 - \sigma_2 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2} = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} - \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2} = \lambda \frac{\partial \hat{W}}{\partial \lambda} \tag{3.228a}$$

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_3 \frac{\partial W}{\partial \lambda_3} = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} \tag{3.228b}$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_3 \frac{\partial W}{\partial \lambda_3} = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2} \tag{3.228c}$$

It is clear that the elastic potential can be also expressed as a function of $\boldsymbol{B}$, and in particular of its invariants

$$I_1 = \operatorname{tr} \boldsymbol{B} \tag{3.229a}$$

$$I_2 = \operatorname{tr} \boldsymbol{B}^2 \tag{3.229b}$$

$$I_3 = \det \boldsymbol{B} \tag{3.229c}$$

where $I_3 = 1$ for an incompressible material. Hence, the elastic potential assumes the form

$$\bar{W}(I_1, I_2) = \hat{W}(\lambda_1, \lambda_2) \tag{3.230}$$

and from the chain rule we have

$$\frac{\partial \hat{W}}{\partial \lambda_i} = \frac{\partial \bar{W}}{\partial I_1} \frac{\partial I_1}{\partial \lambda_i} + \frac{\partial \bar{W}}{\partial I_2} \frac{\partial I_2}{\partial \lambda_i} \tag{3.231}$$

Employing this expression, we can rewrite relations (3.228) in the following alternative form

$$\sigma_1 - \sigma_2 = 2\left(\lambda_1^2 - \lambda_2^2\right) \frac{\partial \bar{W}}{\partial I_1} + 4\left(\lambda_1^4 - \lambda_2^4\right) \frac{\partial \bar{W}}{\partial I_2} \tag{3.232a}$$

$$\sigma_1 - \sigma_3 = 2\left(\lambda_1^2 - \frac{1}{\lambda_1^2 \lambda_2^2}\right) \left[\frac{\partial \bar{W}}{\partial I_1} + 2\left(\lambda_1^2 + \frac{1}{\lambda_1^2 \lambda_2^2}\right) \frac{\partial \bar{W}}{\partial I_2}\right] \tag{3.232b}$$

$$\sigma_2 - \sigma_3 = 2\left(\lambda_2^2 - \frac{1}{\lambda_1^2 \lambda_2^2}\right) \left[\frac{\partial \bar{W}}{\partial I_1} + 2\left(\lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2}\right) \frac{\partial \bar{W}}{\partial I_2}\right] \tag{3.232c}$$

3.6.3 Some elastic potentials

Mooney-Rivlin material

The explicit form of the deformation energy function for an isotropic elastic material proposed by Mooney [3] is

$$\bar{W}(I_1, I_2) = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_2}{4}\left(I_1^2 - I_2 - 6\right) \tag{3.233}$$

or, analogously,

$$\hat{W}(\lambda_1, \lambda_2) = \frac{\mu_1}{2}\left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3\right) - \frac{\mu_2}{2}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2\lambda_2^2 - 3\right) \tag{3.234}$$

or, in the case of plane strain,

$$\breve{W}(\lambda) = \frac{\mu_1 - \mu_2}{2}\left(\lambda^2 + \frac{1}{\lambda^2} - 2\right) \tag{3.235}$$

where μ_1 and μ_2 are material parameters, while $\mu_0 = \mu_1 - \mu_2$ represents the shear modulus. In this case, referring to equation (3.200), we simply obtain the identifications

$$\beta_0 = \mu_1 \tag{3.236a}$$

$$\beta_1 = \mu_2 \tag{3.236b}$$

Ogden material

The following class of deformation energy functions has been given by Ogden [4] to model experimental analyses on rubber:

$$\hat{W}(\lambda_1, \lambda_2) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}\left[\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \frac{1}{(\lambda_1\lambda_2)^{\alpha_i}} - 3\right] \tag{3.237}$$

or, for the case of plane strain,

$$\breve{W}(\lambda) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}\left[\lambda^{\alpha_i} + \frac{1}{\lambda^{\alpha_i}} - 2\right] \tag{3.238}$$

where μ_i and α_i are material parameters subject to the constraints

$$2\mu_0 = \sum_{i=1}^{N} \mu_i \alpha_i \qquad \text{with} \quad \mu_i \alpha_i > 0 \quad i = 1, \ldots, N \tag{3.239}$$

where $N \in \mathbb{N} \setminus \{ 0 \}$ determines the number of terms in the deformation energy function, μ_i have the dimension shear moduli, and α_i are dimensionless parameters. In particular, the Mooney-Rivlin material can be obtained if $N = 2$, $\alpha_1 = 2$, and $\alpha_2 = -2$.

The following values yield a good fitting with some experimental data from uniaxial and equi-biaxial tension tests, and pure shear tests on vulcanized rubber (Treloar [24, 25] and Ogden [4]):

$$\begin{aligned} \alpha_1 &= 1.3 & \mu_1 &= 6.3 \times 10^5\,\mathrm{N\,m^{-2}} \\ \alpha_2 &= 5.0 & \mu_2 &= 1.2 \times 10^3\,\mathrm{N\,m^{-2}} \\ \alpha_3 &= -2.0 & \mu_3 &= 0.1 \times 10^5\,\mathrm{N\,m^{-2}} \end{aligned} \tag{3.240}$$

and $\mu_0 = 4.225 \times 10^5\,\mathrm{N\,m^{-2}}$; some limitations are given by Ogden *et al.* [26].

The coefficients of relations (3.200) can be obtained from equation (3.237) in the form

$$\beta_0 = \frac{1}{\lambda_1^2 - \lambda_2^2} \sum_{i=1}^{N} \mu_i \left[\frac{\lambda_1^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_1^2 - (\lambda_1 \lambda_2)^{-2}} \lambda_1^2 - \frac{\lambda_2^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_2^2 - (\lambda_1 \lambda_2)^{-2}} \lambda_2^2 \right] \tag{3.241a}$$

$$\beta_1 = \frac{1}{\lambda_1^2 - \lambda_2^2} \sum_{i=1}^{N} \mu_i \left[\frac{\lambda_1^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_1^2 - (\lambda_1 \lambda_2)^{-2}} - \frac{\lambda_2^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_2^2 - (\lambda_1 \lambda_2)^{-2}} \right] \tag{3.241b}$$

or

$$\beta_0 = \frac{\lambda^2}{\lambda^4 - 1} \sum_{i=1}^{N} \mu_i \left[\frac{\lambda^{\alpha_i} - 1}{\lambda^2 - 1} \lambda^2 - \frac{1 - \lambda^{\alpha_i}}{1 - \lambda^2} \frac{1}{\lambda^{\alpha_i}} \right] \tag{3.242a}$$

$$\beta_1 = \frac{\lambda^2}{\lambda^4 - 1} \sum_{i=1}^{N} \mu_i \left[\frac{\lambda^{\alpha_i} - 1}{\lambda^2 - 1} - \frac{1 - \lambda^{\alpha_i}}{1 - \lambda^2} \frac{\lambda^2}{\lambda^{\alpha_i}} \right] \tag{3.242b}$$

for the case of plane strain.

J_2-deformation theory of plasticity

Within hyperelasticity, Hutchinson & Neale [17] defined the J_2-deformation theory of plasticity (see also Hutchinson & Tvergaard [27]). Referring to equation (3.200), the corresponding constitutive equation is

$$\sigma_i = \frac{2}{3} E_s \varepsilon_i + \hat{p} \tag{3.243}$$

for $i = 1, 2, 3$, and, analogously, the incompressibility constraint is expressed as

$$\sum_{i=1}^{3} \varepsilon_i = 0 \tag{3.244}$$

where $\varepsilon_i = \ln \lambda_i$ represents the logarithmic deformation, $\hat{p} = \operatorname{tr} \boldsymbol{\sigma}/3$, while E_s is the secant modulus to the curve giving the behaviour of the effective stress σ_e as a function of the effective strain ε_e, defined as

$$\varepsilon_\mathrm{e} = \sqrt{\frac{2}{3}\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2\right)} \tag{3.245a}$$

$$\sigma_\mathrm{e} = \sqrt{\frac{3}{2}\left(S_1^2 + S_2^2 + S_3^2\right)} \tag{3.245b}$$

in which S_i (for $i = 1, 2, 3$) are the principal components of the deviatoric stress tensor. Furthermore, we assume

$$E_\mathrm{s} = K \varepsilon_\mathrm{e}^{N-1} \tag{3.246}$$

where $N \in (0; 1]$ is the hardening exponent and K is a (positive) constitutive stiffness parameter. Therefore, the deformation energy function is expressed as

$$W^\varepsilon = \frac{K}{N+1} \varepsilon_\mathrm{e}^{N+1} \tag{3.247}$$

where we adopted[1]:

$$q = -\hat{p} + \frac{2}{3} E_{\mathrm{s}} \frac{\left(\lambda_2^4 - \lambda_2^{-2}\right)\left(\lambda_1^2 + \lambda_3^2\right)\varepsilon_1 - \left(\lambda_1^4 - \lambda_1^{-2}\right)\left(\lambda_2^2 + \lambda_3^2\right)\varepsilon_2}{\left(\lambda_1^2 - \lambda_2^2\right)\left(\lambda_1^2 - \lambda_3^2\right)\left(\lambda_2^2 - \lambda_3^2\right)} \tag{3.248a}$$

$$\beta_0 = \frac{2}{3} E_{\mathrm{s}} \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{\left(\varepsilon_1 - \varepsilon_3\right)\lambda_1^2}{\lambda_1^2 - \lambda_3^2} - \frac{\left(\varepsilon_2 - \varepsilon_3\right)\lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right] \tag{3.248b}$$

$$\beta_1 = \frac{2}{3} E_{\mathrm{s}} \frac{1}{\lambda_1^2 - \lambda_2^2}\left(\frac{\varepsilon_1 - \varepsilon_3}{\lambda_1^2 - \lambda_3^2} - \frac{\varepsilon_2 - \varepsilon_3}{\lambda_2^2 - \lambda_3^2}\right) \tag{3.248c}$$

In the case of plane strain we have

$$\varepsilon_3 = \ln \lambda_3 = 0 \tag{3.249}$$

while the incompressibility contraint implies

$$\varepsilon_1 = -\varepsilon_2 = \varepsilon = -\ln \lambda \tag{3.250}$$

so that the effective strain ε_{e} becomes

$$\varepsilon_{\mathrm{e}} = \frac{2\varepsilon}{\sqrt{3}} \tag{3.251}$$

If we introduce the parameter $\tilde{K}$, defined as

$$\tilde{K} = K\left(\frac{2}{\sqrt{3}}\right)^{N-1} \tag{3.252}$$

equations (3.246) and (3.246) can be manipulated and rewritten as follows

$$E_{\mathrm{s}} = \tilde{K}\varepsilon_{\mathrm{e}}^{N-1} \tag{3.253a}$$

$$W^{\varepsilon} = \frac{4}{3}\frac{\tilde{K}}{N+1}\varepsilon^{N+1} \tag{3.253b}$$

[1] See equations (3.200) and (3.243).

In this case, referring to equation (3.200), we obtain

$$q = -\hat{p} \tag{3.254a}$$

$$\beta_0 = \frac{4}{3} E_\mathrm{s} \frac{\lambda^2}{\lambda^4 - 1} \varepsilon \tag{3.254b}$$

$$\beta_1 = \frac{2}{3} E_\mathrm{s} \frac{\lambda^2}{1 - \lambda^4} \varepsilon \tag{3.254c}$$

3.7 Boundary problems

In the Lagrangian formulation of the elastic boundary problems, the independent variables are $(\boldsymbol{x}_0, t)$.

The objective is to determine the motion $\boldsymbol{\phi}(\boldsymbol{x}_0, t)$ of a body subject to given initial and boundary conditions, once the reference configuration and the density $\rho_0(\boldsymbol{x}_0)$ are known. The motion equation in terms of reference coordinates, as seen in Section 3.4.2, is

$$\operatorname{div} \boldsymbol{S} + \rho_0 \boldsymbol{b} = \rho_0 \ddot{\boldsymbol{x}} \tag{3.255}$$

with $\boldsymbol{S}$ depending on $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0, t)$ through the constitutive equation

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F}) \tag{3.256}$$

where

$$\boldsymbol{F} = \operatorname{Grad} \boldsymbol{\phi}(\boldsymbol{x}_0, t) \tag{3.257}$$

In general, the body forces $\boldsymbol{b}$ are functions of the spatial points $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{x}_0, t)$. If we assume, without any loss of generality, that the body occupies the reference configuration at the beginning of the motion ($t = t_0$), the *initial conditions (IC)* are:

$$\boldsymbol{\phi}(\boldsymbol{x}_0, t_0) = \boldsymbol{x}_0 \tag{3.258a}$$

$$\dot{\boldsymbol{\phi}}(\boldsymbol{x}_0, t_0) = \boldsymbol{v}_0(\boldsymbol{x}_0) \tag{3.258b}$$

where $\boldsymbol{v}_0$ is a function assigned on $\mathcal{B}_0$.

To specify the *boundary conditions (BC)*, we consider two subsets $\partial\mathcal{B}_0^x \subseteq \partial\mathcal{B}_0$ and $\partial\mathcal{B}_0^s \subseteq \partial\mathcal{B}_0$ in which, respectively, displacements and surface tensions are assigned.[2] Then, it is possible to write:

$$\boldsymbol{\phi}(\boldsymbol{x}_0, t) = \boldsymbol{\xi}(\boldsymbol{x}_0, t) \qquad \text{on } \partial\mathcal{B}_0^x \tag{3.259a}$$

$$\boldsymbol{S}\boldsymbol{n}_0 = \boldsymbol{s}(\boldsymbol{x}_0, \boldsymbol{x}, \boldsymbol{F}, t) \qquad \text{on } \partial\mathcal{B}_0^s \tag{3.259b}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{s}$ are assigned vector fields. The condition (3.259b) is rather general and holds for a wide range of prescribed tensions, including static loads and pressures.

[2] The following relations hold for the two boundaries above defined:

$$\partial\mathcal{B}_0^x \cup \partial\mathcal{B}_0^s = \partial\mathcal{B}^0$$
$$\partial\mathcal{B}_x^0 \cap \partial\mathcal{B}_s^0 = \emptyset$$

Chapter 4

INCREMENTAL DEFORMATIONS

This Chapter is focused on the definition of incremental quantities and the incremental formulation of the elastic problem. The constitutive operator, the elastic moduli, the stresses and the strains are derived in the incremental form.

IN ORDER TO STUDY THE RESPONSE of a solid body subject to prestress, a boundary problem and a set of linear incremental relations should be introduced. Even though there are general theories on this topic, here the attention is limited to the case of linear increment. Finally, the main constitutive equations, that can perfectly describe the behaviour of many elastic materials, are illustrated in an incremental formulation. In the following Sections, an elastic, initially isotropic (in absence of load) material is considered with known initial conditions, for which explicit expressions for the stress are available in every phase of the deformation, as well as incremental equations directly obtained from those associated to finite deformations.

4.1 Definition of an incremental quantity

Let $\boldsymbol{\phi}(\boldsymbol{x}_0, t)$ be a solution of the boundary problem (3.255)-(3.259). We want to find solutions near $\boldsymbol{\phi}$ when the boundary conditions are perturbed.

Let $\bar{\boldsymbol{\phi}}(\boldsymbol{x}_0, t)$ be a solution of the perturbed boundary and assume that $\bar{\boldsymbol{x}} =$

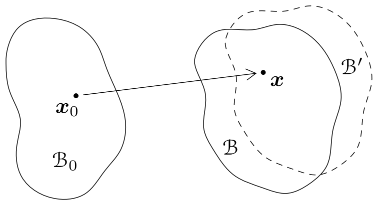


Figure 4.1: Incremental deformations.

$\bar{\boldsymbol{\phi}}(\boldsymbol{x}_0, t)$. The displacement of a material particle is

$$\dot{\boldsymbol{x}} = \bar{\boldsymbol{x}} - \boldsymbol{x} = \bar{\boldsymbol{\phi}}(\boldsymbol{x}_0, t) - \boldsymbol{\phi}(\boldsymbol{x}_0, t) = \dot{\boldsymbol{\phi}}(\boldsymbol{x}_0, t) \tag{4.1}$$

which is assumed as the definition of $(\dot{})$.

If the displacement $\dot{\boldsymbol{x}}$ is *small* for all $\boldsymbol{x}_0 \in \mathcal{B}_0$, so that the terms of order $|\dot{\boldsymbol{x}}|^2$ can be neglected, then we can refer to $\dot{\boldsymbol{x}}$ as a (*linear*) *incremental deformation* of the current configuration $\mathcal{B}_t$.

The increment of the deformation gradient due to the incremental deformation $\dot{\boldsymbol{x}}$ is given by

$$\dot{\boldsymbol{F}} = (\mathrm{Grad}\,\boldsymbol{\phi})^{\cdot} = \mathrm{Grad}\,\bar{\boldsymbol{\phi}} - \mathrm{Grad}\,\boldsymbol{\phi} = \mathrm{Grad}\,\dot{\boldsymbol{\phi}} \tag{4.2}$$

an expression in which the linearity property of the operator Grad has been used.

In general, the increment of a scalar (vector or tensor) function $f(\boldsymbol{F})$ regular enough is given by its Taylor series expansion limited to the linear term

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{F}}[\dot{\boldsymbol{F}}] + o(\dot{\boldsymbol{F}}) \tag{4.3}$$

4.2 Incremental motions equations

According with (4.3), the incremental linearized constitutive equation for an elastic material becomes

$$\dot{\boldsymbol{S}} = \mathbb{E}_0 \dot{\boldsymbol{F}} \tag{4.4a}$$

$$\mathbb{E}_0 = \frac{\partial \hat{\boldsymbol{S}}(\boldsymbol{F})}{\partial \boldsymbol{F}} \tag{4.4b}$$

or, similarly,

$$\dot{\boldsymbol{S}} = \mathbb{E}_0 \dot{\boldsymbol{F}} + q\boldsymbol{F}^{-\mathrm{T}} \dot{\boldsymbol{F}} \boldsymbol{F}^{-\mathrm{T}} - \dot{q} \boldsymbol{F}^{-\mathrm{T}} \tag{4.5}$$

when the material is incompressible. $\mathbb{E}_0$ is the fourth order tensor of the *elastic moduli* in the reference configuration. When the material is hyperelastic, we have $\boldsymbol{S} = \partial W / \partial \boldsymbol{F}$ and from (4.4b) we obtain

$$\mathbb{E}_0 = \frac{\partial^2 W(\boldsymbol{F})}{\partial \boldsymbol{F}^2} \tag{4.6}$$

Usually, in incremental boundary problems, it is convenient to choose the known current configuration as a reference configuration. In this case $\boldsymbol{F} = \boldsymbol{I}$ while $\dot{\boldsymbol{F}}$, which represents the incremental deformation gradient, is different from zero since

$$\dot{\boldsymbol{F}} = \boldsymbol{F}\boldsymbol{L} = \boldsymbol{L} \tag{4.7}$$

Note that isotropy is a property associated to a particular configuration that should be assumed as a reference configuration in order to formulate the constitutive equation.

4.2.1 Incremental boundary problem

Referring to Chapter 3, the set of equations governing the quasi-static incremental equation for a compressible material expressed in a material reference system is the following:

$$\begin{cases} \operatorname{div}\dot{\boldsymbol{S}} + \dot{\boldsymbol{b}}_0 = \boldsymbol{0} \\ \dot{\boldsymbol{S}} = \mathbb{E}\dot{\boldsymbol{F}} \\ \dot{\boldsymbol{F}} = \operatorname{Grad}\dot{\boldsymbol{x}} \\ \dot{\boldsymbol{S}}\boldsymbol{n}_0 = \dot{\boldsymbol{s}}(\boldsymbol{x},t) & \text{on } \partial\mathcal{B}_0^s \\ \dot{\boldsymbol{x}} = \dot{\boldsymbol{\xi}}(\boldsymbol{x},t) & \text{on } \partial\mathcal{B}_0^x \end{cases} \tag{4.8}$$

4.3 Incremental constitutive equations

The constitutive equation for an elastic, isotropic and incompressible Cauchy material (Section 3.6.1) is

$$\boldsymbol{\sigma} = -q\boldsymbol{I} + \beta_0\boldsymbol{B} + \beta_1\boldsymbol{B}^{-1} \tag{4.9}$$

where β_0 and β_1 are generic functions of the two invariants of $\boldsymbol{B}$. The material derivative of (4.9) is

$$\dot{\boldsymbol{\sigma}} = -\dot{q}\boldsymbol{I} + \beta_0\dot{\boldsymbol{B}} + \beta_1\left(\boldsymbol{B}^{-1}\right)^{\cdot} + \dot{\beta}_0\boldsymbol{B} + \dot{\beta}_1\boldsymbol{B}^{-1} \tag{4.10}$$

in which

$$\dot{\boldsymbol{B}} = \boldsymbol{D}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{D} + \boldsymbol{W}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{W} \tag{4.11}$$

and

$$\left(\boldsymbol{B}^{-1}\right)^{\cdot} = -\boldsymbol{B}^{-1}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{B}^{-1} - \boldsymbol{B}^{-1}\boldsymbol{W} + \boldsymbol{W}\boldsymbol{B}^{-1} \tag{4.12}$$

where $\boldsymbol{D}$ is the Eulerian deformation velocity tensor and $\boldsymbol{W}$ is the spin tensor. Once the equation (4.9) is known, and with the definition of the Jaumann derivative

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{W} \tag{4.13}$$

the constitutive equation (4.9) becomes

$$\overset{\triangledown}{\boldsymbol{\sigma}} + \dot{q}\boldsymbol{I} = \beta_0(\boldsymbol{D}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{D}) - \beta_1(\boldsymbol{B}^{-1}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{B}^{-1}) + \dot{\beta}_0\boldsymbol{B} + \dot{\beta}_1\boldsymbol{B}^{-1} \tag{4.14}$$

Note that

$$\dot{\beta}_i = \frac{\partial\beta_i(I_1,I_2)}{\partial I_1}\operatorname{tr}\dot{\boldsymbol{B}} + 2\frac{\partial\beta_i}{\partial I_2(I_1,I_2)}\boldsymbol{B}\cdot\dot{\boldsymbol{B}} \tag{4.15}$$

where

$$\operatorname{tr} \dot{\boldsymbol{B}} = 2\boldsymbol{B} \cdot \boldsymbol{D} \tag{4.16a}$$

$$\boldsymbol{B} \cdot \dot{\boldsymbol{B}} = 2\boldsymbol{B}^2 \cdot \boldsymbol{D} \tag{4.16b}$$

or, similarly,

$$\dot{\hat{\beta}}_i = \frac{\partial \hat{\beta}_i(\lambda_1, \lambda_2)}{\partial \lambda_1} \dot{\lambda}_1 + \frac{\partial \hat{\beta}_i(\lambda_1, \lambda_2)}{\partial \lambda_2} \dot{\lambda}_2 \tag{4.17}$$

where $\hat{\beta}_i = \hat{\beta}_i(\lambda_1, \lambda_2)$ (per $i = 0, 1$) are the coefficients β_0 and β_1 expressed in function of the principal stretches. One can obtain

$$\begin{aligned}\overset{\nabla}{\boldsymbol{\sigma}} + \dot{q}\boldsymbol{I} = {} & \beta_0(\boldsymbol{D}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{D}) - \beta_1 \left(\boldsymbol{B}^{-1}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{B}^{-1}\right) + +2\Bigg(\boldsymbol{B} \cdot \boldsymbol{D}\frac{\partial \beta_0}{\partial I_1} + \\ & + 2\boldsymbol{B}^2 \cdot \boldsymbol{D}\frac{\partial \beta_0}{\partial I_2}\Bigg)\boldsymbol{B} + 2\left(\boldsymbol{B} \cdot \boldsymbol{D}\frac{\partial \beta_1}{\partial I_1} + 2\boldsymbol{B}^2 \cdot \boldsymbol{D}\frac{\partial \beta_1}{\partial I_2}\right)\boldsymbol{B}^{-1}\end{aligned} \tag{4.18}$$

or, similarly,

$$\begin{aligned}\overset{\nabla}{\boldsymbol{\sigma}} + \dot{q}\boldsymbol{I} = {} & \beta_0(\boldsymbol{D}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{D}) - \beta_1(\boldsymbol{B}^{-1}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{B}^{-1}) + \\ & + \left(\frac{\partial \hat{\beta}_0}{\partial \lambda_1}\dot{\lambda}_1 + \frac{\partial \hat{\beta}_0}{\partial \lambda_2}\dot{\lambda}_2\right)\boldsymbol{B} + \left(\frac{\partial \hat{\beta}_1}{\partial \lambda_1}\dot{\lambda}_1 + \frac{\partial \hat{\beta}_1}{\partial \lambda_2}\dot{\lambda}_2\right)\boldsymbol{B}^{-1}\end{aligned} \tag{4.19}$$

The incremental constitutive equation (4.18) holds for the three-dimensional incompressible Cauchy elasticity.

4.3.1 Incremental moduli

We want to apply equation (4.18) to the case of *incremental in-plane deformations* by imposing the generic *homogeneous* deformation condition. In the principal Eulerial reference system we have

$$\boldsymbol{B} = \lambda_1^2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda_2^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \frac{1}{\lambda_1^2 \lambda_2^2} \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \tag{4.20}$$

where $D_{i3} = D_{3i} = 0$ (per $i = 1, 2, 3$), so that the out-of-plane stress components are

$$\overset{\triangledown}{\boldsymbol{\sigma}}_{3i} = \overset{\triangledown}{\boldsymbol{\sigma}}_{i3} = 0 \qquad i = 1, 2 \tag{4.21}$$

and

$$\overset{\triangledown}{\boldsymbol{\sigma}}_{33} = -\dot{q} + \left(\lambda_1^2 - \lambda_2^2\right)\left\{ \frac{1}{\lambda_1^2\lambda_2^2}\left[\frac{\partial\beta_0}{\partial I_1} + 2\left(\lambda_1^2 + \lambda_2^2\right)\frac{\partial\beta_0}{\partial I_2}\right] + \right. \\ \left. + \lambda_1^2\lambda_2^2\left[\frac{\partial\beta_1}{\partial I_1} + 2\left(\lambda_1^2 + \lambda_2^2\right)\frac{\partial\beta_1}{\partial I_2}\right]\right\}(D_{11} - D_{22}) \tag{4.22}$$

or, alternatively,

$$\overset{\triangledown}{\boldsymbol{\sigma}}_{33} = -\dot{q} + \left[\frac{1}{\lambda_1^2\lambda_2^2}\left(\lambda_1\frac{\partial\hat{\beta}_0}{\partial\lambda_1} - \lambda_2\frac{\partial\hat{\beta}_0}{\partial\lambda_2}\right) + \right. \\ \left. + \lambda_1^2\lambda_2^2\left(\lambda_1\frac{\partial\hat{\beta}_1}{\partial\lambda_1} - \lambda_2\frac{\partial\hat{\beta}_1}{\partial\lambda_2}\right)\right]\frac{D_{11} - D_{22}}{2} \tag{4.23}$$

The in-plane stress components can be expressed in the Biot [2] form as

$$\begin{cases} \overset{\triangledown}{\sigma}_{12} = 2\mu D_{12} \\ \overset{\triangledown}{\sigma}_{11} - \overset{\triangledown}{\sigma}_{22} = 2\mu_*\left(D_{11} - D_{22}\right) \\ D_{11} + D_{22} = 0 \end{cases} \tag{4.24}$$

where μ and μ_* are two incremental moduli corresponding, respectively, to a shear parallel to and inclined at $\pi/4$ with respect the principal Eulerial axes. These can be expressed in function of the invariants of $\boldsymbol{B}$

$$\mu = \frac{\lambda_1^2 + \lambda_2^2}{2}\left(\beta_0 - \frac{\beta_1}{\lambda_1^2\lambda_2^2}\right) \tag{4.25a}$$

$$\mu_* = \frac{\lambda_1^2 + \lambda_2^2}{2}\beta_0 + \frac{(\lambda_1^2 - \lambda_2^2)^2}{2}\left[\frac{\partial\beta_0}{\partial I_1} + 2(\lambda_1^2 + \lambda_2^2)\frac{\partial\beta_0}{\partial I_2}\right] + \\ - \frac{1}{\lambda_1^2\lambda_2^2}\left\{\frac{\lambda_1^2 + \lambda_2^2}{2}\beta 1 + \frac{(\lambda_1^2 - \lambda_2^2)^2}{2}\left[\frac{\partial\beta_1}{\partial I_1} + 2(\lambda_1^2 + \lambda_2^2)\frac{\partial\beta_1}{\partial I_2}\right]\right\} \tag{4.25b}$$

or of the principal stretches

$$\mu = \frac{\lambda_1^2 + \lambda_2^2}{2}\left(\hat{\beta}_0 - \frac{\hat{\beta}_1}{\lambda_1^2\lambda_2^2}\right) \tag{4.26a}$$

$$\begin{aligned}\mu_* = {} & \frac{\lambda_1^2 + \lambda_2^2}{2}\hat{\beta}_0 + \frac{\lambda_1^2 - \lambda_2^2}{4}\left(\lambda_1\frac{\partial\hat{\beta}_0}{\partial\lambda_1} - \lambda_2\frac{\partial\hat{\beta}_0}{\partial\lambda_2}\right) + \\ & - \frac{1}{\lambda_1^2\lambda_2^2}\left[\frac{\lambda_1^2 + \lambda_2^2}{2}\hat{\beta}_1 + \frac{\lambda_1^2 - \lambda_2^2}{4}\left(\lambda_1\frac{\partial\hat{\beta}_1}{\partial\lambda_1} - \lambda_2\frac{\partial\hat{\beta}_1}{\partial\lambda_2}\right)\right]\end{aligned} \tag{4.26b}$$

being

$$\dot{\lambda}_i = \lambda_i D_{ii} \tag{4.27}$$

where index i is not summed. An alternative expression for the two incremental moduli μ e μ_*, related to the existence of the deformation energy function, has been proposed by Biot [2] (see Appendix A) in the form:

$$\mu = \frac{1}{2}\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2}\left(\lambda_1\frac{\partial W}{\partial\lambda_1} - \lambda_2\frac{\partial W}{\partial\lambda_2}\right) \tag{4.28a}$$

$$\mu_* = \frac{1}{4}\left(\lambda_1\frac{\partial W}{\partial\lambda_1} + \lambda_2\frac{\partial W}{\partial\lambda_2} + \lambda_1^2\frac{\partial^2 W}{\partial\lambda_1^2} + \lambda_2^2\frac{\partial^2 W}{\partial\lambda_2^2} - \lambda_1\lambda_2\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}\right) \tag{4.28b}$$

Plane strain incremental moduli

We want to apply equations (4.25) and (4.26) to the case of incremental plane strain superimposed to a state of homogeneous plane strain. In the case of plane strain, the principal stretch directed orthogonally out of the plane is always unit. For instance, for an incompressible material, if $\lambda_1 = \lambda$ then we must have $\lambda_2 = 1/\lambda$. Consequently the left Cauchy-Green deformation tensor and its inverse in the principal Eulerian reference system become

$$\boldsymbol{B} = \lambda^2\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{1}{\lambda^2}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \tag{4.29a}$$

$$\boldsymbol{B}^{-1} = \frac{1}{\lambda^2}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda^2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \tag{4.29b}$$

A substitution of (4.29) into (4.18) leads to

$$\overset{\nabla}{\sigma}_{3i} = \overset{\nabla}{\sigma}_{i3} = 0 \tag{4.30}$$

with $i = 1, 2$ and

$$\overset{\nabla}{\sigma}_{33} = -\dot{q} + \left(\lambda^2 - \frac{1}{\lambda^2}\right)\left[\frac{\partial \beta 0}{\partial I_1} + 2\left(\lambda^2 + \frac{1}{\lambda^2}\right)\frac{\partial \beta 0}{\partial I_2} + \frac{\partial \beta 1}{\partial I_1} + \right.$$
$$\left. + 2\left(\lambda^2 + \frac{1}{\lambda^2}\right)\frac{\partial \beta 1}{\partial I_2}(D_{11} - D_{22})\right] \quad (4.31)$$

while the incremental moduli μ e μ_* become functions of the invariants of $\boldsymbol{B}$

$$\mu = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)(\beta_0 - \beta_1) \quad (4.32a)$$

$$\mu_* = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)(\beta_0 - \beta_1) + \frac{1}{2}\left(\lambda^2 - \frac{1}{\lambda^2}\right)^2\left[2\left(\lambda^2 + \frac{1}{\lambda^2}\right)\frac{\partial \beta_0}{\partial I_2} + \right.$$
$$\left. + \frac{\partial \beta_0}{\partial I_1} - \frac{\partial \beta_1}{\partial I_1} + 2\left(\lambda^2 + \frac{1}{\lambda^2}\right)\frac{\partial \beta_1}{\partial I_2}(D_{11} - D_{22})\right] \quad (4.32b)$$

If we introduce the coefficients

$$\tilde{\beta}_i(\lambda) = \hat{\beta}_i\left(\lambda, \frac{1}{\lambda}\right) \quad (4.33)$$

with $i = 0, 1$ and if we use the chain rule for differentiation, we obtain

$$\lambda \frac{\partial \tilde{\beta}_i}{\partial \lambda} = \frac{\partial \hat{\beta}_i}{\partial \lambda_1} - \frac{1}{\lambda}\frac{\partial \hat{\beta}_i}{\partial \lambda_2} \quad (4.34)$$

Therefore, the incremental moduli, functions of the principal stretch, are equal to

$$\mu = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)(\tilde{\beta}_0 - \tilde{\beta}_1) \quad (4.35a)$$

$$\mu_* = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)(\tilde{\beta}_0 - \tilde{\beta}_1) + \frac{\lambda}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\left(\frac{\partial \tilde{\beta}_0}{\partial \lambda} - \frac{\partial \tilde{\beta}_1}{\partial \lambda}\right) \quad (4.35b)$$

The expressions of μ and μ_* in terms of deformation energy $\breve{W}(\lambda)$ are

$$\mu = \frac{\lambda}{2}\left(\frac{\lambda^4+1}{\lambda^4-1}\right)\frac{\partial \breve{W}}{\partial \lambda} \tag{4.36a}$$

$$\mu_* = \frac{\lambda}{4}\frac{\partial}{\partial \lambda}\left(\lambda \frac{\partial \breve{W}}{\partial \lambda}\right) \tag{4.36b}$$

4.3.2 Examples of incremental constitutive equations

Mooney-Rivlin material

For the Mooney-Rivlin [3] material we have

$$\mu = \mu_* = \frac{1}{2}\left(\lambda_1^2+\lambda_2^2\right)\left(\mu_1 - \frac{\mu_2}{\lambda_1^2\lambda_2^2}\right) \tag{4.37}$$

and, analogously,

$$\mu = \mu_* = \frac{1}{2}\left(\frac{\lambda^4+1}{\lambda^2}\right)(\mu_1-\mu_2) \tag{4.38}$$

for plane strain. Furthermore, we have

$$\overset{\triangledown}{\sigma}_{33} = -\dot{q} \tag{4.39}$$

Ogden material

For the Ogden [4] material the coefficients in the relations (4.24) can be obtained from equation (3.237) in the form

$$\mu = \frac{1}{2}\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2-\lambda_2^2}\sum_{i=1}^{N}\mu_i\left(\lambda_1^{\alpha_i}-\lambda_2^{\alpha_i}\right) \tag{4.40a}$$

$$\mu_* = \frac{1}{4}\sum_{i=1}^{N}\alpha_i\mu_i\left(\lambda_1^{\alpha_i}+\lambda_2^{\alpha_i}\right) \tag{4.40b}$$

and, analogously,

$$\mu = \frac{1}{2}\frac{\lambda^4+1}{\lambda^4-1}\sum_{i=1}^{N}\mu_i\left(\frac{\lambda^{2\alpha_i}-1}{\lambda^{\alpha_i}}\right) \tag{4.41a}$$

$$\mu_* = \frac{1}{4}\sum_{i=1}^{N}\alpha_i\mu_i\left(\frac{\lambda^{2\alpha_i}+1}{\lambda^{\alpha_i}}\right) \tag{4.41b}$$

for plane stress. The unique component different from zero in the Jaumann derivative of the Cauchy stress in the direction out of the plane is:

$$\overset{\nabla}{\sigma}_{33} = \sum_{i=1}^{N}\left[\mu_i\frac{g_1^i(\lambda_1,\lambda_2)+g_2^i(\lambda_1,\lambda_2)+g_3^i(\lambda_1,\lambda_2)}{(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)\left(\lambda_1^2-\lambda_3^2\right)^2\left(\lambda_2^2-\lambda_3^2\right)^2}\,\lambda_3^{\alpha_i+1}\right] \tag{4.42}$$

where the expressions of $g_j^i(\lambda_1,\lambda_2)$ are given by the following relations:

$$\begin{aligned} g_1^i(\lambda_1,\lambda_2) &= \lambda_1^{\alpha_i}\lambda_2^{2(\alpha_i+1)}\left(\lambda_1^2-\lambda_3^2\right)^2\Big[\lambda_1(\lambda_1-\lambda_2)\left(\lambda_1^2+\lambda_3^2\right) \\ &\times\left[\alpha_i\left(\lambda_2^2-\lambda_3^2\right)-2\left(\lambda_2^2+\lambda_3^2\right)\right]-2\lambda_3^4\left[\left(\lambda_1^2+\lambda_2^2\right)-\lambda_3\left(1+\lambda_3^{-6}\right)\right]\Big] \end{aligned} \tag{4.43}$$

for $j=1$, while for $j=2$

$$\begin{aligned} g_2^i(\lambda_1,\lambda_2) &= (\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^3\lambda_3^6\Big[\alpha_i\left(\lambda_1^2-\lambda_3^2\right)\left(\lambda_2^2-\lambda_3^2\right)\left(1+\lambda_3^6\right)+ \\ &+2\lambda_3^{-1}\left[\left(\lambda_3^3-1\right)+\lambda_3^{-9}\left(\lambda_3^3+1\right)+\lambda_3^{-5}\left(\lambda_1^2+\lambda_2^2\right)\right]\Big] \end{aligned} \tag{4.44}$$

and, finally, for $j=3$

$$\begin{aligned} g_3^i(\lambda_1,\lambda_2) &= \lambda_1^{(1+2\alpha_i)}\lambda_2^{\alpha_i}\left(\lambda_2^2-\lambda_3^2\right)\Big[2\lambda_1^{-4}\lambda_2^{-4}(\lambda_1-\lambda_2)^2+\alpha_i\lambda_2(\lambda_2-\lambda_1) \\ &\times\left[\left(\lambda_3^2-\lambda_2^2\right)+\left(\lambda_1^{-2}-\lambda_3^{-2}\right)\lambda_3^{-2}\right]+2\lambda_2(\lambda_2-\lambda_1)\left[\left(\lambda_1^2+\lambda_3^2\right)+\left(\lambda_2^2+\lambda_3^2\right)\right]+ \\ &+2\lambda_2^4\left(1-\lambda_1^5\lambda_2\right)+2\lambda_2^{-2}\left(1-\lambda_1^5\lambda_2^7\right)\Big] \end{aligned} \tag{4.45}$$

in which, due to incompressibility constraint, we have

$$\lambda_3 = \frac{1}{\lambda_1\lambda_2} \tag{4.46}$$

Hypoelastic material

If we identify $\boldsymbol{T} \in \mathrm{Sym}$ with the Cauchy tensor $\boldsymbol{\sigma}$ and if we denote with $\overset{\diamond}{(\cdot)}$ the Eulerian objective derivative of $\boldsymbol{T}$ (assumed to be an isotropic function of $\boldsymbol{T}$ and $\boldsymbol{D}$), the representation

$$\begin{aligned}\overset{\diamond}{\sigma} = -\dot{q}\boldsymbol{I} + \gamma_1\boldsymbol{D} + (\gamma_2\boldsymbol{T}\cdot\boldsymbol{D} + \gamma_3\boldsymbol{T}^2\cdot\boldsymbol{D})\boldsymbol{I} + (\gamma_4\boldsymbol{T}\cdot\boldsymbol{D} + \gamma_5\boldsymbol{T}^2\cdot\boldsymbol{D})\boldsymbol{T} + \\ + (\gamma_6\boldsymbol{T}\cdot\boldsymbol{D} + \gamma_7\boldsymbol{T}^2\cdot\boldsymbol{D})\boldsymbol{T}^2 + \gamma_8(\boldsymbol{D}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{D}) + \gamma_9(\boldsymbol{D}\boldsymbol{T}^2 + \boldsymbol{T}^2\boldsymbol{D})\end{aligned} \tag{4.47}$$

(where coefficients γ_i, con $i = 1, \dots, 9$, are polynomial functions of the invariants of $\boldsymbol{T}$) describes an incompressible and *hypoelastic* material (Truesdell & Noll [20]). However, even if $\boldsymbol{T}$ does not represent the Cauchy stress and coefficients γ_i (per $i = 1, \dots, 9$) remain completely arbitrary (but independent from $\boldsymbol{D}$), in a principal reference system of $\boldsymbol{T}$, assuming incremental plane strain, we obtain

$$\overset{\nabla}{\sigma}_{i3} = \overset{\nabla}{\sigma}_{3i} = 0 \tag{4.48}$$

for $i = 1, 2$, and

$$\begin{aligned}\overset{\nabla}{\sigma}_{33} = -\dot{q} + (T_1 - T_2)\big[\gamma_2 + \gamma_4 T_3 + \gamma_6 T_3^2 + \\ + (T_1 + T_2)(\gamma_3 + \gamma_5 T_3 + \gamma_7 T_3^2)\big] D_{11}\end{aligned} \tag{4.49}$$

where T_i (per $i = 1, 2, 3$) are the principal values of $\boldsymbol{T}$, whereas $\overset{\nabla}{\sigma}_{12}$, $\overset{\nabla}{\sigma}_{21}$ and $\overset{\nabla}{\sigma}_{11} - \overset{\nabla}{\sigma}_{22}$ are expressed by relation (4.24) with shear moduli taken equal to

$$\mu = \frac{1}{2}\big[\gamma_1 + \gamma_8(T_1 + T_2) + \gamma_9(T_1^2 + T_2^2)\big] \tag{4.50}$$

and

$$\begin{aligned}\mu_* = \frac{1}{2}\Big[\gamma_1 + (T_1 - T_2)^2\big[\gamma_4 + (T_1 + T_2)(\gamma_5 + \gamma_6) + (T_1 + T_2)^2\gamma_7\big] + \\ + (T_1 + T_2)\,\gamma_8 + \left(T_1^2 + T_2^2\right)\gamma_9\Big]\end{aligned} \tag{4.51}$$

J_2-deformation theory of plasticity: hyperelastic and hypoelastic approaches

Within the J_2-deformation theory of plasticity, the coefficients of (4.24) can be obtained from equation (3.243) in the form

$$\mu = \frac{1}{3} E_\mathrm{s} (\varepsilon_1 - \varepsilon_2) \coth(\varepsilon_1 - \varepsilon_2) \tag{4.52a}$$

$$\mu_* = \frac{1}{9} \frac{E_\mathrm{s}}{\varepsilon_\mathrm{e}^2} \left[3(\varepsilon_1 + \varepsilon_2)^2 + N(\varepsilon_1 - \varepsilon_2)^2 \right] \tag{4.52b}$$

or

$$\mu = \frac{2}{3} E_\mathrm{s}\, \varepsilon \coth(2\varepsilon) \tag{4.53a}$$

$$\mu_* = \frac{1}{3} E_\mathrm{s} N \tag{4.53b}$$

for plane strain. The out-of-plane stress increment is

$$\overset{\triangledown}{\sigma}_{33} = \dot{\hat{p}} \tag{4.54}$$

Differently from Hutchinson & Neale [17], Stören & Rice [28] presented the following hypoelastic law to describe elastoplastic materials subject to proportional loads:

$$\overset{\triangledown}{\boldsymbol{S}} = 2 h_1 \boldsymbol{D} - \frac{1-N}{N} \frac{\boldsymbol{S} \cdot \overset{\triangledown}{\boldsymbol{\sigma}}}{\boldsymbol{S} \cdot \boldsymbol{S}} \boldsymbol{S} \tag{4.55}$$

where $\boldsymbol{S} = \boldsymbol{\sigma} - \operatorname{tr} \boldsymbol{\sigma}/3$ is the deviatoric stress, N is the hardening parameter and h_1 is the modulus secant to the stress-strain curve. Equation (4.55) can be rewritten as

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\hat{p}} \boldsymbol{I} + 2 h_1 \left[\boldsymbol{D} - (1-N) \frac{\boldsymbol{S} \cdot \boldsymbol{D}}{\boldsymbol{S} \cdot \boldsymbol{S}} \boldsymbol{S} \right] \tag{4.56}$$

where $\dot{\hat{p}} = \operatorname{tr} \dot{\boldsymbol{\sigma}}/3$. Equation (4.56) is a particular case of equation (4.47) and can be solved through the system (4.24).

4.3.3 General form of the incremental constitutive equations for incompressible plane strain

It is convenient to rewrite the Biot constitutive equation (4.24) as a function of the material derivative of the nominal stress tensor $\boldsymbol{t}$. The incremental nominal stress tensor is defined as $\dot{\boldsymbol{t}} = \dot{\boldsymbol{S}}^{\mathrm{T}}$. Recalling the definition of the first Piola-Kirchhoff tensor ($\boldsymbol{S} = J\boldsymbol{\sigma}\boldsymbol{F}^{-\mathrm{T}}$), we obtain

$$\dot{\boldsymbol{S}} = \dot{J}\boldsymbol{\sigma}\boldsymbol{F}^{-\mathrm{T}} + J\dot{\boldsymbol{\sigma}}\boldsymbol{F}^{-\mathrm{T}} + J\boldsymbol{\sigma}\big(\boldsymbol{F}^{-\mathrm{T}}\big)^{\cdot} \tag{4.57}$$

where

$$\big(\boldsymbol{F}^{-\mathrm{T}}\big)^{\cdot} = -\boldsymbol{L}^{\mathrm{T}}\boldsymbol{F}^{-\mathrm{T}} \tag{4.58}$$

Since we consider incompressible incremental deformations (namely, with $J = 1$ and $\dot{J} = 0$) and we assume the current configuration as the reference configuration (meaning that $\boldsymbol{F} = \boldsymbol{I}$), the relation (4.57) becomes

$$\dot{\boldsymbol{S}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{L}^{\mathrm{T}} \tag{4.59}$$

and, introducing the Jaumann derivative,

$$\dot{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}} + \boldsymbol{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{W} \tag{4.60}$$

Therefore,

$$\dot{\boldsymbol{t}} = \big(\overset{\nabla}{\boldsymbol{\sigma}}\big)^{\mathrm{T}} - \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{W} - \boldsymbol{D}\boldsymbol{\sigma}^{\mathrm{T}} \tag{4.61}$$

where

$$\boldsymbol{\sigma} = \sigma_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i \qquad i = 1, 2, 3 \tag{4.62}$$

in the principal reference system. Substituting equation (4.24) into (4.61) one obtains

$$\dot{t}_{12} = (2\mu - \sigma_2)D_{12} - \sigma_1 W_{12} \tag{4.63a}$$
$$\dot{t}_{21} = (2\mu - \sigma_1)D_{21} - \sigma_2 W_{21} \tag{4.63b}$$
$$\dot{t}_{11} - \dot{t}_{22} = 2\mu_*(D_{11} - D_{22}) - D_{11}\sigma_1 + D_{22}\sigma_2 = (4\mu_* - \sigma_1 - \sigma_2)D_{11} \tag{4.63c}$$
$$\dot{t}_{i3} = \dot{t}_{3i} = 0 \tag{4.63d}$$
$$\dot{t}_{33} = \overset{\nabla}{\sigma}_3 = \dot{\sigma}_3 \tag{4.63e}$$

Furthermore, from equation (4.59) we have

$$\dot{t}_{11} + \dot{t}_{22} = \dot{\sigma}_1 + \dot{\sigma}_2 - \sigma_1 D_{11} - \sigma_2 D_{22} = \dot{\sigma}_1 + \dot{\sigma}_2 + (\sigma_2 - \sigma_1)D_{11} \tag{4.64}$$

so that

$$\dot{t}_{11} = (2\mu_* - \sigma_1)D_{11} + \frac{\dot{\sigma}_1 + \dot{\sigma}_2}{2} \tag{4.65a}$$

$$\dot{t}_{22} = (2\mu_* - \sigma_2)D_{22} + \frac{\dot{\sigma}_1 + \dot{\sigma}_2}{2} \tag{4.65b}$$

Since $\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W} = \mathrm{grad}\,\boldsymbol{v}$, the constitutive equations (4.24) become

$$\dot{t}_{ij} = \mathbb{K}_{ijkl}\, v_{l,k} + \dot{p}\delta_{ij} = \tilde{\mathbb{K}}_{ijkl}\, v_{l,k} + \dot{\pi}\delta_{ij} \tag{4.66a}$$

$$v_{i,i} = 0 \tag{4.66b}$$

where δ_{ij} is the Kronecker delta, and

$$\dot{p} = \frac{\dot{\sigma}_1 + \dot{\sigma}_2}{2} \tag{4.67a}$$

$$\dot{\pi} = \frac{\dot{t}_{11} + \dot{t}_{22}}{2} = \dot{p} - \frac{\sigma_1 - \sigma_2}{2} v_{1,1} \tag{4.67b}$$

are the in-plane hydrostatic stresses related, respectively, to the Cauchy stresses (*Cauchy in-plane hydrostatic stress*) and to nominal stresses (*nominal in-plane hydrostatic stress*), v_i are the velocity components and $\mathbb{K}_{ijkl}$ represents the instantaneous moduli; in particular, $\mathbb{K}_{ijkl}$ possesses the major symmetry

$$\mathbb{K}_{ijkl} = \mathbb{K}_{klij} \tag{4.68}$$

and the non null components of $\mathbb{K}$ are (Hill & Hutchinson [5])

$$\begin{aligned}
\mathbb{K}_{1111} &= \mu_* - \frac{\sigma}{2} - p & \qquad \mathbb{K}_{1122} &= \mathbb{K}_{2211} = -\mu_* \\
\mathbb{K}_{2222} &= \mu_* + \frac{\sigma}{2} - p & \qquad \mathbb{K}_{1221} &= \mathbb{K}_{2112} = \mu - p \\
\mathbb{K}_{1212} &= \mu + \frac{\sigma}{2} & \qquad \mathbb{K}_{2121} &= \mu - \frac{\sigma}{2}
\end{aligned} \tag{4.69}$$

where p and σ are defined as

$$p = \frac{\sigma_1 + \sigma_2}{2} \tag{4.70a}$$

$$\sigma = \sigma_1 - \sigma_2 \tag{4.70b}$$

which represent, respectively, the principal and the deviatoric part of the pre-stress. Note that the unique components of $\tilde{\mathbb{K}}_{ijkl}$ different from $\mathbb{K}_{ijkl}$ are

$$\tilde{\mathbb{K}}_{1111} = \tilde{\mathbb{K}}_{2222} = \mu_* - p \tag{4.71}$$

so that even $\tilde{\mathbb{K}}_{ijkl}$ possesses the major symmetry. Consequently, the components of the material derivative of the nominal stress written explicitly are

$$\dot{t}_{11} = \left(2\mu_* - \frac{\sigma}{2} - p\right) v_{1,1} + \dot{p} \tag{4.72a}$$

$$\dot{t}_{12} = (\mu - p) v_{1,2} + \left(\mu + \frac{\sigma}{2}\right) v_{2,1} \tag{4.72b}$$

$$\dot{t}_{21} = (\mu - p) v_{2,1} + \left(\mu - \frac{\sigma}{2}\right) v_{1,2} \tag{4.72c}$$

$$\dot{t}_{22} = \left(2\mu_* + \frac{\sigma}{2} - p\right) v_{2,2} + \dot{p} \tag{4.72d}$$

$$\dot{t}_{i3} = \dot{t}_{3i} = 0 \tag{4.72e}$$

$$\dot{t}_{33} = \dot{\sigma}_3 \tag{4.72f}$$

with $i = 1, 2$.

4.3.4 Incremental von Mises stress

The incremental von Mises stress s_{vm} is defined as

$$s_{\mathrm{vm}} = \sqrt{\frac{3}{2} \overset{\triangledown}{\boldsymbol{S}} \cdot \overset{\triangledown}{\boldsymbol{S}}} \tag{4.73}$$

where

$$\overset{\triangledown}{\boldsymbol{S}} = \overset{\triangledown}{\boldsymbol{\sigma}} - \frac{1}{3}\left(\operatorname{tr} \overset{\triangledown}{\boldsymbol{\sigma}}\right) \boldsymbol{I} \tag{4.74}$$

In fact

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{t}} + \boldsymbol{L}\boldsymbol{\sigma} \tag{4.75}$$

and

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{t}} + \boldsymbol{D}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{W} \tag{4.76}$$

in which the following incremental plane deformations are equal to zero

$$D_{i3} = D_{3i} = W_{i3} = W_{3i} = 0 \qquad i = 1, 2, 3 \tag{4.77}$$

while $\dot{\boldsymbol{t}}$ is given by equation (4.72). If we consider the case of plane strain, for which $\sigma_3 = 0$, the components of the incremental deviatoric stress in the principal system of $\boldsymbol{\sigma}$ are:

$$\overset{\nabla}{\sigma}_{11} = \frac{1}{3}\big(2\dot{t}_{11} + 2v_{1,1}\sigma_1 - \dot{t}_{22} - v_{2,2}\sigma_2 - \dot{t}_{33}\big) \tag{4.78a}$$

$$\overset{\nabla}{\sigma}_{12} = \dot{t}_{12} + \frac{v_{1,2} + v_{2,1}}{2}\sigma_2 + \frac{v_{1,2} - v_{2,1}}{2}\sigma_1 \tag{4.78b}$$

$$\overset{\nabla}{\sigma}_{21} = \dot{t}_{21} + \frac{v_{2,1} + v_{1,2}}{2}\sigma_1 + \frac{v_{2,1} - v_{1,2}}{2}\sigma_2 \tag{4.78c}$$

$$\overset{\nabla}{\sigma}_{22} = \frac{1}{3}\big(2\dot{t}_{22} + 2v_{2,2}\sigma_2 - \dot{t}_{11} - v_{1,1}\sigma_1 - \dot{t}_{33}\big) \tag{4.78d}$$

$$\overset{\nabla}{\sigma}_{33} = \frac{1}{3}\big(2\dot{\sigma}_3 - \dot{t}_{22} - \dot{t}_{11} + v_{2,2}\sigma_2\big) \tag{4.78e}$$

$$\overset{\nabla}{\sigma}_{3i} = \overset{\nabla}{\sigma}_{i3} = 0 \tag{4.78f}$$

for $i = 1, 2$.

4.4 Uniqueness of the solution in incremental boundary problems

Let us consider, ab absurdo, the existence of two solutions $\dot{\boldsymbol{\phi}}_1$ and $\dot{\boldsymbol{\phi}}_2$ of the incremental boundary problem described in Section 4.2.1. Let $\dot{\boldsymbol{S}}_1$ and $\dot{\boldsymbol{S}}_2$ be the corresponding stresses. If $\Delta(\cdot) = (\cdot)_1 - (\cdot)_2$ defines the difference between the fields, we have

$$\begin{cases} \operatorname{div}\big(\Delta\dot{\boldsymbol{S}}\big) = \boldsymbol{0} & \\ \Delta\dot{\boldsymbol{\phi}} = \boldsymbol{0} & \text{on } \partial\mathcal{B}_0^x \\ \big(\Delta\dot{\boldsymbol{S}}\big)\boldsymbol{n}_0 = \boldsymbol{0} & \text{on } \partial\mathcal{B}_0^s \end{cases} \tag{4.79}$$

in which the body forces are considered independent from $\dot{\boldsymbol{\phi}}$. On application of the divergence theorem (2.77), we obtain

$$\int_{\partial B_0} \big(\Delta\dot{\boldsymbol{S}}\big)^{\mathrm{T}}\boldsymbol{n}_0 \cdot \Delta\dot{\boldsymbol{\phi}}\,\mathrm{d}S_0 = \int_{B_0} \operatorname{div}\big(\Delta\dot{\boldsymbol{S}}\Delta\dot{\boldsymbol{\phi}}\big)\,\mathrm{d}V_0 = \boldsymbol{0} \tag{4.80}$$

and, using equation (2.76e), we have

$$\int_{B_0} (\Delta \dot{\boldsymbol{S}}) \cdot \mathrm{Grad}(\Delta \dot{\boldsymbol{\phi}})\, \mathrm{d}V_0 + \int_{B_0} \Delta \dot{\boldsymbol{\phi}} \cdot \mathrm{div}(\Delta \dot{\boldsymbol{S}})\, \mathrm{d}V_0 = \mathbf{0} \tag{4.81}$$

from which, being the second integral equal to zero, a sufficient condition proving the uniqueness of the incremental problem is

$$\int_{B_0} (\Delta \dot{\boldsymbol{S}}) \cdot (\Delta \dot{\boldsymbol{F}})\, \mathrm{d}V_0 > 0 \tag{4.82}$$

for every pairs of incremental displacement fields compatible with the fifth relation of the system (4.8).

Let us consider now a deformation path governed by a load parameter, which starts from a configuration in which the *uniqueness condition* (4.82) is verified, and let assume

$$\int_{B_0} (\Delta \dot{\boldsymbol{S}}) \cdot (\Delta \dot{\boldsymbol{F}})\, \mathrm{d}V_0 = 0 \tag{4.83}$$

for some $\dot{\boldsymbol{\phi}} \neq \mathbf{0}$. This means that we found a bifurcation point or a limit load point. $\dot{\boldsymbol{\phi}}$ is called *eigenmode*, while the critic configuration is a *primary eigenstate* and represents a *bifurcation point* on the deformation path.

Chapter 5

TWO-DIMENSIONAL GREEN'S FUNCTIONS

Within a plane strain Biot constitutive framework, the Green's functions set is derived for the incremental displacements and mean stress fields for the problem of a concentrated force acting on a point belonging to an infinite, incremental, non-linear elastic, medium in presence of prestress. In a first step, the load configuration is considered static, and then the problem is solved for the dynamic case.

THE DETERMINATION of Green's functions and related integral representations of elastic states are classical problems in the linear theory of elasticity. In this Chapter we extend this theory to the case of non-linear incompressible elasticity, and within this framework, we consider an infinite body subject to a plane and homogeneous deformation [1]. We assume that the material obeys the constitutive framework of Biot [2] through two incremental shear moduli, that are functions of the in-plane stretches. At a generic stage of the deformation path, when the material is still in the elliptic range, an incremental point load is acting at an arbitrary location in the body, remaining in a plane strain condition. A detailed analysis of the characteristic equation associated to the equilibrium equations is provided in order to determine the regime classification.

The problem is solved in terms of incremental displacement and mean stress fields, and the nominal stress rate distribution is obtained; therefore, we provide the Green's function set for an infinite, non-linear elastic, incompressible material. The singular solutions are treated through the plane-wave expansion method in the stream function formulation of Hill & Hutchinson [5] and have been obtained in a closed form. These solutions can be used to analyze the effect of a perturbation superimposed to a given homogeneous strain of an elastic medium; in particular, we can quantify the decay rate of self-equilibrated loads in a homogeneously stretched elastic solid. We can build many self-equilibrated load conditions by superimposing the solution for the concentrated force; the simplest system corresponds to the concentrated force dipole. We will show that, approaching the elliptic regime boundary, the incremental solution tends to self-organize along well defined shear band patterns.

The infinite-body Green's function is used to obtain a boundary integral formulation for the incremental displacement and mean stress fields, which constitute a rigorous approach to boundary element techniques in finite strain elasticity [29, 30]. Furthermore, we consider also the possibility that the concentrated load be dynamic; in particular we extend the above solutions to the case of dynamic, time-harmonic incremental loading [9, 10]. The Green's functions set is derived and generalized to this new condition and the solutions are decoupled in quasi-static and dynamic terms.

5.1 Constitutive equations

We consider an hyperelastic, incompressible, initially isotropic body subject to an homogeneous plane strain [1]. The most general constitutive equation is given by Biot [2] and can be expressed in the principal reference system of the Cauchy stress. Using the Lagrangian formulation of the field equations in the current configuration, the relation between the material derivative of the nominal stress t_{ij} with respect to time and the velocity gradient $v_{i,j}$ can be written as

$$\dot{t}_{ij} = \mathbb{K}_{ijkl}\, v_{l,k} + \dot{p}\delta_{ij} \tag{5.1}$$

where $\dot{p}$ is the *hydrostatic in-plane stress-rate* (or *in-plane mean stress*), δ_{ij} is the Kronecker delta, and $\mathbb{K}_{ijkl}$ are the instantaneous moduli. As we have seen in Section 4.3.3, we observe that $\mathbb{K}$ possesses the major symmetry and is a function of the principal components σ_1, σ_2 of the Cauchy stress and of the incremental shear moduli μ and μ_*, where μ is the shear modulus corresponding to a shear parallel to the principal stress axes, while μ_* the shear modulus corresponding to a shear inclined at $\pi/4$ with respect to these axes. The non null components of $\mathbb{K}$ are given by (4.69) while the incompressibility constraint becomes:

$$v_{i,i} = 0 \tag{5.2}$$

namely, the velocity field v_i is solenoidal. Equations (5.1) and (5.2) are general and include Mooney-Rivlin and Ogden materials, as well as elastoplastic materials following the J_2-deformation theory. In particular, for the Mooney-Rivlin material we have:

$$\sigma = \mu_0 \left(\lambda^2 - \lambda^{-2}\right) \tag{5.3a}$$

$$\mu_* = \mu = \frac{\mu_0}{2} \left(\lambda^2 + \lambda^{-2}\right) \tag{5.3b}$$

where μ_0 is the *ground-state shear modulus* and λ is the *maximum current stretch* with the constraint $\lambda > 1$. These two equations can be rewritten in a more general form that includes Ogden material:

$$\sigma = \sum_{i=1}^{N} \mu_i \left(\lambda^{\beta_i} - \lambda^{-\beta_i}\right) \tag{5.4a}$$

$$\mu_* = \frac{1}{4} \sum_{i=1}^{N} \mu_i \beta_i \left(\lambda^{\beta_i} + \lambda^{-\beta_i}\right) \tag{5.4b}$$

$$\mu = \frac{1}{2} \frac{\lambda^4 + 1}{\lambda^4 - 1} \sum_{i=1}^{N} \mu_i \left(\lambda^{\beta_i} - \lambda^{-\beta_i}\right) \tag{5.4c}$$

where μ_i and β_i are material parameters.

Equations (5.1) and (5.2) hold, in general, for anisotropic materials, which include the cases of initially orthotropic materials with respect to the principal directions x_1 and x_2 and the case of hyperelastic materials (that admit an elastic potential).

Equation (5.1) is written in the principal reference system of the Cauchy stress using a Lagrangian formulation (*current state*) and can be rewritten in the normal version (Bigoni *et al*. [10, 29]) as:

$$\dot{t}_{ij} = \tilde{\mathbb{K}}_{ijkl}\, v_{l,k} + \dot{\pi}\delta_{ij} \tag{5.5}$$

where π is the *nominal in-plane hydrostatic stress* defined by (4.67b), which can be rewritten as

$$\dot{\pi} = \dot{p} - \frac{\sigma}{2} v_{1,1} \tag{5.6}$$

5.2 Forces acting on the system

Two types of force systems can be considered, the dynamic and the static system; in both cases we consider incremental forces.

In the case of dynamic forces, we consider the incremental dynamic body forces $\dot{f}_i$, the mass density ρ, and the dependence of time00. For the *time harmonic* functions of time and space we have the following relations:

$$f(\boldsymbol{x}, t) = \hat{f}(\boldsymbol{x}) e^{-\mathrm{i}\,\Omega t} \tag{5.7a}$$

$$v_j(\boldsymbol{x}, t) = \hat{v}_j(\boldsymbol{x}) e^{-\mathrm{i}\,\Omega t} \tag{5.7b}$$

where Ω is the circular frequency. In the hypothesis of *time harmonic* motion, every field can be expressed in the form given by (5.7a) and (5.7b).

In the case of static forces, we consider an incremental force acting on the point $\boldsymbol{x} = \boldsymbol{0}$ and with components $\dot{f}_1$ and $\dot{f}_2$ along the principal stress axes; the force system is composed by a load line extending orthogonally to the deformation plane. The incremental concentrated force can be expressed as

$$\dot{\boldsymbol{f}} = \dot{f}_j \delta(\boldsymbol{x}) \tag{5.8}$$

where $\delta(\boldsymbol{x})$ is the two-dimensional Dirac delta and $\boldsymbol{x}$ describes the generic material point.

5.3 Incremental equilibrium equations

We refer to the case of incompressible, anisotropic and prestressed elasticity, for which the most general equilibrium equations are:

$$\dot{t}_{ij,i} + \dot{f}_j \delta(\boldsymbol{x}) = \rho v_{j,tt} \tag{5.9}$$

For *time harmonic* functions we have

$$v_{j,t} = \frac{\partial}{\partial t}\left[\widehat{v}_j(\boldsymbol{x})e^{-\mathrm{i}\,\Omega t}\right] = -\mathrm{i}\,\Omega\widehat{v}_j(\boldsymbol{x})e^{-\mathrm{i}\,\Omega t} \tag{5.10a}$$

$$v_{j,tt} = \frac{\partial}{\partial t}\left[-\mathrm{i}\,\Omega\widehat{v}_j(\boldsymbol{x})e^{-\mathrm{i}\,\Omega t}\right] = \mathrm{i}^2\,\Omega^2\,\underbrace{\widehat{v}_j(\boldsymbol{x})e^{-\mathrm{i}\,\Omega t}}_{=v_j(\boldsymbol{x},t)} = -\Omega^2 v_j \tag{5.10b}$$

so that equilibrium equations become:

$$\dot{t}_{ij,i} + \dot{f}_j = -\rho\Omega^2 v_j \tag{5.11}$$

Now we proceed with the evaluation of the nominal stress. The expressions of t_{ij} must be expanded using equation (5.1) and substituting the values of $\mathbb{K}$ defined by (4.69), and we obtain

$$v_{l,k} = \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} \tag{5.12}$$

and

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{1111} & \mathbb{K}_{1112} & \mathbb{K}_{1211} & \mathbb{K}_{1212} \\ \mathbb{K}_{1121} & \mathbb{K}_{1122} & \mathbb{K}_{1221} & \mathbb{K}_{1222} \\ \mathbb{K}_{2111} & \mathbb{K}_{2112} & \mathbb{K}_{2211} & \mathbb{K}_{2212} \\ \mathbb{K}_{2121} & \mathbb{K}_{2122} & \mathbb{K}_{2221} & \mathbb{K}_{2222} \end{bmatrix} \tag{5.13}$$

or, substituting the values of the components,

$$\mathbb{K} = \begin{bmatrix} \mu_* - \dfrac{\sigma}{2} - p & 0 & 0 & \mu + \dfrac{\sigma}{2} \\ 0 & -\mu_* & \mu - p & 0 \\ 0 & \mu - p & -\mu_* & 0 \\ \mu - \dfrac{\sigma}{2} & 0 & 0 & \mu_* + \dfrac{\sigma}{2} - p \end{bmatrix} \tag{5.14}$$

After some calculations we arrive at these four equations:

$$\dot{t}_{11} = \mathbb{K}_{1111}v_{1,1} + \mathbb{K}_{1112}v_{2,1} + \mathbb{K}_{1121}v_{1,2} + \mathbb{K}_{1122}v_{2,2} + \dot{p} \tag{5.15a}$$

$$\dot{t}_{12} = \mathbb{K}_{1211}v_{1,1} + \mathbb{K}_{1212}v_{2,1} + \mathbb{K}_{1221}v_{1,2} + \mathbb{K}_{1222}v_{2,2} \tag{5.15b}$$

$$\dot{t}_{21} = \mathbb{K}_{2111}v_{1,1} + \mathbb{K}_{2112}v_{2,1} + \mathbb{K}_{2121}v_{1,2} + \mathbb{K}_{2122}v_{2,2} \tag{5.15c}$$

$$\dot{t}_{22} = \mathbb{K}_{2211}v_{1,1} + \mathbb{K}_{2212}v_{2,1} + \mathbb{K}_{2221}v_{1,2} + \mathbb{K}_{2222}v_{2,2} + \dot{p} \tag{5.15d}$$

from which

$$\dot{t}_{11} = \left(\mu_* - \frac{\sigma}{2} - p\right) v_{1,1} - \mu_* v_{2,2} + \dot{p} \tag{5.16a}$$

$$\dot{t}_{12} = \left(\mu + \frac{\sigma}{2}\right) v_{2,1} + (\mu - p)\, v_{1,2} \tag{5.16b}$$

$$\dot{t}_{21} = (\mu - p)\, v_{2,1} + \left(\mu + \frac{\sigma}{2}\right) v_{1,2} \tag{5.16c}$$

$$\dot{t}_{22} = -\mu_* v_{1,1} + \left(\mu_* + \frac{\sigma}{2} - p\right) v_{2,2} + \dot{p} \tag{5.16d}$$

But from the incompressibility constraint (5.2), namely $v_{i,i} = 0$, we have

$$v_{2,2} = -v_{1,1} \tag{5.17a}$$

$$v_{1,1} = -v_{2,2} \tag{5.17b}$$

A substitution of the above relations into (5.16) yields:

$$\dot{t}_{11} = \left(\mu_* - \frac{\sigma}{2} - p\right) v_{1,1} + \mu_* v_{2,2} + \dot{p} = \left(2\mu_* - p - \frac{\sigma}{2}\right) v_{1,1} + \dot{p} \tag{5.18a}$$

$$\dot{t}_{22} = \mu_* v_{1,1} + \left(\mu_* + \frac{\sigma}{2} - p\right) v_{2,2} + \dot{p} = \left(2\mu_* - p + \frac{\sigma}{2}\right) v_{2,2} + \dot{p} \tag{5.18b}$$

It is possible to introduce dimensionless parameters of material anisotropy (ξ) and prestress (η, κ), defined as [1] :

$$\xi = \frac{\mu_*}{\mu} \tag{5.19a}$$

$$\eta = \frac{p}{\mu} = \frac{\sigma_1 + \sigma_2}{2\mu} \tag{5.19b}$$

$$\kappa = \frac{\sigma}{2\mu} = \frac{\sigma_1 - \sigma_2}{2\mu} \tag{5.19c}$$

[1] Sometimes the prestress parameter η is denoted with letter χ (for instance, see Bigoni & Capuani [1]).

with which equations (5.16) and (5.18) can be rewritten as

$$\dot{t}_{11} = \mu\,(2\xi - \kappa - \eta)\,v_{1,1} + \dot{p} \tag{5.20a}$$
$$\dot{t}_{12} = \mu\,[(1+\kappa)\,v_{2,1} + (1-\eta)\,v_{1,2}] \tag{5.20b}$$
$$\dot{t}_{21} = \mu\,[(1-\eta)\,v_{2,1} + (1-\kappa)\,v_{1,2}] \tag{5.20c}$$
$$\dot{t}_{22} = \mu\,(2\xi + \kappa - \eta)\,v_{2,2} + \dot{p} \tag{5.20d}$$

The equilibrium equations require the calculation of $\dot{t}_{ij,i}$, that yields:

$$\dot{t}_{ij,i} = \begin{bmatrix} \dot{t}_{11,1} + \dot{t}_{21,2} \\ \dot{t}_{12,1} + \dot{t}_{22,2} \end{bmatrix} \tag{5.21}$$

so that

$$\dot{t}_{i1,i} = (2\mu_* - p)v_{1,11} - \frac{\sigma}{2}v_{1,11} + (\mu - p)v_{2,12} + \left(\mu - \frac{\sigma}{2}\right)v_{1,22} + \dot{p}_{,1} \tag{5.22a}$$
$$\dot{t}_{i2,i} = (2\mu_* - p)v_{2,22} + \frac{\sigma}{2}v_{2,22} + (\mu - p)v_{1,12} + \left(\mu + \frac{\sigma}{2}\right)v_{2,11} + \dot{p}_{,2} \tag{5.22b}$$

but, due to incompressibility constraint ($v_{2,2} = -v_{1,1}$) and taking into account equation (5.6), we have the following relations:

$$\dot{p}_{,1} - \frac{\sigma}{2}v_{1,11} = \left(\dot{p} - \frac{\sigma_1 - \sigma_2}{2}v_{1,1}\right)_{,1} = \dot{\pi}_{,1} \tag{5.23a}$$
$$\dot{p}_{,2} + \frac{\sigma}{2}v_{2,22} = \left(\dot{p} - \frac{\sigma_1 - \sigma_2}{2}v_{1,1}\right)_{,2} = \dot{\pi}_{,2} \tag{5.23b}$$

The general equilibrium equations become:

$$(2\mu_* - p)v_{1,11} + (\mu - p)v_{2,12} + \left(\mu - \frac{\sigma}{2}\right)v_{1,22} + \dot{f}_1\delta(\boldsymbol{x}) = -\dot{\pi}_{,1} + \\ -\rho\Omega^2 v_1 \tag{5.24a}$$
$$(2\mu_* - p)v_{2,22} + (\mu - p)v_{1,12} + \left(\mu + \frac{\sigma}{2}\right)v_{2,11} + \dot{f}_2\delta(\boldsymbol{x}) = -\dot{\pi}_{,1} + \\ -\rho\Omega^2 v_2 \tag{5.24b}$$

and now, due to the *time-harmonic* assumption, these are function only of the spatial variable. From the above relations we can obtain two further equations. If we differentiate equation (5.24a) with respect to x_2 and (5.24b) with respect to x_1:

$$(2\mu_* - p)v_{1,112} + (\mu - p)v_{2,212} + \left(\mu - \frac{\sigma}{2}\right) v_{1,222} + \dot{f}_1\delta_{,2}(\boldsymbol{x}) + \\ + \dot{\pi}_{,12} + \rho\Omega^2 v_{1,2} = 0 \tag{5.25a}$$

$$(2\mu_* - p)v_{2,212} + (\mu - p)v_{1,112} + \left(\mu + \frac{\sigma}{2}\right) v_{2,111} + \dot{f}_2\delta_{,1}(\boldsymbol{x}) + \\ + \dot{\pi}_{,12} + \rho\Omega^2 v_{2,1} = 0 \tag{5.25b}$$

but

$$v_{2,212} = (v_{2,2})_{,12} = (-v_{1,1})_{,12} = -v_{1,112} \tag{5.26}$$

so that

$$(2\mu_* - p)v_{1,112} - (\mu - p)v_{1,112} + \left(\mu - \frac{\sigma}{2}\right) v_{1,222} + \dot{f}_1\delta_{,2}(\boldsymbol{x}) + \\ + \dot{\pi}_{,12} + \rho\Omega^2 v_{1,2} = 0 \tag{5.27a}$$

$$-(2\mu_* - p)v_{1,112} + (\mu - p)v_{1,112} + \left(\mu + \frac{\sigma}{2}\right) v_{2,111} + \dot{f}_2\delta_{,1}(\boldsymbol{x}) + \\ + \dot{\pi}_{,12} + \rho\Omega^2 v_{2,1} = 0 \tag{5.27b}$$

and a simplification yields

$$(2\mu_* - \mu)v_{1,112} + \left(\mu - \frac{\sigma}{2}\right) v_{1,222} + \dot{f}_1\delta_{,2}(\boldsymbol{x}) + \dot{\pi}_{,12} + \rho\Omega^2 v_{1,2} = 0 \tag{5.28a}$$

$$-(2\mu_* - \mu)v_{1,112} + \left(\mu + \frac{\sigma}{2}\right) v_{2,111} + \dot{f}_2\delta_{,1}(\boldsymbol{x}) + \dot{\pi}_{,12} + \rho\Omega^2 v_{2,1} = 0 \tag{5.28b}$$

If we subtract equation (5.28b) to (5.28a) we obtain:

$$-\left(\mu + \frac{\sigma}{2}\right) v_{2,111} + 2(2\mu_* - \mu)v_{1,112} + \left(\mu - \frac{\sigma}{2}\right) v_{1,222} + \\ + \dot{f}_1\delta_{,2}(\boldsymbol{x}) - \dot{f}_2\delta_{,1}(\boldsymbol{x}) = \rho\Omega^2 \left(v_{2,1} - v_{1,2}\right) \tag{5.29}$$

If we differentiate equation (5.24a) with respect to x_1 and (5.24b) with respect to x_2:

$$(2\mu_* - p)v_{1,111} + (\mu - p)v_{2,211} + \left(\mu - \frac{\sigma}{2}\right) v_{1,122} + \dot{f}_1\delta_{,1}(\boldsymbol{x}) + \\ + \dot{\pi}_{,11} + \rho\Omega^2 v_{1,1} = 0 \tag{5.30a}$$

$$(2\mu_* - p)v_{2,222} + (\mu - p)v_{1,122} + \left(\mu + \frac{\sigma}{2}\right) v_{2,211} + \dot{f}_2\delta_{,2}(\boldsymbol{x}) + \\ + \dot{\pi}_{,22} + \rho\Omega^2 v_{2,2} = 0 \tag{5.30b}$$

but

$$v_{2,211} = \left(v_{2,2}\right)_{,11} = \left(-v_{1,1}\right)_{,11} = -v_{1,111} \tag{5.31a}$$

$$v_{1,122} = \left(v_{1,1}\right)_{,22} = \left(-v_{2,2}\right)_{,22} = -v_{2,222} \tag{5.31b}$$

so that

$$(2\mu_* - p)v_{1,111} - (\mu - p)v_{1,111} - \left(\mu - \frac{\sigma}{2}\right) v_{2,222} + \dot{f}_1\delta_{,1}(\boldsymbol{x}) + \\ + \dot{\pi}_{,11} + \rho\Omega^2 v_{1,1} = 0 \tag{5.32a}$$

$$(2\mu_* - p)v_{2,222} - (\mu - p)v_{2,222} - \left(\mu + \frac{\sigma}{2}\right) v_{1,111} + \dot{f}_2\delta_{,2}(\boldsymbol{x}) + \\ + \dot{\pi}_{,22} - \rho\Omega^2 v_{2,2} = 0 \tag{5.32b}$$

and a simplification yields

$$(2\mu_* - \mu)v_{1,111} - \left(\mu - \frac{\sigma}{2}\right) v_{2,222} + \dot{f}_1\delta_{,1}(\boldsymbol{x}) + \dot{\pi}_{,11} + \rho\Omega^2 v_{1,1} = 0 \tag{5.33a}$$

$$(2\mu_* - \mu)v_{2,222} - \left(\mu + \frac{\sigma}{2}\right) v_{1,111} + \dot{f}_2\delta_{,2}(\boldsymbol{x}) + \dot{\pi}_{,22} - \rho\Omega^2 v_{2,2} = 0 \tag{5.33b}$$

A summation of equations (5.33a) and (5.33b) yields:

$$\left(2\mu_* - 2\mu - \frac{\sigma}{2}\right) v_{1,111} + \left(2\mu_* - 2\mu + \frac{\sigma}{2}\right) v_{2,222} + \dot{f}_1\delta_{,1}(\boldsymbol{x}) + \dot{f}_2\delta_{,2}(\boldsymbol{x}) + \\ + \dot{\pi}_{,11} + \dot{\pi}_{,22} = 0 \tag{5.34}$$

and if we isolate the *incremental in-plane hydrostatic nominal stress*:

$$\dot{\pi}_{,11} + \dot{\pi}_{,22} = -2(\mu_* - \mu)\left(v_{1,111} + v_{2,222}\right) + \frac{\sigma}{2}\left(v_{1,111} - v_{2,222}\right) + \\ - \dot{f}_1 \delta_{,1}(\boldsymbol{x}) - \dot{f}_2 \delta_{,2}(\boldsymbol{x}) \quad (5.35)$$

We introduce now the *stream function* $\psi(x_1, x_2)$ that defines a solenoidal, but arbitrary, velocity field such that

$$v_1 = \psi_{,2} \quad (5.36a)$$

$$v_2 = -\psi_{,1} \quad (5.36b)$$

and we rewrite equations (5.29) and (5.35), thus obtaining the following equilibrium equation system:

$$\begin{cases} \left(\mu + \dfrac{\sigma}{2}\right)\psi_{,1111} + 2(2\mu_* - \mu)\psi_{,1122} + \left(\mu - \dfrac{\sigma}{2}\right)\psi_{,2222} = \dot{f}_2\delta_{,1}(\boldsymbol{x}) + \\ \qquad - \dot{f}_1\delta_{,2}(\boldsymbol{x}) - \rho\Omega^2\left(\psi_{,11} - \psi_{,22}\right) \\ \dot{\pi}_{,11} + \dot{\pi}_{,22} = -2(\mu_* - \mu)\left(\psi_{,2111} - \psi_{,1222}\right) + \dfrac{\sigma}{2}(\psi_{,2111} + \psi_{,1222}) + \\ \qquad - \dot{f}_1\delta_{,1}(\boldsymbol{x}) - \dot{f}_2\delta_{,2}(\boldsymbol{x}) \end{cases} \quad (5.37)$$

Note that if $\sigma = 0$ and $\mu_* = \mu$ the first equation of (5.37) becomes:

$$\mu\psi_{,1111} + 2\mu\psi_{,1122} + \mu\psi_{,2222} = \dot{f}_2\delta_{,1}(\boldsymbol{x}) - \dot{f}_1\delta_{,2}(\boldsymbol{x}) - \rho\Omega^2\left(\psi_{,11} - \psi_{,22}\right) \quad (5.38)$$

that is formally identical to the Navier-Stokes equation for incompressible plane viscous flow.

5.4 Standard regime classification

The first equation of (5.37) yields the regime classification, which is based on its characteristic associated equation [9]. The homogeneous associated equation of (5.37) is:

$$\left(\mu + \frac{\sigma}{2}\right)\psi_{,1111} + 2(2\mu_* - \mu)\psi_{,1122} + \left(\mu - \frac{\sigma}{2}\right)\psi_{,2222} = 0 \quad (5.39)$$

and, dividing by $\mu\psi_{,2222}$,

$$\mu\psi_{,2222}\left[\left(1+\frac{\sigma}{2\mu}\right)\frac{\psi_{,1111}}{\psi_{,2222}}+2\left(\frac{2\mu_*}{\mu}-1\right)\frac{\psi_{,1122}}{\psi_{,2222}}+\left(1-\frac{\sigma}{2\mu}\right)\right]=0 \tag{5.40}$$

We introduce the prestress parameter[2]

$$\frac{\sigma}{2\mu}=\kappa \tag{5.41}$$

and we assume, without any loss of generality, $\kappa \geq 0$ since it is sufficient to choose the axes 1 and 2 in appropriate manner.

Assuming that the solution has the following structure:

$$\psi(x_1, x_2) = Ae^{\boldsymbol{\omega}\cdot\boldsymbol{x}} \tag{5.42}$$

where $A \in \mathbb{R}$, $\boldsymbol{\omega} \in \mathbb{C}^2$, and $\boldsymbol{x} \in \mathbb{R}^2$, we can calculate the derivatives of this solution:

$$\psi_{,1} = \omega_1 Ae^{\boldsymbol{\omega}\cdot\boldsymbol{x}} \tag{5.43a}$$

$$\psi_{,2} = \omega_2 Ae^{\boldsymbol{\omega}\cdot\boldsymbol{x}} \tag{5.43b}$$

so that the fourth order derivatives can be expressed as:

$$\psi_{,1111} = \omega_1^4 Ae^{\boldsymbol{\omega}\cdot\boldsymbol{x}} \tag{5.44a}$$

$$\psi_{,2222} = \omega_2^4 Ae^{\boldsymbol{\omega}\cdot\boldsymbol{x}} \tag{5.44b}$$

$$\psi_{,1122} = \omega_1^2\omega_2^2 Ae^{\boldsymbol{\omega}\cdot\boldsymbol{x}} \tag{5.44c}$$

and if we substitute these relations in (5.40), we obtain:

$$\mu\omega_2^4\left[(1+\kappa)\frac{\omega_1^4}{\omega_2^4}+2\left(\frac{2\mu_*}{\mu}-1\right)\frac{\omega_1^2}{\omega_2^2}+(1-\kappa)\right]=0 \tag{5.45}$$

[2] See equation (5.19c). Furthermore, the following relation may apply:

$$\kappa = \frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2+\lambda_2^2}$$

where λ_1,λ_2 are the *in-plane prestretches*; however, this expression holds only if an elastic potential exists.

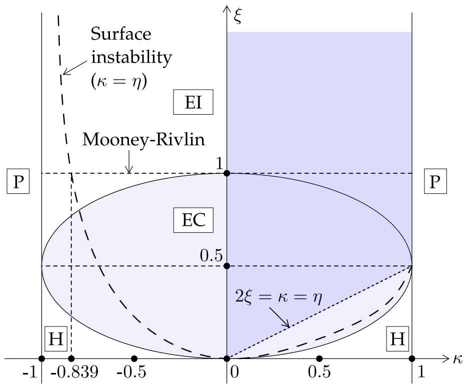


Figure 5.1: Regime classification in the plane of parameters $\kappa = (T_1 - T_2)/(2\mu)$ and $\xi = \mu_*/\mu$.

which is the most suitable formulation of the characteristic equation associated to (5.37) that can be used for the regime classification. We consider (5.45) as a quadratic equation with the unknown ω_1^2/ω_2^2 so that can be solved as follows:

$$\left(\frac{\omega_1^2}{\omega_2^2}\right)_{1/2} = \frac{1 - \frac{2\mu_*}{\mu}}{1+\kappa} \pm \frac{1}{1+\kappa}\sqrt{\left(\frac{2\mu_*}{\mu} - 1\right)^2 - (1+\kappa)(1-\kappa)} \tag{5.46}$$

but the root of the discriminant is

$$\sqrt{\Delta} = \sqrt{4\frac{\mu_*^2}{\mu^2} - 4\frac{\mu_*}{\mu} + 1 - (1-\kappa^2)} = \sqrt{\kappa^2 - 4\frac{\mu_*}{\mu} + 4\frac{\mu_*^2}{\mu^2}} \tag{5.47}$$

so that the square of the solutions can be expressed as

$$\left(\frac{\omega_1^2}{\omega_2^2}\right)_{1/2} = \frac{1 - \frac{2\mu_*}{\mu}}{1+\kappa} \pm \frac{1}{1+\kappa}\sqrt{\kappa^2 - 4\frac{\mu_*}{\mu} + 4\frac{\mu_*^2}{\mu^2}} \tag{5.48}$$

and from these equations we can obtain the ratios ω_1/ω_2 by calculating the roots $\pm\sqrt{\omega_1^2/\omega_2^2}$. It is suitable to write the above solutions in the following form:

$$\gamma_1 = \frac{1 - \frac{2\mu_*}{\mu}}{1+\kappa} - \frac{\sqrt{\Delta}}{1+\kappa} \tag{5.49a}$$

$$\gamma_2 = \frac{1 - \frac{2\mu_*}{\mu}}{1+\kappa} + \frac{\sqrt{\Delta}}{1+\kappa} \tag{5.49b}$$

in order to rewrite equation (5.45), by exploiting the fundamental algebra theorem, as the product:

$$\mu\omega_2^4(1+\kappa)\left(\frac{\omega_1^2}{\omega_2^2} - \gamma_1\right)\left(\frac{\omega_1^2}{\omega_2^2} - \gamma_2\right) = 0 \tag{5.50}$$

Furthermore, from this equation, we define the operator $L(\boldsymbol{\omega})$ as:

$$L(\boldsymbol{\omega}) = \mu\omega_2^4(1+\kappa)\left(\frac{\omega_1^2}{\omega_2^2} - \gamma_1\right)\left(\frac{\omega_1^2}{\omega_2^2} - \gamma_2\right) \tag{5.51}$$

that will be used in the following, in order to represent in a easy and compact way the first member of the characteristic equation (5.45).

From the resolution method described above, we deduce that the possible solutions of (5.45) are:

- 4 real solutions within the hyperbolic regime (H), namely $\gamma_1, \gamma_2 \in \mathbb{R}^+$ and $\Delta > 0$;
- 2 real solutions within the parabolic regime (P), namely $\gamma_1 = \gamma_2 \in \mathbb{R}^+$ and $\Delta = 0$;
- no real solutions within the elliptic regime (E), namely when $\kappa < 1$.

The elliptic regime (E) can be subdivided into:

- elliptic complex regime (EC), in which we have 2 conjugated pairs of complex solutions ω_i; from equation (5.50) we deduce that γ_1, γ_2 are conjugated pairs and $\Delta < 0$;
- elliptic imaginary regime (EI), in which we have 4 pure imaginary conjugated solutions ω_i; from equation (5.50) we deduce that $\gamma_1, \gamma_2 \in \mathbb{R}^-$ are conjugated pairs and $\Delta > 0$.

There are two ways to exit from the elliptic regime (E):

1. to cross the (EI)/(P) threshold, namely to pass from $\Delta > 0$ to $\Delta = 0$ and from $\kappa < 1$ to $\kappa > 1$; if $\kappa = 1$ we have:

$$\sqrt{\Delta} = \sqrt{1 - 4\frac{\mu_*}{\mu} + 4\frac{\mu_*^2}{\mu^2}} = \sqrt{\left(1 - \frac{2\mu_*}{\mu}\right)^2} = 1 - \frac{2\mu_*}{\mu} \tag{5.52}$$

so that

$$\gamma_1 = (1+\kappa)^{-1}\left[1 - \frac{2\mu_*}{\mu} - \left(1 - \frac{2\mu_*}{\mu}\right)\right] = 0 \tag{5.53a}$$

$$\gamma_2 = \frac{2}{1+\kappa}\frac{2\mu_*}{\mu} \tag{5.53b}$$

therefore, if we reach the (EI)/(P) threshold starting from (E), then $\Delta > 0$ e $\mu < 2\mu_*$, namely:

$$\frac{2\mu_*}{\mu} > 1 \tag{5.54}$$

2. to cross the (EC)/(H) threshold, namely to pass from $\Delta < 0$ to $\Delta > 0$ so that $\Delta = 0$; obviously we have:

$$\gamma_1 = \gamma_2 = \frac{1}{1+\kappa}\frac{2\mu_*}{\mu} \tag{5.55}$$

if we reach the (EC)/(H) starting from (E), then $\Delta < 0$ e $\mu > 2\mu_*$, namely:

$$\frac{2\mu_*}{\mu} < 1 \tag{5.56}$$

Now we can do some observations on the regime classification; for instance, for the Mooney-Rivlin material we have $\mu = \mu_*$, so that

$$\gamma_{1/2} = -\frac{1}{1+\kappa} \pm \frac{1}{1+\kappa}\sqrt{\kappa^2 - 4 + 4} = \frac{-1 \pm \kappa}{1+\kappa} \tag{5.57}$$

from which

$$\gamma_1 = -1 \tag{5.58a}$$

$$\gamma_2 = \frac{\kappa - 1}{1 + \kappa} \tag{5.58b}$$

Incompressible elastic materials, initially isotropic and subject to plane strain deformations, can not enter in the parabolic regime (P), so that the threshold (EI)/(P) can be reached only at the limit of infinite stretches. In this text we assume to remain within the elliptic regime.

5.5 Velocity field for the static problem

We focus on the resolution of the static problem, so that the equilibrium equation we are considering becomes:

$$\begin{aligned} \left(\mu + \frac{\sigma}{2}\right) \psi_{,1111} + 2(2\mu_* - \mu)\psi_{,1122} + \left(\mu - \frac{\sigma}{2}\right) \psi_{,2222} = \\ = \dot{f}_2 \delta_{,1}(\boldsymbol{x}) - \dot{f}_1 \delta_{,2}(\boldsymbol{x}) \end{aligned} \tag{5.59}$$

which is obtained from the first of equation (5.37) by taking the dynamic term

$$-\rho\Omega^2 \left(\psi_{,11} - \psi_{,22}\right) = -\rho\Omega^2 \nabla^2 \psi \tag{5.60}$$

to be zero in the right-hand side. Since the problem is linear, the solution for a concentrated force can be obtained from superposition of the solutions for two forces, one acting on the x_1 axis and one acting on the x_2 axis. We consider here a force $\dot{f}$ with unit components such that $\dot{f}_i = \delta_{ig}$.

Introducing the linear differential operator $\mathcal{L}$ with constant coefficients:

$$\mathcal{L}[\ ] = \left(\mu + \frac{\sigma}{2}\right) \frac{\partial^4[\ ]}{\partial x_1^4} + 2(2\mu_* - \mu) \frac{\partial^4[\ ]}{\partial x_1^2 \partial x_2^2} + \left(\mu - \frac{\sigma}{2}\right) \frac{\partial^4[\ ]}{\partial x_2^4} \tag{5.61}$$

or, taking into account the dimensionless prestress parameter κ,

$$\mathcal{L}[\ ] = \mu(1+\kappa) \frac{\partial^4[\ ]}{\partial x_1^4} + 2\mu \left(\frac{2\mu_*}{\mu} - 1\right) \frac{\partial^4[\ ]}{\partial x_1^2 \partial x_2^2} + \mu(1-\kappa) \frac{\partial^4[\ ]}{\partial x_2^4} \tag{5.62}$$

we can write equation (5.59) as

$$\mathcal{L}\psi^g + \left(\delta_{1g}\frac{\partial[\ ]}{\partial x_2} - \delta_{2g}\frac{\partial[\ ]}{\partial x_1}\right)\delta(\boldsymbol{x}) = 0 \tag{5.63}$$

The Green's function is the general solution of the equilibrium equation for the concentrated force. For the generic function $h(\boldsymbol{x})$ the plane wave expansion yields:

$$h(\boldsymbol{x}) = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\tilde{h}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\omega \tag{5.64}$$

In the transformed domain we have

$$\tilde{h}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} \tag{5.65}$$

which is a derivative with respect to $(\boldsymbol{\omega}\cdot\boldsymbol{x})$, so that

$$\delta(\boldsymbol{x}) = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\tilde{\delta}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\omega = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\frac{\mathrm{d}\omega}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} \tag{5.66}$$

and in the above relation we must calculate $\delta_{,1}$ and $\delta_{,2}$:

$$\tilde{\delta}_{,1} = \frac{\partial}{\partial x_1}\tilde{\delta}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{\partial}{\partial x_1}\left[\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2}\right] = \frac{-2(\boldsymbol{\omega}\cdot\boldsymbol{x})}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^4}\omega_1 = \frac{-2\omega_1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.67a}$$

$$\tilde{\delta}_{,2} = \frac{\partial}{\partial x_2}\tilde{\delta}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{\partial}{\partial x_2}\left[\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2}\right] = \frac{-2(\boldsymbol{\omega}\cdot\boldsymbol{x})}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^4}\omega_2 = \frac{-2\omega_2}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.67b}$$

Using plane wave expansion, equation (5.59) becomes:

$$\mathcal{L}\tilde{\psi}^g(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.68}$$

where:

$$\psi^g(\boldsymbol{x}) = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\tilde{\psi}^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\omega \tag{5.69}$$

Taking into account the chain rule for differentiation:

$$\left[\tilde{\psi}^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\right]_{,k} = \tilde{\psi}^g_{,(\boldsymbol{\omega}\cdot\boldsymbol{x})}(\boldsymbol{\omega}\cdot\boldsymbol{x})_{,k} = \tilde{\psi}^g_{,(\boldsymbol{\omega}\cdot\boldsymbol{x})}\omega_{,k} = \omega_{,k}\left(\tilde{\psi}^g\right)' \tag{5.70}$$

where $\tilde{\psi}^g_{,(\boldsymbol{\omega}\cdot\boldsymbol{x})} = \left(\tilde{\psi}^g\right)'$. Now equation (5.59) can be rewritten as:

$$L(\boldsymbol{\omega})\left(\tilde{\psi}^g\right)'''' = \frac{2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.71}$$

where the operator $L(\boldsymbol{\omega})$ is given by (5.51) and, if we remain within the elliptic regime, we have:

$$L(\boldsymbol{\omega}) = \mu\omega_2^4(1+\kappa)\left(\frac{\omega_1^2}{\omega_2^2} - \gamma_1\right)\left(\frac{\omega_1^2}{\omega_2^2} - \gamma_2\right) > 0 \qquad \text{in (E)} \tag{5.72}$$

Equation (5.71) can be directly integrated in order to calculate $\tilde{\psi}^g$:

$$\left(\tilde{\psi}^g\right)'''' = \frac{2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.73}$$

but, taking into account that:

$$\int \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = -\frac{1}{2}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} \tag{5.74a}$$

$$-\frac{1}{2}\int \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{1}{2}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.74b}$$

$$\frac{1}{2}\int \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{1}{2}\ln|\boldsymbol{\omega}\cdot\boldsymbol{x}| - \frac{1}{2}\ln|\boldsymbol{\omega}\cdot\boldsymbol{x}_0| = \frac{1}{2}\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| \tag{5.74c}$$

$$\frac{1}{2}\int \ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}|\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{\omega}\cdot\boldsymbol{x})\big[\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| - 1\big] \tag{5.74d}$$

and neglecting the integration constants, and introducing a dimensionless $\hat{\boldsymbol{x}}$, namely:

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{x}_0|} \tag{5.75}$$

the solution of equation (5.71) is

$$\tilde{\psi}^g = \frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})}(\boldsymbol{\omega}\cdot\boldsymbol{x})\big[\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| - 1\big] \tag{5.76}$$

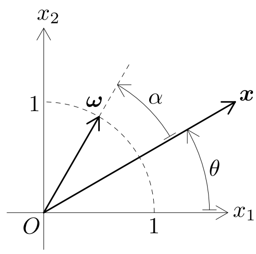


Figure 5.2: Reference system, vectors $\boldsymbol{\omega}$, $\boldsymbol{x}$ and angles θ and α.

The antitransform of (5.76) yields the Green's *stream function* ψ^g

$$\psi^g = -\frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} \frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})} (\boldsymbol{\omega} \cdot \boldsymbol{x}) \big[\ln|\boldsymbol{\omega} \cdot \hat{\boldsymbol{x}}| - 1 \big] \, \mathrm{d}\omega \tag{5.77}$$

Taking into account the reference system illustrated in Figure 5.2, we want to see in detail the terms involved in equation (5.77). The vectors $\boldsymbol{\omega}$, $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ have, respectively, the following components:

$$\boldsymbol{\omega} = \{ \cos(\alpha + \theta) \, , \sin(\alpha + \theta) \, \} \tag{5.78a}$$

$$\boldsymbol{x} = \{ \, r \cos\theta \, , r \sin\theta \, \} \tag{5.78b}$$

$$\hat{\boldsymbol{x}} = \{ \, \cos\theta \, , \sin\theta \, \} \tag{5.78c}$$

Now the terms involved in (5.77) can be expressed as

$$\begin{aligned} \boldsymbol{\omega} \cdot \boldsymbol{x} &= \cos(\alpha + \theta) r \cos\theta + \sin(\alpha + \theta) r \sin\theta = \\ &= (\cos\alpha \cos\theta - \sin\alpha \sin\theta) r \cos\theta + (\sin\alpha \cos\theta - \cos\alpha \sin\theta) r \sin\theta = \\ &= r \cos\alpha \cos^2\theta + r \cos\alpha \sin^2\theta = r \cos\alpha \end{aligned} \tag{5.79}$$

and, analogously,

$$\boldsymbol{\omega} \cdot \hat{\boldsymbol{x}} = \cos\alpha \tag{5.80}$$

and also

$$\begin{aligned} L(\boldsymbol{\omega}) &= \mu\omega_2^4(1+\kappa)\left(\frac{\omega_1^2}{\omega_2^2}-\gamma_1\right)\left(\frac{\omega_1^2}{\omega_2^2}-\gamma_2\right) = \\ &= \mu(1+\kappa)\sin^4(\alpha+\theta)\left[\cot^2(\alpha+\theta)-\gamma_1\right]\left[\cot^2(\alpha+\theta)-\gamma_2\right] = \\ &= \mu(1+\kappa)\Lambda(\alpha+\theta) \end{aligned} \tag{5.81}$$

where:

$$\Lambda(\alpha+\theta) = \sin^4(\alpha+\theta)\left[\cot^2(\alpha+\theta)-\gamma_1\right]\left[\cot^2(\alpha+\theta)-\gamma_2\right] \tag{5.82}$$

Furthermore,

$$\begin{aligned} \delta_{1g}\omega_2 - \delta_{2g}\omega_1 &= \delta_{1g}\sin(\alpha+\theta) - \delta_{2g}\cos(\alpha+\theta) = \\ &= \sin\left[\alpha+\theta+(1-g)\frac{\pi}{2}\right] \end{aligned} \tag{5.83}$$

being

$$\sin\left(\gamma-\frac{\pi}{2}\right) = \cos\gamma \tag{5.84}$$

If we assume that

$$\oint_{|\boldsymbol{\omega}|=1} \mathrm{d}\omega = \int_0^{2\pi} \mathrm{d}\alpha \tag{5.85}$$

equation (5.77) can be rewritten as:

$$\psi^g = -\frac{1}{4\pi^2}\int_0^{2\pi} \frac{\sin\left[\alpha+\theta+(1-g)\frac{\pi}{2}\right] r\cos\alpha\left[\ln|\cos\alpha|-1\right]}{\mu(1+\kappa)\Lambda(\alpha+\theta)}\,\mathrm{d}\alpha \tag{5.86}$$

namely

$$\begin{aligned} \psi^g = &-\frac{r}{4\pi^2\mu(1+\kappa)} \\ &\times\int_0^{2\pi} \frac{\cos\alpha\sin\left[\alpha+\theta+(1-g)\frac{\pi}{2}\right]\left[\ln|\cos\alpha|-1\right]}{\Lambda(\alpha+\theta)}\,\mathrm{d}\alpha \end{aligned} \tag{5.87}$$

The Green tensor for the infinite body represents the velocity field associated to the *stream function* (5.77); if we define

$$v_1 = \psi_{,2} \tag{5.88a}$$

$$v_2 = -\psi_{,1} \tag{5.88b}$$

we have

$$v_1^g = \psi_{,2}^g \tag{5.89a}$$

$$v_2^g = -\psi_{,1}^g \tag{5.89b}$$

Inserting the last two equations in (5.77), we can observe that $v_1^2 = v_2^1$, but this is simply a consequence of the major symmetry of $\mathbb{K}_{ijkl}$.

We show now in detail how the velocity can be calculated:

$$v_1^g = \psi_{,2}^g = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\left[\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})}(\boldsymbol{\omega}\cdot\boldsymbol{x})\big[\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| - 1\big]\right]_{,2}\mathrm{d}\omega \tag{5.90}$$

namely

$$\begin{aligned} v_1^g = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})}\Big[(\boldsymbol{\omega}\cdot\boldsymbol{x})_{,2}\big[\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| - 1\big] + \\ + (\boldsymbol{\omega}\cdot\boldsymbol{x})\big[\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| - 1\big]_{,2}\Big]\mathrm{d}\omega \end{aligned} \tag{5.91}$$

but it is clear that

$$(\boldsymbol{\omega}\cdot\boldsymbol{x})_{,2} = \omega_2 \tag{5.92a}$$

$$\big[\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| - 1\big]_{,2} = \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\omega_2 \tag{5.92b}$$

and therefore, substituting these terms, we obtain:

$$\begin{aligned} v_1^g = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})}\Big[\omega_2\ln|\boldsymbol{\omega}\cdot\hat{\boldsymbol{x}}| - \omega_2 + \\ + (\boldsymbol{\omega}\cdot\boldsymbol{x})\frac{\omega_2}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\Big]\mathrm{d}\omega \end{aligned} \tag{5.93}$$

A simplification yields the expression for the first component of the velocity:

$$v_1^g = -\frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} \frac{\omega_2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})} \ln|\boldsymbol{\omega} \cdot \hat{\boldsymbol{x}}| \, \mathrm{d}\omega \tag{5.94}$$

In the same way, being

$$(\boldsymbol{\omega} \cdot \boldsymbol{x})_{,1} = \omega_1 \tag{5.95a}$$

$$\big[\ln|\boldsymbol{\omega} \cdot \hat{\boldsymbol{x}}| - 1\big]_{,1} = \frac{1}{(\boldsymbol{\omega} \cdot \boldsymbol{x})}\omega_1 \tag{5.95b}$$

we obtain the expression for the second component of the velocity:

$$v_2^g = \frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} \frac{\omega_1(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})} \ln|\boldsymbol{\omega} \cdot \hat{\boldsymbol{x}}| \, \mathrm{d}\omega \tag{5.96}$$

Equations (5.94) and (5.96) can be rewritten in a more compact form as follows:

$$v_i^g = -\frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} (\delta_{1i}\omega_2 - \delta_{2i}\omega_1) \frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})} \ln|\boldsymbol{\omega} \cdot \hat{\boldsymbol{x}}| \, \mathrm{d}\omega \tag{5.97}$$

and if we assume

$$\tilde{v}_i^g(\boldsymbol{\omega} \cdot \boldsymbol{x}) = (\delta_{1i}\omega_2 - \delta_{2i}\omega_1) \frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})} \ln|\boldsymbol{\omega} \cdot \hat{\boldsymbol{x}}| \tag{5.98}$$

we obtain

$$v_i^g = -\frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} \tilde{v}_i^g(\boldsymbol{\omega} \cdot \boldsymbol{x}) \, \mathrm{d}\omega \tag{5.99}$$

A substitution of the expression for $(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)$ and for $L(\boldsymbol{\omega})$ yields the following expression for the velocity:

$$\begin{aligned} v_m^g = -\frac{r}{4\pi^2} \int_0^{2\pi} \sin\left[\alpha + \theta + (1-m)\frac{\pi}{2}\right] \sin\left[\alpha + \theta + (1-g)\frac{\pi}{2}\right] \\ \times \frac{\ln|\cos\alpha|}{\mu(1+\kappa)\Lambda(\alpha+\theta)} \, \mathrm{d}\alpha \end{aligned} \tag{5.100}$$

and, taking into account that

$$\sin\left[\alpha+\theta+(1-g)\frac{\pi}{2}\right]=\cos\left[\alpha+\theta+(2-g)\frac{\pi}{2}\right] \tag{5.101}$$

we obtain the general expression for the velocity field:

$$v_m^g=-\frac{1}{4\pi^2\mu(1+\kappa)}\int_0^{2\pi}\frac{(\delta_{1i}\omega_2-\delta_{2i}\omega_1)(\delta_{1g}\omega_2-\delta_{2g}\omega_1)}{\Lambda(\alpha+\theta)}\ln|\cos\alpha|\,\mathrm{d}\alpha \tag{5.102}$$

In the limit case of infinitesimal ($\kappa=0$) and isotropic ($\mu=\mu^*$, $\gamma_1=\gamma_2=-1$) elasticity, equation (5.102) yields the well-known Green's functions for Stokes flow (see Ladyzhenskaya [31, Sezione 3.4]):

$$v_1^1=\frac{1}{8\pi\mu(1-\nu)}\left[(4\nu-3)\ln r+\cos^2\theta\right] \tag{5.103a}$$

$$v_2^2=\frac{1}{8\pi\mu(1-\nu)}\left[(4\nu-3)\ln r+\sin^2\theta\right] \tag{5.103b}$$

$$v_1^2=v_2^1=\frac{1}{8\pi\mu(1-\nu)}\sin\theta\cos\theta \tag{5.103c}$$

for the compressible material, while for the incompressible one we have:

$$v_m^m=-\frac{1}{4\pi\mu}\ln\hat{r}+\frac{3-2m}{8\pi\mu}\cos(2\theta) \tag{5.104a}$$

$$v_1^2=v_2^1=-\frac{1}{4\pi\mu}\sin\theta\cos\theta=-\frac{1}{8\pi\mu}\sin(2\theta) \tag{5.104b}$$

where ν is the Poisson ratio and μ is the shear modulus.

5.6 Velocity field for the dynamic problem

Now we focus on the dynamic part of the problem; the equilibrium equation we are considering is the first of (5.37), namely:

$$\left(\mu+\frac{\sigma}{2}\right)\psi_{,1111}+2(2\mu_*-\mu)\psi_{,1122}+\left(\mu-\frac{\sigma}{2}\right)\psi_{,2222}=$$
$$=\dot{f}_2\delta_{,1}(\boldsymbol{x})-\dot{f}_1\delta_{,2}(\boldsymbol{x})-\rho\Omega^2\left(\psi_{,11}-\psi_{,22}\right) \tag{5.105}$$

We follow the same procedure described in the previous Section, which means to rewrite equation (5.105) by introducing the operator $\mathcal{L}$, and using the plane wave expansion allows us to write:

$$L(\boldsymbol{\omega})\big(\tilde{\psi}^g\big)'''' + \rho\Omega^2\big(\tilde{\psi}^g\big)'' = \frac{2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.106}$$

If we divide the above expression by $L(\boldsymbol{\omega})$ we obtain the equation governing the dynamic problem in the canonical form:

$$\big(\tilde{\psi}^g\big)'''' + \frac{\rho\Omega^2}{L(\boldsymbol{\omega})}\big(\tilde{\psi}^g\big)'' = \frac{2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.107}$$

and we note that the velocity of a transverse plane wave propagating in the direction defined by the unit vector $\boldsymbol{\omega}$ is:

$$\sqrt{\frac{L(\boldsymbol{\omega})}{\rho}} \tag{5.108}$$

The characteristic equation associated to (5.107) is:

$$\lambda^4 + \eta^2\lambda^2 = R(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.109}$$

where

$$\eta^2 = \Omega\sqrt{\frac{\rho}{L(\boldsymbol{\omega})}} \tag{5.110}$$

is the wave number in the direction $\boldsymbol{\omega}$.

We integrate equation (5.107) with respect to the variable $\boldsymbol{\omega}\cdot\boldsymbol{x}$ and we exploit the parameter variation method. The associated characteristic equation yields the homogeneous solution:

$$\lambda^4 + \eta^2\lambda^2 = \lambda^2\big(\lambda^2 + \eta^2\big) = 0 \tag{5.111}$$

from which we obtain the following four solutions:

$$\lambda_1 = 0 \tag{5.112a}$$
$$\lambda_2 = 0 \tag{5.112b}$$
$$\lambda_3 = -\mathrm{i}\,\eta \tag{5.112c}$$
$$\lambda_4 = \mathrm{i}\,\eta \tag{5.112d}$$

that allows us to generate the following solution functions:

$$\tilde{\psi}^g_{\mathrm{om},1} = e^{0\cdot(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 1 \tag{5.113a}$$
$$\tilde{\psi}^g_{\mathrm{om},2} = (\boldsymbol{\omega}\cdot\boldsymbol{x})e^{0\cdot(\boldsymbol{\omega}\cdot\boldsymbol{x})} = \boldsymbol{\omega}\cdot\boldsymbol{x} \tag{5.113b}$$
$$\tilde{\psi}^g_{\mathrm{om},3} = e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.113c}$$
$$\tilde{\psi}^g_{\mathrm{om},4} = e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.113d}$$

Since the sum of solutions is a solution, the homogeneous solution can be expressed as

$$\tilde{\psi}^g_{\mathrm{om}} = A_1 + A_2(\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_3 e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + A_4 e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.114}$$

and we are looking for a particular solution of the form:

$$\tilde{\psi}^g_{\mathrm{p}} = B_1 + B_2(\boldsymbol{\omega}\cdot\boldsymbol{x}) + B_3 e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + B_4 e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.115}$$

where B_1, B_2, B_3, B_4 are unknown functions of $(\boldsymbol{\omega}\cdot\boldsymbol{x})$ that must be determined. For simplicity, we write

$$R(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} = \frac{\beta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.116}$$

where

$$\beta = \frac{2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})} \tag{5.117}$$

When using the variation of constants method, the following system must be solved in order to determinate the constants B_1, B_2, B_3, B_4:

$$\begin{cases} B_1^{'}\tilde{\psi}^{g}_{\mathrm{om},1} + B_2^{'}\tilde{\psi}^{g}_{\mathrm{om},2} + B_3^{'}\tilde{\psi}^{g}_{\mathrm{om},3} + B_4^{'}\tilde{\psi}^{g}_{\mathrm{om},4} = 0 \\ B_1^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},1}\big)^{'} + B_2^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},2}\big)^{'} + B_3^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},3}\big)^{'} + B_4^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},4}\big)^{'} = 0 \\ B_1^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},1}\big)^{''} + B_2^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},2}\big)^{''} + B_3^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},3}\big)^{''} + B_4^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},4}\big)^{''} = 0 \\ B_1^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},1}\big)^{'''} + B_2^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},2}\big)^{'''} + B_3^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},3}\big)^{'''} + B_4^{'}\big(\tilde{\psi}^{g}_{\mathrm{om},4}\big)^{'''} = R(\boldsymbol{\omega}\cdot\boldsymbol{x}) \end{cases} \tag{5.118}$$

where the apostrophe denotes the derivative with respect to the argument $(\boldsymbol{\omega}\cdot\boldsymbol{x})$. If we substitute in this system the preceding functions, we obtain:

$$\begin{cases} B_1^{'} + (\boldsymbol{\omega}\cdot\boldsymbol{x})B_2^{'} + B_3^{'}e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 0 \\ 0 + B_2^{'} - \mathrm{i}\,\eta B_3^{'}e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \mathrm{i}\,\eta B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 0 \\ 0 + 0 - \eta^2 B_3^{'}e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \eta^2 B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 0 \\ 0 + 0 + \mathrm{i}\,\eta^3 B_3^{'}e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \mathrm{i}\,\eta^3 B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = R(\boldsymbol{\omega}\cdot\boldsymbol{x}) \end{cases} \tag{5.119}$$

From the third equation of the above system we have:

$$B_3^{'}e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 0 \quad\longrightarrow\quad B_3^{'} = -B_4^{'}e^{2\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.120}$$

From the second, taking into account the previous relation,

$$B_2^{'} + \mathrm{i}\,\eta B_4^{'}e^{2\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \mathrm{i}\,\eta B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 0 \tag{5.121}$$

from which

$$B_2^{'} + 2\,\mathrm{i}\,\eta B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 0 \quad\longrightarrow\quad B_2^{'} = -2\,\mathrm{i}\,\eta B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.122}$$

A substitution of the values of $B_2^{'}$ and $B_3^{'}$ into the first equation of (5.119) yields:

$$B_1^{'} - 2\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - B_4^{'}e^{2\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = 0 \tag{5.123}$$

from which

$$B_1^{'} = 2\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})B_4^{'}e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.124}$$

Now is possible to rewrite the fourth equation of (5.119) as a function of the constant $B_4^{'}$:

$$-\,\mathrm{i}\,\eta^3 B_4^{'} e^{2\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x}} - \mathrm{i}\,\eta^3 B_4^{'} e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = R(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.125}$$

from which

$$-\,2\,\mathrm{i}\,\eta^3 B_4^{'} e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = R(\boldsymbol{\omega}\cdot\boldsymbol{x}) \quad \longrightarrow \quad B_4^{'} = \frac{\mathrm{i}\,R(\boldsymbol{\omega}\cdot\boldsymbol{x})}{2\eta^3 e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}} \tag{5.126}$$

namely

$$B_4^{'} = \frac{\mathrm{i}(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{\eta^3 L(\boldsymbol{\omega})(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = \frac{\mathrm{i}\,\beta}{2\eta^3}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.127}$$

An integration of the above equation yields the constant B_4:

$$B_4 = \frac{\mathrm{i}\,\beta}{2\eta^3}\int \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{\mathrm{i}\,\beta}{2\eta^3} I_1 \tag{5.128}$$

where the integral I_1 is calculated by parts as follows:

$$\begin{aligned} I_1 &= \int \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \\ &= \frac{-1}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \int \frac{-1}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2}(-\,\mathrm{i}\,\eta) e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \\ &= \frac{-e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} - \frac{\mathrm{i}\,\eta}{2}\left[\frac{-e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \int \frac{-1(-\,\mathrm{i}\,\eta)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})} e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x})\right] \end{aligned} \tag{5.129}$$

and collecting some terms

$$\begin{aligned} I_1 = &\left[\frac{-1}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} + \frac{\mathrm{i}\,\eta}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \\ &+ \left(\frac{-\,\mathrm{i}\,\eta}{2}\right)(-1)(-1)(-\,\mathrm{i}\,\eta)\int \frac{e^{-\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \end{aligned} \tag{5.130}$$

The last integral of the above relation can not be analytically solved, so that we exploit the special functions as follows:

$$I_1 = \frac{1}{2}\left[\frac{-1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} + \frac{\mathrm{i}\,\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \frac{1}{2}\eta^2 \underbrace{\int \frac{e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x})}_{-\,\mathrm{Ei}(1,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x}))} =$$

$$= \frac{1}{2}\left[\frac{-1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} + \frac{\mathrm{i}\,\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \frac{1}{2}\eta^2\,\mathrm{Ei}\big(1,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) \tag{5.131}$$

where Ei denotes the *exponential integral* function, that is defined as[3]:

$$\mathrm{Ei}\big(1,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) = \int (\boldsymbol{\omega}\cdot\boldsymbol{x})^{-1} e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.132}$$

Finally, the constant B_4 is given by:

$$B_4 = \frac{-\beta}{4\eta^3}\left[\frac{\mathrm{i}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} + \frac{\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \frac{\mathrm{i}\,\beta}{4\eta}\,\mathrm{Ei}\big(1,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) \tag{5.133}$$

[3] The *exponential integral* function is defined as:

$$\mathrm{Ei}(a,z) = \int_0^1 e^{-k_1 z} k_1^{-a}\,\mathrm{d}k_1 \quad \text{con} \quad \Re(z) > 0$$

and can be generalized as:

$$\mathrm{Ei}(a,z) = z^{a-1}\Gamma(1-a,z)$$

which applies on $\mathbb{C}$ except the point zero, if we use the function $\mathrm{Ei}(1,z)$, while the Gamma function is defined as:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t$$

We have the following relation:

$$\Gamma(a,z) = \Gamma(a) - \frac{z^a}{a}\,\mathrm{hypergeom}(a, l+a, z)$$

expressed as a function of the hypergeometric function, and if $\Re(a) > 0$ then

$$\Gamma(a,z) = \int_z^\infty e^{-t} t^{a-1}\,\mathrm{d}t$$

For the constants B_1 and B_2 we have:

$$B_1' = 2\,\mathrm{i}\,\eta \left[\frac{\mathrm{i}\,\eta}{2\eta^3} \frac{e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \right] (\boldsymbol{\omega}\cdot\boldsymbol{x}) e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = -\frac{\beta}{\eta^2} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} \tag{5.134a}$$

$$B_2' = -2\,\mathrm{i}\,\eta \left[\frac{\mathrm{i}\,\eta}{2\eta^3} \frac{e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \right] e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = \frac{\beta}{\eta^2} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \tag{5.134b}$$

that can be directly integrated, so that we obtain:

$$B_1 = -\frac{\beta}{\eta^2} \int \frac{\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x})}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} = \frac{\beta}{\eta^2} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.135a}$$

$$B_2 = \frac{\beta}{\eta^2} \int \frac{\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x})}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} = -\frac{\beta}{2\eta^2} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} \tag{5.135b}$$

To calculate the constant B_3 we use the expression (5.120), into which a substitution of $B_4^{'}$ yields:

$$B_3^{'} = -\left[\frac{\mathrm{i}\,\beta}{2\eta^3} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \right] e^{2\,\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = -\frac{\mathrm{i}\,\beta}{2\eta^3} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.136}$$

An integration of the above equation yields the constant B_3:

$$B_3 = -\frac{\mathrm{i}\,\beta}{2\eta^3} \int \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\, \mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = -\frac{\mathrm{i}\,\beta}{2\eta^3} I_2 \tag{5.137}$$

where the integral I_2 is calculated by parts as follows:

$$\begin{aligned}
I_2 &= \int \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\, \mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \\
&= \frac{-1}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \int \frac{-1}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2}\, \mathrm{i}\,\eta e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\, \mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \\
&= \frac{-e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} + \frac{\mathrm{i}\,\eta}{2} \left[\frac{-e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \int \frac{-1(\mathrm{i}\,\eta)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})} e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\, \mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \right]
\end{aligned} \tag{5.138}$$

which leads to

$$I_2 = \left[\frac{-1}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} - \frac{\mathrm{i}\,\eta}{2(\boldsymbol{\omega}\cdot\boldsymbol{x})} \right] e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \frac{\mathrm{i}\,\eta}{2}(\mathrm{i}\,\eta) \int \frac{e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\, \mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.139}$$

In the last integral of the above relation we must exploit the special functions, in a way similar to the procedure for the evaluation of I_1:

$$
\begin{aligned}
I_2 &= \frac{1}{2}\left[\frac{-1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} - \frac{\mathrm{i}\,\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \frac{1}{2}\eta^2 \int \frac{e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \\
&= \frac{1}{2}\left[\frac{-1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} - \frac{\mathrm{i}\,\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \frac{1}{2}\eta^2\,\mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)
\end{aligned}
\tag{5.140}
$$

where $\mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)$ denotes the exponential integral function, which, in this case, is defined as[4]:

$$
\mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) = -\int (\boldsymbol{\omega}\cdot\boldsymbol{x})^{-1} e^{-(-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x}))}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.141}
$$

Finally, the constant B_3 is given by:

$$
B_3 = \frac{\beta}{4\eta^3}\left[\frac{\mathrm{i}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} - \frac{\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \frac{\mathrm{i}\,\beta}{4\eta}\,\mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) \tag{5.142}
$$

The particular solution (5.115) becomes

$$
\begin{aligned}
\tilde{\psi}^g_{\mathrm{p}} = {} & \frac{\beta}{\eta^2}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \frac{\beta}{2\eta^2}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + \frac{\beta}{4\eta^3}\left[\frac{\mathrm{i}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} - \frac{\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] + \\
& - \frac{\mathrm{i}\,\beta}{4\eta}\,\mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} - \frac{\beta}{4\eta^3}\left[\frac{\mathrm{i}}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} + \frac{\eta}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\right] + \\
& + \frac{\mathrm{i}\,\beta}{4\eta}\,\mathrm{Ei}\big(1, \mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}
\end{aligned}
\tag{5.143}
$$

namely

$$
\begin{aligned}
\tilde{\psi}^g_{\mathrm{p}} = {} & \frac{\beta}{\eta^2}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{x})}\left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{4}\right) + \frac{\mathrm{i}\,\beta}{4\eta}\Big[\mathrm{Ei}\big(1, \mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + \\
& - \mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})}\Big]
\end{aligned}
\tag{5.144}
$$

[4] See note 3 on page 117.

An expansion of the above relation yields

$$\tilde{\psi}^g_{\mathrm{p}} = \frac{\mathrm{i}\,\beta}{4\eta}\Big\{\big[\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + \mathrm{i}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\big]\,\mathrm{Ei}\big(1, \mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) + \\ - \big[\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{i}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\big]\,\mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)\Big\} \tag{5.145}$$

and collecting the exponential integral functions we obtain:

$$\tilde{\psi}^g_{\mathrm{p}} = \frac{\mathrm{i}\,\beta}{4\eta}\Big\{\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\Big[\mathrm{Ei}\big(1, \mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) - \mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)\Big] + \\ + \mathrm{i}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\Big[\mathrm{Ei}\big(1, \mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) + \mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)\Big]\Big\} \tag{5.146}$$

If we introduce two special functions, in particular the *cosine integral*[5] and the

[5] The *cosine integral* function is defined for all the complex variables x as:

$$\mathrm{Ci}(x) = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t}\,\mathrm{d}t$$

or, with the *exponential integral* functions, as

$$\mathrm{Ci}(a, z) = -\frac{1}{2}\big[\mathrm{Ei}(1, x) + \mathrm{Ei}(1, -x)\big] + \mathrm{i}\,\frac{\pi}{2}\big[\mathrm{csign}(x) - 1\big]\,\mathrm{csign}(\mathrm{i}\,x)$$

which, for $x \in \mathbb{R}$, reduces to:

$$\mathrm{Ci}(a, z) = -\frac{1}{2}\big[\mathrm{Ei}(1, x) + \mathrm{Ei}(1, -x)\big]$$

where γ is the *Euler-Mascheroni constant*, defined as:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \approx 0.57721\ldots$$

sine integral[6], we have:

$$\tilde{\psi}^g_{\mathrm{p}} = \frac{\beta}{2\eta}\left\{\underbrace{-\frac{1}{2}\Big[\mathrm{Ei}\big(1, \mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) + \mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)\Big]}_{\mathrm{Ci}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + \right.$$
$$\left. + \underbrace{\frac{1}{2}\,\mathrm{i}\Big[\mathrm{Ei}\big(1, \mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big) - \mathrm{Ei}\big(1, -\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})\big)\Big]}_{-\,\mathrm{Si}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)}\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\right\} \tag{5.147}$$

so that the expression of the particular solution becomes:

$$\tilde{\psi}^g_{\mathrm{p}} = \frac{\beta}{2\eta}\big[\mathrm{Ci}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{Si}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\big] \tag{5.148}$$

where, taking into account the definition of β given in (5.117), we have

$$\frac{\beta}{2\eta} = \frac{\delta_{1g}\omega_2 - \delta_{2g}\omega_1}{\eta L(\boldsymbol{\omega})} \tag{5.149}$$

Recalling the expression for the wave number η given by (5.110)

$$\eta L(\boldsymbol{\omega}) = L(\boldsymbol{\omega})\Omega\sqrt{\frac{\rho}{L(\boldsymbol{\omega})}} = \Omega\sqrt{\rho L(\boldsymbol{\omega})} = \sqrt{\rho\Omega^2 L(\boldsymbol{\omega})} \tag{5.150}$$

[6] The *sine integral* function is defined for all the complex variables x as:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$$

or, with the *exponential integral* functions, as

$$\mathrm{Si}(a, z) = -\,\mathrm{i}\,\frac{1}{2}\big[\mathrm{Ei}(1, x) - \mathrm{Ei}(1, -x)\big] + \frac{\pi}{2}\,\mathrm{csign}(x)$$

(where $\mathrm{csign}(x)$ is the complex sign function), which, for $x \in \mathbb{R}$, reduces to:

$$\mathrm{Si}(a, z) = -\,\mathrm{i}\,\frac{1}{2}\big[\mathrm{Ei}(1, x) - \mathrm{Ei}(1, -x)\big]$$

we obtain the final expression for the solution of equation (5.107):

$$\tilde{\psi}^g = \frac{\delta_{1g}\omega_2 - \delta_{2g}\omega_1}{\sqrt{\rho\Omega^2 L(\boldsymbol{\omega})}}\big[\mathrm{Ci}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{Si}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\big]+ \\ + A_1 + A_2(\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_3 e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} + A_4 e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} \tag{5.151}$$

The coefficient

$$\frac{\delta_{1g}\omega_2 - \delta_{2g}\omega_1}{\sqrt{\rho\Omega^2 L(\boldsymbol{\omega})}} \tag{5.152}$$

can be included inside the constants A_1, A_2, A_3, A_4. Being $\tilde{\psi}^g$ a *stream function*, for simplicity we choose $A_1 = 0$ and $A_2 = 0$ since the constant terms are superfluous (in fact we only need the derivatives of $\tilde{\psi}^g$), while the linear terms represent rigid body motions, that are negligible in an infinite medium. Taking into account that

$$e^{\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = \cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{i}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.153a}$$

$$e^{-\mathrm{i}\,\eta(\boldsymbol{\omega}\cdot\boldsymbol{x})} = +\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + \mathrm{i}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.153b}$$

the final solution can be written as:

$$\tilde{\psi}^g = \frac{\delta_{1g}\omega_2 - \delta_{2g}\omega_1}{\sqrt{\rho\Omega^2 L(\boldsymbol{\omega})}}\Big[\mathrm{Ci}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{Si}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})+ \\ + A_1^*\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_2^*\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})+ \\ + \mathrm{i}\big[A_3^*\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_4^*\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\big]\Big] \tag{5.154}$$

In order to calculate the constants A_i^* (with $i = 1, 2, 3, 4$) we consider the *far-field approximation*[7] in the variable $\boldsymbol{\omega}\cdot\boldsymbol{x}$ of the following term:

$$\mathrm{Ci}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{Si}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) = \\ = -\frac{\pi}{2}\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + o\left(\frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{x}}\right) \tag{5.155}$$

for $\boldsymbol{\omega}\cdot\boldsymbol{x} \to +\infty$. Neglecting an arbitrary harmonic solution, from equa-

[7] The asymptotic expansion.

tions (5.154) and (5.155) we obtain the *far-field approximation* for $\tilde{\psi}^g$:

$$\tilde{\psi}^g(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{\delta_{1g}\omega_2 - \delta_{2g}\omega_1}{\sqrt{\rho\Omega^2 L(\boldsymbol{\omega})}}\left[\frac{\pi}{2}\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + \mathrm{i}\,\frac{\pi}{2}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\right] + \\ + o\left(\frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{x}}\right) \quad (5.156)$$

which represents outgoing waves[8]. As a consequence of the asymptotic representation (5.156), the constants A_i^* have been determined, and we have:

$$\tilde{\psi}^g = \frac{\delta_{1g}\omega_2 - \delta_{2g}\omega_1}{\sqrt{\rho\Omega^2 L(\boldsymbol{\omega})}}\Big[\mathrm{Ci}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{Si}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + \\ - \mathrm{i}\,\frac{\pi}{2}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\Big] \quad (5.157)$$

The plane wave expansion of the above relations yields the *stream function*, which, expressed in polar coordinates, is

$$\tilde{\psi}^g(r,\theta) = -\frac{1}{2\pi^2\rho\Omega c}\int_0^\pi \frac{\sin\left(\alpha+\theta-\delta_{2g}\frac{\pi}{2}\right)}{\Lambda(\alpha+\theta)}\,\Xi\left(\frac{\omega r}{c}\frac{\cos\alpha}{\sqrt{\Lambda(\alpha+\theta)}}\right)\mathrm{d}\alpha \quad (5.158)$$

where

$$\Xi(x) = \mathrm{Ci}(|x|)\sin x\,\mathrm{Si}(x)\cos x - \frac{\pi}{2}\sin x \quad (5.159a)$$

$$\Lambda(\alpha) = \sin^4\alpha\left(\cot^2\alpha - \gamma_1\right)\left(\cot^2\alpha - \gamma_2\right) > 0 \quad (5.159b)$$

while

$$c = \sqrt{\frac{\mu(1+\kappa)}{\rho}} \quad (5.160a)$$

$$c\sqrt{\Lambda(\alpha)} \quad (5.160b)$$

are, respectively, the propagation velocity of a transverse wave moving parallel to the x_1 axis and the the propagation velocity in the direction defined by the

[8] The term $\mathrm{i}\,\frac{\pi}{2}\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})$ must be added to the term $\frac{\pi}{2}\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})$ of equation (5.156) in order to avoid ingoing waves, that can not exist in an infinite medium.

angle α. Note that $\Lambda(\alpha)$ is always strictly positive within the elliptic regime and has the following properties:

$$\Lambda(0) = \Lambda(\pi) = 1 \tag{5.161a}$$

$$\Lambda\left(\frac{\pi}{2}\right) = \Lambda\left(\frac{3}{2}\pi\right) = \gamma_1\gamma_2 = \frac{1-\kappa}{1+\kappa} \tag{5.161b}$$

while $\Lambda(\alpha) = 1$ in the special case in which the incremental response becomes isotropic ($\kappa = 0$, $\mu = \mu_*$). The *stream function* (5.157) is not singular in $r = 0$ and depends on θ and on the dimensionless variable

$$\hat{r} = \frac{\Omega r}{c} \tag{5.162}$$

Since $v_1 = \psi_{,2}$ and $v_2 = -\psi_{,1}$, we introduce the Green's function for the incremental displacements for the infinite body:

$$v_1^g = \psi_{,2}^g \tag{5.163a}$$

$$v_2^g = -\psi_{,1}^g \tag{5.163b}$$

so that from (5.157) and from the above relations we have:

$$\tilde{v}_j^g(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{(\delta_{1j}\omega_2 - \delta_{2j}\omega_1)(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{L(\boldsymbol{\omega})}\Bigg[\mathrm{Ci}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + \\ - \mathrm{Si}(\eta|\boldsymbol{\omega}\cdot\boldsymbol{x}|)\sin(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) - \mathrm{i}\,\frac{\pi}{2}\cos(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\Bigg] \tag{5.164}$$

and finally, the Green's tensor is given by:

$$v_j^g(\boldsymbol{x}) = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\tilde{v}_j^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\omega \tag{5.165}$$

As a conclusion, we observe that the expression of v_j^g has a logarithmic singularity when $\hat{r}$ goes to zero. Therefore, near the singularity we have $r \to 0$ and the asymptotic behaviour of the dynamic Green's function is the same as that of the quasi-static case.

5.7 Gradient of the velocity field

The gradient of the velocity associated to the Green's tensor is:

$$\frac{\partial v_j^g}{\partial x_1} = \cos\theta \frac{\partial v_j^g}{\partial r} - \sin\theta \frac{\partial v_j^g}{\partial \theta} \tag{5.166a}$$

$$\frac{\partial v_j^g}{\partial x_2} = \sin\theta \frac{\partial v_j^g}{\partial r} + \cos\theta \frac{\partial v_j^g}{\partial \theta} \tag{5.166b}$$

Note that, due to the symmetry of the Green's tensor, the incompressibility constraint allows us to write:

$$v_{2,2}^1 = v_{1,2}^2 = -v_{1,1}^1 \qquad v_{2,1}^1 = v_{1,1}^2 = -v_{2,2}^2 \tag{5.167}$$

so that only four components of the velocity gradient need to be determined.

The expressions of v_j^g have a form in which the dependence on r is explicit, while the dependence on θ involves an integral; the derivative with respect to r produces a $\frac{1}{r}$ singularity, while the derivative with respect to θ does not modify the singularity. With these considerations, the velocity gradient for the static problem can be expressed through the following relations:

$$\begin{aligned} v_{1,g}^1 = \frac{1}{2\pi^2\mu(1+\kappa)r} \Bigg\{ & \frac{\pi \cos\left[\theta + (1-g)\frac{\pi}{2}\right]}{\gamma_1\sqrt{-\gamma_2} + \gamma_2\sqrt{-\gamma_1}} + \sin\left[\theta + (1-g)\frac{\pi}{2}\right] \\ & \times \int_0^{\frac{\pi}{2}} \Big[\ln(\cos\alpha)\,\Sigma\left(\alpha+\theta, \alpha+\theta\right) \\ & + \ln(\sin\alpha)\,\Sigma\left(\alpha+\theta+\frac{\pi}{2}, \alpha+\theta+\frac{\pi}{2}\right)\Big]\,\mathrm{d}\alpha \Bigg\} \end{aligned} \tag{5.168}$$

and

$$\begin{aligned} v_{2,g}^2 = -\frac{1}{2\pi^2\mu(1+\kappa)r} \Bigg\{ & \frac{\pi \cos\left[\theta + (1-g)\frac{\pi}{2}\right]}{\gamma_1\sqrt{-\gamma_2} + \gamma_2\sqrt{-\gamma_1}} - \sin\left[\theta + (1-g)\frac{\pi}{2}\right] \\ & \times \int_0^{\frac{\pi}{2}} \Big[\ln(\cos\alpha)\,\Sigma\left(\alpha+\theta+\frac{\pi}{2}, \alpha+\theta\right) \\ & + \ln(\sin\alpha)\,\Sigma\left(\alpha+\theta, \alpha+\theta+\frac{\pi}{2}\right)\Big]\,\mathrm{d}\alpha \Bigg\} \end{aligned} \tag{5.169}$$

where:

$$\Sigma(\gamma,\eta) = \frac{\left[2\cos\gamma\Lambda(\eta) - \sin\gamma\Lambda^{'}(\eta)\right]}{\Lambda^2(\eta)}\sin\gamma \tag{5.170a}$$

$$\Lambda^{'}(\eta) = \frac{\partial}{\partial\eta}\Lambda(\eta) \tag{5.170b}$$

For the dynamic problem we have:

$$\begin{aligned}
v^s_{g,k} = &-\frac{(2\delta_{sg}-1)}{2\pi^2\mu(1+\kappa)}\left\{v^{s*}_{g,k} - \left(\ln\hat{r} + \gamma - \mathrm{i}\,\frac{\pi}{2}\right)\left[\int_0^\pi K^g_s(\alpha+\theta)\xi(\alpha,\alpha+\theta)\right.\right.\\
&\times\sin\big(\hat{r}\xi(\alpha,\alpha+\theta)\big)\frac{\Omega}{c}\zeta\left(\alpha+\theta,\theta-\delta_{2k}\frac{\pi}{2}\right)\mathrm{d}\alpha + \frac{\sin\left(\theta-\delta_{2k}\frac{\pi}{2}\right)}{r}\\
&\times\left.\int_0^\pi\Sigma\Big(\alpha+\theta-\delta_{2s}\frac{\pi}{2},\alpha+\theta\Big)\cos\big(\hat{r}\xi(\alpha,\alpha+\theta)\big)\,\mathrm{d}\alpha\right]+\\
&-\int_0^{\frac{\pi}{2}}\ln\xi(\alpha,\alpha+\theta)K^g_s(\alpha+\theta)\sin\big(\hat{r}\xi(\alpha,\alpha+\theta)\big)\frac{\Omega}{c}\xi(\alpha,\alpha+\theta)\\
&\times\zeta\left(\alpha+\theta,\theta-\delta_{2k}\frac{\pi}{2}\right)\mathrm{d}\alpha + \int_0^{\frac{\pi}{2}}\ln\xi(\alpha,\alpha-\theta)K^g_s(\alpha-\theta)\xi(\alpha,\alpha-\theta)\\
&\times\sin\big(\hat{r}\xi(\alpha,\alpha-\theta)\big)\frac{\Omega}{c}\zeta\left(\alpha-\theta,-\theta-\delta_{2k}\frac{\pi}{2}\right)\mathrm{d}\alpha + K^g_s(\alpha+\theta)\\
&\left.\times\wp\big(\hat{r}\xi(\alpha,\alpha+\theta)\big)\frac{\Omega}{c}\xi(\alpha,\alpha+\theta)\zeta\left(\alpha+\theta,\theta-\delta_{2k}\frac{\pi}{2}\right)\mathrm{d}\alpha\right\}
\end{aligned} \tag{5.171}$$

where, using the definitions of Σ and $\Lambda^{'}$ given in (5.170) and assuming

$$\wp(x) = -\sin x\int_0^x\frac{\cos t - 1}{t}\,\mathrm{d}t + \cos x\,\mathrm{Si}(x) \tag{5.172}$$

and observing that for the dynamic case we have

$$\Lambda^{'}(\beta) = \sin(2\beta)\big[(2\gamma_1\gamma_2 + \gamma_1 + \gamma_2)\sin^2\beta - (2+\gamma_1+\gamma_2)\cos^2\beta\big] \tag{5.173}$$

the strong singular term $v^{s*}_{g,k}$ is defined as follows:

$$
\begin{aligned}
v^{s*}_{g,k} = \frac{1}{r}\Bigg\{ & \cos\left(\theta - \delta_{2k}\frac{\pi}{2}\right)\int_0^{\pi} K^g_s(\alpha+\theta)\cos\big(\hat{r}\xi(\alpha,\alpha+\theta)\big)\,\mathrm{d}\alpha + \\
& + \sin\left(\theta - \delta_{2k}\frac{\pi}{2}\right)\int_0^{\frac{\pi}{2}}\left[\frac{\Lambda'(\alpha+\theta)}{2\Lambda(\alpha+\theta)}K^g_s(\alpha-\theta)\cos\big(\hat{r}\xi(\alpha,\alpha-\theta)\big) + \right. \\
& - \ln\big(\xi(\alpha,\alpha+\theta)\big)\Sigma\left(\alpha+\theta-\delta_{2s}\frac{\pi}{2},\alpha+\theta\right)\cos\big(\hat{r}\xi(\alpha,\alpha+\theta)\big) + \\
& + (1-2\delta_{1g}\delta_{1s})\bigg(\frac{\Lambda'(\alpha-\theta)}{2\Lambda(\alpha-\theta)}K^g_s(\alpha-\theta)\cos\big(\hat{r}\xi(\alpha,\alpha-\theta)\big) + \\
& \left. - \ln\big(\xi(\alpha,\alpha-\theta)\big)\Sigma\left(\alpha-\theta-\delta_{2s}\frac{\pi}{2},\alpha+\theta\right)\cos\big(\hat{r}\xi(\alpha,\alpha-\theta)\big)\bigg)\right]\mathrm{d}\alpha \\
& - \int_0^{\pi} K^g_s(\alpha+\theta)\left[\cos\big(\hat{r}\xi(\alpha,\alpha+\theta)\big) - 1\right]\zeta\left(\alpha+\theta,\theta-\delta_{2k}\frac{\pi}{2}\right)\mathrm{d}\alpha + \\
& - \sin\left(\theta - \delta_{2k}\frac{\pi}{2}\right)\int_0^{\pi}\Sigma\left(\alpha+\theta-\delta_{2s}\frac{\pi}{2},\alpha+\theta\right)\Im\big(\hat{r}\xi(\alpha,\alpha+\theta)\big)\,\mathrm{d}\alpha\Bigg\}
\end{aligned}
\tag{5.174}
$$

The plane wave expansion of the incremental displacement gradient is:

$$
v^g_{j,k} = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\tilde{v}^g_{j,k}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\omega \tag{5.175}
$$

where the unit vector $\boldsymbol{\omega}$ defines the unit circle in the plane Ox_1x_2, illustrated in Figure 5.2, and

$$
\tilde{v}^g_{j,k} = \omega_k\frac{\delta_{jg}-\omega_j\omega_g}{L(\boldsymbol{\omega})}\left[\frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{x}} - \eta\Xi(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\right] \tag{5.176}
$$

A comparison of equation (5.174) with the gradient of $\tilde{v}^g_i$ in the quasi-static case (see equation (5.98)), shows that the singular term of (5.174) coincides with that of the quasi-static solution, so that is possible to write the following expression:

$$
\tilde{v}^g_{j,k} = \left(\tilde{v}^g_{j,k}\right)_{\text{static}} - \omega_k\frac{\delta_{jg}-\omega_j\omega_g}{L(\boldsymbol{\omega})}\eta\Xi(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{5.177}
$$

in which the quasi-static and the dynamic terms appear separately.

5.8 Incremental stress field for the static problem

Once the velocity gradient is known, the part of the nominal *stress-rate* linearly related to this gradient can be obtained from (5.1), but the nominal hydrostatic stress $\dot{\pi}$ is unknown. To evaluate $\dot{\pi}$ we need equation (5.35), which we rewrite below

$$\begin{aligned}\dot{\pi}_{,11} + \dot{\pi}_{,22} = -2(\mu_* - \mu)\left(\psi_{,2111} - \psi_{,1222}\right) + \frac{\sigma}{2}\left(\psi_{,2111} + \psi_{,1222}\right) + \\ - \dot{f}_1 \delta_{,1}(\boldsymbol{x}) - \dot{f}_2 \delta_{,2}(\boldsymbol{x}) \quad (5.178)\end{aligned}$$

in which the velocity field must be written exploiting the plane wave expansion (5.99). From the chain rule for differentiation, we have:

$$\begin{aligned}v^g_{1,1} &= \frac{\partial}{\partial x_1}\left[-\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1} \tilde{v}^g_1(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\omega\right] = \\ &= -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1} \frac{\partial \tilde{v}^g_1(\boldsymbol{\omega}\cdot\boldsymbol{x})}{\partial(\boldsymbol{\omega}\cdot\boldsymbol{x})}\frac{\partial(\boldsymbol{\omega}\cdot\boldsymbol{x})}{\partial x_1}\,\mathrm{d}\omega = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1} \omega_1\left(\tilde{v}^g_1\right)'\,\mathrm{d}\omega \quad (5.179)\end{aligned}$$

and similarly

$$v^g_{2,2} = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1} \omega_2\left(\tilde{v}^g_2\right)'\,\mathrm{d}\omega \quad (5.180)$$

while, for the plane wave expansion of $\delta(\boldsymbol{x})$, we have:

$$- \dot{f}_1 \delta_{,1}(\boldsymbol{x}) - \dot{f}_2 \delta_{,2}(\boldsymbol{x}) = \frac{2(\delta_{1g}\omega_1 + \delta_{2g}\omega_2)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3} \quad (5.181)$$

where $\dot{f}_i = \delta_{ig}$. The expansion of the hydrostatic nominal Green's stress is:

$$\dot{\pi}^g = -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1} \tilde{\dot{\pi}}^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\omega \quad (5.182)$$

from which we can deduce

$$\begin{aligned}\dot{\pi}_{,11} + \dot{\pi}_{,22} &= -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\left[\omega_1^2\left(\tilde{\dot{\pi}}^g\right)'' + \omega_2^2\left(\tilde{\dot{\pi}}^g\right)''\right]\mathrm{d}\omega = \\ &= -\frac{1}{4\pi^2}\oint_{|\boldsymbol{\omega}|=1}\left(\tilde{\dot{\pi}}^g\right)''\,\mathrm{d}\omega \end{aligned} \tag{5.183}$$

because $|\boldsymbol{\omega}|^2 = \omega_1^2 + \omega_2^2 = 1$. A substitution of the above terms into equation (5.178) yields:

$$\begin{aligned}\left(\tilde{\dot{\pi}}^g\right)'' = -2(\mu_* - \mu)\left[\omega_1^3(\tilde{v}_1^g)''' + \omega_2^3(\tilde{v}_2^g)'''\right] + \frac{\sigma}{2}\left[\omega_1^3(\tilde{v}_1^g)''' - \omega_2^3(\tilde{v}_2^g)'''\right] + \\ + \frac{2(\delta_{1g}\omega_1 + \delta_{2g}\omega_2)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3}\end{aligned} \tag{5.184}$$

Taking into account that:

$$\int \frac{2(\delta_{1g}\omega_1 + \delta_{2g}\omega_2)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^3}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = -\frac{(\delta_{1g}\omega_1 + \delta_{2g}\omega_2)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2} \tag{5.185a}$$

$$\int -\frac{(\delta_{1g}\omega_1 + \delta_{2g}\omega_2)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})^2}\,\mathrm{d}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{\delta_{1g}\omega_1 + \delta_{2g}\omega_2}{\boldsymbol{\omega}\cdot\boldsymbol{x}} \tag{5.185b}$$

equation (5.184) can be directly integrated, yielding:

$$\begin{aligned}\tilde{\dot{\pi}}^g = -2(\mu_* - \mu)\left[\omega_1^3(\tilde{v}_1^g)' + \omega_2^3(\tilde{v}_2^g)'\right] + \frac{\sigma}{2}\left[\omega_1^3(\tilde{v}_1^g)' - \omega_2^3(\tilde{v}_2^g)'\right] + \\ + \frac{\delta_{1g}\omega_1 + \delta_{2g}\omega_2}{\boldsymbol{\omega}\cdot\boldsymbol{x}}\end{aligned} \tag{5.186}$$

where we neglected the integration constants. Now only the expressions of $\left(\tilde{v}_i^g\right)'$ must be rewritten; using equations (5.98) and their derivatives:

$$\left(\tilde{v}_i^g\right)' = (\delta_{1i}\omega_2 - \delta_{2i}\omega_1)\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})} = \frac{\delta_{ig} - \omega_i\omega_g}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})} \tag{5.187}$$

namely

$$\left(\tilde{v}_1^g\right)' = \omega_2 \frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})} \tag{5.188a}$$

$$\left(\tilde{v}_2^g\right)' = -\omega_1 \frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})} \tag{5.188b}$$

a substitution of the above relations into equation (5.186) yields:

$$\begin{aligned}\tilde{\dot{\pi}}^g = &-2(\mu_* - \mu)\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})}\left[\omega_1^3\omega_2 - \omega_1\omega_2^3\right] + \\ &+ \frac{\sigma}{2}\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})}\left[\omega_1^3\omega_2 + \omega_1\omega_2^3\right] + \frac{\delta_{1g}\omega_1 + \delta_{2g}\omega_2}{\boldsymbol{\omega}\cdot\boldsymbol{x}}\end{aligned} \tag{5.189}$$

and collecting the terms

$$\begin{aligned}\tilde{\dot{\pi}}^g = &\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})}\omega_1\omega_2\left[-2\left(\mu_* - \mu\right)\left(\omega_1^2 - \omega_2^2\right) + \frac{\sigma}{2}(\omega_1^2 + \omega_2^2)\right] + \\ &+ \frac{\omega_g}{\boldsymbol{\omega}\cdot\boldsymbol{x}}\end{aligned} \tag{5.190}$$

namely

$$\tilde{\dot{\pi}}^g = \frac{\omega_g}{\boldsymbol{\omega}\cdot\boldsymbol{x}} + \omega_1\omega_2\frac{(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})}\left[-2\left(\mu_* - \mu\right)\left(\omega_1^2 - \omega_2^2\right) + \frac{\sigma}{2}\right] \tag{5.191}$$

The term $\omega_1\omega_2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1)$ can be rewritten in a more compact form as follows:

$$\begin{aligned}\omega_1\omega_2(\delta_{1g}\omega_2 - \delta_{2g}\omega_1) &= \omega_g(\omega_2^2\delta_{1g} - \omega_1^2\delta_{2g}) = \omega_g(1-\omega_g^2)(\delta_{1g} - \delta_{2g}) = \\ &= (3-2g)\omega_g(1-\omega_g^2)\end{aligned} \tag{5.192}$$

so that equation (5.191), taking into account the definition of k given in (5.19c), becomes:

$$\tilde{\dot{\pi}}^g = \frac{\omega_g}{\boldsymbol{\omega}\cdot\boldsymbol{x}} + (2g-3)\frac{\mu\omega_g(1-\omega_g^2)}{(\boldsymbol{\omega}\cdot\boldsymbol{x})L(\boldsymbol{\omega})}\left[2\left(\frac{\mu_*}{\mu} - 1\right)(\omega_1^2 - \omega_2^2) - \kappa\right] \tag{5.193}$$

Note that in the infinitesimal ($\kappa = 0$) and isotropic ($\mu = \mu_*$) elasticity cases, the antitransform of $\tilde{\dot{\pi}}^g$ yields two Cauchy integral principal values that can be easily calculated:

$$\dot{\pi}^g = -\frac{1}{4\pi^2} \oint_{|\boldsymbol{\omega}|=1} \frac{\omega_g}{\boldsymbol{\omega} \cdot \boldsymbol{x}} \, \mathrm{d}\omega \tag{5.194}$$

Referring to Figure 5.2:

$$\dot{\pi}^1 = -\frac{1}{4\pi^2} \int_0^{2\pi} \frac{\cos(\alpha + \theta)}{r \cos \alpha} \, \mathrm{d}\alpha = -\frac{\cos \theta}{2\pi r} \tag{5.195a}$$

$$\dot{\pi}^2 = -\frac{1}{4\pi^2} \int_0^{2\pi} \frac{\sin(\alpha + \theta)}{r \cos \alpha} \, \mathrm{d}\alpha = -\frac{\sin \theta}{2\pi r} \tag{5.195b}$$

In general, the antitransform of $\tilde{\dot{\pi}}^g$ is:

$$\begin{aligned} \dot{\pi}^g = -\frac{1}{4\pi^2} \int_0^{2\pi} \frac{\omega_g}{\boldsymbol{\omega} \cdot \boldsymbol{x}} \, \mathrm{d}\alpha - \frac{1}{4\pi^2} \int_0^{2\pi} (2g-3) \frac{\mu \omega_g (1 - \omega_g^2)}{(\boldsymbol{\omega} \cdot \boldsymbol{x}) L(\boldsymbol{\omega})} \\ \times \left[2 \left(\frac{\mu_*}{\mu} - 1 \right) (\omega_1^2 - \omega_2^2) - \kappa \right] \mathrm{d}\alpha \end{aligned} \tag{5.196}$$

but if we let

$$\boldsymbol{\omega} \cdot \boldsymbol{x} = r \cos \alpha \tag{5.197a}$$

$$L(\boldsymbol{\omega}) = \mu(1 + \kappa) \Lambda(\alpha + \theta) \tag{5.197b}$$

and

$$2 \left(\frac{\mu_*}{\mu} - 1 \right) (\omega_1^2 - \omega_2^2) - \kappa = 2 \left(\frac{\mu_*}{\mu} - 1 \right) (2\omega_1^2 - 1) - \kappa \tag{5.198}$$

since $\boldsymbol{\omega}$ is a unit vector, then

$$\begin{aligned} 2 \left(\frac{\mu_*}{\mu} - 1 \right) (\omega_1^2 - \omega_2^2) - \kappa &= 2 \left(\frac{\mu_*}{\mu} - 1 \right) \left[2 \cos^2(\alpha + \theta) - 1 \right] - \kappa = \\ &= \Gamma(\alpha + \theta) \end{aligned} \tag{5.199}$$

and we have

$$\dot{\pi}^g = -\frac{1}{4\pi^2 r}\int_0^{2\pi} \frac{\omega_g}{\cos\alpha}\,\mathrm{d}\alpha - \frac{(2g-3)}{4\pi^2(1+\kappa)r}\int_0^{2\pi} \frac{\omega_g(1-\omega_g^2)\Gamma(\alpha+\theta)}{\cos\alpha\Lambda(\alpha+\theta)}\,\mathrm{d}\alpha \quad (5.200)$$

Finally, the hydrostatic nominal stress is:

$$\dot{\pi}^1 = -\frac{\cos\theta}{2\pi r} + \frac{1}{4\pi^2(1+k)r}\int_0^{2\pi} \frac{\sin^2(\alpha+\theta)\cos(\alpha+\theta)\Gamma(\alpha+\theta)}{\cos\alpha\Lambda(\alpha+\theta)}\,\mathrm{d}\alpha \quad (5.201a)$$

$$\dot{\pi}^2 = -\frac{\sin\theta}{2\pi r} - \frac{1}{4\pi^2(1+k)r}\int_0^{2\pi} \frac{\sin(\alpha+\theta)\cos^2(\alpha+\theta)\Gamma(\alpha+\theta)}{\cos\alpha\Lambda(\alpha+\theta)}\,\mathrm{d}\alpha \quad (5.201b)$$

5.9 Incremental stress field for the dynamic problem

To complete the Green's function set, the hydrostatic incremental mean stress for the dynamic problem must be determined. To calculate $\dot{\pi}$ we exploit equation (5.35) (as for the static problem, see (5.178)). The procedure is analogous to that used for the static case, but now the velocity field is given by (5.176). Inserting the plane wave expansion of the incremental hydrostatic mean stress (5.182), we obtain:

$$\tilde{\dot{\pi}}^g = \frac{\omega_g}{\boldsymbol{\omega}\cdot\boldsymbol{x}} + (2g-3)\frac{\omega_g(1-\omega_g^2)}{L(\boldsymbol{\omega})}\left[2(\mu_*-\mu)\left(\omega_1^2-\omega_2^2\right)-\frac{\sigma}{2}\right] \times \left[\frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{x}} - \eta\Xi(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\right] \quad (5.202)$$

where the function Ξ is given by (5.159a).

An alternative form of (5.202), more suitable for the next calculations, is:

$$\tilde{\dot{\pi}}^g = \omega_g\eta\Xi(\eta\boldsymbol{\omega}\cdot\boldsymbol{x}) + \omega_g\frac{(2\mu_*-\mu)(1-\omega_g^2)+\left[\mu-(2g-3)\frac{\sigma}{2}\right]\omega_g^2}{L(\boldsymbol{\omega})} \times \left[\frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{x}} - \eta\Xi(\eta\boldsymbol{\omega}\cdot\boldsymbol{x})\right] \quad (5.203)$$

A comparison betweeen (5.203) with the analogous equation for the static case (5.193), shows that the singular terms are the same, and therefore we can write:

$$
\begin{aligned}
\tilde{\dot{\pi}}^g = \left(\tilde{\dot{\pi}}^g\right)_{\text{static}} + \omega_g \eta \Xi(\eta \boldsymbol{\omega} \cdot \boldsymbol{x}) \\
\times \left[1 - \frac{(2\mu_* - \mu)(1 - \omega_g^2) + \left[\mu - (2g-3)\frac{\sigma}{2}\right]\omega_g^2}{L(\boldsymbol{\omega})}\right]
\end{aligned}
\tag{5.204}
$$

A substitution of equation (5.204) into the expansion (5.182) yields the Green's function for the incremental hydrostatic mean stress in the following form:

$$
\begin{aligned}
\dot{\pi}^1(r,\theta) = \left(\dot{\pi}^1\right)_{\text{static}}(r,\theta) - \frac{\Omega}{2\pi^2(1+k)c}\int_0^\pi \frac{\sin^2(\alpha+\theta)}{\Lambda^{\frac{3}{2}}}\Xi\left(\hat{r}\frac{\cos\alpha}{\sqrt{\Lambda(\alpha+\theta)}}\right) \\
\times \cos(\alpha+\theta)\,\Gamma(\alpha+\theta)\,\mathrm{d}\alpha
\end{aligned}
\tag{5.205a}
$$

$$
\begin{aligned}
\dot{\pi}^2(r,\theta) = \left(\dot{\pi}^2\right)_{\text{static}}(r,\theta) - \frac{\Omega}{2\pi^2(1+k)c}\int_0^\pi \frac{\cos^2(\alpha+\theta)}{\Lambda^{\frac{3}{2}}}\Xi\left(\hat{r}\frac{\cos\alpha}{\sqrt{\Lambda(\alpha+\theta)}}\right) \\
\times \sin(\alpha+\theta)\,\Gamma(\alpha+\theta)\,\mathrm{d}\alpha
\end{aligned}
\tag{5.205b}
$$

where, according to (5.199),

$$
\Gamma(\alpha+\theta) = 2\left(\frac{\mu_*}{\mu} - 1\right)\left[2\cos^2(\alpha+\theta) - 1\right] - \kappa \tag{5.206}
$$

The integrals we obtained are non-singular functions of r, while $(\cdot)_{\text{static}}$ denotes the Green's functions for the static problem, given by (5.201).

It is important to note that the incremental hydrostatic Green stress reduces to the static case both in the limit of low frequency and in the case of isotropy ($\mu = \mu_*$ and $\kappa = 0$, namely $\Gamma(\alpha) = 0$). Furthermore, we can note that the Green's function in terms of the increment of the in-plane Cauchy mean stress, $\dot{p}^g$, can be obtained from (5.205) by exploiting the following relation

$$
\dot{p}^g = \dot{\pi}^g - \frac{\sigma}{2}v^g_{1,1} \tag{5.207}
$$

Chapter 6

Inclusion and dislocation problems

The inclusion and dislocation problems are generalized to the case of incompressible, anisotropic, elastic and prestressed materials. The solutions are given in terms of incremental displacements and mean stress fields. Some simple example are reported in order to show the role and the effects of prestress and anisotropy, and, in particular, the formation of shear bands.

The theory of dislocations and inclusions in solids has been thoroughly developed for elastic materials, unloaded in their natural state. We extend this theory to cover the possibility that the material is prestressed, through a generalization of solutions found by Eshelby [12–14] and Willis [15], by introducing an incremental formulation for incompressible materials, in which the nominal stress is related to the incremental displacement gradient, within a general constitutive framework (which embraces Mooney-Rivlin and Ogden materials and also material models describing softening [32]) under the plane strain constraint. We note that several of the presented results remain valid within a three-dimensional context.

Anisotropy strongly influences near dislocation stress fields, as shown in Figure 1.1 on page 4 (see **Appendix B** for more details). Almost all crystals are anisotropic, so that anisotropy has been the subject of an intense research effort [33–35] and has been recently advocated as a way to study dislocation core properties [36, 37].

Our interest is to analyze the effect of orthotropy induced by prestress on dislocation (and inclusions) fields, within the general framework of incremental non-linear elasticity, but with a special emphasis on a certain material model for metals (the J_2-deformation theory [17, 27])[1] . Hence our investigation is relevant for ductile metals subject to extreme strain, where the nucleation of a clustering of dislocations into a 'super dislocation' perturbs a material that has a low stiffness, so low that the differential equations governing the incremental equilibrium are close to the boundary of ellipticity loss.

When this boundary is approached (from the interior of the elliptic region), our solution for edge dislocations (but also, in general, for inclusions) reveals features of severely deformed metals near the shear band formation. In this situation we show that emission of a dislocation (which can be also viewed as a 'super dislocation') dipole produces incremental fields strongly localized along the directions of the shear bands, formally excluded *within* the elliptic region. This may induce a cascade of dislocation clustering, which may explain the fact that the amount of slip that takes place on an active shear band is three orders of magnitude greater than could be produced by the passage of a single dislocation [38].

6.1 The inclusion problem

Let us consider a region D defining an infinite, incompressible elastic medium subject to an homogeneous prestress[2] and containing an inclusion with a generic shape, volume D_{in} and surface ∂D_{in}. We imagine that we remove the inclusion from the medium and let it undergo an incremental uniform displacement gradient $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\text{P}}$, that can be thought as an inelastic deformation (for instance,

[1] It should be noted that the stress-induced anisotropy is not merely represented by a specific form of constitutive operator, but introduces asymmetries not present when prestress is absent. These asymmetric effects are related to the fact that, differently from the usual small strain context, a small rigid-body rotation produces a stress increment in a prestressed body.

[2] The assumption that the prestress is uniform (essential to the achievement of analytical solutions) may seem more restrictive than it really is. In fact, dislocation nucleation occurs always in a prestressed material and, though the existing state of stress is non-uniform, this inhomogeneity is in first approximation negligible when the scale of the dislocation is considered. Moreover, as noticed by Bigoni *et al*. [39–41], the incremental deformation fields found near the elliptic boundary in a homogeneously prestressed material are a sort of 'ultimate deformation modes', dominating the previously developed inhomogeneous, but small, strain and thus correctly representing the near-failure deformation.

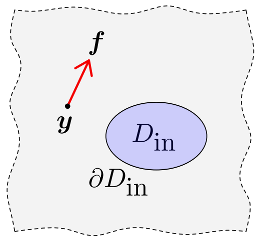


Figure 6.1: Infinite medium D containing an inclusion of volume D_{in} and surface ∂D_{in}; a concentrated force is applied in point $\boldsymbol{y}$ outside the inclusion.

plastic or thermal deformation).[3]

We apply a layer of forces and pressure on ∂D_{in} in order to restore the inclusion to its original form, and then we put it back inside the medium. If now we release the inclusion, this forces and pressure layer acts on the matrix: the body is not subject to external forces, but has an inner stress, due to the inclusion transformation. Inside the inclusion the incremental displacement field[4] is

$$v_i = v_i^{\text{inc}} = v_i^{\text{E}} + v_i^{\text{P}} \tag{6.1}$$

and the gradient of incremental deformation is:

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_i^{\text{inc}}}{\partial x_j} = \frac{\partial v_i^{\text{E}}}{\partial x_j} + \frac{1}{3}\frac{\partial v_k^{\text{P}}}{\partial x_k}\delta_{ij} \tag{6.2}$$

where the first term of the second member represents the elastic deformation generated at the release of the inclusion. It is important to note that even if the

[3] Usually the Eshelby inclusion problem in linear elasticity (Eshelby [12–14] and Willis [15]) is formulated by prescribing an inelastic *strain*, not a displacement gradient. If a displacement gradient is assigned instead, the skew-symmetric part of this, representing a rigid-body infinitesimal rotation, produces null fields outside the inclusion, meaning that the solution for the infinite body containing the inclusion consists in a pure, uniform, rigid-body rotation. The situation is different for incremental non-linear elasticity, when a prestress is present. In this case, a rigid-body rotation alters the incremental stress and therefore yields non-null fields outside the inclusion. These have to be added to a uniform incremental rotation field outside the inclusion, in order to generate a solution.

[4] The incremental quantities can be understood either as rates taken with respect to a time-like parameter, or, more interestingly, as incremental quantities related to a small displacement field superimposed upon a given configuration strained homogeneously but arbitrarily (see Ogden [22]).

material is incompressible, the incremental displacement $\boldsymbol{v}^{\mathrm{P}}$ does not necessarily satisfy the incompressibility constraint, while the field v_i^{E} does.[5] In particular we have:

$$\frac{\partial v_k^{\mathrm{E}}}{\partial x_k} = 0 \quad \longrightarrow \quad \frac{\partial v_k}{\partial x_k} = \frac{\partial v_k^{\mathrm{P}}}{\partial x_k} \tag{6.3}$$

The Green incremental nominal stresses are

$$\dot{t}_{ij}^{g} = \mathbb{K}_{ijkl}\frac{\partial v_l^{g}}{\partial x_k} + \dot{p}^{g}\delta_{ij} \tag{6.4}$$

In the infinite medium (the matrix) the incremental displacement field coincides with the field $\boldsymbol{v}^{\mathrm{E}}$, so that the gradient of incremental displacement is:

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_i^{\mathrm{E}}}{\partial x_j} \tag{6.5}$$

while the incremental nominal stress is given only by the elastic part of the incremental deformation as:

$$\dot{t}_{ij} = \dot{t}_{ij}^{\mathrm{E}} = \mathbb{K}_{ijkl}\frac{\partial v_l^{\mathrm{E}}}{\partial x_k} + \dot{p}^{\mathrm{E}}\delta_{ij} = \mathbb{K}_{ijkl}\frac{\partial}{\partial x_k}(v_l - v_l^{\mathrm{P}}) + (\dot{p} - \dot{p}^{\mathrm{P}})\delta_{ij} \tag{6.6}$$

namely, the initial transformation does not produce stress. The incremental mean stress $\dot{p}^{\mathrm{P}}$, defined inside the inclusion, is homogeneous and is associated to the deformation $\nabla_{\boldsymbol{x}}\boldsymbol{v}^{\mathrm{P}}$. We will show later that results will be independent of this, but it is better for the moment to keep track of a part of the stress that is related to the inclusion transformation, defined as

$$\dot{t}_{ij}^{\mathrm{P}} = \mathbb{K}_{ijkl}\frac{\partial v_l^{\mathrm{P}}}{\partial x_k} + \dot{p}^{\mathrm{P}}\delta_{ij} \tag{6.7}$$

which is uniform since $\nabla_{\boldsymbol{x}}\boldsymbol{v}^{\mathrm{P}}$ is uniform.

Here we exclude body forces inside the medium and the inclusion, therefore the equilibrium equations for the infinite body containing a concentrated unit force are written as:

$$\frac{\partial \dot{t}_{ij}^{g}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + \delta_{gj}\delta(\boldsymbol{x}-\boldsymbol{y}) = 0 \tag{6.8}$$

[5] In this treatise we assume that the initial deformation is accompanied by a volume change.

where $\dot{t}^g_{ij}$ is the Green function for the incremental nominal stress, namely the ij-component of the stress at the point $\boldsymbol{x}$ produced by a concentrated unit force applied in point $\boldsymbol{y}$ and in direction g, while $\delta(\boldsymbol{x}-\boldsymbol{y})$ is the Dirac delta. The Green's function set that can be used in this problem has been presented in the previous Chapter. For the points $\boldsymbol{x} \neq \boldsymbol{y}$ the equilibrium equations become:

$$\frac{\partial \dot{t}_{ij}(\boldsymbol{x})}{\partial x_i} = 0 \tag{6.9}$$

We assume that the singularity at point $\boldsymbol{y}$ is enclosed by a disk C_ε centered in $\boldsymbol{y}$, with radius ε and surface ∂C_ε. We define a closed, finite and simply connected domain D_{out} outside both the inclusion and the disk C_ε as:

$$D_{\text{out}} = \mathbb{R}^n \setminus \{\ D_{\text{in}} \cup C_\varepsilon\ \} \tag{6.10}$$

where $n = 2, 3$ depending on the dimensions of the space we are considering (for instance, $n = 2$ denotes the two-dimensional case); an alternative expression is

$$D_{\text{out}} = \complement_{\mathbb{R}^n}\ \{\ D_{\text{in}} \cup C_\varepsilon\ \} \tag{6.11}$$

where $\complement_{\mathbb{R}^n}$ denotes the complement with respect to the universe $\mathbb{R}^n$. The external boundary ∂D_{out} of this domain can be regarded as the union of the surfaces of the inclusion, of the disk and of and external boundary ∂D_{ext} as follows:

$$\partial D_{\text{out}} = \partial D_{\text{in}} \cup \partial C_\varepsilon \cup \partial D_{\text{ext}} \tag{6.12}$$

On the region D_{out} and on its boundary ∂D_{out} defined above, we can apply the Betti identity, thus yielding

$$\int_{D_{\text{out}}} \left[\frac{\partial \dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} v_j(\boldsymbol{x}) - \frac{\dot{t}_{ij}(\boldsymbol{x})}{\partial x_i} v^g_j(\boldsymbol{x}-\boldsymbol{y}) \right] \mathrm{d}V_{\boldsymbol{x}} = 0 \tag{6.13}$$

Using the product rule for differentiation, the above equation can be rewrit-

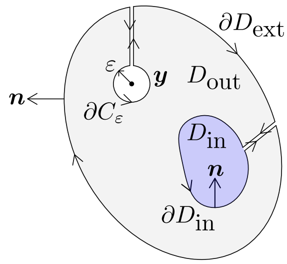


Figure 6.2: Integration domain D_{out} for an infinite body containing an inclusion (shown in purple) of volume D_{in} and surface ∂D_{in}; the singularity at $\boldsymbol{y}$ is enclosed in a disk C_ε of radius ε and surface ∂C_ε. D_{out} represents the simply connected region outside the inclusion and excluding the disk surrounding the singularity, whereas its external boundary is represented by ∂D_{ext}.

ten as follows:

$$\underbrace{\int_{D_{\text{out}}} \frac{\partial}{\partial x_i}\left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right]\mathrm{d}V_{\boldsymbol{x}}}_{=\text{Int}_1} + \\ - \underbrace{\int_{D_{\text{out}}}\left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})\frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \dot{t}_{ij}(\boldsymbol{x})\frac{\partial v^g_j}{\partial x_i}(\boldsymbol{x}-\boldsymbol{y})\right]\mathrm{d}V_{\boldsymbol{x}}}_{=\text{Int}_2} = 0 \quad (6.14)$$

The second integral of (6.14) can be calculated as

$$\text{Int}_2 = \int_{D_{\text{out}}}\left[\mathbb{K}_{ijkl}\frac{\partial v^g_l(\boldsymbol{x}-\boldsymbol{y})}{\partial x_k}\frac{\partial v_j(\boldsymbol{x})}{\partial x_i} + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})\delta_{ij}\frac{\partial v_j(\boldsymbol{x})}{\partial x_i} + \right. \\ \left. - \mathbb{K}_{ijkl}\frac{\partial v_l(\boldsymbol{x})}{\partial x_k}\frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} - \dot{p}(\boldsymbol{x})\delta_{ij}\frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}\right]\mathrm{d}V_{\boldsymbol{x}} \quad (6.15)$$

but, due to the incompressibility constraint[6] and the major symmetry of the constitutive tensor $\mathbb{K}$, we have $\text{Int}_2 = 0$. On application of the divergence

[6] We recall that

$$\frac{\partial v_j(\boldsymbol{x})}{\partial x_j} = \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j} = 0$$

theorem to the term Int_1, equation (6.14) reduces to

$$\int_{\partial D_{\text{out}}} \left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right] n_i \,\mathrm{d}S_{\boldsymbol{x}} = 0 \tag{6.16}$$

where $\boldsymbol{n}$ is the normal outward vector of the integration boundary D_{out}.

By decomposing the integration boundary according to (6.12), and taking the limit for $\varepsilon \to 0$ (namely, we reduce the disk until it degenerates in a point), and moving the external boundary ∂D_{ext} to the infinite, we obtain:

$$\begin{aligned}\int_{\partial D_{\text{in}}} &\left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right] n_i \,\mathrm{d}S_{\boldsymbol{x}}+ \\ &+ \lim_{\varepsilon\to 0}\int_{\partial C_\varepsilon} \left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right] n_i \,\mathrm{d}S_{\boldsymbol{x}}+ \\ &\quad + \int_{\partial D_{\text{ext}}} \left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right] n_i \,\mathrm{d}S_{\boldsymbol{x}} = 0\end{aligned} \tag{6.17}$$

The third integral of the above expression is equal to zero, since we assume that incremental stress and displacement field are decaying at infinity. Since $v^g_i \sim \ln r$, on the disk C_ε we have:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial C_\varepsilon} &\dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})n_i \,\mathrm{d}S_{\boldsymbol{x}} = \\ &= \lim_{\varepsilon\to 0}\int_{C_\varepsilon} -\frac{\partial}{\partial x_i}\left[\dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right] \mathrm{d}V_{\boldsymbol{x}} = 0\end{aligned} \tag{6.18}$$

if the volume of C_ε (and therefore its surface) tends to zero, while

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial C_\varepsilon} &\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x})n_i \,\mathrm{d}S_{\boldsymbol{x}} = \\ &= \lim_{\varepsilon\to 0}\int_{C_\varepsilon} -\left[\frac{\partial \dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}v_j(\boldsymbol{x}) + \dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})\frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right] \mathrm{d}V_{\boldsymbol{x}}\end{aligned} \tag{6.19}$$

where the second addend in the second integral gives a null contribution for $\varepsilon \to 0$, when the volume of C_ε tends to zero. Note that in expressions (6.18) and (6.19) there is a change of sign, since the first member in both expressions has the normal vector n_i pointing inside the region C_ε, while the divergence theorem is applied with the outward normal. Taking into account the properties of the Dirac delta function, we can write:

$$\lim_{\varepsilon\to 0}\int_{\partial C_\varepsilon} t^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x})n_i\,\mathrm{d}S_{\boldsymbol{x}} = \\ = \lim_{\varepsilon\to 0}\int_{C_\varepsilon} \delta_{gj}\delta(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x})\,\mathrm{d}V_{\boldsymbol{x}} = v_g(\boldsymbol{y}) \qquad (6.20)$$

Note that the normal vector of the first integral (6.18) and the first of (6.19) are pointing inside the region C_ε, while the minus appearing in the second integrals of the same equations arises from the change of direction of the normal (outward from the region C_ε).[7]

Now equation (6.17) becomes

$$\int_{\partial D_{\text{in}}} \left[t^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right] n_i\,\mathrm{d}S_{\boldsymbol{x}} + v_g(\boldsymbol{y}) = 0 \qquad (6.21)$$

where the normal $\boldsymbol{n}$ of the first integral is pointing inside the inclusion. With a change of sign in the normal vector, we obtain:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right] n_i\,\mathrm{d}S_{\boldsymbol{x}} \qquad (6.22)$$

where now the normal is outward (with respect to the inclusion).

A further application of the divergence theorem yields

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \frac{\partial}{\partial x_i}\left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})v_j(\boldsymbol{x}) - \dot{t}_{ij}(\boldsymbol{x})v^g_j(\boldsymbol{x}-\boldsymbol{y})\right]\mathrm{d}V_{\boldsymbol{x}} = \\ = \int_{D_{\text{in}}}\left[\frac{\partial \dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}v_j(\boldsymbol{x}) - \frac{\partial \dot{t}_{ij}(\boldsymbol{x})}{\partial x_i}v^g_j(\boldsymbol{x}-\boldsymbol{y})\right]\mathrm{d}V_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}}\left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})\frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \dot{t}_{ij}(\boldsymbol{x})\frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}\right]\mathrm{d}V_{\boldsymbol{x}} \qquad (6.23)$$

[7] Note that equations (6.18)-(6.20) can be obtained starting from equilibrium equations.

but inside the inclusion we have e

$$\frac{\partial \dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} = 0 \tag{6.24a}$$

$$\frac{\partial \dot{t}_{ij}(\boldsymbol{x})}{\partial x_i} = 0 \tag{6.24b}$$

so that (6.23) reduces to

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \left[\dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \dot{t}_{ij}(\boldsymbol{x}) \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \right] \mathrm{d}V_{\boldsymbol{x}} \tag{6.25}$$

Furthermore, within the inclusion equations (6.2) and (6.6) are still valid, and they allow us to write the following relations:

$$\begin{aligned} \dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} &= \mathbb{K}_{ijkl} \frac{\partial v^g_l(\boldsymbol{x}-\boldsymbol{y})}{\partial x_k} \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} + \\ &\quad + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \delta_{ij} \left[\frac{\partial v^{\mathrm{E}}_j(\boldsymbol{x})}{\partial x_i} + \frac{\partial v^{\mathrm{P}}_j(\boldsymbol{x})}{\partial x_i} \right] = \\ &= \mathbb{K}_{ijkl} \frac{\partial v^g_l(\boldsymbol{x}-\boldsymbol{y})}{\partial x_k} \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v^{\mathrm{P}}_m(\boldsymbol{x})}{\partial x_m} \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} \dot{t}_{ij}(\boldsymbol{x}) \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} &= \mathbb{K}_{ijkl} \frac{\partial}{\partial x_k} \left[v_l(\boldsymbol{x}) - v^{\mathrm{P}}_l(\boldsymbol{x}) \right] \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + \\ &\quad + \left[\dot{p}(\boldsymbol{x}) - \dot{p}^{\mathrm{P}}(\boldsymbol{x}) \right] \delta_{ij} \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} = \\ &= \mathbb{K}_{ijkl} \frac{\partial v_l(\boldsymbol{x})}{\partial x_k} \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} - \mathbb{K}_{ijkl} \frac{\partial v^{\mathrm{P}}_l(\boldsymbol{x})}{\partial x_k} \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \end{aligned} \tag{6.27}$$

in which we used the incompressibility constraint for the fields $v^{\mathrm{E}}_j(\boldsymbol{x})$ and $v^g_j(\boldsymbol{x}-\boldsymbol{y})$ respectively. Exploiting the relations (6.26) and (6.27), we can rewrite (6.25) as:

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v^{\mathrm{P}}_l(\boldsymbol{x})}{\partial x_k} \frac{\partial v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v^{\mathrm{P}}_m(\boldsymbol{x})}{\partial x_m} \right] \mathrm{d}V_{\boldsymbol{x}} \tag{6.28}$$

On application of the divergence theorem to the first addend of the integrand in (6.28), and taking into account that $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\mathrm{P}}$ is uniform[8] and the incremental displacement field is solenoidal, we obtain the integral equation for the incremental displacement field outside the inclusion produced by the uniform inelastic field $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\mathrm{P}}$:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_l^{\mathrm{P}}(\boldsymbol{x})}{\partial x_k} v_j^g(\boldsymbol{x}-\boldsymbol{y}) n_i \, \mathrm{d}S_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}} \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_m^{\mathrm{P}}(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \tag{6.29}$$

where $\boldsymbol{n}$ is the outward normal of the inclusion. If we want to express the Green's functions as $G_{ij}(\boldsymbol{y}-\boldsymbol{x})$ instead of $G_{ij}(\boldsymbol{x}-\boldsymbol{y})$, we obtain:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_l^{\mathrm{P}}(\boldsymbol{x})}{\partial x_k} v_j^g(\boldsymbol{y}-\boldsymbol{x}) n_i \, \mathrm{d}S_{\boldsymbol{x}} + \\ - \int_{D_{\text{in}}} \dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) \frac{\partial v_m^{\mathrm{P}}(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \tag{6.30}$$

so that there is a change of sign in the volume integral, since the mean stress is an odd function, while the incremental displacements v_j^g are even functions and therefore do not have a change of sign.[9] Note that in equation (6.29) there is a volume integral involving both deviatoric and volumetric part of $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\mathrm{P}}$; in particular we observe that this integral is equal to zero for pure deviatoric incremental displacement gradients.

If we introduce a potential $P_i^g(\boldsymbol{x}-\boldsymbol{y})$ such that

$$\dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) = \frac{\partial P_i^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \tag{6.31}$$

[8] This condition implies that the following relation holds:

$$\frac{\partial^2 v_i^{\mathrm{P}}(\boldsymbol{x})}{\partial x_j \partial x_k} = 0$$

[9] We recall that $\dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) = -\dot{p}^g(\boldsymbol{x}-\boldsymbol{y})$, while $v_j^g(\boldsymbol{y}-\boldsymbol{x}) = v_j^g(\boldsymbol{x}-\boldsymbol{y})$.

the equation (6.28) can be rewritten as

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + \frac{\partial P_i^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \right] \mathrm{d}V_{\boldsymbol{x}} \tag{6.32}$$

and we can use the product rule for differentiation to obtain

$$\begin{aligned} v_g(\boldsymbol{y}) = & \int_{D_{\text{in}}} \frac{\partial}{\partial x_i} \left[\mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} v_j^g(\boldsymbol{x}-\boldsymbol{y}) + P_i^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \right] \mathrm{d}V_{\boldsymbol{x}} + \\ & - \int_{D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial^2 v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k \partial x_i} v_j^g(\boldsymbol{x}-\boldsymbol{y}) + P_i^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial^2 v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m \partial x_i} \right] \mathrm{d}V_{\boldsymbol{x}} \end{aligned} \tag{6.33}$$

However, because $v_l^{\text{P}}(\boldsymbol{x})$ is homogeneous and on application of divergence theorem, the above expression reduces to:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} v_j^g(\boldsymbol{x}-\boldsymbol{y}) + P_i^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \right] n_i \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.34}$$

and in this way the incremental displacement field for the inclusion is expressed only in terms of boundary integrals.

Nevertheless, in equation (6.34) the potential $P_i^g(\boldsymbol{x}-\boldsymbol{y})$ is unknown. An easy way to define it is the following:

$$P_i^g(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{n} \int \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \, \mathrm{d}x_i \tag{6.35}$$

where n is the dimension of the space considered (for instance, $n = 2$ for a two-dimensional problem and $n = 3$ for a three-dimensional one); therefore $P_i^g(\boldsymbol{x}-\boldsymbol{y})$ is a tensor containing the primitives of the Green's mean stress. Within the two-dimensional case, we can introduce a scalar coefficient $R_i(\hat{\alpha})$ defined as:

$$R_i(\hat{\alpha}) = \delta_{i1}\hat{\alpha} + (1-\hat{\alpha})\delta_{i2}, \tag{6.36}$$

where δ_{ij} is the Kronecker delta, $i, j = 1, 2$ and $\hat{\alpha} \in [0, 1]$, so that we can obtain a family of potentials $P_i^g(\boldsymbol{x}-\boldsymbol{y})$ dependent on the arbitrary choice of the coefficient $\hat{\alpha} \in [0,1]$ in the form

$$P_i^g(\boldsymbol{x}-\boldsymbol{y}) = R_i(\hat{\alpha}) \int \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \, \mathrm{d}x_i, \tag{6.37}$$

where index i is not summed.

The gradient of the incremental displacement produced by the inclusion is:

$$\frac{\partial v_g(\boldsymbol{y})}{\partial y_r} = \int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial y_r} n_i \, \mathrm{d}S_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}} \frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial y_r} \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \tag{6.38}$$

since $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\text{P}}$ is homogeneous. Furthermore, for the Green's functions the following rule for the change of differentiation variable applies:

$$\frac{\partial G_{ij}(\boldsymbol{x}-\boldsymbol{y})}{\partial y_i} = -\frac{\partial G_{ij}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \tag{6.39}$$

so that we can write:

$$\frac{\partial v_g(\boldsymbol{y})}{\partial y_r} = -\int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r} n_i \, \mathrm{d}S_{\boldsymbol{x}} + \\ - \int_{D_{\text{in}}} \frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r} \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \tag{6.40}$$

In an analogous manner, the second gradient of the incremental displacement produced by the inclusion can be expressed as:

$$\frac{\partial^2 v_g(\boldsymbol{y})}{\partial y_r \partial y_s} = \int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} \frac{\partial^2 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial y_r \partial y_s} n_i \, \mathrm{d}S_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}} \frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial y_r \partial y_s} \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \tag{6.41}$$

because $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\text{P}}$ is homogeneous. To calculate the second order derivative, the rule (6.39) has been applied twice[10], so that the second gradient becomes:

$$\frac{\partial^2 v_g(\boldsymbol{y})}{\partial y_r \partial y_s} = \int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} \frac{\partial^2 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} n_i \, \mathrm{d}S_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}} \frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \quad (6.42)$$

6.2 The dislocation problem

We consider the final configuration as the result of the application of a dislocation with intensity $d_i(\boldsymbol{x}) = v_i^{\text{P}}(\boldsymbol{x})$, where $v_i^{\text{P}}(\boldsymbol{x})$ is the displacement field corresponding to the uniform deformation $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\text{P}}$ defined in the previous Section.

Within this context, this type of dislocation is often called 'Somigliana dislocation'; for instance, it is required that the elastic displacements v_i^{E} be discontinuous by a quantity d_i through ∂D_{in}, while ∂D_{in} remains in equilibrium in absence of external forces; furthermore, the surface ∂D_{in} can be open or closed, and d_i may vary on ∂D_{in}.

Within the inclusion, we have:

$$v_i^{\text{E}} = v_i - v_i^{\text{P}} \quad (6.43)$$

since the incremental elastic displacements must be measured in the unconstrained material configuration, while in the matrix we have $v_i^{\text{E}} = v_i$. We require that the total incremental displacements v_i are continuous, and ∂D_{in} must be in equilibrium.

Taking into account that

$$\frac{\partial}{\partial x_i} \left[\mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} v_j^g(\boldsymbol{x}-\boldsymbol{y}) \right] = \mathbb{K}_{ijkl} \frac{\partial^2 v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k \partial x_i} v_j^g(\boldsymbol{x}-\boldsymbol{y}) + \\ + \mathbb{K}_{ijkl} \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_k} \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \quad (6.44)$$

[10] A double application of the rule (6.39) implies a double change of sign, so that the second order derivative has no change of sign whenever we change the variable of differentiation.

namely,

$$\frac{\partial}{\partial x_i}\left[\mathbb{K}_{ijkl}\frac{\partial v_l^{\mathrm{P}}(\boldsymbol{x})}{\partial x_k}v_j^g(\boldsymbol{x}-\boldsymbol{y})\right]=\mathbb{K}_{ijkl}\frac{\partial v_l^{\mathrm{P}}(\boldsymbol{x})}{\partial x_k}\frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \tag{6.45}$$

but since $\boldsymbol{v}^{\mathrm{P}}$ is uniform (see note 8 on page 144), we can apply the divergence theorem to equation (6.29)

$$v_g(\boldsymbol{y})=\int_{D_{\text{in}}}\mathbb{K}_{ijkl}\frac{\partial v_l^{\mathrm{P}}(\boldsymbol{x})}{\partial x_k}\frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}\,\mathrm{d}V_{\boldsymbol{x}}+\int_{D_{\text{in}}}\dot{p}^g(\boldsymbol{x}-\boldsymbol{y})\frac{\partial v_m^{\mathrm{P}}(\boldsymbol{x})}{\partial x_m}\,\mathrm{d}V_{\boldsymbol{x}} \tag{6.46}$$

and collecting a derivative with respect to x_k yields:

$$\begin{aligned}v_g(\boldsymbol{y})=\int_{D_{\text{in}}}\Bigg\{\frac{\partial}{\partial x_k}\left[\mathbb{K}_{ijkl}v_l^{\mathrm{P}}(\boldsymbol{x})\frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}\right]-\mathbb{K}_{ijkl}v_l^{\mathrm{P}}(\boldsymbol{x})\frac{\partial^2 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_k\partial x_i}+\\+\dot{p}^g(\boldsymbol{x}-\boldsymbol{y})\frac{\partial v_m^{\mathrm{P}}(\boldsymbol{x})}{\partial x_m}\Bigg\}\,\mathrm{d}V_{\boldsymbol{x}}\end{aligned} \tag{6.47}$$

From the equilibrium equations, within the inclusion we have:

$$\frac{\partial \dot{t}_{ij}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}=0 \tag{6.48}$$

from which follows that

$$\mathbb{K}_{ijkl}\frac{\partial^2 v_l^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i\partial x_k}+\frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}\delta_{ij}=0 \tag{6.49}$$

and recalling that $\mathbb{K}_{ijkl}$ has the major symmetry

$$\mathbb{K}_{ijkl}\frac{\partial^2 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i\partial x_k}+\frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}\delta_{il}=0 \tag{6.50}$$

so that

$$\mathbb{K}_{ijkl}v_l^{\mathrm{P}}(\boldsymbol{x})\frac{\partial^2 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i\partial x_k}=-v_l^{\mathrm{P}}(\boldsymbol{x})\frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_l} \tag{6.51}$$

If we insert the above expression into (6.47) we obtain:

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \left\{ \frac{\partial}{\partial x_k} \left[\mathbb{K}_{ijkl} v_l^{\mathrm{P}}(\boldsymbol{x}) \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \right] + v_l^{\mathrm{P}}(\boldsymbol{x}) \frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_l} + \right.$$
$$\left. + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_m^{\mathrm{P}}(\boldsymbol{x})}{\partial x_m} \right\} \mathrm{d}V_{\boldsymbol{x}} \quad (6.52)$$

Here we can collect the derivative with respect to x_l inside the integrand as:

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \left\{ \frac{\partial}{\partial x_k} \left[\mathbb{K}_{ijkl} v_l^{\mathrm{P}}(\boldsymbol{x}) \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \right] + \frac{\partial}{\partial x_l} \left[v_l^{\mathrm{P}}(\boldsymbol{x}) \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \right] + \right.$$
$$\left. - \frac{\partial v_l^{\mathrm{P}}(\boldsymbol{x})}{\partial x_l} \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_m^{\mathrm{P}}(\boldsymbol{x})}{\partial x_m} \right\} \mathrm{d}V_{\boldsymbol{x}} \quad (6.53)$$

and a simplification yields:

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \frac{\partial}{\partial x_k} \left[\mathbb{K}_{ijkl} v_l^{\mathrm{P}}(\boldsymbol{x}) \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + v_l^{\mathrm{P}}(\boldsymbol{x}) \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \delta_{kl} \right] \mathrm{d}V_{\boldsymbol{x}} \quad (6.54)$$

If we apply again the divergence theorem, we obtain the integral equation for the incremental displacement field outside the inclusion produced by the inelastic uniform field $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\mathrm{P}}$:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \delta_{kl} \right] v_l^{\mathrm{P}}(\boldsymbol{x}) n_k \, \mathrm{d}S_{\boldsymbol{x}} \quad (6.55)$$

which is an expression fully equivalent to (6.29) and is expressed as a function of the incremental inelastic displacement $\boldsymbol{v}^{\mathrm{P}}$ of the initial transformation.

If we want to express the Green's functions as $G_{ij}(\boldsymbol{y}-\boldsymbol{x})$ instead of $G_{ij}(\boldsymbol{x}-\boldsymbol{y})$, in an analogous manner to what illustrated in the previous section, we must go back to equation (6.52), which now reads as:

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \left\{ \frac{\partial}{\partial x_k} \left[\mathbb{K}_{ijkl} v_l^{\text{P}}(\boldsymbol{x}) \frac{\partial v_j^g(\boldsymbol{y}-\boldsymbol{x})}{\partial x_i} \right] - v_l^{\text{P}}(\boldsymbol{x}) \frac{\partial \dot{p}^g(\boldsymbol{y}-\boldsymbol{x})}{\partial x_l} + \right. \\ \left. - \dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \right\} \mathrm{d}V_{\boldsymbol{x}} \tag{6.56}$$

Note that the signs of the second and third addend of the integrand changed;[11] If we collect a derivative with respect to x_l inside the integrand, we have

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \left\{ \frac{\partial}{\partial x_k} \left[\mathbb{K}_{ijkl} v_l^{\text{P}}(\boldsymbol{x}) \frac{\partial v_j^g(\boldsymbol{y}-\boldsymbol{x})}{\partial x_i} \right] - \frac{\partial}{\partial x_l} \left[v_l^{\text{P}}(\boldsymbol{x}) \dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) \right] + \right. \\ \left. + \frac{\partial v_l^{\text{P}}(\boldsymbol{x})}{\partial x_l} \dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) - \dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \right\} \mathrm{d}V_{\boldsymbol{x}} \tag{6.57}$$

and simplifying:

$$v_g(\boldsymbol{y}) = \int_{D_{\text{in}}} \frac{\partial}{\partial x_k} \left[\mathbb{K}_{ijkl} v_l^{\text{P}}(\boldsymbol{x}) \frac{\partial v_j^g(\boldsymbol{y}-\boldsymbol{x})}{\partial x_i} - v_l^{\text{P}}(\boldsymbol{x}) \dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) \delta_{kl} \right] \mathrm{d}V_{\boldsymbol{x}} \tag{6.58}$$

On application of the divergence theorem we obtain the integral equation for the incremental displacement field outside the inclusion:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v_j^g(\boldsymbol{y}-\boldsymbol{x})}{\partial x_i} - \dot{p}^g(\boldsymbol{y}-\boldsymbol{x}) \delta_{kl} \right] v_l^{\text{P}}(\boldsymbol{x}) n_k \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.59}$$

and we observe that with a change in the representation of the Green's functions we have a change of sign in the second addend of the integrand.

If we introduce the following notation for the incremental Green's traction along the surface with unit normal n_i

$$\tau_j^g(\boldsymbol{x}-\boldsymbol{y}) = \dot{t}_{ij}^g(\boldsymbol{x}-\boldsymbol{y}) n_i \tag{6.60}$$

[11] We recall that the mean stress is an odd function, while the incremental displacements v_j^g are even functions, so that they do not change the sign. See note 9 on page 144.

the equation (6.55) becomes:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \tau_m^g(\boldsymbol{x}-\boldsymbol{y}) v_m^{\text{P}}(\boldsymbol{x}) \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.61}$$

Note that the expressions for the components of τ_j^g are given both in singular and regularized forms by Bigoni *et al*. [10] and can be used to calculate the integral equation (6.61).

The gradient of the incremental displacement produced by the dislocation is:

$$\frac{\partial v_g(\boldsymbol{y})}{\partial y_r} = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial^2 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial y_r} + \frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial y_r} \delta_{kl} \right] v_l^{\text{P}} n_k \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.62}$$

since $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\text{P}}$ is homogeneous, and on application of the rule (6.39) we obtain:

$$\frac{\partial v_g(\boldsymbol{y})}{\partial y_r} = -\int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial^2 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_r} + \frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r} \delta_{kl} \right] v_l^{\text{P}} n_k \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.63}$$

Similarly, the second gradient of the incremental displacement produced by the dislocation writes as:

$$\frac{\partial^2 v_g(\boldsymbol{y})}{\partial y_r \partial y_s} = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial^3 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial y_r \partial y_s} + \frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial y_r \partial y_s} \delta_{kl} \right] v_l^{\text{P}} n_k \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.64}$$

being $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\text{P}}$ homogeneous; if we apply twice the rule (6.39), the second gradient becomes:[12]

$$\frac{\partial^2 v_g(\boldsymbol{y})}{\partial y_r \partial y_s} = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial^3 v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_r \partial x_s} + \frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} \delta_{kl} \right] v_l^{\text{P}} n_k \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.65}$$

[12] See note 10 on page 147.

6.3 The incremental mean stress of the inclusion

The incremental equilibrium equations are:

$$\frac{\partial \dot{t}_{ij}(\boldsymbol{x})}{\partial x_i} = 0 \tag{6.66a}$$

$$\frac{\partial \dot{t}^g_{ij}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} = 0 \tag{6.66b}$$

and the stresses are expressed as:

$$\dot{t}_{ij} = \mathbb{K}_{ijkl}\frac{\partial v_l(\boldsymbol{x})}{\partial x_k} + \dot{p}(\boldsymbol{x})\delta_{ij} \tag{6.67a}$$

$$\dot{t}^g_{ij} = \mathbb{K}_{ijkl}\frac{\partial v^g_l(\boldsymbol{x}-\boldsymbol{y})}{\partial x_k} + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})\delta_{ij} \tag{6.67b}$$

From equation (6.66a) we have:

$$\mathbb{K}_{ijkl}\frac{\partial^2 v_l(\boldsymbol{x})}{\partial x_i \partial x_k} + \frac{\partial \dot{p}(\boldsymbol{x})}{\partial x_i}\delta_{ij} = 0 \quad \longrightarrow \quad \frac{\partial \dot{p}(\boldsymbol{x})}{\partial x_i}\delta_{ij} = -\mathbb{K}_{ijkl}\frac{\partial^2 v_l(\boldsymbol{x})}{\partial x_i \partial x_k} \tag{6.68}$$

and changing the dummy indexes:

$$\frac{\partial \dot{p}(\boldsymbol{x})}{\partial x_i} = -\mathbb{K}_{sirg}\frac{\partial^2 v_g(\boldsymbol{x})}{\partial x_r \partial x_s} \tag{6.69}$$

We apply the equilibrium equation (6.69) to the incremental displacements (6.29):

$$\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = -\mathbb{K}_{sirg}\frac{\partial^2 v_g(\boldsymbol{y})}{\partial y_r \partial y_s} \tag{6.70}$$

A substitution of the second order derivative of the displacements (6.42) into the above relation provides:

$$\begin{aligned}\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = &-\int_{\partial D_{\text{in}}} \mathbb{K}_{sirg}\mathbb{K}_{jklm}\frac{\partial v^{\mathrm{P}}_m(\boldsymbol{x})}{\partial x_l}\frac{\partial^2 v^g_j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} n_j \,\mathrm{d}S_{\boldsymbol{x}} + \\ &-\int_{D_{\text{in}}} \mathbb{K}_{sirg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s}\frac{\partial v^{\mathrm{P}}_m(\boldsymbol{x})}{\partial x_m}\,\mathrm{d}V_{\boldsymbol{x}}\end{aligned} \tag{6.71}$$

but from the equilibrium equations (6.66b):

$$\mathbb{K}_{ijkl}\frac{\partial^2 v_l^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_k} + \frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i}\delta_{ij} = 0 \tag{6.72}$$

and we obtain:

$$\mathbb{K}_{sirk}\frac{\partial^2 v_k^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = \mathbb{K}_{sirg}\frac{\partial^2 v_g^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = -\frac{\partial \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \tag{6.73}$$

In this way, equation (6.71) can be rewritten as:

$$\begin{aligned}\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = \int_{\partial D_{\text{in}}} \mathbb{K}_{jklm}\frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_l}\frac{\partial \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} n_j \,\mathrm{d}S_{\boldsymbol{x}} + \\ - \int_{D_{\text{in}}} \mathbb{K}_{sirg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s}\frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m}\,\mathrm{d}V_{\boldsymbol{x}}\end{aligned} \tag{6.74}$$

We derive the equilibrium equations (6.66b) with respect to x_s:

$$\mathbb{K}_{trpq}\frac{\partial^3 v_q^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_p \partial x_s \partial x_t} + \frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_s}\delta_{ir} = 0 \tag{6.75}$$

and multiplying by $\mathbb{K}_{sirg}$ provides:

$$\mathbb{K}_{sirg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = -\mathbb{K}_{sirg}\mathbb{K}_{trpq}\frac{\partial^3 v_q^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_p \partial x_s \partial x_t} \tag{6.76}$$

Now we must calculate all the components of the relation (6.76): note that only the index i is free and we must take into account the incompressibility constraint and the reciprocity of the Green's functions ($v_i^g(\boldsymbol{x}-\boldsymbol{y}) = v_g^i(\boldsymbol{x}-\boldsymbol{y})$). For simplicity, we omit the arguments $(\boldsymbol{x}-\boldsymbol{y})$ of the Green's functions in some of the following steps.

$$\begin{aligned}\mathbb{K}_{sirg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} = \mathbb{K}_{1irg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_1} + \mathbb{K}_{2irg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_2} = \\ = \mathbb{K}_{1i1g}\frac{\partial^2 \dot{p}^g}{\partial x_1^2} + \mathbb{K}_{1i2g}\frac{\partial^2 \dot{p}^g}{\partial x_2 \partial x_1} + \mathbb{K}_{2i1g}\frac{\partial^2 \dot{p}^g}{\partial x_1 \partial x_2} + \mathbb{K}_{2i2g}\frac{\partial^2 \dot{p}^g}{\partial x_2^2}\end{aligned} \tag{6.77}$$

namely:

$$\mathbb{K}_{sirg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} = \mathbb{K}_{1i11}\frac{\partial^2 \dot{p}^1}{\partial x_1^2} + \mathbb{K}_{1i12}\frac{\partial^2 \dot{p}^2}{\partial x_1^2} + (\mathbb{K}_{1i21} + \mathbb{K}_{2i11})\frac{\partial^2 \dot{p}^1}{\partial x_1 \partial x_2} + \\ + (\mathbb{K}_{1i22} + \mathbb{K}_{2i12})\frac{\partial^2 \dot{p}^2}{\partial x_1 \partial x_2} + \mathbb{K}_{2i21}\frac{\partial^2 \dot{p}^1}{\partial x_2^2} + \mathbb{K}_{2i22}\frac{\partial^2 \dot{p}^2}{\partial x_2^2} \quad (6.78)$$

Since $\mathbb{K}_{ijkl} = 0$ if three indexes are the same and one is different, we have:

$$\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} = \mathbb{K}_{1111}\frac{\partial^2 \dot{p}^1}{\partial x_1^2} + (\mathbb{K}_{1122} + \mathbb{K}_{2112})\frac{\partial^2 \dot{p}^2}{\partial x_1 \partial x_2} + \mathbb{K}_{2121}\frac{\partial^2 \dot{p}^1}{\partial x_2^2} \quad (6.79)$$

so that

$$\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} = \left(\mu_* - \frac{\sigma}{2} - p\right)\frac{\partial^2 \dot{p}^1}{\partial x_1^2} + (\mu - \mu_* - p)\frac{\partial^2 \dot{p}^2}{\partial x_1 \partial x_2} + \\ + \left(\mu_* - \frac{\sigma}{2}\right)\frac{\partial^2 \dot{p}^1}{\partial x_2^2} \quad (6.80)$$

and analogously

$$\mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} = \mathbb{K}_{1212}\frac{\partial^2 \dot{p}^2}{\partial x_1^2} + (\mathbb{K}_{1221} + \mathbb{K}_{2211})\frac{\partial^2 \dot{p}^1}{\partial x_1 \partial x_2} + \mathbb{K}_{2222}\frac{\partial^2 \dot{p}^2}{\partial x_2^2} \quad (6.81)$$

so that

$$\mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} = \left(\mu_* + \frac{\sigma}{2}\right)\frac{\partial^2 \dot{p}^2}{\partial x_1^2} + (\mu - \mu_* - p)\frac{\partial^2 \dot{p}^1}{\partial x_1 \partial x_2} + \\ + \left(\mu_* + \frac{\sigma}{2} - p\right)\frac{\partial^2 \dot{p}^2}{\partial x_2^2} \quad (6.82)$$

Now we need the expressions of the second order derivatives of the Green's mean stress $\dot{p}^g(\boldsymbol{x} - \boldsymbol{y})$ with respect to variables x_r and x_s, evaluated as

$$\frac{\partial^2 \dot{p}^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_r \partial x_s} = -\mathbb{K}_{trpq}\frac{\partial^3 v_q^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_p \partial x_s \partial x_t} \quad (6.83)$$

and these derivatives must be substituted into (6.80) and (6.82). Omitting for simplicity the arguments of the functions, the relation (6.83) can be expanded into:

$$
\begin{aligned}
-\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} &= \mathbb{K}_{1rpq} \frac{\partial^3 v_q^g}{\partial x_p \partial x_s \partial x_1} + \mathbb{K}_{2rpq} \frac{\partial^3 v_q^g}{\partial x_p \partial x_s \partial x_2} = \\
&= \mathbb{K}_{1r1q} \frac{\partial^3 v_q^g}{\partial x_1^2 \partial x_s} + \mathbb{K}_{1r2q} \frac{\partial^3 v_q^g}{\partial x_1 \partial x_2 \partial x_s} + \mathbb{K}_{2r1q} \frac{\partial^3 v_q^g}{\partial x_1 \partial x_2 \partial x_s} + \\
&\quad + \mathbb{K}_{2r2q} \frac{\partial^3 v_q^g}{\partial x_2^2 \partial x_s}
\end{aligned}
\tag{6.84}
$$

so that

$$
\begin{aligned}
-\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} &= \mathbb{K}_{1r11} \frac{\partial^3 v_1^g}{\partial x_1^2 \partial x_s} + \mathbb{K}_{1r12} \frac{\partial^3 v_2^g}{\partial x_1^2 \partial x_s} + (\mathbb{K}_{1r21} + \mathbb{K}_{2r11}) \frac{\partial^3 v_1^g}{\partial x_1 \partial x_2 \partial x_s} + \\
&\quad + (\mathbb{K}_{1r22} + \mathbb{K}_{2r12}) \frac{\partial^3 v_2^g}{\partial x_1 \partial x_2 \partial x_s} + \mathbb{K}_{2r21} \frac{\partial^3 v_1^g}{\partial x_2^2 \partial x_s} + \mathbb{K}_{2r22} \frac{\partial^3 v_2^g}{\partial x_2^2 \partial x_s}
\end{aligned}
\tag{6.85}
$$

When calculating the second order derivatives of $\dot{p}^g(\boldsymbol{x}-\boldsymbol{y})$, it is possible to change the order of differentiation if we assume the continuity of the displacement field $\boldsymbol{v}^g(\boldsymbol{x}-\boldsymbol{y})$.[13] Taking into account the incompressibility constraint,

[13] We recall that the integration is performed outside the inclusion.

equation (6.85) can be written through the following expressions:

$$-\frac{\partial^2 \dot{p}^g}{\partial x_1^2} = \mathbb{K}_{1111}\frac{\partial^3 v_1^g}{\partial x_1^3} + (\mathbb{K}_{1122}+\mathbb{K}_{2112})\frac{\partial^3 v_2^g}{\partial x_1^2 \partial x_2} + \mathbb{K}_{2121}\frac{\partial^3 v_1^g}{\partial x_1 \partial x_2^2} =$$
$$= (\mathbb{K}_{1111}-\mathbb{K}_{1122}-\mathbb{K}_{2112})\frac{\partial^3 v_1^g}{\partial x_1^3} + \mathbb{K}_{2121}\frac{\partial^3 v_1^g}{\partial x_1 \partial x_2^2} \tag{6.86a}$$

$$-\frac{\partial^2 \dot{p}^g}{\partial x_1 \partial x_2} = \mathbb{K}_{1111}\frac{\partial^3 v_1^g}{\partial x_1^2 \partial x_2} + (\mathbb{K}_{1122}+\mathbb{K}_{2112})\frac{\partial^3 v_2^g}{\partial x_1 \partial x_2^2} + \mathbb{K}_{2121}\frac{\partial^3 v_1^g}{\partial x_2^3} =$$
$$= (\mathbb{K}_{1111}-\mathbb{K}_{1122}-\mathbb{K}_{2112})\frac{\partial^3 v_1^g}{\partial x_1^2 \partial x_2} + \mathbb{K}_{2121}\frac{\partial^3 v_1^g}{\partial x_2^3} \tag{6.86b}$$

$$-\frac{\partial^2 \dot{p}^g}{\partial x_1 \partial x_2} = \mathbb{K}_{1212}\frac{\partial^3 v_2^g}{\partial x_1^3} + (\mathbb{K}_{1221}+\mathbb{K}_{2211})\frac{\partial^3 v_1^g}{\partial x_1^2 \partial x_2} + \mathbb{K}_{2222}\frac{\partial^3 v_2^g}{\partial x_1 \partial x_2^2} =$$
$$= \mathbb{K}_{1212}\frac{\partial^3 v_2^g}{\partial x_1^3} + (\mathbb{K}_{1221}+\mathbb{K}_{2211}-\mathbb{K}_{2222})\frac{\partial^3 v_1^g}{\partial x_1^2 \partial x_2} \tag{6.86c}$$

$$-\frac{\partial^2 \dot{p}^g}{\partial x_2^2} = \mathbb{K}_{1212}\frac{\partial^3 v_2^g}{\partial x_1^2 \partial x_2} + (\mathbb{K}_{1221}+\mathbb{K}_{2211})\frac{\partial^3 v_1^g}{\partial x_1 \partial x_2^2} + \mathbb{K}_{2222}\frac{\partial^3 v_2^g}{\partial x_2^3} =$$
$$= \mathbb{K}_{1212}\frac{\partial^3 v_2^g}{\partial x_1^2 \partial x_2} + (\mathbb{K}_{1221}+\mathbb{K}_{2211}-\mathbb{K}_{2222})\frac{\partial^3 v_1^g}{\partial x_1 \partial x_2^2} \tag{6.86d}$$

namely

$$-\frac{\partial^2 \dot{p}^g}{\partial x_1^2} = -\left(2\mu_* - \mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^g}{\partial x_1^3} - \left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^g}{\partial x_1 \partial x_2^2} \tag{6.87a}$$

$$-\frac{\partial^2 \dot{p}^g}{\partial x_1 \partial x_2} = -\left(2\mu_* - \mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^g}{\partial x_1^2 \partial x_2} - \left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^g}{\partial x_2^3} \tag{6.87b}$$

$$-\frac{\partial^2 \dot{p}^g}{\partial x_1 \partial x_2} = -\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^g}{\partial x_1^3} + \left(2\mu_* - \mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_1^g}{\partial x_1^2 \partial x_2} \tag{6.87c}$$

$$-\frac{\partial^2 \dot{p}^g}{\partial x_2^2} = -\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^g}{\partial x_1^2 \partial x_2} + \left(2\mu_* - \mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_1^g}{\partial x_1 \partial x_2^2} \tag{6.87d}$$

A substitution of equations (6.87a), (6.87b) and (6.87d) into (6.80) yields:

$$\begin{aligned}\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} &= \left(\mu - \frac{\sigma}{2}\right)\left[-\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^1}{\partial x_1^2 \partial x_2} + \left(2\mu_* - \mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_1^1}{\partial x_1 \partial x_2^2}\right] + \\ &- \left(\mu_* - \frac{\sigma}{2} - p\right)\left[\left(2\mu_* - \mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^1}{\partial x_1^3} + \left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^1}{\partial x_1 \partial x_2^2}\right] + \\ &- (\mu - \mu_* - p)\left[\left(2\mu_* - \mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2}{\partial x_1^2 \partial x_2} + \left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2}{\partial x_2^3}\right]\end{aligned} \tag{6.88}$$

The incompressibility constraint allows us to rewrite the following terms:

$$\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1 \partial x_2^2} = -\frac{\partial^3 v_2^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2^3} \tag{6.89a}$$

$$\frac{\partial^3 v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2 \partial x_2} = -\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} \tag{6.89b}$$

Furthermore, recalling that $v_i^g(\boldsymbol{x}-\boldsymbol{y}) = v_g^j(\boldsymbol{x}-\boldsymbol{y})$, we have:

$$\begin{aligned}\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} &= \left(4\mu\mu_* - 4\mu_*^2 - \mu\sigma + 2\mu_*\sigma - \frac{\sigma^2}{2}\right)\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} + \\ &- \sigma\left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_2^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2^3}\end{aligned} \tag{6.90}$$

Matching equations (6.87b) and (6.87c) for $g = 2$:

$$\begin{aligned}&-\left(2\mu_* - \mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2 \partial x_2} - \left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2^3} = \\ &\qquad = \left(2\mu_* - \mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2 \partial x_2} - \left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3}\end{aligned} \tag{6.91}$$

and recalling equations (6.89), we obtain the following condition:

$$\begin{aligned}&(2\mu_* - \mu)\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} - \left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_2^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2^3} = \\ &\qquad = -(2\mu_* - \mu)\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} - \left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3}\end{aligned} \tag{6.92}$$

A substitution of equation (6.92) into (6.90) provides:

$$\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = \left(4\mu\mu_* - 4\mu_*^2 + \mu\sigma - 2\mu_*\sigma - \frac{\sigma^2}{2}\right)\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} + \\ - \sigma\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} \quad (6.93)$$

and collecting the derivative with respect to x_1 we obtain:

$$\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = \frac{\partial}{\partial x_1}\left[-\sigma\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} + \right. \\ \left. + \left(4\mu\mu_* - 4\mu_*^2 + \mu\sigma - 2\mu_*\sigma - \frac{\sigma^2}{2}\right)\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3}\right] \quad (6.94)$$

We can repeat this procedure by substituting equations (6.87a), (6.87c) and (6.87d) into (6.82):

$$\mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g}{\partial x_r \partial x_s} = -\left(\mu + \frac{\sigma}{2}\right)\left[\left(2\mu_* - \mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2}{\partial x_1^3} + \left(\mu - \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2}{\partial x_1 \partial x_2^2}\right] + \\ + (\mu - \mu_* - p)\left[\left(2\mu_* - \mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_1^1}{\partial x_1^2 \partial x_2} - \left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^1}{\partial x_1^3}\right] + \\ + \left(\mu_* + \frac{\sigma}{2} - p\right)\left[\left(2\mu_* - \mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_1^2}{\partial x_1 \partial x_2^2} - \left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^2}{\partial x_1^2 \partial x_2}\right] \quad (6.95)$$

The incompressibility constraint allows us to rewrite the following terms:

$$\frac{\partial^3 v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^3} = -\frac{\partial^3 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2 \partial x_2} \quad (6.96a)$$

$$\frac{\partial^3 v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1 \partial x_2^2} = -\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2 \partial x_2} \quad (6.96b)$$

thus yielding

$$\mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = \left(4\mu\mu_* - 4\mu_*^2 + \mu\sigma - 2\mu_*\sigma - \frac{\sigma^2}{2}\right)\frac{\partial^3 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2 \partial x_2} + \\ - \sigma\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^3 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2 \partial x_2} \tag{6.97}$$

and collecting the derivative with respect to x_2 we obtain:

$$\mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = \frac{\partial}{\partial x_2}\left[-\sigma\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^2 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2} + \right. \\ \left. + \left(4\mu\mu_* - 4\mu_*^2 + \mu\sigma - 2\mu_*\sigma - \frac{\sigma^2}{2}\right)\frac{\partial^2 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2}\right] \tag{6.98}$$

If we introduce the function $F(\boldsymbol{x}-\boldsymbol{y})$ defined as:

$$F(\boldsymbol{x}-\boldsymbol{y}) = \left(4\mu\mu_* - 4\mu_*^2 + \mu\sigma - 2\mu_*\sigma - \frac{\sigma^2}{2}\right)\frac{\partial^2 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2} + \\ - \sigma\left(\mu + \frac{\sigma}{2}\right)\frac{\partial^2 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2} \tag{6.99}$$

or, in terms of dimensionless parameters (5.19a) and (5.19c), we can write the function $F(\boldsymbol{x}-\boldsymbol{y})$ as:

$$F(\boldsymbol{x}-\boldsymbol{y}) = 2\mu^2\left\{\left[(1-\kappa)(\kappa+2\xi) - 2\xi^2\right]\frac{\partial^2 v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2} + \right. \\ \left. - \kappa(1+\kappa)\frac{\partial^2 v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1^2}\right\} \tag{6.100}$$

Hence, from the expressions obtained for (6.94) and (6.98) we can conclude that

$$\mathbb{K}_{sirg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} = \frac{\partial F(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \tag{6.101}$$

Observation 6.1. The function $F(\boldsymbol{x}-\boldsymbol{y})$ is an even function, since it contains the second order derivatives of the functions $v_i^g(\boldsymbol{x}-\boldsymbol{y})$, which are even functions too.

Thanks to expression (6.101) we can rewrite (6.74) as:

$$\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = \int_{\partial D_{\text{in}}} \mathbb{K}_{jklm} \frac{\partial v^{\text{P}}_m(\boldsymbol{x})}{\partial x_l} \frac{\partial \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} n_j \, \mathrm{d}S_{\boldsymbol{x}} + \\ - \int_{D_{\text{in}}} \frac{\partial F(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \frac{\partial v^{\text{P}}_m(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \quad (6.102)$$

which now can be directly integrated with respect to variable y_i through a change of sign[14] , allowing us to obtain the incremental mean stress field produced by an inclusion:

$$\dot{p}(\boldsymbol{y}) = -\int_{\partial D_{\text{in}}} \mathbb{K}_{jklm} \frac{\partial v^{\text{P}}_m(\boldsymbol{x})}{\partial x_l} \dot{p}^k(\boldsymbol{x}-\boldsymbol{y}) n_j \, \mathrm{d}S_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}} F(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v^{\text{P}}_m(\boldsymbol{x})}{\partial x_m} \, \mathrm{d}V_{\boldsymbol{x}} \quad (6.103)$$

where $\dot{p}^k(\boldsymbol{x}-\boldsymbol{y})$ are the Green's mean stresses.

6.4 The incremental mean stress of the dislocation

The calculation of the incremental mean stress field for the dislocation problem is similar to the procedure illustrated in the previous Section. We start from equilibrium equations (6.66a) from which we obtain (6.69). If we apply the equilibrium equatio (6.69) to the displacement field (6.55), we have:

$$\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = -\mathbb{K}_{sirg} \frac{\partial^2 v_g(\boldsymbol{y})}{\partial y_r \partial y_s} \quad (6.104)$$

[14] The derivative of the first member is performed with respect to y_i, while in the second is performed with respect to x_i: as a consequence, the integration with respect to y_i requires a change of sign in the second member. (We recall that the Green's functions have the form: $G_{ij}(\boldsymbol{x}-\boldsymbol{y})$).

A substitution of the second order derivative of the displacement field (6.65) into the above relation yields:

$$\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = -\int_{\partial D_{\text{in}}} \mathbb{K}_{sirg} \left[\mathbb{K}_{jklm} \frac{\partial^3 v_k^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j \partial x_r \partial x_s} + \right. \\ \left. + \frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} \delta_{lm} \right] v_m^{\text{P}}(\boldsymbol{x}) n_l \, \mathrm{d}S_{\boldsymbol{x}} \quad (6.105)$$

but from equilibrium equations (6.66b)

$$\mathbb{K}_{ijkl} \frac{\partial^2 v_l^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_k} + \frac{\partial \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \delta_{ij} = 0 \quad (6.106)$$

we obtain:

$$\mathbb{K}_{sirk} \frac{\partial^3 v_k^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j \partial x_r \partial x_s} = \mathbb{K}_{sirg} \frac{\partial^3 v_g^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j \partial x_r \partial x_s} = -\frac{\partial^2 \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_j} \quad (6.107)$$

In this way, equation (6.105) can be rewritten as:

$$\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{jklm} \frac{\partial^2 \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_j} + \right. \\ \left. - \mathbb{K}_{sirg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_r \partial x_s} \delta_{lm} \right] v_m^{\text{P}}(\boldsymbol{x}) n_l \, \mathrm{d}S_{\boldsymbol{x}} \quad (6.108)$$

and using the function $F(\boldsymbol{x}-\boldsymbol{y})$ defined by (6.99) (or equivalently by (6.100)), the above relation becomes

$$\frac{\partial \dot{p}(\boldsymbol{y})}{\partial y_i} = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{jklm} \frac{\partial^2 \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i \partial x_j} - \frac{\partial F(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} \delta_{lm} \right] v_m^{\text{P}}(\boldsymbol{x}) n_l \, \mathrm{d}S_{\boldsymbol{x}} \quad (6.109)$$

which now can be directly integrated with respect to the variable y_i through a change of sign[15] , allowing us to obtain the incremental mean stress field produced by a dislocation:

$$\dot{p}(\boldsymbol{y}) = -\int_{\partial D_{\text{in}}} \left[\mathbb{K}_{jklm} \frac{\partial \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j} - F(\boldsymbol{x}-\boldsymbol{y}) \delta_{lm} \right] v_m^{\text{P}}(\boldsymbol{x}) n_l \, \mathrm{d}S_{\boldsymbol{x}} \quad (6.110)$$

[15] See note 14 on the preceding page.

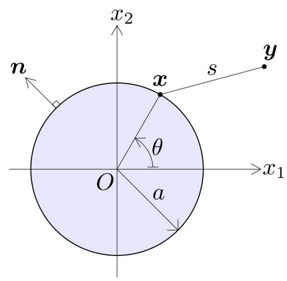


Figure 6.3: Reference system for the circular inclusion with radius a, subject to an initial transformation corresponding to an homothetic expansion (or contraction) $v_i^{\mathrm{P}}(\boldsymbol{x}) = \beta x_i$, meaning a purely volumetric dilatational Eulerian strain.

6.5 Example: the circular inclusion

We consider an infinite, incompressible, anisotropic medium subject to a prestress and including a circular inclusion with radius a, defined by the region

$$D_{\text{in}} = \left\{ (x_1, x_2) \mid x_1^2 + x_2^2 \leq a^2 \right\} \tag{6.111}$$

and its boundary (surface) is

$$\partial D_{\text{in}} = \left\{ (x_1, x_2) \mid x_1^2 + x_2^2 = a^2 \right\} \tag{6.112}$$

Such inclusion is subject to an initial, incremental, uniform, and volume changing deformation gradient $\nabla_{\boldsymbol{x}} \boldsymbol{v}^{\mathrm{P}}$. We assume that this gradient corresponds to an homothetic expansion (or contraction) of the inclusion, meaning a purely volumetric dilatational Eulerian strain, namely:

$$\frac{\partial v_i^{\mathrm{P}}(\boldsymbol{x})}{\partial x_j} = \beta \delta_{ij} \tag{6.113}$$

with $\beta \in \mathbb{R}^+ \setminus \{ 0 \}$ and the initial incremental displacement associated to this transformation is

$$v_i^{\mathrm{P}}(\boldsymbol{x}) = \beta x_i \tag{6.114}$$

Since this is a problem in the plane, we observe that the trace of the displacement gradient is equal to:

$$\frac{\partial v_i^{\mathrm{P}}(\boldsymbol{x})}{\partial x_i} = 2\beta \tag{6.115}$$

The reference system is illustrated in Figure 6.3. The inclusion has its centre coincident with origin of the reference system Ox_1x_2; the source point $\boldsymbol{x}$, lying on the surface of the inclusion, and its relative normal vector have the following coordinates

$$\boldsymbol{x} = \{\, a\cos\theta\,, a\sin\theta\,\} \tag{6.116a}$$

$$\boldsymbol{n} = \{\, \cos\theta\,, \sin\theta\,\} \tag{6.116b}$$

The point $\boldsymbol{y}$ lies outside the inclusion and we want to evaluate the incremental displacement and mean stress fields at this points. The distance between the source point $\boldsymbol{x}$ and the generic point of the matrix $\boldsymbol{y}$ is defined as:

$$s = d(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}| = \sqrt{(a\cos\theta - y_1)^2 + (a\sin\theta - y_2)^2} \tag{6.117}$$

6.5.1 Incremental displacement field

The inclusion formulation

The incremental displacement field of the inclusion described in Section 6.1 is given by (6.29), reported here:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_l^{\mathrm{P}}(\boldsymbol{x})}{\partial x_k} v_j^g(\boldsymbol{x} - \boldsymbol{y}) n_i \,\mathrm{d}S_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}} \dot{p}^g(\boldsymbol{x} - \boldsymbol{y}) \frac{\partial v_m^{\mathrm{P}}(\boldsymbol{x})}{\partial x_m} \,\mathrm{d}V_{\boldsymbol{x}} \tag{6.118}$$

Taking into account the geometry of the problem (equations (6.111), (6.112), (6.116)) and relations (6.113) and (6.115), the above expression becomes

$$v_g(\boldsymbol{y}) = \int_0^{2\pi} \mathbb{K}_{ijkl}\beta\delta_{lk} v_j^g(\boldsymbol{x}-\boldsymbol{y}) n_i a \, \mathrm{d}\theta + \\ + \int_0^a \int_0^{2\pi} \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})(2\beta) a \, \mathrm{d}\theta \, \mathrm{d}a \quad (6.119)$$

namely

$$v_g(\boldsymbol{y}) = \beta a \int_0^{2\pi} \mathbb{K}_{ijll} v_j^g(\boldsymbol{x}-\boldsymbol{y}) n_i \, \mathrm{d}\theta + \beta a^2 \int_0^{2\pi} \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \, \mathrm{d}\theta \quad (6.120)$$

since

$$\int_0^a \int_0^{2\pi} \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) a \, \mathrm{d}\theta \, \mathrm{d}a = \frac{a^2}{2} \int_0^{2\pi} \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \, \mathrm{d}\theta \quad (6.121)$$

The first integrand can be manipulated as follows:

$$\begin{aligned} \mathbb{K}_{ijll} v_j^g n_i &= (\mathbb{K}_{ij11} + \mathbb{K}_{ij22}) v_j^g n_i = \\ &= (\mathbb{K}_{1j11} + \mathbb{K}_{1j22}) v_j^g n_1 + (\mathbb{K}_{2j11} + \mathbb{K}_{2j22}) v_j^g n_2 = \\ &= (\mathbb{K}_{1111} + \mathbb{K}_{1122}) v_1^g n_1 + (\mathbb{K}_{1211} + \mathbb{K}_{1222}) v_2^g n_1 + \\ &\quad + (\mathbb{K}_{2111} + \mathbb{K}_{2122}) v_1^g n_2 + (\mathbb{K}_{2211} + \mathbb{K}_{2222}) v_2^g n_2 \end{aligned} \quad (6.122)$$

but from constitutive equations we have $\mathbb{K}_{1211}, \mathbb{K}_{1222}, \mathbb{K}_{2111}, \mathbb{K}_{2122} = 0$, so that the above relation reduces to:

$$\mathbb{K}_{ijll} v_j^g(\boldsymbol{x}-\boldsymbol{y}) n_i = -\left(\frac{\sigma}{2} + p\right) v_1^g(\boldsymbol{x}-\boldsymbol{y}) n_1 + \left(\frac{\sigma}{2} - p\right) v_2^g(\boldsymbol{x}-\boldsymbol{y}) n_2 \quad (6.123)$$

Using the dimensionless parameters η and κ we observe that

$$\frac{\sigma}{2} \pm p = \mu(\kappa \pm \eta) \quad (6.124)$$

so that equation (6.123) can be rewritten as

$$\mathbb{K}_{ijll} v_j^g(\boldsymbol{x}-\boldsymbol{y}) n_i = \mu\big[-(\kappa+\eta) v_1^g(\boldsymbol{x}-\boldsymbol{y}) n_1 + (\kappa-\eta) v_2^g(\boldsymbol{x}-\boldsymbol{y}) n_2\big] \quad (6.125)$$

Consequently, the general expression for the incremental displacement field produced by the circular inclusion is

$$v_g(\boldsymbol{y}) = \mu\beta a \int_0^{2\pi} \left[-(\kappa+\eta) v_1^g(\boldsymbol{x}-\boldsymbol{y}) \cos\theta + (\kappa-\eta) v_2^g(\boldsymbol{x}-\boldsymbol{y}) \sin\theta \right] \mathrm{d}\theta + \\ + \beta a^2 \int_0^{2\pi} \dot{p}^g(\boldsymbol{x}-\boldsymbol{y})\, \mathrm{d}\theta \quad (6.126)$$

The dislocation formulation

The displacement field produced by the inclusion can be determined by exploiting the dislocation formulation illustrated in Section 6.2; in particular, the incremental displacement field is given by (6.55), namely:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \delta_{kl} \right] v_l^{\mathrm{P}}(\boldsymbol{x}) n_k \, \mathrm{d}V_{\boldsymbol{x}} \quad (6.127)$$

If we take into account the geometry of the problem, we have

$$v_g(\boldsymbol{y}) = \beta a \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v_j^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} + \dot{p}^g(\boldsymbol{x}-\boldsymbol{y}) \delta_{kl} \right] x_l n_k \, \mathrm{d}\theta \quad (6.128)$$

Since

$$\mathbb{K}_{ijkl} \frac{\partial v_j^g}{\partial x_i} = \mathbb{K}_{1jkl} \frac{\partial v_j^g}{\partial x_1} + \mathbb{K}_{2jkl} \frac{\partial v_j^g}{\partial x_2} = \\ = \mathbb{K}_{11kl} \frac{\partial v_1^g}{\partial x_1} + \mathbb{K}_{12kl} \frac{\partial v_2^g}{\partial x_1} + \mathbb{K}_{21kl} \frac{\partial v_1^g}{\partial x_2} + \mathbb{K}_{22kl} \frac{\partial v_2^g}{\partial x_2} \quad (6.129)$$

the integrand can be manipulated as follows:

$$\left[\mathbb{K}_{ijkl} \frac{\partial v_j^g}{\partial x_i} + \dot{p}^g \delta_{kl} \right] x_l n_k = \left[\mathbb{K}_{111l} \frac{\partial v_1^g}{\partial x_1} + \mathbb{K}_{121l} \frac{\partial v_2^g}{\partial x_1} + \mathbb{K}_{211l} \frac{\partial v_1^g}{\partial x_2} + \right. \\ \left. + \mathbb{K}_{221l} \frac{\partial v_2^g}{\partial x_2} + \dot{p}^g \delta_{1l} \right] x_l n_1 + \left[\mathbb{K}_{112l} \frac{\partial v_1^g}{\partial x_1} + \mathbb{K}_{122l} \frac{\partial v_2^g}{\partial x_1} + \right. \\ \left. + \mathbb{K}_{212l} \frac{\partial v_1^g}{\partial x_2} + \mathbb{K}_{222l} \frac{\partial v_2^g}{\partial x_2} + \dot{p}^g \delta_{2l} \right] x_l n_2 \quad (6.130)$$

A further expansion provides

$$\begin{aligned}\left[\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}+\dot{p}^g\delta_{kl}\right]x_l n_k = &\left[\mathbb{K}_{1111}\frac{\partial v_1^g}{\partial x_1}+\mathbb{K}_{1211}\frac{\partial v_2^g}{\partial x_1}+\mathbb{K}_{2111}\frac{\partial v_1^g}{\partial x_2}+\right.\\ &\left.+\mathbb{K}_{2211}\frac{\partial v_2^g}{\partial x_2}+\dot{p}^g\right]x_1 n_1+\left[\mathbb{K}_{1112}\frac{\partial v_1^g}{\partial x_1}+\mathbb{K}_{1212}\frac{\partial v_2^g}{\partial x_1}+\mathbb{K}_{2112}\frac{\partial v_1^g}{\partial x_2}+\right.\\ &\left.+\mathbb{K}_{2212}\frac{\partial v_2^g}{\partial x_2}\right]x_2 n_1+\left[\mathbb{K}_{1121}\frac{\partial v_1^g}{\partial x_1}+\mathbb{K}_{1221}\frac{\partial v_2^g}{\partial x_1}+\mathbb{K}_{2121}\frac{\partial v_1^g}{\partial x_2}+\right.\\ &\left.+\mathbb{K}_{2221}\frac{\partial v_2^g}{\partial x_2}\right]x_1 n_2+\left[\mathbb{K}_{1122}\frac{\partial v_1^g}{\partial x_1}+\mathbb{K}_{1222}\frac{\partial v_2^g}{\partial x_1}+\mathbb{K}_{2122}\frac{\partial v_1^g}{\partial x_2}+\right.\\ &\left.+\mathbb{K}_{2222}\frac{\partial v_2^g}{\partial x_2}+\dot{p}^g\right]x_2 n_2\end{aligned} \quad (6.131)$$

but from constitutive equations we have $\mathbb{K}_{1211}, \mathbb{K}_{1222}, \mathbb{K}_{2111}, \mathbb{K}_{2122} = 0$, so that the above relation reduces to:

$$\begin{aligned}\left[\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}+\dot{p}^g\delta_{kl}\right]x_l n_k = &\left[\mathbb{K}_{1111}\frac{\partial v_1^g}{\partial x_1}+\mathbb{K}_{2211}\frac{\partial v_2^g}{\partial x_2}+\dot{p}^g\right]x_1 n_1+\\ &+\left[\mathbb{K}_{1212}\frac{\partial v_2^g}{\partial x_1}+\mathbb{K}_{2112}\frac{\partial v_1^g}{\partial x_2}\right]x_2 n_1+\left[\mathbb{K}_{1221}\frac{\partial v_2^g}{\partial x_1}+\mathbb{K}_{2121}\frac{\partial v_1^g}{\partial x_2}\right]x_1 n_2+\\ &+\left[\mathbb{K}_{1122}\frac{\partial v_1^g}{\partial x_1}+\mathbb{K}_{2222}\frac{\partial v_2^g}{\partial x_2}+\dot{p}^g\right]x_2 n_2\end{aligned} \quad (6.132)$$

namely

$$\begin{aligned}\left[\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}+\dot{p}^g\delta_{kl}\right]x_l n_k = &\left[\left(\mu_*-\frac{\sigma}{2}-p\right)\frac{\partial v_1^g}{\partial x_1}-\mu_*\frac{\partial v_2^g}{\partial x_2}\right]a\cos^2\theta+\\ &+\left[\left(2\mu+\frac{\sigma}{2}-p\right)\frac{\partial v_2^g}{\partial x_1}+\left(2\mu-\frac{\sigma}{2}-p\right)\frac{\partial v_1^g}{\partial x_2}\right]a\cos\theta\sin\theta+\\ &+\left[\left(\mu_*+\frac{\sigma}{2}-p\right)\frac{\partial v_2^g}{\partial x_2}-\mu_*\frac{\partial v_1^g}{\partial x_1}\right]a\sin^2\theta+\dot{p}^g a\end{aligned} \quad (6.133)$$

If we simplify the above expression, we have

$$\left[\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}+\dot{p}^g\delta_{kl}\right]x_l n_k = a\left[\left(\mu_* - \frac{\sigma}{2}-p\right)\cos^2\theta-\mu_*\sin^2\theta\right]\frac{\partial v_1^g}{\partial x_1}+ \\ +\frac{a}{2}\left[\left(2\mu+\frac{\sigma}{2}-p\right)\frac{\partial v_2^g}{\partial x_1}+\left(2\mu-\frac{\sigma}{2}-p\right)\frac{\partial v_1^g}{\partial x_2}\right]\sin(2\theta)+ \\ +a\left[\left(\mu_*+\frac{\sigma}{2}-p\right)\sin^2\theta-\mu_*\cos^2\theta\right]\frac{\partial v_2^g}{\partial x_2}+\dot{p}^g a \quad (6.134)$$

and using the dimensionless parameters:

$$\left[\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}+\dot{p}^g\delta_{kl}\right]x_l n_k = \mu a\left[(\xi-\kappa-\eta)\cos^2\theta-\xi\sin^2\theta\right]\frac{\partial v_1^g}{\partial x_1}+ \\ +\frac{\mu a}{2}\left[(2+\kappa-\eta)\frac{\partial v_2^g}{\partial x_1}+(2-\kappa-\eta)\frac{\partial v_1^g}{\partial x_2}\right]\sin(2\theta)+ \\ +\mu a\left[(\xi+\kappa-\eta)\sin^2\theta-\xi\cos^2\theta\right]\frac{\partial v_2^g}{\partial x_2}+\dot{p}^g a \quad (6.135)$$

Finally, the general expression for the incremental displacement field produced by a circular inclusion, obtained through the dislocation formulation, is

$$v_g(\boldsymbol{y}) = \mu\beta a^2\int_0^{2\pi}\left\{\left[(\xi-\kappa-\eta)\cos^2\theta-\xi\sin^2\theta\right]\frac{\partial v_1^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1}+\right. \\ +\left[(\xi+\kappa-\eta)\sin^2\theta-\xi\cos^2\theta\right]\frac{\partial v_2^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2}+\frac{2+\kappa-\eta}{2}\sin(2\theta)\frac{\partial v_2^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1}+ \\ \left.+\frac{2-\kappa-\eta}{2}\sin(2\theta)\frac{\partial v_1^g(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2}+\frac{\dot{p}^g(\boldsymbol{x}-\boldsymbol{y})}{\mu}\right\}\mathrm{d}\theta \quad (6.136)$$

and this is an alternative expression to (6.126).

6.5.2 Incremental mean stress field

The inclusion formulation

The incremental mean stress field of the inclusion described in Section 6.3 is given by (6.103), namely:

$$\dot{p}(\boldsymbol{y}) = -\int_{\partial D_{\text{in}}} \mathbb{K}_{jklm} \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_l} \dot{p}^k(\boldsymbol{x}-\boldsymbol{y}) n_j \,\mathrm{d}S_{\boldsymbol{x}} + \\ + \int_{D_{\text{in}}} F(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial v_m^{\text{P}}(\boldsymbol{x})}{\partial x_m} \,\mathrm{d}V_{\boldsymbol{x}} \quad (6.137)$$

Taking into account the geometry of the problem (equations (6.111), (6.112), (6.116)) and relations (6.113) and (6.115), the above expression becomes

$$\dot{p}(\boldsymbol{y}) = -\beta a \int_0^{2\pi} \mathbb{K}_{jklm} \delta_{ml} \dot{p}^k(\boldsymbol{x}-\boldsymbol{y}) n_j \,\mathrm{d}\theta + \\ + 2\beta \int_0^a \int_0^{2\pi} F(\boldsymbol{x}-\boldsymbol{y}) a \,\mathrm{d}\theta \,\mathrm{d}a \quad (6.138)$$

Recalling equation (6.121) and observing that the first integrand can be rewritten as[16]

$$\mathbb{K}_{jkmm} \dot{p}^k(\boldsymbol{x}-\boldsymbol{y}) n_j = -\mu(\kappa+\eta) \dot{p}^1(\boldsymbol{x}-\boldsymbol{y}) n_1 + \mu(\kappa-\eta) \dot{p}^2(\boldsymbol{x}-\boldsymbol{y}) n_2 \quad (6.139)$$

In this way, the general expression for the incremental mean stress field produced by a circular inclusion is

$$\dot{p}(\boldsymbol{y}) = -\mu\beta a \int_0^{2\pi} \left[-(\kappa+\eta) \dot{p}^1(\boldsymbol{x}-\boldsymbol{y}) \cos\theta + (\kappa-\eta) \dot{p}^2(\boldsymbol{x}-\boldsymbol{y}) \sin\theta \right] \mathrm{d}\theta + \\ + \beta a^2 \int_0^{2\pi} F(\boldsymbol{x}-\boldsymbol{y}) \,\mathrm{d}\theta \quad (6.140)$$

[16] To understand this point, one can observe the steps between expressions (6.122) and (6.125), and then adapt them to the first integrand of equation (6.138).

The dislocation formulation

If we use the dislocation formulation described in Section 6.4, the incremental mean stress field given by (6.110) is

$$\dot{p}(\boldsymbol{y}) \quad = \quad -\int_{\partial D_{\text{in}}} \left[\mathbb{K}_{jklm} \frac{\partial \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j} \; - \; F(\boldsymbol{x} \; - \; \boldsymbol{y})\delta_{lm} \right] v^{\mathrm{P}}_m(\boldsymbol{x}) n_l \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.141}$$

and if we adapt it to the geometry of this problem, we have

$$\dot{p}(\boldsymbol{y}) = -\beta a \int_0^{2\pi} \left[\mathbb{K}_{jklm} \frac{\partial \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j} - F(\boldsymbol{x}-\boldsymbol{y})\delta_{lm} \right] x_m n_l \, \mathrm{d}\theta \tag{6.142}$$

Similarly to the previous Subsection, we can observe that the integrand can be rewritten as:

$$\begin{aligned} \left[\mathbb{K}_{jklm} \frac{\partial \dot{p}^k}{\partial x_j} - F\delta_{lm} \right] x_m n_l = a\mu \left[(\xi-\kappa-\eta)\cos^2\theta - \xi\sin^2\theta \right] \frac{\partial \dot{p}^1}{\partial x_1} + \\ + \frac{a\mu}{2} \left[(2+\kappa-\eta)\frac{\partial \dot{p}^2}{\partial x_1} + (2-\kappa-\eta)\frac{\partial \dot{p}^1}{\partial x_2} \right] \sin(2\theta) + \\ + a\mu \left[(\xi+\kappa-\eta)\sin^2\theta - \xi\cos^2\theta \right] \frac{\partial \dot{p}^2}{\partial x_2} - aF \end{aligned} \tag{6.143}$$

so that the alternative general expression for the incremental mean stress field produced by a circular inclusion is

$$\begin{aligned} \dot{p}(\boldsymbol{y}) = -\mu\beta a^2 \int_0^{2\pi} \Bigg\{ \left[(\xi-\kappa-\eta)\cos^2\theta - \xi\sin^2\theta \right] \frac{\partial \dot{p}^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1} + \\ + \frac{1}{2} \left[(2+\kappa-\eta)\frac{\partial \dot{p}^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1} + (2-\kappa-\eta)\frac{\partial \dot{p}^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2} \right] \sin(2\theta) + \\ + \left[(\xi+\kappa-\eta)\sin^2\theta - \xi\cos^2\theta \right] \frac{\partial \dot{p}^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2} - \frac{F(\boldsymbol{x}-\boldsymbol{y})}{\mu} \Bigg\} \mathrm{d}\theta \end{aligned} \tag{6.144}$$

6.5.3 A particular case: the absence of prestress

We can immediately observe that in case of null prestress ($p = 0$ and $\sigma = 0$, namely $\kappa = 0$ and $\eta = 0$) the solutions (6.126) and (6.140) reduce to:

$$v_g(\boldsymbol{y}) = \beta a^2 \int_0^{2\pi} \dot{p}^g(\boldsymbol{x} - \boldsymbol{y})\,\mathrm{d}\theta \tag{6.145a}$$

$$\dot{p}(\boldsymbol{y}) = 4\mu^2\beta a^2 \xi(1-\xi) \int_0^{2\pi} \frac{\partial^2 v_1^1(\boldsymbol{x} - \boldsymbol{y})}{\partial x_1^2}\,\mathrm{d}\theta \tag{6.145b}$$

when using the inclusion formulation, while using the dislocation formulation the equations (6.136) and (6.144) reduce to:

$$\begin{aligned} v_g(\boldsymbol{y}) = \mu\beta a^2 \int_0^{2\pi} \Bigg\{ & \xi \left[\frac{\partial v_1^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_1} - \frac{\partial v_2^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_2} \right] \cos(2\theta) + \\ & + \left[\frac{\partial v_1^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_2} + \frac{\partial v_2^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_1} \right] \sin(2\theta) + \frac{\dot{p}^g(\boldsymbol{x} - \boldsymbol{y})}{\mu} \Bigg\}\,\mathrm{d}\theta \end{aligned} \tag{6.146}$$

and

$$\begin{aligned} \dot{p}(\boldsymbol{y}) = -\mu\beta a^2 \int_0^{2\pi} \Bigg\{ & \xi \left[\frac{\partial \dot{p}^1(\boldsymbol{x} - \boldsymbol{y})}{\partial x_1} - \frac{\partial \dot{p}^2(\boldsymbol{x} - \boldsymbol{y})}{\partial x_2} \right] \cos(2\theta) + \\ & + \left[\frac{\partial \dot{p}^2(\boldsymbol{x} - \boldsymbol{y})}{\partial x_1} + \frac{\partial \dot{p}^1(\boldsymbol{x} - \boldsymbol{y})}{\partial x_2} \right] \sin(2\theta) - \frac{F(\boldsymbol{x} - \boldsymbol{y})}{\mu} \Bigg\}\,\mathrm{d}\theta \end{aligned} \tag{6.147}$$

respectively.

6.5.4 A particular case: linear isotropic elasticity without prestress

In the case of linear isotropic elasticity without prestress ($\kappa = 0$, $\eta = 0$, $\xi = 1$) we obtain the following incremental displacement

$$v_g(\boldsymbol{y}) = -\frac{\beta a^2}{2\pi} \int_0^{2\pi} \frac{x_g - y_g}{s^2}\,\mathrm{d}\theta \tag{6.148}$$

but

$$\int_0^{2\pi} \frac{y_g - (\delta_{1g}\cos\theta + \delta_{2g}\sin\theta)a}{(a\cos\theta - y_1)^2 + (a\sin\theta - y_2)^2}\,\mathrm{d}\theta = \frac{\pi y_g}{y_1^2 + y_2^2} \tag{6.149}$$

so that

$$v_g(\boldsymbol{y}) = \frac{\beta a^2}{2}\frac{y_g}{y_1^2 + y_2^2} = \frac{\beta a^2}{2}\frac{y_g}{|\boldsymbol{y}|^2} \tag{6.150}$$

Introducing a polar coordinate reference system, the point $\boldsymbol{y}$ is described as:

$$\boldsymbol{y} = \{\, |\boldsymbol{y}|\cos\phi\,, |\boldsymbol{y}|\sin\phi \,\} \tag{6.151}$$

from which

$$\cos\phi = \frac{y_1}{|\boldsymbol{y}|} \tag{6.152a}$$

$$\sin\phi = \frac{y_2}{|\boldsymbol{y}|} \tag{6.152b}$$

The radial displacement is defined as

$$v_r = v_1\cos\phi + v_2\sin\phi \tag{6.153}$$

and inserting equations (6.150) and (6.152)

$$v_r = \frac{\beta a^2}{2}\left(\frac{y_1}{|\boldsymbol{y}|^2}\frac{y_1}{|\boldsymbol{y}|} + \frac{y_2}{|\boldsymbol{y}|^2}\frac{y_2}{|\boldsymbol{y}|}\right) = \frac{\beta a^2}{2|\boldsymbol{y}|} \tag{6.154}$$

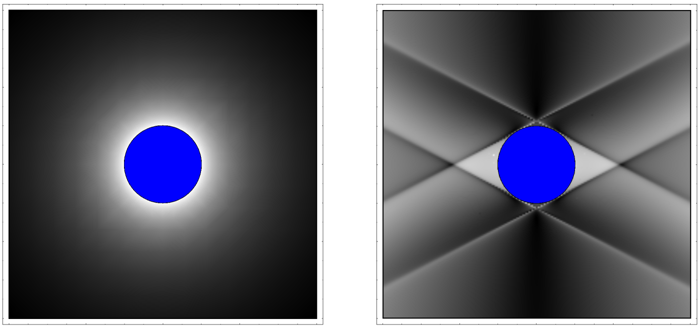
The tangential displacement is defined as

$$v_\theta = -v_1\sin\phi + v_2\cos\phi \tag{6.155}$$

and inserting equations (6.150) and (6.152)

$$v_\theta = \frac{\beta a^2}{2}\left(-\frac{y_1}{|\boldsymbol{y}|^2}\frac{y_2}{|\boldsymbol{y}|} + \frac{y_2}{|\boldsymbol{y}|^2}\frac{y_1}{|\boldsymbol{y}|}\right) = 0 \tag{6.156}$$

Therefore, the displacement field produced by a circular inclusion in the case of linear isotropic (incompressible) elasticity without prestress is a radial field.



(a) *Incompressible isotropic material.* **(b)** *J_2-deformation theory of plasticity.*

Figure 6.4: Modulus of the incremental displacement field near a circular inclusion in an infinite medium. The inclusion has radius a and is subject to a purely dilatational strain. **(a)** Isotropic, incompressible matrix material without prestress. **(b)** J_2-deformation theory of plasticity matrix material, prestressed near the elliptic boundary ($N = 0.380$; $\widehat{\varepsilon} = 0.657$; $\widehat{\varepsilon}^{\mathrm{E}} = 0.658$). Note the strong effect of prestress, determining four localizations of deformation, parallel to the shear band inclinations corresponding to ellipticity loss.

The elastic solution of Lamé is

$$u_r(r) = \varepsilon_0 \frac{a^2}{r} \tag{6.157}$$

which coincides with solution (6.154).[17] From the definition of the function $F(\boldsymbol{x} - \boldsymbol{y})$, given by (6.100), we immediately observe that it vanishes for $\kappa = 0$, $\eta = 0$, $\xi = 1$, so that the incremental mean stress in the case of linear isotropic (incompressible) elasticity without prestress is equal to zero:

$$\dot{p}(\boldsymbol{y}) = 0 \tag{6.158}$$

Equation (6.136) has been used to generate the incremental solution shown in Figure 6.4, for an isotropic elastic (with null prestress) matrix material (6.4a) and for a J_2-deformation theory matrix material uniformly deformed near the boundary (but still within) the elliptic region (6.4b). The latter material,

[17] Just define $\beta = 2\varepsilon_0$ and $r = |\boldsymbol{y}| = \sqrt{y_1^2 + y_2^2}$.

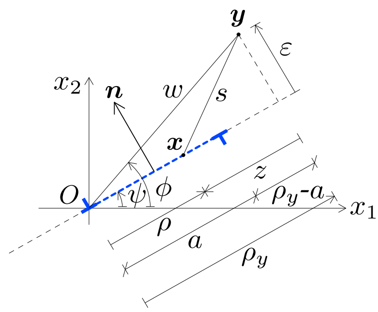


Figure 6.5: Infinite medium, including a straight *edge dislocation dipole* of finite length a and inclined at a constant angle ψ with respect to the x_1 axis.

with a hardening exponent $N = 0.380$, is pre-deformed at a logarithmic strain $\widehat{\varepsilon} = 0.657$, a value close to loss of ellipticity, occurring at $\widehat{\varepsilon}^{\mathrm{E}} = 0.658$.

The strong effect of prestress is evident from Figure 6.4 on the preceding page, so that *the incremental displacement fields are completely different* and the situation near the ellipticity loss shows the emergence of strongly localized fields, focused parallel to the four shear band directions (for a J_2-material with $N = 0.380$ the shear bands are inclined at $\pm 27.37°$ with respect to the x_1 axis).

6.6 Example: the edge dislocation dipole

We consider an infinite, incompressible, anisotropic medium subject to prestress and containing a *straight edge dislocations* dipole of finite length a and Burgers vector b constant along the dislocation line[18]; we imagine that one of the two dislocations is centered at the origin of the reference system O.

The geometry of the problem is illustrated in Figure 6.5, and we immediately observe that the source point $\boldsymbol{x}$, lying on the dipole line, has the following coordinates

$$\boldsymbol{x} = \{\, \rho \cos\psi \,, \rho \sin\psi \,\} \tag{6.159}$$

where $\rho \in [0, a]$ is the local coordinate starting from the origin O and following

[18] Note that for the dislocation the initial transformation $\boldsymbol{v}^{\mathrm{P}}$ corresponds to the Burgers vector b, namely, a jump in terms of incremental displacement along the dislocation line and parallel tho this direction.

the dislocation line, while the normal vector to the dipole line and the Burgers vector can be expressed as:

$$\boldsymbol{n} = \{\, -\sin\psi \,, \cos\psi \,\} \tag{6.160a}$$

$$\boldsymbol{v}^{\mathrm{P}} = \boldsymbol{b} = b\,\{\, \cos\psi \,, \sin\psi \,\} \tag{6.160b}$$

where $b = |\boldsymbol{b}|$. Since the Burgers vector is assumed constant, we have a condition of orthogonality between the normal $\boldsymbol{n}$ and $\boldsymbol{b}$, namely

$$\boldsymbol{v}^{\mathrm{P}} \perp \boldsymbol{n} \quad \longrightarrow \quad v_i^{\mathrm{P}} n_i = 0 \tag{6.161}$$

The point $\boldsymbol{y}$ lies outside the dipole and in this point we want to evaluate the incremental displacement and mean stess field. The distance between the source point $\boldsymbol{x}$ and the generic point of the matrix $\boldsymbol{y}$ is defined as:

$$s = d(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}| = \sqrt{(\rho\cos\psi - y_1)^2 + (\rho\sin\psi - y_2)^2} \tag{6.162}$$

6.6.1 Incremental displacement field

The incremental displacement field of the dislocation described in Section 6.2 is given by (6.55) which is reported below:

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \left[\mathbb{K}_{ijkl} \frac{\partial v_j^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_i} + \dot{p}^g(\boldsymbol{x} - \boldsymbol{y})\delta_{kl} \right] v_l^{\mathrm{P}}(\boldsymbol{x}) n_k \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.163}$$

but due to the orthogonality between $\boldsymbol{n}$ and $\boldsymbol{b}$, the above expression reduces to

$$v_g(\boldsymbol{y}) = \int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial v_j^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_i} v_l^{\mathrm{P}}(\boldsymbol{x}) n_k \, \mathrm{d}S_{\boldsymbol{x}} \tag{6.164}$$

since

$$\delta_{kl} v_l^{\mathrm{P}}(\boldsymbol{x}) n_k = v_k^{\mathrm{P}}(\boldsymbol{x}) n_k = 0 \tag{6.165}$$

The integrand of equation (6.164) is similar to that illustrated in the previous Section, therefore the relations (6.129)-(6.133) may still apply, but here the term related to the Green's mean stress vanishes and we must replace x_l with v_l^{P}. In particular, we have:

$$\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}v_l^{\mathrm{P}}n_k = \frac{\partial v_j^g}{\partial x_i}\Big[\mathbb{K}_{ij11}v_1^{\mathrm{P}}n_1 + \mathbb{K}_{ij12}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{ij21}v_1^{\mathrm{P}}n_2 + \\ + \mathbb{K}_{ij22}v_1^{\mathrm{P}}n_2\Big] \quad (6.166)$$

namely

$$\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}v_l^{\mathrm{P}}n_k = \frac{\partial v_1^g}{\partial x_1}\Big[\mathbb{K}_{1111}v_1^{\mathrm{P}}n_1 + \mathbb{K}_{1112}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{1121}v_1^{\mathrm{P}}n_2 + \mathbb{K}_{1122}v_1^{\mathrm{P}}n_2\Big] + \\ + \frac{\partial v_1^g}{\partial x_2}\Big[\mathbb{K}_{2111}v_1^{\mathrm{P}}n_1 + \mathbb{K}_{2112}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{2121}v_1^{\mathrm{P}}n_2 + \mathbb{K}_{2122}v_1^{\mathrm{P}}n_2\Big] + \\ + \frac{\partial v_2^g}{\partial x_1}\Big[\mathbb{K}_{1211}v_1^{\mathrm{P}}n_1 + \mathbb{K}_{1212}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{1221}v_1^{\mathrm{P}}n_2 + \mathbb{K}_{1222}v_1^{\mathrm{P}}n_2\Big] + \\ + \frac{\partial v_2^g}{\partial x_2}\Big[\mathbb{K}_{2211}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{2212}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{2221}v_1^{\mathrm{P}}n_2 + \mathbb{K}_{2222}v_1^{\mathrm{P}}n_2\Big] \quad (6.167)$$

but from constitutive equations we have $\mathbb{K}_{1211} = \mathbb{K}_{1222} = \mathbb{K}_{2111} = \mathbb{K}_{2122} = 0$, so that the above expression reduces to:

$$\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}v_l^{\mathrm{P}}n_k = \frac{\partial v_1^g}{\partial x_1}\Big[\mathbb{K}_{1111}v_1^{\mathrm{P}}n_1 + \mathbb{K}_{1122}v_1^{\mathrm{P}}n_2\Big] + \\ + \frac{\partial v_1^g}{\partial x_2}\Big[\mathbb{K}_{2112}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{2121}v_1^{\mathrm{P}}n_2\Big] + \frac{\partial v_2^g}{\partial x_1}\Big[\mathbb{K}_{1212}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{1221}v_1^{\mathrm{P}}n_2\Big] + \\ + \frac{\partial v_2^g}{\partial x_2}\Big[\mathbb{K}_{2211}v_2^{\mathrm{P}}n_1 + \mathbb{K}_{2222}v_1^{\mathrm{P}}n_2\Big] \quad (6.168)$$

Recalling equations (6.160), we have

$$\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}v_l^{\mathrm{P}}n_k = -\frac{b}{2}\sin(2\psi)\left(2\mu_* - \frac{\sigma}{2} - p\right)\frac{\partial v_1^g}{\partial x_1} + \\ + \left[\left(\mu - \frac{\sigma}{2}\right)b\cos^2\psi - (\mu - p)b\sin^2\psi\right]\frac{\partial v_1^g}{\partial x_2} + \Big[(\mu - p)b\cos^2\psi + \\ - \left(\mu + \frac{\sigma}{2}\right)b\sin^2\psi\Big]\frac{\partial v_2^g}{\partial x_1} + \frac{b}{2}\sin(2\psi)\left(2\mu_* + \frac{\sigma}{2} - p\right)\frac{\partial v_2^g}{\partial x_2} \quad (6.169)$$

and taking into account the incompressibility constraint[19],

$$\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}v_l^{\mathrm{P}}n_k = b\sin(2\psi)(p-2\mu_*)\frac{\partial v_1^g}{\partial x_1}+ \\ +b\left[\left(\mu-\frac{\sigma}{2}\right)\cos^2\psi-(\mu-p)\sin^2\psi\right]\frac{\partial v_1^g}{\partial x_2}+ \\ +b\left[(\mu-p)\cos^2\psi-\left(\mu+\frac{\sigma}{2}\right)\sin^2\psi\right]\frac{\partial v_2^g}{\partial x_1} \quad (6.170)$$

If we use the dimensionless parameters ξ, η and κ, we observe that

$$\mu - p = \mu(1-\eta) \quad (6.171a)$$
$$\mu \pm \frac{\sigma}{2} = \mu(1\pm\kappa) \quad (6.171b)$$
$$p - 2\mu_* = \mu(\eta - 2\xi) \quad (6.171c)$$

so that equation (6.170) can be rewritten as

$$\mathbb{K}_{ijkl}\frac{\partial v_j^g}{\partial x_i}v_l^{\mathrm{P}}n_k = \mu b(\eta-2\xi)\sin(2\psi)\frac{\partial v_1^g}{\partial x_1}+\mu b\Big[(1-\kappa)\cos^2\psi+ \\ -(1-\eta)\sin^2\psi\Big]\frac{\partial v_1^g}{\partial x_2}+\mu b\Big[(1-\eta)\cos^2\psi-(1+\kappa)\sin^2\psi\Big]\frac{\partial v_2^g}{\partial x_1} \quad (6.172)$$

We define the following coefficients as functions of the inclination angle ψ of the dipole, of the prestress and of the material parameters:

$$\Omega_1(\psi) = (\eta-2\xi)\sin(2\psi) \quad (6.173a)$$
$$\Omega_2(\psi) = (1-\kappa)\cos^2\psi-(1-\eta)\sin^2\psi \quad (6.173b)$$
$$\Omega_3(\psi) = (1-\eta)\cos^2\psi-(1+\kappa)\sin^2\psi \quad (6.173c)$$

[19] We recall that the incompressibility constraint is expressed as

$$\frac{\partial v_2^g}{\partial x_2} = -\frac{\partial v_1^g}{\partial x_1}$$

and, finally, the incremental displacement field produced by a *straight edge dislocations* dipole is:

$$v_g(\boldsymbol{y},\psi) = \mu b \int_0^a \left[\Omega_1(\psi) \frac{\partial v_1^g(\boldsymbol{y},\psi,\rho)}{\partial x_1} + \Omega_2(\psi) \frac{\partial v_1^g(\boldsymbol{y},\psi,\rho)}{\partial x_2} + \right. \\ \left. + \Omega_3(\psi) \frac{\partial v_2^g(\boldsymbol{y},\psi,\rho)}{\partial x_1} \right] \mathrm{d}\rho \quad (6.174)$$

6.6.2 Incremental mean stress field

The incremental mean stress field of the dislocation described in Section 6.4 is given by (6.110), namely:

$$\dot{p}(\boldsymbol{y}) = -\int_{\partial D_{\text{in}}} \left[\mathbb{K}_{jklm} \frac{\partial \dot{p}^k(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j} - F(\boldsymbol{x}-\boldsymbol{y})\delta_{lm} \right] v_m^{\text{P}}(\boldsymbol{x}) n_l \, \mathrm{d}S_{\boldsymbol{x}} \quad (6.175)$$

which, due to the orthogonality between $\boldsymbol{n}$ and $\boldsymbol{b}$, reduces to

$$\dot{p}(\boldsymbol{y}) = -\int_{\partial D_{\text{in}}} \mathbb{K}_{ijkl} \frac{\partial \dot{p}^j(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} v_l^{\text{P}}(\boldsymbol{x}) n_k \, \mathrm{d}S_{\boldsymbol{x}} \quad (6.176)$$

In an analogous manner to that followed in (6.166)-(6.169), we can manipulate the integrand of the above expression, thus obtaining:

$$\mathbb{K}_{ijkl} \frac{\partial \dot{p}^j}{\partial x_i} v_l^{\text{P}} n_k = -\frac{b}{2} \sin(2\psi) \left(2\mu_* - \frac{\sigma}{2} - p \right) \frac{\partial \dot{p}^1}{\partial x_1} + \\ + b \left[\left(\mu - \frac{\sigma}{2} \right) \cos^2\psi - (\mu - p)\sin^2\psi \right] \frac{\partial \dot{p}^1}{\partial x_2} + b \left[(\mu - p)\cos^2\psi + \right. \\ \left. - \left(\mu + \frac{\sigma}{2} \right) \sin^2\psi \right] \frac{\partial \dot{p}^2}{\partial x_1} + \frac{b}{2} \sin(2\psi) \left(2\mu_* + \frac{\sigma}{2} - p \right) \frac{\partial \dot{p}^2}{\partial x_2} \quad (6.177)$$

If we use the dimensionless parameters ξ, η and κ, we should add to relations (6.171) the following term:

$$2\mu_* \pm \frac{\sigma}{2} - p = \mu(2\xi \pm \kappa - \eta) \quad (6.178)$$

so that equation (6.177) rewrites as:

$$\mathbb{K}_{ijkl}\frac{\partial \dot{p}^j}{\partial x_i}v_l^{\mathrm{P}}n_k = \frac{\mu b}{2}(\kappa+\eta-2\xi)\sin(2\psi)\frac{\partial \dot{p}^1}{\partial x_1} + \mu b\Big[(1-\kappa)\cos^2\psi + \\ -(1-\eta)\sin^2\psi\Big]\frac{\partial \dot{p}^1}{\partial x_2} + \mu b\Big[(1-\eta)\cos^2\psi - (1+\kappa)\sin^2\psi\Big]\frac{\partial \dot{p}^2}{\partial x_1} + \\ + \frac{\mu b}{2}(\kappa-\eta+2\xi)\sin(2\psi)\frac{\partial \dot{p}^2}{\partial x_2} \quad (6.179)$$

As in the previous Subsection, we introduce some other coefficients as functions of the inclination angle ψ of the dipole, of the prestress and of the material parameters:

$$\Omega_4(\psi) = \frac{1}{2}(\kappa+\eta-2\xi)\sin(2\psi) \quad (6.180a)$$

$$\Omega_5(\psi) = \frac{1}{2}(\kappa-\eta+2\xi)\sin(2\psi) \quad (6.180b)$$

and, finally, the incremental mean stress field produced by a *straight edge dislocations* dipole is:

$$\dot{p}(\boldsymbol{y},\psi) = -\mu b\int_0^a \Bigg[\Omega_2(\psi)\frac{\partial \dot{p}^1(\boldsymbol{y},\psi,\rho)}{\partial x_2} + \Omega_3(\psi)\frac{\partial \dot{p}^2(\boldsymbol{y},\psi,\rho)}{\partial x_1} + \\ + \Omega_4(\psi)\frac{\partial \dot{p}^1(\boldsymbol{y},\psi,\rho)}{\partial x_1} + \Omega_5(\psi)\frac{\partial \dot{p}^2(\boldsymbol{y},\psi,\rho)}{\partial x_2}\Bigg]\,\mathrm{d}\rho \quad (6.181)$$

6.6.3 A particular case: the absence of prestress

We immediately observe that in the case of null prestress ($p = 0$ and $\sigma = 0$, namely $\kappa = 0$ and $\eta = 0$) the solutions (6.174) and (6.181) reduce to:

$$v_g(\boldsymbol{y},\psi) = \mu b\int_0^a \Bigg\{-2\xi\sin(2\psi)\frac{\partial v_1^g(\boldsymbol{y},\psi,\rho)}{\partial x_1} + \\ + \left[\frac{\partial v_1^g(\boldsymbol{y},\psi,\rho)}{\partial x_2} + \frac{\partial v_2^g(\boldsymbol{y},\psi,\rho)}{\partial x_1}\right]\cos(2\psi)\Bigg\}\,\mathrm{d}\rho \quad (6.182)$$

and

$$\dot{p}(\boldsymbol{y},\psi) = -\mu b \int_0^a \left\{ \left[\frac{\partial \dot{p}^1(\boldsymbol{y},\psi,\rho)}{\partial x_2} + \frac{\partial \dot{p}^2(\boldsymbol{y},\psi,\rho)}{\partial x_1} \right] \cos(2\psi) + \right.$$
$$\left. - \xi \left[\frac{\partial \dot{p}^1(,\psi)}{\partial x_1} + \frac{\partial \dot{p}^2(\boldsymbol{y},\psi,\rho)}{\partial x_2} \right] \sin(2\psi) \right\} \mathrm{d}\rho \quad (6.183)$$

respectively.

6.6.4 A particular case: linear isotropic elasticity without prestress

In the case of linear isotropic elasticity without prestress ($\kappa = 0$, $\eta = 0$, $\xi = 1$) and considering the dipole lying on the x_1 axis, namely $\psi = 0$, we obtain:

$$v_g(\boldsymbol{y},\psi) = \mu b \int_0^a \left[\frac{\partial v_1^g(\boldsymbol{y},\psi,\rho)}{\partial x_2} + \frac{\partial v_2^g(\boldsymbol{y},\psi,\rho)}{\partial x_1} \right] \mathrm{d}\rho \quad (6.184a)$$

$$\dot{p}(\boldsymbol{y},\psi) = -\mu b \int_0^a \left[\frac{\partial \dot{p}^1(\boldsymbol{y},\psi,\rho)}{\partial x_2} + \frac{\partial \dot{p}^2(\boldsymbol{y},\psi,\rho)}{\partial x_1} \right] \mathrm{d}\rho \quad (6.184b)$$

The derivatives of the Green's functions for the linear elastic isotropic incompressible material, in a *plane strain* condition, are:

$$\frac{\partial v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2} = -\frac{(x_2-y_2)}{4\pi\mu s^4}\left[3(x_1-y_1)^2 + (x_2-y_2)^2\right] \quad (6.185a)$$

$$\frac{\partial v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1} = -\frac{(x_1-y_1)}{4\pi\mu s^4}\left[(x_1-y_1)^2 + 3(x_2-y_2)^2\right] \quad (6.185b)$$

$$\frac{\partial v_2^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1} = -\frac{(x_2-y_2)}{4\pi\mu s^4}\left[(x_1-y_1)^2 - (x_2-y_2)^2\right] = \frac{\partial v_1^2}{\partial x_1} \quad (6.185c)$$

$$\frac{\partial v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2} = \frac{(x_1-y_1)}{4\pi\mu s^4}\left[(x_1-y_1)^2 - (x_2-y_2)^2\right] = \frac{\partial v_2^1}{\partial x_2} \quad (6.185d)$$

for the displacement, while for the mean stress we have:

$$\frac{\partial \dot{p}^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2} = \frac{\partial \dot{p}^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1} = \frac{(x_1-y_1)(x_2-y_2)}{\pi s^4} \quad (6.186)$$

Noting that

$$\frac{\partial v_1^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2}+\frac{\partial v_2^1(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1}=-\frac{(x_1-y_1)^2(x_2-y_2)}{\pi s^4} \tag{6.187a}$$

$$\frac{\partial v_1^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_2}+\frac{\partial v_2^2(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1}=-\frac{(x_1-y_1)(x_2-y_2)^2}{\pi s^4} \tag{6.187b}$$

and taking into account that $\psi=0$, equations (6.184a) become:

$$\begin{aligned} v_1(\boldsymbol{y}) &= -\frac{b}{\pi}\int_0^a \frac{(x_1-y_1)^2(x_2-y_2)}{s^4}\,\mathrm{d}x_1 = \\ &= \frac{by_2}{\pi}\int_0^a \frac{(x_1-y_1)^2}{\left[(x_1-y_1)^2+y_2^2\right]^2}\,\mathrm{d}x_1 \end{aligned} \tag{6.188}$$

and

$$\begin{aligned} v_2(\boldsymbol{y}) &= -\frac{b}{\pi}\int_0^a \frac{(x_1-y_1)(x_2-y_2)^2}{s^4}\,\mathrm{d}x_1 = \\ &= -\frac{by_2^2}{\pi}\int_0^a \frac{x_1-y_1}{\left[(x_1-y_1)^2+y_2^2\right]^2}\,\mathrm{d}x_1 \end{aligned} \tag{6.189}$$

for the components along x_1 and x_2 respectively. The primitives of the integrands appearing in (6.188) and in (6.189) are:

$$\begin{aligned} \int \frac{(x_1-y_1)^2}{\left[(x_1-y_1)^2+y_2^2\right]^2}\,\mathrm{d}x_1 = \frac{1}{2y_2}\arctan\left(\frac{x_1-y_1}{y_2}\right)+ \\ -\frac{x_1-y_1}{(x_1-y_1)^2+y_2^2}+\mathrm{const}_1 \end{aligned} \tag{6.190}$$

and

$$\int \frac{x_1-y_1}{\left[(x_1-y_1)^2+y_2^2\right]^2}\,\mathrm{d}x_1 = -\frac{1}{2}\frac{1}{(x_1-y_1)^2+y_2^2}+\mathrm{const}_2 \tag{6.191}$$

so that

$$v_1 = \frac{b}{2\pi}\left[\arctan\left(\frac{x_1 - y_1}{y_2}\right) - \frac{2(x_1 - y_1)y_2}{(x_1 - y_1)^2 + y_2^2} + \text{const}_3\right]_{x_1=0}^{x_1=a} \tag{6.192a}$$

$$v_2 = \frac{b}{2\pi}\left[\frac{y_2^2}{(x_1 - y_1)^2 + y_2^2} + \text{const}_4\right]_{x_1=0}^{x_1=a} \tag{6.192b}$$

where

$$\text{const}_3 = \frac{by_2}{\pi}\text{const}_1 \tag{6.193a}$$

$$\text{const}_4 = -\frac{by_2^2}{\pi}\text{const}_2 \tag{6.193b}$$

We require that $v_1(y_1, 0) = 0$, so that

$$\frac{b}{2\pi}\left(\frac{\pi}{2} + \text{const}_3\right) = 0 \quad \longrightarrow \quad \text{const}_3 = -\frac{b}{4} \tag{6.194}$$

Furthermore, we note that

$$\arctan\left(\frac{x_1 - y_1}{y_2}\right) - \frac{\pi}{2} = \arctan\left(\frac{y_2}{x_1 - y_1}\right) \tag{6.195}$$

In equation (6.192b) we choose the constant

$$\text{const}_4 = -\frac{1}{2} \tag{6.196}$$

that corresponds to a rigid body motion, and yields:

$$\frac{y_2^2}{(x_1 - y_1)^2 + y_2^2} - \frac{1}{2} = \frac{1}{2}\frac{y_2^2 - (x_1 - y_1)^2}{(x_1 - y_1)^2 + y_2^2} \tag{6.197}$$

which is an expression symmetric in the variables y_1 and y_2. Concerning the mean stress, we observe that

$$\frac{\partial \dot{p}^1(\boldsymbol{x} - \boldsymbol{y})}{\partial x_2} + \frac{\partial \dot{p}^2(\boldsymbol{x} - \boldsymbol{y})}{\partial x_1} = 2\frac{(x_1 - y_1)(x_2 - y_2)}{\pi s^4} \tag{6.198}$$

therefore equation (6.184b) becomes:

$$\dot{p}(\boldsymbol{y}) = \frac{2\mu b y_2}{\pi} \int_0^a \frac{(x_1 - y_1)}{\left[(x_1 - y_1)^2 + y_2^2\right]^2} \, \mathrm{d}x_1 \tag{6.199}$$

which can be easily integrated. Finally, the displacement field produced by a *straight edge dislocations* dipole, aligned with x_1 axis and included inside a linear elastic isotropic incompressible material, is

$$v_1 = \frac{b}{2\pi} \left[\arctan\left(\frac{y_2}{x_1 - y_1} \right) - \frac{2(x_1 - y_1) y_2}{(x_1 - y_1)^2 + y_2^2} \right]_{x_1=0}^{x_1=a} \tag{6.200a}$$

$$v_2 = -\frac{b}{4\pi} \left[\frac{(x_1 - y_1)^2 - y_2^2}{(x_1 - y_1)^2 + y_2^2} \right]_{x_1=0}^{x_1=a} \tag{6.200b}$$

while the mean stress field is

$$\dot{p}(\boldsymbol{y}) = -\frac{\mu b y_2}{\pi} \left[\frac{1}{(x_1 - y_1)^2 + y_2^2} \right]_{x_1=0}^{x_1=a} \tag{6.201}$$

Note that equations (6.200) and (6.201) correspond to the solutions given in literature [14] through superposition of the effects of two single dislocations embedded in an infinite medium.

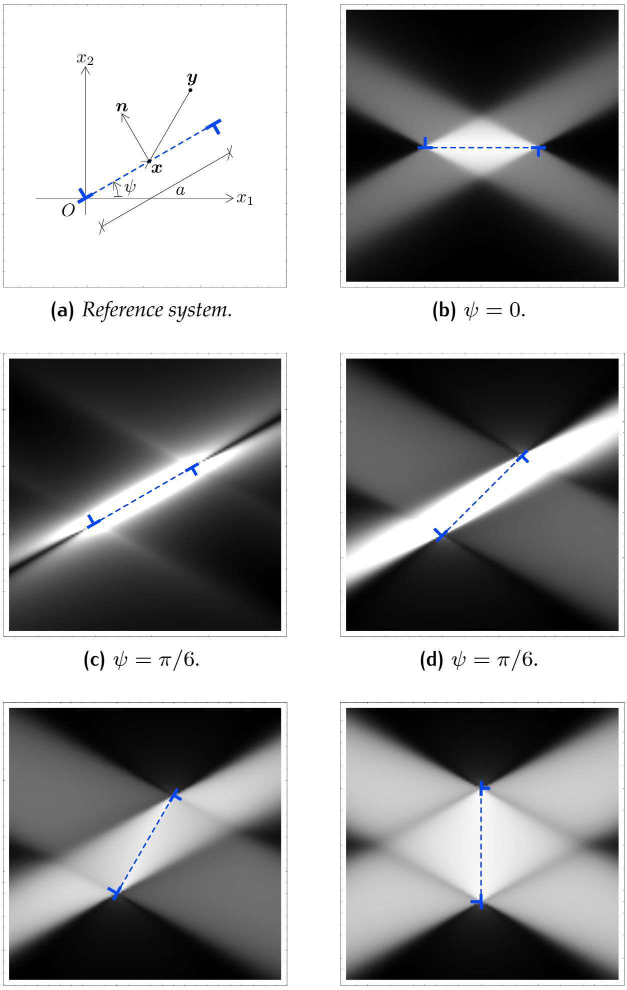
6.6.5 The solution along the dislocation line

The displacement and mean stress fields can be explicitly evaluated along the dislocation line through equations (6.174) and (6.181) and the following considerations on the Green's function structure. From Figure 6.5 on page 173, the point $\boldsymbol{y}$ admits this representation:

$$\boldsymbol{y} = \{ \, w \cos \phi \, , w \sin \phi \, \} \tag{6.202}$$

and if we go on the dislocation line, namely for $\phi = \psi$ with ψ constant, the new coordinates are

$$\boldsymbol{y} = (\rho + z) \{ \, \cos \psi \, , \sin \psi \, \} = \rho_y \{ \, \cos \psi \, , \sin \psi \, \} \tag{6.203}$$

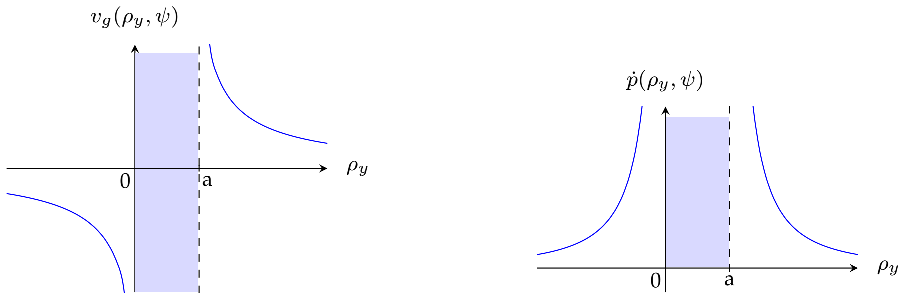


(a) *Reference system.* **(b)** $\psi = 0$.

(c) $\psi = \pi/6$. **(d)** $\psi = \pi/6$.

(e) $\psi = \pi/6$. **(f)** $\psi = \pi/6$.

Figure 6.6: The (level sets of the modulus of incremental) displacement field produced by *the emission of a straight edge dislocation dipole* (of length a and inclination ψ with respect to the x_1 axis of orthotropy, see the geometrical setting in **(a)**) in a (J_2-deformation theory) plastic material, homogeneously deformed until near the elliptic boundary. These images have been obtained assuming a hardening parameters $N = 0.363$ and a deformation $\widehat{\varepsilon} = 0.610$.



(a) *Qualitative behaviour of* $v_g(\rho_y, \psi)$.

(b) *Qualitative behaviour of* $\dot{p}(\rho_y, \psi)$.

Figure 6.7: Qualitative behaviour representation of the solutions in terms of incremental displacement (6.209) and of incremental mean stress (6.210) along the dislocation line.

where $\rho_y = \rho + z$ denotes the distance of point $\boldsymbol{y}$ along the dislocation line, starting from the origin O; in this particular case (corresponding to the assumption $\varepsilon = 0$) the mutual distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$ defined by (6.162) can be expressed as

$$s = |\boldsymbol{x} - \boldsymbol{y}| = z \tag{6.204}$$

These elements allow us to rewrite the gradients of the Green's functions for the incremental displacement and mean stress as functions of the mutual distance z as

$$\frac{\partial v_i^g(z, \psi)}{\partial x_j} = \frac{1}{z} \frac{\partial \bar{v}_i^g(\psi)}{\partial x_j} \tag{6.205a}$$

$$\frac{\partial \dot{p}^g(z, \psi)}{\partial x_j} = \frac{1}{z^2} \frac{\partial \dot{\bar{p}}^g(\psi)}{\partial x_j} \tag{6.205b}$$

where $\bar{v}_i^g$ and $\dot{\bar{p}}^g$ are functions only of the variable ψ. In this form, namely with an explicit dependence on z, it is possible to perform the integration on ρ through a change of variable. In particular, we observe that:

$$\int_0^a \frac{\mathrm{d}\rho}{z} = \int_0^a \frac{\mathrm{d}\rho}{\rho_y - \rho} = \Big[- \ln(\rho_y - \rho) \Big]_{\rho=0}^{\rho=a} = \ln \left(\frac{\rho_y}{\rho_y - a} \right) \tag{6.206}$$

and, analogously,

$$\int_0^a \frac{\mathrm{d}\rho}{z^2} = \int_0^a \frac{\mathrm{d}\rho}{(\rho_y - \rho)^2} = \frac{1}{\rho_y - \rho}\bigg|_{\rho=0}^{\rho=a} = \frac{a}{(\rho_y - a)\rho_y} \tag{6.207}$$

in which we assume the following definition interval:

$$\rho_y \in (-\infty, 0) \cup (a, +\infty) \tag{6.208}$$

Finally, the incremental displacement and mean stress fields along the dislocation line are:

$$v_g(\rho_y, \psi) = \mu b \ln\left(\frac{\rho_y}{\rho_y - a}\right)\left[\Omega_1(\psi)\frac{\partial \bar{v}_1^g(\psi)}{\partial x_1} + \Omega_2(\psi)\frac{\partial \bar{v}_1^g(\psi)}{\partial x_2} + \right. \\ \left. + \Omega_3(\psi)\frac{\partial \bar{v}_2^g(\psi)}{\partial x_1}\right] \tag{6.209}$$

and

$$\dot{p}(\rho_y, \psi) = -\frac{\mu b a}{(\rho_y - a)\rho_y}\left[\Omega_2(\psi)\frac{\partial \dot{\bar{p}}^1(\psi)}{\partial x_2} + \Omega_3(\psi)\frac{\partial \dot{\bar{p}}^2(\psi)}{\partial x_1} + \Omega_4(\psi)\frac{\partial \dot{\bar{p}}^1(\psi)}{\partial x_1} + \right. \\ \left. + \Omega_5(\psi)\frac{\partial \dot{\bar{p}}^2(\psi)}{\partial x_2}\right] \tag{6.210}$$

respectively. Note that the incremental displacement and mean stress fields exhibit essentially different asymptotic behaviours at the dislocation tips (near both points, $\rho_y = 0$ and $\rho_y = a$). In fact, as one can expect, the displacement field shows a logarithmic singularity (similar to that found by Eshelby [14], Figure 6.7a on the preceding page), whereas the mean stress displays a $1/z$ singularity (Figure 6.7b on the facing page).

6.7 The numerical treatment of the boundary integral equations

The numerical treatment of the boundary integral equations (6.174) and (6.181) involves a Cauchy-type integral, for equation (6.174), and a hypersingular integral, for equation (6.181). The use of these equations implies the knowledge of the gradient of the Green's function for incremental displacement and for incremental in-plane mean stress; the former has been given by Bigoni & Capuani [1] and will not be repeated, while the latter can be obtained using equations (48) and (62) given by Bigoni & Capuani [1], so that we arrive at the final expressions

$$\begin{aligned}\dot{p}^g_{,j} = \frac{1}{2\pi s^4}\left[1 - \frac{k}{1+k}\frac{1}{\gamma_1^{\delta_{1g}}\sqrt{-\gamma_2} + \gamma_2^{\delta_{1g}}\sqrt{-\gamma_1}}\right]\Big\{(\delta_{1g}\delta_{1j} - \delta_{2g}\delta_{2j}) \\ \times\left[(x_1-y_1)^2 - (x_2-y_2)^2\right] + 2(\delta_{1g}\delta_{2j} + \delta_{2g}\delta_{1j})(x_1-y_1)(x_2-y_2)\Big\} + \\ + \frac{1}{2\pi^2(1+k)}\int_0^{\pi}\zeta_{gj}(\boldsymbol{x},\boldsymbol{y},\alpha)\,\mathrm{d}\alpha + \frac{k}{\pi^2(1+k)}\int_0^{\frac{\pi}{2}}\Xi_{gj}(\boldsymbol{x},\boldsymbol{y},\alpha)\,\mathrm{d}\alpha \end{aligned} \quad (6.211)$$

where $\zeta_{gj}(\boldsymbol{x},\boldsymbol{y},\alpha)$ and $\Xi_{gj}(\boldsymbol{x},\boldsymbol{y},\alpha)$ are functions (not reported for brevity) of the distance between the source point $\boldsymbol{x}$, the generic point $\boldsymbol{y}$ and the angle α, as defined by Bigoni & Capuani [1, Figure 1]; coefficients γ_1 and γ_2 are also defined in Bigoni & Capuani [1, equation (15)]. Note also that δ_{1g}, δ_{2g}, δ_{1j} and δ_{2j} are all Kronecker deltas (taking the values 0 and 1). Note that the term $\zeta_{gj}(\boldsymbol{x},\boldsymbol{y},\alpha)$ is related to the gradient of the Green's hydrostatic nominal stress, whereas the term $\Xi_{gj}(\boldsymbol{x},\boldsymbol{y},\alpha)$ is related to the second gradient of the Green's velocity.

The numerical evaluation of the boundary integral equation (6.174) requires the following treatment. First, we introduce the reference system shown in Figure 6.5 on page 173, where

$$\boldsymbol{x} = \{\,\rho\cos\psi\,,\rho\sin\psi\,\} \quad (6.212a)$$

$$\boldsymbol{y} = \{\,w\cos\phi\,,w\sin\phi\,\} \quad (6.212b)$$

and

$$\phi = \arctan\left(\frac{y_2}{y_1}\right) \quad (6.213a)$$

$$w = \sqrt{y_1^2 + y_2^2} \quad (6.213b)$$

so that

$$y_1 - x_1 = (\rho_y - \rho)\cos\psi - \varepsilon\sin\psi \quad (6.214a)$$

$$y_2 - x_2 = (\rho_y - \rho)\sin\psi + \varepsilon\cos\psi \quad (6.214b)$$

where ε can become a small parameter, and

$$\varepsilon = w \sin(\phi - \psi) \tag{6.215a}$$

$$\rho_y = w \cos(\phi - \psi) \tag{6.215b}$$

We introduce the change of variables

$$z = \rho_y - \rho \tag{6.216}$$

so that

$$y_1 - x_1 = z \cos\psi - \varepsilon \sin\psi \tag{6.217a}$$

$$y_2 - x_2 = z \sin\psi + \varepsilon \cos\psi \tag{6.217b}$$

Note from Figure 6.5 on page 173 that, whereas the source point $\boldsymbol{x}$ ranges along the dislocation line $\rho \in [0, a]$, point $\boldsymbol{y}$ is arbitrary. Therefore the variable z (does not) vanishes for all $\boldsymbol{y}$ whose projections lie (out-) in-side the dislocation line ($\rho_y \notin (0, a)$) $\rho_y \in (0, a)$, so that the problem in managing equations (6.174) and (6.181) occurs when $\rho_y \in (0, a)$. In this situation, ε can be made arbitrarily small, but different from zero, whereas variable z can be expanded around zero.

Using (6.216), the integrals involved in (6.174) can be written in the following form:

$$\tilde{v} = \int_{\rho_y - a}^{\rho_y} \frac{z \pm \varepsilon}{z^2 + \varepsilon^2}\, G(\varepsilon, z)\, \mathrm{d}z \tag{6.218}$$

where

$$G(\varepsilon, z) = \int_0^{\frac{\pi}{2}} \Delta(\varepsilon, z, \alpha)\, \mathrm{d}\alpha \tag{6.219}$$

in which function $\Delta(\varepsilon, z, \alpha)$ takes a complicated expression, not reported for brevity. Therefore, a Taylor series expansion of function $\Delta(\varepsilon, z, \alpha)$ in the variable z yields

$$\Delta(\varepsilon, z, \alpha) = \tilde{\Delta}(\varepsilon, z, \alpha) + O(z^2) \tag{6.220}$$

where

$$\tilde{\Delta}(\varepsilon, z, \alpha) = \Delta(\varepsilon, 0, \alpha) + \left.\frac{\partial \Delta(\varepsilon, z, \alpha)}{\partial z}\right|_{z=0} z \tag{6.221}$$

so that function $G(\varepsilon, z)$ can be regularized as

$$G(\varepsilon, z) = \int_0^{\frac{\pi}{2}} \left[\Delta(\varepsilon, z, \alpha) - \tilde{\Delta}(\varepsilon, z, \alpha)\right] \mathrm{d}\alpha + \int_0^{\frac{\pi}{2}} \Delta(\varepsilon, 0, \alpha)\, \mathrm{d}\alpha + \\ + z \int_0^{\frac{\pi}{2}} \left. \frac{\partial \Delta(\varepsilon, z, \alpha)}{\partial z} \right|_{z=0} \mathrm{d}\alpha \quad (6.222)$$

where the derivative of function $\Delta(\varepsilon, z, \alpha)$ in the variable z can be easily calculated (though it takes a complicated expression, which is not reported for conciseness).

As a conclusion, instead of working with the integral (6.218), we can work with its regularized version, written as

$$\tilde{v}(\varepsilon, \rho_y) = \int_{\rho_y - a}^{\rho_y} \frac{z \pm \varepsilon}{z^2 + \varepsilon^2} \left\{ \int_0^{\frac{\pi}{2}} \left[\Delta(\varepsilon, z, \alpha) - \tilde{\Delta}(\varepsilon, z, \alpha)\right] \mathrm{d}\alpha \right\} \mathrm{d}z + \\ + \left[z - \varepsilon \arctan\left(\frac{z}{\varepsilon}\right) \pm \varepsilon \log \sqrt{z^2 + \varepsilon^2} \right]_{z=\rho_y - a}^{z=\rho_y} \int_0^{\frac{\pi}{2}} \left. \frac{\partial \Delta(\varepsilon, z, \alpha)}{\partial z} \right|_{z=0} \mathrm{d}\alpha + \\ + \left[\pm \arctan\left(\frac{z}{\varepsilon}\right) + \log \sqrt{z^2 + \varepsilon^2} \right]_{z=\rho_y - a}^{z=\rho_y} \int_0^{\frac{\pi}{2}} \Delta(\varepsilon, 0, \alpha)\, \mathrm{d}\alpha \quad (6.223)$$

in which the singular terms have been explicitly evaluated. The integral equation for in-plane mean stress increment (6.181) can be treated in a way similar to that used to obtain (6.223), which is used in Section 6.8 to produce numerical values for the incremental displacement fields near a dislocation dipole.

6.8 Dislocation clustering in a metal near the elliptic border

We are now in a position to explore the effect of prestress on a metal deformed near the elliptic boundary. For this purpose, we can use equation (6.174), in the regularized version (6.223) for an edge dislocation dipole (which may be thought of as a 'super dislocation', i.e. a collection of dislocations smeared out along a certain direction) of length a, which is assumed to be nucleated in a

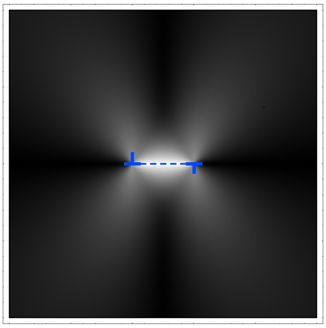


Figure 6.8: Level sets of the modulus of the displacement field around a *straight edge dislocations* dipole of length a in an infinite, incompressible elastic and isotropic medium without prestress. Note the strong difference with the behaviour near the elliptic threshold (Figure 6.6 on page 183).

J_2–deformation theory material with a hardening parameter $N = 0.363$

Incremental displacement fields for a unit length Burgers vector, at a prestrain $\widehat{\varepsilon} = 0.610$ (so that the material is close to the ellipticity threshold $\widehat{\varepsilon}^{\mathrm{E}} = 0.642$, but still within the elliptic region) are plotted in Figure 6.6 on page 183 for different inclinations ψ of the dipole with respect to the orthotropy axes (see the sketch in Figure 6.6a on page 183).

Note that the dislocation solution depends on the parameter η, which has been assumed equal to 0.490. The following inclinations have been considered: $\psi = \{\, 0, \pi/6, \pi/4, \pi/3, \pi/2 \,\}$.

It is worth observing that the perturbation induced by a dislocation dipole is different from that induced by a force dipole (as considered by Bigoni and Capuani [1]). In fact force and dislocation dipoles produce similar effects only in the far fields and only under the assumption that the prestress is absent [77]. A study on the connection between force and dislocation dipole is presented in Appendix C.

In all cases reported in figure 6.6 on page 183, we observe the formation of zones of intense deformation, aligned parallel to the inclination of the shear bands ($\pm 27.37°$ with respect to the x_1-axis), formally possible only at loss of ellipticity. The response of the material far from the elliptic boundary is completely different, as shown in Figure 6.8, pertaining to an isotropic incompressible material at null prestress.

Because the dislocation activity is triggered by a rise in the shear stress, and this occurs for highly prestressed materials along the preferred directions shown in Figure 6.6 on page 183, along these the dislocation activity tends to be strongly promoted. Therefore, this activation will again generate an increment in shear stress along the same directions, thus producing a sort of 'cascade effect', which will cluster dislocation formation along shear bands. This effect may explain the fact that the amount of slip taking place on active shear bands may be up to three orders of magnitude greater than that produced by a single dislocation [38].

We can now summarize the above discussion. Prestress has been shown to be an important factor in the mechanics of dislocation clustering, in ductile metals deformed near the shear band formation. A new solution for an edge dislocation, valid for incremental non-linear elasticity, with the current state taken as homogeneous, shows in fact emergence of highly localized deformation patterns, when the material is deformed near the boundary of ellipticity loss, which may trigger 'cascade' dislocation activation along shear band directions. Although this conclusion is limited by the assumption of homogeneity of the prestress (which is the only way to arrive at analytical solutions), it may correctly model the situation when a dislocation dipole is emitted.

Chapter 7

THREE-DIMENSIONAL GREEN'S FUNCTIONS

The problem of the concentrated force in an infinite, incompressible, anisotropic, elastic and prestressed material is generalized to the three dimensional case. Some simple examples of force dipoles acting on an axisymmetric infinite medium is reported in order to show the localization of the deformation, arising from high level of prestress.

THE RESPONSE of a homogeneously deformed non-linear elastic solid to a perturbing agent is the key to the investigation of several important problems, such as for instance, bifurcation of an elastic block [2, 5], or layered structures [42–45], wave propagation [46, 47], near-crack stress field determination [48–51], and shear band development [32, 39]. In these investigations the availability of an infinite-body Green's function allows the treatment of 'complex' problems (for instance, the stress field near a dislocation in a prestressed solid; Argani *et al*. [16]) and permits the development of boundary integral equations and the related boundary element techniques. Despite its importance, the first Green's function set for incompressible homogeneously deformed elastic solids has been provided by Willis [52] and explicitly derived for two-dimensional elasticity by Bigoni & Capuani [1]. Later, Green's functions have been derived for time-harmonic problems [9, 53] and boundary element formulations have been proposed [10, 29, 30]. However, all these results are restricted to plane elasticity, so that the only contribution valid for a three-dimensional context still remains that of Willis [52].

In this Chapter, infinite-body Green's functions are derived for incremental displacement and incremental mean stress for a non-linear elastic incompressible solid deformed homogeneously. Based on these Green's functions, the boundary integral equation set for the incremental response of a homogeneously prestressed elastic solid is derived, which provides the basis for boundary element techniques. These results [1] generalize Bigoni & Capuani [1] to three-dimensional elasticity and apply, as particular cases to Mooney-Rivlin elasticity and J_2-deformation theory of plasticity. The latter case allows for the analysis of localized shear deformation as induced by a perturbing force dipole in a three-dimensional elastic context. In this case, the incremental displacements are shown to be localized along a cone (only in special cases with circular cross section) of concentrated incremental shear strains, which differs from the well-known planar shear band. This result may explain the well-known cup-cone failure of ductile metal bars (see for instance Dieter [54]) and the conical failure zone observed by Desrues *et al*. [55] in cylindrical specimen of granular material. Moreover, it may be related to the mechanisms of conical fracturing observed in brittle materials (glass, see Lawn [56], polycarbonate, Figure 1.2 on page 5) and rocks subject to impact (for instance, the shatter cones found in shocked rocks near meteorite impact or underground nuclear test sites, see French [57] and Sagy *et al*. [58]).

7.1 The Green's functions set

In a relative Lagrangian description, an elastic body subject to prestress is characterized by a linear relation (see Bigoni [77] for details) between the increment[2] of the nominal stress t_{ij} and the gradient of the incremental displacement $v_{i,j}$

$$\dot{t}_{ij} = \mathbb{K}_{ijkl} \frac{\partial v_l}{\partial x_k} + \dot{p}\delta_{ij} \tag{7.1}$$

where δ_{ij} is the Kronecker delta and $\dot{p}$ denotes the incremental mean stress ($p = \operatorname{tr} \boldsymbol{T}/3$, where $\boldsymbol{T}$ is the Cauchy stress), which plays the role of a Lagrange multiplier related to the incompressibility constraint, so that we require that the

[1] The results presented in this Chapter can be applied as a particular case to the linearized viscous flow of an incompressible *orthotropic* fluid, for which we give for the first time infinite body Green's function and boundary equation sets.

[2] The increments are denoted with a superscript dot.

velocity field v_i is solenoidal

$$\frac{\partial v_k}{\partial x_k} = 0 \tag{7.2}$$

Note that the incremental constitutive tensor $\mathbb{K}_{ijkl}$ does not possess the minor symmetries, but the major symmetry, which comes from the existence of a potential $W(\nabla \boldsymbol{v})$ such that

$$\mathbb{K}_{ijkl} = \frac{\partial^2 W}{\partial[\nabla \boldsymbol{v}]_{ji} \partial[\nabla \boldsymbol{v}]_{lk}} \tag{7.3}$$

where

$$[\nabla \boldsymbol{v}]_{ji} = \frac{\partial v_j}{\partial x_i} \tag{7.4}$$

and this assumption will be valid within this Chapter. The incremental equilibrium equations are

$$\frac{\partial \dot{t}_{ij}}{\partial x_i} + \dot{f}_j = 0 \tag{7.5}$$

where $\dot{f}$ is the increment of the body force.

The Green's functions set is composed by the Green's functions for the incremental displacements v_i^g and for the incremental mean stress $\dot{p}^g$. The Green's stress is defined as

$$\dot{t}_{ij}^g = \mathbb{K}_{ijkl} \frac{\partial v_l^g}{\partial x_k} + \dot{p}^g \tag{7.6}$$

and must satisfy the equilibrium equations

$$\frac{\dot{t}_{ij}^g}{\partial x_i} + \delta_{jg} \delta(\boldsymbol{x}) = 0 \tag{7.7}$$

where $\delta(\boldsymbol{x})$ is the three-dimensional delta function and $\boldsymbol{x}$ is the generic material point.

Taking into account equation (7.6), the equilibrium equation (7.7) can be rewritten as

$$\mathbb{K}_{ijkl} \frac{\partial^2 v_l^g}{\partial x_k \partial x_i} + \frac{\partial \dot{p}^g}{\partial x_j} + \delta_{jg} \delta(\boldsymbol{x}) = 0 \tag{7.8}$$

We exploit the plane wave expansion of the delta function, expressed as

$$\delta(\boldsymbol{x}) = -\frac{1}{8\pi^2}\int_{|\boldsymbol{\omega}|=1} \delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} \tag{7.9}$$

and in an analogous manner, the plane wave expansion of the incremental displacements and mean stresses are:

$$v_k^g(\boldsymbol{x}) = -\frac{1}{8\pi^2}\int_{|\boldsymbol{\omega}|=1} \hat{v}_k^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} \tag{7.10a}$$

$$\dot{p}^g(\boldsymbol{x}) = -\frac{1}{8\pi^2}\int_{|\boldsymbol{\omega}|=1} \hat{p}^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} \tag{7.10b}$$

where $(\cdot)^{'}$ denotes the derivative with respect to the argument $(\boldsymbol{\omega}\cdot\boldsymbol{x})$. Due to relations (7.9) and (7.10), the equilibrium equations (7.8) can be rewritten as:

$$\mathbb{K}_{ijkl}\frac{\partial^2}{\partial x_k \partial x_i}\left[\hat{v}_l^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\right] + \frac{\partial}{\partial x_j}\left[\hat{p}^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\right] + \delta_{jg}\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0 \tag{7.11}$$

But since

$$\frac{\partial(\omega_k x_k)}{\partial x_j} = \frac{\partial \omega_k}{\partial x_j}x_k + \omega_k\delta_{jk} = \omega_j \tag{7.12}$$

it follows that

$$\frac{\partial}{\partial x_i}\left[\hat{v}_l^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\right] = \omega_j\left(\hat{v}_l^g\right)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{7.13}$$

and then

$$\frac{\partial^2}{\partial x_k \partial x_i}\left[\hat{v}_l^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\right] = \omega_i\frac{\partial}{\partial x_k}\left[\left(\hat{v}_l^g\right)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x})\right] = \omega_i\omega_k\left(\hat{v}_l^g\right)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{7.14}$$

while

$$\frac{\partial}{\partial x_j}\left[\hat{p}^g(\boldsymbol{\omega}\cdot\boldsymbol{x})\right] = \omega_j\left(\hat{p}^g\right)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{7.15}$$

A substitution of equations (7.14) ans (7.15) into (7.11) yields:

$$\omega_i\mathbb{K}_{ijkl}\,\omega_k\left(\hat{v}_l^g\right)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + \omega_j\left(\hat{p}^g\right)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + \delta_{jg}\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0 \tag{7.16}$$

We define the acoustic tensor $A_{jl}(\boldsymbol{\omega})$ as

$$A_{jl}(\boldsymbol{\omega}) = \omega_i \mathbb{K}_{ijkl}\, \omega_k \tag{7.17}$$

which is symmetric if $\mathbb{K}_{ijkl}$ has the major symmetry; note that the Biot constitutive framework satisfies this condition. Equation (7.16) can be rewritten as:

$$A_{jl}(\boldsymbol{\omega})\big(\hat{v}_l^g\big)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + \omega_j\big(\hat{p}^g\big)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + \delta_{jg}\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0 \tag{7.18}$$

The incompressibility constraint (7.2) in the transformed domain becomes:

$$-\frac{1}{8\pi^2}\int_{|\boldsymbol{\omega}|=1} \omega_k\big(\hat{v}_k^g\big)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} = 0 \tag{7.19}$$

If the above relation holds for every vector $\boldsymbol{\omega}$, then we have

$$\omega_k\big(\hat{v}_k^g\big)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0 \tag{7.20}$$

The derivative of (7.20) with respect to x_s yields:

$$\frac{\partial}{\partial x_s}\Big[\omega_k\big(\hat{v}_k^g\big)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x})\Big] = \omega_k\omega_s\big(\hat{v}_k^g\big)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0 \tag{7.21}$$

Here we conclude that

$$\omega_k\big(\hat{v}_k^g\big)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0 \tag{7.22}$$

since equation (7.21) must hold for every x_s.

Within the elliptic regime, the acoustic tensor can be inverted ($A_{kj}(\boldsymbol{\omega}) \in \mathrm{Inv}$), so that we can multiply equation (7.18) by $A_{kj}(\boldsymbol{\omega})^{-1}$:

$$\begin{aligned} A_{kj}(\boldsymbol{\omega})^{-1}A_{jl}(\boldsymbol{\omega})\big(\hat{v}_l^g\big)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_{kj}(\boldsymbol{\omega})^{-1}\omega_j\big(\hat{p}^g\big)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) +& \\ + A_{kj}(\boldsymbol{\omega})^{-1}\delta_{jg}\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0& \end{aligned} \tag{7.23}$$

namely

$$\big(\hat{v}_l^g\big)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_{kj}(\boldsymbol{\omega})^{-1}\omega_j\big(\hat{p}^g\big)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_{kg}(\boldsymbol{\omega})^{-1}\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0 \tag{7.24}$$

Exploiting condition (7.22), which comes from incompressibility, we do a projection of the (7.24) on $\boldsymbol{\omega}$:

$$\underbrace{\omega_k \left(\hat{v}_l^g\right)^{''} (\boldsymbol{\omega} \cdot \boldsymbol{x})}_{=0} + \omega_k A_{kj}(\boldsymbol{\omega})^{-1} \omega_j \left(\hat{p}^g\right)^{'} (\boldsymbol{\omega} \cdot \boldsymbol{x}) + \\ + \omega_k A_{kg}(\boldsymbol{\omega})^{-1} \delta^{''} (\boldsymbol{\omega} \cdot \boldsymbol{x}) = 0 \quad (7.25)$$

so that

$$\omega_k A_{kj}(\boldsymbol{\omega})^{-1} \omega_j \left(\hat{p}^g\right)^{'} (\boldsymbol{\omega} \cdot \boldsymbol{x}) + \omega_k A_{kg}(\boldsymbol{\omega})^{-1} \delta^{''} (\boldsymbol{\omega} \cdot \boldsymbol{x}) = 0 \quad (7.26)$$

from which we can obtain the expression for the derivative of the incremental mean stress

$$\left(\hat{p}^g\right)^{'} (\boldsymbol{\omega} \cdot \boldsymbol{x}) = -\frac{\omega_k A_{kg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega}) \omega_s} \delta^{''} (\boldsymbol{\omega} \cdot \boldsymbol{x}) \quad (7.27)$$

A substitution of equation (7.27) into (7.24) yields the second derivative of the incremental displacement

$$\left(\hat{v}_k^g\right)^{''} (\boldsymbol{\omega} \cdot \boldsymbol{x}) = \left[\frac{A_{kj}^{-1}(\boldsymbol{\omega}) \omega_j \, \omega_t A_{tg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega}) \omega_s} - A_{kg}^{-1}(\boldsymbol{\omega}) \right] \delta^{''} (\boldsymbol{\omega} \cdot \boldsymbol{x}) \quad (7.28)$$

The integration of equations (7.27) and (7.28) and their antitransform leads to the *Green's functions set for the incremental displacement field of an elastic, anisotropic, incompressible material subject to prestress*

$$v_k^g(\boldsymbol{x}) = -\frac{1}{8\pi^2 r} \int_{|\boldsymbol{\omega}|=1} \left[\frac{A_{kj}^{-1}(\boldsymbol{\omega}) \omega_j \, \omega_t A_{tg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega}) \omega_s} - A_{kg}^{-1}(\boldsymbol{\omega}) \right] \delta(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \, \mathrm{d}\boldsymbol{\omega} \quad (7.29)$$

and for the *incremental mean stress*

$$\dot{p}^g(\boldsymbol{x}) = \frac{1}{8\pi^2 r^2} \int_{|\boldsymbol{\omega}|=1} \frac{\omega_k A_{kg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega}) \omega_s} \delta^{'} (\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \, \mathrm{d}\boldsymbol{\omega} \quad (7.30)$$

where

$$r = |\boldsymbol{x}| \quad (7.31)$$

while $\boldsymbol{e}_r$ is the unit vector associated to the vector $\boldsymbol{x}$, namely

$$\boldsymbol{e}_r = \frac{\boldsymbol{x}}{r} \quad (7.32)$$

and we used the well-known property of the delta function

$$\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{1}{r^2}\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) \tag{7.33}$$

reported, for instance, in Gel'fand e Shilov [59, p. 213, equation (20)].

From equation (7.29) we can immediately note the symmetry between the indexes k and g that holds for the incremental displacement field:

$$v_k^g(\boldsymbol{x}) = v_g^k(\boldsymbol{x}) \tag{7.34}$$

and this is a consequence of the symmetry of the acoustic tensor $A_{ij}(\boldsymbol{\omega})$, which arises from the major symmetry of $\mathbb{K}_{ijkl}$.

If we employed the following:

$$\dot{S}_{ij} = \mathbb{G}_{ijkl}\frac{\partial v_k}{\partial x_l} + \dot{p}\delta_{ij} \tag{7.35a}$$

$$\frac{\partial \dot{S}_{ij}}{\partial x_j} + \dot{f}_i = 0 \tag{7.35b}$$

(where $\dot{S}_{ij} = \dot{t}_{ji}$ is the increment of the fist Piola-Kirchhoff stress tensor) instead of the constitutive equation (7.3) and of the equilibrium equation (7.1), the Green's functions (7.29) and (7.30) would not change, but the acoustic tensor would chang its definition to:

$$A_{ik}(\boldsymbol{\omega}) = \omega_j \mathbb{G}_{ijkl}\,\omega_l \tag{7.36}$$

In the transformed domain, the Green's function for the incremental displacement field can be written as

$$\hat{v}_k^g = \frac{1}{r}V_{gk}(\boldsymbol{\omega})\delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) \tag{7.37}$$

where

$$V_{gk}(\boldsymbol{\omega}) = \frac{A_{kj}^{-1}(\boldsymbol{\omega})\omega_j\,\omega_t A_{tg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega})\,\omega_s} - A_{kg}^{-1}(\boldsymbol{\omega}) \tag{7.38}$$

The gradient of the incremental displacements can be calculated as

$$\frac{\partial \hat{v}_k^g}{\partial x_l} = \frac{1}{r^2} D_{gkl}(\boldsymbol{\omega}) \delta^{'}(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \tag{7.39}$$

where

$$D_{gkl}(\boldsymbol{\omega}) = \omega_l V_{gk}(\boldsymbol{\omega}) \tag{7.40}$$

so that, the antitransform yields the *integral equation for the gradient of the incremental displacement field*

$$\frac{\partial v_k^g(\boldsymbol{x})}{\partial x_l} = -\frac{1}{8\pi^2 r^2} \int_{|\boldsymbol{\omega}|=1} D_{gkl}(\boldsymbol{\omega}) \delta^{'}(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \, \mathrm{d}\boldsymbol{\omega} \tag{7.41}$$

7.2 Boundary integral equations for homogeneously prestressed three-dimensional solids

The boundary integral equation for the incremental displacement of a uniformly prestressed non-linear elastic body subject to mixed boundary conditions has been given by Bigoni & Capuani [1], with reference to a two-dimensional deformation. However, their result can be immediately generalized to three-dimensional deformation. In fact, with reference to a uniformly prestressed body subject to the following incremental boundary conditions holding on non-overlapping regions ∂B_{v} and ∂B_{τ} of the boundary ∂B

$$\begin{cases} v = \bar{v} & \text{on} \quad \partial B_{\mathrm{v}} \\ \dot{t}_{ij} n_i = \dot{\tau}_j & \text{on} \quad \partial B_{\tau} \end{cases} \tag{7.42}$$

the Betti identity written on incremental fields yields for the incremental displacement of point $\boldsymbol{y}$ on the boundary ∂B

$$v_j(\boldsymbol{y}) C_j^g(\boldsymbol{y}) = \int_{\partial B} \left[\dot{t}_{ij} n_i v_j^g(\boldsymbol{x}, \boldsymbol{y}) - \dot{t}_{ij}^g(\boldsymbol{x}, \boldsymbol{y}) n_i v_j \right] \mathrm{d}S_{\boldsymbol{x}} \tag{7.43}$$

where

$$C_j^g(\boldsymbol{y}) = \lim_{\varepsilon \to 0} \int_{\partial C_\varepsilon} \dot{t}_{ij}^g(\boldsymbol{x}, \boldsymbol{y}) n_i \, \mathrm{d}S_{\boldsymbol{x}} \tag{7.44}$$

is the C-matrix defined in the limit of vanishing radius ε of the sphere C_ε. Note that for points $\boldsymbol{y}$ interior to the region B we have $C_j^g = \delta_{gj}$, so that the boundary integral equation for incremental displacements is obtained

$$v_g(\boldsymbol{y}) = \int_{\partial B} \left[\dot{t}_{ij} n_i v_j^g(\boldsymbol{x}, \boldsymbol{y}) - \dot{t}_{ij}^g(\boldsymbol{x}, \boldsymbol{y}) n_i v_j\right] \mathrm{d}S_{\boldsymbol{x}} \tag{7.45}$$

Although equations (7.43)–(7.45) are formally identical to equations (57)–(59) of Bigoni & Capuani [1], so that their derivation is immediate, the boundary integral equation for the incremental mean stress $\dot{p}(\boldsymbol{y})$ requires a complex derivation (since the result shown in Appendix B of Bigoni & Capuani [1] is strictly limited to two-dimensional deformation), which is provided in the following through the introduction of an *ad hoc* potential Φ.

7.2.1 Boundary integral equations for the incremental mean stress

The boundary integral equation for the mean stress increment $\dot{p}(\boldsymbol{y})$ is the necessary complement to the equation for incremental displacements (7.45). This can be obtained with reference to the incremental mixed boundary value problem (7.42), through a double differentiation of equation (7.45) with respect to $\boldsymbol{y}$ and use of the incremental equilibrium equations (7.5) with null body forces to obtain

$$\dot{p}_{,h}(\boldsymbol{y}) = -\int_{\partial B} \mathbb{K}_{nhsg} \left[\dot{t}_{ij} n_i v_{j,sn}^g(\boldsymbol{x}, \boldsymbol{y}) - \dot{t}_{ij,sn}^g(\boldsymbol{x}, \boldsymbol{y}) n_i v_j\right] \mathrm{d}S_{\boldsymbol{x}} \tag{7.46}$$

Repeated use of the incremental equilibrium equations (7.5) yields

$$\begin{aligned} \dot{p}_{,h}(\boldsymbol{y}) = \int_{\partial B} \Big[& \dot{t}_{ig} n_i \dot{p}_{,h}^g(\boldsymbol{x}, \boldsymbol{y}) - n_i v_j \mathbb{K}_{ijkg} \dot{p}_{,hk}^g(\boldsymbol{x}, \boldsymbol{y}) + \\ & + n_i v_i \mathbb{K}_{nhsg} \dot{p}_{,sn}^g(\boldsymbol{x}, \boldsymbol{y}) \Big] \mathrm{d}S_{\boldsymbol{x}} \end{aligned} \tag{7.47}$$

which, introducing the potential Φ defined as

$$\mathbb{K}_{sirg}\, \dot{p}_{,rs}^g = \Phi_{,i} \tag{7.48}$$

becomes

$$\dot{p}_{,h}(\boldsymbol{y}) = \int_{\partial B} \Big[\dot{t}_{ig} n_i \dot{p}^g_{,h}(\boldsymbol{x}, \boldsymbol{y}) - n_i v_j \mathbb{K}_{ijkg} \dot{p}^g_{,hk}(\boldsymbol{x}, \boldsymbol{y}) + \\ + n_i v_i \Phi_{,h}(\boldsymbol{x}, \boldsymbol{y}) \Big] \, \mathrm{d}S_{\boldsymbol{x}} \quad (7.49)$$

Equation (7.49) can be integrated to obtain *the boundary integral equation for the incremental mean stress*

$$\dot{p}(\boldsymbol{y}) = \int_{\partial B} \Big[\dot{t}_{ig} n_i \dot{p}^g(\boldsymbol{x}, \boldsymbol{y}) - n_i v_j \mathbb{K}_{ijkg} \dot{p}^g_{,k}(\boldsymbol{x}, \boldsymbol{y}) + n_i v_i \Phi(\boldsymbol{x}, \boldsymbol{y}) \Big] \, \mathrm{d}S_{\boldsymbol{x}} \quad (7.50)$$

complementing equation (7.45) and thus providing the boundary integral equation set for incompressible, prestressed elasticity. Now the existence of potential Φ and its form have to be determined.

7.3 Existence of a scalar potential

In Section 6.3 we demonstrated that within a plane constitutive framework the following relation holds:

$$\mathbb{K}_{sirg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x} - \boldsymbol{y})}{\partial x_r \partial x_s} = \frac{\partial F(\boldsymbol{x} - \boldsymbol{y})}{\partial x_i} \quad (7.51)$$

where the function $F(\boldsymbol{x} - \boldsymbol{y})$ is given by (6.99) or (6.100), where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. We want to generalize this expression to the three-dimensional case. To this purpose, we search for a potential $\Phi(\boldsymbol{x})$, with $\boldsymbol{x} \in \mathbb{R}^3$, that satisfies the following relation:

$$\mathbb{K}_{sirg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} = \frac{\partial \Phi(\boldsymbol{x})}{\partial x_i} \quad (7.52)$$

A necessary and sufficient condition for the existence of this potential is

$$\nabla \times (\nabla \Phi) = \operatorname{curl} \operatorname{grad} \Phi = \mathbf{0} \quad (7.53)$$

which in indexes becomes:

$$\varepsilon_{jti} \frac{\partial}{\partial x_t} \left[\mathbb{K}_{sirg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \right] = 0 \quad (7.54)$$

where ε_{jti} is the Ricci or Levi-Civita tensor[3], and which corresponds to the following equations system:

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathbb{K}_{s1rg}\dfrac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} & \mathbb{K}_{s2rg}\dfrac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} & \mathbb{K}_{s3rg}\dfrac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \end{vmatrix} = \mathbf{0} \tag{7.55}$$

If we expand the determinant, we obtain

$$\begin{aligned} \nabla \times (\nabla\Phi) = & \left\{ \frac{\partial}{\partial x_2}\left[\mathbb{K}_{s3rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\right] - \frac{\partial}{\partial x_3}\left[\mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\right]\right\}\boldsymbol{e}_1 + \\ & + \left\{ \frac{\partial}{\partial x_3}\left[\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\right] - \frac{\partial}{\partial x_1}\left[\mathbb{K}_{s3rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\right]\right\}\boldsymbol{e}_2 + \\ & + \left\{ \frac{\partial}{\partial x_1}\left[\mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\right] - \frac{\partial}{\partial x_2}\left[\mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\right]\right\}\boldsymbol{e}_3 = \mathbf{0} \end{aligned} \tag{7.56}$$

from which

$$\begin{aligned} \nabla \times (\nabla\Phi) = & \left[\mathbb{K}_{s3rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2} - \mathbb{K}_{s2rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3}\right]\boldsymbol{e}_1 + \\ & + \left[\mathbb{K}_{s1rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3} - \mathbb{K}_{s3rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1}\right]\boldsymbol{e}_2 + \\ & + \left[\mathbb{K}_{s2rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1} - \mathbb{K}_{s1rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2}\right]\boldsymbol{e}_3 = \mathbf{0} \end{aligned} \tag{7.57}$$

and a simplification leads to

$$\mathbb{K}_{sirg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_u} - \mathbb{K}_{surg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_i} = 0 \tag{7.58}$$

namely

$$\mathbb{K}_{sirg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_u} = \mathbb{K}_{surg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_i} \tag{7.59}$$

[3] See Section 2.1.10 and the definition given by (2.29).

Note that if $i = u$ we have the identity relation. On application of the derivative of the equilibrium equations with respect to $\boldsymbol{x}$, we can write

$$\mathbb{K}_{sirg}\frac{\partial^2 v_g(\boldsymbol{x})}{\partial x_r \partial x_s} + \frac{\partial \dot{p}(\boldsymbol{x})}{\partial x_i} = 0 \tag{7.60}$$

while for the Green's functions we have

$$\mathbb{K}_{sirg}\frac{\partial^2 v_k^g(\boldsymbol{x})}{\partial x_r \partial x_s} = \mathbb{K}_{sirg}\frac{\partial^2 v_g^k(\boldsymbol{x})}{\partial x_r \partial x_s} = -\frac{\partial \dot{p}^k(\boldsymbol{x})}{\partial x_i} \tag{7.61}$$

If we change the indexes:

$$\mathbb{K}_{tipq}\frac{\partial^2 v_q^g(\boldsymbol{x})}{\partial x_p \partial x_t} + \frac{\partial \dot{p}^g(\boldsymbol{x})}{\partial x_i} = 0 \tag{7.62}$$

and differentiate twice this expression with respect to$\boldsymbol{x}$:

$$\mathbb{K}_{tipq}\frac{\partial^4 v_q^g(\boldsymbol{x})}{\partial x_p \partial x_t \partial x_r \partial x_s} + \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_i \partial x_r \partial x_s} = 0 \tag{7.63}$$

the application to both members of the constitutive tensor $\mathbb{K}$ yields

$$\mathbb{K}_{surg}\mathbb{K}_{tipq}\frac{\partial^4 v_q^g(\boldsymbol{x})}{\partial x_p \partial x_t \partial x_r \partial x_s} = -\mathbb{K}_{surg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_i \partial x_r \partial x_s} \tag{7.64}$$

Note that in this relation the free indexes are i and u. Now we can do a comparison between (7.64) and (7.59). Since for the Green's functions we have $v_k^g(\boldsymbol{x}) = v_g^k(\boldsymbol{x})$, namely we can interchange the indexes, we observe that the first member of equation (7.64) is symmetric with respect to indexes i and u. In fact

$$\begin{aligned}\mathbb{K}_{surg}\mathbb{K}_{tipq}\frac{\partial^4 v_q^g(\boldsymbol{x})}{\partial x_p \partial x_t \partial x_r \partial x_s} &= \mathbb{K}_{surg}\mathbb{K}_{tipq}\frac{\partial^4 v_g^q(\boldsymbol{x})}{\partial x_p \partial x_t \partial x_r \partial x_s} = \\ &= \mathbb{K}_{tipq}\mathbb{K}_{surg}\frac{\partial^4 v_g^q(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_p \partial x_t} = \\ &= \mathbb{K}_{surg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_i \partial x_r \partial x_s}\end{aligned} \tag{7.65}$$

but, if we assume the continuity of $\dot{p}^g(\boldsymbol{x})$, we have

$$\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} = \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_s \partial x_r} \quad \longrightarrow \quad \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_i \partial x_r \partial x_s} = \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_i} \tag{7.66}$$

In the second equality of equation (7.65) we used the major symmetry of the constitutive tensor $\mathbb{K}$, so that we can write $\mathbb{K}_{pqti} = \mathbb{K}_{tipq}$, thus yielding

$$\mathbb{K}_{surg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_i \partial x_r \partial x_s} = \mathbb{K}_{surg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_i} \tag{7.67}$$

and in this way we demonstrated the equivalence between equation (7.67) and the second member of (7.59). Now we have to demonstrate the equivalence between the first members of (7.59) and (7.64), namely the symmetry between indexes i and u, but to verify this equality it is sufficient to observe the first and the third members of (7.65).

7.4 The form of the scalar potential

The equation (7.52) allows us to obtain the $\Phi(\boldsymbol{x})$ through a direct integration with respect to variables x_1, x_2 or x_3; in particular we obtain the following three forms, depending on the integration variable adopted:

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s1rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_1 + f_1(x_2, x_3) \tag{7.68a}$$

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_2 + f_2(x_1, x_3) \tag{7.68b}$$

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_3 + f_3(x_1, x_2) \tag{7.68c}$$

If we consider (7.68a) and (7.68c) and we apply a derivative with respect to x_2, we obtain

$$\begin{aligned} \frac{\partial \Phi(\boldsymbol{x})}{\partial x_2} &= \frac{\partial}{\partial x_2} \left[\int \mathbb{K}_{s1rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_1 + f_1(x_2, x_3) \right] = \\ &= \int \mathbb{K}_{s1rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2} \, \mathrm{d}x_1 + \frac{\partial f_1(x_2, x_3)}{\partial x_2} \end{aligned} \tag{7.69}$$

and

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_2} = \frac{\partial}{\partial x_2}\left[\int \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\, \mathrm{d}x_3 + f_3(x_1, x_2)\right] =$$
$$= \int \mathbb{K}_{s3rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2}\, \mathrm{d}x_3 + \frac{\partial f_3(x_1, x_2)}{\partial x_2} \tag{7.70}$$

respectively, but from (7.59) we have

$$\mathbb{K}_{s1rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2} = \mathbb{K}_{s2rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1} \tag{7.71}$$

and, in an analogous manner,

$$\mathbb{K}_{s3rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2} = \mathbb{K}_{s2rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3} \tag{7.72}$$

Equations (7.71) and (7.71) allow us to rewrite (7.69) and (7.70) as

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_2} = \int \mathbb{K}_{s2rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1}\, \mathrm{d}x_1 + \frac{\partial f_1(x_2, x_3)}{\partial x_2} \tag{7.73a}$$
$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_2} = \int \mathbb{K}_{s2rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3}\, \mathrm{d}x_3 + \frac{\partial f_3(x_1, x_2)}{\partial x_2} \tag{7.73b}$$

so that their primitives can be calculated. Therefore, we obtain

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_2} = \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} + \frac{\partial f_1(x_2, x_3)}{\partial x_2} \tag{7.74a}$$
$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_2} = \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} + \frac{\partial f_3(x_1, x_2)}{\partial x_2} \tag{7.74b}$$

The above equations can be integrated:

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\, \mathrm{d}x_2 + f_1(x_2, x_3) + F_{21}(x_1, x_3) \tag{7.75a}$$
$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\, \mathrm{d}x_2 + f_3(x_1, x_2) + F_{23}(x_1, x_3) \tag{7.75b}$$

where $F_{21}(x_1, x_3)$ and $F_{23}(x_1, x_3)$ are scalar valued functions arising from the integration[4]. A comparison between (7.75) and (7.68b) leads to

$$f_1(x_2, x_3) + F_{21}(x_1, x_3) = f_2(x_1, x_3) \tag{7.76a}$$

$$f_3(x_1, x_2) + F_{23}(x_1, x_3) = f_2(x_1, x_3) \tag{7.76b}$$

from which we deduce that $f_1 = f_1(x_3)$ only and $f_3 = f_3(x_1)$ only, namely f_1 and f_3 are functions only of the variable x_3 and x_1 respectively; furthermore, from (7.74) we obtain

$$\frac{\partial f_1(x_2, x_3)}{\partial x_2} = \frac{\partial f_3(x_1, x_2)}{\partial x_2} \tag{7.77}$$

If we consider (7.68a) and (7.68b) and we apply a derivative with respect to x_3, we obtain

$$\begin{aligned}\frac{\partial \Phi(\boldsymbol{x})}{\partial x_3} &= \frac{\partial}{\partial x_3}\left[\int \mathbb{K}_{s1rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\, \mathrm{d}x_1 + f_1(x_2, x_3)\right] = \\ &= \int \mathbb{K}_{s1rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3}\, \mathrm{d}x_1 + \frac{\partial f_1(x_2, x_3)}{\partial x_3}\end{aligned} \tag{7.78}$$

and

$$\begin{aligned}\frac{\partial \Phi(\boldsymbol{x})}{\partial x_3} &= \frac{\partial}{\partial x_3}\left[\int \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\, \mathrm{d}x_2 + f_2(x_1, x_3)\right] = \\ &= \int \mathbb{K}_{s2rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3}\, \mathrm{d}x_2 + \frac{\partial f_2(x_1, x_3)}{\partial x_3}\end{aligned} \tag{7.79}$$

respectively, but from (7.59) we have

$$\mathbb{K}_{s1rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3} = \mathbb{K}_{s3rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1} \tag{7.80}$$

and, in an analogous manner,

$$\mathbb{K}_{s2rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3} = \mathbb{K}_{s3rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2} \tag{7.81}$$

[4] The functions we introduced are *not* tensors; the subscripts have been chosen to recall the integration variable (the first index) and the choice of the initial form of the function $\Phi(\boldsymbol{x})$ (the second index).

Equations (7.80) and (7.81) allow us to rewrite (7.78) and (7.79) as

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_3} = \int \mathbb{K}_{s3rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1} \, \mathrm{d}x_1 + \frac{\partial f_1(x_2, x_3)}{\partial x_3} \tag{7.82a}$$

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_3} = \int \mathbb{K}_{s3rg} \frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2} \, \mathrm{d}x_2 + \frac{\partial f_3(x_1, x_3)}{\partial x_3} \tag{7.82b}$$

so that their primitives can be calculated. Therefore, we obtain

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_3} = \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} + \frac{\partial f_1(x_2, x_3)}{\partial x_3} \tag{7.83a}$$

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_3} = \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} + \frac{\partial f_2(x_1, x_3)}{\partial x_3} \tag{7.83b}$$

The above equations can be integrated:

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_3 + f_1(x_2, x_3) + F_{31}(x_1, x_2) \tag{7.84a}$$

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_3 + f_2(x_1, x_3) + F_{32}(x_1, x_2) \tag{7.84b}$$

where $F_{31}(x_1, x_2)$ and $F_{32}(x_1, x_2)$ are scalar valued functions arising from the integration[5]. A comparison between (7.84) and (7.68c) leads to

$$f_1(x_2, x_3) + F_{31}(x_1, x_2) = f_3(x_1, x_2) \tag{7.85a}$$

$$f_2(x_1, x_3) + F_{32}(x_1, x_2) = f_3(x_1, x_2) \tag{7.85b}$$

from which we deduce that $f_1 = f_1(x_2)$ only and $f_2 = f_2(x_1)$ only, namely f_1 and f_2 are functions only of the variable x_2 and x_1 respectively; furthermore, from (7.83) we obtain

$$\frac{\partial f_1(x_2, x_3)}{\partial x_3} = \frac{\partial f_2(x_1, x_3)}{\partial x_3} \tag{7.86}$$

If we consider (7.68b) and (7.68c) and we apply a derivative with respect to

[5] As in the previous case, the functions we introduced are *not* tensors (see note 4 on the previous page).

x_1, we obtain

$$\begin{aligned}\frac{\partial \Phi(\boldsymbol{x})}{\partial x_1} &= \frac{\partial}{\partial x_1}\left[\int \mathbb{K}_{s2rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_2 + f_2(x_1,x_3)\right] = \\ &= \int \mathbb{K}_{s2rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1}\,\mathrm{d}x_2 + \frac{\partial f_2(x_1,x_3)}{\partial x_1}\end{aligned} \tag{7.87}$$

and

$$\begin{aligned}\frac{\partial \Phi(\boldsymbol{x})}{\partial x_1} &= \frac{\partial}{\partial x_1}\left[\int \mathbb{K}_{s3rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_3 + f_3(x_1,x_2)\right] = \\ &= \int \mathbb{K}_{s3rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1}\,\mathrm{d}x_3 + \frac{\partial f_3(x_1,x_2)}{\partial x_1}\end{aligned} \tag{7.88}$$

respectively, but from (7.59) we have

$$\mathbb{K}_{s2rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1} = \mathbb{K}_{s1rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2} \tag{7.89}$$

and, in an analogous manner,

$$\mathbb{K}_{s3rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_1} = \mathbb{K}_{s1rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3} \tag{7.90}$$

Equations (7.89) and (7.90) allow us to rewrite (7.87) and (7.88) as

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_1} = \int \mathbb{K}_{s1rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_2}\,\mathrm{d}x_2 + \frac{\partial f_2(x_1,x_3)}{\partial x_1} \tag{7.91a}$$

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_1} = \int \mathbb{K}_{s1rg}\frac{\partial^3 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s \partial x_3}\,\mathrm{d}x_3 + \frac{\partial f_3(x_1,x_2)}{\partial x_1} \tag{7.91b}$$

so that their primitives can be calculated. Therefore, we obtain

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_1} = \mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} + \frac{\partial f_2(x_1,x_3)}{\partial x_1} \tag{7.92a}$$

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_1} = \mathbb{K}_{s1rg}\frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} + \frac{\partial f_3(x_1,x_2)}{\partial x_1} \tag{7.92b}$$

The above equations can be integrated:

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s1rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_1 + f_2(x_1, x_3) + F_{12}(x_2, x_3) \tag{7.93a}$$

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{s1rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \, \mathrm{d}x_1 + f_3(x_1, x_2) + F_{13}(x_2, x_3) \tag{7.93b}$$

where $F_{12}(x_2, x_3)$ and $F_{13}(x_2, x_3)$ are scalar valued functions arising from the integration[6]. A comparison between (7.93) and (7.68a) leads to

$$f_2(x_1, x_3) + F_{12}(x_2, x_3) = f_1(x_2, x_3) \tag{7.94a}$$

$$f_3(x_1, x_2) + F_{13}(x_2, x_3) = f_1(x_2, x_3) \tag{7.94b}$$

from which we deduce that $f_2 = f_2(x_3)$ only and $f_3 = f_3(x_2)$ only, namely f_2 and f_3 are functions only of the variable x_3 and x_2 respectively; furthermore, from (7.92) we obtain

$$\frac{\partial f_2(x_1, x_3)}{\partial x_1} = \frac{\partial f_3(x_1, x_2)}{\partial x_1} \tag{7.95}$$

A comparison between (7.76), (7.85), (7.94) with (7.77), (7.86), (7.95), respectively, allows us to observe that the functions f_1, f_2, f_3 must necessarily be constant (with arbitrary values). We can verify this statement through a comparison between the conclusions we did every time from (7.76), (7.85), and (7.94), following which, we have:

$$f_1 = f_1(x_3) = f_1(x_2) \tag{7.96a}$$

$$f_2 = f_2(x_1) = f_2(x_3) \tag{7.96b}$$

$$f_3 = f_3(x_1) = f_3(x_2) \tag{7.96c}$$

If the functions f_1, f_2, f_3 are constant, from (7.76), (7.85), and (7.94) we have that even the functions F_{ij} (with $i, j = 1, 2, 3$) are arbitrary constants[7]. But, for the purpose of the calculation of the stress field, we need only the gradient of the potential $\Phi(\boldsymbol{x})$, so that we can assume all the integration functions (that are constant) equal to zero, which means to assume $F_{ij} = f_i = 0$ ($\forall i, j = 1, 2, 3$).

As a conclusion, the potential $\Phi(\boldsymbol{x})$ can be expressed as a simple primitive of (7.52) by omitting every integration constant; in this way, we obtain three

[6] As in the previous cases, the functions we introduced are *not* tensors (see note 4 on page 205).

[7] We refer to scalar valued functions obtained from the integrations.

possible principal forms, fully equivalent to each other and similar to (7.68), for the expression of the potential as follows:

$$\begin{aligned}\Phi(\boldsymbol{x}) &= \int \mathbb{K}_{s1rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_1 = \int \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_2 = \\ &= \int \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_3\end{aligned} \tag{7.97}$$

which can be rewritten in a more compact form as

$$\Phi(\boldsymbol{x}) = \int \mathbb{K}_{sirg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_i \tag{7.98}$$

where index i should be chosen from the values $1, 2, 3$ and is *not* summed. In analogy to Section 6.1, we can rewrite the form of the potential $\Phi(\boldsymbol{x})$ in a more general way that takes account, at the same time, the three primitives given in (7.98) through the introduction of a 'distribution' coefficient $R_i\big(\hat{\alpha}, \hat{\beta}\big)$ defined as[8]:

$$R_i\big(\hat{\alpha}, \hat{\beta}\big) = \hat{\alpha}\delta_{i1} + \hat{\beta}\delta_{i2} + \big(1 - \hat{\alpha} - \hat{\beta}\big)\delta_{i3} \tag{7.99}$$

where δ_{ij} is the Kronecker delta, $i, j = 1, 2, 3$ and $\hat{\alpha}, \hat{\beta} \in [0, 1]$; furthermore, the constraint $\hat{\alpha} + \hat{\beta} \leq 1$ must be verified. In this way, we obtain a family of potentials that depend on the arbitrary choice of the coefficients $\hat{\alpha}$ and $\hat{\beta}$, that writes as

$$\Phi(\boldsymbol{x}) = \sum_{i,g,r,s=1}^{3} R_i\big(\hat{\alpha}, \hat{\beta}\big) \int \mathbb{K}_{sirg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_i \tag{7.100}$$

where now the sum of the indexes is explicit;[9] the above relation corresponds to

$$\begin{aligned}\Phi(\boldsymbol{x}) = \hat{\alpha} \int \mathbb{K}_{s1rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_1 + \hat{\beta} \int \mathbb{K}_{s2rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_2 + \\ + (1 - \hat{\alpha} - \hat{\beta}) \int \mathbb{K}_{s3rg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s}\,\mathrm{d}x_3\end{aligned} \tag{7.101}$$

The equation defining the potential Φ starting from the equilibrium equa-

[8] See for instance the expressions (6.35)-(6.37).

[9] This is necessary since in the argument of the sum (7.100) there is an index repeated three times.

tions is:

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial x_i} = \mathbb{K}_{sirg} \frac{\partial^2 \dot{p}^g(\boldsymbol{x})}{\partial x_r \partial x_s} \tag{7.102}$$

In the transformed domain we have $\hat{\Phi} = \hat{\Phi}(\boldsymbol{\omega} \cdot \boldsymbol{x})$ and $\hat{p}^g = \hat{p}^g(\boldsymbol{\omega} \cdot \boldsymbol{x})$, and therefore their derivatives are expressed as

$$\frac{\partial \hat{\Phi}(\boldsymbol{\omega} \cdot \boldsymbol{x})}{\partial x_i} = \omega_i \hat{\Phi}'(\boldsymbol{\omega} \cdot \boldsymbol{x}) \tag{7.103}$$

and

$$\frac{\partial^2 \hat{p}^g(\boldsymbol{\omega} \cdot \boldsymbol{x})}{\partial x_r \partial x_s} = \omega_r \omega_s (\hat{p}^g)''(\boldsymbol{\omega} \cdot \boldsymbol{x}) \tag{7.104}$$

so that, in the transformed domain, equation (7.102) can be rewritten as

$$\omega_i \hat{\Phi}'(\boldsymbol{\omega} \cdot \boldsymbol{x}) = \mathbb{K}_{sirg}\, \omega_r \omega_s (\hat{p}^g)''(\boldsymbol{\omega} \cdot \boldsymbol{x}) \tag{7.105}$$

but from the definition of the acoustic tensor (7.17) we observe that

$$\mathbb{K}_{sirg}\, \omega_r \omega_s = A_{ig}(\boldsymbol{\omega}) \tag{7.106}$$

and therefore

$$\omega_i \hat{\Phi}'(\boldsymbol{\omega} \cdot \boldsymbol{x}) = A_{ig}(\boldsymbol{\omega}) (\hat{p}^g)''(\boldsymbol{\omega} \cdot \boldsymbol{x}) \tag{7.107}$$

If we multiply the above relation by ω_i, which corresponds to a projection on $\boldsymbol{\omega}$, we obtain:

$$\underbrace{\omega_i \omega_i}_{=1} \hat{\Phi}'(\boldsymbol{\omega} \cdot \boldsymbol{x}) = \hat{\Phi}'(\boldsymbol{\omega} \cdot \boldsymbol{x}) = \omega_i A_{ig}(\boldsymbol{\omega}) (\hat{p}^g)''(\boldsymbol{\omega} \cdot \boldsymbol{x}) \tag{7.108}$$

The second order derivative of the mean stress with respect to the argument $(\boldsymbol{\omega} \cdot \boldsymbol{x})$ is:

$$\begin{aligned} \frac{\partial}{\partial(\boldsymbol{\omega} \cdot \boldsymbol{x})} (\hat{p}^g)'(\boldsymbol{\omega} \cdot \boldsymbol{x}) &= \frac{\partial}{\partial(\boldsymbol{\omega} \cdot \boldsymbol{x})} \left[-\frac{\omega_k A^{-1}_{kg}(\boldsymbol{\omega})}{\omega_r A^{-1}_{rs}(\boldsymbol{\omega}) \omega_s} \delta''(\boldsymbol{\omega} \cdot \boldsymbol{x}) \right] = \\ &= -\frac{\omega_k A^{-1}_{kg}(\boldsymbol{\omega})}{\omega_r A^{-1}_{rs}(\boldsymbol{\omega}) \omega_s} \delta'''(\boldsymbol{\omega} \cdot \boldsymbol{x}) \end{aligned} \tag{7.109}$$

so that equation (7.108) becomes:

$$\hat{\Phi}'(\boldsymbol{\omega}\cdot\boldsymbol{x}) = -\frac{\omega_i A_{ig}(\boldsymbol{\omega})\,\omega_k A^{-1}_{kg}(\boldsymbol{\omega})}{\omega_r A^{-1}_{rs}(\boldsymbol{\omega})\omega_s}\delta'''(\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{7.110}$$

The integration and the antitransform of the above equation can isolate the potential:

$$\hat{\Phi}(\boldsymbol{x}) = \frac{1}{8\pi^2 r^3}\int_{|\boldsymbol{\omega}|=1}\frac{\omega_i A_{ig}(\boldsymbol{\omega})\,\omega_k A^{-1}_{kg}(\boldsymbol{\omega})}{\omega_r A^{-1}_{rs}(\boldsymbol{\omega})\omega_s}\delta''(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} \tag{7.111}$$

But since $A \in \mathrm{Sym}$, the numerator of the preceding expression can be rewritten as follows:

$$\omega_i A_{ig}\,\omega_k A^{-1}_{kg} = \omega_k A^{-1}_{kg} A_{gi}\omega_i = \omega_k \delta_{ki}\,\omega_i = 1 \tag{7.112}$$

and finally, the expression of the potential is:

$$\hat{\Phi}(\boldsymbol{x}) = \frac{1}{8\pi^2 r^3}\int_{|\boldsymbol{\omega}|=1}\frac{\delta''(\boldsymbol{\omega}\cdot\boldsymbol{x})}{\omega_r A^{-1}_{rs}(\boldsymbol{\omega})\omega_s}\,\mathrm{d}\boldsymbol{\omega} \tag{7.113}$$

7.5 Constitutive model

The objective of this Section is to define a constitutive model to be used in the implementation of the Green's function set given by (7.29) and (7.30). We start with an accurate description of general constitutive models, and then we focus on some particular case.

The first Piola Kirchhoff stress tensor is:

$$\boldsymbol{S} = J\boldsymbol{T}\boldsymbol{F}^{-\mathrm{T}} \tag{7.114}$$

where $\boldsymbol{T}$ is the Cauchy stress, $\boldsymbol{F}$ is the displacement gradient and J its determinant. The material derivative with respect to time is:

$$\dot{\boldsymbol{S}} = J\big[\dot{\boldsymbol{T}}\boldsymbol{F}^{-\mathrm{T}} + \boldsymbol{T}\big(\boldsymbol{F}^{-\mathrm{T}}\big)^{\dot{}}\big] \tag{7.115}$$

within the hypothesis of constant J. Since $\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{I}$, it follows that

$$\left(\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\right)^{\cdot} = \left(\boldsymbol{F}^{-\mathrm{T}}\right)^{\cdot}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{F}^{-\mathrm{T}}\left(\boldsymbol{F}^{\mathrm{T}}\right)^{\cdot} = 0 \tag{7.116}$$

namely

$$\left(\boldsymbol{F}^{-\mathrm{T}}\right)^{\cdot} = -\boldsymbol{F}^{-\mathrm{T}}\left(\boldsymbol{F}^{\mathrm{T}}\right)^{\cdot}\boldsymbol{F}^{-\mathrm{T}} \tag{7.117}$$

but since $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$ (where $\boldsymbol{L}$ is the incremental displacement gradient)

$$\dot{\boldsymbol{F}}^{\mathrm{T}} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}} = \left(\boldsymbol{F}^{\mathrm{T}}\right)^{\cdot} \tag{7.118}$$

namely

$$\boldsymbol{L}^{\mathrm{T}} = \boldsymbol{F}^{-\mathrm{T}}\left(\boldsymbol{F}^{\mathrm{T}}\right)^{\cdot} \tag{7.119}$$

so that

$$\left(\boldsymbol{F}^{-\mathrm{T}}\right)^{\cdot} = -\boldsymbol{L}^{\mathrm{T}}\boldsymbol{F}^{-\mathrm{T}} \tag{7.120}$$

the derivative of the first Piola-Kirchhoff stress tensor writes as

$$\dot{\boldsymbol{S}} = J\left(\dot{\boldsymbol{T}}\boldsymbol{F}^{-\mathrm{T}} - \boldsymbol{T}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{F}^{-\mathrm{T}}\right) = J\left(\dot{\boldsymbol{T}} - \boldsymbol{T}\boldsymbol{L}^{\mathrm{T}}\right)\boldsymbol{F}^{-\mathrm{T}} \tag{7.121}$$

in which the following relations are employed (see Section 3.1.4):

$$\boldsymbol{L} = (\nabla\boldsymbol{u})^{\cdot} = \boldsymbol{D} + \boldsymbol{W} \tag{7.122}$$

and

$$\boldsymbol{D} = (\boldsymbol{I} \boxdot \boldsymbol{I})[\boldsymbol{L}] = \mathbb{S}[\boldsymbol{L}] \tag{7.123a}$$

$$\boldsymbol{W} = (\boldsymbol{I} \boxtimes \boldsymbol{I} - \boldsymbol{I} \boxdot \boldsymbol{I})[\boldsymbol{L}] = \mathbb{W}[\boldsymbol{L}] \tag{7.123b}$$

where $\boldsymbol{D}$ is the Eulerian strain increment, $\boldsymbol{W}$ is the *spin tensor*, while $\mathbb{S}$ and $\mathbb{W}$ are, respectively, the symmetric and the skew-symmetric operators for second-order tensors (defined by equations (2.52) in Section 2.1.14).

From the definition of the Jaumann (or corotational) derivative (3.185b), we have:

$$\dot{\boldsymbol{T}} = \overset{\triangledown}{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W} \tag{7.124}$$

The incompressibility constraint ($J = 1$) and the above relation allow us to write

$$
\begin{aligned}
\dot{\boldsymbol{S}} = (\dot{\boldsymbol{T}} - \boldsymbol{T}\boldsymbol{L}^{\mathrm{T}})\boldsymbol{F}^{-\mathrm{T}} &= \left(\overset{\triangledown}{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W} - \boldsymbol{T}\boldsymbol{L}^{\mathrm{T}}\right)\boldsymbol{F}^{-\mathrm{T}} = \\
&= \left[\overset{\triangledown}{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W} - \boldsymbol{T}(\boldsymbol{D} - \boldsymbol{W})\right]\boldsymbol{F}^{-\mathrm{T}} = \\
&= \left(\overset{\triangledown}{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{D}\right)\boldsymbol{F}^{-\mathrm{T}}
\end{aligned}
\tag{7.125}
$$

The constitutive equations (7.3) for a linear, incremental, time-independent, incompressible ($\operatorname{tr}\boldsymbol{D} = 0$) material, transversely isotropic about the $\boldsymbol{e}_z$ axis, can be expressed as functions of an incremental stress potential U as follows

$$
\overset{\triangledown}{\boldsymbol{T}} = \frac{\partial U(\boldsymbol{D}, \boldsymbol{G}, h_i)}{\partial \boldsymbol{D}} + \overset{\triangledown}{p}\boldsymbol{I} \tag{7.126}
$$

where $\boldsymbol{G} = \boldsymbol{e}_z \otimes \boldsymbol{e}_z$, while h_i denote a generic set of invariants representing the current state, and

$$
p = \frac{1}{3}\operatorname{tr}\boldsymbol{T} \tag{7.127}
$$

is the mean Cauchy stress, which plays a role of a Lagrange multiplier, so that $\overset{\triangledown}{p} = \dot{p}$. Different choices of invariants h_i are possible, like the principal stretches, to describe the anisotropy induced by the stress state, or scalar values, describing the instantaneous hardening strain for the plastic deformation theory. The constitutive law (7.126) describes a wide behaviour class of materials, for instance, hyperelasticity and a subset of hypoelastic relations when $\boldsymbol{e}_z$ denotes a principal stress direction [20].

The material frame invariance principle requires that the potential U is an isotropic function of the arguments $\boldsymbol{D}$ e $\boldsymbol{G}$, namely

$$
U(\boldsymbol{D}, \boldsymbol{G}) = U\left(\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}, \boldsymbol{Q}\boldsymbol{G}\boldsymbol{Q}^{\mathrm{T}}\right) \tag{7.128}
$$

for every rotation $\boldsymbol{Q}$. This allows us to use the Spencer & Rivlin representation theorem [60] for a scalar function of two second order symmetric tensors. Following Zysset and Curnier [61], and considering only the quadratic term of $\boldsymbol{D}$

in the potential U, we can write the constitutive equation (7.126) as:

$$\overset{\triangledown}{\boldsymbol{T}} = \mathbb{E}[\boldsymbol{D}] + \overset{\triangledown}{p}\boldsymbol{I} \tag{7.129}$$

and now equation (7.125) becomes:

$$\dot{\boldsymbol{S}} = \big[\mathbb{E}[\boldsymbol{D}] + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{D} + \dot{p}\big]\boldsymbol{F}^{-\mathrm{T}} \tag{7.130}$$

where the $\mathbb{E}$ tensor is given by the following function of $\boldsymbol{G} = \boldsymbol{e}_z \otimes \boldsymbol{e}_z$:

$$\mathbb{E} = \Gamma_1 \boldsymbol{I} \boxdot \boldsymbol{I} + \Gamma_2 \boldsymbol{G} \otimes \boldsymbol{G} + \Gamma_3(\boldsymbol{G} \boxdot \boldsymbol{I} + \boldsymbol{I} \boxdot \boldsymbol{G}) + \Gamma_4 \boldsymbol{I} \otimes \boldsymbol{G} \tag{7.131}$$

in which the parameters Γ_i (con $i = 1, 2, 3, 4$) are functions of the current state; the previous relation can be expressed with indexes as:

$$\mathbb{E}_{ijkl} = \frac{\Gamma_1}{2}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) + \Gamma_2 G_{ij} G_{kl} + \Gamma_3\big(G_{ik}\delta_{jl} + \delta_{ik}G_{jl}\big) + \Gamma_4 \delta_{ij} G_{kl} \tag{7.132}$$

We observe that $\mathbb{E} \in \mathrm{Sym}$ and, in particular, possesses both minor and major symmetries. Actually, the parameter Γ_4 should multiply the term $(\boldsymbol{I}\otimes\boldsymbol{G}+\boldsymbol{G}\otimes\boldsymbol{I})$, but, due to incompressibility constraint, we have $(\boldsymbol{G} \otimes \boldsymbol{I})[\boldsymbol{D}] = 0$ (being $\mathrm{tr}\,\boldsymbol{D} = 0$). The condition

$$\overset{\triangledown}{p} = \frac{1}{3}\,\mathrm{tr}\,\overset{\triangledown}{\boldsymbol{T}} \tag{7.133}$$

requires the following relation between parameters Γ_i (whith $i = 1, 2, 3, 4$):

$$\Gamma_2 + 2\Gamma_3 + 3\Gamma_4 = 0 \tag{7.134}$$

namely, only three parameters Γ_i are sufficient to represent $\mathbb{E}$. As a consequence, the parameters Γ_i (with $i = 1, 2, 3, 4$) can be expressed as functions of a set of three independent incremental moduli μ_i (with $i = 1, 2, 3$):

$$\Gamma_1 = 4\mu_2 - 2\mu_1 \tag{7.135a}$$

$$\Gamma_2 = 2\mu_1 + 2\mu_2 - 4\mu_3 \tag{7.135b}$$

$$\Gamma_3 = 2\mu_1 - 4\mu_2 + 2\mu_3 \tag{7.135c}$$

$$\Gamma_4 = 2\mu_2 - 2\mu_1 \tag{7.135d}$$

and we observe that these assumptions satisfy the constraint (7.134), since

$$2\mu_1 + 2\mu_2 - 4\mu_3 + 4\mu_1 - 8\mu_2 + 4\mu_3 + 6\mu_2 - 6\mu_1 = 0 \tag{7.136}$$

The nominal stress $\dot{\boldsymbol{t}}$ is defined as

$$\dot{\boldsymbol{t}} = \dot{\boldsymbol{S}}^{\mathrm{T}} = \boldsymbol{F}^{-1}\left[\mathbb{E}[\boldsymbol{D}] - \boldsymbol{T}\boldsymbol{W} - \boldsymbol{D}\boldsymbol{T}^{\mathrm{T}} + \dot{p}\right] \tag{7.137}$$

and, furthermore, equation (7.1) still apply, while (7.130) is correlated to (7.35a).

We want now to find the relation between the moduli $\mathbb{E}$ and $\mathbb{K}$. The first term between brackets in equation (7.137) can be expressed as

$$\mathbb{E}[\boldsymbol{D}] = \mathbb{E}\mathbb{S}[\boldsymbol{L}] = \mathbb{E}(\boldsymbol{I} \boxdot \boldsymbol{I})[\boldsymbol{L}] = \mathbb{E}\mathbb{S}\left[\boldsymbol{L}^{\mathrm{T}}\right] \tag{7.138}$$

or, in index notation,

$$\mathbb{E}_{ijkl} D_{kl} = \mathbb{E}_{ijkl}\mathbb{S}_{klmn} L_{mn} = \mathbb{E}_{ijkl}\frac{1}{2}\big(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}\big) L_{mn} \tag{7.139}$$

The second term can be expressed as

$$\boldsymbol{T}\boldsymbol{W} = \boldsymbol{T}\mathbb{W}[\boldsymbol{L}] = \boldsymbol{T}(\boldsymbol{I} \boxtimes \boldsymbol{I} - \boldsymbol{I} \boxdot \boldsymbol{I})[\boldsymbol{L}] \tag{7.140}$$

or, in index notation,

$$\begin{aligned} T_{iq} W_{qj} = T_{iq}\mathbb{W}_{qjkl} L_{kl} = T_{iq}\left[\delta_{qk}\delta_{jl} - \frac{1}{2}\big(\delta_{qk}\delta_{jl} + \delta_{ql}\delta_{jk}\big)\right] L_{kl} = \\ = T_{iq}\frac{1}{2}\big(\delta_{qk}\delta_{jl} - \delta_{ql}\delta_{jk}\big) L_{kl} \end{aligned} \tag{7.141}$$

and we can note that

$$-\boldsymbol{T}\boldsymbol{W} = \boldsymbol{T}\boldsymbol{W}^{\mathrm{T}} = \boldsymbol{T}\mathbb{W}\left[\boldsymbol{L}^{\mathrm{T}}\right] \tag{7.142}$$

Furthermore,

$$\boldsymbol{D}\boldsymbol{T} = \mathbb{S}[\boldsymbol{L}]\boldsymbol{T} = (\boldsymbol{I} \boxdot \boldsymbol{I})[\boldsymbol{L}]\boldsymbol{T} = \frac{1}{2}\big(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{T} + \boldsymbol{L}\boldsymbol{T}\big) \tag{7.143}$$

or, in index notation,

$$D_{iq}T_{qj} = \mathbb{S}_{iqkl}L_{kl}T_{qj} = \frac{1}{2}\left(\delta_{ik}\delta_{ql} + \delta_{il}\delta_{qk}\right)L_{kl}T_{qj} \tag{7.144}$$

and since

$$\boldsymbol{D}\boldsymbol{T}^{\mathrm{T}} = \mathbb{S}[\boldsymbol{L}]\boldsymbol{T}^{\mathrm{T}} = \mathbb{S}\left[\boldsymbol{L}^{\mathrm{T}}\right]\boldsymbol{T} \tag{7.145}$$

The third term between brackets in equation (7.137) can be rewritten as

$$D_{iq}T_{jq} = \mathbb{S}_{iqkl}L_{kl}T_{jq} = \frac{1}{2}\left(\delta_{ik}\delta_{ql} + \delta_{il}\delta_{qk}\right)L_{kl}T_{jq} \tag{7.146}$$

In an analogous manner, the second and the third term between brackets in equation (7.130) can be rewritten as

$$\boldsymbol{W}\boldsymbol{T} = \mathbb{W}[\boldsymbol{L}]\boldsymbol{T} \tag{7.147a}$$

$$\boldsymbol{T}\boldsymbol{D} = \boldsymbol{T}\mathbb{S}[\boldsymbol{L}] \tag{7.147b}$$

and their index versions become

$$W_{iq}T_{qj} = \mathbb{W}_{iqkl}L_{kl}T_{qj} = \frac{1}{2}\left(\delta_{ik}\delta_{ql} - \delta_{il}\delta_{qk}\right)L_{kl}T_{qj} \tag{7.148}$$

and

$$T_{iq}D_{qj} = T_{iq}\mathbb{S}_{qjkl}L_{kl} = T_{iq}\frac{1}{2}\left(\delta_{qk}\delta_{jl} + \delta_{ql}\delta_{jk}\right)L_{kl} \tag{7.149}$$

respectively. Now the expressions of $\dot{\boldsymbol{S}}$ (7.130) and $\dot{\boldsymbol{t}}$ (7.137) can be rewritten as follows:

$$\dot{\boldsymbol{S}} = \Big\{\dot{p} + \left[\mathbb{E}(\boldsymbol{I} \boxdot \boldsymbol{I}) + \boldsymbol{T}(\boldsymbol{I} \boxtimes \boldsymbol{I} - \boldsymbol{I} \boxdot \boldsymbol{I}) - \boldsymbol{T}(\boldsymbol{I} \boxdot \boldsymbol{I})\right][\boldsymbol{L}]\Big\}\boldsymbol{F}^{-\mathrm{T}} \tag{7.150}$$

and

$$\dot{\boldsymbol{t}} = \boldsymbol{F}^{-1}\Big\{\dot{p} + \left[\mathbb{E}(\boldsymbol{I} \boxdot \boldsymbol{I}) + \boldsymbol{T}(\boldsymbol{I} \boxtimes \boldsymbol{I} - \boldsymbol{I} \boxdot \boldsymbol{I}) - \boldsymbol{T}^{-\mathrm{T}}(\boldsymbol{I} \boxdot \boldsymbol{I})\right]\left[\boldsymbol{L}^{-\mathrm{T}}\right]\Big\} \tag{7.151}$$

A comparison between equation (7.151) and (7.1) we obtain

$$\mathbb{K}_{ijkl} = \mathbb{E}_{ijkl}\frac{1}{2}\big(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}\big) + T_{iq}\frac{1}{2}\big(\delta_{qk}\delta_{jl} - \delta_{ql}\delta_{jk}\big) + \\ - T_{jq}\frac{1}{2}\big(\delta_{ik}\delta_{ql} + \delta_{il}\delta_{qk}\big) \quad (7.152)$$

so that the relation we are looking for, namely the definition of $\mathbb{K}$ as a function of $\mathbb{E}$, is

$$\mathbb{K}_{ijkl} = \mathbb{E}_{ijkl} + \frac{1}{2}\big(T_{ik}\delta_{jl} - T_{il}\delta_{jk}\big) - \frac{1}{2}\big(T_{jl}\delta_{ik} + T_{jk}\delta_{il}\big) \quad (7.153)$$

Note that, within a relative Lagrangian description we have $\boldsymbol{F}^{-1} = \boldsymbol{I}$, so that expressions (7.150) and (7.151) reduce to

$$\dot{\boldsymbol{S}} = \dot{p} + \big[\mathbb{E}(\boldsymbol{I} \boxdot \boldsymbol{I}) + \boldsymbol{T}(\boldsymbol{I} \boxtimes \boldsymbol{I} - \boldsymbol{I} \boxdot \boldsymbol{I}) - \boldsymbol{T}(\boldsymbol{I} \boxdot \boldsymbol{I})\big][\boldsymbol{L}] \quad (7.154)$$

and

$$\dot{\boldsymbol{t}} = \dot{p} + \big[\mathbb{E}(\boldsymbol{I} \boxdot \boldsymbol{I}) + \boldsymbol{T}(\boldsymbol{I} \boxtimes \boldsymbol{I} - \boldsymbol{I} \boxdot \boldsymbol{I}) - \boldsymbol{T}^{-\mathrm{T}}(\boldsymbol{I} \boxdot \boldsymbol{I})\big]\,\big[\boldsymbol{L}^{-\mathrm{T}}\big] \quad (7.155)$$

respectively.

For the elastic incompressible Cauchy material (isotropic in the initial reference system) the Cauchy stress can be expressed as:

$$\boldsymbol{T} = -\pi\boldsymbol{I} + \alpha_1\boldsymbol{B} + \alpha_{-1}\boldsymbol{B}^{-1} \quad (7.156)$$

where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ is the left Cauchy-Green tensor, while π is an arbitrary hydrostatic pressure. The scalar quantities α_1 and α_{-1}, that are called *response functions*, are defined as:

$$\alpha_1 = \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{(\sigma_1 - \sigma_3)\lambda_1^2}{\lambda_1^2 - \lambda_3^2} - \frac{(\sigma_2 - \sigma_3)\lambda_2^2}{\lambda_2^2 - \lambda_3^2}\right] > 0 \quad (7.157a)$$

$$\alpha_{-1} = \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{\sigma_1 - \sigma_3}{\lambda_1^2 - \lambda_3^2} - \frac{\sigma_2 - \sigma_3}{\lambda_2^2 - \lambda_3^2}\right] \leq 0 \quad (7.157b)$$

which means that these are functions of the principal stretches λ_i (where $i = 1, 2, 3$), that satisfy the incompressibility constraint ($\lambda_1\lambda_2\lambda_3 = 1$), and of the principal components σ_i (where $i = 1, 2, 3$) of the Cauchy stress $\boldsymbol{T}$.

If the material is hyperelastic with a unit volume potential $W(\lambda_1, \lambda_2, \lambda_3)$, the differences between the principal stresses σ_i are equal to:

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \tilde{W}(\lambda_1, \lambda_2)}{\partial \lambda_1} \tag{7.158a}$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \tilde{W}(\lambda_1, \lambda_2)}{\partial \lambda_2} \tag{7.158b}$$

where

$$\tilde{W}(\lambda_1, \lambda_2) = W\big(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}\big) \tag{7.159}$$

Being $\lambda_3 = (\lambda_1 \lambda_2)^{-1}$, due to incompressibility constraint, we have that

$$\lambda_1 \frac{\partial \tilde{W}(\lambda_1, \lambda_2)}{\partial \lambda_1} = \lambda_1 \frac{\partial}{\partial \lambda_1} W\big(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}\big) = \lambda_1 \left(\frac{\partial W}{\partial \lambda_1} + \frac{\partial W}{\partial \lambda_3} \frac{\partial \lambda_3}{\partial \lambda_1} \right) \tag{7.160}$$

but

$$\frac{\partial \lambda_3}{\partial \lambda_1} = \frac{\partial}{\partial \lambda_1} \left(\frac{1}{\lambda_1 \lambda_2} \right) = \frac{1}{\lambda_2} \left(-\frac{1}{\lambda_1^2} \right) = -\lambda_1^{-2} \lambda_2^{-1} \tag{7.161}$$

hence

$$\lambda_1 \frac{\partial \tilde{W}(\lambda_1, \lambda_2)}{\partial \lambda_1} = \lambda_1 \left(\frac{\partial W}{\partial \lambda_1} - \frac{1}{\lambda_1^2 \lambda_2} \frac{\partial W}{\partial \lambda_3} \right) = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_3 \frac{\partial W}{\partial \lambda_3} \tag{7.162}$$

In an analogous manner we have

$$\lambda_2 \frac{\partial \tilde{W}(\lambda_1, \lambda_2)}{\partial \lambda_2} = \lambda_2 \frac{\partial}{\partial \lambda_2} W\big(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}\big) = \lambda_2 \left(\frac{\partial W}{\partial \lambda_2} + \frac{\partial W}{\partial \lambda_3} \frac{\partial \lambda_3}{\partial \lambda_2} \right) \tag{7.163}$$

but

$$\frac{\partial \lambda_3}{\partial \lambda_2} = \frac{\partial}{\partial \lambda_2} \left(\frac{1}{\lambda_1 \lambda_2} \right) = \frac{1}{\lambda_1} \left(-\frac{1}{\lambda_2^2} \right) = -\lambda_1^{-1} \lambda_2^{-2} \tag{7.164}$$

so that

$$\lambda_2 \frac{\partial \tilde{W}(\lambda_1, \lambda_2)}{\partial \lambda_2} = \lambda_2 \left(\frac{\partial W}{\partial \lambda_2} - \frac{1}{\lambda_1 \lambda_2^2} \frac{\partial W}{\partial \lambda_3} \right) = \lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_3 \frac{\partial W}{\partial \lambda_3} \tag{7.165}$$

Within the J_2-deformation theory, the elastic energy expressed as a function of the strain is

$$W = \frac{K}{N+1}\varepsilon_{\mathrm{e}}^{N+1} \tag{7.166}$$

where K is a stiffness constitutive parameter, $N \in (0,1]$ is the hardening index (or exponent), while ε_{e} is the effective strain, given by

$$\varepsilon_{\mathrm{e}} = \sqrt{\frac{2}{3}\varepsilon_i\varepsilon_i} \tag{7.167}$$

where $\varepsilon_i = \ln\lambda_i$ within the Lagrangian principal reference system, so that

$$\varepsilon_{\mathrm{e}} = \sqrt{\frac{2}{3}\left[(\ln\lambda_1)^2 + (\ln\lambda_2)^2 + (\ln\lambda_3)^2\right]} \tag{7.168}$$

Referring to an axisymmetric problem, with symmetry axis $z = x_3$, the response functions α_1 and α_{-1} depend only on the axial stretch λ_3. If we apply the material time derivative to equation (7.156) and we use the definition of the Jaumann derivative, the incremental constitutive law can be expressed through relations (7.129) and (7.131), in which we use the following values of parameters Γ_i (whith $i = 1, 2, 3, 4$):

$$\Gamma_1 = 2\left(\frac{\alpha_1}{\lambda_3} - \alpha_{-1}\lambda_3\right) \tag{7.169a}$$

$$\Gamma_2 = \left(\lambda_3^2 - \frac{1}{\lambda_3}\right)\left(\lambda_3\frac{\mathrm{d}\alpha_1}{\mathrm{d}\lambda_3} - \frac{\mathrm{d}\alpha_{-1}}{\mathrm{d}\lambda_3}\right) \tag{7.169b}$$

$$\Gamma_3 = \left(\lambda_3^2 - \frac{1}{\lambda_3}\right)\left(\alpha_1 + \frac{\alpha_{-1}}{\lambda_3}\right) \tag{7.169c}$$

$$\Gamma_4 = -\frac{1}{3}(\Gamma_2 + 2\Gamma_3) \tag{7.169d}$$

while the incremental moduli μ_i (where $i = 1, 2, 3$) have the form:

$$\mu_1 = \frac{1}{3}\bigg[\alpha_1 \left(\frac{1}{\lambda_3} + 2\lambda_3^2\right) - \alpha_{-1}\left(\frac{2}{\lambda_3^2} + \lambda_3\right) + (\lambda_3^3 - 1)\frac{\mathrm{d}\alpha_1}{\mathrm{d}\lambda_3} + \\ + \left(\frac{1}{\lambda_3} - \lambda_3^2\right)\frac{\mathrm{d}\alpha_{-1}}{\mathrm{d}\lambda_3}\bigg] \tag{7.170a}$$

$$\mu_2 = \frac{1}{3}\bigg[\alpha_1 \left(\frac{2}{\lambda_3} + \lambda_3^2\right) - \alpha_{-1}\left(\frac{1}{\lambda_3^2} + 2\lambda_3\right) + \frac{1}{2}(\lambda_3^3 - 1)\frac{\mathrm{d}\alpha_1}{\mathrm{d}\lambda_3} + \\ + \frac{1}{2}\left(\frac{1}{\lambda_3} - \lambda_3^2\right)\frac{\mathrm{d}\alpha_{-1}}{\mathrm{d}\lambda_3}\bigg] \tag{7.170b}$$

$$\mu_3 = \frac{1}{2}\left[\alpha_1 \left(\frac{1}{\lambda_3} + \lambda_3^2\right) - \alpha_{-1}\left(\frac{1}{\lambda_3^2} + \lambda_3\right)\right] \tag{7.170c}$$

Within the J_2-deformation theory, the expressions for moduli α_1 and α_{-1} reduce to:

$$\alpha_1 = K\varepsilon_{\mathrm{e}}^{N-1}\frac{\lambda_3}{\lambda_3^3 - 1}\left(\frac{\lambda_3^3}{\lambda_3^3 - 1}\ln\lambda_3 - \frac{1}{3}\right) \tag{7.171a}$$

$$\alpha_{-1} = K\varepsilon_{\mathrm{e}}^{N-1}\frac{\lambda_3^2}{\lambda_3^3 - 1}\left(\frac{1}{\lambda_3^3 - 1}\ln\lambda_3 - \frac{1}{3}\right) \tag{7.171b}$$

while the shear moduli reduce to:

$$\mu_1 = \frac{KN}{3}\varepsilon_{\mathrm{e}}^{N-1} \tag{7.172a}$$

$$\mu_2 = \frac{K}{6}(N+1)\varepsilon_{\mathrm{e}}^{N-1} \tag{7.172b}$$

$$\mu_3 = \frac{K}{2}\varepsilon_{\mathrm{e}}^{N-1}\frac{\lambda_3^3 + 1}{\lambda_3^3 - 1}\ln\lambda_z \tag{7.172c}$$

where $\varepsilon_{\mathrm{e}} = |\ln\lambda_3|$. Note that the stress-strain relation that can be obtained from equation (7.156) is

$$\sigma = K\varepsilon_{\mathrm{e}}^{N-1}\ln\lambda_3 \tag{7.173}$$

where σ is the Cauchy current stress in the axial direction, so that the uniaxial strain and stress states are governed only by the axial stretch λ_3. Taking into account that the effective strain is given by equation (7.168) and assuming $\lambda_1\lambda_2\lambda_3 = 1$, due to incompressibility constraint, and $\lambda_1 = \lambda_2$ for the axisymmetric problem, we have

$$\lambda_1^2 \lambda_3 = 1 \quad \longrightarrow \quad \lambda_1 = \frac{1}{\sqrt{\lambda_3}} \tag{7.174}$$

and, exploiting again incompressibility constraint,

$$\ln\left(\lambda_1^2\right) + \ln \lambda_3 = 2 \ln \lambda_1 + \ln \lambda_3 = 0 \quad \longrightarrow \quad \ln \lambda_1 = -\frac{1}{2} \ln \lambda_3 \tag{7.175}$$

therefore

$$\varepsilon_1 = -\frac{1}{2} \varepsilon_3 \tag{7.176}$$

With these assumptions, the effective strain becomes

$$\varepsilon_{\mathrm{e}} = \sqrt{\frac{2}{3}(2\varepsilon_1^2 + \varepsilon_3^2)} = \sqrt{\frac{2}{3}\left[2\left(-\frac{\varepsilon_3}{2}\right)^2 + \varepsilon_3^2\right]} = \sqrt{\frac{2}{3}\left(\frac{\varepsilon_3^2}{2} + \varepsilon_3^2\right)} = |\varepsilon_3| \tag{7.177}$$

The deformation gradient has the form

$$\boldsymbol{F} = \begin{bmatrix} \lambda_3^{-1/2} & 0 & 0 \\ 0 & \lambda_3^{-1/2} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{7.178}$$

so that the Cauchy stress tensor has also a diagonal form, namely

$$\boldsymbol{T} = \begin{bmatrix} T_1^* & 0 & 0 \\ 0 & T_2^* & 0 \\ 0 & 0 & T_3^* \end{bmatrix} \tag{7.179}$$

In particular, for the axisymmetric problem we have

$$T_1 - T_2 = \lambda_1 \frac{\partial \tilde{W}(\lambda_1, \lambda_2)}{\partial \lambda_1} = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_3 \frac{\partial W}{\partial \lambda_3} \tag{7.180a}$$

$$T_1 - T_2 = 0 \tag{7.180b}$$

7.5.1 Constitutive equations for the axisymmetric problem

After the discussions presented in the previous Section, we are now in the position to write the final form of the constitutive equations for a transversely isotropic material. Referring to an axisymmetric problem, and introducing a cylindrical coordinate reference system with the $z = x_3$ axis coincident with the symmetry axis, the relation between the incremental Piola-Kirchhoff stress and the velocity gradient $\boldsymbol{L}$ is:

$$
\begin{aligned}
\dot{\boldsymbol{S}}_{rr} &= \dot{p} + 2\mu_2 L_{rr} + 2(\mu_1 - \mu_2)L_{\theta\theta} \qquad & \dot{\boldsymbol{S}}_{r\theta} &= \dot{\boldsymbol{S}}_{\theta r} = (2\mu_2 - \mu_1)(L_{r\theta} + L_{\theta r}) \\
\dot{\boldsymbol{S}}_{\theta\theta} &= \dot{p} + 2\mu_2 L_{\theta\theta} + 2(\mu_1 - \mu_2)L_{rr} \qquad & \dot{\boldsymbol{S}}_{zz} &= \dot{p} + (2\mu_1 - \sigma)L_{zz} \\
\dot{\boldsymbol{S}}_{rz} &= \left(\mu_3 + \frac{\sigma}{2}\right) L_{rz} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{zr} \\
\dot{\boldsymbol{S}}_{zr} &= \left(\mu_3 - \frac{\sigma}{2}\right) L_{rz} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{zr} \\
\dot{\boldsymbol{S}}_{\theta z} &= \left(\mu_3 + \frac{\sigma}{2}\right) L_{\theta z} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{z\theta} \\
\dot{\boldsymbol{S}}_{z\theta} &= \left(\mu_3 - \frac{\sigma}{2}\right) L_{\theta z} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{z\theta}
\end{aligned}
\tag{7.181}
$$

where σ is the current Cauchy stress in the axial direction ($x_3 = z$) and the velocity gradient is:

$$
\boldsymbol{L} = \begin{bmatrix} L_{rr} & L_{r\theta} & L_{rz} \\ L_{\theta r} & L_{\theta\theta} & L_{\theta z} \\ L_{zr} & L_{z\theta} & L_{zz} \end{bmatrix} = \begin{bmatrix} v_{r,r} & \frac{1}{r}(v_{r,\theta} - v_\theta) & v_{r,z} \\ v_{\theta,r} & \frac{1}{r}(v_r + v_{\theta,\theta}) & v_{\theta,z} \\ v_{z,r} & \frac{1}{r} v_{z,\theta} & v_{z,z} \end{bmatrix}
\tag{7.182}
$$

The nominal stress tensor $\dot{\boldsymbol{t}}$ is defined as the transposition of the tensor $\dot{\boldsymbol{S}}$, thus yielding

$$
\begin{aligned}
&\dot{\boldsymbol{t}}_{rr} = \dot{p} + 2\mu_2 L_{rr} + 2(\mu_1 - \mu_2)L_{\theta\theta} && \dot{\boldsymbol{t}}_{r\theta} = \dot{\boldsymbol{t}}_{\theta r} = (2\mu_2 - \mu_1)(L_{r\theta} + L_{\theta r}) \\
&\dot{\boldsymbol{t}}_{\theta\theta} = \dot{p} + 2\mu_2 L_{\theta\theta} + 2(\mu_1 - \mu_2)L_{rr} && \dot{\boldsymbol{t}}_{zz} = \dot{p} + (2\mu_1 - \sigma)L_{zz} \\
&\dot{\boldsymbol{t}}_{rz} = \left(\mu_3 - \frac{\sigma}{2}\right) L_{rz} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{zr} \\
&\dot{\boldsymbol{t}}_{zr} = \left(\mu_3 + \frac{\sigma}{2}\right) L_{rz} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{zr} \\
&\dot{\boldsymbol{t}}_{\theta z} = \left(\mu_3 - \frac{\sigma}{2}\right) L_{\theta z} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{z\theta} \\
&\dot{\boldsymbol{t}}_{z\theta} = \left(\mu_3 + \frac{\sigma}{2}\right) L_{\theta z} + \left(\mu_3 - \frac{\sigma}{2}\right) L_{z\theta}
\end{aligned}
\tag{7.183}
$$

The constitutive law can be expressed using the first Piola-Kirchhoff stress tensor $\dot{\boldsymbol{S}}$ (7.35a) or the nominal stress tensor $\dot{\boldsymbol{t}}$ (7.1) (we recall that in the second case we use the transposed velocity gradient). For the first formulation (Piola-Kirchhoff stress tensor) we obtain

$$
\begin{aligned}
&\dot{\boldsymbol{S}}_{rr} = \mathbb{G}_{rrrr} L_{rr} + \mathbb{G}_{rr\theta\theta} L_{\theta\theta} + \dot{p} && \dot{\boldsymbol{S}}_{\theta\theta} = \mathbb{G}_{\theta\theta rr} L_{rr} + \mathbb{G}_{\theta\theta\theta\theta} L_{\theta\theta} + \dot{p} \\
&\dot{\boldsymbol{S}}_{zz} = \mathbb{G}_{zzzz} L_{zz} + \dot{p} && \dot{\boldsymbol{S}}_{r\theta} = \mathbb{G}_{r\theta r\theta} L_{r\theta} + \mathbb{G}_{r\theta\theta r} L_{\theta r} \\
&\dot{\boldsymbol{S}}_{rz} = \mathbb{G}_{rzrz} L_{rz} + \mathbb{G}_{rzzr} L_{zr} && \dot{\boldsymbol{S}}_{\theta z} = \mathbb{G}_{\theta z\theta z} L_{\theta z} + \mathbb{G}_{\theta zz\theta} L_{z\theta} \\
&\dot{\boldsymbol{S}}_{\theta r} = \mathbb{G}_{\theta rr\theta} L_{r\theta} + \mathbb{G}_{\theta r\theta r} L_{\theta r} && \dot{\boldsymbol{S}}_{zr} = \mathbb{G}_{zrrz} L_{rz} + \mathbb{G}_{zrzr} L_{zr} \\
&\dot{\boldsymbol{S}}_{z\theta} = \mathbb{G}_{z\theta\theta z} L_{\theta z} + \mathbb{G}_{z\theta z\theta} L_{z\theta}
\end{aligned}
\tag{7.184}
$$

from which we obtain the non null components of the tensor $\mathbb{G}$:

$$
\begin{aligned}
&\mathbb{G}_{rrrr} = \mathbb{G}_{\theta\theta\theta\theta} = 2\mu_2 \\
&\mathbb{G}_{rr\theta\theta} = \mathbb{G}_{\theta\theta rr} = 2(\mu_1 - \mu_2) \\
&\mathbb{G}_{zzzz} = 2\mu_1 - \sigma \\
&\mathbb{G}_{r\theta r\theta} = \mathbb{G}_{r\theta\theta r} = \mathbb{G}_{\theta rr\theta} = \mathbb{G}_{\theta r\theta r} = 2\mu_2 - \mu_1 \\
&\mathbb{G}_{rzrz} = \mathbb{G}_{\theta z\theta z} = \mu_3 + \frac{\sigma}{2} \\
&\mathbb{G}_{rzzr} = \mathbb{G}_{zrrz} = \mathbb{G}_{zrzr} = \mathbb{G}_{\theta zz\theta} = \mathbb{G}_{z\theta\theta z} = \mathbb{G}_{z\theta z\theta} = \mu_3 - \frac{\sigma}{2}
\end{aligned}
\tag{7.185}
$$

and we note that there are only seventeen non null terms. In an analogous manner, for the second formulation (nominal stress tensor) we obtain

$$
\begin{aligned}
\dot{\boldsymbol{t}}_{rr} &= \mathbb{K}_{rrrr} L_{rr} + \mathbb{K}_{rr\theta\theta} L_{\theta\theta} + \dot{p} & \dot{\boldsymbol{t}}_{\theta\theta} &= \mathbb{K}_{\theta\theta rr} L_{rr} + \mathbb{K}_{\theta\theta\theta\theta} L_{\theta\theta} + \dot{p} \\
\dot{\boldsymbol{t}}_{zz} &= \mathbb{K}_{zzzz} L_{zz} + \dot{p} & \dot{\boldsymbol{t}}_{r\theta} &= \mathbb{K}_{r\theta r\theta} L_{\theta r} + \mathbb{K}_{r\theta\theta r} L_{r\theta} \\
\dot{\boldsymbol{t}}_{rz} &= \mathbb{K}_{rzrz} L_{zr} + \mathbb{K}_{rzzr} L_{rz} & \dot{\boldsymbol{t}}_{\theta z} &= \mathbb{K}_{\theta z\theta z} L_{z\theta} + \mathbb{K}_{\theta zz\theta} L_{\theta z} \\
\dot{\boldsymbol{t}}_{\theta r} &= \mathbb{K}_{\theta rr\theta} L_{\theta r} + \mathbb{K}_{\theta r\theta r} L_{r\theta} & \dot{\boldsymbol{t}}_{zr} &= \mathbb{K}_{zrrz} L_{zr} + \mathbb{K}_{zrzr} L_{rz} \\
\dot{\boldsymbol{t}}_{z\theta} &= \mathbb{K}_{z\theta\theta z} L_{z\theta} + \mathbb{K}_{z\theta z\theta} L_{\theta z}
\end{aligned}
\tag{7.186}
$$

from which we obtain the non null components of the tensor $\mathbb{K}$:

$$
\begin{aligned}
&\mathbb{K}_{rrrr} = \mathbb{K}_{\theta\theta\theta\theta} = 2\mu_2 \\
&\mathbb{K}_{rr\theta\theta} = \mathbb{K}_{\theta\theta rr} = 2(\mu_1 - \mu_2) \\
&\mathbb{K}_{zzzz} = 2\mu_1 - \sigma \\
&\mathbb{K}_{r\theta r\theta} = \mathbb{K}_{r\theta\theta r} = \mathbb{K}_{\theta rr\theta} = \mathbb{K}_{\theta r\theta r} = 2\mu_2 - \mu_1 \\
&\mathbb{K}_{zrzr} = \mathbb{K}_{z\theta z\theta} = \mu_3 + \frac{\sigma}{2} \\
&\mathbb{K}_{rzzr} = \mathbb{K}_{zrrz} = \mathbb{K}_{rzrz} = \mathbb{K}_{z\theta\theta z} = \mathbb{K}_{\theta zz\theta} = \mathbb{K}_{\theta z\theta z} = \mu_3 - \frac{\sigma}{2}
\end{aligned}
\tag{7.187}
$$

7.6 (Strong) ellipticity conditions

We still refer to an axisymmetric problem with symmetry axis $z = x_3$, as already described in the previous Section. The strong ellipticity condition is given by the following relation:

$$
(\boldsymbol{g} \otimes \boldsymbol{n})\mathbb{G}[\boldsymbol{g} \otimes \boldsymbol{n}] > 0 \quad \longleftrightarrow \quad g_i n_j \mathbb{G}_{ijkl}\, g_k n_l > 0 \tag{7.188}
$$

where $\boldsymbol{n}$ is a unit vector denoting the normal to the sliding plane (or *shear band* plane), while $\boldsymbol{g} \perp \boldsymbol{n}$ is a unit vector belonging to this plane and denotes the sliding direction (see Figure 7.1). Since we refer to an axisymmetric problem, based on an infinite body subject to concentrated loads applied along the axis of axisymmetry and with its direction, it is convenient to represent the vectors

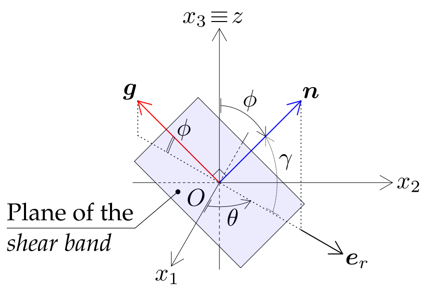


Figure 7.1: Definition of the sliding plane (*shear band*) and its normal vector $\boldsymbol{n}$; the unit vector $\boldsymbol{g} \perp \boldsymbol{n}$ belongs to this plane and denotes the sliding direction.

$\boldsymbol{n}$ and $\boldsymbol{g}$ in the plane Orz as follows:

$$\boldsymbol{n} = \{\, \sin\phi \,, \cos\phi \,\} \tag{7.189a}$$

$$\boldsymbol{g} = \{\, -\cos\phi \,, \sin\phi \,\} \tag{7.189b}$$

where ϕ represents the inclination of $\boldsymbol{g}$, measured from the z axis; we recall that we can neglect the dependence on the angle θ because of the axisymmetry. The incremental displacement gradient can be expressed as

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W} = \boldsymbol{g} \otimes \boldsymbol{n} \tag{7.190}$$

while the first Piola-Kirchhoff tensor becomes

$$\dot{\boldsymbol{S}} = \mathbb{G}[\boldsymbol{L}] + \dot{p}\boldsymbol{I} \quad \longleftrightarrow \quad \dot{S}_{ij} = \mathbb{G}_{ijkl} L_{kl} + \dot{p}\delta_{ij} = \mathbb{G}_{ijkl}\, g_k n_l + \dot{p}\delta_{ij} \tag{7.191}$$

With this definitions, the strong ellipticity condition (7.188) rewrites as:

$$\boldsymbol{L} \cdot \dot{\boldsymbol{S}} > 0 \tag{7.192}$$

and can be expressed as a function of the moduli $\mathbb{G}$ and of the angle ϕ. Equation (7.192), taking into account the major symmetry of $\mathbb{G}$, becomes:

$$\begin{aligned} L_{ij}\dot{S}_{ij} &= g_i g_k n_r^2 \mathbb{G}_{irkr} + g_i g_k n_z^2 \mathbb{G}_{izkz} + g_i g_k n_r n_z (\mathbb{G}_{irkz} + \mathbb{G}_{izkr}) = \\ &= g_r^2 n_r^2 \mathbb{G}_{rrrr} + g_z^2 n_r^2 \mathbb{G}_{zrzr} + g_r^2 n_z^2 \mathbb{G}_{rzrz} + g_z^2 n_z^2 \mathbb{G}_{zzzz} + \\ &\quad + 2 g_r g_z n_r n_z (\mathbb{G}_{rrzz} + \mathbb{G}_{rzzr}) > 0 \end{aligned} \tag{7.193}$$

being $\boldsymbol{L} \cdot \boldsymbol{I} = 0$, and referring to the components shown in (7.189)

$$g_r^2 n_r^2 = \cos^2\phi \sin^2\phi \cos^4\theta \tag{7.194a}$$
$$g_z^2 n_z^2 = \cos^2\phi \sin^2 \tag{7.194b}$$
$$g_r^2 n_z^2 = \phi \sin^4\phi \cos^2\theta \tag{7.194c}$$
$$g_z^2 n_r^2 = \phi \cos^4\phi \cos^2\theta \tag{7.194d}$$
$$g_r g_z n_r n_z = -\cos^2\phi \sin^2\phi \cos^2\theta \tag{7.194e}$$

equation (7.193) reduces to

$$\begin{aligned} L_{ij}\dot{S}_{ij} = \cos^2\phi \sin^2\phi \big[\mathbb{G}_{rrrr} + \mathbb{G}_{zzzz} - 2(\mathbb{G}_{rrzz} + \mathbb{G}_{rzzr})\big] + \\ + \sin^4\phi\, \mathbb{G}_{zrzr} + \cos^4\phi\, \mathbb{G}_{rzrz} > 0 \end{aligned} \tag{7.195}$$

If the inclination of $\boldsymbol{n}$ is measured with respect to the r axis through an angle γ (complementary to ϕ), we obtain a similar expression

$$\begin{aligned} L_{ij}\dot{S}_{ij} = \cos^2\phi \sin^2\phi \big[\mathbb{G}_{rrrr} + \mathbb{G}_{zzzz} - 2(\mathbb{G}_{rrzz} + \mathbb{G}_{rzzr})\big] + \\ + \cos^4\phi\, \mathbb{G}_{zrzr} + \sin^4\phi\, \mathbb{G}_{rzrz} > 0 \end{aligned} \tag{7.196}$$

namely, sine and cosine of the angle have their role interchanged with each other. Using the moduli of $\mathbb{G}$ given in (7.185), the regime equation (7.196) for the axisymmetric three-dimensional problem becomes

$$\begin{aligned} \mu_3 \cos^4\phi \left[\left(1 - \frac{T_z}{2\mu_3}\right)\tan^4\phi + \left(\frac{\mu_1}{\mu_3} + \frac{\mu_2}{\mu_3} - 1\right)\tan^2\phi + \right. \\ \left. + \left(1 + \frac{T_z}{2\mu_3}\right)\right] > 0 \qquad \forall \phi \end{aligned} \tag{7.197}$$

when expressed as a function of angle ϕ, or

$$\mu_3 \cos^4 \gamma \left[\left(1 + \frac{T_z}{2\mu_3}\right) \tan^4 \gamma + \left(\frac{\mu_1}{\mu_3} + \frac{\mu_2}{\mu_3} - 1\right) \tan^2 \gamma + \right.$$
$$\left. + \left(1 - \frac{T_z}{2\mu_3}\right)\right] > 0 \qquad \forall\gamma \quad (7.198)$$

when angle γ is used. For simplicity, and in order to maintain an analogy with respect to Section 5.4, we assume: $T_z = \sigma$, $\tan\gamma = \rho$ and $\mu_3 > 0$. With these assumption, expression (7.198) rewrites as

$$\left(\mu_3 + \frac{\sigma}{2}\right) \rho^4 + 2(\mu_1 + \mu_2 - \mu_3)\rho^2 + \left(\mu_3 - \frac{\sigma}{2}\right) > 0 \qquad \forall\rho \tag{7.199}$$

which is a second degree equation in the unknown ρ^2. The solutions of the equation associated to (7.199) are

$$\rho^2_{1/2} = \frac{\mu_3 - \mu_1 - \mu_2 \pm \sqrt{\Delta}}{\mu_3 + \frac{\sigma}{2}} \tag{7.200}$$

where Δ is the discriminant of (7.199), and is equal to

$$\begin{aligned} \Delta &= (\mu_1 + \mu_2 - \mu_3)^2 - \left(\mu_3 + \frac{\sigma}{2}\right)\left(\mu_3 - \frac{\sigma}{2}\right) = \\ &= \mu_1^2 + \mu_2^2 + 2\mu_1\mu_2 - 2\mu_1\mu_3 - 2\mu_2\mu_3 + \frac{\sigma^2}{4} = \\ &= (\mu_1 + \mu_2)^2 - 2\mu_3(\mu_1 + \mu_2) + \frac{\sigma^2}{4} \end{aligned} \tag{7.201}$$

namely

$$\Delta = (\mu_1 + \mu_2)(\mu_1 + \mu_2 - 2\mu_3) + \frac{\sigma^2}{4} \tag{7.202}$$

so that equation (7.200) can be rewritten as

$$\rho^2_{1/2} = \frac{\mu_3 - \mu_1 - \mu_2}{\mu_3 + \frac{\sigma}{2}} \pm \frac{\sqrt{4(\mu_1 + \mu_2)(\mu_1 + \mu_2 - 2\mu_3) + \sigma^2}}{2\mu_3 + \sigma} \tag{7.203}$$

We can easily note that the solutions of the inequality depend on the term

$\mu_3 + \frac{\sigma}{2}$; in particular, we have

$$\begin{cases} \rho^2 < \rho_1^2 \quad \vee \quad \rho^2 > \rho_2^2 & \text{if } \mu_3 + \frac{\sigma}{2} > 0 \\ \rho_1^2 < \rho^2 < \rho_2^2 & \text{if } \mu_3 + \frac{\sigma}{2} < 0 \end{cases} \tag{7.204}$$

If $\Delta < 0$, the ellipticity condition is always verified when

$$\left(\mu_3 + \frac{\sigma}{2}\right) > 0 \quad \longleftrightarrow \quad \frac{\sigma}{2\mu_3} > -1 \tag{7.205}$$

Furthermore, if $\rho = 0$, necessarily we have

$$\left(\mu_3 - \frac{\sigma}{2}\right) > 0 \quad \longleftrightarrow \quad \frac{\sigma}{2\mu_3} < 1 \tag{7.206}$$

The conditions given by (7.205) and (7.206) can be synthesized as follows:

$$\left|\frac{\sigma}{2\mu_3}\right| < 1 \tag{7.207}$$

while the discriminant Δ is negative if

$$\Delta = \left(\frac{\mu_1 + \mu_2}{\mu_3} - 1\right)^2 - \left(1 + \frac{\sigma}{2\mu_3}\right)\left(1 - \frac{\sigma}{2\mu_3}\right) < 0 \tag{7.208}$$

namely, when

$$\left(\frac{\mu_1 + \mu_2}{\mu_3} - 1\right)^2 < 1 - \left(\frac{\sigma}{2\mu_3}\right)^2 < 0 \tag{7.209}$$

The stationary points can be calculated as the roots of the first order derivative of (7.199), thus yielding the following equation:

$$4\left[\left(\mu_3 + \frac{\sigma}{2}\right)\rho^2 + (\mu_1 + \mu_2 - \mu_3)\right]\rho = 0 \tag{7.210}$$

from which

$$\rho = 0 \quad \vee \quad \rho^2 = \frac{\mu_3 - \mu_1 - \mu_2}{\mu_3 + \frac{\sigma}{2}} \tag{7.211}$$

If the numerator of the second solution is a negative number, namely if

$$\mu_3 - \mu_1 - \mu_2 < 0 \quad \longleftrightarrow \quad \mu_1 + \mu_2 - \mu_3 > 0 \tag{7.212}$$

then the form (7.210) is positive-definite, being $\mu_3 + \frac{\sigma}{2} > 0$. If we have

$$\mu_1 + \mu_2 - \mu_3 < 0 \tag{7.213}$$

then inequality (7.199) becomes:

$$\left(\mu_3 + \frac{\sigma}{2}\right)\left(\frac{\mu_3 - \mu_1 - \mu_2}{\mu_3 + \frac{\sigma}{2}}\right)^2 + 2(\mu_1 + \mu_2 - \mu_3)\frac{(\mu_3 - \mu_1 - \mu_2)}{\mu_3 + \frac{\sigma}{2}} + \\ + \left(\mu_3 - \frac{\sigma}{2}\right) > 0 \tag{7.214}$$

namely

$$\frac{(\mu_1 + \mu_2 - \mu_3)^2}{\mu_3 + \frac{\sigma}{2}} - 2\frac{(\mu_1 + \mu_2 - \mu_3)^2}{\mu_3 + \frac{\sigma}{2}} + \mu_3 - \frac{\sigma}{2} > 0 \tag{7.215}$$

and simplifying:

$$-\frac{(\mu_1 + \mu_2 - \mu_3)^2}{\mu_3 + \frac{\sigma}{2}} + \mu_3 - \frac{\sigma}{2} > 0 \tag{7.216}$$

from which we obtain

$$(\mu_1 + \mu_2 - \mu_3)^2 < \mu_3^2 - \frac{\sigma^2}{4} > 0 \tag{7.217}$$

hence

$$\mu_1 + \mu_2 - \mu_3 > -\sqrt{\mu_3^2 - \frac{\sigma^2}{4}} \tag{7.218}$$

since we assumed $\mu_1 + \mu_2 - \mu_3 < 0$. If we divide the previous expression by μ_3, we obtain an other condition to be satisfied in order to have ellipticity:

$$\frac{\mu_1 + \mu_2}{\mu_3} > 1 - \sqrt{1 - \left(\frac{\sigma}{2\mu_3}\right)^2} \tag{7.219}$$

As a conclusion, the conditions on the shear moduli for the (strong)ellipticity are given by (7.206) and (7.219) with $\mu_3 > 0$, namely

$$\mu_3 > 0 \tag{7.220a}$$

$$\left|\frac{\sigma}{2\mu_3}\right| < 1 \tag{7.220b}$$

$$\frac{\mu_1 + \mu_2}{\mu_3} > 1 - \sqrt{1 - \left(\frac{\sigma}{2\mu_3}\right)^2} \tag{7.220c}$$

7.6.1 Regime classification

The regime equation is (7.198) and has the following form

$$A\rho^4 + B\rho^2 + C = 0 \tag{7.221}$$

with $A, B, C \in \mathbb{R}$, namely, a quadratic equation in the unknown ρ^2 whose solutions have the form:

$$\rho^2_{1/2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B \pm \sqrt{\Delta}}{2A} \tag{7.222}$$

from which

$$\rho^2_{1/2} = \pm\sqrt{\frac{-B - \sqrt{\Delta}}{2A}} \tag{7.223a}$$

$$\rho^2_{3/4} = \pm\sqrt{\frac{-B + \sqrt{\Delta}}{2A}} \tag{7.223b}$$

In an analogous manner of Section 5.4, the solutions (7.223) characterize the regime of (7.221). The three principal cases are distinguished by the positivity, nullity or negativity of the discriminant Δ as follows:

- if $\Delta < 0$, then $\rho^2_{1/2} \in \mathbb{C}$, so that $\rho_i \in \mathbb{C}$ (for $i = 1, 2, 3, 4$), corresponding to the elliptic complex regime (EC);
- if $\Delta > 0$, then $\rho^2_{1/2} \in \mathbb{R}$;
  - if $\rho^2_{1/2} \in \mathbb{R}^- \setminus \{ 0 \}$, then $\rho_i \in \mathbb{C}$ with $\Re(\rho_i) = 0$ (for $i = 1, 2, 3, 4$), corresponding to the elliptic imaginary regime (EI);

- if $\rho^2_{1/2} \in \mathbb{R}^+ \setminus \{ 0 \}$, then $\rho_i \in \mathbb{R}$ (per $i = 1, 2, 3, 4$), corresponding to the hyperbolic regime (H);
- if $\rho^2_1 \in \mathbb{R}^+ \setminus \{ 0 \}$ and $\rho^2_2 \in \mathbb{R}^- \setminus \{ 0 \}$, then $\rho_{1/2} \in \mathbb{R}$ and $\rho_{3/4} \in \mathbb{C}$ with $\Re(\rho_{3/4}) = 0$, corresponding to the parabolic regime (P);

- if $\Delta = 0$, then $\rho^2_1 = \rho^2_2 \in \mathbb{R}$, so that only two solutions exist: $\rho_1 = \sqrt{\frac{-B}{2A}}$ and $\rho_2 = -\sqrt{\frac{-B}{2A}}$
  - if $\frac{-B}{2A} \in \mathbb{R}^- \setminus \{ 0 \}$, then $\rho_i \in \mathbb{C}$ with $\Re(\rho_i) = 0$ (for $i = 1, 2$);
  - if $\frac{-B}{2A} \in \mathbb{R}^+ \setminus \{ 0 \}$, then $\rho_i \in \mathbb{R}$ (for $i = 1, 2$);
  - if $B = 0$, then the equation is degenerate.

The threshold dividing the elliptic complex and the hyperbolic regimes (EC/H) is determined by imposing $\Delta = 0$; in particular, if this threshold is reached from (EC), the limit condition is given by $\Delta \leq 0$.

7.6.2 Regime classification for the axisymmetric J_2-deformation theory

We want to calculate the limits defining the threshold between the regimes (EC/H), starting from (EC), within the axisymmetric J_2-deformation theory. If we consider equation (7.208), the imposition $\Delta \leq 0$ corresponds to

$$\Delta = \left(\frac{\mu_1 + \mu_2}{\mu_3} - 1 \right)^2 - \left[1 - \left(\frac{\sigma}{2\mu_3} \right)^2 \right] \leq 0 \tag{7.224}$$

For the constitutive model we are considering, the shear moduli are

$$\mu_1 = \frac{KN}{3} \varepsilon_{\text{e}}^{N-1} \tag{7.225a}$$

$$\mu_2 = \frac{K}{6} (N+1) \varepsilon_{\text{e}}^{N-1} \tag{7.225b}$$

$$\mu_3 = \frac{K}{2} \frac{\lambda^3 + 1}{\lambda^3 - 1} \varepsilon_{\text{e}}^{N-1} \ln \lambda_3 \tag{7.225c}$$

while the prestress is

$$\sigma = K \varepsilon_{\text{e}}^{N-1} \ln \lambda_3 \tag{7.226}$$

We observe that the following relations, between the moduli above defined, hold:

$$\mu_1 + \mu_2 = \frac{K}{6}(3N+1)\varepsilon_{\mathrm{e}}^{N-1} \tag{7.227a}$$

$$\frac{\mu_1 + \mu_2}{\mu_3} = \frac{K(3N+1)\varepsilon_{\mathrm{e}}^{N-1} 2(\lambda_3^3 - 1)}{6K(\lambda_3^3+1)\varepsilon_{\mathrm{e}}^{N-1}\ln\lambda_3} = \frac{3N+1}{3\ln\lambda_3}\frac{\lambda_3^3-1}{\lambda_3^3+1} \tag{7.227b}$$

$$\frac{\sigma}{2\mu_3} = \frac{\lambda_3^3-1}{\lambda_3^3+1} \tag{7.227c}$$

so that the inequality (7.224) can be rewritten as

$$\Delta = \left(\frac{3N+1}{3\ln\lambda_3}\frac{\lambda_3^3-1}{\lambda_3^3+1} - 1\right)^2 - \left[1 - \left(\frac{\lambda_3^3-1}{\lambda_3^3+1}\right)^2\right] \leq 0 \tag{7.228}$$

namely

$$\frac{\left[(3N+1)(\lambda_3^3-1) - 3(\lambda_3^3+1)\ln\lambda_3\right]^2}{(3\ln\lambda_3)^2(\lambda_3^3+1)^2} + \\ - \frac{(3\ln\lambda_3)^2\left[(\lambda_3^3+1)^2 - (\lambda_3^3-1)^2\right]}{(3\ln\lambda_3)^2(\lambda_3^3+1)^2} \leq 0 \tag{7.229}$$

Since the denominator is always positive, we can focus on the numerator:

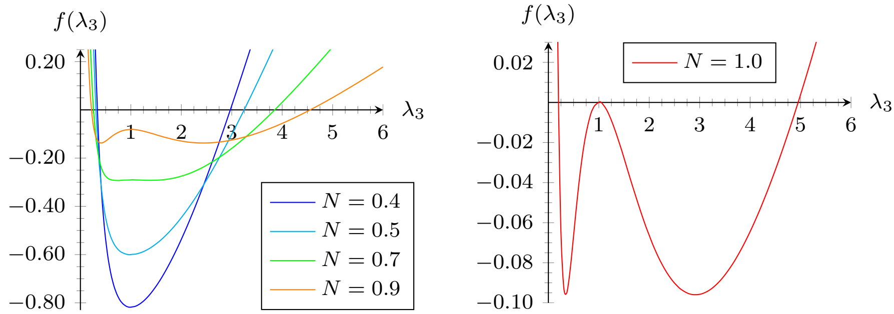
$$(3N+1)^2(\lambda_3^3-1)^2 - 6(3N+1)(\lambda_3^3-1)(\lambda_3^3+1)\ln\lambda_3 + (\lambda_3^3+1)^2(3\ln\lambda_3)^2 + \\ - (3\ln\lambda_3)^2(\underbrace{\lambda_3^6 + 2\lambda_3^3 + 1 - \lambda_3^6 + 2\lambda_3^3 - 1}_{=4\lambda_3^3}) \leq 0 \tag{7.230}$$

but

$$(\lambda_3^3+1)^2 - 4\lambda_3^3 = \lambda_3^6 + 2\lambda_3^3 + 1 - 4\lambda_3^3 = (\lambda_3^3-1)^2 \tag{7.231}$$

so that we can rewrite (7.230) as

$$(3N+1)^2(\lambda_3^3-1)^2 - 6(3N+1)(\lambda_3^3-1)(\lambda_3^3+1)\ln\lambda_3 + \\ + (\lambda_3^3-1)^2(3\ln\lambda_3)^2 \leq 0 \tag{7.232}$$



(a) *Characteristic values of N.* **(b)** *Particular case for $N = 1$.*

Figure 7.2: Representation of the curves (7.233) for some characteristic values of the hardening parameter N. The interval in which we have the elliptic complex regime (EC) corresponds to the regions in between the roots of the illustrated functions. In the case of $N = 1$, we exclude from this interval the value $\lambda_3 = 1$, as expressed by equation (7.234).

and we divide this expression by $(3N + 1)^2(\lambda_3^3 - 1)^2$, which is a positive quantity[10] , thus yielding the regime equation providing the limit (EC/H) starting from (EC):

$$f(\lambda_3) = 1 - 6\frac{\lambda_3^3 + 1}{\lambda_3^3 - 1}\frac{\ln \lambda_3}{3N + 1} + \left(\frac{3 \ln \lambda_3}{3N + 1}\right)^2 \leq 0 \tag{7.233}$$

The values of λ satisfying the equality correspond to the limit values . For typical values of the material parameter N, namely for $0 < N < 1$, from equation (7.233) we obtain the families of curves[11] illustrated in Figure 7.2a. Consequently, the solutions of the equation associated to (7.233) are always two: $\lambda_3 < 1$ (namely, the prestretch corresponding to compression) and $\lambda_3 > 1$ (namely, the prestretch corresponding to tension). Furthermore, we observe that for $N = 1$ the graph of equation (7.233) corresponds to that shown in Figure 7.2b, namely $\lambda_3 = 1$ is a root of the function. As a conclusion, the interval in which we have the elliptic

[10] We recall that, within this Section, we consider a non zero prestrain, namely $\lambda_3 \neq 1$, while $N \in (0, 1]$.
[11] We consider the first member of (7.233) as a parametric function (with parameter N) whose independent variable is λ_3.

regime (EC), until the hyperbolic (H) threshold, is defined as

$$I^{\mathrm{EC}} = \left(\lambda_3^{(1)}, 1\right) \cup \left(1, \lambda_3^{(2)}\right) \tag{7.234}$$

We can reach the threshold (EC/H) starting from (EC) for the following values of λ_3:

$$\lambda_3 \to \lambda_3^{(1)\,+} \quad \vee \quad \lambda_3 \to \lambda_3^{(2)\,-} \tag{7.235}$$

7.6.3 Regime classification for the isotropic material

For an isotropic material we have $\mu_1 = \mu_2 = \mu_3 = \mu$, so that the regime equation for the three-dimensional problem (7.199) reduces to

$$\left(\mu + \frac{\sigma}{2}\right)\rho^4 + 2\mu\rho^2 + \mu - \frac{\sigma}{2} > 0 \qquad \forall \rho \tag{7.236}$$

and if we define

$$\kappa = \frac{\sigma}{2\mu} \tag{7.237}$$

in an analogous manner to Section 5.4, the inequality (7.236) becomes

$$(1+\kappa)\rho^4 + 2\rho^2 + 1 - \kappa > 0 \qquad \forall \rho \tag{7.238}$$

The square of the solutions of the associated equation are

$$\rho_{1/2}^2 = \frac{-1 \pm \sqrt{1-(1-\kappa)^2}}{1+\kappa} = \frac{-1 \pm |\kappa|}{1+\kappa} \tag{7.239}$$

which should be distinguished by the sign of κ, since it appears as a modulus:

$$\begin{cases} \rho_{1/2}^2 = -1 \quad \vee \quad \rho_{1/2}^2 = \dfrac{\kappa-1}{1+\kappa} & \text{if } \kappa \geq 0 \\ \rho_{1/2}^2 = \dfrac{\kappa-1}{1+\kappa} \quad \vee \quad \rho_{1/2}^2 = -1 & \text{if } \kappa < 0 \end{cases} \tag{7.240}$$

but these expressions reduce to

$$\rho_{1/2}^2 = -1 \quad \vee \quad \rho_{1/2}^2 = \frac{\kappa-1}{1+\kappa} \tag{7.241}$$

independently from the sign of κ. The sign of the square of the solution as a function of κ must be analyzed; in particular, we have

$$\frac{\kappa - 1}{1+\kappa} > 0 \quad \longleftrightarrow \quad \kappa < -1 \quad \vee \quad \kappa > 1 \tag{7.242}$$

The solutions of the equation associated to (7.238) are:

$$\begin{cases} \rho_{1/2} = \pm\,\mathrm{i} \quad \in \mathbb{C} \\ \rho_{3/4} = \begin{cases} \pm\sqrt{\dfrac{\kappa-1}{1+\kappa}} & \in \mathbb{R}^+ \cup \{\, 0 \,\} & \text{if } |\kappa| > 1 \\ \pm\,\mathrm{i}\sqrt{\dfrac{1-\kappa}{1+\kappa}} & \in \mathbb{C} & \text{if } |\kappa| < 1 \end{cases} \end{cases} \tag{7.243}$$

Consequently, if $|\kappa| > 1$ we have two real ($\rho_{3/4}$) and two imaginary ($\rho_{1/2}$) solutions, namely, we are within the parabolic regime (P); if $|\kappa| < 1$, we have four pure imaginary and conjugated solutions, so that the regime is elliptic imaginary (EI). Since we exclude the value $\kappa = -1$, the interval in which we have the elliptic regime (EI) until the parabolic (P) threshold, is defined as

$$I^{\mathrm{EI}} = (-1, 1) \tag{7.244}$$

and we can reach the threshold (EC/P) starting from (EI) for the following values of κ:

$$\kappa \to -1^+ \quad \vee \quad \kappa \to 1^- \tag{7.245}$$

7.7 Integration technique of the 3D Green's functions: the geometric description

The Green's functions set found in Section 7.1 consists in integral functions. We want to find a technique for describing in an easy and suitable way the geometry of the problem, in order to make explicit the integration variables. The objective is to implement equations (7.29), (7.30) and (7.113), namely, to do numeric simulations allowing us to analyze the effect of anisotropy and prestress on an infinite body, on which a system of concentrated forces is acting.

We consider a generic point in the space defined by the position vector

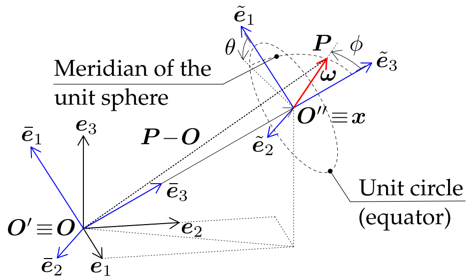


Figure 7.3: Reference system used for the implementation of formulae (7.29), (7.30) and (7.113). The unit vector ω (shown red) defines the surface of a unit sphere centred at point O'' (defined by the position vector x). The dashed curves denote the equator and the meridian defining the vector ω within the local reference system $\{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \}$.

$x \in \mathbb{E}^3$, in which we want to calculate the incremental displacement and mean stress fields generated by a unit force applied in point O, and we denote with $r = |x|$ the length of the position vector x. First we introduce three reference systems, as illustrated in Figure 7.3.

The first reference system is centred at O and is given by the counterclockwise orthonormal basis $\{ e_1, e_2, e_3 \}$; this system is assumed as the *initial reference system* and, as we will see later, the implementation of equations (7.29), (7.30) and (7.113) will be expressed only in terms of quantities described in the basis $\{ e_1, e_2, e_3 \}$. The second reference system is centred at $O' \equiv O$ and is given by the counterclockwise orthonormal basis $\{ \bar{e}_1, \bar{e}_2, \bar{e}_3 \}$; this system has the same origin as the first, but is rotated in a way such that the unit vector $\bar{e}_3$ is aligned with the position vector x. The third reference system, that for simplicity is called *local reference system*, is centred at $O'' \equiv x$ and is given by the counterclockwise orthonormal basis $\{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \}$; this system has the unit vector $\tilde{e}_3$ aligned with the position vector x. As a consequence, the second and the third reference systems are coaxial and their difference consists in a translation equal to $\overline{OO''} = r$.

To the determine the basis $\{ \bar{e}_1, \bar{e}_2, \bar{e}_3 \}$ we assume the following directions:

$$\bar{e}_1 = \alpha_1 \bar{e}_3 \times e_3 \tag{7.246a}$$

$$\bar{e}_2 = \alpha_2 \bar{e}_3 \times \bar{e}_1 \tag{7.246b}$$

$$\bar{e}_3 = \alpha_3 x \tag{7.246c}$$

where $\alpha_i \in \mathbb{R}$ (for $i = 1, 2, 3$) are normalization coefficients that must be determined. The choice of using the unit vector $\bar{\boldsymbol{e}}_3$ parallel to $\boldsymbol{x}$ allows us to find immediately the coefficient α_3

$$\bar{\boldsymbol{e}}_3 = \frac{\boldsymbol{x}}{r} \quad \longrightarrow \quad \alpha_3 = r^{-1} \tag{7.247}$$

To calculate the coefficient α_1, we initially search for a vector parallel to $\bar{\boldsymbol{e}}_1$, defined as

$$\boldsymbol{e}_1' = \boldsymbol{x} \times \boldsymbol{e}_3 = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ 0 & 0 & 1 \end{vmatrix} = x_2 \boldsymbol{e}_1 - x_1 \boldsymbol{e}_2 \tag{7.248}$$

and whose modulus is

$$|\boldsymbol{e}_1'| = \sqrt{x_1^2 + x_2^2} \tag{7.249}$$

while the coefficient α_1 is simply the reciprocal of this modulus, namely

$$\alpha_1 = \frac{1}{\sqrt{x_1^2 + x_2^2}} \tag{7.250}$$

In an analogous manner, we can calculate the coefficient α_2. We search for a vector parallel to $\bar{\boldsymbol{e}}_2$, defined as

$$\boldsymbol{e}_2' = \boldsymbol{x} \times \boldsymbol{e}_1' = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ x_2 & -x_1 & 0 \end{vmatrix} = x_1 x_3 \boldsymbol{e}_1 + x_2 x_3 \boldsymbol{e}_2 - (x_1^2 + x_2^2) \boldsymbol{e}_3 \tag{7.251}$$

and whose squared modulus is

$$\begin{aligned} |\boldsymbol{e}_2'|^2 &= x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^4 + 2 x_1^2 x_2^2 + x_2^4 \\ &= x_1^2 (x_1^2 + x_2^2 + x_3^2) + x_2^2 (x_1^2 + x_2^2 + x_3^2) = (x_1^2 + x_2^2) r^2 \end{aligned} \tag{7.252}$$

being $r^2 = x_1^2 + x_2^2 + x_3^2$, and the coefficient α_2 the reciprocal of the modulus of $\boldsymbol{e}_2'$, namely

$$\alpha_2 = \frac{1}{r\sqrt{x_1^2 + x_2^2}} \tag{7.253}$$

The basis $\{\, \bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3 \,\}$ assumes the following representation as a function of the initial reference system $\{ \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \}$:

$$\bar{\boldsymbol{e}}_1 = \frac{x_2\boldsymbol{e}_1 - x_1\boldsymbol{e}_2}{\sqrt{x_1^2 + x_2^2}} \tag{7.254a}$$

$$\bar{\boldsymbol{e}}_2 = \frac{x_1x_3\boldsymbol{e}_1 + x_2x_3\boldsymbol{e}_2 - (x_1^2 + x_2^2)\boldsymbol{e}_3}{r\sqrt{x_1^2 + x_2^2}} \tag{7.254b}$$

$$\bar{\boldsymbol{e}}_3 = \frac{x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3}{r} \tag{7.254c}$$

We observe that the unit vectors above defined have effectively unit length, namely $|\bar{\boldsymbol{e}}_1| = |\bar{\boldsymbol{e}}_2| = |\bar{\boldsymbol{e}}_3| = 1$, and we can verify that they constitute a counter-clockwise system because the following products are satisfied:

$$\bar{\boldsymbol{e}}_1 \times \bar{\boldsymbol{e}}_2 = \bar{\boldsymbol{e}}_3 \tag{7.255a}$$

$$\bar{\boldsymbol{e}}_2 \times \bar{\boldsymbol{e}}_3 = \bar{\boldsymbol{e}}_1 \tag{7.255b}$$

$$\bar{\boldsymbol{e}}_3 \times \bar{\boldsymbol{e}}_1 = \bar{\boldsymbol{e}}_2 \tag{7.255c}$$

The integration of equations (7.29), (7.30) and (7.113) requires the definition of a unit vector $\boldsymbol{\omega}$ whose origin is the point $\boldsymbol{x} \equiv \boldsymbol{O}''$. To this purpose, we define a point $\boldsymbol{P}$ such that $\boldsymbol{\omega} = \boldsymbol{P} - \boldsymbol{O}''$. In the third reference system, centred at $\boldsymbol{O}''$ and with basis $\{ \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3 \}$, we can introduce a local spherical coordinate system defined by two angles $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$ and by a radial coordinate. In this reference system the point $\boldsymbol{P}$ is described through the two angles θ and ϕ, while its radial coordinate is fixed and equal to 1, since $\overline{\boldsymbol{PO}''} = |\boldsymbol{\omega}| = 1$. Consequently, the point $\boldsymbol{P}$ belongs to the the surface of a unit sphere centred at $\boldsymbol{x} \equiv \boldsymbol{O}''$.

We define now the following vectors, represented in the second reference system (centred at $\boldsymbol{O}' \equiv \boldsymbol{O}$ and with basis $\{ \bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3 \}$), and related to the description of the vector $\boldsymbol{\omega}$:

$$\boldsymbol{P} - \boldsymbol{O}' = \bar{\omega}_1\bar{\boldsymbol{e}}_1 + \bar{\omega}_2\bar{\boldsymbol{e}}_2 + \bar{\omega}_3\bar{\boldsymbol{e}}_3 \tag{7.256a}$$

$$\boldsymbol{O}'' - \boldsymbol{O}' = \bar{r}_1\bar{\boldsymbol{e}}_1 + \bar{r}_2\bar{\boldsymbol{e}}_2 + \bar{r}_3\bar{\boldsymbol{e}}_3 \tag{7.256b}$$

$$\boldsymbol{\omega} = \boldsymbol{P} - \boldsymbol{O}'' = (\boldsymbol{P} - \boldsymbol{O}') - (\boldsymbol{O}'' - \boldsymbol{O}') \tag{7.256c}$$

but, since $\bar{\boldsymbol{e}}_3$ is aligned with $\boldsymbol{x}$, we have $\bar{r}_1, \bar{r}_2 = 0$ and $\bar{r}_3 = r$. Hence, equation (7.256c) can be rewritten as

$$\begin{aligned}\boldsymbol{\omega} &= (\bar{\omega}_1 - \bar{r}_1)\bar{\boldsymbol{e}}_1 + (\bar{\omega}_2 - \bar{r}_2)\bar{\boldsymbol{e}}_2(\bar{\omega}_3 - \bar{r}_3)\bar{\boldsymbol{e}}_3 = \\ &= \bar{\omega}_1\bar{\boldsymbol{e}}_1 + \bar{\omega}_2\bar{\boldsymbol{e}}_2 + (\bar{\omega}_3 - r)\bar{\boldsymbol{e}}_3 \end{aligned} \tag{7.257}$$

If $\boldsymbol{P}$ belongs to the spherical surface above introduced, the components $\bar{\omega}_i$ (con $i = 1, 2, 3$) of (7.256a) can be expressed as[12]:

$$\bar{\omega}_1 = \cos\theta \sin\phi \tag{7.258a}$$

$$\bar{\omega}_1 = \sin\theta \sin\phi \tag{7.258b}$$

$$\bar{\omega}_1 = \bar{r}_3 + \cos\phi \tag{7.258c}$$

so that the representation of $\boldsymbol{\omega}$ in the second reference system reduces to

$$\boldsymbol{\omega} = \cos\theta \sin\phi\, \bar{\boldsymbol{e}}_1 + \sin\theta \sin\phi\, \bar{\boldsymbol{e}}_2 + \cos\phi\, \bar{\boldsymbol{e}}_3 \tag{7.259}$$

If we express the unit vectors $\bar{\boldsymbol{e}}_i$ as functions of the unit vectors $\boldsymbol{e}_i$ (con $i = 1, 2, 3$) through equations (7.254), we obtain the representation of $\boldsymbol{\omega}$ in the initial (Cartesian) reference system:

$$\boldsymbol{\omega} = \frac{1}{r\sqrt{x_1^2 + x_2^2}} \begin{bmatrix} (rx_2 \cos\theta + x_1 x_3 \sin\theta)\sin\phi + x_1\sqrt{x_1^2 + x_2^2}\cos\phi \\ (x_2 x_3 \sin\theta - r x_1 \cos\theta)\sin\phi + x_2\sqrt{x_1^2 + x_2^2}\cos\phi \\ -\left(x_1^2 + x_2^2\right)\sin\theta\sin\phi + x_3\sqrt{x_1^2 + x_2^2}\cos\phi \end{bmatrix} \tag{7.260}$$

Note that for $\phi = \pi/2$ we obtain the components of $\boldsymbol{\omega}$ in the case that $\boldsymbol{\omega}$ belongs to the plane determined by $\tilde{e}_1$ and $\tilde{e}_2$, namely

$$\boldsymbol{\omega} = \frac{1}{r\sqrt{x_1^2 + x_2^2}} \begin{bmatrix} rx_2 \cos\theta + x_1 x_3 \sin\theta \\ x_2 x_3 \sin\theta - r x_1 \cos\theta \\ -\left(x_1^2 + x_2^2\right)\sin\theta \end{bmatrix} \tag{7.261}$$

Before seeing the application of these definitions regarding the geometry of the problem, we must do some further observations on the Dirac delta distribution and its derivatives, that will be exposed in the next Section.

[12] We recall that we are using the representation in the reference system centred at $\boldsymbol{O}' \equiv \boldsymbol{O}$ and with basis $\{\, \bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3 \,\}$.

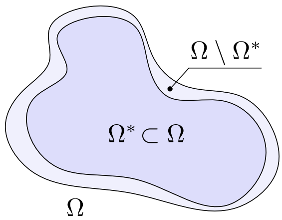


Figure 7.4: Representation of a compact support Ω^* of a function $\phi(\boldsymbol{x})$.

7.8 The distributional derivative and the Dirac delta

In order to calculate explicitly the integral equations defining the incremental displacement and mean stress fields, some definitions and further details about the Dirac delta and the evaluation of its derivative must be illustrated.

7.8.1 The distributional derivative

Let $\phi(x)$ be a continuous test function. The application of the delta function to the test functions yields:

$$\big(\delta(x-a), \phi(x)\big) = \phi(a) \tag{7.262}$$

where $\phi(x)$ belongs to the Schwartz space[13] and we consider $\delta(x)$ as a distribution (or continuous linear functional). The principal properties of the Dirac delta are the following:

$$\delta(x) = 0 \quad \forall x \neq 0 \tag{7.263a}$$

$$\int_{-\infty}^{+\infty} \delta(x)\,\mathrm{d}x = 1 \tag{7.263b}$$

$$\delta(\boldsymbol{x}) = \delta(x_1)\delta(x_2)\ldots\delta(x_n) \quad \text{se } \boldsymbol{x} \in \mathbb{R}^n \tag{7.263c}$$

Let $f(\boldsymbol{x})$ be a generalized function (a functional) and $\phi(\boldsymbol{x})$ be a smooth C^∞ class test function with compact support Ω^*. Ω^* is a region strictly included in

[13] The Schwartz space is the space of the functions with rapid decay at infinity and infinitely differentiable, whose partial derivatives decay rapidly too.

Ω, on which $\phi(\boldsymbol{x})$ can assume any value (see Figure 7.4). On the region $\Omega \setminus \Omega^*$ we have $\phi(\boldsymbol{x}) = 0$ and we assume that the volume (and/or surface) of this region is not infinitesimal, namely, the distance between the boundaries $\partial\Omega^*$ and $\partial\Omega$ is not infinitesimal, but finite and non null in every point. It follows that

$$\phi^{(n)}(\boldsymbol{x}) = \frac{\partial \phi(\boldsymbol{x})}{\partial x_n} = 0 \quad \text{on } \partial\Omega \tag{7.264}$$

namely, all the n-th order derivatives of $\phi(\boldsymbol{x})$ are null on the boundary of Ω.

Let us imagine that $f(\boldsymbol{x})$ is defined as $f\colon \phi \to \mathbb{R}$ and we define

$$\int_\Omega f(\boldsymbol{x})\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (f, \phi) \tag{7.265}$$

namely, the function $f(\boldsymbol{x})$ applied to $\phi(\boldsymbol{x})$. To apply the derivative of $f(\boldsymbol{x})$ to $\phi(\boldsymbol{x})$ is equivalent to:

$$\begin{aligned} \left(\frac{\partial f}{\partial x_i}, \phi\right) &= \int_\Omega \frac{\partial f(\boldsymbol{x})}{\partial x_i} \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \\ &= \int_\Omega \frac{\partial}{\partial x_i}\big[f(\boldsymbol{x})\phi(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x} - \int_\Omega f(\boldsymbol{x}) \frac{\partial \phi(\boldsymbol{x})}{\partial x_i}\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{7.266}$$

where the product rule has been used. On application of the divergence theorem, we obtain:

$$\left(\frac{\partial f}{\partial x_i}, \phi\right) = \int_{\partial\Omega} f(\boldsymbol{x})\phi(\boldsymbol{x}) n_i\,\mathrm{d}S_{\boldsymbol{x}} - \int_\Omega f(\boldsymbol{x}) \frac{\partial \phi(\boldsymbol{x})}{\partial x_i}\,\mathrm{d}\boldsymbol{x} \tag{7.267}$$

but $\phi(\boldsymbol{x})$ is zero on the boundary $\partial\Omega$ as well as its derivatives, so that the first integral is null. Consequently, the above relations can be simplified as

$$\left(\frac{\partial f}{\partial x_i}, \phi\right) = -\int_\Omega f(\boldsymbol{x}) \frac{\partial \phi(\boldsymbol{x})}{\partial x_i}\,\mathrm{d}\boldsymbol{x} = -\left(f, \frac{\partial \phi}{\partial x_i}\right) \tag{7.268}$$

namely, the derivative of the distribution is related only to the derivative of the test function. Regarding the higher order derivatives, we proceed similarly and we obtain the following generalization. Let α be a multi-index:

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{7.269}$$

where $\alpha_i \in \mathbb{N} \cup \{ 0 \}$. Let us define:

$$|\alpha| = \sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 + \cdots + \alpha_n \tag{7.270}$$

while we represent the n-th order derivatives as

$$\mathrm{D}^\alpha f(\boldsymbol{x}) = \frac{\partial^{|\alpha|} f(\boldsymbol{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \tag{7.271}$$

then the distributional derivative of order α is:

$$(\mathrm{D}^\alpha f, \phi) = (-1)^{|\alpha|} (f, \mathrm{D}^\alpha \phi) \tag{7.272}$$

7.8.2 The derivative of the Dirac delta

The uni-dimensional case

If we apply the concept of distributional derivative to the Dirac delta, we have

$$\big(\delta^{'}(x-a), \phi(x)\big) = \big(\delta(x-a), \phi^{'}(x)\big) = -\phi^{'}(a) \tag{7.273}$$

and, more in general,

$$\big(\delta^{(\alpha)}(x-a), \phi(x)\big) = (-1)^{|\alpha|} \big(\delta(x-a), \phi^{\alpha}(x)\big) = (-1)^{|\alpha|} \phi^{\alpha}(a) \tag{7.274}$$

while for scalar functions

$$\big(\delta^{(n)}(x-a), \phi(x)\big) = (-1)^n \phi^n(a) \tag{7.275}$$

recalling that

$$\frac{\mathrm{d}}{\mathrm{d}x} \delta(-x) = -\frac{\mathrm{d}}{\mathrm{d}x} \delta(x) \tag{7.276}$$

If we want to make explicit and implement the above relations, we can exploit the following approximation:

$$\delta(x) \approx \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)} = \delta^*(x) \tag{7.277}$$

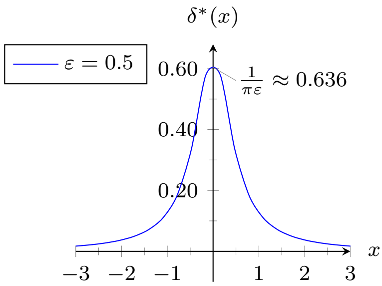


Figure 7.5: Example of the graph of the function $\delta^*(x)$ approximating the Dirac delta.

with $\varepsilon \in \mathbb{R}^+$, and in particular we assume $\varepsilon \to 0^+$ (without any loss of generality). We observe that

$$\delta^*(0) = \frac{1}{\pi\varepsilon} \tag{7.278}$$

which becomes singular for $\varepsilon = 0$, exactly as $\delta(x)$. Furthermore, we observe the following property:

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta^*(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}\,\mathrm{d}x = \frac{\varepsilon}{\pi}\left[\frac{1}{\varepsilon}\arctan\left(\frac{x}{\varepsilon}\right)\right]_{-\infty}^{+\infty} = \\ = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\right] = 1 \end{aligned} \tag{7.279}$$

as from the definition of $\delta(x)$ and we observe that this result is independent of ε.

The two-dimensional case

We consider a unit vector $\boldsymbol{\omega}$, applied in $\boldsymbol{x}$, described in a local reference system defined by the basis $\{\,\tilde{e}_1, \tilde{e}_2\,\}$; the representation of $\boldsymbol{\omega}$ is:

$$\boldsymbol{\omega} = \cos\theta\,\tilde{\boldsymbol{e}}_1 + \sin\theta\,\tilde{\boldsymbol{e}}_2 \tag{7.280}$$

where $\theta \in [0, 2\pi]$, and we want to calculate the following integral:

$$\int_{|\boldsymbol{\omega}|=1} g(\boldsymbol{\omega})\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} \tag{7.281}$$

where $\boldsymbol{e}_r$ corresponds to the unit vector defined by the position vector $\boldsymbol{x}$, while $g(\boldsymbol{\omega})$ is a generic continuous function. The increment of the vector $\boldsymbol{\omega}$ can be expressed as:

$$\omega_1 + \Delta\omega_1 = \cos(\theta + \Delta\theta) \tag{7.282a}$$

$$\omega_2 + \Delta\omega_2 = \sin(\theta + \Delta\theta) \tag{7.282b}$$

The increment of the argument of g is

$$(\boldsymbol{\omega} + \Delta\boldsymbol{\omega}) \cdot \boldsymbol{e}_r = \boldsymbol{\omega} \cdot \boldsymbol{e}_r + \Delta(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \tag{7.283}$$

while for the function itself we have

$$g\big((\boldsymbol{\omega} + \Delta\boldsymbol{\omega}) \cdot \boldsymbol{e}_r\big) = g(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) + \Delta g(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \tag{7.284}$$

From relations (7.283) and (7.284) we can obtain the increments:

$$\begin{aligned}\Delta g(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) = g\big(&\cos(\theta + \Delta\theta)(\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_r) + \sin(\theta + \Delta\theta)(\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_r)\big) + \\ &- g\big(\cos\theta(\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_r) + \sin\theta(\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_r)\big)\end{aligned} \tag{7.285}$$

and

$$\begin{aligned}\Delta(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) = &\big[\cos(\theta + \Delta\theta) - \cos\theta\big](\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_r) + \\ &+ \big[\sin(\theta + \Delta\theta) - \sin\theta\big](\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_r)\end{aligned} \tag{7.286}$$

If we use the Taylor series expansion for the increments of the vector ω, given by (7.282), we obtain:

$$\cos(\theta + \Delta\theta) = \cos\theta \underbrace{\cos(\Delta\theta)}_{=1} - \sin\theta \underbrace{\sin(\Delta\theta)}_{=\Delta\theta} \approx \cos\theta - \Delta\theta \sin\theta \tag{7.287a}$$

$$\sin(\theta + \Delta\theta) = \sin\theta \underbrace{\cos(\Delta\theta)}_{=1} + \cos\theta \underbrace{\sin(\Delta\theta)}_{=\Delta\theta} \approx \sin\theta + \Delta\theta \cos\theta \tag{7.287b}$$

so that the increment $\Delta(\boldsymbol{\omega} \cdot \boldsymbol{e}_r)$ can be rewritten as

$$\Delta(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) = -\Delta\theta \sin\theta\, (\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_r) + \Delta\theta \cos\theta\, (\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_r) \tag{7.288}$$

namely,

$$\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) = \Delta\theta\big[-\sin\theta\,(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{e}_r) + \cos\theta(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{e}_r)\big] \tag{7.289}$$

and calculating the scalar products, we obtain:

$$\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) = \Delta\theta\big(-\sin\theta\,x_1 + \cos\theta\,x_2\big) \tag{7.290}$$

and we note that the quantity between the brackets is the equation of a line perpendicular to the vector $\boldsymbol{\omega}$. In an analogous manner, we can rewrite the increment $\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)$, which reduces to:

$$\begin{aligned}\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) &= g\big(\cos(\theta+\Delta\theta)x_1 + \sin(\theta+\Delta\theta)x_2\big) - g\big(\cos\theta\,x_1 + \sin\theta\,x_2\big) = \\ &= \Delta g(\theta)\end{aligned} \tag{7.291}$$

Now we can write the incremental ratio

$$\frac{\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)}{\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)} = -\frac{\Delta g(\theta)}{\Delta\theta}\frac{1}{-\sin\theta\,x_1 + \cos\theta\,x_2} \tag{7.292}$$

The limit $\Delta\theta \to 0$ yields:

$$\lim_{\Delta\theta\to 0}\frac{\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)}{\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)} = \frac{1}{-\sin\theta\,x_1 + \cos\theta\,x_2}\frac{\partial g(\theta)}{\partial\theta} = g^{'}(\theta) \tag{7.293}$$

and this derivative is singular for all the points lying on the line perpendicular to $\boldsymbol{\omega}$.

The three-dimensional case

We consider the unit vector $\boldsymbol{\omega}$, applied in $\boldsymbol{x}$, described in the local reference system defined by the basis $\{\ \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\ \}$ illustrated in Figure 7.3 on page 236:

$$\boldsymbol{\omega} = \cos\theta\sin\phi\,\tilde{\boldsymbol{e}}_1 + \sin\theta\sin\phi\,\tilde{\boldsymbol{e}}_2 + \cos\phi\,\tilde{\boldsymbol{e}}_3 \tag{7.294}$$

where $\theta\in[0,2\pi]$, $\phi\in[0,\pi]$ and we want to calculate the following integral:

$$\int_{|\boldsymbol{\omega}|=1} g(\boldsymbol{\omega})\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} \tag{7.295}$$

where $\boldsymbol{e}_r$ corresponds to the unit vector defined by the position vector $\boldsymbol{x}$, while $g(\boldsymbol{\omega})$ is a generic continuous function. The increment of the vector $\boldsymbol{\omega}$ can be expressed as:

$$\omega_r + \Delta\omega_r = \cos(\phi + \Delta\phi) \tag{7.296a}$$

$$\omega_\perp + \Delta\omega_\perp = \sin(\phi + \Delta\phi) \tag{7.296b}$$

The increment of the argument of g is

$$(\boldsymbol{\omega} + \Delta\boldsymbol{\omega}) \cdot \boldsymbol{e}_r = \boldsymbol{\omega} \cdot \boldsymbol{e}_r + \Delta(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \tag{7.297}$$

while for the function itself we have

$$g\big((\boldsymbol{\omega} + \Delta\boldsymbol{\omega}) \cdot \boldsymbol{e}_r\big) = g(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) + \Delta g(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \tag{7.298}$$

From relations (7.297) and (7.298) we can obtain the increments:

$$\begin{aligned}\Delta g(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) = g\big(&\cos\theta \sin(\phi + \Delta\phi)(\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_r) + \sin\theta \sin(\phi + \Delta\phi)(\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_r) + \\ &+ \cos(\phi + \Delta\phi)(\tilde{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_r)\big) - g\big(\cos\theta \sin\phi\,(\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_r) + \\ &+ \sin\theta \sin\phi\,(\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_r) + \cos\phi\,(\tilde{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_r)\big)\end{aligned} \tag{7.299}$$

and

$$\begin{aligned}\Delta(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) = \cos\theta\big[\sin(\phi + \Delta\phi) - \sin\phi\big](\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_r) + \sin\theta\big[\sin(\phi + \Delta\phi) - \sin\phi\big] \\ \times (\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_r) + \big[\cos(\phi + \Delta\phi) - \cos\phi\big](\tilde{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_r)\end{aligned} \tag{7.300}$$

If we use the Taylor series expansion for the increments of the vector ω, given by (7.296), we obtain:

$$\cos(\phi + \Delta\phi) = \cos\phi \underbrace{\cos(\Delta\phi)}_{=1} - \sin\phi \underbrace{\sin(\Delta\phi)}_{=\Delta\phi} \approx \cos\phi - \Delta\phi \sin\phi \tag{7.301a}$$

$$\sin(\phi + \Delta\phi) = \sin\phi \underbrace{\cos(\Delta\phi)}_{=1} + \cos\phi \underbrace{\sin(\Delta\phi)}_{=\Delta\phi} \approx \sin\phi + \Delta\phi \cos\phi \tag{7.301b}$$

so that the increment $\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)$ can be rewritten as

$$\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) = \cos\theta\Delta\phi\cos\phi\,(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{e}_r) + \sin\theta\Delta\phi\cos\phi\,(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{e}_r) + \\ - \Delta\phi\sin\phi\,(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{e}_r) \quad (7.302)$$

namely,

$$\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) = \Delta\phi\big[\cos\theta\,\Delta\phi\cos\phi\,(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{e}_r) + \sin\theta\cos\phi\,(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{e}_r) + \\ - \sin\phi\,(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{e}_r)\big] \quad (7.303)$$

and since $\boldsymbol{e}_r \parallel \tilde{\boldsymbol{e}}_3$:

$$\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) = -\Delta\phi\sin\phi \quad (7.304)$$

In an analogous manner, we can rewrite the increment $\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)$, which reduces to:

$$\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) = g(\cos\phi - \Delta\phi\sin\phi) - g(\cos\phi) = \Delta g(\phi) \quad (7.305)$$

Now we can write the incremental ratio

$$\frac{\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)}{\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)} = -\frac{\Delta g(\phi)}{\Delta\phi\sin\phi} \quad (7.306)$$

The limit $\Delta\phi \to 0$ yields:

$$\lim_{\Delta\phi\to 0}\frac{\Delta g(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)}{\Delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)} = -\frac{1}{\sin\phi}\frac{\partial g(\phi)}{\partial\phi} = g^{'}(\phi) \quad (7.307)$$

Using the spherical coordinate system ($\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$) and taking into account that the vector $\boldsymbol{\omega}$ has unit length, the integral (7.295) corresponds to the following surface integral

$$\int_{|\boldsymbol{\omega}|=1} g(\boldsymbol{\omega})\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} = \int_0^{2\pi}\int_0^{\pi} g\big(\boldsymbol{\omega}(\phi,\theta)\big)\delta^{'}(\cos\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \quad (7.308)$$

being the determinant of the Jacobian matrix equal to $\sin\phi$, since the radius has unit length. On application of the rule (7.307) to the function $\delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r) = \delta(\cos\phi)$, we obtain

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} g\big(\boldsymbol{\omega}(\phi,\theta)\big)\delta^{'}(\cos\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta &= \\ = \int_0^{2\pi}\int_0^{\pi} g\big(\boldsymbol{\omega}(\phi,\theta)\big)\left[-\frac{1}{\sin\phi}\frac{\partial\delta(\cos\phi)}{\partial\phi}\right]\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta &= \\ = \int_0^{2\pi}\int_0^{\pi} -g\big(\boldsymbol{\omega}(\phi,\theta)\big)\frac{\partial\delta(\cos\phi)}{\partial\phi}\,\mathrm{d}\phi\,\mathrm{d}\theta &\end{aligned} \tag{7.309}$$

Taking into account the property of the Dirac delta, the derivative shifts to the function g and the integral reduces to a line integral[14] so that the integral in (7.309) can be calculated as:

$$\int_0^{2\pi}\int_0^{\pi} -g\big(\boldsymbol{\omega}(\phi,\theta)\big)\frac{\partial\delta(\cos\phi)}{\partial\phi}\,\mathrm{d}\phi\,\mathrm{d}\theta = \int_0^{2\pi} \frac{\partial g\big(\boldsymbol{\omega}(\phi,\theta)\big)}{\partial\phi}\bigg|_{\phi=\frac{\pi}{2}}\mathrm{d}\theta \tag{7.310}$$

We must recall that the previous equation *does not* represent the result of the integral (7.295) described in the initial reference system, but yields it in the local reference system $\{\,\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\,\}$; consequently, once we have calculated equation (7.310), we must apply a change of coordinates.

An alternative method for the three-dimensional case

We illustrate in this Subsection an alternative method for the calculation of the Dirac delta derivative for the three-dimensional case. Instead of defining the derivative as the limit of the incremental ratio, we apply the technique of the change of variable to the integral (7.308), here reported:

$$\int_0^{2\pi}\int_0^{\pi} g(\phi,\theta)\delta^{'}(\cos\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{7.311}$$

If we define $y = \cos\phi$ (so that $\mathrm{d}y = -\sin\phi\,\mathrm{d}\phi$), the above integral can be rewritten as:

$$\int_0^{2\pi}\int_{-1}^{1} g(\arccos y,\theta)\delta^{'}(y)\,\mathrm{d}y\,\mathrm{d}\theta \tag{7.312}$$

[14] Note that the argument of the delta function (namely, $\cos\phi$) vanishes only for $\phi = \frac{\pi}{2}$, being $\phi \in [0,\pi]$.

and, taking into account the Dirac delta property,

$$\int_0^{2\pi} \int_{-1}^{1} g(\arccos y, \theta) \delta^{'}(y) \, \mathrm{d}y \, \mathrm{d}\theta = -\int_0^{2\pi} \left. \frac{\partial g(\arccos y, \theta)}{\partial y} \right|_{y=0} \mathrm{d}\theta \tag{7.313}$$

However, the integrand can be manipulated as follows:

$$\begin{aligned} \frac{\partial g(\arccos y, \theta)}{\partial y} &= \frac{\partial g(\arccos y, \theta)}{\partial \arccos y} \frac{\partial \arccos y}{\partial y} = \frac{-1}{\sqrt{1-y^2}} \frac{\partial g(\phi, \theta)}{\partial \phi} = \\ &= \frac{-1}{\sqrt{1-\cos^2 \phi}} \frac{\partial g(\phi, \theta)}{\partial \phi} = -\frac{1}{\sin \phi} \frac{\partial g(\phi, \theta)}{\partial \phi} \end{aligned} \tag{7.314}$$

and in this way we arrive to the rule (7.307). Now equation (7.313) becomes

$$-\int_0^{2\pi} \left. \frac{\partial g(\arccos y, \theta)}{\partial y} \right|_{y=0} \mathrm{d}\theta = \int_0^{2\pi} \left[\frac{1}{\sin \phi} \frac{\partial g(\phi, \theta)}{\partial \phi} \right]_{\phi = \frac{\pi}{2}} \mathrm{d}\theta \tag{7.315}$$

which corresponds to (7.310). Consequently, we must also bring the description of the above equation back to the initial reference system.

7.9 Final implementation of the three-dimensional Green's functions

The plane wave expansion requires the integration on the surface of a unit sphere centred at point $\boldsymbol{x}$. To this aim, we use the expressions for $\boldsymbol{\omega}$ given by (7.260) and we perform the integration with respect to the variables θ and ϕ, an in particular we obtain:

$$\int_{|\boldsymbol{\omega}|=1} [\quad] \, \mathrm{d}\boldsymbol{\omega} = \int_0^{2\pi} \int_0^{\pi} [\quad] \sin \phi \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{7.316}$$

being

$$\mathrm{d}A = (1 \sin \phi \, \mathrm{d}\phi)(1 \, \mathrm{d}\theta) = \sin \phi \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{7.317}$$

Note that in the above expressions we must take into account in an appropriate manner the change of the integrand functions representation, which are functions of the vector $\boldsymbol{\omega}$ (and how it is described). We can formalize the law governing the representation change of the vector $\boldsymbol{\omega}$ from the local reference system $\{\, \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3 \,\}$ to the initial system $\{\, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \,\}$, through the following rotation tensor:

$$[\boldsymbol{Q}] = \frac{1}{r\sqrt{x_1^2 + x_2^2}} \begin{bmatrix} r x_2 & x_1 x_3 & x_1\sqrt{x_1^2 + x_2^2} \\ -r x_1 & x_2 x_3 & x_2\sqrt{x_1^2 + x_2^2} \\ 0 & -(x_1^2 + x_2^2) & x_3\sqrt{x_1^2 + x_2^2} \end{bmatrix} \tag{7.318}$$

A generic vector $\boldsymbol{u}$ can be expressed in the initial reference system $\{\, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \,\}$ starting from its components in the local reference system (denoted with a tilde) as follows:

$$\boldsymbol{u} = \boldsymbol{Q}\tilde{\boldsymbol{u}} \tag{7.319}$$

and the inverse law is

$$\tilde{\boldsymbol{u}} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{u} \tag{7.320}$$

which yields the components of $\boldsymbol{u}$ in the local reference system as functions of those in the initial system. In an analogous manner, a second order tensor can be represented as

$$\boldsymbol{U} = U_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \tilde{U}_{ij}\, \tilde{\boldsymbol{e}}_i \otimes \tilde{\boldsymbol{e}}_j \tag{7.321}$$

depending on the choice of the basis. The components U_{ij} and $\tilde{U}_{ij}$ are related as follows

$$\tilde{U}_{ij} = \tilde{\boldsymbol{e}}_i \cdot \boldsymbol{U}\tilde{\boldsymbol{e}}_j = \tilde{\boldsymbol{e}}_i \left[U_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l\right] \tilde{\boldsymbol{e}}_j = U_{kl}(\boldsymbol{e}_k \cdot \tilde{\boldsymbol{e}}_i)(\boldsymbol{e}_l \cdot \tilde{\boldsymbol{e}}_j) \tag{7.322}$$

Assuming

$$Q_{ki} = \boldsymbol{e}_k \cdot \tilde{\boldsymbol{e}}_i \tag{7.323}$$

we have

$$\tilde{U}_{ij} = Q_{ik} U_{kl} Q_{lj} \tag{7.324}$$

namely,

$$\tilde{\boldsymbol{U}} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{U}\boldsymbol{Q} \quad \longleftrightarrow \quad \boldsymbol{U} = \boldsymbol{Q}\tilde{\boldsymbol{U}}\boldsymbol{Q}^{\mathrm{T}} \tag{7.325}$$

Regarding the incremental displacement field (7.29), in the integrand appears the Dirac delta as a post-multiplier for all the terms inside the integral, so that we have the following form:

$$\int_{|\boldsymbol{\omega}|=1} G(\boldsymbol{\omega},\boldsymbol{x})\delta(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} \tag{7.326}$$

and it is clear that the integrand does not vanish only if $\boldsymbol{\omega}\cdot\boldsymbol{x} = 0$, namely, when the vectors $\boldsymbol{\omega}$ and $\boldsymbol{x}$ are mutually perpendicular.[15] This implies that the region of the space on which the integrand gives a non null contribution is the intersection of the spherical surface (with unit radius) and the plane generated by the unit vectors $\tilde{\boldsymbol{e}}_1$ and $\tilde{\boldsymbol{e}}_2$. Consequently, the surface integral reduces to a line integral on the unit circle lying on the plane generated by $\tilde{\boldsymbol{e}}_1$ and $\tilde{\boldsymbol{e}}_2$, namely, it has the form:

$$\int_{|\boldsymbol{\omega}|=1} [\quad]\delta(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} = \int_0^{2\pi} [\quad]\,\mathrm{d}\theta \tag{7.327}$$

being the line element of the unit circle equal to $\mathrm{d}l = 1\,\mathrm{d}\theta$ (so that the area reduces to a length).

To simplify the distinction in the representation that must be used in the following equations, we denote with with latin letters the components in the *initial reference system* $\{\,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\,\}$, while we denote with greek letters (and a tilde on the relative quantity) those in the *local reference system* $\{\,\tilde{\boldsymbol{e}}_1,\tilde{\boldsymbol{e}}_2,\tilde{\boldsymbol{e}}_3\,\}$; in this way, the integral equations for the incremental displacement (7.29) can be expressed as:

$$\begin{aligned} v_k^g(\boldsymbol{x}) &= -\frac{1}{8\pi^2 r}\int_{|\boldsymbol{\omega}|=1} V_{gk}(\boldsymbol{\omega})\delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} = \\ &= -\frac{Q_{g\alpha}Q_{k\beta}}{8\pi^2 r}\int_0^{2\pi}\int_0^{\pi} \tilde{V}_{\alpha\beta}(\theta,\phi)\,\delta(\cos\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \end{aligned} \tag{7.328}$$

where

$$\boldsymbol{V}(\boldsymbol{\omega}) = \frac{\boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}\otimes\boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}}{\boldsymbol{\omega}\cdot\boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}} - \boldsymbol{A}^{-1}(\boldsymbol{\omega}) \tag{7.329}$$

The difference between V_{gk} and $\tilde{V}_{\alpha\beta}$ consists only in the representation: the first

[15] We recall that the Dirac delta function has the following property:

$$\int_{a-\varepsilon}^{a+\varepsilon} g(t)\delta(t-a)\,\mathrm{d}t = g(a)$$

where $a\in\mathbb{R}$ and $\varepsilon\in\mathbb{R}^+$.

is described in the initial system, while the second in the local one; consequently, the following relation for the change of the reference system holds:

$$\tilde{\boldsymbol{V}} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{Q} \quad \longleftrightarrow \quad \boldsymbol{V} = \boldsymbol{Q}\tilde{\boldsymbol{V}}\boldsymbol{Q}^{\mathrm{T}} \tag{7.330}$$

On application of the rule (7.327) to (7.328) yields the *Green's function for the incremental displacement field*

$$v_k^g(\boldsymbol{x}) = -\frac{Q_{g\alpha}Q_{k\beta}}{8\pi^2 r}\int_0^{2\pi}\tilde{V}_{\alpha\beta}(\theta,\pi/2)\,\mathrm{d}\theta \tag{7.331}$$

Regarding the mean stress field (7.30), its integral formulation has a form similar to (7.308), namely

$$\begin{aligned}\dot{p}^g(\boldsymbol{x}) &= \frac{1}{8\pi^2 r^2}\int_{|\boldsymbol{\omega}|=1} P_g(\boldsymbol{\omega})\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} = \\ &= \frac{Q_{g\alpha}}{8\pi^2 r^2}\int_0^{2\pi}\int_0^{\pi}\tilde{P}_\alpha(\theta,\phi)\delta^{'}(\cos\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\end{aligned} \tag{7.332}$$

where

$$\boldsymbol{P}(\boldsymbol{\omega}) = \frac{\boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}}{\boldsymbol{\omega}\cdot\boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}} \tag{7.333}$$

so that P_g and $\tilde{P}_\alpha$ are represented, respectively, in the initial and in the local reference systems; the rule that allows us to switch from a system to the other is

$$\tilde{\boldsymbol{P}} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P} \quad \longleftrightarrow \quad \boldsymbol{P} = \boldsymbol{Q}\tilde{\boldsymbol{P}} \tag{7.334}$$

Taking into account the (7.310), the *Green's function for the incremental mean stress field* reduces to

$$\dot{p}^g(\boldsymbol{x}) = \frac{Q_{g\alpha}}{8\pi^2 r^2}\int_0^{2\pi}\left.\frac{\partial\tilde{P}_\alpha(\theta,\phi)}{\partial\phi}\right|_{\phi=\frac{\pi}{2}}\mathrm{d}\theta \tag{7.335}$$

The integral involved in the expression of the gradient of the incremental displacement field (7.41) can be written as

$$\int_{|\boldsymbol{\omega}|=1} D_{gkl}(\boldsymbol{\omega})\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} = Q_{g\alpha}Q_{k\beta}Q_{l\gamma}\int_0^{2\pi}\int_0^{\pi}\tilde{D}_{\alpha\beta\gamma}(\theta,\phi)\delta^{'}(\cos\phi) \times \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \quad (7.336)$$

where

$$\boldsymbol{D}(\boldsymbol{\omega}) = \boldsymbol{V}(\boldsymbol{\omega})\otimes\boldsymbol{\omega} \quad (7.337)$$

so that the *Green's function for the gradient of the incremental displacement field* becomes

$$v^g_{k,l}(\boldsymbol{x}) = -\frac{Q_{g\alpha}Q_{k\beta}Q_{l\gamma}}{8\pi^2r^2}\int_0^{2\pi}\left.\frac{\partial\tilde{D}_{\alpha\beta\gamma}(\theta,\phi)}{\partial\phi}\right|_{\phi=\pi/2}\mathrm{d}\theta \quad (7.338)$$

Finally, the potential (7.113) has the form

$$\begin{aligned}\hat{\Phi}(\boldsymbol{x}) &= \frac{1}{8\pi^2r^3}\int_{|\boldsymbol{\omega}|=1} Z(\boldsymbol{\omega})\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} = \\ &= \frac{1}{8\pi^2r^3}\int_0^{2\pi}\int_0^{\pi}\tilde{Z}(\theta,\phi)\delta^{''}(\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta\end{aligned} \quad (7.339)$$

where

$$Z(\boldsymbol{\omega}) = \frac{1}{\boldsymbol{\omega}\cdot\boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}} \quad (7.340)$$

and the difference between Z and $\tilde{Z}$ consists in their representations. Taking into account the (7.310), we obtain

$$\hat{\Phi}(\boldsymbol{x}) = \frac{1}{8\pi^2r^3}\int_0^{2\pi}\left.\frac{\partial^2\tilde{Z}(\theta,\phi)}{\partial\phi^2}\right|_{\phi=\frac{\pi}{2}}\mathrm{d}\theta \quad (7.341)$$

Once established the integration domain for the integral functions describing the incremental displacement and mean stress fields and the geometry of the problem (namely, the choice of the initial reference system and the consequent representation of $\boldsymbol{\omega}$), the components of the acoustic tensor $\boldsymbol{A}(\boldsymbol{\omega})$ must be calculated.

7.9.1 Application to incompressible isotropic elasticity or Stokes flow

We consider the special case of incompressible isotropic elasticity, for which the stress tensor reduces to

$$\sigma_{ij} = p\delta_{ij} + \mu\big(v_{i,j} + v_{j,i}\big) \tag{7.342}$$

so that the constitutive tensor becomes

$$\mathbb{K}_{ijkl} = \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) \tag{7.343}$$

where μ is the shear modulus and, consequently, the acoustic tensor and its inverse are expressed as

$$A_{jl}(\boldsymbol{\omega}) = \mu(\omega_j\omega_l + \delta_{jl}) \tag{7.344a}$$

$$A^{-1}_{jl}(\boldsymbol{\omega}) = -\frac{1}{2\mu}\omega_j\omega_l + \frac{1}{\mu}\delta_{jl} \tag{7.344b}$$

and, in addition, we have

$$\omega_j A^{-1}_{jl}(\boldsymbol{\omega}) = -\frac{1}{2\mu}\omega_j\omega_j\omega_l + \frac{1}{\mu}\omega_j\delta_{jl} = \frac{1}{2\mu}\omega_l \tag{7.345}$$

and

$$\omega_j A^{-1}_{jl}(\boldsymbol{\omega})\omega_l = \frac{1}{2\mu}\omega_l\omega_l = \frac{1}{2\mu} \tag{7.346}$$

It follows that the Green's function set for the velocity and mean stress fields becomes

$$v^g_k(\boldsymbol{x}) = \frac{\delta_{gk}}{4\pi r} - \frac{1}{8\pi^2 r}\int_{|\boldsymbol{\omega}|=1} \omega_g\omega_k\delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} \tag{7.347a}$$

$$\dot{p}^g(\boldsymbol{x}) = \frac{1}{8\pi^2 r^2}\int_{|\boldsymbol{\omega}|=1} \omega_g\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} \tag{7.347b}$$

while the potential Φ reduces to

$$\Phi(\boldsymbol{x}) = \frac{\mu}{4\pi^2 r^3}\int_{|\boldsymbol{\omega}|=1} \delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega} = -2\mu\delta(\boldsymbol{x}) \tag{7.348}$$

The potential (7.348) provides a null contribution in the boundary integral (7.50), while the Green's functions (7.347a) and (7.347b) can be calculated through equations (7.331) and (7.335). The representation of $\boldsymbol{\omega}$ in the local reference system $\{\, \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3 \,\}$ is

$$\boldsymbol{\omega} = \cos\theta \sin\phi \, \tilde{\boldsymbol{e}}_1 + \sin\theta \sin\phi \, \tilde{\boldsymbol{e}}_2 + \cos\phi \, \tilde{\boldsymbol{e}}_3 \tag{7.349}$$

so that the product $\boldsymbol{\omega} \otimes \boldsymbol{\omega}$ is

$$\boldsymbol{\omega} \otimes \boldsymbol{\omega} = \begin{bmatrix} \cos^2\theta \sin^2\phi & \cos\theta \sin\theta \sin^2\phi & \cos\theta \cos\phi \sin\phi \\ \cos\theta \sin\theta \sin^2\phi & \sin^2\theta \sin^2\phi & \sin\theta \sin\phi \cos\phi \\ \cos\theta \cos\phi \sin\phi & \sin\theta \cos\phi \sin\phi & \cos^2\phi \end{bmatrix} \tag{7.350}$$

while the derivative of $\boldsymbol{\omega}$ with respect to ϕ is

$$\frac{\partial \boldsymbol{\omega}}{\partial \phi} = \{\, \cos\theta \cos\phi \,, \sin\theta \cos\phi \,, -\sin\phi \,\} \tag{7.351}$$

The integrand of equation (7.331) can be expressed as

$$\tilde{V}_{\alpha\beta} = \frac{1}{\mu}(\boldsymbol{\omega} \otimes \boldsymbol{\omega} - \boldsymbol{I}) \tag{7.352}$$

and, if it is evaluated for $\phi = \pi/2$, reduces to

$$\tilde{V}_{\alpha\beta}(\theta, \pi/2) = \frac{1}{\mu} \begin{bmatrix} \cos^2\theta - 1 & \cos\theta \sin\theta & 0 \\ \cos\theta \sin\theta & \sin^2\theta - 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{7.353}$$

whose integration with respect to variable θ, as indicated in equation (7.331), yields

$$\int_0^{2\pi} \tilde{V}_{\alpha\beta}(\theta, \phi) = -\frac{\pi}{\mu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \tilde{v}_{\alpha\beta} \tag{7.354}$$

The displacement field is obtained on application of the transformation from local to initial coordinates, defined by the rule (7.330), as follows:

$$v_g^k(\boldsymbol{x}) = -\frac{Q_{g\alpha}\tilde{v}_{\alpha\beta}Q_{k\beta}}{8\pi^2 r} = \frac{1}{8\pi\mu r}\boldsymbol{Q}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\boldsymbol{Q}^{\mathrm{T}} \tag{7.355}$$

and finally, we obtain

$$v_g^k(\boldsymbol{x}) = \frac{1}{8\pi\mu r^3}\begin{bmatrix}x_1^2+r^2 & x_1x_2 & x_1x_3\\ x_2x_1 & x_2^2+r^2 & x_2x_3\\ x_3x_1 & x_3x_2 & x_3^2+r^2\end{bmatrix} \tag{7.356}$$

The integrand of equation (7.335) can be calculated by imposing $\phi = \pi/2$ in equation (7.351)

$$\left.\frac{\partial\boldsymbol{\omega}}{\partial\phi}\right|_{\phi=\frac{\pi}{2}} = \{\,0\,,0\,,-1\,\} \tag{7.357}$$

and its integration with respect to variable θ, as indicated in equation (7.335), yields

$$\int_0^{2\pi}\left.\frac{\partial\boldsymbol{\omega}}{\partial\phi}\right|_{\phi=\frac{\pi}{2}} = -2\pi\,\{\,0\,,0\,,1\,\} = \tilde{p}_\alpha \tag{7.358}$$

The mean stress field can be obtained on application of the transformation from local to initial coordinates, defined by the rule (7.334), as follows:

$$\dot{p}^g(\boldsymbol{x}) = \frac{Q_{g\alpha}\tilde{p}_\alpha}{8\pi^2 r^2} = -\frac{1}{4\pi r^2}\boldsymbol{Q}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \tag{7.359}$$

and finally, we obtain

$$\dot{p}^g(\boldsymbol{x}) = -\frac{1}{4\pi r^3}\{\,x_1\,,x_2\,,x_3\,\} \tag{7.360}$$

The transformation law for the mean stress can also be applied as follows. The mean stress must be represented in the initial reference system; recalling that

$$\dot{\boldsymbol{p}} = (\tilde{\boldsymbol{p}}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 + (\tilde{\boldsymbol{p}}\cdot\boldsymbol{e}_2)\boldsymbol{e}_2 + (\tilde{\boldsymbol{p}}\cdot\boldsymbol{e}_3)\boldsymbol{e}_3 \tag{7.361}$$

the components of $\dot{p}^\alpha$ in the initial reference system are

$$\dot{p}^1 = -\frac{\tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1}{4\pi r^2} = -\frac{x_1}{4\pi r^3} \tag{7.362a}$$

$$\dot{p}^2 = -\frac{\tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2}{4\pi r^2} = -\frac{x_2}{4\pi r^3} \tag{7.362b}$$

$$\dot{p}^3 = -\frac{\tilde{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3}{4\pi r^2} = -\frac{x_3}{4\pi r^3} \tag{7.362c}$$

From equations (7.356) and (7.360) we obtain the following expression for the displacement and mean stress field for incompressible isotropic elasticity:

$$v_k^g(\boldsymbol{x}) = \frac{1}{8\pi\mu r}\left(\delta_{kg} + \frac{x_k x_g}{r^2}\right) \tag{7.363a}$$

$$\dot{p}^g(\boldsymbol{x}) = \frac{x_g}{4\pi r^3} \tag{7.363b}$$

representing, for the Stokes flow, the well-known *Stokeslet*.

7.9.2 Linear elastic isotropic compressible material

The Green's functions are presented in literature only in terms of displacement field for the case of linear elastic isotropic compressible materials without prestress; using the Lamé constants, the displacement field is expressed as (see Hirth & Lothe [132])

$$u_{ij}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{8\pi\mu}\left[\delta_{ij}\nabla^2 r - \frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2 r}{\partial x_i \partial x_j}\right] \tag{7.364}$$

or, by introduction of the Poisson's ratio ν, as [63, 64]

$$u_{ij}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ij}\nabla^2 r + \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right] \tag{7.365}$$

In these two expressions we consider the distance r defined as

$$r = d(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}| = \sqrt{(x_i - y_i)^2} \tag{7.366}$$

namely, the distance between the source point $\boldsymbol{y}$ and the generic point $\boldsymbol{x}$, in which we measure the field. It follows that the Green's functions (7.364) and (7.364) are translated functions, so that they have a structure similar to those employed in the previous Chapter, namely, in the inclusions and dislocations problems.

The first derivative of r is

$$\frac{\partial r}{\partial x_i} = \frac{x_i - y_i}{r} \tag{7.367}$$

while its second order derivatives are expressed as

$$\frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{x_i - y_i}{r}\right) = \frac{\delta_{ij}}{r} - \frac{(x_i - y_i)(x_j - y_j)}{r^3} \tag{7.368a}$$

$$\frac{\partial^2 r}{\partial x_i^2} = \frac{r^2 - (x_i - y_i)^2}{r^3} \tag{7.368b}$$

so that the Laplacian becomes

$$\nabla^2 r = \frac{2}{r} \tag{7.369}$$

Recalling the constitutive laws relating the Lamé constants to the elastic modulus and the Poisson's ratio

$$\mu = \frac{E}{2(1+\nu)} \tag{7.370a}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{1-2\nu} \tag{7.370b}$$

we can write the following relations

$$\lambda + \mu = \mu\left(\frac{2\nu}{1-2\nu} + 1\right) = \frac{\mu}{1-2\nu} \tag{7.371a}$$

$$\lambda + 2\mu = \mu\left(\frac{2\nu}{1-2\nu} + 2\right) = \frac{2\mu(1-\nu)}{1-2\nu} \tag{7.371b}$$

yielding

$$\frac{\lambda+\mu}{\lambda+2\mu} = \frac{\mu}{1-2\nu}\frac{1-2\nu}{2\mu(1-\nu)} = \frac{1}{2(1-\nu)} \tag{7.372}$$

The quantity between brackets in the second member of (7.364) can be rewritten as

$$\delta_{ij}\nabla^2 r - \frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{2}{r}\delta_{ij} - \frac{1}{2(1-\nu)}\left[\frac{\delta_{ij}}{r} - \frac{(x_i-y_i)(x_j-y_j)}{r^3}\right] =$$
$$= \frac{1}{r}\left[2 - \frac{1}{2(1-\nu)}\right]\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{2(1-\nu)r^3} =$$
$$= \frac{1}{2(1-\nu)r}\left[(3-4\nu)\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{r^2}\right] \tag{7.373}$$

Now equations (7.364) and (7.365) can be expressed as functions of the polar coordinates of the points $\boldsymbol{x}$ and $\boldsymbol{y}$

$$u_{ij}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{r^2}\right] \tag{7.374}$$

and if we assume that the point $\boldsymbol{y}$ is centred at the origin, we have:

$$u_{ij}(\boldsymbol{x}) = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ij} + \frac{x_i x_j}{r^2}\right] \tag{7.375}$$

For the incompressible material we have $\nu = 0.5$, so that (7.374) and (7.375) reduce to

$$u_{ij}(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{8\pi\mu r}\left[\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{r^2}\right] \tag{7.376}$$

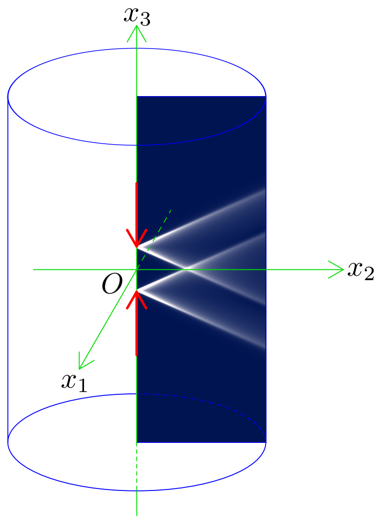
and

$$u_{ij}(\boldsymbol{x}) = \frac{1}{8\pi\mu r}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right) \tag{7.377}$$

respectively, and we observe that this last expression exactly coincides with (7.363a).

7.10 Conical localization of deformation

We are now in a position to analyze the effect of a force dipole as an agent perturbing an infinite elastic incompressible media, prestrained with a stretch λ_3 and obeying the J_2–deformation theory of plasticity within the axisymmetric stress state described in Section 7.5.1 and with a value of the hardening param-



(a) *Modulus of the incremental displacements in the plane Orx_3.*

(b) *Volume levels of the modulus of the incremental displacements.*

Figure 7.6: Level sets of the modulus of the incremental displacement field generated by a convergent force dipole subject to a compressive prestretch near the elliptic boundary; the field is focused along four *conical* shear bands.

eter $N = 0.4$, for which the critical stretches for failure of ellipticity, given in Section 7.6.2, are $0.336\,519$ (for the initial state of compressive prestress) and 2.9716 (corresponding to a tensile prestress).

We consider first the situation in which the dipole is aligned parallel to the axis x_3, as sketched in Figure 7.6a, with the forces (shown red) orientated in a convergent direction; furthermore, an initial state of compressive stretch is assumed. The incremental displacement field can be obtained through superposition and subsequent implementation of equation (7.331). Due to the axisymmetric conditions, it is possible to plot the modulus of the incremental displacements only in the plane Orx_3, as shown in Figure 7.6. Note that the modulus of the displacement field is focused along four cones or *conical shear bands*.

For $N = 0.4$ and $\lambda_3 = 0.336\,519$, the regime equation (7.199) provides an inclination of the shear cone surfaces equal to $66.16°$ with respect to the x_3-axis[16], clearly visible in Figure 7.6.

[16] The regime equation (7.199) can be evaluated with the following substitutions: the material parameters (7.225), the prestress (7.226), and the effective strain (7.177); in this way, the regime equation becomes a function of the material parameter N and of the prestretch λ_3, so that it is possible to calculate the inclination of the shear cone surfaces.

Part II

NON-STANDARD ELASTOPLASTIC CONSTITUTIVE LAWS

Chapter 8

Introduction

GRANULAR AND GEOLOGICAL MATERIALS are employed for many industrial purposes: shock and vibration absorbers, fire protection, thermal barriers, refractory products, wear protectors, electric isolators, and catalysts. They are characterized by pressure-sensitive yielding, and dilatant/contractant inelastic behaviour[1]. Several yield functions have been introduced for the mechanical description of these materials, which have to satisfy different requirements, among which, the most important are convexity and smoothness, two requisites met by the yield function proposed by Bigoni & Piccolroaz [77, 115, 116], henceforth referred to as the 'BP yield function', or simply 'BP'. Moreover, this function has an extreme 'deformability', thus proves particularly appropriate to describe the granular/solid transition occurring during forming of ceramic powders [117, 118], a crucial process in the production of many ceramic products.

Used in the context of elastoplastic modelling, the BP yield function introduces the problem that to be convex, it has been defined $+\infty$ outside certain regions in the stress-space. Therefore, in its original form, the BP yield function cannot be implemented within an elastoplastic integration scheme, if a gradient-based return mapping algorithm is used, for which the gradient of the yield function is needed everywhere in the stress-space [119]. If a non-convex version of the BP yield function (obtained by squaring the terms) is implemented with a

[1] These mechanical behaviours are observed in: ceramic and metal powders [78–82], concrete [83], geomaterials [84–94], masonry [95–97], but also metals [98–105], high strength alloys [106], and shape memory alloys [107–114].

return mapping algorithm, wrong results can be produced, as a specific example will demonstrate.

The problem of the BP yield surface is also common to other yield surfaces for geomaterials [120], so that Penasa, Piccolroaz, Argani & Bigoni [121][2] have overcome the difficulty by proposing two algorithms: one is based on a forward Euler technique with a correction based on a 'centre-of-mass' return scheme, fully applicable to the original form of the BP yield function (defined $+\infty$ outside the yield surface), and another based on a cutoff-substepping return mapping algorithm that can be applied on the squared (and non-convex) version of the BP yield function. Iso-error maps and comparisons with two model problems allowing for a semi-analytical solution (the forming of a ceramic powder pressed against a rigid spherical cup and the expansion of a green body spherical shell subject to internal pressure) show that both algorithms perform correctly, with an accuracy comparable in certain regions of the stress state, even if there are regions where each algorithm is superior to the other.

In particular, the 'centre-of-mass' algorithm is faster than the other, but less accurate near vertices of the deviatoric yield surface, while the cutoff-substepping return mapping algorithm is always more accurate than the other, but can become slow for stress states near the vertices of the meridian yield surface. Finally, we may conclude that, although both algorithms have their advantages and limitations, generally speaking the cutoff-substepping return mapping algorithm can eventually be preferred.

Outline of Part II

Part II is organized as follows. A brief overview of the Bigoni & Piccolroaz yield surface and *two novel integration algorithms* are described in the **ninth Chapter**, while in the **tenth Chapter** the classic problems of elastoplastic spherical shells are *extended* to the case of the Bigoni & Piccolroaz yield surface, showing *new features* of the solutions.

[2] M. Penasa, A. Piccolroaz, L.P. Argani, and D. Bigoni. "Integration algorithms of elastoplasticity for ceramic powder compaction". In Press on: J. Europ. Ceramic Soc. (2014).

Chapter 9

Integration algorithms of elastoplasticity for ceramic powder compaction

A brief overview on the Bigoni & Piccolroaz yield surface is given in this Chapter. Two algorithms are described: an explicit integration scheme based on a forward Euler technique with a 'centre-of-mass' return correction and an implicit integration scheme based on a 'cutoff-substepping' return algorithm.

As mentioned in the introduction, the problem with the BP yield function is that it is defined as $+\infty$ in some regions outside the elastic domain (for $p \notin [-c, p_c]$), Figure 9.1. Therefore, an integration algorithm based on a standard return mapping technique cannot work, so that the purpose of this Chapter (Section 9.2) is to introduce an explicit forward Euler algorithm to solve this problem, while an implicit algorithm will be presented in the Section 9.3 defined on a 'squared version' of the yield function.

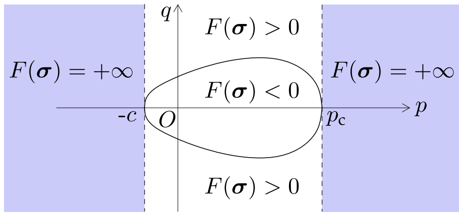


Figure 9.1: The BP yield function represented as a surface in the p–q plane.

9.1 The BP yield surface and its centre of mass

Bigoni & Piccolroaz [115, 116] have introduced a new yield function (BP) for *isotropic* materials, defined in terms of the stress tensor $\boldsymbol{\sigma}$ by

$$F(\boldsymbol{\sigma}) = f(p) + \frac{q}{g(\theta)} \tag{9.1}$$

where, defining the parameter Φ as

$$\Phi = \frac{p + c}{p_\mathrm{c} + c} \tag{9.2}$$

the meridian and deviatoric functions are respectively written as[1]

$$f(p) = \begin{cases} -Mp_c\sqrt{(\Phi - \Phi^m)\,[2(1-\alpha)\Phi + \alpha]} & \text{if } \Phi \in [0,1] \\ +\infty & \text{if } \Phi \notin [0,1] \end{cases} \tag{9.3a}$$

$$\frac{1}{g(\theta)} = \cos\left[\beta\frac{\pi}{6} - \frac{\cos^{-1}(\gamma\cos 3\theta)}{3}\right] \tag{9.3b}$$

[1] The expression (9.3b) was proposed by Podgórski [122, 123] and independently by Bigoni & Piccolroaz [115].

in which p, q and θ (the Lode's angle) are the following stress invariants

$$p = -\frac{\operatorname{tr}\boldsymbol{\sigma}}{3} \tag{9.4a}$$

$$q = \sqrt{3J_2} \tag{9.4b}$$

$$\theta = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right) \tag{9.4c}$$

functions of the second and third invariant of the deviatoric stress $\boldsymbol{S}$

$$J_2 = \frac{1}{2}\operatorname{tr}\boldsymbol{S}^2 \tag{9.5a}$$

$$J_3 = \frac{1}{3}\operatorname{tr}\boldsymbol{S}^3 \tag{9.5b}$$

$$\boldsymbol{S} = \boldsymbol{\sigma} - \frac{\operatorname{tr}\boldsymbol{\sigma}}{3}\boldsymbol{I} \tag{9.5c}$$

$\boldsymbol{I}$ being the identity tensor.

The yield function (9.1)–(9.3) is convex when the seven material parameters defining the meridian shape function $f(p)$ and the deviatoric shape function $g(\theta)$ lie within the following intervals

$$\begin{array}{llll} M > 0 & p_c > 0 & c \geq 0 & 0 < \alpha < 2 \\ m > 1 & 0 \leq \beta \leq 2 & 0 \leq \gamma \leq 1 & \end{array} \tag{9.6}$$

Centre of mass of the yield surface

The numerical integration algorithm that will be developed later is based on the knowledge of the centre of mass of the yield surface. This, with reference to Figure 9.2, can be obtained as follows.

We begin by noting that the yield surface possesses the isotropy symmetries in the deviatoric plane (see Bigoni & Piccolroaz [115]), therefore, the centre of mass of the yield surface lies on the hydrostatic axis. The infinitesimal area of the deviatoric section can be evaluated as

$$\mathrm{d}A = \frac{1}{2}\rho^2(\theta)\,\mathrm{d}\theta \tag{9.7}$$

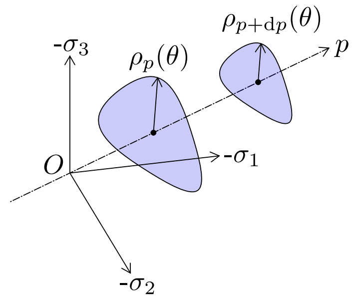


Figure 9.2: Radius $\rho(\theta)$ and the centre of mass of two indicative deviatoric sections (located at different mean stresses p) of the BP yield surface. Due to the isotropy symmetries of the deviatoric sections, the centres of mass lie on the hydrostatic axis.

where

$$\rho(\theta) = \sqrt{\frac{2}{3}}q = -\sqrt{\frac{2}{3}}f(p)g(\theta) \tag{9.8}$$

is the radius of the surface boundary evaluated with respect to the hydrostatic axis, so that the area of the deviatoric section is expressed as

$$A(p) = 2f^2(p)\int_0^{\frac{\pi}{3}} g^2(\theta)\,\mathrm{d}\theta \tag{9.9}$$

On application of the definition of the centre of mass

$$p_\mathrm{G} = \frac{\int_{-c}^{p_\mathrm{c}} p\,A(p)\,\mathrm{d}p}{\int_{-c}^{p_\mathrm{c}} A(p)\,\mathrm{d}p} \tag{9.10}$$

provides the *coordinate of the centre of mass of the BP yield surface along the hydrostatic axis*

$$p_\mathrm{G} = \frac{(m+1)p_\mathrm{c}\left[(\alpha-3)m-6\right] + c\left[6(\alpha+1)+m(m+7)\right]}{(m+3)\left[(\alpha-4)m-2(\alpha+2)\right]} \tag{9.11}$$

a formula involving all the *meridian* parameters of the yield function, except M.

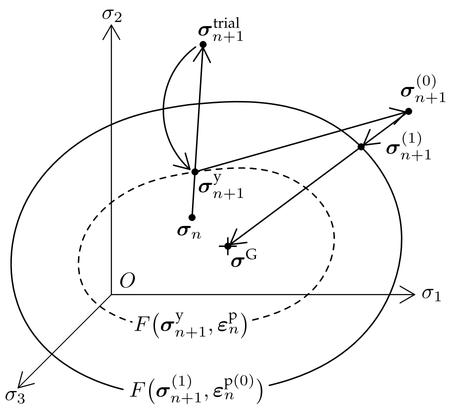


Figure 9.3: Geometrical sketch of the 'centre of mass return algorithm' for the integration of rate elastoplastic constitutive equations.

9.2 The 'centre of mass' return algorithm

We propose a numerical integration procedure for rate elastoplastic constitutive equations based on a return algorithm which is geometrically sketched in Figure 9.3 and can be syntetically described with reference to Box 9.2.

In particular, starting from a given state at a step n [point (1) in Box 9.2] and after the usual trial elastic step [point (2)], the stress point at yielding is found along the line joining the trial and the initial state [point (3)]; from this point, after the purely elastic strain is eliminated from the strain increment [point (4)], a new stress increment is found using the tangent elastoplastic operator [point (5)]; the plastic strain increment is updated [point (6)]; and finally, a return on the updated yield surface is performed along the line joining with the centre of mass of the yield surface [points (7)–(8)].

Box 9.2: The 'centre of mass' return algorithm

(1) Given an initial state at step n, described by the variables $\boldsymbol{\sigma}_n$, $\boldsymbol{\varepsilon}^{\mathrm{e}}_n$, $\boldsymbol{\varepsilon}^{\mathrm{p}}_n$ and given a strain increment $\Delta\boldsymbol{\varepsilon}$;

(2) evaluate the elastic trial solution

$$\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1} = \boldsymbol{\sigma}_n + \mathbb{E}[\Delta\boldsymbol{\varepsilon}]$$

(3) along the line from $\boldsymbol{\sigma}_n$ to $\boldsymbol{\sigma}^{\mathrm{trial}}_{n+1}$ find the stress point $\boldsymbol{\sigma}^{\mathrm{y}}_{n+1}$ at yielding (for fixed $\boldsymbol{\varepsilon}^{\mathrm{p}}_n$), so that

$$F\left(\boldsymbol{\sigma}^{\mathrm{y}}_{n+1}, \boldsymbol{\varepsilon}^{\mathrm{p}}_n\right) = 0$$

(4) evaluate the elastic deformation increment corresponding to $\boldsymbol{\sigma}^{\mathrm{y}}_{n+1} - \boldsymbol{\sigma}_n$, namely,

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{y}}_{n+1} = \mathbb{E}^{-1}\left[\boldsymbol{\sigma}^{\mathrm{y}}_{n+1} - \boldsymbol{\sigma}_n\right]$$

(5) evaluate the stress increment via the tangent elastoplastic operator (for fixed $\boldsymbol{\varepsilon}^{\mathrm{p}}_n$) as

$$\boldsymbol{\sigma}^{(0)}_{n+1} = \boldsymbol{\sigma}^{\mathrm{y}}_{n+1} + \mathbb{C}\left[\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{\mathrm{y}}_{n+1}\right]$$

(6) update the plastic deformation

$$\boldsymbol{\varepsilon}^{\mathrm{p}\,(0)}_{n+1} = \boldsymbol{\varepsilon}^{\mathrm{p}}_n + \Delta\boldsymbol{\varepsilon} - \mathbb{E}^{-1}\left[\boldsymbol{\sigma}^{(0)}_{n+1}\right]$$

(7) find the stress $\boldsymbol{\sigma}^{(1)}_{n+1}$ on the updated yield surface

$$F\left(\boldsymbol{\sigma}^{(1)}_{n+1}, \boldsymbol{\varepsilon}^{\mathrm{p}\,(0)}_{n+1}\right) = 0$$

(8) update the plastic deformation for the final stress state on the yield surface

$$\boldsymbol{\varepsilon}^{\mathrm{p}\,(1)}_{n+1} = \boldsymbol{\varepsilon}^{\mathrm{p}}_n + \Delta\boldsymbol{\varepsilon} - \mathbb{E}^{-1}\left[\boldsymbol{\sigma}^{(1)}_{n+1} - \boldsymbol{\sigma}_n\right]$$

(9) EXIT.

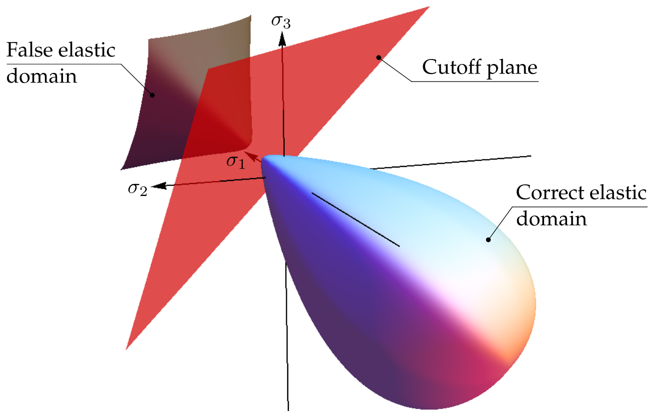


Figure 9.4: Cutoff plane for the BP yield surface. Stress points where the return mapping algorithm works correctly are on the side of the plane where the yield surface lies. The 'false elastic domain' is shown brown.

There are two 'find' in the procedure explained in Box 9.2: the first is at line (3) and the second is at point (7). Both correspond to a root-finding procedure for a scalar function (the yield function) of tensor variable (the stress), that can be pursued with different numerical techniques, so that we employed a bisection method. Regarding the 'find' at point (3), the zero of F is sought along the segment joining $\boldsymbol{\sigma}_n$ with $\boldsymbol{\sigma}_{n+1}^{\text{trial}}$, while no directions are *a-priori* prescribed for returning on the yield surface from the stress state $\boldsymbol{\sigma}_{n+1}^{(0)}$ at point (5). We propose to find the zero of $F\big(\boldsymbol{\sigma}_{n+1}^{(1)}, \boldsymbol{\varepsilon}_{n+1}^{\text{p}\,(0)}\big) = 0$ along the segment drawn from $\boldsymbol{\sigma}_{n+1}^{(0)}$ to the centre of mass of the yield surface, $\boldsymbol{\sigma}^{\text{G}}$ [defined by parameter p_{G}, equation (9.11)].

9.3 The cutoff-substepping integration algorithm

As an alternative to the forward Euler procedure with 'centre of mass' return correction introduced in the previous Section, we propose an implicit integration scheme. Since the standard return mapping algorithm does not work in a zone of the stress-space, this zone can be delimited by introducing a cutoff plane orthogonal to the hydrostatic axis, so that a new algorithm can be set up in which the return mapping scheme is augmented of a substepping when the trial elastic stress falls within that zone. In particular, if the trial elastic solution $\boldsymbol{\sigma}^{\text{trial}}$ falls on the same side of the plane as the starting point, the return mapping algorithm correctly converges (as demonstrated in Section 9.3.1), while, if it falls beyond the cutoff plane, an iterative subincrementation is performed, in which the strain increment $\Delta\varepsilon$ is subdivided and the return mapping is iteratively applied with successive updates of the BP yield function, so that, eventually, the entire initial step will be performed remaining within the correct stress zone.

The position of this cutoff plane depends on shape and size of the BP yield surface (see Figure 9.4) and can be determined as follows.

9.3.1 The squared BP yield function and the cutoff plane

The squared BP yield function is obtained by squaring the terms in equation (9.3), so that its meridian part (divided by p_c) can be written as

$$\tilde{f}(\Phi) = M^2 \left(\Phi - \Phi^m\right) \left[2(1-\alpha)\Phi + \alpha\right] \tag{9.12}$$

The first and second order derivatives of this function with respect to Φ are

$$\frac{\mathrm{d}\tilde{f}(\Phi)}{\mathrm{d}\Phi} = M^2 \left\{2(1-\alpha)\left[2\Phi - (1+m)\Phi^m\right] + \alpha\big(1 - m\Phi^{m-1}\big)\right\} \tag{9.13}$$

and

$$\frac{\mathrm{d}^2\tilde{f}(\Phi)}{\mathrm{d}\Phi^2} = M^2 \left\{2(1-\alpha)\left[2 - m(1+m)\Phi^{m-1}\right] - \alpha m(m-1)\Phi^{m-2}\right\} \tag{9.14}$$

respectively. Note that the squared BP yield function is differentiable (its first and second order derivatives are defined everywhere), but, in general, is no longer convex and displays a so-called 'false elastic domain' (a nomenclature introduced by Brannon & Leelavanichkul [120]), visible in Figure 9.4. For this reason, the Newton-Raphson algorithm

$$\Phi_{n+1} = \Phi_n - \frac{\tilde{f}(\Phi_n)}{\left.\frac{\mathrm{d}\tilde{f}(\Phi)}{\mathrm{d}\Phi}\right|_{\Phi_n}} \tag{9.15}$$

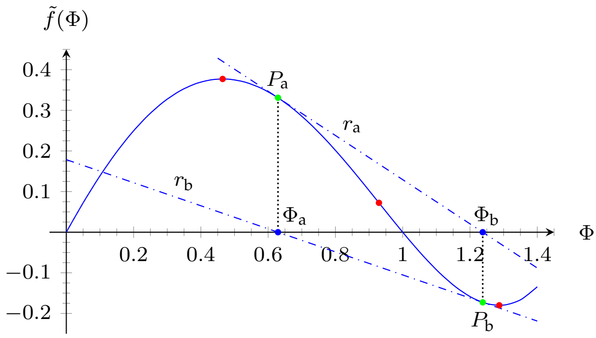


Figure 9.5: Determination of the position of the cutoff plane. Local stationary (maximum and minimum) and inflection points are denoted by red spots, while the bounds of the non-convex region (in which the Newton-Raphson algorithm can be still used) are shown green. The dashed lines r_a and r_b are the tangent lines to the meridian function at the points P_a and P_b, respectively. The graph $\tilde{f}(\Phi)$ has been obtained with the following set of parameters: $M = 1$, $m = 3$, $\alpha = 1.5$, $p_\mathrm{c} = 100\,\mathrm{MPa}$, and $c = 10\,\mathrm{MPa}$.

in general fails to converge. Nevertheless, it is possible to demonstrate that, for the squared BP yield function, a non-convex region exists in which the Newton-Raphson method still converges, despite the non-convexity. The region is delimited by the above-introduced cutoff plane, which position can be determined as follows.

Position of the cutoff plane

Let us consider the situation sketched in Figure 9.5. The generic points $P_\mathrm{a} = \big(\Phi_a, \tilde{f}(\Phi_a)\big)$ and $P_\mathrm{b} = \big(\Phi_b, \tilde{f}(\Phi_b)\big)$ lie on the meridian function, so that it is possible to calculate in those points the tangents r_a and r_b as

$$r_\mathrm{a}\colon\ \tilde{f}(\Phi) = \tilde{f}(\Phi_\mathrm{a}) + \tilde{f}^{'}(\Phi_\mathrm{a})(\Phi - \Phi_\mathrm{a}) \tag{9.16}$$

and

$$r_\mathrm{b}\colon\ \tilde{f}(\Phi) = \tilde{f}(\Phi_\mathrm{b}) + \tilde{f}^{'}(\Phi_\mathrm{b})(\Phi - \Phi_\mathrm{b}) \tag{9.17}$$

referred to the points P_a and P_b respectively, and where a prime denotes the derivative with respect to Φ. If we impose that $\big(\Phi_\mathrm{a}, 0\big) \in r_\mathrm{b}$ and $\big(\Phi_\mathrm{b}, 0\big) \in r_\mathrm{a}$, we obtain the following non-linear algebraic system:

$$\begin{cases} \tilde{f}(\Phi_\mathrm{a}) + \tilde{f}'(\Phi_\mathrm{a})(\Phi - \Phi_\mathrm{a}) = 0 \\ \tilde{f}(\Phi_\mathrm{b}) + \tilde{f}'(\Phi_\mathrm{b})(\Phi - \Phi_\mathrm{b}) = 0 \end{cases} \tag{9.18}$$

with the unknowns Φ_a and Φ_b; these values, that can be calculated numerically, define the region $[\Phi_\mathrm{a}, \Phi_\mathrm{b}]$ in which the Newton-Raphson algorithm can be still used, even though the squared BP yield function is not convex.

As a conclusion, Φ_b is the value defining the position of the cutoff plane, to be used in the sub-incrementation scheme, as shown in Box 9.3.1 (note that Φ_a is not needed, since in the integration algorithm the trial elastic stress always lies outside the elastic domain).

Box 9.3.1: The 'cutoff-substepping' integration algorithm

(1) Given an initial state at step n, described by the variables $\boldsymbol{\sigma}_n$, $\boldsymbol{\varepsilon}^\mathrm{e}_n$, $\boldsymbol{\varepsilon}^\mathrm{p}_n$ and given a strain increment $\Delta\boldsymbol{\varepsilon}$;

(2) Set $\Delta\boldsymbol{\varepsilon}_i = \Delta\boldsymbol{\varepsilon}$ and $m = 1$ (where m defines the substep interval);

(3) INITIALIZE: all variables are set equal to the value at the initial step n;

(4) DO $i = 1, m$;

(5) Evaluate the elastic trial solution

$$\boldsymbol{\sigma}^\text{trial}_{n+1,i} = \boldsymbol{\sigma}_n + \mathbb{E}[\Delta\boldsymbol{\varepsilon}_i]$$

(6) Calculate $\Phi^\text{trial}_{n+1,i} = \frac{p_{n+1,i}+c_n}{p_{\mathrm{c},n}+c_n}$ and Φ_b by solving equation (9.18);

(7) Check position with respect to the cutoff plane
IF $\Phi^\text{trial}_{n+1,i} \leq \Phi_\mathrm{b}$ GOTO Standard Return Mapping;

(8) Substepping procedure
ELSE $m = 2m$ AND $\Delta\boldsymbol{\varepsilon}_i = \frac{\Delta\boldsymbol{\varepsilon}}{m}$;

(9) GOTO (3)

Table 9.1: Deformation steps $\Delta\varepsilon$ used for comparing the performance of the centre-of-mass integration algorithm with the return mapping, the latter performed on the squared version of the BP yield function.

	Test specifications	Deformation $\Delta\varepsilon_1$	Deformation $\Delta\varepsilon_2 = \Delta\varepsilon_3$
Test 1	Isotropic compression	-0.024	-0.024
Test 2	Isotropic traction	$0.000\,137\,14$	$0.000\,137\,14$
Test 3	Negative uniaxial deformation	$-0.008\,072\,8$	0
Test 4	Positive uniaxial deformation	$0.000\,373\,12$	0
Test 5	Triaxial compression	$-0.009\,283\,9$	$-0.018\,567\,8$
Test 6	Triaxial extension	$-0.006\,091$	$-0.012\,182$
		$\Delta\varepsilon_1 = -\Delta\varepsilon_2$	$\Delta\varepsilon_3$
Test 7	Shear	$0.000\,784\,08$	0

9.4 The numerical performance: finite step accuracy

The numerical performance of the centre-of-mass integration technique has been tested by comparing results obtained for a prescribed finite step of deformation (taken in different directions in the space of symmetric tensors as elucidated in Table 9.1) with those obtained with the cutting-plane return-mapping technique [119] applied to the 'squared-version' of the BP yield surface, without sub-incrementation. In this way, it will become evident that for certain values of the trial elastic stress convergence will not occur for the latter algorithm.

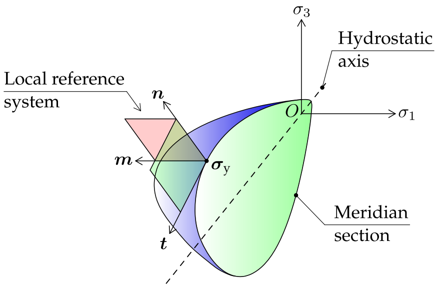
The comparison between the two integration algorithms has been performed by assuming:

- a form of the yield surface, namely,

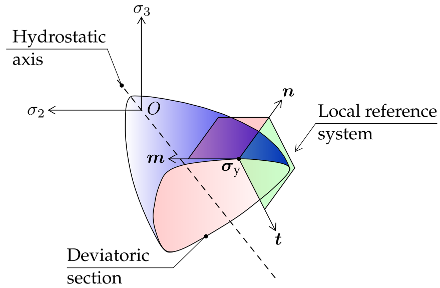
$$\begin{array}{llll} M = 0.26 & m = 2 & \alpha = 1.99 & \beta = 0.12 \\ \gamma = 0.98 & p_{\mathrm{c}} = 350\,\mathrm{MPa} & c = 2\,\mathrm{MPa} & \end{array} \tag{9.19}$$

- elastic parameters in terms of Lamé constants $\lambda = 2669.49\,\mathrm{MPa}$ and $\mu = 4745.76\,\mathrm{MPa}$;

- linear strain-hardening.



(a) *View of a meridian section.*



(b) *View of a deviatoric section.*

Figure 9.6: Sections of the yield surface and local reference system employed for the construction of the iso-error maps.

Note that the above parameters have been selected to be representative of a concrete-like material and the linear-hardening elastoplastic model has been implemented as a UMAT routine for Simulia Abaqus®.

The strain steps prescribed in Table 9.1 for testing the capability of the integration algorithms and the corresponding trial elastic stresses are reported together with the strain-space and stress-space representations of the BP meridian sections, respectively in the upper and lower parts of Figure 9.7, where θ is the Lode's angle (9.4c). The trial elastic stresses in the deviatoric plane of the BP yield surface are reported in the central part of Figure 9.7.

Note that the prescribed trial stresses have been given so that, in all cases, exactly the 20% of its norm lies outside the elastic domain, $\left\|\boldsymbol{\sigma}^{\mathrm{trial}}\right\| = 1.2\left\|\boldsymbol{\sigma}_{\mathrm{y}}\right\|$. Results, in terms of stress and plastic strain reached at the end of the procedure, are reported in Table 9.2 for Tests 1 to 6, while results of the Test 7 are reported in tables 9.3 and 9.4. In addition to the two algorithms under testing, a so-called 'exact' result has also been included. This is obtained through successive subdivision of the strain increment into a sufficiently large number of sub-increments to achieve convergence within a high tolerance (so that the relative error between the last two sub-increments lies below 1×10^{-6}).

For the isotropic compression deformation path ('test 1') the return mapping algorithm fails to converge, as a consequence of the lack of convexity of the squared-version of the BP yield function, and therefore results are not reported

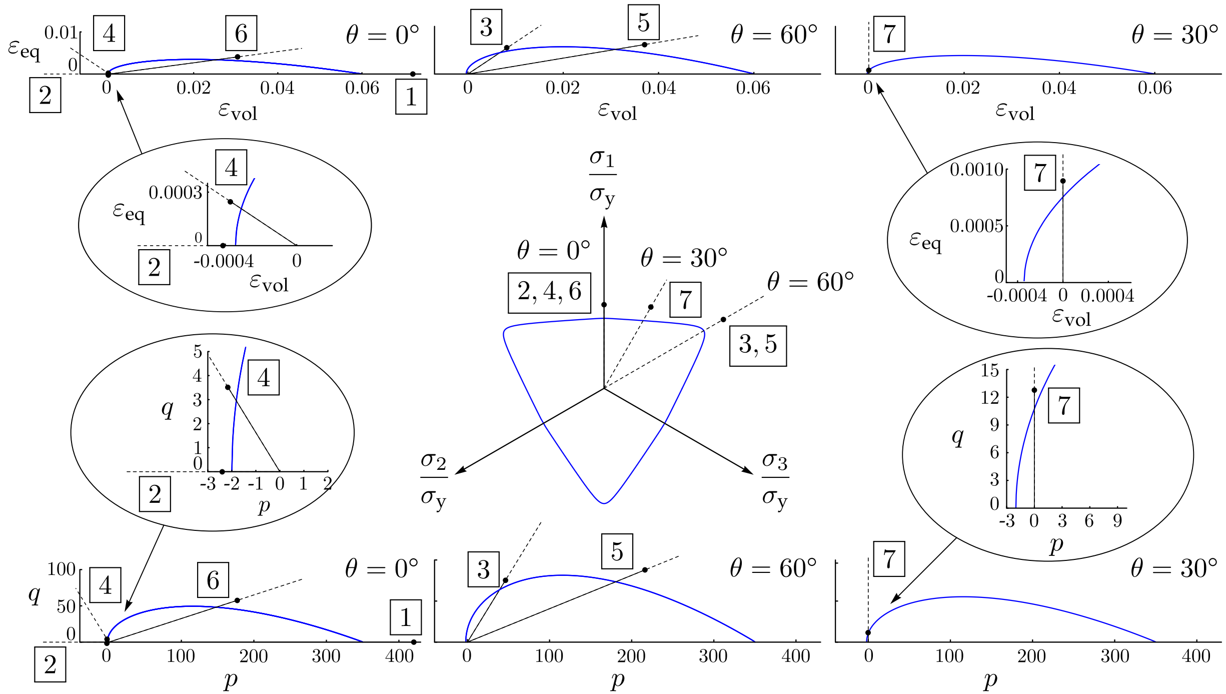


Figure 9.7: Prescribed finite strain steps in the strain-space (upper part) and corresponding elastic trial stresses in the deviatoric plane (central part) and meridian plane (lower part) of the stress-space, for Tests 1 to 7 reported in Table 9.1. Finite steps are prescribed in such a way that the norm of the trial stress exceeds by 20% the norm of the corresponding yield stress along the radial path from the origin to the trial stress.

Table 9.2: Stress and plastic strain at the end of the finite step calculated with different algorithms for the strain and stress paths 1-6 of Table 9.1, graphically represented in Figure 9.7.

	Method	Stress [MPa]		Error	Plastic strain		Error
		σ_1	$\sigma_2 = \sigma_3$	%	$\varepsilon_1^{\mathrm{p}}$	$\varepsilon_2^{\mathrm{p}} = \varepsilon_3^{\mathrm{p}}$	%
Test 1	Centre of mass	-384.8	-384.8	0.05	-2.0103×10^{-3}	-2.0103×10^{-3}	0.53
	Return mapping	-420.0	-420.0	9.15	0	0	100
	Exact	-384.8	-384.8		-2.0210×10^{-3}	-2.0210×10^{-3}	
Test 2	Centre of mass	2.002	2.002	0.00	2.2726×10^{-5}	2.2726×10^{-5}	0.00
	Return mapping	2.002	2.002	0.00	2.2726×10^{-5}	2.2726×10^{-5}	0.00
	Exact	2.002	2.002		2.2726×10^{-5}	2.2726×10^{-5}	
Test 3	Centre of mass	-95.56	-25.88	0.89	-3.7303×10^{-4}	3.5937×10^{-4}	8.56
	Return mapping	-94.56	-25.23	0.45	-4.4081×10^{-4}	3.2748×10^{-4}	4.25
	Exact	-94.89	-25.45		-4.1833×10^{-4}	3.3810×10^{-4}	
Test 4	Centre of mass	4.029	0.616	0.23	3.3229×10^{-5}	1.9628×10^{-5}	2.14
	Return mapping	4.015	0.628	0.31	3.4796×10^{-5}	1.8526×10^{-5}	2.93
	Exact	4.023	0.621		3.3895×10^{-5}	1.9159×10^{-5}	
Test 5	Centre of mass	-191.4	-261.2	0.04	5.2867×10^{-4}	-1.3881×10^{-3}	0.16
	Return mapping	-191.5	-261.4	0.09	5.3819×10^{-4}	1.0881×10^{-3}	1.08
	Exact	-191.3	-261.2		5.2705×10^{-4}	-1.3947×10^{-3}	
Test 6	Centre of mass	-193.0	-146.1	0.11	-4.0150×10^{-4}	7.4916×10^{-4}	1.52
	Return mapping	-192.8	-145.9	0.00	-4.1020×10^{-4}	7.3847×10^{-4}	0.22
	Exact	-192.8	-145.9		-4.1020×10^{-4}	7.3847×10^{-4}	

Table 9.3: Stress at the end of the finite step calculated with different algorithms for the strain and stress path 7 of Table 9.1, graphically represented in Figure 9.7.

	Method	Stress [MPa]			Error
		σ_1	σ_2	σ_3	%
Test 7	Centre of mass	6.460	-7.826	-0.433	0.61
	Return mapping	6.441	-7.741	-0.354	0.54
	Exact	6.450	-7.782	-0.391	

Table 9.4: Plastic strain at the end of the finite step calculated with different algorithms for the strain and stress path 7 of Table 9.1, graphically represented in Figure 9.7.

	Method	Plastic strain			Error
		$\varepsilon_1^{\mathrm{p}}$	$\varepsilon_2^{\mathrm{p}}$	$\varepsilon_3^{\mathrm{p}}$	%
Test 7	Centre of mass	7.4599×10^{-5}	1.1561×10^{-5}	1.6671×10^{-5}	6.64
	Return mapping	7.8884×10^{-5}	4.9236×10^{-4}	1.0745×10^{-5}	5.92
	Exact	7.6856×10^{-5}	8.0822×10^{-6}	1.3524×10^{-5}	

in the table.

Iso-error maps have been plotted to display the error trend of the two algorithms in the stress-space for a set of different strain increments, chosen with the condition that the trial elastic solutions $\boldsymbol{\sigma}^{\text{trial}}$ lie respectively in the meridian (defined by the basis $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ in Figure 9.6a) and deviatoric (defined by the basis $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$ in Figure 9.6b) planes.

The iso-error maps plotting ranges have been chosen as follows:

$$0 \leq \frac{\Delta\sigma_n^{\text{trial}}}{|\boldsymbol{\sigma}_{\mathrm{y}}|} \leq 0.2 \tag{9.20a}$$

$$-0.2 \leq \frac{\Delta\sigma_t^{\text{trial}}}{|\boldsymbol{\sigma}_{\mathrm{y}}|} \leq 0.2 \tag{9.20b}$$

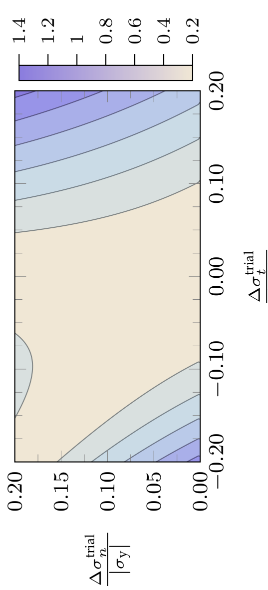
$$-0.2 \leq \frac{\Delta\sigma_m^{\text{trial}}}{|\boldsymbol{\sigma}_{\mathrm{y}}|} \leq 0.2 \tag{9.20c}$$

where $\boldsymbol{\sigma}_{\mathrm{y}}$ is the considered stress at yielding, and

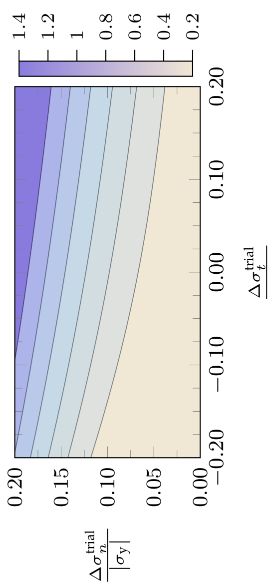
$$\Delta\boldsymbol{\sigma}^{\text{trial}} = \Delta\sigma_t^{\text{trial}}\boldsymbol{t} + \Delta\sigma_n^{\text{trial}}\boldsymbol{n} + \Delta\sigma_m^{\text{trial}}\boldsymbol{m} \tag{9.21}$$

The iso-error maps are reported in figures 9.8-9.11, assuming as yield stresses $\boldsymbol{\sigma}_{\mathrm{y}}$ those corresponding to the Tests 3-6 of Table 9.1, graphically represented in Figure 9.7.

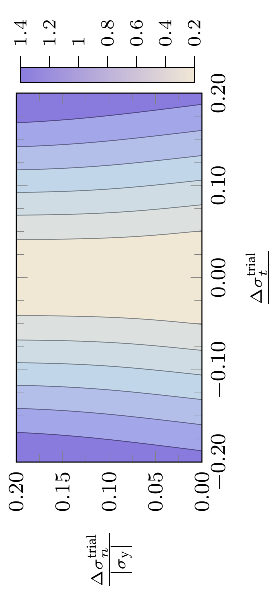
It can be noted from figures 9.8c and 9.10c that the centre-of-mass algorithm has a low accuracy when the yield stress $\boldsymbol{\sigma}_{\mathrm{y}}$ lies near the corner of the deviatoric section (see Figure 9.7, central part, Tests 3 and 5) and the stress increment is not radial. On the other hand, the accuracy is high in both planes defined by the basis $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ and $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$, when the yield stress $\boldsymbol{\sigma}_{\mathrm{y}}$ lies near the flat parts of this section (see Figure 9.7, central part, Tests 4 and 6), as shown in figures 9.9 and 9.11.



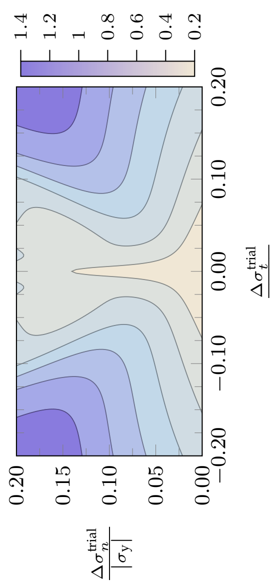
(a) *Centre of mass,* $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ *plane.*



(b) *Return mapping,* $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ *plane.*

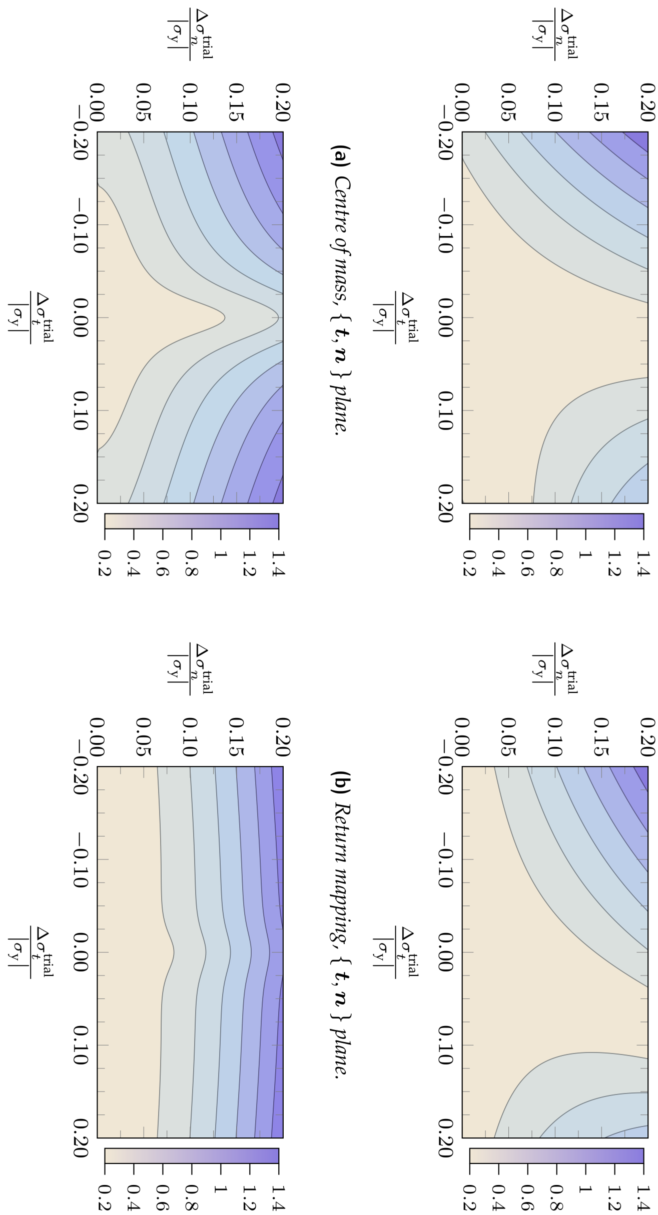


(c) *Centre of mass,* $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$ *plane.*



(d) *Return mapping,* $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$ *plane.*

Figure 9.8: Iso-error maps for Test 3 (see Table 9.1 and Figure 9.7).



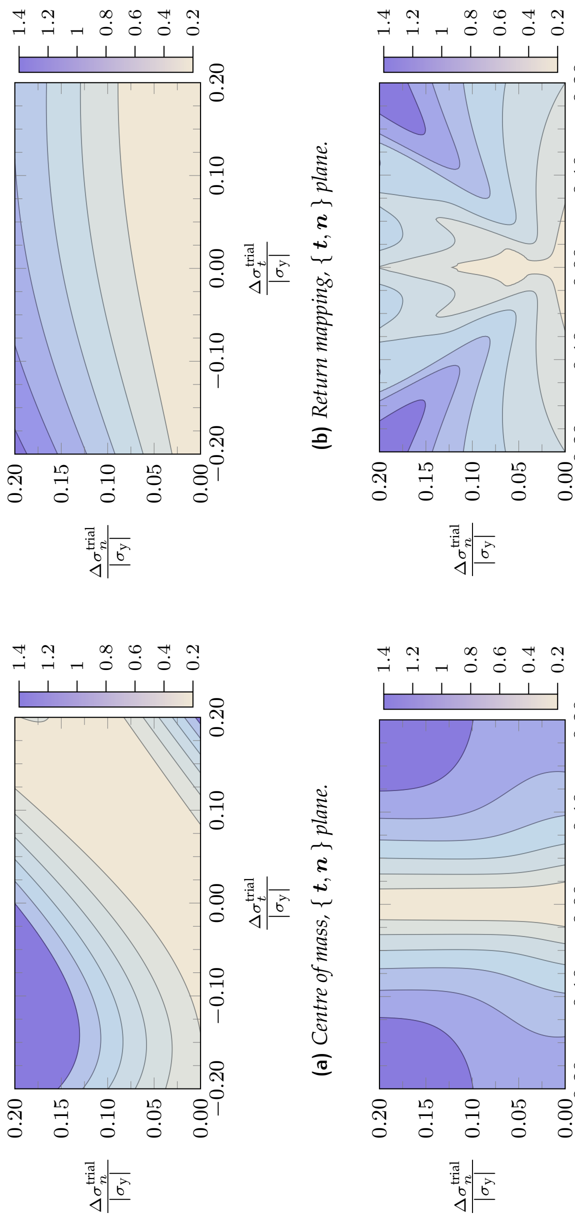
(a) *Centre of mass, $\{ \boldsymbol{t}, \boldsymbol{n} \}$ plane.*

(b) *Return mapping, $\{ \boldsymbol{t}, \boldsymbol{n} \}$ plane.*

(c) *Centre of mass, $\{ \boldsymbol{m}, \boldsymbol{n} \}$ plane.*

(d) *Return mapping, $\{ \boldsymbol{m}, \boldsymbol{n} \}$ plane.*

Figure 9.9: Iso-error maps for Test 4 (see Table 9.1 and Figure 9.7).



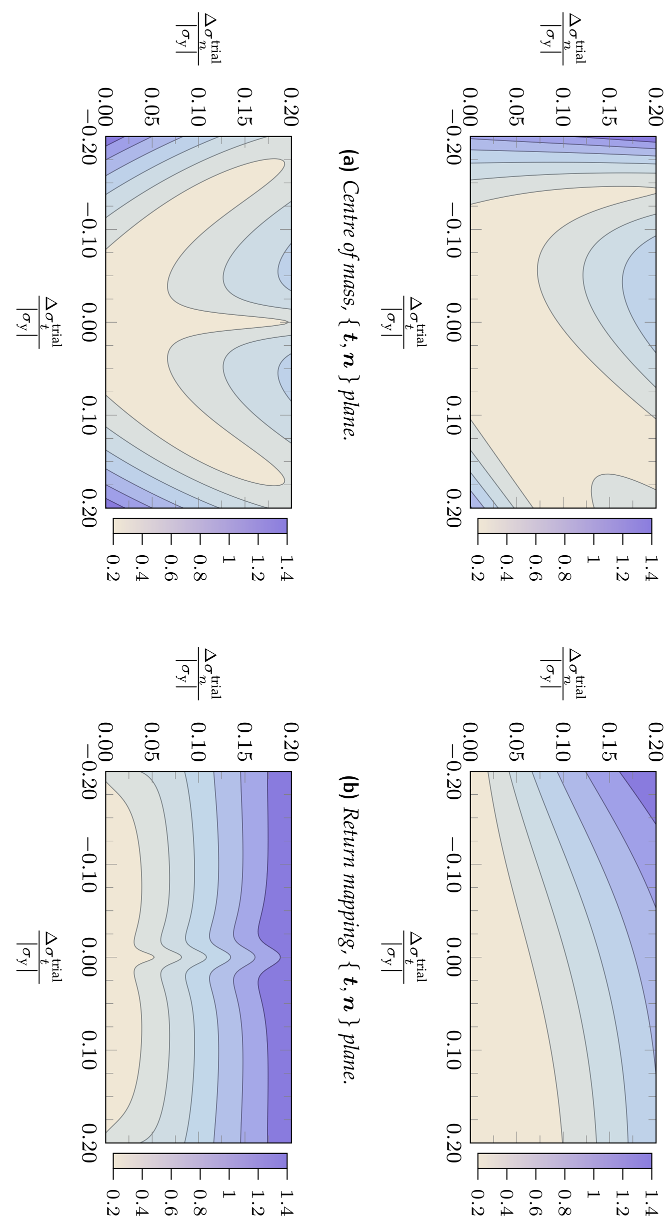
(a) *Centre of mass, $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ plane.*

(b) *Return mapping, $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ plane.*

(c) *Centre of mass, $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$ plane.*

(d) *Return mapping, $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$ plane.*

Figure 9.10: Iso-error maps for Test 5 (see Table 9.1 and Figure 9.7).



(a) *Centre of mass, $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ plane.*

(b) *Return mapping, $\{\, \boldsymbol{t}, \boldsymbol{n} \,\}$ plane.*

(c) *Centre of mass, $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$ plane.*

(d) *Return mapping, $\{\, \boldsymbol{m}, \boldsymbol{n} \,\}$ plane.*

Figure 9.11: Iso-error maps for Test 6 (see Table 9.1 and Figure 9.7).

Chapter 10

Comparison with semi-analytical solutions

Two interesting applications for the algorithms described in the previous Chapter are the problems of the compaction of a ceramic powder against a rigid spherical cup and the expansion of a thick spherical shell made up of a green body. The exact solutions are derived, and a comparison in terms of iso-error maps and the results provided by the two algorithms show that both the proposed algorithms perform correctly and accurately.

Numerical results obtained by employing the proposed algorithms have been compared with semi-analytical solutions of a simple compaction problem and a deformation of a green body. In particular, in Section 10.2, the forming of a thick perfectly-plastic layer of ceramic powder is considered, pressed against a rigid spherical cup, see Figure 10.2a on page 289. Moreover, a thick spherical shell of a green body is considered in Section 10.3, subjected to an internal uniform pressure with a traction-free external boundary and expanded until collapse, corresponding to complete plasticization, see Figure 10.2b on page 289. Due to the spherical symmetry of both the problems, it is possible in both cases to obtain accurate semi-analytical solutions for the stress field by direct numerical integration of the equilibrium equations.

These benchmark problems, differing only in the boundary conditions, are used to check the accuracy and efficiency of the proposed algorithms. They represent only model problem simulations of industrial processes and cannot

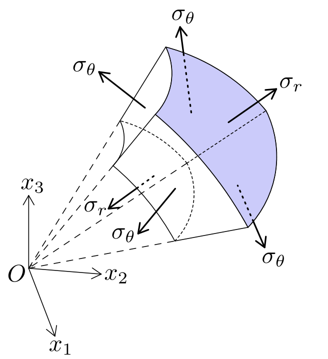


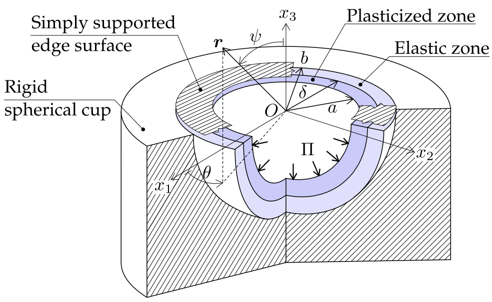
Figure 10.1: Reference system and stress components for the spherical shell.

be considered fully realistic, since hardening (and therefore the evolution of the yield surface) is neglected, so that the increase in cohesion is not taken into account.

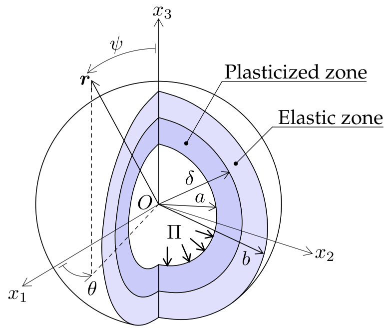
The problem of the expansion of a thick spherical shell is interesting in itself, due to the applications in geotechnics, and it has been previously solved under a number of hypotheses [124–128], although never with the BP yield function. The problem of compaction of a layer of powder against a rigid cup was previously not addressed in analytically.

For both problems, the inner and outer radii of the shell are denoted with a and b respectively, while the internal pressure is Π, which is assumed to increase from zero to the maximum value corresponding to the full plasticization of the shell. Since the geometry shows radial symmetry, we assume a spherical coordinate system (r, θ, ϕ). The solution is known in the case of perfect plasticity with the Tresca yield criterion [126], so that our objective is to generalize the solution to the BP yield criterion.

Due to the spherical symmetry, the stress and deformation depend only on the radius r, see Figure 10.1. The non-vanishing radial, azimuthal, and polar



(a) *Compaction of a thick perfectly-plastic layer of ceramic powder against a rigid cup.*



(b) *Expansion of a thick perfectly-plastic spherical shell under internal pressure.*

Figure 10.2: Geometry and reference system for the benchmark problems of compaction (a) and expansion (b) of perfectly-plastic spherical shells under internal pressure. In both cases, the boundary of the plasticized zone is represented by the radius δ which moves from $r = a$ to $r = b$ at increasing internal pressure Π.

deformation components are respectively

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} \tag{10.1a}$$

$$\varepsilon_\theta = \varepsilon_\phi = \frac{u}{r} \tag{10.1b}$$

where u is the radial displacement. The compatibility equation is

$$\varepsilon_r = \frac{\mathrm{d}}{\mathrm{d}r}(r\varepsilon_\theta) \tag{10.2}$$

while the equilibrium equation in spherical coordinates is

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \tag{10.3}$$

to be complemented by the boundary conditions

The elastic constitutive equations are

$$\varepsilon_r = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta), \tag{10.4a}$$

$$\varepsilon_\theta = \frac{1}{E}\big[(1-\nu)\sigma_\theta - \nu\sigma_r\big] \tag{10.4b}$$

where E is the elastic Young modulus and ν the Poisson's ratio. The Tresca yield criterion coincides (under the current assumptions) with the von Mises criterion, which can be written as

$$|\sigma_\theta - \sigma_r| - \sigma_{\mathrm{y}} = 0 \tag{10.5}$$

where σ_{y} is the uniaxial yield stress, while the BP yield criterion (9.1) writes now in the following form

$$F(\boldsymbol{\sigma}) = f\Big(\frac{\sigma_r + 2\sigma_\theta}{3}\Big) + \frac{|\sigma_r - \sigma_\theta|}{g(\frac{\pi}{3})} = 0 \tag{10.6}$$

10.1 The elastic solution

Using equations (10.2), (10.3) and (10.4) we obtain

$$\frac{1-\nu}{2}\frac{\mathrm{d}}{\mathrm{d}r}(\sigma_r + 2\sigma_\theta) = 0 \tag{10.7}$$

This equation together with (10.3) forms a system of ODEs, that can be solved exactly and the solution is given by

$$\sigma_r(r) = \frac{C_1}{3} + \frac{C_2}{r^3} \tag{10.8a}$$

$$\sigma_\theta(r) = \frac{C_1}{3} - \frac{C_2}{2r^3} \tag{10.8b}$$

where C_1 and C_2 are constants to be defined through the boundary conditions (BC). The associated deformation and displacement fields are obtained

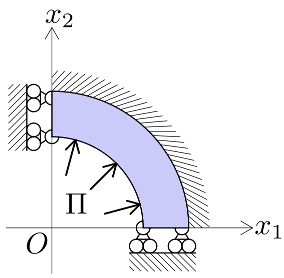


Figure 10.3: Schematic representation of the model of the rigid spherical cup. Due to the spherical symmetry, the problem can be reduced to the analysis of a quarter of anulus; note the disposition of the constraints: the anulus is simply supported at the edges and is confined (clamped) along the external surface.

from (10.4) and (10.1) and read

$$\varepsilon_r(r) = \frac{1}{E}\left[(1-2\nu)\frac{C_1}{3} + (1+\nu)\frac{C_2}{r^3}\right] \tag{10.9a}$$

$$\varepsilon_\theta(r) = \frac{1}{E}\left[(1-2\nu)\frac{C_1}{3} - (1+\nu)\frac{C_2}{2r^3}\right] \tag{10.9b}$$

$$u(r) = \frac{1}{E}\left[(1-2\nu)\frac{C_1}{3}r - (1+\nu)\frac{C_2}{2r^2}\right] \tag{10.9c}$$

10.2 Compaction of a thick layer of perfectly-plastic material obeying the BP yield condition against a rigid spherical cup

For the compaction problem of a thick layer against a rigid spherical cup, figures 10.2a and 10.3, the boundary conditions write as follows

$$\sigma_r\Big|_{r=a} = -\Pi \tag{10.10a}$$

$$u\Big|_{r=b} = 0 \tag{10.10b}$$

where Π is the internal pressure. The material parameters defining the shape of the BP yield surface have been chosen to be representative of alumina powder [81, 82], namely

$$
\begin{aligned}
&M = 1.1 \qquad && m = 2 \qquad && \alpha = 0.1 \qquad && \beta = 0.19 \\
&\gamma = 0.9 \qquad && p_c = 40\,\mathrm{MPa} \qquad && c = 1.5\,\mathrm{MPa} &&
\end{aligned}
\tag{10.11}
$$

Note that, since hardening and increasing of cohesion are neglected, we assume an initial state corresponding to an intermediate stage of a densification process.

10.2.1 The elastic solution

Initially the problem is purely elastic, which occurs when the internal pressure is sufficiently small, say, $\Pi \leq \Pi_y$, where Π_y is defined as the inner pressure producing the initiation of yielding at the inner radius of the shell.

The solution (10.8)–(10.9c) together with boundary conditions (10.10), provides the following stress field within the thick spherical layer, $a \leq r \leq b$,

$$\sigma_r^e(\Pi, r) = -\frac{a^3(1+\nu)\Pi}{a^3(1+\nu) + 2b^3(1-2\nu)} - \frac{2a^3b^3(1-2\nu)\Pi}{a^3(1+\nu) + 2b^3(1-2\nu)}\frac{1}{r^3} \tag{10.12a}$$

$$\sigma_\theta^e(\Pi, r) = -\frac{a^3(1+\nu)\Pi}{a^3(1+\nu) + 2b^3(1-2\nu)} + \frac{a^3b^3(1-2\nu)\Pi}{a^3(1+\nu) + 2b^3(1-2\nu)}\frac{1}{r^3} \tag{10.12b}$$

For the von Mises yield criterion (VM), the critical yield pressure Π_y is represented by the stress state satisfying

$$|\sigma_r^e - \sigma_\theta^e| = \sigma_y \tag{10.13}$$

and can be evaluated as

$$\Pi_y = \frac{\sigma_0}{3}\left[2 + \frac{1+\nu}{1-2\nu}\left(\frac{a}{b}\right)^3\right] \tag{10.14}$$

In the following calculations $\nu = 0.26$ has been assumed. For the BP yield criterion, the critical yield pressure Π_y corresponds to a stress state satisfying

$$\max_{a \leq r \leq b} F\big(\sigma_r^e(\Pi_y, r), \sigma_\theta^e(\Pi_y, r)\big) = 0 \tag{10.15}$$

so that Π_{y} can be evaluated as the numerical solution of the above equation and it can be numerically shown that the plasticization starts from the inner surface of the layer, $r = a$.

10.2.2 The elastoplastic solution

The elastoplastic solution holds for an internal pressure $\Pi > \Pi_{\mathrm{y}}$, which implies both elastic and plastic deformation of the layer. The plastic flow starts from the inner surface of the layer and propagates within a spherical region with inner radius a and outer δ and moving toward b. The remaining part of the layer, namely, for $\delta \leq r \leq b$, behaves as an elastic layer with inner radius δ and outer b, subject to an internal pressure Π_δ at the interface with the plasticized zone.

Assuming that the yield pressure at the interface $r = \delta$ is Π_δ, a generic yield criterion writes as

$$F\big(\sigma_r^{\mathrm{e}}(\Pi_\delta, \delta), \sigma_\theta^{\mathrm{e}}(\Pi_\delta, \delta)\big) = 0 \tag{10.16}$$

which provides a relation between δ and Π_δ. For example, the pressure at the interface for the von Mises criterion can be obtained from equation (10.14) imposing $a = \delta$ as

$$\Pi_\delta = \frac{\sigma_0}{3}\left[2 + \frac{1+\nu}{1-2\nu}\left(\frac{\delta}{b}\right)^3\right] \tag{10.17}$$

whereas for the BP criterion the pressure Π_δ has to be evaluated numerically.

The solution for the elastic zone ($\delta \leq r \leq b$) can be obtained from equations (10.12a) and (10.12b) where a and Π are replaced, respectively, by δ and Π_δ which are given by (10.16), so that the stresses become

$$\sigma_r^{\mathrm{ep}}(r) = -\frac{\delta^3(1+\nu)\Pi_\delta}{\delta^3(1+\nu) + 2b^3(1-2\nu)} - \frac{2\delta^3 b^3(1-2\nu)\Pi_\delta}{\delta^3(1+\nu) + 2b^3(1-2\nu)}\frac{1}{r^3} \tag{10.18a}$$

$$\sigma_\theta^{\mathrm{ep}}(r) = -\frac{\delta^3(1+\nu)\Pi_\delta}{\delta^3(1+\nu) + 2b^3(1-2\nu)} + \frac{\delta^3 b^3(1-2\nu)\Pi_\delta}{\delta^3(1+\nu) + 2b^3(1-2\nu)}\frac{1}{r^3} \tag{10.18b}$$

Hence the elastic part of the solution is known as the relation between the radius δ and the pressure Π_δ is known.

The solution for the plasticized zone ($a \leq r \leq \delta$) is obtained from the algebraic-differential system composed by the equilibrium equations (10.3), the boundary conditions (10.10), and the yield condition (10.5) or (10.6) (depending on the criterion assumed). This system writes as

$$\begin{cases} \dfrac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \dfrac{2}{r}\left(\sigma_r - \sigma_\theta\right) = 0 \\ F\big(\sigma_r(r), \sigma_\theta(r)\big) = 0 \\ \sigma_r\Big|_{r=a} = -\Pi \\ \sigma_r\Big|_{r=\delta} = -\Pi_\delta \end{cases} \tag{10.19}$$

which has been solved analytically for von Mises yield and numerically for the BP yield function. In particular, the system (10.19) admits for von Mises the following solution

$$\sigma_r^{\mathrm{ep}}(r) = -\frac{\sigma_{\mathrm{y}}}{3}\left[2 + \frac{1+\nu}{1-2\nu}\left(\frac{\delta}{b}\right)^3 + 6\ln\left(\frac{\delta}{r}\right)\right] \tag{10.20a}$$

$$\sigma_\theta^{\mathrm{ep}}(r) = -\frac{\sigma_{\mathrm{y}}}{3}\left[-1 + \frac{1+\nu}{1-2\nu}\left(\frac{\delta}{b}\right)^3 + 6\ln\left(\frac{\delta}{r}\right)\right] \tag{10.20b}$$

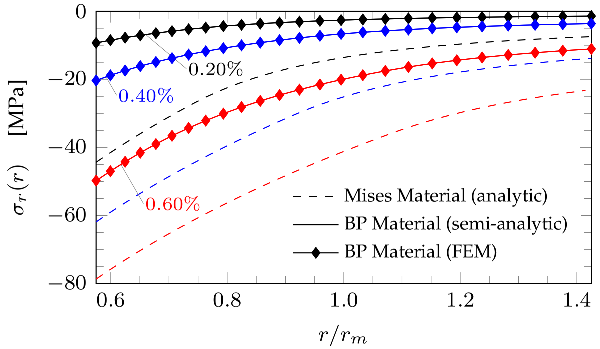
and the relation between δ and the internal pressure Π writes as

$$\Pi = \frac{\sigma_{\mathrm{y}}}{3}\left[2 + \frac{1+\nu}{1-2\nu}\left(\frac{\delta}{b}\right)^3 + 6\ln\left(\frac{\delta}{a}\right)\right] \tag{10.21}$$

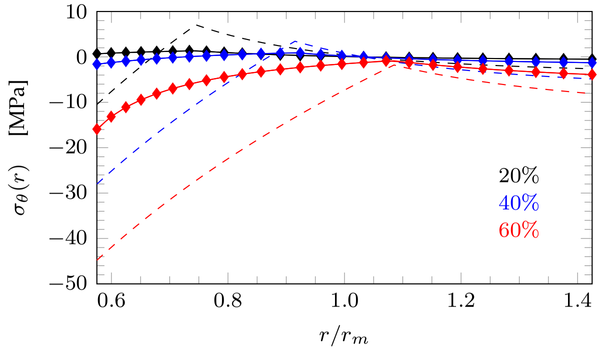
which is a nonlinear relation. Once a fixed value of the radius δ, representing the amplitude of the plasticized zone, is chosen, it is possible to obtain the internal pressure Π and the stresses in every part of the layer, namely for $a \leq r \leq b$.

Results in terms of radial and polar stress components and the two stress invariants p and q are reported in Figure 10.4 as functions of the through-thickness radius (divided by the mean radius $r_m = (a+b)/2$ of the spherical layer)[1], together with the numerical results obtained with the two proposed algorithms. Three different plastic boundaries δ have been considered (corresponding to the 20%, 40% and 60% of the thickness) for both von Mises and the BP yield criterion. Results presented in the figure fully support the validity of the proposed numerical algorithms, which have given coincident results, superimposed on the semi-analytical solution.

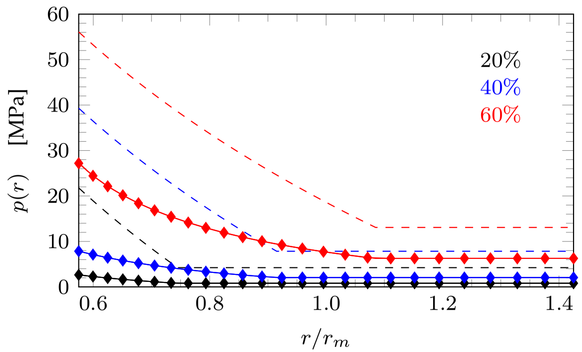
[1] More details on the geometric parameters are given in Section 10.4.



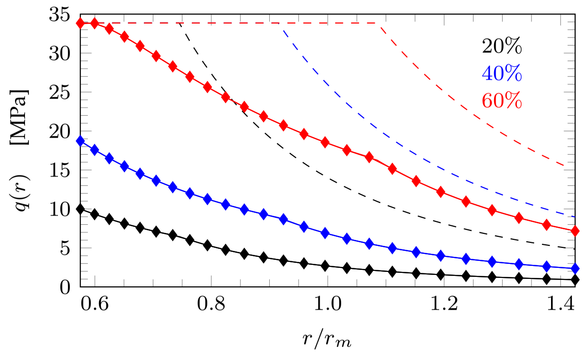
(a) *Radial stress component* σ_r.



(b) *Polar stress component* σ_θ.



(c) *Mean stress* p.



(d) *Deviatoric invariant* q.

Figure 10.4: Compaction of a perfectly-plastic thick layer, obeying von Mises and BP yield conditions, against a rigid spherical cup, representative of a ceramic powder. The quantities are expressed as functions of the dimensionless radial position. Note that for the von Mises criterion $\sigma_y = 33.86$ MPa has been chosen, so that the von Mises cylinder is circumscribed around the BP surface in the stress space.

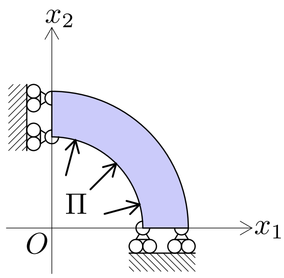


Figure 10.5: Schematic representation of the model of the expansion of a thick spherical shell. Due to the spherical symmetry, the problem can be reduced to the analysis of a quarter of anulus; note the disposition of the constraints: the anulus is simply supported at the edges and its external surface is free, differently from the previous benchmark example.

10.3 The expansion of a perfectly plastic thick shell obeying the BP yield condition

For the problem of expansion of a thick spherical shell subjected to an internal uniform pressure, figures 10.2b and 10.5, the boundary conditions are as follows

$$\sigma_r\Big|_{r=a} = -\Pi \tag{10.22a}$$

$$\sigma_r\Big|_{r=b} = 0 \tag{10.22b}$$

where Π is the internal pressure and the outer boundary is assumed traction-free. The material parameters defining the shape of the BP yield surface have been chosen to be representative of a partially densified ceramic powder, namely

$$\begin{array}{llll} M = 1.33 & m = 2 & \alpha = 1 & \beta = 1 \\ \gamma = 0 & p_\mathrm{c} = 150\,\mathrm{MPa} & c = 150\,\mathrm{MPa} & \end{array} \tag{10.23}$$

The solution of this problem can be obtained with the same method as that described in Section 10.2, since only the boundary conditions are different.

The elastic solution, valid until the internal pressure is sufficiently small,

$\Pi \le \Pi_{\mathrm{y}}$, is given by

$$\sigma_r^{\mathrm{e}}(r) = \frac{\Pi}{\left(\frac{b}{a}\right)^3 - 1}\left[1 - \left(\frac{b}{r}\right)^3\right] \tag{10.24a}$$

$$\sigma_\theta^{\mathrm{e}}(r) = \frac{\Pi}{\left(\frac{b}{a}\right)^3 - 1}\left[1 + \frac{1}{2}\left(\frac{b}{r}\right)^3\right] \tag{10.24b}$$

For the von Mises yield criterion, $|\sigma_r^{\mathrm{e}} - \sigma_\theta^{\mathrm{e}}| = \sigma_{\mathrm{y}}$, the critical yield pressure Π_{y} is obtained as

$$\Pi_{\mathrm{y}} = \frac{2}{3}\sigma_0\left[1 - \left(\frac{a}{b}\right)^3\right] \tag{10.25}$$

whereas for the BP yield criterion, the critical yield pressure Π_{y} is obtained by solving equation (10.15) and it can be numerically proven that the plasticization starts from the inner surface of the shell.

The elastoplastic solution holds for an internal pressure $\Pi > \Pi_{\mathrm{y}}$, which implies both elastic and plastic deformation of the shell. The plastic flow starts from the inner surface of the shell and propagates within a spherical region with inner radius a and outer δ and moving toward b. The remaining part of the shell, namely, for $\delta \le r \le b$, behaves as an elastic shell with inner radius δ and outer b, subject to an internal pressure Π_δ at the interface with the plasticized zone.

The relation between δ and Π_δ is obtained by solving equation (10.16). For the von Mises criterion Π_δ is obtained as

$$\Pi_\delta = \frac{2}{3}\sigma_{\mathrm{y}}\left[1 - \left(\frac{\delta}{b}\right)^3\right] \tag{10.26}$$

whereas for the BP criterion the pressure Π_δ has to be evaluated numerically.

The solution for the elastic zone, $\delta \le r \le b$, is given by

$$\sigma_r^{\mathrm{ep}}(r) = \frac{\Pi_\delta}{\left(\frac{b}{\delta}\right)^3 - 1}\left[1 - \left(\frac{b}{r}\right)^3,\right] \tag{10.27a}$$

$$\sigma_\theta^{\mathrm{ep}}(r) = \frac{\Pi_\delta}{\left(\frac{b}{\delta}\right)^3 - 1}\left[1 + \frac{1}{2}\left(\frac{b}{r}\right)^3\right] \tag{10.27b}$$

The solution for the plasticized zone, $a \le r \le \delta$, is obtained from the algebraic-differential system (10.19). This system has a solution with closed form for the simple case of von Mises yield criterion; in this case the stresses take the form

$$\sigma_r = -\frac{2}{3}\sigma_{\mathrm{y}} \left[1 - \left(\frac{\delta}{b}\right)^3 + \ln\left(\frac{\delta}{r}\right)^3 \right] \tag{10.28a}$$

$$\sigma_\theta = \frac{1}{3}\sigma_{\mathrm{y}} \left[1 + 2\left(\frac{\delta}{b}\right)^3 - 2\ln\left(\frac{\delta}{r}\right)^3 \right] \tag{10.28b}$$

and the relation between δ and the internal pressure Π writes as

$$\Pi = \frac{2}{3}\sigma_{\mathrm{y}} \left[1 - \left(\frac{\delta}{b}\right)^3 + \ln\left(\frac{\delta}{a}\right)^3 \right] \tag{10.29}$$

Once a fixed value of the radius δ representing the amplitude of the plasticized zone is chosen, it is possible to obtain the internal pressure Π and the stresses in every part of the shell, namely for $a \le r \le b$.

Results in terms of radial and polar stress components and the two stress invariants p and q are reported in Figure 10.6 as functions of the through-thickness radius (divided by the mean radius $r_m = (a + b)/2$ of the thick shell)[2], together with the numerical results obtained with the two proposed algorithms. Three different plastic boundaries δ have been considered (corresponding to the 28%, 55% and 86% of the thickness) for both von Mises and the BP yield criterion. Again the two proposed algorithms have given coincident values, superimposed with the semi-analytical solution, thus confirming once more the validity of the presented numerical approaches.

10.4 Notes on the geometric parameters of the benchmarks

The geometry of the problems analyzed in the previous Sections is characterized by three parameters: the inner (a) and the outer (b) radii of the shells and

[2] More details on the geometric parameters are given in Section 10.4.

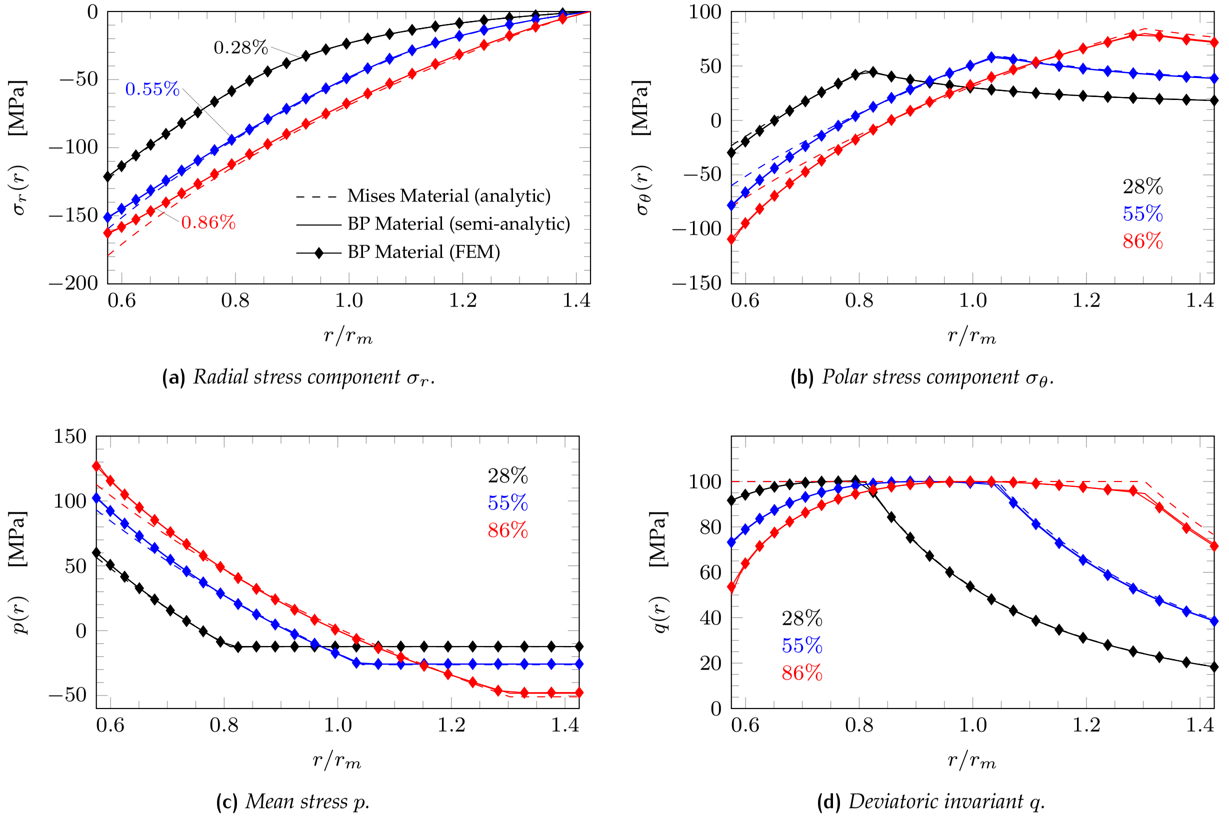


Figure 10.6: Expansion of a perfectly-plastic thick spherical shell, obeying von Mises and BP yield conditions, representative of a green body. The quantities are expressed as functions of the dimensionless radial position. Note that for the von Mises criterion $\sigma_y = 100$ MPa has been chosen, so that the von Mises cylinder is circumscribed around the BP surface in the stress space.

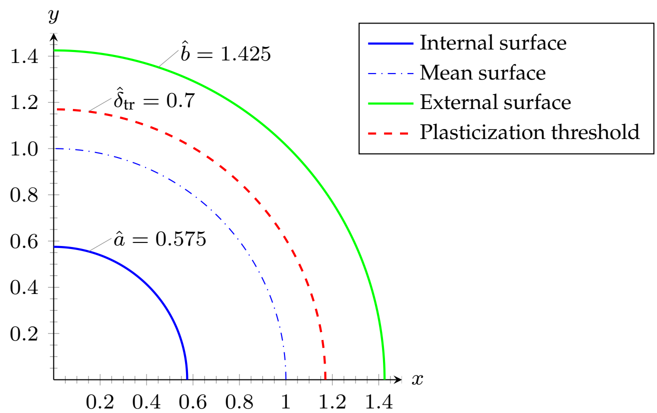


Figure 10.7: Example of a (quarter of) section of a spherical shell with unit mean radius r_{m} and dimensionless thickness $\hat{t} = 0.85$; it follows that the dimensionless inner and outer radii of the shell are equal to $\hat{a} = 0.575$ and $\hat{b} = 1.425$. Here is depicted a plasticization state equal to 70% of thickness of the wall thickness, which corresponds to $\hat{\delta}_{\mathrm{tr}} = 0.7$.

the radius denoting the plasticization boundary (δ). We observe that, independently from the material parameters, the behaviour of the shells (for instance, the possibility to have the complete plasticization of the shells) depends on the slenderness of their walls. Therefore, it is useful to refer to the mean radius r_{m} of the shell and to its thickness $\hat{t}$, whose relations with the inner and outer radii are:

$$r_{\mathrm{m}} = \frac{a+b}{2} \tag{10.30a}$$

$$t = b - a \tag{10.30b}$$

We can reformulate the problems of the compaction and expansion with dimensionless parameters, by dividing equations (10.30) by the mean radius, namely, referring to a *unit mean radius spherical shell*, so that:

$$r_{\mathrm{m}} = 1 \tag{10.31a}$$

$$\hat{t} = \frac{t}{r_{\mathrm{m}}} \qquad \hat{t} \in [0, 2] \tag{10.31b}$$

Note that the limit (but only theoretically admissible) cases $\hat{t} = 0$ and $\hat{t} = 2$ correspond, respectively, to a spherical surface with zero thickness and to a ball. With the above definitions, the inner and outer radii of the unit radius shell can be expressed as:

$$\hat{a} = 1 - \frac{\hat{t}}{2} \qquad \hat{a} \in [0, 1] \tag{10.32a}$$

$$\hat{b} = 1 + \frac{\hat{t}}{2} \qquad \hat{b} \in [1, 2] \tag{10.32b}$$

Furthermore, the plasticization boundary can be described as a fraction of the wall thickness of the shell, starting from the inner surface; the transformation law relating the radius δ to its propagation fraction $\hat{\delta}$ is:

$$\hat{\delta} = \frac{\delta - a}{b - a} \qquad \hat{\delta} \in [0, 1] \tag{10.33}$$

and, consequently, the radius $\hat{\delta}_{\text{tr}}$ corresponding to $\hat{\delta}$ on the unit mean radius shell is:

$$\hat{\delta}_{\text{tr}} = \hat{a} + \hat{t}\hat{\delta} = 1 - \frac{\hat{t}}{2} + \hat{t}\hat{\delta} \qquad \hat{\delta}_{\text{tr}} \in [\hat{a}, \hat{b}] \tag{10.34}$$

After these considerations, we can observe that for the dimensionless problems the solutions in terms of stresses $\sigma_r$and σ_θ are functions of the (dimensionless) radius r belonging to the interval $[\hat{a}, \hat{b}]$. By introducing these dimensionless parameters, we can solve the problems in terms of only two relevant parameters: the state of the plasticization propagation $\hat{\delta} \in [0, 1]$ and the slenderness of the shell, represented by the dimensionless thickness $\hat{t} \in [0, 2]$.

Part III

APPENDIXES AND GLOSSARY

Appendix A

EXPRESSIONS OF INCREMENTAL BIOT MODULI

IN this Appendix we derive the alternative expression for the two incremental moduli μ and μ_*, related to the existence of the deformation energy function, given in Section 4.3.1. To obtain relation (4.28) we consider the spectral representations of the left Cauchy-Green tensor $\boldsymbol{B}$ and of the Cauchy stress $\boldsymbol{\sigma}$:

$$\boldsymbol{B} = \lambda_1^2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda_2^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \lambda_3^2 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \tag{A.1a}$$

$$\boldsymbol{\sigma} = \sigma_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \tag{A.1b}$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ denotes a triplet of eulerian principal axes. If we differentiate equation (A.1) with respect to time, we obtain:

$$\begin{aligned}\dot{\boldsymbol{B}} = 2\lambda_1\dot{\lambda}_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + 2\lambda_2\dot{\lambda}_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + 2\lambda_3\dot{\lambda}_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \\ + \lambda_1^2 \left(\dot{\boldsymbol{e}}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \dot{\boldsymbol{e}}_1\right) + \lambda_2^2 \left(\dot{\boldsymbol{e}}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \dot{\boldsymbol{e}}_2\right) + \\ + \lambda_3^2 \left(\dot{\boldsymbol{e}}_3 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \dot{\boldsymbol{e}}_3\right)\end{aligned} \tag{A.2}$$

and

$$\dot{\boldsymbol{\sigma}} = \dot{\sigma}_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \dot{\sigma}_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \dot{\sigma}_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \\ + \dot{\sigma}_1 \left(\dot{\boldsymbol{e}}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \dot{\boldsymbol{e}}_1\right) + \dot{\sigma}_2 \left(\dot{\boldsymbol{e}}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \dot{\boldsymbol{e}}_2\right) + \\ + \dot{\sigma}_3 \left(\dot{\boldsymbol{e}}_3 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \dot{\boldsymbol{e}}_3\right) \tag{A.3}$$

For an incremental plane strain condition we have $\dot{\boldsymbol{e}}_3 = \boldsymbol{0}$ and $\boldsymbol{e}_1 \cdot \dot{\boldsymbol{e}}_2 = -\dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2$, so that the in-plane components $\dot{B}_{12}$ and $\dot{\sigma}_{12}$ reduce to:

$$\dot{B}_{12} = \left(\lambda_1^2 - \lambda_2^2\right) \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 \tag{A.4a}$$

$$\dot{\sigma}_{12} = \left(\sigma_1 - \sigma_2\right) \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 \tag{A.4b}$$

while equations (4.11) and (4.13) yield

$$\dot{B}_{12} = \left(\lambda_1^2 + \lambda_2^2\right) D_{12} - \left(\lambda_1^2 - \lambda_2^2\right) W_{12} \tag{A.5a}$$

$$\dot{\sigma}_{12} = \overset{\triangledown}{\sigma}_{12} - \left(\sigma_1 - \sigma_2\right) W_{12} \tag{A.5b}$$

If we consider equations (A.4) and (A.5), we have

$$D_{12} = \frac{\left(\dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 + W_{12}\right)\left(\lambda_1^2 - \lambda_2^2\right)}{\lambda_1^2 + \lambda_2^2} \tag{A.6a}$$

$$\overset{\triangledown}{\sigma}_{12} = \left(\dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 + W_{12}\right)\left(\sigma_1 - \sigma_2\right) \tag{A.6b}$$

from which, eliminating the term $\dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 + W_{12}$, we obtain

$$\overset{\triangledown}{\sigma}_{12} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \left(\sigma_1 - \sigma_2\right) D_{12} \tag{A.7}$$

A comparison between (5.20a) and (A.7) yields the Biot expression for μ, equation (4.28a).

The incremental modulus μ_* can be obtained from the derivative with respect to time of equation (3.220) when it is written as a function of the in-plane components and taking account that $\dot{\lambda}_3 = 0$ for incremental deformations in plane stress. Finally we arrive at

$$\dot{\sigma}_1 - \dot{\sigma}_2 = \dot{\lambda}_1 \frac{\partial W}{\partial \lambda_1} + \lambda_1 \dot{\lambda}_1 \frac{\partial^2 W}{\partial \lambda_1^2} + \lambda_1 \dot{\lambda}_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} - \dot{\lambda}_2 \frac{\partial W}{\partial \lambda_2} + \\ - \lambda_2 \dot{\lambda}_1 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} - \lambda_2 \dot{\lambda}_2 \frac{\partial^2 W}{\partial \lambda_2^2} \quad \text{(A.8)}$$

If we use the potential function $\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \frac{1}{\lambda_1 \lambda_2})$ instead of $W(\lambda_1, \lambda_2, \lambda_3)$, we obtain the same relation, with $\hat{W}$ instead of W.

Considering equations (4.11), (A.2), (A.3) and (4.13) for the 11 and 22 components, we obtain the following relations:

$$\dot{\lambda}_i = D_{ii} \lambda_i \qquad i = 1, 2 \quad \text{(A.9a)}$$

$$\overset{\triangledown}{\sigma}_{22} - \overset{\triangledown}{\sigma}_{11} = \dot{\sigma}_{11} - \dot{\sigma}_{22} \quad \text{(A.9b)}$$

which, substituted in equation (A.8), allow us to write

$$\overset{\triangledown}{\sigma}_{22} - \overset{\triangledown}{\sigma}_{11} = \left(\lambda_1 \frac{\partial W}{\partial \lambda_1} + \lambda_2 \frac{\partial W}{\partial \lambda_2} + \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} + \\ - 2 \lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right) \frac{D_{11} - D_{22}}{2} \quad \text{(A.10)}$$

A comparison between equations (5.20d) and (A.10) yields the Biot expression for μ_*,equation (4.28b).

Appendix B

Note on the photoelastic experiment reported in Figure 1.1

PHOTOELASTIC EXPERIMENTS have been performed with a circular (with quarterwave retarders for 560 nm) polariscope (dark field arrangement and equipped with a white and sodium vapor lightbox at $\lambda = 589.3$ nm, purchased from Tiedemann & Betz), designed and manufactured at the Laboratory for Physical Modeling of Structure and Photoelasticity of the Solid and Structure Mechanics Group at University of Trento[1].

Photos have been taken with a Nikon D200 digital camera equipped with a AF-S micro Nikkor (70-180 mm, 1:4.55.6D) lens. The photoelastic material is a 5 mm thick platelet obtained from a commercial two-part epoxy resin (Crystal Resins© by Gedeo, 305 Avenue du pic de Bretagne, 13420 Gemenos, France).

The orthotropic material has been obtained by cutting (with a circular saw, blade HSS-DMo5 $63 \times 0.3 \times 16$ Z128 A) 0.3 mm thick and 2 mm deep parallel grooves (at a distance 2.5 mm) in the resin sample, a technique previously used by O'Regan [133] on photoelastic coatings. The dislocation has been created with two 0.5 mm thick steel platelets in contact to each other at one side and attached to the resin on the other side. The platelets (placed horizontally and aligned parallel to the dashed line in the figure) have been forced to slide each

[1] More details are available on the Research Group website: `http://www.ing.unitn.it/dims/ssmg/`.

against the other to generate the stress field near an edge dislocation.

Appendix C

A relationship between a dislocation dipole and a force dipole

In this Appendix we prove that when the prestress is absent and for isotropic incompressible elasticity, the far-fields induced by a dislocation dipole on a single glide plane and a force dipole inclined at $\pi/4$ are identical.

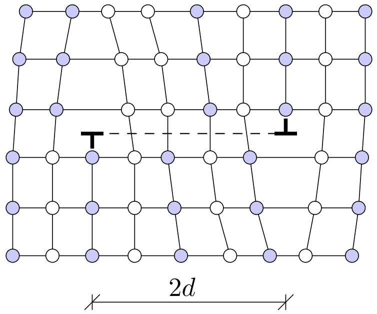
In a Oxy reference system, the in-plane stress field produced by *a single straight edge dislocation* in a linearly elastic isotropic medium is [132]

$$\left\{ \sigma^{\mathrm{d}}_{xx},\, \sigma^{\mathrm{d}}_{yy},\, \sigma^{\mathrm{d}}_{xy} \right\} = \frac{b\,\mu}{2\pi(1-\nu)\,r^4} \times \left\{ -y\left(3x^2+y^2\right),\, y\left(x^2-y^2\right),\, x\left(x^2-y^2\right) \right\} \tag{C.1}$$

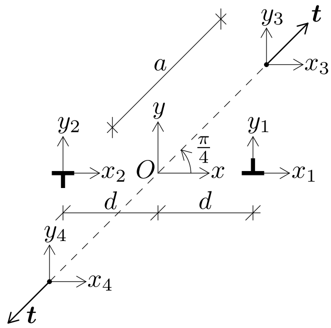
where b is the Burgers vector and μ and ν are the elastic shear modulus and the Poisson ratio. The corresponding strain field follows from equation (C.1) in the form

$$\left\{ \varepsilon^{\mathrm{d}}_{xx},\, \varepsilon^{\mathrm{d}}_{yy},\, \varepsilon^{\mathrm{d}}_{xy} \right\} = \frac{1}{\bar{E}} \left\{ \sigma^{\mathrm{d}}_{xx} - \bar{\nu}\sigma^{\mathrm{d}}_{yy},\, \sigma^{\mathrm{d}}_{yy} - \bar{\nu}\sigma^{\mathrm{d}}_{xx},\, (1+\bar{\nu})\sigma^{\mathrm{d}}_{xy} \right\} \tag{C.2}$$

where $\bar{E}$ and $\bar{\nu}$ are the modified Young modulus and Poisson's ratio, while the



(a) *Lattice distorsion induced by a dislocation dipole.*



(b) *Dislocation and force dipoles.*

Figure C.1: Sketch of the lattice distorsion induced by a dislocation dipole lying on a single slip plane **(a)** and its conventional representation **(b)**, together with a force dipole (inclined at $\pi/4$ with respect to the x_1 axis). In linear isotropic *incompressible* elasticity, the far-fields induced by the dislocation dipole and by the force dipole are identical.

displacement field can be written as

$$u_x = \frac{b}{2\pi}\left[\arctan\frac{y}{x} + \frac{xy}{2(1-\nu)r^2}\right] \tag{C.3a}$$

$$u_y = -\frac{b}{8\pi(1-\nu)}\left[(1-2\nu)\ln r^2 + \frac{x^2-y^2}{r^2}\right] \tag{C.3b}$$

A dislocation dipole on a single glide plane consists of two parallel edge dislocations lying in the same slip plane at a distance $2d$ and having opposite sign (see Figure C.1a, for a sketch of the distortion induced in a crystal lattice, and Figure C.1b, for its graphical conventional representation). In quasi-static conditions, such a simple dislocation structure is not stable, so that the dislocations, unless pinned, 'attract' each other to reduce their total elastic energy. In this way they move towards each other until they combine and annihilate.

Leaving aside issues on stability, the stress field produced by the dislocation dipole can simply be obtained through superposition of solution (C.1), which can be used with reference to the 'local' coordinates (Figure C.1b)

$$x_i = x + (-1)^i d \tag{C.4a}$$

$$y_i = y \tag{C.4b}$$

$$r_i^2 = \left[x + (-1)^i d\right]^2 + y^2 = r^2 + 2(-1)^i x d + d^2 \tag{C.4c}$$

where $i = 1, 2$.

At a large distance form the dislocation dipole, parameter d/r can be considered small and the stress fields can be expanded into a Taylor series, to obtain the far field approximation

$$\left\{ \sigma_{xx}^{\mathrm{d}},\, \sigma_{yy}^{\mathrm{d}},\, \sigma_{xy}^{\mathrm{d}} \right\} \sim \frac{\mu\, d\, b}{\pi(1-\nu)\, r^6} \times \left\{ 2xy\left(r^2 - 4x^2\right),\, 2xy\left(r^2 - 4y^2\right),\, r^4 - 8x^2y^2 \right\} \tag{C.5}$$

which satisfies equilibrium equations.

The far-field stress field for a force dipole in linear elasticity can be obtained by superimposing the Green's stress field

$$\begin{aligned}
T_{11}^1 &= -\frac{[2(1-\nu) + \cos(2\theta)]\cos\theta}{4\pi(1-\nu)\, r} & T_{11}^2 &= -\frac{[2\nu + \cos(2\theta)]\sin\theta}{4\pi(1-\nu)\, r} \\
T_{12}^1 &= -\frac{[2(1-\nu) + \cos(2\theta)]\sin\theta}{4\pi(1-\nu)\, r} & T_{12}^2 &= -\frac{[2(1-\nu) - \cos(2\theta)]\cos\theta}{4\pi(1-\nu)\, r} \\
T_{22}^1 &= \frac{(1-4\nu)\cos\theta + \cos(3\theta)}{8\pi(1-\nu)\, r} & T_{22}^2 &= \frac{[\cos(2\theta) - 2(1-\nu)]\sin\theta}{4\pi(1-\nu)\, r}
\end{aligned} \tag{C.6}$$

each expressed in the local coordinate system shown in Figure C.1b. Eventually we arrive at

$$x_i = x + (-1)^i a\cos\alpha \tag{C.7a}$$

$$y_i = y + (-1)^i a\sin\alpha \tag{C.7b}$$

$$\begin{aligned}
r_i^2 &= \left[x + (-1)^i a\cos\alpha\right]^2 + \left[y + (-1)^i a\sin\alpha\right]^2 \\
&= r^2 + 2(-1)^i a\left(x\cos\alpha + y\sin\alpha\right) + a^2
\end{aligned} \tag{C.7c}$$

where $i = 3, 4$ and $\alpha = \pi/4$.

At a large distance from the force dipole, the dimensionless parameter a/r

becomes small, so that a Taylor series expansion of the stress fields again gives representation (C.5) with the correspondence

$$t\,a = 2d\,b\,\mu \tag{C.8}$$

Appendix D

List of symbols

MATHEMATICAL symbols, functions, operations and operators used in this manuscript are reported and described in this Appendix. These quantities are listed in a schematic way through descriptive tables and subdivided into different categories. The most common symbols and notations used in mathematical books have been adopted in order to have a clear and understandable text.

Table D.1: Groups, numeric sets and imaginary units.

Symbol	Description
Lin	Linear group
Orth	Orthogonal group
Sym	Symmetric group
Skw	Skew-symmetric group
Inv	Set of invertible tensors (general linear group)
$\mathbb{N}$	Natural number set
$\mathbb{Z}$	Integer number set
$\mathbb{Q}$	Rational number set
$\mathbb{R}$	Real number set
$\mathbb{C}$	Complex number set
$\mathbb{H}$	Hypercomplex number set (quaternions)
i	(First) imaginary unit ($\mathbb{C}$, $\mathbb{H}$)
j	Second imaginary unit ($\mathbb{H}$)
k	Third imaginary unit ($\mathbb{H}$)
$\lvert\cdot\rvert$	Absolute value, modulus
$\arg{(\cdot)}$	Argument
$\overline{(\cdot)}$	Conjugated
$\Re(\cdot)$	Real part
$\Im(\cdot)$	Imaginary part ($\mathbb{C}$)
$\mathbb{E}^n$	n-dimensional Euclidean space

Table D.2: Set symbols.

Symbol	Description
$\in, \ni$	Set membership
$\notin$	Non membership
$\subset$	Proper subset
$\supset$	Proper superset
$\subseteq$	Subset
$\supseteq$	Superset
$\cup$	Set-theoretic union
$\cap$	Set-theoretic intersection
$\setminus$	Relative complement or set theoretic difference
$\emptyset$	Empty set
$\complement_{(\cdot)}$	Absolute complement with respect to universe $(\cdot)$
$\wp(\cdot)$	Set of the parts
$\min$	Minimum
$\max$	Maximum
$\inf$	Lower bound
$\sup$	Upper bound

Table D.3: Logic symbols.

Symbol	Description
$\exists$	Existential quantifier
$\exists!$	Uniqueness quantifier
$\nexists$	Existence negation
$\forall$	Universal quantifier
$\vee$	Logic disjunction
$\wedge$	Logic conjunction
$\longrightarrow$	Material implication
$\longleftrightarrow$	Material coimplication
$\Longrightarrow$	Logic implication
$\Longleftrightarrow$	Logic coimplication
$\mid$, :	Such that

Table D.4: Geometric symbols.

Symbol	Description
$A\widehat{B}C$	Angle ABC with vertex in B
$\mathrm{d}(\cdot\,,\cdot)$	Distance between two elements
$\overline{AB}$	Length of the orientated segment AB
$\parallel$	Parallel
$\nparallel$	Non parallel
$\perp$	Perpendicular
$\not\perp$	Non perpendicular
$\equiv$	Equivalence, coincidence
$\cong$	Congruence
$\ncong$	Non congruence
$\approx$	Similarity
$\not\approx$	Non similarity

Table D.5: Trigonometric functions.

Symbol	Function
$\cos(\cdot)$	Cosine
$\sin(\cdot)$	Sine
$\tan(\cdot)$	Tangent
$\sec(\cdot)$	Secant
$\csc(\cdot)$	Cosecant
$\cot(\cdot)$	Cotangent
$\arccos(\cdot)$	Arc cosine
$\arcsin(\cdot)$	Arc sine
$\arctan(\cdot)$	Arc tangent
$\mathrm{arcsec}(\cdot)$	Arc secant
$\mathrm{arccosec}(\cdot)$	Arc cosecant
$\mathrm{arccot}(\cdot)$	Arc cotangent

Table D.6: Exponential functions.

Symbol	Function
$\exp(\cdot)$	Exponential
$\ln(\cdot)$	Natural logarithm
$\cosh(\cdot)$	Hyperbolic cosine
$\sinh(\cdot)$	Hyperbolic sine
$\tanh(\cdot)$	Hyperbolic tangent
$\mathrm{sech}(\cdot)$	Hyperbolic secant
$\mathrm{cosech}(\cdot)$	Hyperbolic cosecant
$\coth(\cdot)$	Hyperbolic cotangent
$\mathrm{arcosh}(\cdot)$	Inverse hyperbolic cosine
$\mathrm{arsinh}(\cdot)$	Inverse hyperbolic sine
$\mathrm{artanh}(\cdot)$	Inverse hyperbolic tangent
$\mathrm{arsech}(\cdot)$	Inverse hyperbolic secant
$\mathrm{arcsch}(\cdot)$	Inverse hyperbolic cosecant
$\mathrm{arcoth}(\cdot)$	Inverse hyperbolic cotangent

Table D.7: Special and non-standard functions.

Symbol	Function
$\mathrm{sign}(\cdot)$	Sign function
$\mathrm{csign}(\cdot)$	Complex sign function
$\mathrm{Ei}(\cdot)$	Exponential integral function
$\mathrm{Ci}(\cdot)$	Cosine integral function
$\mathrm{Si}(\cdot)$	Sine integral function
$\mathrm{hypergeom}(\cdot)$	Hypergeometric function
$\mathrm{Heaviside}(\cdot)$	Heaviside function
$\mathrm{Volume}(\cdot)$	Volume
$\delta(\cdot)$	Dirac delta (distribution)

Table D.8: Vector operators.

Symbol	Operation
$\mathrm{div}(\cdot)$	Divergence (with respect to spatial coordinates)
$\mathrm{Div}(\cdot)$	Divergence (with respect to material coordinates)
$\mathrm{curl}(\cdot)$	Curl (with respect to spatial coordinates)
$\mathrm{Curl}(\cdot)$	Curl (with respect to material coordinates)
$\mathrm{grad}(\cdot)$	Gradient (with respect to spatial coordinates)
$\mathrm{Grad}(\cdot)$	Gradient (with respect to material coordinates)

Table D.9: Vectors, matrices and tensors.

Symbol	Description
$\lvert\cdot\rvert$	Euclidean norm, determinant
$\lVert\cdot\rVert$	Norm
$[\![\cdot]\!]$	Jump
$\langle\cdot\rangle$	Macaulay brakets
$\det(\cdot)$	Determinant
$\mathrm{dev}(\cdot)$	Deviator
$\mathrm{diag}(\cdot)$	Diagonal
$\dim(\cdot)$	Dimension
$\ker(\cdot)$	Kernel
$\mathrm{rank}(\cdot)$	Rank
$\mathrm{tr}(\cdot)$	Trace
$(\cdot)^{-1}$	Inversion (tensor or matrix)
$(\cdot)^{\mathrm{T}}$	Transposition (tensor or matrix)
$\cdot$	Scalar product between vectors or tensors
$\times$	Vector product between vectors
$\otimes$	Tensor product between vectors and/or tensors
$\boxdot$	Tensor product between tensors
$\boxtimes$	Tensor product between tensors
δ_{ij}	Kronecker delta tensor
ε_{ijk}	Ricci or Levi-Civita tensor

Glossary

BC boundary conditions. 70, 290, 321

bijection (or bijective function or one-to-one correspondence) A function between the elements of two sets, where every element of one set is paired with exactly one element of the other set, and every element of the other set is paired with exactly one element of the first set. There are no unpaired elements. 321,

BP Bigoni-Piccolroaz yield function. 265, 268, 274, 288, 291, 296, 321

cardinality Number of objects in a set. 321,

chain rule The chain rule for differentiating a function $L\big(x(t), y(t)\big)$ with respect to t is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial L}{\partial y}\frac{\partial y}{\partial t}$$

. 38, 64, 65, 80, 106, 128, 321,

characteristic equation (of a square matrix) The characteristic equation of a square matrix $\boldsymbol{A}$ is $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = 0$; for a 2×2 matrix, this reduces to $\lambda^2 - \lambda \operatorname{tr} \boldsymbol{A} + \det \boldsymbol{A} = 0$. 21, 321,

codomain (or target set) The set into which all of the output of the function is constrained to fall. 321,

complex conjugate The complex conjugate $\bar{z}$ of a complex number $z = a + \mathrm{i}\, b$ is defined as $\bar{z} = a - \mathrm{i}\, b$ (see complex number). 103, 235, 321,

complex number A number that can be expressed in the form $a + \mathrm{i}\, b$, where a and b are real numbers and i is the imaginary unit (see imaginary unit). 103, 120, 121, 321,

connected space A topological space that cannot be represented as the union of two or more disjoint nonempty open subsets. 139, 140, 321,

domain of definition (or domain of a function) The set of 'input' or argument values for which the function is defined. 321,

E elliptic regime. 103, 321
EC elliptic complex regime. 103, 230, 233, 321
EI elliptic imaginary regime. 103, 230, 321
eigenvalue The eigenvalues of a square matrix $\boldsymbol{A}$ are the scalars λ for which $\boldsymbol{Av} = \lambda\boldsymbol{v}$ for some nonzero vector $\boldsymbol{v}$ (which is called an eigenvector). The eigenvalues are the solutions of the characteristic equation of the square matrix $\boldsymbol{A}$. 19, 20, 27, 31, 321,
eigenvector An eigenvector of a square matrix $\boldsymbol{A}$ is a nonzero vector $\boldsymbol{v}$ satisfying $\boldsymbol{Av} = \lambda\boldsymbol{v}$ for some eigenvalue λ. 19, 27, 31, 321,

Greek letters 321
π Ratio of the circumference of a circle to its diameter. 321

H hyperbolic regime. 103, 231, 321
homothety (or homothecy) A transformation of an affine space determined by a point O (its center) and a nonzero number k (its ratio), which sends any point M to another point N such that the segment $\overline{ON}$ is on the same line as $\overline{OM}$, but scaled by a factor k in accordance with the rule: $M \mapsto O + k\,\overline{OM}$. In Euclidean geometry, homotheties are the similarities that fix a point and either preserve (if $k > 0$) or reverse (if $k < 0$) the direction of all vectors; the one-to-one correspondence between M and N writes as $\overline{ON} = k\,\overline{OM}$. 5, 162, 321,

IC initial conditions. 70, 321
image The subset of a function's codomain which is the output of the function on a subset of its domain. 321,
imaginary unit (i) A mathematical concept which extends the real number system $\mathbb{R}$ to the complex-number system $\mathbb{C}$. The imaginary unit's core property is that $\mathrm{i}^2 = -1$, so that it can be considered as the square root of -1. 321,

Jacobian matrix (of a function) The matrix of all first-order partial derivatives of a function (whose domain and codomain belong to the Euclidean space). Let $F\colon U \to \mathbb{R}^m$ be a function defined on an open set U of the Euclidean space $\mathbb{R}^n$;

the Jacobian matrix calculated at $(x_1, \ldots, x_n)$ is:

$$J_F = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix} = \frac{\partial(F_1, \ldots, F_m)}{\partial(x_1, \ldots, x_n)}$$

namely, is the tensor product between the vector differential operator Nabla (∇) and the function itself. The Jacobian is the determinant of the Jacobian matrix when $m = n$ (see matrix and Nabla). 247, 321,

LaTeX A typesetting language. ii, 321,

matrix A $n \times m$ matrix $\boldsymbol{A}$, where $n, m \in \mathbb{N} \setminus \{ 0 \}$, is an array of quantities arranged into n rows and m columns. The entry where the i-th row meets the j-th column is denoted by a_{ij}. 14, 321, 323,

multi-index An n-dimensional multi-index is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of natural numbers ($\alpha_i \in \mathbb{N} \cup \{ 0 \} \quad \forall i = 1, \ldots, n$). 241, 321,

Nabla (or del) (∇) Vector differential operator which can be used for a compact notation of the Jacobian, gradient, divergence, and curl differential operators. In the n-dimensional Euclidean space $\mathbb{R}^n$ described through a Cartesian coordinate system, del is defined as

$$\nabla = \sum_{i=1}^{n} \boldsymbol{e}_i \frac{\partial}{\partial x_i}$$

where $\{ \boldsymbol{e}_i \mid 1 \leq i \leq n \}$ is the standard basis in this space. 321, 323,

ordinary differential equation (ODE) An equation containing an unknown function of one independent variable and its derivatives of any order. The order of the ODE is the order of the highest derivative in the ODE. 290, 321,

P parabolic regime. 103, 231, 321

partial differential equation (PDE) A differential equation that contains unknown multivariable functions and their partial derivatives. 321,

product rule A formula used to find the derivatives of products of two or more functions; in Leibniz notation this can be written as

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

. 139, 145, 241, 321,

real number A value that represents a quantity along a continuous line. 321,
Roman letters 321
e Neper number. 321

set A collection of distinct objects. 11, 19, 26, 321,

Taylor series A representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $n!$ denotes the factorial of n and $f^{(n)}(a)$ denotes the n-th derivative of f evaluated at the point a. 21, 74, 187, 244, 246, 313, 321,
tensor Geometric object that describes linear relations between vectors, scalars, and other tensors. An nth-rank tensor in m-dimensional space is a mathematical object that has n indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of space. A type (r, s) tensor on the vector space $\mathcal{V}$, contravariant of order r and covariant order s, is defined as an element of a vector space $T^r_s(\mathcal{V}) = \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{r} \otimes \underbrace{\mathcal{V}^* \otimes \cdots \otimes \mathcal{V}^*}_{s}$, where $\mathcal{V}^*$ is the dual space. A general representation with respect to bases $\boldsymbol{e}_i$ and $\boldsymbol{e}^j$ of $\mathcal{V}$ and $\mathcal{V}^*$, respectively, is $\boldsymbol{T} = T^{i_1 \dots i_n}_{j_1 \dots j_m}\, \boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_n} \otimes \boldsymbol{e}^{j_1} \otimes \cdots \otimes \boldsymbol{e}^{j_n}$. 9, 11, 17, 321,

vector space ($\mathcal{V}$) A set that is closed under finite vector addition and scalar multiplication. 9, 11, 321,
VM Von Mises yield function. 292, 321

yield surface A manifold in the space of stresses, which separates the elastic state ('inner points') from the yielding state (points lying on the manifold), while 'outer' points are not admissible (within the rate-independent plasticity). 265, 268, 321,

Bibliography

References for Part I

[1] D. Bigoni and D. Capuani. "Green's function for incremental nonlinear elasticity: shear bands and boundary integral formulation". In: *J. Mech. Phys. Solids* 50.3 (2002), pp. 471–500. DOI: 10.1016/S0022-5096(01)00090-4 (cit. on pp. 3, 4, 91, 92, 96, 186, 189, 191, 192, 198, 199).

[2] M.A. Biot. *Mechanics of incremental deformations*. New York, NY: John Wiley and Sons, 1965 (cit. on pp. 3, 78, 79, 91, 92, 191).

[3] M. Mooney. "A theory of large elastic deformation". In: *J. Appl. Phys.* 11.9 (1940), pp. 582–592. DOI: 10.1063/1.1712836 (cit. on pp. 3, 61, 66, 81).

[4] R.W. Ogden. "Large deformation isotropic elasticity: on the correlation of theory and experiment for incompressible rubberlike solids". In: *Proc. Royal Soc. A* 326.1567 (1972), pp. 565–583. DOI: 10.1098/rspa.1972.0026 (cit. on pp. 3, 62, 66, 67, 81).

[5] R. Hill and J.W. Hutchinson. "Bifurcation phenomena in the plane tension test". In: *J. Mech. Phys. Solids* 23.4-5 (1975), pp. 239–264. DOI: 10.1016/0022-5096(75)90027-7 (cit. on pp. 3, 86, 92, 191).

[6] A.E. Green and W. Zerna. *Theoretical elasticity*. Vol. 1. 2nd edition. London: Oxford University Press, 1968 (cit. on p. 3).

[7] A.E. Green, R.S. Rivlin, and R.T. Shield. "General theory of small elastic deformations superposed on large elastic deformations". In: *Proc. Royal Soc. A* 211.1104 (1952), pp. 128–154. DOI: 10.1098/rspa.1952.0030 (cit. on p. 3).

[8] I.M. Lifshitz and L.N. Rozentsveig. "Construction on the Green tensor for the fundamental equation of elasticity theory in the case of an unbounded elastically anisotropic medium". In: *Zh. Eksp. Teor. Fiz.* 17 (1947), pp. 783–791 (cit. on p. 3).

[9] D. Bigoni and D. Capuani. "Time-harmonic Green's function and boundary integral formulation for incremental non-linear elasticity: dynamics of wave patterns and shear bands". In: *J. Mech. Phys. Solids* 53.5 (2005), pp. 1163–1187. DOI: 10.1016/j.jmps.2004.11.007 (cit. on pp. 3, 92, 100, 191).

[10] D. Bigoni et al. "A novel boundary element approach to time-harmonic dynamics of incremental non-linear elasticity: the role of pre-stress on structural vibrations and dynamic shear banding". In: *Comp. Methods Appl. Mech. Eng.* 196.41-44 (2007), pp. 4222–4249. DOI: 10.1016/j.cma.2007.04.013 (cit. on pp. 3, 92, 93, 151, 191).

[77] D. Bigoni. *Nonlinear solid mechanics. Bifurcation theory and material instability*. Cambridge, UK: Cambridge University Press, 2012 (cit. on pp. 3, 189, 192, 265).

[12] J.D. Eshelby. "The continuum theory of lattice defects". In: *Solid State Phys.* 3.C (1956), pp. 79–144. DOI: 10.1016/S0081-1947(08)60132-0 (cit. on pp. 4, 135, 137).

[13] J.D. Eshelby. "The determination of the elastic field of an ellipsoidal inclusion, and related problems". In: *Proc. Royal Soc. A* 241.1226 (1957), pp. 376–396. DOI: 10.1098/rspa.1957.0133 (cit. on pp. 4, 135, 137).

[14] J.D. Eshelby. "A simple derivation of the elastic field of an edge dislocation". In: *British Journal of Applied Physics (B.J.A.P.)* 17.9 (1966), pp. 1131–1135. DOI: 10.1088/0508-3443/17/9/303 (cit. on pp. 4, 135, 137, 182, 185).

[15] J.R. Willis. "Dislocations and inclusions". In: *J. Mech. Phys. Solids* 13.6 (1965). Ed. by Pergamon Press Ltd., pp. 377–395. DOI: 10.1016/0022-5096(65)90038-4 (cit. on pp. 4, 135, 137).

[16] L.P. Argani, D. Bigoni, and G. Mishuris. "Dislocations and inclusions in prestressed metals". In: *Proc. Roy. Soc. A* 469.2154 (2013). DOI: 10.1098/rspa.2012.0752 (cit. on pp. 4, 191).

[17] J.W. Hutchinson and K.W. Neale. “Finite strain J_2-deformation theory”. In: *Proc. IUTAM Symp. on Finite Elasticity*. Ed. by D. E. Carlson and R.T. Shield. The Hague, The Netherlands: Martinus Nijhoff, 1979, pp. 237–247. DOI: 10.1007/978-94-009-7538-5_14 (cit. on pp. 5, 68, 84, 136).

[18] M.E. Gurtin. *An introduction to continuum mechanics*. Vol. 158. London: Academic Press Inc., 1981 (cit. on pp. 22, 25).

[19] M.E. Gurtin. “The linear theory of Elasticity”. In: *Encyclopedia of Physics*. Ed. by Springer. Vol. VIa/2. S (Ed.) Berlin: Flügge, 1972, pp. 1–295 (cit. on p. 22).

[20] C. Truesdell and W. Noll. “The non-linear field of mechanics”. In: *Encyclopedia of Physics*. Ed. by S. Flügge. Vol. III/3. Berlin: Springer-Verlag, 1965. DOI: 10.1007/978-3-642-46015-9_1 (cit. on pp. 22, 60, 83, 213).

[21] R. Hill. “Aspects of invariance in solid mechanics”. In: *Advances in Applied Mechanics*. Ed. by C.-S. Yih. Vol. 18. New York: Academic Press, 1978, pp. 1–75 (cit. on p. 22).

[22] R.W. Ogden. *Non-linear elastic deformations*. Vol. 158. Chichester: Ellis Horwood, 1984 (cit. on pp. 22, 137).

[23] G.A. Holzapfel. *Nonlinear Solid Mechanics. A Continuum Approach for Engineering*. New York: J. Wiley & Sons, 2000 (cit. on p. 25).

[24] L.R.G. Treolar. *The physics of rubber elasticity*. Vol. 158. Third Edition. Oxford University Press, 1975 (cit. on p. 67).

[25] L.R.G. Treolar. “Stress-strain data for vulcanized rubber under various types of deformation”. In: *Transactions of the Faraday Society* 40 (1944), pp. 59–70 (cit. on p. 67).

[26] R.W. Ogden, G. Saccomandi, and I. Sgura. “Fitting hyperelastic models to experimental data”. In: *Computational Mechanics* 34.6 (2004), pp. 484–502. DOI: 10.1007/s00466-004-0593-y (cit. on p. 67).

[27] J.W. Hutchinson and V. Tvergaard. “Shear band formation in plane strain”. In: *Int. J. Solids Structures* 17.5 (1981), pp. 451–470. DOI: 10.1016/0020-7683(81)90053-6 (cit. on pp. 68, 136).

[28] S. Stören and J.R. Rice. “Localized necking in thin sheets”. In: *J. Mech. Phys. Solids* 23.6 (1975), pp. 421–441. DOI: 10.1016/0022-5096(75)90004-6 (cit. on p. 84).

[29] M. Brun, D. Bigoni, and D. Capuani. “A boundary technique for incremental, non-linear elasticity - Part I: Formulation”. In: *Comput. Methods Appl. Mech. Eng.* 192.22-23 (2003), pp. 2461–2479. DOI: `10.1016/S0045-7825(03)00268-8` (cit. on pp. 92, 93, 191).

[30] M. Brun, D. Bigoni, and D. Capuani. “A boundary technique for incremental, non-linear elasticity - Part II: Bifurcation and shear bands”. In: *Comput. Methods Appl. Mech. Eng.* 192.22-23 (2003), pp. 2481–2499. DOI: `10.1016/S0045-7825(03)00272-X` (cit. on pp. 92, 191).

[31] O.A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. New York: Gordon & Breach, 1963 (cit. on p. 112).

[32] F. Dal Corso and D. Bigoni. “Growth of slip surfaces and line inclusions along shear bands in a softening material”. In: *Int. J. Fracture* 166.1-2 (2010), pp. 225–237. DOI: `10.1007/s10704-010-9534-1` (cit. on pp. 135, 191).

[33] V. Mantič and F. Paris. “Symmetrical representation of stresses in the Stroh formalism and its application to a dislocation and a dislocation dipole in an anisotropic elastic medium”. In: *J. Elasticity* 47.2 (1997), pp. 101–120. DOI: `10.1023/A:1007400325896` (cit. on p. 135).

[34] J.W. Steeds and J.R. Willis. “Dislocations in anisotropic media”. In: *Dislocations in Solids* 1 (1979). Ed. by F.R.N. Nabarro. New York, NY: North-Holland, pp. 143–166 (cit. on p. 135).

[35] T.C.T. Ting. “Green’s functions for a bimaterial consisting of two orthotropic quarter planes subjected to an antiplane force and a screw dislocation”. In: *Math. Mech. Solids* 10.2 (2005), pp. 197–211. DOI: `10.1177/1081286505036318` (cit. on p. 135).

[36] W. Cai et al. “Anisotropic elastic interactions of a periodic dislocation array”. In: *Phys. Rev. Letters* 86.25 (2001), pp. 5727–5730. DOI: `10.1103/PhysRevLett.86.5727` (cit. on p. 135).

[37] E. Clouet, L. Ventelon, and F. Willaime. “Dislocation core energies and core fields from first principles”. In: *Phys. Rev. Letters* 102.5 (2009). DOI: `10.1103/PhysRevLett.102.055502` (cit. on p. 135).

[38] A.H. Cottrell. *Dislocations and plastic flow in crystals*. Oxford, UK: Oxford University Press, 1953 (cit. on pp. 136, 190).

[39] D. Bigoni and F. Dal Corso. "The unrestrainable growth of a shear band in a prestressed material". In: *Proc. Royal Soc. A* 464.2097 (2008), pp. 2365–2390. DOI: `10.1098/rspa.2008.0029` (cit. on pp. 136, 191).

[40] F. Dal Corso, D. Bigoni, and M. Gei. "The stress concentration near a rigid line inclusion in a prestressed, elastic material. Part I. Full-field solution and asymptotics". In: *J. Mech. Phys. Solids* 56.3 (2008), pp. 815–838. DOI: `10.1016/j.jmps.2007.07.002` (cit. on p. 136).

[41] F. Dal Corso, D. Bigoni, and M. Gei. "The stress concentration near a rigid line inclusion in a prestressed, elastic material. Part II. Implications on shear band nucleation, growth and energy release rate". In: *J. Mech. Phys. Solids* 56.3 (2008), pp. 839–857. DOI: `10.1016/j.jmps.2007.07.003` (cit. on p. 136).

[42] D. Bigoni, D. Ortiz, and A. Needleman. "Effect of interfacial compliance on bifurcation of a layer bonded to a substrate". In: *Int. J. Solids Structures* 34.33-34 (1997), pp. 4305–4326. DOI: `10.1016/S0020-7683(97)00025-5` (cit. on p. 191).

[43] D.J. Steigmann and R.W. Ogden. "Plane deformations of elastic solids with intrinsic boundary elasticity". In: *Proc. Roy. Soc. London A* 453.1959 (1997), pp. 853–877. DOI: `10.1098/rspa.1997.0047` (cit. on p. 191).

[44] N. Triantafyllidis and F.K. Lehner. "Interfacial instability of density-stratified two-layer systems under initial stress". In: *J. Mech. Phys. Solids* 41.1 (1993), pp. 117–142. DOI: `10.1016/0022-5096(93)90066-O` (cit. on p. 191).

[45] N. Triantafyllidis and Y.M. Leroy. "Stability of a frictional material layer resting on a viscous half-space". In: *J. Mech. Phys. Solids* 42.1 (1994), pp. 51–110. DOI: `10.1016/0022-5096(94)90050-7` (cit. on p. 191).

[46] M.A. Dowaikh and R.W. Ogden. "On surface waves and deformations in a pre-stressed incompressible elastic solids". In: *IMA J. Appl. Math.* 44.3 (1990), pp. 261–284. DOI: `10.1093/imamat/44.3.261` (cit. on p. 191).

[47] M.A. Dowaikh and R.W. Ogden. "Interfacial waves and deformations in pre-stressed elastic media". In: *Proc. Roy. Soc. London A* 443.1888 (1991), pp. 313–328. DOI: `10.1098/rspa.1991.0050` (cit. on p. 191).

[48] N.D. Cristescu, E.M. Craciun, and E. Soós. *Mechanics of elastic composites*. Boca Raton. Chapman & Hall/CRC, 2004 (cit. on p. 191).

[49] A.N. Guz. *Fundamentals of the three-dimensional theory of stability of deformable bodies*. Berlin: Springer-Verlag, 1999 (cit. on p. 191).

[50] E.M. Cračiun and E. Soós. "Interaction of two unequal cracks in a prestressed fiber reinforced composite". In: *Int. J. Fracture* 94.2 (1998), pp. 137–159 (cit. on p. 191).

[51] E. Radi, D. Bigoni, and D. Capuani. "Effects of pre-stress on crack-tip fields in elastic, incompressible solids". In: *Int. J. Solids Structures* 39.15 (2002), pp. 3971–3996. DOI: 10.1016/S0020-7683(02)00252-4 (cit. on p. 191).

[52] J.R. Willis. "Inclusions and cracks in constrained anisotropic media". In: *Modern Theory of Anisotropic Elasticity and Applications*. Ed. by J.J. Wu, T.C.T. Ting, and D.M. Barnett. Philadelphia: SIAM, 1991, pp. 87–102 (cit. on p. 191).

[53] A. Piccolroaz, D. Bigoni, and J.R. Willis. "A dynamical interpretation of flutter instability in a continuous medium". In: *J. Mech. Phys. Solids* 54.11 (2006), pp. 2391–2417. DOI: 10.1016/j.jmps.2006.05.005 (cit. on p. 191).

[54] G.E. Dieter. *Mechanical Metallurgy*. New York: McGraw-Hill, 1961 (cit. on p. 192).

[55] J. Desrues et al. "Void ratio evolution inside shear bands in triaxial sand specimens studied by computed tomography". In: *Geotechnique* 46.3 (1996), pp. 529–546. DOI: 10.1680/geot.1996.46.3.529 (cit. on p. 192).

[56] B.R. Lawn. "Indentation of Ceramics with Spheres: A Century after Hertz". In: *J. Am. Ceram. Soc.* 81.8 (1998), pp. 1977–1994. DOI: 10.1111/j.1151-2916.1998.tb02580.x (cit. on p. 192).

[57] B.M. French. *Traces of Catastrophe: A Handbook of Shock-Metamorphic Effects in Terrestrial Meteorite Impact Structures*. Ed. by Houston. LPI Contribution 954. Lunar and Planetary Institute, 1998 (cit. on p. 192).

[58] A. Sagy, J. Fineberg, and Z. Reches. "Shatter cones: Branched, rapid fractures formed by shock impact". In: *J. Geophys. Res.* 109.10 (2004), B10209. DOI: 10.1029/2004JB003016 (cit. on p. 192).

[59] I.M. Gel'fand and G.E. Shilov. *Generalized Functions, Properties and Operations*. Vol. 1. New York: Academic Press, 1964 (cit. on p. 197).

[60] A.J.M. Spencer and R.S. Rivlin. "Further results on the theory of matrix polynomials". In: *Arch. Rat. Mech. Anal.* 4.1 (1959), pp. 214–230. DOI: 10.1007/BF00281388 (cit. on p. 213).

[61] P.K. Zysset and A. Curnier. "An alternative model for anisotropic elasticity based on fabric tensors". In: *Mech. Mater.* 21.4 (1995), pp. 243–250. DOI: 10.1016/0167-6636(95)00018-6 (cit. on p. 213).

[132] J.P. Hirth and J. Lothe. *Theory of dislocations*. New York: J. Wiley & Sons, 1968 (cit. on pp. 257, 311).

[63] J. Balaš, J. Sládek, and V. Sládek. *Stress Analysis by Boundary Element Methods*. Vol. 23. Studies in Applied Mechanics. Bratislava, CZE: Elsevier, 1989 (cit. on p. 257).

[64] F. Hartmann and C. Katz. *Structural Analysis with Finite Elements*. Berlin: Springer-Verlag, 2004 (cit. on p. 257).

Further reading for Part I

[65] R. Courant and D. Hilbert. *Methods of Mathematical Physics*. Vol. II. New York: J. Wiley and Sons, 1962.

[66] A.E.H. Love. *A treatise on the mathematical theory of elasticity*. Vol. 158. Fourth Edition. Cambridge: Cambridge university Press, 1927.

[67] K. Bertoldi, M. Brun, and D. Bigoni. "A new boundary element technique without domain integrals for elastoplastic solids". In: *Int. J. Numer. Meth. Engng.* 64.7 (2005), pp. 877–906. DOI: 10.1002/nme.1385.

[68] D. Bigoni and M. Gei. "Bifurcation of a coated, elastic cylinder". In: *Int. J. Solids Structures* 38.30-31 (2001), pp. 5117–5148. DOI: 10.1016/S0020-7683(00)00322-X.

[69] S. Colli, M. Gei, and D. Bigoni. "A boundary formulation for incremental nonlinear elastic deformation of compressible solids". In: *C.M.E.S.* 40.1 (2009), pp. 29–62. DOI: 10.3970/cmes.2009.040.029.

[70] A. Gajo, D. Bigoni, and D. Muir Wood. "Multiple shear band development and related instabilities in granular materials". In: *J. Mech. Phys. Solids* 52.12 (2004), pp. 2683–2724. DOI: 10.1016/j.jmps.2004.05.010.

[71] R. Hill. "On constitutive inequalities for simple materials". In: *J. Mech. Phys. Solids* 16 (1968), pp. 229–242. DOI: 10.1016/0022-5096(68)90031-8.

[72] G. Noselli, F. Dal Corso, and D. Bigoni. “The stress intensity near a stiffener disclosed by photoelasticity”. In: *International Journal of Fracture* 166.1-2 (2010), pp. 91–103. DOI: 10.1007/s10704-010-9502-9.

[73] Y.C. Pan and T.W. Chou. “Point force solution for an infinite transversely isotropic solid”. In: *J. Appl. Mech.* 43.4 (1976), pp. 608–612. DOI: 10.1115/1.3423941.

[74] T.C.T. Ting and V.-G. Lee. “The three-dimensional elastostatic Green’s function for general anisotropic linear elastic solids”. In: *J. Mech. Appl. Math.* 50.3 (1997), pp. 404–423. DOI: 10.1093/qjmam/50.3.407.

[75] V. Volterra. “Sur l’ équilibre des corps élastiques multiplement connexes”. In: *Annales Scientifiques de l’ É.N.S.* 3e série 24 (1907), pp. 401–517.

[76] J.R. Willis. “The Elastic Interaction Energy Of Dislocation Loops In Anisotropic Media”. In: *J. Mech. Appl. Math.* 18.4 (1965). Ed. by Pergamon Press Ltd., pp. 419–433. DOI: 10.1093/qjmam/18.4.419.

References for Part II

[77] D. Bigoni. *Nonlinear solid mechanics. Bifurcation theory and material instability*. Cambridge, UK: Cambridge University Press, 2012 (cit. on pp. 3, 189, 192, 265).

[78] W. Bier and S. Hartmann. “A finite strain constitutive model for metal powder compaction using a unique and convex single surface yield function”. In: *Eur. J. Mechanics-A/Solids* 25.6 (2006), pp. 1009–1030. DOI: 10.1016/j.euromechsol.2006.01.002 (cit. on p. 265).

[79] S. Hartmann and W. Bier. “High-order time integration applied to metal powder plasticity”. In: *Int. J. Plasticity* 24.1 (2008), pp. 17–54. DOI: 10.1016/j.ijplas.2007.01.014 (cit. on p. 265).

[80] U. Heisserer et al. “p-FEM for finite deformation powder compaction”. In: *Comput. Meth. Appl. Mech. Engrg.* 197.6-8 (2008), pp. 727–740. DOI: 10.1016/j.cma.2007.09.001 (cit. on p. 265).

[81] A. Piccolroaz, D. Bigoni, and A. Gajo. “An elastoplastic framework for granular materials becoming cohesive through mechanical densification. Part. I - small strain formulation”. In: *Eur. J. Mech. A/Solids* 25.2 (2006), pp. 334–357. DOI: 10.1016/j.euromechsol.2005.10.001 (cit. on pp. 265, 292).

[82] A. Piccolroaz, D. Bigoni, and A. Gajo. “An elastoplastic framework for granular materials becoming cohesive through mechanical densification. Part. II - the formulation of elastoplastic coupling at large strain”. In: *Eur. J. Mech. A/Solids* 25.2 (2006), pp. 358–369. DOI: 10.1016/j.euromechsol.2005.10.002 (cit. on pp. 265, 292).

[83] R.R. Babua, G.S. Benipal, and A.K. Singh. “Constitutive modelling of concrete: an overview”. In: *Asian J. Civil Eng. (Building and Housing)* 6 (2005), pp. 211–246 (cit. on p. 265).

[84] R. Conti, C. Tamagnini, and A. DeSimone. “Critical softening in Cam-Clay plasticity: Adaptive viscous regularization, dilated time and numerical integration across stress-strain jump discontinuities”. In: *Comput. Method. Appl. M.* 258 (2013), pp. 118–133. DOI: 10.1016/j.cma.2013.02.002 (cit. on p. 265).

[85] G. Dal Maso, A. DeSimone, and F. Solombrino. “Quasistatic evolution for Cam-Clay plasticity: properties of the viscosity solution”. In: *Calculus Variat. Part. Diff. Equat.* 44.3-4 (2012), pp. 495–541. DOI: 10.1007/s00526-011-0443-6 (cit. on p. 265).

[86] G. Dal Maso, A. Demyanov, and A. DeSimone. “Quasistatic Evolution Problems for Pressure-sensitive Plastic Materials”. In: *Milan J. Math.* 75.1 (2007), pp. 117–134. DOI: 10.1007/s00032-007-0071-y (cit. on p. 265).

[87] R. Dal Pino, P. Narducci, and G. Royer-Carfagni. “A SEM investigation on fatigue damage of marble”. In: *J. Materials Sci. Letters* 18.19 (1999), pp. 1619–1622. DOI: 10.1023/A:1006689022297 (cit. on p. 265).

[88] H. DorMohammadi and A.R. Khoei. “A three-invariant cap model with isotropic-kinematic hardening rule and associated plasticity for granular materials”. In: *Int. J. Solids Structures* 45.2 (2008), pp. 631–656. DOI: 10.1016/j.ijsolstr.2007.08.019 (cit. on p. 265).

[89] S. Maiolino. “Proposition of a general yield function in geomechanics”. In: *Comptes Rendus Mecanique* 333.3 (2005), pp. 279–284. DOI: 10.1016/j.crme.2004.11.008 (cit. on p. 265).

[90] G. Mortara. “A new yield and failure criterion for geomaterials”. In: *Geotechnique* 58.2 (2008), pp. 125–132. DOI: 10.1680/geot.2008.58.2.125 (cit. on p. 265).

[91] A. Paluszny and S.K. Matthai. “Numerical modeling of discrete multi-crack growth applied to pattern formation in geological brittle media”. In: 46.18-19 (2009), pp. 3383–3397. DOI: 10.1016/j.ijsolstr.2009.05.007 (cit. on p. 265).

[92] H.A. Sheldon, A.C. Barnicoat, and A. Ord. “Numerical modelling of faulting and fluid flow in porous rocks: An approach based on critical state soil mechanics”. In: *J. Struct. Geol.* 28.8 (2006), pp. 1468–1482. DOI: 10.1016/j.jsg.2006.03.039 (cit. on p. 265).

[93] J. Zhou et al. “Micromechanics damage modeling of brittle rock failure processes under compression”. In: *Int. J. Comput. Meth.* 10.6 (2013). 1350034. DOI: 10.1142/S0219876213500345 (cit. on p. 265).

[94] F. Descamps and J.P. Tshibangu. “Modelling the Limiting Envelopes of Rocks in the Octahedral Plane”. In: *Oil Gas Sci. Technol.* Vol. 62. 5. Rev. IFP. 2007, pp. 683–694. DOI: 10.2516/ogst:2007072 (cit. on p. 265).

[95] M. Angelillo, L. Cardamone, and A. Fortunato. “A numerical model for masonry-like structures”. In: *J. Mech. Materials Structures* 5.4 (2010), pp. 583–615. DOI: 10.2140/jomms.2010.5.583 (cit. on p. 265).

[96] S. De Faveri, L. Freddi, and R. Paroni. “No-tension bodies: A reinforcement problem”. In: *Europ. J. Mech. A/Solids* 39 (2013), pp. 163–169. DOI: 10.1016/j.euromechsol.2012.11.006 (cit. on p. 265).

[97] S.Y. Wang et al. “Numerical simulation of the failure process of unreinforced masonry walls due to concentrated static and dynamic loading”. In: *Int. J. Solids Structures* 49.2 (2012), pp. 377–394. DOI: 10.1016/j.ijsolstr.2011.10.016 (cit. on p. 265).

[98] A. Bolchoun, V.A. Kolupaev, and H. Altenbach. “Convex and Non-convex Flow Surfaces”. In: *Forschung im Ingenieurwesen-Engineering Research* 75.2 (2011), pp. 73–92. DOI: 10.1007/s10010-011-0135-6 (cit. on p. 265).

[99] T. Coppola, L. Cortese, and P. Folgarait. “The effect of stress invariants on ductile fracture limit in steels”. In: *Eng. Fract. Mech.* 76.9 (2009), pp. 1288–1302. DOI: 10.1016/j.engfracmech.2009.02.006 (cit. on p. 265).

[100] F. Ebnoether and D. Mohr. “Predicting ductile fracture of low carbon steel sheets: Stress-based versus mixed stress/strain-based Mohr-Coulomb model”. In: *Int. J. Solids Structures* 50.7-8 (2013), pp. 1055–1066. DOI: 10.1016/j.ijsolstr.2012.11.026 (cit. on p. 265).

[101] W. Hu and Z.R. Wang. “Multiple-factor dependence of the yielding behavior to isotropic ductile materials”. In: *Comput. Mat. Sci.* 32.1 (2005), pp. 31–46. DOI: 10.1016/j.commatsci.2004.06.002 (cit. on p. 265).

[102] G. Mirone and D. Corallo. “Stress-strain and ductile fracture characterization of an X100 anisotropic steel: Experiments and modelling”. In: *Eng. Fract. Mech.* 102 (2013), pp. 118–145. DOI: 10.1016/j.engfracmech.2013.02.027 (cit. on p. 265).

[103] M.P. Pietryga, I.N. Vladimirov, and S. Reese. “A finite deformation model for evolving flow anisotropy with distortional hardening including experimental validation”. In: *Mech. Materials* 44 (2012), pp. 163–173. DOI: 10.1016/j.mechmat.2011.07.014 (cit. on p. 265).

[104] T. Wierzbicki et al. “Calibration and evaluation of seven fracture models”. In: *Int. J. Mech. Sci.* 47.4-5 SPEC. ISS. (2005), pp. 719–743. DOI: 10.1016/j.ijmecsci.2005.03.003 (cit. on p. 265).

[105] T. Coppola and P. Folgarait. “The influence of stress invariants on ductile fracture strain in steels”. In: *Proc. XXXVI AIAS Congress*. Sept. 4-8, In italian. 2007, pp. 237–247 (cit. on p. 265).

[106] Y. Bai and T. Wierzbicki. “A new model of metal plasticity and fracture with pressure and Lode dependence”. In: *Int. J. Plasticity* 24.6 (2008), pp. 1071–1096. DOI: 10.1016/j.ijplas.2007.09.004 (cit. on p. 265).

[107] M.R. Laydi and C. Lexcellent. “Yield criteria for shape memory materials: convexity conditions and surface transport”. In: *Math. Mech. Solids* 15.2 (2010), pp. 165–208. DOI: 10.1177/1081286508095324 (cit. on p. 265).

[108] K. Lavernhe-Taillard et al. “Multiaxial Shape Memory Effect and Superelasticity”. In: *Strain* 45.1 (2009), pp. 77–84. DOI: 10.1111/j.1475-1305.2008.00528.x (cit. on p. 265).

[109] C. Lexcellent et al. “Experimental and numerical determinations of the initial surface of phase transformation under biaxial loading in some polycrystalline shape-memory alloys”. In: *J. Mech. Phys. Solids* 50.12 (2002), pp. 2717–2735. DOI: 10.1016/S0022-5096(02)00007-8 (cit. on p. 265).

[110] B. Raniecki and C. Lexcellent. “Thermodynamics of isotropic pseudoelasticity in shape memory alloys”. In: *Eur. J. Mechanics-A/Solids* 17.2 (1998), pp. 185–205. DOI: `10.1016/S0997-7538(98)80082-X` (cit. on p. 265).

[111] B. Raniecki and Z. Mróz. “Yield or martensitic phase transformation conditions and dissipation functions for isotropic, pressure-insensitive alloys exhibiting SD effect”. In: *Acta Mech.* 195.1-4 (2008), pp. 81–102. DOI: `10.1007/s00707-007-0544-7` (cit. on p. 265).

[112] L. Saint-Sulpice, S.A. Arbab Chirani, and S. Calloch. “A 3D super-elastic model for shape memory alloys taking into account progressive strain under cyclic loadings”. In: *Mech. Materials* 41.1 (2009), pp. 12–26. DOI: `10.1016/j.mechmat.2008.07.004` (cit. on p. 265).

[113] P. Sedlák et al. “Thermomechanical model for NiTi-based shape memory alloys including R-phase and material anisotropy under multi-axial loadingss”. In: *Int. J. Plasticity* 39 (2012), pp. 132–151. DOI: `10.1016/j.ijplas.2012.06.008` (cit. on p. 265).

[114] K. Taillard et al. “Equivalent transformation strain and its relation with martensite volume fraction for isotropic and anisotropic shape memory alloys”. In: *Mech. Materials* 40.4-5 (2008), pp. 151–170. DOI: `10.1016/j.mechmat.2007.07.005` (cit. on p. 265).

[115] D. Bigoni and A. Piccolroaz. “Yield criteria for quasibrittle and frictional materials”. In: *Int. J. Solids Structures* 41.11-12 (2004), pp. 2855–2878. DOI: `10.1016/j.ijsolstr.2003.12.024` (cit. on pp. 265, 268, 269).

[116] A. Piccolroaz and D. Bigoni. “Yield criteria for quasibrittle and frictional materials: a generalization to surfaces with corners”. In: *Int. J. Solids Structures* 46.20 (2009), pp. 3587–3596. DOI: `10.1016/j.ijsolstr.2009.06.006` (cit. on pp. 265, 268).

[117] F. Bosi et al. “Experimental investigation of the elastoplastic response of aluminum silicate spray dried powder during cold compaction”. In: *J. Europ. Ceramic Soc.* (2013). In Press. DOI: `10.1016/j.jeurceramsoc.2013.11.037` (cit. on p. 265).

[118] S. Stupkiewicz, A. Piccolroaz, and D. Bigoni. “Elastoplastic coupling to model cold ceramic powder compaction”. In: *J. Europ. Ceramic Soc.* (2013). In Press. DOI: `10.1016/j.jeurceramsoc.2013.11.017` (cit. on p. 265).

[119] J. Simo and T.J.R. Hughes. "General return mapping algorithms for rate-independent plasticity". In: *Constitutive Laws for Engineering Materials: Theory and Applications*. Ed. by C.S. et al. Desai. Elsevier Science Publ. Co., 1987, pp. 221–231 (cit. on pp. 265, 277).

[120] R.M. Brannon and S. Leelavanichkul. "A multi-stage return algorithm for solving the classical damage component of constitutive models for rocks, ceramics, and other rock-like media". In: *Int. J. Fracture* 163.1-2 (2010), pp. 133–149. DOI: `10.1007/s10704-009-9398-4` (cit. on pp. 266, 274).

[121] M. Penasa et al. "Integration algorithms of elastoplasticity for ceramic powder compaction". In: *J. Europ. Ceramic Soc.* (2014). In Press. DOI: `10.1016/j.jeurceramsoc.2014.01.041` (cit. on p. 266).

[122] J. Podgórski. "Limit state condition and the dissipation function for isotropic materials". In: *Arch. Mech. Soc.* 36 (1984), pp. 323–342 (cit. on p. 268).

[123] J. Podgórski. "General failure criterion for isotropic media". In: *J. Eng. Mech. ASCE* 111.2 (1985), pp. 188–201. DOI: `10.1061/(ASCE)0733-9399(1985)111:2(188)` (cit. on p. 268).

[124] D. Bigoni and F. Laudiero. "The quasi-static finite cavity expansion in a non-standard elasto-plastic medium". In: *Int. J. Mech. Sci.* 31.11-12 (1989), pp. 825–837. DOI: `10.1016/0020-7403(89)90027-1` (cit. on p. 288).

[125] Cohen T., R. Masri, and Durban D. "Analysis of circular hole expansion with generalized yield criteria". In: *Int. J. Solids Structures* 46.20 (2009), pp. 3643–3650. DOI: `10.1016/j.ijsolstr.2009.06.013` (cit. on p. 288).

[126] R. Hill. "On the inhomogeneous deformation of a plastic lamina in a compression test". In: *Philos. Mag.* 7th ser. 41.319 (1950), pp. 733–744. DOI: `10.1080/14786445008561006` (cit. on p. 288).

[127] L. Rapoport, Z. Katzir, and M.B. Rubin. "Termination of the starting problem of dynamic expansion of a spherical cavity in an infinite elastic-perfectly-plastic medium". In: *Wave Motion* 48.5 (2011), pp. 441–452. DOI: `10.1016/j.wavemoti.2011.03.001` (cit. on p. 288).

[128] K.Y. Volokh. "Cavitation instability in rubber". In: *Int. J. Appl. Mech.* 3.2 (2011), pp. 299–311. DOI: `10.1142/S1758825111001007` (cit. on p. 288).

Further reading for Part II

[129] R. Hill. *The Mathematical Theory of Plasticity*. Oxford: Clarendon Press, 1950.

[130] E.A. De Souza Neto, D. Perić, and D.R.J. Owen. *Computational methods for plasticity. Theory and applications*. UK: John Wiley and Sons, 2008.

[131] A. Piccolroaz, A. Gajo, and D. Bigoni. "Forming of advanced ceramics". In: *Selected Mechanical Problems in Structural Ceramics*. Ed. by D. Bigoni. Polish Academy of Sciences. Warsaw, 2002, pp. 81–115.

References for Part III

[132] J.P. Hirth and J. Lothe. *Theory of dislocations*. New York: J. Wiley & Sons, 1968 (cit. on pp. 257, 311).

[133] O'Regan R. "New method for determining strain on the surface of a body with photoelastic coatings". In: *Experimental Mech.* 5.8 (1965), pp. 241–246. DOI: `10.1007/BF02327147` (cit. on p. 309).

List of websites

[134] *MiKTEX distribution official page*. MiKTEX. 2014. URL: `http://miktex.org/`.

[135] *TEXnicCenter editor official page*. TEXnicCenter. 2014. URL: `http://www.texniccenter.org/`.

[136] *TEXmaker editor official page*. TEXmaker. 2014. URL: `http://www.xm1math.net/texmaker/`.

[137] *TEXstudio editor official page*. TEXstudio. 2014. URL: `http://texstudio.sourceforge.net/`.

[138] *FreeCAD official page*. FreeCAD. 2014. URL: `http://www.freecadweb.org/`.

[139] *GIMP official page*. GIMP. 2014. URL: `http://www.gimp.org/`.

[140] *Inkscape official page*. Inkscape. 2014. URL: `http://inkscape.org/`.

[141] *LibreCAD official page*. LibreCAD. 2014. URL: http://librecad.org/cms/home.html.

[142] *Abaqus® official page*. Simulia (Dassault Systèmes). 2014. URL: http://www.3ds.com/products-services/simulia/portfolio/abaqus/latest-release/.

[143] *CAELinux official page*. CAELinux. 2014. URL: http://www.caelinux.com/CMS/.

[144] *Dassault Systèmes official page*. Dassault Systèmes. 2014. URL: http://www.3ds.com/.

[145] *DraftSight official page*. Dassault Systèmes. 2014. URL: http://www.3ds.com/products-services/draftsight/.

[146] *Wolfram® official page*. Wolfram®. 2014. URL: http://www.wolfram.com/mathematica/.

[147] *Wolfram Mathematica® documentation official page*. Wolfram®. 2014. URL: http://reference.wolfram.com/mathematica/guide/Mathematica.html.

[148] *DICAM official page*. DICAM. 2014. URL: http://www.unitn.it/dicam.

[149] *University of Trento official page*. UniTN. 2014. URL: http://www.unitn.it/.

[150] *Computational Lab official page*. Computational Solid and Structural Mechanics. 2014. URL: http://www.unitn.it/dicam/7282/computational-solid-and-structural-mechanics.

[151] *Experimental Lab official page*. Laboratory for Physical Modeling of Structures and Photoelasticity. 2014. URL: http://www.unitn.it/dicam/7286/laboratory-for-physical-modeling-of-structures-and-photoelasticity.

INDEX

INDEX OF NAMES

W

Z

DECLARATION

I, Luca Prakash Argani, hereby declare that this thesis titled "Dislocations and Green's Functions In Prestressed Solids" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

Daiano, Thursday 8th May, 2014

Luca Prakash Argani