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Philosophiæ Doctor Thesis

Constrained Calculus of Variations and Geometric Optimal Control Theory

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To my mother, in her loving memory.

"If every individual student follows the same current fashion in expressing and thinking [..], then the variety of hypothesis being generated [..] is limited. Perhaps rightly so, for possibly the chance is high that the truth lies in the fashionable direction. But, on the off-chance that it is in another direction — a direction obvious from an unfashionable view of field-theory — who will find it? Only someone who has sacrificed himself by teaching himself [..] from a peculiar and unusual point of view; one that he may have to invent for himself. I say sacrificed himself because he most likely will get nothing from it, because the truth may lie in another direction, perhaps even the fashionable one.

But, if my own experience is any guide, the sacrifice is really not great because if the peculiar viewpoint taken is truly experimentally equivalent to the usual in the realm of the known there is always a range of applications and problems in this realm for which the special viewpoint gives one a special power and clarity of thought, which is valuable in itself. Furthermore, in the search for new laws, you always have the psychological excitement of feeling that possible nobody has yet thought of the crazy possibility you are looking at right now."

Richard P. Feynman

(Richard P. Feynman)

"Le véritable voyage de découverte ne consiste pas à chercher de nouveaux paysages, mais à avoir de nouveaux yeux."

Marcel Proust

(Marcel Proust)

Preface

The present work provides a fresh approach to the calculus of variations in the presence of non–holonomic constraints.

The whole topic has been extensively studied since the beginning of the twentieth century and has been recently revived by its close links with optimal control theory. It is actually of great interest because of its several applications in a wide range of fields such as Physics, Engineering [24] and Economics [12]. Among others, we mention here the pioneering works of Bolza and Bliss [5], the contribution of Pontryagin [17] and the more recent developments by Sussman, Agrachev, Hsu, Montgomery and Griffiths [35, 1, 27, 15, 9], characterized by a differential geometric approach.

Consider an abstract system $\mathfrak{B}$ subject to a set of differentiable conditions, restricting the set of both its admissible configurations and velocities. We shall tackle the following problem: how do we pick out among all the admissible evolutions of $\mathfrak{B}$ connecting two fixed configurations, the ones (if any) that minimize a given action functional?

In broaching the matter, we will make use of the tools provided by jet–bundle geometry, non–holonomic geometry and gauge theory. The abstract system $\mathfrak{B}$ is viewed as a dynamical system whose state can be specified by a finite number of degrees of freedom. Denoted by $\mathcal{V}_{n+1}$ its configuration space–time, having local coordinates $t, q^1, \ldots, q^n$, the admissible evolutions of $\mathfrak{B}$ are then characterized by the solutions of the parametric system of differential equations

$$\frac{dq^i}{dt} = \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \quad , \quad r \leqslant n \tag{1}$$

expressing the derivatives of the state variables in terms of a smaller number of *control* variables.

Equations (1) are interpreted as the local representation of a set of kinetic constraints. More precisely, they are regarded as the local expression of the condition under which an evolution γ is kinematically admissible. Geometrically, the request is that the jet–extension of γ must belong to a submanifold $i\colon \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ which describes the totality of admissible *kinetic states*. Given the system (1), by Cauchy theorem, every assignment of the functions $z^A(t)$ and of a point in $\mathcal{V}_{n+1}$ determines an evolution of $\mathfrak{B}$ as the solution of the given ordinary differential equations with the given initial conditions. However, in the absence of specific assumptions on the nature of $\mathcal{A}$, the functions $z^A(t)$, in themselves, have no invariant geometrical meaning. To pursue the idea of the z^A's as the controllers of the evolution, attention should be rather shifted on *sections* $\sigma : \mathcal{V}_{n+1} \to \mathcal{A}$. Hence, every such section is called a *control*.

Besides the constraints (1), it is also given an *action functional*

$$\mathcal{I}[\gamma] := \int_{t_0}^{t_1} \mathscr{L}(t, q^1(t), \ldots, q^n(t), z^1(t), \ldots, z^r(t))\, dt \tag{2}$$

expressed as the integral of a suitable "cost function", or *Lagrangian* $\mathscr{L}(t, q, z)$ along the admissible evolutions of the system. As stated above, our goal is to find, among these, the ones connecting the fixed end–points $q^i(t_0)$, $q^i(t_1)$ which minimize the functional (2). Exactly as in ordinary function theory, the first step in the solution of the problem consists in investigating the stationarity conditions for the action functional through the analysis of its first variation.

The infinitesimal deformations of an admissible section are discussed via a revisitation of the familiar *variational equation*. The novelty of the approach relies on the introduction of a transport law for vertical vector fields along γ, yielding a covariant characterization of the "true" degrees of freedom.

The analysis is subsequently extended to arbitrary *piecewise differentiable* evolutions consisting of families of contiguous closed arcs $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$. No restrictions are posed on the deformability of the intervals or on the mobility of the "corners" $\gamma(a_s)$, $s = 1, \ldots, N-1$.

The argument allows to assign to every admissible evolution a corresponding *abnormality index*, rephrasing in a geometrical context the traditional attributes of normality and abnormality commonly found in the literature [10].

Furthermore, the abnormality index of an evolution is seen to be related to its *ordinariness*, that is to the property that *every* admissible infinitesimal deformation vanishing at the end–points is tangent to some finite deformations with fixed end–points.

Within the stated framework, the search for the (local) stationary curves of $\mathcal{I}$ with respect to the admissible deformations leaving the end–points fixed results in a fully covariant algorithm, summarizing the content of Pontryagin's maximum principle. The resulting equations are shown to provide sufficient conditions for *any* evolution, and necessary and sufficient conditions for an *ordinary* evolution to be an extremal.

A major breakthrough consists in the possibility of lifting the given constrained variational problem to a corresponding *free* one in the contact bundle $\mathcal{C}(\mathcal{A}) \to \mathcal{A}$, defined as the pull–back of the dual space $V^*(\mathcal{V}_{n+1})$ over $\mathcal{A}$.

This solution method relies on the capability to establish a canonical correspondence between the input data of the problem, namely the kinetic constraints and the Lagrangian, and a distinguished 1–form Θ_{PPC} in the contact manifold such that every stationary curve of the variational problem based on it projects onto a stationary curve of the corresponding problem in $\mathcal{A}$ related to the functional (2).

The canonical characterization of the form Θ_{PPC} — called the *Pontryagin–Poincaré–Cartan form* — within the manifold $\mathcal{C}(\mathcal{A})$ is actually intimately connected with the gauge structure of the whole theory: as it is well known, two different Lagrangians differing by a term $\frac{df}{dt}$, being $f = f(t, q)$ any smooth function over the configuration manifold, give rise to two equivalent variational problems. In this sense, the real information isn't brought so much by the Lagrangian as by the action functional.

In order to analyze the implications of this fact, keeping all differences into account, we take advantage of the geometrical setting introduced some years ago for a gauge–invariant formulation of Classical Mechanics [31, 32].

The construction is based on the introduction of a principal fibre bundle over the configuration space–time $\mathcal{V}_{n+1}$, with structural group $(\mathbb{R}, +)$, referred to as the bundle of *affine scalars.* This is seen to induce two principal bundles $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ over the velocity space $j_1(\mathcal{V}_{n+1})$, respectively called the *Lagrangian* and *co–Lagrangian* bundle, as well as the further *Hamiltonian* and *co–Hamiltonian* bundles over the phase space $\Pi(\mathcal{V}_{n+1})$. In the presence of non–holonomic constraints, the Lagrangian bundles are easily adapted to the submanifold $\mathcal{A}$, through a straightforward pull–back procedure.

Gauge–equivalent Lagrangians are then naturally interpreted as different representations of one and the same section $\ell\colon j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ of the Lagrangian bundle, defined up to an action of the gauge group.

A crucial role in the construction of the canonical *Pontryagin–Poincaré–Cartan form* over the contact manifold $\mathcal{C}(\mathcal{A})$ is then seen to be played by the locus of zeroes of a distinguished pairing in the product manifold $\mathcal{L}(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} \mathcal{H}(\mathcal{V}_{n+1})$.

In the resulting scheme, a gauge–independent *free* variational problem over $\mathcal{C}(\mathcal{A})$ is proved to be equivalent to the original constrained one.

The last part of the present work is devoted to establishing whether a given piecewise differentiable extremal γ, which is supposed to be *normal* even on closed subintervals, gives rise to a minimum for the action functional (2).

The issue is worked out analyzing the so–called second variation of $\mathcal{I}$. Actually, the subject proves to be much harder than one could ever expect. First of all, the expression in local coordinates of the second variation evidently involves the second derivatives of the Lagrangian function, evaluated along the extremal curve. These last are easily seen to undergo a non–tensorial transformation law whenever the first derivatives of $\mathscr{L}$ don't vanish along γ. This, of course, represents an actual obstruction to a geometric approach. Apparently, the natural way out should consist in making use of the gauge structure of the theory, by means of which it is possible to replace the original Lagrangian by an equivalent one, characterized by its being *critical* along the curve.

However, this "adaptation" method looks beforehand to be strictly connected with the time intervals over which the arcs constituting the evolution γ are individually defined. Therefore, it unavoidably fails whenever the deformation process varies such intervals.

The combination of both the request for the tensorial nature of all results and the will to deal with piecewise differentiable curves made up of closed arcs whose reference intervals are possibly changed by the deformation process is thus the cause of much trouble.

Even so, it is actually possible to get over this standoff by resorting to a family of *local* gauge transformations instead of a single *global* one. Pursuing this strategy enables to get a plainly covariant expression of the second variation in terms of a quadratic form made up of an integral part and an algebraic one, related only to the "jumps" of the curve.

It is now possible to break up the remaining part of the problem into consecutive logical steps. First of all, each single closed arc constituting the evolution is requested to give rise to a minimum with respect to the special class of deformations which leave its own end–points fixed. This involves uniquely the behaviour of the integral part of the quadratic form.

Focussing attention on a single arc, we'll first prove a sufficient condition for minimality. This will turn out to be intimately related to the solvability of a non–linear differential equation throughout the definition interval of the arc itself.

In the second instance, *Jacobi vector fields* are taken into account. They represent a special class of infinitesimal deformations such that each of them links families of extremal curves. They are used to investigate the processes of focalization and, by means of the further concept of *conjugate point*, to give a necessary condition for minimality.

Both the sufficient and necessary conditions are eventually glued together, showing that the lack of conjugate points along the arc implies the solvability of the above non–linear differential equation on the whole of it.

At this point, it only remains to establish how the previous results can be converted into a global one, applicable to the whole evolution.

We will show how this can be done by investigating the definiteness property of the second variation restricted to the infinitesimal deformations vanishing at the corners and of a further quadratic form, defined on a suited quotient space.

Contents

Chapter 1

Geometric setup

1.1 Preliminaries

For the sake of convenience, we review here a few basic aspects of jet–bundle geometry [20, 29] which will play a major role in the subsequent discussion. The terminology is borrowed from Mechanics[1].

Let $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ denote an $(n+1)$–dimensional fibre bundle, henceforth called the *event space* and referred to local fibred coordinates $t, q^1, \ldots, q^n$. Every section $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$, locally described as $q^i = q^i(t)$, will be interpreted as an *evolution* of an abstract system $\mathfrak{B}$, parameterized in terms of the independent variable t.

The first jet–space $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ is then an affine bundle over $\mathcal{V}_{n+1}$, modelled on the vertical space $V(\mathcal{V}_{n+1})$ and called the *velocity space.* Both spaces $j_1(\mathcal{V}_{n+1})$ and $V(\mathcal{V}_{n+1})$ may be viewed as submanifolds of the tangent space $T(\mathcal{V}_{n+1})$ according to the identifications[2]

$$j_1(\mathcal{V}_{n+1}) = \{ z \in T(\mathcal{V}_{n+1}) \mid \langle z, dt \rangle = 1 \} \tag{1.1.1a}$$

$$V(\mathcal{V}_{n+1}) = \{ \mathrm{v} \in T(\mathcal{V}_{n+1}) \mid \langle \mathrm{v}, dt \rangle = 0 \} \tag{1.1.1b}$$

In view of equation (1.1.1a), every $z \in j_1(\mathcal{V}_{n+1})$ determines a projection operator $\mathcal{P}_z\colon T_{\pi(z)}(\mathcal{V}_{n+1}) \to V_{\pi(z)}(\mathcal{V}_{n+1})$, sending each vector $X \in T_{\pi(z)}(\mathcal{V}_{n+1})$ into the vertical vector

$$\mathcal{P}_z(X) := X - \big\langle X, (dt)_{\pi(z)} \big\rangle\, z \tag{1.1.2}$$

Given any set of local coordinates $t, q^1, \ldots, q^n$ on $\mathcal{V}_{n+1}$, the corresponding local jet–coordinate system on $j_1(\mathcal{V}_{n+1})$ is denoted by $t, q^1, \ldots, q^n, \dot q^1, \ldots, \dot q^n$, with

[1] Although this is a natural choice, it may be somehow misleading. Just to avoid any possible misunderstanding, it is therefore advisable to recall that, although formulated making use of mechanical terms, constrained calculus of variations doesn't satisfy the principle of determinism and, as such, it can't by no means be considered as belonging under Classical Mechanics.

[2] Property (1.1.1a) is peculiar of those jet–spaces which are built on fibre bundles having a 1–dimensional base space.

transformation laws

$$\bar t = t + c \quad , \qquad \bar q^i = \bar q^i(t, q^1, \dots, q^n) \quad , \qquad \dot{\bar q}^i = \frac{\partial \bar q^i}{\partial t} \; + \; \frac{\partial \bar q^i}{\partial q^k}\, \dot q^k \tag{1.1.3}$$

The vertical bundle $V(\mathcal{V}_{n+1})$ is similarly referred to coordinates $t, q^1, .., q^n, v^1, .., v^n$. In this way, the content of equations (1.1.1a,b) is summarized into the relations

$$z \;=\; \left(\frac{\partial}{\partial t} + \dot q^i(z)\,\frac{\partial}{\partial q^i}\right)_{\pi(z)} \qquad\qquad \forall\; z \in j_1(\mathcal{V}_{n+1}) \tag{1.1.4a}$$

$$\mathrm{v} \;=\; v^i(\mathrm{v})\,\left(\frac{\partial}{\partial q^i}\right)_{\pi(\mathrm{v})} \qquad\qquad \forall\; \mathrm{v} \in V(\mathcal{V}_{n+1}) \tag{1.1.4b}$$

while the projection operator (1.1.2) is expressed in coordinates as

$$\begin{aligned} \mathcal{P}_z\left(X^0\left(\frac{\partial}{\partial t}\right)_{\pi(z)} + X^i\left(\frac{\partial}{\partial q^i}\right)_{\pi(z)}\right) &= \left(X^i \;-\; X^0\,\dot q^i(z)\right)\left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} = \\ &= \left\langle X, \left(dq^i - \dot q^i(z)\,dt\right)_{\pi(z)}\right\rangle \left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \end{aligned} \tag{1.1.5}$$

By the very definition of jet–bundle, every section $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$ may be lifted to a section $j_1(\gamma)\colon \mathbb{R} \to j_1(\mathcal{V}_{n+1})$, simply by assigning to each $t \in \mathbb{R}$ the tangent vector to γ, namely

$$\gamma \,:\, q^i = q^i(t) \quad \longrightarrow \quad j_1(\gamma) \;:\; \begin{cases} q^i = q^i(t) \\ \dot q^i = \frac{dq^i}{dt} \end{cases} \tag{1.1.6}$$

The section $j_1(\gamma)$ will be called the *jet–extension* of γ on $j_1(\mathcal{V}_{n+1})$. The annihilator of the distribution tangent to the totality of the jet–extensions of sections γ is a subspace $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ of $T^*(j_1(\mathcal{V}_{n+1}))$, called the *contact bundle*. The tangent space to the curve $j_1(\gamma) \subset T(j_1(\mathcal{V}_{n+1}))$ is spanned by the vector field

$$(j_1(\gamma))_*\left(\frac{\partial}{\partial t}\right) \;=\; \frac{\partial}{\partial t} + \left(\frac{dq^i}{dt}\right)\frac{\partial}{\partial q^i} + \left(\frac{d\dot q^i}{dt}\right)\frac{\partial}{\partial \dot q^i} \;=\; \frac{\partial}{\partial t} + \dot q^i\,\frac{\partial}{\partial q^i} + \left(\frac{d^2 q^i}{dt^2}\right)\frac{\partial}{\partial \dot q^i}$$

The request for the curve $j_1(\gamma)$ to pass through an arbitrarily chosen point z in $j_1(\mathcal{V}_{n+1})$ fixes exclusively the values of the functions $q^i(t)$ and of their first derivatives but it doesn't affect the second derivatives $\frac{d^2q^i}{dt^2}$. Therefore, a vector $Y \in T_z\,(j_1(\mathcal{V}_{n+1}))$ is tangent to the jet–extension of some section γ if and only if it is represented in coordinate as

$$Y \;=\; Y^0\left[\left(\frac{\partial}{\partial t}\right)_z + \dot q^i(z)\left(\frac{\partial}{\partial q^i}\right)_z\right] + Y^i\left(\frac{\partial}{\partial \dot q^i}\right)_z \qquad \forall\; Y^0, Y^i \in \mathbb{R} \tag{1.1.7}$$

From this it is easily seen that the contact bundle is locally generated by the 1–forms

$$\omega^i = dq^i - \dot{q}^i\,dt \tag{1.1.8}$$

Every section $\sigma\colon j_1(\mathcal{V}_{n+1}) \to \mathcal{C}(j_1(\mathcal{V}_{n+1}))$ is called a *contact* 1*–form*.

We now address ourselves to the vertical bundle[3] $V(j_1(\mathcal{V}_{n+1})) \xrightarrow{\zeta} j_1(\mathcal{V}_{n+1})$. Given any jet–coordinate system $t, q^i, \dot{q}^i$ in $j_1(\mathcal{V}_{n+1})$, we refer $V(j_1(\mathcal{V}_{n+1}))$ to fibred coordinates $t, q^i, \dot{q}^i, \dot{v}^i$ according to the prescription

$$\mathrm{V} \in V(j_1(\mathcal{V}_{n+1})) \quad \Longleftrightarrow \quad \mathrm{V} = \dot{v}^i(\mathrm{V}) \left(\frac{\partial}{\partial \dot{q}^i}\right)_{\zeta(\mathrm{V})}$$

The affine character of the fibration $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ provides a canonical identification of $V(j_1(\mathcal{V}_{n+1}))$ with the pull–back of $V(\mathcal{V}_{n+1})$ under the projection $\pi\colon j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$, giving rise to the vector bundle homomorphism

$$\begin{array}{ccc} V(j_1(\mathcal{V}_{n+1})) & \xrightarrow{\ \varrho\ } & V(\mathcal{V}_{n+1}) \\ {\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{\ \pi\ } & \mathcal{V}_{n+1} \end{array} \tag{1.1.9}$$

For each $z \in j_1(\mathcal{V}_{n+1})$, the fibre $\Sigma_z = \pi^{-1}(\pi(z))$ through z is actually an affine submanifold of $j_1(\mathcal{V}_{n+1})$, modelled on the vertical space $V_{\pi(z)}(\mathcal{V}_{n+1})$. Every pair (z, v), $\mathrm{v} \in V_{\pi(z)}(\mathcal{V}_{n+1})$ is therefore an "applied vector" at z in Σ_z, that is an element of the tangent space $T_z(\Sigma_z)$. On the other hand, by definition, $T_z(\Sigma_z)$ is canonically isomorphic to the vertical space $V_z(J_1(\mathcal{V}_{n+1}))$. By varying z, we conclude that the totality of pairs $(z, \mathrm{v}) \in j_1(\mathcal{V}_{n+1}) \times V(\mathcal{V}_{n+1})$ satisfying $\pi(z) = \pi(v)$ is in bijective correspondence with the points of $V(j_1(\mathcal{V}_{n+1}))$, thereby establishing diagram (1.1.9).

In fibre coordinates, the representation of the map ϱ takes the simple form

$$\varrho\left(V^i\left(\frac{\partial}{\partial \dot{q}^i}\right)_z\right) = V^i\left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \quad \Longleftrightarrow \quad v^i\,\varrho(\mathrm{V}) = \dot{v}^i(\mathrm{V}) \qquad \forall\, \mathrm{V} \in V(j_1(\mathcal{V}_{n+1})) \tag{1.1.10}$$

[3] Since $j_1(\mathcal{V}_{n+1})$ is fibred on both $\mathcal{V}_{n+1}$ and the real line $\mathbb{R}$, there exist *two* vertical fibre bundles over $j_1(\mathcal{V}_{n+1})$. In the following, $V(E;B)$ will stand for the bundle of vertical vectors associated with the fibration $E \to B$. Moreover, in order to make the notation as easy as possible, the symbol $V(j_1(\mathcal{V}_{n+1}))$ will denote — by a little abuse of language — the vertical bundle with respect to the fibration $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$.

In the same manner, every vertical vector v at $\pi(z)$ determines a corresponding vertical vector v^v at z, which is tangent to the curve $\xi \rightarrow z + \xi\mathrm{v}$. The correspondence $\mathrm{v} \rightarrow \mathrm{v}^v$ is known as *vertical lift* of vectors and is expressed in local coordinates as

$$\mathrm{v} \,=\, V^i \left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \quad\longrightarrow\quad \mathrm{v}^v \,=\, V^i \left(\frac{\partial}{\partial \dot{q}^i}\right)_z \tag{1.1.11}$$

On account of equation (1.1.1b), the dual space of $V(\mathcal{V}_{n+1})$ under the pairing $\langle\ ,\ \rangle : T(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} T^*(\mathcal{V}_{n+1}) \rightarrow \mathscr{F}(\mathcal{V}_{n+1})$ is a vector bundle, henceforth denoted by $V^*(\mathcal{V}_{n+1})$, which is canonically isomorphic to the quotient of the cotangent space $T^*(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ by the equivalence relation

$$\sigma \sim \sigma' \quad\Longleftrightarrow\quad \begin{cases} \pi(\sigma) = \pi(\sigma') \\ \sigma - \sigma' \propto dt_{|\pi(\sigma)} \end{cases} \tag{1.1.12}$$

Every local coordinate system t, q^i in $\mathcal{V}_{n+1}$ induces fibred coordinates t, q^i, p_i in $V^*(\mathcal{V}_{n+1})$, with

$$p_i(\hat{\sigma}) \,:=\, \left\langle \hat{\sigma}, \left(\frac{\partial}{\partial q^i}\right)_{\pi(\hat{\sigma})} \right\rangle \qquad \forall\ \hat{\sigma} \in V^*(\mathcal{V}_{n+1})$$

and transformation laws

$$\bar{t} = t + c\ , \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n)\ , \qquad \bar{p}_i = p_k\,\frac{\partial q^k}{\partial \bar{q}^i} \tag{1.1.13}$$

The pull–back of $V^*(\mathcal{V}_{n+1})$ through the map $\pi\colon j_1(\mathcal{V}_{n+1}) \rightarrow \mathcal{V}_{n+1}$ provides another equivalent definition of the contact bundle $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$. By this point of view, a contact 1–form σ is essentially a pair $(z, \hat{\sigma}) \in j_1(\mathcal{V}_{n+1}) \times V^*(\mathcal{V}_{n+1})$, with $\hat{\sigma} \in V^*_{\pi(z)}(\mathcal{V}_{n+1})$. Now, by equation (1.1.2), every such pair determines a linear functional on the tangent space $T_z(j_1(\mathcal{V}_{n+1}))$ according to the prescription

$$\langle \sigma, X \rangle \,:=\, \langle \hat{\sigma}, \mathcal{P}_z \pi_*(X) \rangle \qquad \forall\ X \in T_z(j_1(\mathcal{V}_{n+1})) \tag{1.1.14}$$

In coordinates, recalling equation (1.1.5), the definition of $p_i(\hat{\sigma})$ and making use of the identification $p_i(\sigma) = p_i(\hat{\sigma})$, equation (1.1.14) takes the explicit form

$$\langle \sigma, X \rangle \,=\, \left\langle \hat{\sigma}, \left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \right\rangle \left\langle \omega^i{}_{|z}, X \right\rangle \,=\, \left\langle p_i(\hat{\sigma})\omega^i{}_{|z}, X \right\rangle \,=\, \left\langle p_i(\sigma)\omega^i{}_{|\zeta(\sigma)}, X \right\rangle$$

From this it easily seen that the knowledge of the functional (1.1.14) is mathematically equivalent to the knowledge of σ. Moreover, we find again that the contact bundle is identical to the vector subbundle of the cotangent space $T^*(j_1(\mathcal{V}_{n+1}))$ locally generated by the forms (1.1.8), while the coordinates p_i coincide with the components involved in the representation

$$\sigma \,=\, p_i(\sigma)\,\omega^i{}_{|\zeta(\sigma)} \qquad \forall\; \sigma \in \mathcal{C}(j_1(\mathcal{V}_{n+1})) \tag{1.1.15}$$

The situation is conveniently summarized into the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(j_1(\mathcal{V}_{n+1})) & \xrightarrow{\kappa} & V^*(\mathcal{V}_{n+1}) \\ {\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \tag{1.1.16}$$

Notice that, by construction, $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ is at the same time a vector bundle over $j_1(\mathcal{V}_{n+1})$ and an affine bundle over $V^*(\mathcal{V}_{n+1})$.

At each $z \in j_1(\mathcal{V}_{n+1})$ the duality between $V_{\pi(z)}(\mathcal{V}_{n+1})$ and $V^*_{\pi(z)}(\mathcal{V}_{n+1})$ determines a bilinear pairing $\langle\ \|\ \rangle : V_z(j_1(\mathcal{V}_{n+1})) \times \mathcal{C}(j_1(\mathcal{V}_{n+1})) \to \mathbb{R}$ based on the prescription

$$\langle \mathrm{V} \,\|\, \sigma\rangle \,:=\, \langle \varrho(\mathrm{V}), \kappa(\sigma)\rangle \qquad \forall\; \mathrm{V} \in V_z(j_1(\mathcal{V}_{n+1})),\, \sigma \in \mathcal{C}(j_1(\mathcal{V}_{n+1})) \tag{1.1.17}$$

In coordinates, setting $\mathrm{V} \,=\, \dot{v}^i(\mathrm{V})\left(\frac{\partial}{\partial \dot{q}^i}\right)_z$, $\sigma \,=\, p_i(\sigma)\,\omega^i{}_{|z}$, equations (1.1.10), (1.1.17) yield the expression

$$\langle \mathrm{V} \,\|\, \sigma\rangle \,=\, \dot{v}^i(\mathrm{V}) \left\langle \left(\frac{\partial}{\partial \dot{q}^i}\right)_z, \kappa(\sigma) \right\rangle \,=\, \dot{v}^i(\mathrm{V})\, p_i(\sigma) \tag{1.1.18}$$

By varying z, we extend it to a bilinear pairing between vertical vectors and contact 1–forms on $j_1(\mathcal{V}_{n+1})$, fulfilling the duality relations

$$\left\langle \frac{\partial}{\partial \dot{q}^i} \,\middle\|\, \omega^j \right\rangle \,=\, \delta_i{}^j \tag{1.1.19}$$

1.2 Non–holonomic constraints

Let $\mathcal{A}$ denote an embedded submanifold of $j_1(\mathcal{V}_{n+1})$, fibred over $\mathcal{V}_{n+1}$. The situation, summarized into the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathcal{V}_{n+1} & = & \mathcal{V}_{n+1} \end{array} \tag{1.2.1}$$

provides the natural setting for the study of non–holonomic constraints.

The manifold $\mathcal{A}$ is referred to local fibred coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$ with transformation laws

$$\bar{t} = t + c\,, \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n)\,, \qquad \bar{z}^A = \bar{z}^A(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \tag{1.2.2}$$

while the imbedding $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ is locally expressed as

$$\dot{q}^i = \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \qquad i = 1, \ldots, n \tag{1.2.3}$$

with rank $\left\| \frac{\partial(\psi^1 \cdots \psi^n)}{\partial(z^1 \cdots z^r)} \right\| = r$. Alternatively, one may adopt an implicit representation

$$g^\sigma\left(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n\right) = 0 \qquad \sigma = 1, \ldots, n - r \tag{1.2.4}$$

with rank $\left\| \frac{\partial(g^1 \cdots g^{n-r})}{\partial(\dot{q}^1 \cdots \dot{q}^n)} \right\| = n - r$. For simplicity, in the following we shall not distinguish between the manifold $\mathcal{A}$ and its image $i(\mathcal{A}) \subset j_1(\mathcal{V}_{n+1})$.

A section $\gamma \colon \mathbb{R} \to \mathcal{V}_{n+1}$ will be called $\mathcal{A}$–*admissible* (admissible for short) if and only if its first jet–extension is contained in $\mathcal{A}$, namely if there exists a section $\hat{\gamma} \colon \mathbb{R} \to \mathcal{A}$ satisfying $j_1(\pi \cdot \hat{\gamma}) = i \cdot \hat{\gamma}$. With this notation, given any section $\hat{\gamma}$ described in coordinates as $q^i = q^i(t)$, $z^A = z^A(t)$, the admissibility requirement takes the explicit form

$$\frac{dq^i}{dt} = \psi^i(t, q^1(t), \ldots, q^n(t), z^1(t), \ldots, z^r(t)) \tag{1.2.5}$$

Equations (1.2.5) indicates that, for any admissible evolution of the system, the knowledge of the functions $z^A(t)$ determines $q^i(t)$ up to initial data. On the other hand, in the absence of specific assumptions on the nature of the manifold $\mathcal{A}$, the functions $z^A(t)$, in themselves, have no invariant geometrical meaning.

To pursue the idea of the z^A's as the controllers of the the evolution of the system, attention should rather be shifted on *sections* $\sigma : \mathcal{V}_{n+1} \to \mathcal{A}$. Henceforth, every such section will be called a *control* for the system; the composite map $i \cdot \sigma : \mathcal{V}_{n+1} \to j_1(\mathcal{V}_{n+1})$ will be called an *admissible velocity field*.
In local coordinates we have the representations

$$\sigma \quad : \quad z^A = z^A(t, q^1, \ldots, q^n) \tag{1.2.6a}$$

$$i \cdot \sigma : \quad \dot{q}^i = \psi^i(t, q^1, \ldots, q^n, z^A(t, q^1, \ldots, q^n)) \tag{1.2.6b}$$

confirming that the knowledge of σ does actually determine the evolution of the system from any given initial event in $\mathcal{V}_{n+1}$, through a well posed Cauchy problem.

A section $\gamma \colon \mathbb{R} \to \mathcal{V}_{n+1}$ and a control $\sigma \colon \mathcal{V}_{n+1} \to \mathcal{A}$ will be said to *belong* to each other if and only if the lift $\hat{\gamma} \colon \mathbb{R} \to \mathcal{A}$ factors into $\hat{\gamma} = \sigma \cdot \gamma$, i.e. if and only if the jet–extension $j_1(\gamma)$ coincides with the composite map $i \cdot \sigma \cdot \gamma : \mathbb{R} \to j_1(\mathcal{V}_{n+1})$.

The concepts of vertical vector and contact 1–form are easily extended to the submanifold $\mathcal{A}$: as usual, the vertical bundle $V(\mathcal{A})$ is the kernel of the push–forward $\pi_*\colon T(\mathcal{A}) \to T(\mathcal{V}_{n+1})$ while the contact bundle $\mathcal{C}(\mathcal{A})$ is the pull–back on $\mathcal{A}$ of the bundle $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$, as expressed by the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\hat{\imath}} & \mathcal{C}(j_1(\mathcal{V}_{n+1})) \\ {\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\zeta} \\ \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \end{array} \tag{1.2.7}$$

The manifolds $V(\mathcal{A})$ and $\mathcal{C}(\mathcal{A})$ will be referred to local coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r, w^1, \ldots, w^r$ and $t, q^1, \ldots, q^n, z^1, \ldots, z^r, p_1, \ldots, p_n$ respectively.
In this way, setting

$$\tilde{\omega}^i \;:=\; i^*(\omega^i) \;=\; dq^i - \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r)\, dt \tag{1.2.8}$$

we have the representations

$$X \in V(\mathcal{A}) \qquad \Longleftrightarrow \qquad X \;=\; w^A(X) \left(\frac{\partial}{\partial z^A}\right)_{\zeta(X)} \tag{1.2.9a}$$

$$\sigma \in \mathcal{C}(\mathcal{A}) \qquad \Longleftrightarrow \qquad \sigma \;=\; p_i(\sigma)\, \tilde{\omega}^i \tag{1.2.9b}$$

The restriction to $V(\mathcal{A})$ of the push–forward $i_*\colon T(\mathcal{A}) \to T(j_1(\mathcal{V}_{n+1}))$ determines a vector bundle homomorphism

$$\begin{array}{ccc} V(\mathcal{A}) & \xrightarrow{i_*} & V(j_1(\mathcal{V}_{n+1})) \\ {\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\zeta} \\ \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \end{array} \tag{1.2.10}$$

Composing the last one with diagram (1.1.9) and introducing the simplified notation $\hat{\varrho} := \varrho \cdot i_*$, we get a homomorphism

$$\begin{array}{ccc} V(\mathcal{A}) & \xrightarrow{\hat{\varrho}} & V(\mathcal{V}_{n+1}) \\ {\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \tag{1.2.11}$$

In coordinates, the previous argument provides the representation

$$\hat{\varrho}\left(V^A \left(\frac{\partial}{\partial z^A}\right)_z \right) \;=\; \varrho\left(V^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_z \left(\frac{\partial}{\partial \dot{q}^i}\right)_{i(z)} \right) \;=\; V^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_z \left(\frac{\partial}{\partial q^i}\right)_{\pi(z)}$$

written more synthetically as

$$v^i\,\hat{\varrho}(V) \;=\; \left(\frac{\partial\psi^i}{\partial z^A}\right)_{z(V)} w^A(V) \tag{1.2.12}$$

In a similar way, composing diagrams (1.1.16) and (1.2.7) and setting $\hat{\kappa} := \kappa\cdot\hat{\imath}$, we get a bundle morphism

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\hat{\kappa}} & V^*(\mathcal{V}_{n+1}) \\ {\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array} \tag{1.2.13}$$

described in coordinates as

$$t\,(\hat{\kappa}(\sigma)) = t(\sigma)\,, \qquad q^i\,(\hat{\kappa}(\sigma)) = q^i(\sigma)\,, \qquad p_i\,(\hat{\kappa}(\sigma)) = p_i(\sigma)$$

The latter allows to regard the contact bundle $\mathcal{C}(\mathcal{A})$ as a fibre bundle over the space $V^*(\mathcal{V}_{n+1})$, identical to the pull-back of $V^*(\mathcal{V}_{n+1})$ through the map $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$.

At each $z \in \mathcal{A}$, diagrams (1.2.11), (1.2.13) determine a bilinear pairing between $V_z(\mathcal{A})$ and $\mathcal{C}_z(\mathcal{A})$, essentially identical to the restriction of the pairing (1.1.17), based on the prescriptions

$$\langle V \parallel \sigma\rangle \;:=\; \langle\hat{\varrho}(V), \hat{\kappa}(\sigma)\rangle \;=\; p_i(\sigma)\left(\frac{\partial\psi^i}{\partial z^A}\right)_z w^A(V) \tag{1.2.14}$$

Once again, by varying z, we get a bilinear pairing between vertical vectors and contact 1–forms satisfying the relations

$$\left\langle \frac{\partial}{\partial z^A}\,\middle\|\, \tilde{\omega}^i\right\rangle_z \;=\; \left(\frac{\partial\psi^i}{\partial z^A}\right)_z \qquad \forall\, z\in\mathcal{A} \tag{1.2.15}$$

It should not pass unnoticed that, unlike the original pairing (1.1.17), the map $V(\mathcal{A})\times_{\mathcal{A}}\mathcal{C}(\mathcal{A}) \to \mathscr{F}(\mathcal{A})$, based on equation (1.2.15), has now a *singular* character. A simple dimensionality argument actually shows that no duality can be established between the spaces $V(\mathcal{A})$ and $\mathcal{C}(\mathcal{A})$, it being self–evident that any contact 1–form $\nu = \nu_i\tilde{\omega}^i$ fulfilling $\nu_i\big(\frac{\partial\psi^i}{\partial z^A}\big)_{\zeta(\nu)} = 0$, $A = 1,\ldots,r$ annihilates all vertical vectors. The totality of these 1–forms generates a vector subbundle $\chi(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$, called the *Chetaev bundle* [30]. Every element $\nu\in\chi(\mathcal{A})$ is called a *Chetaev 1–form* on $\mathcal{A}$.

At last, it is worth remarking the presence on $\mathcal{C}(\mathcal{A})$ of a distinguished 1–form θ_L, called the *Liovulle 1–form*, defined by the relation

$$\big\langle X, \theta_{L|\sigma}\big\rangle \;=\; \langle\zeta_*(X), \sigma\rangle \qquad \forall\;\sigma\in\mathcal{C}(\mathcal{A}),\;\; X\in T_\sigma(\mathcal{C}(\mathcal{A})) \tag{1.2.16}$$

and expressed in coordinates as

$$\Theta_L = p_i \tilde{\omega}^i = p_i (dq^i - \psi^i dt) \tag{1.2.17}$$

1.3 Fibre bundles along sections

Let us now see how the geometric setup developed so far looks like when restricted to a given section. The argument will play an important role in the variational context as it provides a suitable framework for dealing with deformations.

The pull–back over the section γ of the vertical space $V(\mathcal{V}_{n+1})$ determines a vector bundle $V(\gamma) \xrightarrow{t} \mathbb{R}$, called the vertical bundle over γ. Given any local coordinate system t, q^i in $\mathcal{V}_{n+1}$, we shall refer $V(\gamma)$ to fibred coordinates t, v^i according to the representation

$$X \in V(\gamma) \qquad \Longleftrightarrow \qquad X = v^i(X) \left(\frac{\partial}{\partial q^i}\right)_{\gamma(t(X))} \tag{1.3.1}$$

Likewise, the *dual* bundle $V^*(\gamma) \xrightarrow{t} \mathbb{R}$ is identical to the pull–back on γ of the space $V^*(\mathcal{V}_{n+1})$. With the notation of §1.1, the situation is expressed by the commutative diagram

$$\begin{array}{ccc} V^*(\gamma) & \longrightarrow & V^*(\mathcal{V}_{n+1}) \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow{\ \gamma\ } & \mathcal{V}_{n+1} \end{array} \tag{1.3.2}$$

The elements of $V^*(\gamma)$ will be called the *virtual* 1*–forms* along γ.

More generally, every element belonging to a fibred tensor product of the form $V(\gamma) \otimes_{\mathbb{R}} V^*(\gamma) \otimes_{\mathbb{R}} \cdots$ will be called a *virtual tensor* along γ.

Notice that, according to the stated definition, a virtual 1–form $\hat{\lambda}$ at a point $\gamma(t)$ is *not* a 1–form in the ordinary sense, but an *equivalence class* of 1–forms under the relation

$$\hat{\lambda} \sim \hat{\lambda}' \qquad \Longleftrightarrow \qquad \hat{\lambda} - \hat{\lambda}' \propto (dt)_{\gamma(t)} \tag{1.3.3}$$

For simplicity, we preserve the notation $\langle\ ,\ \rangle$ for the pairing between $V(\gamma)$ and $V^*(\gamma)$. Also, given any local coordinate system t, q^i in $\mathcal{V}_{n+1}$, we refer $V^*(\gamma)$ to fiber coordinates t, p_i, with $p_i(\hat{\lambda}) = \big\langle \hat{\lambda}, \big(\frac{\partial}{\partial q^i}\big)_{\gamma(t(\hat{\lambda}))} \big\rangle$.

The virtual 1–forms along γ determined by the differentials dq^i will be denoted by $\hat{\omega}^i$, $i = 1, \ldots, n$. In this way, every section $W \colon \mathbb{R} \to V(\gamma) \otimes_{\mathbb{R}} V^*(\gamma) \otimes_{\mathbb{R}} \cdots$ is locally expressed as

$$W = W^{i\,\cdots\cdots}_{\;j\,\cdots}(t) \left(\frac{\partial}{\partial q^i}\right)_\gamma \otimes \hat{\omega}^j \otimes \cdots \tag{1.3.4}$$

We remark that, according to diagram (1.3.2), each fiber $V^*(\gamma)_{|t}$ is isomorphic to the subspace of the cotangent space $T^*_{\gamma(t)}(\mathcal{V}_{n+1})$ annihilating the tangent vector to the curve γ at the point $\gamma(t)$. This had to be expected as it was implicit in the two equivalent definitions of the contact bundle we stated early. Formally, this viewpoint is implemented by setting $\hat{\omega}^i = \left(dq^i - \frac{dq^i}{dt}\,dt\right)_\gamma$. Although apparently simpler, this characterization of $V^*(\gamma)$ has some drawbacks in the case of piecewise differentiable sections and so we shall preferably stick to the original definition.

We recall from §1.1 that every section $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$ admits a jet–extension $j_1(\gamma)\colon \mathbb{R} \to j_1(\mathcal{V}_{n+1})$, expressed in coordinates as $q^i = q^i(t)$, $\dot{q}^i = \frac{dq^i}{dt}$. In a similar way, every vertical vector field $X = X^i \frac{\partial}{\partial q^i}$ over $\mathcal{V}_{n+1}$ may be lifted to a field $J(X) = X^i \frac{\partial}{\partial q^i} + \left(\frac{\partial X^i}{\partial t} + \frac{\partial X^i}{\partial q^k}\dot{q}^k\right)\frac{\partial}{\partial \dot{q}^i} := X^i \frac{\partial}{\partial q^i} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i}$ over $j_1(\mathcal{V}_{n+1})$. The argument is entirely standard (see, for instance, [20]) and is based on the following construction:

- the local 1–parameter group of diffeomorphisms $\varphi_\xi\colon \mathcal{V}_{n+1} \to \mathcal{V}_{n+1}$ generated by X induces, by push–forward, a one parameter group of diffeomorphisms $(\varphi_\xi)_*\colon T(\mathcal{V}_{n+1}) \to T(\mathcal{V}_{n+1})$
- the infinitesimal generator of $(\varphi_\xi)_*$ is a vector field $T(X)$ over $T(\mathcal{V}_{n+1})$
- the field $T(X)$ is tangent to the submanifold $j_1(\mathcal{V}_{n+1}) \subset T(\mathcal{V}_{n+1})$ locally described by the equation $\dot{t} = 1$. As such, it defines a vector field $J(X)$ over $j_1(\mathcal{V}_{n+1})$

Proposition 1.3.1. *The first jet space $j_1(V(\gamma))$ is canonically isomorphic to the vector bundle over $\mathbb{R}$ formed by the totality of vectors Z along $j_1(\gamma)$ annihilating the 1–form dt. With this identification, the fibration $\pi_*\colon j_1(V(\gamma)) \to V(\gamma)$ coincides with the restriction to $j_1(V(\gamma))$ of the push–forward of the projection $\pi\colon j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$.*

Proof. Fix any $t^* \in \mathbb{R}$ and a section $X\colon \mathbb{R} \to V(\gamma)$, then choose any vector field Y defined in a neighborhood $U \ni \gamma(t^*)$ and such that $Y_{|\gamma(t)} = X(t)\ \ \forall\, t \in \gamma^{-1}(U)$.

In coordinates, setting $\gamma : q^i = q^i(t)\,,\ X = X^i \left(\frac{\partial}{\partial q^i}\right)_\gamma$, the lift of the field Y at the point $j_1(\gamma)(t^*)$ takes the form

$$J(Y)_{|\,j_1(\gamma)(t^*)} \;=\; X^i(t^*)\left(\frac{\partial}{\partial q^i}\right)_{j_1(\gamma)(t^*)} + \left.\frac{dX^i}{dt}\right|_{t=t^*}\left(\frac{\partial}{\partial \dot q^i}\right)_{j_1(\gamma)(t^*)} \tag{1.3.5}$$

It's now an easy matter to verify that all assertion of Proposition 1.3.1 follow as a direct result of equation (1.3.5). □

Consistently with equation (1.3.5), given any section $X\colon \mathbb{R} \to V(\gamma)$, the jet–extension $j_1(X)$ will be called the *lift* of X to the curve $j_1(\gamma)$. In local coordinates, we have the representation

$$j_1(X) \;=\; X^i\left(\frac{\partial}{\partial q^i}\right)_{j_1(\gamma)} + \frac{dX^i}{dt}\left(\frac{\partial}{\partial \dot q^i}\right)_{j_1(\gamma)} \tag{1.3.6}$$

Both manifolds $j_1(V(\gamma))$ and $V(\gamma)$ have an obvious nature of vector bundles over $\mathbb{R}$. With respect to this structure, the map $\pi_*\colon j_1(V(\gamma)) \to V(\gamma)$ is clearly an homomorphism with kernel identical to the restriction of the vertical bundle $V(j_1(\mathcal{V}_{n+1}))$ to the curve $j_1(\gamma)$. We set $\ker(\pi_*) := V(j_1(\gamma))$ and call it the *vertical subbundle* of $j_1(V(\gamma))$.

The manifold $j_1(V(\gamma))$ will be referred to jet–coordinates $t, v^i, \dot v^i$, based on the identification

$$Z \in j_1(V(\gamma)) \quad\Longleftrightarrow\quad Z = v^i(Z)\left(\frac{\partial}{\partial q^i}\right)_{j_1(\gamma)(t(Z))} + \dot v^i(Z)\left(\frac{\partial}{\partial \dot q^i}\right)_{j_1(\gamma)(t(Z))} \tag{1.3.7}$$

In terms of these, the jet–extension of a section $v^i = v^i(t)$ takes the standard form $v^i = v^i(t)\,,\ \dot v^i = \frac{dv^i}{dt}$, while the projection $\pi_* : j_1(V(\gamma)) \to V(\gamma)$ is described by $v^i(\pi_*(Z)) = v^i(Z)$. In particular, the vertical subbundle $V(j_1(\gamma))$ coincides with the submanifold of $j_1(V(\gamma))$ locally described by the equation $v^i = 0,\ i = 1, \ldots, n$.

Corollary 1.3.0.1. *The vector bundles* $V(j_1(\gamma)) \xrightarrow{t} \mathbb{R}$ *and* $V(\gamma) \xrightarrow{t} \mathbb{R}$ *are canonically isomorphic*

Proof. As pointed out in §1.1, for each $z \in j_1(\mathcal{V}_{n+1})$ the affine character of the fibration $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ determines an isomorphism between the vertical spaces $V_z(j_1(\mathcal{V}_{n+1}))$ and $V_{\pi(z)}(\mathcal{V}_{n+1})$, expressed in coordinates as

$$\varrho\left[V^i\left(\frac{\partial}{\partial \dot q^i}\right)_z\right] \;=\; V^i\left(\frac{\partial}{\partial q^i}\right)_{\pi(z)}$$

In particular, for $z = j_1(\gamma)(t)$, our previous definitions imply the identifications $\pi(z) = \gamma(t)\,,\ V_{\pi(z)}(\mathcal{V}_{n+1}) = V(\gamma)_{|t}\,,\ V_z(j_1(\mathcal{V}_{n+1})) = V(j_1(\gamma))_{|t}$. By varying t, this gives rise to a vector bundle isomorphism

$$
\begin{array}{ccc}
V(j_1(\gamma)) & \xrightarrow{\varrho} & V(\gamma) \\
{\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t} \\
\mathbb{R} & =\!= & \mathbb{R}
\end{array}
\tag{1.3.8}
$$

expressed in coordinates as

$$
\varrho\left[V^i\left(\frac{\partial}{\partial \dot{q}^i}\right)_{j_1(\gamma)}\right] = V^i\left(\frac{\partial}{\partial q^i}\right)_{\gamma}
\tag{1.3.9}
$$

□

In the presence of non–holonomic constraints, given any admissible section $\hat{\gamma}$, let $A(\hat{\gamma}) \xrightarrow{t} \mathbb{R}$ denote the vector bundle formed by the totality of vectors along $\hat{\gamma}$ annihilating the 1–form dt. On account of Proposition 1.3.1, the push–forward $i_* : T(\mathcal{A}) \to T(j_1(\mathcal{V}_{n+1}))$ gives rise to a bundle morphism

$$
\begin{array}{ccc}
A(\hat{\gamma}) & \xrightarrow{i_*} & j_1(V(\gamma)) \\
{\scriptstyle \pi_*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_*} \\
V(\gamma) & =\!= & V(\gamma)
\end{array}
\tag{1.3.10}
$$

making $A(\hat{\gamma})$ into a subbundle of $j_1(V(\gamma))$ fibred over $V(\gamma)$.

Once again all arrows in diagram (1.3.10), regarded as maps between vector bundles over $\mathbb{R}$, have the nature of homomorphisms. The kernel of the projection $A(\hat{\gamma}) \xrightarrow{\pi_*} V(\gamma)$, clearly identical to the restriction of the vertical bundle $V(\mathcal{A})$ to the curve $\hat{\gamma}$, will be denoted by $V(\hat{\gamma})$, and will be called the *vertical subbundle* along $\hat{\gamma}$.

Every fibred coordinate system t, q^i, z^A in $\mathcal{A}$ induces coordinates t, v^i, w^A in $A(\hat{\gamma})$ according to the prescription

$$
\hat{X} \;=\; v^i(\hat{X})\left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}(t(\hat{X}))} + \; w^A(\hat{X})\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}(t(\hat{X}))} \qquad \forall\, \hat{X} \in A(\hat{\gamma})
\tag{1.3.11}
$$

In terms of these, and of the jet–coordinates $t, v^i, \dot{v}^i$ on $j_1(V(\gamma))$, the morphism (1.3.10) is locally described by the system

$$
t = t\,, \qquad v^i = v^i\,, \qquad \dot{v}^i = \left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}} v^k + \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} w^A
\tag{1.3.12}
$$

while the vertical subbundle $V(\hat{\gamma})$ coincides with the slice $v^i = 0$ in $A(\hat{\gamma})$.

For later use, let us finally observe that the morphism (1.3.10) maps $V(\hat{\gamma})$ into the vertical subbundle $V(j_1(\gamma)) \subset j_1(V(\gamma))$. Composing with the morphism (1.3.8), and recalling the definition of the composite map $\hat{\varrho} := \varrho \cdot i_*$, this gives rise to an injective homomorphism

$$\begin{tikzcd} V(\hat{\gamma}) \arrow[r, "\hat{\varrho}"] \arrow[d, "t"'] & V(\gamma) \arrow[d, "t"] \\ \mathbb{R} \arrow[r, equal] & \mathbb{R} \end{tikzcd} \tag{1.3.13}$$

In coordinates, equations (1.3.9), (1.3.12) provide the representation

$$\hat{\varrho}\left[Y^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}\right] = Y^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \left(\frac{\partial}{\partial q^i}\right)_{\gamma} \tag{1.3.14}$$

1.4 The gauge setup

In the sequel we will take advantage of a geometrical setting that was initially introduced almost ten years ago in order to develop a gauge–invariant formulation of Lagrangian Mechanics. Hence, for convenience of the reader, we outline here its main features.

1.4.1 The Lagrangian bundles

Given any system subject to (smooth) positional constraints, we introduce a double fibration $P \xrightarrow{\pi} \mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, where:

i) $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ is the configuration space–time of the system;

ii) $P \xrightarrow{\pi} \mathcal{V}_{n+1}$ is a principal fibre bundle with structural group $(\mathbb{R}, +)$.

As a consequence of the stated definition, each fibre $P_x := \pi^{-1}(x)$, $x \in \mathcal{V}_{n+1}$ is an affine 1–space. The total space P is therefore a trivial bundle, diffeomorphic in a non canonical way to the Cartesian product $\mathcal{V}_{n+1} \times \mathbb{R}$, called the bundle of *affine scalars* over $\mathcal{V}_{n+1}$.

The action of $(\mathbb{R}, +)$ on P results into a 1–parameter group of diffeomorphisms $\psi_\xi \colon P \to P$, conventionally expressed through the additive notation

$$\psi_\xi(\nu) := \nu + \xi \qquad \forall\, \xi \in \mathbb{R},\ \nu \in P \tag{1.4.1}$$

Every map $u\colon P \to \mathbb{R}$ satisfying the requirement

$$u(\nu + \xi) \;=\; u(\nu) + \xi$$

is called a (global) trivialization of P. If u, u' is any pair of trivializations, the difference $u - u'$ is then (the pull–back of) a function over $\mathcal{V}_{n+1}$. Moreover, every section $\varsigma\colon \mathcal{V}_{n+1} \to P$ determines a trivialization $u_\varsigma \in \mathscr{F}(P)$ and conversely, the relation between ς and u_ς being expressed by the condition

$$\nu \;=\; \varsigma(\pi(\nu)) \;+\; u_\varsigma(\nu) \qquad \forall\, \nu \in P \tag{1.4.2}$$

Therefore, once a (global) trivialization $u\colon P \to \mathbb{R}$ has been chosen, every section $\varsigma\colon \mathcal{V}_{n+1} \to P$ is completely characterized by the knowledge of the function $f = \varsigma^*(u) \in \mathscr{F}(\mathcal{V}_{n+1})$.

The assignment of u allows to lift every local coordinate system $t, q^1, \ldots, q^n$ over $\mathcal{V}_{n+1}$ to a corresponding fibred one $t, q^1, \ldots, q^n, u$ over P. The most general transformation between fibred coordinates has the form

$$\bar t = t + c\;, \qquad \bar q^i = \bar q^i(t, q^1, \ldots, q^n)\;, \qquad \bar u = u + f(t, q^1 \ldots, q^n)$$

The action of the group $(\mathbb{R}, +)$ on the manifold P is expressed in fibred coordinates by the relations

$$t(\nu + \xi) = t(\nu)\;, \qquad q^i(\nu + \xi) = q^i(\nu)\;, \qquad u(\nu + \xi) = u(\nu) + \xi$$

As a result, the generator of the group action (1.4.1), usually referred to as the *fundamental vector field* of P, is canonically identified with the field $\frac{\partial}{\partial u}$.

The (pull–back of) absolute time function determines a fibration $P \xrightarrow{t} \mathbb{R}$ whose associated first jet–space will be indicated by $j_1(P, \mathbb{R}) \xrightarrow{\pi} P$. As usual, this will be referred to local jet–coordinates $t, q^i, u, \dot q^i, \dot u$ subject to transformation laws

$$\bar t = t + c\;, \qquad \bar q^i = \bar q^i(t, q^1, \ldots, q^n)\;, \qquad \bar u = u + f(t, q^1 \ldots, q^n) \tag{1.4.3a}$$

$$\dot{\bar q}^i = \frac{\partial \bar q^i}{\partial q^k}\,\dot q^k + \frac{\partial \bar q^i}{\partial t}\;, \qquad \dot{\bar u} = \dot u + \frac{\partial f}{\partial q^k}\,\dot q^k + \frac{\partial f}{\partial t} := \dot u + \dot f \tag{1.4.3b}$$

The manifold $j_1(P, \mathbb{R})$ is naturally embedded into the tangent space $T(P)$ through the identification

$$j_1(P, \mathbb{R}) \;=\; \{z \in T(P) \mid \langle z, dt\rangle \;=\; 1\}$$

expressed in local coordinate as

$$z \in j_1(P, \mathbb{R}) \quad\Longleftrightarrow\quad z \;=\; \left[\frac{\partial}{\partial t} + \dot q^i(z)\frac{\partial}{\partial q^i} + \dot u^i(z)\frac{\partial}{\partial u}\right]_{\pi(z)} \tag{1.4.4}$$

Every section $\gamma\colon \mathbb{R} \to P$ may be lifted to a section $\hat{\gamma}\colon \mathbb{R} \to j_1(P,\mathbb{R})$ by assigning to each $t \in \mathbb{R}$ the tangent vector to γ, namely

$$\gamma \;:\; \begin{cases} q^i = q^i(t) \\ u = u(t) \end{cases} \quad \longrightarrow \quad \hat{\gamma} \;:\; \begin{cases} q^i = q^i(t) \\ u = u(t) \\ \dot{q}^i = \frac{dq^i}{dt} \\ \dot{u}^i = \frac{du}{dt} \end{cases} \tag{1.4.5}$$

In addition to the jet attributes, the space $j_1(P,\mathbb{R})$ inherits from P two distinguished actions of the group $(\mathbb{R},+)$, related in a straightforward way to the identification (1.4.4).

The first one is simply the push–forward of the action (1.4.1), restricted to the submanifold $j_1(P,\mathbb{R}) \subset T(P)$. In jet–coordinates, a comparison with equation (1.4.4) provides the local representation

$$(\psi_\xi)_*\,(z) \;=\; \left[\frac{\partial}{\partial t} \,+\, \dot{q}^i(z)\,\frac{\partial}{\partial q^i} \,+\, \dot{u}^i(z)\,\frac{\partial}{\partial u}\right]_{\pi(z)+\xi} \tag{1.4.6a}$$

expressed symbolically as

$$(\psi_\xi)_* \;:\; (t, q^i, u, \dot{q}^i, \dot{u}^i) \;\longrightarrow\; (t, q^i, u+\xi, \dot{q}^i, \dot{u}^i) \tag{1.4.6b}$$

The quotient of $j_1(P,\mathbb{R})$ by this action is a $(2n+2)$–dimensional manifold, henceforth denoted by $\mathcal{L}(\mathcal{V}_{n+1})$. As shown in [31], the quotient map makes $j_1(P,\mathbb{R})$ into a principal fibre bundle over $\mathcal{L}(\mathcal{V}_{n+1})$, with structural group $(\mathbb{R},+)$. Furthermore, equation (1.4.6b) shows that $\mathcal{L}(\mathcal{V}_{n+1})$ is an affine fibre bundle over $\mathcal{V}_{n+1}$ with local coordinates $t, q^i, \dot{q}^i, \dot{u}$.

The second action of $(\mathbb{R},+)$ on $j_1(P,\mathbb{R})$ follows from the invariant character of the field $\frac{\partial}{\partial u}$ and is expressed in local coordinates by the addition

$$\phi_\xi(z) \;:=\; z + \xi\,\left(\frac{\partial}{\partial u}\right)_{\pi(z)} \;=\; \left[\frac{\partial}{\partial t} + \dot{q}^i(z)\,\frac{\partial}{\partial q^i} + \left(\dot{u}^i(z) + \xi\right)\,\frac{\partial}{\partial u}\right]_{\pi(z)} \tag{1.4.7a}$$

summarized into the symbolic relation

$$\phi_\xi \;:\; (t, q^i, u, \dot{q}^i, \dot{u}^i) \;\longrightarrow\; (t, q^i, u, \dot{q}^i, \dot{u}^i + \xi) \tag{1.4.7b}$$

The quotient of $j_1(P,\mathbb{R})$ by this action is once again a $(2n+2)$–dimensional manifold, henceforth denoted by $\mathcal{L}^c(\mathcal{V}_{n+1})$. As before, equation (1.4.7b) points out that $\mathcal{L}^c(\mathcal{V}_{n+1})$ is a fibre bundle over P (as well as on $\mathcal{V}_{n+1}$), with coordinates $t, q^i, u, \dot{q}^i$. The quotient map makes $j_1(P,\mathbb{R}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$ into a principal fibre bundle, with structural group $(\mathbb{R},+)$ and group action (1.4.7a).

The last step in the construction relies on the observation that the group actions (1.4.6a), (1.4.7a) do *commute*. Each of them may be then used to induce a group action on the quotient space generated by the other one. As illustrated in [31], this makes both $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ into principal fibre bundles over a common "double quotient" space, canonically diffeomorphic to the velocity space $j_1(\mathcal{V}_{n+1})$. The situation is summarized into the commutative diagram

$$\begin{array}{ccc} j_1(P,\mathbb{R}) & \longrightarrow & \mathcal{L}^c(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{L}(\mathcal{V}_{n+1}) & \longrightarrow & j_1(\mathcal{V}_{n+1}) \end{array} \tag{1.4.8}$$

in which all arrows denote principal fibrations, with structural groups isomorphic to $(\mathbb{R},+)$ and group actions obtained in a straightforward way from equations (1.4.6b), (1.4.7b). The principal fibre bundles $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ are respectively called the *Lagrangian* and the *co–Lagrangian bundle* over $j_1(\mathcal{V}_{n+1})$.

The advantage of this framework is exploited to the utmost by giving up the traditional approach, based on the interpretation of the Lagrangian function $\mathscr{L}(t,q^i,\dot q^i)$ as the representation of a (gauge–dependent) *scalar field* over $j_1(\mathcal{V}_{n+1})$ and introducing instead the concept of *Lagrangian section*, meant as a section $\ell\colon j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ of the Lagrangian bundle.

For each choice of the trivialization u of P, the description of ℓ takes the local form

$$\dot u \,=\, \mathscr{L}(t,q^i,\dot q^i) \tag{1.4.9}$$

and so it does still rely on the assignment of a function $\mathscr{L}(t,q^i,\dot q^i)$ over $j_1(\mathcal{V}_{n+1})$. However, as soon as the trivialization is changed into $\bar u = u + f$, the representation (1.4.9) undergoes the transformation law

$$\dot{\bar u} \,=\, \dot u + \dot f \,=\, \mathscr{L}(t,q^i,\dot q^i) + \dot f := \mathscr{L}'(t,q^i,\dot q^i) \tag{1.4.10}$$

involving a different, gauge–equivalent Lagrangian.

1.4.2 The non–holonomic Lagrangian bundles

Let us return to diagram (1.2.1), with the base manifold explicitly identified with the configuration space–time $\mathcal{V}_{n+1}$ of an abstract system $\mathfrak{B}$ and with the imbedding $i\colon \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ taken as a description of the *kinetic constraints* acting on it [28, 30]. The construction of the Lagrangian bundles is easily adapted to the submanifold $\mathcal{A}$, through a straightforward pull-back process.

The situation is conveniently illustrated by means of a commutative diagram

$$
\begin{tikzcd}
 & j_1(P,\mathbb{R}) \arrow[rr] \arrow[dd] & & \mathcal{L}^c(\mathcal{V}_{n+1}) \arrow[dd] \\
j_1^{\mathcal{A}}(P,\mathbb{R}) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{L}^c(\mathcal{A}) \arrow[ur] \arrow[dd] & \\
 & \mathcal{L}(\mathcal{V}_{n+1}) \arrow[rr] & & j_1(\mathcal{V}_{n+1}) \\
\mathcal{L}(\mathcal{A}) \arrow[ur] \arrow[rr] & & \mathcal{A} \arrow[ur] &
\end{tikzcd}
\tag{1.4.11}
$$

where:

- $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^c(\mathcal{A})$ are respectively the pull–back of $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ on the submanifold $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$;

- $j_1^{\mathcal{A}}(P,\mathbb{R})$ may be alternatively seen as the pull–back of $j_1(P,\mathbb{R}) \to \mathcal{L}(\mathcal{V}_{n+1})$ on the submanifold $\mathcal{L}(\mathcal{A}) \to \mathcal{L}(\mathcal{V}_{n+1})$ or as the pull–back of $j_1(P,\mathbb{R}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$ on $\mathcal{L}^c(\mathcal{A}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$.

As usual, we refer $\mathcal{A}$ to local fibred coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$ with transformation laws

$$
\bar{t} = t + c \;, \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n) \;, \qquad \bar{z}^A = \bar{z}^A(t, q^1, \ldots, q^n, z^1, \ldots, z^r)
\tag{1.4.12}
$$

and express the imbedding $i \colon \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ in the form

$$
\dot{q}^i \;=\; \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r)
\tag{1.4.13}
$$

The geometrical properties of the above–defined pull–back bundles are straightforwardly inherited from their respective holonomic counterparts. In particular:

- Every choice of a trivialization u of P allows to lift any coordinate system of $\mathcal{A}$ to coordinates t, q^i, z^A, u on $\mathcal{L}^c(\mathcal{A})$, $t, q^i, z^A, \dot{u}$ on $\mathcal{L}(\mathcal{A})$ and $t, q^i, u, z^A, \dot{u}$ on $j_1^{\mathcal{A}}(P,\mathbb{R})$. The resulting coordinate transformations are obtained by completing equations (1.4.12) with (the significant part of) the system

$$
\bar{u} = u + f(t, q^i) \;, \quad \bar{\dot{u}} = \dot{u} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k}\, \psi^k(t, q^i, z^A) := \dot{u} + \dot{f}
\tag{1.4.14}
$$

- Equation (1.4.13) locally describes all the embeddings $\mathcal{L}(\mathcal{A}) \to \mathcal{L}(\mathcal{V}_{n+1})$, $\mathcal{L}^c(\mathcal{A}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$ and $j_1^{\mathcal{A}}(P,\mathbb{R}) \to j_1(P,\mathbb{R})$.

- Both actions (1.4.6a), (1.4.7a) of the group $(\mathbb{R},+)$ on $j_1(P,\mathbb{R})$ preserve the submanifold $j_1^{\mathcal{A}}(P,R)$ thereby inducing two corresponding actions $(\psi_\xi)_*$ and ϕ_ξ on $j_1^{\mathcal{A}}(P,R)$, expressed in coordinate as

$$(\psi_\xi)_* : \ \big(t,q^i,u,z^A,\dot u\big) \ \longrightarrow \ \big(t,q^i,u+\xi,z^A,\dot u\big) \tag{1.4.15a}$$

$$\phi_\xi : \ \big(t,q^i,u,z^A,\dot u\big) \ \longrightarrow \ \big(t,q^i,u,z^A,\dot u+\xi\big) \tag{1.4.15b}$$

Acting in the same way as before, it is easily seen that the manifold $j_1^{\mathcal{A}}(P,\mathbb{R})$ is a principal fibre bundle over $\mathcal{L}(\mathcal{A})$ under the action $(\psi_\xi)_*$, as well as a principal fibre bundle over $\mathcal{L}^c(\mathcal{A})$ under the action ϕ_ξ. Moreover, both $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^c(\mathcal{A})$ are principal fibre bundles over $\mathcal{A}$ under the (induced) actions $(\psi_\xi)_*$ and ϕ_ξ respectively. Accordingly, all arrows in the front and rear faces of the diagram (1.4.11) express principal fibrations, while those in the left and right–hand faces are principal bundle homomorphisms.

Preserving the terminology, the principal fibre bundles $\mathcal{L}(\mathcal{A}) \to \mathcal{A}$ and $\mathcal{L}^c(\mathcal{A}) \to \mathcal{A}$ will be respectively called the *non–holonomic Lagrangian* and *non–holonomic co–Lagrangian bundle* over $\mathcal{A}$. A section $\ell\colon \mathcal{A} \to \mathcal{L}(\mathcal{A})$ will be called a (non–holonomic) Lagrangian section. Once a trivialization u of P has been fixed, any such section is locally expressed as

$$\dot u \ = \ \mathscr{L}(t,q^i,z^A) \tag{1.4.16}$$

Under an arbitrary change $u \to u+f$ of the trivialization, the representation (1.4.16) undergoes the transformation law

$$\bar{\dot u} \ = \ \dot u + \dot f \ = \ \mathscr{L}(t,q^i,z^A) + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\,\psi^i \ := \ \mathscr{L}'(t,q^i,z^A) \tag{1.4.17}$$

1.4.3 The Hamiltonian bundles

Parallelling the discussion in §1.4.1, we shall now deal with the construction of the *Hamiltonian bundles* on $\mathcal{V}_{n+1}$. To this end, we focus on the fibration $P \to \mathcal{V}_{n+1}$, and denote by $\pi\colon j_1(P,\mathcal{V}_{n+1}) \to P$ the associated first jet–space.
Every fibred coordinate system t,q^i,u on P induces local coordinates t,q^i,u,p_0,p_i on $j_1(P,\mathcal{V}_{n+1})$, with transformation group

$$\bar t = t + c\,, \qquad \bar q^i = \bar q^i(t,q^1,\dots,q^n)\,, \qquad \bar u = u + f(t,q^1,\dots,q^n) \tag{1.4.18a}$$

$$\bar p_0 = p_0 + \frac{\partial f}{\partial t} + \left(p_k + \frac{\partial f}{\partial q^k}\right)\,\frac{\partial q^k}{\partial t}\ , \qquad \bar p_i = \left(p_k + \frac{\partial f}{\partial q^k}\right)\,\frac{\partial q^k}{\partial \bar q^i} \tag{1.4.18b}$$

The manifold $j_1(P,\mathcal{V}_{n+1})$ is naturally imbedded into the cotangent space $T^*(P)$ through the identification

$$j_1(P,\mathcal{V}_{n+1}) \;=\; \left\{\eta \in T^*(P) \,\middle|\, \left\langle \eta, \frac{\partial}{\partial u}\right\rangle \;=\; 1\right\}$$

expressed in local coordinate as

$$\eta \in j_1(P,\mathcal{V}_{n+1}) \quad\Longleftrightarrow\quad \eta \;=\; \big[du \,-\, p_0(\eta)\,dt \,-\, p_i(\eta)\,dq^i\big]_{\pi(\eta)} \tag{1.4.19}$$

Furthermore, equations (1.4.18 a,b) ensure the invariance of the contact 1–form

$$\tilde{\Theta} \;=\; du \,-\, p_0\,dt \,-\, p_i\,dq^i \tag{1.4.20}$$

henceforth referred to as the *Liouville* 1*–form* of $j_1(P,\mathcal{V}_{n+1})$.

Exactly as in the Lagrangian case, one can easily establish two distinguished actions of the group $(\mathbb{R},+)$ on $j_1(P,\mathcal{V}_{n+1})$, expressed locally as

$$(\psi_\xi)_*(\eta) \;:=\; (\psi_{-\xi})_*{}^*(\eta) = \big[du - p_0(\eta)\,dt - p_i(\eta)\,dq^i\big]_{\pi(\eta)+\xi} \tag{1.4.21 a}$$

$$\phi_\xi(\eta) \;:=\; \eta - \xi\,(dt)_{\pi(\eta)} = \big[du - (p_0(\eta)+\xi)\;dt - p_i(\eta)\,dq^i\big]_{\pi(\eta)} \tag{1.4.21 b}$$

Referring once again to [31] for the necessary details, we point out that:

- The direct product of the actions (1.4.21 a,b) makes $j_1(P,\mathcal{V}_{n+1})$ into a principal fibre bundle over a $(2n+1)$–dimensional base space $\Pi(\mathcal{V}_{n+1})$, with coordinates t, q^i, p_i, called the *phase space.*

- In view of equations (1.1.13), (1.4.18 a,b), it is readily seen that the phase space $\Pi(\mathcal{V}_{n+1})$ is an affine bundle over $\mathcal{V}_{n+1}$, modelled on $V^*(\mathcal{V}_{n+1})$.

- The quotient of $j_1(P,\mathcal{V}_{n+1})$ by the action (1.4.21 a), denoted by $\mathcal{H}(\mathcal{V}_{n+1})$, is an affine bundle over $\mathcal{V}_{n+1}$, modelled on the cotangent space $T^*(\mathcal{V}_{n+1})$ and called the *Hamiltonian bundle.*

- Any trivialization $u\colon P \to \mathbb{R}$ allows to lift every local coordinate system $t, q^1, \ldots, q^n$ on $\mathcal{V}_{n+1}$ to a corresponding one $t, q^1, \ldots, q^n, p_0, p_1, \ldots, p_n$ on $\mathcal{H}(\mathcal{V}_{n+1})$, subject to the transformation law

$$\bar{p}_0 = p_0 + \frac{\partial f}{\partial t}\;, \qquad \bar{p}_i = p_i + \frac{\partial f}{\partial q^i} \tag{1.4.22}$$

  further to a change of u into $\bar{u} = u + f(t,q)$.

- The quotient map makes $j_1(P,\mathcal{V}_{n+1})$ into a principal fibre bundle over $\mathcal{H}(\mathcal{V}_{n+1})$, with structural group $(\mathbb{R},+)$ and fundamental vector $\frac{\partial}{\partial u}$.

- The canonical 1–form (1.4.20) endows $j_1(P,\mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ with a distinguished connection, called the *canonical connection*. At the same time, the action (1.4.21 b) "passes to the quotient", thereby making $\mathcal{H}(\mathcal{V}_{n+1})$ into a principal fibre bundle over the phase space $\Pi(\mathcal{V}_{n+1})$.

- The quotient of $j_1(P,\mathcal{V}_{n+1})$ by the action (1.4.21 b), denoted by $\mathcal{H}^c(\mathcal{V}_{n+1})$, is a $(2n+2)$–dimensional manifold, with coordinates t, q^i, u, p_i, called the *co–Hamiltonian bundle*. The quotient map makes $j_1(P,\mathcal{V}_{n+1})$ into a principal fibre bundle over $\mathcal{H}^c(\mathcal{V}_{n+1})$. At the same time, the action (1.4.21 a), suitably transferred to $\mathcal{H}^c(\mathcal{V}_{n+1})$, makes the latter into a principal fibre bundle over $\Pi(\mathcal{V}_{n+1})$.

The previous discussion is summarized into the commutative diagram

$$\begin{array}{ccc} j_1(P,\mathcal{V}_{n+1}) & \longrightarrow & \mathcal{H}^c(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{H}(\mathcal{V}_{n+1}) & \longrightarrow & \Pi(\mathcal{V}_{n+1}) \end{array} \tag{1.4.23}$$

in which all arrows denote principal fibrations, with structural group isomorphic to $\mathbb{R}$. As implicit in the notation, it may be easily showed that the manifold $j_1(P,\mathcal{V}_{n+1})$ is indeed identical to the pull–back of $\mathcal{H}^c(\mathcal{V}_{n+1})$ over $\mathcal{H}(\mathcal{V}_{n+1})$, as well as the pull–back on $\mathcal{H}(\mathcal{V}_{n+1})$ over $\mathcal{H}^c(\mathcal{V}_{n+1})$.

1.4.4 Further developments

The identifications (1.4.4), (1.4.19) provide a natural pairing between the fibres of the first jet–spaces $j_1(P,\mathbb{R}) \xrightarrow{\pi} P$ and $j_1(P,\mathcal{V}_{n+1}) \xrightarrow{\pi} P$, expressed in coordinate as

$$\langle z,\eta\rangle \;=\; \left\langle \left[\frac{\partial}{\partial t} + \dot q^i(z)\frac{\partial}{\partial q^i} + \dot u^i(z)\frac{\partial}{\partial u}\right]_{\pi(z)},\, \left[du - p_0(\eta)\,dt - p_i(\eta)\,dq^i\right]_{\pi(\eta)} \right\rangle \tag{1.4.24}$$

for all $z \in j_1(P,\mathbb{R}), \eta \in j_1(P,\mathcal{V}_{n+1})$ satisfying $\pi(z) = \pi(\eta)$.

In view of equations (1.4.6 a), (1.4.21 a), the correspondence (1.4.24) satisfies the invariance property

$$\Big\langle (\psi_\xi)_*(z)\,,\,(\psi_\xi)_*(\eta)\Big\rangle \;=\; \big\langle z,\eta\big\rangle \tag{1.4.25}$$

thereby inducing an analogous pairing operation between the fibres of the bundles $\mathcal{L}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ and $\mathcal{H}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$, or — just the same — giving rise to a bi–affine map of the fibred product $\mathcal{L}(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} \mathcal{H}(\mathcal{V}_{n+1})$ onto $\mathbb{R}$, expressed in coordinates as

$$\zeta,\mu \quad\longrightarrow\quad F(\zeta,\mu) \,:=\, \dot u(\zeta) - p_0(\mu) - p_i(\mu)\,\dot q^i(\zeta) \tag{1.4.26}$$

Let $\mathcal{S}$ denote the submanifold of $\mathcal{L}(\mathcal{V}_{n+1})\times_{\mathcal{V}_{n+1}}\mathcal{H}(\mathcal{V}_{n+1})$ described by the equation

$$\mathcal{S} \;=\; \left\{(\zeta,\mu)\in\mathcal{L}(\mathcal{V}_{n+1})\times_{\mathcal{V}_{n+1}}\mathcal{H}(\mathcal{V}_{n+1}) \;\middle|\; F(\zeta,\mu)\;=\;0\right\} \tag{1.4.27}$$

A straightforward argument, based on equation (1.4.26), shows that the submanifold $\mathcal{S}$ is at the same time a fibre bundle over $\mathcal{L}(\mathcal{V}_{n+1})$ and over $\mathcal{H}(\mathcal{V}_{n+1})$. The former case is made explicit by referring $\mathcal{S}$ to local coordinates $t,q^i,\dot q^i,\dot u,p_i$, the p_i's been regarded as fibre coordinates. The latter circumstance is similarly accounted for by referring $\mathcal{S}$ to coordinates $t,q^i,\dot q^i,p_0,p_i$, related to the previous ones by the transformation

$$\dot u \;=\; p_0\,+\,p_i\,\dot q^i$$

and with the $\dot q^i$'s playing the role of fibre coordinates.

Recalling the definition of the contact bundle $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$, the situation is summarized into the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{S} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{H}(\mathcal{V}_{n+1}) \arrow[dl] \arrow[dd, equal] \\
\mathcal{L}(\mathcal{V}_{n+1}) \arrow[rr] \arrow[dd, equal] & & \mathcal{V}_{n+1} \arrow[dd, equal] & \\
 & \mathcal{L}(\mathcal{V}_{n+1})\times_{\mathcal{V}_{n+1}}\mathcal{H}(\mathcal{V}_{n+1}) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{H}(\mathcal{V}_{n+1}) \arrow[dl] \arrow[dd] \\
\mathcal{L}(\mathcal{V}_{n+1}) \arrow[rr] \arrow[dd] & & \mathcal{V}_{n+1} \arrow[dd, equal] & \\
 & \mathcal{C}(j_1(\mathcal{V}_{n+1})) \arrow[rr] \arrow[dl] & & V^*(\mathcal{V}_{n+1}) \arrow[dl] \\
j_1(\mathcal{V}_{n+1}) \arrow[rr] & & \mathcal{V}_{n+1} &
\end{tikzcd} \tag{1.4.28}$$

in which the nature of $\mathcal{S}$ as a principal fibre bundle over $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$ stands out. Depending on the choice of the local coordinates over $\mathcal{S}$, the group action may be expressed symbolically either as

$$\phi_\xi \;:\; (t,q^i,\dot q^i,\dot u^i,p_i) \;\longrightarrow\; (t,q^i,\dot q^i,\dot u^i+\xi,p_i) \tag{1.4.29a}$$

or

$$\phi_\xi \;:\; (t,q^i,\dot q^i,p_0,p_i) \;\longrightarrow\; (t,q^i,\dot q^i,p_0+\xi,p_i) \tag{1.4.29b}$$

Furthermore, it's worth pointing out that the canonical contact 1–form (1.4.20) of $j_1(P, \mathcal{V}_{n+1})$ can be pulled–back onto the fibred product $j_1(P, \mathbb{R}) \times_P j_1(P, \mathcal{V}_{n+1})$. The principal fibre bundle $j_1(P, \mathbb{R}) \times_P j_1(P, \mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} \mathcal{H}(\mathcal{V}_{n+1})$ is consequently endowed with a canonical connection.
For every choice of the trivialization u of $P \to \mathcal{V}_{n+1}$, the difference $du - \tilde{\Theta}$ is (the pull–back of) a 1–form $\tilde{\Theta}_u$ on $\mathcal{L}(\mathcal{V}_{n+1}) \times_{\mathcal{V}_{n+1}} \mathcal{H}(\mathcal{V}_{n+1})$, locally expressed as

$$\tilde{\Theta}_u \;=\; p_0\, dt + p_i\, dq^i \tag{1.4.30}$$

and subject to the transformation law

$$\tilde{\Theta}_{\bar{u}} = \left(p_0 + \frac{\partial f}{\partial t}\right) dt + \left(p_i + \frac{\partial f}{\partial q^i}\right) dq^i = \tilde{\Theta}_u + df \tag{1.4.31}$$

under an arbitrary transformation $u \to \bar{u} = u + f(t, q)$.

Eventually, the form $\tilde{\Theta}_u$ can be once again pulled–back onto $\mathcal{S}$. In this last step, depending on the choice of the coordinates over $\mathcal{S}$, the resulting 1–form can be locally expressed as

$$\Theta_u \;=\; p_0\, dt + p_i\, dq^i \;\equiv\; \dot{u}\, dt + p_i\, \left(dq^i - \dot{q}^i\, dt\right) \tag{1.4.32}$$

Hence, the submanifold $\mathcal{S}$ is provided with a distinguished 1–form Θ_u which is defined up to the choice of the trivialization of P.

1.5 The variational setup

1.5.1 Deformations

Given a section $\gamma \colon \mathbb{R} \to \mathcal{V}_{n+1}$, locally described as $q^i = q^i(t)$, a *finite deformation* of γ is, by definition, a continuous map $\varphi \colon \Delta \subset \mathbb{R} \times \mathbb{R} \to \mathcal{V}_{n+1}$, defined on the subset $\Delta = \{(t, \xi) \mid t_0 \leqslant t \leqslant t_1,\ -\varepsilon < \xi < \varepsilon\}$ and satisfying the condition $\varphi(t, 0) = \gamma(t)$. By varying the parameter ξ within its definition domain, we get a 1–parameter family of sections γ_ξ, satisfying $\gamma_0 = \gamma$.

Actually, it is usually made a distinction between the so called *weak* and *strong* variations. In order to understand this difference we need to introduce some topology in the space of sections of $\mathcal{V}_{n+1}$.

Definition 1.5.1. *Let $\gamma \colon (c, d) \to \mathcal{V}_{n+1}$ be a differentiable section, $[a, b] \subset (c, d)$ be any closed interval and (U, h), $h = (t, q^1, \ldots, q^n)$ a corresponding fibred local chart such that $\gamma(t) \subset U$ for any $t \in [a, b]$. Let also ε and α be a positive number and a non–negative integer respectively. Then $\mathcal{N}_{(\varepsilon,\alpha)}(\gamma)$ is the set of all differentiable sections $\gamma' \colon \mathbb{R} \to \mathcal{V}_{n+1}$ such that the following two conditions hold for any $t \in [a, b]$:*

1) $\gamma'(t) \subset U$

2) $\left| \frac{d^k\left(q^i \cdot \gamma'(t)\right)}{dt^k} - \frac{d^k\left(q^i \cdot \gamma(t)\right)}{dt^k} \right| < \varepsilon \qquad \forall\ k = 0, \ldots, \alpha$

We let the reader verify that the sets $\mathcal{N}_{(\varepsilon,\alpha)}(\gamma)$ form a system of neighborhoods of γ for a topology on the space of sections of $\mathcal{V}_{n+1}$. In particular, the topology related to the sets $\mathcal{N}_{(\varepsilon,0)}(\gamma)$ is called the *strong* topology while the one related to the sets $\mathcal{N}_{(\varepsilon,1)}(\gamma)$ is referred to as the *weak* topology.

By abuse of language, a deformation γ_ξ is also said to be *weak* (or *strong*) if, for any $\delta > 0$, there exists an $\varepsilon > 0$ such that $\gamma_\xi \in \mathcal{N}_{(\varepsilon,1)}(\gamma)$ (or $\gamma_\xi \in \mathcal{N}_{(\varepsilon,0)}(\gamma)$) for any $\xi < \delta$. We point up that, as a consequence of the previous definitions, any weak deformation is also *always* a strong one while the converse may not occur.

Example 1.5.1: In the one–dimensional case, consider the variation

$$\gamma_\xi(t) \quad : \quad q(\varphi_\xi(t)) \;=\; q(t) \,+\, \xi \sin\left(\frac{t}{\xi^2}\right)$$

As ξ goes to zero, γ_ξ tends to γ by the squeeze rule. However, we have

$$\frac{dq(\varphi_\xi(t))}{dt} \;=\; \frac{dq}{dt} \,+\, \frac{1}{\xi} \cos\left(\frac{t}{\xi^2}\right)$$

and so $\frac{1}{\xi}$ tends to infinity while the cosine oscillates, generating increasingly large variations in the slope — a typical strong, not weak, variation.

For each $t \in \mathbb{R}$, the curve $\xi \to \gamma_\xi(t)$ is called the *orbit* of the deformation γ_ξ through the point $\gamma(t)$. The vector field along γ tangent to the orbits at $\xi = 0$, whenever defined, is called the *infinitesimal deformation* associated with γ_ξ.

In the presence of non–holonomic constraints, care must be taken of the requirement of kinematical admissibility. A deformation γ_ξ is called *admissible* if and only if each section $\gamma_\xi : \mathbb{R} \to \mathcal{V}_{n+1}$ is admissible. In a similar way, a deformation $\hat{\gamma}_\xi$ of an admissible section $\hat{\gamma} : \mathbb{R} \to \mathcal{A}$ is called admissible if and only if all sections $\hat{\gamma}_\xi : \mathbb{R} \to \mathcal{A}$ are admissible.

As pointed out in §1.2, the admissible sections $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ are in 1–1 correspondence with the admissible sections $\hat{\gamma} : \mathbb{R} \to \mathcal{A}$ through the relations

$$\gamma = \pi \cdot \hat{\gamma}\,, \qquad j_1(\gamma) = i \cdot \hat{\gamma} \tag{1.5.1}$$

Every admissible deformation of γ may therefore be expressed as

$$\gamma_\xi = \pi \cdot \hat{\gamma}_\xi$$

$\hat{\gamma}_\xi : \mathbb{R} \to \mathcal{A}$ denoting an admissible deformation of $\hat{\gamma}$.

In coordinates, preserving the representation $\hat{\gamma} : q^i = q^i(t)\,,\ z^A = z^A(t)\,,$ the admissible deformations of $\hat{\gamma}$ are described by equations of the form

$$\hat{\gamma}_\xi : \qquad q^i = \varphi^i(\xi,t)\,, \quad z^A = \zeta^A(\xi,t) \tag{1.5.2}$$

subject to the conditions

$$\varphi^i(0,t) = q^i(t)\,, \quad \zeta^A(0,t) = z^A(t) \tag{1.5.3a}$$

$$\frac{\partial \varphi^i}{\partial t} = \psi^i(t, \varphi^i(\xi,t), \zeta^A(\xi,t)) \tag{1.5.3b}$$

We now dwell upon the fact that any (admissible) *not weak* finite deformation $\varphi \colon \Delta \to \mathcal{V}_{n+1}$ can by no means be lifted to a corresponding (admissible) deformation $\hat{\varphi} \colon \Delta \to \mathcal{A}$, since the continuity of $\hat{\varphi}$ is lacking. On this account, from now on we will restrict ourselves to consider weak variations only. Variational problems with respect to strong variations can be dealt by means of a more general method, based on the so–called Weierstrass Excess function. The argument is beyond the purposes of the present work and will not be pursued.

Setting

$$\begin{aligned} X^i(t) &:= \left(\frac{\partial \varphi^i}{\partial \xi}\right)_{\xi=0}\,, \qquad & \Gamma^A(t) &:= \left(\frac{\partial \zeta^A}{\partial \xi}\right)_{\xi=0} \\ Z^i(t) &:= \left(\frac{\partial^2 \varphi^i}{\partial \xi^2}\right)_{\xi=0}\,, \qquad & K^A(t) &:= \left(\frac{\partial^2 \zeta^A}{\partial \xi^2}\right)_{\xi=0} \end{aligned} \tag{1.5.4}$$

the infinitesimal deformation tangent to $\hat{\gamma}_\xi$ is described by the vector field

$$\hat{X} = X^i(t)\left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + \Gamma^A(t)\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \tag{1.5.5}$$

while equation (1.5.3b) is reflected into the relations

$$\frac{dX^i}{dt} = \frac{\partial}{\partial t}\frac{\partial \varphi^i}{\partial \xi}\bigg|_{\xi=0} = \left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}} X^k + \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \Gamma^A \tag{1.5.6a}$$

$$\begin{aligned} \frac{dZ^i}{dt} = \frac{\partial}{\partial t}\frac{\partial^2 \varphi^i}{\partial \xi^2}\bigg|_{\xi=0} &= \left(\frac{\partial^2 \psi^i}{\partial q^k\,\partial q^r}\right)_{\hat{\gamma}} X^k X^r + 2\left(\frac{\partial^2 \psi^i}{\partial q^k\,\partial z^A}\right)_{\hat{\gamma}} X^k \Gamma^A + \\ &+ \left(\frac{\partial^2 \psi^i}{\partial z^A\,\partial z^B}\right)_{\hat{\gamma}} \Gamma^A \Gamma^B + \left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}} Z^k + \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} K^A \end{aligned} \tag{1.5.6b}$$

the first of which is commonly referred to as the *variational equation*.

The infinitesimal deformation tangent to the projection $\gamma_\xi = \pi \cdot \hat{\gamma}_\xi$ is similarly described by the field

$$X = \pi_* \hat{X} = \left(\frac{\partial \varphi^i}{\partial \xi}\right)_{\xi=0} \left(\frac{\partial}{\partial q^i}\right)_\gamma = X^i(t)\, \frac{\partial}{\partial q^i} \tag{1.5.7}$$

Collecting all previous results and recalling the definitions of the vector bundles $V(\gamma)$ and $A(\hat{\gamma})$ we get the following

Proposition 1.5.1. *Let* $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ *and* $\hat{\gamma} : \mathbb{R} \to \mathcal{A}$ *denote two admissible sections, related by equation (1.5.1). Then:*

i) the infinitesimal deformations of γ *and of* $\hat{\gamma}$ *are respectively expressed as sections* $X : \mathbb{R} \to V(\gamma)$ *and* $\hat{X} : \mathbb{R} \to A(\hat{\gamma})$*;*

ii) a section $X : \mathbb{R} \to V(\gamma)$ *represents an* admissible *infinitesimal deformation of* γ *if and only if its first jet–extension factors through* $A(\hat{\gamma})$*, i.e. if and only if there exists a section* $\hat{X} : \mathbb{R} \to A(\hat{\gamma})$ *satisfying* $j_1(X) = i_* \hat{X}$*; conversely, a section* $\hat{X} : \mathbb{R} \to A(\hat{\gamma})$ *represents an admissible infinitesimal deformation of* $\hat{\gamma}$ *if and only if it projects into an admissible infinitesimal deformation of* γ*, i.e. if and only if* $i_* \hat{X} = j_1(\pi_* \hat{X})$*.*

The proof is entirely straightforward, and is left to the reader.

From a structural viewpoint, Proposition 1.5.1 establishes a complete symmetry between the roles of diagram (1.2.1) in the study of the admissible *evolutions* and of diagram (1.3.10) in the study of the admissible infinitesimal *deformations*, thus enforcing the intuitive idea that the latter context is essentially a “linearized counterpart” of the former one.

1.5.2 Infinitesimal controls

According to Proposition 1.5.1, the admissible infinitesimal deformations of an admissible section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ are in 1–1 correspondence with the sections $\hat{X} : \mathbb{R} \to A(\hat{\gamma})$ satisfying the consistency requirement $i_* \hat{X} = j_1(\pi_* \hat{X})$.

In local coordinates, setting $\hat{X} = X^i(t)\, \frac{\partial}{\partial q^i} + \Gamma^A(t)\, \frac{\partial}{\partial z^A}$, the stated requirement is expressed by the variational equation

$$\frac{dX^i}{dt} = \frac{\partial \psi^i}{\partial q^k}\, X^k + \frac{\partial \psi^i}{\partial z^A}\, \Gamma^A \tag{1.5.8}$$

all coefficients being evaluated along the curve $\hat{\gamma}$.

Exactly as it happened in §1.2 with regard to the admissibility of evolutions, equation (1.5.8) indicates that, for each admissible $\hat{X}$, the knowledge of the functions $\Gamma^A(t)$ determines the remaining $X^i(t)$ up to initial data, through the solution of a well posed Cauchy problem.

Once again, however, the drawback is that the components Γ^A, in themselves, have no invariant geometrical meaning, but obey the non–homogeneous transformation law

$$\begin{aligned}\bar{\Gamma}^A = \left\langle \hat{X}, d\bar{z}^A \right\rangle &= \frac{\partial \bar{z}^A}{\partial t} \left\langle \hat{X}, dt \right\rangle + \frac{\partial \bar{z}^A}{\partial q^i} \left\langle \hat{X}, dq^i \right\rangle + \frac{\partial \bar{z}^A}{\partial z^B} \left\langle \hat{X}, z^B \right\rangle = \\ &= \frac{\partial \bar{z}^A}{\partial q^i} X^i + \frac{\partial \bar{z}^A}{\partial z^B} \Gamma^B\end{aligned} \tag{1.5.9}$$

under an arbitrary coordinate transformation. Therefore, if $\hat{\gamma}$ is covered by several local charts, assigning the functions $\Gamma^A(t)$ on each of them doesn't even allow to verify if they link up properly except by integrating the variational equation.

The difficulty is overcome introducing a linearized version of the idea of *control*. Referring to diagram (1.3.10), we thus state the following

Definition 1.5.2. *Let $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ denote an admissible evolution. Then:*

- *a linear section $h : V(\gamma) \to A(\hat{\gamma})$, meant as a vector bundle homomorphism satisfying $\pi_* \cdot h = id$, is called an* infinitesimal control *along γ;*
- *the image $\mathcal{H}(\hat{\gamma}) := h(V(\gamma))$, viewed as a vector subbundle of $A(\hat{\gamma}) \to \mathbb{R}$, is called the* horizontal distribution *along $\hat{\gamma}$ induced by h; every section $\hat{X} : \mathbb{R} \to A(\hat{\gamma})$ satisfying $\hat{X}(t) \in \mathcal{H}(\hat{\gamma})\ \forall\ t \in \mathbb{R}$ is called a horizontal section.*

Remark 1.5.1: The term *infinitesimal control* is intuitively clear: given an admissible section γ, let $\sigma : \mathcal{V}_{n+1} \to \mathcal{A}$ denote any control belonging to γ, that is satisfying $\sigma \cdot \gamma = \hat{\gamma}$. Then, on account of the identity $\pi_* \cdot \sigma_* = (\pi \cdot \sigma)_* = id$, the restriction to $V(\gamma)$ of the tangent map $\sigma_* : T(\mathcal{V}_{n+1}) \to T(\mathcal{A})$ determines a linear section $\sigma_* : V(\gamma) \to A(\hat{\gamma})$. The infinitesimal controls may therefore be thought of as equivalence classes of ordinary controls belonging to the same curve and having a first order contact along it.

Given an infinitesimal control $h : V(\gamma) \to A(\hat{\gamma})$, on account of Definition 1.5.2 and of the canonicity of the vertical subbundle $V(\hat{\gamma}) = \ker \pi_*$, it is easily seen that the horizontal distribution $\mathcal{H}(\hat{\gamma})$ does indeed provide a splitting of the vector bundle $A(\hat{\gamma})$ into the fibred direct sum

$$A(\hat{\gamma}) = \mathcal{H}(\hat{\gamma}) \oplus_{\mathbb{R}} V(\hat{\gamma}) \tag{1.5.10}$$

This gives rise to a couple of homomorphisms $\mathcal{P}_H : A(\hat{\gamma}) \to \mathcal{H}(\hat{\gamma})$ (horizontal projection) and $\mathcal{P}_V : A(\hat{\gamma}) \to V(\hat{\gamma})$ (vertical projection), uniquely defined by the relations

$$\mathcal{P}_{\mathcal{H}} = h \cdot \pi_* \qquad ; \qquad \mathcal{P}_V = id - \mathcal{P}_{\mathcal{H}} \tag{1.5.11}$$

In fibre coordinates, preserving the notation (1.3.1), (1.3.11), every infinitesimal control $h : V(\gamma) \to A(\hat{\gamma})$ is represented by a linear system of the form

$$w^A = h_i{}^A(t)\, v^i \tag{1.5.12}$$

In this way:

- the horizontal distribution $\mathcal{H}(\hat{\gamma})$ is locally spanned by the vector fields
$$\tilde{\partial}_i := h\left[\left(\frac{\partial}{\partial q^i}\right)_{\gamma}\right] = \left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + h_i{}^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \tag{1.5.13}$$

- every vertical vector field $X = X^i(t)\left(\frac{\partial}{\partial q^i}\right)_{\gamma}$ along γ may be lifted to a horizontal field $h(X)$ along $\hat{\gamma}$, expressed in components as
$$h(X) = X^i(t)\,\tilde{\partial}_i = X^i(t)\left[\left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + h_i{}^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}\right] \tag{1.5.14}$$

- every vector $\hat{X} = X^i\left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + \Gamma^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \in A(\hat{\gamma})$ admits a unique representation of the form $\hat{X} = \mathcal{P}_{\mathcal{H}}(\hat{X}) + \mathcal{P}_V(\hat{X})$, with
$$\mathcal{P}_{\mathcal{H}}(\hat{X}) = X^i\,\tilde{\partial}_i\,, \quad \mathcal{P}_V(\hat{X}) = \left(\Gamma^A - X^i h_i{}^A\right)\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} := Y^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \tag{1.5.15}$$

- denoting by $\mathcal{C}(\hat{\gamma})$ the contact bundle along $\hat{\gamma}$, meant as the restriction of the contact bundle $\mathcal{C}(\mathcal{A})$ to the curve $\hat{\gamma}$, in view of equation (1.5.13) we have the relations
$$\left\langle \tilde{\partial}_i\,,\,\tilde{\omega}^k\right\rangle = \left\langle \left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + h_i{}^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}\,,\,\left(dq^k - \psi^k\,dt\right)_{\hat{\gamma}}\right\rangle = \delta_i^k \tag{1.5.16}$$
showing that the bundle $\mathcal{C}(\hat{\gamma})$ and $\mathcal{H}(\hat{\gamma})$ are dual of each other under ordinary pairing.

Remark 1.5.2: As implicit in the previous discussion, the advantage of the newer formulation comes from the fact that, unlike $\Gamma^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}$, the vector field $\mathcal{P}_V(\hat{X}) = Y^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}$ has now an invariant geometrical meaning.

It follows from the request $\bar{\tilde{\partial}}_i = \frac{\partial q^j}{\partial \bar{q}^i}\tilde{\partial}_j$ that
$$\frac{\partial}{\partial \bar{q}^i} + \bar{h}_i{}^A\frac{\partial}{\partial \bar{z}^A} = \frac{\partial q^j}{\partial \bar{q}^i}\left(\frac{\partial}{\partial q^j} + h_j{}^B\frac{\partial}{\partial z^B}\right) \;\Rightarrow\; \frac{\partial z^B}{\partial \bar{q}^i}\frac{\partial}{\partial z^B} + \bar{h}_i{}^A\frac{\partial z^B}{\partial \bar{z}^A}\frac{\partial}{\partial z^B} = \frac{\partial q^j}{\partial \bar{q}^i}h_j{}^B\frac{\partial}{\partial z^B}$$
that is
$$\bar{h}_i{}^A = \frac{\partial \bar{z}^A}{\partial z^B}\frac{\partial q^j}{\partial \bar{q}^i}h_j{}^B - \frac{\partial \bar{z}^A}{\partial z^B}\frac{\partial z^B}{\partial \bar{q}^i} \tag{1.5.17}$$
Therefore, the components Y^A undergo the *homogeneous* transformation law
$$\begin{aligned}\bar{Y}^A &= \bar{\Gamma}^A - \bar{X}^i\,\bar{h}_i{}^A = X^j\frac{\partial \bar{z}^A}{\partial q^j} + \Gamma^B\frac{\partial \bar{z}^A}{\partial z^B} - X^j\frac{\partial \bar{q}^i}{\partial q^j}\left(\frac{\partial \bar{z}^A}{\partial z^B}\frac{\partial q^j}{\partial \bar{q}^i}h_j{}^B - \frac{\partial \bar{z}^A}{\partial z^B}\frac{\partial z^B}{\partial \bar{q}^i}\right) = \\ &= \left(\Gamma^B - X^j\,h_j{}^B\right)\frac{\partial \bar{z}^A}{\partial z^B} + X^j\left(\cancel{\frac{\partial \bar{z}^A}{\partial q^j} + \frac{\partial \bar{q}^i}{\partial q^j}\frac{\partial \bar{z}^A}{\partial z^B}\frac{\partial z^B}{\partial \bar{q}^i}}\right) = \\ &= Y^B\,\frac{\partial \bar{z}^A}{\partial z^B}\end{aligned} \tag{1.5.18}$$

the cancelation coming from the identity

$$\frac{\partial \bar{z}^A}{\partial \bar{q}^i} = 0 \;\Rightarrow\; \frac{\partial \bar{z}^A}{\partial z^B}\frac{\partial z^B}{\partial \bar{q}^i} = -\frac{\partial \bar{z}^A}{\partial q^j}\frac{\partial q^j}{\partial \bar{q}^i}$$

The role of Definition 1.5.2 in the study of the variational equation (1.5.8) is further enhanced by the following

Definition 1.5.3. *Let h be an infinitesimal control for the (admissible) section γ. A section $X : \mathbb{R} \to V(\gamma)$ is said to be h–transported along γ if and only if its horizontal lift $h(X) : \mathbb{R} \to A(\hat{\gamma})$ is an admissible infinitesimal deformation of $\hat{\gamma}$, namely if and only if $i_* \cdot h(X) = j_1(X)$.*

In view of equations (1.5.8), (1.5.14), setting $X = X^i(t)\big(\frac{\partial}{\partial q^i}\big)_\gamma$, the condition for h–transport is expressed in coordinates by the linear system of ordinary differential equations

$$\frac{dX^i}{dt} = \left[\left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}} + h_k{}^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}}\right] X^k = X^k\, \tilde{\partial}_k \psi^i \tag{1.5.19}$$

From the latter, recalling Cauchy theorem, we conclude that the h–transported sections of $V(\gamma)$ form an n–dimensional vector space V_h, isomorphic to each fibre $V(\gamma)_{|t}$ through the evaluation map $X \to X(t)$. We have thus proved:

Proposition 1.5.2. *Every infinitesimal control $h : V(\gamma) \to A(\hat{\gamma})$ determines a trivialization of the vector bundle $V(\gamma) \xrightarrow{t} \mathbb{R}$.*

Proposition 1.5.2 provides an identification between sections $X : \mathbb{R} \to V(\gamma)$ and vector valued functions $X : \mathbb{R} \to V_h$ and therefore — by duality — also an identification between sections $\hat{\lambda} : \mathbb{R} \to V^*(\gamma)$ and vector valued functions $\hat{\lambda} : \mathbb{R} \to V_h^*$, thus allowing the introduction of an *absolute time derivative* $\frac{D}{Dt}$ for vertical vector fields and virtual 1–forms along γ.

The algorithm is readily implemented in components. To this end, let $\big\{e_{(a)}\big\}$, $\big\{e^{(a)}\big\}$ denote any pair of dual bases for the spaces V_h, V_h^*. By definition, each $e_{(a)}$ is a vertical vector field along γ, obeying the transport law (1.5.19).
In coordinates, setting $e_{(a)} = e_{(a)}^{\,i}\big(\frac{\partial}{\partial q^i}\big)_\gamma$, this implies the relation

$$\frac{de_{(a)}^{\,i}}{dt} = e_{(a)}^{\,k}\, \tilde{\partial}_k\, \psi^i \tag{1.5.20a}$$

In a similar way, each $e^{(a)}$ is a virtual 1–form along γ, expressed on the basis $\hat{\omega}^i$ as $e^{(a)} = e^{(a)}_{\;i}\, \hat{\omega}^i$, with $e^{(a)}_{\;i}\, e_{(b)}^{\,i} = \delta^a{}_b$.

On account of equation (1.5.20a), the components $e^{(a)}_{\;i}$ obey the transport law

$$\frac{d}{dt}\left(e^{(a)}_{\;i}\, e_{(a)}^{\,j}\right) = 0 \qquad \Longrightarrow \qquad \frac{de^{(a)}_{\;i}}{dt} = -\, e^{(a)}_{\;j}\, \tilde{\partial}_i \psi^j \tag{1.5.20b}$$

The functions

$$\tau_i{}^j \,:=\, \frac{de_i^{(a)}}{dt}\, e^{\,j}_{(a)} \,=\, -\, e_i^{(a)}\, \frac{de^{\,j}_{(a)}}{dt} \tag{1.5.21a}$$

will be called the *temporal connection coefficients* associated with the infinitesimal control h in the coordinate system t, q^i. Comparison with equations (1.5.13), (1.5.20a,b) provides the representation

$$\tau_i{}^j \,=\, -\,\tilde{\partial}_i \psi^j \,=\, -\left(\frac{\partial \psi^j}{\partial q^i}\right)_{\hat\gamma} -\, h_i{}^A \left(\frac{\partial \psi^j}{\partial z^A}\right)_{\hat\gamma} \tag{1.5.21b}$$

Given any vertical vector field $X = X^i \big(\frac{\partial}{\partial q^i}\big)_\gamma$ along γ, the definition of the operator $\frac{D}{Dt}$ is then summarized into the expression

$$\begin{aligned} \frac{DX}{Dt} &= \frac{d}{dt}\left\langle X, e^{(a)}\right\rangle e_{(a)} \,=\, \frac{d}{dt}\left\langle X, e_i^{(a)}\,\hat\omega^i\right\rangle e_{(a)} \,=\, \frac{d}{dt}\left(X^i e_i^{(a)}\right) e^{\,j}_{(a)} \left(\frac{\partial}{\partial q^j}\right)_\gamma = \\ &= \left(\frac{dX^j}{dt} + X^i \tau_i{}^j\right)\left(\frac{\partial}{\partial q^j}\right)_\gamma \end{aligned} \tag{1.5.22a}$$

with the coefficients $\tau_i{}^j$ given by equation (1.5.21b). In a similar way, given any virtual 1–form $\hat\lambda = \lambda_i\,\hat\omega^i$, the same argument provides the evaluation

$$\frac{D\hat\lambda}{Dt} = \frac{d}{dt}\left\langle \hat\lambda, e_{(a)}\right\rangle e^{(a)} = \frac{d}{dt}\left(\lambda_i\, e^{\,i}_{(a)}\right) e_j^{(a)}\,\hat\omega^j = \left(\frac{d\lambda_j}{dt} - \lambda_i\,\tau_j{}^i\right)\hat\omega^j \tag{1.5.22b}$$

By a little abuse of notation we shall henceforth systematically use the following symbology:

$$\frac{DX^i}{Dt} \,:=\, \left(\frac{DX}{Dt}\right)^i, \qquad \frac{D\lambda_i}{Dt} \,:=\, \left(\frac{D\hat\lambda}{Dt}\right)_i$$

The operation $\frac{D}{Dt}$ is immediately extended to a derivation of the algebra of virtual tensor fields along γ, commuting with contractions. In coordinates, we have the representation

$$\frac{D}{Dt}\left[W^i{}_{j\ldots}(t)\left(\frac{\partial}{\partial q^i}\right)_\gamma \otimes \hat\omega^j \otimes \cdots\right] := \frac{DW^i{}_{j\ldots}}{Dt}\left(\frac{\partial}{\partial q^i}\right)_\gamma \otimes \hat\omega^j \otimes \cdots \tag{1.5.23a}$$

with

$$\frac{DW^i{}_{j\ldots}}{Dt} \,=\, \frac{dW^i{}_{j\ldots}}{dt} \,+\, \tau_k{}^i\, W^k{}_{j\ldots} \,-\, \tau_j{}^k\, W^i{}_{k\ldots} \,+\, \cdots \tag{1.5.23b}$$

After these preliminaries, let us go back to the variational equation (1.5.8). By means of the projections (1.5.11), every section $\hat X = X^i \big(\frac{\partial}{\partial q^i}\big)_{\hat\gamma} + \Gamma^A \big(\frac{\partial}{\partial z^A}\big)_{\hat\gamma}$ splits into the sum

$$\hat X \,=\, \mathcal{P}_{\mathcal{H}}(\hat X) \,+\, \mathcal{P}_V(\hat X) \,=\, h(X) \,+\, Y \tag{1.5.24}$$

with

$$X = \pi_*(X) \quad , \qquad Y = \mathcal{P}_V(\hat{X}) = \left(\Gamma^A - h_i{}^A X^i\right) \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}$$

On the other hand, on account of equation (1.5.13), the variational equation (1.5.8) is mathematically equivalent to the relation

$$\frac{dX^i}{dt} - \tilde{\partial}_k(\psi^i)\, X^k \;=\; \left(-\, h_k{}^A X^k + \Gamma^A\right) \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}}$$

Recalling equations (1.5.21 b), (1.5.22 a), (1.5.24), as well as the representation (1.3.14) of the homomorphism $V(\hat{\gamma}) \xrightarrow{\hat{\varrho}} V(\gamma)$, the latter may be written synthetically as

$$\frac{DX}{Dt} \;=\; \hat{\varrho}\big(Y\big) \;=\; \hat{\varrho}\big(\mathcal{P}_V(\hat{X})\big) \tag{1.5.25 a}$$

or also, setting $X = X^a e_{(a)}$, $Y = Y^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}$, and expressing everything in components in the basis $e_{(a)}$

$$\frac{dX^a}{dt} = \left\langle e^{(a)},\, \hat{\varrho}(Y) \right\rangle \;=\; e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} Y^A \tag{1.5.25 b}$$

Exactly as its original counterpart (1.5.8), equation (1.5.25 a) points out that every infinitesimal deformation X is determined by the knowledge of a vertical vector field $Y = Y^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}$ through the solution of a well posed Cauchy problem.

As we noticed earlier, the advantage is that, in the newer formulation, all quantities have a precise geometrical meaning relative to the horizontal distribution $\mathcal{H}(\hat{\gamma})$ induced by the infinitesimal control h. On the other hand, one should not overlook the fact that, in the standard formulation of the problem, no distinguished section $h : V(\gamma) \to A(\hat{\gamma})$ is provided, and none is needed in order to formulate the results. In this respect, the infinitesimal control h plays the role of a *gauge field*, useful for covariance purposes, but unaffecting the evaluation of the extremals. Accordingly, in the subsequent analysis we shall employ h as a user–defined object, eventually checking the invariance of the results under arbitrary changes $h \to h'$.

1.5.3 Corners

In order to address a more and more vast class of problems, we actually shall not deal with sections in the ordinary sense but with *piecewise differentiable evolutions*, defined on *closed* intervals. To account for this aspect, we adopt the following standard terminology:

- an admissible closed arc $\bigl(\gamma,[a,b]\bigr)$ in $\mathcal{V}_{n+1}$ is the restriction to a closed interval $[a,b]$ of an admissible section $\gamma:(c,d)\to\mathcal{V}_{n+1}$ defined on some open interval $(c,d)\supset[a,b]$;

- a piecewise differentiable evolution of the system in the interval $[t_0,t_1]$ is a finite collection

$$\bigl(\gamma,[t_0,t_1]\bigr):=\bigl\{\bigl(\gamma^{(s)},[a_{s-1},a_s]\bigr),\ s=1,\ldots,N,\ t_0=a_0<a_1<\cdots<a_N=t_1\bigr\}$$

  of admissible closed arcs satisfying the matching conditions

$$\gamma^{(s)}(a_s)\;=\;\gamma^{(s+1)}(a_s)\qquad\forall\,s=1,\ldots,N-1 \tag{1.5.26}$$

On account of equation (1.5.26), the image $\gamma(t)$ is well defined and continuous for all $t_0\leqslant t\leqslant t_1$, thus allowing to regard the map $\gamma:[t_0,t_1]\to\mathcal{V}_{n+1}$ as a section in a broad sense. The points $\gamma(t_0)$, $\gamma(t_1)$ are called the end–points of γ, while the points $c_s:=\gamma(a_s)$, $s=1,\ldots,N-1$ are called the *corners* of γ.

Consistently with the stated definitions, the lift of an admissible closed arc $\bigl(\gamma,[a,b]\bigr)$ is the restriction to $[a,b]$ of the lift $\hat\gamma:(c,d)\to\mathcal{A}$, while the lift $\hat\gamma$ of a piecewise differentiable evolution $\bigl\{\bigl(\gamma^{(s)},[a_{s-1},a_s]\bigr)\bigr\}$ is the family of lifts $\hat\gamma^{(s)}$, each restricted to the interval $[a_{s-1},a_s]$. The image $\hat\gamma(t)$ is well defined for all $t\neq a_1,\ldots,a_{N-1}$, thus allowing to regard $\hat\gamma:[t_0,t_1]\to\mathcal{A}$ as a (generally discontinuous) section of the velocity space. In particular, since the map $i:\mathcal{A}\to j_1(\mathcal{V}_{n+1})$ is an imbedding of $\mathcal{A}$ into an *affine* bundle over $\mathcal{V}_{n+1}$, each difference

$$\bigl[\hat\gamma\bigr]_{a_s}=i\bigl(\hat\gamma^{(s+1)}(a_s)\bigr)-i\bigl(\hat\gamma^{(s)}(a_s)\bigr),\quad s=1,\ldots,N-1$$

identifies a vertical vector in $T_{c_s}(\mathcal{V}_{n+1})$, henceforth called the *jump* of $\hat\gamma$ at the corner c_s.

In local coordinates, setting $q^i(\gamma^{(s)}(t)):=q^{\,i}_{(s)}(t)$, equations (1.2.5), (1.5.26) provide the representation

$$\bigl[\hat\gamma\bigr]_{a_s}=\left(\left(\frac{dq^{\,i}_{(s+1)}}{dt}\right)_{a_s}-\left(\frac{dq^{\,i}_{(s)}}{dt}\right)_{a_s}\right)\left(\frac{\partial}{\partial q^i}\right)_{c_s}=\Bigl[\psi^i(\hat\gamma)\Bigr]_{a_s}\left(\frac{\partial}{\partial q^i}\right)_{c_s} \tag{1.5.27}$$

with $\bigl[\psi^i(\hat\gamma)\bigr]_{a_s}:=\psi^i(\hat\gamma^{(s+1)}(a_s))-\psi^i(\hat\gamma^{(s)}(a_s))$ denoting the jump of the function $\psi^i(\hat\gamma(t))$ at $t=a_s$.

Pursuing the generalization process, an admissible deformation of an admissible closed arc $\bigl(\gamma,[a,b]\bigr)$ is a 1–parameter family $\bigl(\gamma_\xi,[a(\xi),b(\xi)]\bigr)$, $|\xi|<\varepsilon$, of admissible closed arcs depending continuously on ξ and satisfying the condition $\bigl(\gamma_0,[a(0),b(0)]\bigr)=\bigl(\gamma,[a,b]\bigr)$. Notice that the definition explicitly includes possible variations of the reference intervals $[a(\xi),b(\xi)]$.

In a similar way, an admissible deformation of a piecewise differentiable evolution $\big(\gamma,[t_0,t_1]\big)$ is a collection $\big\{\big(\gamma_\xi^{(s)},[a_{s-1}(\xi),a_s(\xi)]\big)\big\}$ of deformations of the various arcs, satisfying the matching conditions

$$\gamma_\xi^{(s)}(a_s(\xi)) \,=\, \gamma_\xi^{(s\,+\,1)}(a_s(\xi)) \qquad \forall\,|\xi|<\varepsilon\,,\ s=1,\ldots,N-1 \tag{1.5.28}$$

Under the stated circumstances, the lifts $\hat\gamma_\xi$ and $\hat\gamma_\xi^{(s)}$, respectively restricted to the intervals $[a(\xi),b(\xi)]$ and $[a_{s-1}(\xi),a_s(\xi)]$ are easily recognized to provide deformations for the lifts $\hat\gamma:[a,b]\to\mathcal{A}$ and $\hat\gamma^{(s)}:[a_{s-1},a_s]\to\mathcal{A}$.

Unless otherwise stated, we shall only consider deformations leaving the interval $[t_0,t_1]$ fixed, namely those satisfying the conditions $a_0(\xi)\equiv t_0\,,\ a_N(\xi)\equiv t_1$. No restriction will be posed on the functions $a_s(\xi)$, $s=1,\ldots,N-1$.

Each curve $c_s(\xi):=\gamma_\xi(a_s(\xi))$ will be called the *orbit* of the corner c_s under the given deformation.

In local coordinates, setting $q^i(\gamma_\xi^{(s)}(t))\,=\,\varphi^{\,i}_{\,(s)}(\xi,t)$, the matching conditions (1.5.26) read

$$\varphi^{\,i}_{\,(s)}(\xi,a_s(\xi))\,=\,\varphi^{\,i}_{\,(s\,+\,1)}(\xi,a_s(\xi)) \tag{1.5.29}$$

while the representation of the orbit $c_s(\xi)$ takes the form

$$c_s(\xi):\quad t\,=\,a_s(\xi)\,,\quad q^i\,=\,\varphi^{\,i}_{\,(s)}(\xi,a_s(\xi)) \tag{1.5.30}$$

The previous arguments are naturally reflected into the definition of the infinitesimal deformations. Thus, an admissible infinitesimal deformation of an admissible closed arc $\big(\gamma,[a,b]\big)$ is a triple (α,X,β), where X is the restriction to $[a,b]$ of an admissible infinitesimal deformation of $\gamma:(c,d)\to\mathcal{V}_{n+1}$, while α,β are the derivatives

$$\alpha\,=\,\frac{da}{d\xi}\bigg|_{\xi=0}\,,\qquad \beta\,=\,\frac{db}{d\xi}\bigg|_{\xi=0} \tag{1.5.31}$$

expressing the *speed of variation* of the interval $[a(\xi),b(\xi)]$ at $\xi=0$.

Likewise, an admissible infinitesimal deformation of a piecewise differentiable evolution $\big(\gamma,[t_0,t_1]\big)$ is a collection $\big\{\cdots\alpha_{s-1}\,,X_{(s)}\,,\alpha_s\cdots\big\}$ of admissible infinitesimal deformations of each single closed arc, with $\alpha_s=\frac{da_s}{d\xi}\big|_{\xi=0}$, and, in particular, with $\alpha_0=\alpha_N=0$ whenever the interval $[t_0,t_1]$ is held fixed.

At the same time, whenever a corner c_s is shifted by the deformation process, the tangent vector to $c_s(\xi)$ is given by

$$W_{(s)}\;=\;\left[\big(c_s(\xi)\big)_*\,\frac{d}{d\xi}\right]_{\xi=0}\;=\;\alpha_s\left(\frac{\partial}{\partial t}\right)_{c_s}\;+\;\Big(\alpha_s\psi^i\,+\,X^i\Big)_{c_s}\left(\frac{\partial}{\partial q^i}\right)_{c_s} \tag{1.5.32}$$

The quantities $\alpha_s, X^i_{(s)}, Z^i_{(s)}$ aren't actually independent: equations (1.5.29) imply the identities

$$\frac{\partial\varphi^{\,i}_{(s)}}{\partial\xi} + \frac{\partial\varphi^{\,i}_{(s)}}{\partial t}\,\frac{da_s}{d\xi} = \frac{\partial\varphi^{\,i}_{(s+1)}}{\partial\xi} + \frac{\partial\varphi^{\,i}_{(s+1)}}{\partial t}\,\frac{da_s}{d\xi} \quad ; \tag{1.5.33a}$$

$$\frac{\partial^2\varphi^{\,i}_{(s)}}{\partial\xi^2} + 2\,\frac{\partial^2\varphi^{\,i}_{(s)}}{\partial t\partial\xi}\,\frac{da_s}{d\xi} + \frac{\partial^2\varphi^{\,i}_{(s)}}{\partial t^2}\left(\frac{da_s}{d\xi}\right)^2 + \frac{\partial\varphi^{\,i}_{(s)}}{\partial t}\,\frac{d^2a_s}{d\xi^2} =$$
$$= \frac{\partial^2\varphi^{\,i}_{(s+1)}}{\partial\xi^2} + 2\,\frac{\partial^2\varphi^{\,i}_{(s+1)}}{\partial t\partial\xi}\,\frac{da_s}{d\xi} + \frac{\partial^2\varphi^{\,i}_{(s+1)}}{\partial t^2}\left(\frac{da_s}{d\xi}\right)^2 + \frac{\partial\varphi^{\,i}_{(s+1)}}{\partial t}\,\frac{d^2a_s}{d\xi^2} \tag{1.5.33b}$$

From these, evaluating everything at $\xi = 0$, recalling definitions (1.5.4) and introducing the notation $\beta_s = \frac{d^2a_s}{d\xi^2}\Big|_{\xi=0}$, we get the *jump relations*

$$\left(X^{\,i}_{(s+1)} - X^{\,i}_{(s)}\right)_{a_s} = -\alpha_s\left(\frac{dq^{\,i}_{(s+1)}}{dt} - \frac{dq^{\,i}_{(s)}}{dt}\right)_{a_s} = -\alpha_s\left[\psi^i(\hat\gamma)\right]_{a_s} \tag{1.5.34a}$$

$$\left(Z^{\,i}_{(s+1)} - Z^{\,i}_{(s)}\right)_{a_s} = 2\,\alpha_s\left[\frac{dX^i_{(s)}}{dt} - \frac{dX^i_{(s+1)}}{dt}\right]_{a_s} + \beta_s\left[\frac{dq^i_{(s)}}{dt} - \frac{dq^i_{(s+1)}}{dt}\right]_{a_s} +$$
$$+\,\alpha_s^2\left[\frac{d^2q^i_{(s)}}{dt^2} - \frac{d^2q^i_{(s+1)}}{dt^2}\right]_{a_s} \tag{1.5.34b}$$

whence also, in view of the variational equation (1.5.8),

$$\left(Z^{\,i}_{(s+1)} - Z^{\,i}_{(s)}\right)_{a_s} = 2\,\alpha_s\left[\left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma^{(s)}} X^k_{(s)} - \left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma^{(s+1)}} X^k_{(s+1)} + \right.$$
$$\left. + \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} \Gamma^A_{(s)} - \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s+1)}} \Gamma^A_{(s+1)}\right]_{a_s} +$$
$$+\,\beta_s\left[\psi^i{}_{|\hat\gamma^{(s)}} - \psi^i{}_{|\hat\gamma^{(s+1)}}\right]_{a_s} + \alpha_s^2\left[\frac{d\psi^i{}_{|\hat\gamma^{(s)}}}{dt} - \frac{d\psi^i{}_{|\hat\gamma^{(s+1)}}}{dt}\right]_{a_s} \tag{1.5.34c}$$

Moreover, the admissibility of each single infinitesimal deformation $X_{(s)}$ requires the existence of a corresponding lift $\hat X_{(s)} = X^{\,i}_{(s)}\left(\frac{\partial}{\partial q^i}\right)_{\hat\gamma^{(s)}} + \Gamma^A_{(s)}\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma^{(s)}}$ satisfying the variational equation (1.5.8).

Both aspects are conveniently accounted for by the assignment to each $\gamma^{(s)}$ of an (arbitrarily chosen) infinitesimal control $h^{(s)} : V(\gamma^{(s)}) \to A(\hat\gamma^{(s)})$. In this way, proceeding as in §1.5.2 and denoting by $\left(\frac{D}{Dt}\right)_{\gamma^{(s)}}$ the absolute time derivative along $\gamma^{(s)}$ induced by $h^{(s)}$, we get the following

Proposition 1.5.3. *Every admissible infinitesimal deformation of an admissible evolution* $\big(\gamma,[t_0,t_1]\big)$ *over a fixed interval* $[t_0,t_1]$ *is determined, up to initial data, by a collection of vertical vector fields* $\big\{Y_{(s)}=Y^A_{(s)}\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma^{(s)}}\big\}$, $s=1,\dots,N$ *and by* $N-1$ *real numbers* $\alpha_1,\dots,\alpha_{N-1}$ *through the covariant variational equations*

$$\left(\frac{DX_{(s)}}{Dt}\right)_{\gamma^{(s)}} = \hat\varrho\,(Y_{(s)}) = Y^A_{(s)}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}}\left(\frac{\partial}{\partial q^i}\right)_{\gamma^{(s)}} \qquad s=1,\dots,N \tag{1.5.35}$$

completed with the jump conditions (1.5.34a)*. The lift of the deformation is described by the family of vector fields*

$$\hat X_{(s)} = h^{(s)}(X_{(s)}) + Y_{(s)}, \qquad s=1,\dots,N \tag{1.5.36}$$

The proof is entirely straightforward, and is left to the reader. Introducing n piecewise differentiable vector fields $\tilde\partial_1,\dots,\tilde\partial_n$ along $\hat\gamma$ according to the prescription

$$\tilde\partial_i(t) = h^{(s)}\left(\frac{\partial}{\partial q^i}\right)_{\gamma^{(s)}(t)} \qquad \forall\, t\in(a_{s-1},a_s)\,,\quad s=1,\dots,N$$

equation (1.5.36) takes the explicit form

$$\hat X_{(s)} = h^{(s)}\left(X^{\,i}_{(s)}\left(\frac{\partial}{\partial q^i}\right)_{\gamma^{(s)}}\right) + Y_{(s)} = X^{\,i}_{(s)}\,\tilde\partial_i + Y^A_{(s)}\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma^{(s)}} \tag{1.5.37}$$

on each open arc $\hat\gamma^{(s)}:(a_{s-1},a_s)\to\mathcal{A}$.

To discuss the implications of equation (1.5.35), resuming the notation $V(\gamma)$ for the totality of vertical vectors along γ[4], we define a transport law in $V(\gamma)$, henceforth called h–transport, gluing $h^{(s)}$–transport along each arc $\big(\gamma^{(s)},[a_{s-1},a_s]\big)$ and continuity at the corners, namely continuity of the components at $t=a_s$.

In view of Proposition 1.5.2, the h–transported fields form an n–dimensional vector space V_h, isomorphic to each fibre $V(\gamma)_{|t}$. This provides a canonical identification of $V(\gamma)$ with the cartesian product $[t_0,t_1]\times V_h$, thus allowing to regard every section $X:[t_0,t_1]\to V(\gamma)$ as a vector valued function $X:[t_0,t_1]\to V_h$.

Exactly as in § 1.5.2, the situation is formalized referring V_h to a basis $\{e_{(a)}\}$ related to the basis $\left(\frac{\partial}{\partial q^i}\right)_\gamma$ by the transformation

$$\left(\frac{\partial}{\partial q^i}\right)_\gamma = e^{(a)}_{\ i}(t)\, e_{(a)}\,, \qquad e_{(a)} = e^{\ i}_{(a)}(t)\left(\frac{\partial}{\partial q^i}\right)_\gamma \tag{1.5.38}$$

Given any admissible infinitesimal deformation $\big\{\big(X_{(s)},[a_{s-1},a_s]\big)\big\}$, we now glue all sections $X_{(s)}:[a_{s-1},a_s]\to V(\gamma^{(s)})$ into a single, piecewise differentiable

[4] Notice that this makes perfectly good sense also at the corners $\gamma(a_s)$.

function $X : [t_0, t_1] \to V_h$, with jump discontinuities at $t = a_s$ expressed in components by equation (1.5.34a). For each $s = 1, \ldots, N$ this provides the representation

$$X_{(s)} = X^a(t)\, e_{(a)}\,, \qquad \left(\frac{DX_{(s)}}{Dt}\right)_{\gamma^{(s)}} = \frac{dX^a}{dt}\, e_{(a)} \qquad \forall\, t \in (a_{s-1}, a_s) \tag{1.5.39}$$

In a similar way, we collect all fields $Y_{(s)}$ into a single object Y, henceforth conventionally called a vertical vector field along $\hat{\gamma}$.

By abuse of language, we also denote by $Y = Y^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}$ the vector field along the open arcs of $\hat{\gamma}$ defined by the prescription

$$Y^A(t) \,=\, Y^A_{(s)}(t) \qquad a_{s-1} < t < a_s\,, \quad s = 1, \ldots, N \tag{1.5.40}$$

In this way, the covariant variational equation (1.5.35) takes the form

$$\frac{dX^a}{dt} \,=\, Y^A\, e^{(a)}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \qquad \forall\, t \neq a_s \tag{1.5.41a}$$

completed with the jump conditions

$$\big[X^a\big]_{a_s} = \big[X^i\big]_{a_s} e^{(a)}_i(a_s) = -\alpha_s\, e^{(a)}_i(a_s) \Big[\psi^i(\hat{\gamma})\Big]_{a_s} \quad s = 1, \ldots, N-1 \tag{1.5.41b}$$

1.5.4 The abnormality index

A deeper insight into the algorithm discussed in §1.5.3 is gained denoting by $\mathfrak{V}$ the infinite dimensional vector space formed by the totality of vertical vector fields $Y = \big\{Y_{(s)},\, s = 1, \ldots, N\big\}$ along $\hat{\gamma}$, and setting $\mathfrak{W} := \mathfrak{V} \oplus \mathbb{R}^{N-1}$. On account of equations (1.5.41a,b), every admissible infinitesimal deformation of γ is then determined, up to initial data, by an element $(Y, \alpha_1, \ldots, \alpha_{N-1}) \in \mathfrak{W}$.

In the following we shall be mainly interested in infinitesimal deformations $X : [t_0, t_1] \to V(\gamma)$ vanishing at the end–points. Setting $X(t_0) = 0$, equations (1.5.41a,b) provide the evaluation

$$X(t) = \left(\int_{t_0}^{t} Y^A\, e^{(a)}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} dt \,-\, \sum_{a_s < t} \alpha_s\, e^{(a)}_i(a_s) \Big[\psi^i(\hat{\gamma})\Big]_{a_s} \right) e_{(a)} \tag{1.5.42}$$

The vanishing of both $X(t_0)$ and $X(t_1)$ is therefore expressed by the condition

$$\left(\int_{t_0}^{t_1} Y^A\, e^{(a)}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} dt \,-\, \sum_{s=1}^{N-1} \alpha_s\, e^{(a)}_i(a_s) \Big[\psi^i(\hat{\gamma})\Big]_{a_s} \right) e_{(a)} \,=\, 0 \tag{1.5.43}$$

The left hand side of equation (1.5.43) defines a linear map $\Upsilon : \mathfrak{W} \to V_h$ whose kernel is therefore isomorphic to the vector space of the admissible infinitesimal deformations vanishing at the end–points of γ.

Depending on the nature of the inclusion $\Upsilon(\mathfrak{W}) \subset V_h$, the evolutions of the system will be classified into *normal*, when $\Upsilon(\mathfrak{W}) = V_h$, and *abnormal*, when $\Upsilon(\mathfrak{W}) \subsetneq V_h$ [5].

The dimension of the annihilator $\big(\Upsilon(\mathfrak{W})\big)^0 \subset V_h^*$ will be called the *abnormality index* of γ.

On this point, a useful characterization is provided by the following

Proposition 1.5.4. *The annihilator $\big(\Upsilon(\mathfrak{W})\big)^0 \subset V_h^*$ coincides with the totality of h–transported virtual 1–forms $\hat\lambda = \lambda_i\,\hat\omega^i$ satisfying the conditions*

$$\lambda_i \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma} = 0 \qquad A = 1,\dots,r \tag{1.5.44a}$$

$$\lambda_i(a_s)\big[\psi^i(\hat\gamma)\big]_{a_s} = 0 \qquad s = 1,\dots,N-1 \tag{1.5.44b}$$

Proof. In view of equation (1.5.43), the subspace $\big(\Upsilon(\mathfrak{W})\big)^0 \subset V_h^*$ consists of the totality of elements $\hat\lambda = \lambda_a\, e^{(a)} = \lambda_a\, e^{(a)}_{\ i}\, \hat\omega^i$ satisfying the relation

$$\lambda_a \left(\int_{t_0}^{t_1} Y^A\, e^{(a)}_{\ i} \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma} dt \;-\; \sum_{s=1}^{N-1} \alpha_s\, e^{(a)}_{\ i}(a_s) \Big[\psi^i(\hat\gamma)\Big]_{a_s} \right) = 0$$

$\forall\,(Y, \alpha_1, \dots, \alpha_{N-1}) \in \mathfrak{W}$, clearly equivalent to equations (1.5.44a,b). $\square$

By equations (1.5.21b), (1.5.22b), the condition of h–transport of $\hat\lambda$ along each arc $\gamma^{(s)}$ is expressed in coordinates as

$$\frac{d\lambda_i}{dt} + \lambda_k \left(\frac{\partial\psi^k}{\partial q^i}\right)_{\hat\gamma} + h_i^{\ A}\, \cancel{\lambda_k \left(\frac{\partial\psi^k}{\partial z^A}\right)_{\hat\gamma}} = 0 \tag{1.5.45}$$

the cancellation arising from the requirement (1.5.44a).

The content of Proposition 1.5.4 is therefore independent of the choice of the infinitesimal controls $h^{(s)} : V(\gamma^{(s)}) \to A(\hat\gamma^{(s)})$.

Remark 1.5.3: According to Proposition 1.5.4, the abnormality index of a piecewise differentiable section γ cannot exceed the abnormality index of each single arc $\gamma^{(s)}$. Thus, for example, if one of the arcs is normal, γ is necessarily normal. More generally, because of the additional restrictions posed by equations (1.5.44b) and by the continuity requirements $[\hat\lambda]_{a_s} = 0$, an evolution may happen to be normal even if *all* its arcs $\gamma^{(s)}$ are abnormal. Typical examples are:

[5] As we shall see, when applied to the extremals of an action functional, this terminology agrees with the current one (see, among others, [10] and references therein).

- $\mathcal{V}_{n+1} = \mathbb{R} \times E_2$, referred to coordinates t, x, y. Constraint: $\dot{x}^2 + \dot{y}^2 = v^2$. Imbedding $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$ expressed in coordinates as $\dot{x} = v\cos z$, $\dot{y} = v \sin z$. Piecewise differentiable evolution γ consisting of two arcs:

$$\begin{aligned} \gamma^{(1)}: &\quad x = 0, \qquad y = vt \qquad t_0 \le t \le 0 \\ \gamma^{(2)}: &\quad x = vt, \qquad y = 0 \qquad 0 \le t \le t_1 \end{aligned}$$

  Equation (1.5.44a) admits h–transported solutions $\hat{\lambda}^{(1)} = \alpha\,\hat{\omega}^2$ along $\gamma^{(1)}$ and $\hat{\lambda}^{(2)} = \beta\,\hat{\omega}^1$ along $\gamma^{(2)}$, $\forall\, \alpha, \beta \in \mathbb{R}$. Both arcs are therefore abnormal. Notwithstanding, γ is normal, since no pair $\hat{\lambda}^{(1)}, \hat{\lambda}^{(2)}$ matches into a continuous non–null virtual 1–form along γ.

- $\mathcal{V}_{n+1} = \mathbb{R} \times E_2$. Coordinates t, x, y. Constraint: $v^3\dot{x} = (\dot{y}^2 - a^2t^2)^2$. Imbedding $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$ expressed in coordinates as $\dot{x} = v^{-3}(z^2 - a^2t^2)^2$, $\dot{y} = z$. Piecewise differentiable evolution γ consisting of two arcs:

$$\begin{aligned} \gamma^{(1)}: &\quad x = 0, && y = \frac{1}{2}\,a(t^2 - t^{*2}) && t_0 \le t \le t^* \\ \gamma^{(2)}: &\quad x = \frac{a^4}{5v^3}\,(t^5 - t^{*5}), && y = 0 && t^* \le t \le t_1 \end{aligned}$$

  $(t^* \neq 0)$. Equation(1.5.44a) admits h–transported solutions of the form $\hat{\lambda} = \alpha\hat{\omega}^1$ along the whole of γ. Both arcs $\gamma^{(1)}$, $\gamma^{(2)}$ are therefore abnormal. Notwithstanding, γ is normal, since no solution satisfies condition (1.5.44b).

Remark 1.5.4: Even in the differentiable case, the normality of an evolution γ is a *global* property. In this sense, a normal arc $\gamma\colon [t_0, t_1] \to \mathcal{V}_{n+1}$ may happen to be abnormal when restricted to a subinterval $[t_0^*, t_1^*] \subset [t_0, t_1]$. An illustrative example may be given by means of a bump function:

- $\mathcal{V}_{n+1} = \mathbb{R} \times E_3$. Coordinates t, q^1, q^2, q^3. Imbedding $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$ expressed in coordinates as $\dot{q}^1 = z^1$, $\dot{q}^2 = z^2$, $\dot{q}^3 = g(t)z^2$, being $g(t)$ a C^∞ function defined as $g(t) := -\frac{2t}{(t^2-1)^2}\,e^{\frac{1}{t^2-1}}$ for any $|t| < 1$ and $g(t) := 0$ otherwise. Differentiable evolution γ consisting of the single arc:

$$\gamma: \quad q^1 = vt^2\ , \quad q^2 = vt\ , \quad q^3 = vf(t) \qquad t_0 \leqslant t \leqslant t_1\ , \quad t_0 < -1\ ,\ t_1 > 1$$

  being

$$f(t) := \begin{cases} e^{\frac{1}{t^2-1}} & |t| < 1 \\ 0 & |t| \geqslant 1 \end{cases}$$

  For any $\alpha \in \mathbb{R}$, equation(1.5.44a) admits therefore h–transported solutions of the form $\hat{\lambda} = \alpha\hat{\omega}^3$ when restricted to the subinterval $[t_0, -1]$. Notwithstanding, γ is normal, since no solution may be found along the whole of it.

In view of the contents of Remark 1.5.4, an evolution $\gamma\colon [t_0, t_1] \to \mathcal{V}_{n+1}$ will be called *locally normal* if its restriction to any closed subinterval $[t_0^*, t_1^*] \subseteq [t_0, t_1]$ is a normal arc, namely if and only if, along any such subinterval, equations (1.5.44) admit the one trivial solution $\lambda_i(t) = 0$.

As a concluding remark, it's worth pointing out that, although geometrically significant, the arguments discussed so far provide only a partial picture of the situation. Actually, rather than the totality of admissible infinitesimal deformations vanishing at the end–points — here identified with the kernel of the map $\Upsilon : \mathfrak{W} \to V_h$ — a variational context involves the (possibly smaller) subfamily $\mathfrak{X}$ of infinitesimal deformations tangent to admissible *finite* deformations with fixed end–points.

The linear span of $\mathfrak{X}$, henceforth denoted by $\Delta(\gamma)$, will be called the *variational space* of γ. The evolutions of the system will be classified into *ordinary*, when $\Delta(\gamma) = \ker(\Upsilon)$ and *exceptional*, when $\Delta(\gamma) \subsetneq \ker(\Upsilon)$.

A hierarchy between the various typologies is provided by the following

Proposition 1.5.5. *The normal evolutions form a subset of the ordinary ones.*

The result is proved in Appendix B. In this connection, see also [27].

Chapter 2

The first variation

2.1 Problem statement

Let $\mathscr{L} \in \mathscr{F}(\mathcal{A})$ denote a differentiable function on the velocity space $\mathcal{A}$, henceforth called the *Lagrangian*. Also, let $\big(\gamma, [t_0, t_1]\big)$ (γ for short) denote an admissible piecewise differentiable evolution of the system, defined on a closed interval $[t_0, t_1] \subset \mathbb{R}$. Indicating by $\hat{\gamma}$ the lift of γ to $\mathcal{A}$, define the *action functional*

$$\mathcal{I}[\gamma] := \int_{\hat{\gamma}} \mathscr{L}\, dt := \sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} \big(\hat{\gamma}^{(s)}\big)^* (\mathscr{L})\, dt \tag{2.1.1}$$

As it was already outlined in the Introduction, the problem we intend to deal with is the one of characterizing, among all the admissible evolutions γ connecting a given pair of points in $\mathcal{V}_{n+1}$, the ones (if any) which *minimize*[1] the functional (2.1.1). More precisely, recalling Definition 1.5.1, we state the following

Definition 2.1.1. *An evolution $\big(\gamma, [t_0, t_1]\big)$ is called a* weak local minimum *for the functional (2.1.1) if there is a neighborhood $\mathcal{N}_{(\varepsilon,1)}(\gamma)$ of γ, such that $\mathcal{I}[\gamma] \leqslant \mathcal{I}[\gamma']$ for all admissible piecewise differentiable $\gamma' \in \mathcal{N}_{(\varepsilon,1)}(\gamma)$ joining the end–points of γ. The evolution γ is likewise called a* strong local minimum *for the functional (2.1.1) if all previous properties hold, with $\mathcal{N}_{(\varepsilon,1)}(\gamma)$ systematically replaced by $\mathcal{N}_{(\varepsilon,0)}(\gamma)$.*

As a direct result of Definitions 1.5.1, 2.1.1, we see that every strong extremum is also a weak one while the converse is generally false. Therefore, once the necessary and sufficient conditions for a weak minimum will have been found out, it will be possible to try to supplement them in such a way as to guarantee a strong minimum as well. However, this will not be carried out in the present work.

[1] For the sake of explicitness, we shall consider only conditions for a minimum. In order to obtain the conditions for a maximum, it is only needed to reverse the direction of all inequalities.

Given an admissible evolution γ, we keep in line with Definition 2.1.1 by considering all weak deformations γ_ξ with fixed end–points.

The first step for the solution of the problem is now to study the stationarity conditions for the functional (2.1.1), through the analysis of its so–called *first variation*.

Definition 2.1.2. *An admissible evolution γ is called an* extremal *for the functional (2.1.1) if and only if, for all admissible deformations with fixed end–points $\gamma_\xi = \big\{\big(\gamma_\xi^{(s)}, [a_{s-1}(\xi), a_s(\xi)]\big)\big\}$, the function*

$$\mathcal{I}[\gamma_\xi] := \int_{\hat\gamma_\xi} \mathscr{L}\,dt = \sum_{s=1}^{N} \int_{a_{s-1}(\xi)}^{a_s(\xi)} \big(\hat\gamma_\xi^{(s)}\big)^*(\mathscr{L})\,dt$$

has a stationarity point at $\xi = 0$.

Remark 2.1.1 (The gauge group)**:** As it is well known, given any pair of 1–forms $\mathscr{L}\,dt$ and $\mathscr{L}'\,dt$ over $\mathcal{A}$, their respective action integrals $\mathcal{I}[\gamma] = \int_{\hat\gamma} \mathscr{L}\,dt$ and $\mathcal{I}'[\gamma] = \int_{\hat\gamma} \mathscr{L}'\,dt$ give rise to the same extremal curves if the difference $\big(\mathscr{L}' - \mathscr{L}\big)dt$ is an *exact* differential. Under this circumstance, the equality $\oint \mathscr{L}\,dt = \oint \mathscr{L}'\,dt$ holds along any closed curve, thereby entailing the relation

$$\mathcal{I}'[\gamma_\xi] - \mathcal{I}[\gamma_\xi] = \int_{\hat\gamma_\xi} \big(\mathscr{L}' - \mathscr{L}\big)\,dt \equiv \int_{\hat\gamma} \big(\mathscr{L}' - \mathscr{L}\big)\,dt$$

for any deformation γ_ξ vanishing at the end–points, whence also

$$\frac{d}{d\xi}\Big(\mathcal{I}'[\gamma_\xi] - \mathcal{I}[\gamma_\xi]\Big) \equiv 0$$

In this particular sense, as far as a variational problem based on the functional (2.1.1) is concerned, the Lagrangian function $\mathscr{L} \in \mathscr{F}(\mathcal{A})$ is defined up to an equivalence relation of the form

$$\mathscr{L} \sim \mathscr{L}' \qquad \Longleftrightarrow \qquad \mathscr{L}' - \mathscr{L} = \frac{df}{dt}\,, \quad f \in \mathscr{F}(\mathcal{V}_{n+1}) \tag{2.1.2}$$

Otherwise stated, the real information isn't brought so much by $\mathscr{L}$ in itself as by a whole family of Lagrangians, equivalent to each other in the sense expressed by equation (2.1.2).

The significance of the arguments developed in §1.4.2 relies actually on the fact, explicitly pointed out by equations (1.4.16), (1.4.17), that the representation of an arbitrary section $\ell\colon \mathcal{A} \to \mathcal{L}(\mathcal{A})$ involves exactly this family of Lagrangians, henceforth denoted by $\Lambda(\ell)$. A straightforward check shows that a necessary and sufficient condition for two sections ℓ and ℓ' to fulfil $\Lambda(\ell) = \Lambda(\ell')$ is that the difference $\ell' - \ell$, viewed as a function over $\mathcal{A}$, be itself of the form

$$\ell' - \ell = \frac{df}{dt}\,, \qquad f \in \mathscr{F}(\mathcal{V}_{n+1}) \tag{2.1.3}$$

Thus we see that, within our geometrical framework, the equivalence relation (2.1.2) between *functions* is replaced by the almost identical relation (2.1.3) between *sections*. Intuitively, the latter is a sort of "active counterpart" of the transformation law (1.4.17) for the *representation* of a given section ℓ under arbitrary changes of the trivialization $u\colon P \to \mathbb{R}$.

This viewpoint is formalized through the introduction of the concept of *gauge group*[2]. By definition, a gauge transformation of the bundle $P \to \mathcal{V}_{n+1}$ is an isomorphism

$$\begin{array}{ccc} P & \xrightarrow{\;g\;} & P \\ \big\downarrow & & \big\downarrow \\ \mathcal{V}_{n+1} & = & \mathcal{V}_{n+1} \end{array}$$

fibred over the identity map, and *equivariant* with respect to the action of the structural group, namely fulfilling

$$g(\nu+\xi) \,=\, g(\nu)+\xi \qquad\qquad \forall\,\nu\in P\,,\ \xi\in\Re \tag{2.1.4}$$

On the basis of equation (2.1.4), it is easily recognized that the group of gauge transformations over P is in 1-1 correspondence with the ring of differentiable functions over $\mathcal{V}_{n+1}$, the relation $f \to g_f$ being given explicitly by

$$f\in\mathscr{F}(\mathcal{V}_{n+1}) \qquad \Rightarrow \qquad g_f(\nu) \,:=\, \nu + f(\pi(\nu)) \qquad \forall\,\nu\in P \tag{2.1.5}$$

In local coordinates, the action of the map g_f is expressed synthetically as

$$g_f \;:\; (t,q^i,u) \to (t,q^i,u+f)$$

Every gauge transformation (2.1.5) may be lifted in a canonical way to a diffeomorphism $g_{f\,*} : j_1^{\mathcal{A}}(P,\mathbb{R}) \to j_1^{\mathcal{A}}(P,\mathbb{R})$, expressed in coordinates as

$$g_{f\,*} \;:\; (t,q^i,u,z^A,\dot u) \to (t,q^i,u+f,z^A,\dot u+\dot f)$$

From this it is easily seen that the map $g_{f\,*}$ commutes with both group actions (1.4.15a), (1.4.15b), thus inducing maps $\hat g_f \,:\, \mathcal{L}(\mathcal{A}) \to \mathcal{L}(\mathcal{A})$, and $\hat g_f^c \,:\, \mathcal{L}^c(\mathcal{A}) \to \mathcal{L}^c(\mathcal{A})$, expressed symbolically as

$$\begin{aligned} \hat g_f \;&:\; (t,q^i,z^A,\dot u) \to (t,q^i,z^A,\dot u+\dot f) \\ \hat g_f^c \;&:\; (t,q^i,u,z^A) \to (t,q^i,u+f,z^A) \end{aligned}$$

The situation is summarized into the commutative diagrams

$$\begin{array}{ccc} j_1^{\mathcal{A}}(P,\mathbb{R}) & \xrightarrow{\;g_{f\,*}\;} & j_1^{\mathcal{A}}(P,\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{L}(\mathcal{A}) & \xrightarrow{\;\hat g_f\;} & \mathcal{L}(\mathcal{A}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & = & \mathcal{A} \end{array} \qquad\qquad \begin{array}{ccc} j_1^{\mathcal{A}}(P,\mathbb{R}) & \xrightarrow{\;g_{f\,*}\;} & j_1^{\mathcal{A}}(P,\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{L}^c(\mathcal{A}) & \xrightarrow{\;\hat g_f^c\;} & \mathcal{L}^c(\mathcal{A}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & = & \mathcal{A} \end{array}$$

in which all horizontal arrows denote bundle isomorphisms.

It is now an easy matter to verify that equation (2.1.3) is mathematically equivalent to the condition

$$\ell' \,=\, \hat g_f \cdot \ell \tag{2.1.6}$$

The geometrical counterpart of an "equivalence class of Lagrangians" on $\mathcal{A}$ is therefore a section $\ell : \mathcal{A} \to \mathcal{L}(\mathcal{A})$, defined up to the action of the gauge group.

---

[2] See, for example, [4]

2.2 The Pontryagin–Poincaré–Cartan form

To begin with, we focus on the left–hand face of diagram (1.4.28)

$$
\begin{array}{ccc}
\mathcal{S} & \longrightarrow & \mathcal{C}(j_1(\mathcal{V}_{n+1})) \\
{\scriptstyle \pi_{\mathcal{S}}}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\
\mathcal{L}(\mathcal{V}_{n+1}) & \longrightarrow & j_1(\mathcal{V}_{n+1})
\end{array}
\tag{2.2.1}
$$

and we complete the state of the play with the two missing ingredients that are needed to address the problem, namely

- the non–holonomic constraints (sometimes improperly called "the dynamics"), described by the imbedding $i\colon \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ and locally expressed by the equations
$$\dot q^i \,=\, \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r)$$
- the non–holonomic Lagrangian section $\ell\,:\,\dot u = \mathscr{L}(t, q^i, z^A)$.

We next pull–back the diagram (2.2.1) through the imbedding $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$, giving rise to the analogous diagram

$$
\begin{array}{ccc}
\mathcal{S}^{\mathcal{A}} & \longrightarrow & \mathcal{C}(\mathcal{A}) \\
{\scriptstyle \pi_{\mathcal{S}}}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\
\mathcal{L}(\mathcal{A}) & \longrightarrow & \mathcal{A}
\end{array}
\tag{2.2.2}
$$

By construction, the manifold $\mathcal{S}^{\mathcal{A}}$ is then a principal fibre bundle over $\mathcal{C}(\mathcal{A})$ under the (induced) action

$$\phi_\xi \;:\; (t, q^i, z^A, \dot u^i, p_i) \;\longrightarrow\; (t, q^i, z^A, \dot u^i + \xi, p_i) \tag{2.2.3}$$

By means of the pull–back procedure, the canonical form (1.4.32) determines a distinguished 1–form on $\mathcal{S}^{\mathcal{A}}$, locally expressed by[3]

$$\Theta_u \,=\, p_0\, dt + p^i\, dq^i \;\equiv\; \dot u\, dt + p_i\, \big(dq^i - \psi^i\, dt\big) \tag{2.2.4}$$

Every non–holonomic Lagrangian section $\ell\colon \mathcal{A} \to \mathcal{L}(\mathcal{A})$ determines a trivialization $\varphi_\ell\colon \mathcal{L}(\mathcal{A}) \to \mathbb{R}$ of the bundle $\mathcal{L}(\mathcal{A}) \to \mathcal{A}$. Let $\hat\varphi_\ell := \pi^*_{\mathcal{S}}(\varphi_\ell)$ denote the pull–back of φ_ℓ to $\mathcal{S}^{\mathcal{A}}$, locally expressed as

$$\hat\varphi_\ell(t, q^i, z^A, \dot u, p_i) \,=\, \varphi_\ell(t, q^i, z^A, \dot u) \,=\, \dot u - \mathscr{L}(t, q^i, z^A) \tag{2.2.5}$$

[3]Aiming for easiness, the same symbol Θ_u will stand for both the form (1.4.32) and its pull–back on $\mathcal{A}$.

From this, taking equation (2.2.3) into account, it is an easy matter to check that the function $\hat{\varphi}_\ell$ is a trivialization of the bundle $\mathcal{S}^{\mathcal{A}} \to \mathcal{C}(\mathcal{A})$ and that, as such, it determines a section $\tilde{\ell} \colon \mathcal{C}(\mathcal{A}) \to \mathcal{S}^{\mathcal{A}}$, locally described by the equation

$$\dot{u} \,=\, \mathscr{L}(t, q^i, z^A) \tag{2.2.6}$$

In brief, every section $\ell \colon \mathcal{A} \to \mathcal{L}(\mathcal{A})$ may be lifted to a section $\tilde{\ell} \colon \mathcal{C}(\mathcal{A}) \to \mathcal{S}^{\mathcal{A}}$. The local representations of both sections are formally identical and they obey the transformation law (1.4.17) for an arbitrary change of the trivialization $u \colon P \to \mathbb{R}$.

The section $\tilde{\ell} \colon \mathcal{C}(\mathcal{A}) \to \mathcal{S}^{\mathcal{A}}$ may now be used to pull–back the form (2.2.4) onto $\mathcal{C}(\mathcal{A})$, hereby getting the 1–form

$$\Theta_{\text{PPC}} \,:=\, \tilde{\ell}^*(\Theta_u) \,=\, \mathscr{L}\, dt + p_i \left(dq^i - \psi^i\, dt\right) \,:=\, -\mathscr{H}\, dt + p_i\, dq^i \tag{2.2.7}$$

henceforth referred to as the *Pontryagin–Poincaré–Cartan form.*
Needless to say, the difference $\mathscr{H} := p_i\, \psi^i - \mathscr{L}$, known in the literature as the *Pontryagin Hamiltonian*, is not an Hamiltonian in the traditional sense but a function on the contact bundle.

2.3 The Pontryagin's "maximum principle"

To understand the role of the Pontryagin–Poincaré–Cartan form in the solution of the addressed variational problem, we focus on the fibration $\mathcal{C}(\mathcal{A}) \xrightarrow{\upsilon} \mathcal{V}_{n+1}$, given by the composite map $\upsilon := \pi \cdot \hat{\kappa}$. A piecewise differentiable section $\big(\tilde{\gamma}, [t_0, t_1]\big)$ consisting of a finite family of closed arcs

$$\tilde{\gamma}^{(s)} : [a_{s-1}, a_s] \to \mathcal{C}(\mathcal{A})\,, \qquad s = 1, \ldots, N, \quad t_0 = a_0 < a_1 < \cdots < a_N = t_1$$

will be called *υ–continuous* if and only if the composite map $\upsilon \cdot \tilde{\gamma}$ is continuous, namely if and only if $\tilde{\gamma}$ projects onto a continuous, piecewise differentiable section $\upsilon \cdot \tilde{\gamma} : [t_0, t_1] \to \mathcal{V}_{n+1}$. A deformation $\tilde{\gamma}_\xi = \big\{\big(\tilde{\gamma}^{(s)}_\xi, [a_{s-1}(\xi), a_s(\xi)]\big)\big\}$ will similarly be called υ–continuous if and only if all sections $\tilde{\gamma}_\xi$ are υ–continuous. A necessary and sufficient condition for this to happen is the validity of the *matching conditions* (1.5.28), synthetically written as

$$\lim_{t \to a_s^+(\xi)} \upsilon \cdot \tilde{\gamma}_\xi(t) \quad = \lim_{t \to a_s^-(\xi)} \upsilon \cdot \tilde{\gamma}_\xi(t) \qquad s = 1, \ldots, N-1 \tag{2.3.1}$$

A υ–continuous deformation $\tilde{\gamma}_\xi$ is said to preserve the end–points of $\upsilon \cdot \tilde{\gamma}$ if and only if $\upsilon \cdot \tilde{\gamma}_\xi$ is a deformation with fixed end–points. A vector field along $\tilde{\gamma}$ tangent to the orbits of a υ–continuous deformation is called an *infinitesimal deformation*.

Notice that, since the stated definitions do not include any admissibility requirement for the sections $\upsilon \cdot \tilde{\gamma}_\xi$, the only condition needed in order for a vector field $X^i\left(\frac{\partial}{\partial q^i}\right)_{\tilde{\gamma}} + \Gamma^A\left(\frac{\partial}{\partial z^A}\right)_{\tilde{\gamma}} + \Pi_i\left(\frac{\partial}{\partial p_i}\right)_{\tilde{\gamma}}$ to represent an infinitesimal deformation of $\tilde{\gamma}$ is the consistency with the matching conditions (2.3.1), expressed in components by the *jump relations*

$$\lim_{t\to a_s^+(\xi)}\left(X^i + \alpha_s\,\frac{dq^i}{dt}\right) = \lim_{t\to a_s^-(\xi)}\left(X^i + \alpha_s\,\frac{dq^i}{dt}\right) \qquad s = 1,\ldots,N-1 \tag{2.3.2}$$

with $\alpha_s = \left(\frac{da_s}{d\xi}\right)_{\xi=0}$. On the same line as in §1.2, any section $\tilde{\gamma}\colon [t_0,t_1] \to \mathcal{C}(\mathcal{A})$, locally described as

$$q^i = q^i(t), \quad z^A = z^A(t), \quad p_i = p_i(t)$$

and satisfying

$$\frac{dq^i}{dt} = \psi^i\left(t, q^1(t),\ldots,q^n(t), z^1(t),\ldots,z^r(t)\right)$$

will henceforth be called *admissible*.

By means of Θ_{PPC} we now define an action integral over $\mathcal{C}(\mathcal{A})$, assigning to each υ–continuous section $\tilde{\gamma} : q^i = q^i(t),\ z^A = z^A(t),\ p_i = p_i(t)$ the real number

$$\mathcal{I}[\tilde{\gamma}] := \int_{\tilde{\gamma}} \Theta_{\text{PPC}} = \int_{t_0}^{t_1}\left(p_i\,\frac{dq^i}{dt} - \mathscr{H}\right)dt \tag{2.3.3}$$

From the foregoing discussion, it should be clear that two different forms Θ_{PPC} and Θ'_{PPC} linked together by a change of the trivialization u of P give rise to two distinct representations of the same variational problem. In other words, the extremal curves of two variational problems differing by the action of the gauge group project onto the very same curve in $\mathcal{V}_{n+1}$. In this connection, the study of the consequences of both the impositions $\dot{u} = \dot{f}$ and — in an extreme case — $\dot{u} = 0$ gains some relevance.

For any υ–continuous deformations $\tilde{\gamma}_\xi$ preserving the end–points of $\upsilon \cdot \tilde{\gamma}$ we have the relation

$$\left.\frac{d\mathcal{I}[\tilde{\gamma}_\xi]}{d\xi}\right|_{\xi=0} = \int_{t_0}^{t_1}\left[\left(\frac{dq^i}{dt} - \frac{\partial\mathscr{H}}{\partial p_i}\right)\Pi_i - \left(\frac{dp_i}{dt} + \frac{\partial\mathscr{H}}{\partial q^i}\right)X^i - \frac{\partial\mathscr{H}}{\partial z^A}\,\Gamma^A\right]dt\ +$$
$$+\sum_{s=1}^{N}\left\{\lim_{t\to a_s^-}\left[\alpha_s\left(p_i\,\frac{dq^i}{dt} - \mathscr{H}\right) + p_i X^i\right] - \lim_{t\to a_{s-1}^+}\left[\alpha_{s-1}\left(p_i\,\frac{dq^i}{dt} - \mathscr{H}\right) + p_i X^i\right]\right\}$$

From the latter, taking equations (2.3.1) and the conditions $X^i(t_0) = X^i(t_1) = 0$ into account, we conclude that the vanishing of $\left.\frac{d\mathcal{I}}{d\xi}\right|_{\xi=0}$ under arbitrary deformations of the given class is mathematically equivalent to the system

$$\frac{dq^i}{dt} = \frac{\partial \mathscr{H}}{\partial p_i} = \psi^i(t, q^i, z^A) \tag{2.3.4a}$$

$$\frac{dp_i}{dt} = -\frac{\partial \mathscr{H}}{\partial q^i} = -p_k \frac{\partial \psi^k}{\partial q^i} + \frac{\partial \mathscr{L}}{\partial q^i} \tag{2.3.4b}$$

$$\frac{\partial \mathscr{H}}{\partial z^A} = p_i \frac{\partial \psi^i}{\partial z^A} - \frac{\partial \mathscr{L}}{\partial z^A} = 0 \tag{2.3.4c}$$

completed with the continuity conditions

$$\big[p_i\big]_{a_s} = \big[\mathscr{H}\big]_{a_s} = 0 \qquad s = 1, \ldots, N-1 \tag{2.3.4d}$$

where, as usual, we are denoting by $[f]_{a_s}$ the jump of the function $f(t)$ at $t = a_s$.

Equation (2.3.4a) shows that the extremal curves for the functional (2.3.3) are admissible. Therefore, whenever any of them is concerned, we have the identification

$$\mathcal{I}[\tilde{\gamma}] := \int_{\tilde{\gamma}} \Theta_{\text{PPC}} = \int_{t_0}^{t_1} \left[\mathscr{L} + p_i \cancel{\left(\frac{dq^i}{dt} - \psi^i \right)} \right] dt = \int_{t_0}^{t_1} \mathscr{L}\left(t, q^i(t), z^A(t)\right) dt \tag{2.3.5}$$

Moreover, their being extremals with respect to arbitrary deformations vanishing at the end–points automatically makes them extremals with respect to the narrower class of admissible deformations as well. As a consequence, we can state that every "free" extremal for the functional (2.3.3) gives rise to an extremal $\gamma\colon\ q^i = q^i(t)$ of the original problem.

Conversely, it is a much more awkward matter to establish if and under which hypotheses an admissible evolution γ is an extremal for the functional (2.1.1) which can be obtained from an extremal $\tilde{\gamma}$ for the functional (2.3.3). Heuristically, the variational problem (2.3.3) can be viewed as the study of the functional (2.1.1) in which the kinematical admissibility condition (1.2.5) plays no more the role of an *a priori* request upon sections but it is retrieved afterwards by the method of Lagrange multipliers. It is therefore reasonable that, under suitable hypotheses, one can prove the equivalence between the variational problem in $\mathcal{A}$ and the one in $\mathcal{C}(\mathcal{A})$. Let us investigate this point.

Given an admissible piecewise differentiable evolution γ, denoting by $\hat{X}_{(s)}$ the infinitesimal deformation associated with each single $\hat{\gamma}^{(s)}_{\xi}$ and recalling the definition $\alpha_s = \left.\frac{da_s}{d\xi}\right|_{\xi=0}$, the search for the extremality conditions for γ passes through the evaluation

$$\left.\frac{d\mathcal{I}[\hat{\gamma}_\xi]}{d\xi}\right|_{\xi=0} = \sum_{s=1}^{N}\left[\frac{d}{d\xi}\int_{a_{s-1}(\xi)}^{a_s(\xi)} \mathscr{L}\big(\hat{\gamma}_\xi^{(s)}\big)\,dt\right]_{\xi=0} =$$

$$= \sum_{s=1}^{N}\left\{\int_{a_{s-1}}^{a_s} \hat{X}_{(s)}(\mathscr{L})\,dt \;+\; \Big[\alpha_s\,\mathscr{L}(\hat{\gamma}^{(s)}(a_s)) - \alpha_{s-1}\,\mathscr{L}(\hat{\gamma}^{(s)}(a_{s-1}))\Big]\right\} \tag{2.3.6a}$$

On account of the assumption $\alpha_0 = \alpha_N = 0$, recalling equation (1.5.37) and denoting by

$$\big[\mathscr{L}(\hat{\gamma})\big]_{a_s} := \big[\mathscr{L}(\hat{\gamma}^{(s+1)}(a_s)) - \mathscr{L}(\hat{\gamma}^{(s)}(a_s))\big]$$

the jump of the function $\mathscr{L}(\hat{\gamma}(t))$ at $t = a_s$, equation (2.3.6a) may be concisely written as

$$\left.\frac{d\mathcal{I}[\hat{\gamma}_\xi]}{d\xi}\right|_{\xi=0} = \sum_{s=1}^{N}\int_{a_{s-1}}^{a_s}\Big(X_{(s)}^{\,i}\,\tilde{\partial}_i(\mathscr{L}) + Y_{(s)}^{A}\,\frac{\partial\mathscr{L}}{\partial z^A}\Big)\,dt \;-\; \sum_{s=1}^{N-1}\alpha_s\big[\mathscr{L}(\hat{\gamma})\big]_{a_s} \tag{2.3.6b}$$

Equation (2.3.6b) is further elaborated by means of the introduction of N virtual 1–forms $\hat{\lambda}^{(s)} = p_i^{(s)}(t)\,\hat{\omega}^i$ (one for each arc $\gamma^{(s)}$) satisfying the transport law

$$\left(\frac{D\hat{\lambda}^{(s)}}{Dt}\right)_{\gamma^{(s)}} = \Big(\tilde{\partial}_i\mathscr{L}\Big)_{\hat{\gamma}^{(s)}}\,\hat{\omega}^i \tag{2.3.7a}$$

as well as the matching conditions

$$\hat{\lambda}^{(s)}\big|_{a_s} = \hat{\lambda}^{(s+1)}\big|_{a_s} \qquad s = 1,\ldots,N-1 \tag{2.3.7b}$$

In order to make the notation as easy as possible we collect all $\hat{\lambda}^{(s)}$ into a continuous, piecewise differentiable section $\hat{\lambda} : [t_0, t_1] \to V^*(\gamma)$ according to the prescription

$$\hat{\lambda}(t) = \hat{\lambda}^{(s)}(t) \qquad \forall\, t \in [a_{s-1}, a_s] \tag{2.3.8}$$

On account of equations (2.3.7a,b), $\hat{\lambda}$ is then uniquely determined by $\mathscr{L}$, up to initial data at $t = t_0$.

Taking the covariant variational equation (1.5.35) as well as the duality relations $\Big\langle \Big(\frac{\partial}{\partial q^i}\Big)_{\gamma^{(s)}}, \hat{\omega}^k \Big\rangle = \delta_i^k$ into account, by equation (2.3.7a) we get the expression

$$X_{(s)}^{\,i}\,\tilde{\partial}_i\mathscr{L} = \Big\langle X_{(s)}, \Big(\frac{D\hat{\lambda}^{(s)}}{Dt}\Big)_{\gamma^{(s)}}\Big\rangle = \frac{d}{dt}\Big\langle X_{(s)}, \hat{\lambda}^{(s)}\Big\rangle - \Big\langle\Big(\frac{DX_{(s)}}{Dt}\Big)_{\gamma^{(s)}}, \hat{\lambda}^{(s)}\Big\rangle =$$

$$= \frac{d}{dt}\Big(X_{(s)}^{\,i}\,p_i^{(s)}\Big) - p_i^{(s)}\Big(\frac{\partial\psi^i}{\partial z^A}\Big)_{\hat{\gamma}^{(s)}} Y_{(s)}^{A}$$

whence also

$$\int_{a_{s-1}}^{a_s} X_{(s)}^{\,i}\, \tilde{\partial}_i(\mathscr{L})\, dt \;=\; \Big[X_{(s)}^{\,i}\, p_i^{(s)} \Big]_{a_{s-1}}^{a_s} \;-\; \int_{a_{s-1}}^{a_s} p_i^{(s)} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}^{(s)}} Y_{(s)}^A\, dt$$

Summing over s, restoring the notations (1.5.40), (2.3.8) and recalling equations (1.5.34a), (2.3.7b) as well as the conditions $X(t_0) = X(t_1) = 0$, this implies the relation

$$\sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} X_{(s)}^{\,i}\, \tilde{\partial}_i(\mathscr{L})\, dt \;=\; -\int_{t_0}^{t_1} p_i \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} Y^A\, dt \;+\; \sum_{s=1}^{N-1} \alpha_s \Big[\psi^i(\hat{\gamma}) \Big]_{a_s} p_i(a_s)$$

In this way, omitting all unnecessary subscripts, equation (2.3.6b) gets the final form

$$\left. \frac{d\mathcal{I}\,[\hat{\gamma}_\xi]}{d\xi} \right|_{\xi=0} = \int_{t_0}^{t_1} \left(\frac{\partial \mathscr{L}}{\partial z^A} - p_i\, \frac{\partial \psi^i}{\partial z^A} \right) Y^A\, dt + \sum_{s=1}^{N-1} \alpha_s \Big[p_i(t)\, \psi^i(\hat{\gamma}) - \mathscr{L}(\hat{\gamma}) \Big]_{a_s} \tag{2.3.9}$$

In the algebraic environment introduced in §1.5.4, the previous discussion is naturally formalized regarding the right hand side of equation (2.3.9) as a linear functional $d\mathcal{I}_\gamma : \mathfrak{W} \to \mathbb{R}$ on the vector space $\mathfrak{W} = \mathfrak{V} \oplus \mathbb{R}^{N-1}$. A necessary and sufficient condition for γ to be an extremal for the functional (2.1.1) is then the vanishing of $d\mathcal{I}_\gamma$ on the subset $\mathfrak{X} \subset \mathfrak{W}$ formed by the totality of elements $Y, \alpha_1, \ldots, \alpha_{N-1}$ arising from finite deformations with fixed end–points. By linearity, the previous condition is mathematically equivalent to the requirement

$$\Delta(\gamma) \subset \ker(d\mathcal{I}_\gamma) \tag{2.3.10}$$

with $\Delta(\gamma) = \mathrm{Span}(\mathfrak{X}) \subseteq \ker(\Upsilon)$ denoting the variational space of γ.

As we shall see, equation (2.3.10) provides an algorithm for the determination of *all* the extremals of the functional (2.1.1) within the class of *ordinary* evolutions.

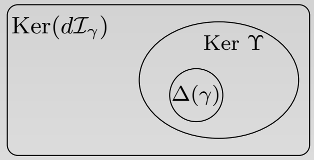
The exceptional case is considerably more complicated, because of the lack of an explicit characterization of the space $\Delta(\gamma)$ in terms of the local properties of the section γ. In this respect, the simplest procedure and, quite often, the only available one, is checking equation (2.3.10) separately on each exceptional evolution.

In what follows we shall adopt an intermediate strategy, namely, rather than dealing with equation (2.3.10) we shall discuss the implications of the stronger requirement

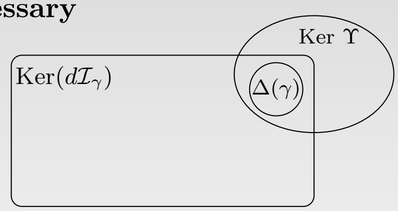
$$\ker(\Upsilon) \subset \ker(d\mathcal{I}_\gamma) \tag{2.3.11a}$$

According to the classification introduced in §1.5.4, the latter is *necessary and sufficient* for an *ordinary* evolution γ to be an extremal of the functional (2.1.1), but merely *sufficient* for an *exceptional* evolution to be an extremal.

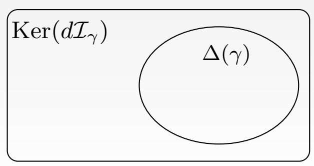
- In the *exceptional* case, condition (2.3.11a) is **sufficient**



but not **necessary**



- In the *ordinary* case condition (2.3.11a) is instead both **necessary** and **sufficient**



By elementary algebra, the requirement (2.3.11) is equivalent to the existence of a (possibly non–unique) linear functional $K : V_h \to \mathbb{R}$ satisfying the relation

$$\begin{tikzcd} \mathfrak{W} \arrow[r, "\Upsilon"] \arrow[dr, "d\mathcal{I}_\gamma"'] & V_h \arrow[d, "K"] \\ & \mathbb{R} \end{tikzcd} \tag{2.3.11b}$$

Setting $K = K_a\, e^{(a)}$, and recalling equations (1.5.43), (2.3.9), the requirement (2.3.11b) is expressed in components as

$$\int_{t_0}^{t_1}\left(\frac{\partial\mathscr{L}}{\partial z^A} - p_i\,\frac{\partial\psi^i}{\partial z^A}\right)Y^A\,dt + \sum_{s=1}^{N-1}\alpha_s\Big[p_i(t)\,\psi^i(\hat\gamma) - \mathscr{L}(\hat\gamma)\Big]_{a_s} =$$
$$K_a\left(\int_{t_0}^{t_1} Y^A\,e^{(a)}_{\;i}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma} dt\; - \sum_{s=1}^{N-1}\alpha_s\,e^{(a)}_{\;i}(a_s)\Big[\psi^i(\hat\gamma)\Big]_{a_s}\right)$$

By the arbitrariness of $Y, \alpha_1, \ldots, \alpha_{N-1}$, the latter condition splits into the system

$$\frac{\partial \mathscr{L}}{\partial z^A} - \Big(p_i + K_a\, e^{(a)}_{\ i}\Big)\,\frac{\partial \psi^i}{\partial z^A} \;=\; 0 \qquad\qquad A = 1, \ldots, r \tag{2.3.12a}$$

$$\Big[\Big(p_i + K_a\, e^{(a)}_{\ i}\Big)\,\psi^i(\hat\gamma) - \mathscr{L}(\hat\gamma)\Big]_{a_s} \;=\; 0 \qquad\qquad s = 1, \ldots, N-1 \tag{2.3.12b}$$

Collecting all results, and recalling Propositions 1.5.4, 1.5.5 we conclude

Theorem 2.3.1. *Given an admissible evolution γ, let $\wp(\gamma)$ denote the totality of piecewise differentiable virtual 1–forms $\hat\lambda = p_i(t)\,\hat\omega^i$ along γ satisfying equations (2.3.7a,b), (2.3.8) as well as the finite relations*

$$p_i\,\frac{\partial \psi^i}{\partial z^A} \;=\; \frac{\partial \mathscr{L}}{\partial z^A} \qquad\qquad A = 1, \ldots, r \tag{2.3.13a}$$

and the matching conditions

$$\Big[p_i \psi^i(\hat\gamma) - \mathscr{L}(\hat\gamma)\Big]_{a_s} \;=\; 0 \qquad\qquad s = 1, \ldots, N-1 \tag{2.3.13b}$$

Then:

a) *the condition $\wp(\gamma) \neq \emptyset$ is* sufficient *for γ to be an extremal for the functional (2.1.1);*

b) *if γ is an* ordinary *evolution, the same condition is also* necessary *for γ to be an extremal;*

c) *γ is a* normal extremal, *namely an extremal belonging to the class of normal evolutions, if and only if the set $\wp(\gamma)$ consists of a single element.*

Proof. In view of equations (2.3.9), (2.3.13a,b), whenever the ansatz $\hat\lambda \in \wp(\gamma)$ is allowed, it implies $\frac{d\mathcal{I}}{d\xi}\big|_{\xi=0} = 0$ for all admissible infinitesimal deformations vanishing at the end–points of γ. Assertion *a*) is then a direct consequence of Definition 2.1.2.

In particular, according to our previous discussion, if γ is an *ordinary* extremal, there exists at least one h–transported 1–form $K = K_a e^{(a)}$ satisfying equations (2.3.12a,b) in correspondence with any continuous virtual 1-form $\hat\lambda = p_i\,\hat\omega^i$ obeying the transport law (2.3.7a). The sum $\hat\lambda + K = \big(p_i + K_a\, e^{(a)}_{\ i}\big)\,\hat\omega^i$ is hence automatically in the class $\wp(\gamma)$, thus proving assertion *b*).

Finally, as pointed out in §1.5.2, the normal evolutions form a subclass of the ordinary ones, uniquely characterized by the requirement $\big(\Upsilon(\mathfrak{W})\big)^0 = \{0\}$. Therefore, according to assertion *b*), a normal evolution γ is an extremal if and only if the class $\wp(\gamma)$ is nonempty. Moreover, by equations (2.3.7a), (2.3.12a), if $\hat{\lambda}, \hat{\lambda}'$ is any pair of elements in the class $\wp(\gamma)$, the difference $\hat{\lambda}-\hat{\lambda}'$ is automatically an h–transported 1–form satisfying equations. (1.5.44a,b). By Proposition 1.5.4 this implies $\hat{\lambda}-\hat{\lambda}' \in \big(\Upsilon(\mathfrak{W})\big)^0 \Rightarrow \hat{\lambda}=\hat{\lambda}'$, thus establishing assertion c). □

In view of equations (1.5.21b), (1.5.22b)), for any $\hat{\lambda} \in \wp(\gamma)$ the transport law (2.3.7a) simplifies to

$$\frac{dp_i}{dt} + p_k \left(\frac{\partial \psi^k}{\partial q^i}\right)_{\hat{\gamma}} + \cancel{h_i{}^A\, p_k \left(\frac{\partial \psi^k}{\partial z^A}\right)_{\hat{\gamma}}} = \left(\frac{\partial L}{\partial q^i}\right)_{\hat{\gamma}} + \cancel{h_i{}^A \left(\frac{\partial L}{\partial z^A}\right)_{\hat{\gamma}}}$$

the cancellation being due to equation (2.3.13a). Exactly as it happened with Proposition 1.5.4, all assertions of Theorem 2.3.1 have therefore an intrinsic meaning, irrespective of the choice of the infinitesimal controls $h^{(s)} : V(\gamma^{(s)}) \to A(\hat{\gamma}^{(s)})$. The previous arguments provide an algorithm for the determination of the *ordinary* extremals of the functional (2.1.1), relying on $2n+r$ equations

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A) \tag{2.3.14a}$$

$$\frac{dp_i}{dt} + \frac{\partial \psi^k}{\partial q^i}\, p_k = \frac{\partial \mathscr{L}}{\partial q^i} \tag{2.3.14b}$$

$$p_i\, \frac{\partial \psi^i}{\partial z^A} = \frac{\partial \mathscr{L}}{\partial z^A} \tag{2.3.14c}$$

for the unknowns $q^i(t), p_i(t), z^A(t)$, completed with the continuity requirements

$$\big[q^i\big]_{a_s} = \big[p_i\big]_{a_s} = \big[p_i\, \psi^i - \mathscr{L}\big]_{a_s} = 0 \qquad s = 1, \ldots, N-1 \tag{2.3.15}$$

Collecting all results, we can now state the following

Theorem 2.3.2. *Every ordinary extremal γ for the functional (2.1.1) is the projection of at least one extremal $\tilde{\gamma}$ for the functional (2.3.3). Moreover, the normality of γ implies the uniqueness of $\tilde{\gamma}$.*

Proof. It is easily seen from the previous discussion that every ordinary extremal $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$ for the functional (2.1.1) determines both a unique admissible section $\hat{\gamma}\colon \mathbb{R} \to \mathcal{A}$ and a section $\hat{\lambda}\colon \mathbb{R} \to V^*(\gamma)$ belonging to $\wp(\gamma)$. Because of the nature of the contact bundle $\mathcal{C}(\mathcal{A})$ of fibre bundle over the space $V^*(\mathcal{V}_{n+1})$, identical to the pull-back of the latter through the map $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$,

$$\begin{array}{ccc} \mathcal{C}(\mathcal{A}) & \xrightarrow{\hat{\kappa}} & V^*(\mathcal{V}_{n+1}) \\ {\scriptstyle \zeta}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{A} & \xrightarrow{\pi} & \mathcal{V}_{n+1} \end{array}$$

the pair $(\hat{\gamma}, \hat{\lambda})$ characterizes a v–continuous section $\tilde{\gamma}\colon \mathbb{R} \to \mathcal{C}(\mathcal{A})$ satisfying

$$\zeta \cdot \tilde{\gamma} = \hat{\gamma} \quad , \qquad \hat{\kappa} \cdot \tilde{\gamma} = \hat{\lambda}$$

The thesis follows now directly from the observation that the equations (2.3.4) coincide exactly with the equations (2.3.14), (2.3.15). The section $\tilde{\gamma}$ is therefore an extremal for the functional (2.3.3) which projects onto γ.

Eventually, whenever γ is *normal*, the uniqueness of $\tilde{\gamma}$ is a straightforward consequence of the fact that — in this case — the set $\wp(\gamma)$ consists of a single element, as shown in Theorem 2.3.1. $\square$

As far as the *ordinary* extremals are concerned, the original constrained variational problem in the event space is therefore equivalent to a *free* variational problem in the contact manifold. This is precisely the essence of Pontryagin's *maximum principle*.

As already pointed out, all equations (2.3.14), (2.3.15) are independent of the choice of the infinitesimal controls, and involve only the "true" data of the problem, namely the Lagrangian section ℓ and the constraint equations (1.2.5). In particular, the last pair of equations (2.3.15) extend to the ordinary evolutions the well known *Erdmann–Weierstrass corner conditions* of holonomic variational calculus [8, 19].

Remark 2.3.1 (Same problem, equivalent solution)**:** There is another possible approach to the problem, slightly different but completely equivalent to the one outlined so far. Apparently, it complicates matters without giving any significant advantage. On the other hand, it seems to be the most faithful translation of the original Pontryagin's treatment of the subject ([17]) into the geometrical context. Hence, at least for historical reasons, it is worth telling about.

A variational problem, based on the functional

$$\mathcal{I}[\gamma] := \int_{\hat{\gamma}} \dot{u}\, dt \tag{2.3.16}$$

is introduced in the manifold $\mathcal{L}(\mathcal{V}_{n+1})$, where $\hat{\gamma}$ stands for the jet–extension of a section $\gamma\colon [t_0, t_1] \to P$. As the 1–form $\dot{u}\, dt$ is well defined in $\mathcal{L}(\mathcal{V}_{n+1})$ up to a term $\dot{f}\, dt$, the functional (2.3.16) is independent of a particular choice of the gauge.
Setting $\gamma\colon\ q^i = q^i(t)\,,\ u = u(t)$, it follows that

$$\int_{\hat{\gamma}} \dot{u}\, dt = u(t_1) - u(t_0)$$

and so, assuming the values of $q^i(t_0)$ and $q^i(t_1)$ as fixed, the problem consists in finding a curve γ which minimizes the increment $u(t_1) - u(t_0)$ and whose projection onto $\mathcal{V}_{n+1}$ leaves the end–points fixed.

We now require the section $\hat{\gamma}$ to belong to the submanifold $\hat{\mathcal{A}}$ of $\mathcal{L}(\mathcal{V}_{n+1})$ locally described by the equations

$$\dot{q}^i = \psi^i(t, q^i, z^A)\,, \qquad \dot{u} = \mathscr{L}(t, q^i, z^A) \tag{2.3.17}$$

In other words, we are making use of the simultaneous assignment of both the kinetic constraints and the Lagrangian section to express the submanifold $\hat{\mathcal{A}}$ as the image $\ell(\mathcal{A})$[4]. In this way, every admissible section $q^i = q^i(t)$ in $\mathcal{V}_{n+1}$ determines, up to a constant, an admissible section $q^i = q^i(t)$, $u = u(t)$ of P.

Compared with the main approach outlined in §2.2, the present formulation just replaces the section $\ell\colon \mathcal{A} \to \mathcal{L}(\mathcal{A})$ with the image space $\hat{\mathcal{A}} = \ell(\mathcal{A})$, considered as a submanifold of $\mathcal{L}(\mathcal{V}_{n+1})$. This submanifold, and consequently the section ℓ, is regarded as set and therefore the representation $\dot{u} = \mathscr{L}(t, q^i, z^A)$ is affected only by passive gauge transformations. The same variational problem is bred by different submanifolds related together by the action of the gauge group.

As well as in § 2.2, the submanifold $\hat{\mathcal{A}} \to \mathcal{L}(\mathcal{V}_{n+1})$ is lifted up onto a submanifold $\mathcal{C}(\hat{\mathcal{A}}) \to \mathcal{S}$ both by identifying $C(\hat{\mathcal{A}})$ with the image space $\tilde{\ell}(\mathcal{C}(\mathcal{A}))$ and by pulling back $\mathcal{S}$ onto $\hat{\mathcal{A}}$ by means of the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\hat{\mathcal{A}}) & \xrightarrow{\tilde{\jmath}} & \mathcal{S} \\ \downarrow & & \downarrow \\ \hat{\mathcal{A}} & \xrightarrow{\jmath} & \mathcal{L}(\mathcal{V}_{n+1}) \end{array} \tag{2.3.18}$$

All the same, the imbedding $\mathcal{C}(\hat{\mathcal{A}}) \xrightarrow{\tilde{\jmath}} \mathcal{S}$ is fibred onto $V^*(\mathcal{V}_{n+1})$ and its expression in coordinate is formally identical to equations (2.3.17) which are involved in the representation of the submanifold $\hat{\mathcal{A}}$.
It is now possible to make use of the form (1.4.32) to induce on $\mathcal{C}(\hat{\mathcal{A}})$ the 1–form

$$\tilde{\jmath}^*\left(\Theta_u\right) = \mathscr{L}\, dt + p_i \left(dq^i - \psi^i\, dt\right) \tag{2.3.19}$$

and, consequently, to define an action integral by the integration of the form (2.3.19) along any section $\tilde{\gamma}\colon [t_0\,,\, t_1] \to \mathcal{C}(\hat{\mathcal{A}})$. Once again, this merely reproduces in the image space $\mathcal{C}(\hat{\mathcal{A}})$ the construction carried on in §2.2. Namely, the 1–form (2.3.19) is simply the image of the *Pontryagin–Poincaré–Cartan form* (2.2.7) under the diffeomorphism $\tilde{\ell}\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\hat{\mathcal{A}})$.

As previously mentioned, a significant role in the study of the variational problem based on the functional (2.3.3) is played by the choice of the Lagrangian section as $\dot{u} = \dot{f}(t, q)$. In this particular situation, the problem under discussion can be made, by means of a gauge transformation, into the study of the action functional

$$\mathcal{I}_0[\tilde{\gamma}] := \int_{\tilde{\gamma}} \Theta_L = \int_{t_0}^{t_1} p_i \left(\frac{dq^i}{dt} - \psi^i\right) dt \tag{2.3.20}$$

induced by the Liouville form (1.2.17) of $\mathcal{C}(\mathcal{A})$. The corresponding extremal curves

[4]Needless to say, the holonomic case is automatically included in the present scheme, the constraints (2.3.17) being reduced to the single request $\dot{u} = \mathscr{L}(t, q^i, \dot{q}^i)$.

are easily seen to satisfy the Euler–Lagrange equations

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A) \tag{2.3.21a}$$

$$\frac{dp_i}{dt} + \frac{\partial \psi^k}{\partial q^i}\, p_k = 0 \tag{2.3.21b}$$

$$p_i \frac{\partial \psi^i}{\partial z^A} = 0 \tag{2.3.21c}$$

$$\big[p_i\big]_{a_s} = \big[p_i\, \psi^i\big]_{a_s} = 0 \qquad\qquad s = 1, \ldots, N-1 \tag{2.3.21d}$$

Equation (2.3.21a) is the admissibility requirement for the section $\upsilon \cdot \tilde{\gamma}$. For this reason, if an extremal $\tilde{\gamma}$ of the functional (2.3.20) satisfies $\upsilon \cdot \tilde{\gamma} = \gamma$, its projection $\zeta \cdot \tilde{\gamma}$ under the map $\zeta : \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ coincides with the lift $\hat{\gamma} : [t_0, t_1] \to \mathcal{A}$.

For any admissible γ, the extremals projecting onto γ are therefore in 1–1 correspondence with the solutions $p_i(t)$ of the homogeneous system (2.3.21b,c,d), with the functions $q^i(t)$, $z^A(t)$ regarded as given. On the other hand, according to Proposition 1.5.4, equations (2.3.21b,c,d) are precisely the relations characterizing the totality of virtual 1–forms $p_i(t)\,\hat{\omega}^i$ belonging to the annihilator $\big(\Upsilon(\mathfrak{W})\big)^0$.

We have thus proved the following

Proposition 2.3.1. *Let $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ denote any continuous, piecewise differentiable section. Then:*

a) γ is admissible if and only if the functional (2.3.20) admits at least one extremal $\tilde{\gamma}$ projecting onto γ, namely satisfying $\upsilon \cdot \tilde{\gamma} = \gamma$;

b) for any such γ, the totality of extremals of $\mathcal{I}_0$ projecting onto γ form a finite dimensional vector space over $\mathbb{R}$, with dimension equal to the abnormality index of γ.

In the language of § 1.5.4, Proposition 2.3.1 asserts that a section $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ describes a *normal* evolution of the system if and only if the functional (2.3.20) admits *exactly one* extremal projecting onto γ, namely the one corresponding to the trivial solution $p_i(t) = 0$. If the extremals projecting onto γ are more than one, γ represents an *abnormal* evolution; if no such extremal exists, γ is not admissible.

We now come back to the study of the variational problem based on functional (2.3.3) and we state

Proposition 2.3.2. *The totality of extremals of the functional (2.3.3) projecting onto a section $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ is an affine space, modelled on the vector space formed by the extremals of the functional (2.3.20) projecting onto γ.*

Proof. The proof is entirely straightforward and is based on the observation that if $\tilde{\gamma}\colon q^i = q^i(t)$, $z^A = z^A(t)$, $p_i = p_i(t)$ and $\tilde{\gamma}'\colon q^i = q^i(t)$, $z^A = z^A(t)$, $p_i = \tau_i(t)$ are both extremals of the functional (2.3.3) projecting onto γ, then the contemporaneous validity of the Euler–Lagrange equations

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A)\,, \qquad \frac{dp_i}{dt} + p_k \frac{\partial \psi^k}{\partial q^i} = \frac{\partial \mathscr{L}}{\partial q^i}\,, \qquad p_k \frac{\partial \psi^k}{\partial z^A} = \frac{\partial \mathscr{L}}{\partial z^A}$$

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A)\,, \qquad \frac{d\tau_i}{dt} + \tau_k \frac{\partial \psi^k}{\partial q^i} = \frac{\partial \mathscr{L}}{\partial q^i}\,, \qquad \tau_k \frac{\partial \psi^k}{\partial z^A} = \frac{\partial \mathscr{L}}{\partial z^A}$$

implies that the curve $q^i = q^i(t)$, $z^A = z^A(t)$, $p_i = p_i(t) - \tau_i(t)$ is an extremal for the functional (2.3.20). $\square$

The previous arguments provide a restatement of Theorem 2.3.1 in the environment $\mathcal{C}(\mathcal{A})$. In particular, it is worth remarking that, in general, the projection algorithm $\tilde{\gamma} \to \upsilon \cdot \tilde{\gamma}$, applied to the totality of extremals of the functional (2.3.3), does not yield back *all* the extremals of the functional (2.1.1), but only a subclass, wide enough to include the *ordinary* ones. The missing extremals may be obtained determining the abnormal evolutions by means of Proposition 2.3.1, finding out which ones have an exceptional character, and analyzing each of them individually.

2.4 Hamiltonian formulation

Temporarily leaving aside all aspects related to the presence of corners, we observe that a differentiable curve $\tilde{\gamma}$ in $\mathcal{C}(\mathcal{A})$ is at the same time a section with respect to the fibration $\mathcal{C}(\mathcal{A}) \xrightarrow{t} \mathbb{R}$ and an extremal for the functional (2.3.3) if and only if its tangent vector field $Z := \tilde{\gamma}_*\big(\frac{\partial}{\partial t}\big)$ satisfies the properties

$$\big\langle Z, dt \big\rangle = 1\,, \qquad Z \,\lrcorner\, d\Theta_{\text{PPC}} = 0 \tag{2.4.1}$$

On account of equation (2.2.7), at any $\varsigma \in \mathcal{C}(\mathcal{A})$ a necessary and sufficient condition for the existence of at least one vector $Z \in T_\varsigma(\mathcal{C}(\mathcal{A}))$ satisfying equations (2.4.1) is the validity of the relations

$$\left(\frac{\partial \mathscr{H}}{\partial z^A}\right)_{\varsigma} = 0 \tag{2.4.2a}$$

Points ς at which equations (2.4.1) admit a *unique* solution Z will be called *regular points* for the functional (2.3.3). In coordinates, the regularity requirement is expressed by the condition

$$\det\left(\frac{\partial^2 \mathscr{H}}{\partial z^A \partial z^B}\right)_{\varsigma} \neq 0 \tag{2.4.2b}$$

In view of equation (2.4.2b), in a neighborhood of each regular point equations (2.4.2a) may be solved for the z^A's, giving rise to a representation of the form

$$z^A = z^A(t, q^1, \ldots, q^n, p_1, \ldots, p_n) \tag{2.4.3}$$

The regular points form therefore a $(2n+1)$–dimensional submanifold $\mathcal{R} \xrightarrow{j} \mathcal{C}(\mathcal{A})$, locally diffeomorphic to the space $V^*(\mathcal{V}_{n+1})$.

When restricted to the submanifold $\mathcal{R}$, the pull–back of the form (2.2.4) by means of the section $\tilde{\ell}\colon \mathcal{C}(\mathcal{A}) \to \mathcal{S}^{\mathcal{A}}$ provides the 1–form

$$\tilde{\Theta}_{\text{PPC}} := \big(j \cdot \tilde{\ell}\big)^*(\Theta_u) = -\mathcal{H}dt + p_i dq^i \tag{2.4.4}$$

having denoted by $\mathcal{H} := j^*(\mathscr{H})$ the pull–back of the Pontryagin Hamiltonian, expressed in coordinates as

$$\mathcal{H} = \mathscr{H}(t, q^r, z^A(t, q^i, p_i), p_r) = p_k\,\psi^k(t, q^r, z^A(t, q^i, p_i)) - \mathscr{L}(t, q^r, z^A(t, q^i, p_i))$$

In view of equations (2.2.7), (2.4.2a) we have then the identifications

$$\frac{\partial \mathcal{H}}{\partial p_i} = \frac{\partial \mathscr{H}}{\partial p_i} + \cancel{\frac{\partial \mathscr{H}}{\partial z^A}}\,\frac{\partial z^A}{\partial p_i} = \psi^i \tag{2.4.5a}$$

$$\frac{\partial \mathcal{H}}{\partial q^i} = \frac{\partial \mathscr{H}}{\partial q^i} + \cancel{\frac{\partial \mathscr{H}}{\partial z^A}}\,\frac{\partial z^A}{\partial q^i} = p_k\,\frac{\partial \psi^k}{\partial q^i} - \frac{\partial \mathscr{L}}{\partial q^i} \tag{2.4.5b}$$

On account of these, equations (2.3.14a,b) gives rise to the following system of ordinary differential equations in normal form for the unknowns $q^i(t), p_i(t)$

$$\frac{dq^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \tag{2.4.6a}$$

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q^i} \tag{2.4.6b}$$

The original constrained Lagrangian variational problem has thus been reduced to a free Hamiltonian problem on the submanifold $j : \mathcal{R} \to \mathcal{C}(\mathcal{A})$, with Hamiltonian $\mathcal{H}(t, q^1, \ldots, q^n, p_1, \ldots, p_n)$ identical to the pull–back $\mathcal{H} = j^*(\mathscr{H})$ [5]. Once again, all this is in full agreement with Pontryagin's principle.

Remark 2.4.1: By virtue of Cauchy theorem, equations (2.4.6a,b) require the assignment of $2n$ initial data in order to give rise to a unique solution. This indicates that, as far as the calculus of variations is concerned, a fixed end–points problem is always well–posed, regardless of its holonomic or non–holonomic nature. In the latter case, however, it is easily seen that the contemporaneous knowledge of both the initial position and velocity of the

[5] Conversely, setting $\mathcal{H} = j^*(\mathscr{H})$, the inverse Legendre transformation $\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}$, together with equation (2.4.5a), yields back the constraint equations $\dot{q}^i = \psi^i(t, q^k, z^A)$.

system (which corresponds to the assignment of $n + r$ data) is not enough to determine its future evolution. In this sense, it becomes apparent that the constrained calculus of variations can't be considered as a branch of Mechanics for the principle of determinism is not fulfilled.

All the same, the subject of the present work is commonly classified in the literature as *vakonomic Mechanics*. This terminology, first introduced by Arnold [2], is an abbreviation for "Mechanics of variational axiomatic kind" and is often thought as a sort of non–holonomic Mechanics that differs from the latter inasmuch as the constraints are *a priori* given. It is in our opinion that, although largely diffused, such a terminology is most inappropriate and misleading for, by definition, any theory which aspire to belong under Mechanics must be deterministic.

A v–continuous extremal of the functional (2.3.3) consisting of a finite family of closed arcs $\tilde{\gamma}^{(s)} : [a_{s-1}, a_s] \to \mathcal{C}(\mathcal{A})$, each contained in (a connected component of) the submanifold $\mathcal{R}$ will be called a *regular extremal*.

Singular extremals, partly, or even totally lying outside $\mathcal{R}$ may also exist. In fact, while equation (2.4.2a) is part of the system (2.3.14), and must therefore be satisfied by any extremal, the requirement (2.4.2b) has only to do with the *well–posedness* of the Cauchy problem for the subsystem (2.3.14).

On the other hand, by construction, the Hamilton equations (2.4.6a,b) determines only the regular extremals. The singular ones, if at all, have therefore to be dealt with directly, looking for solutions of equations (2.3.14) not arising from a well posed Cauchy problem.

In principle, this could be done extending to the non–holonomic context the concepts and methods commonly adopted in the study of singular Lagrangians [26]. The argument is beyond the purposes of the present work, and will not be pursued.

To complete our analysis, let us finally discuss the role of equations (2.3.15) in the study of *corners*. To this end, we consider the rear face of diagram (1.4.28), now suitably pulled-back onto $\mathcal{A}$:

$$\begin{array}{ccc} \mathcal{S}^{\mathcal{A}} & \longrightarrow & \mathcal{H}(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}(\mathcal{A}) & \longrightarrow & V^*(\mathcal{V}_{n+1}) \end{array} \tag{2.4.7}$$

and we recall that, for each choice of the trivialization $u \colon P \to \mathbb{R}$, the Liouville 1–form (1.4.20) of $j_1(P, \mathcal{V}_{n+1})$ provides the space $\mathcal{H}(\mathcal{V}_{n+1})$ with the form

$$\tilde{\Theta}_u \;=\; p_0\, dt \,+\, p_i\, dq^i \tag{2.4.8}$$

By making use of it, as well as of the Pontryagin–Poicaré–Cartan form (2.2.7), we now introduce a morphism $\mathcal{C}(\mathcal{A}) \xrightarrow{\Psi} \mathcal{H}(\mathcal{V}_{n+1})$ fibred over $V^*(\mathcal{V}_{n+1})$ and based on the prescription

$$\Psi^*(\tilde{\Theta}_u) \;=\; \Theta_{\text{PPC}}$$

In coordinates, we have the explicit representation

$$\Psi : \qquad t = t\,, \quad q^i = q^i, \quad p_i = p_i\,, \quad p_0 = -\mathscr{H}\big(t, q^i, p_i\,, z^A\big) \tag{2.4.9}$$

The content of equations (2.3.4d) is then summarized into the following

Proposition 2.4.1. *For any υ–continuous extremal $\big(\tilde\gamma, [t_0, t_1]\big)$ of the functional* (2.3.3)*, the composite map $\Psi \cdot \tilde\gamma : [t_0, t_1] \to \mathcal{H}(\mathcal{V}_{n+1})$ is necessarily continuous.*

The previous arguments provide a simple characterization of the jumps that may possibly occur along a *regular* extremal $\tilde\gamma : \big\{\big(\tilde\gamma^{(s)}, [a_{s-1}, a_s]\big)\big\}$. To this end we observe that the restriction of the map (2.4.9) to the submanifold $\mathcal{R} \subset \mathcal{C}(\mathcal{A})$ determines an immersion $\Psi : \mathcal{R} \to \mathcal{H}(\mathcal{V}_{n+1})$ and that, as already pointed out, at each $\varsigma \in \mathcal{R}$ there exists, locally, one and only one differentiable extremal of the functional (2.3.3) through ς.

On the other hand, by Proposition 2.4.1, for each $s = 1, \ldots, N-1$, the arcs $\tilde\gamma^{(s)}$ and $\tilde\gamma^{(s+1)}$ are related by the condition $\Psi\big(\tilde\gamma^{(s)}(a_s)\big) = \Psi\big(\tilde\gamma^{(s+1)}(a_s)\big)$. From this, it is readily seen that the admissible discontinuities of $\tilde\gamma$ or, all the same, the admissible corners in the projection $\gamma := \upsilon \cdot \tilde\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ may only occur at points in which the immersion $\Psi : \mathcal{R} \to \mathcal{H}(\mathcal{V}_{n+1})$ is not injective.

Chapter 3

The second variation

The object of the present Chapter is to establish whether a given *locally normal* extremal gives rise to a local minimum for the functional (2.1.1). As the totality of these extremals has already been characterized, we will now address ourselves to the analysis of the second derivative $\frac{d^2\mathcal{I}}{d\xi^2}\Big|_{\xi=0}$, commonly referred to as the *second variation* of the action functional at γ.

In local coordinates, a simple calculation yields the result

$$
\begin{aligned}
\frac{d^2\mathcal{I}[\hat\gamma_\xi]}{d\xi^2}\bigg|_{\xi=0} = \sum_{s=1}^{N}\Bigg\{ \int_{a_{s-1}}^{a_s} \Bigg[& \left(\frac{\partial^2\mathscr{L}}{\partial q^i \partial q^j}\right)_{\hat\gamma^{(s)}} X^i_{(s)} X^j_{(s)} + 2\left(\frac{\partial^2\mathscr{L}}{\partial q^i \partial z^A}\right)_{\hat\gamma^{(s)}} X^i_{(s)} \Gamma^A_{(s)} + \\
& + \left(\frac{\partial^2\mathscr{L}}{\partial z^A \partial z^B}\right)_{\hat\gamma^{(s)}} \Gamma^A_{(s)} \Gamma^B_{(s)} + \left(\frac{\partial\mathscr{L}}{\partial q^i}\right)_{\hat\gamma^{(s)}} Z^i_{(s)} + \left(\frac{\partial\mathscr{L}}{\partial z^A}\right)_{\hat\gamma^{(s)}} K^A_{(s)} \Bigg] dt + \\
& + 2\,\alpha_s \left(\frac{\partial\mathscr{L}}{\partial q^i} X^i_{(s)} + \frac{\partial\mathscr{L}}{\partial z^A}\Gamma^A_{(s)}\right)_{\hat\gamma^{(s)}(a_s)} + \alpha_s^2 \left(\frac{d\mathscr{L}}{dt}\right)_{\hat\gamma^{(s)}(a_s)} + \\
& - 2\,\alpha_{s-1} \left(\frac{\partial\mathscr{L}}{\partial q^i} X^i_{(s)} + \frac{\partial\mathscr{L}}{\partial z^A}\Gamma^A_{(s)}\right)_{\hat\gamma^{(s)}(a_{s-1})} - \alpha_{s-1}^2 \left(\frac{d\mathscr{L}}{dt}\right)_{\hat\gamma^{(s)}(a_{s-1})} + \\
& + \beta_s\,\mathscr{L}(\hat\gamma^{(s)}(a_s)) - \beta_{s-1}\,\mathscr{L}(\hat\gamma^{(s)}(a_{s-1}))\Bigg\}
\end{aligned}
\tag{3.0.1}
$$

Besides the serious difficulties which lie in the determination of its definiteness, the previous expression hasn't apparently a tensorial character because of the contemporaneous presence of both the first and the second derivatives of the Lagrangian, which entails these last to undergo a transformation law like the following one

$$\frac{\partial^2 \mathscr{L}}{\partial \bar{q}^i \partial \bar{q}^j} = \frac{\partial^2 \mathscr{L}}{\partial q^k \partial q^r} \frac{\partial q^k}{\partial \bar{q}^i} \frac{\partial q^r}{\partial \bar{q}^j} + 2 \frac{\partial^2 \mathscr{L}}{\partial q^k \partial z^A} \frac{\partial q^k}{\partial \bar{q}^i} \frac{\partial z^A}{\partial \bar{q}^j} + \frac{\partial^2 \mathscr{L}}{\partial z^A \partial z^B} \frac{\partial z^A}{\partial \bar{q}^i} \frac{\partial z^B}{\partial \bar{q}^j} + \\ + \frac{\partial \mathscr{L}}{\partial q^k} \frac{\partial^2 q^k}{\partial \bar{q}^i \partial \bar{q}^j} + \frac{\partial \mathscr{L}}{\partial z^A} \frac{\partial^2 z^A}{\partial \bar{q}^i \partial \bar{q}^j}$$

This, of course, makes it unfit to be dealt with in a geometrical framework. Therefore, before getting to the heart of the matter, we ought to take the necessary steps in order to guarantee the tensorial character of all results.

3.1 Adapted Lagrangians

Generally speaking, a function f on a differentiable manifold M is said to be *critical* at a point $x \in M$ if and only if its differential vanishes at x.

Furthermore, the *Hessian* of f at a critical point x is a symmetric bilinear functional $(d^2 f)_x \colon T_x(M) \times T_x(M) \to \mathbb{R}$ which is defined by the following construction: for any $X, Y \in T_x(M)$, denoting by $\tilde{X}, \tilde{Y}$ their respective extensions to vector fields, we let $\big\langle (d^2 f)_x \,,\, X \otimes Y \big\rangle := \tilde{X}_x(\tilde{Y}(f))$, where $\tilde{X}_x$ is of course just X. Its symmetry is a direct consequence of f being critical at x, as we can readily see from the relation

$$\tilde{X}_x(\tilde{Y}(f)) - \tilde{Y}_x(\tilde{X}(f)) = [\tilde{X}, \tilde{Y}]_x(f) = \mathcal{L}_{\tilde{X}}(\tilde{Y})_{|\,x}(f) = 0$$

It is also clearly well–defined inasmuch as $\tilde{X}_x(\tilde{Y}(f)) = X(\tilde{Y}(f))$ is independent of the extension $\tilde{X}$ of X, while $\tilde{Y}_x(\tilde{X}(f))$ is independent of $\tilde{Y}$.

If the manifold M is referred to a local coordinate system $x^1, \ldots, x^n$ and $X = X^i \left(\frac{\partial}{\partial x^i}\right)_x$, $Y = Y^i \left(\frac{\partial}{\partial x^i}\right)_x$, we can set $\tilde{X} = X^i \left(\frac{\partial}{\partial x^i}\right)$, with $X^i = \text{const}$. Then

$$\big\langle (d^2 f)_x \,,\, X \otimes Y \big\rangle = X(\tilde{Y}(f)) = X\left(Y^j \frac{\partial f}{\partial x^j}\right) = X^i Y^j \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_x$$

so we have the representation

$$(d^2 f)_x = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_x (dx^i)_x \otimes (dx^j)_x$$

Under the stated circumstance, the Hessian of f at x has therefore a tensorial character. Similar conclusions hold if f is critical at each point of a submanifold $N \subset M$, in which case we write $(df)_N = 0$ and denote by $(d^2 f)_N$ the Hessian of f along N. Given any function $f = f(t, q^i) \in \mathscr{F}(\mathcal{V}_{n+1})$ we have then the following properties:

i) if f is critical on an admissible section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$, its *symbolic time derivative* $\dot{f} := \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k}\,\psi^k \in \mathscr{F}(\mathcal{A})$ is itself critical on the lift $\hat{\gamma}$ of γ, and satisfies $\dot{f}_{|\hat{\gamma}} = 0$;

ii) under the same assumption, for any admissible deformation $X : \mathbb{R} \to V(\gamma)$ the quadratic form associated[1] to the Hessian $(d^2 f)_\gamma$ fulfils the relation

$$\frac{d}{dt}\,\big\langle (d^2 f)_\gamma\,,\, X \otimes X \big\rangle \;=\; \big\langle (d^2 \dot{f})_{\hat{\gamma}}\,,\, \hat{X} \otimes \hat{X} \big\rangle \tag{3.1.1}$$

Remark 3.1.1: Both properties may be easily verified by observing that the condition $(df)_\gamma = 0$ implies the identities

$$(d\dot{f})_{\hat{\gamma}} \;=\; \left[\frac{\partial \dot{f}}{\partial t}\,dt \,+\, \frac{\partial \dot{f}}{\partial q^k}\,dq^k \,+\, \frac{\partial \dot{f}}{\partial z^A}\,dz^A\right]_{\hat{\gamma}} \;=\; \frac{d}{dt}\left(\frac{\partial f}{\partial t}\right)_{\gamma} dt \,+\, \frac{d}{dt}\left(\frac{\partial f}{\partial q^k}\right)_{\gamma} dq^k \;=\; 0$$

$$\left(\frac{\partial^2 \dot{f}}{\partial q^i \partial q^j}\right)_{\hat{\gamma}} = \left[\frac{\partial^3 f}{\partial q^i \partial q^j \partial t} + \frac{\partial^3 f}{\partial q^i \partial q^j \partial q^k}\,\psi^k + \frac{\partial^2 f}{\partial q^i \partial q^k}\,\frac{\partial \psi^k}{\partial q^j} + \frac{\partial^2 f}{\partial q^j \partial q^k}\,\frac{\partial \psi^k}{\partial q^i}\right]_{\hat{\gamma}} =$$

$$= \frac{d}{dt}\left(\frac{\partial^2 f}{\partial q^i \partial q^j}\right)_{\gamma} + \left(\frac{\partial^2 f}{\partial q^i \partial q^k}\right)_{\gamma}\left(\frac{\partial \psi^k}{\partial q^j}\right)_{\hat{\gamma}} + \left(\frac{\partial^2 f}{\partial q^j \partial q^k}\right)_{\gamma}\left(\frac{\partial \psi^k}{\partial q^i}\right)_{\hat{\gamma}}$$

$$\left(\frac{\partial^2 \dot{f}}{\partial q^i \partial z^A}\right)_{\hat{\gamma}} = \left(\frac{\partial^2 f}{\partial q^i \partial q^k}\right)_{\gamma}\left(\frac{\partial \psi^k}{\partial z^A}\right)_{\hat{\gamma}}, \qquad \left(\frac{\partial^2 \dot{f}}{\partial z^A \partial z^B}\right)_{\hat{\gamma}} = 0$$

The conclusion then follows by direct computation, expressing the derivatives $\frac{dX^i}{dt}$ in terms of the components X^i, Γ^A through the variational equation (1.5.8).

The previous arguments may avail in our variational context. In this respect, we recall the following results from the previous Chapters:

- as far as the *ordinary* evolutions are concerned, the variational problem in the event space based on the functional (2.1.1) is equivalent to the (free) one in the contact manifold based on the functional (2.3.3);

- for each *normal* extremal $\gamma = \big\{\big(\gamma^{(s)}, [a_{s-1}, a_s]\big),\ s = 1, \ldots, N,\ \big\}$ of the action integral (2.1.1) there exists a unique extremal $\tilde{\gamma} : [t_0, t_1] \to \mathcal{C}(\mathcal{A})$ of the functional (2.3.3) projecting onto γ, i.e. satisfying $\zeta \cdot \tilde{\gamma} = \hat{\gamma}$, whence also $\upsilon \cdot \tilde{\gamma} = \pi \cdot \hat{\gamma} = \gamma$;

- in coordinates, setting $\tilde{\gamma}_{(s)} : q^i = q^i_{(s)}(t)$, $z^A = z^A_{(s)}(t)$, $p_i = p^{(s)}_i(t)$, the algorithm for the determination of $\tilde{\gamma}$ relies both on Pontryagin's equations

$$\frac{dq^i_{(s)}}{dt} = \psi^i(t, q^i_{(s)}, z^A_{(s)})\ , \quad \frac{dp^{(s)}_i}{dt} + p^{(s)}_k\,\frac{\partial \psi^k}{\partial q^i} = \frac{\partial \mathscr{L}}{\partial q^i}\ , \quad p^{(s)}_k\,\frac{\partial \psi^k}{\partial z^A} = \frac{\partial \mathscr{L}}{\partial z^A}$$

[1] See Appendix D.

and on Erdmann–Weierstrass matching conditions

$$q^i_{(s+1)}(a_s) = q^i_{(s)}(a_s)\,,\ p_i^{(s+1)}(a_s) = p_i^{(s)}(a_s)\,,\ (\mathscr{H})_{\tilde{\gamma}^{(s+1)}}(a_s) = (\mathscr{H})_{\tilde{\gamma}^{(s)}}(a_s)$$

- under an arbitrary change of the trivialization u of the bundle P into $u' = u - f(t, q^1, \dots, q^n)$, the *Pontryagin–Poincaré–Cartan form* (2.2.7) obeys the transformation law

$$\Theta_{\rm PPC} \to \Theta'_{\rm PPC} = \left(\mathscr{L}(t, q^i, z^A) - \dot{f}\right) dt + \left(p_i - \frac{\partial f}{\partial q^i}\right) \tilde{\omega}^i = \Theta_{\rm PPC} - df$$

- the extremals of the functional $\int_{\tilde{\gamma}} \Theta'_{\rm PPC}$ differ from those of $\int_{\tilde{\gamma}} \Theta_{\rm PPC}$ by a translation $p_i(t) \to \bar{p}_i(t) = p_i(t) - \frac{\partial f(t, q^i(t))}{\partial t}$ along the fibres of $\mathcal{C}(\mathcal{A}) \xrightarrow{\zeta} \mathcal{A}$;
- as it was to be expected on account of the gauge invariance of the projections $\hat{\gamma} = \zeta \cdot \tilde{\gamma}$ and $\gamma = \upsilon \cdot \tilde{\gamma}$, the corresponding action integrals $\int_{\hat{\gamma}} \mathscr{L}'\, dt$ and $\int_{\hat{\gamma}} \mathscr{L}\, dt$ have actually the same extremals with respect to fixed end–points deformations; in particular, every extremal γ yielding a minimum for the first integral, does the same for the second one and conversely.

The idea is now to take advantage of the gauge structure of the theory so as to make every point of the section $\hat{\gamma}$ into a *critical* point for the Lagrangian.

However, in pursuing this strategy, we should not overlook we are extending the class of admissible sections to piecewise differentiable ones. Furthermore, as far as these are concerned, our definition of deformation of an admissible evolution of the system explicitly includes possible variations of the reference intervals.

Whenever both of the previous circumstances occur, the intention of replacing the original Lagrangian by a gauge equivalent and critical one, becomes extremely awkward. This is because, in order to achieve its goal, the function $f \in \mathscr{F}(\mathcal{V}_{n+1})$ which takes part in the gauge transformation $u \to u - f(t, q^1, \dots, q^n)$, should be "tailored" along the section γ and, therefore, with respect to the intervals $[a_{s-1}\,,\, a_s]$. On the other hand, the evaluation of the second variation of the action integral passes through integrations on the *different* intervals $[a_{s-1}(\xi)\,,\, a_s(\xi)]$. In this connection, it is even thinkable an extreme case in which, as ξ varies, the value $t = a_s(\xi)$ swings between the intervals $[a_{s-1}, a_s]$ and $[a_s, a_{s+1}]$.

Remark 3.1.2: These kind of troubles instantly vanish whenever at most *only one* of the above–named circumstances occurs, namely every time we happen to be in one of the following particular situations:

a) section γ is differentiable and so is γ_ξ for any ξ;

b) section γ is differentiable while γ_ξ is just piecewise–differentiable for any ξ; time intervals $[a_{s-1}(\xi)\,,\, a_s(\xi)]$ may be modified by the deformation process[2];

[2]The reader is referred to Appendix C for the proof of the actual existence of this kind of deformations.

c) section γ is piecewise–differentiable and so is γ_ξ for any ξ; time intervals $[a_{s-1}\,,\,a_s]$ remain unchanged during the deformation process.

Whenever b) occurs, the function f is well-defined and differentiable along the entire interval $[t_0, t_1]$ and, as such, may be easily restricted to any interval $[a_{s-1}(\xi)\,,\,a_s(\xi)]$, no matter how the values $a_{s-1}\,,\ a_s$ vary with ξ. On the other hand, in the circumstance c), the "tailoring" on the function f along the section γ holds good along every deformation γ_ξ. Needless to say, situation a) is the easiest one, as it combines all the simplifications brought by b) and c).

Remark 3.1.3: A further pleasantness regarding the particular circumstances described in the previous Remark lies in the fact that, in all cases a), b), and c), the expression of the second variation turns out to be quite simplified. In order to see this, taking equations (1.5.6 b), (1.5.34 c) into account, we first rewrite relation (3.0.1) more suitably as

$$
\begin{aligned}
\left.\frac{d^2\mathcal{I}[\hat\gamma_\xi]}{d\xi^2}\right|_{\xi=0} = \sum_{s=1}^{N}\int_{a_{s-1}}^{a_s} -\Bigg\{ &\left(\frac{\partial^2\mathscr{H}}{\partial q^i\partial q^j}\right)_{\hat\gamma^{(s)}} X^i_{(s)}X^j_{(s)} + 2\left(\frac{\partial^2\mathscr{H}}{\partial q^i\partial z^A}\right)_{\hat\gamma^{(s)}} X^i_{(s)}\Gamma^A_{(s)} + \\
&+\left(\frac{\partial^2\mathscr{H}}{\partial z^A\partial z^B}\right)_{\hat\gamma^{(s)}} \Gamma^A_{(s)}\Gamma^B_{(s)} + \left(\frac{\partial\mathscr{H}}{\partial q^i}\right)_{\hat\gamma^{(s)}} Z^i_{(s)} + \left(\frac{\partial\mathscr{H}}{\partial z^A}\right)_{\hat\gamma^{(s)}} K^A_{(s)}\Bigg\}dt + \\
&+\sum_{s=1}^{N-1}\left\{\alpha_s^2\left[\frac{d\mathscr{H}}{dt}+\frac{dp_i}{dt}\psi^i\right]_{a_s} - 2\,\alpha_s\left(X^i+\alpha_s\psi^i\right)_{a_s}\left[\frac{dp_i}{dt}\right]_{a_s}\right\}
\end{aligned}
\tag{3.1.2}
$$

where, as usual, $\big[g\big]_{a_s}$ stands for the *jump* of the function g at the corner c_s. It is now readily seen that both in the situation b), in which $\frac{d\mathscr{H}}{dt}$, $\frac{dp_i}{dt}$ and ψ^i don't jump at any of the points $\hat\gamma(a_s)$, and in the situation c), in which $\alpha_s = 0$ for any s, the above expression reduces to the only integral term.

In order to cope with these intricacies, we will try a slightly different approach, in line with the nature of the evolution γ as a finite collection of admissible closed arcs $\gamma^{(s)}$, each viewed as the restriction to the closed interval $[a_{s-1}\,,\,a_s]$ of an admissible section (still denoted by $\gamma^{(s)}$) defined on some open neighborhood $(b_{s-1}\,,\,b_s) \supset [a_{s-1}\,,\,a_s]$.

We begin by introducing a family $\{(U_s, h_s),\, s = 1,\dots,N\}$ of local charts in $\mathcal{V}_{n+1}$ such that each U_s is an open neighborhood of the admissible section $\gamma^{(s)}\colon (b_{s-1}\,,\,b_s) \to \mathcal{V}_{n+1}$. Then, careless about P being a trivial bundle, for any s we make use of a differentiable function $f_{(s)}\colon U_s \to \mathbb{R}$ to change, in each $\pi^{-1}(U_s)$, the global trivialization u into a *local* one $u'_{(s)} = u - f_{(s)}$.

As a consequence, the Lagrangian section (1.4.16) is now locally expressed as

$$
\dot u'_{(s)} = \dot u - \dot f_{(s)} = \mathscr{L}(t, q^i_{(s)}, z^A_{(s)}) - \dot f_{(s)} := \mathscr{L}'_{(s)}(t, q^i_{(s)}, z^A_{(s)}) \tag{3.1.3}
$$

and so it relies on the assignment of s different functions $\mathscr{L}'_{(s)}$, each of them defined over the open set $\pi^{-1}(U_s)$, π here denoting the projection $\mathcal{A} \xrightarrow{\pi} \mathcal{V}_{n+1}$.

Likewise, instead of a unique and globally defined Pontryagin–Poincaré–Cartan form (2.2.7), we have now a collection of local 1–forms $\Theta^{(s)}_{\rm PPC}$ whose representation in coordinates reads

$$\Theta^{(s)}_{\rm PPC} \;=\; \Theta_{\rm PPC} - df_{(s)} = \mathscr{L}'_{(s)}\,dt + \left(p_i^{(s)} - \frac{\partial f_{(s)}}{\partial q^i}\right)\tilde{\omega}^i_{(s)} \tag{3.1.4}$$

The idea is to make good use of the above construction, simply by choosing "suitable" functions $f_{(s)}$. In this regard we state

Definition 3.1.1. *Given a normal extremal γ, a function $S^{(s)} \in \mathscr{F}(U_s)$ is said to be* adapted *to the section $\gamma^{(s)}$ if and only if it fulfils the condition*[3]

$$(dS^{(s)})_{\tilde{\gamma}^{(s)}} \;=\; (\Theta^{(s)}_{PPC})_{\tilde{\gamma}^{(s)}} \tag{3.1.5}$$

By a little abuse of language, whenever a function $f_{(s)}\colon U_s \to \mathbb{R}$ is adapted to $\gamma^{(s)}$, the same terminology will be used to denote the corresponding Lagrangian function $\mathscr{L}'_{(s)}$ which takes part in the representation (3.1.3).

Theorem 3.1.1. *For any $s = 1,\dots,N$, there exists (at least) a differentiable function $S^{(s)} \in \mathscr{F}(U_s)$* adapted *to the section $\gamma^{(s)}\colon (b_{s-1}\,,\,b_s) \to \mathcal{V}_{n+1}$.*

Proof. As it is showed in Appendix A, each arc $\gamma^{(s)}$ may be locally made into the coordinate line $\bar{q}^i_{(s)}(t, q^1,\dots,q^n) = 0$, for instance by setting $\bar{q}^i_{(s)} := q^i - q^i_{(s)}(t)$.

A possible local solution of equation (3.1.5) is now easily recognized to be

$$S_0^{(s)}(t,q^i) \;=\; \bar{p}_k^{(s)}(t)\,\bar{q}^k_{(s)} \;+\; \int_{t_0}^{t} \mathscr{L}_{|\hat{\gamma}}\;dt \tag{3.1.6}$$

$\bar{p}_k^{(s)}(t)$ being any functions satisfying $\bar{p}_k^{(s)}(t)\,\frac{\partial \bar{q}^k_{(s)}}{\partial q^i}\Big|_{\gamma^{(s)}(t)} = p_i^{(s)}(t)$.

Then, as a direct consequence of the vanishing of $\bar{q}^i_{(s)}$ along $\gamma^{(s)}$, we have:

$$0 \;=\; \frac{d}{dt}\Big(\bar{q}^i_{(s)\,|\,\gamma^{(s)}}\Big) \;=\; \frac{\partial \bar{q}^i_{(s)}}{\partial t}\bigg|_{\gamma^{(s)}} \;+\; \frac{\partial \bar{q}^i_{(s)}}{\partial q^k}\bigg|_{\gamma^{(s)}} \psi^k_{\,|\hat{\gamma}^{(s)}} \tag{3.1.7a}$$

$$\frac{\partial S_0^{(s)}}{\partial q^i}\bigg|_{\gamma^{(s)}} \;=\; \bar{p}_k^{(s)}(t)\,\frac{\partial \bar{q}^k_{(s)}}{\partial q^i}\bigg|_{\gamma^{(s)}} \;=\; p_i^{(s)}(t) \tag{3.1.7b}$$

$$\begin{aligned}\frac{\partial S_0^{(s)}}{\partial t}\bigg|_{\gamma^{(s)}} \;&=\; \bar{p}_k^{(s)}(t)\,\frac{\partial \bar{q}^k_{(s)}}{\partial t}\bigg|_{\gamma^{(s)}} + \mathscr{L}_{|\hat{\gamma}^{(s)}} \;=\; -\bar{p}_k^{(s)}(t)\,\frac{\partial \bar{q}^i_{(s)}}{\partial q^k}\bigg|_{\gamma^{(s)}} \psi^k_{\,|\hat{\gamma}^{(s)}} + \mathscr{L}_{|\hat{\gamma}^{(s)}} \;= \\ &=\; \Big(\mathscr{L} \,-\, p_k^{(s)}(t)\,\psi^k\Big)_{\hat{\gamma}^{(s)}}\end{aligned} \tag{3.1.7c}$$

□

[3] As usual, we are not distinguish between functions on $\mathcal{V}_{n+1}$ and their pull–back on $\mathcal{C}(\mathcal{A})$.

By linearity, the most general solution of equation (3.1.5) has therefore the local expression

$$S^{(s)} \;=\; S_0^{(s)} \;+\; C^{(s)}(t,q^i) \tag{3.1.8}$$

being $C^{(s)}(t,q^i)$ any function satisfying $(dC^{(s)})_{\gamma^{(s)}} \;=\; 0$. Henceforth, every transformation $S^{(s)} \to S^{(s)} + C^{(s)}$ will be called a *restricted gauge transformation.*

Theorem 3.1.2. *Equation (3.1.5) implies the relations*

$$\mathscr{L}'_{(s)\,|\,\hat\gamma^{(s)}} \;=\; d\mathscr{L}'_{(s)\,|\,\hat\gamma^{(s)}} \;=\; 0 \tag{3.1.9}$$

Proof. Because of Theorem 3.1.1, we have only to check the validity of equations (3.1.9) for the function (3.1.6). In view of equations (3.1.7b,c), denoting by $\dot S_0^{(s)}$ its symbolic time derivative, we have the evaluation

$$\dot S_0^{(s)}{}_{|\,\hat\gamma^{(s)}} \;=\; \left.\frac{\partial S_0^{(s)}}{\partial t}\right|_{\gamma^{(s)}} \;+\; p_k^{(s)}\,\psi^k{}_{|\,\hat\gamma^{(s)}} \;=\; \mathscr{L}_{|\,\hat\gamma^{(s)}} \tag{3.1.10}$$

yielding back the relation $\mathscr{L}'_{(s)\,|\,\hat\gamma^{(s)}} \;=\; \big(\mathscr{L} \;-\; \dot S_0^{(s)}\big)_{|\,\hat\gamma^{(s)}} \;=\; 0$.

Moreover, we have the further relations

$$\left.\frac{\partial \dot S_0^{(s)}}{\partial q^k}\right|_{\hat\gamma^{(s)}} \;=\; \left.\frac{\partial^2 S_0^{(s)}}{\partial q^k \partial q^r}\right|_{\hat\gamma^{(s)}} \psi^r{}_{|\,\hat\gamma^{(s)}} \;+\; \left.\frac{\partial^2 S_0^{(s)}}{\partial q^k \partial t}\right|_{\hat\gamma^{(s)}} \;+\; p_r^{(s)} \left.\frac{\partial \psi^r}{\partial q^k}\right|_{\hat\gamma^{(s)}} \;=$$

$$=\; \frac{dp_k^{(s)}}{dt} \;+\; p_r^{(s)} \left.\frac{\partial \psi^r}{\partial q^k}\right|_{\hat\gamma^{(s)}} \;=\; \left.\frac{\partial \mathscr{L}}{\partial q^k}\right|_{\hat\gamma^{(s)}} \tag{3.1.11a}$$

$$\left.\frac{\partial \dot S_0^{(s)}}{\partial z^A}\right|_{\hat\gamma^{(s)}} \;=\; p_k^{(s)} \left.\frac{\partial \psi^k}{\partial z^A}\right|_{\hat\gamma^{(s)}} \;=\; \left.\frac{\partial \mathscr{L}}{\partial z^A}\right|_{\hat\gamma^{(s)}} \tag{3.1.11b}$$

whence also

$$\left.\frac{\partial \dot S_0^{(s)}}{\partial t}\right|_{\hat\gamma^{(s)}} \;=\; \frac{d\dot S^{(s)}_{0\,|\,\hat\gamma^{(s)}}}{dt} \;-\; \left.\frac{\partial S_0^{(s)}}{\partial q^k}\right|_{\hat\gamma^{(s)}} \psi^k{}_{|\,\hat\gamma^{(s)}} \;-\; \left.\frac{\partial \dot S_0^{(s)}}{\partial z^A}\right|_{\hat\gamma^{(s)}} \frac{dz^A}{dt} \;=$$

$$=\; \frac{d\mathscr{L}_{|\,\hat\gamma^{(s)}}}{dt} \;-\; \left.\frac{\partial \mathscr{L}}{\partial q^k}\right|_{\hat\gamma^{(s)}} \psi^k{}_{|\,\hat\gamma^{(s)}} \;-\; \left.\frac{\partial \mathscr{L}}{\partial z^A}\right|_{\hat\gamma^{(s)}} \frac{dz^A}{dt} \;=\; \left.\frac{\partial \mathscr{L}}{\partial t}\right|_{\hat\gamma^{(s)}} \tag{3.1.11c}$$

□

On account of Theorems 3.1.1, 3.1.2, the Hessian $\big(d^2\mathscr{L}'_{(s)}\big)_{\hat\gamma^{(s)}}$ provides a tensorial field along $\hat\gamma^{(s)}$ for any $s = 1,\dots,N$. Its local representation is most easily expressed in terms of non–holonomic bases. Denoting by $\odot$ the symmetrized tensor product and setting $\tilde{\omega}^i_{(s)} = \left(dq^i - \psi^i\,dt\right)_{\hat{\gamma}^{(s)}} = \omega^i_{\,|\,\hat{\gamma}^{(s)}}$, $\tilde{\nu}^A_{(s)} = \left(dz^A - \frac{dz^A}{dt}\,dt\right)_{\hat{\gamma}^{(s)}}$, a straightforward calculation yields the result

$$\left(d^2\mathscr{L}'_{(s)}\right)_{\hat{\gamma}^{(s)}} = \left[\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^r\partial q^k}\right]_{\hat{\gamma}^{(s)}} \tilde{\omega}^r_{(s)}\otimes\tilde{\omega}^k_{(s)} + 2\left[\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial q^k}\right]_{\hat{\gamma}^{(s)}} \tilde{\nu}^A_{(s)}\odot\tilde{\omega}^k_{(s)} + \\ + \left[\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial z^B}\right]_{\hat{\gamma}^{(s)}} \tilde{\nu}^A_{(s)}\otimes\tilde{\nu}^B_{(s)} \tag{3.1.12}$$

Remark 3.1.4: The components $G^{(s)}_{AB} := \left[\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial z^B}\right]_{\hat{\gamma}^{(s)}}$ are invariant under arbitrary restricted gauge transformations and may therefore be evaluated arbitrarily choosing $S^{(s)}$ within the class of solutions of equation (3.1.5). Making use of the ansatz (3.1.6), we obtain the representation

$$G^{(s)}_{AB} = \left[\frac{\partial^2(\mathscr{L}-\dot{S}^{(s)})}{\partial z^A\partial z^B}\right]_{\hat{\gamma}^{(s)}} = \left[\frac{\partial^2\mathscr{L}}{\partial z^A\partial z^B}\right]_{\hat{\gamma}^{(s)}} - p^{(s)}_i(t)\left[\frac{\partial^2\psi^i}{\partial z^A\partial z^B}\right]_{\hat{\gamma}^{(s)}}$$

or equivalently

$$G^{(s)}_{AB} = -\left[\frac{\partial^2 K_{(s)}}{\partial z^A\partial z^B}\right]_{\hat{\gamma}^{(s)}} \tag{3.1.13}$$

with $K_{(s)} := p^{(s)}_i(t)\,\psi^i(t,q^i,z^A) - \mathscr{L}(t,q^i,z^A)$, henceforth referred to as the *restricted Pontryagin Hamiltonian.*

In view of the identification $\left[\frac{\partial^2 K_{(s)}}{\partial z^A\partial z^B}\right]_{\hat{\gamma}^{(s)}(t)} = \left[\frac{\partial^2\mathscr{H}_{(s)}}{\partial z^A\partial z^B}\right]_{\tilde{\gamma}^{(s)}(t)}$, the matrix (3.1.13) is automatically non singular along any *regular* extremal.

Remark 3.1.5: Whenever $\det G^{(s)}_{AB}\neq 0$, the Hessian (3.1.12) determines an *infinitesimal control* along $\hat{\gamma}^{(s)}$, namely a linear section $h^{(s)} : V(\gamma^{(s)}) \to A(\hat{\gamma}^{(s)})$, uniquely defined by the condition

$$\left\langle \left(d^2\mathscr{L}'_{(s)}\right)_{\hat{\gamma}^{(s)}},\, h^{(s)}(X_{(s)})\otimes Y_{(s)}\right\rangle = 0 \quad \forall\, X_{(s)}\in V(\gamma^{(s)})\,,\ Y_{(s)}\in V(\hat{\gamma}^{(s)}) \tag{3.1.14a}$$

In view of equations (1.5.13), (3.1.12), the requirement (3.1.14 a) is locally expressed the relations

$$\left\langle \left(d^2\mathscr{L}'_{(s)}\right)_{\hat{\gamma}^{(s)}},\, \tilde{\partial}_i\otimes\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}^{(s)}}\right\rangle = \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^A}\right)_{\hat{\gamma}^{(s)}} + G^{(s)}_{AB}\,h^{(s)}{}_i^B = 0 \tag{3.1.14b}$$

Under the assumption $\det G^{(s)}_{AB}\neq 0$, these may be solved for the components $h^{(s)}{}_i^B$, thereby providing the representation

$$h^{(s)}{}_i^A = -G^{AB}_{(s)}\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^B}\right)_{\hat{\gamma}^{(s)}}$$

whence also

$$\tilde{\partial}_i := h^{(s)}\left[\left(\frac{\partial}{\partial q^i}\right)_{\gamma^{(s)}}\right] = \left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}^{(s)}} - G^{AB}_{(s)}\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^B}\right)_{\hat{\gamma}^{(s)}}\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} \tag{3.1.15}$$

with $G^{(s)}_{AB}\, G^{BC}_{(s)} = \delta_A{}^C$.

The absolute time derivative along $\gamma^{(s)}$ induced by $h^{(s)}$ will be denoted by $\left(\frac{D}{Dt}\right)_{\gamma^{(s)}}$. The expression (1.5.21 b) for the temporal connection coefficients takes now the form

$$\tau_i{}^j \;=\; -\,\tilde{\partial}_i \psi^j \;=\; -\left(\frac{\partial \psi^j}{\partial q^i}\right)_{\hat{\gamma}^{(s)}} + G^{AB}_{(s)} \left(\frac{\partial \psi^j}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} \left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial z^B}\right)_{\hat{\gamma}^{(s)}} \tag{3.1.16}$$

Unlike the components $G^{(s)}_{AB}$, the full Hessian (3.1.12) and therefore also the associated infinitesimal control and its corresponding time derivative, are *not* gauge invariant, but explicitly depend on the particular choice of the Lagrangian $\mathscr{L}'_{(s)}$.

In view of Erdmann-Weierstrass conditions (2.3.15), the following identity is easily seen to hold at each corner c_s

$$\left.\frac{\partial S^{(s)}}{\partial q^i}\right|_{c_s} = \left.\frac{\partial S^{(s+1)}}{\partial q^i}\right|_{c_s}, \quad \left.\frac{\partial S^{(s)}}{\partial t}\right|_{c_s} = \left.\frac{\partial S^{(s+1)}}{\partial t}\right|_{c_s} \quad \Longrightarrow \quad d\big(S^{(s+1)} - S^{(s)}\big)_{c_s} = 0$$

and so the Hessian of the difference $S^{(s+1)} - S^{(s)}$, evaluated at the point $c_s = \big(a_s\,,\, \gamma^{(s)}(a_s)\big)$, is itself a tensor, hereby denoted by $\big[d^2 S\big]_{c_s}$.

We now introduce the quantity

$$\begin{aligned} \sigma_s(\xi) \;:=\; & \Big(S^{(s+1)} \,-\, S^{(s)}\Big)_{c_s(\xi)} \;= \\ &= S^{(s+1)}\Big(a_s(\xi)\,,\, \varphi^i_{(s+1)}\big(a_s(\xi),\xi\big)\Big) \,-\, S^{(s)}\Big(a_s(\xi)\,,\, \varphi^i_{(s)}\big(a_s(\xi),\xi\big)\Big) \end{aligned} \tag{3.1.17}$$

and, in view of (1.5.32) and (1.5.33 b), we point up the relation

$$\begin{aligned} \left.\frac{d^2 \sigma_s(\xi)}{d\xi^2}\right|_{\xi=0} =\; & \alpha_s^2 \left.\frac{\partial^2 \big(S^{(s+1)} - S^{(s)}\big)}{\partial t^2}\right|_{c_s} + 2\alpha_s\Big(\alpha_s \psi^i + X^i\Big)_{c_s} \left.\frac{\partial^2 \big(S^{(s+1)} - S^{(s)}\big)}{\partial t\, \partial q^i}\right|_{c_s} + \\ & + \Big(\alpha_s \psi^i + X^i\Big)_{c_s} \Big(\alpha_s \psi^j + X^j\Big)_{c_s} \left.\frac{\partial^2 \big(S^{(s+1)} - S^{(s)}\big)}{\partial q^i\, \partial q^j}\right|_{c_s} \end{aligned} \tag{3.1.18}$$

written more suitably as

$$\left.\frac{d^2 \sigma_s(\xi)}{d\xi^2}\right|_{\xi=0} \;=\; \Big\langle \big[d^2 S\big]_{c_s}\,,\; W_s \otimes W_s \Big\rangle \tag{3.1.19}$$

From this, collecting all the previous results, we get the following identity

$$\begin{aligned} \sum_{s=1}^{N} \int_{a_{s-1}(\xi)}^{a_s(\xi)} \mathscr{L}'_{(s)|\hat{\gamma}_\xi}\, dt \;-\; \sum_{s=1}^{N-1} \sigma_s(\xi) \;=\; & \int_{\hat{\gamma}_\xi} \mathscr{L}\, dt \;-\; \sum_{s=1}^{N} \int_{a_{s-1}(\xi)}^{a_s(\xi)} \dot{S}^{(s)}{}_{|\hat{\gamma}_\xi}\, dt \;+ \\ - \sum_{s=1}^{N-1} \Big(S^{(s+1)} \,-\, S^{(s)}\Big)_{c_s(\xi)} \;=\; & \int_{\hat{\gamma}_\xi} \mathscr{L}\, dt \;-\; S^{(N)}(t_1) \;+\; S^{(0)}(t_0) \;= \\ =\; & \int_{\hat{\gamma}_\xi} \mathscr{L}\, dt \;-\; \int_{\hat{\gamma}} \mathscr{L}\, dt \end{aligned} \tag{3.1.20}$$

For later use, before moving on to the analysis of the second variation of the action functional, we observe that, whenever γ is a normal extremal, the use of an adapted Lagrangian provides a canonical splitting of $T(\mathcal{C}(\mathcal{A}))$.

As already remarked, if the trivialization u of P is changed in each $\pi^{-1}(U_s)$ into the local one $u'_{(s)} = u - f_{(s)}(t, q^1, \ldots, q^n)$, then the extremals of the corresponding functional $\sum_{s=1}^N \int_{\tilde{\gamma}^{(s)}} \Theta^{(s)}_{\mathrm{PPC}}$ differ locally from those of $\int_{\tilde{\gamma}} \Theta_{\mathrm{PPC}}$ by a translation $p_i(t) \to \bar{p}^{(s)}_i(t) = p_i(t) - \frac{\partial f_{(s)}(t, q^i(t))}{\partial q^i}$ along the fibres of $\mathcal{C}(\mathcal{A})$. In particular:

- if $f_{(s)}$ is adapted to $\gamma^{(s)}$, the local representation of $\tilde{\gamma}^{(s)}$ satisfies the condition $\bar{p}^{(s)}_i(t) \equiv 0$;
- because of the condition $(dC^{(s)})_{\gamma^{(s)}} = 0$, the lift $\gamma \to \tilde{\gamma}$ is evidently invariant under restricted gauge transformations.

Moreover, in view of the nature of $\mathcal{C}(\mathcal{A}) \xrightarrow{\zeta} \mathcal{A}$ as a vector bundle over $\mathcal{A}$, each local section $\mathcal{O}_{(s)} : \pi^{-1}(U_s) \subset \mathcal{A} \to \mathcal{C}(\mathcal{A})$ given by $p^{(s)}_i(t, q^i, z^A) = 0$ has an invariant geometrical meaning.

For each $\sigma_{(s)} \in \mathcal{O}_{(s)}(\pi^{-1}(U_s))$, every vector $\tilde{Z}_{(s)} \in T_{\sigma_{(s)}}(\mathcal{C}(\mathcal{A}))$ may therefore be split into a "horizontal" part $\tilde{Z}^h_{(s)} \in \mathcal{O}_{(s)*}\zeta_*(\tilde{Z}_{(s)})$ tangent to the submanifold $\mathcal{O}_{(s)}(\pi^{-1}(U_s))$, and a "vertical" part $\tilde{Z}^v_{(s)}$, tangent to the fiber $\zeta^{-1}(\zeta(\sigma_{(s)}))$ at $\sigma_{(s)}$. In coordinates, we have the explicit representation

$$\tilde{Z}^v_{(s)} = \left\langle \tilde{Z}, (dp^{(s)}_i)_{\sigma_{(s)}} \right\rangle \left(\frac{\partial}{\partial p_i} \right)_{\sigma_{(s)}}, \qquad \tilde{Z}^h_{(s)} = \tilde{Z}_{(s)} - \tilde{Z}^v_{(s)}$$

The previous algorithm can be get interacted with another intrinsic attribute of $\mathcal{C}(\mathcal{A})$, represented by the Liouville 1–form (1.2.17). In this way, to each vector $\tilde{Z}_{(s)} \in T_{\sigma_{(s)}}(\mathcal{C}(\mathcal{A}))$ we may associate a 1–form in $T^*_{\sigma_{(s)}}(\mathcal{C}(\mathcal{A}))$ according to the prescription

$$\begin{aligned} \tilde{Z}_{(s)} \to \tilde{Z}^v_{(s)} \to \tilde{Z}^v_{(s)} \lrcorner \left(d\Theta_L\right)_{\sigma_{(s)}} &= \left\langle \tilde{Z}_{(s)}, (dp^{(s)}_i)_{\sigma_{(s)}} \right\rangle \left(\frac{\partial}{\partial p_i} \lrcorner\, d\Theta_L \right)_{\sigma_{(s)}} = \\ &= \left\langle \tilde{Z}_{(s)}, (dp^{(s)}_i)_{\sigma_{(s)}} \right\rangle \omega^i_{\sigma_{(s)}} \end{aligned} \tag{3.1.21}$$

At last, we observe that every element of the cotangent space $T^*_{\sigma_{(s)}}(\mathcal{C}(\mathcal{A}))$ generated by the above process may be uniquely expressed as the pull–back of a 1–form $\lambda_{(s)} \in T^*_{\upsilon(\sigma_{(s)})}(\mathcal{V}_{n+1})$.

Collecting all results and recalling the definition of virtual 1–forms introduced in § 1.3, we conclude that the lift algorithm $\gamma \to \tilde{\gamma}$ determines a bijective correspondence between vector fields $\tilde{Z}_{(s)}$ along each $\tilde{\gamma}^{(s)}$ and pairs $(\hat{Z}_{(s)}, \lambda_{(s)})$, in which $\hat{Z}_{(s)} = \zeta_*(\tilde{Z}_{(s)})$ is a vector field along $\hat{\gamma}^{(s)} = \zeta \cdot \tilde{\gamma}^{(s)}$, while $\lambda_{(s)} = \left\langle \tilde{Z}_{(s)}, (dp_i)_{\tilde{\gamma}^{(s)}} \right\rangle \hat{\omega}^i$

is a virtual 1–form along $\gamma^{(s)}$. This will play a crucial role in the subsequent discussion.

3.2 The second variation of the action functional

Let $\gamma\colon [t_0,t_1] \to \mathcal{V}_{n+1}$ be a normal (not necessarily regular) extremal of the action functional (2.1.1). In view of the identity (3.1.20), the analysis of the second variation of $\mathcal{I}[\gamma]$ may be better carried out by evaluating the second derivative

$$\left.\frac{d^2\mathcal{I}[\hat\gamma_\xi]}{d\xi^2}\right|_{\xi=0} = \frac{d^2}{d\xi^2}\left[\sum_{s=1}^{N}\int_{a_{s-1}(\xi)}^{a_s(\xi)} \mathscr{L}'_{(s)\,|\,\hat\gamma_\xi}\,dt \;-\; \sum_{s=1}^{N-1}\sigma_s(\xi)\right]_{\xi=0}$$

In this connection, being each $\mathscr{L}'_{(s)}$ adapted to the corresponding arc $\hat\gamma^{(s)}$, a simple calculation yields the result

$$\begin{aligned}\frac{d^2}{d\xi^2}\left[\int_{a_{s-1}(\xi)}^{a_s(\xi)} \mathscr{L}'_{(s)\,|\,\hat\gamma_\xi}\,dt\right]_{\xi=0} &= \int_{a_{s-1}}^{a_s}\left[\left(\frac{\partial^2\mathscr{L}}{\partial q^i\partial q^j}\right)_{\hat\gamma^{(s)}} X^i_{(s)}X^j_{(s)} + \right.\\ &\left. + 2\left(\frac{\partial^2\mathscr{L}}{\partial q^i\partial z^A}\right)_{\hat\gamma^{(s)}} X^i_{(s)}\Gamma^A_{(s)} + \left(\frac{\partial^2\mathscr{L}}{\partial z^A\partial z^B}\right)_{\hat\gamma^{(s)}} \Gamma^A_{(s)}\Gamma^B_{(s)}\right] = \\ &= \int_{a_{s-1}}^{a_s}\left\langle\left(d^2\mathscr{L}'_{(s)}\right)_{\hat\gamma^{(s)}}, \hat X_{(s)}\otimes\hat X_{(s)}\right\rangle dt\end{aligned}\tag{3.2.1}$$

which, together with equation (3.1.18), provides the final (plainly covariant) expression

$$\begin{aligned}\left.\frac{d^2\mathcal{I}[\hat\gamma_\xi]}{d\xi^2}\right|_{\xi=0} &= \sum_{s=1}^{N}\int_{a_{s-1}}^{a_s}\left\langle\left(d^2\mathscr{L}'_{(s)}\right)_{\hat\gamma^{(s)}}\,,\;\hat X_{(s)}\otimes\hat X_{(s)}\right\rangle dt \;+\\ &- \sum_{s=1}^{N-1}\left\langle\left[d^2S\right]_{c_s}\,,\;W_s\otimes W_s\right\rangle\end{aligned}\tag{3.2.2}$$

Remark 3.2.1: In view of equation (3.1.8), the Lagrangian $\mathscr{L}'_{(s)}$ is not unique, but is defined up to a restricted gauge transformation $\mathscr{L}'_{(s)} \to \mathscr{L}'_{(s)} - \dot C^{(s)}$, with $(dC^{(s)})_{\gamma^{(s)}} = 0$. Therefore, as an internal consistency check, we ought to prove that the expression (3.2.2) does not depend on any specific choice of the functions $S^{(s)}(t,q^i)$.

We start by noticing the following identities, which are a straightforward consequence of the condition $(dC^{(s)})_{\gamma(s)} = 0$:

$$0 = \frac{d}{dt}\left(\frac{\partial C^{(s)}}{\partial q^i}\right)_{\gamma(s)} = \left(\frac{\partial^2 C^{(s)}}{\partial q^i \partial t}\right)_{\gamma(s)} + \left(\frac{\partial^2 C^{(s)}}{\partial q^i \partial q^j}\right)_{\gamma(s)} \psi^j_{\,|\hat{\gamma}(s)}$$

$$0 = \frac{d}{dt}\left(\frac{\partial C^{(s)}}{\partial t}\right)_{\gamma(s)} = \left(\frac{\partial^2 C^{(s)}}{\partial t^2}\right)_{\gamma(s)} + \left(\frac{\partial^2 C^{(s)}}{\partial t \partial q^j}\right)_{\gamma(s)} \psi^j_{\,|\hat{\gamma}(s)} =$$

$$= \left(\frac{\partial^2 C^{(s)}}{\partial t^2}\right)_{\gamma(s)} - \left(\frac{\partial^2 C^{(s)}}{\partial q^i \partial q^j}\right)_{\gamma(s)} \psi^i_{\,|\hat{\gamma}(s)} \psi^j_{\,|\hat{\gamma}(s)}$$

In view of these, denoting by $\left[d^2 C\right]_{c_s}$ the tensor provided at the point c_s by the Hessian of the difference $C^{(s+1)} - C^{(s)}$, we now evaluate

$$\begin{aligned}
\Big\langle \left[d^2 C\right]_{c_s}, W_s \otimes W_s \Big\rangle &= \alpha_s^2 \left[\frac{\partial^2 C}{\partial t^2}\right]_{c_s} + 2\alpha_s \Big(\alpha_s \psi^i + X^i\Big)_{c_s} \left[\frac{\partial^2 C}{\partial t\, \partial q^i}\right]_{c_s} + \\
&+ \Big(\alpha_s \psi^i + X^i\Big)_{c_s} \Big(\alpha_s \psi^j + X^j\Big)_{c_s} \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j}\right]_{c_s} = \\
&= \alpha_s^2 \left[\frac{\partial^2 C}{\partial q^i \partial q^j} \psi^i \psi^j\right]_{c_s} - 2\alpha_s \Big(\alpha_s \psi^i + X^i\Big)_{c_s} \left[\frac{\partial^2 C}{\partial q^i \partial q^j} \psi^j\right]_{c_s} + \\
&+ \Big(\alpha_s \psi^i + X^i\Big)_{c_s} \Big(\alpha_s \psi^j + X^j\Big)_{c_s} \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j}\right]_{c_s}
\end{aligned}$$

but, by the jump relations (1.5.34a),

$$\begin{aligned}
\left[\frac{\partial^2 C}{\partial q^i \partial q^j} \psi^j\right]_{c_s} \Big(\alpha_s \psi^i + X^i\Big)_{c_s} &= \left[\frac{\partial^2 C}{\partial q^i \partial q^j} \Big(\alpha_s \psi^i + X^i\Big) \psi^j\right]_{c_s} = \\
&= \left[\frac{\partial^2 C}{\partial q^i \partial q^j} X^i \psi^j\right]_{c_s} + \alpha_s \left[\frac{\partial^2 C}{\partial q^i \partial q^j} \psi^i \psi^j\right]_{c_s}
\end{aligned}$$

and also

$$\begin{aligned}
\left[\frac{\partial^2 C}{\partial q^i\, \partial q^j}\right]_{c_s} \Big(\alpha_s \psi^i + X^i\Big)_{c_s} \Big(\alpha_s \psi^j + X^j\Big)_{c_s} &= \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j} \Big(\alpha_s \psi^i + X^i\Big) \Big(\alpha_s \psi^j + X^j\Big)\right]_{c_s} = \\
&= \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j} X^i X^j\right]_{c_s} + 2\alpha_s \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j} X^i \psi^j\right]_{c_s} + \alpha_s^2 \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j} \psi^i \psi^j\right]_{c_s}
\end{aligned}$$

whence finally

$$\sum_{s=1}^{N-1} \Big\langle \left[d^2 C\right]_{c_s}, W_s \otimes W_s \Big\rangle = \sum_{s=1}^{N-1} \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j} X^i X^j\right]_{c_s}$$

On the other hand, by equation (3.1.1), we have

$$\begin{aligned}
\sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} \Big\langle \big(d^2 \dot{C}^{(s)}\big)_{\hat{\gamma}(s)}, \hat{X}_{(s)} \otimes \hat{X}_{(s)} \Big\rangle\, dt &= \sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} \frac{d}{dt} \Big\langle \big(d^2 C^{(s)}\big)_{\gamma(s)}, X_{(s)} \otimes X_{(s)} \Big\rangle\, dt = \\
&= - \sum_{s=1}^{N-1} \left[\frac{\partial^2 C}{\partial q^i\, \partial q^j} X^i X^j\right]_{c_s}
\end{aligned}$$

and so we see that each single term of the right–hand side of equation (3.2.2) actually depends on how the function $S^{(s)}$ has been chosen, while the entire expression is (as hoped) gauge–invariant.

The problem of establishing whether a locally normal extremal constitutes a minimum for the functional (2.1.1), now based on the analysis of the expression (3.2.2), may be conveniently broken up into two consecutive logical steps:

i) first of all, each single arc $\big(\gamma^{(s)}, [a_{s-1}, a_s]\big)$ is requested to give rise to a minimum with respect to the special class of deformations which leave the points $\gamma^{(s)}(a_{s-1})$, $\gamma^{(s)}(a_s)$ fixed;

ii) afterwards, it still remains to figure out how to link up the previous results in order to make them globally applicable to the entire evolution γ.

This way of going about the matter surely makes the treatment a little bit longer than what it would be in case the problem is tackled as a whole at once. However, in return, the discussion will turn out to be more clear as various difficulties are faced separately. Moreover, the analysis of i), that will henceforth called the *associated single–arc problem*, is evidently equivalent to the one that would be drawn when dealing with the (not infrequent) situation[4] in which the section γ is differentiable as well as γ_ξ for any ξ.

3.3 The associated single–arc problem

From now on we shall thus momentarily focus our attention on a single specific admissible closed arc $\big(\gamma^{(s)}, [a_{s-1}, a_s]\big)$, which is supposed to represent a *normal* extremal of the action functional $\int_{\hat\gamma^{(s)}} \mathscr{L}\, dt$. Collecting all the previous results, we see that the analysis of its second variation involves uniquely the behavior of the integral

$$\left.\frac{d^2\mathcal{I}\,[\hat\gamma^{(s)}_\xi]}{d\xi^2}\right|_{\xi=0} = \int_{a_{s-1}}^{a_s} \Big\langle \big(d^2\mathscr{L}'_{(s)}\big)_{\hat\gamma^{(s)}} \,,\ \hat X_{(s)} \otimes \hat X_{(s)} \Big\rangle\, dt \tag{3.3.1}$$

In particular, when $\gamma^{(s)}$ is a *regular* extremal, introducing the horizontal basis (3.1.15) associated with the hessian $\big(d^2\mathscr{L}'_{(s)}\big)_{\hat\gamma^{(s)}}$ and expressing $\hat X_{(s)}$ in components as $\hat X_{(s)} = X^i_{(s)}\,\tilde\partial_i + Y^A_{(s)}\big(\frac{\partial}{\partial z^A}\big)_{\hat\gamma^{(s)}}$, equation (3.1.14b) provides the identification

$$\Big\langle \big(d^2\mathscr{L}'_{(s)}\big)_{\hat\gamma^{(s)}} \,,\, \hat X_{(s)} \otimes \hat X_{(s)} \Big\rangle = N^{(s)}_{kr}\, X^k_{(s)}\, X^r_{(s)} + G^{(s)}_{AB}\, Y^A_{(s)}\, Y^B_{(s)} \tag{3.3.2}$$

[4]See Remark 3.1.2.

with

$$N^{(s)}_{kr} := \left\langle \left(d^2\mathscr{L}'_{(s)}\right)_{\hat\gamma^{(s)}}, \tilde\partial_k \otimes \tilde\partial_r \right\rangle = \left[\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^k \partial q^r}\right) - G^{AB}_{(s)}\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^k \partial z^A}\right)\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^r \partial z^B}\right)\right]_{\hat\gamma^{(s)}} \tag{3.3.3}$$

As already pointed out, unlike the integral (3.3.1), the Hessian $\left(d^2\mathscr{L}'_{(s)}\right)_{\hat\gamma^{(s)}}$ is not a gauge–invariant object. The effect of the restricted gauge group on the representation (3.3.2) is therefore reflected into the fact that the integrand at the right-hand-side of equation (3.3.1) is defined up to an arbitrary transformation of the form

$$\begin{aligned}
\left\langle \left(d^2\mathscr{L}'_{(s)}\right)_{\hat\gamma^{(s)}}, \hat X_{(s)} \otimes \hat X_{(s)} \right\rangle \longrightarrow \left\langle \left(d^2\left(\mathscr{L}'_{(s)} - \dot C^{(s)}\right)\right)_{\hat\gamma^{(s)}}, \hat X_{(s)} \otimes \hat X_{(s)} \right\rangle = \\
= \left\langle \left(d^2\mathscr{L}'_{(s)}\right)_{\hat\gamma^{(s)}}, \hat X_{(s)} \otimes \hat X_{(s)} \right\rangle - \frac{d}{dt}\left\langle \left(d^2 C^{(s)}\right)_{\gamma^{(s)}}, X_{(s)} \otimes X_{(s)} \right\rangle = \\
= \left(N^{(s)}_{ij} - \frac{DC^{(s)}_{ij}}{Dt}\right) X^i_{(s)} X^j_{(s)} - 2\, C^{(s)}_{ij}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} X^j_{(s)}\, Y^A_{(s)} + G^{(s)}_{AB}\, Y^A_{(s)} Y^B_{(s)}
\end{aligned} \tag{3.3.4}$$

where we have introduced the simplified notation $C^{(s)}_{ij} := \left(\frac{\partial^2 C^{(s)}}{\partial q^i \partial q^j}\right)_{\gamma^{(s)}}$ and with the components $\frac{DC_{ij}}{Dt}$ expressed by equation (1.5.23) in terms of the ordinary derivatives $\frac{dC_{ij}}{dt}$ and of the temporal connection coefficients ${\tau_i}^k$.

On this basis we state

Theorem 3.3.1. *Let $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$ be a normal extremal. Then, if the matrix $G^{(s)}_{AB}(t)$ is non singular at a point $t^* \in (a_{s-1}, a_s)$, there exist $\varepsilon > 0$ and a restricted gauge transformation $\mathscr{L}'_{(s)} \to \mathscr{L}'_{(s)} - \dot C^{(s)}$ such that the Hessian $\left(d^2(\mathscr{L}'_{(s)} - \dot C^{(s)})\right)_{\hat\gamma^{(s)}(t)}$ has algebraic rank equal to r for $t \in (t^* - \varepsilon, t^* + \varepsilon)$.*

Proof. By continuity, there exists an interval $[c, d] \ni t^*$ where $\det G^{(s)}_{AB} \neq 0$. We focus on that interval, and apply equation (3.3.4) to the arc $\gamma^{(s)}\left([c, d]\right)$. Setting

$$\tilde Y^A_{(s)} := Y^A_{(s)} - G^{AB}_{(s)}\, C^{(s)}_{ir}\left(\frac{\partial\psi^r}{\partial z^B}\right)_{\hat\gamma^{(s)}} X^i_{(s)}$$

and taking the symmetry of C_{ij} into account, equation (3.3.4) may be rewritten as

$$\begin{aligned}
\left\langle \left(d^2(\mathscr{L}'_{(s)} - \dot C^{(s)})\right)_{\hat\gamma^{(s)}}, \hat X_{(s)} \otimes \hat X_{(s)} \right\rangle = \\
= \left[N^{(s)}_{ij} - \frac{DC_{ij}}{Dt} - G^{AB}_{(s)}\left(\frac{\partial\psi^r}{\partial z^A}\right)_{\hat\gamma^{(s)}}\left(\frac{\partial\psi^l}{\partial z^B}\right)_{\hat\gamma^{(s)}} C^{(s)}_{ir}\, C^{(s)}_{lj}\right] X^i_{(s)} X^j_{(s)} + G^{(s)}_{AB}\, \tilde Y^A_{(s)} \tilde Y^B_{(s)}
\end{aligned}$$

The thesis is therefore established as soon as we prove that the Riccati–type differential equation

$$\frac{DC^{(s)}_{ij}}{Dt} + G^{AB}_{(s)}\left(\frac{\partial\psi^r}{\partial z^A}\right)_{\hat\gamma^{(s)}}\left(\frac{\partial\psi^l}{\partial z^B}\right)_{\hat\gamma^{(s)}} C_{ir}\,C_{lj} - N^{(s)}_{ij} = 0 \tag{3.3.5}$$

admits at least one symmetric solution $C^{(s)}_{ij} = C^{(s)}_{ij}(t)$ in a neighborhood of $t = t^*$. To this end, we set

$$M^{rl}_{(s)} := G^{AB}_{(s)}\left(\frac{\partial\psi^r}{\partial z^A}\right)_{\hat\gamma^{(s)}}\left(\frac{\partial\psi^l}{\partial z^B}\right)_{\hat\gamma^{(s)}} \tag{3.3.6}$$

and denote by $C^{(S)}_{ij}$ and $C^{(A)}_{ij}$ respectively the symmetric and antisymmetric part of $C^{(s)}_{ij}$. Due to the symmetry of $M^{ij}_{(s)}$ and $N^{(s)}_{ij}$, equation (3.3.5) then splits into the system

$$\frac{DC^{(S)}_{ij}}{Dt} + M^{rl}_{(s)}\left[C^{(S)}_{ir}\,C^{(S)}_{lj} + C^{(A)}_{ir}\,C^{(A)}_{lj}\right] - N^{(s)}_{ij} = 0 \tag{3.3.7a}$$

$$\frac{DC^{(A)}_{ij}}{Dt} + M^{rl}_{(s)}\,C^{(S)}_{ir}\,C^{(A)}_{lj} + M^{rl}_{(s)}\,C^{(A)}_{ir}\,C^{(S)}_{lj} = 0 \tag{3.3.7b}$$

Being the second equation linear and homogeneous in $C^{(A)}_{ij}$, by Cauchy theorem we conclude that, if we choose $C^{(s)}_{ij}$ symmetric at $t = t^*$, there exists $\varepsilon > 0$ such that the solution of the Cauchy problem for equation (3.3.5) exists and is symmetric for $|t - t^*| < \varepsilon$. $\square$

In view of Theorem 3.3.1, whenever $\det G^{(s)}_{AB}(t^*) \neq 0$, by a proper choice of the gauge around the point $\gamma(t^*)$, the quadratic polynomial (3.3.4) can be reduced to the canonical form

$$\left\langle \left(d^2(\mathscr{L}'_{(s)} - \dot C^{(s)})\right)_{\hat\gamma^{(s)}}, \hat X_{(s)}\otimes\hat X_{(s)}\right\rangle = G^{(s)}_{AB}\,\tilde Y^A_{(s)}\,\tilde Y^B_{(s)} = -\left(\frac{\partial^2 K_{(s)}}{\partial z^A\partial z^B}\right)_{\hat\gamma^{(s)}}\tilde Y^A_{(s)}\,\tilde Y^B_{(s)} \tag{3.3.8}$$

in a neighborhood of t^*, $K_{(s)}$ denoting the restricted Pontryagin Hamiltonian.

Unfortunately, the purely *local* validity of equation (3.3.8) makes it unsuited to the study of the second variation (3.3.1), which involves an integration over the whole interval $[a_{s-1}, a_s]$. We shall return on this point later. At present, we shall concentrate on the role of Theorem 3.3.1 in the identification of *sufficient* conditions for a regular extremal γ to yield a (relative) minimum for the action functional. In this connection, a preliminary result is provided by the following

Corollary 3.3.1.1. *Under the same assumptions as in Theorem 3.3.1, given any vertical vector field* $V_{(s)} = V^A_{(s)}\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma^{(s)}}$ *along* $\hat\gamma^{(s)}$ *with compact support*

$[a,b] \subset (t^*-\varepsilon, t^*+\varepsilon)$, *there exist a differentiable function* $g=g(t)$ *not identically zero on* $[a,b]$ *and an infinitesimal deformation* $\hat{X}_{(s)} = X^i_{(s)}\big(\frac{\partial}{\partial q^i}\big)_{\hat{\gamma}^{(s)}} + Y^A_{(s)}\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}^{(s)}}$ *with support contained in* $[a,b]$ *satisfying the relation*

$$Y^A_{(s)} - G^{AB}_{(s)}\, C^{(s)}_{rs} \left(\frac{\partial \psi^s}{\partial z^B}\right)_{\hat{\gamma}^{(s)}} X^r_{(s)} \,=\, g\, V^A_{(s)}$$

Proof. Using the variational equation in the form (1.5.25a), the required conditions are summarized into the pair of relations

$$\frac{DX^i_{(s)}}{Dt} \,=\, \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} \left[g\, V^A_{(s)} + G^{AB}_{(s)}\, C^{(s)}_{rl} \left(\frac{\partial \psi^l}{\partial z^B}\right)_{\hat{\gamma}^{(s)}} X^r_{(s)} \right] \tag{3.3.9a}$$

$$X^i_{(s)}(a) \,=\, X^i_{(s)}(b) \,=\, 0 \tag{3.3.9b}$$

For any choice of $g(t)$, equation (3.3.9) is a first order linear differential equation for the unknowns $X^i_{(s)}(t)$; integrating it with initial data $X^i_{(s)}(a)=0$ yields the solution

$$X^i_{(s)}(t) \,=\, W^i{}_k(t) \int_a^t (W^{-1})^k{}_r \left(\frac{\partial \psi^r}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} g\, V^A_{(s)}\, d\xi$$

$W^i{}_k$ being the *Wronskian* of the equation. In order to ensure $X^i_{(s)}(b)=0$ it is then sufficient to choose $g(t)$ within the (infinite–dimensional) vector space of differentiable functions over $(t^*-\varepsilon, t^*+\varepsilon)$ satisfying the conditions

$$\int_a^b (W^{-1})^k{}_r \left(\frac{\partial \psi^r}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} g\, V^A_{(s)}\, d\xi \,=\, 0\,, \qquad k = 1\dots n$$

□

Corollary 3.3.1.2. *The positive semidefiniteness of the matrix* $G^{(s)}_{AB}(t)$ *at all* $t\in[a_{s-1},a_s]$ *is a necessary condition for a normal extremal* $\gamma^{(s)}:[a_{s-1},a_s]\to\mathcal{V}_{n+1}$ *to yield a minimum for the action functional.*

Proof. Suppose that $G^{(s)}_{AB}$ is not positive semidefinite at some $t^*\in[a_{s-1},a_s]$. Depending on the value of $\det G^{(s)}_{AB}(t^*)$ we have then two possible alternatives:

i) if $\det G^{(s)}_{AB}(t^*)\neq 0$, on account of Theorem 3.3.1 there exist a restricted gauge transformation $\mathscr{L}'_{(s)} \to \mathscr{L}'_{(s)} - \dot{C}^{(s)}$ such that a representation like (3.3.8) holds in a neighborhood $(t^*-\varepsilon, t^*+\varepsilon)$.

Then, given any vertical vector field $V_{(s)}$ with support contained in the interval $(t^*-\varepsilon, t^*+\varepsilon)$ and satisfying $G^{(s)}_{AB}\, V^A_{(s)}\, V^B_{(s)} < 0$ (for instance, the eigenvector corresponding to the negative eigenvalue of $G^{(s)}_{AB}$ in $(t^*-\varepsilon, t^*+\varepsilon)$, multiplied by a suitable function with compact support), Corollary 3.3.1.1 implies the existence of at least one infinitesimal deformation $\hat{X}_{(s)}$ satisfying

$$\left.\frac{d^2\mathcal{I}}{d\xi^2}\right|_{\xi=0} = \int_{a_{s-1}}^{a_s} \big\langle \big(d^2(\mathscr{L}'_{(s)} - \dot{C}^{(s)})\big)_{\hat{\gamma}^{(s)}}\,,\, \hat{X}_{(s)}\otimes\hat{X}_{(s)} \big\rangle\, dt = \int_a^b g^2\, G^{(s)}_{AB}\, V^A_{(s)}\, V^B_{(s)}\, dt < 0$$

Therefore, γ does not provide a minimum for the action functional.

ii) if $\det G^{(s)}_{AB}(t^*) = 0$, choose $\varepsilon > 0$ in such a way that

- $-\varepsilon$ is not a root of the secular equation $\det(G^{(s)}_{AB} - \lambda\,\delta_{AB}) = 0$;
- at least one root of the secular equation is smaller than $-\varepsilon$.

Let $M \in \mathscr{F}(\mathcal{A})$ be a differentiable function globally defined on $\mathcal{A}$ and having local expression[5] $M = \varepsilon\,\delta_{AB}\,(z^A - z^A(t))(z^B - z^B(t))$ in a neighborhood U of the point $\hat{\gamma}^{(s)}(t^*)$. Also, let $[c, d] \ni t^*$ be a closed interval, satisfying $\hat{\gamma}^{(s)}([c, d]) \subset U$. Setting $\mathscr{L}^*_{(s)} := \mathscr{L}'_{(s)} + M$, one can then easily verify the properties:

a) the section $\gamma^{(s)} : [c, d] \to \mathcal{V}_{n+1}$ is a normal extremal for the action integral $\int_{\hat{\gamma}^{(s)}} \mathscr{L}^*_{(s)}\,dt$;

b) the matrix $\left(\frac{\partial^2 \mathscr{L}^*_{(s)}}{\partial z^A \partial z^B}\right)_{\hat{\gamma}^{(s)}(t^*)} = G^{(s)}_{AB} + \varepsilon\,\delta_{AB}$ is both non singular and non positive (semi)–definite.

In view of a) and b), the analysis developed in point *i)* ensures the existence of at least one infinitesimal deformation $\hat{X}_{(s)}$ having support in $[a, b] \subset [c, d]$ and satisfying $\int_c^d \left\langle (d^2\mathscr{L}^*_{(s)})_{\hat{\gamma}^{(s)}}, \hat{X}_{(s)} \otimes \hat{X}_{(s)} \right\rangle dt < 0$. On the other hand, by construction, this implies also

$$
\begin{aligned}
&\int_c^d \left\langle \left(d^2\mathscr{L}'_{(s)}\right)_{\hat{\gamma}^{(s)}}, \hat{X}_{(s)} \otimes \hat{X}_{(s)} \right\rangle dt \;= \\
&= \int_c^d \left\langle \left(d^2\mathscr{L}^*_{(s)}\right)_{\hat{\gamma}^{(s)}}, \hat{X}_{(s)} \otimes \hat{X}_{(s)} \right\rangle dt \;-\; \varepsilon \int_c^d \delta_{AB} \left\langle dz^A, \hat{X}_{(s)} \right\rangle \left\langle dz^B, \hat{X}_{(s)} \right\rangle dt \;\leqslant \\
&\leqslant \int_c^d \left\langle \left(d^2\mathscr{L}^*_{(s)}\right)_{\hat{\gamma}^{(s)}}, \hat{X}_{(s)} \otimes \hat{X}_{(s)} \right\rangle dt \;<\; 0
\end{aligned}
$$

once again proving that γ does not yield a minimum for the action functional. □

3.3.1 The matrix Riccati equation and the sufficient conditions

From now on we shall concentrate on the class of *regular* normal extremals. The role of regularity in the solution of the Pontryagin equations (2.3.4) — more specifically, in the conversion of these into a system of ordinary differential equations in Hamiltonian forms for the unknowns $q^i(t), p_i(t)$ — should be well known from § 2.4. However, when the problem is not *finding* the extremals, but working with a given extremal $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$, regularity is merely an *attribute* of γ,

[5] As usual, we are writing $z^A(t)$ for $z^A(\hat{\gamma}(t))$.

ensuring the existence of an expression of the form (3.3.8) in a neighborhood of each $t^* \in [a_{s-1}, a_s]$.

On the other hand, as already pointed out, the purely local validity of equation (3.3.8) is of little help in the evaluation of the second variation (3.3.1): it should therefore be investigated to what extent equation (3.3.8) may be converted into a global result, valid over the whole interval $[a_{s-1}, a_s]$. On account of equations (3.3.5), (3.3.6), this means analyzing the interval of existence of the solutions of the Riccati–like differential equation [6]

$$\frac{DC^{(s)}_{ij}}{Dt} + M^{rl}_{(s)}\, C^{(s)}_{ir}\, C^{(s)}_{lj} - N^{(s)}_{ij} = 0 \tag{3.3.10}$$

The main difficulty with the latter comes from its non–linearity. To overcome this aspect, we introduce two auxiliary virtual tensors $E^{(s)}_{ij}(t)$ and $K_{(s)}{}^{i}{}_{j}(t)$ along $\gamma^{(s)}$, satisfying the transport laws

$$\frac{DK_{(s)}{}^{i}{}_{j}}{Dt} = M^{ir}_{(s)}\, E^{(s)}_{rj} \tag{3.3.11a}$$

$$\frac{DE^{(s)}_{ij}}{Dt} = N^{(s)}_{ir}\, K_{(s)}{}^{r}{}_{j} \tag{3.3.11b}$$

On any interval (a,b) on which $\det K_{(s)}{}^{i}{}_{j}(t) \neq 0$, the (generally non symmetric) tensor

$$C^{(s)}_{ij} = E^{(s)}_{ir}\left(K^{-1}_{(s)}\right)^{r}{}_{j} \tag{3.3.12}$$

is then well defined, and satisfies the relation

$$\frac{DE^{(s)}_{ip}}{Dt} = \frac{DC^{(s)}_{ir}}{Dt}\, K_{(s)}{}^{r}{}_{p} + C^{(s)}_{ir}\, \frac{DK_{(s)}{}^{r}{}_{p}}{Dt}$$

Substituting from equations (3.3.11a), (3.3.12) and multiplying by $(K^{-1}_{(s)})^{p}{}_{j}$ the latter may be rewritten in the form

$$N^{(s)}_{ij} = \frac{DC^{(s)}_{ij}}{Dt} + C^{(s)}_{ir}\, M^{rl}_{(s)}\, E^{(s)}_{lp}\, (K^{-1}_{(s)})^{p}{}_{j} = \frac{DC^{(s)}_{ij}}{Dt} + C^{(s)}_{ir}\, M^{rl}_{(s)}\, C^{(s)}_{lj} \tag{3.3.10'}$$

formally identical to equation (3.3.10).

Needless to say, the symmetry property $C^{(s)}_{ij} = C^{(s)}_{ji}$ is also needed in order for the tensor (3.3.12) to represent the Hessian of a function $C^{(s)}$ along $\gamma^{(s)}$. An argument similar to the one exploited in the proof of Theorem 3.3.1 shows that this aspect relies entirely on the choice of the initial data. Indeed, on account of equation (3.3.10'), the antisymmetric part of $C^{(s)}_{ij}$ obeys a linear homogeneous

[6] The regularity assumption is once again crucial in ensuring the *global* character of the absolute time derivative $\frac{D}{Dt}$ induced by the hessian $(d^2\mathscr{L}'_{(s)})_{\hat\gamma^{(s)}}$ along $\gamma^{(s)}$.

system of the form (3.3.7b). Once again, by Cauchy Theorem we conclude that if $C^{(s)}_{ij}$ turns out to be symmetric say at $t = a_{s-1}$ (as it happens e.g. choosing $E^{(s)}_{ir}(a_{s-1}) = 0$, $K_{(s)}{}^{r}{}_{j}(a_{s-1}) = \delta^{r}_{j}$), it will remain symmetric up to the first value $t^* > a_{s-1}$ (if any) at which $\det K_{(s)}{}^{r}{}_{j}(t^*) = 0$.

Remark 3.3.1: The analysis of equations (3.3.10), (3.3.11a, b) is considerably simplified referring the virtual tensor algebra along $\gamma^{(s)}$ to an $h^{(s)}$–transported basis $\{e^{(a)}, e_{(a)}\}$ and recalling that, in this way, the components of the absolute time derivative of a field T coincide with the ordinary derivatives $\frac{dT^{a}{}_{b\,\cdots}}{dt}$. Equation (3.3.10) reduces then to the ordinary matrix Riccati equation

$$\frac{dC^{(s)}_{ab}}{dt} + M^{rs}_{(s)}\, C^{(s)}_{ar}\, C^{(s)}_{sb} - N^{(s)}_{ab} = 0 \tag{3.3.13}$$

while equations (3.3.11a, b) take the simpler form

$$\frac{dK_{(s)}{}^{a}{}_{b}}{dt} = M^{ac}_{(s)}\, E^{(s)}_{cb} \tag{3.3.14a}$$

$$\frac{dE^{(s)}_{ab}}{dt} = N^{(s)}_{ac}\, K_{(s)}{}^{c}{}_{b} \tag{3.3.14b}$$

Collecting all results, we can now state

Theorem 3.3.2 (Sufficient conditions). *Let $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$ be a normal extremal for the action functional. Also, let $\mathscr{H}_{(s)} = p^{(s)}_{i}\, \psi^i - \mathscr{L}$ denote the Pontryagin Hamiltonian associated with the given Lagrangian. Then, if the matrix*

$$G^{(s)}_{AB}(t) := -\left(\frac{\partial^2 K_{(s)}}{\partial z^A \partial z^B}\right)_{\hat{\gamma}^{(s)}} = -\left(\frac{\partial^2 \mathscr{H}_{(s)}}{\partial z^A \partial z^B}\right)_{\tilde{\gamma}^{(s)}}$$

is positive definite at each $t \in [a_{s-1}, a_s]$ and if the system (3.3.11a, b) admits at least one solution $E^{(s)}_{ij}(t), K_{(s)}{}^{i}{}_{j}(t)$ satisfying the conditions

- $E^{(s)}_{ir}\left(K^{-1}_{(s)}\right)^{r}{}_{j}$ *symmetric,*
- $\det K_{(s)}{}^{i}{}_{j} \neq 0$ *everywhere on $[a_{s-1}, a_s]$,*

the section $\gamma^{(s)}$ yields a weak local minimum for the action functional.

Proof. The stated assumptions imply both the regularity of the extremal $\gamma^{(s)}$ and the existence of a global solution of the Riccati–like equation (3.3.10) along $\gamma^{(s)}$, thus ensuring the validity of an expression like equation (3.3.8) on the whole interval $[a_{s-1}, a_s]$. So, if the matrix $G^{(s)}_{AB}$ is positive definite on $[a_{s-1}, a_s]$, this provides the evaluation

$$\frac{d^2\mathcal{I}[\hat{\gamma}^{(s)}_\xi]}{d\xi^2}\bigg|_{\xi=0} = \int_{a_{s-1}}^{a_s} \Big\langle \big(d^2(\mathscr{L}'_{(s)} - \dot{C}^{(s)})\big)_{\hat{\gamma}^{(s)}}\,,\, \hat{X}_{(s)} \otimes \hat{X}_{(s)} \Big\rangle\, dt =$$

$$= \int_{a_{s-1}}^{a_s} G^{(s)}_{AB}\, \tilde{Y}^A_{(s)} \tilde{Y}^B_{(s)}\, dt \;>\; 0$$

for every non–zero admissible deformation $\hat{X} : [a_{s-1}, a_s] \to A(\hat{\gamma})$ vanishing at the end–points. □

A deeper insight into the meaning of the condition $\det K_{(s)}{}^i{}_j \neq 0$ is provided by the study of the *Jacobi vector fields*, reviewed and adapted to the present geometrical context.

3.3.2 Jacobi fields

Given a regular normal extremal $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$, we now consider the (unique) extremal $\tilde{\gamma}^{(s)} : [a_{s-1}, a_s] \to \mathcal{C}(\mathcal{A})$ of the functional $\int_{\tilde{\gamma}^{(s)}} \Theta^{(s)}_{\mathrm{PPC}}$ projecting onto $\gamma^{(s)}$. Also, as usual, we preserve the notation $\upsilon = \pi \cdot \zeta : \mathcal{C}(\mathcal{A}) \to \mathcal{V}_{n+1}$ for the composite fibration $\mathcal{C}(\mathcal{A}) \to \mathcal{A} \to \mathcal{V}_{n+1}$.

Let us now introduce a special class of deformations $\tilde{\gamma}^{(s)}_\xi$ of $\tilde{\gamma}^{(s)}$ in which every section $\tilde{\gamma}^{(s)}_\xi : [a_{s-1}, a_s] \to \mathcal{C}(\mathcal{A})$ is itself an *extremal* of $\int_{\tilde{\gamma}^{(s)}} \Theta^{(s)}_{\mathrm{PPC}}$.

In this way, the 1–parameter family $\gamma^{(s)}_\xi := \upsilon \cdot \tilde{\gamma}^{(s)}_\xi$ is a deformation of the original section $\gamma^{(s)}$, consisting of extremals of the functional $\int_{\hat{\gamma}^{(s)}} \mathscr{L}\, dt$.

At this stage, we do not impose any restriction on the behavior of the end–points $\gamma^{(s)}_\xi(a_{s-1})$, $\gamma^{(s)}_\xi(a_s)$. In coordinates, setting

$$\tilde{\gamma}^{(s)}_\xi :\; q^i = \varphi^i_{(s)}(t,\xi)\,, \quad z^A = \zeta^A_{(s)}(t,\xi)\,, \quad p_i = \bar{\rho}^{(s)}_i(t,\xi) \qquad a_{s-1}(\xi) \leqslant t \leqslant a_s(\xi) \tag{3.3.15}$$

our assumptions are summarized into the request that, for each value of ξ, the functions at the right–hand–side of equations (3.3.15) satisfy Pontryagin's equations

$$\frac{\partial \varphi^i_{(s)}}{\partial t} = \psi^i(t, \varphi^i_{(s)}, \zeta^A_{(s)}) \tag{3.3.16a}$$

$$\frac{\partial \bar{\rho}^{(s)}_i}{\partial t} + \bar{\rho}^{(s)}_k \left(\frac{\partial \psi^k}{\partial q^i}\right)_{\hat{\gamma}^{(s)}_\xi} = \left(\frac{\partial \mathscr{L}'_{(s)}}{\partial q^i}\right)_{\hat{\gamma}^{(s)}_\xi} \tag{3.3.16b}$$

$$\bar{\rho}^{(s)}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}_\xi} = \left(\frac{\partial \mathscr{L}'_{(s)}}{\partial z^A}\right)_{\hat{\gamma}^{(s)}_\xi} \tag{3.3.16c}$$

As a check of inner consistency it is worth observing that, in view of the condition $(d\mathscr{L}'_{(s)})_{\hat\gamma^{(s)}} = 0$, equations (3.3.16b, c) and the normality of $\gamma^{(s)}$ yield back the relation $\bar\rho_i^{(s)}(t,0) = 0$.

Strictly associated with $\tilde\gamma_\xi^{(s)}$ is a corresponding *infinitesimal deformation*, locally expressed as $\tilde X_{(s)} = X^i_{(s)}\big(\frac{\partial}{\partial q^i}\big)_{\tilde\gamma^{(s)}} + \Gamma^A_{(s)}\big(\frac{\partial}{\partial z^A}\big)_{\tilde\gamma^{(s)}} + \bar\pi_i^{(s)}\big(\frac{\partial}{\partial p_i}\big)_{\tilde\gamma^{(s)}}$, with

$$X^i_{(s)} = \left(\frac{\partial \varphi^i_{(s)}}{\partial\xi}\right)_{\xi=0}, \quad \Gamma^A_{(s)} = \left(\frac{\partial \zeta^A_{(s)}}{\partial\xi}\right)_{\xi=0}, \quad \bar\pi_i^{(s)} = \left(\frac{\partial \bar\rho_i^{(s)}}{\partial\xi}\right)_{\xi=0} \tag{3.3.17}$$

Taking equations (3.3.16) and the relation $\bar\rho_i^{(s)}(t,0) = 0$ into account, it is easily seen that the components (3.3.17) satisfy the following system of differential–algebraic equations

$$\frac{dX^i_{(s)}}{dt} = \left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma^{(s)}} X^k_{(s)} + \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} \Gamma^A_{(s)} \tag{3.3.18a}$$

$$\frac{d\bar\pi_i^{(s)}}{dt} + \bar\pi_k^{(s)}\left(\frac{\partial\psi^k}{\partial q^i}\right)_{\hat\gamma^{(s)}} = \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial q^k}\right)_{\hat\gamma^{(s)}} X^k_{(s)} + \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^A}\right)_{\hat\gamma^{(s)}} \Gamma^A_{(s)} \tag{3.3.18b}$$

$$\bar\pi_i^{(s)}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} = \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial q^k}\right)_{\hat\gamma^{(s)}} X^k_{(s)} + \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial z^B}\right)_{\hat\gamma^{(s)}} \Gamma^B_{(s)} \tag{3.3.18c}$$

Given any vector field $\tilde X_{(s)}$ satisfying equations (3.3.17), its push–forward $\upsilon_*\tilde X_{(s)}$ will be called a *Jacobi field* along $\gamma^{(s)}$. By definition, a Jacobi field $X = X^i_{(s)}\big(\frac{\partial}{\partial q^i}\big)_{\gamma^{(s)}}$ is therefore the infinitesimal deformation tangent to a finite deformation consisting of a 1–parameter family of extremals of the action functional.

Remark 3.3.2 (The accessory problem)**:** The resemblance between equations (3.3.18) and Pontryagin's ones (2.3.4) sticks out a mile. This aspect can be made explicit by replacing the imbedding (1.2.3) by its linearized counterpart (1.3.12) , namely regarding the vector bundle $V(\gamma^{(s)})$ as the configuration space–time of an abstract system $\mathfrak{B}'$, and the bundle $A(\hat\gamma^{(s)}) \to V(\gamma^{(s)})$ as the associated space of admissible velocities. In this way, the admissible evolutions of $\mathfrak{B}'$ are in 1-1 correspondence with the admissible infinitesimal deformations of $\gamma^{(s)}$.

Referring $V(\gamma^{(s)})$ and $A(\hat\gamma^{(s)})$ to coordinates t, v^i and t, v^i, w^A respectively, according to the prescriptions (1.3.1) and (1.3.11), the imbedding $i_* : A(\hat\gamma^{(s)}) \to j_1(V(\gamma^{(s)})$ is locally expressed by

$$\dot v^i = \left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma^{(s)}} v^k + \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} w^A := \Psi^i(t, v^i, w^A) \tag{3.3.19}$$

To complete the picture, we adopt the quadratic form

$$\mathfrak{L}_{(s)}(\hat X_{(s)}) := \frac{1}{2}\left\langle \left(d^2\mathscr{L}'_{(s)}\right)_{\hat\gamma^{(s)}}, \hat X_{(s)}\otimes\hat X_{(s)}\right\rangle \tag{3.3.20}$$

as a Lagrangian on $A(\hat{\gamma}^{(s)})$, whose representation in coordinates reads

$$\mathfrak{L}_{(s)}(t,v^i,w^A)=\frac{1}{2}\left[\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial q^j}\right)_{\hat{\gamma}^{(s)}}v^iv^j+2\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^A}\right)_{\hat{\gamma}^{(s)}}v^iw^A+\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial z^B}\right)_{\hat{\gamma}^{(s)}}w^Aw^B\right]$$

and denote by $\mathfrak{I}$ the functional assigning the action integral $\mathfrak{I}[X_{(s)}]:=\int_{\hat{X}_{(s)}}\mathfrak{L}_{(s)}\,dt$ to each admissible section $X_{(s)}:[a_{s-1},a_s]\to V(\gamma^{(s)})$. In this way, for any finite deformation $\gamma^{(s)}_\xi$ of $\gamma^{(s)}$ tangent to $X_{(s)}$, equation (3.2.1) provides the identification

$$\mathfrak{I}[X_{(s)}]\;=\;\frac{1}{2}\,\frac{d^2\mathcal{I}[\hat{\gamma}^{(s)}_\xi]}{d\xi^2}\bigg|_{\xi=0}$$

It may be now easily verified that the equations (3.3.18) involved in the definition of the Jacobi fields, now more suitably rewritten as

$$\begin{aligned}
&\frac{dX^i_{(s)}}{dt}\;=\;X^k_{(s)}\,\frac{\partial\Psi^k}{\partial v^i}\;+\;\Gamma^A_{(s)}\,\frac{\partial\Psi^k}{\partial w^A}\\
&\frac{d\bar{\pi}^{(s)}_i}{dt}\;+\;\bar{\pi}^{(s)}_k\,\frac{\partial\Psi^k}{\partial v^i}\;=\;\frac{\partial\mathfrak{L}_{(s)}}{\partial v^i}\\
&\bar{\pi}^{(s)}_i\,\frac{\partial\Psi^i}{\partial w^A}\;=\;\frac{\partial\mathfrak{L}_{(s)}}{\partial w^A}
\end{aligned}$$

are formally identical to the Pontryagin's equations for the determination of the extremals $v^i=X^i_{(s)}(t)$, $w^A=\Gamma^A_{(s)}(t)$ of the functional $\mathfrak{I}$ subject to the constraints (3.3.19), which is commonly referred to as the *accessory variational problem*.

Coming back to the system (3.3.18) and recalling the discussion at the end of §3.1, we decompose the field $\tilde{X}_{(s)}$ into the pair

$$\hat{X}_{(s)}\;=\;X^i_{(s)}\left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}^{(s)}}+\;\Gamma^A_{(s)}\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}^{(s)}}\quad,\qquad\lambda^{(s)}\;=\;\bar{\pi}^{(s)}_i\,\hat{\omega}^i_{|\gamma^{(s)}}$$

in which $\hat{X}_{(s)}$ is a vector field along $\hat{\gamma}^{(s)}$ while $\lambda_{(s)}$ is a virtual 1–form along $\gamma^{(s)}$. By a little abuse of language, this will be called a *Jacobi pair* belonging to $X_{(s)}=\upsilon_*\tilde{X}_{(s)}$.

Under the further (crucial) hypothesis of regularity of $\gamma^{(s)}$, we next make use of the infinitesimal control $h^{(s)}\colon V(\gamma^{(s)})\to A(\hat{\gamma}^{(s)})$ induced[7] by the Lagrangian $\mathscr{L}'_{(s)}$ to express the field $\hat{X}_{(s)}$ in terms of the Jacobi field $X_{(s)}$ and of a vertical vector $Y_{(s)}$ in the form $\hat{X}_{(s)}=h^{(s)}(X_{(s)})+Y_{(s)}=X^i_{(s)}\tilde{\partial}_i+Y^A_{(s)}\frac{\partial}{\partial z^A}$. On account of equations (3.1.15) we have then the relation

$$\Gamma^A_{(s)}\;=\;Y^A_{(s)}\;-\;G^{AB}_{(s)}\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^B}\right)_{\hat{\gamma}^{(s)}}X^i_{(s)}\tag{3.3.22}$$

[7] See Remark 3.1.5

Together with the identification $G^{(s)}_{AB} = \left(\dfrac{\partial^2 \mathscr{L}'_{(s)}}{\partial z^A \partial z^B}\right)_{\hat\gamma^{(s)}}$, the latter allows to write equation (3.3.18c) into the form

$$\bar\pi_i^{(s)}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} = G^{(s)}_{AB}\, Y^B_{(s)} \quad\Longrightarrow\quad Y^A_{(s)} = G^{AB}_{(s)}\,\bar\pi_i^{(s)}\left(\frac{\partial\psi^i}{\partial z^B}\right)_{\hat\gamma^{(s)}} \tag{3.3.23}$$

From this, substituting into equations (3.3.18 a,b), recalling the definitions (3.3.3), (3.3.6) and expressing everything in terms of the absolute time derivative, we eventually obtain the following system of differential equations for the unknowns $X^i_{(s)}(t), \bar\pi_i^{(s)}(t)$:

$$\left(\frac{DX^i_{(s)}}{Dt}\right)_{\gamma^{(s)}} = G^{AB}_{(s)}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}}\left(\frac{\partial\psi^j}{\partial z^B}\right)_{\hat\gamma^{(s)}}\bar\pi_j^{(s)} = M^{ij}_{(s)}\,\bar\pi_j^{(s)} \tag{3.3.24a}$$

$$\left(\frac{D\bar\pi_i^{(s)}}{Dt}\right)_{\gamma^{(s)}} = \left[\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial q^j}\right)_{\hat\gamma^{(s)}} - G^{AB}_{(s)}\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^A}\right)_{\hat\gamma^{(s)}}\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^j\partial z^B}\right)_{\hat\gamma^{(s)}}\right]X^j_{(s)} =$$

$$= N^{(s)}_{ij}\,X^j_{(s)} \tag{3.3.24b}$$

As the attentive reader will have certainly already noticed, these are formally identical to the linearized form of Riccati equation (3.3.11): a result that will prove to be fundamental in the sequel in order to determine the necessary and sufficient conditions for a weak local minimum.

Remark 3.3.3: Keeping in line with Remark 3.3.1, if the virtual tensor algebra along $\gamma^{(s)}$ is referred to an $h^{(s)}$–transported basis $\left\{e^{(a)}, e_{(a)}\right\}$, the set of $2n$ differential equations (3.3.24) are written in the form

$$\frac{dX^a_{(s)}}{dt} = M^{ab}_{(s)}\,\bar\pi_b^{(s)} \tag{3.3.25a}$$

$$\frac{d\bar\pi_a^{(s)}}{dt} = N^{(s)}_{ab}\,X^b_{(s)} \tag{3.3.25b}$$

Once again, these are easily seen to represent the Hamilton equations for the function

$$\mathfrak{H}_{(s)}(t, X^a_{(s)}, \bar\pi_b^{(s)}) = \bar\pi_a\,\Psi^a - \mathfrak{L}_{(s)} = \frac{1}{2}M^{ab}\bar\pi_a^{(s)}\bar\pi_b^{(s)} - \frac{1}{2}N^{(s)}_{ab}\,X^a_{(s)}X^b_{(s)} \tag{3.3.26}$$

which is the linearized counterpart of the Hamiltonian function on the extremal curve $\gamma^{(s)}$.

The relationship between Jacobi fields and the second variation is made evident by the following

Proposition 3.3.1. *Given a Jacobi pair* $\big(\hat{X}_{(s)}, \lambda^{(s)}\big)$*, for any arbitrary vector field* $\hat{Z}_{(s)} \in A(\hat{\gamma}^{(s)})$*, the following identity holds:*

$$\Big\langle \big(d^2 \mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}} \,,\, \hat{X}_{(s)} \otimes \hat{Z}_{(s)} \Big\rangle \;=\; \frac{d(\bar{\pi}^{(s)}_i Z^i_{(s)})}{dt} \tag{3.3.27}$$

Proof. The thesis immediately follows by direct computation, in view of equations (3.1.12), (3.3.18). Setting $\hat{Z}_{(s)} = Z^i \big(\frac{\partial}{\partial q^i}\big)_{\hat{\gamma}^{(s)}} + Z^A \big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}^{(s)}}$, we have

$$\begin{aligned} \frac{d(\bar{\pi}^{(s)}_i Z^i_{(s)})}{dt} &= \left(\frac{\partial^2 \mathscr{L}}{\partial q^i \partial q^j}\right)_{\hat{\gamma}^{(s)}} X^j_{(s)} Z^i_{(s)} + \left(\frac{\partial^2 \mathscr{L}}{\partial q^i \partial z^A}\right)_{\hat{\gamma}^{(s)}} \Gamma^A_{(s)} Z^i_{(s)} + \\ &+ \left(\frac{\partial^2 \mathscr{L}}{\partial z^A \partial q^i}\right)_{\hat{\gamma}^{(s)}} X^i_{(s)} Z^A_{(s)} + \left(\frac{\partial^2 \mathscr{L}}{\partial z^A \partial z^B}\right)_{\hat{\gamma}^{(s)}} \Gamma^B_{(s)} Z^A_{(s)} \end{aligned}$$

□

Remark 3.3.4: Hitherto, our treatment of Jacobi fields has uniquely involved the adapted Lagrangian $\mathscr{L}'_{(s)}$. This choice was suggested both by consistency with the previous analysis and also by the simplified calculations. However, it goes without saying that it's not at all necessary in order to cover the subject. We could actually have considered $\tilde{\gamma}^{(s)}$ as extremal of the functional $\int_{\tilde{\gamma}^{(s)}} \Theta_{\rm PPC}$ instead of $\int_{\tilde{\gamma}^{(s)}} \Theta^{(s)}_{\rm PPC}$. In this way equations (3.3.18) would have been directly written in terms of the Pontryagin Hamiltonian $\mathscr{H}$, with the quantities $\bar{\pi}^{(s)}_i$ replaced by $\pi^{(s)}_i := \big(\frac{\partial \rho^{(s)}_i}{\partial \xi}\big)_{\xi=0}$, related to the previous ones by the relation

$$\bar{\pi}^{(s)}_i(t) \;=\; \left(\frac{\partial \bar{\rho}^{(s)}_i(t,\xi)}{\partial \xi}\right)_{\xi=0} \;=\; \frac{\partial}{\partial \xi}\left(\rho^{(s)}_i(t,\xi) \;-\; \frac{\partial S^{(s)}}{\partial q^i}\right)_{\xi=0} \;=\; \pi^{(s)}_i(t) \;-\; \frac{\partial^2 S^{(s)}}{\partial q^i \partial q^j} X^j_{(s)}$$

The argument is almost identical to the one developed so far and will be omitted.

3.3.3 Conjugate points and the necessary conditions

Jacobi fields are related to the necessary conditions for (local) minimality through the concept of *conjugate point*.

Definition 3.3.1 (Conjugate point)**.** *A point* $\gamma^{(s)}(\tau)$*,* $\tau \in (a_{s-1}, a_s]$*, along a given extremal curve* $\gamma^{(s)}$ *is said to be* conjugate *to* $\gamma^{(s)}(a_{s-1})$ *if there exists a non–zero Jacobi field* $X_{(s)} : [a_{s-1}, a_s] \to V(\gamma^{(s)})$ *such that* $X_{(s)}(a_{s-1}) = X_{(s)}(\tau) = 0$.

It is easily seen that the search for conjugate points can be performed by looking for a solution of equations (3.3.24) with $X^i_{(s)}(a_{s-1}) = 0$ and $\bar{\pi}^{(s)}_i(a_{s-1})$ varying amongst all the possible values in $\mathbb{R}^n$.

Because of the linearity of equations (3.3.24), their solution will depend on the initial data through a set of time–dependent matrices in the form

$$X^i_{(s)}(t) \;=\; \mathscr{A}^i_{\;j}(t,a_{s-1})\,X^j_{(s)}(a_{s-1}) \;+\; \mathscr{B}^{ij}(t,a_{s-1})\,\bar{\pi}^{(s)}_j(a_{s-1}) \tag{3.3.28a}$$

$$\bar{\pi}^{(s)}_i(t) \;=\; \mathscr{C}_{ij}(t,a_{s-1})\,X^j_{(s)}(a_{s-1}) \;+\; \mathscr{D}_i^{\;j}(t,a_{s-1})\,\bar{\pi}^{(s)}_j(a_{s-1}) \tag{3.3.28b}$$

with $\mathscr{A}^i_{\;j}(a_{s-1},a_{s-1}) = \delta^i_{\;j}$, $\mathscr{B}^{ij}(a_{s-1},a_{s-1}) = 0$, $\mathscr{C}_{ij}(a_{s-1},a_{s-1}) = 0$ and $\mathscr{D}_i^{\;j}(a_{s-1},a_{s-1}) = \delta_i^{\;j}$. Conjugate points can be therefore determined by means of equation (3.3.28a) restricted to the choice $X^i_{(s)}(a_{s-1}) = 0$, namely

$$X^i_{(s)}(t) \;=\; \mathscr{B}^{ij}(t,a_{s-1})\,\bar{\pi}^{(s)}_j(a_{s-1}) \tag{3.3.29}$$

Hence, a point $\gamma^{(s)}(\tau)$ is *conjugate* to $\gamma^{(s)}(a_{s-1})$ whenever $\bar{\pi}^{(s)}_j(a_{s-1})$ belongs to the kernel of $\mathscr{B}^{ij}(\tau,a_{s-1})$ and this can only happen when $\det(\mathscr{B}^{ij}(\tau,a_{s-1}))$ vanishes.

The link between conjugate points and the analysis of the second variation is clarified by the following generalization of a classical result of Bliss ([19]):

Theorem 3.3.3. *Consider an extremal closed arc $\gamma^{(s)} : [a_{s-1},a_s] \to \mathcal{V}_{n+1}$ and suppose there exists a value $\tau \in (a_{s-1},a_s)$ such that the point $\gamma^{(s)}(\tau)$ is conjugate to $\gamma^{(s)}(a_{s-1})$. Then the quadratic form*

$$\left.\frac{d^2\mathcal{I}\,[\hat{\gamma}^{(s)}_\xi]}{d\xi^2}\right|_{\xi=0} \;=\; \int_{a_{s-1}}^{a_s} \Big\langle \big(d^2\mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}} \;,\; \hat{X}_{(s)} \otimes \hat{X}_{(s)} \Big\rangle\, dt$$

is necessarily indefinite.

Proof. Let us define a symmetric bilinear functional $\big(d^2\mathcal{I}\big)_{\hat{\gamma}^{(s)}}$ over $A(\hat{\gamma}^{(s)})$ as

$$\Big\langle \big(d^2\mathcal{I}\big)_{\hat{\gamma}^{(s)}} \;,\; \hat{V}_{(s)} \otimes \hat{W}_{(s)} \Big\rangle \;:=\; \int_{a_{s-1}}^{a_s} \Big\langle \big(d^2\mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}} \;,\; \hat{V}_{(s)} \otimes \hat{W}_{(s)} \Big\rangle\, dt$$

for any $\hat{V}_{(s)}$, $\hat{W}_{(s)}$ in $A(\hat{\gamma}^{(s)})$. Then, by a well–known result in the theory of quadratic forms[8], the thesis is proved as soon as we show that, in the presence of a point $\gamma^{(s)}(\tau)$ conjugated to $\gamma^{(s)}(a_{s-1})$, the kernel of $\big(d^2\mathcal{I}\big)_{\hat{\gamma}^{(s)}}$ does not coincide with the locus of zeroes of its associated quadratic form.

Under the stated hypothesis, there exists a Jacobi field $J_{(s)} \in V(\gamma^{(s)})$ such that $J_{(s)}(a_{s-1}) = J_{(s)}(\tau) = 0$. By means of this, we now define a continuous infinitesimal deformation vanishing at the end–points $X_{(s)} : [a_{s-1},a_s] \to V(\gamma^{(s)})$ in the following manner:

$$X_{(s)}(t) \;:=\; \begin{cases} J_{(s)}(t) & a_{s-1} \leqslant t \leqslant \tau \\ 0 & \tau \leqslant t \leqslant a_s \end{cases}$$

[8]See Appendix D, Lemma D.1.

Then, denoting by $\hat{X}_{(s)} \in A(\hat{\gamma}^{(s)})$ the lift of $X_{(s)}$ and by $\big(\hat{J}_{(s)}, \lambda^{(s)}\big)$ a Jacobi pair belonging to $J_{(s)}$, in view of equation (3.3.27) we have

$$\Big\langle \big(d^2\mathcal{I}\big)_{\hat{\gamma}^{(s)}} \,,\, \hat{X}_{(s)} \otimes \hat{X}_{(s)} \Big\rangle \;=\; \int_{a_{s-1}}^{\tau} \Big\langle \big(d^2\mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}} \,,\, \hat{J}_{(s)} \otimes \hat{J}_{(s)} \Big\rangle \, dt \;= \\ =\; \Big[\, \bar{\pi}^{(s)}_i \, \hat{J}^i_{(s)} \Big]_{a_{s-1}}^{\tau} \;=\; 0$$

At the same time, if $W_{(s)}$ is any infinitesimal deformation of $\gamma^{(s)}$ vanishing at the end–points and such that $W_{(s)}(\tau) \neq 0$, we have also

$$\Big\langle \big(d^2\mathcal{I}\big)_{\hat{\gamma}^{(s)}} \,,\, \hat{X}_{(s)} \otimes \hat{W}_{(s)} \Big\rangle \;=\; \int_{a_{s-1}}^{\tau} \Big\langle \big(d^2\mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}} \,,\, \hat{J}_{(s)} \otimes \hat{W}_{(s)} \Big\rangle \, dt \;= \\ =\; \bar{\pi}^{(s)}_i(\tau)\, \hat{W}^i_{(s)}(\tau)$$

Since, by hypothesis, $J_{(s)}(t) \neq 0$ for every $t \in (a_{s-1}, \tau)$, the uniqueness of the solution of the "time–reversed" Cauchy problem (3.3.24) in $\gamma^{(s)}(\tau)$ implies that $\bar{\pi}^{(s)}_i(\tau) \neq 0$ for at least one value of the index i. Therefore

$$\Big\langle \big(d^2\mathcal{I}\big)_{\hat{\gamma}^{(s)}} \,,\, \hat{X}_{(s)} \otimes \hat{W}_{(s)} \Big\rangle \;\neq\; 0$$

showing that $\hat{X}_{(s)}$ does not belong to the kernel of $\big(d^2\mathcal{I}\big)_{\hat{\gamma}^{(s)}}$. $\square$

As a direct consequence of Theorem 3.3.3, we can now state the following

Proposition 3.3.2 (Necessary conditions). *Suppose the extremal closed arc $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$ is a (local) minimum for the functional $\int_{\hat{\gamma}^{(s)}} \mathscr{L}\, dt$. Then, for every $\tau \in (a_{s-1}, a_s)$, there cannot be any point $\gamma^{(s)}(\tau)$ conjugate to $\gamma^{(s)}(a_{s-1})$.*

3.3.4 The necessary and sufficient conditions

So far we have separately proved a sufficient and a necessary condition for a given extremal $\gamma^{(s)}$ to be a minimum; we shall now glue them together into a necessary and sufficient one. However, in order to do so, we shall need to strengthen the hypothesis of normality of $\gamma^{(s)}$ by requiring the latter to be *locally normal*.

In the event, we will prove that, whenever no conjugate point is present, the solutions of equations (3.3.24) can be used to build a global solution of the Riccati equation (3.3.5), valid along the whole interval $[a_{s-1}, a_s]$, thus satisfying some of the hypothesis of Theorem 3.3.2. To this purpose, we first need a technical argument.

Lemma 3.3.3.1. *Let $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$*[9] *be a locally normal extremal and suppose the matrix $G^{(s)}_{AB}$ is non–singular at each $t \in [a_{s-1}, a_s]$. If along $\gamma^{(s)}$ there is no point conjugate to $\gamma^{(s)}(a_{s-1})$, then there exists a $t^* > a_s$ such that the absence of conjugate points may be extended over a wider interval $[a_{s-1}, t^*]$.*

Proof. Consider the family of Jacobi pairs $\big(\hat{X}_{(s)}, \lambda^{(s)}\big)_{(k)}$, $k = 1, \ldots, n$, obtained as solutions of equations (3.3.24) with initial data

$$(X_{(s)})^i_{(k)}(a_{s-1}) \,=\, 0 \quad , \qquad (\bar{\pi}^{(s)})_{i(k)}(a_{s-1}) \,=\, \delta_{ik}$$

The non–existence of conjugate points along $\gamma^{(s)}$ is easily seen to be equivalent to the condition $\det\big((X_{(s)})^i_{(k)}(t)\big) \neq 0$ for all $t \in (a_{s-1}, a_s]$.

If that is not the case, there would be some $\tau \in (a_{s-1}, a_s]$ at which the homogenous system $a^k\,(X_{(s)})^i_{(k)}(\tau) \,=\, 0$ would admit a non–null solution $a^1, .., a^n$. The fields $\hat{X}_{(s)} := a^k\,(\hat{X}_{(s)})_{(k)}$, $\lambda^{(s)} \,=\, a^k\,(\lambda^{(s)})_{(k)}$ would then constitute a Jacobi pair satisfying the conditions $\lambda^{(s)}(a_{s-1}) \neq 0$, $\hat{X}_{(s)}(a_{s-1}) \,=\, \hat{X}_{(s)}(\tau) \,=\, 0$. On the other hand, $\hat{X}_{(s)}$ cannot be identically zero over the whole interval $[a_{s-1}, \tau]$: if it were so, the 1–form $\lambda^{(s)}$ would satisfy the equations

$$\left.\begin{aligned} &\left(\frac{DX^i_{(s)}}{Dt}\right)_{\gamma^{(s)}} \,=\, M^{ij}_{(s)}\,\bar{\pi}^{(s)}_j = 0 \quad \Longrightarrow \quad \bar{\pi}^{(s)}_j \left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat{\gamma}^{(s)}} = 0 \\ &\left(\frac{D\bar{\pi}^{(s)}_i}{Dt}\right)_{\gamma^{(s)}} \,=\, 0 \end{aligned}\right. \qquad \forall\; a_{s-1} \leqslant t \leqslant \tau$$

contradicting the local normality of $\gamma^{(s)}$.

To sum up, $\hat{X}_{(s)}$ would be a non–zero Jacobi vector field vanishing at both a_{s-1} and τ, which clashes with the assumption of non–existence of conjugate points along $\gamma^{(s)}$.

By continuity, this implies $\det\big((X_{(s)})^i_{(k)}(t)\big) \neq 0$ for all $t \in (a_{s-1}, t^*]$ with $t^* \in (a_s, b_s)$ sufficiently close to a_s. The absence of conjugate points holds therefore in a wider interval $[a_{s-1}, t^*]$. $\square$

We are now ready to take the conclusive step towards the formulation of the necessary and sufficient conditions for minimality, which is provided by the following

Proposition 3.3.3. *Let $\gamma^{(s)} : [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$ be a locally normal extremal and suppose the matrix $G^{(s)}_{AB}$ is non–singular at each $t \in [a_{s-1}, a_s]$. If no pair of conjugate points exists on $\gamma^{(s)}$, the Riccati equation (3.3.5) admits a symmetric solution throughout the interval $[a_{s-1}, a_s]$.*

[9] We recall that the closed arc $\gamma^{(s)}$ is the restriction to the closed interval $[a_{s-1}, a_s]$ of an admissible section defined on some open neighborhood $(b_{s-1}, b_s) \supset [a_{s-1}, a_s]$.

Proof. As usual, we regard $\gamma^{(s)}$ as the restriction of an extremal defined on an open interval $(b_{s-1}, b_s) \supset [a_{s-1}, a_s]$. Let $t^* \in (a_s, b_s)$ and consider a family of solutions $\big(\hat{X}_{(s)}, \lambda^{(s)}\big)_{(k)}$ of equations (3.3.24), obtained imposing the initial conditions $(X_{(s)})^i_{(k)}(t^*) = 0$ and $(\bar{\pi}^{(s)})_{i(k)}(t^*) = \delta_{ik}$.

In view of Lemma 3.3.3.1, whenever t^* is chosen sufficiently close to a_s, the absence of conjugate points implies the requirement $\det\big((X_{(s)})^i_{(k)}(t)\big) \neq 0$ for all $t \in [a_{s-1}, t^*)$.

A comparison between the Hamiltonian system (3.3.24) and the linearization (3.3.11) of Riccati equation shows that we can now assume the identifications

$$K_{(s)}{}^i{}_j(t) := (X_{(s)})^i_{(j)}(t)\,, \qquad E^{(s)}_{ij}(t) := (\bar{\pi}^{(s)})_{i(j)}(t) \tag{3.3.30}$$

As a consequence, the matrix $K_{(s)}{}^i{}_j(t)$ is non–singular everywhere on $[a_{s-1}, t^*)$ and therefore, as we've seen in §3.3.1, the tensor $C^{(s)}_{ij} = E^{(s)}_{ir}\big(K^{-1}_{(s)}\big)^r{}_j$ represents a solution of the Riccati equation (3.3.5) all over the interval $[a_{s-1}, t^*) \supset [a_{s-1}, a_s]$.

In order to complete the proof, we now only need to show that this $C^{(s)}_{ij}$ is also symmetric. To this end we observe that the matrix $R^{ip}_{(s)} := K_{(s)}{}^i{}_j\big(E^{-1}_{(s)}\big)^{jp}$ is perfectly meaningful in a neighborhood $(t^* - \delta, t^*]$ and satisfies the relations

$$R^{ip}_{(s)}(t^*) = 0 \quad , \qquad R^{ip}_{(s)}\,C^{(s)}_{pq} = K_{(s)}{}^i{}_j\big(E^{-1}_{(s)}\big)^{jp} E^{(s)}_{pr}\big(K^{-1}_{(s)}\big)^r{}_q = \delta^i{}_p \quad \forall\, t < t^*$$

The matrix $R^{ip}_{(s)}$ is therefore symmetric at $t = t^*$. Moreover, on account of equations (3.3.24), it satisfies the equation

$$\frac{DR^{ip}_{(s)}}{Dt} = \frac{DK_{(s)}{}^i{}_j}{Dt}\big(E^{-1}_{(s)}\big)^{jp} + K_{(s)}{}^i{}_j\,\frac{D\big(E^{-1}_{(s)}\big)^{jp}}{Dt} = M^{ir}_{(s)} - R^{il}_{(s)}\,N^{(s)}_{lk}\,R^{kp}_{(s)}$$

which is again of the Riccati–type (3.3.5), with the roles of the matrices $M^{ij}_{(s)}, N^{(s)}_{ij}$ interchanged. Exactly as in Theorem 3.3.1, this establishes the symmetry of $R^{ip}_{(s)}$ in a neighborhood of $t = t^*$.

For each $t \in (t^* - \delta, t^*)$ the matrix $C^{(s)}_{ij}(t) = \big(R^{ij}_{(s)}(t)\big)^{-1}$ is therefore symmetric. Once again, on account of the linearity of equation (3.3.7b), we conclude that $C^{(s)}_{ij}(t)$ is symmetric over the whole interval $[a_{s-1}, t^*) \supset [a_{s-1}, a_s]$. $\square$

Collecting all the above arguments, we are now able to state the following

Theorem 3.3.4 (Necessary and sufficient conditions). *Suppose the closed arc $\gamma^{(s)} \colon [a_{s-1}, a_s] \to \mathcal{V}_{n+1}$ is a locally normal extremal of the functional $\int_{\hat{\gamma}^{(s)}} \mathscr{L}\, dt$ with respect to the class of deformations vanishing at the end–points and let $\tilde{\gamma}^{(s)}$ be its (unique) lift to $\mathcal{C}(\mathcal{A})$ solving Pontryagin's equations (2.3.4 a, b, c). Denote* *by* $\mathscr{H}_{(s)}(t, q^i, z^A, p_i) = p_i^{(s)}\,\psi^i(t, q^i, z^A) - \mathscr{L}(t, q^i, z^A)$ *the Pontryagin Hamiltonian associated with the given Lagrangian and let*

$$G^{(s)}_{AB}(t) \;:=\; -\left(\frac{\partial^2 \mathscr{H}_{(s)}}{\partial z^A \partial z^B}\right)_{\tilde{\gamma}^{(s)}}$$

Then, the arc $\gamma^{(s)}$ *is a minimum for the action functional if and only if, for every* $t \in [a_{s-1}, a_s]$*, the matrix* $G^{(s)}_{AB}$ *is positive definite and there is no point conjugate to* $\gamma^{(s)}(a_{s-1})$*.*

The proof should, at this time, be quite straightforward and is left to the reader.

3.4 The induced quadratic form

With Theorem 3.3.4, the former step of the stated resolution strategy can be said brought off. From now onwards, we shall thus embrace the hypothesis of each arc $\gamma^{(s)}$ being a *locally normal* extremal of the functional $\int_{\hat{\gamma}^{(s)}} \mathscr{L}\,dt$ and a minimum with respect to the fixed end–points deformations and we'll devote ourselves to the further task of finding out whether it is possible to combine the previous results in order to make them globally applicable to the entire evolution γ. This will involve the study of the definiteness properties of the quadratic form (3.2.2) and will be carried out by making use of the results of Appendix D and, in particular, along the lines of Theorem D.1.

To start with, we observe that, under the present hypothesis, we are supposed to be able to find N restricted gauge transformations in such a way that

$$\left\langle \left(d^2 \mathscr{L}'_{(s)}\right)_{\hat{\gamma}^{(s)}} \,,\; \hat{X}_{(s)} \otimes \hat{X}_{(s)} \right\rangle \;=\; G^{(s)}_{AB}\, Y^A_{(s)}\, Y^B_{(s)} \qquad \forall\, s \;=\; 1, \ldots, N$$

and so the quadratic form (3.2.2) can be written more suitably as

$$\left.\frac{d^2 \mathcal{I}[\hat{\gamma}_\xi]}{d\xi^2}\right|_{\xi=0} \;=\; \sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} G^{(s)}_{AB}\, Y^A_{(s)}\, Y^B_{(s)}\; dt \;-\; \sum_{s=1}^{N-1} \left\langle \left[d^2 S\right]_{c_s} \,,\; W_s \otimes W_s \right\rangle \tag{3.4.1}$$

Moreover, being each matrix $G^{(s)}_{AB}$ positive definite, the Lagrangian functions $\mathscr{L}'_{(s)}$, $s = 1, \ldots, N$ provide their respective arcs $\gamma^{(s)}$ with an infinitesimal control $h^{(s)}$, therefore assigning a transport law to the vertical space $V(\gamma)$ or, all the same, a canonical trivialization of the latter into the cartesian product $\mathbb{R} \times V_h$.

We recall that, in the algebraic environment developed in §1.5.4, the vector space of the admissible infinitesimal deformations vanishing at the end–points of γ was seen to be isomorphic to the kernel of the linear map $\Upsilon \colon \mathfrak{W} \to V_h$ whose representation in an h–transported basis $\{e_{(a)}\}$ of V_h reads

$$\Upsilon\big(Y, \alpha_1, \ldots, \alpha_{N-1}\big) \;=\; \left(\int_{t_0}^{t_1} Y^A\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma} dt \;-\; \sum_{s=1}^{N-1} \alpha_s\, k^a_{(s)}\right) e_{(a)}$$

being $k^a_{(s)} := e^{(a)}_{\;i}(a_s)\,\big[\psi^i(\hat\gamma)\big]_{a_s}$.

In view of this, we now introduce one more linear map

$$\mathcal{W}\colon\ \ker(\Upsilon) \;\to\; T_{c_1}(\mathcal{V}_{n+1}) \times \cdots \times T_{c_{N-1}}(\mathcal{V}_{n+1})$$

which maps each element $\big(Y, \alpha_1, \ldots, \alpha_{N-1}\big) \in \ker(\Upsilon)$ into the corresponding collection $W_{(1)}, \ldots, W_{(N\,-\,1)}$ of tangent vectors to the orbits of the corners. The subspace $\ker(\mathcal{W}) \subset \ker(\Upsilon)$ is therefore formed by the totality of the admissible infinitesimal deformations for which $W_{(1)} = \cdots = W_{(N\,-\,1)} = 0$, namely the ones that vanish at the corners.

Setting $\chi^a_{(s)}\big(W_{(1)}, \ldots, W_{(N\,-\,1)}\big) := \big\langle X_{(s)}, e^{(a)}\big\rangle_{c_s}$, we'll henceforth refer the space $T_{c_1}(\mathcal{V}_{n+1}) \times \cdots \times T_{c_{N-1}}(\mathcal{V}_{n+1})$ to the coordinate system $\{\alpha_s, \chi^a_{(s)}\}$. Recalling the expression (1.5.32), this results in the representation

$$W_{(s)} \;=\; \alpha_s \left(\frac{\partial}{\partial t}\right)_{c_s} \;+\; \left(\chi^a_{(s)} \;+\; \alpha_s\, \psi^i_{\;|\hat\gamma^{(s)}(a_s)}\, e^{(a)}_{\;i}(a_s)\right)(e_{(a)})_{c_s} \tag{3.4.2}$$

Theorem 3.4.1. *If each arc $\gamma^{(s)}$ is normal, the map $\mathcal{W}$ is surjective.*

Proof. We first observe the following identity

$$\begin{aligned} X^a_{(s\,+\,1)}(a_s) \;&=\; \chi^a_{(s)}\big(W_{(1)}, \ldots, W_{(N\,-\,1)}\big) \;+\; \big\langle \big[X\big]_{a_s}, e^{(a)}(a_s)\big\rangle \;= \\ &=\; \chi^a_{(s)}\big(W_{(1)}, \ldots, W_{(N\,-\,1)}\big) \;-\; \alpha_s\, k^a_{(s)} \end{aligned}$$

which is a direct consequence of the jump conditions (1.5.41b). The request for any arbitrary element $\big(W_{(1)}, \ldots, W_{(N\,-\,1)}\big)$ to be the image under $\mathcal{W}$ of a corresponding $\big(Y, \alpha_1, \ldots, \alpha_{N-1}\big) \in \ker(\Upsilon)$ makes the vertical vector fields Y to be subject to the conditions

$$\int_{a_{s-1}}^{a_s} Y^A_{(s)}\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} dt \;=\; X^a_{(s)}(a_s) - X^a_{(s)}(a_{s-1}) \;=\; \chi^a_{(s)} - \chi^a_{(s\,-\,1)} + \alpha_{s-1} k^a_{(s\,-\,1)} \tag{3.4.3}$$

The conclusion follows at once simply by observing that the above equation admits (at least) a solution Y for any possible choice of the variables $\alpha_s, \chi^a_{(s)}$ if and only if the mappings

$$Y_{(s)} \;\to\; \int_{a_{s-1}}^{a_s} Y^A_{(s)}\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} dt \qquad s = 1, \ldots, N$$

are surjective, which is totally equivalent to the normality of each arc $\gamma^{(s)}$. $\square$

On account of Theorem 3.4.1, under the stated hypothesis, the quotient space $\ker(\Upsilon)/\ker(\mathcal{W})$ coincides with the cartesian product $T_{c_1}(\mathcal{V}_{n+1}) \times \cdots \times T_{c_{N-1}}(\mathcal{V}_{n+1})$. Each element $\big(W_{(1)}, \ldots, W_{(N-1)}\big)$, thought as an equivalence class in $\ker(\Upsilon)$, is then formed by the totality of $\big(Y, \alpha_1, \ldots, \alpha_{N-1}\big)$ such that, for any s, $Y_{(s)}$ fulfils the condition (3.4.3) while $\alpha_s = \big\langle W_{(s)}, dt_{|c_s}\big\rangle$.

Coming back to the study of the quadratic form (3.4.1), it is readily seen that its restriction to the subspace $\ker(\mathcal{W})$ is positive definite, being the sum of N positive definite quadratic forms. Moreover, its restriction to any equivalence class $\mathcal{W}^{-1}\big(W_{(1)}, \ldots, W_{(N-1)}\big)$ has a single stationarity point. In order to find it out, it is possible to make use of the method of Lagrange multipliers by considering the functional

$$\sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} G^{(s)}_{AB}\, Y^A_{(s)}\, Y^B_{(s)}\, dt \;-\; \sum_{s=1}^{N-1} \Big\langle \big[d^2 S\big]_{c_s}\,,\; W_s \otimes W_s \Big\rangle \;+$$

$$+ \sum_{s=1}^{N} \nu^{(s)}_a \left[\int_{a_{s-1}}^{a_s} Y^A_{(s)}\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} dt \,-\, \chi^a_{(s)} \,+\, \chi^a_{(s-1)} \,-\, \alpha_{s-1} k^a_{(s-1)} \right] \tag{3.4.4}$$

with independent variables $Y^A_{(s)}$, $\nu^{(s)}_a$ and fixed α_s, $\chi^a_{(s)}$.

The vanishing of the first derivatives with respect to the $\nu^{(s)}_a$'s obviously gives back the constraints (3.4.3), while the variation with respect to the $Y^A_{(s)}$'s provides the relations

$$2\, G^{(s)}_{AB}\, Y^B_{(s)} \,+\, \nu^{(s)}_a\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} \;=\; 0 \quad\Longrightarrow\quad \bar{Y}^A_{(s)} \;=\; \frac{1}{2}\, G^{AB}_{(s)}\, \nu^{(s)}_a\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^B}\right)_{\hat{\gamma}^{(s)}} \tag{3.4.5}$$

Substituting into equations (3.4.3), we get

$$\frac{1}{2}\, \nu^{(s)}_b \int_{a_{s-1}}^{a_s} G^{AB}_{(s)} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} \left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat{\gamma}^{(s)}} e^{(a)}_{\;i}\, e^{(b)}_{\;j}\, dt \;=$$

$$= \frac{1}{2}\, \nu^{(s)}_b \int_{a_{s-1}}^{a_s} M^{ab}_{(s)}\, dt \;=\; \chi^a_{(s)} \,-\, \chi^a_{(s-1)} \,+\, \alpha_{s-1} k^a_{(s-1)} \tag{3.4.6}$$

Because of the non–singularity of $G^{(s)}_{AB}$, the local normality of each arc $\gamma^{(s)}$ implies the positive definiteness of the corresponding matrix

$$g^{ab}_{(s)} \;:=\; \int_{a_{s-1}}^{a_s} M^{ab}_{(s)}\, dt \qquad\qquad s \,=\, 1, \ldots\, N$$

and therefore, denoting by $g^{(s)}_{ab}$ its inverse matrix, we can solve equations (3.4.6) for the unknowns $\nu^{(s)}_a$'s in the form

$$\nu^{(s)}_a \;=\; 2\; g^{(s)}_{ab} \left[\chi^b_{(s)} \,-\, \chi^b_{(s-1)} \,+\, \alpha_{s-1}\, k^b_{(s-1)} \right] \tag{3.4.7}$$

The expression of the stationarity point $\bar{Y}^A_{(s)}$ can now be rewritten as

$$\bar{Y}^A_{(s)} = G^{AB}_{(s)}\, g^{(s)}_{ab} \left[\chi^b_{(s)} - \chi^b_{(s-1)} + \alpha_{s-1}\, k^b_{(s-1)}\right] e^{(a)}_i \left(\frac{\partial \psi^i}{\partial z^B}\right)_{\hat{\gamma}^{(s)}}$$

which is actually a *minimum* point, once again on account of the positive definiteness of the matrixes $G^{(s)}_{AB}$.

At last, we may induce a quadratic form $f\colon {}^{\ker(\Upsilon)}/_{\ker(\mathcal{W})} \to \mathbb{R}$ by mapping each equivalence class $\big(W_{(1)}, \ldots, W_{(N-1)}\big)$ in $\ker(\Upsilon)$ into the real number given by the evaluation of the quadratic form (3.4.1) at the corresponding (unique) minimum point $\bar{Y}^A_{(s)}$. In local coordinates, we have the representation

$$\begin{aligned} f\big(\alpha_s, \chi^a_{(s)}\big) = & \sum_{s=1}^{N} g^{(s)}_{ab} \big(\chi^a_{(s)} - \chi^a_{(s-1)} + \alpha_{s-1} k^a_{(s-1)}\big)\big(\chi^b_{(s)} - \chi^b_{(s-1)} + \alpha_{s-1} k^b_{(s-1)}\big) + \\ & - \sum_{s=1}^{N-1} \Big\langle \big[d^2 S\big]_{c_s} \,,\; W_s \otimes W_s \Big\rangle \end{aligned} \tag{3.4.8}$$

being the vectors W_s implicitly expressed in terms of the variables $\alpha_s, \chi^a_{(s)}$ by means of equation (3.4.2).

Collecting all previous results, we have thus proved

Theorem 3.1. *Let $\big(\gamma, [t_0, t_1]\big) := \big\{\big(\gamma^{(s)}, [a_{s-1}, a_s]\big),\ s = 1, \ldots, N\big\}$ be a piecewise differentiable locally normal extremal of the functional $\int_{\hat{\gamma}} \mathscr{L}\, dt$. Suppose the matrix $G^{(s)}_{AB}$ is positive definite along each arc $\gamma^{(s)}$ and suppose there is no point conjugate to $\gamma^{(s)}(a_{s-1})$. Then, a necessary and sufficient condition for the minimality of γ is the positive definiteness of the quadratic form (3.4.8).*

Appendix A

Adapted local charts

The aim of the present Appendix is to single out a distinguished finite family of local charts in $\mathcal{A}$ that covers the section $\hat{\gamma}$ and makes its representation as easy as possible. The use of these charts will turn out to be most useful especially when the discussion itself is already rather entangled, as it helps in easing the notation and reduces the effort needed to carry out all calculations. It goes without saying that, in order to preserve the generality of all results, one should always take care of checking their independence of any particular choice of coordinates.

Lemma A.1. *Let $\gamma : (c,d) \to \mathcal{V}_{n+1}$ be a differentiable section and $m,n \in (c,d)$. Then, for every closed interval $[a,b] \subset (c,d)$ there exist an open neighborhood $(m,n) \supset [a,b]$ and a differentiable vector field X such that $\gamma_* \left(\frac{\partial}{\partial t}\right) = X_{|\gamma(t)}$ for any $t \in (m,n)$.*

Proof. Let $m \in (c,a)$ and $n \in (b,d)$. Being compact, the arc $\gamma([m,n])$ is covered by a finite family of local charts with compact closure $(V_1, k_1), \dots (V_r, k_r)$ that we order timewise. In each local chart, where γ is represented in coordinates by $q^i = \varphi^i(t)$, it is always possible to arrange a straightforward transformation $\bar{q}^i = q^i - \varphi^i(t)$ such that γ reduces to the coordinate line $\bar{q}^i = 0$, which is therefore tangent to the field $\frac{\partial}{\partial t}$. We now sort out, among all the partitions of unity that are subordinate to the covering $\{V_1, \dots, V_r, \mathcal{V}_{n+1} - \gamma\left([c,d]\right)\}$, the (finite) family of functions whose supports intersect $\gamma\left([c,d]\right)$ and define as g_α, $\alpha = 1, \dots, r$, the sum of the ones whose supports are contained in V_α *but not* in V_β, $\beta < \alpha$. In this way, we've provided every open set V_α with a function g_α having support in V_α and globally defined on $\mathcal{V}_{n+1}$ in such a way that $\sum_\alpha g_\alpha\left(\gamma(t)\right) = 1$ for every $t \in [m,n]$.

It is now an easy matter to see that, if we define a field $X_{(\alpha)}$ as

$$X_{(\alpha)}\big|_x = \begin{cases} g_\alpha(x) \left(\frac{\partial}{\partial t}\right)_x & \forall\, x \in V_\alpha \\ 0 & \forall\, x \notin V_\alpha \end{cases}$$

the vector field $X := \sum_{\alpha=1}^{r} X_{(\alpha)}$ fulfils all the required properties. $\square$

According to Lemma A.1, the integral line of X passing through the point $\gamma(m)$ is defined at least up to $t = n$. By well–known theorems in differential equations (see e.g. [11, 22]), this in turn implies that the same will happen if the initial data are chosen in an open neighborhood of $\gamma(m)$. In particular, if we denote by W the intersection of this open set with the hyperplane $\Sigma_a : t = a$ and by Ω the flow tube containing all the integral lines of X that spit out of W, then:

- all the lines contained in Ω are defined (at least) up to $m \leqslant t \leqslant n$;
- every local coordinate system $q^1, \ldots, q^n$ on W may be used to refer Ω to local coordinates $t, q^1, \ldots, q^n$. Moreover, it is always possible, without any loss of generality, to make the choice $q^i(\gamma(a)) = 0$ which makes the curve γ into the coordinate line $q^i = 0$.

In the presence of piecewise differentiable sections it is possible to apply the previous construction in each single arc and then to combine the results into a global one. We first provide the arc $\gamma^{(1)}$ with a local chart $(\Omega_1, t, q^1_{(1)}, \ldots, q^n_{(1)})$ as above. We then choose $W_1 \subset \Omega_1 \cap \Sigma_{a_1}$ and refer it to local coordinates $q^1_{(1)}, \ldots, q^n_{(1)}$. In doing so, we should be wise enough to take it as small as to be used as initial data set for a second flow tube Ω_2 which will contain (not strictly) the closed interval $[a_1, a_2]$. Pursuing this process till the end, we obtain a finite family of local charts (one for every differentiable arc $\gamma^{(s)}$) with the following properties:

(i) each single arc $\gamma^{(s)}$ is contained in Ω_s and is represented there as the coordinate line $q^i_{(s)} = 0$;

(ii) in the intersection $\Omega_s \cap \Omega_{s+1}$, the transformation

$$q^i_{(s+1)} = q^i_{(s+1)}(t, q^1_{(s)}, \ldots, q^n_{(s)})$$

is such that

$$q^i_{(s+1)}(a_s, q^1_{(s)}, \ldots, q^n_{(s)}) = q^i_{(s)} \tag{A.1}$$

Lemma A.2. *Let $\hat{\gamma} : (c, d) \to \mathcal{A}$ be the lift of an admissible differentiable section $\gamma : (c, d) \to \mathcal{V}_{n+1}$. Then, for any closed interval $[a, b] \subset (c, d)$ there exists a fibred local chart $(\hat{U}, \hat{h})$, $\hat{h} = (t, q^1, \ldots, q^n, z^1, \ldots, z^r)$ satisfying the properties*

$$\text{(i)} \qquad \hat{\gamma}(t) \in \hat{U} \quad \forall\, t \in [a, b]; \tag{A.2a}$$

$$\text{(ii)} \qquad \hat{\gamma}\big((c, d)\big) \cap \hat{U} \text{ coincides with the curve } q^i = z^A = 0; \tag{A.2b}$$

$$\text{(iii)} \qquad \psi^i\big(\hat{\gamma}(t)\big) = \left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}(t)} = 0 \qquad \forall\, \hat{\gamma}(t) \in \hat{U}. \tag{A.2c}$$

Proof. The construction carried out at the end of Lemma A.1 ensures the existence of "tubular" local charts (Ω, h), $h = (t, \bar{q}^1, \ldots, \bar{q}^n)$ in $\mathcal{V}_{n+1}$ and $(\hat{\Omega}, \hat{k}')$, $\hat{k}' = (t, x^1, \ldots, x^{n+r})$ in $\mathcal{A}$ satisfying the conditions

$$\begin{aligned} &\gamma([a,b]) \subset \Omega, \qquad \bar{q}^i(\gamma(t)) = 0 \qquad \forall\, t \in (c,d) \cap \gamma^{-1}(\Omega) \\ &\hat{\gamma}([a,b]) \subset \hat{\Omega}, \qquad x^\alpha(\hat{\gamma}(t)) = 0 \qquad \forall\, t \in (c,d) \cap \hat{\gamma}^{-1}(\hat{\Omega}) \end{aligned}$$

Without loss of generality we may assume $\pi(\hat{\Omega}) \subset \Omega$. The restriction to $\hat{\Omega}$ of the projection $\pi : \mathcal{A} \to \mathcal{V}_{n+1}$ is then described in coordinates as

$$\bar{q}^i = \bar{q}^i(t, x^1, \ldots, x^{n+r})$$

with $\operatorname{rank} \left\| \frac{\partial(\bar{q}^1 \cdots \bar{q}^n)}{\partial(x^1 \cdots x^{n+r})} \right\| = n$. In particular, the differentials $dt, d\bar{q}^1, \ldots, d\bar{q}^n$ are linearly independent everywhere on $\hat{\Omega}$.

Let $\mu^A := {\mu^A}_\alpha(t)\, dx^\alpha{}_{|\hat{\gamma}(t)}$ denote r linear differential forms along $\hat{\gamma}$, depending differentiably on t, and completing $dt_{|\hat{\gamma}(t)}, d\bar{q}^i{}_{|\hat{\gamma}(t)}$ to a basis of $T^*_{\hat{\gamma}(t)}(\mathcal{A})$.

Define r differentiable functions on $\hat{\Omega}$ by

$$\bar{z}^A = \sum_{\alpha=1}^{n+r} {\mu^A}_\alpha(t)\, x^\alpha$$

Then, by construction, the Jacobian $\left\| \frac{\partial(\bar{q}^1 \cdots \bar{q}^n\, \bar{z}^1 \cdots \bar{z}^r)}{\partial(x^1 \cdots x^{n+r})} \right\|$ is non singular at each point $\hat{\gamma}(t)$. The functions $t, \bar{q}^i, \bar{z}^A$ form therefore a coordinate system in a neighborhood $\hat{U}$ of the intersection $\hat{\gamma}\big((c,d)\big) \cap \hat{\Omega}$. The system is automatically fibred over Ω, and satisfies both properties (A.2a, b), and the first condition (A.2c).

To complete the proof, let $\bar{\dot{q}}^i = \bar{\psi}^i(t, \bar{q}^i, \bar{z}^A)$ denote the representation of the imbedding $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$ in the coordinates $t, \bar{q}^i, \bar{z}^A$. Under an arbitrary linear transformation $q^i = {\alpha^i}_j(t)\, \bar{q}^j$, $z^A = \bar{z}^A$ we have then the transformation laws

$$\psi^i = \frac{d{\alpha^i}_j}{dt}\, \bar{q}^j + {\alpha^i}_j\, \bar{\psi}^j, \qquad \frac{\partial \psi^i}{\partial q^k} = \left(\frac{d{\alpha^i}_j}{dt} + {\alpha^i}_r \frac{\partial \bar{\psi}^r}{\partial \bar{q}^j} \right) \left(\alpha^{-1}\right)^j{}_k$$

In particular, if the matrix ${\alpha^i}_j(t)$ is a solution of the differential equation

$$\frac{d{\alpha^i}_j}{dt} + {\alpha^i}_r \left(\frac{\partial \bar{\psi}^r}{\partial \bar{q}^j} \right)_{\hat{\gamma}(t)} = 0$$

the coordinates t, q^i, z^A satisfies all stated requirements. $\square$

Every local chart $(\hat{U}, \hat{h})$ satisfying equations (A.2a, b, c) will be said to be *adapted* to the closed arc $\big(\hat{\gamma}, [a,b]\big)$.

Corollary A.1. *Let* $\hat{\gamma} = \big\{\big(\hat{\gamma}^{(s)}, [a_{s-1}, a_s]\big),\ s = 1, \ldots, N\big\}$ *be the lift of an admissible piecewise differentiable section* $\big(\gamma, [t_0, t_1]\big)$. *Then, there exist fibred local charts* $(\hat{U}_s, \hat{h}_s)$, $\hat{h}_s = \big(t, q^1_{(s)}, \ldots, q^n_{(s)}, z^1_{(s)}, \ldots, z^r_{(s)}\big)$ *adapted to the arcs* $\hat{\gamma}^{(s)}$ *such that, in each intersection* $\pi(\hat{U}_s) \cap \pi(\hat{U}_{s+1})$, *the coordinate transformation* $q^i_{(s+1)} = q^i_{(s+1)}(t, q^1_{(s)}, \ldots, q^n_{(s)})$ *satisfies the condition (A.1)*

$$q^i_{(s+1)}(a_s, q^1_{(s)}, \ldots, q^n_{(s)}) = q^i_{(s)}$$

Proof. The result follows at once by applying Lemma A.2 arc by arc and setting

$$\alpha_{(s)}{}^i{}_j(a_s) \;=\; \alpha_{(s+1)}{}^i{}_j(a_s)$$

for all $s = 1, \ldots, N-1$. $\square$

Every family of local charts $\big\{(\hat{U}_s, \hat{h}_s),\ s = 1, \ldots, N\big\}$ satisfying the requirements of Corollary A.1 will be said to be *adapted* to the lift $\hat{\gamma}$.

Assigning an adapted family of local charts automatically singles out a distinguished infinitesimal control $h^{(s)}$ along each arc $\gamma^{(s)}$, uniquely defined by the requirement

$$h^{(s)}\left[\left(\frac{\partial}{\partial q^i_{(s)}}\right)_{\gamma^{(s)}(t)}\right] = \left(\frac{\partial}{\partial q^i_{(s)}}\right)_{\hat{\gamma}^{(s)}(t)} \qquad \Longleftrightarrow \qquad h_i{}^A(t) = 0$$

In view of equations (1.5.21b), (1.5.22a) and (A.2c), the absolute time derivative associated with $h^{(s)}$ is described in coordinates as

$$\frac{D}{Dt}\left(\frac{\partial}{\partial q^i_{(s)}}\right)_{\gamma^{(s)}(t)} =\; 0 \qquad\qquad s = 1, \ldots, N \tag{A.3}$$

Since, by Corollary A.1, the fields $\left(\frac{\partial}{\partial q^i_{(s)}}\right)_{\gamma^{(s)}(t)}$ are continuous at the corners, then the sections $e_{(i)} : [t_0, t_1] \to V(\gamma)$ given by

$$e_{(i)}(t) \;=\; \left(\frac{\partial}{\partial q^i_{(s)}}\right)_{\gamma^{(s)}(t)} \qquad \forall\ t \in [a_{s-1}, a_s], \quad s = 1, \ldots, N \tag{A.4}$$

form a basis for the space V_h of h–transported vector fields along γ.

On account of equation (A.2c), the corresponding dual basis for the space V_h^* is given by $e^{(i)}(t) \;=\; \hat{\omega}^i{}_{|\gamma^{(s)}(t)} \;=\; dq^i_{(s)|\gamma^{(s)}(t)}$ $\ \forall\ t \in [a_{s-1}, a_s]$, $s = 1, \ldots, N$. By definition, together with equations (A.3) we have therefore the dual relations

$$\frac{D}{Dt}\,\hat{\omega}^i{}_{|\gamma^{(s)}(t)} \;=\; 0 \tag{A.5}$$

Appendix B

Finite deformations with fixed end–points: an existence theorem

According to Proposition 1.5.1, the admissible infinitesimal deformations of an admissible, piecewise differentiable section $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ are in bijective correspondence with the sections $\hat{X} : \mathbb{R} \to A(\hat{\gamma})$ fulfilling the consistency requirement locally expressed by the variational equation (1.5.8).

In the event, this bijective correspondence is actually considered as a full *identification* between them. It was just in this particular sense that in §1.5.4 we claimed that the most general admissible infinitesimal deformation X of γ vanishing at $t = t_0$ is determined by an element $(Y, \underset{\sim}{\alpha}) \in \mathfrak{W}$, namely by a vertical vector field Y along $\hat{\gamma}$ and by a collection of real numbers $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_{N-1})$ and that, in particular, a necessary and sufficient condition for X to satisfy $X(t_1) = 0$ is expressed by the requirement (1.5.43) which, in adapted coordinates, reads

$$\int_{t_0}^{t_1} Y^A \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} dt \; - \; \sum_{s=1}^{N-1} \alpha_s \left[\psi^i(\hat{\gamma}) \right]_{a_s} \; = \; 0 \tag{B.1}$$

This is, for the most part, a right way of acting but care must be taken inasmuch there now may be pathological circumstances in which one can find admissible infinitesimal deformations of γ vanishing at its end–points that are *not* tangent to any admissible finite deformation γ_ξ with fixed end–points.

Example B.1. *Consider a system* $\mathfrak{B}$ *in* $\mathcal{V}_{n+1} = \mathbb{R} \times E_2$ *(referred to coordinates* t, x, y *) and subject to the constraint* $\dot{x}^2 + \dot{y}^2 = v^2$*. We seek those evolutions which join the end–points* $(t_0 = 0, x_0 = 0, y_0 = 0)$ *and* $(t_1 = \bar{t}, x_1 = v\bar{t}, y_1 = 0)$ *and minimize a given action functional.*

It is now apparent that, regardless of the nature of the functional, the problem has a unique solution, represented by the curve $\gamma : x(t) = vt,\ y(t) = 0$*.*

Such a solution is therefore a "rigid" curve, completely lacking in admissible finite deformations with fixed end–points. Even so, there could be admissible infinitesimal *deformations vanishing at the end–points. To see this, we express the imbedding* $i\colon \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ *in the form*

$$i \;:\; \begin{cases} \dot{x} &= v\,\cos z \\ \dot{y} &= v\,\sin z \end{cases}$$

We then require the admissibility of γ *by making the condition*

$$\begin{cases} v &= v\,\cos z \\ 0 &= v\,\sin z \end{cases}$$

whence we get $z = 0$. *A possible lift of* γ *is therefore represented by the curve* $\hat{\gamma} : x(t) = vt,\ y(t) = 0,\ z(t) = 0.$

The variational equations (1.5.8) *are now expressed by*

$$\begin{cases} \frac{dX^1}{dt} &= -v\,(\sin z)_{\hat{\gamma}}\,\Gamma \;=\; 0 \\ \frac{dX^2}{dt} &= v\,(\cos z)_{\hat{\gamma}}\,\Gamma \;=\; v\Gamma \end{cases}$$

the first of which, together with the request $X^1(t_1) = 0$, *entails*

$$X^1(t) \;\equiv\; 0$$

In like manner, the second one becomes

$$X^2(t) \;=\; v\int_0^t \Gamma(\tau)\,d\tau$$

completed by the condition

$$X^2(t_1) \;=\; v\int_0^{t_1} \Gamma(\tau)\,d\tau \;=\; 0$$

To sum up, a possible particular solution is given by

$$X^1(t) \;=\; 0\,, \quad X^2(t) \;=\; v\sin\left(\frac{\pi t}{t_1}\right)\,, \quad \Gamma(t) \;=\; \frac{\pi}{t_1}\cos\left(\frac{\pi t}{t_1}\right)$$

and so we've found an admissible infinitesimal deformation which vanishes at the end–points of γ, *regardless of the latter being a rigid curve that admits no finite deformations.*

Therefore, given an admissible, piecewise differentiable section γ, a crucial question is establishing under what circumstances *every* admissible infinitesimal deformation vanishing at its end–points is tangent to an admissible finite deformation γ_ξ with fixed end–points. If this is the case, the evolution γ is called *ordinary*, otherwise *exceptional*. We will now try to get sufficient conditions for ordinariness.

For this purpose, recalling the contents of Appendix A, we introduce a family of local charts $\{(U_s, k_s)\}$ adapted to $\hat\gamma$ and denote by $\{e_{(i)}\}$, $\{e^{(i)}\}$ the corresponding dual bases for the spaces V_h, V_h^*.

We also bring in, as an auxiliary tool, a positive metric on V_h, described by a symmetric tensor $\Phi = g_{ij}\, e^{(i)} \otimes e^{(j)}$. In view of the identification of $V(\gamma)$ with $[t_0, t_1] \times V_h$, this automatically sets up a scalar product along the fibres of $V(\gamma)$ which, in turn, determines a scalar product between vertical vector fields along $\hat\gamma$, based on the prescription

$$\big(Y, Z\big) \;:=\; \big(\hat\varrho(Y),\, \hat\varrho(Z)\big) \tag{B.2}$$

$\hat\varrho : V(\hat\gamma) \to V(\gamma)$ denoting the homomorphism (1.3.13). In adapted coordinates, equations (1.3.14), (B.2) provide the evaluation $(Y, Z) = G_{AB}\, Y^A Z^B$, with

$$G_{AB} = \left(\hat\varrho\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma},\, \hat\varrho\left(\frac{\partial}{\partial z^B}\right)_{\hat\gamma}\right) = g_{ij}\left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma}\left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat\gamma} \tag{B.3}$$

As usual, the inverse of the matrix G_{AB} will be denoted by G^{AB}.

In a similar manner, by the affine character of the fibration $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$, assigning Φ induces an "orthogonal projection" from the fibers of $V(j_1(\gamma))$ to the ones of $V(\hat\gamma)$ whose representation in local coordinates reads

$$\begin{aligned}
\left(\frac{\partial}{\partial \dot q^i}\right)_{j_1(\gamma)} \;\longrightarrow\;\; & G^{AB}\left(\left(\frac{\partial}{\partial \dot q^i}\right)_{j_1(\gamma)},\, i_*\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma}\right)\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma} \;= \\
&= \; G^{AB}\left(\varrho\left(\frac{\partial}{\partial q^i}\right)_{j_1(\gamma)},\, \varrho\left(\frac{\partial}{\partial q^j}\right)_{j_1(\gamma)}\right)\left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat\gamma}\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma} \;= \\
&= \; G^{AB}\, g_{ij}\left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat\gamma}\left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma}
\end{aligned} \tag{B.4}$$

being now $\varrho : V(j_1(\gamma)) \to V(\gamma)$ the homomorphism (1.3.8).

By means of Φ, to every $\underset{\sim}{\alpha} = (\alpha_1, \dots, \alpha_{N-1}) \in \mathbb{R}^{N-1}$ we associate $N-1$ functions $a_s(\xi)$ according to the prescription

$$a_s(\xi) \;:=\; a_s \,+\, \alpha_s\,\xi \,-\, \tfrac{1}{2}\,\alpha_s^{\,2}\,\xi^{\,2} g_{ij}\,\nu^i\left[\psi^j(\hat\gamma)\right]_{a_s} \qquad s = 1, \dots, N-1 \tag{B.5}$$

For notational convenience, the family is completed by the constant functions $a_0(\xi) = t_0$, $a_N(\xi) = t_1$.

In a similar way, given any vertical vector field Y along $\hat{\gamma}$, meant as a family of fields $Y_{(s)} = Y^A_{(s)}\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}(s)}$ along the arcs of $\hat{\gamma}$, for each $\nu \in V_h$ we denote by $\sigma^{(s)}_{(\xi,\nu)} : \pi(U_s) \to U_s$, $s = 1, \ldots, N$ the $(n+1)$–parameter families of sections described in coordinates as

$$z^{\,A}_{(s)} = \xi\, Y^A_{(s)}(t) + \tfrac{1}{2}\,\xi^2\,\chi^{\,A}_{(s)\,i}(t)\,\nu^i \tag{B.6}$$

with

$$\chi^{\,A}_{(s)\,i}(t) := g_{ik}\,G^{AB}\left(\frac{\partial\psi^k}{\partial z^B}\right)_{\hat{\gamma}} \tag{B.7}$$

It goes without saying that, being strictly coordinate–dependent, equation (B.7) has no invariant geometrical meaning, but is merely a technical tool, whose usefulness will be clear in the subsequent discussion.

Theorem B.1. *Let γ be an admissible, piecewise differentiable evolution and denote by $(Y, \underset{\sim}{\alpha})$ an admissible infinitesimal deformation of γ which vanishes at the end–points. Define the metric Φ and the functions $\chi^{\,A}_{(s)\,i}(t)$, $a_s(\xi)$ as above. Then, given any open subset $\Delta \subset V_h$ with compact closure, there exist an $\varepsilon > 0$ and a family $\gamma_{(\xi,\nu)} = \big\{\big(\gamma^{(s)}_{(\xi,\nu)}, [a_{s-1}(\xi), a_s(\xi)]\big)\big\}$ of piecewise differentiable admissible sections defined for $|\xi| < \varepsilon$, $\nu \in \Delta$ and fulfilling the following properties:*

a) $\gamma_{(0,\nu)}(t) = \gamma(t) \quad \forall\,\nu$;

b) $\gamma_{(\xi,\nu)}(t_0) = \gamma(t_0) \quad \forall\,\xi, \nu$;

c) $\gamma^{(s)}_{(\xi,\nu)}(a_s(\xi)) = \gamma^{(s+1)}_{(\xi,\nu)}(a_s(\xi)) \quad \forall\, s = 1, \ldots, N-1$

d) each arc $\gamma^{(s)}_{(\xi,\nu)}(t)$, expressed in coordinates as $q^{\,i}_{(s)} = \varphi^{\,i}_{(s)}(t, \xi, \nu^i)$, satisfies the control equation

$$\frac{\partial\varphi^{\,i}_{(s)}}{\partial t} = \psi^i\Big(t,\ \varphi^{\,i}_{(s)},\ \xi\, Y^A_{(s)}(t) + \tfrac{1}{2}\,\xi^2\,\chi^{\,A}_{(s)\,i}\,\nu^i\Big) \tag{B.8}$$

Proof. Let $\{(U_s, k_s)\}$ be a family of local charts adapted to $\hat{\gamma}$ and $A \subset V_h$ denote an open set with compact closure containing $\bar{\Delta}$. A straightforward argument shows the existence of an $m > 0$ such that the image $\sigma^{(s)}_{(\xi,\nu)}(\pi(U_s))$ is entirely contained in U_s for all $\nu \in A$, $|\xi| < m$, $s = 1, \ldots, N$.

We choose such an $m \in \mathbb{R}_+$ and examine the situation separately in each chart (U_s, k_s). There, solving equation (B.8) amounts to determining the integral curves of the $(n+1)$–parameter family of vector fields $Z^{(s)}_{(\xi,\nu)} = \dfrac{\partial}{\partial t} + Z^{\,i}_{(s)}\dfrac{\partial}{\partial q^i}$ on $\pi(U_s)$, with $Z^{\,i}_{(s)} = \psi^i\big(t, q^k, \xi\, Y^A_{(s)}(t) + \frac{1}{2}\,\xi^2\,\chi^{\,A}_{(s)\,h}(t)\,\nu^h\big)$.

This, in turn, is equivalent to determining the integral curves of a single vector field $\tilde{Z}_{(s)} = \dfrac{\partial}{\partial t} + Z^{\,i}_{(s)}\dfrac{\partial}{\partial q^i}$ in the product manifold $(-m,m)\times A\times\pi(U_s)$.

Let $\zeta^{(s)}_{(\xi,\nu)}(t;x)$ denote the integral curve of $\tilde{Z}_{(s)}$ through the point (ξ,ν,x). Also, let c_{s-1} denote the corner $\gamma(a_{s-1})$. Then, on account of equations (A.2c), chosen any $\nu^*\in A$, the curve $\zeta^{(s)}_{(0,\nu^*)}(t;c_{s-1})$ coincides with the coordinate line $q^i=0, \xi=0,\ \nu=\nu^*$ and is therefore defined for all t in an open interval $(b_{s-1},b_s)\supset[a_{s-1},a_s]$.

By well–known theorems in ordinary differential equations [11, 22] this implies the existence of an open neighborhood $W_{s-1}\ni(0,\nu^*,c_{s-1})$ such that the curve $\zeta^{(s)}_{(\xi,\nu)}(t;x)$ is defined for all $(\xi,\nu,x)\in W_{s-1}$ and all t in the closed interval $\big[t(x),a_s(\xi)\big]\subset(b_{s-1},b_s)$.

In particular, denoting by Σ_s the slice $t=a_s(\xi)$ in $(-m,m)\times A\times\pi(U_s)$, we conclude that the 1–parameter group of diffeomorphisms determined by the field $\tilde{Z}_{(s)}$ maps the intersection $W_{s-1}\cap\Sigma_{s-1}$ into an open neighborhood of the point $(0,\nu^*,c_s)$ in Σ_s. Without loss of generality we may always arrange for the image of each $W_{s-1}\cap\Sigma_{s-1}$ to be contained in $W_s\cap\Sigma_s$, $s=1,\dots,N$.

The rest is now entirely straightforward: let U and $\varepsilon_U>0$ respectively denote an open neighborhood of ν^* in A and a positive number such that[1] $(\xi,\nu,x_0)\in W_0\cap\Sigma_0\ \forall\ |\xi|<\varepsilon_U,\ \nu\in U$. For each $|\xi|<\varepsilon_U,\ \nu\in U$ consider the sequence of closed arcs $\gamma^{(s)}_{(\xi,\nu)}:[a_{s-1}(\xi),a_s(\xi)]\to\pi(U_s)$ defined inductively by

$$
\begin{aligned}
\gamma^{(1)}_{(\xi,\nu)}(t) &= \zeta^{(1)}_{(\xi,\nu)}(t;x_0) && t\in[t_0,a_1(\xi)]\\
\gamma^{(s+1)}_{(\xi)}(t) &= \zeta^{(s+1)}_{(\xi,\nu)}\big(t;\gamma^{(s)}_{(\xi)}(a_s(\xi))\big) && t\in[a_s(\xi),a_{s+1}(\xi)]
\end{aligned}
$$

The collection $\gamma_{(\xi,\nu)}:=\big\{\big(\gamma^{(s)}_{(\xi,\nu)},[a_{s-1}(\xi),a_s(\xi)]\big),\ s=1,\dots,N\big\}$ is then easily recognized to define an $(n+1)$–parameter family of continuous, piecewise differentiable sections fulfilling all Theorem's requirements. To complete our proof let us finally recall that, for any $\nu^*\in A$, the family $\gamma_{(\xi,\nu)}$ exists for all ν in an open neighborhood $U\ni\nu^*$ and all $|\xi|<\varepsilon_U$. On the other hand, by the assumed compactness of $\bar{\Delta}$, the subset $\Delta\subset A$ may be covered by a finite number of subsets $\{U_1,\dots,U_k\}$ of the required type.

The conclusion thus follows by choosing $\varepsilon=\min\{\varepsilon_{U_1},\dots,\varepsilon_{U_k}\}$. $\square$

According to Theorem B.1, for any open subset $\Delta\subset V_h$ with compact closure, the correspondence $\nu\to\gamma_{(\xi,\nu)}(t_1)$ sets up a 1–parameter family of differentiable maps of Δ into the hypersurface $t=t_1$, with values in a neighborhood of the point $\gamma(t_1)$. Moreover, given any differentiable curve $\nu=\nu(\xi)$ in Δ, the 1–parameter family of sections $\gamma_{(\xi,\nu(\xi))}(t),\ |\xi|<\varepsilon,\ t\in[t_0,t_1]$ is a defor-

[1] Notice that, according to our thesis, we are “freezing” the choice of the point x_0.

mation of γ tangent to the original infinitesimal deformation X determined by $(Y, \alpha_1, \ldots, \alpha_{N-1})$ and leaving the first end–point $\gamma(t_0)$ fixed.

Therefore, in order to find an answer for our opening question, it just remains to establish the existence of a curve $\nu(\xi)$ satisfying $\gamma_{(\xi,\nu(\xi))}(t_1) \equiv \gamma(t_1)$ in some open neighborhood of $\xi = 0$.

In adapted coordinates, setting for simplicity $\varphi^i(\xi,\nu) := \varphi^{\,i}_{(N)}(t_1,\xi,\nu)$, the required condition reads

$$\varphi^i\big(\xi, \nu^1(\xi), \ldots, \nu^n(\xi)\big) \,=\, 0 \qquad\qquad i = 1, \ldots, n \tag{B.9a}$$

Taking the relations $\varphi^i(0,\nu) = q^{\,i}_{(N)}(\gamma(t_1)) = 0$, $\big(\frac{\partial \varphi^i}{\partial \xi}\big)_{\xi=0} = X^i(t_1)$ into account, a straightforward application of Taylor's theorem shows that, whenever the condition $X(t_1) = 0$ holds true, namely whenever the field Y and the coefficients α_s fulfil equation (B.1), the functions φ^i are necessarily of the form $\varphi^i(\xi,\nu) = \xi^2\,\theta^i(\xi,\nu)$, with $\theta^i(\xi,\nu)$ regular at $\xi = 0$. Under the stated assumptions, equation (B.9) is therefore equivalent to the condition

$$\theta^i(\xi, \nu^1, \ldots, \nu^n) \,=\, 0 \qquad\qquad i = 1, \ldots, n \tag{B.9b}$$

We will now discuss its solvability for the ν^i's as functions of ξ in a neighborhood of $\xi = 0$. To start with, we observe that the matching conditions *c*) of Theorem B.1 give rise to relations of the form

$$\varphi^{\,i}_{(s\,+\,1)}(a_s(\xi),\xi,\nu) \,=\, q^{i}_{(s\,+\,1)}\Big(a_s(\xi)\,,\,\varphi^{\,1}_{(s)}\big(a_s(\xi),\xi,\nu\big)\,,\ldots,\,\varphi^{\,n}_{(s)}\big(a_s(\xi),\xi,\nu\big)\Big)$$

$q^{i}_{(s\,+\,1)} = q^{i}_{(s\,+\,1)}(t, q^{1}_{(s)}, \ldots, q^{n}_{(s)})$ denoting the transformation between adapted coordinates in the intersection $\pi(U_s \cap U_{s+1})$. From these, deriving with respect to ξ we get the expressions

$$\frac{\partial \varphi^{\,i}_{(s\,+\,1)}}{\partial t}\,\frac{da_s}{d\xi} + \frac{\partial \varphi^{\,i}_{(s\,+\,1)}}{\partial \xi} \,=\, \frac{\partial q^{i}_{(s\,+\,1)}}{\partial t}\,\frac{da_s}{d\xi} + \frac{\partial q^{i}_{(s\,+\,1)}}{\partial q^{\,k}_{(s)}}\left(\frac{\partial \varphi^{\,k}_{(s)}}{\partial t}\,\frac{da_s}{d\xi} + \frac{\partial \varphi^{\,k}_{(s)}}{\partial \xi}\right) \tag{B.10}$$

At $\xi = 0$, recalling equations (1.5.34a), (A.1), (B.5) as well as the identification $X^{\,i}_{(s)} = \frac{\partial \varphi^{\,i}_{(s)}}{\partial \xi}\Big|_{\xi=0}$ the latter provide the relation

$$X^{\,i}_{(s\,+\,1)}(a_s) = \alpha_s\,\frac{\partial q^{i}_{(s\,+\,1)}}{\partial t}\bigg|_{c_s} + X^{\,i}_{(s)}(a_s) \;\Longrightarrow\; \frac{\partial q^{i}_{(s\,+\,1)}}{\partial t}\bigg|_{c_s} = -\Big[\psi^i(\hat\gamma)\Big]_{a_s} \tag{B.11}$$

In a similar way, on account of equations (A.1), (B.5), (B.11), deriving equation (B.10) with respect to ξ and evaluating everything at $\xi = 0$, a straightforward calculation yields the result

$$\left[\frac{\partial^2\varphi^{\,i}_{(s+1)}}{\partial\xi^2}-\frac{\partial^2\varphi^{\,i}_{(s)}}{\partial\xi^2}\right]_{c_s}=$$

$$=\alpha_s^2\,\frac{\partial^2 q^i_{(s+1)}}{\partial t^2}+2\alpha_s\,\frac{\partial^2 q^i_{(s+1)}}{\partial t\,\partial q^k_{(s)}}\,X^{\,k}_{(s)}+\frac{\partial^2 q^i_{(s+1)}}{\partial q^h_{(s)}\partial q^k_{(s)}}\,X^{\,h}_{(s)}X^{\,k}_{(s)}+$$

$$-\,2\,\alpha_s\left[\frac{dX^{\,i}_{(s+1)}}{dt}-\frac{dX^{\,i}_{(s)}}{dt}\right]_{c_s}+\alpha_s^{\,2}\left[\psi^i(\hat\gamma)\right]_{a_s}g_{rk}\left[\psi^r(\hat\gamma)\right]_{a_s}\nu^k \tag{B.12}$$

expressing the jumps $\left[\frac{\partial^2\varphi^{\,i}_{(s+1)}}{\partial\xi^2}-\frac{\partial^2\varphi^{\,i}_{(s)}}{\partial\xi^2}\right]_{c_s}$ in terms of the section γ, of the infinitesimal deformation and of the variables ν^i.

In addition to this let us now make use of the fact that, exactly as it happened in §1.5.1 with equation (1.5.6b), in each adapted chart, by derivation of (B.8), we get the evolution equations

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\varphi^{\,i}_{(s)}}{\partial\xi^2}\right)_{\xi=0}=\left(\frac{\partial^2\psi^i}{\partial q^k\partial q^r}\right)_{\hat\gamma^{(s)}}X^kX^r+2\left(\frac{\partial^2\psi^i}{\partial q^k\partial z^A}\right)_{\hat\gamma^{(s)}}X^kY^A+$$

$$+\left(\frac{\partial^2\psi^i}{\partial z^A\partial z^B}\right)_{\hat\gamma^{(s)}}Y^AY^B+\cancel{\left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma^{(s)}}\left(\frac{\partial^2\varphi^{\,k}_{(s)}}{\partial\xi^2}\right)_{\xi=0}}+\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}}\chi^{\,A}_{(s)\,k}\,\nu^k$$

the cancelation arising from equation (A.2c).

From the latter, restoring the notation $\varphi^i(\xi,\nu)$ for $\varphi^{\,i}_{(N)}(t_1,\xi,\nu)$ and recalling equations (B.7), (B.12), as well as the components g_{ij} being — by definition — *constant* along γ, we get an expression of the form

$$\theta^i\big|_{\xi=0}=\left(\frac{\partial^2\varphi^i}{\partial\xi^2}\right)_{\xi=0}=$$

$$=b^i+\left(\sum_{s=1}^{N}\int_{a_{s-1}}^{a_s}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}(t)}\chi^{\,A}_{(s)\,k}(t)\;dt+\sum_{s=1}^{N-1}\alpha_s^2\left[\psi^i(\hat\gamma)\right]_{a_s}g_{jk}\left[\psi^j(\hat\gamma)\right]_{a_s}\right)\nu^k=$$

$$=b^i+\left(\int_{t_0}^{t_1}G^{AB}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma}\left(\frac{\partial\psi^j}{\partial z^B}\right)_{\hat\gamma}dt+\sum_{s=1}^{N-1}\alpha_s^2\left[\psi^i(\hat\gamma)\right]_{a_s}\left[\psi^j(\hat\gamma)\right]_{a_s}\right)g_{jk}\,\nu^k \tag{B.13}$$

with $b^i\in\mathbb{R}$ depending solely on the section γ and on the original infinitesimal deformation. Collecting all results we can therefore state

Proposition B.1. *Let $\gamma:[t_0,t_1]\to\mathcal{V}_{n+1}$ be a continuous, piecewise differentiable, admissible section. Then, if the matrix*

$$S^{ij}:=\int_{t_0}^{t_1}G^{AB}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma}\left(\frac{\partial\psi^j}{\partial z^B}\right)_{\hat\gamma}dt+\sum_{s=1}^{N-1}\alpha_s^2\left[\psi^i(\hat\gamma)\right]_{a_s}\left[\psi^j(\hat\gamma)\right]_{a_s} \tag{B.14}$$

is non–singular, every infinitesimal deformation of γ vanishing at the end–points is tangent to a finite deformation with fixed end–points.

Proof. The conclusion follows at once simply by observing that, on account of equation (B.13), the non–singularity of the matrix (B.14) ensures the solvability of equations (B.9b) in a neighborhood of $\xi = 0$. $\square$

Proposition B.1 may be rephrased in the language of §1.5.4: whenever the section γ is abnormal, Proposition 1.5.4 and equation (A.4) imply actually the existence of at least one non–zero virtual 1–form $\lambda_i\, \hat{\omega}^i{}_{|\gamma}$ with *constant* components λ_i fulfilling the relations

$$\lambda_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}(t)} = 0\,, \qquad \lambda_i \left[\psi^i(\hat{\gamma})\right]_{a_s} = 0 \tag{B.15}$$

and therefore automatically satisfying $\lambda_i\, S^{ij} = 0$, completely equivalent to the singularity of the matrix (B.14).

More specifically, denoting by p the *abnormality index* of γ, we have the following

Theorem B.2. *The matrix* (B.14) *has rank $n - p$.*

Proof. By definition, the index p coincides with the dimension of the annihilator $\big(\Upsilon(\mathfrak{W})\big)^0 \subset V_h^*$, which is identical to the dimension of the space of constant solutions of equations (B.15).

On the other hand, by equations (B.3), (B.14), the matrix S^{ij} is positive semidefinite. Its kernel is therefore identical to the totality of zeroes of the quadratic form[2] $S^{ij}\lambda_i\lambda_j$, that is to the totality of n–tuples $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ fulfilling the relation

$$\begin{aligned} 0 &= \left(\int_{t_0}^{t_1} G^{AB} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat{\gamma}} dt \,+\, \sum_{s=1}^{N-1} {\alpha_s}^2 \left[\psi^i(\hat{\gamma})\right]_{a_s} \left[\psi^j(\hat{\gamma})\right]_{a_s}\right) \lambda_i \lambda_j = \\ &= \int_{t_0}^{t_1} G^{AB} \left(\lambda_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}}\right) \left(\lambda_j \left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat{\gamma}}\right) dt \,+\, \sum_{s=1}^{N-1} {\alpha_s}^2 \left(\lambda_i \left[\psi^i(\hat{\gamma})\right]_{a_s}\right)^2 \end{aligned}$$

Because of the positive definiteness of $G^{AB}(t)$, the last condition is equivalent to equations (B.15). This proves $\dim\big(\ker(S^{ij})\big) = p$ which, in turn, entails $\operatorname{rank}\big(S^{ij}\big) = n - p$. $\square$

In the language of §1.5.4, Proposition B.1 and Theorem B.2 show that the normal evolutions form a subset of the ordinary ones, thus establishing Proposition 1.5.5.

Along the same lines, a deeper result is provided by the following

[2] See Appendix D, Lemma D.1.

Theorem B.3. *Let p $(\geqslant 0)$ denote the abnormality index of the evolution γ. Then a sufficient condition for the ordinariness of γ is the existence of both an $(n-p)$–dimensional submanifold $S \subset \mathcal{V}_{n+1}$, contained in the slice $t = t_1$ and including the point $\gamma(t_1)$, and an $\varepsilon > 0$ such that every deformation γ_ξ which leaves $\gamma(t_0)$ fixed fulfils the relation $\gamma_\xi(t_1) \in S$ for all $|\xi| < \varepsilon$.*

Proof. We assume the existence of both a submanifold $S \xrightarrow{i} \mathcal{V}_{n+1}$ and an $\varepsilon > 0$ with the stated properties. We also denote by $(V, \zeta^1, \dots, \zeta^{n-p})$ a local chart in S centered at the point $\gamma(t_1)$ and by

$$t = t_1, \qquad q^i_{(N)} = \varrho^i\big(\zeta^1, \dots, \zeta^{n-p}\big) \tag{B.16}$$

the representation of S in adapted coordinates.

By hypothesis, the correspondence $(\xi, \nu) \to \gamma_{(\xi,\nu)}(t_1)$ factors through S for any open subset $\Delta \subset V_h$ with compact closure and for any $\xi \in (-\varepsilon, \varepsilon)$. This gives rise to a differentiable map $g : (-\varepsilon, \varepsilon) \times \Delta \to S$ satisfying the relation $\gamma_{(\xi,\nu)}(t_1) = i \cdot g(\xi, \nu)$.

In coordinates, setting $\zeta^\alpha(g(\xi,\nu)) = g^\alpha(\xi, \nu^1, \dots, \nu^n)$ and resuming the notation $\varphi^i(\xi, \nu^1, \dots, \nu^n)$ for $q^i_{(N)}(\gamma_{(\xi,\nu)}(t_1))$, this provides the identification

$$\varphi^i(\xi, \nu^1, \dots, \nu^n) \;=\; \varrho^i\big(g^1(\xi, \nu^1, \dots, \nu^n), \dots,\, g^{n-p}(\xi, \nu^1, \dots, \nu^n)\big) \tag{B.17}$$

From this, recalling the relation $g^\alpha(0, \nu^1, \dots, \nu^n) = \zeta^\alpha(\gamma(t_1)) = 0$ as well as the rank of the Jacobian $\frac{\partial(\varrho^1 \cdots \varrho^n)}{\partial(\zeta^1 \cdots \zeta^{n-p})}$ being maximal, it is easily seen that the equalities $\varphi^i(0, \nu^1, \dots, \nu^n) = \frac{\partial \varphi^i}{\partial \xi}(0, \nu^1, \dots, \nu^n) = 0$ are reflected into analogous properties of the functions g^α.

By Taylor's theorem we have therefore an expression of the form

$$g^\alpha \;=\; \xi^2 \mu^\alpha(\xi, \nu^1, \dots, \nu^n) \tag{B.18}$$

with the functions μ^α regular at $\xi = 0$.

The proof is thus reduced to establishing the solvability of the system

$$\mu^\alpha(\xi, \nu^1, \dots, \nu^n) \;=\; 0 \tag{B.19}$$

for the ν^i's as functions of ξ in a neighborhood of $\xi = 0$.

To this end, by direct computation, from equations (B.17), (B.18) we derive the relation

$$\left(\frac{\partial^2 \varphi^i}{\partial \xi^2}\right)_{\xi=0} = \cancel{\left(\frac{\partial^2 \varrho^i}{\partial \zeta^\alpha \partial \zeta^\beta}\, \frac{\partial g^\alpha}{\partial \xi}\, \frac{\partial g^\beta}{\partial \xi}\right)_{\xi=0}} + \left(\frac{\partial \varrho^i}{\partial \zeta^\alpha}\, \frac{\partial^2 g^\alpha}{\partial \xi^2}\right)_{\xi=0} =$$

$$= \, 2 \left(\frac{\partial \varrho^i}{\partial \zeta^\alpha}\right)_{(0,\dots,0)} \mu^\alpha{}_{|\xi=0} \;=\; 2 \left(\frac{\partial \varrho^i}{\partial \zeta^\alpha}\right)_{\gamma(t_1)} \mu^\alpha(0, \nu^1, \dots, \nu^n)$$

Together with equations (B.13), (B.14), the latter provides the identification

$$b^i + S^{ir}\, g_{rk}\, \nu^k \;=\; 2\left(\frac{\partial \varrho^i}{\partial \zeta^\alpha}\right)_{\gamma(t_1)} \mu^\alpha(0, \nu^1, \ldots, \nu^n) \tag{B.20}$$

In view of this, the functions $\mu^\alpha(0, \nu^1, \ldots, \nu^n)$ are therefore linear polynomials

$$\mu^\alpha(0, \nu^1, \ldots, \nu^n) \;=\; M^\alpha{}_k\, \nu^k \,+\, c^\alpha \tag{B.21}$$

with coefficients $M^\alpha{}_k$, c^α uniquely determined in terms of b^i, S^{ir}, g_{rk} and of the imbedding (B.16). In particular, by equation (B.20), the rank of the matrix $M^\alpha{}_k$ cannot be smaller than the one of S^{ij} and, of course, cannot exceed $n-p$. According to Theorem B.2, we have therefore $\operatorname{rank} M^\alpha{}_k = n-p$.
Collecting all results, we conclude:

- the system (B.19) admits ∞^p solutions of the form $(0, \nu^{*1}, \ldots, \nu^{*n})$;
- on account of equation (B.21), the Jacobian $\left\| \frac{\partial(\mu^1 \cdots \mu^{n-p})}{\partial(\nu^1 \cdots \nu^n)} \right\|$ has rank $n-p$ at each point $(0, \nu^1, \ldots, \nu^n)$. By continuity, it has therefore rank $n-p$ in a neighborhood of every solution $(0, \nu^{*1}, \ldots, \nu^{*n})$ of equations (B.19).

By the implicit function theorem, this proves that the system (B.19) admits at least a solution of the form $\nu^i = \nu^i(\xi)$ in a neighborhood of $\xi = 0$ (actually, infinitely many solutions whenever $p > 0$). $\square$

Appendix C

Admissible angular deformations

Let $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ be a normal differentiable evolution. If $\hat{\gamma} : [t_0, t_1] \to \mathcal{A}$ is the lift of γ we can refer $\mathcal{A}$ to a system of local fibred coordinates $(\hat{U}, \{t, q^i, z^A\})$ adapted to $\hat{\gamma}$, as discussed in Appendix A.

Chosen both an arbitrary point $t^* \in (t_0, t_1)$ as well as point $\bar{z} = (t^*, 0, \bar{z}^A)$ on the fibre $\pi^{-1}(\gamma(t^*)) \subset \mathcal{A}$, for every $\xi^* \in (0, t^* - t_0)$ we can take into account the control $\sigma : U \to \mathcal{A}$, locally described as:

$$z^A \sigma(t, q) = \begin{cases} 0 & t_0 \leqslant t < t^* - \xi \\ \bar{z}^A & t^* - \xi \leqslant t < t^* \\ 0 & t^* \leqslant t \leqslant t_1 \end{cases}$$

Theorem C.1. *There exists $\varepsilon > 0$ such that for every $\xi < \varepsilon$ the equation*

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A \sigma(t, q^i))$$

with initial data $q^i(t_0) = 0$ admits a unique solution $q^i(t, \xi)$ which is continuous over the interval $[t_0, t_1]$ and piecewise–differentiable over (t_0, t_1), with corners located in $t^* - \xi$ and t^*.

Proof. As far as the interval $[t_0, t^* - \xi)$ is concerned, the required solution is evidently $q^i(t, \xi) = 0$. Then, moving onto $[t^* - \xi, t^*)$ and here considering the differential equation

$$\frac{dq^i}{dt} = \psi^i(t, q^i, \bar{z}^A) \tag{C.1}$$

we can readily prove the existence of an $\varepsilon > 0$ such that equation (C.1) admits a unique solution fulfilling the condition $q^i(t^* - \xi, \xi) = 0$ for every $\xi < \varepsilon$. The values $\bar{q}^i$ taken by this solution when evaluated in $t = t^*$ can be assumed “small” (namely of the same order as ξ) and may be used as initial data in t^* for the differential equation

$$\frac{dq^i}{dt} = \psi^i(t, q^i, 0)$$

Therefore, by well–known theorems in ordinary differential equations, such equation is solvable up to the point $t = t_1$, taking care of decreasing the value of ε if necessary. As a result, we are given an admissible deformation $q^i = \varphi^i(t, \xi)$ of the curve γ that is irreversible (since it is defined for $\xi > 0$ only), that fulfils the condition $\lim_{\xi \to 0^+} \gamma_\xi = \gamma$ and that, unlike the original evolution γ, is endowed with a pair of corners. $\square$

A great improvement of Theorem C.1 is provided by the following:

Corollary C.1. *If γ is a* normal *curve, then it is possible to alter the control σ in the interval $[t^*, t_1]$ in such a way that all the curves γ_ξ pass through the same point $\gamma_\xi(t_1) = \gamma(t_1)$.*

Proof. Let $t = t^*$, $q^i = \bar{q}^i(\xi)$ be the orbit of the second corner of the deformation γ_ξ and let $\hat{X} = X^i(t)\big(\frac{\partial}{\partial q^i}\big)_{\hat{\gamma}} + Y^A(t)\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}}$ be an infinitesimal deformation of the arc $(\hat{\gamma}, [t^*, t_1])$, such that $X^i(t^*) = \frac{d\bar{q}^i}{d\xi}\big|_{\xi=0}$. Chosen a system of local coordinates adapted to γ, the variational equation reads

$$X^i(t) \ = \ X^i(t^*) \ + \int_{t^*}^{t} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} Y^A \, dt$$

Therefore, among the above described infinitesimal deformations, the ones which vanish in $t = t_1$ are in bijective correspondence with the vector fields $Y^A(t)\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}}$ satisfying:

$$\int_{t^*}^{t_1} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} Y^A \, dt \ = \ -X^i(t^*)$$

Now let $\hat{X}$ be an infinitesimal deformation with the above properties. Following the guidelines provided in Appendix B, in the interval $[t^*, t_1]$ we substitute the original control $z^A \sigma(t, q^i) = 0$ with

$$z^A \sigma(t, q^i) \ = \ \xi \, Y^A(t) \ + \ \frac{1}{2} \xi^2 \, \chi^A_i(t) \, \nu^i$$

where, passing over all the useless details, $\chi^A_i(t)$ is an $n \times r$ matrix while $\underset{\sim}{\nu} = (\nu^1, \ldots, \nu^n)$ is a vector in $\mathbb{R}^n$. The quantities $q^i(t)$ are required to fulfill the differential equation

$$\frac{dq^i}{dt} \ = \ \psi^i(t, q^i, \xi \, Y^A \ + \ \frac{1}{2} \xi^2 \, \chi^A_i(t) \, \nu^i) \quad , \tag{C.2}$$

with initial data $q^i(t^*,\xi) = \bar{q}^i(\xi)$. Recalling the results of Appendix B, for sufficiently small values of ξ, the solution of the system (C.2) exists up to $t = t_1$ thus determining a trajectory $q^i = q^i(t_1, \xi, \nu^1, \dots, \nu^n) := \varphi^i(\xi, \nu^1, \dots, \nu^n)$.

Once again, we only need to determine a set of functions $\nu^i = \nu^i(\xi)$ such that $\varphi^i(\xi, \nu^1(\xi), \dots, \nu^n(\xi)) = 0$. By Dini's theorem, this is only possible if the Jacobian matrix $\frac{\partial \varphi^i}{\partial \nu^j}$ is non–singular. In this connection, the following facts can be proved:

• the relations $\varphi^i(0, \underset{\sim}{\nu}) = q^i(\gamma(t_1)) = 0$, $\left(\frac{\partial \varphi^i}{\partial \xi}\right)_{\xi=0} = X^i(t_1) = 0$, entail that $\varphi^i(\xi, \underset{\sim}{\nu}) = \xi^2\, \theta^i(\xi, \underset{\sim}{\nu})$, $\theta^i(\xi, \underset{\sim}{\nu})$ being regular for $\xi \to 0^+$. The required identity can be therefore expressed in the form:

$$\theta^i(\xi, \nu^1, \dots, \nu^n) \;=\; 0 \tag{C.3}$$

• in a system of adapted coordinates, equation (C.2) yields the evolution equation

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial^2 q^i}{\partial \xi^2}\right)_{\xi=0} = \left(\frac{\partial^2 \psi^i}{\partial q^k \partial q^r}\right)_{\hat{\gamma}} X^k X^r + 2\left(\frac{\partial^2 \psi^i}{\partial q^k \partial z^A}\right)_{\hat{\gamma}} X^k Y^A \\
+ \left(\frac{\partial^2 \psi^i}{\partial z^A \partial z^B}\right)_{\hat{\gamma}} Y^A Y^B + \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \chi^A_k \nu^k
\end{aligned}$$

whence

$$\frac{\partial}{\partial t}\left(\frac{\partial \theta^i}{\partial \nu^k}\right)_{\xi=0} = \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \chi^A_k \;\Rightarrow\; \frac{\partial \theta^i}{\partial \nu^k} = \int \frac{\partial \psi^i}{\partial z^A}\, \chi^A_k \; dt \tag{C.4}$$

• the solvability of (C.3) is then equivalent to the non–singularity of the matrix (C.4) for at least one choice of the functions χ^A_i, which is automatically guaranteed by the normality of γ. $\square$

Appendix D

A touch of theory of quadratic forms

Let V be a linear space[1] over $\mathbb{R}$ and let $\psi\colon V \times V \to \mathbb{R}$ be a symmetric bilinear functional. The mapping $v \mapsto \psi(v,v)$ of V into $\mathbb{R}$ is called the *quadratic form* associated to ψ. If V is referred to a basis $\{e_1, \ldots, e_n\}$, we then have the representation

$$\psi(v,v) \,=\, v^i\, v^j\, \psi(e_i, e_j) \,=\, \psi_{ij}\, v^i\, v^j$$

which actually shows the nature of $\psi(v,v)$ as a quadratic form in the variables v^i.

Depending on the properties of their associated quadratic form, symmetric bilinear functionals are classified as

- *indefinite* if, varying v, the quantity $\psi(v,v)$ may assume arbitrary real values;
- *positive (negative) semidefinite* if $\psi(v,v) \geqslant 0 \;\; (\leqslant 0) \;\forall\, v \in V$.

In this connection, we have the following

Lemma D.1. *A symmetric bilinear functional $\psi\colon V \times V \to \mathbb{R}$ is semidefinite if and only if then its kernel*[2] $\ker(\psi)$ *coincides with the locus of zeroes of the quadratic form $\psi(v,v)$.*

Proof. The vanishing of $\psi(u,u)$ for all $u \in \ker(\psi)$ is quite obvious. Let's see the converse. If $u \in V$ is such that $\psi(u,u) = 0$, then

$$\psi(\alpha u + v, \alpha u + v) \,=\, 2\,\alpha\,\psi(u,v) \,+\, \psi(v,v)$$

[1] For the time being, we suppose V to be finite–dimensional. In case of need, however, we may straightforwardly make all results that are drawn here fit an infinite–dimensional context.

[2] We recall that the kernel of a bilinear functional $\psi\colon V \times V \to \mathbb{R}$ is defined as the set $\ker(\psi) = \{\, u \,|\, u \in V, \psi(u,v) = 0 \,\forall\, v \in V\,\}$.

for all $v \in V$, $\alpha \in \mathbb{R}$. Because of the arbitrariness of α, if the functional ψ has to be semidefinite — as it is by hypothesis — the quantity $\psi(u,v)$ is necessarily zero for all $v \in V$. This in turn implies $u \in \ker(\psi)$. $\square$

Another possible way of looking at Lemma D.1 is that if we are given a symmetric bilinear functional ψ on V and if we find $u, v \in V$ such that $\psi(u,u) = 0$ *but* $\psi(u,v) \neq 0$, then we can assert that ψ is necessarily *indefinite*.

A not singular semidefinite symmetric bilinear functional is said to be *definite*. According to Lemma D.1, this entails

$$\psi \textit{ positive (negative) definite} \quad \Longleftrightarrow \quad \psi(v,v) > 0 \ (< 0) \quad \forall\, v \in V,\ v \neq 0$$

We now conclude this brief Appendix by proving how the knowledge of the definite character of the functional ψ on both a subspace and a quotient space enables to give a statement about its definiteness on the entire space.

Theorem D.1. *Let $K \subset V$ be a linear subspace and $W := {}^{V}\!/_{K}$ the quotient space of V by K. If the restriction of the symmetric bilinear functional $\psi\colon V \times V \to \mathbb{R}$ onto the subspace K is not singular, then:*

i) for any $v \in V$, the restriction to the equivalence class $[v]$ of the quadratic form associated with ψ has a single stationarity point v^*;

ii) defining a map $f\colon W \to \mathbb{R}$ as $f([v]) := \psi(v^*, v^*)$ automatically sets up a quadratic form on the quotient space W;

iii) if ψ is positive definite, so is f; conversely, the positive definiteness of both f on W and ψ on K implies the positive definiteness of ψ on the whole of V.

Proof. We consider a basis $\{\kappa_\alpha\}$, $\alpha = 1, \ldots, r = \dim K$, in the subspace K and complete it to a basis $\{\kappa_\alpha, e_i\}$ of V. Every element $v \in V$ is then represented in components as $v = \xi^\alpha \kappa_\alpha + v^i e_i$, while its equivalence class $[v]$ is the affine space formed by the totality of vectors $u = \xi^\alpha \kappa_\alpha + v^i e_i$ with fixed v^i's and arbitrary ξ^α's. The restriction to $[v]$ of the quadratic form associated to the functional ψ is thus written in coordinates as

$$\psi(u,u) \;=\; \psi_{\alpha\beta}\, \xi^\alpha \xi^\beta \;+\; 2\, \psi_{\alpha i}\, \xi^\alpha v^i \;+\; \psi_{ij}\, v^i v^j$$

whilst the search for its stationarity points is carried out by means of the equation

$$0 \;=\; \frac{\partial \psi}{\partial \xi^\alpha} \;= 2\left(\psi_{\alpha\beta}\, \xi^\beta \;+\; \psi_{\alpha i}\, v^i\right) \tag{D.1}$$

Hence, because of the non–singularity of the matrix $\psi_{\alpha\beta}$, denoting by $\psi^{\alpha\beta}$ its inverse, we find out

$$v^* \;=\; -\psi^{\alpha\beta}\,\psi_{\beta i}\,v^i\,\kappa_\alpha \;+\; v^i e_i \;:=\; \xi^{*\alpha}\kappa_\alpha \;+\; v^i e_i \tag{D.2}$$

This proves i). Assertion ii) is then easily seen to be self–evident simply by pointing out that each element $[v]$ has components v^i with respect to the basis $\{[e_i]\}$ of W and that the function f is represented in coordinates as

$$\psi(v^*, v^*) = \psi_{\alpha\beta}\,\xi^{*\alpha}\xi^{*\beta} \;+\; 2\,\psi_{\alpha i}\,\xi^{*\alpha}v^i \;+\; \psi_{ij}\,v^i v^j \;=\; \Big(\psi_{ij} \;-\; \psi^{\alpha\beta}\,\psi_{\alpha i}\,\psi_{\beta j}\Big)v^i v^j \tag{D.3}$$

At last, if ψ is positive definite, then

$$\psi(v,v) > 0 \;\; \forall\, v \neq 0 \quad \Rightarrow \quad \psi(v^*,v^*) > 0 \;\; \forall\, v^* \neq 0 \quad \Rightarrow \quad f([v]) > 0 \;\; \forall\,[v] \neq 0$$

showing the positivity of f.

Conversely, if ψ is positive definite when restricted to K, the stationarity point v^* that we worked out by means of equations (D.1), (D.2) is clearly a *minimum*, the Hessian $\frac{\partial^2\psi}{\partial\xi^\alpha\partial\xi^\beta}$ being positive definite by hypothesis. Thus, if f is also positive definite, for any $v \in V$, $\psi(v,v) \geqslant \psi(v^*,v^*) = f([v])$ which, in particular, entails $\psi(v,v) > 0\;\forall\, v \notin K$. On the other hand, by hypothesis, $\psi(v,v) > 0\;\forall\, v \in K - \{0\}$ whence the conclusion. $\square$

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