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SPECIAL RATIONALLY CONNECTED MANIFOLDS

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Contents

Introduction

In the study of complex algebraic surfaces, which is a classical topic in algebraic geometry, rational and ruled surfaces turned out to be very special, and, in some sense, the simplest to understand. Going to higher dimensions, the situation becomes different and much more complicated. For instance, rationality of a variety is a very difficult property to establish. Moreover, many varieties that behave very similarly to ruled (respectively rational) varieties fail to be ruled (respectively rational). Thus, from many points of view, rational and ruled varieties are not the right higher dimensional analogues of rational and ruled surfaces.

At the end of the last century, the two new concepts of uniruled and rationally connected varieties were introduced as suitable higher dimensional analogues of ruled and rational surfaces. Uniruled varieties are algebraic varieties that are covered by rational curves, i.e. varieties that contain a rational curve through a general point. Among uniruled varieties, those that contain a rational curve through two general points are especially important. Varieties satisfying this property are called rationally connected and were introduced by Kollár, Miyaoka and Mori in [KMM92c], and independently by Campana in [Cam92].

Uniruled and rationally connected varieties have intensely been studied since their introduction. A natural problem about rationally connected varieties is to characterize them by means of bounding degrees of rational curves connecting points. This is the main topic of this dissertation and our main tools are taken from the theory of rational curves on varieties (see [Kol96] or [Deb01] for a general reference).

Recently Ionescu and Russo have studied this problem, and in particular they focused their attention on *conic (or conically) connected* manifolds embedded in projective space, i.e. projective manifolds such that two general points may be joined by a rational curve of degree 2 with respect to a fixed very ample line bundle L.
In [IR07], they proved a classification theorem for these manifolds. Their result shows that conic connected manifolds $X \subset \mathbb{P}^N$ of dimension n are Fano and have

Picard number ρ_X less than or equal to 2. Additionally, if $\rho_X = 1$, then, unless X is projectively equivalent to the Veronese variety $v_2(\mathbb{P}^n)$, the Picard group is generated by the hyperplane section and the index of the manifold is at least $\frac{n+1}{2}$; if $\rho_X = 2$, they obtained the following list of possibilities: Segre products of two projective spaces and their hyperplane sections, or the inner projections, from a linear space, of the Veronese variety $v_2(\mathbb{P}^n)$.
In this thesis, we will reconsider the work of Ionescu and Russo, proving that their classification result holds true assuming just the ampleness of the fixed line bundle L, and, as we will seen later, we will carry on a similar investigation for rationally connected manifolds with respect to rational curves of degree 3.

Conic connected manifolds were studied also by Kachi and Sato who characterized a special subclass of these manifolds.
More precisely, in [KS99], Kachi and Sato considered projective varieties with at worst $\mathbb{Q}$-factorial singularities such that a fixed non-singular point $x \in X$ and two general points of X may be joined by an irreducible rational curve on X of degree 2 with respect to a fixed ample Cartier divisor on X. It is clear from the definition that projective varieties that satisfy the above property are conic connected, and a Kachi-Sato's theorem states that the only possibilities are $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$, where $\mathbb{Q}^n$ is a (possibly singular) hyperquadric in $\mathbb{P}^{n+1}$.
Only for the smooth case, we will give a different proof of Kachi-Sato's result using the theory of rational curves on projective varieties.

After conic connected manifolds, as already said before, we will consider smooth complex projective varieties X which are rationally connected by rational curves of degree 3 with respect to a fixed ample line bundle L, or equivalently which admits a covering family V of rational curves of degree 3 with respect to L such that two general points of X may be joined by a curve parametrized by V. We will call X *rationally cubic connected*.
The study of rationally cubic connected manifolds is the main subject of this thesis. A first step towards the understanding of these manifolds could be to establish a bound on the Picard number.

First of all we will study rationally cubic connected manifolds that are covered by "lines", i.e. by rational curves of degree 1 with respect to L. For these manifolds it is possible to find an upper bound on their Picard number, namely we will prove that the Picard number is equal to or less than 3.
Among these manifolds we will concentrate on those which have the Picard number equal to 3; we will show that if $\rho_X = 3$ then there is a covering family of "lines" whose numerical class spans a negative extremal ray of the Kleiman-Mori cone of X.

Unfortunately, for rationally cubic connected manifolds which don't admit a cove-

ring family of "lines" there isn't an upper bound on the Picard number.
In fact, for every positive integer m we can construct a rationally cubic connected manifold which is not covered by "lines" and whose Picard number is equal to m; these rationally cubic connected manifolds are obtained by the blow up of $\mathbb{P}^n$ at $(m-1)$ distinct points and they are such that if their Picard number is greater than 3 and $n > 2$ then they are not Fano.
For that reason we will consider rationally cubic connected manifolds which are not covered by "lines" but are Fano. We will show that up to a few exceptions in dimension 2 also the Picard number of these manifolds is equal to or less than 3. More precisely, we will prove that either the Picard number is equal to or less than 2 or X is the blow up of $\mathbb{P}^n$ along two disjoint subvarieties that are linear subspaces or quadrics.

The thesis is organized as follows:

In the first chapter we recall the terminology and the main results of intersection theory. Moreover, we define Fano manifolds and we briefly discuss what is known about their classification.

Chapters 2 to 4 are dedicated to the theory of rational curves on projective varieties. In Chapter 2, we introduce the parameter spaces of rational curves on a projective variety and we define uniruled, rationally connected and rationally chain connected varieties. In the following chapter, we gather the basic results of Mori's theory, bend and break lemmas and the Cone Theorem, and we describe Fano-Mori contractions. In chapter 4, we talk about families of rational curves and Chow families of rational 1-cycles, and we prove some important estimates for the dimension of the locus of a family of rational curves or of the locus of chains of rational curves.

The aim of Chapter 5 is to define a relation of rational connectedness with respect to k Chow families (we claim that two points are equivalent with respect to this relation if there exists a chain of rational 1-cycles, parametrized by the fixed Chow families, which joins the points) and to study this relation. We call this relation $rc(\mathcal{V}^1, \dots, \mathcal{V}^k)$-relation.
Rational connectedness was introduced by Kollár, Miyaoka, and Mori and independently by Campana, and so, we introduce the two different notations and we cite their fundamental results.
Their main theorem claim that to the $rc(\mathcal{V}^1, \dots, \mathcal{V}^k)$-relation we can associate a fibration at least on an open subset of variety. To understand how the fibration is defined, in the last section, we give a sketch of Campana's construction of the fibration.

In Chapter 6 we list some conditions under which the numerical class of every curve lying in some subvariety S of a projective variety X is contained in a linear subspace of $N_1(X)$ or in a subcone of $NE(X)$. Moreover we prove some properties of fibrations associated to the $rc(\mathcal{V}^1, \dots, \mathcal{V}^k)$-relation. These results give some important informations about the structure of the Kleiman-Mori cone of the variety and about the extremality of quasi unsplit Chow families considered for the fibration.

Chapter 7 deals with conic connected manifolds. First of all, we generalize the classification theorem of Ionescu and Russo without assuming that conic connected manifolds are embedded in projective space, and we show a different proof of Kachi-Sato's theorem in the smooth case.

In Chapter 8 we study rationally cubic connected manifolds.

In the first section we consider rationally cubic connected manifolds that admit a covering family of rational curves of degree 1 with respect to a fixed ample line bundle. As already said before, we give the proof of the existence of an upper bound on the Picard number of these rationally cubic connected manifolds, and, in particular, we prove that $\rho_X = 3$ if and only if X is $rc(\mathcal{W}, \mathcal{W}', \mathcal{W}'')$-connected with respect to three families of "lines", W, W' and W''.

Theorem *Let (X, L) be a polarized manifold. Assume that X is rationally cubic connected by a family V and admits a covering family of lines. Then $\rho_X \leq 3$, equality holding if and only if there exist three families of lines W, W', W'' with $[V] = [W] + [W'] + [W'']$ such that W is covering, W' is horizontal and dominating with respect to the $rc(\mathcal{W})$-fibration and W'' is horizontal and dominating with respect to the $rc(\mathcal{W}, \mathcal{W}')$-fibration.*

Moreover we show that if $\rho_X = 3$ then there is a covering family of "lines" whose numerical class spans a negative extremal ray of the Kleiman-Mori cone of X.

The last section is devoted to the study of Fano rationally cubic connected manifolds that are not covered by lines. After having proved that every Del Pezzo surface is rationally cubic connected, we assume that manifolds have dimension greater than 2. We prove that the Picard number is less than or equal to 3 and if the equality holds we obtain a precise classification of manifolds.

Theorem *Let (X, L) be a polarized manifold. Assume that X is rationally cubic connected by a family V and doesn't admit a covering family of lines. Assume that X is a Fano manifold and has dimension $n > 2$. Then either $\rho_X \leq 2$ or we have the following list of possibilities:*

(1) $(X, L) \simeq (Bl_{\Lambda_1,\Lambda_2}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$, *where $Bl_{\Lambda_1,\Lambda_2}(\mathbb{P}^n)$ is the blow up of $\mathbb{P}^n$*

along two linear subspaces Λ_1, Λ_2 *such that*

$$\Lambda_1 \cap \Lambda_2 = \emptyset, \quad \dim \Lambda_1 + \dim \Lambda_2 = n - 2$$

and E_1, E_2 *are the exceptional divisors of the blow up* π, $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$*;*

(2) $(X, L) \simeq (Bl_{\Lambda_1, Z_1}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$*, where* $Bl_{\Lambda_1, Z_1}(\mathbb{P}^n)$ *is the blow up of* $\mathbb{P}^n$ *along a linear subspaces* Λ_1 *and along a quadric* $Z_1 \subset \Lambda_2 \simeq \mathbb{P}^{\dim Z_1 + 1}$ *such that*

$$\Lambda_1 \cap \Lambda_2 = \emptyset, \quad \dim Z_1 \geq \tfrac{n}{2} - 1, \quad \dim \Lambda_1 + \dim Z_1 = n - 2$$

and E_1, E_2 *are the exceptional divisors of the blow up* π, $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$*;*

(3) $(X, L) \simeq (Bl_{Z_1, Z_2}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$*, where* $Bl_{Z_1, Z_2}(\mathbb{P}^n)$ *is the blow up of* $\mathbb{P}^n$ *along two quadrics* $Z_1 \subset \Lambda_1 \simeq \mathbb{P}^{\frac{n}{2}}$ *and* $Z_2 \subset \Lambda_2 \simeq \mathbb{P}^{\frac{n}{2}}$ *such that*

$$\dim \Lambda_1 \cap \Lambda_2 = 0, \quad \dim Z_1 = \dim Z_2 = \tfrac{n}{2} - 1$$

and E_1, E_2 *are the exceptional divisors of the blow up* π, $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ *(clearly* n *is even).*

Finally, we study the Kleiman-Mori cone of rationally cubic connected manifolds that are listed in the previous theorem. Moreover, after having described $NE(X)$, we characterize the family V and we show how cycles parametrized by the Chow family $\mathcal{V}$ associated to V can split.

Unless otherwise stated, we work over the field $\mathbb{C}$ of complex numbers, and our notation is consistent with the usual one, as for instance in [Har77], [Kol96] and [Deb01].

Chapter 1

Background material

For all the material in this chapter the main references are the first chapters of [Kol96] and [Deb01].

1.1 Intersection number

Let X be a proper scheme of dimension n and let $D_1, \ldots, D_r$ be Cartier divisors on X with $r \geq n$.

Definition 1.1. The *intersection number* $D_1 \cdot \ldots \cdot D_r$ is the coefficient of $m_1 \cdots m_r$ in the polynomial

$$\chi(X, m_1 D_1 + \ldots + m_r D_r) := \Sigma_i (-1)^i h^i(X, m_1 D_1 + \ldots + m_r D_r)$$

If Y is a closed subscheme of X of dimension at most s, we also set

$$D_1 \cdot \ldots \cdot D_s \cdot Y = D_{1|Y} \cdot \ldots \cdot D_{s|Y}$$

Remark 1.2. If $r > n$ then $D_1 \cdot \ldots \cdot D_r = 0$.

Remark 1.3. If D is a Cartier divisor and C is a complete curve on X (i.e. C is a integral proper one-dimensional subscheme of X), we can consider the intersection number $D \cdot C$ which is the leading coefficient of the polynomial $\chi(X, mD_{|C})$. Note that the Riemann-Roch theorem ([Har77, IV.1.3])

$$\chi(X, mD_{|C}) = m \deg(\mathcal{O}_C(D)) + \chi(C, \mathcal{O}_C)$$

implies

$$D \cdot C = \deg(\mathcal{O}_C(D))$$

Definition 1.4. A *1-cycle* Γ on X is a formal linear combination of irreducible curves with integral coefficients:

$$\Gamma = \sum_{i=1}^{s} n_i C_i$$

In particular, if all the coefficients are nonnegative the 1-cycle Γ is called *effective*.

Notation 1.5. We denote by $Z_1(X)$ the free abelian group of the 1-cycles on X and by $Div(X)$ the group of the Cartier divisors on X.

Definition 1.6. Two Cartier divisors $D, D' \in Div(X)$ are *numerically equivalent* if $D \cdot C = D' \cdot C$ for every curve $C \subset X$. We write $D \equiv D'$.
The quotient of $Div(X)$ by this equivalence relation is denoted by $N^1(X)_{\mathbb{Z}}$, and we can also consider the $\mathbb{R}$-vector space

$$N^1(X) := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$$

Definition 1.7. Two 1-cycles $\Gamma, \Gamma' \in Z_1(X)$ are *numerically equivalent* if $D \cdot \Gamma = D' \cdot \Gamma$ for every Cartier divisor $D \in Div(X)$.
The quotient of $Z_1(X)$ by this equivalence relation is denoted by $N_1(X)_{\mathbb{Z}}$, and we can also consider the $\mathbb{R}$-vector space

$$N_1(X) := N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$$

Definition 1.8. Let $f : X \to Y$ be a proper morphism, let Γ an irreducible 1-cycle on X and set $\Gamma' := f(\Gamma)$.
We define the *push-forward* $f_* : Z_1(X) \to Z_1(Y)$ as follows:

$$f_*\Gamma = \begin{cases} 0 & \text{if } \dim \Gamma' = 0 \\ \\ \deg(f_{|\Gamma})\Gamma' & \text{if } \dim \Gamma' = 1 \end{cases}$$

Remark 1.9. If $D \in Div(Y)$ and C is a curve on X, we have so-called *projection formula*:

$$f^*D \cdot C = D \cdot f_*C$$

where $f^* : Div(Y) \to Div(X)$ is the pull-back.

Definition 1.10. Let S be a normal surface and X a proper scheme.
Two effective 1-cycles $\Delta, \Delta' \in Z_1(S)$ are *effectively algebraically equivalent* if there exist a proper flat morphism $p : S \to C$ onto a smooth curve C and two points $x, x' \in C$ such that $\Delta = p^{-1}(x)$ and $\Delta' = p^{-1}(x')$.
Two effective 1-cycles $\Gamma, \Gamma' \in Z_1(X)$ are *effectively algebraically equivalent* if there exist a normal surface S, a proper morphism $g : S \to X$ and two effectively algebraically equivalent 1-cycles $\Delta, \Delta' \in Z_1(S)$ such that $\Gamma = g_*\Delta$ and $\Gamma' = g_*\Delta'$.
The transitive hull of this relation defines an equivalence relation on $Z_1(X)$, which we call *effective algebraic equivalence*.

Definition 1.11. Two effective 1-cycles $\Gamma, \Gamma' \in Z_1(X)$ are *algebraically equivalent* if there exists a 1-cycle E such that $\Gamma + E$ and $\Gamma' + E$ are effectively algebraically equivalent.

Remark 1.12. Note that if $\Gamma, \Gamma' \in Z_1(X)$ are algebraically equivalent then they are also numerically equivalent.

The intersection form induces a nondegenerate pairing

$$N^1(X) \times N_1(X) \quad \to \quad \mathbb{R}$$

which makes these vector spaces canonically dual. Moreover, they are finite-dimensional by the Néron-Severi theorem, and the number

$$\rho_X = \dim N^1(X) = \dim N_1(X)$$

is called the *Picard number* of X.

Definition 1.13. The *cone of curves* $NE(X) \subset N_1(X)$ is the convex cone generated by the numerical equivalence classes $[\Gamma]$ of effective 1-cycles Γ on X. The *Kleiman-Mori cone* $\overline{NE}(X)$ is the closure of the cone of curves.

Notation 1.14. If $D \in N^1(X)$ is a Cartier divisor, we set

$$NE(X)_{D\geq 0} = \{\Gamma \in NE(X) | D \cdot \Gamma \geq 0\}$$

and similarly $NE(X)_{D\leq 0}$, $NE(X)_{D>0}$, $NE(X)_{D<0}$, $N_1(X)_{D\geq 0}$, etc.

For a projective variety, we have the following numerical characterization of ampleness:

Theorem 1.15. ***(Kleiman's Criterion)*** *Let X be a projective variety.*

(1) *A Cartier divisor D on X is ample if and only if $D \cdot z > 0$ for every $z \in \overline{NE}(X) \backslash \{0\}$.*

(2) *For every ample divisor H and for every integer k, the set*

$$\{z \in \overline{NE}(X) : H \cdot z \leq k\}$$

is compact, hence it contains a finite number of numerical classes of irreducible curves.

Definition 1.16. A Cartier divisor D on a proper scheme X is *numerically effective*, or *nef*, if $D \cdot \Gamma \geq 0$ for every $\Gamma \in \overline{NE}(X) \backslash \{0\}$ or equivalently $D \cdot C \geq 0$ for every curve $C \subseteq X$.

The Kleiman's criterion implies that ampleness is a numerical property, and so is nefness, so we talk about ample and nef classes of Cartier divisors in $N^1(X)$.
Moreover, it follows easily from Kleiman's criterion that the ample classes generate an open cone in $N^1(X)$, which is called the *ample cone* and whose closure coincides with the *nef cone*, i.e. the cone generated by the classes of nef divisors on X.

Corollary 1.17. *The Kleiman-Mori cone $\overline{NE}(X)$ of a projective variety X contains no lines, i.e. it is entirely contained in an open half-space plus the origin.*

1.2 Fano manifolds

Definition 1.18. A smooth complex projective manifold is called *Fano* if its anticanonical bundle $-K_X$ is ample.

Definition 1.19. Let X be a Fano manifold of dimension n. We define the *index* of X as

$$r_X = \max\{t \in \mathbb{N} : -K_X \equiv tL\}$$

where L is a ample divisor on X. We also define the *pseudoindex* of X as

$$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m \text{ for some rational curve } C \subset X\}$$

Remark 1.20. Since X is smooth, $Pic(X)$ is torsion free and therefore the divisor H satisfying $-K_X = r_X L$ is uniquely determined and called the *fundamental divisor* of X.

It is easy to see that r_X divides i_X, and that $i_X \leq n+1$.
The characterization of Fano manifolds of index $r_X \geq n$ is due to Kobayashi and Ochiai:

Theorem 1.21. *([KO73]) Let X be a Fano manifold of dimension n and let L be the fundamental divisor of X. Then:*

1. $r_X \leq n+1$;
2. $r_X = n+1 \Leftrightarrow (X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
3. $r_X = n \Leftrightarrow (X, L) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

In particular, Fano manifolds of index $n-1$ are called Del Pezzo manifolds and have been classified in [Fuj90] using the Apollonius method, i.e. proving that the linear system $|L|$ contains a smooth divisor and constructing a ladder down to the well-known case of surfaces.
By the classification of Fano manifolds of dimension 3, due to Fano, Iskovskikh

([Isk77], [Isk78]), Mori and Mukai ([MM82],[MM03] and [Muk04]), using the same method, Fano varieties of index $n-2$, called Mukai varieties, have been classified; in [Muk89] Mukai announced the classification assuming the existence of a smooth member in $|L|$, and this is proved by Mella in [Mel99].

Chapter 2

Rational Curves on Varieties

In this chapter we introduce the theory of rational curves on varieties, and we define uniruled and rationally connected varieties.

2.1 Parameter Spaces

In order to study the geometry of rational curves on a variety X, it is crucial to consider some space parametrizing such objects. There are several notions of "parameter space for rational curves on X". We can view rational curves as subschemes of X, effective 1-cycles or even as morphisms from $\mathbb{P}^1$ to X.

2.1.1 Chow Schemes

Let X be a projective variety. The Chow scheme $\mathrm{Chow}(X)$ is a scheme parametrizing effective cycles on X. There is a subscheme $\mathcal{U}_{\mathrm{Chow}(X)} \subset \mathrm{Chow}(X) \times X$ satisfying the following properties.

- Every connected component V of $\mathrm{Chow}(X)$ is a reduced, projective scheme and $V \times_{\mathrm{Chow}(X)} \mathcal{U}_{\mathrm{Chow}(X)}$ is an effective cycle on $V \times X$.
- For any normal scheme T, and any family $\mathcal{C} \to T$ of effective cycles on X parametrized by T, there is a unique morphism $T \to \mathrm{Chow}(X)$ such that $\mathcal{C}$ is the pullback of $\mathcal{U}_{\mathrm{Chow}(X)}$ to $T \times X$.

To study rational curves, we want to consider the subset of $\mathrm{Chow}(X)$ which parametrizes effective 1-cycles on X.

Theorem 2.1. *There exists a projective scheme $Chow_1(X)$, which parametrizes effective 1-cycles on X, with the property that if two 1-cycles belong to the same irreducible component of $Chow_1(X)$ then they are effectively algebraically equivalent.*

2.1.2 Hom Schemes

Let Y and X be varieties. If Y is projective and X is smooth and quasi-projective, then there exists a locally Noetherian scheme $Hom(Y, X)$ parametrizing morphisms $f : Y \to X$ (we denote by $[f]$ the corresponding points in $Hom(Y, X)$). This scheme has the following universal property:

- for any scheme T and for every morphism $F : Y \times T \to X$ there exists a unique morphism $F' : T \to Hom(Y, X)$ such that the diagram

$$\begin{tikzcd} Y \times Hom(Y,X) \arrow[r, "e"] & X \\ Y \times T \arrow[u, "Id \times F'"] \arrow[ur, "F"'] & \end{tikzcd}$$

  commutes, where $e : Y \times Hom(Y, X) \to X$ denotes the evaluation map which sends $(y, [f])$ to $f(y)$.

In general the scheme $Hom(Y, X)$ has countably many components, but each irreducible component is in fact a quasi-projective variety.
The following theorem ([Kol96, II.1.7]) provides very important informations about its local structure:

Theorem 2.2. *Let $f_0 : Y \to X$ be a morphism from a projective variety Y to a smooth quasi-projective variety X. Then*

(1) *the Zariski tangent space to $Hom(Y, X)$ is*

$$T_{[f_0]}Hom(Y, X) \simeq H^0(Y, f_0^*TX)$$

where TX denotes the tangent bundle of X;

(2) $\dim_{[f_0]} Hom(Y, X) \geq h^0(Y, f_0^*TX) - h^1(Y, f_0^*TX)$*;*

(3) *if $H^1(Y, f_0^*TX) = 0$ then $Hom(Y, X)$ is smooth at $[f_0]$ and has dimension $h^0(Y, f_0^*TX)$.*

The same construction holds if we consider morphisms from Y to X which fix a closed subscheme $B \subset Y$; more precisely, if $g : B \to X$ is a given morphism we can consider the scheme $Hom(Y, X; g)$ which parametrizes morphisms $f : Y \to X$ such that $f_{|B} = g$.
Clearly $Hom(Y, X; g)$ is a subscheme of $Hom(Y, X)$, and it has similar properties:

(1) $T_{[f_0]}Hom(Y, X; g) \simeq H^0(Y, f_0^*TX \otimes \mathcal{I}_B)$ where $\mathcal{I}_B$ denotes the ideal sheaf of B in Y;

(2) $\dim_{[f_0]} Hom(Y, X; g) \geq h^0(Y, f_0^*TX \otimes \mathcal{I}_B) - h^1(Y, f_0^*TX \otimes \mathcal{I}_B)$;

(3) if $H^1(Y, f_0^*TX \otimes \mathcal{I}_B) = 0$ then $Hom(Y, X; g)$ is smooth at $[f_0]$ and has dimension $h^0(Y, f_0^*TX \otimes \mathcal{I}_B)$.

2.1.3 Parametrizing curves on varieties

Now we consider the scheme $Hom(Y, X)$ in the special case when Y is a proper curve C without embedded points. In this case the previous theorems get simpler:

Theorem 2.3. *Let X be a smooth quasi-projective variety, C a proper curve without embedded points of genus $g(C)$, and $f : C \to X$ a morphism. Then*

(1) $T_{[f]}Hom(C, X) \simeq H^0(C, f^*TX)$;

(2) $\dim_{[f]} Hom(C, X) \geq -K_X \cdot f_*C + \dim X(1 - g(C))$.

Theorem 2.4. *Let X be a smooth quasi-projective variety, C a proper curve without embedded points of genus $g(C)$, and $f : C \to X$ a morphism. Let B be a closed subscheme of C of finite length $l(B)$ and $g : B \to X$ a morphism. Then*

(1) $T_{[f]}Hom(C, X; g) \simeq H^0(C, f^*TX \otimes \mathcal{I}_B)$;

(2) $\dim_{[f]} Hom(C, X; g) \geq -K_X \cdot f_*C + \dim X(1 - g(C) - l(B))$.

2.1.4 Parametrizing rational curves

Let X be a normal projective variety and let $Hom(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f : \mathbb{P}^1 \to X$.
We consider $Hom_{bir}(\mathbb{P}^1, X) \subset Hom(\mathbb{P}^1, X)$, the open subscheme corresponding to those morphisms $f : \mathbb{P}^1 \to X$ which are birational onto their image, and its normalization $Hom^n_{bir}(\mathbb{P}^1, X)$.
By the Lüroth theorem, every nonconstant morphism $f : \mathbb{P}^1 \to X$ can be written as

$$\mathbb{P}^1 \xrightarrow{w} \mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$$

where g is birational onto its image. Thus, at least set-theoretically, $Hom^n_{bir}(\mathbb{P}^1, X)$ contains all information about $Hom(\mathbb{P}^1, X)$.
Moreover, if h is any automorphism of $\mathbb{P}^1$ and $f \in Hom^n_{bir}(\mathbb{P}^1, X)$, then $f \circ h$ is counted as a different morphism, while for our purposes they should be considered as the same rational curve. For this reason we consider the group action of $Aut(\mathbb{P}^1)$ on $Hom^n_{bir}(\mathbb{P}^1, X)$ and we define the quotient:

Definition 2.5. The space $Ratcurves^n(X)$ is the quotient of $Hom^n_{bir}(\mathbb{P}^1, X)$ by $Aut(\mathbb{P}^1)$, and the space $Univ(X)$ is the quotient of the product action of $Aut(\mathbb{P}^1)$ on the space $Hom^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1$.

There is a natural commutative diagram

$$\begin{array}{ccc} Hom^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 & \xrightarrow{U} & Univ(X) \\ \downarrow & & \downarrow{\scriptstyle p} \\ Hom^n_{bir}(\mathbb{P}^1, X) & \xrightarrow{u} & Ratcurves^n(X) \end{array} \tag{2.1}$$

where u and U are principal $Aut(\mathbb{P}^1)$-bundles and p is a $\mathbb{P}^1$-bundle.

If we fix a point $x \in X$, then we can consider the scheme $Hom(\mathbb{P}^1, X; 0 \mapsto x)$ which parametrized morphisms $f : \mathbb{P}^1 \to X$ sending $0 \in \mathbb{P}^1$ to x. Again we have a commutative diagram

$$\begin{array}{ccc} Hom^n_{bir}(\mathbb{P}^1, X; 0 \mapsto x) \times \mathbb{P}^1 & \xrightarrow{U} & Univ(X, x) \\ \downarrow & & \downarrow{\scriptstyle p} \\ Hom^n_{bir}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{u} & Ratcurves^n(X, x) \end{array} \tag{2.2}$$

Remark 2.6. For every integer $d \geq 0$ we can consider the quasi-projective scheme $Hom_d(\mathbb{P}^1, X)$ which parametrizes morphisms $\mathbb{P}^1 \to X$ of degree d with respect to a given ample divisor, and the space $Hom(\mathbb{P}^1, X)$ can be written as the disjoint union

$$\bigcup_{d \geq 0} Hom_d(\mathbb{P}^1, X).$$

This implies that on a projective variety X there exist only countably many numerical classes of rational curves. Moreover, for every positive integer d and any ample divisor H there exists only finite number of numerical classes of rational curves of H-degree $\leq d$.

2.2 Uniruled and rationally connected varieties

In this section we briefly discuss the theory of uniruled, rationally connected and rationally chain connected varieties.
We refer to [Kol96, IV] and to [Deb01, Chapter 4] for proofs.

Definition 2.7. A variety X of dimension n is called *uniruled* if there exist a variety Y of dimension $n-1$ and a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$.

If X is a proper variety defined over an algebraically closed field k, X uniruled obviously implies that X is covered by rational curves, and the converse holds if k is uncountable; in particular, we can give this equivalent definition:

Definition 2.8. Let X be a proper variety of dimension n over an uncountable algebraically closed field k. X is called *uniruled* if there is a rational curve through a general point.

Moreover, if we assume that X is a smooth projective variety over an uncountable algebraically closed field of characteristic zero, there is another characterization of uniruledness in terms of rational curves on X, i.e. the uniruledness is equivalent to the existence of a single rational curve over which the tangent bundle TX is generated by global sections.
Let X be a smooth projective variety of dimension n and let $f : \mathbb{P}^1 \to X$ be a morphism. By Grothendieck's theorem, the vector bundle f^*TX decomposes as a sum of line bundles

$$f^*TX \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

where we assume $a_1 \geq ... \geq a_n$. If f is nonconstant, then there is a sheaf inclusion $\mathcal{O}_{\mathbb{P}^1}(2) \cong T\mathbb{P}^1 \hookrightarrow f^*TX$, and thus $a_1 \geq 2$.

Remark 2.9. Note that f^*TX is generated by its global sections if and only if $a_n \geq 0$

Definition 2.10. Let r be a nonnegative integer. A rational curve $f : \mathbb{P}^1 \to X$ on a smooth variety X is *r-free* if $f^*TX \otimes \mathcal{O}_{\mathbb{P}^1}(-r)$ is generated by its global sections, i.e. if $a_n \geq r$.
If f is 0-free (respectively 1-free), then f is called *free* (respectively *very free*).

Proposition 2.11. *Let X be a smooth projective variety over an uncountable algebraically closed field of characteristic zero. X is uniruled if and only if X admits a free rational curve.*

Another important notion is that of rational connectedness. Rationally connected varieties were introduced by Kollár, Miyaoka and Mori in [KMM92c], and independently by Campana in [Cam92].

Definition 2.12. A variety X of dimension n is called *rationally connected* if it is proper and if there exist a variety M and a rational map $e : \mathbb{P}^1 \times M \dashrightarrow X$ such that the rational map

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 \times M & \dashrightarrow & X \times X \\ (t, t', z) & \mapsto & (e(t,z), e(t',z)) \end{array}$$

is dominant.

As before, if we assume that the ground field k is algebraically closed and uncountable, the definition of rationally connected variety becomes very simple:

Definition 2.13. Let X be a variety of dimension n over an uncountable algebraically closed field k. X is called *rationally connected* if it is proper and there is a irreducible rational curve through a general pair of points.

Indeed, over $\mathbb{C}$ or over any other uncountable algebraically closed field of characteristic zero, there is a relationship between rational connectedness and the existence of a very free rational curve:

Theorem 2.14. *Let X be a smooth projective variety over an uncountable algebraically closed field of characteristic zero.*
X is rationally connected if and only if X contains a very free rational curve.

Now we study and define rationally chain connected varieties over an algebraically closed field:

Definition 2.15. Let X be a variety of dimension n over an algebraically closed field k. X is *rationally chain connected* if it is proper and there exist a variety T and a subscheme $\mathcal{C}$ of $T \times X$ such that:

- the fibers of the projection $\mathcal{C} \to T$ are (connected proper) 1-cycles with only rational components;
- the projection $\mathcal{C} \times_T \mathcal{C} \to X \times X$ is dominant.

If the ground field k is uncountable, we have the following equivalent definition:

Definition 2.16. Let X be a variety of dimension n over an uncountable algebraically closed field k. X is *rationally chain connected* if and only if for very general closed points $x_1, x_2 \in X$ there is a connected 1-cycle $\Gamma \subset X$ which contains x_1 and x_2 such that every irreducible component of Γ is rational.

Note that being rationally connected is stronger that being rationally chain connected. But if we consider an uncountable algebraically closed field of characteristic zero (for example $k = \mathbb{C}$), and if X is a smooth proper variety, these definitions are equivalent:

Theorem 2.17. *Let X be a smooth proper variety over an uncountable algebraically closed field of characteristic zero. Then X is rationally chain connected if and only if X is rationally connected.*

Chapter 3

Mori theory for smooth varieties

3.1 Bend and Break technique

Mori's "bend and break" argument was originally introduced in his famous paper [Mor79] in order to prove Hartshorne's conjecture about varieties with ample tangent bundle. His techniques have turn out to be a very powerful tool for investigating the birational geometry of algebraic varieties.
Mori's main idea is the following: if a curve (of positive genus) on a variety X deforms nontrivially while keeping a point fixed, then it breaks up into an effective 1-cycle with a rational component passing through the fixed point.
The first bend and break lemma (Lemma (3.1)) proves exactly that, given a curve, if its space of deformations is sufficiently big, then a rational curve is produced.
Moreover, the second bend and break lemma (Lemma (3.2)) says that a curve deforming nontrivially, while keeping two point fixed, must degenerate into an effective 1-cycle with rational components.

Lemma 3.1. ***(Bend and Break I)*** *Let $f : C \to X$ be a smooth curve on a projective variety X, and let c be a point on C. If*

$$\dim_{[f]} Hom(C, X; f_{|\{c\}}) \geq 1$$

then there exists a curve $f' : C' \to X$ and a connected effective nonzero rational 1-cycle Γ on X which passes through $f(c)$ and satisfies

$$f_* C \sim f'_* C' + \Gamma$$

According to Theorem (2.4), *when X is smooth along $f(C)$*, the hypothesis is fulfilled whenever

$$-K_X \cdot f_* C - g(C) \dim X \geq 1$$

Lemma 3.2. ***(Bend and Break II)*** *Let X be a projective variety and let $f : \mathbb{P}^1 \to X$ be a rational curve. If*

$$\dim_{[f]} Hom(\mathbb{P}^1, X; f_{|\{0,\infty\}}) \geq 2$$

then there exists a connected nonintegral effective rational 1-cycle Γ passing through $f(0)$ and $f(\infty)$ and such that $f_*\mathbb{P}^1 \sim \Gamma$. In particular they are numerically equivalent.

According to Theorem (2.4), *when X is smooth along $f(\mathbb{P}^1)$*, the hypothesis is fulfilled whenever

$$-K_X \cdot f_*\mathbb{P}^1 - \dim X \geq 2$$

The bend and break lemmas are very important to the study of Fano manifolds. In fact, using the bend and break lemmas, Mori proved that a Fano manifold X is covered by rational curves.

Theorem 3.3. *Let X be a smooth projective variety of dimension n such that $-K_X$ is ample. Then through any point of X there exists a rational curve $\Gamma \subset X$ satisfying $-K_X \cdot \Gamma \leq n + 1$.*

3.2 The Cone theorem

In this section we state Mori's theorem on the structure of the Kleiman-Mori cone of a smooth projective variety X.
Mori showed that the part of cone contained in $N_1(X)_{K_X<0}$ is generated by countable many extremal rays and that these rays can only accumulate on the hyperplane $N_1(X)_{K_X=0}$.
This result is known as the Cone theorem and it was proved by Mori in [Mor82].

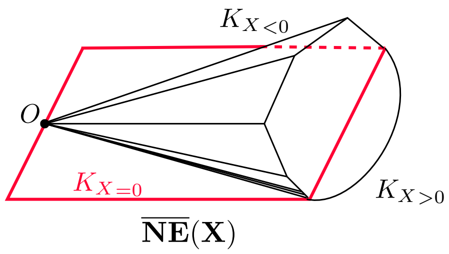
Theorem 3.4. *Let X be a smooth projective variety of dimension n. Then there exist on X countably many rational curves $\{C_i\}_{i\in\mathbb{N}}$ such that*

$$0 < -K_X \cdot C_i \leq n + 1$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X\geq 0} + \sum_{i\in\mathbb{N}} \mathbb{R}^+[C_i]$$

where the $\mathbb{R}^+[C_i]$ are all the (distinct) extremal rays of $\overline{NE}(X)$ that meet the half-space of $N_1(X)$ given by $\{z \in N_1(X) \mid K_X \cdot z < 0\}$. These rays are locally discrete in that half-space.



Remark 3.5. Since the Kleiman-Mori cone of a Fano manifold is entirely contained in the half-space $N_1(X)_{K_X<0}$ by definition, the Cone theorem immediately yields that *the Kleiman-Mori cone of a Fano manifold is polyhedral.*

Definition 3.6. A subcone σ of $\overline{NE}(X)$ is called an *extremal face* if it satisfies the following condition:

$$a, b \in \overline{NE}(X) \text{ and } (a+b) \in \sigma \Rightarrow a, b \in \sigma$$

An extremal face of dimension one is called an *extremal ray*, and a curve whose numerical class belongs to an extremal ray is called a *extremal curve.*

Definition 3.7. An extremal face σ of $\overline{NE}(X)_{K_X<0}$ is called a *negative extremal face* of $\overline{NE}(X)$; a negative extremal face of dimension one is called a *negative extremal ray.*

3.2.1 Fano-Mori contractions

Definition 3.8. A *contraction* $f : X \to Y$ is a proper morphism with connected fibers between two normal varieties X and Y.

If X and Y are projective, we define the relative cone of f as the convex subcone $NE(f)$ of $NE(X)$ generated by the classes of curves contracted by f. Since Y is projective, an irreducible curve C on X is contracted by f if and only if $f_*C = 0$. It follows that $NE(f)$ is the intersection of $NE(X)$ with the vector space $\ker(f_*)$. Moreover, by [Deb01, Proposition 1.14], $NE(f)$ is extremal and the morphism f is uniquely determinated by $NE(f)$ up to isomorphism.

Definition 3.9. A *Fano-Mori contraction* $\varphi : X \to Y$ of a smooth variety X is a contraction such that $-K_X \cdot C > 0$ for any contracted curve.

Remark 3.10. Note that if φ is a Fano-Mori contraction, then the relative cone $NE(\varphi)$ is a negative extremal face of $NE(X)$.

Theorem 3.11. ***(Rationality Theorem)*** *Let X be a smooth complex projective variety such that K_X is not nef. Let H be a nef and big Cartier divisor on X. Then the number*

$$r = \sup\{t \in \mathbb{R} \mid H + tK_X \text{ is nef}\}$$

is rational.

Corollary 3.12. *Let X be a smooth projective variety and let σ be a negative extremal face of $\overline{NE}(X)$. Then there exists a nef divisor H on X such that:*

(1) $\sigma = \{z \in \overline{NE}(X) : H \cdot z = 0\}$;

(2) *the divisor $mH - K_X$ is ample for all integers $m \gg 0$.*

The divisor H is called a supporting divisor of the face σ.

Theorem 3.13. ***(Base-point free Theorem)*** *Let X be a smooth projective variety and let H be a nef divisor on X such that $aH - K_X$ is nef and big for some positive integer a. Then the linear system $|mH|$ is base-point free for all integers $m \gg 0$.*

Combining these two results, we have that to a negative extremal face σ of $\overline{NE}(X)$ we can associate a nef divisor H, one multiple of which induces a morphism $\varphi_{|mH|} : X \to Y \subseteq \mathbb{P}^N$. The part with connected fibers of the Stein factorization of $\varphi_{|mH|}$ is a Fano-Mori contraction; namely the following theorem holds:

Theorem 3.14. ***(Contraction Theorem)*** *Let X be a smooth variety and let H be a nef divisor on X such that*

$$\sigma := H^{\perp} \cap \overline{NE}(X)$$

is entirely contained in $\{z \in N_1(X) : K_X \cdot z < 0\}$ (i.e. H is a supporting divisor of σ). Then there exists a projective morphism

$$\varphi : X \to Y$$

onto a normal projective variety Y, which is characterized by the following properties:

(1) *a curve $C \subset X$ is contracted to a point by φ if and only if $H \cdot C = 0$;*

(2) *φ has connected fibers;*

(3) *$H = \varphi^* A$ for some ample Cartier divisor $A \in Div(Y)$.*

The map φ is usually called the Fano-Mori contraction (or the extremal contraction) associated to the face σ and the Cartier divisor H is called a supporting divisor of the map φ (or of the face σ).

Definition 3.15. An extremal contraction associated to a face of dimension one, i.e. a negative extremal ray, is called an *elementary contraction.*

Notation 3.16. Since the Cone theorem and Contraction theorem give us no informations about positive part of $NE(X)_{K_X\geq 0}$ of $NE(X)$, we will focus our attention on negative extremal faces and rays, and from now on we will simply call them "extremal".

Definition 3.17. We denote by

$$E = E(\varphi) = \{x \in X : \dim(\varphi^{-1}\varphi(x)) > 0\}$$

the *exceptional locus* of φ; it coincides with the union of all curves in X which are contracted by φ, and from this reason it is sometimes called *locus* of φ.

Definition 3.18. If $E = X$, i.e. $\dim X > \dim Y$, then φ is called *of fiber type*. Moreover, if $\dim X = \dim Y$, φ is called *birational*. In particular:

- if the codimension of E is equal to 1, then φ is *divisorial*;
- if the codimension of E is at least 2, then φ is *small*.

Definition 3.19. let X be a smooth complex projective variety. Let $\varphi : X \to Z$ be an elementary extremal contraction of fiber type. φ is called a *scroll* (respectively a *quadric fibration*) if there exists a line bundle $L \in Pic(X)$ such that it is φ-ample (i.e. $L \cdot C > 0$ for any curve contracted by φ) and $K_X + (\dim X - \dim Z + 1)L$ (respectively $K_X + (\dim X - \dim Z)L$) is a supporting divisor of φ.
φ is called a $\mathbb{P}$-*bundle* if Z is smooth and there exists a vector bundle $\mathcal{F}$ of rank $(\dim X - \dim Z + 1)$ on Z such that $X \simeq \mathbb{P}_Z(\mathcal{F})$.

Remark 3.20. An equidimensional scroll is a $\mathbb{P}$-bundle by [Fuj87, Lemma 2.12]. Moreover some special scroll contractions arise from projectivization of Bănică sheaves; a Bănică sheaf is a coherent sheaf ε of rank $r \geq 2$ over a normal variety Y whose projectivization is a smooth variety.
In particular, if φ is a scroll such that every fiber has dimension $\leq \dim X - \dim Z + 1$, then Z is smooth and X is the projectivization of a Bănică sheaf on Z ([BW96, Proposition 2.5]). We will call these contractions *special Bănică scrolls*.

Chapter 4

Families of rational curves

Let X be a smooth complex projective variety of dimension n.

4.1 Families of rational curves

Definition 4.1. A *family of rational curves* V on X is an irreducible component of the scheme $Ratcurves^n(X)$.
Given a rational curve $f : \mathbb{P}^1 \to C \subset X$, a *family of deformations* of that curve is any irreducible component of $Ratcurves^n(X)$ containing the point parametrizing that curve.
We define the *locus of the family* V to be the set of points of X through which there is a curve among those parametrized by V. We denote it by $\mathrm{Locus}(V)$.
We say that V is a *covering family* if $\mathrm{Locus}(V) = X$ and that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.
We denote by V_x the subscheme of V parametrizing rational curves passing through a point $x \in \mathrm{Locus}(V)$ and by $\mathrm{Locus}(V_x)$ the set of the points of X through which there is a curve among those parametrized by V_x.

Definition 4.2. Let V be a family of rational curves on X. Then

(1) V is *unsplit* if it is proper;

(2) V is *locally unsplit* if, for a general point $x \in \mathrm{Locus}(V)$, V_x is proper;

(3) V is *generically unsplit* if the fiber of the double-evaluation map

$$\begin{array}{rccc} \Pi : & V & \to & X \times X \\ & [f] & \mapsto & (f(q), f(p)) \end{array}$$

over the general point of its image has dimension at most 0.

Remark 4.3. By [Kol96, II.2.11], $Ratcurves^n(X)$ has a natural inclusion into the scheme $\mathrm{Chow}(X)$, so we can consider the image of V in $\mathrm{Chow}(X)$. We denote it by $\widetilde{V}$.
By [Kol96, II.2.2], V is proper if and only if $\widetilde{V}$ is closed in $\mathrm{Chow}(X)$.
We denote by $\mathcal{V}$ the closure of $\widetilde{V}$ in $\mathrm{Chow}(X)$. A point $w \in \mathcal{V} \setminus \widetilde{V}$ corresponds to a 1-cycle $\sum a_i[C_i]$ where C_i are (irreducible) rational curves on X, $a_i \in \mathbb{N}$ and $\sum a_i \geq 2$.
Then, if V is not an unsplit family, the general rational curve in V degenerates and in the limit it splits up into a reducible 1-cycle.

Remark 4.4. Note that (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Remark 4.5. If V is an unsplit dominating family of rational curves, then it is covering.

Notation 4.6. By abuse of notation, given a line bundle $L \in Pic(X)$, we denote by $L \cdot V$ the intersection number $L \cdot C$, with C any curve among those parametrized by V.

Definition 4.7. Let U be an open dense subset of X and $\pi : U \to Z$ a proper surjective morphism to a quasi-projective variety; we say that a family of rational curves V is a *horizontal dominating family with respect to* π if $\mathrm{Locus}(V)$ dominates Z and curves parametrized by V are not contracted by π.

Remark 4.8. Let R be an extremal ray of $NE(X)$ and let C be an extremal curve such that $[C] \in R$ and the anticanonical degree of C is minimal in R; C is often called a *minimal extremal rational curve.*
If we denote by V a family of deformations of C, then V is unsplit: in fact, if C degenerates into a reducible cycle, its components must belong to the ray R, since R is extremal; but in R the curve C has the minimal intersection with the anticanonical bundle, hence this is impossible.

Proposition 4.9. *Let V be a family of rational curves on X and $x \in Locus(V)$ a point such that every component of V_x is proper. Then*

$$\dim V \leq \dim Locus(V) + \dim Locus(V_x) - 2$$

Proof. Let x be a point of $\mathrm{Locus}(V)$ satisfying our assumptions. Diagram (2.1) can be restricted to the family V, and we obtain the basic diagram

$$\begin{tikzcd} p^{-1}(V) =: U \arrow[r, "i"] \arrow[d, "p"] & X \\ V & \end{tikzcd} \tag{4.1}$$

where i is the map induced by the evaluation map $e : Hom^n_{bir}(\mathbb{P}^1, X) \to X$ and p is a $\mathbb{P}^1$-bundle.
Then $V_x = \{[f] \in V : f(0) = x\} = p(i^{-1}(x))$, and since i doesn't contract any fiber of p we have

$$\dim V_x = \dim i^{-1}(x).$$

By upper semi-continuity of the fiber dimension,

$$\dim i^{-1}(x) \geq \dim U - \dim \mathrm{Locus}(V)$$

$$\begin{aligned} \Rightarrow \dim V_x = \dim i^{-1}(x) &\geq \dim U - \dim \mathrm{Locus}(V) \\ &\geq \dim V - \dim \mathrm{Locus}(V) + 1. \end{aligned}$$

Similarly, if $y \in \mathrm{Locus}(V_x)$, we can consider

$$V_{x,y} = \{[f] \in V : f(0) = x, f(\infty) = y\}$$

and the pointed version of the previous diagram

$$\begin{tikzcd} U_x \arrow[r, "i_x"] \arrow[d, "p_x"'] & \mathsf{Locus}(V_x) \subseteq X \\ V_x & \end{tikzcd}$$

Then $V_{x,y} = p_x(i_x^{-1}(y))$ and, as before

$$\begin{aligned} \dim V_{x,y} &= \dim i_x^{-1}(y) \geq \dim V_x - \dim \mathrm{Locus}(V_x) + 1 \\ \Rightarrow \dim V_{x,y} &\geq \dim V_x - \dim \mathrm{Locus}(V_x) + 1 \\ &\geq \dim V - \dim \mathrm{Locus}(V) - \dim \mathrm{Locus}(V_x) + 2 \end{aligned}$$

If V_x is proper or if V is generically unsplit and x, y are general points in $\mathrm{Locus}(V)$, then $\dim V_{x,y} = 0$, so it follows that

$$\dim V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) - 2$$

□

Proposition 4.10. ***(Ionescu-Wiśniewski Inequality)*** *Let V be a family of rational curves on X and $x \in Locus(V)$ a point such that every component of V_x is proper. Then*

(1) $\dim X - K_X \cdot V \leq \dim Locus(V) + \dim Locus(V_x) + 1$*;*

(2) *every irreducible component of $Locus(V)_x$ has dimension $\geq -K_X \cdot V - 1$.*

Proof. By Theorem (2.3), if V is a family of deformations of f, we have

$$\dim_{[f]} Hom(\mathbb{P}^1, X) \geq -K_X \cdot V + \dim X.$$

But

$$\dim_{[f]} Hom(\mathbb{P}^1, X) = \dim V + \dim Aut(\mathbb{P}^1) = \dim V + 3$$

so, by the previous proposition, we conclude that

$$\dim X - K_X \cdot V \leq \dim V + 3 \leq (\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) - 2) + 3$$

$$\Rightarrow \dim X - K_X \cdot V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1.$$

The second inequality follows from $\dim \mathrm{Locus}(V) \leq \dim X$. □

Remark 4.11. If V is locally unsplit and $-K_X \cdot V = \dim \mathrm{Locus}(V_x) + 1$ for a general $x \in \mathrm{Locus}(V)$, then V is a dominating family.

Proposition 4.12. ***(Fiber Locus Inequality)*** *Let σ be an extremal face of $\overline{NE}(X)$ and let φ be the Fano-Mori contraction associated to σ. Denote by E the exceptional locus of φ and let F be an irreducible component of a (non trivial) fiber of φ. Then*

$$\dim E + \dim F \geq \dim X + l - 1$$

where

$$l = \min\{-K_X \cdot C |\ C \textit{ is a rational curve s.t. } [C] \in \sigma\}$$

If φ is the contraction of an extremal ray R, then $l = l(R)$ is called the length of the ray.

Proof. Note that if V is the family of deformations of a rational curve C which is contained in F then E contains $\mathrm{Locus}(V)$ and F contains $\mathrm{Locus}(V_x)$ for some point $x \in X$. Thus $\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V)_x \leq \dim E + \dim F$, and the claim follows from the Ionescu-Wiśniewski inequality. □

4.2 Chow families of rational 1-cycles

Definition 4.13. A *Chow family of rational 1-cycles* $\mathcal{V}$ is an irreducible component of $\mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles.
We define $\mathrm{Locus}(\mathcal{V})$ to be the set of points of X through which there is a cycle among those parametrized by $\mathcal{V}$.
Notice that $\mathrm{Locus}(\mathcal{V})$ is a closed subset of X ([Kol96, II.2.3]). We say that $\mathcal{V}$ is a *covering family* if $\mathrm{Locus}(\mathcal{V}) = X$.

Definition 4.14. If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$, denoted by $\mathcal{V}$, is called the *Chow family associated to V*.

Definition 4.15. Let V be a family of rational curves and let $\mathcal{V}$ be the Chow family associated to V. We say that V (and also $\mathcal{V}$) is *quasi unsplit* if every component of any reducible cycle parametrized by $\mathcal{V}$ has numerical class proportional to the numerical class of a curve parametrized by V.

Definition 4.16. Let $\mathcal{V}$ be a Chow family of rational 1-cycles.
Let $\Gamma = \sum_{i=1}^{k} \Gamma_i$ be a reducible cycle in $\mathcal{V}$. We denote the family of deformations of the irreducible component Γ_i of Γ by V_i for every i.
The families $\{V_i\}_{i=1,\dots,k}$ are called *fellow families with respect to* $\mathcal{V}$.
The families $\{V_j\}_{j \neq i}$ are called *fellow families of* V_i with respect to $\mathcal{V}$.

4.2.1 Chow families and prerelations

In the language of [Kol96, II.4], a Chow family of rational 1-cycles $\mathcal{V}$ defines a proper prerelation and an algebraic relation.

Definition 4.17. Let U, V, X be schemes. The collection of schemes and morphisms $(V \xleftarrow{s} U \xrightarrow{w} X, V \xrightarrow{\sigma} U)$ such that $s \circ \sigma = Id_V$ is called a *prerelation.*
A prerelation is called *proper* if the morphisms $s, w, w \circ \sigma$ are all proper.

Definition 4.18. Let X/S be a scheme. An *algebraic relation* on X is a scheme R together with a pair of morphism $w : R \to X$ and $u : R \to X$.
Set $\tilde{R} := Im[R \xrightarrow{(w,u)} X \times X] \subset X \times X$ and $\tilde{R}(x) := u(w^{-1}(x)) \subset X$ for $x \in X$.
$\tilde{R}$ is the *set-theoretic relation generated by* R, and $\tilde{R}(x)$ is the *relation class of* x.

Let X be a smooth complex projective variety and let $\mathcal{V}$ be a Chow family of rational 1-cycles on X. We can consider the following diagram, coming from the universal family over $\mathrm{Chow}(X)$:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "i"] \arrow[d, "p"'] & X \\ \mathcal{V} & \end{tikzcd} \tag{4.2}$$

where i is the map induced by the evaluation and the fiber of p are connected and have rational components. Both i and p are proper ([Kol96, II.2.2]).
Taking the normal form, defined in [Kol96, IV.4.4.5], we obtain a proper prerelation $(\mathcal{V}' \xleftarrow{p'} \mathcal{U}' \xrightarrow{i'} X, \mathcal{V}' \xrightarrow{\sigma} \mathcal{U}')$:

$$\begin{tikzcd} \mathcal{U}' := \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \arrow[r, "i'"] \arrow[d, shift right, "p'"'] & X \\ \mathcal{V}' := \mathcal{U} \arrow[u, shift right, "\sigma"'] & \end{tikzcd}$$

Moreover, $(R = \mathcal{U}', w = i' \circ \sigma \circ p', u = i')$ is an algebraic relation.

Remark 4.19. In particular, if V is an unsplit family of rational curves, then V corresponds to the normalization of the associated Chow family $\mathcal{V}$, and V itself defines a proper prerelation.

4.3 Chains of rational curves

Let X be a smooth complex projective variety and Y an irreducible subset of X. Let $\mathcal{V}^1, ..., \mathcal{V}^k$ be Chow families of rational 1-cycles on X.

Definition 4.20. Locus$(\mathcal{V}^1, ..., \mathcal{V}^k)$ is the set of points which belong to a connected chain of k 1-cycles belonging respectively to the families $\mathcal{V}^1, ..., \mathcal{V}^k$.

$$x \in \text{Locus}(\mathcal{V}^1, ..., \mathcal{V}^k) \;\Leftrightarrow\; \begin{array}{l} \exists \;\Gamma_1 \in \mathcal{V}^1, ..., \Gamma_k \in \mathcal{V}^k \text{ such that} \\ \Gamma_i \cap \Gamma_{i+1} \neq \emptyset, \;\; \mathbf{x \in \Gamma_1 \cup ... \cup \Gamma_k} \end{array}$$

In particular, Locus$(\mathcal{V}^1, ..., \mathcal{V}^k) \subset \bigcup_{i=1}^k \text{Locus}(\mathcal{V}^i)$.

Definition 4.21. Locus$(\mathcal{V}^1, ..., \mathcal{V}^k)_Y$ is the set of points that can be jointed to Y by a connected chain of k 1-cycles belonging respectively to the families $\mathcal{V}^1, ..., \mathcal{V}^k$.

$$x \in \text{Locus}(\mathcal{V}^1, ..., \mathcal{V}^k)_Y \;\Leftrightarrow\; \begin{array}{l} \exists \;\Gamma_1 \in \mathcal{V}^1, ..., \Gamma_k \in \mathcal{V}^k \text{ such that} \\ \Gamma_i \cap \Gamma_{i+1} \neq \emptyset, \;\; \Gamma_1 \cap Y \neq \emptyset, \;\; \mathbf{x \in \Gamma_k} \end{array}$$

In particular, Locus$(\mathcal{V}^1, ..., \mathcal{V}^k)_Y \subset \text{Locus}(\mathcal{V}^k)$.

Remark 4.22. If Y is a closed subset, then Locus$(\mathcal{V}^1, ..., \mathcal{V}^k)_Y$ is closed.
Since

$$\text{Locus}(\mathcal{V}^1, ..., \mathcal{V}^k)_Y = \text{Locus}(\mathcal{V}^k)_{\text{Locus}(\mathcal{V}^1,...,\mathcal{V}^{k-1})_Y}$$

it is enough to prove that if Y is a closed subset then Locus$(\mathcal{V})_Y$ is closed.
Let $\mathcal{V}_Y = p(i^{-1}(Y \cap \text{Locus}(\mathcal{V})))$ be the subset of $\mathcal{V}$ parametrizing cycles of $\mathcal{V}$ meeting Y. Then Locus$(\mathcal{V})_Y = i(p^{-1}(\mathcal{V}_Y))$, so it is closed by the properness of the morphisms i and p.

Remark 4.23. Analogously we define Locus$(V^1, ..., V^k)_Y$ for $V^1, \ldots, V^k$ families of rational curves. Notice also that from our definition it follows that Locus$(V_x) =$ Locus$(V)_x$.

If we consider unsplit families $V^1, \ldots, V^k$ on X and Y is a point of X, there exists an lower bound for the dimension of Locus$(V^1, \ldots, V^k)_x$:

Theorem 4.24. *([BCDD03, Théorème 5.2]) Let $V^1, \ldots, V^k$ be k unsplit families of rational curves on X. If the corresponding classes in $N_1(X)$ are independent, then either $Locus(V^1, ..., V^k)_x$ is empty or*

$$\dim Locus(V^1, ..., V^k)_x \geq \sum_{i=1}^{k} -K_X \cdot V^i - k$$

Using the same techniques as in proof of Theorem (4.24), we can show the following:

Lemma 4.25. *([ACO04, Lemma 5.4]) Let $Y \subset X$ be an irreducible closed subset and let V be an unsplit family of rational curves.*
Assume that curves contained in Y are numerically independent from curves parametrized by V, and that $Y \cap Locus(V) \neq \emptyset$.
Then for a general point $y \in Y \cap Locus(V)$

(1) $\dim Locus(V)_Y \geq \dim(Y \cap Locus(V)) + \dim Locus(V)_y$;

(2) $\dim Locus(V)_Y \geq \dim Y - K_X \cdot V - 1$.

Moreover, if $V^1, ..., V^k$ are numerically independent unsplit families such that curves contained in Y are numerically independent from curves parametrized by $V^1, ..., V^k$ then either $Locus(V^1, ..., V^k)_Y = \emptyset$ or

(3) $\dim Locus(V^1, ..., V^k)_Y \geq \dim Y + \sum_{i=1}^{k}(-K_X \cdot V^i) - k$.

Proof. Consider the diagram (4.1). Since V is unsplit, for a point $y \in Y \cap \mathrm{Locus}(V)$ we have

$$\dim i^{-1}(y) = \dim V_y = \dim \mathrm{Locus}(V)_y - 1.$$

We define $V_Y := p(i^{-1}(Y))$ and $U_Y := p^{-1}(V)_y$ and we denote by n the dimension of X. For a general $y \in Y \cap \mathrm{Locus}(V)$, we obtain that

$$\begin{aligned} \dim U_Y &= \dim(Y \cap \mathrm{Locus}(V)) + \dim \mathrm{Locus}(V)_y \\ &\geq (\dim Y + \dim \mathrm{Locus}(V) - n) + \dim \mathrm{Locus}(V)_y \end{aligned}$$

But, by the Ionescu-Wiśniewski inequality, we have that

$$\dim \mathrm{Locus}(V) - n + \dim \mathrm{Locus}(V)_y \geq -K_X \cdot V - 1$$

$$\Rightarrow \dim U_Y \geq \dim Y - K_X \cdot V - 1.$$

Since $\mathrm{Locus}(V)_Y = i(U_Y)$, it is enough to prove that $i : U_Y \to X$ is generically finite.
To show that we take a point $x \in i(U_Y) \setminus Y$ and we assume that, by contradiction, $i^{-1}(x) \cap U_Y$ contains a curve C' which is not contained in any fiber of p. Let B' be the curve $p(C') \subset V_Y$ and let $\nu : B \to B'$ be the normalization of B'. By base change we have the following diagram:

$$\begin{tikzcd} S_B \arrow[r, "j"] \arrow[d, "p_B"'] & X \\ B & \end{tikzcd}$$

Let C_Y be a curve in S_B which dominates B and whose image via j is contained in Y; such a curve exists since the image via j of every fiber of p_B meets Y.
We observe that $j(C_Y)$ is a point or is a curve in $Y \cap \mathrm{Locus}(V)_y$.
If $j(C_Y)$ is a point, we have a one-parameter family of curves passing through two fixed points, and it is impossible because V is an unsplit family.
If $j(C_Y)$ is a curve in $Y \cap \mathrm{Locus}(V)_y$, then from Corollary (6.7) it follows that there exists a curve in Y such that it is numerically proportional to a curve parametrized by V, against the assumptions.
To prove the claim (3) it is enough to recall that

$$\mathrm{Locus}(V^1, ..., V^k)_Y = \mathrm{Locus}(V^k)_{\mathrm{Locus}(V^1,...,V^{k-1})_Y}.$$

□

Remark 4.26. If in the previous theorem V^1 is not a covering family and moreover $\mathrm{Locus}(V^1, ..., V^k)_x$ is nonempty, then

$$\dim \mathrm{Locus}(V^1, ..., V^k)_x \geq \sum_{i=1}^{k} -K_X \cdot V^i - k + 1$$

In fact, by definition

$$\mathrm{Locus}(V^1, ..., V^k)_x = \mathrm{Locus}(V^2, ..., V^k)_{\mathrm{Locus}(V^1)_x}$$

$$\Rightarrow \dim \mathrm{Locus}(V^1, ..., V^k)_x \geq \dim \mathrm{Locus}(V^1)_x + \sum_{i=2}^{k} -K_X \cdot V^i - (k-1)$$

But from Remark (4.11) it follows that $\dim \mathrm{Locus}(V^1)_x \geq -K_X \cdot V^1$ because V^1 is not covering.

Definition 4.27. $\mathrm{ChLocus}_m(\mathcal{V}^1, ..., \mathcal{V}^k)_Y$ is the set of points that can be jointed to Y by a connected chain of at most m rational 1-cycles belonging to the families $\mathcal{V}^1, ..., \mathcal{V}^k$.

$$\begin{aligned} x \in \mathrm{ChLocus}_m(\mathcal{V}^1, ..., \mathcal{V}^k)_Y \quad \Leftrightarrow \quad & \exists \ \Gamma_1, ..., \Gamma_s (s \leq m) \text{ such that } \Gamma_i \in \mathcal{V}^j \\ & \Gamma_i \cap \Gamma_{i+1} \neq \emptyset, \quad \Gamma_1 \cap Y \neq \emptyset, \quad \mathbf{x \in \Gamma_s} \end{aligned}$$

In particular

$$\mathrm{ChLocus}_m(\mathcal{V}^1, ..., \mathcal{V}^k)_Y = \bigcup_{\substack{1 \leq i(j) \leq k \\ 1 \leq s \leq m}} \mathrm{Locus}(\mathcal{V}^{i(1)}, ..., \mathcal{V}^{i(s)})_Y$$

and if Y is a closed subset, then $\mathrm{ChLocus}_m(\mathcal{V}^1, ..., \mathcal{V}^k)_Y$ is closed.

Chapter 5

Rational connectedness with respect to k Chow families

If X is rationally chain connected, then any two points can be connected by a chain of rational curves. In general we can define a relation of *rational connectedness with respect to k Chow families* $\mathcal{V}^1, ..., \mathcal{V}^k$ in the following way:

Definition 5.1. Two points $x, y \in X$ are in the $rc(\mathcal{V}^1, ..., \mathcal{V}^k)$-relation if there exists a chain of rational 1-cycles in $\mathcal{V}^1, ..., \mathcal{V}^k$ which joins x to y, i.e. if $y \in \mathrm{ChLocus}_m(\mathcal{V}^1, ..., \mathcal{V}^k)_x$ for some m.

The aim of this chapter is to study this relation, that was introduced by Kollár, Miyaoka, and Mori and independently by Campana, and to construct a quotient of the variety by this relation.
We will introduce the two different notations (Kollár, Miyaoka, and Mori work in the algebraic category, while Campana works in the analytic context). Moreover we will recall the main results that are due to Kollár-Miyaoka-Mori ([KMM92a], [KMM92b] and [Kol96, IV.4] for proofs) and to Campana ([Cam94] for proofs). In particular, we will give a sketch of the Campana's construction of the quotient.

5.1 Kollár-Miyaoka-Mori's construction

Since every Chow family $\mathcal{V}^j$ defines a proper prerelation $(\mathcal{V}'_j \xleftarrow{p'_j} \mathcal{U}'_j \xrightarrow{i'_j} X, \mathcal{V}'_j \xrightarrow{\sigma_j} \mathcal{U}'_j)$, as we have already observed in the section 4.2.1, we can define the rational connectedness with respect to $\mathcal{V}^1, ..., \mathcal{V}^k$ using the language of [Kol96, IV.4.8]:

Definition 5.2. Let $(\mathcal{V}'_j \xleftarrow{p'_j} \mathcal{U}'_j \xrightarrow{i'_j} X, \mathcal{V}'_j \xrightarrow{\sigma_j} \mathcal{U}'_j)$, $j = 1, \ldots k$ be a collection of prerelations and $\mathcal{U}'_0 = X$, $p'_0 = i'_0 = \sigma_0 = Id_X$ the identity prerelation.
Fix an integer $m > 0$. Let $x_1, x_2 \in X$ be points. We say that x_1 and x_2 can be *connected by a $(\mathcal{U}'_1, \ldots, \mathcal{U}'_k)$-chain of length m* if and only if there are

- points $y_1 = x_1, y_2, \ldots, y_m, y_{m+1} = x_2 \in X$
- a function $\tau : \{1, \ldots, m\} \to \{0, 1, \ldots, k\}$, and
- points $v_{\tau(s)} \in \mathcal{V}'_{\tau(s)}$ such that

$$\begin{cases} y_s = i'_{\tau(s)} \circ \sigma_{\tau(s)}(v_{\tau(s)}) \\ \\ y_{s+1} \in \widetilde{\mathcal{U}'}_{\tau(s)}(v_{\tau(s)}) \end{cases} \quad \forall s = 1, \ldots, m$$

where $\widetilde{\mathcal{U}'}_{\tau(s)}(v_{\tau(s)}) := i'_{\tau(s)}((p'_{\tau(s)})^{-1}(v_{\tau(s)}))$ is called the relation class of $v_{\tau(s)}$.

We say that x_1 and x_2 can be *connected by a* $(\mathcal{U}'_1, \ldots, \mathcal{U}'_k)$*-chain* if and only if x_1 and x_2 can be connected by a $(\mathcal{U}'_1, \ldots, \mathcal{U}'_k)$-chain of length m for some m.

Theorem 5.3. *Let* $(\mathcal{V}'_j \xleftarrow{p'_j} \mathcal{U}'_j \xrightarrow{i'_j} X, \mathcal{V}'_j \xrightarrow{\sigma_j} \mathcal{U}'_j)$, $j = 1, \ldots k$ *be prerelations. For every* k, m *there is an algebraic relation* $Chain_m(\mathcal{U}'_1, \ldots, \mathcal{U}'_k)$ *such that for every* $x_1, x_2 \in X$, x_1 *and* x_2 *can be connected by a* $(\mathcal{U}'_1, \ldots, \mathcal{U}'_k)$*-chain of length at most* m *if and only if*

$$(x_1, x_2) \in \widetilde{Chain}_m(\mathcal{U}'_1, \ldots, \mathcal{U}'_k).$$

Proof. If $m = 1$, we set $R_j := Chain_1(\mathcal{U}'_j) = \mathcal{U}'_j$ with morphisms $w_j = i'_j \circ \sigma_j \circ p'_j$ and $u_j = i'_j$. R_0 denotes the identity relation.
If $m > 1$, to construct $Chain_m(\mathcal{U}'_1, \ldots, \mathcal{U}'_k)$ we have to define the product $R_j * R_s$ of $R_j := Chain_1(\mathcal{U}'_j)$ and $R_s := Chain_1(\mathcal{U}'_s)$.
$R_j * R_s$ is the fiber product $R_j \times_X R_s$, and $R_j * R_s := R_j \times_X R_s$ with the morphisms $w_j \circ u_j^* w_s$ and $u_s \circ w_s^* u_j$ is an algebraic relation which is called the product of R_j and R_s.

$$\begin{tikzcd} & R_j * R_s := R_j \times_X R_s \arrow[dl, "u_j^* w_s"'] \arrow[dr, "w_s^* u_j"] & \\ R_j \arrow[dr, "u_j"'] & & R_s \arrow[dl, "w_s"] \\ & X & \end{tikzcd}$$

In particular this relation is such that $\widetilde{R_j * R_s} = \widetilde{R_j} * \widetilde{R_s}$, i.e. (x, y) belongs to the set-theoretic relation generated by $R_j * R_s$ ($(x, y) \in \widetilde{R_j * R_s}$) if and only if there exists $z \in X$ such that $(x, z) \in \widetilde{R_j}$ and $(z, y) \in \widetilde{R_s}$.
Now, let $\tau : \{1, \ldots, m\} \to \{0, 1, \ldots, k\}$ be any function. By induction on m we obtain that there is an algebraic relation

$$R_{\tau(1)} * R_{\tau(2)} * \ldots * R_{\tau(m)}.$$

It is clear that the following relation has the required properties:

$$\mathrm{ChLocus}_m(U_1,\dots,U_k) := \bigcup_{\tau:\{1,\dots,m\}\to\{0,1,\dots,k\}} R_{\tau(1)} * R_{\tau(2)} * \dots * R_{\tau(m)}$$

□

Thus the $rc(\mathcal{V}^1,...,\mathcal{V}^k)$-relation is the set-theoretic relation $\widetilde{Chain}(\mathcal{U'}_1,\dots,\mathcal{U'}_k)$ associated to the proper proalgebraic relation

$$Chain(\mathcal{U'}_1,\dots,\mathcal{U'}_k) = \bigcup_{m\in\mathbb{N}} Chain_m(\mathcal{U'}_1,\dots,\mathcal{U'}_k).$$

To the $rc(\mathcal{V}^1,...,\mathcal{V}^k)$-relation we can associate a fibration, at least on an open subset, and we will call it $rc(\mathcal{V}^1,...,\mathcal{V}^k)$-fibration:

Theorem 5.4. *Let $\mathcal{V}^1,...,\mathcal{V}^k$ be Chow families of rational curves on a normal proper variety X. Then there exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that*

(1) *the $rc(\mathcal{V}^1,...,\mathcal{V}^k)$-relation restricts to an equivalence relation on X^0;*

(2) *the fibers of π are equivalence classes for the $rc(\mathcal{V}^1,...,\mathcal{V}^k)$-relation;*

(3) *for every $z \in Z^0$ any two points in $\pi^{-1}(z)$ can be connected by a chain of at most $2^{\dim X - \dim Z} - 1$ cycles in $\mathcal{V}^1,...,\mathcal{V}^k$.*

Definition 5.5. In the above assumptions, if π is the constant map, we say that X is $rc(\mathcal{V}^1,...,\mathcal{V}^k)$-connected.

5.2 Campana's construction

In [Cam81], Campana considers not necessarily compact Kähler manifolds and his setting is different from the one presented in [KMM92a], [KMM92b] and [Kol96]. For this reason, first of all we introduce the Campana's notation.
For all the material in this section the main reference is [Cam04].

Definition 5.6. Let X be a complex space and $d \in \mathbb{N}$. A *d-cycle* on X is a finite linear combination $Z = \sum n_i Z_i$ where $n_i \in \mathbb{N}$ (n_i is called the *multiplicity of Z_i* in the cycle Z) and the Z_i's are compact irreducible analytic subsets of X of pure dimension d which are pairwise distinct. The *support of Z*, denoted $|Z|$, is the union of the (reduced) Z_i's.

Definition 5.7. Let S be a normal complex space and $(Z_s)_{s\in S}$ be a family of d-cycles of X parametrized by S (i.e. for each $s \in S$, Z_s is a element of $\mathrm{Chow}_d(X)$). Let

$$|G_S| := \{(s,t) : x \in |Z_s|\} \subset S \times X$$

The set $|G_S|$ is called the *incidence graph* of the family S. We denote by p_S and by p_X the restriction of the first and second projections of $S \times X$ to G_S.
Then this family is said to be *analytic* if:

(1) the incidence graph $|G_S|$ is a closed analytic subset of $S \times X$;

(2) the restriction of the first projection p_S of $S \times X$ to $|G_S|$ is proper, surjective and its fibers have pure dimension d;

(3) for any irreducible component $|G_S^j|$ of $|G_S|$, there exists a positive integer n^j such that for s generic in $S^j := p_S(|G_{S^j}|)$ all irreducible components of $|Z_S|$, contained in $|G_S^j|$, have multiplicity n^j. (The closed analytic cycle $G_S = \sum_j n^j G_S^j$ is called the *graph* of the analytic family parametrized by S).

(4) for any $s \in S$, any j, and any local multisection $\sigma : S' \to |G_S^j|$, defined on a small open neighborhood S' of s in S, if the image of σ meets Z_s at a single point x, contained in a unique irreducible component Z_s^i of Z_s, the multiplicity of $|Z_s^i|$ in Z_s is $m \cdot n^j$, where m is the degree of the restriction of p_X to the image of σ.

In particular, by the following theorem, an analytic family S corresponds to a subset of the Chow scheme $\mathrm{Chow}(X)$:

Theorem 5.8. *Let $G \subset S \times X$ be an irreducible compact analytic subset such that the restriction $p : G \to S$ is surjective. There exists a unique meromorphic map $f : S \dashrightarrow Chow(X)$ sending a generic $s \in S$ to the reduced cycle of X with support $p^{-1}(s)$. In particular, the image of f is compact since S is. If moreover the fibers of p are all of the same dimension, and if S is normal, then f is holomorphic.*

Definition 5.9. Let X be a compact connected normal complex space. Then $S \subset \mathrm{Chow}(X)$ is said to be a *covering family* of X if the following conditions are satisfied:

(1) S is an at most countable disjoint union of compact irreducible subvarieties $S_i \subset \mathrm{Chow}(X)$;

(2) if $s \in S_i$ is a generic point, then Z_s is irreducible and reduced, this for any irreducible component S_i of S;

(3) X is the union of all $|Z_s|$'s for $s \in S$.

Remarks 5.10.

(a) Our conditions on X and on Z_s imply that X is irreducible and that the multiplicity n^j associated to $|G^j_S|$ will be equal to one (it is not necessary to distinguish between $|G_S|$ and G_S).

(b) We can replace conditions (2), (3) equivalently by: the restriction of the projection p_S (respectively p_X) to the incidence graph has irreducible reduced generic fibers (respectively is surjective).

(c) If S is a covering family, then at least one of its irreducible components is a covering family.

(d) We could prove that Z_s is connected for $s \in S$ and that the incidence graph G_{S_i} of S_i is irreducible and compact for every i.

We give the Campana's definition of the chain connectedness:

Definition 5.11. Let $S \subset \mathrm{Chow}(X)$ be a covering family of X. For $s_1, ..., s_n \in S$ we say that $Z_{s_1}, ..., Z_{s_n}$ form an *n-chain* of S if the union of their supports is connected. Two points $x, x' \in X$ are called *S-equivalent* if and only if x and x' can be joined by a n-chain for some $n \in \mathbb{N}$, depending on x, x'. In which case we say that x and x' are *n-equivalent*.
As every point $x \in X$ is connected to itself by a 1-chain, this defines an equivalence relation $R(S)$ on X.
X is called *S-connected* if $R(S)$ has a single equivalence class (i.e. any two points can be connected by some n-chain).
We define

$$R_n(S) := \{(x, y) \in X \times X : x, y \text{ are } n\text{-equivalent}\}, \quad R(S) := \sum_{n \in \mathbb{N}} R_n(S).$$

In particular, let R^0 be an irreducible component of $R_n(S)$. Then R^0 is said to be *significant* if it contains Δ_X (Δ_X is the diagonal of $X \times X$). We denote by R_n^Δ the union of the significant irreducible components of $R_n(S)$.

Remarks 5.12.

(a) Campana's relation is not the rational connectedness which we defined at the beginning of this chapter. In fact the families that Campana considers are not strictly contained in $\mathrm{Chow}_1(X)$, but they can parametrize effective d-cycles with $d > 1$.
However, if S is contained in $\mathrm{Chow}_1(X)$, the two definitions are equivalent.

(b) If Z_s is a member of the family S, that does not meet any other member of the family S (i.e. $|Z_s| \cap |Z_{s'}| \neq \emptyset \Rightarrow s = s'$), then $|Z_s|$ is an equivalence class for $R(S)$.

(c) For an irreducible covering family, the non-significant components have no influence on the graph of the equivalence relation.

(d) If S is not normal, let $\nu : S' \to S$ be its normalization. Then the morphism ν corresponds to an analytic family of n-cycles parametrized by S'. In fact this family contains the same cycles as S, but the same cycles Z_s will appear several times, if $s \in S$ is not a normal point. This also shows that normalizing does not change the equivalence relation $R(S)$ induced on X.

Definition 5.13. Let X be a compact connected normal complex space, S an irreducible covering family for X and G_S the incidence graph. Then S is said to be *stationary* if

(1) $p_X : G_S \to X$ is a modification

(2) $\dim(R_1^{\Delta}(S)) = \dim(R_2^{\Delta}(S))$

Now we define the fibrations in the analytic context:

Definition 5.14. A *fibration* $f : X \dashrightarrow Y$ is a surjective meromorphic map between irreducible compact complex spaces such that the generic fiber of f is irreducible.

Remark 5.15. A fibration induces an equivalence relation on X in the following way: two points $x, x' \in X$ are 1-equivalent if there exists a $y \in Y$ such that $x, x' \in f^{-1}(y)$, where $f^{-1}(y)$ is a fiber of f ([Cam04, Definition 1.2]).
As every point x is connected to itself, the graph of 1-chain $R_1(f) \subset X \times X$ is symmetric and contains the diagonal, hence induces an equivalence relation on X, whose graph will be denoted $R(f) \subset X \times X$

Definition 5.16. Let S be a covering family for a compact connected normal complex space X, and let $S_i \subset S$ be an irreducible compact component.
A fibration $f : X \dashrightarrow Y$ on a compact connected normal complex space Y is S-*subordinate* if a general fiber of f is contained in an S-equivalence class. We denote by $\mathcal{F}(X, S)$ the set of S-subordinate almost holomorphic fibrations of X. For $f \in \mathcal{F}(X, S)$, S_i is called f-*covering* if $f \circ p_X^i : G_{S_i} \dashrightarrow Y$ is surjective, where G_{S_i} is the incidence graph of S_i and $p_X^i : G_{S_i} \dashrightarrow X$ is the projection to X.

Remark 5.17. Notice that the fibration f is subordinate to S if and only if $R(f) \subset R(S)$. Moreover, S_i is f-covering if and only if a general fiber of f meets some cycle parametrized by S_i.

Campana proves the following result:

Theorem 5.18. *Let X be a compact connected normal complex space and let $S \subset Chow(X)$ be a covering family for X. Then there exists a fibration $q_S : X \dashrightarrow X_S$ such that its general fiber is an equivalence class for $R(S)$.*
Furthermore q_S is almost holomorphic and unique up to equivalence of meromorphic fibrations. The map q_S is called the S-quotient of X.

It is the analytic version of Theorem (5.4). We don't show the proof, but we give a sketch of the construction of the S-quotient.

Step 1. S is an irreducible stationary covering family for X.
Since S is an irreducible covering family for X, we have the following base diagram:

$$\begin{array}{ccc} G_S & \xrightarrow{p_X} & X \\ \downarrow{\scriptstyle p_S} & & \\ S & & \end{array}$$

In this particular case, Campana shows that

Theorem 5.19. *The map $q_S := p_S \circ (p_X)^{-1} : X \dashrightarrow S$ is an almost holomorphic fibration which is the S-quotient of X.*

Step 2. S is an irreducible covering family for X, but it is not stationary.
Campana constructs an irreducible stationary compact covering family S' that induces the same equivalence relation of S on X, so he can apply the previous result and find the S-quotient. The S-quotient is the S'-quotient.
In particular, the generic member of this stationary family consists of the set $R_n(x)$ of all $y \in X$ which are n-equivalent to x, for a general point $x \in X$, and n sufficiently large, but independent of x (see [Cam04, Section 1.6]).

Step 3. S is a covering family for X.
By definition, S is an at most countable disjoint union of compact irreducible subvarieties $S_i \subset \mathrm{Chow}(X)$, i.e. $S = \bigcup_i S_i$.
Consider a S-subordinate almost holomorphic fibration $f : X \dashrightarrow Y$ and define

$$\widetilde{S} := \{S_i \subset S \mid S_i \text{ is a } f\text{-covering irreducible component of } S\}$$

Then there are two possibilities:

(a) For every $S_i \in \widetilde{S}$, a general cycle of S_i is contained in a fiber of f. Then f is the S-quotient.

(b) Assume that there exists $S_{\mathbf{i}} \in \widetilde{S}$ such that a general cycle of $S_{\mathbf{i}}$ is not contained in a fiber of f. Then Campana constructs an irreducible compact covering family S' of X such that the S'-quotient $q_{S'} : X \dashrightarrow X_{S'}$ satisfies the following properties:

- $\exists\ g : Y \dashrightarrow X_{S'}$ an almost holomorphic mapping such that $q_{S'} = g \circ f$ (i.e. f is subordinate to S');
- $R(S_{\mathbf{i}}) \subset R(S')$ and $R(S') \subset R(S)$.

In particular, $q_{S'} \in \mathcal{F}(X, S)$, so he can replace f by $q_{S'}$ and repeat the construction.

Campana continues until all f-covering irreducible components satisfy the case **(a)**. Then the "last" f is the S-quotient.

Now, following the Campana's steps, we define the quotient in a particular case: we consider two quasi unsplit families of rational curves and we construct the rationally connected fibration with respect to them.

Example 5.20. Let X be a smooth complex projective variety.
Let $V, W \in \mathit{Ratcurves}^n(X)$ be quasi unsplit families of rational curve on X. We consider the Chow families $\mathcal{V}, \mathcal{W} \subset \mathrm{Chow}(X)$ associated to V, W: in particular, $\mathcal{V}, \mathcal{W}$ are irreducible components of $\mathrm{Chow}(X)$.
We assume that V is a dominating family and W is a horizontal dominating family with respect to the $rc(\mathcal{V})$-fibration (or $\mathcal{V}$-quotient), and we define

$$S := \mathcal{V} \cup \mathcal{W}.$$

Using the Campana's method, as already said, we want to construct q_S, the S-quotient (in the language of [Kol96], q_S is the $rc(\mathcal{V}, \mathcal{W})$-fibration).
Note that $\mathcal{V}$ is an irreducible covering family of X, so we can consider the $\mathcal{V}$-quotient (or $rc(\mathcal{V})$-fibration) q_V:

$$\begin{tikzcd} G_{\mathcal{V}'} \arrow[r, "p_X"] \arrow[d, "p_{\mathcal{V}'}"'] & X \arrow[dl, dashed, "q_V"] \\ \mathcal{V}' & \end{tikzcd} \tag{5.1}$$

$$\boxed{q_V = p_{\mathcal{V}'} \circ (p_X)^{-1} \text{ is the } \mathcal{V}\text{-quotient}}$$

By Campana's construction, $\mathcal{V}'$ is an irreducible stationary compact covering family of X such that the generic member of $\mathcal{V}'$ consists of the set $R_n(x)$ of all $y \in X$ which

are n-equivalent to x with respect to $\mathcal{V}$, for a general point $x \in X$, and n sufficiently large, but independent of x.
We denote by I_{q_V} the indeterminacy locus of q_V.[1]
The $\mathcal{V}$-quotient is a S-subordinate almost holomorphic fibration because $R(q_V) \subset R(S)$.
Moreover, since W is a horizontal dominating family for the $\mathcal{V}$-quotient, the general fiber of q_V meets some cycle parametrized by $\mathcal{W}$, i.e. $\mathcal{W}$ is a q_V-covering irreducible component of S.
Hence we can construct an irreducible compact covering family S^1 of X such that the S^1-quotient $q_{S^1} : X \dashrightarrow X_{S^1}$ is the S-quotient.
By our assumptions, for a generic $w \in \mathcal{W}$, $\dim(q_V(|Z_w|)) > 0$ and the generic cycle Z_w is not contained in I_{q_V} nor in any exceptional fiber of the fibration q_V. We set $|Z_w^*| := |Z_w| \setminus I_{q_V}$ and we define $q_V(|Z_w|) := \overline{q_V(|Z_w^*|)}$. By construction, $q_V(|Z_w|)$ is an irreducible compact cycle in $\mathcal{V}'$ and up to restricting to a Zariski open set $\mathcal{W}^* \subset \mathcal{W}$ we can suppose the family of cycles to be equidimensional.
In particular, $\{q_V(|Z_w|)\}_{w \in \mathcal{W}^*}$ is an analytic family in $\mathcal{V}'$.
If $\Gamma \subset \mathcal{W} \times \mathcal{V}'$ is the closure of the incidence graph of this family, then the general fibers of the projection of Γ on $\mathcal{W}$ are equidimensional, hence there exists a meromorphic map $\varphi : \mathcal{W} \dashrightarrow \mathrm{Chow}(\mathcal{V}')$.
Let $\overline{\mathcal{W}^*}$ be the normalization of the closure of the image of φ. $\overline{\mathcal{W}^*}$ parametrizes an analytic family on $\mathcal{V}'$ which is irreducible, compact and covering.
If we restrict $\mathcal{W}^*$ a bit further, we can prove that $Z'_w := q_V^{-1}(q_V(|Z_w|))$ for $w \in \mathcal{W}^*$ defines an analytic family in X. As before, it follows that there exists a meromorphic map $\phi : \mathcal{W} \dashrightarrow \mathrm{Chow}(X)$.
Normalizing its image, we obtain an irreducible analytic family S^1 which is compact and covering.
In particular, $Z'_w := q_V^{-1}(q_V(|Z_w|))$ is a generic member of the family S^1, and the S^1-quotient is the S-quotient.
Since S^1 is not necessary stationary, we consider the associated irreducible stationary compact covering family $\mathcal{S}^1 \subset \mathrm{Chow}(X)$ (see the second step of the Campana's construction). So we have that

$$\begin{tikzcd} G_{\mathcal{S}^1} \arrow[r, "p_X"] \arrow[d, "p_{\mathcal{S}^1}"'] & X \arrow[dl, dashed, "q_{\mathcal{S}^1}"] \\ \mathcal{S}^1 & \end{tikzcd} \tag{5.2}$$

$$\boxed{q_{\mathcal{S}^1} = p_{\mathcal{S}^1} \circ (p_X)^{-1} \text{ is the } S\text{-quotient}}$$

[1] In [Cam04], the indeterminacy locus of q_V is the set of points $x \in X$ where the fiber of the first projection $p_X^{-1}(x) \subset \Gamma$ (Γ is the closure of the graph of q_V) is not a singleton.

such that

(1) p_X is birational (i.e. p_X is a modification) and $q_{\mathcal{S}^1}$ is almost holomorphic;

(2) a general fiber of $q_{\mathcal{S}^1}$ is a $\mathcal{S}^1$-equivalence class;

(3) a general fiber of $q_{\mathcal{S}^1}$, hence of $p_{\mathcal{S}^1}$, is irreducible.

Remark 5.21. Note that this example is very helpful to understand the general case: let $V^1, \ldots, V^k$ $(k \in \mathbb{N})$ be k quasi unsplit families on X such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient (or the $rc(\mathcal{V}^1, \cdots, \mathcal{V}^k)$-fibration) and we want to construct the quotient with respect to these families.
Since we assume that $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient $q_{\mathcal{S}^{i-1}}$ for every $i = 2, \ldots, k$, $\mathcal{V}^i$ is $q_{\mathcal{S}^{i-1}}$-covering.
So, to construct the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^k)$-quotient, it is enough to iterate $(k-1)$ times the construction. In particular, we obtain an irreducible compact stationary family $\mathcal{S} \subset \mathrm{Chow}(X)$ such that

$$\begin{tikzcd} G_{\mathcal{S}} \arrow[r, "p_X"] \arrow[d, "p_{\mathcal{S}}"'] & X \arrow[dl, dashed, "q_{\mathcal{S}}"] \\ \mathcal{S} & \end{tikzcd} \tag{5.3}$$

$$\boxed{q_{\mathcal{S}} = p_{\mathcal{S}} \circ (p_X)^{-1} \text{ is the } S\text{-quotient}}$$

Chapter 6

Families of rational curves and the Kleiman-Mori cone

In this chapter we list some conditions under which the numerical class of every curve lying in some subvariety S of a projective variety X is contained in a linear subspace of $N_1(X)$ or in a subcone of $NE(X)$.
Moreover we will prove some properties of the rationally connected fibrations with respect to k quasi unsplit Chow families.
The results that we will show will give some important informations about the structure of the Kleiman-Mori cone of the variety and about the extremality of the quasi unsplit families which we consider for the fibration.

Notation 6.1. Let X be a smooth complex projective variety.
If Γ is a 1-cycle on X then we will denote by $[\Gamma]$ its numerical equivalence class in $N_1(X)$; if V is a family of rational curves, we will denote by $[V]$ the numerical equivalence class of any curve among those parametrized by V.
If $\mathcal{V}$ is a Chow family of rational 1-cycles, we will denote by $[\mathcal{V}]$ the numerical class in $NE(X)$ of the general cycle of the family $\mathcal{V}$.
If $S \subset X$, we will denote by $N_1(S, X) \subseteq N_1(X)$ the vector subspace generated by numerical classes of curves of X contained in S; moreover, we will denote by $NE(S, X) \subseteq NE(X)$ the subcone generated by numerical classes of curves of X contained in S.
Let $V^1, \ldots, V^k$ be k unsplit family of rational curves. We write by abuse of notation

$$N_1(S, X) = \langle [V^1], \ldots, [V^k] \rangle \text{ or } N_1(S, X) = \langle [\Gamma_1], \ldots, [\Gamma_k] \rangle$$

if the numerical class in X of every curve $\Gamma \subset S$ can be written as $[\Gamma] = \sum_i a_i [\Gamma_i]$ with $a_i \in \mathbb{Q}$ and Γ_i is a curve parametrized by V^i, and similarly

$$NE(S, X) = \langle [V^1], \ldots, [V^k] \rangle \text{ or } NE(S, X) = \langle [\Gamma_1], \ldots, [\Gamma_k] \rangle$$

if the numerical class in X of every curve $\Gamma \subset S$ can be written as $[\Gamma] = \sum_i a_i[\Gamma_i]$ with $a_i \in \mathbb{Q}_{\geq 0}$ and Γ_i is a curve parametrized by V^i.

6.1 Chow families and the Picard number

Let X be a smooth complex projective variety of dimension n.

Lemma 6.2. *Let $Y \subset X$ a closed subset and $\mathcal{V}$ a Chow family of rational 1-cycles. Then every curve contained in Locus$(\mathcal{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles parametrized by $\mathcal{V}$ which intersect Y.*

Proof. We define $\mathcal{V}_Y := p(i^{-1}(Y \cap \mathrm{Locus}(\mathcal{V})))$ and $\mathcal{U}_Y := p^{-1}(\mathcal{V}_Y)$, and we consider the following diagram

$$\begin{tikzcd} \mathcal{U}_Y \arrow[r, "i"] \arrow[d, "p"'] & X \\ \mathcal{V}_Y & \end{tikzcd}$$

Let C be a curve in Locus$(\mathcal{V})_Y$ which is not an irreducible component of a cycle parametrized by $\mathcal{V}$. Then $i^{-1}(C)$ contains an irreducible curve C' which is not contained in any fiber of p and dominates C via i.
Let $B := p(C')$ and let S be the ruled surface $p^{-1}(B)$.
Let C'_Y be a curve in S which dominates B and whose image via i is contained in Y; such a curve exists since the image via i of every fiber of $p_{|S}$ meets Y.
By [Kol96, II.4.19] every curve in S is algebraically equivalent to a linear combination with rational coefficients of C'_Y and of the irreducible components of fibers of $p_{|S}$.
Thus any curve in $i(S)$, and in particular C, is algebraically equivalent in $i(\mathcal{U}_Y) = \mathrm{Locus}(\mathcal{V})_Y$ (and hence in X) to a linear combination with rational coefficients of $i_*(C'_Y)$ and of irreducible components of cycles parametrized by $\mathcal{V}_Y$. $\square$

Corollary 6.3. *Let $Y \subset X$ be a closed subset, $\mathcal{V}^1, ..., \mathcal{V}^k$ Chow families of rational 1-cycles, m a positive integer.*
Then every curve contained in ChLocus$_m(\mathcal{V}^1, ..., \mathcal{V}^k)_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles parametrized by $\mathcal{V}^1, ..., \mathcal{V}^k$.

Proof. Recalling that

$$\mathrm{ChLocus}_m(\mathcal{V}^1, ..., \mathcal{V}^k)_Y = \bigcup_{\substack{1\leq i(j)\leq k \\ 1\leq s\leq m}} \mathrm{Locus}(\mathcal{V}^{i(1)}, ..., \mathcal{V}^{i(s)})_Y$$

we have that every irreducible component of $\mathrm{ChLocus}_m(\mathcal{V}^1, ..., \mathcal{V}^k)_Y$ is contained in $\mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(s)})_Y$ for some s-uple $(i(1), \ldots, i(s))$.
Thus we must describe the numerical classes of the curves contained in the locus $\mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(s)})_Y$. Applying s times Lemma (6.2) to $\mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(s)})_Y$ with $Y_0 = Y$ and $Y_j = \mathrm{Locus}(\mathcal{V}^{i(1)}, ..., \mathcal{V}^{i(j)})_Y$, we obtain that every curve contained in $\mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(s)})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles parametrized by $\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(s)}$. □

Proposition 6.4. *Let $\mathcal{V}^1, ..., \mathcal{V}^k$ be Chow families of rational 1-cycles on X and let $\pi : X^0 \to Z^0$ be the $rc(\mathcal{V}^1, ..., \mathcal{V}^k)$-fibration.*
Let $Y \subset X$ be a closed subset which dominates Z^0 via π; then every curve in X is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles in $\mathcal{V}^1, ..., \mathcal{V}^k$.

Proof. By Theorem (5.4), every pair of points in a general fiber of π can be connected by a chain of 1-cycles belonging to $\mathcal{V}^1, \ldots, \mathcal{V}^k$ of length at most $M = 2^{\dim X - \dim Z} - 1$.
Then, since Y is closed and dominates Z^0 via π, $\mathrm{ChLocus}_M(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is dense in X and is closed.
Thus $X = \mathrm{ChLocus}_M(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ and the statement follows from the previous corollary. □

Corollary 6.5. *Suppose that X is rationally connected with respect to some Chow families $\mathcal{V}^1, ..., \mathcal{V}^k$; then every curve in X is numerically equivalent to a linear combination with rational coefficients of the irreducible components of cycles parametrized by $\mathcal{V}^1, ..., \mathcal{V}^k$. In particular, if X is rationally connected with respect to k quasi unsplit families then $\rho_X \leq k$.*

Proof. Since the $rc(\mathcal{V}^1, ..., \mathcal{V}^k)$-fibration $\pi : X \to \{*\}$ is the constant map, the claim follows from Proposition (6.4) taking Y to be any point of X. To show the second part it is enough to recall that all cycles parametrized by a quasi unsplit family are numerically proportional by definition. □

6.2 Unsplit families and the Picard number

Let X be a smooth complex projective variety of dimension n.

Lemma 6.6. *Let $Y \subset X$ be a closed subset and V an unsplit family of rational curves. Then every curve contained in $\mathrm{Locus}(V)_Y$ is numerically equivalent to a linear combination with rational coefficients*

$$\lambda \Gamma_Y + \mu \Gamma_V$$

where Γ_Y is a curve in Y, Γ_V belongs to the family V and $\lambda \geq 0$.

Proof. Let Γ be a curve contained in $\mathrm{Locus}(V)_Y$. If $\Gamma \subset Y$ or Γ is a curve parametrized by V we have nothing to prove, so we can suppose that this is not the case. Using the notation of diagram (4.1), we define $V_Y := p(i^{-1}(Y \cap \mathrm{Locus}(V))$ and we have that $i^{-1}(\Gamma)$ contains an irreducible curve Γ' which is not contained in a fiber of p and dominates Γ via i. Let $B' := p(\Gamma') \subset V_Y$, let $\nu : B \to B'$ be the normalization of B' and let S be the normalization of $B \times_V U$.
By standard arguments (see for instance [Wis89, 1.14]), it can be shown that S is a ruled surface over the curve B. Then, we can consider the following diagram:

$$\begin{array}{ccc} S & \xrightarrow{j} & X \\ {\scriptstyle\pi}\downarrow & & \\ B & & \end{array}$$

Let f be a fiber of π and let Γ_Y be a curve in S which dominates B and whose image via j is contained in Y; such a curve exists since the image via j of every fiber of p meets Y.
Since S is a ruled surface, every curve in S is algebraically equivalent to a linear combination with rational coefficients of Γ_Y and f.
Therefore every curve in $j(S)$ is algebraically, hence numerically, equivalent in X to a linear combination with rational coefficients

$$\lambda j_*(\Gamma_Y) + \delta j_*(f)$$

where $j_*(\Gamma_Y)$ is a curve in Y or is the zero cycle, and $j_*(f)$ is a curve of the family V.
Note that the proof actually yields that $\lambda \geq 0$; in fact, let Γ_S be an irreducible curve in S which dominates Γ via j. In S we can write $\Gamma_S \equiv \lambda \Gamma_Y + \delta f$ and, intersecting with f we have $\lambda \geq 0$. $\square$

Corollary 6.7. *Let V be a family of rational curves and x a point in X such that V_x is proper. Then $N_1(Locus(V)_x, X) = \langle [V] \rangle$ and $NE(Locus(V)_x, X) = \langle [V] \rangle$.*

Corollary 6.8. *Let V^1 be a locally unsplit family of rational curves and $V^2, \ldots, V^k$ unsplit families of rational curves. Then, for a general point $x \in Locus(V^1)$ either $Locus(V^1, \ldots, V^k)_x = \emptyset$ or*

$$N_1(Locus(V^1, \ldots, V^k)_x, X) = \langle [V^1], \ldots, [V^k] \rangle.$$

Corollary 6.9. *Let R be an extremal ray of X, W_R a family of deformations of a minimal extremal curve in R, x a point in $Locus(W_R)$ and V an unsplit family of rational curves, numerically independent from W_R.*
Then $NE(ChLocus_m(V)_{Locus(W_R)_x}, X) = \langle [V], [W_R] \rangle$.

Proof. Note that

$$\mathrm{ChLocus}_m(V)_{\mathrm{Locus}(W_R)_x} = \mathrm{Locus}(V)_{\mathrm{ChLocus}_{m-1}(V)_{\mathrm{Locus}(W_R)_x}}$$

Then, iterating Lemma (6.6) m times, we have that any curve Γ, which is contained in $\mathrm{ChLocus}_m(V)_{\mathrm{Locus}(W_R)_x}$, is algebraically equivalent to a linear combination with rational coefficients

$$\Gamma \equiv \lambda\Gamma_1 + \delta\Gamma_V$$

where $[\Gamma_1] \in W_R$, $[\Gamma_V] \in V$ and $\lambda \geq 0$. We want to prove that $\delta \geq 0$.
Suppose that $\delta < 0$. Then we can write $\Gamma_1 \equiv \alpha\Gamma_V + \beta C$ with $\alpha, \beta \geq 0$.
But, since Γ_1 is extremal, it follows that both $[\Gamma]$ and $[\Gamma_V]$ belong to the extremal ray R, and this a contradiction. □

Remark 6.10. More generally, if σ is an extremal face of $NE(X)$, F is a fiber of the associated contraction and V is an unsplit family whose numerical class doesn't belong to σ, then $NE(\mathrm{Locus}(V)_F, X) = \langle [V], \sigma \rangle$.

Lemma 6.11. *Let $Z \subset X$ be a closed subset and let $V^1, \ldots, V^k$ be unsplit families of rational curves.*
Then every curve contained in ChLocus$(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Z$ is numerically equivalent to a linear combination with rational coefficients

$$\lambda\Gamma_Z + \mu_1\Gamma_{V^1} + \ldots + \mu_k\Gamma_{V^k}$$

where Γ_Z is a curve in Z, Γ_{V^i} is a curve parametrized by V^i $(i = 1, \ldots, k)$ and $\lambda \geq 0$.

6.3 Properties of rationally connected fibrations

In this section, we will use the Campana's notation (see Example (5.20)). For all the material in this section the references are the articles [BCD07], [NO08], where the following results are proved for a covering quasi unsplit Chow family of rational 1-cycles.

Let X be a normal irreducible complex projective variety.
Let $V^1, \ldots, V^k$ be k quasi unsplit families of rational curves. Let $\mathcal{V}^1, \ldots, \mathcal{V}^k$ be the Chow families of rational 1-cycles associated to $V^1, \ldots, V^k$.
We define $S = \mathcal{V}^1 \cup \ldots \cup \mathcal{V}^k$.
Assume that V^1 is dominating and $\mathcal{V}^i$ is a horizontal dominating family with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient (or the $rc(\mathcal{V}^1, \ldots, \mathcal{V}^{i-1})$-fibration) for every $i = 2, \ldots, k$.
Then, as already observed in Remark (5.21), we can construct an irreducible compact stationary family $\mathcal{S} \subset \mathrm{Chow}(X)$ and the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^k)$-quotient

$$\begin{tikzcd} G_{\mathcal{S}} \arrow[r, "p_X"] \arrow[d, "p_{\mathcal{S}}"'] & X \arrow[dl, dashed, "q_{\mathcal{S}}"] \\ \mathcal{S} & \end{tikzcd} \tag{6.1}$$

$$\boxed{q_{\mathcal{S}} = p_{\mathcal{S}} \circ (p_X)^{-1} \text{ is the } S\text{-quotient}}$$

such that

(1) p_X is birational (i.e. p_X is a modification) and $q_{\mathcal{S}}$ is almost holomorphic;

(2) a general fiber of $q_{\mathcal{S}}$ is a $\mathcal{S}$-equivalence class;

(3) a general fiber of $q_{\mathcal{S}}$, hence of $p_{\mathcal{S}}$, is irreducible.

Notation 6.12. Now, we fix the following notation:

$$e := p_X, \quad p := p_{\mathcal{S}}, \quad q := q_{\mathcal{S}}.$$

We denote by E the exceptional locus of $e := p_X$ and $B := p_X(E) = e(E) \subset X$.
We denote by $f_{\mathcal{S}}$ the dimension of the general fiber of q.
Note that $f_{\mathcal{S}} = \dim X - \dim \mathcal{S}$.

Definition 6.13. A subset Z of X is *$\mathcal{S}$-rationally connected* if every connected component of Z is contained in some $\mathcal{S}$-equivalence class.

Lemma 6.14. *Let X be a normal projective variety. Let $V^1, \ldots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \ldots, k$. Consider the diagram (6.1). Then $e(p^{-1}(s))$ is $\mathcal{S}$-rationally connected for any $s \in \mathcal{S}$.*

Proof. (See [BCD07, Lemma 1.]) Let $R(\mathcal{S}) \subset X \times X$ the graph of the equivalence relation defined by $\mathcal{S}$: by [Cam04, Lemma 1.14] it is a countable union of Zariski closed and compact subsets. The fiber product $G_{\mathcal{S}} \times_{\mathcal{S}} G_{\mathcal{S}}$ is irreducible and from the properties (1),(2),(3) of the S-quotient it follows that $(e \times e)(G_{\mathcal{S}} \times_{\mathcal{S}} G_{\mathcal{S}}) \subset R(\mathcal{S})$. Therefore, for any $x \in e(p^{-1}(s))$, the cycle $e(p^{-1}(s))$ is contained in the $\mathcal{S}$-equivalence class of x. $\square$

Proposition 6.15. *Let X be a normal and $\mathbb{Q}$-factorial projective variety.*
Let $V^1, \ldots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \ldots, k$.
Consider the diagram (6.1). Then:

(1) *$e(p^{-1}(s))$ is a $\mathcal{S}$-equivalence class of dimension $f_{\mathcal{S}}$ $\forall s \in \mathcal{S} \setminus p(E)$;*

(2) *B is the union of all $\mathcal{S}$ -equivalence classes of dimension greater than $f_{\mathcal{S}}$.*

Proof. (See [BCD07, Proposition 1.]) Set $X^0 := X \setminus B$ and $\mathcal{S}^0 := \mathcal{S} \setminus p(E)$. Choose a very ample line bundle L on $\mathcal{S}$, and let $U \subset |L|$ be the open subset of divisors H that are irreducible and such that $H \cap \mathcal{S}^0 \neq \emptyset$. For any H in U, we define $\widehat{H} := \overline{q^{-1}(H \cap \mathcal{S}^0)}$, which is a Weil divisor in X. Since X is $\mathbb{Q}$-factorial, some multiple of $\widehat{H}$ defines a line bundle $\widehat{L}$ on X.
Let now $N := h^0(L)$, and let $g_1, ..., g_N$ be general global sections generating L. For each $j = 1, ..., N$, let $H_j \in |L|$ be the divisor of zeros of g_j and $\widehat{H}_j$ in X as defined above.

We want to prove that $B = \widehat{H}_1 \cap ... \cap \widehat{H}_N$.

($\supseteq$) If $x \notin B$, then e^{-1} is defined in x and so, by definition, $q = p \circ e^{-1}$ is defined in x. Moreover there is some $j_0 \in \{1, ..., N\}$ such that $q(x) \notin H_{j_0}$. Then $x \notin \widehat{H}_{j_0}$, i.e. $x \notin \widehat{H}_1 \cap ... \cap \widehat{H}_N$.

($\subseteq$) Let $x \in B$ and fix $j \in \{1, ..., N\}$. Then $e^{-1}(x)$ has positive dimension.
Let $C \subset G_{\mathcal{S}}$ be an irreducible curve such that $e(C) = x$. Then $p(C)$ is a curve in $\mathcal{S}$, hence $H_j \cap p(C) \neq \emptyset$ and $p^{-1}(H_j) \cap C \neq \emptyset$.
Since $p^{-1}(H_j)$ doesn't contain any component of E, $e(p^{-1}(H_j))$ is a divisor in X which coincides with $\widehat{H}_j$ over $X \setminus B$. This implies that $\widehat{H}_j = e(p^{-1}(H_j))$ and $x \in \widehat{H_j}$ for every j. Then we have that $x \in \widehat{H}_1 \cap ... \cap \widehat{H}_N$.

Let C be a irreducible curve in X such that $\widehat{H} \cdot C = 0$ for some $H \in U$. We want to prove that either $C \subseteq B$, or $C \cap B = \emptyset$ and $q(C)$ is a point.

Suppose that $C \nsubseteq B$. This implies that there is some $j \in \{1, ..., N\}$ such that C is not contained in $\widehat{H}_j$. Then, as $\widehat{H} \cdot C = 0$, we have that $\widehat{H}_j \cdot C = 0$, and hence $C \cap \widehat{H}_j = \emptyset$. As $B = \widehat{H}_1 \cap ... \cap \widehat{H}_N$, we have $C \cap B = \emptyset$.
Now we want to prove that $q(C)$ is a point. Suppose by contradiction that $q(C)$ is a curve. Then there exists $H_0 \in U$ such that H_0 intersects $q(C)$ in a finite number of points. Then $\widehat{H}_0$ intersects C without containing it. Hence $\widehat{H}_0 \cdot C > 0$. But $\widehat{H}$ and $\widehat{H}_0$ are numerically equivalent, so $\widehat{H} \cdot C > 0$, and this is a contradiction.

Now we want to prove that B is closed with respect to $\mathcal{S}$-equivalence.

To show this statement, first of all, we claim that $\widehat{H}_j \cdot V^i = 0$ for every $i = 1, \ldots, k$. Observe that $V^1, \ldots, V^k$ are quasi unsplit and a general cycle of the family $\mathcal{S}$ is contained in a fiber of q disjoint from $\widehat{H}_j$. In particular, by the definition of $\mathcal{S}$, for every i a general cycle parametrized by $\mathcal{V}^i$ is contained in a fiber of q disjoint from $\widehat{H}_j$. This implies that $\widehat{H}_j \cdot V^i = 0$ for every $i = 1, \ldots, k$ and, as $\mathcal{V}^i$ is quasi unsplit, we have that every irreducible component C of a cycle parametrized by V^i is such

$\widehat{H}_j \cdot C = 0$.
We consider an irreducible component C_i of a cycle of $\mathcal{V}^i$ such that $C_i \cap B \neq \emptyset$. We know that $\widehat{H} \cdot C_i = 0$ and so, by what we proved above, we can conclude that $C \subseteq B$.
Hence B is is closed with respect to $\mathcal{S}$-equivalence, and in particular, for a $\mathcal{S}$-equivalence class F, either $F \subseteq B$ or $F \cap B = \emptyset$.

Let F be a $\mathcal{S}$-equivalence class such that $F \cap B = \emptyset$. Consider an irreducible component C of a cycle of $\mathcal{V}^i$ for some i such that $C \subseteq F$.
Since $V^1, \dots, V^k$ are quasi unsplit, we have $\widehat{H} \cdot C = 0$, hence $q(C)$ is a point by what we proved above. By definition of $\mathcal{S}$-equivalence, we get $q(F) = s_0 \in \mathcal{S}$ and $F \subseteq e(p^{-1}(s_0))$.
But, from Lemma (6.14) it follows that $e(p^{-1}(s_0))$ is contained in a $\mathcal{S}$-equivalence class, and then we have that $F = e(p^{-1}(s_0))$. Clearly $s_0 \in \mathcal{S}_0$ because $F \cap B = \emptyset$, and so F is a proper fiber of q of dimension $f_{\mathcal{S}}$.

For any $x \in X$, we define $\mathcal{S}_x := p(e^{-1}(x))$ be the family of cycles parametrized by $\mathcal{S}$ and passing through x, and we consider $e(p^{-1}(\mathcal{S}_x))$.
Notice that, for any $s \in \mathcal{S}_x$, $x \in e(p^{-1}(s))$ and $e(p^{-1}(s))$ is contained in a $\mathcal{S}$-equivalence class. Hence $e(p^{-1}(\mathcal{S}_x))$ is $\mathcal{S}$-rationally connected for any $x \in X$.
Note that $\dim \mathcal{S}_x = \dim e^{-1}(x)$. By Zariski's Main Theorem

$$\dim \mathcal{S}_x > 0 \Leftrightarrow x \in B$$

Hence, if $x \in B$, then

$$\dim e(p^{-1}(\mathcal{S}_x)) \geq f_{\mathcal{S}} + 1$$

Now let F be a $\mathcal{S}$-equivalence class contained in B, and $x \in F$. Then $e(p^{-1}(\mathcal{S}_x))$ has dimension at least $f_{\mathcal{S}} + 1$ and is contained in F, hence $\dim F \geq f_{\mathcal{S}} + 1$, i.e. B is the union of all $\mathcal{S}$-equivalence classes of dimension bigger than $f_{\mathcal{S}}$. $\square$

Proposition 6.16. *Let X be a normal and $\mathbb{Q}$-factorial projective variety. Let $V^1, \dots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \dots, k$. Consider the diagram (6.1).*
If B is $\mathcal{S}$-rationally connected, then $[V^1], \dots, [V^k]$ belong to an extremal face Σ of $NE(X)$ and $\dim \Sigma = k$.

Proof. (See [BCD07, Proposition 2.]) Set $X^0 := X \setminus B$ and $\mathcal{S}^0 := \mathcal{S} \setminus p(E)$. Choose a very ample line bundle L on $\mathcal{S}$, and let $U \subset |L|$ be the open subset of divisors H that are irreducible and such that $H \not\subseteq p(E)$. For any H in U, we define $\widehat{H} := \overline{q^{-1}(H \cap \mathcal{S}^0)}$.
Recall that, as already proved in the proof of Proposition (6.15), $B = \widehat{H}_1 \cap \ldots \cap \widehat{H}_N$ for some $H_1, \dots, H_N \in U$. We want to prove that $\widehat{H}$ is nef. Suppose by contradiction

that there exists an irreducible curve $C \subset X$ such that $\widehat{H} \cdot C < 0$.
This implies that $\widehat{H}_j \cdot C < 0$ for every $j = 1, \dots, N$, and so we can conclude that $C \subseteq B$. By hypothesis, B is $\mathcal{S}$-rationally connected, hence C is numerically equivalent to a linear combination of irreducible components of cycles parametrized by the Chow families $\mathcal{V}^1, \dots, \mathcal{V}^k$ which are quasi unsplit. Therefore $[C] \in \langle [V^1], \dots, [V^k] \rangle$. From this it follows that $\widehat{H} \cdot C = 0$ because, as already shown in the proof of Proposition (6.15), $\widehat{H} \cdot V^i = 0$ for every $i = 1, \dots, k$. But this is a contradiction.
Finally, we show that

$$\widehat{H} \cdot C = 0 \Leftrightarrow [C] \in \langle [V^1], \dots, [V^k] \rangle.$$

Note that if $\widehat{H} \cdot C = 0$, then, by the proof of the previous proposition, $C \subseteq B$ or C is contained in a fiber of q, both are $\mathcal{S}$-rationally connected. Then, as already observed before, $[C] \in \langle [V^1], \dots, [V^k] \rangle$. □

Lemma 6.17. *Let X be a normal and $\mathbb{Q}$-factorial projective variety.*
Let $V^1, \dots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \dots, k$.
If $\dim B = f_{\mathcal{S}} + 1$, then every connected component of B is a $\mathcal{S}$-equivalence class.

Proof. (See [BDC07, Lemma 3]). □

The previous results can be summarized in the following proposition:

Proposition 6.18. *Let X be a normal and $\mathbb{Q}$-factorial projective variety.*
Let $V^1, \dots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \dots, k$. Then

(1) *either $B = \emptyset$ or $\dim B \geq f_{\mathcal{S}} + 1$;*

(2) *if $B = \emptyset$ or if $\dim B = f_{\mathcal{S}} + 1$ then $[V^1], \dots, [V^k]$ belong to an extremal face Σ of $NE(X)$ and $\dim \Sigma = k$.*

Theorem 6.19. *Let X be a normal and $\mathbb{Q}$-factorial complex projective variety. Let $V^1, \dots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \dots, k$.*
If $f_{\mathcal{S}} \geq n - 3$ then $[V^1], \dots, [V^k]$ belong to an extremal face Σ of $NE(X)$ and $\dim \Sigma = k$.

Proof. Suppose that B is not empty. By Proposition (6.18) (1), we have that $\dim B \geq f_{\mathcal{S}} + 1 \geq n - 2$.
But, since X is normal, $\dim B \leq n - 2$. Hence $\dim B = n - 2 = f_{\mathcal{S}} + 1$. Then, by Proposition (6.18) (2), $[V^1], \dots, [V^k]$ belong to an extremal face Σ of $NE(X)$ and $\dim \Sigma = k$. □

Lemma 6.20. *Let X be a smooth complex projective variety. Let $V^1, \ldots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \ldots, k$.*
Consider the diagram (6.1). Let D be a very ample divisor on $q(X \setminus B)$ and let $\widehat{D} := \overline{q^{-1}D}$. Then

(1) $\widehat{D} \cdot V^i = 0$ *for every* $i = 1, \ldots, k$*;*

(2) *if $C \not\subset B$ is a curve whose numerical class doesn't belong to $\langle [V^1], \ldots, [V^k] \rangle$, then $\widehat{D} \cdot C > 0$;*

(3) *if $[V^1], \ldots, [V^k]$ don't belong to a k-dimensional extremal face Σ of $NE(X)$, then there exists a curve $C \subset B$ such that*

$$[C] \notin \langle [V^1], \ldots, [V^k] \rangle \text{ and } \widehat{D} \cdot C \leq 0$$

Proof. (See [NO08, Lemma 2.2]) By Campana's construction, a general cycle of $\mathcal{V}^i$ is contained in a fiber of q disjoint from $\widehat{D}$, so $\widehat{D} \cdot V^i = 0$ for every $i = 1, \ldots, k$.
If C is as in (2), then $q(C)$ is a curve in $\mathcal{S}$ and the result follows from projection formula.
Finally, if $[V^1], \ldots, [V^k]$ don't belong to a k-dimensional extremal face Σ of $NE(X)$, then either $\widehat{D}$ is not nef or $\widehat{D}$ is nef but

$$\widehat{D}_{=0} \cap NE(X) \supsetneq \langle [V^1], \ldots, [V^k] \rangle.$$

In both cases there exists a curve $C \subset X$ such that $[C] \notin \langle [V^1], \ldots, [V^k] \rangle$ and $\widehat{D} \cdot C \leq 0$. This curve must be contained in B by the proof of Proposition (6.15). □

Lemma 6.21. *Let X be a smooth complex projective variety. Let $V^1, \ldots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \ldots, k$.*
Consider the diagram (6.1). Let $Z \subset X$ be an irreducible subvariety such that $\langle [V^1], \ldots, [V^k] \rangle \cap NE(Z, X) = \{0\}$.
Then there exists an irreducible component X_Z of ChLocus$(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Z$ containing Z and such that

(1) *if $Z \not\subset B$, then* $\dim X_Z \geq \dim Z + f_{\mathcal{S}}$*;*

(2) *if $Z \subset B$, then* $\dim X_Z \geq \dim Z + f_{\mathcal{S}} + 1$.

Proof. (See [NO08, Lemma 2.4]). □

Proposition 6.22. *Let X be a Fano manifold. Let $V^1, \ldots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \ldots, k$.*
Let $q : X \dashrightarrow \mathcal{S}$ be the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^k)$-quotient and suppose that $\dim \mathcal{S} > 0$.
Then either $[V^1], \ldots, [V^k]$ are contained in a k-dimensional extremal face of $NE(X)$ or there exists a small extremal ray R whose exceptional locus is contained in the indeterminacy locus of q.

Proof. Let $\widehat{D}$ be a divisor as in Lemma (6.20). Suppose that $[V^1], \ldots, [V^k]$ are not contained in a k-dimensional extremal face of $NE(X)$. Since X is Fano, by Lemma (6.20) (3), there exists an extremal ray R, whose exceptional locus is contained in B and such that $\widehat{D} \cdot R \leq 0$.
Note that $\dim B \leq n-2$, and so R is a small extremal ray. □

Remark 6.23. Denote by F the general fiber of the extremal contraction associated to R. By the Fiber Locus inequality, $\dim F \geq (i_X + 1)$. In particular $F \subset B$, and we can consider $\mathrm{ChLocus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_F$.
By Lemma (6.21), we have

$$\dim \mathrm{ChLocus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_F \geq (i_X + 1) + f_{\mathcal{S}} + 1.$$

By Proposition (6.15), B is closed with respect to $\mathcal{S}$-equivalence, and so

$$\mathrm{ChLocus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_F \subset B.$$

This implies that

$$\dim \mathrm{ChLocus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_F \leq \dim B \leq n-2$$

$$\Rightarrow f_{\mathcal{S}} \leq n - 4 - i_X$$

Corollary 6.24. *Let X be a Fano manifold. Let $V^1, \ldots, V^k$ be k quasi unsplit families of rational curves such that V^1 is dominating and $\mathcal{V}^i$ is horizontal and dominating with respect to $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^{i-1})$-quotient for every $i = 2, \ldots, k$. Let $q : X \dashrightarrow \mathcal{S}$ be the $(\mathcal{V}^1 \cup \ldots \cup \mathcal{V}^k)$-quotient and let $f_{\mathcal{S}}$ the dimension of the general $\mathcal{S}$-equivalence class.*
If $f_{\mathcal{S}} \geq n - 3 - i_X$, then $[V^1], \ldots, [V^k]$ belong to a k-dimensional negative extremal face Σ of $NE(X)$.

Chapter 7

Conic connected manifolds

In this chapter, we will study polarized manifolds (X, L) which are rationally connected with respect to rational curves of degree 2 with respect to a fixed ample line bundle L. These manifolds are called conic connected.
Conic connected manifolds were studied by Paltin Ionescu and Francesco Russo in [IR07]. They considered conic connected manifolds embedded in $\mathbb{P}^N$, i.e. the line bundle L which give the polarization is taken to be very ample. The main result of their paper is a classification theorem for these manifolds.
We want to show that their classification result holds true assuming just the ampleness of L.
Moreover, in the last section, we give a different proof of a theorem due to Kachi and Sato; this result characterize a special subclass of conic connected manifolds.

7.1 Conic connected manifolds

Definition 7.1. Let (X, L) be a polarized manifold (X is a smooth complex projective variety of dimension n and L is an ample line bundle on X). Suppose that two general points $x, x' \in X$ may be joined by a rational curve $C \subset X$, i.e. X is a rationally connected manifold. Define $d := L \cdot C$.

- If $d = 1$ then X is called **line connected**.
- If $d = 2$ then X is called **conic connected**.
- if $d = 3$ then X is called **rationally cubic connected**.

Remark 7.2. X is line connected if and only if $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

For conic connected manifolds, assuming L to be very ample, as already said, Ionescu and Russo proved the following classification result:

Theorem 7.3. *([IR07, Theorem 2.2]) Let $X \subset \mathbb{P}^N$ be a smooth irreducible linearly normal non-degenerate conic connected manifold of dimension n.*
Then either $X \subset \mathbb{P}^N$ is a Fano manifold with $Pic(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ and of index $r_X \geq \frac{n+1}{2}$, or it is projectively equivalent to one of the following:

(1) *the Veronese variety $\upsilon_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$;*

(2) *the projection of $\upsilon_2(\mathbb{P}^n)$ from the linear space $\langle \upsilon_2(\mathbb{P}^s)\rangle$, where $\mathbb{P}^s \subset \mathbb{P}^n$ is a linear subspace; equivalently $X \simeq Bl_{\mathbb{P}^s}(\mathbb{P}^n)$ embedded in $\mathbb{P}^N$ by the linear system of quadric hypersurfaces of $\mathbb{P}^n$ passing through $\mathbb{P}^s$; alternatively $X \simeq \mathbb{P}_{\mathbb{P}^r}(\varepsilon)$ with $\varepsilon \simeq \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(n-r)} \oplus \mathcal{O}_{\mathbb{P}^r}(2)$, $r = 1, 2, ..., n-1$ embedded by $|\mathcal{O}_{\mathbb{P}(\varepsilon)}(1)|$. Here $N = \frac{n(n+3)}{2} - \binom{s+2}{2}$ and s is an integer such that $0 \leq s \leq n-2$;*

(3) *a hyperplane section of the Segre embedding $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{N+1}$. Here $n \geq 3$ and $N = ab + a + b - 1$, where $a \geq 2$ and $b \geq 2$ are such that $a + b = n + 1$;*

(4) *$\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$ Segre embedded, where a, b are positive integers such that $a + b = n$.*

Now we give a generalization of Theorem (7.3): we consider polarized manifolds which are conic connected, and we show that we obtain the same classification of Ionescu and Russo.

Theorem 7.4. *Let (X, L) be a polarized manifold of dimension n. Assume that X is a conic connected manifold. Then either X is a Fano manifold with $Pic(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ and of index $r_X \geq \frac{n+1}{2}$, and $L \simeq \mathcal{O}_X(1)$, or*

(1) *$(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$;*

(2) *$(X, L) \simeq (\mathbb{P}_{\mathbb{P}^{n-r+1}}(\mathcal{O}_{\mathbb{P}^{n-r+1}}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^{n-r+1}}(2)), \mathcal{O}_X(1))$ where $2 \leq r \leq n$;*

(3) *$(X, L) \simeq (\mathbb{P}_{\mathbb{P}^s}(T\mathbb{P}^s \oplus \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(r-s)}), \mathcal{O}_X(1))$ where $s \leq \frac{n+1}{2}$, $r \geq 2$, $s \geq 2$ and $r + s = n + 1 \geq 4$;*

(4) *$(X, L) \simeq (\mathbb{P}^r \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s}(1))$ where r, s are positive integers such that $r + s = n$.*

Proof. By Remark (2.6) there exists a dominating family V of rational curves such that two general points $x, x' \in X$ may be joined by a rational curve parametrized by V and $L \cdot V = 2$.
Let $x \in X$ be a general point and let V_x be the subscheme of V parametrizing rational curves among those parametrized by V passing through x.
By [Deb01, Proposition 4.9], if $f : \mathbb{P}^1 \to X$ is a general curve parametrized by V_x, then f is a 1-free curve, i.e.

$$f^*TX \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with $a_1 \geq a_2 \geq ... \geq a_n$ and $a_1 \geq 2$, $a_n \geq 1$.
This implies that

$$-K_X \cdot f_*\mathbb{P}^1 = \sum_1^n a_i \geq n+1.$$

But $-K_X \cdot f_*\mathbb{P}^1 = -K_X \cdot V$, then the anticanonical degree of the family V is greater than or equal to $n+1$.

Suppose that V *is locally unsplit*: for the general point $x \in X$, V_x is proper.
Since $\mathrm{Locus}(V_x) = X$, every curve $C \subset X$ is numerically proportional to V, and so $\rho_X = 1$. Moreover, by Proposition (4.10), we have

$$-K_X \cdot V \leq \dim \mathrm{Locus}(V)_x + 1 = n+1$$

$$\Rightarrow -K_X \cdot V = n+1$$

In [Keb02] Kebekus proved that $\mathbb{P}^n$ is the only projective variety which admits a dominating locally unsplit family of rational curves which has anticanonical degree equal to $n+1$. Thus, *if* V *is locally unsplit, then* $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$.

Suppose that V *is not locally unsplit but it is quasi unsplit.*
Let $\mathcal{V}$ be the Chow family associated to V. Since $\mathrm{Locus}(\mathcal{V})_x = X$ and V is quasi unsplit, from Lemma (6.2) it follows that every curve $C \subset X$ is numerically proportional to V, and so $\rho_X = 1$. In particular, since $-K_X \cdot V \geq n+1$, X is a Fano manifold. Note that $-K_X \equiv r_X L$, and hence

$$n+1 \leq -K_X \cdot V = r_X L \cdot V = 2\, r_X$$

$$\Rightarrow r_X \geq \frac{n+1}{2}.$$

Thus, *if* V *is not locally unsplit but is quasi unsplit, then* X *is a Fano manifold with* $Pic(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ *and of index* $r_X \geq \frac{n+1}{2}$.

Suppose that V *is not locally unsplit and not quasi unsplit.*
Now we divide the proof into three steps:

Step 1. Bound on $\rho_{\mathbf{X}}$

Let $\{(W_i^1, W_i^2)\}_{i=1,\dots,k}$ be pairs of fellow families with respect to $\mathcal{V}$: every family W_i^j is a family of deformations of an irreducible component of a reducible cycle of $\mathcal{V}$ and, since $L \cdot V = L \cdot (W_i^1 + W_i^2) = 2$, W_i^j is unsplit.
For every $i \in \{1, \dots, k\}$ we consider the following subsets of X:

$$\mathrm{Locus}(W_i^1)_{\mathrm{Locus}(W_i^2)} \cup \mathrm{Locus}(W_i^2)_{\mathrm{Locus}(W_i^1)}$$

As W_i^1 and W_i^2 are unsplit, $\mathrm{Locus}(W_i^1)$ and $\mathrm{Locus}(W_i^2)$ are closed subset of X, and therefore

$$\mathrm{Locus}(W_i^1)_{\mathrm{Locus}(W_i^2)} \cup \mathrm{Locus}(W_i^2)_{\mathrm{Locus}(W_i^1)}$$

are closed subset of X for every $i \in \{1, \dots, k\}$.
Since V is not locally unsplit, we have

$$X = \bigcup_{i=1,\dots,k} \left(\mathrm{Locus}(W_i^1)_{\mathrm{Locus}(W_i^2)} \cup \mathrm{Locus}(W_i^2)_{\mathrm{Locus}(W_i^1)}\right)$$

Then there exists a pair of fellow families, which we denote by (W^1, W^2), such that $X = \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)}$ or $X = \mathrm{Locus}(W^2)_{\mathrm{Locus}(W^1)}$.
Assume without loss of generality that

$$X = \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)}.$$

In particular, W^1 is covering. We can consider the $rc(\mathcal{W}^1)$-fibration $\pi : X \dashrightarrow Z$. Since V is not quasi unsplit, $\dim Z > 0$ and, as $X = \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)}$, W^2 is a horizontal dominating family with respect to π.
Hence, we can consider the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration $\pi' : X \dashrightarrow Z'$.
Since W^1, W^2 are fellow families with respect to $\mathcal{V}$, the map π' contracts curves parametrized by V, and so π' is the constant map. Then, by Corollary (6.5), $\rho_X \leq 2$, and, since we are assuming that V is not quasi unsplit, this implies that $\rho_X = 2$.

We proved that $\rho_X = 2$, and now we study the Kleiman-Mori cone of X.

Step 2. Extremality of $\mathbf{W^1}$

We want to prove that an extremal ray of $NE(X)$ is generated by the numerical class of the family W^1.

Let x be a point in $\mathrm{Locus}(W^2)$, and we consider $\mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x}$.
Take $Y = \mathrm{Locus}(W^2)_x$. By Lemma (6.6), every curve in $\mathrm{Locus}(W^1)_Y$ is numerically equivalent to a linear combination with rational coefficients

$$\lambda\Gamma_Y + \mu\Gamma_{W^1}$$

where Γ_Y is a curve contained in Y, Γ_{W^1} belongs to W^1 and $\lambda \geq 0$.
From Corollary (6.7), $NE(Y, X) = \langle [W^2] \rangle$, and so every curve in X is numerically equivalent to a linear combination with rational coefficients

$$\alpha\Gamma_{W^2} + \mu\Gamma_{W^1}$$

where Γ_{W^2} is a curve parametrized by W^2, Γ_{W^1} belongs to W^1 and $\alpha \geq 0$.
This implies that $[W^1]$ belongs to an extremal ray of $NE(\mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x}, X)$.

Note also that

$$\begin{aligned} n \geq \dim \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x} &\geq \dim \mathrm{Locus}(W^2)_x - K_X \cdot W^1 - 1 \\ &\geq (-K_X \cdot W^2 - 1) - K_X \cdot W^1 - 1 \\ &\geq (n+1) - 2 = n - 1 \end{aligned}$$

$$\Rightarrow \dim \mathrm{Locus}(W_1)_{\mathrm{Locus}(W^2)_x} = \begin{cases} n \\ n-1 \end{cases}$$

If $\dim \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x} = n$, then the numerical class of W^1 belongs to an extremal ray of $NE(X)$.

Remark 7.5. If W^2 is not covering, from Remark (4.26), it follows that

$$\dim \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x} = n$$

and $-K_X \cdot V = n+1$.

Therefore, we suppose that $\dim \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x} = n-1$. This implies that $-K_X \cdot V = n+1$, and from Remark (4.11) it follows that W^2 is covering.
Set $D = \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x}$. D is an effective divisor and, since W^1, W^2 are covering, $D \cdot W^1 \geq 0$ and $D \cdot W^2 \geq 0$.
If $D \cdot W^1 > 0$, then

$$X = \mathrm{ChLocus}_2(W^1)_{\mathrm{Locus}(W^2)_x} = \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^1)_{\mathrm{Locus}(W^2)_x}}.$$

Thus, by Lemma (6.6) and Corollary (6.7), every curve in X is numerically equivalent to a linear combination with rational coefficients

$$\lambda \Gamma_{W^2} + \mu \Gamma_{W^1}$$

where Γ_{W^2} belongs to W^2, Γ_{W^1} is parametrized by W^1 and $\lambda \geq 0$. This implies that the numerical class of W^1 belongs to an extremal ray of $NE(X)$.
Assume now that $D \cdot W^1 = 0$.
Recalling that every curve in D is numerically equivalent to a linear combination

$$\alpha \Gamma_{W^2} + \mu \Gamma_{W^1}$$

with $\alpha \geq 0, \alpha, \mu \in \mathbb{Q}$, we can conclude that $D_{|D}$ is nef.
We observe that $D \cdot C \geq 0$ for every curve C such that C meets D but it is not contained in D. Thus D is nef, and therefore the numerical class of W^1 generates an extremal ray of the Kleiman-Mori cone of X.

We claimed that $[W^1]$ generates an extremal ray of the Kleiman-Mori cone of X, and now we want to describe X.

Step 3. Classification

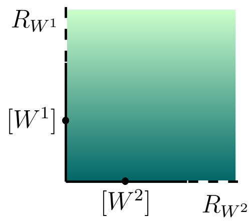
To describe X, we divide our study into two cases:

1. W^2 is covering;

2. W^2 is not covering.

Case 1. We suppose that W^2 is a covering family.

From our assumptions, as already observed for W^1, it follows that the numerical class of W^2 generates an extremal ray of the Kleiman-Mori cone of the variety X. Therefore W^1, W^2 are covering unsplit families of rational curves whose numerical classes span the extremal rays of $NE(X)$ $(NE(X) = \langle [W^1], [W^2] \rangle)$.



Since W^1, W^2 are covering, by [Deb01, Corollary 4.11] we have that

$$-K_X \cdot W^1 \geq 2 \text{ and } -K_X \cdot W^2 \geq 2$$

and so X is a Fano manifold.
Let φ be the Fano-Mori contraction of fiber type associated to the extremal ray R_{W^1}, and let ψ be the Fano-Mori contraction of fiber type associated to the extremal ray R_{W^2}.

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ Y & & Z \end{tikzcd}$$

Denote by F_φ a fiber of φ and by F_φ^g a general fiber of φ. Analogously, denote by F_ψ a fiber of ψ and by F_ψ^g a general fiber of ψ.
By the Fiber Locusinequality, we have that

$$\dim F_\varphi \geq -K_X \cdot W^1 - 1$$

$$\dim F_\psi \geq -K_X \cdot W^2 - 1$$

$$\Rightarrow \dim F_\varphi + \dim F_\psi \geq -K_X \cdot (W^1 + W^2) - 2$$
$$\Rightarrow \dim F_\varphi + \dim F_\psi \geq (n+1) - 2 = n - 1$$

Note that

$$\begin{aligned} \dim Y &= \dim X - \dim F_\varphi^g \\ \dim Z &= \dim X - \dim F_\psi^g \end{aligned}$$

Moreover, $\dim F_\varphi \leq \dim Z$ and $\dim F_\psi \leq \dim Y$ because, if these inequalities are not true, then there exists a curve whose numerical class belongs to R_{W^1} and R_{W^2}, and it is impossible. Thus

$$n \geq \dim F_\varphi + \dim F_\psi^g \geq \dim F_\varphi^g + \dim F_\psi^g \geq n - 1$$
$$\Rightarrow \dim Z - 1 \leq \dim F_\varphi^g \leq \dim F_\varphi \leq \dim Z$$

Therefore either φ is an equidimensional Fano-Mori contraction or there exist special fibers such that $\dim F_\varphi = \dim F_\varphi^g + 1$.
Similarly, we can prove that

$$\dim Y - 1 \leq \dim F_\psi^g \leq \dim F_\psi \leq \dim Y$$

It follows that at least one of these elementary extremal contractions must be equidimensional: suppose that φ is not equidimensional. We want to show that ψ is equidimensional.
Let $\widetilde{F}$ be a fiber of φ such that $\widetilde{F}$ dominates Z ($\dim \widetilde{F} = \dim Z$ and $\dim F_\varphi^g = \dim Z - 1$). Then $\widetilde{F}$ meets every fiber of ψ, but the dimension of the intersection $\widetilde{F} \cap F_\psi$ must be equal to 0. Therefore

$$\begin{aligned} n \geq \dim \widetilde{F} + \dim F_\psi &= \dim Z + \dim F_\psi \\ &= (n - \dim F_\psi^g) + \dim F_\psi \end{aligned}$$

$$\Rightarrow \dim F_\psi - \dim F_\psi^g \leq 0$$
$$\Rightarrow \dim F_\psi = \dim F_\psi^g$$

i.e. ψ is equidimensional.

Case 1.1 **We assume that $-\mathbf{K_X} \cdot \mathbf{V} \geq \mathbf{n+2}$.**

We consider $\mathrm{Locus}(W^1)_{F_\psi}$. From Lemma (4.25) it follows that

$$\begin{aligned} n \geq \dim \mathrm{Locus}(W^1)_{F_\psi} &\geq \dim F_\psi - K_X \cdot W^1 - 1 \\ &\geq (-K_X \cdot W^2 - 1) - K_X \cdot W^1 - 1 = n \end{aligned}$$

$$\Rightarrow \dim F_\psi = -K_X \cdot W^2 - 1 \text{ and } -K_X \cdot V = n + 2.$$

Similarly, if we consider Locus$(W^2)_{F_\varphi}$, we can prove that

$$\dim F_\varphi = -K_X \cdot W^1 - 1$$

Now, note that

$$K_X + (\dim F_\psi + 1)L = K_X + (n - \dim Z + 1)L$$

is a supporting divisor of ψ and

$$K_X + (\dim F_\varphi + 1)L = K_X + (n - \dim Y + 1)L$$

is a supporting divisor of φ.
Then, by Definition (3.19), ψ and φ are two equidimensional scrolls, and therefore, by Remark (3.20), ψ is a $\mathbb{P}$-bundle onto a smooth variety Z ($(X, L) \simeq (\mathbb{P}(\varepsilon_Z), \mathcal{O}_X(1))$ where ε_Z is an ample vector bundle on Z) and φ is a $\mathbb{P}$-bundle onto a smooth variety Y ($(X, L) \simeq (\mathbb{P}(\varepsilon_Y), \mathcal{O}_X(1))$ where ε_Y is an ample vector bundle on Y).
Thus X is a smooth variety endowed with two different $\mathbb{P}$-bundle structures, φ and ψ. Since fibers of different extremal ray contractions can meet only in points we have that $\dim X \leq \dim Y + \dim Z$. But

$$\begin{aligned} \dim Y + \dim Z &= (n - \dim F_\varphi) + (n - \dim F_\psi) \\ &= 2n - (-K_X \cdot W^1 - K_X \cdot W^2 - 2) \\ &= 2n - (n + 2 - 2) = n \end{aligned}$$

i.e. $\dim X = \dim Y + \dim Z$, and this is possible if and only if $X = \mathbb{P}^r \times \mathbb{P}^s$, where $r = \dim Y$ and $s = \dim Z$ (it is a corollary of [La84, Theorem 4.1]). Hence we get case (4) of the theorem.

***Case 1.2* From now on we can assume that $-\mathbf{K_X} \cdot \mathbf{V} = \mathbf{n} + \mathbf{1}$.**
Suppose that φ, ψ are equidimensional and $\dim \mathbf{F}_\varphi = \dim \mathbf{Z} - \mathbf{1}$.
Then

$$\begin{aligned} \dim F_\varphi + \dim F_\psi &= (\dim Z - 1) + \dim F_\psi \\ &= n - 1 \end{aligned}$$

$$\Rightarrow \dim F_\psi = n - \dim F_\varphi - 1 = \dim Y - 1$$

Note that $\dim Y + \dim Z = n + 1$ and in particular, since $-K_X \cdot V = n + 1$, we know that

$$\dim F_\varphi = -K_X \cdot W^1 - 1$$

$$\dim F_\psi = -K_X \cdot W^2 - 1$$

In particular, we have that

$$\dim Y = \dim F_\psi + 1 \geq 2$$

$$\dim Z = \dim F_\varphi + 1 \geq 2$$

$$\Rightarrow \dim X = \dim Y + \dim Z - 1 \geq 3.$$

As already proved in the case 1.1, φ is a $\mathbb{P}$-bundle onto a smooth variety Y and ψ is a $\mathbb{P}$-bundle onto a smooth variety Z.
By [OW02, Theorem 2], we have two possibilities:

1. $Y = Z = \mathbb{P}^m$ with $m = \frac{\dim X+1}{2} = \frac{n+1}{2}$ and $X = \mathbb{P}(T\mathbb{P}^m) = \mathbb{P}(T\mathbb{P}^{\frac{n+1}{2}})$;

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}^{\frac{n+1}{2}} & & \mathbb{P}^{\frac{n+1}{2}} \end{tikzcd}$$

2. Y, Z have a $\mathbb{P}$-bundle structure over a smooth curve C and $X = Y \times_C Z$:

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & C \end{array}$$

But, since $\rho_X = 2$, the second case is ruled-out, and hence $X = \mathbb{P}(T\mathbb{P}^{\frac{n+1}{2}})$.
Consider the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^m} \to \mathcal{O}_{\mathbb{P}^m}(1)^{\oplus(m+1)} \to T\mathbb{P}^m \to 0.$$

Set $\varepsilon = \mathcal{O}_{\mathbb{P}^m}(1)^{\oplus(m+1)}$. The surjection $\varepsilon \to T\mathbb{P}^m$ induces the inclusion $\mathbb{P}(T\mathbb{P}^m) \hookrightarrow \mathbb{P}(\varepsilon) = \mathbb{P}^m \times \mathbb{P}^m$ and $X = \mathbb{P}(T\mathbb{P}^m)$ is a divisor of degree $(1,1)$ of the product of projective spaces $\mathbb{P}^m \times \mathbb{P}^m$ ([Sa85, Lemma 1.15]). This leads to case (3) of the theorem.

Suppose that φ, ψ are equidimensional and $\dim \mathbf{F}_\varphi = \dim \mathbf{Z}$.
Then

$$\begin{aligned} \dim F_\varphi + \dim F_\psi &= \dim Z + \dim F_\psi \\ &= n \end{aligned}$$

$$\Rightarrow \dim F_\psi = n - \dim F_\varphi = \dim Y$$

Note that $\dim Y + \dim Z = n$.
In particular, since $-K_X \cdot V = n+1$, this implies that

$$\begin{cases} \dim F_\varphi = -K_X \cdot W^1 \\ \dim F_\psi = -K_X \cdot W^2 - 1 \end{cases} \text{or} \begin{cases} \dim F_\varphi = -K_X \cdot W^1 - 1 \\ \dim F_\psi = -K_X \cdot W^2 \end{cases}$$

Suppose that

$$\begin{cases} \dim F_\varphi = -K_X \cdot W^1 \\ \dim F_\psi = -K_X \cdot W^2 - 1 \end{cases}$$

Then ψ is a $\mathbb{P}$-bundle onto a smooth variety Z ($(X, L) \simeq (\mathbb{P}(\varepsilon_Z), \mathcal{O}_X(1))$ where ε_Z is an ample vector bundle on Z), and for every fiber F_φ

$$(K_X + (\dim F_\varphi)L) \cdot W^1 = 0$$

By adjunction we have $K_{F_\varphi} = (K_X)_{|F_\varphi}$, so $K_{F_\varphi} = -(\dim F_\varphi)L_{F_\varphi}$.
Since L is ample, F_φ is Fano and $r_{F_\varphi} = \dim F_\varphi$. From Theorem (1.21), $(F_\varphi, L_F) \simeq (\mathbb{Q}^{\dim F_\varphi}, \mathcal{O}_{\mathbb{Q}^{\dim F_\varphi}}(1))$.
Since a fiber of φ dominates Z, from [NO07, Lemma 4.1] it follows that $X \simeq \mathbb{P}^r \times Z \simeq \mathbb{P}^r \times \mathbb{Q}^s$ where $r = \dim Y$ and $s = \dim Z$.
Note that it is impossible, because if $X = \mathbb{P}^r \times \mathbb{Q}^s$, then there exist pairs of points of X such that they can not be joined by a rational curve of degree 2 with respect to $L \simeq \mathcal{O}_{\mathbb{P}^r \times \mathbb{Q}^s}(1)$.

Suppose that φ is not equidimensional, i.e. there exist special fibers $\widetilde{\mathbf{F}}_\varphi$ of φ such that $\dim \mathbf{Z} = \dim \widetilde{\mathbf{F}}_\varphi = \dim \mathbf{F}^{\mathbf{g}}_\varphi + \mathbf{1}$.
This implies that

$$\dim F^g_\varphi = -K_X \cdot W^1 - 1$$

$$\dim F_\psi = -K_X \cdot W^2 - 1$$

Recalling that W^1, W^2 are covering, we have that

$$\dim F^g_\varphi = -K_X \cdot W^1 - 1 \geq 1 \Rightarrow \dim Z \geq 2$$

$$\dim F_\psi = -K_X \cdot W^2 - 1 \geq 1 \Rightarrow \dim Y \geq 2$$

$$\Rightarrow \dim X = n = \dim Y + \dim Z - 1 \geq 3$$

Denote $r = \dim Y$ and $s = \dim Z$.
Then ψ is a $\mathbb{P}$-bundle onto a smooth variety Z ($(X, L) \simeq (\mathbb{P}(\varepsilon_Z), \mathcal{O}_X(1))$ where ε_Z is an ample vector bundle of rank r on Z), and φ is a scroll because φ is supported by

$$K_X + (\dim F^g_\varphi + 1)L = K_X + (\dim X - \dim Y + 1)L.$$

Since every fiber of φ has dimension $\leq \dim X - \dim Y + 1$, Y is a smooth variety and X is the projectivization of a Bănică sheaf on Y.
Moreover, by [AW93, Proposition 4.3], every fiber of φ is a projective space:

$$F_\varphi^g \simeq \mathbb{P}^{n-r}, \widetilde{F}_\varphi \simeq \mathbb{P}^{n-r+1}$$

and the dimension of $\widetilde{F}_\varphi$ is less then or equal to $\frac{n}{2}$, i.e. $r \geq \frac{n}{2} + 1$.
Note that there exists a surjective morphism from $\widetilde{F}_\varphi$ to Z. Then by [La84, Theorem 4.1], $Z \simeq \mathbb{P}^{n-r+1} \simeq \mathbb{P}^s$ where $s \leq \frac{n}{2}$:

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ Y & & \mathbb{P}^s \end{tikzcd}$$

As $(X, L) \simeq (\mathbb{P}(\varepsilon_Z), \mathcal{O}_X(1))$, we want to describe the ample vector bundle ε_Z on Z. By the canonical bundle formula

$$K_X + r\xi_\varepsilon = \psi^*(K_Z + \det \varepsilon)$$

Let $\tilde{l}$ be a curve parametrized by W^1. Then

$$\begin{aligned} K_X \cdot \tilde{l} + r\xi_\varepsilon \cdot \tilde{l} = K_X \cdot \tilde{l} + rL \cdot \tilde{l} &= \psi^*(K_Z + \det \varepsilon) \cdot \tilde{l} \\ &= K_Z \cdot \psi_*(\tilde{l}) + \psi^*(\det \varepsilon) \cdot \tilde{l} \end{aligned}$$

Since $\dim Z = -K_X \cdot W^1$ and $Z \simeq \mathbb{P}^s$, it follows that

$$K_X \cdot \tilde{l} + r = (K_X \cdot \tilde{l} - 1)\mathcal{O}_{\mathbb{P}^s}(1)\psi_*(\tilde{l}) + \psi^*(\det \varepsilon) \cdot \tilde{l}$$

$$\Rightarrow \det \varepsilon \cdot \psi_*(\tilde{l}) = r + 1$$

Let $l \subset Z$ be a line and let ε_l be the restriction of ε_Z to l. ε_l is a vector bundle on $\mathbb{P}^1$ and then it is decomposable in the direct sum of line bundles on $\mathbb{P}^1$:

$$\varepsilon_l = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

Since ε is ample and $\det \varepsilon \cdot \psi_*(\tilde{l}) = r + 1$, we have

$$\varepsilon_l = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(2),$$

i.e. ε is a uniform vector bundle on $\mathbb{P}^s$ of rank r with the splitting type $(1, \ldots, 1, 2)$. Since $r \geq s$, from [Wi93, Proposition 1.9] it follows that:

$$\varepsilon \simeq \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^s}(2) \text{ or } \varepsilon \simeq T\mathbb{P}^s \oplus \mathcal{O}(1)^{\oplus(r-s)}.$$

Suppose that $\varepsilon \simeq \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^s}(2)$. Then X is the blow up of $\mathbb{P}^n$ along a linear subspace Λ of dimension $r - 2$. But it is impossible, because we are assuming that there are two elementary extremal contractions of fiber type.
Therefore $\varepsilon \simeq T\mathbb{P}^s \oplus \mathcal{O}(1)^{\oplus(r-s)}$, and hence $X \simeq \mathbb{P}(T\mathbb{P}^s \oplus \mathcal{O}(1)^{\oplus(r-s)})$ is a divisor of degree $(1, 1)$ in $\mathbb{P}^r \times \mathbb{P}^s$ and

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}^r & & \mathbb{P}^s \end{tikzcd}$$

So we get case (3) of the theorem.

Case 2. We suppose that W^2 is not a covering family.

We consider the Fano-Mori contraction $\varphi : X \to Y$ associated to the extremal ray R_{W^1} which is generated by the numerical class of W^1.
Since W^1 is covering, φ is of fiber type. Denote by F_φ a fiber of φ.
By the Fiber Locus inequality, we have that

$$\dim F_\varphi \geq -K_X \cdot W^1 - 1.$$

We consider $\text{Locus}(W^2)_{F_\varphi} \subset \text{Locus}(W^2) \subsetneq X$. By Remark (7.5), $-K_X \cdot V = n+1$, and then, from Lemma (4.25), it follows that

$$\begin{aligned} n > \dim \text{Locus}(W^2)_{F_\varphi} &\geq \dim F_\varphi - K_X \cdot W^2 - 1 \\ &\geq (-K_X \cdot W^1 - 1) - K_X \cdot W^2 - 1 = n - 1. \end{aligned}$$

Therefore $\dim \text{Locus}(W^2)_{F_\varphi} = n-1$ and $\dim F_\varphi = -K_X \cdot W^1 - 1$. Hence φ is an equidimensional Fano-Mori contraction, and

$$K_X + (\dim F_\varphi + 1)L = K_X + (n - \dim Y + 1)L$$

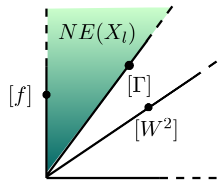
is a supporting divisor of φ. By Definition (3.19), φ is a equidimensional scroll, and so, by Remark (3.20), φ is a $\mathbb{P}$-bundle onto a smooth variety Y: $(X, L) \simeq (\mathbb{P}(\varepsilon_Y), \mathcal{O}_X(1))$ where ε_Y is an ample vector bundle of rank $r = (n - \dim Y + 1)$ on Y.
Note that $2 \leq r \leq n$. In fact $\dim Y > 0$ and so $r \leq n$. Moreover, since W^1 is covering, by [Deb01, Corollary 4.11] we have that $-K_X \cdot W^1 \geq 2$. Hence $\dim F_\varphi \geq 1$ and so

$$r = \dim F_\varphi + 1 \geq 2.$$

Now we want to describe ε_Y.
Let $x \in X$ be a point that doesn't belong to the union of the loci of the fellow families of W^1; such a point exists because we are assuming that the fellow families of W^1 are not covering.
Since X is conic connected, there is a irreducible curve Γ such that $x \in \Gamma$ and Γ is parametrized by V.
Let $\varphi(\Gamma) = l' \subset Y$ and let l be the normalization of l', $l \simeq \mathbb{P}^1 \xrightarrow{\nu} l'$.
Let $\varepsilon_{l'}$ be the restriction of ε to l'. Note that $\Gamma \subset \mathbb{P}(\varepsilon_{l'})$.
Let $X_l \to l \simeq \mathbb{P}^1$ be the projectivization of the pull-back of the vector bundle. We have the following diagram:

$$\begin{tikzcd} X_l \arrow[r, "\bar{\nu}"] \arrow[d] & \mathbb{P}(\varepsilon_{l'}) \arrow[d] \\ l \simeq \mathbb{P}^1 \arrow[r, "\nu"] & l' \subset Y \end{tikzcd}$$

The variety X_l is a projective bundle over $\mathbb{P}^1$ $(X_l \simeq \mathbb{P}_{\mathbb{P}^1}(\nu^*\varepsilon))$, so its cone of curves $NE(X_l)$ is generated by the class of a line (denote it by f) in a fiber of the natural projection $X_l \to \mathbb{P}^1$ and the class of a section (denote it by C_0) whose intersection with the tautological line bundle $\xi_{\nu^*\varepsilon}$ is minimal: $NE(X_l) = \langle [f], [C_0] \rangle$.
Since we have an identification $NE(X_l) \simeq NE(\mathbb{P}(\varepsilon_{l'}), X)$, the numerical class of f belongs to the extremal ray of $NE(X)$ generated by $[W^1]$ and we can write $[\Gamma] = a[C_0] + b[f]$ with $a, b \in \mathbb{Z}^+$.
Since Γ is not contracted by φ, the numerical class of Γ can not be a multiple of $[f]$, and so $a \neq 0$.
To show that $b \neq 0$, suppose by contradiction that $b = 0$. Then $[\Gamma]$ is a multiple of $[C_0]$ and the cone of X_l has the following structure:



But in $\mathbb{P}(\varepsilon_{l'})$ there are curves whose numerical classes are multiple of $[W^2]$, and thus $[W^2] \in NE(X_l) \simeq NE(\mathbb{P}(\varepsilon_{l'}), X)$. Therefore, $[\Gamma]$ is not a multiple of $[C_0]$ and $b \neq 0$.
Since $L \cdot f = 1$, we have that

$$2 = L \cdot \Gamma = aL \cdot C_0 + bL \cdot f = aL \cdot C_0 + b$$

As $a, b > 0$ and $L \cdot C_0 > 0$, this implies that $a = b = 1$ and $L \cdot C_0 = 1$.
Hence $[\Gamma] = [C_0] + [f]$ and $[C_0] = [W^2]$.
$\nu^*\varepsilon$ is a vector bundle on $\mathbb{P}^1$ and then it is decomposable in the direct sum of line bundles on $\mathbb{P}^1$:

$$\nu^*\varepsilon = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

Since $\dim \mathrm{Locus}(W^2)_{F_\varphi} = n - 1$, we have

$$\nu^*\varepsilon = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(c)$$

Consider the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(c) \to \nu^*\varepsilon \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)} \to 0$$

Denote by p the natural projection $X_l \to \mathbb{P}^1$. Let $D := \xi_{\nu^*\varepsilon} \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-c)$.
Note that

$$\begin{cases} \xi_{\nu^*\varepsilon} \cdot C_0 = 1 \\ \xi_{\nu^*\varepsilon} \cdot f = 1 \end{cases} \text{ and } \begin{cases} p^*\mathcal{O}_{\mathbb{P}^1}(-c) \cdot C_0 = -c \\ p^*\mathcal{O}_{\mathbb{P}^1}(-c) \cdot f = 0 \end{cases}$$

$$\Rightarrow \begin{cases} D \cdot C_0 = 1 - c \\ D \cdot f = 1 \end{cases} \Rightarrow D \cdot \Gamma = 2 - c$$

Since $\Gamma \not\subset D$, $c < 3$. In fact, if $c \geq 3$ then $D \cdot \Gamma < 0$.
Hence $c = 1$ or $c = 2$. If $c = 1$ then X_l is the product of two projective spaces and so there exists a horizontal line with respect to the extremal contraction φ passing through x. But it is a contradiction of our assumptions.
Therefore $c = 2$ and

$$\nu^*\varepsilon = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(2).$$

By the canonical bundle formula, we have that

$$K_X + r\xi_\varepsilon = \varphi^*(K_Y + \det \varepsilon)$$

Then

$$\begin{aligned} K_X \cdot \Gamma + r\xi_\varepsilon \cdot \Gamma &= K_Y \cdot \varphi_*\Gamma + \det \varepsilon \cdot \varphi_*\Gamma \\ &= K_Y \cdot l' + \det \varepsilon \cdot l' \\ &= K_Y \cdot l' + (r+1) \end{aligned}$$

Since $-K_X \cdot \Gamma = -K_X \cdot V = (n+1)$ and $\xi_\varepsilon \cdot \Gamma = 2$, it follows that

$$-(n+1) + 2r = K_Y \cdot l' + (r+1)$$

$$\begin{aligned} \Rightarrow -K_Y \cdot l' &= (n+1) - r + 1 \\ &= (n+1) - (n - \dim Y + 1) + 1 \\ &= \dim Y + 1 \end{aligned}$$

Now we want to prove that l' belongs to an unsplit family of rational curves of Y. Let V_Y be a family of rational curves of Y such that l' is parametrized by V_Y. Suppose by contradiction that V_Y is not unsplit and let $(l_1 + l_2)$ be a reducible cycle in $\mathcal{V}_Y$. As ε is ample, it follows that

$$r + 1 = \det \varepsilon \cdot l = \det \varepsilon \cdot l_1 + \det \varepsilon \cdot l_2 \geq r + r = 2r$$

Thus $r \leq 1$, but it is impossible because $r \geq 2$.
Therefore V_Y is unsplit, its anticanonical degree is equal to $\dim Y + 1$ and it is covering. This implies that $Y \simeq \mathbb{P}^{n-r+1}$ and l' is a line.
Then ε is a uniform vector bundle on $\mathbb{P}^{n-r+1}$ of rank r with the splitting type $(1, \ldots, 1, 2)$.

By [Wi93, Proposition 1.9], ε is either decomposable into a sum of line bundle or (if $r \geq n-r+1$) isomorphic to $\varepsilon \simeq T\mathbb{P}^{n-r+1} \oplus \mathcal{O}(1)^{\oplus(2r-n-1)}$.

(1.) If $\varepsilon \simeq \mathcal{O}_{\mathbb{P}^{n-r+1}}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^{n-r+1}}(2)$, X is the blow up of $\mathbb{P}^n$ along a linear subspace Λ of dimension $r-2$ and there are two elementary extremal contractions:

$$\begin{tikzcd} & Bl_\Lambda(\mathbb{P}^n) \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\ \mathbb{P}^n & & \mathbb{P}^{n-r+1} \end{tikzcd}$$

ψ is divisorial and the extremal ray associated to ψ is generated by the numerical class of curves which are contracted by the blow down along Λ. φ is of fiber type and the extremal ray associated to φ is generated by the numerical class of the strict transforms of lines of $\mathbb{P}^n$ which meet Λ in a point.
The family of the strict transforms of lines of $\mathbb{P}^n$ which meet Λ in a point is W^1, and the family of curves which are contracted by the blow down along Λ is W^2 and therefore $[W^2]$ belongs to an extremal ray.
We get the case (2) of the theorem.

(2.) Suppose that $\varepsilon \simeq T\mathbb{P}^{n-r+1} \oplus \mathcal{O}(1)^{\oplus(2r-n-1)}$ where $r \geq \frac{n+1}{2}$.
Then $X = \mathbb{P}(T\mathbb{P}^{n-r+1} \oplus \mathcal{O}(1)^{\oplus(2r-n-1)})$ is a divisor of degree $(1,1)$ in $\mathbb{P}^{n-r+1} \times \mathbb{P}^r$ and there are two extremal contractions of fiber type:

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}^{n-r+1} & & \mathbb{P}^r \end{tikzcd}$$

We want to prove that it is impossible. First of all we show that the numerical class of W^2 belongs to an extremal ray.
Consider the following nef divisors: $\varphi^*\mathcal{O}_{\mathbb{P}^{n-r+1}}(1)$ and $\psi^*\mathcal{O}_{\mathbb{P}^r}(1)$.
Let γ be a minimal extremal rational curve of the extremal ray associated to ψ. Note that

$$\begin{cases} \varphi^*\mathcal{O}_{\mathbb{P}^{n-r+1}}(1) \cdot W^1 = 0 \\ \psi^*\mathcal{O}_{\mathbb{P}^r}(1) \cdot \gamma = 0 \end{cases} \quad \text{and} \quad \begin{cases} L \cdot W^1 = 1 \\ L \cdot \gamma = 1 \end{cases}$$

Then $L = \varphi^*\mathcal{O}_{\mathbb{P}^{n-r+1}}(1) + \psi^*\mathcal{O}_{\mathbb{P}^r}(1)$.
Since $L \cdot W^2 = 1$ and $\varphi^*\mathcal{O}_{\mathbb{P}^{n-r+1}}(1)$, $\psi^*\mathcal{O}_{\mathbb{P}^r}(1)$ are nef, the numerical class of the family W^2 must belong to an extremal ray and $[W^2] = [\gamma]$.
It is a contradiction because we are assuming that W^2 is not covering and the Fano-Mori contraction associated to the extremal ray generated by $[W^2]$ is of fiber type. $\square$

Corollary 7.6. *If X is a conic connected manifold, then X is a Fano manifold whose Picard number is equal to or less than 2.*

7.2 Kachi-Sato's Theorem

In [KS99], Kachi and Sato found an exact condition that isolates $\mathbb{P}^n$ and $\mathbb{Q}^n$ within the class of conic connected manifolds.
They proved that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ are the only polarized manifolds that satisfy the following property: a fixed point $x \in X$ and two general points of X may be joined by an irreducible rational curve on X of degree 2 with respect to a fixed ample line bundle.
More precisely, they showed a slightly more general result; they considered projective varieties with at worst $\mathbb{Q}$-factorial singularities such that through a fixed non-singular point $x \in X$ and through two general points of X there is an irreducible rational curve on X of degree 2 with respect to a fixed ample Cartier divisor on X, and they proved that these projective varieties are isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$, where $\mathbb{Q}^n$ is a (possibly singular) hyperquadric in $\mathbb{P}^{n+1}$ (see [KS99, Theorem 5.1]).
In this section, we show a different proof of Kachi-Sato's result in the smooth case.

Theorem 7.7. *([KS99, Theorem 5.1]) Let (X, L) be a polarized manifold. Let $x \in X$ be a point. Assume that for two general points $y_1, y_2 \in X$ there is a rational curve $C \subset X$ passing through x, y_1, y_2 such that $L \cdot C = 2$. Then*

$$(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \text{ or } (X, L) \simeq (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$$

Remark 7.8. If $n = 2$ then by Theorem (7.4) X is a Del Pezzo surface whose Picard number is equal to or less than 2, and so the statement of previous theorem can be easily proved.

Proof. We can assume that $n \geq 3$.
By Remark (2.6) it follows that there exists a dominating family V of rational curves such that $L \cdot V = 2$, and $x, y_1, y_2 \in X$ may be joined by a rational curve parametrized by V for every general points $y_1, y_2 \in X$.
By [Deb01, Proposition 4.9], if $f : \mathbb{P}^1 \to X$ is a general curve parametrized by V, then f is a 2-free curve, i.e.

$$f^*TX \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with $a_1 \geq a_2 \geq \ldots \geq a_n$ and $a_n \geq 2$.
This implies that

$$-K_X \cdot f_*\mathbb{P}^1 = \sum_1^n a_i \geq 2n.$$

But $-K_X \cdot f_*\mathbb{P}^1 = -K_X \cdot V$, then the anticanonical degree of the family V is greater than or equal to $2n$.

Let $\mathcal{V}$ be the Chow family associated to V. Note that every reducible cycle in $\mathcal{V}$ has two irreducible components. We consider a pair (V^1, V^2) of fellow families with respect to $\mathcal{V}$ (V^1 and V^2 are families of rational curves which are families of deformations of two irreducible components of a reducible cycle in $\mathcal{V}$).
Since $[V^1] + [V^2] = [V]$ and $L \cdot V = 2$, V^1 and V^2 are unsplit families such that $L \cdot V^1 = L \cdot V^2 = 1$.
By Proposition (4.10) for any $x \in \mathrm{Locus}(V^j)$ ($j = 1, 2$) we have

$$\dim \mathrm{Locus}(V^j)_x \geq -K_X \cdot V^j - 1$$

$$\Rightarrow -K_X \cdot V^j \leq \dim \mathrm{Locus}(V^j)_x + 1 \leq n + 1$$

Since $-K_X \cdot (V^1 + V^2) \geq 2n$ and $-K_X \cdot V^j \leq (n + 1)$ with $j = 1, 2$, there are the following possibilities:

$-K_X \cdot V^1$	$-K_X \cdot V^2$
n+1	n+1, n, n-1
n	n, n+1
n-1	n+1

Therefore we can divide the proof into two cases.

Case 1. At least one family of deformations of a irreducible component of a reducible cycle of $\mathcal{V}$ has anticanonical degree equal to $\mathbf{n+1}$.

Denote by $\widetilde{V}$ this family. From Proposition (4.10) it follows that

$$\dim \mathrm{Locus}(\widetilde{V})_x = n.$$

Applying Corollary (6.7) we obtain that $\rho_X = 1$.
Note that $-K_X = (n + 1)L$. Then X is a Fano manifold, and by the classification of Kobaiashi-Ochiai (Theorem (1.21)) we have that $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Case 2. Every family of deformations of a irreducible component of a reducible cycle of $\mathcal{V}$ has anticanonical degree equal to $\mathbf{n}$.

Consider a pair (V^1, V^2) of fellow families with respect to $\mathcal{V}$. V^1 and V^2 are unsplit and covering.
Moreover, for a general point $x \in X$ and for $j = 1, 2$, we have that

$$\dim \mathrm{Locus}(V^j)_x \geq n - 1.$$

Hence, since $n \geq 3$, we get

$$\dim(\mathrm{Locus}(V^1)_x \cap \mathrm{Locus}(V^2)_x) \geq 1.$$

This implies that there exists a curve such that is numerically proportional to V^1 and V^2 because by Corollary (6.7) we have that $NE(\text{Locus}(V^1)_x, X) = \langle [V^1] \rangle$ and $NE(\text{Locus}(V^2)_x, X) = \langle [V^2] \rangle$. Then the families V^1 and V^2 are numerically proportional.
But $[V] = [V^1] + [V^2]$, so every component of any reducible cycle in $\mathcal{V}$ is numerically proportional to V, i.e. V is a quasi unsplit family.
We can consider the $rc(\mathcal{V})$-fibration, $\pi : X \dashrightarrow Z$. By the properties of V, π is the constant map, and so by Corollary (6.5) $\rho_X = 1$.
Note that $-K_X = nL$, and this implies that X is Fano. By Theorem (1.21), we can conclude that $(X, L) \simeq (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$. □

Chapter 8

Rationally cubic connected manifolds

In this chapter we will consider rationally cubic connected manifolds.
A first step towards the understanding of these manifolds could be to establish a bound on the Picard number. Unfortunately, differently from the case of conic connected manifolds, there isn't an upper bound on the Picard number. In fact, as we will shown in the following example, for every integer $m > 0$ we can construct a rationally cubic connected manifold whose Picard number is equal to m.

Example 8.1. Let $P_1, ..., P_k$ be general points of $\mathbb{P}^n$ and let X be the blow up of $\mathbb{P}^n$ at $P_1, ..., P_k$, with $n \geq 2$ and

$$0 \leq k \leq \left(\begin{array}{c} n+3 \\ 3 \end{array} \right) - (2n+2).$$

Let $\varphi : X \to \mathbb{P}^n$ be the blow up and let $\{E_i = \varphi^{-1}(P_i)\}_{i=1,...,k}$ be the exceptional divisors. X is rationally connected with respect to the family V of deformations of the strict transform of a general line in $\mathbb{P}^n$.
By [Cop02], the line bundle

$$L = \varphi^* \mathcal{O}_{\mathbb{P}^n}(3) - \left(\sum_{i=1}^{k} E_i \right)$$

is very ample and $L \cdot V = 3$. Then X is rationally cubic connected and $\rho_X = k + 1$.

However we will prove that if rationally connected manifolds are covered by "lines", i.e. by curves of degree 1 with respect to a fixed ample line bundle, then the Picard number is equal to or less than 3.
In particular this implies that for $n \geq 3$ and $k \geq 3$ rationally cubic connected manifolds described in Example (8.1) are not covered by "lines" because they have

Picard number greater than 3. Moreover, we observe that these manifols are not Fano. In fact, if we consider the strict transform l of a line in $\mathbb{P}^n$ passing through two points P_1 and P_2 that are centers of the blow up, then, by the canonical bundle formula of the blow up, $-K_X \cdot l \leq 0$.
For that reason we will consider rationally cubic connected manifolds which are not covered by "lines" but are Fano. We will show that up to a few exceptions in dimension 2 the Picard number of these manifolds is equal to or less than 3; moreover, if it is equal to 3 we will obtain a precise classification of these manifolds.

First of all, we recall the definition of rationally cubic connected manifolds.

Definition 8.2. Let (X, L) be a polarized manifold. X is rationally cubic connected - RCC for short - if two general points $x, x' \in X$ may be joined by a rational curve $\gamma \subset X$ of degree 3 with respect to L, or equivalently, if there exists a dominating family V of rational curves such that $L \cdot V = 3$ and through two general points of X there is a curve parametrized by V.

Remark 8.3. Let $x \in X$ be a general point and let V_x be the subscheme of V parametrizing rational curves among those parametrized by V passing through x. By [Deb01, Proposition 4.9], if $f : \mathbb{P}^1 \to X$ is a general curve parametrized by V_x, then f is a 1-free curve, i.e.

$$f^*TX \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with $a_1 \geq a_2 \geq ... \geq a_n$ and $a_1 \geq 2$, $a_n \geq 1$.
This implies that

$$-K_X \cdot f_*\mathbb{P}^1 = \sum_1^n a_i \geq n+1.$$

But $-K_X \cdot f_*\mathbb{P}^1 = -K_X \cdot V$, then the anticanonical degree of the family V is greater than or equal to $n+1$.

If V is locally unsplit, for the general point $x \in X$, V_x is proper.
Since $\mathrm{Locus}(V_x) = X$, every curve $C \subset X$ is numerically proportional to V, and so $\rho_X = 1$. Moreover, by Proposition (4.10), we have

$$-K_X \cdot V \leq \dim \mathrm{Locus}(V)_x + 1 = n+1$$

$$\Rightarrow -K_X \cdot V = n+1$$

In [Keb02] Kebekus proved that $\mathbb{P}^n$ is the only projective variety which admits a dominating locally unsplit family of rational curves which has anticanonical degree equal to $n+1$. Thus, if V is locally unsplit, then $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(3))$.

Suppose that V is not locally unsplit but for a general point $x \in X$ every component of any reducible cycle, which passes through x and is parametrized by the Chow

family $\mathcal{V}$ associated to V, has numerical class proportional to the numerical class of a curve parametrized by V.
Since $\text{Locus}(\mathcal{V})_x = X$, every curve $\gamma \subset X$ is numerically proportional to V, and so $\rho_X = 1$. In particular, since $-K_X \cdot V \geq n+1$, X is a Fano manifold.
Note that $-K_X \equiv r_X L$, and hence

$$n+1 \leq -K_X \cdot V = r_X L \cdot V = 3\, r_X$$

$$\Rightarrow r_X \geq \frac{n+1}{3}.$$

Thus X is a Fano manifold of Picard number one and of index $r_X \geq \frac{n+1}{3}$ with fundamental divisor L.

From now on we can assume that V is not locally unsplit and that through a general point $x \in X$ there is a reducible cycle in $\mathcal{V}$ such that at least one of its irreducible components is numerically independent to V. In particular this implies that V is not quasi unsplit and $\rho_X > 1$.
Moreover, we observe that a cycle in $\mathcal{V}$ can split into two or three irreducible rational components since $L \cdot V = 3$; we will call a component of degree one a line and a component of degree two a conic.

8.1 RCC-manifolds covered by lines

As said in the introduction of this chapter, we will start by considering RCC-manifolds which are covered by lines and we will prove the following theorem:

Theorem 8.4. *Let (X, L) be a polarized manifold. Suppose that X is RCC by a family V and that X admits a covering family of lines. Then $\rho_X \leq 3$, equality holding if and only if there exist three families of lines W, W', W'' with $[V] = [W] + [W'] + [W'']$ such that W is covering, W' is horizontal and dominating with respect to the $rc(\mathcal{W})$-fibration and W'' is horizontal and dominating with respect to the $rc(\mathcal{W}, \mathcal{W}')$-fibration.*

First of all, we state some results which will be used throughout the proof of the theorem.

Lemma 8.5. *Let (X, L) be a polarized manifold. Let $Z \subset X$ be a closed subset and let W_1 be a covering unsplit family of rational curves. Suppose that $N_1(Z, X)$ is two-dimensional and there is a irreducible curve Γ whose numerical class $[\Gamma]$ is extremal in $NE(Z, X)$.*
Moreover assume that $[W^1] \notin N_1(Z, X)$ and that for some integer m we have

$$X = ChLocus_m(\mathcal{W}^1)_Z.$$

Then the numerical classes $[W^1], [\Gamma]$ lie in a (two-dimensional) extremal face of $NE(X)$ and $[W^1]$ spans a negative extremal ray.

Proof. By our assumptions, there is a curve Γ' such that $[\Gamma'] \in N_1(Z, X)$ and every curve in Z is numerically equivalent to a linear combination $a\Gamma + b\Gamma'$ with $b \geq 0$. Then, from Lemma (6.11) it follow that every curve contained in X is numerically equivalent to a linear combination

$$\alpha\Gamma_Z + \beta\Gamma_{W^1}$$

where Γ_Z is a curve in Z and Γ_{W^1} is parametrized by W^1, and $\alpha \geq 0$; therefore the numerical class of every curve in X can be written as

$$\delta_1[\Gamma] + \delta_2[\Gamma'] + \beta[\Gamma_{W^1}]$$

with $\delta_2 \geq 0$.
Let Γ_1 and Γ_2 be two curves in X; we can write

$$[\Gamma_1] = \alpha_1[\Gamma_Z] + \beta_1[\Gamma_{W^1}] = \delta_1^1[\Gamma] + \delta_2^1[\Gamma'] + \beta_1[\Gamma_{W^1}]$$

and

$$[\Gamma_2] = \alpha_2[\Gamma'_Z] + \beta_2[\Gamma_{W^1}] = \delta_1^2[\Gamma] + \delta_2^2[\Gamma'] + \beta_2[\Gamma_{W^1}]$$

where $\alpha_1, \alpha_2 \geq 0$ and $\delta_2^1, \delta_2^2 \geq 0$.
First of all we claim that $[W^1], [\Gamma]$ belong to a (two-dimensional) extremal face of $NE(X)$.
To prove the statement it is enough to suppose that $[\Gamma_1] + [\Gamma_2] \in \Pi := \langle [W^1], [\Gamma] \rangle$ and to show that $[\Gamma_1] \in \Pi$ and $[\Gamma_2] \in \Pi$.
Asking for $[\Gamma_1] + [\Gamma_2]$ to be in Π amounts to impose $\delta_2^1 + \delta_2^2 = 0$, hence $\delta_2^1 = \delta_2^2 = 0$ and both $[\Gamma_1]$ and $[\Gamma_2]$ belong to Π.
Now we want to prove that the numerical class $[W^1]$ generates an extremal ray. As before, assuming that $[\Gamma_1] + [\Gamma_2] \in \langle [W^1] \rangle$ we must show that $[\Gamma_1], [\Gamma_2] \in \langle [W^1] \rangle$.
If $[\Gamma_1] + [\Gamma_2] \in \langle [W^1] \rangle$ then $\alpha_1[\Gamma_Z] + \alpha_2[\Gamma'_Z] = \underline{0}$.
We can assume that $[\Gamma_Z] \neq \underline{0}$ and $[\Gamma'_Z] \neq \underline{0}$. Clearly, if $\alpha_1 = 0$ or $\alpha_2 = 0$ then $[\Gamma_1] = \beta_1[\Gamma_{W^1}]$ and $[\Gamma_2] = \beta_2[\Gamma_{W^1}]$, and so we have that $[\Gamma_1], [\Gamma_2] \in \langle [W^1] \rangle$.
Hence we suppose by contradiction that $\alpha_1 > 0$ and $\alpha_2 > 0$. We know that

$$\alpha_1[\Gamma_Z] = \delta_1^1[\Gamma] + \delta_2^1[\Gamma'], \quad \delta_2^1 \geq 0$$

and

$$\alpha_2[\Gamma'_Z] = \delta_1^2[\Gamma] + \delta_2^2[\Gamma'], \quad \delta_2^2 \geq 0;$$

hence

$$\alpha_1[\Gamma_Z] + \alpha_2[\Gamma'_Z] \quad = \quad (\delta_1^1 + \delta_1^2)[\Gamma] + (\delta_2^1 + \delta_2^2)[\Gamma'].$$

Notice that $\delta_2^1 + \delta_2^2 \geq 0$, $\delta_2^1 \geq 0$ and $\delta_2^1 \geq 0$.
From $\alpha^1[\Gamma_Z] + \alpha^2[\Gamma'_Z] = \underline{0}$, it follows that

$$\delta_2^1 + \delta_2^2 = 0$$

and so we have that $\delta_2^1 = \delta_2^1 = 0$. Thus

$$\alpha_1[\Gamma_Z] = \delta_1^1[\Gamma] \text{ and } \alpha_2[\Gamma'_Y] = \delta_1^2[\Gamma]$$

and hence

$$\delta_1^1 = -\delta_1^2$$

Therefore $\delta_1^1\delta_1^2 < 0$ and we have that the line $\langle[\Gamma]\rangle$ is contained in $NE(X)$. But it is impossible. Thus $[\Gamma_1], [\Gamma_2] \in \langle[W^1]\rangle$. □

Analogously we can prove the following lemma:

Lemma 8.6. *Let (X, L) be a polarized manifold. Let W^1, W^2, W^3 be three unsplit families of rational curves which are numerically independent. Assume that for some $x \in X$ and some integers m_1, m_2 we have*

$$X = \mathit{ChLocus}_{m_1}(\mathcal{W}^1, \mathcal{W}^2)_{\mathit{ChLocus}_{m_2}(\mathcal{W}^3)_x}.$$

Then the numerical classes $[W^1], [W^2]$ lie in a (two-dimensional) extremal face of $NE(X)$.

Proof. By Lemma (6.11) $N_1(\mathrm{ChLocus}_{m_2}(\mathcal{W}^3)_x, X) = \langle[W^3]\rangle$ and so

$$NE(\mathrm{ChLocus}_{m_2}(\mathcal{W}^3)_x, X) = \langle[W^3]\rangle.$$

Then, again from Lemma (6.11) it follows that every curve in X is numerically equivalent to a linear combination with rational coefficients $\sum_{j=1}^3 a_j W^j$, with $a_3 \geq 0$.
Let Π be the plane defined by $[W^1]$ and $[W^2]$ and let Γ^1 and Γ^2 be two curves such that $[\Gamma_1] + [\Gamma_2] \in \Pi$; write $[\Gamma_i] = \sum c_j^i [W^j]$, with $c_3^i \geq 0$.
To prove that $[W^1], [W^2]$ lie in a (two-dimensional) extremal face of $NE(X)$, we must show that $[\Gamma_1]$ and $[\Gamma_2]$ belong to Π.
Asking for $[\Gamma_1] + [\Gamma_2]$ to be in Π amounts to impose $c_3^1 + c_3^2 = 0$, hence $c_3^1 = c_3^2 = 0$ and both $[\Gamma_1]$ and $[\Gamma_2]$ belong to Π. □

Proposition 8.7. *Assume that (X, L) is RCC by a family V and that through a general point of X there is a connected rational 1-cycle whose numerical class is $[V]$, consisting of three lines. Then $\rho_X \leq 3$. Moreover, if $\rho_X = 3$ then there exist three families of lines W^1, W^2, W^3 with $[V] = [W^1] + [W^2] + [W^3]$ such that W^1 is covering, W^2 is horizontal and dominating with respect to the $rc(\mathcal{W}^1)$-fibration and W^3 is horizontal and dominating with respect to the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration.*

Proof. Let $\mathcal{P} = \{(W_i^1, W_i^2, W_i^3)\}_{i=1,\dots,s}$ be the set of unsplit families such that

$$[W_i^1] + [W_i^2] + [W_i^3] = [V].$$

Denote by B_i the set of points which are contained in a connected chain $l_i^1 \cup l_i^2 \cup l_i^3$, with l_i^j is parametrized by W_i^j and $l_i^j \cap l_i^{j+1} \neq \emptyset$ for $j = 1, 2$. The set B_i can be written as the union of three closed subset:

$$\begin{aligned} B_i^1 &:= \mathrm{Locus}(W_i^2, W_i^1)_{\mathrm{Locus}(W_i^3)} \\ B_i^2 &:= i_2(p_2^{-1}(p_2(i_2^{-1}(\mathrm{Locus}(W_i^1))) \cap p_2(i_2^{-1}(\mathrm{Locus}(W_i^3))))) \\ B_i^3 &:= \mathrm{Locus}(W_i^2, W_i^3)_{\mathrm{Locus}(W_i^1)} \end{aligned}$$

where i_2, p_2 are proper morphisms associated to the Chow family $\mathcal{W}_i^2$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "i_2"] \arrow[d, "p_2"'] & X \\ \mathcal{W}_i^2 & \end{tikzcd}$$

Notice that B_i^j is the set of points on curves parametrized by W_i^j belonging to the chain. Since through the general point of X there is a reducible cycle consisting of three lines, we have that

$$X = \bigcup_{i=1}^{s} B_i = \bigcup_{i=1}^{s} \left(\bigcup_{j=1}^{3} B_i^j \right).$$

Since B_i^j are a finite number and each of them is closed there is a pair of index (i_0, j_0) such that X is contained in $B_0 := B_{i_0}^{j_0}$. By construction the set B_i^j is contained in $\mathrm{Locus}(W_i^j)$, therefore the family $W_{i_0}^{j_0}$ is covering.
To simplify notation we denote from now on by W^1, W^2 and W^3 the families corresponding to the index i_0. Whitout loss of generality we also assume that $j_0 = 1$, i.e. W^1 is a covering family of rational curves.
Then we can consider the $rc(\mathcal{W}^1)$-fibration $\pi_1 : X \dashrightarrow Z_1$. If $\dim Z_1 = 0$ then $\rho_X = 1$ by Corollary (6.5); otherwise we claim that either W^2 or W^3 is horizontal and dominating with respect to π_1.
Notice that connected cycles $l^1 \cup l^2 \cup l^3$ are not contracted by π_1, otherwise also curves parametrized by V would be contracted, and Z_1 should be a point. Then at least one of the irreducible components of $l^2 \cup l^3$ is not contracted by π and it meets the general fiber F of π_1.
Suppose without loss of generality that this irreducible component is parametrized by W^2. Thus W^2 is a horizontal dominating family with respect to π_1. Take the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration $\pi_2 : X \dashrightarrow Z_2$.

If $\dim Z_2 = 0$, from Corollary (6.5) it follows that $\rho_X = 2$; otherwise we can prove, arguing as above, that W^3 is a horizontal dominating family for π_2.
Thus we can consider the $rc(\mathcal{W}^1, \mathcal{W}^2, \mathcal{W}^3)-$fibration π_3. π_3 contracts the cycles $l^1 \cup l^2 \cup l^3$, hence contracts curve parametrized by V. This implies that π_3 is the constant map, and so, by Corollary (6.5) we have that $\rho_X = 3$. $\square$

Proof. (of Theorem (8.4)) As already observed before, we can assume that the family V is not locally unsplit and not quasi unsplit since otherwise we can conclude that $\rho_X = 1$. We divide our proof into two cases.

Case 1. Assume that for two general points $\mathbf{x}, \mathbf{x}' \in \mathbf{X}$***, there exists a reducible cycle in*** $\mathcal{V}$ ***passing through*** $\mathbf{x}, \mathbf{x}'$***.***

We consider the following sets:

- $\mathcal{G}_1 = \{(W^j, C^j)\}_{j=1,\dots,k}$, the set of the fellow families with respect to $\mathcal{V}$ such that $L \cdot W^j = 1$ and $L \cdot C^j = 2$;

- $\mathcal{G}_2 = \{(W_s^1, W_s^2, W_s^3)\}_{s=1,\dots,m}$, the set of of the fellow families with respect to $\mathcal{V}$ such that $L \cdot W_s^1 = L \cdot W_s^2 = L \cdot W_s^3 = 1$;

- $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$.

For every pair of fellow families in $\mathcal{G}_1$, we denote by $\mathcal{W}^j$ the Chow family associated to W^j, with universal family $\mathcal{U}_j$, and by $\mathcal{C}^j$ the Chow family associated to C^j, with universal family $\mathcal{F}_j$.
We recall that, as already seen in section (4.2.1), these Chow families define proper prerelations, $(\mathcal{U}_j \xleftarrow{p'_j} \mathcal{U}'_j \xrightarrow{i'_j} X, \mathcal{U}_j \xrightarrow{\sigma_j} \mathcal{U}'_j)$ and $(\mathcal{F}_j \xleftarrow{p''_j} \mathcal{F}'_j \xrightarrow{i''_j} X, \mathcal{F}_j \xrightarrow{\sigma'_j} \mathcal{F}'_j)$:

$$\begin{tikzcd} \mathcal{U}'_j := \mathcal{U}_j \times_{\mathcal{W}^j} \mathcal{U}_j \arrow[r, "i'_j"] \arrow[d, shift right, "p'_j"'] & X \\ \mathcal{U}_j \arrow[u, shift right, "\sigma_j"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}'_j := \mathcal{F}_j \times_{\mathcal{C}^j} \mathcal{F}_j \arrow[r, "i''_j"] \arrow[d, shift right, "p''_j"'] & X \\ \mathcal{F}_j \arrow[u, shift right, "\sigma'_j"'] & \end{tikzcd}$$

and algebraic relations, $R_{1,j}$ and $R_{2,j}$:

$$\begin{tikzcd} & R_{1,j} = \mathcal{U}_j \times_{\mathcal{W}^j} \mathcal{U}_j \arrow[dl, "w_j"'] \arrow[dr, "u_j"] & \\ X & & X \end{tikzcd} \qquad \begin{tikzcd} & R_{2,j} = \mathcal{F}_j \times_{\mathcal{C}^j} \mathcal{F}_j \arrow[dl, "w'_j"'] \arrow[dr, "u'_j"] & \\ X & & X \end{tikzcd}$$

Similarly, for every triplets of fellow families in $\mathcal{G}_2$, we denote by $\mathcal{W}_s^r$ the Chow family associated to W_s^r, with universal family $\mathcal{M}_{r,s}$, and by $\widetilde{R}_{r,s}$ the algebraic relations defined by $\mathcal{W}_s^r$ ($r = 1, 2, 3$ and $s = 1, \dots, m$):

$$\begin{tikzcd} \mathcal{M}'_{r,s} := \mathcal{M}_{r,s} \times_{\mathcal{W}^r_s} \mathcal{M}_{r,s} \arrow[r, "i'_{r,s}"] \arrow[d, shift right, "p'_{r,s}"'] & X \\ \mathcal{M}_{r,s} \arrow[u, shift right, "\sigma_{r,s}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & \widetilde{R}_{r,s} = \mathcal{M}_{r,s} \times_{\mathcal{W}^r_s} \mathcal{M}_{r,s} \arrow[dl, "w_{r,s}"'] \arrow[dr, "u_{r,s}"] & \\ X & & X \end{tikzcd}$$

Since we are assuming that two general points $x, x' \in X$ may be joined by a reducible cycle in $\mathcal{V}$ whose irreducible components belong to fellow families in $\mathcal{G}$, we consider $Chain_2(\mathcal{U}'_j, \mathcal{F}'_j)$ and $Chain_3(\mathcal{M}'_{1,s}, \mathcal{M}'_{2,s}, \mathcal{M}'_{3,s})$.
By Theorem (5.3), for every j and for every s, we know that

$$Chain_2(\mathcal{U}'_j, \mathcal{F}'_j) = \bigcup_{\tau:\{1,2\}\to\{0,1,2\}} R_{\tau(1),j} * R_{\tau(2),j}$$

$$Chain_3(\mathcal{M}'_{1,s}, \mathcal{M}'_{2,s}, \mathcal{M}'_{3,s}) = \bigcup_{\sigma:\{1,2,3\}\to\{0,1,2,3\}} \widetilde{R}_{\sigma(1),s} * \widetilde{R}_{\sigma(2),s} * \widetilde{R}_{\sigma(3),s}$$

where $R_{0,j}$ and $\widetilde{R}_{0,s}$ denote the identity relations.
Set

$$Y_1 := \bigcup_j \left(\bigcup_{\substack{\tau:\{1,2\}\to\{1,2\} \\ \tau \ injective}} R_{\tau(1),j} * R_{\tau(2),j} \right)$$

and

$$Y_2 := \bigcup_s \left(\bigcup_{\substack{\sigma:\{1,2,3\}\to\{1,2,3\} \\ \sigma \ injective}} \widetilde{R}_{\sigma(1),s} * \widetilde{R}_{\sigma(2),s} * \widetilde{R}_{\sigma(3),s} \right)$$

Using these constructions, we want to prove that if we consider $Y := Y_1 \cup Y_2$ then there exists a proper surjective morphism $\Phi : Y \to X \times X$.
Recalling that every algebraic relation which we are considering is a proper algebraic relation, we have that every product of these relations is a proper algebraic relation with proper morphisms into X. Hence there are two proper morphisms from Y to X, $q : Y \to X$ and $u : Y \to X$, which make a commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[ddl, "q"'] \arrow[ddr, "u"] & \\
 & X \times X \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\
 & \{*\} &
\end{tikzcd}$$

$$f \circ q = f \circ u$$

where $\{*\} = Spec(\mathbb{C})$, and π, π' are the projection morphisms of the fibred product onto its factors.
By the universal property of the fibred product, there exists a unique morphism $\Phi : Y \to X \times X$ such that $q = \pi \circ \Phi$ and $u = \pi' \circ \Phi$:

$$\begin{tikzcd}
 & Y \arrow[ddl, "q"'] \arrow[d, "\Phi"] \arrow[ddr, "u"] & \\
 & X \times X \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
X \arrow[dr] & & X \arrow[dl] \\
 & \{*\} &
\end{tikzcd}$$

Therefore we constructed the morphism Φ from Y to $X \times X$, and now we want to prove that Φ is surjective and proper.
To show that Φ is proper, it is enough to prove that π is separated.
In fact, since q is proper, if π is separated, then from [Har77, Corollary II.4.8] it follows that Φ is proper.
By [Har77, Corollary II.4.6], π is separated if $f \circ \pi : X \times X \to \{*\}$ is separated. Note that, since $X \times X$ is a separated scheme over $\mathbb{C}$, the diagonal morphism $\Delta : X \times X \to (X \times X) \times (X \times X)$ is a closed immersion, and hence, by definition, $f \circ \pi : X \times X \to \{*\}$ is separated.
To prove that Φ is surjective, first of all we claim that Φ must be dominant. We suppose by contradiction that Φ is not dominant, i.e. $\overline{(Im\Phi)} \subsetneq X \times X$. Then for two general points $x, x' \in X$, there doesn't exist a reducible cycle in $\mathcal{V}$ passing through x, x' whose irreducible rational components are parametrized by fellow families in $\mathcal{G}$. But it is impossible, and so Φ is dominant.
We know that the image of a proper scheme is proper (see [Hart77, Exercise II.4.4]), and thus, as Y is the union of proper schemes, we have that the image of Φ is a closed subset of $X \times X$. But this closed subset must be dense, hence it is $X \times X$ and so Φ is surjective.
In particular, this implies there exists a morphism from a product relation $\overline{R} \subset Y$ into $X \times X$ which is surjective.

Suppose that this product relation $\overline{R}$ belongs to Y_1. Then there is a pair, denoted by $(W^{\mathbf{i}}, C^{\mathbf{i}})$, of fellow families in $\mathcal{G}_1$ such that at least one of these morphisms

$$\Phi_1 : R_{1,\mathbf{i}} * R_{2,\mathbf{i}} \to X \times X,\ \Phi_2 : R_{2,\mathbf{i}} * R_{1,\mathbf{i}} \to X \times X$$

is surjective. Actually, by construction, both Φ_1 and Φ_2 are surjective.
For (x, x') to be in the image of Φ_1 (respectively Φ_2) means that there is a cycle $l \cup \gamma$ with l and γ parametrized by $\mathcal{W}^{\mathbf{i}}$ and $\mathcal{C}^{\mathbf{i}}$ such that $x \in l$ and $x' \in \gamma$ (respectively $x' \in l$ and $x \in \gamma$). So, by the surjectivety of Φ_1 and Φ_2 we have that, for every $x \in X$

$$X = \mathrm{Locus}(W^{\mathbf{i}}, \mathcal{C}^{\mathbf{i}})_x = \mathrm{Locus}(\mathcal{C}^{\mathbf{i}}, W^{\mathbf{i}})_x.$$

If C is not locally unsplit, then through a general point $x \in X$ there is a reducible cycle with numerical class $[C]$, consisting of two lines. But, since W is unsplit and covering, there is a line parametrized by W which passes through x. Therefore, since x is general, we have that X is covered by triplets of lines and so to conclude we can apply Proposition (8.7).
If else C is locally unsplit, then $X = \mathrm{Locus}(C^{\mathbf{i}}, W^{\mathbf{i}})_x$ for a general point $x \in X$; by Lemma (6.2) this implies that $N_1(X) = \langle [W^{\mathbf{i}}], [C^{\mathbf{i}}] \rangle$, and $\rho_X = 2$.
If $\overline{R}$ belongs to Y_2, then there exists a triplet of families (W_s^1, W_s^2, W_s^3) in $\mathcal{G}_2$ such that $\overline{R}$ is a product relation which is obtained from the algebraic relations associated to W_s^1, W_s^2 and W_s^3. In particular, since this morphism is surjective, for a general point $x \in X$ there is a reducible cycle in $\mathcal{V}$ whose three irreducible components are parametrized by W_s^1, W_s^2 and W_s^3; hence from Proposition (8.7) it follows that either $\rho_X = 2$ or $\rho_X = 3$ and families W_s^1, W_s^2 and W_s^3 satisfy the statement of the theorem.

Case 2. Suppose that there is not a reducible cycle in $\mathcal{V}$ passing through two general points $\mathbf{x}, \mathbf{x}' \in \mathbf{X}$.

This implies that

$$-K_X \cdot V = n + 1.$$

In fact, we assume by contradiction that $-K_X \cdot V \geq n + 2$. Let $f : \mathbb{P}^1 \to X$ be a general curve in V. Since X is smooth and $-K_X \cdot V \geq n + 2$, we have that

$$\begin{aligned} \dim_{[f]} Hom(\mathbb{P}^1, X; f_{\{0,\infty\}}) &\geq -K_X \cdot f_*\mathbb{P}^1 - n \\ &= -K_X \cdot V - n \\ &\geq 2 \end{aligned}$$

From Lemma (3.2), it follows that there exists a reducible cycle in $\mathcal{V}$ passing through $f(0)$ and $f(\infty)$.
Hence, as V is a dominating family, two general points $x, x' \in X$ may be joined by a reducible cycle in $\mathcal{V}$, but this is a contradiction.

We can assume that, through a general point of X there is not a reducible cycle consisting of three lines such that its numerical class is $[V]$, since otherwise to conclude we can apply Proposition (8.7).
Consider the set $\mathcal{B}' = \{(W^i, C^i)\}$ of pairs of fellow families (W^i, C^i) with respect to $\mathcal{V}$ such that $L \cdot W^i = 1$, $L \cdot C^i = 2$ and through a general point of X there is a reducible cycle $\ell \cup \gamma$, with ℓ and γ parametrized respectively by W^i and C^i.
Let $\mathcal{B} = \{(W^i, C^i)\}_{i=1,\dots,k}$ be a maximal set of pairs as above such that the families $V, W^1, \dots, W^k$ are numerically independent (or equivalently $V, C^1, \dots, C^k$ are numerically independent).
As $X = \mathrm{Locus}(\mathcal{V})_x$ for a general point $x \in X$, by Lemma (6.2)

$$N_1(X) = \langle [V], [W^1], [C^1], \dots, [W^k], [C^k] \rangle = \langle [V], [W^1], \dots, [W^k] \rangle,$$

hence the Picard number of X is $k+1$.
If (W^1, C^1) is the only pair of families which belongs to $\mathcal{B}$, then we have that $\rho_X = 2$. Hence we can assume $k \geq 2$, and to prove the statement it is enough to show that $k = 2$, i.e. $\rho_X = 3$, and there exists three families of rational curves W, W', W'' with $[V] = [W] + [W'] + [W'']$ such that W is covering, W' is horizontal and dominating with respect to the rc$(\mathcal{W})$-fibration and W'' is horizontal and dominating with respect to the rc$(\mathcal{W}, \mathcal{W}')$-fibration.
First of all, we collect some properties of these pairs in $\mathcal{B}$ in the following lemmas; we will used these results throughout the proof.

Lemma 8.8. *For every i W^i or C^i is dominating.*

Proof. Suppose that W^i is not covering. Since for a general point $x \in X$ there exists a reducible cycle $\ell \cup \gamma$, with ℓ and γ parametrized respectively by W^i and C^i, we have

$$X = \mathrm{Locus}(W^i) \cup \overline{\mathrm{Locus}(C^i)}.$$

As $\mathrm{Locus}(W^i) \subsetneq X$, we get $X = \overline{\mathrm{Locus}(C^i)}$, i.e. C^i is dominating.
Analogously, if C^i is not dominating, W^i is covering. □

Lemma 8.9. *If $(W^i, C^i) \in \mathcal{B}$ and W^i is covering then C^i is dominating and locally unsplit, but not quasi unsplit.*

Proof. If C^i is quasi unsplit, then we can consider the $rc(\mathcal{W}^i, \mathcal{C}^i)$-fibration. This map contracts curves parametrized by V, hence it is the constant map. This implies that $\rho_X = 2$, a contradiction. Therefore we can assume that C^i is not quasi unsplit.
Let $x \in X$ be general and consider $\mathrm{Locus}(W^i)_x$. Notice that by our assumption $\mathrm{Locus}(W^i)_x \cap \mathrm{Locus}(C^i) \neq \emptyset$.
If there is a point $y \in \mathrm{Locus}(W^i)_x \cap \mathrm{Locus}(C^i)$ such that C^i_y is not proper, then through x there exists a reducible cycle with numerical class $[C^i]$, consisting of

two lines. Clearly, as $y \in \text{Locus}(W^i)_x$, there is a line parametrized by W^i that passes through y. But it is a contradiction because we are assuming that through a general point of X there is not a reducible cycle consisting of three lines such that its numerical class is $[V]$.
Hence we can suppose that, for every $y \in \text{Locus}(W^i)_x \cap \text{Locus}(C^i)$, C^i_y is proper. As W^i is covering, this implies that C^i is locally unsplit.
By Lemma (6.2) we have that $N_1(\text{Locus}(W^i, C^i)_x, X) = \langle [W^i], [C^i] \rangle$ and, recalling the proof of Lemma (4.25), we get that

$$\dim \text{Locus}(W^i, C^i)_x \geq n - 1.$$

Moreover, C^i is not an unsplit family, but, since for every $y \in \text{Locus}(W^i)_x \cap \text{Locus}(C^i)$ C^i_y is proper, we can apply Lemma (6.6) and we have that $[C^i]$ is extremal in $NE(\text{Locus}(W^i, C^i)_x, X)$.
If $X = \text{Locus}(W^i, C^i)_x$, then $\rho_X = 2$, a contradiction; therefore an irreducible component of $\text{Locus}(W^i, C^i)_x$ is a divisor, that we will call D^i_x.
Since W^i is covering we know that $D^i_x \cdot W^i \geq 0$; if the intersection number is positive then we have that $X = \text{Locus}(W^i)_{D^i_x}$ and so from Lemma (6.6) it follows that $\rho_X = 2$, a contradiction.
Hence $D^i_x \cdot W^i = 0$; then every curve of W^i which meets D^i_x is contained in it, and in particular this implies that $x \in D^i_x$.
This has two important consequences: the first one is that $D^i_x \cdot V > 0$; in fact being general, x can be joined to another general point $x' \notin D^i_x$ by a curve parametrized by V. The second one is that, since $x \in D^i_x \subset \text{Locus}(C^i)$ and x is general, then C^i is a dominating family. $\square$

Remark 8.10. From Lemma (8.8) and from Lemma (8.9) it follows that if C^i is locally unsplit then C^i is dominating.
In fact if we suppose by contradiction that C^i is not dominating then by Lemma (8.8) W^i is covering. But this implies that C^i is locally unsplit and dominating, a contradiction.

Lemma 8.11. *Suppose that $k \geq 2$. If $(W^i, C^i) \in \mathcal{B}$ and C^i is not locally unsplit then W^i is not covering and C^i is dominating. Moreover, $\rho_X = 3$ and there exist three families of lines W, W', W'' with $[V] = [W] + [W'] + [W'']$ such that W is covering, W' is horizontal and dominating with respect to the rc$(\mathcal{W})$-fibration and W'' is horizontal and dominating with respect to the rc$(\mathcal{W}, \mathcal{W}')$-fibration.*

Proof. By Lemma (8.11) the family W^i is not covering, and so from Lemma (8.8) it follows that C^i is dominating.
As C^i is not locally unsplit, for a general point $x \in X$ there is a connected 1-cycle $\ell' \cup \ell''$ parametrized by the Chow family $\mathcal{C}^i$ associated to C^i.

This implies that there is a covering family T' of lines which is a family of deformation of a irreducible component of a reducible cycle in $\mathcal{C}^i$ such that for a general point $x \in X$ there is a reducible cycle $\ell' \cup \ell''$, with ℓ' and ℓ'' parametrized respectively by T' and by the fellow family of T' with respect to $\mathcal{C}^i$.
Denote by T'' the fellow family of T'. In particular $[T'] + [T''] = [C^i]$.
Notice that the family T'' is horizontal and dominating with respect to the $rc(\mathcal{T}')$-fibration, and so we can consider the $rc(\mathcal{T}', \mathcal{T}'')$-fibration $\pi : X \dashrightarrow Z$ that contracts curves parametrized by C^i.
If $\dim Z = 0$ then $\rho_X = 2$ by Corollary (6.5), a contradiction. Hence $\dim Z > 0$. We claim that W^i is horizontal and dominating with respect to π.
Curves parametrized by W^i are not contracted by π since otherwise also curves parametrized by V would be contracted, and Z should be a point.
Therefore, since we are assuming that through a general point of X there is a reducible cycle $\gamma \cup l$, with γ and l parametrized respectively by C^i and W^i, a general fiber of π meets a line l and does not contain it, and so W^i is horizontal and dominating with respect to π.
We can consider the $rc(\mathcal{T}', \mathcal{T}'', \mathcal{W}^i)$-fibration π'; π' contracts curves parametrized by V because $[T'] + [T''] + [W^i] = [V]$, and thus π is the constant map.
Then we have that $\rho_X \leq 3$ and if $\rho_X = 3$ there are three families of rational curves T', T'', W^i such that T' is covering, T'' is horizontal and dominating with respect to the rc$(\mathcal{T}')$-fibration and W^i is horizontal and dominating with respect to the rc$(\mathcal{T}', \mathcal{T}'')$-fibration, and $[T'] + [T''] + [W^i] = [V]$. $\square$

Now, we divide our proof into two cases:

(2.1) At least one of families $W^1, \dots, W^k$ is covering.

(2.2) For every $i = 1, \dots, k$ W^i is not covering.

Case 2.1 At least one of families $W^1, \dots, W^k$ is covering.

Without loss of generality we can assume that W^1 is covering. By Lemma (8.9) C^1 is a dominating family of conics which is locally unsplit and not quasi unsplit. In particular, as already shown in the proof of Lemma (8.9), for a general point $x \in X$ we can consider the divisor D^1_x (D^1_x is an irreducible component of $\mathrm{Locus}(W^1, C^1)_x$).
Moreover, we can assume that $n > 2$. In fact if $n = 2$ then for a general point $x \in X$ we can consider $\mathrm{Locus}(C^1, W^1)_x$; by Lemma (4.25) we get that

$$\begin{aligned} \dim \mathrm{Locus}(C^1, W^1)_x &\geq \dim \mathrm{Locus}(C^1)_x - K_X \cdot W^1 - 1 \\ &\geq 1 + 2 - 1 = 2. \end{aligned}$$

This implies that $\rho_X = 2$, a contradiction.

Therefore $n \geq 3$ and we consider the pair (W^2, C^2) in $\mathcal{B}$.

If C^2 is not locally unsplit, then to conclude we can apply Lemma (8.11).

Hence from now on we assume that C^2 is locally unsplit. From Remark (8.10) it follows that C^2 is dominating.
Since x is general and C^2 is dominating, we have that D^1_x meets a general curve of C^2. The intersection number $D^1_x \cdot C^2$ is nonnegative because C^2 is dominating and it cannot be zero since otherwise D^1_x would contain a general curve of C^2, but $[C^2] \notin N_1(D^1_x, X) = \langle [W^1], [C^1] \rangle$. Thus $D^1_x \cdot C^2 > 0$ and, by the same reason, $\dim \mathrm{Locus}(C^2)_x = 1$. Hence $-K_X \cdot C^2 = 2$.
Recalling that $-K_X \cdot (C^2 + W^2) = -K_X \cdot V = n+1$, we have $-K_X \cdot W^2 = n-1$.
If W^2 is not covering then by Lemma (4.25)

$$\dim \mathrm{Locus}(W^2, W^1)_x = n$$

and by Lemma (6.6) $\rho_X = 2$, a contradiction.
If else W^2 is covering, then we can consider $\mathrm{Locus}(W^2)_x \cap D^1_x$ that is not empty; as $-K_X \cdot W^2 = n-1$, we have that

$$\begin{aligned} \dim(\mathrm{Locus}(W^2)_x \cap D^1_x) &\geq (-K_X \cdot W^2 - 1) + (n-1) - n \\ &= (n-2) + (n-1) - n \\ &= n-3. \end{aligned}$$

But, $\dim(\mathrm{Locus}(W^2)_x \cap D^1_x) = 0$ because $[W^2] \notin N_1(D^1_x, X) = \langle [W^1], [C^1] \rangle$. Hence if $n > 3$ we obtain a contradiction.
So we have just to study $n = 3$. First of all we observe that $X = \mathrm{Locus}(W^2)_{D^1_x}$; in fact by Lemma (4.25)

$$\begin{aligned} \dim \mathrm{Locus}(W^2)_{D^1_x} &\geq \dim D^1_x - K_X \cdot W^2 - 1 \\ &= 2 + 2 - 1 \\ &= 3. \end{aligned}$$

Therefore by Lemma (6.6) we have that $\rho_X = 3$.
We recall that $N_1(D^1_x, X) = \langle [W^1], [C^1] \rangle$ and that the numerical class $[C^1]$ is extremal in $NE(D^1_x, X)$. Therefore by Lemma (8.5) $[W^2]$ and $[C^1]$ lie in a two-dimensional extremal face σ of $NE(X)$ and $[W^2]$ spans a negative extremal ray of $NE(X)$.
By Lemma (8.9) the family C^1 is not quasi unsplit, and so there is a connected cycle $l' \cup l''$ that belongs to the Chow family $\mathcal{C}^1$ associated to C^1 such that $[l'] \notin \langle [C^1] \rangle$ and $[l''] \notin \langle [C^1] \rangle$. Let T_1' be a family of deformations of l' and let T_1'' be a family of deformations of l''. Clearly $[T_1'] + [T_1''] = [C^1]$. Since $[C^1]$ belongs to σ, we have that $[T_1'] \in \sigma$ and $[T_1''] \in \sigma$.

Notice that either T_1' or T_1'' is such that every curve parametrized by this family is numerically equivalent to a linear combination with rational coefficients

$$a\Gamma_{W^2} + b\Gamma_{C^1}$$

where Γ_{W^2} belongs to W^2, Γ_{C^1} is a curve parametrized by C^1 and $a, b \geq 0$.
Without loss of generality assume that curves parametrized by T' have this property.
Since $-K_X \cdot W^2 = -K_X \cdot C^1 = 2$, we have that $-K_X \cdot T_1' > 0$.
First of all assume that $b \neq 0$.
Suppose that $-K_X \cdot T_1' = 2$. If T_1' is not covering then $\dim \operatorname{Locus}(T_1', W^2)_x = 3$ and so $\rho_X = 2$, a contradiction; if else T_1' is covering then

$$X = \operatorname{Locus}(T_1', W^1, W^2)_x = \operatorname{Locus}(W^2, T_1', W^1)_x = \operatorname{Locus}(W^1, W^2, T_1')_x;$$

hence from Lemma (8.5) it follows that the Kleiman-Mori cone has three negative extremal rays which are spanned by $[W^1]$, $[W^2]$ and $[T_1']$.
This implies that $[C^1] \in \langle [T_1'] \rangle$, a contradiction.
Assume that $-K_X \cdot T_1' = 1$. Recalling that $L \cdot T_1' = 1$, we have that

$$\begin{aligned} -K_X \cdot T_1' = & \quad 1 \quad = 2a + 2b \\ L \cdot T_1' = & \quad 1 \quad = a + 2b. \end{aligned}$$

Therefore $a = 0$, i.e. curves parametrized by T_1' is numerically proportional to curves parametrized by C^1, a contradiction.
If $b = 0$ then, since (W^1, C^1) and (W^2, C^2) have the same properties, we can consider families T_2', T_2'' that are families of deformations of irreducible components of a reducible cycle in $\mathcal{C}^2$. Arguing as above, we have that $[W^1]$ and $[C^2]$ lie in a two-dimensional extremal face of $NE(X)$ and $[W^1]$ spans a negative extremal ray.
As before, if $[T_2'] \notin \langle [W^1] \rangle$, we obtain a contradiction.
If else $[T_2'] = [W^1]$ then $[T_1''] = [T_2'']$ spans a extremal ray of $NE(X)$. Moreover we can consider the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration $\pi : X \dashrightarrow Z$; we observe that $\dim Z = 1$ and curves parametrized by T_1'' (or by T_2'') are not contracted by π. Moreover

$$[W^2] + [T_1''] + [W^1] = [C^1] + [W^1] = [V].$$

Then T_1'' is horizontal and dominating with respect to the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration. Therefore we have that the $rc(\mathcal{W}^1, \mathcal{W}^2, \mathcal{T}_1'')$-fibration is the constant map.

Case 2.2 For every $i = 1, \ldots, k$ W^i is not covering.

Since W^i is not covering, C^i is a dominating family of conics. Moreover, we recall that by our assumption there is a covering family $\widetilde{W}$ of lines.
We can suppose that for every i C^i is locally unsplit, since otherwise we can apply Lemma (8.11).

For every $i = 1, \ldots, k$ denote by E^i the set $\mathrm{Locus}(C^i, W^i)_x$; by Lemma (6.6) it has dimension $\dim E_i \geq n-1$; equality holds, since $E_i \subseteq \mathrm{Locus}(W^i)$, so the inclusion is an equality and E^i is irreducible.
Denote by $\Pi_i \subset N_1(X)$ the two-dimensional plane spanned by the numerical classes of W^i and C^i.
If $[\widetilde{W}] \in \Pi_i$ for every i, then $[\widetilde{W}] \in \langle [V] \rangle$, and therefore the $rc(\widetilde{\mathcal{W}})$-fibration is the constant map and $\rho_X = 1$, a contradiction.
Hence there exists a plane Π_i such that $[\widetilde{W}] \notin \Pi_i$; without loss of generality we can assume that $\Pi_i = \Pi_1$.
Recalling that by Lemma (6.6) $N_1(E_1, X) \subseteq \langle [W^1], [C^1] \rangle$, we have that $E_1 \cdot \widetilde{W} > 0$, and so W^1 is horizontal and dominating with respect to the $rc(\widetilde{\mathcal{W}})$-fibration. Consider the $rc(\widetilde{\mathcal{W}}, \mathcal{W}^1)$-fibration, whose general fiber has dimension

$$\begin{aligned} \dim F \geq \dim \mathrm{Locus}(W^1, \widetilde{W})_x &\geq \dim \mathrm{Locus}(W^1)_x - K_X \cdot \widetilde{W} - 1 \\ &\geq -K_X \cdot W^1 - K_X \cdot \widetilde{W} - 1 \\ &\geq -K_X \cdot W^1 + 1. \end{aligned}$$

Curves parametrized by C^1 are not contracted by the $rc(\widetilde{\mathcal{W}}, \mathcal{W}^1)$-fibration, since otherwise the fibration goes to a point and $\rho_X = 2$.
But $\dim \mathrm{Locus}(C^1)_x \geq -K_X \cdot C^1 - 1$ and then

$$\begin{aligned} \dim F + \dim \mathrm{Locus}(C^1)_x &\geq (-K_X \cdot W^1 + 1) + (-K_X \cdot C^1 - 1) \\ &\geq n+1 \end{aligned}$$

and this is a contradiction. □

8.1.1 Extremality of covering families of lines

This section will be devoted to the proof of the following:

Theorem 8.12. *Assume that (X, L) is RCC by a family V. Suppose that $\rho_X = 3$ and X is covered by lines. Then there is a covering family of lines whose numerical class belongs to a negative extremal ray of $NE(X)$.*

Proof. Recalling the proof of Theorem (8.4), since $\rho_X = 3$ there exist three families of lines W^1, W^2, W^3 with $[V] = [W^1] + [W^2] + [W^3]$ such that W^1 is covering, W^2 is horizontal and dominating with respect to the $\mathrm{rc}(\mathcal{W}^1)$-fibration and W^3 is horizontal and dominating with respect to the $\mathrm{rc}(\mathcal{W}^1, \mathcal{W}^2)$-fibration.

Case 1. Suppose that W^1, W^2, W^3 are covering.

We want to prove that at least one of families W^1, W^2, W^3 is such that its numerical class spans an extremal ray.

Assume that $[W^1]$ doesn't generate a negative extremal ray. We claim that $[W^2]$ and $[W^3]$ belong to an extremal face of $NE(X)$.
If $[W^1]$ is not extremal, then by Proposition (6.18) there is an equivalence class with respect to the $rc(\mathcal{W}^1)$-relation of dimension greater that the general one. Since the general equivalence class has dimension greater than or equal to $-K_X \cdot W^1 - 1$ because it contains $\mathrm{Locus}(W^1)_x$, there exists an irreducible component Z of this special class of dimension

$$\dim Z \geq -K_X \cdot W^1.$$

Consider $\mathrm{Locus}(W^2, W^3)_Z$; by Lemma (4.25)

$$\begin{aligned} \dim \mathrm{Locus}(W^2, W^3)_Z &\geq -K_X \cdot (W^1 + W^2 + W^3) - 2 \\ &= n - 1. \end{aligned}$$

Moreover we observe that by Lemma (6.6) every curve in $\mathrm{Locus}(W^2, W^3)_Z$ is numerically equivalent to a linear combination with rational coefficients

$$\alpha\Gamma_{W^1} + \beta\Gamma_{W^2} + \delta\Gamma_{W^3}$$

where Γ_{W^1}, Γ_{W^2} are curves parametrized by W^1 and by W^2, Γ_{W^3} belongs to W^3 and $\alpha \geq 0$.
If $X = \mathrm{Locus}(W^2, W^3)_Z$ then, as already observed in Lemma (8.6), $[W^2]$ and $[W^3]$ belong to an extremal face and $[W^3]$ is extremal.
Therefore we can suppose that an irreducible component of $\mathrm{Locus}(W^2, W^3)_Z$ is a divisor D. Notice that $D \cdot W^i \geq 0$ for every i because every family W^i is covering. If D is positive either on W^2 or on W^3 we have $X = \mathrm{ChLocus}_{m_1}(\mathcal{W}^2, \mathcal{W}^3)_Z$; from Lemma (8.6) it follows that $[W^2], [W^3]$ belong to an extremal face of $NE(X)$.
Hence $D \cdot W^2 = D \cdot W^3 = 0$. This implies that $D_{|D}$ is nef, and hence D is nef and is a supporting divisor of a face which contains $[W^2]$ and $[W^3]$.
We can repeat the same argument starting from another family, say W^2; therefore we prove that, if neither $[W^1]$ nor $[W^2]$ span an extremal ray, then $[W^3]$ belongs to two different extremal faces of $NE(X)$, hence it spans an extremal ray.

Case 2. Two families among W^1, W^2, W^3 are covering.

If W^3 is a covering family, then it is horizontal and dominating with respect to the $\mathrm{rc}(\mathcal{W}^1)$-fibration; moreover, since X is $\mathrm{rc}(\mathcal{W}^1, \mathcal{W}^2, \mathcal{W}^3)$-connected, W^2 will be horizontal and dominating with respect to the $\mathrm{rc}(\mathcal{W}^1, \mathcal{W}^3)$-fibration, so, without loss of generality we can assume that W^2 is covering and W^3 is not.
First of all we observe that

$$\dim \mathrm{Locus}(W^3) = \begin{cases} n-1 \\ n-2 \end{cases}$$

In fact, if we suppose that $\dim \mathrm{Locus}(W^3) \le n-3$, then for a general point $x \in \mathrm{Locus}(W^3)$, we have that

$$\dim \mathrm{Locus}(W^3)_x \ge -K_X \cdot W^3 + 2$$

and therefore

$$\dim \mathrm{Locus}(W^3, W^1, W^2)_x \ge -K_X \cdot (W^1 + W^2 + W^3) \ge n+1.$$

Case 2.1 Suppose that $\dim \mathbf{Locus}(\mathbf{W^3}) = \mathbf{n-2}$**.**

This implies that $\dim \mathrm{Locus}(W^3)_x \ge -K_X \cdot W^3 + 1$ and so

$$X = \mathrm{Locus}(W^1)_{\mathrm{Locus}(W^3,W^2)_x}.$$

We want to prove that $[W^1]$ and $[W^2]$ span two negative extremal rays that are contained in a (two-dimensional) extremal face σ of $NE(X)$.
Set $Z := \mathrm{Locus}(W^3, W^2)_x$. Z is a closed subset of X.
Moreover $N_1(Z, X) = \langle [W^3], [W^2] \rangle$ and $[W^2]$ is extremal in $NE(Z, X)$.
Then by Lemma (8.5) $[W^1]$ and $[W^2]$ lie in an extremal face σ of $NE(X)$ and $[W^1]$ generates a negative extremal ray.
Now we observe that $X = \mathrm{Locus}(W^2)_{\mathrm{Locus}(W^3,W^1)_x}$ and so from Lemma (8.5) it follows that the numerical class of W^2 spans a negative extremal ray of σ.

Case 2.2 Suppose that $\dim \mathbf{Locus}(\mathbf{W^3}) = \mathbf{n-1}$**.**

This implies that $\dim \mathrm{Locus}(W^3)_x \ge -K_X \cdot W^3$ for a general $x \in \mathrm{Locus}(W^3)$. Moreover, we know that $-K_X \cdot W^3 \ge 0$ since otherwise $-K_X \cdot (W^1 + W^2) > n+1$ and so $\dim \mathrm{Locus}(W^3, W^1, W^2)_x > n$.
Set $H := \mathrm{Locus}(W^3)$. Since W^1, W^2 are covering, $H \cdot W^1 \ge 0$ and $H \cdot W^2 \ge 0$.
Recalling that W^3 is horizontal and dominating with respect to the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration, we have that $H \cdot W^1 > 0$ or $H \cdot W^2 > 0$.
Assume without loss of generality that $H \cdot W^2 > 0$.
We claim that the numerical classes of W^1 and W^2 belong to an extremal face σ.
If this is not the case, then by Lemma (6.20) there is a divisor D such that $D \cdot W^1 = D \cdot W^2 = 0$ and $D \cdot W^3 > 0$.
Moreover, since $[W^1], [W^2], [W^3] \in \overline{NE}(X)_{K_X \le 0}$, we have that there is a negative extremal ray R such that $D \cdot R < 0$ and it is small.
Denote by F a fiber of the elementary contraction associated to R, which, by Proposition (4.12) has dimension $\dim F \ge 2$. By Lemma (4.25) we have

$$\dim \mathrm{Locus}(W^1, W^2)_F \ge -K_X \cdot (W^1 + W^2).$$

Since $H \cdot W^2 > 0$ for some x the intersection $\mathrm{Locus}(W^3)_x \cap \mathrm{Locus}(W^1, W^2)_F$ is not empty and moreover

$$\begin{aligned}\dim(\mathrm{Locus}(W^3)_x \cap \mathrm{Locus}(W^1, W^2)_F) &\ge -K_X \cdot (W^1 + W^2 + W^3) - n \\ &= 1.\end{aligned}$$

Hence there is an irreducible curve γ in X such that it is numerically proportional to W^3, i.e. $[\gamma] = \alpha[W^3]$ with $\alpha > 0$, and it is numerically proportional to a linear combination with rational coefficients

$$\mu\Gamma_R + \delta\Gamma_{W^1} + \beta\Gamma_{W^2}$$

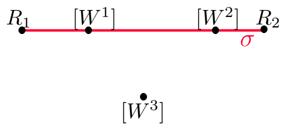
where Γ_{W^1} and Γ_{W^2} are parametrized by W^1 and W^2, $[\Gamma_R] \in R$ and $\mu \geq 0$. But it is impossible because

$$D \cdot \gamma = \alpha D \cdot W^3 > 0$$

and

$$D \cdot \gamma = \mu D \cdot \Gamma_R + \delta D \cdot \Gamma_{W^1} + \beta D \cdot \Gamma_{W^2} \leq 0.$$

Therefore $[W^1]$ and $[W^2]$ belong to an extremal face σ of $NE(X)$.
Denote by R_1 and by R_2 two extremal rays that belong to σ; at least one of these two extremal rays belongs to $NE(X)_{K_X<0}$ since $-K_X \cdot W^1 \geq 2$ and $-K_X \cdot W^2 \geq 2$.



Assume that R_1 is a negative extremal ray; suppose by contradiction that $[W^1] \notin R_1$. Then by Lemma (6.20) there is a divisor $\widetilde{D}$ such that $\widetilde{D} \cdot W^1 = 0$, $\widetilde{D} \cdot W^2 > 0$, $\widetilde{D} \cdot W^3 > 0$ and $\widetilde{D} \cdot R_1 < 0$.
Therefore the exceptional locus of R_1 is contained in the indeterminacy locus of the $rc(\mathcal{W}^1)$-fibration and so it is small. Denote by F' a fiber of the elementary contraction associated to R_1, which, by Proposition (4.12) has dimension $\dim F' \geq 2$. By Lemma (4.25) we have

$$\dim \operatorname{Locus}(W^1, W^2)_{F'} \geq -K_X \cdot (W^1 + W^2).$$

Since $H \cdot W^2 > 0$ for some x the intersection $\operatorname{Locus}(W^3)_x \cap \operatorname{Locus}(W^1, W^2)_{F'}$ is not empty, and moreover

$$\begin{aligned} \dim(\operatorname{Locus}(W^3)_x \cap \operatorname{Locus}(W^1, W^2)_{F'}) &\geq -K_X \cdot (W^1 + W^2 + W^3) - n \\ &= 1. \end{aligned}$$

But it is impossible because $NE(\operatorname{Locus}(W^1, W^2)_{F'}, X) \subset \sigma$ and $[W^3] \notin \sigma$.
If $R_1 \not\subseteq \overline{NE}(X)_{K_X<0}$ then R_2 is a negative extremal ray. Assume by contradiction that $[W^2] \notin R_2$. As above R_2 is small because its exceptional locus is contained in the indeterminacy locus of the $rc(\mathcal{W}^2)$-fibration. Denote by F'' a fiber of the contraction associated to R_2. Then

$$\dim \operatorname{Locus}(W^2, W^1)_{F''} \geq -K_X \cdot (W^1 + W^2).$$

As W^1 is covering, we have that $\mathrm{Locus}(W^2)_{F''} \subseteq \mathrm{Locus}(W^2, W^1)_{F''}$, and so, recalling that $H \cdot W^2 > 0$, for some x $\mathrm{Locus}(W^3)_x \cap \mathrm{Locus}(W^2, W^1)_{F''} \neq \emptyset$, getting a contradiction as before. Thus $R_2 = \langle [W^2] \rangle$.
Therefore we proved that either the numerical class of W^1 or the numerical class of W^2 spans a negative extremal ray of $NE(X)$.

Case 3. Only W^1 is covering.

Let $F_{1,2}$ be the general fiber of the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration $\varphi : X \dashrightarrow Z$ and let x be a general point of $F_{1,2}$.
Clearly, $\mathrm{Locus}(W^2, W^1)_x \subseteq F_{1,2}$. Hence, as W^2 is not covering

$$\dim F_{1,2} \geq -K_X \cdot (W^1 + W^2) - 1.$$

Since W^3 is a horizontal family with respect to φ, we consider $\mathrm{Locus}(W^3)_x$ that is non empty. We know that $\dim \mathrm{Locus}(W^3)_x \geq -K_X \cdot W^3$ because W^3 is not covering, and so

$$\begin{aligned} \dim \mathrm{Locus}(W^3)_x + \dim F_{1,2} &\geq -K_X \cdot (W^1 + W^2 + W^3) - 1 \\ &\geq -K_X \cdot V - 1 \\ &\geq n. \end{aligned}$$

Notice that

$$\dim(\mathrm{Locus}(W^3)_x \cap F_{1,2}) = 0$$

because $NE(\mathrm{Locus}(W^3)_x, X) = \langle [W^3] \rangle$ and φ doesn't contract curves parametrized by W^3. Therefore

$$\dim \mathrm{Locus}(W^3)_x + \dim F_{1,2} = -K_X \cdot (W^1 + W^2 + W^3) - 1 = n \tag{8.1}$$

This implies that $\mathrm{Locus}(W^3)_x$ is a closed subset of X which dominates Z via φ. Then by Lemma (8.6), $[W_1], [W^2]$ belong to an extremal face σ of $NE(X)$. Denote by R_1 and by R_2 two extremal rays that belong to σ.
By (8.1) we have that

$$\dim F_{1,2} = \dim \mathrm{Locus}(W^2, W^1)_x = -K_X \cdot (W^1 + W^2) - 1 \tag{8.2}$$

and

$$\begin{cases} \dim \mathrm{Locus}(W^2)_x = -K_X \cdot W^2 \\ \dim \mathrm{Locus}(W^3)_x = -K_X \cdot W^3 \end{cases}$$

In particular, from Proposition (4.10) it follows that

$$\dim \mathrm{Locus}(W^2) = \dim \mathrm{Locus}(W^3) = n - 1.$$

Set $D_3 = \text{Locus}(W^3)$ and consider $\text{Locus}(W^3)_{F_{1,2}} \subseteq \text{Locus}(W^3)$. From Lemma (4.25) and (8.1) it follows that

$$\begin{aligned} \dim \text{Locus}(W^3)_{F_{1,2}} &\geq -K_X \cdot (W^1 + W^2 + W^3) - 2 \\ &\geq n - 1. \end{aligned}$$

$$\Rightarrow D_3 = \text{Locus}(W^3) = \text{Locus}(W^3)_{F_{1,2}}.$$

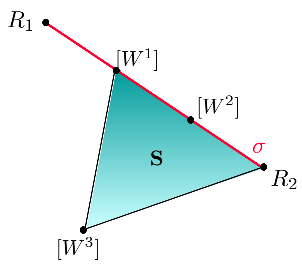
By Lemma (6.6) every curve in D_3 is numerically equivalent to a linear combination with rational coefficients

$$\alpha \Gamma_{F_{1,2}} + \beta \Gamma_{W^3}$$

where $\Gamma_{F_{1,2}}$ is a curve contained in $F_{1,2}$, Γ_{W^3} is parametrized by W^3 and $\alpha \geq 0$.
Since $F_{1,2} = \text{Locus}(W^2, W^1)_x$, by Lemma (6.6) $N_1(F_{1,2}, X) = \langle [W^1], [W^2] \rangle$ and moreover $[W^1]$ is extremal in $NE(F_{1,2}, X)$.
Then every curve in D_3 is numerically equivalent to a linear combination with rational coefficients

$$\mu \Gamma_{W^2} + \delta \Gamma_{W^1} + \beta \Gamma_{W^3}$$

where Γ_{W^1} and Γ_{W^2} are parametrized by W^1 and W^2, Γ_{W^2} belongs to W^3 and $\mu \geq 0$.



Therefore we have that $NE(D_3, X) \subseteq S$. Since W^1 is covering, $D_3 \cdot W^1 \geq 0$.
If $D_3 \cdot W^1 > 0$, then W^3 is a horizontal dominating family with respect to the $rc(\mathcal{W}^1)$-fibration. Thus we can consider $\text{Locus}(W^2)_x$ and the general fiber $F_{1,3}$ of the $rc(\mathcal{W}^1, \mathcal{W}^3)$-fibration $\varphi' : X \dashrightarrow Z'$; we can prove that $\text{Locus}(W^2)_x$ dominates Z' via φ'. Then from Lemma (8.6) it follows that $[W^1], [W^3]$ belong to an extremal face, and so $[W^1]$ belongs to two different extremal faces, hence it spans a negative extremal ray.
If else $D_3 \cdot W^1 = 0$, then $D_3 \cdot W^2 > 0$ since W^3 is horizontal and dominating with respect to the $rc(\mathcal{W}^1, \mathcal{W}^2)$-fibration.
This implies that every curve γ in D_3 whose numerical class belongs to the extremal face σ is such that $D_3 \cdot \gamma \geq 0$.
Suppose by contradiction that $[W^1] \notin R_1$. Then there exists an irreducible curve Γ in X such that $[\Gamma] \in \sigma$ and $D_3 \cdot \Gamma < 0$. Hence $\Gamma \subset D_3$, a contradiction.
Thus the numerical class of W^1 generates the negative extremal ray R_1. $\square$

8.2 Fano RCC-manifolds not covered by lines

Now we will study RCC-manifolds that are not covered by lines. As already observed in the introduction, it is impossible to find an upper bound on their Picard number. However assuming that they are Fano, we will prove that up to a few exceptions in dimension 2 the Picard number is equal to or less than 3.

First of all we consider Del Pezzo surfaces and we will show that, choosing a suitable polarization, every Del Pezzo surface is rationally cubic connected.

Clearly if X is a Del Pezzo surface then $\rho_X \leq 9$.

If $X \simeq \mathbb{P}^2$ or $X \simeq Bl_{P_1,...,P_k}(\mathbb{P}^2)$, where $Bl_{P_1,...,P_k}(\mathbb{P}^2)$ is the blow up of $\mathbb{P}^2$ at k general points $P_1, ..., P_k$ $(1 \leq k \leq 8)$, then X is rationally cubic connected with respect to the family V of deformations of the strict transform of a general line in $\mathbb{P}^2$ and $L = -K_X$.

If $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ then X is rationally connected with respect to the family V of deformations of a smooth curve on X of bidegree $(1,1)$ and L has type $(1,2)$.

Therefore we have that

Proposition 8.13. $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, $(Bl_{P_1,...,P_k}(\mathbb{P}^2), -K_{Bl_{P_1,...,P_k}(\mathbb{P}^2)})$ *(with* $1 \leq k \leq 8$*) and* $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2))$ *are rationally cubic connected manifolds.*

From now on we will assume that $n > 2$ and we will prove the following theorem:

Theorem 8.14. *Let* (X, L) *be a polarized manifold. Suppose that* X *is RCC by a family* V *which doesn't admit a covering family of lines. Assume that* X *is a Fano manifold and has dimension* $n > 2$.
Then either $\rho_X \leq 2$ *or we have the following list of possibilities*

(1) $(X, L) \simeq (Bl_{\Lambda_1,\Lambda_2}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$, *where* $Bl_{\Lambda_1,\Lambda_2}(\mathbb{P}^n)$ *is the blow up of* $\mathbb{P}^n$ *along two linear subspaces* Λ_1, Λ_2 *such that*

$$\Lambda_1 \cap \Lambda_2 = \emptyset, \quad \dim \Lambda_1 + \dim \Lambda_2 = n - 2$$

and E_1, E_2 *are the exceptional divisors of the blow up* π, $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$*;*

(2) $(X, L) \simeq (Bl_{\Lambda_1,Z_1}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$, *where* $Bl_{\Lambda_1,Z_1}(\mathbb{P}^n)$ *is the blow up of* $\mathbb{P}^n$ *along a linear subspaces* Λ_1 *and along a quadric* $Z_1 \subset \Lambda_2 \simeq \mathbb{P}^{\dim Z_1 + 1}$ *such that*

$$\Lambda_1 \cap \Lambda_2 = \emptyset, \quad \dim Z_1 \geq \tfrac{n}{2} - 1, \quad \dim \Lambda_1 + \dim Z_1 = n - 2$$

and E_1, E_2 *are the exceptional divisors of the blow up* π, $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$*;*

(3) $(X, L) \simeq (Bl_{Z_1,Z_2}(\mathbb{P}^n), 3\mathcal{H} - E_1 - E_2)$, *where* $Bl_{Z_1,Z_2}(\mathbb{P}^n)$ *is the blow up of* $\mathbb{P}^n$ *along two quadrics* $Z_1 \subset \Lambda_1 \simeq \mathbb{P}^{\frac{n}{2}}$ *and* $Z_2 \subset \Lambda_2 \simeq \mathbb{P}^{\frac{n}{2}}$ *such that*

$$\dim \Lambda_1 \cap \Lambda_2 = 0, \quad \dim Z_1 = \dim Z_2 = \tfrac{n}{2} - 1$$

and E_1, E_2 are the exceptional divisors of the blow up π, $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ (clearly n is even).

Proof. Recalling the proof of case 1. of Theorem (8.4), since we are assuming that X is not covered by lines, we can suppose that through two general points $x, x' \in X$, there is not a reducible cycle in $\mathcal{V}$, and so $-K_X \cdot V = n+1$.
We can assume that V is not locally unsplit and not quasi unsplit since otherwise we have that $\rho_X = 1$.
Consider the set $\mathcal{B}' = \{(W^i, C^i)\}$ of pairs of fellow families (W^i, C^i) with respect to $\mathcal{V}$ such that $L \cdot W^i = 1$, $L \cdot C^i = 2$ and through a general point of X there is a reducible cycle $\ell \cup \gamma$, with ℓ and γ parametrized respectively by W^i and C^i.
Let $\mathcal{B} = \{(W^i, C^i)\}_{i=1,\dots,k}$ be a maximal set of pairs as above such that the families $V, W^1, \dots, W^k$ are numerically independent (or equivalently $V, C^1, \dots, C^k$ are numerically independent).
Denote by $\Pi_i \subset N_1(X)$ the two-dimensional plane spanned by the numerical classes of W^i and C^i.
Notice that for every i W^i is not covering and C^i is dominating and locally unsplit, since otherwise there is a covering family of lines.
This implies that

$$-K_X \cdot C^i \geq 2 \ \text{ and } \ -K_X \cdot W^i \leq (n-1).$$

But $-K_X \cdot V = n+1$ and X is Fano, and so

$$2 \leq -K_X \cdot C^i \leq n \quad \text{and} \quad 1 \leq -K_X \cdot W^i \leq (n-1)$$

Moreover, for every $i = 1, \dots, k$ and for a general point $x \in X$ we can consider $\mathrm{Locus}(C^i, W^i)_x$ that is contained in $\mathrm{Locus}(W^i)$; set $E_i = \mathrm{Locus}(C^i, W^i)_x$.
By Lemma (6.6) it has dimension $\dim E_i \geq n-1$. Hence $E_i = \mathrm{Locus}(W^i)$ and E_i is irreducible.
Let us divide the divisors E_i in the following way:

- if $-K_X \cdot W^i = n-1$ we will call E_i a divisor of the first kind;
- if $-K_X \cdot W^i = 1$ we will call E_i a divisor of the second kind;
- if $2 \leq -K_X \cdot W^i \leq (n-2)$ we will call E_i a divisor of the third kind.

Notice that if E_i is of the first kind, then $E_i = \mathrm{Locus}(W^i)_x$ for any $x \in \mathrm{Locus}(W^i)$ and $N_1(E_i, X) = \langle [W^i] \rangle$; if else E_i is either of the second or of the third kind then $N_1(E_i, X) = \langle [C^i], [W^i] \rangle$ and moreover $[W^i]$ is extremal in $NE(E_i, X)$ by Lemma (6.6).

As $\mathrm{Locus}(\mathcal{V})_x = X$, by Lemma (6.2) we have that

$$N_1(X) = \langle [V], [W^1], [C^1], \dots, [W^k], [C^k] \rangle = \langle [V], [W^1], \dots, [W^k],$$

hence the Picard number of X is $k+1$.
Clearly if there exists only one pair of fellow families in $\mathcal{B}$, then $\rho_X = 2$; hence we can assume $k \geq 2$, and to prove the statement it is enough to show that $k = 2$ and X is the blow up of $\mathbb{P}^n$ along two disjoint subvarieties of degree 1 or 2.

First of all we observe that as $k > 1$ and the families of conics are dominating, for a general point $x \in X$ there are two rational curves γ_i and γ_j which pass through x and are parametrized respectively by C^i and by C^j. But for every $i \neq j$

$$\dim(\mathrm{Locus}(C^i)_x \cap \mathrm{Locus}(C^j)_x) = 0$$

since C^i and C^j are numerically independent, and therefore

$$n \geq \dim \mathrm{Locus}(C^i)_x + \dim(\mathrm{Locus}(C^j)_x \geq -K_X \cdot (C^i + C^j) - 2$$

$$\Rightarrow -K_X \cdot (C^i + C^j) \leq n+2 \quad \forall i \neq j. \tag{8.3}$$

In particular this implies that if there is a divisor of the second kind then all the other divisors are of the first kind.

Case 1. There exists a divisor E_i of the first kind.

Consider the line bundle $K_X + (n-1)L$; if it is not nef, then it is not nef on an extremal ray R which has length greater than or equal to n. By [Wi89, Proposition 2.4] this implies that $\rho_X \leq 2$, and it is a contradiction.
So $K_X+(n-1)L$ is nef, and defines an extremal face σ which contains the numerical class of every W^i such that $-K_X \cdot W^i = n-1$.
Let φ be the extremal contraction associated to σ. We claim that φ is birational. Suppose by contradiction that φ is of fiber type. Then there is a dominating family T of rational curves whose numerical class belongs to σ. We can assume that T is locally unsplit; in fact, if T is not locally unsplit there exists a family T' of deformations of a irreducible component of a reducible cycle in $\mathcal{T}$ which is dominating and $[T'] \in \sigma$.
Set $d := L \cdot T$. Then for a general point $x \in X$ we have that

$$\begin{aligned} \dim \mathrm{Locus}(T)_x &\geq -K_X \cdot T - 1 \\ &= (n-1)L \cdot T - 1 \\ &= d(n-1) - 1 \end{aligned}$$

Notice that $\dim \mathrm{Locus}(T)_x < n$; otherwise $\rho_X = 1$ and it is a contradiction. Therefore $d(n-1) - 1 < n$ and so $d = 1$, namely T is an unsplit covering family of lines such that $-K_X \cdot T = n-1$. But this is a contradiction because we are assuming that X is not covered by lines.
Thus we have proved that φ is birational, and so by [BS95, Theorem 7.3.2] and [AO02, Theorem 1.2] all rays in $\overline{NE}(X)_{K_X+(n-1)L=0}$ are birational and they can be

simultaneously contracted into a smooth variety X', with the morphism $\varphi : X \to X'$ expressing X as blow up of X' at a finite set of points Z.
Since X is a Fano manifold and we are supposing that $\rho_X > 2$, by [BCW01, Théorème 1] we have that X is the blow up of $X' \simeq Bl_Y(\mathbb{P}^n)$ at a point $a \in X'$, where $Bl_Y(\mathbb{P}^n)$ is the blow up of $\mathbb{P}^n$ along a subvariety Y of dimension $n-2$ and of degree $1 \leq d \leq n$ which is contained in an hyperplane H such that $a \notin H$.
Now, we want to prove that the subvariety Y has degree d equal to 1 or to 2.
As X is rationally cubic connected, there must exist a family V of rational curves such that two general points $x, x' \in X$ may be joined by a rational curve parametrized by this family, and an ample line bundle L such that $L \cdot V = 3$.
In particular, we have that the anticanonical degree of V must be equal to $n+1$, and so, the family V is the family of deformations of the strict transform of a general line of $\mathbb{P}^n$.
Let $\pi : X \to \mathbb{P}^n$ be the blow up. Denote by E_a and E_Y the exceptional divisors and set $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$.
Since $\rho_X = 3$, the ample divisor L is numerically equivalent to a linear combination

$$L \equiv \alpha\pi^*\mathcal{O}_{\mathbb{P}^n}(1) - \beta E_a - \mu E_Y$$

In particular, $\alpha = 3$ because $L \cdot V = 3$, and from the ampleness of L it follows that $\beta = 1$ and $\mu = 1$, and $d = 2$. In fact if we denote by

- Γ_a a minimal curve which is contracted by the blow down at a
- Γ_Y a minimal curve which is contracted by the blow down along Y
- l a curve which is the strict transform of a line of $\mathbb{P}^n$ which meets Y in a point and passes through a

then

$$\begin{cases} L \cdot \Gamma_a > 0 \\ L \cdot \Gamma_Y > 0 \\ L \cdot l > 0 \end{cases}$$

and therefore

$$\begin{cases} \beta \in \mathbb{N} \\ \mu \in \mathbb{N} \\ 3 - \beta - \mu > 0 \end{cases} \Rightarrow \begin{cases} \beta = 1 \\ \mu = 1 \end{cases} \Rightarrow L \equiv 3\pi^*\mathcal{O}_{\mathbb{P}^n}(1) - E_a - E_Y.$$

Moreover, if $d > 1$, we can consider a curve γ which is the strict transform of a line of $\mathbb{P}^n$ that is contained in the hyperplane H. Then

$$\begin{aligned} L \cdot \gamma &= (3\pi^*\mathcal{O}_{\mathbb{P}^n}(1) - E_a - E_Y) \cdot \gamma \\ &= 3 - d > 0 \end{aligned}$$

$$\Rightarrow d = 2.$$

Therefore, if $d = 1$ we are in one of the cases described in (1), and, if $d = 2$ we are in one of the cases described in (2) of theorem.

Case 2. Every divisor E_i is either of the second kind or of the third kind.

As we are assuming that $\rho_X > 2$, this implies that all the divisors are of the third kind; in fact, as already observed before, if there is a divisor of the second kind then all the other divisor are of the first kind, and this is impossible.
Hence every divisor is of the third kind and so for every $i = 1, \dots, k$ we have that

$$\begin{cases} 3 \leq & -K_X \cdot C^i & \leq (n-1) \\ 2 \leq & -K_X \cdot W^i & \leq (n-2) \end{cases}$$

Step 1. The divisors E_i are pairwise disjoint.

Suppose by contradiction that $E_i \cap E_j \neq \emptyset$ for some i, j. Then we can consider $\mathrm{Locus}(W^j)_x$ for $x \in E_i \cap E_j$ and it is such that

$$\dim \mathrm{Locus}(W^j)_x \geq -K_X \cdot W^j \geq 2.$$

This implies that $\dim(E_i \cap \mathrm{Locus}(W^j)_x) \geq 1$ and so there is an irreducible curve which is numerically proportional to W^j and whose numerical class belongs to $N_1(E_i, X) = \langle [C^j], [W^j] \rangle$. But it is impossible because $[W^j] \notin \Pi_i = \langle [C^i], [W^i] \rangle$.

Step 2. The numerical class of every family W^i spans an extremal ray of $NE(X)$.

First of all we show that from our assumptions it follows that for every $i \neq j$

$$\begin{cases} E_i \cdot C^j = 0 \\ E_i \cdot W^j = 0 \end{cases}$$

(A) Let us start assuming by contradiction that there exist two pairs of fellow families (W^i, C^i) and (W^j, C^j) in $\mathcal{B}$ such that $E_i \cdot C^j > 0$. Then

$$\begin{aligned} \dim E_i + \dim \mathrm{Locus}(C^j)_x &\geq (n-1) - K_X \cdot C^j - 1 \\ &\geq (n-1) + 2 = n+1 \end{aligned}$$

$$\Rightarrow \dim(E_i \cap \mathrm{Locus}(C^j)_x) \geq 1$$

But this is impossible because $[C^j] \notin \Pi_i$.

(B) Now we assume that there exist two pairs of fellow families (W^i, C^i) and (W^j, C^j) such that $E_i \cdot W^j > 0$. Then

$$\begin{aligned} \dim E_i + \dim \mathrm{Locus}(W^j)_x &\geq (n-1) - K_X \cdot W^j \\ &\geq (n-1) + 2 = n+1 \end{aligned}$$

$$\Rightarrow \dim(E_i \cap \mathrm{Locus}(W^j)_x) \geq 1$$

But this is impossible because $[W^j] \notin \Pi_i$.

Therefore we proved the statement, and so, from now on we have that for every $i \neq j$

$$\begin{cases} E_i \cdot C^j = 0 \\ E_i \cdot W^j = 0 \end{cases}$$

In particular, this implies that $E_i \cdot V = 0$ for every i.
We have thus proved that E_i is trivial outside of the plane Π_i and in the plane Π_i has intersection number zero with V. Being effective, E_i cannot be trivial also on Π_i, and, recalling that $E_i \cdot C^i \geq 0$ since C^i is a dominating family, we deduce $E \cdot W^i < 0$.
Since $E_i \cdot W^i < 0$ and X is Fano, there is an extremal ray R_i of $NE(X)$ such that $E_i \cdot R_i < 0$. In particular, $\mathrm{Locus}(R_i) \subseteq E_i$ and so $R_i \subset NE(E_i, X)$. Recalling that $[W^i]$ is extremal in $NE(E_i, X)$ we have that $R_i = \langle [W^i] \rangle$ and $\mathrm{Locus}(R_i) = E_i$.

Step 3. For every i the elementary contraction associated to R_i is the blow down of a smooth divisor to a smooth subvariety.

Let $\varphi_i : X \to Z_i$ be the elementary contractions associated to R_i. Denote by F_i the general fiber of φ_i and let x be a general point of F_i; set $c_i = -K_X \cdot C^i$ and $w_i = -K_X \cdot W^i$.
As $\mathrm{Locus}(W^i)_x \subseteq F_i$ and $\dim(F_i \cap \mathrm{Locus}(C^i)_x) = 0$, we get

$$\begin{aligned} n &\geq \dim F_i + \dim \mathrm{Locus}(C^i)_x \\ &\geq -K_X \cdot W^i - K_X \cdot C^i - 1 \\ &= n \end{aligned}$$

$$\Rightarrow \dim F_i = \dim \mathrm{Locus}(W^i)_x = -K_X \cdot W^i = w_i$$

From [AW93, Theorem 4.1 (iii)] it follows that φ is a blow down of a smooth divisor $E_i \subset X$ to a smooth subvariety of dimension $(n - w_i - 1)$ of Z_i, with $1 \leq (n - w_i - 1) \leq n - 3$.
Moreover, we claim that $E_i \cdot W^i = -1$.
Since for a general point $x \in \mathrm{Locus}(W^i)$ $\dim \mathrm{Locus}(W^i)_x = w_i$, we know that

$$\dim W^i = n + w_i - 3.$$

Let $f : \mathbb{P}^1 \to \Gamma \subset E_i$ be a curve of the family W^i which intersects the smooth locus of E_i; since every element of $Hom_{[f]}(\mathbb{P}^1, E_i)$ is also an element of $Hom_{[f]}(\mathbb{P}^1, X)$ and $\mathrm{Locus}(W^i) = E_i$ we can take an irreducible component of $Hom_{[f]}(\mathbb{P}^1, E_i)$, call it T, which is contained in W^i; this implies that

$$\dim T \leq \dim W^i = n + w_i - 3.$$

Since E_i is a divisor in a smooth variety, E_i is a locally complete intersection and so we can apply [Kol96, Theorem II.1.3] which gives

$$\dim T \geq -K_{E_i} \cdot \Gamma + \dim E_i - 3;$$

combining the two inequalities we get

$$-K_{E_i} \cdot \Gamma \leq w_i + 1.$$

Recalling that $E_i \cdot W^i < 0$, by the adjunction formula $K_{E_i} = (K_X + E_i)_{|E_i}$, we have that $E_i \cdot \Gamma = -1$ and therefore $E_i \cdot W^i = -1$ and $E_i \cdot C^i = 1$.

Step 4. All the extremal rays R_i belong to an extremal face of $NE(X)$ of dimension $(\rho_X - 1)$.
Consider the divisor $L + \sum E_i$. We prove that it is nef and it vanishes only on curves whose numerical class belong to one of the R_i.
First of all notice that the properties are true by construction for the restriction of $L + \sum E_i$ to Π_i for every i.
Let γ be an irreducible curve in X such that $(L + \sum E_i) \cdot \gamma \leq 0$. Then there exists an index j such that $E_j \cdot \gamma < 0$; hence $\gamma \subset E_j$ and $[\gamma] \in \Pi_j$. This implies that $[\gamma] \notin \Pi_i$ for every $i \neq j$ and hence $E_i \cdot \gamma = 0$.
Therefore we have that $(L + \sum E_i) \cdot \gamma = 0$ and $[\gamma] \in \langle [W^j] \rangle$.
Thus $(L + \sum E_i)$ is nef and there is a $(\rho_X - 1)$-dimensional face σ of $NE(X)$ generated by the R_i. Let $\varphi_\sigma : X \to X'$ be the associated contraction; the variety X' is smooth and φ_σ is a blow up along smooth disjoint centers Y_i.

Step 5. X is the blow up of $\mathbb{P}^n$ along two smooth disjoint centers Y_i.
Consider the Fano-Mori contraction $\varphi_\sigma : X \to X'$ associated to the extremal face σ. As already proved before, φ_σ is the blow up of a smooth variety X' along k disjoint centers Y_i. As $\rho_X = k + 1$, we have that $\rho_{X'} = 1$.
We claim that $X' = \mathbb{P}^n$ and to show this statement we prove that there exists a minimal dominating family of rational curves in X' which has anticanonical degree equal to $\dim X' + 1$.
Let V' be a family of deformation of the image of a general curve parametrized by V; clearly V' is a dominating family for X'.
The divisor $L + \sum E_i$ is nef and supports the face contracted by φ_σ, hence there exists an ample divisor L' on X' such that $\varphi_\sigma^* L' = L + \sum E_i$.
Notice that $(L + \sum E_i) \cdot V = 3$, and so from the projection formula it follows that $L' \cdot V' = 3$. Moreover, since $E_i \cdot V = 0$, by the canonical bundle formula we have that

$$\begin{aligned} -K_X \cdot V &= -\varphi_\sigma^* K_{X'} \cdot V - \sum_{i=1}^{k} (codim(Y_i) - 1) E_i \cdot V \\ &= -\varphi_\sigma^* K_{X'} \cdot V. \end{aligned}$$

Hence, again by the projection formula $-K_{X'} \cdot V' = \dim X' + 1 = n + 1$.
Suppose by contradiction that V' is not minimal, i.e. there is a dominating family V'' of rational curves in X' such that $-K_{X'} \cdot V'' < n + 1$.
Since $\rho_{X'} = 1$, curves parametrized by V'' are numerically proportional to curves in V', and so we have two possibilities:

1. $L' \cdot V'' = 1$ and $-K_{X'} \cdot V'' = \frac{1}{3}(n + 1)$;

2. $L' \cdot V'' = 2$ and $-K_{X'} \cdot V'' = \frac{2}{3}(n + 1)$.

Let W be the dominating family of deformations of the strict transform of a general curve in V''; since V'' is dominating, a general curve parametrized by V'' does not meet $\cup Y_i$, hence $E_i \cdot W = 0$ for every i and W is numerically proportional to V. Moreover, $L \cdot W = L' \cdot V''$.
Since we are assuming that X is not covered by lines, we can suppose that $L \cdot W = 2$ and W is locally unsplit.
First of all, we want to prove that there is a family of conics in $\mathcal{B}$ whose anticanonical degree is equal to or greater than $\frac{(n+2)}{2}$.
Since E_i cannot contain curves of C^j for $j \neq i$, but C^j is dominating, it follows that there exists a reducible cycle $l_j + \bar{l}_j$ in $\mathcal{C}_j$ such that $E_i \cdot l_j < 0$.
Notice that $L + E_i$ is nef on Π_i, hence $E_i \cdot l_j = -1$; both L and E_i have the same intersection number, then $[l_j] = [W^i]$.
From this it follows that $-K_X \cdot C^j \geq -K_X \cdot W^i + 1$; hence

$$-K_X \cdot (C^j + C^i) \geq -K_X \cdot (W^j + W^i) + 2.$$

By equation (8.3) we have $-K_X \cdot (W^j + W^i) \leq n$. Recalling that for every i $-K_X \cdot (C^i + W^i) = n + 1$, we get that

- if $k = 2$ then $-K_X \cdot (W^1 + W^2) = n$ and $-K_X \cdot (C^1 + C^2) = n + 2$;

- if $k \geq 3$ then for every i $-K_X \cdot W^i = \frac{n}{2}$ and $-K_X \cdot C^i = \frac{n+2}{2}$.

In particular, there is a family of conics in $\mathcal{B}$ whose anticanonical degree is equal to or greater than $\frac{(n+2)}{2}$, and without loss of generality we can assume that this family is C^1.
As W and C^1 are dominating families of X, for a general point $x \in X$ we can consider $\mathrm{Locus}(C^1)_x$ and $\mathrm{Locus}(W)_x$, and we get

$$\begin{aligned}
\dim(\mathrm{Locus}(C^1)_x \cap \mathrm{Locus}(W)_x) &\geq \dim \mathrm{Locus}(C^1)_x + \dim \mathrm{Locus}(W)_x - n \\
&\geq (-K_X \cdot C^1 - 1) + (-K_X \cdot W - 1) - n \\
&\geq \frac{(n+2)}{2} + \frac{2}{3}(n+1) - 2 - n \\
&= \frac{1}{6}n - \frac{1}{3}
\end{aligned}$$

But $n > 2$ and so $\dim(\mathrm{Locus}(C^1)_x \cap \mathrm{Locus}(W)_x) > 0$, i.e. there is an irreducible curve γ which is numerically proportional to C^1 and to W; but this is impossible because $E_1 \cdot C^1 = 1$ and $E_1 \cdot W = 0$.
Therefore we have proved that V' is a minimal dominating family of X', and so $X' \simeq \mathbb{P}^n$ and $L' \simeq \mathcal{O}_{\mathbb{P}^n}(3)$.

Step 6. There are only two centers Y_1 *and* Y_2 *such that* $\dim Y_1 + \dim Y_2 = n - 2$.
Let Y_1 and Y_2 be two centers of the blow up $\varphi_\sigma : X \to \mathbb{P}^n$. Consider the joint of Y_1 and Y_2 and denote it by $J(Y_1, Y_2)$.
Recalling that $\dim Y_i = n - 1 + K_X \cdot W^i$ and $-K_X \cdot (W^1 + W^2) = n$, we have that

$$\dim Y_1 + \dim Y_2 = n - 2$$

and this implies that $J(Y_1, Y_2)$ has dimension $n - 1$.
Suppose by contradiction that there exists another center Y_3 (or equivalently that $\rho_X > 3$). Then $J(Y_1, Y_2)$ meets Y_3 since $\dim Y_3 \geq 1$, and so there is a line $\ell \subset \mathbb{P}^n$ which meets Y_1, Y_2 and Y_3. We consider its strict transform l'; then

$$L \cdot l' = (\varphi_\sigma^* \mathcal{O}_{\mathbb{P}^n}(3) - \sum E_i) \cdot l' \leq 0,$$

contradicting the ampleness of L.

Step 7. For $i = 1, 2$, Y_i *is a quadric or a linear subspace of* $\mathbb{P}^n$.
Let $\mathcal{S}(Y_1)$ be the secant variety of Y_1. Suppose that $\dim \mathcal{S}(Y_1) \geq \dim Y_1 + 2$. Then

$$\begin{aligned} \dim(\mathcal{S}(Y_1) \cap Y_2) &\geq \dim \mathcal{S}(Y_1) + \dim Y_2 - n \\ &\geq \dim Y_1 + 2 + \dim Y_2 - n \\ &= 0 \end{aligned}$$

i.e. there is a line l in $\mathbb{P}^n$ which meets Y_1 in two points and Y_2 in a point.
Consider the strict transform l' in X of l. Then $L \cdot l' = 0$ contradicting the ampleness of L. Therefore $\dim \mathcal{S}(Y_1) \leq \dim Y_1 + 1$ and analogously $\dim \mathcal{S}(Y_2) \leq \dim Y_2 + 1$.
We recall that a nonsingular variety $Z \subset \mathbb{P}^n$ of dimension k can be isomorphically projected to $\mathbb{P}^{n-1}$ if and only if $\mathcal{S}(Z) \neq \mathbb{P}^n$, where $\mathcal{S}(Z)$ is the secant variety of Z, and, the minimal number m such that Z can be isomorphically projected to $\mathbb{P}^m$ is equal to the dimension of the secant variety of Z.
Moreover, by [Zak93, Corollary II.2.11], if a nondegenerate nonsingular variety $Z \subset \mathbb{P}^n$ of dimension k can be isomorphically projected to $\mathbb{P}^m$ $(m < n)$, then

$$k \leq \frac{2}{3}(m - 1).$$

Assume by contradiction that Y_i is nondegenerate. Y_i can be isomorphically projected to $\mathbb{P}^m$ where

$$m = \dim \mathcal{S}(Y_i) \leq \dim Y_i + 1$$

and hence
$$\dim Y_i \le \frac{2}{3}(m-1) \le \frac{2}{3}\dim Y_i.$$
Clearly it is impossible. Therefore Y_i is a degenerate subvariety of $\mathbb{P}^n$.
In particular, if $\dim \mathcal{S}(Y_i) = \dim Y_i$ then Y_i is a linear subspace of $\mathbb{P}^n$.
Otherwise, if $\dim \mathcal{S}(Y_i) = \dim Y_i + 1$ then Y_i is a hypersurface of $\Lambda_i \simeq \mathbb{P}^{\dim Y_i + 1} \subset \mathbb{P}^n$.
Notice also that from the ampleness of L it follows that there cannot exist trisecant lines of Y_i in $\mathbb{P}^n$, and hence Y_i is a quadric, and we can prove that $\dim Y_i \ge \frac{n}{2} - 1$.
In fact, considering the strict transform l of a secant line of Y_i and recalling that X is Fano, by the canonical bundle formula, we get

$$1 \le -K_X \cdot l = (n+1) - 2w_i$$

$$\Rightarrow w_i = codim Y_i - 1 \le \frac{n}{2}$$

$$\Rightarrow \dim Y_i \ge \frac{n}{2} - 1.$$

Therefore

- if $\dim \mathcal{S}(Y_1) = \dim Y_1$ and $\dim \mathcal{S}(Y_2) = \dim Y_2$, then X is the blow up of $\mathbb{P}^n$ along two disjoint linear subspaces Y_1, Y_2 such that
$$\begin{cases} 1 \le \dim Y_i \le (n-3) \\ \dim Y_1 + \dim Y_2 = n-2 \end{cases}$$
This leads to case (1).

- If $\dim \mathcal{S}(Y_1) = \dim Y_1$ and $\dim \mathcal{S}(Y_2) = \dim Y_2 + 1$, then X is the blow up of $\mathbb{P}^n$ along a linear subspaces Y_1 and along a quadric $Y_2 \subset \Lambda_2 \simeq \mathbb{P}^{\dim Y_2 + 1}$ such that
$$\begin{cases} 1 \le \dim Y_1 \le (\frac{n}{2} - 1) \\ (\frac{n}{2} - 1) \le \dim Y_2 \le (n-3) \\ \dim Y_1 + \dim Y_2 = n-2 \end{cases}$$
Moreover Λ_2 and Y_1 must be disjoint, because there cannot exist lines in $\mathbb{P}^n$ which meet Y_1 in a point and Y_2 in two points. Thus we get case (2) of the theorem.

- If $\dim \mathcal{S}(Y_1) = \dim Y_1 + 1$ and $\dim \mathcal{S}(Y_2) = \dim Y_2 + 1$, then X is the blow up of $\mathbb{P}^n$ along two quadrics Y_1, Y_2 such that $\dim Y_1 = \dim Y_2 = \left(\frac{n}{2} - 1\right)$ (clearly n is even). Notice also that $Y_i \subset \Lambda_i \simeq \mathbb{P}^{\frac{n}{2}}$, and $\dim(\Lambda_1 \cap \Lambda_2) = 0$ because there cannot exist trisecant lines of $Y_1 \cup Y_2$. Then we get case (3) of the theorem.

□

8.2.1 Description of Blow-ups

Now we study Fano RCC-manifolds which are not covered by lines and have Picard number equal to 3; in particular we want to describe the Kleiman-Mori cone and the family V, and to find the fellow families with respect to $\mathcal{V}$ that we used in the proof of Theorem (8.14).

X is the blow up of $\mathbb{P}^n$ along two linear subspaces $\Lambda_1 \simeq \mathbb{P}^r$, $\Lambda_2 \simeq \mathbb{P}^s$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $r + s = n - 2$.

Canonical Bundle

$$K_X = \pi^*\mathcal{O}_{\mathbb{P}^n}(-n-1) + (n-r-1)E_1 + (n-s-1)E_2$$

where $\pi : X \to \mathbb{P}^n$ is the blow up of $\mathbb{P}^n$, E_1 and E_2 are the exceptional divisors.

Description of the Kleiman-Mori cone of X

Denote by

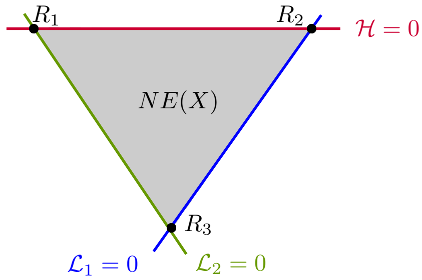
- l_1 a minimal curve which is contracted by the blow down along Λ_1;
- l_2 a minimal curve which is contracted by the blow down along Λ_2;
- l a curve which is the strict transform of a line of $\mathbb{P}^n$ that meets Λ_i in a point for $i = 1, 2$;
- $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$;
- $\mathcal{L}_1 = \varphi_1^*\mathcal{O}_{\mathbb{P}^{n-r-1}}(1)$, where $\varphi_1 : X \to \mathbb{P}^{n-r-1}$;
- $\mathcal{L}_2 = \varphi_2^*\mathcal{O}_{\mathbb{P}^{n-s-1}}(1)$, where $\varphi_2 : X \to \mathbb{P}^{n-s-1}$;

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] \arrow[dd, "\pi"] & \\
Bl_{\Lambda_1}(\mathbb{P}^n) \arrow[d] \arrow[dr] & & Bl_{\Lambda_2}(\mathbb{P}^n) \arrow[d] \arrow[dl] \\
\mathbb{P}^{n-r-1} & \mathbb{P}^n & \mathbb{P}^{n-s-1}
\end{tikzcd}$$

Note that $\mathcal{H}, \mathcal{L}_1$ and $\mathcal{L}_2$ are nef divisors on X, and we have that:

$$\begin{cases} \mathcal{H} \cdot l_1 = 0 \\ \mathcal{H} \cdot l_2 = 0 \end{cases} \quad \begin{cases} \mathcal{L}_1 \cdot l_2 = 0 \\ \mathcal{L}_1 \cdot l = 0 \end{cases} \quad \begin{cases} \mathcal{L}_2 \cdot l_1 = 0 \\ \mathcal{L}_2 \cdot l = 0 \end{cases}$$

and hence $\mathcal{H}, \mathcal{L}_1$ and $\mathcal{L}_2$ are the supporting divisors of the extremal faces of $NE(X)$:



$[l_1] \in R_1$ $\quad$ $[l_2] \in R_2$ $\quad$ $[l] \in R_3$

Clearly the elementary contraction ψ_1 associated to the extremal ray R_1 is the blow down of E_1, and the elementary contraction ψ_2 associated to the extremal ray R_2 is the blow down of E_2. Moreover, the elementary contraction ψ_3 associated to R_3 is divisorial, and it is the blow up of $\mathbb{P}^{n-r-1} \times \mathbb{P}^{n-s-1}$ along a smooth subvariety $Y \simeq \mathbb{P}^{n-r-2} \times \mathbb{P}^{n-s-2}$:

$$
\begin{tikzcd}
 & & \mathbb{P}^{n-r-1} & & \\
Bl_{\Lambda_1}(\mathbb{P}^n) \arrow[urr] \arrow[ddd] & & & & \mathbb{P}^{n-r-1} \times \mathbb{P}^{n-s-1} \arrow[ull] \arrow[ddd] \\
 & & X = Bl_{\Lambda_1,\Lambda_2}(\mathbb{P}^n) \arrow[uu] \arrow[ull, "\psi_2"] \arrow[urr, "\psi_3"'] \arrow[d, "\psi_1"] \arrow[ddll, "\pi"'] \arrow[ddrr] & & \\
 & & Bl_{\Lambda_2}(\mathbb{P}^n) \arrow[dll] \arrow[drr] & & \\
\mathbb{P}^n & & & & \mathbb{P}^{n-s-1}
\end{tikzcd}
$$

Description of fellow families with respect to $\mathcal{V}$

As already observed in the proof of Theorem (8.14), the family V is the family of deformations of the strict transform of a general line of $\mathbb{P}^n$ which has anticanonical degree equal to $n+1$, and $L = 3\mathcal{H} - E_1 - E_2$.
By Kleiman's criterion L is ample and such that $L \cdot V = 3$.

Now we study how cycles in $\mathcal{V}$ can split. There are three possibilities:

- a cycle in $\mathcal{V}$ splits into two irreducible components, Γ_1 and γ_1, where Γ_1 is parametrized by a family C^1 of deformations of the strict transform of a line of $\mathbb{P}^n$ which meets Λ_1 in a point, and γ_1 is parametrized by a family W^1 of deformations of a minimal curve l_1;

- a cycle in $\mathcal{V}$ splits into two irreducible components, Γ_2 and γ_2, where Γ_2 is parametrized by a family C^2 of deformations of the strict transform of a line of $\mathbb{P}^n$ which meets Λ_2 in a point, and γ_2 is parametrized by a family W^2 of deformations of a minimal curve l_2;

- a reducible cycle in $\mathcal{V}$ can have three irreducible components $\gamma_1, \gamma_2, \gamma$, where γ_1 is parametrized by a family W^1 of deformations of a minimal curve l_1, γ_2 is parametrized by a family W^2 of deformations of a minimal curve l_2 and γ belongs to a family W of deformations of the strict transform of a line of $\mathbb{P}^n$ that meets Λ_i in a point for $i = 1, 2$.

Therefore

- W^1, C^1 are fellow families with respect to $\mathcal{V}$; in particular C^1 is dominating and locally unsplit, and W^1 is unsplit but not covering. Moreover, they are such that

$$\begin{cases} -K_X \cdot C^1 = r+2 \\ L \cdot C^1 = 2 \end{cases} \qquad \begin{cases} -K_X \cdot W^1 = n-r-1 \\ L \cdot W^1 = 1 \end{cases}$$

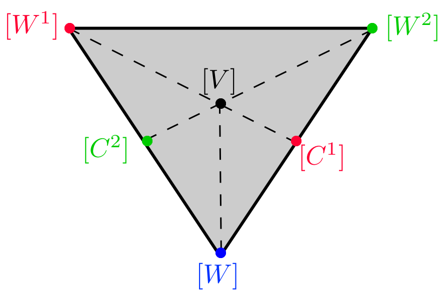
- W^2, C^2 are fellow families with respect to $\mathcal{V}$; in particular C^2 is dominating and locally unsplit, and W^2 is unsplit but not covering. Moreover, they are such that

$$\begin{cases} -K_X \cdot C^2 = s+2 \\ L \cdot C^2 = 2 \end{cases} \qquad \begin{cases} -K_X \cdot W^2 = n-s-1 \\ L \cdot W^2 = 1 \end{cases}$$

- W^1, W^2, W are fellow families with respect to $\mathcal{V}$; in particular W is unsplit but not covering and

$$\begin{cases} -K_X \cdot W = 1 \\ L \cdot W = 1 \end{cases}$$

Notice also that W, W^2 are fellow families with respect to the Chow family associated to C^1, and W, W^1 are fellow families with respect to the Chow family associated to C^2.



X is the blow up of $\mathbb{P}^n$ along a linear subspaces $\Lambda_1 \simeq \mathbb{P}^r$ and a quadric $Z_1 \simeq \mathbb{Q}^s$ such that $Z_1 \subset \Lambda_2 \simeq \mathbb{P}^{s+1}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, $s \geq \frac{n}{2} - 1$ and $r + s = n - 2$.

Canonical Bundle

$$K_X = \pi^*\mathcal{O}_{\mathbb{P}^n}(-n-1) + (n-r-1)E_1 + (n-s-1)E_2$$

where $\pi : X \to \mathbb{P}^n$ is the blow up of $\mathbb{P}^n$, E_1 and E_2 are the exceptional divisors.

Description of the Kleiman-Mori cone of X

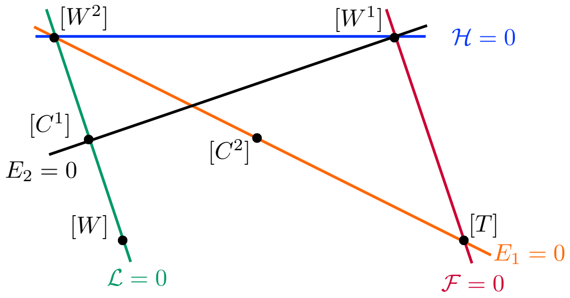
Denote by

- l_1 a minimal curve which is contracted by the blow down along Λ_1 and by W^1 a family of deformations of l_1;
- l_2 a minimal curve which is contracted by the blow down along Z_1 and by W^2 a family of deformations of l_2;
- l a curve which is the strict transform of a line of $\mathbb{P}^n$ that meets Λ_1 in a point and meets Z_1 in another point, and by W a family of deformations of l;
- γ a curve which is the strict transform to a line of $\mathbb{P}^n$ that is contained in Λ_2 and by T a family of deformations of γ;
- Γ_1 a curve that is the strict transform of a line of $\mathbb{P}^n$ which meets Λ_1 in a point, and by C^1 a family of deformations of Γ_1;
- Γ_2 a curve that is the strict transform of a line of $\mathbb{P}^n$ which meets Z_1 in a point, and by C^2 a family of deformations of Γ_2;
- $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$;
- $\mathcal{L} = \varphi_1^*\mathcal{O}_{\mathbb{P}^{n-r-1}}(1)$, where $\varphi_1 : X \to \mathbb{P}^{n-r-1}$;
- $\mathcal{F} = 2\mathcal{H} - E_2$.

Note that $\mathcal{H}, \mathcal{L}$ and $\mathcal{F}$ are nef divisors on X, and we have the following intersection numbers:

	W^1	W^2	W	T	C^1	C^2
E_1	-1	0	1	0	1	0
E_2	0	-1	1	2	0	1
$\mathcal{H}$	0	0	1	1	1	1
$\mathcal{L}$	1	0	0	1	0	1
$\mathcal{F}$	0	1	1	0	2	1

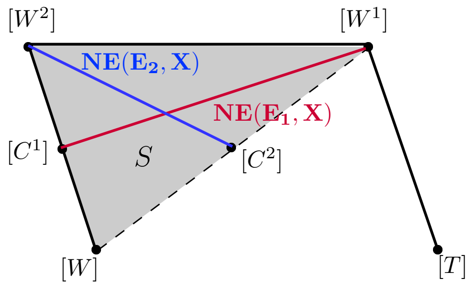
and therefore $\mathcal{H}, \mathcal{L}$ and $\mathcal{F}$ are the supporting divisors of three extremal faces of $NE(X)$.



We want to prove that there is an extremal face of $NE(X)$ which contains $[W]$ and $[T]$.
Consider the joint of Λ_1 and Z_1 and denote it by $J(\Lambda_1, Z_1)$; $J(\Lambda_1, Z_1)$ is the union of the lines of $\mathbb{P}^n$ jointing Λ_1 to Z_1 and it is a projective variety of dimension $n-1$ and of degree 2.
Let E be the strict transform of $J(\Lambda_1, Z_1)$ under π; E is an effective divisor in X. We observe that

$$E = \mathrm{Locus}(W)_{E_1} = \mathrm{Locus}(W)_{E_2}.$$

Since $E_1 = \mathrm{Locus}(W^1)_{\mathrm{Locus}(C^1)_x}$ and $E_2 = \mathrm{Locus}(W^2)_{\mathrm{Locus}(C^2)_x}$, from Lemma (6.6) it follows that $N_1(E_i, X) = \langle [W^i], [C^i] \rangle$ and $[W^i]$ is extremal in $NE(E_i, X)$ for $i = 1, 2$. Then $N_1(E, X) = \langle [W], [W^1], [W^2] \rangle$



and by Lemma (8.6) $NE(E, X) \subseteq S$.
Now we want to prove that $E = 2\mathcal{H} - 2E_1 - E_2$. Consider the following maps:

$$\begin{tikzcd}
X = Bl_{\Lambda_1, Z}(\mathbb{P}^n) \arrow[d, "\varepsilon"] & \\
Bl_{\Lambda_1}(\mathbb{P}^n) \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\
\mathbb{P}^n & \mathbb{P}^{n-r-1}
\end{tikzcd}$$

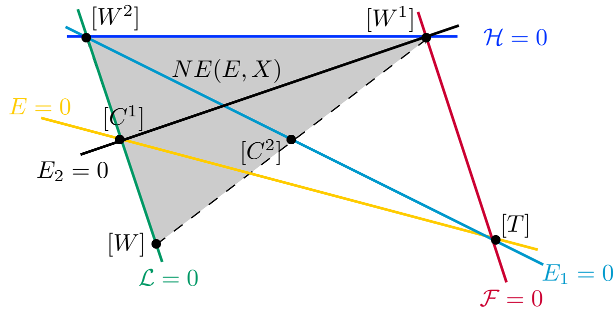
and denote by E' the strict transform of $J(\Lambda_1, Z)$ under φ.
Fix a point $x \in Z_1$. Then the union of the lines of $\mathbb{P}^n$ which pass through x and meet Λ_1 in a point is a linear subspace of $\mathbb{P}^n$ of dimension $r+1$ that contains Λ_1. Recalling that fibers of ψ are the strict transforms of the linear subspaces of $\mathbb{P}^n$ of dimension $r+1$ which contain Λ_1, we have that

$$E' = \psi^* \mathcal{O}_{\mathbb{P}^{n-r-1}}(2)$$

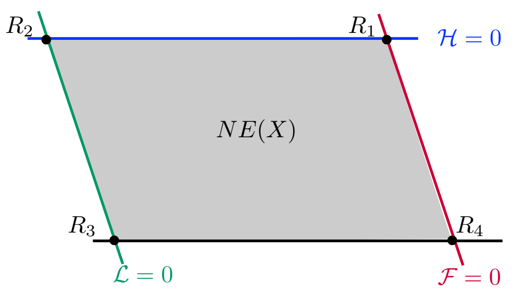
and therefore

$$\begin{aligned} E &= \varepsilon^* E' - E_2 \\ &= \varepsilon^*(\psi^* \mathcal{O}_{\mathbb{P}^{n-r-1}}(2)) - E_2 \\ &= 2(\mathcal{H} - E_1) - E_2 \\ &= 2\mathcal{H} - 2E_1 - E_2. \end{aligned}$$

From this it follows that $E \cdot T = E \cdot C^1 = 0$, and hence



Let R be an extremal ray of $NE(X)$ such that $E \cdot R < 0$. Then there is an irreducible curve $\tilde{l}$ whose numerical class belongs to R and which is contained in E. But this implies that $[\tilde{l}] \in NE(E, X)$, and so $R = \langle [W] \rangle$. Hence we can conclude that $R = \langle [W] \rangle$ is the only extremal ray which is contained in $N_1(X)_{E<0}$ and that there is an extremal face of $NE(X)$ which contains $[W]$ and $[T]$.



Clearly the elementary contraction ψ_1 associated to the extremal ray R_1 is the blow down of E_1, and the elementary contraction ψ_2 associated to the extremal ray R_2 is the blow down of E_2. Moreover, the elementary contraction ψ_3 associated to R_3 is divisorial because its exceptional locus is E.
Instead, the exceptional locus of the elementary contraction ψ_4 associated to R_4 has dimension equal to $s+1$; hence if $\dim \Lambda_1 = 0$ ψ_4 is divisorial, otherwise it is small.

Description of fellow families with respect to $\mathcal{V}$

As already observed in the proof of Theorem (8.14), the family V is the family of deformations of the strict transform of a general line of $\mathbb{P}^n$ which has anticanonical degree equal to $n+1$, and $L = 3\mathcal{H} - E_1 - E_2$.
By Kleiman's criterion L is ample and such that $L \cdot V = 3$.

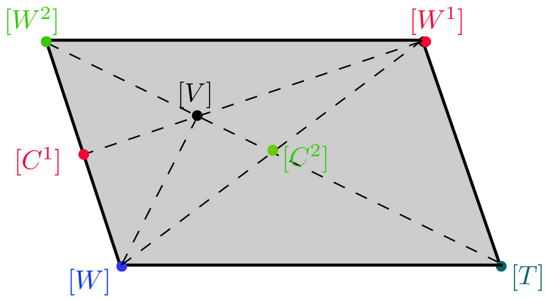
Now we study how cycles in $\mathcal{V}$ can split. There are three cases:

- a cycle in $\mathcal{V}$ splits into two irreducible components, Γ_1 and γ_1, where Γ_1 is parametrized by C^1 and γ_1 is parametrized by W^1;
- a cycle in $\mathcal{V}$ splits into two irreducible components, Γ_2 and γ_2, where Γ_2 is parametrized by C^2 and γ_2 is parametrized by W^1;
- a reducible cycle in $\mathcal{V}$ can have three irreducible components $\gamma_1, \gamma_2, \gamma$, where γ_1 is parametrized by W^1, γ_2 is parametrized by W^2 and γ belongs to W.

Therefore

- W^1, C^1 are fellow families with respect to $\mathcal{V}$; in particular C^1 is dominating locally unsplit and W^1 is unsplit but not covering. Moreover, they are such that
$$\begin{cases} -K_X \cdot C^1 = r+2 \\ L \cdot C^1 = 2 \end{cases} \qquad \begin{cases} -K_X \cdot W^1 = n-r-1 \\ L \cdot W^1 = 1 \end{cases}$$
- W^2, C^2 are fellow families with respect to $\mathcal{V}$; in particular C^2 is dominating locally unsplit and W^2 is unsplit but not covering. Moreover, they are such that
$$\begin{cases} -K_X \cdot C^2 = s+2 \\ L \cdot C^2 = 2 \end{cases} \qquad \begin{cases} -K_X \cdot W^2 = n-s-1 \\ L \cdot W^2 = 1 \end{cases}$$
- W^1, W^2, W are fellow families with respect to $\mathcal{V}$; in particular W is unsplit but not covering and
$$\begin{cases} -K_X \cdot W = 1 \\ L \cdot W = 1 \end{cases}$$

Notice also that W, W^2 are fellow families with respect to the Chow family associated to C^1. Moreover, (W, W^1) and (T, W^2) are two pairs of fellow families with respect to the Chow family associated to C^2.



X is the blow up of $\mathbb{P}^n$ along two quadrics Z_1, Z_2 such that $Z_i \subset \Lambda_i \simeq \mathbb{P}^{\frac{n}{2}}$ $(i = 1, 2)$, $\dim \Lambda_1 \cap \Lambda_2 = 0$ and $\dim Z_1 = \dim Z_2 = \frac{n}{2} - 1$ (n is even).

Canonical Bundle

$$K_X = \pi^*\mathcal{O}_{\mathbb{P}^n}(-n-1) + \frac{n}{2}E_1 + \frac{n}{2}E_2$$

where $\pi : X \to \mathbb{P}^n$ is the blow up of $\mathbb{P}^n$, E_1 and E_2 are the exceptional divisors.

Description of the Kleiman-Mori cone of X

Denote by

- l_1 a minimal curve which is contracted by the blow down along Z_1 and by W^1 a family of deformations of l_1;
- l_2 a minimal curve which is contracted by the blow down along Z_2 and by W^2 a family of deformations of l_2;
- γ_1 a curve which is the strict transform of a line of $\mathbb{P}^n$ that is contained in Λ_1 and by T^1 a family of deformations of γ_1;
- γ_2 a curve which is the strict transform of a line of $\mathbb{P}^n$ that is contained in Λ_2 and by T^2 a family of deformations of γ_1;
- $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$;
- $\mathcal{L}_1 = 2\mathcal{H} - E_1$;
- $\mathcal{L}_2 = 2\mathcal{H} - E_2$;

- $\mathcal{F} = 2\mathcal{H} - E_2 - E_1$.

Note that $\mathcal{H}, \mathcal{L}_1$ and $\mathcal{L}_2$ are nef divisors on X, and we have the following intersection numbers:

	W^1	W^2	T^1	T^2
E_1	-1	0	2	0
E_2	0	-1	0	2
$\mathcal{H}$	0	0	1	1
$\mathcal{L}_1$	1	0	0	2
$\mathcal{L}_2$	0	1	2	0
$\mathcal{F}$	1	1	0	0

and therefore $\mathcal{H}, \mathcal{L}_1$ and $\mathcal{L}_2$ are the supporting divisors of three extremal faces of $NE(X)$. We want to show that $\mathcal{F}$ is nef, and hence it is a supporting divisor of the last extremal face of $NE(X)$.
Suppose by contradiction that there is a irreducible curve $l \subset X$ such that $\mathcal{F} \cdot l < 0$. Then $(\mathcal{H} - E_1) \cdot l < 0$ or $(\mathcal{H} - E_2) \cdot l < 0$.
Assume without loss of generality that $(\mathcal{H} - E_1) \cdot l < 0$, and consider the following maps:

$$\begin{array}{c} X = Bl_{Z_1,Z_2}(\mathbb{P}^n) \\ \varepsilon \downarrow \\ Bl_{Z_1}(\mathbb{P}^n) \\ \varphi \downarrow \\ \mathbb{P}^n \end{array}$$

Let H be an hyperplane of $\mathbb{P}^n$ which contain Λ_1 and let H' be the strict transform of H under φ. In particular, we have that

$$\mathcal{H} - E_1 = \varepsilon^* H'$$

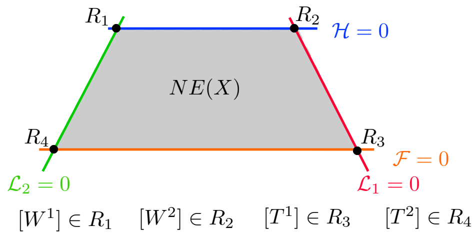
and hence, by the projection formula, we get

$$(\mathcal{H} - E_1) \cdot l = H' \cdot \varepsilon_* l < 0.$$

This implies that the curve l is not contracted by ε and that $\varepsilon(l)$ is contained in H'. Then $\varphi(\varepsilon(l)) \subset \Lambda_1$ and so $l \subset \widetilde{\Lambda_1} \cup E_1$, where $\widetilde{\Lambda_1}$ is the strict transform of Λ_1 under the blow up $\pi : X \to \mathbb{P}^n$. Thus $[l] \in NE(\widetilde{\Lambda_1}, X) \cup NE(E_1, X)$.
We observe that $\widetilde{\Lambda_1} = \mathrm{Locus}(T^1)_x$ and, by Lemma (6.6), we have that $NE(\widetilde{\Lambda_1}, X) = \langle [T^1] \rangle$. Moreover $\mathrm{Locus}(W^1)_{\widetilde{\Lambda_1}} \subseteq E_1$ and we get

$$\begin{aligned} \dim \mathrm{Locus}(W^1)_{\widetilde{\Lambda_1}} &\geq \dim \widetilde{\Lambda_1} - K_X \cdot W^1 - 1 \\ &= \frac{n}{2} + \frac{n}{2} - 1 = n - 1. \end{aligned}$$

Therefore $E_1 = \mathrm{Locus}(W^1)_{\widetilde{\Lambda_1}}$ and so, $NE(E_1, X) = \langle [T^1], [W^1] \rangle$ (both $[T^1]$ and $[W^1]$ are extremal in $NE(E_1, X) = \langle [T^1], [W^1] \rangle$).
This implies that $[l] \in N_1(X)_{\mathcal{F}\geq 0}$ and it is a contradiction. Hence $\mathcal{F}$ is nef and the numerical classes of W^1, W^2, T^1 and T^2 belong to four extremal rays of $NE(X)$:



Clearly the elementary contraction associated to the extremal ray R_1 is the blow down of E_1, and the elementary contraction associated to the extremal ray R_2 is the blow down of E_2.
Moreover, the elementary contractions associated to R_3 and to R_4 are small because their exceptional loci have dimension $\frac{n}{2}$, and the Fano-Mori contraction whose supporting divisor is $\mathcal{F}$ is of fiber type.

Description of fellow families with respect to $\mathcal{V}$

As already observed in the proof of Theorem (8.14), the family V is the family of deformations of the strict transform of a general line of $\mathbb{P}^n$ which has anticanonical degree equal to $n + 1$, and $L = 3\mathcal{H} - E_1 - E_2$.
By Kleiman's criterion L is ample and such that $L \cdot V = 3$.

Now we study how cycles in $\mathcal{V}$ can split. There are three cases:

- a cycle in $\mathcal{V}$ splits into two irreducible components, Γ_1 and $\tilde{l}_1$, where Γ_1 is parametrized by a family C^1 of deformations of the strict transform of a line of $\mathbb{P}^n$ which meets Z_1 in a point, and $\tilde{l}_1$ is parametrized by W^1;
- a cycle in $\mathcal{V}$ splits into two irreducible components, Γ_2 and $\tilde{l}_2$, where Γ_2 is parametrized by a family C^2 of deformations of the strict transform of a line of $\mathbb{P}^n$ which meets Z_2 in a point, and $\tilde{l}_2$ is parametrized by W^2;
- a reducible cycle in $\mathcal{V}$ can have three irreducible components $\tilde{l}_1, \tilde{l}_2, l$, where $\tilde{l}_1$ is parametrized by the family W^1, $\tilde{l}_2$ is parametrized by the family W^2 and l belongs to the family W of deformations of the strict transform of a line of $\mathbb{P}^n$ that meets Z_i in a point for $i = 1, 2$.

Therefore

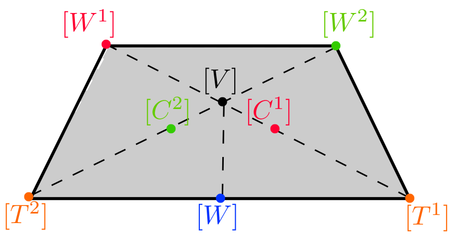
- W^1, C^1 are fellow families with respect to $\mathcal{V}$; in particular C^1 is dominating and locally unsplit, and W^1 is unsplit but not covering. Moreover, they are

such that
$$\begin{cases} -K_X \cdot C^1 = \frac{n}{2}+1 \\ L \cdot C^1 = 2 \end{cases} \quad \begin{cases} -K_X \cdot W^1 = \frac{n}{2} \\ L \cdot W^1 = 1 \end{cases}$$

- W^2, C^2 are fellow families with respect to $\mathcal{V}$; in particular C^2 is dominating and locally unsplit, and W^2 is unsplit but not covering. Moreover, they are such that
$$\begin{cases} -K_X \cdot C^2 = \frac{n}{2}+1 \\ L \cdot C^2 = 2 \end{cases} \quad \begin{cases} -K_X \cdot W^2 = \frac{n}{2} \\ L \cdot W^2 = 1 \end{cases}$$

- W^1, W^2, W are fellow families with respect to $\mathcal{V}$; in particular W is unsplit but not covering and
$$\begin{cases} -K_X \cdot W = 1 \\ L \cdot W = 1 \end{cases}$$
Moreover $\mathcal{F} \cdot W = 0$ and so the numerical class of W is equal to a linear combination $a[T^1] + b[T^2]$; by the intersection numbers we have that $[W] = \frac{1}{2}[T^1] + \frac{1}{2}[T^2]$.

Notice also that (W, W^2) and (T^1, W^1) are two pairs of fellow families with respect to the Chow family associated to C^1. Moreover, (W, W^1) and (T^2, W^2) are two pairs of fellow families with respect to the Chow family associated to C^2.



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