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Fractional diffusion: biological models and nonlinear problems driven by the s-power of the Laplacian

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Introduction

At the beginning of the last century, many problems and many models were described by the diffusion in terms of the Laplace operator. One of the most popular equations in Biology, and in particular in population dynamics, is the logistic equation, which we write as

$$cu_t(x,t) = \Delta u(x,t) + \lambda(\beta(x)u(x,t) - u^2(x,t)) \;\; x \in \Omega \subset \mathbb{R}^N, t > 0. \tag{1}$$

Here Ω is a domain in $\mathbb{R}^N$ where the population having nonnegative density u lives and c, λ are positive constants. It is important to stress that logistic-type equations have been extensively studied and, in addition to the papers that we will cite below, any bibliography would be incomplete. Equation (1) was born to describe the dynamic of populations, and in particular the first works were studied in the framework of Genetics: for instance, the problem of gene dispersion with R. A. Fisher in [23], and the extinction or the diffusion of allelic forms with D. G. Aronson and H. F. Weimberger in [5]. Recently, we can find interesting applications to Ecology of reaction-diffusion systems in [17] or [18].

A first natural analytic step consists in studying steady states for these equations (see [7]). The problem of proving the existence of solutions for stationary Laplace equations with subcritical nonlinearities has been studied by many authors ([**?**] and the reference there in, and [47] for a well- known multiplicity result), while, in presence of critical nonlinearities, we cite the important contribute of Brezis and Niremberg in [12].

However, it is well known that, the heat operator may be too rigid to describe the possible interaction of the species in the whole of Ω and for this fact a nonlocal operator may be more useful than a local one (see [7], [8], [34], [36] and the references therein). We also recall [35], where the solutions of the stationary version of (1) were found when $\Omega = \mathbb{R}^N$. This kind of operators is known since many years (for instance, see [9]), but only in the early 2000's they had a huge development, mainly when related to biological problems. In this context, it is mandatory to mention the pioneering works of Caffarelli, Silvestre [15], Caffarelli, Vasseur [16] and Cabré, Tan [13]. Starting with these contributions, the study of related mathematical problems was significantly expanded in many directions. The nonlocal diffusion is actually largely used in many fields: anomalous diffusion, crystal dislocation, minimal surfaces, water waves, flame propagation. In this setting, it is important to consider the existence of solutions both for stationary problems (see [37], [38], [39], [40], [41], [46] and the reference therein), and for parabolic equations with KPP or different nonlinearities ([7], [8], [14] and the reference therein).

When we want to describe biological phenomena, it is natural to consider the population dynamics, and it us standard to use as a general model the so–called transport equation, namely

$$p_t + v \cdot \nabla p = L(p) \tag{2}$$

where L is an operator depending on the studied problem.

Talking about dynamics, we could refer also to the motion of a singular biological individual. It can be represented using processes in the space, or in the velocity, obtaining respectively the space-jump processes and the velocity jump processes. We stress that, though this is not the topic of this thesis, these processes are stochasticallly regulated. For example, in the classical case [28], the authors starting with a velocity jump processes managed by brownian motion, got the following equation

$$\frac{\partial p}{\partial t}(t,x,v) + v \cdot \nabla p(t,x,v) = -\mu p(t,x,v) + \mu \int_V T(v,v')p(t,x,v')dv', \tag{3}$$

where T is the kernel of the integral operator. From (3) they used the so called "Hilbert expansion method" (see [29]) or "Cattaneo Method" (see [27]), reaching the classical diffusion equation

$$\partial \tilde{p}_t = \nabla(D\nabla\tilde{p}),$$

where D is a finite diffusion matrix and $\tilde{p}$ is the density. The differences between the two methods are partially discussed in [30].

Finally, it is important to mention the fractional equation as the result of the position jump processes (see [45]).

Let us now describe the organization of this thesis.

In the first chapter we summarize some classical and useful results for a better understanding of the subsequent chapters. We start showing some analytical results on the Min Max theory and general compactness condition. We give our functional setting and we present the two classical nonlocal operators we shall consider, stressing the differences between them. After that, we introduce the transport equation, the biological meaning and the relations with the classical diffusion.

The last two chapters include original results. In partcular, in the second chapter we start from the transport equation in presence of the Generalized Velocity Jump processes recalling the work of J. T. King (see [44]), i.e.

$$\begin{aligned} p_t + v\cdot\nabla p = &-\int_0^t \phi(t-s)p(s,x-(t-s)v,v)ds \\ &+\int_0^t \phi(t-s)\int_{\mathbb{R}^N} T(v,v')p(s,x-(t-s)v',v')dv'ds. \end{aligned} \tag{4}$$

Then, we rescale the equation (with a fractional power of the parameter ϵ) and we look the limit when ϵ goes to 0. Since it is impossible to use the classical methods cited above (the reason will be explained in that chapter), we use some ideas appearing in the work of A. Mellet, S. Mischler, C. Mouhot [6], (see for instance also, [33]). In particular, when we consider a peculiar asymptotic behavior for T, i.e. choosing

$$T(v) \approx |v|^{N+2s} \text{ for } |v| \to \infty$$

we find the fractional diffusion equation (Theorem 2.1.1).

Finally, in the third and last chapter, we look for the existence and the multiplicity of solutions for the problem

$$\begin{cases} (-\Delta)^s u(x) = \lambda(\beta(x)u(x) - g(x,u(x))) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \mathbb{R}^N\setminus\Omega, \end{cases} \tag{P_λ^β}$$

where $N \geq 2$, $s \in (0,1)$, $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\alpha}$ for some $0 < \alpha \leq 1$ and $g : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a self-limiting factor for the population which generalizes the classical quadratic nonlinearity in (1). By $\beta : \mathbb{R}^N \to \mathbb{R}$, like for the classical logistic equation, we denote the birth rate and for us it will be a **sign changing** and (generally) **unbounded** function, so contributions to both development and limitation of the population are possible. Of course, this is the most general setting for the problem, but, as far as we know, this is the first time that such a general situation is considered. The chapter is organized as follows. In the first part we introduce the main object of this paper (i.e. the bifurcation parameter λ^*), and, in Section 3.1, we prove a regularity result (see Proposition 3.1.1) of independent interest. In Section 3.2 we give the existence theorem and we prove a bifurcation theorem for problem P_λ^β: via critical point theory and truncation techniques we show that such a problem admits a nontrivial solution if and only if $\lambda > \lambda^*$ (see Theorem 3.2.1) and, using additional but coherent conditions, we will show that nontrivial solutions bifurcate from λ^* (see Proposition 3.2.4). After that, in Section 3.3, we return to study the eigenvalue problem in order to define another parameter (i.e. the second eigenvalue λ_V) and we give a multiplicity result: using some additional assumptions and with the help of the Liapunov-Schmidt reduction theory, we show that the problem admits three nontrivial solutions if $\lambda > \lambda_V$ (see Theorem 3.3.2). To our knowledge, they are the first results concerning the logistic equation in the presence of $(-\Delta)^s$.

Chapter 1

Mathematical background

1.1 Critical point theory

In this first section we give a brief overview of some important definitions, topological conditions and results useful for the Chapter 3. To begin we give the following theorem (see [21]).

Theorem 1.1.1. *(Ekeland Variational principle) Let M be a complete metric space and let $\Phi : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional, bounded from below. Fixed $\epsilon > 0$ and $u \in M$ such that*

$$\Phi(u) \leq \inf_M \Phi + \epsilon.$$

Then, there exists $v \in M$ such that

$$\Phi(v) \leq \Phi(u) \text{ with } d(v, u) \leq \sqrt{\epsilon},$$

and, for all $w \in M$ with $w \neq v$

$$\Phi(w) > \Phi(v) - \sqrt{\epsilon} d(v, w).$$

Definition 1.1.1. Let X be a Banach space, let $\phi \in C^1(X)$. We say that ϕ satisfies the Palais-Smale Condition (or PS-condition) if for every sequence $\{u_n\}_n \subset X$, such that

$$|\phi(u_n)| \leq K \text{ for some } K > 0,$$

and

$$\phi'(u_n) \to 0 \text{ in } X^*, \text{ for } n \to \infty,$$

admits a strongly convergent subsequence.

We observe that the PS-condition is a sort of a compactness condition for sequences of functions. In particular it is a sufficient condition for the existence of a minimizer for a differentiable functional which is bounded from below on X, thanks to Ekeland Variational Principle that we have seen before.

Now, let X be a Banach space, we introduce the following sets

$\phi^c = \{u \in X$ such that $\phi(u) \leq c\}$ the sublevel of ϕ in c,
$K = \{u \in X$ such that $\phi'(u) = 0\}$ the critical point set,
and
$K^c = \{u \in X$ such that $\phi'(u) = 0$ and $\phi(u) = c\}$ the critical point set with critical value c.

Theorem 1.1.2. *(Mountain Pass Theorem, [4]) Let X be a Banach space and $f \in C^1(X, \mathbb{R})$ be such that $f(0) = 0$. Assume that f satisfes the PS-condition and that exist $\rho, \alpha > 0$ such that*

$$\inf_{||u||=\rho} f(u) = \alpha,$$

and there exists $e \in X \setminus B_\rho$ such that $f(e) \leq 0$. Set $\Gamma := \{\gamma \in C([0,1], X\}$ such that $\gamma(0) = 0$ and $\gamma(1) = e$ and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)\}.$$

Then c is a critical value, i.e. there exists an $x \in X$ such that $f(x) = c$ and $f'(x) = 0$.

We stress that the theorem holds with the point 0 but the result still holds for a different point using a simple traslation. The level c given above is called the mountain pass critical level. The points on K^c will be called mountain pass critical points.

Similarly we can give a more general compactness condition.

Definition 1.1.2. Let X a Banach space, let $\phi \in C^1(X)$. We say that ϕ satisfies the Cerami Condition (or C-condition) if for every sequence $\{u_n\}_n \subset X$, such that

$$|\phi(u_n)| \leq K \text{ for some } K > 0,$$

and

$$(1 + ||u_n||_X)\phi'(u_n) \to 0 \text{ in } X^*, \text{ for } n \to \infty,$$

admits a strongly convergent subsequence.

Now, we give another Min-Max result that is based on a particular topological idea, i.e. indentifying a couple of sets separated by a well defined level set. Rigorously speaking we need of a Linked structural definition.

Definition 1.1.3. Let X be a Hausdorff topological space and let D_0, D and S be nonempty closed subsets of Y such that $D_0 \subset D$. The pair $\{D_0, D\}$ is linking with S in Y if and only if

1. $D_0 \cap S = \{0\}$;
2. for all $h \in C(D, Y)$ having D_0 like fixed point, i.e. satisfying $h|_{D_0} = Id|_{D_0}$, we have that D and S have an interseption, i.e. $h(D) \cap S \neq \{0\}$.

Remark 1.1.1. The classic Mountain Pass Theorem fits in this settings choosing $D_0 = \{0, e\}$, $D = [0, e] = \{u \in X$ such that $u = te$ for $t \in [0, 1]\}$ and $S = \{u \in X$ such that $||u|| = \rho\}$.

Therefore, a more general Min Max theorem is the following

Theorem 1.1.3. *(Linking Theorem) Let X be a Banach Space, D_0, D and S are nonempty closed subsets of X, where the couple $\{D_0, D\}$ is linking with S in X. Let $\phi \in C^1(X)$ be a functional satisfies the C-condition, and*

$$\sup_{D_0} \phi \leq \inf_{S} \phi,$$

set $\Gamma = \{\gamma \in C(D, X)$ such that $\gamma|_{D_0} = Id|_{D_0}\}$, and

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in D} \phi(\gamma(u)),$$

then $c \geq \inf_S \phi$ and c is a critical value for ϕ. Moreover, if $c = \inf_S \phi$, then we can find a critical point of ϕ in S.

Definition 1.1.4. Let H be an Hilbert space and let M be a Hilbert Riemannian manifold (i.e. a manifold modeled on the Hilbert space) and suppose that $\phi \in C^2(M, \mathbb{R})$. Let $p \in M$ be a critical point for ϕ. p is called nondegenerate if $d^2\phi(p)$ has a bounded inverse, or equivalently, if the matrix

$$d^2\phi(p)\langle u, v\rangle = \phi''(p)\langle u, v\rangle, \text{ for all } u, v \in T_pM$$

is non singular, i.e. all the eigenvalues are not trivial and with constant sign.

Furthermore, we define the Morse index of p, $m(p)$, as the dimension of the maximal negative definite subspace of $\phi''(p)$, i.e. the number of negative eigenvalues of the Hessian matrix.

Definition 1.1.5. The dimension of the kernel of $d^2\phi(p)$ is called the nullity of ϕ at p.

Remark 1.1.2. From Definition 1.1.5, p is nondegenerate if and only if the nullity of ϕ at p is zero.

In order to study the local behavior of ϕ near a nondegenerate critical points we have

Lemma 1.1.1. *(Morse Lemma) Let M be a Hilbert Riemannian manifold and suppose that $\phi \in C^2(M, \mathbb{R})$ with p a nondegenerate critical point; then there exists a neighborhood U_p of p and a local diffeomorphism $\psi : U_p \to T_p(M)$ with $\psi(p) = 0$, such that*

$$\phi \cdot \psi^{-1}(v) = \phi(p) + \frac{1}{2}\left\langle \phi''(p)v, v \right\rangle, \text{ for all } v \in \psi(U_p).$$

Then, for the sake of completeness, we give also the following definition though we won't use for our results.

Definition 1.1.6. Let X be a Banach space and let $\phi \in C^1(X, \mathbb{R})$ be a functional with p an isolated critical point of ϕ with $\phi(p) = c$. We define the $q - th$ critical group of ϕ as

$$C_q(\phi, p) = H_q(\phi^c \cap U_p, (\phi^c \setminus \{p\}) \cap U_p, G)$$

where U_p is a neighborhood of p such that $K \cap U_p \cap \phi^c = \{p\}$, and $H_q(X, Y)$ is the singular relative homology group with coefficients in the group G.

Remark 1.1.3. If p is an isolated local minimum of $\phi \in C^2(X, \mathbb{R})$ then by definition

$$C_q(\phi, p) = H_q(\{p\}, 0) = \begin{cases} G & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

Theorem 1.1.4 shows the relation between the Morse index and the critical group.

Theorem 1.1.4. *Let $X = H$ be a Hilbert space and $\phi \in C^2(H, \mathbb{R})$ with p a nondegenerate critical point for ϕ with m(p)=j. Then*

$$C_q(\phi, p) = \begin{cases} \mathbb{Z} & \text{if } q = j \\ 0 & \text{if } q \neq j. \end{cases}$$

in which we have choosen $G = \mathbb{Z}$ as coefficient of the group.

1.2 The Fractional Sobolev spaces

In order to introduce our problem in the non local frame, let us define the fractional spaces. We can talk about them as a intermediate Banach spaces between $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Definition 1.2.1. (Fractional Sobolev space). The fractional Sobolev spaces $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \quad : \quad \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+s}}. \in L^p(\Omega \times \Omega) \right\}, \tag{1.1}$$

The norm considered in this space is the following:

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{\frac{1}{p}},$$

where $[u]_{W^{s,p}(\Omega)}$ is the *Gagliardo* seminorm of u

$$[u]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{\frac{1}{p}}$$

Theorem 1.2.1. *For all $p \in [1, +\infty)$, $W^{s,p}(\Omega)$ is a Banach space. Moreover $W^{s,2}(\Omega)$ is a Hilbert space with scalar product*

$$(u, v)_{W^{s,2}(\Omega)} = \int_\Omega uv \, dx + \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy.$$

[[1], pag.213-235].

Give the following simple result

Lemma 1.2.1. *Let $u \in L^p(\Omega)$. $\forall s \in (0, 1)$ then*

$$\int\int_{\{(x,y) \in \Omega \times \Omega \,:\, |x-y| \geq 1\}} \frac{|u(x)|^p}{|x - y|^{N+sp}} \, dxdy$$

is bounded.

Proof. Let ω_{N-1} the $(N-1)$-dimensional measure of the unit sphere $\mathcal{S}^{N-1}$,

$$\begin{aligned}
\int\int_{|x-y|\geq 1} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy =& \int_{y\in\Omega}\int_{x\in\Omega-B_1(y)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
\leq& 2^{p-1}\int_{y\in\Omega}\int_{x\in\Omega-B_1(y)} \frac{|u(x)|^p+|u(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
\leq& 2^p\int_{y\in\Omega}|u(y)|^p\,dy\int_{x\in\Omega-B_1(y)}\frac{1}{|x-y|^{sp+N}}\,dxdy \\
\leq& 2^p\,\|u\|^p_{L^p(\Omega)}\int_{z\in\mathbb{R}^N-B_1(y)}\frac{1}{|z|^{sp+N}}\,dz \\
=& 2^p\,\|u\|^p_{L^p(\Omega)}\,\omega_{N-1}\int_{t=1}^{\infty}\frac{1}{t^{sp+N}}t^{N-1}\,dt \\
=& 2^pC(N,s,p)\,\|u\|^p_{L^p(\Omega)} < \infty.
\end{aligned}$$

□

We give a first relation between the fractional Sobolev Spaces in the following Proposition.

Proposition 1.2.1. *Let $p\in[1,+\infty)$ and $s,s'\in(0,1)$ such that $s\leq s'$. Let $\Omega\subset\mathbb{R}^N$ be an open set and $u\in L^p(\Omega)$. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\,\|u\|_{W^{s',p}(\Omega)}$$

where C is a constant that depends only on N and p. In particular

$$W^{s',p}(\Omega)\subseteq W^{s,p}(\Omega).$$

Proof. Since $s\leq s'$,

$$\int\int_{|x-y|<1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy \leq \int\int_{|x-y|<1}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s'p}}\,dxdy.$$

By this bound and by Proposition 1.2.1, then

$$\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \leq 2^pC(N,s,p)\,\|u\|^p_{L^p(\Omega)} + \int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+s'p}}\,dxdy.$$

Therefore, adding in both sides the quantity $\|u\|^p_{L^p(\Omega)}$, we obtain

$$\begin{aligned}
\|u\|^p_{L^p(\Omega)} + \int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \leq& \|u\|^p_{L^p(\Omega)} + 2^pC(N,s,p)\,\|u\|^p_{L^p(\Omega)} + \\
&+\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+s'p}}\,dxdy,
\end{aligned}$$

that is

$$\begin{aligned}\|u\|^p_{W^{s,p}(\Omega)} \leq &(2^p C(N,s,p)+1)\,\|u\|^p_{L^p(\Omega)} + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+s'p}}\,dxdy \leq \\ \leq &C'(N,s,p)\,\|u\|^p_{W^{s',p}(\Omega)}\,.\end{aligned}$$

□

Using the same notation like in the classical Sobolev spaces, we will write $H^s(\mathbb{R}^N)$ instead of $W^{s,2}(\mathbb{R}^N)$ and in this way we can introduce an alternative definition of the fractional Sobolev spaces for the case $p=2$, using the Fourier transform

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\Omega) : \int_{R^N} (1+|\xi|^{2s})|\mathcal{F}u(\xi)|^2 dxi \right\}, \tag{1.2}$$

Proposition 1.2.2. *The definition of $H^s(\Omega)$ with $\Omega = \mathbb{R}^N$ in* (1.1) *and in* (1.2) *are equivalent.*

It is only necessary to proof wheter

$$[u]_{H^s(\mathbb{R}^N)} = C\int_{R^N} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi, \tag{1.3}$$

for some $C = C(n,s)$. This follow by simple computation:

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x+y)-u(y)|^2}{|x|^{N+2s}}dxdy \\ &= \int_{\mathbb{R}^N} \frac{1}{|x|^{N+2s}}\left(\int_{\mathbb{R}^N} |u(x+y)-u(y)|^2 dy\right)dx \\ &= \int_{\mathbb{R}^N} \frac{1}{|x|^{N+2s}}||u(x+\cdot)-u(\cdot)||_{L^2(\mathbb{R}^N)}dx.\end{aligned}$$

Now we use the Plancherel's theorem which states that the inner product in $L^2(\mathbb{R}^N)$ is invariant under Fourier transforms and we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1}{|x|^{N+2s}}||u(x+\cdot)-u(\cdot)||_{L^2(\mathbb{R}^N)} =& \int_{\mathbb{R}^N} \frac{1}{|x|^{N+2s}}||\mathcal{F}(u(x+\cdot)-u(\cdot))||_{L^2(\mathbb{R}^N)}dx \\ =& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|e^{i\xi\cdot x}-1|^2|\mathcal{F}u(\xi)|^2}{|x|^{N+2s}}d\xi dx \\ =&2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot x)}{|x|^{N+2s}}|\mathcal{F}u(\xi)|^2 d\xi dx \\ =&2C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi dx.\end{aligned}$$

where

$$C = \int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta. \tag{1.4}$$

It is important to check that the integral is rotationally invariant. Infact, changing ξ by $R\xi$ for some rotation matrix R, the scalar product becomes $(R\xi)\cdot x = \xi \cdot (R^T x)$. In particular:

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1-\cos(R\xi\cdot x)}{|x|^{N+2s}} dx &= \int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot Rx)}{|x|^{N+2s}} dx \\ &= \int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot z)}{|z|^{N+2s}} dz\end{aligned}$$

Then, choosing $R(|\xi|e_1) = \xi$ with $(e_1, \ldots, e_N) \in \mathbb{R}^N$ we can write

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot y)}{|y|^{N+2s}} dy &= \int_{\mathbb{R}^N} \frac{1-\cos((R|\xi|e_1)\cdot y)}{|y|^{N+2s}} dy \\ &= \int_{\mathbb{R}^N} \frac{1-\cos(|\xi|e_1\cdot z)}{|y|^{N+2s}} dz \qquad (\text{let } \zeta = |\xi|z) \\ &= \int_{\mathbb{R}^N} \frac{(1-\cos(\zeta_1))|\xi|^{-N+2s}}{|\zeta|^{N+2s}} |\xi^N| d\zeta \\ &= |\xi|^{2s} \int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta\end{aligned} \tag{1.5}$$

so we have proved (1.4). We observe, in last analysis, that the last integral (1.4) is bounded, infact

$$\frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} \leq \frac{|\zeta_1|^2}{|\zeta|^{N+2s}} \leq \frac{1}{|\zeta|^{N+2(s-1)}}$$

hence, for ϵ small enough,

$$\int_{B_\epsilon} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} |dz < +\infty$$

while, for all ϵ,

$$\int_{\mathbb{R}^N\setminus B_\epsilon} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} |dz < +\infty.$$

Now, we give some important immersions for the fractional Soboloev spaces and we start with the " Fractional Sobolev inequality". Let $p^\sharp = \frac{Np}{N-sp}$ the critical fractional exponent, we have the following result.

Theorem 1.2.2. *Let $s \in (0,1)$, $p \in [1, \frac{N}{s})$ and $u \in H^s(\mathbb{R}^N)$. Then*

$$\|u\|^p_{L^{p^\sharp}(\mathbb{R}^N)} \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dxdy$$

where C is a constant that depends only on n and p.

It is note that, with some hypothesis on $\Omega \subseteq \mathbb{R}^N$, we can extend the previous result to the open set (see Theorem 6.7, Theorem 6.9 in [19]).

Theorem 1.2.3. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Let $\Omega \subseteq \mathbb{R}^N$ be an open of $C^{0,1}$ class with bounded boundary, then exists a constant $C = C(N,p,s,\Omega) > 0$ such that for all $q \in [p, p^\sharp]$ and for all $u \in W^{s,p}(\Omega)$ we have*

$$\|u\|_{L^q(\Omega)} \leq C\, \|u\|_{W^{s,p}(\Omega)}\,.$$

Moreover, if $\Omega \subseteq \mathbb{R}^N$ is bounded, then the immersion hold for all $L^q(\Omega)$ with $q \in [1, p^\sharp]$.

If $sp = n$ the exponent $p^\sharp$ tends to ∞, so we have to give analogous continuity immersion theorem.

Theorem 1.2.4. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp = N$. Let $\Omega \subseteq \mathbb{R}^N$ be a domain of $C^{0,1}$ class, there exists a constant $C = C(N,p,s,\Omega) > 0$ such that $\forall q \in [p,\infty)$ and for all $u \in W^{s,p}(\Omega)$ we have*

$$\|u\|_{L^q(\Omega)} \leq C\, \|u\|_{W^{s,p}(\Omega)}\,.$$

Moreover, if $\Omega \subseteq \mathbb{R}^N$ is bounded then $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ $\forall q \in [1,+\infty)$.

To conclude as in the case of the classical Soboloev spaces, some continuous immersions are compact immersions. So we introduce the last result of the section.

Theorem 1.2.5 (Teorema 7.1,[19])**.** *Let $s \in (0,1)$, $p \in [1,+\infty)$ such that $sp < N$. Let $q \in [1, p^\sharp)$, $\Omega \subset \mathbb{R}^N$ be a bounded domain of $C^{0,1}$ class and $F \subset L^p(\Omega)$ bounded. If*

$$\sup_{f \in F} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\, dxdy < \infty,$$

then F is relative compact in $L^q(\Omega)$.

1.3 The Fractional Laplacian

Recently in the mathematic field the study of the fractional spaces and operators became very important. Infact, many biological phenomenas are strictly connect with the nonlocal diffusion, certainly more rapresentative respect the local interpretation. In particular in lecterature exist two different typologies of Fractional operators. It is not will of this section clarify the difference between the two types, inasmuch the difference was clarified many years ago, but to give a general presentation and summarize the properties. This presentation will permit us to introduce subsequently two different existence theorem for the two fractional Laplacian.

1.3.1 Integral definition

A first fractional Laplacian is defined via integral form. Because it is a nonlocal operator, local PDE techniques can not be applied to solve nonlinear problems for $(-\Delta)^s$. To overcome this difficulty, how we will see in this section, Caffarelli and Silvestre [15] proved that it is possible to reduce any fractional problem to a local problem in a one more dimension space.

Definition 1.3.1. Let $\mathcal{S}$ be the Schwartz space and $u \in \mathcal{S}$. Let $s \in (0,1)$, then the fractional Laplacian can be written as

$$(-\Delta)^s u(x) = C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy \tag{1.6}$$

where

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta \right)^{-1}.$$

When $s \in (0, \frac{1}{2})$, the integral in (1.6) can be split near x into two different parts by considering

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy &\le C_1 \int_{B_r} \frac{|x-y|}{|x-y|^{N+2s}} dy + ||u||_\infty \int_{\mathcal{C}B_r} \frac{1}{|x-y|^{N+2s}} dy \\ &= C_2 \int_0^r \frac{1}{\rho^{N+2s-1}} dy + C_3 \int_r^\infty \frac{1}{\rho^{N+2s}} dy < \infty \end{aligned} \tag{1.7}$$

A different approach is via Fourier transfom.
Let $u : \mathbb{R}^N \to \mathbb{R}$, the Fourier transform is defined by

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi} dx \tag{1.8}$$

$$\mathcal{F}^{-1}(\hat{u})(x) = u(x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \hat{u}(\xi) e^{-i\xi\cdot x} d\xi$$

therefore the fractional Laplacian can be viewed as a pseudo-differential operator with symbol $|\xi|^{2s}$,

Proposition 1.3.1. *Let $(-\Delta)^s$ the operator defined in 1.3.1 with $s \in (0,1)$, then*

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)) \text{ for all } \xi \in \mathbb{R}^N. \tag{1.9}$$

In order to prove the equivalence between the two definition, we use the following result

Lemma 1.3.1. *Let $s \in (0,1)$, $u \in H^s(\mathbb{R}^N)$, then*

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|x-y|^{N+2s}}\, dy \ \forall x \in \mathbb{R}^N. \tag{1.10}$$

Proof. We take the following change of variables $z = y - x$.

$$\begin{aligned}(-\Delta)^s u(x) =& - C(N,s) \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(y)-u(x)}{|x-y|^{N+2s}} dy = \quad (y = z+x)\\ =& - C(N,s) \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x+z)-u(x)}{|z|^{N+2s}}\, dz.\end{aligned} \tag{1.11}$$

Relating $\tilde{z} = -z$ we have

$$\lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x+z)-u(x)}{|z|^{N+2s}}\, dz = \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x-\tilde{z})-u(x)}{|\tilde{z}|^{N+2s}}\, d\tilde{z}.$$

so

$$\begin{aligned}2C \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x+z)-u(x)}{|z|^{N+2s}}\, dz =& C \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x+z)-u(x)}{|z|^{N+2s}}\, dz+\\ &+ C \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x-z)-u(x)}{|z|^{N+2s}}\, dz =\\ =& C \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{N+2s}}\, dz.\end{aligned}$$

We can rewrite (1.11), sobstituting z with y,

$$(-\Delta)^s u(x) = -\frac{1}{2} \lim_{\epsilon\to 0^+} \int_{\mathcal{C}B_\epsilon(x)} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} dz.$$

This rappresentation is useful to remove the integral singularity at the origin, infact if we consider the Taylor second order approximation of u, we obtain that

$$\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \leq \frac{\|D^2u\|_{L^\infty}}{|y|^{N+2s-2}}$$

is integrable near 0. □

In general if we take a linear operator $\mathcal{L}$, we define the symbol like a function $\mathcal{S}:\mathbb{R}^N \to \mathbb{R}$ such that

$$\mathcal{L}u(x) = C\mathcal{F}^{-1}(\mathcal{S}\mathcal{F}u(\xi)) \text{ with } \xi \in \mathbb{R}^N.$$

At this point we can say that, using Lemma 1.3.1,

$$\mathcal{F}((-\Delta)^s u)(\xi) = \mathcal{F}\left(\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|x-y|^{N+2s}} dy\right).$$

Using the properties of the Fourier transform we have

$$\begin{aligned}\mathcal{F}((-\Delta)^s u)(\xi) &= \int_{\mathbb{R}^N} \frac{\mathcal{F}\left(u(x+y)+u(x-y)-2u(x)\right)}{|x-y|^{N+2s}} dy \\ &= \int_{\mathbb{R}^N} \frac{\left(e^{-i\xi\cdot y}+e^{i\xi\cdot y}-2\right)}{|x-y|^{N+2s}} \mathcal{F}u(\xi) dy = 2\mathcal{F}u(\xi)\int_{\mathbb{R}^N} \frac{(\cos(\xi\cdot y)-1)}{|x-y|^{N+2s}} dy.\end{aligned}$$

By (1.5) then

$$\mathcal{F}((-\Delta)^s u)(\xi) = C|\xi|^{2s}\mathcal{F}u(\xi),$$

therefore, from the definition of symbol,

$$(-\Delta)^s u(x) = C\mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)(\xi)$$

i.e.

$$\mathcal{S}(\xi) = |\xi|^{2s}.$$

Therefore, the Proposition 1.3.1 is proved.

How we have mentioned, the fractional power of the Laplacian can be determined like an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem.

Thus, we consider a function $u:\mathbb{R}^N \to \mathbb{R}$, let us take the extension function $\overline{u}:\mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$, so that $\overline{u}(x,0) = u(x)$ and the relative harmonic problem

$$div(y^a \nabla \overline{u}) = 0 \tag{1.12}$$

or equivalently

$$\Delta\overline{u} + \frac{a}{y}\overline{u}_y + \overline{u}_{yy} = 0 \tag{1.13}$$

.

Lemma 1.3.2. *Let $\phi : [0, +\infty) \to \mathbb{R}$ be the minimizer of the functional*

$$J(\phi) = \int_0^{+\infty} (|\phi'(y)|^2 + |\phi|^2) y^a dy,$$

then ϕ solves the following equation:

$$\begin{cases} -\hat{\phi}(y) + \frac{a}{y}\hat{\phi}_y(y) + \hat{\phi}_{yy}(y) = 0 \\ \phi(0) = 1 \\ \lim_{y\to\infty} \phi(y) = 0 \end{cases}$$

Then the following theorem shows that the extension of this problem to the upper half space is strictly related to the fractional Laplacian.

Theorem 1.3.1. *Under the framework introduced above, then*

$$-\lim_{y\to 0} y^a \overline{u}_y(x, y) = (-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(x) - u(\tilde{x})}{|x - \tilde{x}|} d\tilde{x},$$

with $a = 1 - 2s$.

Proof. First of all, note that the functional of the equation in (1.12) can be written as

$$\int_{R^N} \int_0^\infty |\nabla \overline{u}|^2 y^a dx dy \tag{1.14}$$

and, through a Fourier transform, it becomes

$$-|\xi|^2 \hat{\overline{u}}(\xi, y) + \frac{a}{y}\hat{\overline{u}}_y(\xi, y) + \hat{\overline{u}}_{yy}(\xi, y) = 0.$$

Thus, by fixing a value for ξ, we are able to obtain an ordinary differential equation in y. Now, if we rescale (1.14) by considering

$$\hat{\overline{u}}(\xi, y) = \hat{u}(\xi)\phi(|\xi|y),$$

we obtain

$$\begin{aligned} &\int_{R^N} \int_0^\infty |\nabla \overline{u}|^2 y^a dx dy \\ &= \int_{R^N} \int_0^\infty |\hat{u}(\xi)|^2 |\xi|^2 \left(|\phi(|\xi|y)|^2 + |\phi'(|\xi|y)|^2\right) y^a d\xi dy \quad \text{(taking } \overline{y} = |\xi|y) \\ &= \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 |\xi|^{1-a} \int_0^\infty \left(|\phi(\overline{y})|^2 + |\phi'(\overline{y})|^2\right) \overline{y}^a d\xi d\overline{y} \qquad \text{(by Lemma 1.3.2)} \\ &= \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 |\xi|^{1-a} J(\phi) d\xi = J(\phi) \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 |\xi|^{2s} d\xi. \end{aligned}$$

the last integral is the functional of the Eulero-Largange problem $(-\Delta)^s u(x) = C\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(u)(\xi))$. Therefore, up to a constant, the functionals coincide.

In particular

$$-\lim_{y\to 0} y^a u_y(x,y) = -\lim_{y\to 0} \frac{u(x,y)-u(x,0)}{y^{1-a}} = C(-\Delta)^{\frac{1-a}{2}} u(x)$$

with $\frac{1-a}{2} = s$. □

Corollary 1.3.1 summarizes what happens if $s = \frac{1}{2}$.

Corollary 1.3.1. *If $s = \frac{1}{2}$, the fractional Laplacian is related to the solutions of the upper half space problem*

$$div(\nabla \overline{u}(x,y)) = \Delta \overline{u}(x,y) = 0$$

where

$$\overline{u} : \mathbb{R}^N \times [0,\infty) \to \mathbb{R} \text{ is the extension function such that } \overline{u}(x,0) = u(x).$$

From what we said, we can define the square root of the Laplacian as

$$-(-\Delta)^{\frac{1}{2}} u(x) = \lim_{y\to 0} \overline{u}_y(x,y),$$

i.e. the partial derivative of the harmonic extension function of u.

1.3.2 The Spectral Fractional Laplacian

Now, we introduce a new kind of operator called "local fractional Laplacian", "regional" or "spectral" in order to differentiate it from the operator defined in the previous section. Although in appearance it is seems very similar to the integral one (specially for the treatment about the related problem), how we can see, it is deeply different.

If $\Omega \subset \mathbb{R}^N$ we can describe the fractional Laplacian through a local extension problem in $\Omega \times (0,+\infty)$.

We consider a function $u : \mathbb{R}^N \to \mathbb{R}$, let us take the extension function $\overline{u} : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$, so that $\overline{u}(x,0) = u(x)$ and the harmonic problem

$$div(y^a \nabla u(x,y)) = 0 \text{ in } \Omega \times [0,+\infty) \tag{1.15}$$

In this section we recall the essential results of Cabré and Tan [13] for $s = \frac{1}{2}$ that hold also for all $s \in (0,1)$.

Definition 1.3.2. For all bounded domain $\Omega \subseteq \mathbb{R}^N$, denote with $\mathcal{C} = \Omega \times (0, +\infty)$ the half cylinder with base $\Omega \times \{0\} = \Omega$, and boundaries $\partial_L \mathcal{C} = \partial\Omega \times [0, +\infty)$.

We taking into account the extension to use new function spaces

$$H^1_{0,L}(\mathcal{C}) = \left\{ v \in H^1(\mathcal{C}) \quad | \quad v = 0 \text{ q.o. su } \partial_L \mathcal{C} \right\},$$

endowed with

$$\|v\|_{H^1_{0,L}(\mathcal{C})} = \left(\int_{\mathcal{C}} |Dv(x,y)|^2 \, dxdy \right)^{\frac{1}{2}}$$

for which holds the following inequality

Lemma 1.3.3 (disuguaglianza di Sobolev per tracce, Lemma 2.3, [13])**.** *Sia $N \geq 2$ e sia $2^\sharp = \frac{2N}{N-1}$. Esiste una costante $C = C(N)$, tale che per ogni $v \in H^1_{0,L}(\mathcal{C})$:*

$$\left(\int_\Omega |v(x,0)|^{2^\sharp} \, dx \right)^{\frac{1}{2^\sharp}} \leq C \left(\int_{\mathcal{C}} |Dv(x,y)|^2 \, dxdy \right)^{\frac{1}{2}}. \tag{1.16}$$

and the extended result for a generic Lesbegue space

Lemma 1.3.4. *Let $N \geq 2$ and $1 \leq q \leq 2^\sharp = \frac{2N}{N-1}$, there exists a positive $C = C(N, q, |\Omega|)$ such that for all $v \in H^1_{0,L}(\mathcal{C})$,*

$$\left(\int_\Omega |v(x,0)|^q \, dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathcal{C}} |Dv(x,y)|^2 \, dxdy \right)^{\frac{1}{2}}. \tag{1.17}$$

If $N = 1$, the inequality (1.17) *holds for all $q \in [1, \infty)$.*

Proof. Using the classic L^p immersions and by Lemma 1.3.3

$$\begin{aligned} \left(\int_\Omega |v(x,0)|^q \, dx \right)^{\frac{1}{q}} &\leq |\Omega|^{1-q/2^\sharp} \left(\int_\Omega |v(x,0)|^{2^\sharp} \, dx \right)^{\frac{1}{2^\sharp}} \\ &\leq C \left(\int_{\mathcal{C}} |Dv(x,y)|^2 \, dxdy \right)^{\frac{1}{2}}. \end{aligned}$$

□

A more strong relation between the classical Lesbegue spaces and the trace space of $H^1_{0,L}(\mathcal{C})$ is the following

Lemma 1.3.5. *Let* $1 \leq q \leq 2^\sharp$ *if* $N \geq 2$ *and* $q \in [1, \infty)$ *if* $N = 1$*; then*

$$Tr_\Omega(H^1_{0,L}(\mathcal{C})) \subset\subset L^q(\Omega).$$

Proof. By the trace operator theory,

$$Tr_\Omega(H^1_{0,L}(\mathcal{C})) \subset H^{\frac{1}{2}}(\Omega).$$

Moreover, since Theorem 1.2.5, we have

$$H^{\frac{1}{2}}(\Omega) \subset\subset L^q(\Omega).$$

From the two inclusion we have the result. □

Finally, we mentioned the trace operator:

$$Tr_\Omega : H^1_{0,L}(\mathcal{C}) \longrightarrow H^{\frac{1}{2}}(\Omega)$$

that is continuous and for all $v \in H^1_{0,L}(\mathcal{C})$

$$\int_\Omega \frac{|v(x,0)|^2}{d(x)} \leq C \int_{\mathcal{C}} |Dv(x,y)|^2 \, dxdy, \text{ (trace Hardy inequality)} \tag{1.18}$$

with $d(x) = dist(x, \partial\Omega)$.

For us, the trace space on $\Omega \times \{0\}$ of the functions $H^1_{0,L}(\mathcal{C})$, will be denoted as

$$\mathcal{V}_0(\Omega) := \left\{u = Tr_\Omega v \text{ such that } v \in H^1_{0,L}(\mathcal{C})\right\} \subset H^{\frac{1}{2}}(\Omega),$$

endowed with

$$\|u\|_{\mathcal{V}_0(\Omega)} = \left\{ \|u\|^2_{H^{\frac{1}{2}}} + \int_\Omega \frac{u^2}{d(x)} \right\}^{\frac{1}{2}}.$$

In particular from (1.18) exists an alternative characterization of $\mathcal{V}_0(\Omega)$, i.e.

$$\mathcal{V}_0(\Omega) := \left\{ u \in H^{\frac{1}{2}}(\Omega) \mid \int_\Omega \frac{u^2}{d(x)} dx < \infty \right\}.$$

(see Lemma 2.7, [13]).

Denoteremo poi con $\mathcal{V}_0^*(\Omega)$ lo spazio duale dotato della norma

$$\|g\|_{\mathcal{V}_0^*(\Omega)} = \sup_{\|u\|_{\mathcal{V}_0(\Omega)} \leq 1} < u, g > .$$

Return to the extension problem in (1.15). In particular for $s = \frac{1}{2}$ the operator $A_{1/2} : \mathcal{V}_0(\Omega) \longrightarrow \mathcal{V}_0^*(\Omega)$, is defined as

$$A_{1/2} u = \left. \frac{\partial v}{\partial \nu} \right|_{\Omega \times \{0\}} \tag{1.19}$$

where $v \in H^1_{0,L}(\mathcal{C})$ is the harmonic extension of u, and this is a bounded and linear operator.

This definition is the best way to explain the name "square root of the Laplacian". Infact, let u a bounded and continuous function in the whole Ω and let v the unique extension in the cylinder, i.e.

$$\begin{cases} -\Delta v(x,y) = 0, \text{ in } \mathcal{C} \\ v(x,0) = u(x) \text{ in } \Omega \times \{0\}. \end{cases} \tag{1.20}$$

and take the operator $T : u \longrightarrow -\partial_y v(.,0)$ map from the Dirichlet data to Neumann data. Since v is harmonic, then $\partial_y v(.,0)$ is an harmonic function again. If we apply twice T we have

$$(T \circ T)u = \partial_{yy} v|_{y=0} = -\Delta_x v|_{y=0} = -\Delta u \quad \text{ in } \Omega,$$

i.e. T is exactly teh square root of Laplacian in Ω ([13], pag.6).

Taking the minimizing problem:

$$\inf \left\{ \int_{\mathcal{C}} |Dv|^2 \, dxdy \text{ such that } v \in H^1_{0,L}(\mathcal{C}) \text{ and } v(.,0) = u \text{ in } \Omega \right\}. \tag{1.21}$$

we have

Proposition 1.3.2. *For all $u \in \mathcal{V}_0(\Omega)$ exists a unique minimum v of (1.21). It is called harmonic extension of u in the whole cylinder and it is trivial in $\partial_L \mathcal{C}$.*

(See Lemma 2.8, [13]).

As the classic Laplacian, we give the following important result

Theorem 1.3.2 (Spectral Decomposition)**.** *Let Ω a bounded domain in $\mathbb{R}^N$ and let $\{e_k\}_k$ the $L^2(\Omega)$ ortonormal base composed by the eigenfunctions of $-\Delta$ with Dirichlet boundary data and the respective eigenvalues $\{\lambda_k\}_k$. Then:*

1. $$\mathcal{V}_0(\Omega) = \left\{ u = \sum_{k=1}^{\infty} b_k e_k \in L^2(\Omega) \text{ such that } \sum_{k=1}^{\infty} b_k^2 \lambda_k^{\frac{1}{2}} < \infty \right\}.$$

2. *Let $u \in \mathcal{V}_0(\Omega)$ with $u = \sum_{k=1}^{\infty} b_k e_k$, then*

$$A_{1/2} u = \sum_{k=1}^{\infty} b_k \lambda_k^{\frac{1}{2}} e_k$$

Proof. 1. We consider $u \in \mathcal{V}_0(\Omega)$. This implies that $u \in L^2(\Omega)$, so we can write $u = \sum_{k=1}^{\infty} b_k e_k$.
Let
$$v(x,y) = \sum_{k=1}^{\infty} b_k e_k(x) e^{-\lambda_k^{\frac{1}{2}} y}.$$
We note that $v(x,0) = u(x)$ in Ω and
$$\Delta v = \sum b_k \left\{ -\lambda_k e_k(x) e^{-\lambda_k^{\frac{1}{2}} y} + \lambda_k e_k(x) e^{-\lambda_k^{\frac{1}{2}} y} \right\} = 0,$$
i.e. v is the harmonic extension of u.

Moreover, we have
$$\begin{aligned} \int_0^\infty \int_\Omega |\nabla v|^2 \, dxdy &= \int_0^\infty \int_\Omega \left\{ |\nabla_x v|^2 + |\partial_y v|^2 \right\} \, dxdy \\ &= \int_0^\infty \int_\Omega \left| \sum_{k=1}^{\infty} b_k \nabla e_k(x) e^{-(\lambda_k)^{1/2} y} \right|^2 \, dxdy \\ &+ \int_0^\infty \int_\Omega \left| \sum_{k=1}^{\infty} b_k e_k(x) \lambda_k^{1/2} e^{-(\lambda_k)^{1/2} y} \right|^2 \, dxdy. \end{aligned} \tag{1.22}$$
but we observe also
$$\begin{aligned} \int_\Omega \left| \sum_{k=1}^{\infty} b_k \nabla e_k(x) e^{-(\lambda_k)^{1/2} y} \right|^2 dx &= \sum_{k=1}^{\infty} b_k^2 \left\| \nabla e_k \right\|^2 e^{-2(\lambda_k)^{1/2} y} = \\ &= \sum_{k=1}^{\infty} b_k^2 (\sqrt{\lambda_k})^2 e^{-2(\lambda_k)^{1/2} y}; \end{aligned}$$
therefore if we return to (1.22)
$$\begin{aligned} \int_0^\infty \int_\Omega |\nabla v^2 \, dxdy =& 2 \sum_{k=1}^{\infty} b_k^2 \lambda_k \int_0^\infty e^{-2(\lambda_k)^{1/2} y} \, dy \\ =& 2 \sum_{k=1}^{\infty} b_k^2 \lambda_k \frac{1}{2(\lambda_k)^{1/2}} \\ =& \sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2}. \end{aligned}$$
In other words $v \in H^1_{0,L}(\mathcal{C})$ if and only if $\sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2} < \infty$.

2. Since the definition (1.19) and from the direct calculous of $-\frac{\partial v}{\partial y}|_{\{y=0\}}$ we obtain
$$A_{1/2}u = -\frac{\partial v}{\partial y}\bigg|_{\{y=0\}} = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} e_k(x).$$

$\square$

Remark 1.3.1. We note that by Proposition 1.3.2, let $u \in \mathcal{V}_0(\Omega)$ there exists a unique $v \in H^1_{0,L}(\mathcal{C})$ harmonic extension.In the light of the last proof, this extension, is undoubtedly

$$v(x,y) = \sum_{k=1}^{\infty} b_k e_k(x) e^{-\lambda_k^{\frac{1}{2}} y}$$

then
$$A_{1/2}u = \frac{\partial v}{\partial \nu}\bigg|_{\Omega\times\{y=0\}} = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} e_k(x) = -\frac{\partial v}{\partial y}\bigg|_{\{y=0\}}.$$

Finally we give

Definition 1.3.3. Let

$$(*) \qquad \begin{cases} A_{1/2}u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

We say $u \in H^1_0(\Omega)$ is a weak solution of $(*)$ if and only if the harmonic extension $v \in H^1_{0,L}(\mathcal{C})$ is a weak solution of

$$(**) \qquad \begin{cases} \Delta v = 0 & \text{in} \quad \mathcal{C} \\ u = 0 & \text{on} \quad \partial_L\mathcal{C} \\ \frac{\partial v}{\partial \nu} = f(x,u) & \text{on} \quad \Omega\times\{0\} \end{cases}$$

i.e. if

$$\int_{\mathcal{C}} \nabla v \cdot \nabla \xi \, dxdy =< f(u), \xi(x,0) >= \int_{\Omega} f(x,u)\xi(x,0)\, dx \quad \forall \xi \in H^1_{0,L}(\mathcal{C}).$$

(see [32] where we study a general problem driven by the square root of the Laplacian).

Let $u \in H^1_0(\Omega)$ with the L^2-decomposition

$$u = \sum_{k\in\mathbb{N}} a_k e_k$$

with $\{e_k\}_k$ the L^2 ortonormal base, we can generalize what we said giving the following definition.

Definition 1.3.4. For all $s \in (0,1)$ and for all $u \in H_0^1(\Omega)$ the Local fractional Laplacian is defined as

$$A_s u = \sum_{k \in \mathbb{N}} u_k \lambda_k^s e_k \tag{1.23}$$

where λ_k and e_k are respectively the k-th eigenvalue and the k-th eigenfunction of the Laplacian operator $-A$ in Ω with zero Dirichlet boundary data in $\partial\Omega$ and $u_k = \int_\Omega u e_k dx$ is the component of u in the $L^2(\Omega)$ in the direction e_k.

Despite the name "local", the spectral operator has a nonlocal nature, but this characteristic is less obvious then in the integral one. Therefore will be important to stress this property.

In general let X a Banach space and $A : X \to X$ be an operator, we denote by $\sigma(A)$ the spectrum of A, the complementar

$$\rho(A) = \mathbb{C} - \sigma(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A : D(A) \to \text{X is bijective}\}$$

the resolvent set of A and with

$$R(\lambda, A) = (\lambda - A)^{-1}$$

the resolvent of A in λ.

Remark 1.3.2. The role of $R(\lambda, A)$ is crucial. Let $\lambda \in \rho(A)$, a solution of a general equation

$$\lambda u - Au = w \text{ with } u, w \in X$$

is rappresented by

$$u = (\lambda - A)^{-1} w = R(\lambda, A) w.$$

We are interested to give some notions and results useful to introduce the s-power of a generic linear operator A. In particular we use a costruction of Balakrishnan in [9] without using the semigroup theory. Infact, like stressed by the author it is not required that the base operator generate a semigroup because its domain is not dense in the space.

Let

$$||R(\lambda, A)|| \leq \frac{M}{|\lambda|} \text{ for all } \lambda > 0 \tag{1.24}$$

for some $M > 0$. For $u \in D(A)$ and for $s \in (0,1)$ we introduce

$$\int_0^\infty \lambda^{s-1} R(\lambda, A) A u d\lambda, \tag{1.25}$$

that is convergent because writing it like

$$\int_0^1 \lambda^{s-1} \left[R(\lambda, A)x - x\right] d\lambda + \int_1^\infty \lambda^{s-1} R(\lambda, A)Aud\lambda,$$

both this integrals are convergent by (1.24) .

Definition 1.3.5. We denote a linear operator with

$$T^s u = \frac{\sin(\pi s)}{\pi} \int_0^\infty \lambda^{s-1} R(\lambda, A)(-A)ud\lambda.$$

Proposition 1.3.3. *This operator can be extended and make it closed and analytic.*

Proof. From the linearity of the operator we have to prove that, taking a sequence $\{u_n\}_n \in D(T^s)$ such that $u_n \to 0$, the sequence $\{T^s(u_n)\}_n$ has zero limit too.

In particular we have that $R(\lambda, A)u_n \in D(T^s)$ and, by 1.3.5 follows that

$$R(\lambda, A)T^s u_n = T^s R(\lambda, A)u_n. \tag{1.26}$$

Moreover, since $AR(\lambda, A)$ is bounded linear, so is $T^s R(\lambda, A)$. Suppose that

$$T^s(u_n) \to y$$

then, passing to limit in (1.26),

$$R(\lambda, A)y = 0,$$

i.e. $y = 0$.

The analiticity follows directly by definition. $\square$

In particular for $u \in D(A)$, $Au \in \overline{D(A)}$ and $s \in (0, 1)$

$$T^s u \to -Au \text{ when s tends to } 1 \tag{1.27}$$

infact, since $Au \in \overline{D(A)}$,

$$\lambda R(\lambda, A)Au = \lambda \frac{1}{\lambda - A} Au \to Au \text{ when } \lambda \to \infty.$$

Prove (1.27) is equivalent to prove

$$T^s u - (-A)u \to 0$$

when s goes to 1. By definition of T^s,

$$\begin{aligned}
T^s u - (-A)u =& \frac{\sin(\pi s)}{\pi} \int_0^\infty \lambda^{s-1} R(\lambda, A)(-A)u d\lambda - (-A)u \\
=& \frac{\sin(\pi s)}{\pi} \int_0^\infty \lambda^{s-1} \left[R(\lambda, A) - \frac{1}{1+\lambda} \right] (-A)u d\lambda \quad \text{(Fixed } K > 0\text{)} \\
=& \frac{\sin(\pi s)}{\pi} \int_0^K \lambda^{s-1} \left[R(\lambda, A) - \frac{1}{1+\lambda} \right] (-A)u d\lambda \\
+& \frac{\sin(\pi s)}{\pi} \int_K^\infty \lambda^{s-1} \left[R(\lambda, A) - \frac{1}{1+\lambda} \right] (-A)u d\lambda \\
<& K^{s-1} \left| \frac{\sin(\pi(1-s))}{\pi(1-s)} \right| \sup || \left[\lambda R(\lambda, A) - I \right] (-A)u || + M \frac{K^{s-1}}{|s-1|} ||Au||
\end{aligned}$$

the last inequality is a bound for the second integral and goes to 0. The first integral is $o(|\sin(\pi s)|)$ and goes to 0 too.

We define

$$(-A)^s \text{ the smallest closed extension of the operator } T^s \tag{1.28}$$

Remark 1.3.3. For $s \leq \frac{1}{2}$, $-(-A)^s$ can be a semigroup generator which is strongly continuous for $t \geq 0$ and uniformly continuous for $t > 0$ see [9], i.e. it is an Feller semigroup infinitesimal generator. If $s > \frac{1}{2}$ this property is not necessary.

Moreover, we have the result that shows the structure of the fractional spectral operator.

Theorem 1.3.3. *(Spectral mapping theorem) If $D(A)$ is dense in X, then*

$$\sigma\left[(-A)^s\right] = \left[\sigma(-A)\right]^s \quad \textit{with } s \in (0,1).$$

We would like to remark that the spectral fractional Laplacian is nonlocal. In particular we can observe this characteristic in the Definition (1.28) and exactly in the integral form of the resolvent. Fixed $x \in \Omega$, for all λ, $R(\lambda, A)$ depends pointly by all the domain (roughly speaking: it seems local!).

We had explain the definition of two fractional operators. In the classic theory, the s-power of the Laplacian is denoted with A_s instead of $-(\Delta)^s$ because this two integral are different, see, e.g. [40] while, often in the literature, they are mistakenly interchanged. It is immediates to observe that the spectral fractional Laplacian is defined only in the bounded domain; condition "si ne qua non" to ensure the existence of the spectrum. Anyway, we can give the eigenvalues for the integral fractional Laplacian, where, as we showed, we defin the operator in the whole space. Therefore it will be less explicity define the eigenvalues and we will appeal to the variational structure of the eigenvalue problems.

We consider the following eigenvalue problems in a smooth domain $\Omega \subset \mathbb{R}^N$ with Dirichlet homogeneous boundary data:

$$\begin{cases} A^s u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.29}$$

$$\begin{cases} (-\Delta)^s u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.30}$$

We introduce a new functional setting (see [39], [41]) very useful to describe the spectrum of the integral fractional operator. We set

$$X_0 = \{g \in H^s(\mathbb{R}^N) \text{ such that } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\};$$

then the relations between this space and the classical fractional spaces is given in the following result.

Proposition 1.3.4. *Let be $u \in X_0$, then*

$$u \in H^s(\mathbb{R}^N)$$

and

$$||u||_{H^s(\Omega)} \leq ||u||_{H^s(\mathbb{R}^N)} \leq ||u||_{X_0}.$$

Furthermore, we have

Proposition 1.3.5. *The embedding of X_0 in $L^{2^*}(\Omega)$ is continuous. Furthermore if Ω has a Lipschitz boundary, then the embedding of X_0 in $L^p(\Omega)$ is compact for any $p \in [1, 2^*)$.*

The problem in (1.30) has a variational structure so we are interest to its weak formulation, that is

$$\begin{cases} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dxdy = \lambda \int_\Omega u(x)\phi(x)dx \text{ for all } \phi \in X_0 \\ u \in X_0. \end{cases}$$

The study of the eigenvalues for the linear non local operator show that

Theorem 1.3.4. *Let $\{\lambda_k\}_{k\in N}$ be the sequence of the eigenvalues of the operator $(-\Delta)^s$ for the problem (1.30), and let $\{e_k\}_{k\in N}$ the eigenfunctions corresponding λ_k. Then the eigenvalues can be characterized as follows:*

$$\lambda_{k,(-\Delta)^s} = \frac{C(n,s)}{2} \min_{u\in P_k\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy}{\int_\Omega |u(x)|^2 dx},$$

where

$$P_k = \{u \in X_0 \text{ such that } \langle u, e_j\rangle_{X_0} = 0 \text{ for all } j = 1,..,k-1\} \quad ,k \geq 2,$$

and $P_1 = X_0$.

In [40] is proved that

$$\lambda_{1,(-\Delta)^s} < \lambda_{1,A^s}.$$

An immediate question could be relatively to the relation between the eigenfunctions and their regularity. It is obvious that, since the definition of A^s, the eigenfunctions $e_{k,s}$ of this operator fully coincide with the eigenfunctions of the classical Laplacian; therefore

$$e_{k,s} \in C^\infty(\Omega) \cap C^m(\overline{\Omega}) \text{ for any } m \in \mathbb{N}.$$

Now, let

$$\delta : \overline{\Omega} \to \mathbb{R}^+ \text{ be a function such that } \delta(x) = dist(x, \partial\Omega)$$

where $x \in \mathbb{R}^N$, then it follows that

$$\tilde{e}_{k,s}/\delta^s|_\Omega \in C^{0,\alpha}(\overline{\Omega})$$

i.e. the eigenfunctions of the integral operator are Holder continuous up to the boundary and this regularity is optimal because they are not Lipschitz continuous (see [40], pag.4).

1.4 The fractional diffusion and the mathematical models

1.4.1 The fractional diffusion and the Position Jump Processes

In the classical theory is known how the local diffusion, in particular the heat equation driven by the Laplacian, is generated starting from the position jump process through the probability notions and the discretization of the space. In the case of the fractional Laplacian, choosing a well defined kernel, there are similar arguments that yield the analogous result. We refer for this result to the discussion in [45].

We consider, to begin, a random walk. We would like to describe the particle movement on the lattice $h\mathbb{Z}^N$.

We suppose that, in the unit of time τ, a particle could jump from one position in the lattice to another position (differently from the classical random walk, not necessarily an adjacent point). Unlike what happens with the classical random walk, we suppose that the probability to jump from a hk point to $h\tilde{k}$, is given by the kernel $\mathcal{L} : \mathbb{R}^N \to [0, +\infty)$ such that:

$$\mathcal{L}(x) = \mathcal{L}(-x) \text{ for all } x \in \mathbb{R}^N$$

and

$$\sum_{k \in \mathbb{R}^N} \mathcal{L}(k) = 1. \tag{1.31}$$

In particular, we have the following intuitive property, that is

$$\mathcal{L}(k - \tilde{k}) = \mathcal{L}(\tilde{k} - k) \quad \text{with } k, \tilde{k} \in \mathbb{Z}^N,$$

i.e. the probability to jump from the point hk to $h\tilde{k}$ is the same to jump from $h\tilde{k}$ to hk and it is decreasing with the distance. In the specific we will obtain the important result considering a special probabilities kernel:

$$\mathcal{L}(k) = \frac{1}{|k|^{N+2s}} \text{ with } \mathcal{L}(0) = 0, \tag{1.32}$$

with

$$s \in (0, 1).$$

Let $u(t, x)$ the probability that the particles stays at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}$, then

$$u(x, t + \tau) = \sum_{k \in \mathbb{Z}^N} \mathcal{L}(k) u(x + hk, t)$$

represent the expected value, i.e. the probability that our particle is found in x at time $t + \tau$ is equals to the sum of all the probabilities of the possible positions $x + hk$ at time t weighted by the probability of jumping from $x + hk$ to x.

Using (1.31),

$$u(x, t+\tau) - u(x, t) = \sum_{k \in \mathbb{Z}^N} \mathcal{L}(k) u(x+hk, t) - u(x, t) = \sum_{k \in \mathbb{Z}^N} \mathcal{L}(k) \left[u(x + hk, t) - u(x, t) \right].$$

Moreover, taking into account (1.32) and $\tau = h^{2s}$, the last equation becomes

$$\begin{aligned}\frac{u(x,t+\tau)-u(x,t)}{\tau} &= \sum_{k\in\mathbb{Z}^N} \frac{\mathcal{L}(k)}{\tau} u(x+hk,t) - u(x,t) \\ &= \sum_{k\in\mathbb{Z}^N} |k|^{-N-2s} h^{-2s} \left[u(x+hk,t)-u(x,t)\right] \big(\text{multiply for } h^{-N}h^{N}\big) \\ &= h^N \sum_{k\in\mathbb{Z}^N} |hk|^{-N-2s} \left[u(x+hk,t)-u(x,t)\right]\end{aligned}$$

that is

$$\frac{u(x,t+\tau)-u(x,t)}{h^{2s}} = h^N \sum_{k\in\mathbb{Z}^N} |hk|^{-N-2s} \left[u(x+hk,t)-u(x,t)\right].$$

Passing to the limit for $h^{2s} \to 0^+$, let $y = x + hk$, the quantity in the right side is the approximating Riemann sum of

$$\int_{\mathbb{R}^N} |y-x|^{-N-2s} \left[u(y,t)-u(x,t)\right] dx,$$

then we obtain

$$\partial_t u(x,t) = \int_{\mathbb{R}^N} \frac{u(y,t)-u(x,t)}{|y-x|^{N+2s}} dx. \tag{1.33}$$

In other words, we can conclude how the heat equation with the fractional Laplacian can be obtained from the limit of the random walk process with long jumps. It is important that $s \in (0,1)$ because, in this way, the integral is determinate in the principal value sense. Make the change of variables $\tilde{y} = y - x$,

$$\int_{\mathbb{R}^N} \frac{u(y,t)-u(x,t)}{|y-x|^{N+2s}} dx = \int_{\mathbb{R}^N} \frac{u(\tilde{y}+x,t)-u(x,t)}{|\tilde{y}|^{N+2s}} dx$$

we can observe that

$$|y|^{N+2s} \quad \text{is integrable at infinity when } s \in (0,1),$$

and, for some $r > 0$, since $\frac{y}{|y|^{N+2s}}$ is odd, we obtain that

$$\int_{B_r} \nabla u(x) \cdot \frac{y}{|y|^{N+2s}} dy = 0 \quad \text{as principal value.}$$

Therefore near 0 we have the integrability of the integral in (1.33).

1.4.2 The diffusion equation and the Velocity Jump Processes

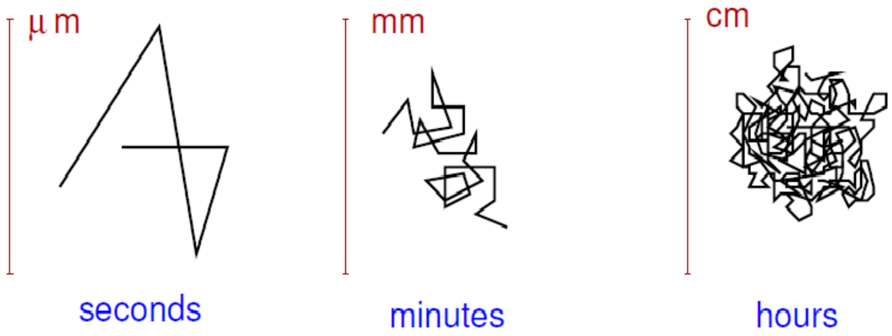

Fig. The three level scales [30]

One of the aims of mathematical models is to describe biological phenomena and species behavior. When we talk about diffusion, we can imagine the macroscopic displeacement of populations (seasonality migrations or migrations due to war, caresty or other factors), or cells, bacteria displeacements in some specific culture. Generally, we can have three different levels of description: microscopic, mesoscopic, and macroscopic (figure above).

The mesoscopic level is an intermediate scale where cells are not rapresented as single units, but as densities, and their movements can be described using microscopic properties. This level of description has a corresponding mathematical model, the so called "transport equation", where important quantities related to the described entities play a central role.
In order to introduce the transport equation we can for instance describe cell movements. In general, this motion is consists in two different phases: speed and rest. After every resting state, the cell enters in the movement phase with a new speed with a scalar and a vector component (intensity and direction). Vice versa, in the successive resting state, we have a reorientation of the movement and so on. We want to stress here how this alternance is typical not only of cells movement, but also it can describe for instance mammalian migrations. It is obviously how, in some cases, the resting states are negligible.

Let now introduce the classic transport equation.

Let $p = p(t, x, v)$ be the density of the population at time $t \in \mathbb{R}^+$, in space $x \in \mathbb{R}^N$ with velocity $v \in V \subset \mathbb{R}^N$, thus

$$\frac{\partial p}{\partial t}(t, x, v) + v \cdot \nabla p(t, x, v) = -\mu p(t, x, v) + \mu \int_V T(v, v')p(t, x, v')dv'. \quad (1.34)$$

The left hand side describes particles movement in the space, while the right hand side

$$\mathcal{K}(v) = -\mu p(t,x,v) + \mu \int_V T(v,v')p(t,x,v')dv' = -\mu p(t,x,v) + \mu \mathcal{I}(p) \quad (1.35)$$

is the so called "turning operator" and describes directions changes. The parameter μ rapresents the "turning rate", that is how often the particles change direction. An important role is played by the kernel $T(v,v')$ called "Turning kernel". This distribution (regulated by stochastic effects) describes the probability that the cell with velocity v' changes its velocity in a new velocity v. In general, the integral part describes the particles turning into a velocity $v \in \mathbb{R}^N$ from all other possibile velocities $v' \in \mathbb{R}^N$. The first term of the rhs, instead, manages the particles that turn out the velocity $v \in \mathbb{R}^N$.

Similarly to Section 1.4.1, one of the classical methods to analyze transport equations is to look to the diffusion limit. Thus, when we pass to the limit, spaces and time scales and direction-change frequencies increase leading to a caotic displeacement comparable with the concept of diffusion.

In order to have a better understanding of it, we have to introduce the rescaling notion of the transport equation (see [29],[30] for the classical cases). As we have said just above, we are able to distinguish three timescale levels:

1. Timescale of turning
2. Drift Timescale
3. Diffusion timescale

each of these levels is relative to a specific time order strictly connected with the characteristic of the subject under examination. By considering for example Escherichia Coli bacteria, if we take laboratory data, we know that it changes direction once per second, that is

$$\mu \approx 1 \cdot s. \quad (1.36)$$

Therefore

$$\frac{1}{\mu} \approx \frac{1}{s}.$$

If we consider the first time scale level, in agreement with (1.36) we can take

$$t_{turn} = o(1).$$

In the same way, we can increase the number of the turns (100 turns for example, equal to look the movement after 100 second) then, let

$$\epsilon \approx 10^{-2} \quad (1.37)$$

we obtain the second and intermediate time scale

$$t_{drift} = o(\epsilon^{-1}).$$

Finally, after 10000 turns (10000 sec), the trajectories look like a random motion, caotic and very similar to diffusion

$$t_{diff} = o(\epsilon^{-2}).$$

In biology, it is important the study of the subject. Through scientific method and sperimental observations, we can arrive to important conclusions useful for the modeling. For example it is well known which are E. Coli characteristics that hold: $s = 10 - 20\frac{\mu m}{s}$, the characteristic speed, $L = 1mm - 1cm$, the characteristic length scale and $\sigma = 1 - 10h$, the macroscopic time scale of observation. Then we can consider the following dimensionless quantities

$$\tilde{v} = \tfrac{v}{s} \qquad \tilde{x} = \tfrac{x}{L} \qquad \tilde{t} = \tfrac{t}{\sigma}.$$

Under this rescalations, the transport equation becomes

$$\frac{1}{\sigma}\partial\tilde{p}_{\tilde{t}}(\tilde{t},\tilde{x},\tilde{v}) + \frac{s}{L}\tilde{v}\cdot\nabla_{\tilde{x}}\tilde{p}(\tilde{t},\tilde{x},\tilde{v}) = -\mu\tilde{p}(\tilde{t},\tilde{x},\tilde{v}) + \mu\int_V T(\tilde{v},\tilde{v}')\tilde{p}(\tilde{t},\tilde{x},\tilde{v}')d\tilde{v}'.$$

Using E. Coli values for s,L,σ and using (1.37), we obtain the rescaled transport equation

$$\epsilon^2\partial\tilde{p}_{\tilde{t}}(\tilde{t},\tilde{x},\tilde{v}) + \epsilon\tilde{v}\cdot\nabla_{\tilde{x}}\tilde{p}(\tilde{t},\tilde{x},\tilde{v}) = -\mu\tilde{p}(\tilde{t},\tilde{x},\tilde{v}) + \mu\int_V T(\tilde{v},\tilde{v}')\tilde{p}(\tilde{t},\tilde{x},\tilde{v}')d\tilde{v}'.$$

Theorem 1.4.1 reassumes a classical result of [29]:

Theorem 1.4.1. *(The diffusion limit) Let* (1.34) *be the transport equation and denote with* $\mathcal{K} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *the Turning operator defined in* (1.35) *with the following basic assumption:*

$$T(v,v') \geq 0, \int_V T(v,v')dv = \int_V T(v,v')dv' = 1, \int_V\int_V T(v,v')dvdv' < \infty$$

There is a strictly positive function $u_0 \in L^2(V)$, *a* $\rho > 0$ *and an* $N > 0$, *such that for all* $(v,v') \in V \times V, u_0(v) \leq T^N(v,v') \leq \rho u_0(v)$.

where $T^N(v,v') = \int_V\int_V T(v,w_1)..T(w_{N-1},v')dw_1..dw_{N-1}$

Moreover for the integral operator $\mathcal{I}$ *in* (1.35),

$$||\mathcal{I}||_{\langle 1\rangle^\perp} < 1 \text{ where } L^2(V) = \langle 1\rangle \oplus \langle 1\rangle^\perp.$$

We can now introduce the macroscopic variables

$$\tilde{t} = \epsilon^2 t \quad \textit{and} \quad \tilde{x} = \epsilon x, \tag{1.38}$$

where ϵ is a small parameter. Then, through the Hilbert expansion method, we can reach the classical diffusion equation.

In fact, we can explain the process through heuristic considerations (taking into account that all the steps are permitted).

Let $p^\epsilon(\tilde{t}, \tilde{x}, v)$ be the rescaled distribution function that solves

$$\epsilon^2 \partial p_{\tilde{t}}(t, x, v) + \epsilon \tilde{v} \cdot \nabla_{\tilde{x}} p(t, x, v) = -\mu p(t, x, v) + \mu \int_V T(v, v') p(t, x, v') dv'.$$

and we consider a regular perturbation, called "Hilbert expansion of p respect to ϵ",

$$p(t, x, v) = p_0(t, x, v) + \epsilon p_1(t, x, v) + \epsilon^2 p_2(t, x, v) + r^\epsilon(t, x, v).$$

Substituting the expansion of p in the rescaled equation, matching the orders of ϵ, then we obtain the following system

$$\mathcal{K}(p_0) = 0 \tag{1.39}$$

$$\mathcal{K}(p_1) = v \cdot \nabla p_0 \tag{1.40}$$

$$\mathcal{K}(p_2) = p_{0t} + v \cdot \nabla p_1 \tag{1.41}$$

while, r^ϵ solves

$$r_t^\epsilon + \epsilon^{-1} v \cdot \nabla r^\epsilon = \epsilon^{-2} \mathcal{K}(r^\epsilon) + \epsilon p_{1t} + \epsilon^2 p_{2t} + \epsilon v \cdot \nabla p^2.$$

By (1.39) we deduce that $p_0 = \tilde{p}(x, t) T(v)$, then by (1.40) we can deduce that $p_1 = \mathcal{K}^{-1}(v \cdot \nabla \tilde{p})$. From third equation (1.41), by integrating with respect to v (and using that $\mathcal{K}$ preserves the total mass), we obtain

$$\partial \tilde{p}_t = \nabla (D \nabla \tilde{p})$$

where D is the diffusion matrix given by

$$D = \int_{\mathbb{R}^N} v \otimes \mathcal{K}^{-1}(v T(v)) dv.$$

Finally we can show that the lower order term converges strongly to 0 in particular Lesbegue spaces. Same result are obtained for different operators chosen in the transport equation, like for the so called "collisional operator" in the case of the Boltzmann equation (assuming for example that $\mathcal{K}$ is conservative, i.e.

$$\int_{\mathbb{R}^N} \mathcal{K}(p)dv = 0 \text{ for all } p \in L^2(\mathbb{R}^N)).$$

In general the problem will be when the Diffusion matrix is finite, conditio sine qua non to get the limit when ϵ goes to 0.

When T is chosen sufficiently decreasing infact, for example like the so called "Maxwellian distribution" in physics, i.e.

$$T(v) = Ce^{-\frac{v^2}{2}} \tag{1.42}$$

the diffusion constant is finite if and only if the second moment

$$\int_{\mathbb{R}^N} |v|^2 T(v)dv$$

is finite.

In the case in which the resting states are not negligible, we introduce coherently a new variable r and, in order to describe a more complicated motion, two new stochastic effects: one for the running phase (τ) and one for the new resting state (ω). Likely to [44], we denote with $f_\tau(t)$ and $f_\omega(t)$ the probability density function used to regulate respectively the time spent during these two phases. Therefore we denote with $p = p(t, x, v)$ the density of the particles at position $x \in \Omega \subset \mathbb{R}^N$ at the time $t \in \mathbb{R}^+$ with velocity $v \in V \subset \mathbb{R}^N$ and with $r = r(t, x, v)$ the density of the particles at position $x \in \Omega \subset \mathbb{R}^N$ at the time $t \in \mathbb{R}^+$ having just finished a jump at velocity $v \in V$. Then this motion is characterized by two equations,

$$\begin{aligned} \frac{\partial p}{\partial t}(t, x, v) + v \cdot \nabla p(t, x, v) = & -\int_0^t \phi_\tau(t-s)p(s, x-(t-s)v, v)ds \\ & + \int_0^t \phi_\omega(t-s)\int_V T(v, v')r(t, x, v')dv'ds, \end{aligned} \tag{1.43}$$

and

$$\frac{\partial r}{\partial t}(t, x, v) = -\int_0^t \phi_\omega(t-s)r(t, x, v)ds + \int_0^t \phi_\tau(t-s)p(s, x-(t-s)v, v)ds, \tag{1.44}$$

where the delay kernel ϕ_i with $i = \tau, \omega$ are strictly connected with the p.d.f. $f_i(t)$. In particular, let $\overline{\phi}_i$ be the Laplace transform of ϕ_i, then we have

$$\overline{\phi}_i(t) = \frac{\lambda \overline{f}_i(\lambda)}{1 - \overline{f}_i(\lambda)} \text{ with } i = \tau, \omega. \tag{1.45}$$

Remark 1.4.1. If we choose for f_i the exponential distribution with mean μ_i, i.e.

$$f_i(t) = \mu_i e^{-\mu_i t} \text{ with } i = \tau, \omega$$

then, by (1.45),

$$\phi_i(t) = \mu_i \delta(t) \text{ with } i = \tau, \omega.$$

where δ is the Dirac Delta.

Moreover, if we suppose that the resting state is negligible, then the equation (1.43) becomes the classic (1.34) .

Analyzing (1.43) we have two different contributions to the rhs. A negative contribution given by

$$\int_0^t \Phi_\tau(t-s)p(s, x-(t-s)v, v)ds$$

and rapresents the density of the outgoing particles from the running phase with velocity v, localized in x and the previous spaces until $x - tv$, and with running time regulated by Φ_τ. After this phase, all the particles will enter in the resting states (by looking at the same integral, in the second equation, it becomes positive). The positive contribution is instead given by

$$\int_0^t \Phi_\omega(t-s) \int_V T(v, v')r(t, x, v')dv'ds$$

and represents the density of the outgoing particles from the resting state, that in the past had a velocity v' and will enter in the running phase at the space x with a new velocity v (stochastically chosen with $T(v, v')$).

This process is called "generalized velocity jump processes with resting states" and also for this process is possibile to obtain the classical diffusion equation. Infact we can arrive to a large time effective diffusion equation for the global density. The method to get the limit is a bit different from the previous and it is called Cattaneo approximation. Without entering in the details we can give the principal steps. The crucial point is analize the large time behaviour of the Delay kernel , correspond to analize the behaviour of the its Laplace transform near 0. In particular, starting to the general equation we can reach a system of k equation and $k+1$ moments. Then the $k+1$th-moment is approximated in order to obtain a final closed system of equations and then get so the classical diffusion equation for the density (see [27]). The convergence, with this method, are more weakly than the Hilbert method, in fact the Hilbert method needs of stronger regularity assumptions in order to find a good bound for p_0, p_1, p_2 (see [6], [28]). We summarize the result in the following general theorem.

Theorem 1.4.2. *Let us consider* (1.43) *and* (1.44) *respectively the running and resting equations. Let* $m_p^0 = \int_V p(t,x,v)dv$ *and* $m_r^0 = \int_V r(t,x,v)dv$ *be the density moments. Putting this quantities in a vector* $u = (m_p^0, m_r^0)^T$*, we consider a regular perturbation*

$$u = u^0 + \epsilon u^1 + \epsilon^2 u^2 + l.o.t$$

where $u^i = m_{p_i} + m_{r_i}$ *for all* $i = 0, 1, 2.$

Then, it is possible to obtain the classical diffusion equation for the total density $m^0 = \int_V p_0(t,x,v)dv + \int_V r_0(t,x,v)dv$*,*

$$\frac{\partial}{\partial t} m^0 = D\nabla_x^2 m^0,$$

where D *costant that depends from the variance of the running time distribution (indipendent of the variance of the waiting time distribution) and from the mean squared speed defined as*

$$S_T^2 = \int_V ||v||^2 T(v,v')dv.$$

Chapter 2

The fractional diffusion and the generalized velocity jump processes

In this section we will consider the equation similar to (1.43), but we will suppose that, in the particles movement, the change of velocity will be istantaneous and the non-presence of the resting states will not have significative influence on the model accuracy (this assumption is very used in many velocity jump model). In order to deduce our equation, we use similar motivations used in [44]. We consider the temporary variable $\eta = \eta(t, x, v)$ representing the density of particles at position xdx, with velocity vdv at time tdt, having just started a jump with velocity v.

In the light of this, we can introduce

$$p(t, x, v) = \int_0^t F(t-s)\eta(s, x-(t-s)v, v)ds \tag{2.1}$$

that is the density of particles at position xdx, with velocity vdv at time tdt being in a running state.

$F(t) = \int_t^\infty f(s)ds$ represents the probability that the velocity phase is longer than t. We suppose, like in [44] that, at the initial time $t = 0$, the particles are initiated in the running at the velocity v with initial distribution $p_0(x, v)$, so we have:

$$\eta(t, x, v) - p_0(x, v)\delta(t) = \int_{\mathbb{R}^N}\int_0^t T(v, v')f(t-s)\eta(s, x-(t-s)v', v')dsdv' \tag{2.2}$$

that is η in the previous time.

Remark 2.0.2. The hypothesis in the initial distribution is very strong. It could be interesting to suppose for example a more natural initial distribution

$$p_0(x, v) = \pi_0(0, x, v) + \eta_0(x, v)$$

in which $\pi_0(t, x, v)$ represents the density of particles that at time 0 were at position x with velocity v and that at time t have not changed velocity. $\pi_0(\cdot, x, v)$ is a non-increasing function, and $\pi_0(0, x, v)$ should represent the initial density of particles at position x with velocity v and, for compatibility with [44] that used an extreme assumption, we could assume that a quantity $\eta_0(x, v)$ had just had a jump at time $t = 0$ to velocity v at position x. Using this, similarly to 2.1, we could write

$$p(t, x, v) = \int_0^t F(t-s)\eta(s, x-(t-s)v, v)ds + \pi_0(t, x-tv, v). \tag{2.3}$$

As for the those that have just started a jump at time t:

$$\begin{aligned}\eta(t, x, v) &= \eta_0(x, v)\delta(t)\\ &+ \int_{\mathbb{R}^N} T(v, v')\left(\int_0^t f(t-s)\eta(s, x-(t-s)v', v')ds - \pi_0'(t, x-tv', v')\right)dv'\end{aligned} \tag{2.4}$$

where the first term is the mass that just jumped at time 0, the second and the third term are those that stopped moving at speed v' at time t.

Back to us, we consider the Fourier transform of the equations (2.1) and (2.2) defined in (1.8). Therefore the Fourier transform of (2.1), becomes

$$(2\pi)^{-\frac{N}{2}}\hat{p}(t, k, v) = (2\pi)^{-\frac{N}{2}}\int_0^t F(t-s)\int_{\mathbb{R}^N} e^{-ik\cdot x}\eta(s, x-(t-s)v, v)d\xi ds.$$

Let $u = x-(t-s)v$, $du = d\xi$, up to a constant, then

$$\begin{aligned}\hat{p}(t, k, v) &= \int_0^t F(t-s)\int_{\mathbb{R}^N} e^{-i\xi\cdot x}\eta(s, x-(t-s)v, v)d\xi ds\\ &= \int_0^t F_\tau(t-s)e^{-i(t-s)k\cdot v}\int_{\mathbb{R}^N} e^{-ik\cdot u}\eta(s, u, v)duds\\ &= \int_0^t F_\tau(t-s)e^{-i(t-s)k\cdot v}\widehat{\eta}(s, k, v)ds.\end{aligned}$$

In the same way we can transform (2.2) and we obtain

$$\hat{\eta}(t, k, v) - \hat{p}_0(k, v)\delta(t) = \int_{\mathbb{R}^N}\int_0^t T(v, v')f(t-s)e^{-i(t-s)k\cdot v}\widehat{\eta}(s, k, v')ds.$$

Taking the Laplace transform (we will denote it with a bar, also when it is applied to a Fourier transform, giving rise to a Laplace-Fourier transform) in time of (2.1), we have

$$\begin{aligned}\overline{p}(\lambda,k,v) &= \int_0^{+\infty}\int_0^{+\infty} F(t-s)e^{-t\lambda}e^{-i(t-s)k\cdot v}\widehat{\eta}^{\epsilon}(s,k,v)ds \text{ (multiplying for } e^{s\lambda}e^{-s\lambda})\\ &= \int_0^{+\infty}\int_0^{+\infty} F(t-s)e^{-(t-s)\lambda}e^{-i(t-s)k\cdot v}e^{-s\lambda}\widehat{\eta}(s,k,v)ds\\ &= \overline{F}(\lambda+ik\cdot v)\overline{\eta}(\lambda,k,v).\end{aligned} \tag{2.5}$$

Analogously, taking the Laplace transform of (2.2),

$$\overline{\eta}(\lambda,k,v)-\hat{p}_0(k,v) = \int_{\mathbb{R}^N} T(v,v')\overline{f}(\lambda+ik\cdot v')\overline{\eta}(\lambda,k,v')dv'. \tag{2.6}$$

From the properties of Laplace transform, it is clear that

$$\overline{F}(\lambda) = \frac{1-\overline{f}(\lambda)}{\lambda},$$

hence

$$\frac{1}{\overline{F}(\lambda)} = \frac{\lambda}{1-\overline{f}(\lambda)} = \frac{\lambda(1-\overline{f}(\lambda))}{1-\overline{f}(\lambda)} + \frac{\lambda\overline{f}(\lambda)}{1-\overline{f}(\lambda)} = \lambda+\overline{\phi}(\lambda). \tag{2.7}$$

We can then substitute (2.5) and (2.6), using (2.7), to obtain

$$\begin{aligned}(\lambda+ik\cdot v)\overline{p}(\lambda,k,v)-\overline{p}_0 &= \overline{\phi}(\lambda+ik\cdot v)\overline{p}(\lambda,x,v)\\ &\quad+\int_{\mathbb{R}^N} T(v,v')\overline{\phi}(\lambda+ik\cdot v')\overline{p}(t,x,v')dv'.\end{aligned} \tag{2.8}$$

Returning to the original variables, we obtain the generalized velocity jump processes without resting states:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+v\cdot\nabla\right)p(t,x,v) &= -\int_0^t \phi(t-s)p(s,x-(t-s)v,v)ds\\ &\quad+\int_0^t \phi(t-s)\int_{\mathbb{R}^N} T(v,v')p(s,x-(t-s)v',v')dv'ds\end{aligned} \tag{2.9}$$

2.1 Fractional limit

How we have seen in the previous Section, we present the Generalized Velocity Jump process like the following equation

$$\begin{aligned} p_t + v \cdot \nabla p = &- \int_0^t \phi(t-s) p(s, x-(t-s)v, v) ds \\ &+ \int_0^t \phi(t-s) \int_{\mathbb{R}^N} T(v,v') p(s, x-(t-s)v', v') dv' ds \end{aligned} \tag{2.10}$$

where the delay kernel ϕ is, in general, a distribution and we have seen that is defined in Laplace space by

$$\overline{\phi}(\lambda) = \frac{\lambda \overline{f}(\lambda)}{1 - \overline{f}(\lambda)} \tag{2.11}$$

where $\overline{f}$ is the Laplace transform of the probability density function of the running state.

First af all we will consider that

$$T(v, v') = T(v)$$

i.e. our kernel, that stochastically manages the velocity, doesn't depend by previous velocity. Therefore every time the Kernel give a new velocity that could be also equal to the previous.

Remark 2.1.1. (The Delay Kernel ϕ)

If the running time is exponential distribuited, we have seen that, our Delay Kernel is a multiple of the Dirac Delta function (it represents the classical istantaneous impulse). In general we can think at ϕ like a distribution having two contributions: the istantaneous impulse at the origin and a well defined function.

Let $f(t)$ be a Laplace-transformable probability density function. In particular, we know from the theory that, to look the small time behaviour of $f(t)$, it corresponds to look the large λ behaviour for the trasformed function $\overline{f}(\lambda)$, and

$$f(0) = \lim_{t \to 0} f(t) = \lim_{\lambda \to \infty} \lambda \overline{f}(\lambda). \tag{2.12}$$

For us ϕ represents a distribution having the following structure:

$$\phi(t) = \delta(t) f(0) + \phi_1(t), \tag{2.13}$$

for which, passing to Laplace transform we obtain

$$\overline{\phi}(\lambda) = f(0) + \overline{\phi}_1(\lambda).$$

Using the Definition of $\phi(\lambda)$ in (2.11), then

$$\overline{\phi}_1(\lambda) = \lambda \frac{\overline{f}(\lambda)}{1-\overline{f}(\lambda)} - f(0) \tag{2.14}$$

Since the existence of $\overline{f}(\lambda)$, we can find the nature of $\phi_1(t)$ taking $g(t) \in C^1(\mathbb{R})$ such that

$$\overline{g}(\lambda) = \frac{\overline{f}(\lambda)}{1-\overline{f}(\lambda)}.$$

Moreover, from (2.12), we deduce $f(0) = g(0)$ and, from the derivative property of the Laplace transform, i.e.

$$\overline{g'}(\lambda) = \lambda \overline{g}(\lambda) - g(0).$$

Finally, comparing this equation with (2.14),

$$\overline{g'}(\lambda) = \overline{\phi}_1(\lambda),$$

hence $g'(t) = \phi_1(t)$ for almost every $t \in \mathbb{R}^+$.

2.1.1 The Rescaling

It is well known from Subsection 1.4.2 that the derivation of diffusion-type equations from the transport equations through the asymptotic analysis it is possible using a particular scaling corresponding to a long time. In the case in which we consider a general rescaling, some problem arise. In fact, it is intuitively that, the matching (1.39), (1.40), (1.41) are not more possible because ϵ had an integer exponent; see (1.38). We refer to [33], [6] in which the authors study a diffusion type equation starting from a kinetic equation. In the following we will use some their techniques in order to obtain our result.

Therefore, we consider two macroscopic variables

$$\tau = \theta(\epsilon)t \quad \text{ and } \quad \xi = \epsilon x \tag{2.15}$$

where $\theta(\epsilon)$ is the appropriate diffusion scaling.

Another question is about the Kernel T. Suppose for example that

$$T(v) \approx |v|^{N+2s} \text{ for } |v| \to \infty$$

i.e. the T are the fundamental solutions of the fractional diffusion equation. It is immediately that the variance is infinite, one more reason to exclude the possibility of using the techniques view in the previus Subsection 1.4.2. We stress that the distributions with infinite variance are called stable (or Lèvy) distributions and the latter plays an important role in probability theory since it can be interpreted as the law of a Lèvy whose law evolution is governed by a fractional diffusion equation.

In order to study the diffusion limit we define, using (2.15),

$$p^\epsilon(\tau,\xi,v) = p(\tau\theta(\epsilon)^{-1}, \xi\epsilon^{-1}, v).$$

Moreover, we can rescaling the problem (2.10) with the new variables and we obtain

$$\begin{aligned}
&\theta(\epsilon)p^\epsilon_\tau + \epsilon v\cdot\nabla p^\epsilon = \\
&-\int_0^{\theta(\epsilon)^{-1}\tau} \phi(\theta(\epsilon)^{-1}\tau - s)p(s,\epsilon^{-1}\xi - (\theta(\epsilon)^{-1}\tau - s)v, v)ds \\
&+\int_0^{\theta(\epsilon)^{-1}\tau} \phi(\theta(\epsilon)^{-1}\tau - s)\int_{\mathbb{R}^N} T(v)p(s,\epsilon^{-1}\xi - (\theta(\epsilon)^{-1}\tau - s)v', v')dv'ds
\end{aligned}$$

with $p^\epsilon(\tau,\xi,v)$ the rescaled function.

Let $\sigma = \theta(\epsilon)s$ then we have

$$\begin{aligned}
&\theta(\epsilon)p^\epsilon_\tau + \epsilon v\cdot\nabla p^\epsilon \\
&= -\theta(\epsilon)^{-1}\int_0^\tau \phi(\theta(\epsilon)^{-1}(\tau-\sigma))p(\theta(\epsilon)^{-1}\sigma, \epsilon^{-1}\xi - \theta(\epsilon)^{-1}(\tau-\sigma)v, v)d\sigma \\
&+\theta(\epsilon)^{-1}\int_0^\tau \phi(\theta(\epsilon)^{-1}(\tau-\sigma))\int_{\mathbb{R}^N} T(v)p(s,\epsilon^{-1}\xi - \theta(\epsilon)^{-1}(\tau-\sigma)v', v')dv'd\sigma \\
&= -\theta(\epsilon)^{-1}\int_0^\tau \phi(\theta(\epsilon)^{-1}(\tau-\sigma))p(\theta(\epsilon)^{-1}\sigma, \epsilon^{-1}(\xi - \theta(\epsilon)^{-1}\epsilon(\tau-\sigma)v), v)d\sigma \\
&+\theta(\epsilon)^{-1}\int_0^\tau \phi(\theta(\epsilon)^{-1}(\tau-\sigma))\int_{\mathbb{R}^N} T(v)p(\theta(\epsilon)^{-1}\sigma,\epsilon^{-1}(\xi - \theta(\epsilon)^{-1}\epsilon(\tau-\sigma)v'), v')dv'd\sigma.
\end{aligned} \tag{2.16}$$

Let

$$p(\theta(\epsilon)^{-1}\sigma, \epsilon^{-1}(\xi - \theta(\epsilon)^{-1}\epsilon(\tau-\sigma)v, v) = p^\epsilon(\sigma, \xi - \epsilon\theta(\epsilon)^{-1}(\tau-\sigma)v, v)$$

then

$$\begin{aligned}
&\theta(\epsilon)p^\epsilon_\tau + \epsilon v\cdot\nabla p^\epsilon \\
&= -\theta(\epsilon)^{-1}\int_0^\tau \phi(\theta(\epsilon)^{-1}(\tau-\sigma))p^\epsilon(\sigma, \xi - \epsilon\theta(\epsilon)^{-1}(\tau-\sigma)v, v)d\sigma \\
&+\theta(\epsilon)^{-1}\int_0^\tau \phi(\theta(\epsilon)^{-1}(\tau-\sigma))\int_{\mathbb{R}^N} T(v)p^\epsilon(\sigma, \xi - \epsilon\theta(\epsilon)^{-1}(\tau-\sigma)v', v')dv'd\sigma.
\end{aligned} \tag{2.17}$$

We stress that, as we said in Remark 2.1.1, our Delay Kernel is thought in the form (2.13). Therefore in the previous equation, the changed term $\phi(\theta(\epsilon)^{-1}(\tau-\sigma))$ is considered in that sense, i.e.

$$\phi(\theta(\epsilon)^{-1}(\tau)) = \delta(\tau)f(0) + \phi_1(\theta(\epsilon)^{-1}(\tau)).$$

Anyway, rewriting our delay kernel as

$$\phi^\epsilon(\tau) = \theta(\epsilon)^{-1}\phi(\theta(\epsilon)^{-1}(\tau)), \tag{2.18}$$

therefore

$$\begin{aligned}\theta(\epsilon)p^\epsilon_\tau + \epsilon v\cdot\nabla p^\epsilon = &-\int_0^\tau \phi^\epsilon(\tau-\sigma)p^\epsilon(\sigma,\xi-\epsilon\theta(\epsilon)^{-1}(\tau-\sigma)v,v)d\sigma\\ &+\int_0^\tau \phi^\epsilon(\tau-\sigma)\int_{\mathbb{R}^N} T(v)p^\epsilon(\sigma,\xi-\epsilon\theta(\epsilon)^{-1}(\tau-\sigma)v'),v')dv'd\sigma.\end{aligned} \tag{2.19}$$

In the next step, we consider the Fourier-Laplace transform of the equation (2.19). Then the Fourier transform of $p^\epsilon(s,\xi-\epsilon\theta(\epsilon)^{-1}(\tau-s)v,v)$ in the first integral on the right side of (2.19), becomes

$$\begin{aligned}&\int_0^\tau \phi^\epsilon(\tau-s)F(p^\epsilon(s,\xi-\epsilon\theta(\epsilon)^{-1}(\tau-s)v,v))ds\\ &=\int_0^\tau \phi^\epsilon(\tau-s)\int_{\mathbb{R}^N} e^{-ik\cdot\xi}p^\epsilon(s,\xi-\epsilon\theta(\epsilon)^{-1}(\tau-s)v,v)d\xi ds.\end{aligned}$$

Let $u=\xi-\epsilon\theta(\epsilon)^{-1}(\tau-s)v$, $du=d\xi$, then

$$\begin{aligned}&=\int_0^{+\infty} \phi^\epsilon(\tau-s)e^{-\epsilon\theta(\epsilon)^{-1}(\tau-s)k\cdot v}\int_{\mathbb{R}^N} e^{-ik\cdot u}p^\epsilon(s,u,v)duds\\ &=\int_0^{+\infty} \phi^\epsilon(\tau-s)e^{-\epsilon\theta(\epsilon)^{-1}(\tau-s)k\cdot v}\widehat{p}^\epsilon(s,k,v)ds.\end{aligned}$$

where $\widehat{p}^\epsilon$ is the Fourier transform of p^ϵ. Like this integral, we pass to the Fourier transform in all other terms in (2.19).

Finally, we take the Laplace transform in time of the first integral on the right side, i.e.

$$\int_0^{+\infty} \phi^\epsilon(\tau-s)e^{-\epsilon\theta(\epsilon)^{-1}(\tau-s)k\cdot v}\widehat{p}^\epsilon(s,k,v)ds$$

and we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \phi^\epsilon(\tau-s)e^{-(\tau-s)\lambda}e^{-\epsilon\theta(\epsilon)^{-1}(\tau-s)k\cdot v}e^{-s\lambda}\widehat{p}^\epsilon(s,k,v)ds\\ &=\int_0^{+\infty}\int_0^{+\infty} \phi^\epsilon(\tau-s)e^{-(\tau-s)\left[\lambda+\epsilon\theta(\epsilon)^{-1}k\cdot v\right]}e^{-s\lambda}\widehat{p}^\epsilon(s,k,v)dsds\\ &=\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)\overline{p}^\epsilon(\lambda,k,v),\end{aligned}$$

where $\overline{\phi}^{\epsilon}$ and $\overline{p}^{\epsilon}$ are the Laplace-Fourier transform of ϕ^{ϵ} and p^{ϵ} (again we will denote with a bar also the Fourier transform, giving rise to a Laplace-Fourier transform). In the same way we trasform the other pieces and we arrive to the following equation

$$\begin{aligned}\lambda\theta(\epsilon)\overline{p}^{\epsilon} + \epsilon ik\cdot v\overline{p}^{\epsilon} - \theta(\epsilon)\overline{p}_0 &= -\overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k\cdot v)\overline{p}^{\epsilon}(\lambda,k,v)\\ &\quad + \int_{\mathbb{R}^N} T(v)\overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k\cdot v')\overline{p}^{\epsilon}(\lambda,k,v')dv'.\end{aligned} \tag{2.20}$$

Remark 2.1.2. It is important to stress that, find the explicit formula for the distribution ϕ is very complicated, sometimes impossible. However, in order to have more information, it becomes foundamental to know the rescaled ϕ behaviour when ϵ goes to 0. In this way, is useful to calculate the Laplace transform of $\phi^{\epsilon}(\tau)$, i.e.

$$\overline{\phi}^{\epsilon}(\lambda) = \theta(\epsilon)^{-1}\int_0^{+\infty} e^{-\lambda\tau}\phi(\theta(\epsilon)^{-1}\tau)d\tau = \int_0^{+\infty} e^{-\lambda\theta(\epsilon)s}\phi(s)ds,$$

then

$$\lim_{\epsilon\to 0}\overline{\phi}^{\epsilon}(\lambda) = \overline{\phi}(0).$$

Now, we look at the following terms

$$\overline{f}(\lambda) = \int_0^{\infty} e^{-\lambda s}f(s)ds \tag{2.21}$$

and

$$\overline{f}'(\lambda) = -\int_0^{\infty} se^{-\lambda s}f(s)ds. \tag{2.22}$$

It is easly to check that $\overline{f}(0) = 1$ and

$$\overline{f}'(0) = -\int_0^{\infty} sf(s)ds = -\tau \tag{2.23}$$

where τ is the expected value and represent the mean running time.

Taking the explicit form of $\overline{\phi}$ in (2.11), by (2.21) and (2.22), then

$$\lim_{\lambda\to 0}\overline{\phi}(\lambda) = \lim_{\lambda\to 0} -\frac{\overline{f}(\lambda) + \lambda\overline{f}'(\lambda)}{\overline{f}'(\lambda)} = -\frac{\overline{f}(0)}{\overline{f}'(0)} = \frac{1}{\tau}. \tag{2.24}$$

Now, we can easly study $\overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k \cdot v')$ when ϵ goes to 0. Infact, we can write

$$\begin{aligned} &\lim_{\epsilon \to 0} \left[\overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k \cdot v') - \frac{1}{\tau} \right] \\ &= \lim_{\epsilon \to 0} \left[\overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k \cdot v') - \overline{\phi}^{\epsilon}(\lambda) \right] + \lim_{\epsilon \to 0} \left[\overline{\phi}^{\epsilon}(\lambda) - \frac{1}{\tau} \right]. \end{aligned} \tag{2.25}$$

From (2.24), the second limit goes to 0. Furthermore we check that

$$\begin{aligned} \overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k \cdot v') &= \int_0^\infty e^{-(\lambda + i\epsilon\theta(\epsilon)^{-1}k \cdot v)\tau} \theta(\epsilon)^{-1} \phi(\theta(\epsilon)^{-1}\tau) d\tau \\ &= \int_0^\infty e^{-\lambda\theta(\epsilon)t} e^{-i\epsilon k \cdot vt} \phi(t) dt. \quad (\text{taking } t = \theta(\epsilon)^{-1}\tau). \end{aligned}$$

Then, by Dominated convergence theorem, we have that

$$\lim_{\epsilon \to 0} \overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k \cdot v') = \overline{\phi}(0)$$

and the first limit in the right side of (2.25) goes to 0 when ϵ goes to 0.

Then, like for $\phi^{\epsilon}(\lambda)$,

$$\lim_{\epsilon \to 0} \overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1}k \cdot v') = \frac{1}{\tau} \text{ for a.e. } k \in \mathbb{R}^N \text{ and a.e. } \lambda \in \mathbb{R}^+. \tag{2.26}$$

2.1.2 The Theorem

We make the following assumption on T:

(HT0) The kernel T is symmetric, $T(v, v') = T(v)$ and is such that

$$\int_{\mathbb{R}^N} T(v) = 1.$$

(HT1) There exist $\alpha > 0$ and $C_1 > 0$ such that

$$T(v)|v|^{N+\alpha} \to C_1 \quad \text{as} \quad |v| \to \infty,$$

We will prove the following result

Theorem 2.1.1. *Assume that hypotheses **(HT0)**, **(HT1)** with* $\alpha \in (0, 2)$ *and* (2.26) *hold. Assume furthermore that the initial distribution* $p_0 \in L^2_x(\mathbb{R}^N; L^1_v(\mathbb{R}^N))$ *and let* p^{ϵ} *be the solution of* (2.19) *with* $\theta(\epsilon) = \epsilon^{\alpha}$*. In particular let* $\rho^{\epsilon} = \int_{\mathbb{R}^N} p^{\epsilon}(t, x, v) dv$ *be the density, when* ϵ *goes to* 0*, then*

$$\rho^{\epsilon}(t, x) \rightharpoonup \rho \text{ in } L^\infty(0, T; L^2_x(\mathbb{R}^N)).$$

Moreover, $\rho(t,x)$ is the solution of the α-fractional diffusion equation:

$$\begin{cases} \partial_t \rho + C(-\Delta)^{\frac{\alpha}{2}}\rho = 0 \ \mathit{in}\ (0,\infty)\times\mathbb{R}^N \\ \rho(0.\cdot) = \rho_0 \ \mathit{in}\ \mathbb{R}^N \end{cases}$$

where

$$C = \frac{C_1}{\tau}\int_{\mathbb{R}^N} \frac{(w\cdot e)^2}{(\overline{\phi}(iw\cdot e))^2 + (w\cdot e)^2}\frac{1}{|w|^{N+\alpha}}dw$$

Proof. In the first step of the proof we would like to find an estimates for the solution p^ϵ. Starting with (2.19), we multiply for p^ϵ both sides and we integrate respect v and x, hence similarly to [33] then

$$\frac{d}{dt}\int_{\mathbb{R}^{2N}} \left(p^\epsilon(t,\cdot)\right)^2 dxdv < 0,$$

from which we deduce that

$$\sup_{t\geq 0}\int_{\mathbb{R}^{2N}} \left(p^\epsilon(t,\cdot)\right)^2 dvdx \leq \int_{\mathbb{R}^{2N}} p_0^2 dxdv. \tag{2.27}$$

Finally we observe that

$$\begin{aligned} \int_{\mathbb{R}^N} \left(\rho^\epsilon(t,\cdot)\right)^2 dx &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} p^\epsilon dv\right)^2 dx \leq \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \left(p^\epsilon\right)^2 dv\right) dx \\ &\leq \int_{\mathbb{R}^{2N}} p_0^2 dxdv, \end{aligned}$$

therefore,

$$\sup_{t\geq 0}\int_{\mathbb{R}^N} \left(\rho^\epsilon(t,\cdot)\right)^2 dx \leq ||p_0||_{L^2_x}, \tag{2.28}$$

i.e. the density is bounded in $L^\infty(0,\infty;L^2(\mathbb{R}^N))$.

In the second step we proceeding to modifying the equation (2.20) (see [33]). In this sense easily we can write that

$$\begin{aligned} \overline{p}^\epsilon(\lambda,k,v) = &\frac{\theta(\epsilon)\overline{p}_0(k,v)}{\lambda\theta(\epsilon) + \epsilon ik\cdot v + \overline{\phi}^\epsilon(\lambda + i\epsilon\theta(\epsilon)^{-1}k\cdot v)} \\ &+ T(v)\int_{\mathbb{R}^N} \frac{\overline{\phi}^\epsilon(\lambda + i\epsilon\theta(\epsilon)^{-1}k\cdot v')\overline{p}^\epsilon(\lambda,k,v')}{\lambda\theta(\epsilon) + \epsilon ik\cdot v + \overline{\phi}^\epsilon(\lambda + i\epsilon\theta(\epsilon)^{-1}k\cdot v)}dv' \end{aligned} \tag{2.29}$$

Let

$$Q_0[\overline{p}^\epsilon](\lambda,k) = \int_{\mathbb{R}^N} \overline{\phi}^\epsilon(\lambda + i\epsilon\theta(\epsilon)^{-1}k\cdot v')\overline{p}^\epsilon(\lambda,k,v')dv' \tag{2.30}$$

so that we can rewrite (2.29) as

$$\overline{p}^{\epsilon}(\lambda,k,v) = \frac{\theta(\epsilon)\overline{p}_0(k,v)}{\lambda\theta(\epsilon)+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)} + \frac{T(v)Q_0[\overline{p}^{\epsilon}](\lambda,k)}{\lambda\theta(\epsilon)+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}. \tag{2.31}$$

Multiplying the equation (2.31) by $\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)$ and integrating respect to v we obtain

$$Q_0[\overline{p}^{\epsilon}](\lambda,k) = \int_{\mathbb{R}^N}\frac{\theta(\epsilon)\overline{p}_0(v)}{\theta(\epsilon)\lambda+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)dv + \int_{\mathbb{R}^N}\frac{T(v)\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{\theta(\epsilon)\lambda+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}dvQ_0[\overline{p}^{\epsilon}](\lambda,k).$$

Now, we can solve for $Q_0[\overline{p}^{\epsilon}](\lambda,k)$ obtaining

$$Q_0\left[\overline{p}^{\epsilon}\right](\lambda,k)\frac{1}{\theta(\epsilon)}\int_{\mathbb{R}^N}\left[1-\frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{\theta(\epsilon)\lambda+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\right]T(v)dv = \int_{\mathbb{R}^N}\frac{\overline{p}_0(k,v)}{\theta(\epsilon)\lambda+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)dv \tag{2.32}$$

where we have used again that $\int_{\mathbb{R}^N} T(v)\,dv = 1$.

The rest of the proof consist to passing to the limit for $\epsilon \to 0$ in the previous equation.

We look at the left side of (2.32), in particular:

$$\frac{1}{\theta(\epsilon)}\int_{\mathbb{R}^N}\left[1-\frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{\theta(\epsilon)\lambda+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\right]T(v)dv \tag{2.33}$$

We write the integrand in the following form

$$1-\frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{\lambda\theta(\epsilon)+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}$$
$$=\frac{\lambda\theta(\epsilon)+\epsilon ik\cdot v}{\lambda\theta(\epsilon)+\epsilon ik\cdot v+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\frac{(\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)+\lambda\theta(\epsilon))-(\epsilon ik\cdot v)}{(\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)+\lambda\theta(\epsilon))-(\epsilon ik\cdot v)}=$$
$$\frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)+\lambda\theta(\epsilon)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda-i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}\lambda\theta(\epsilon)$$
$$+\frac{(\epsilon k\cdot v)^2}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}$$
$$+\frac{\epsilon ik\cdot v\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}. \tag{2.34}$$

So (2.33) becomes a sum of three integrals

$$\frac{1}{\theta(\epsilon)}\int_{\mathbb{R}^N}\frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)+\lambda\theta(\epsilon)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}\lambda\theta(\epsilon)T(v)dv$$
$$+\frac{1}{\theta(\epsilon)}\int_{\mathbb{R}^N}\frac{(\epsilon k\cdot v)^2}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}T(v)dv$$
$$+\frac{1}{\theta(\epsilon)}\int_{\mathbb{R}^N}\frac{\epsilon(ik\cdot v)\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}T(v)dv.$$

By **(HT0)** T is symmetric, then the third integral, with the term $\epsilon ik\cdot v$, is 0.

Then, we look at the first term, that is

$$I_1(\lambda,k)=\lambda\int_{\mathbb{R}^N}\frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)+\lambda\theta(\epsilon)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}T(v)dv. \tag{2.35}$$

We observe that, by (2.26), for a.e. $\lambda\in\mathbb{R}^+$, a.e. $k\in\mathbb{R}^N$ and for η small enough,

$$\left|\frac{T(v)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}\right|\leq\frac{T(v)}{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)^2}$$
$$\leq\frac{\tau^2T(v)}{(1-2\tau\eta)^2}\in L^1_v(\mathbb{R}^N).$$

Therefore, using again (2.26), for a.e. $\lambda\in\mathbb{R}^+$, a.e. $k\in\mathbb{R}^N$ and with η small enough we have

$$\left|\frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)T(v)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}\right|\leq\frac{\tau T(v)}{1-2\tau\eta}\in L^1_v(\mathbb{R}^N),$$

Therefore by Dominated convergence theorem, we can pass to the limit in (2.35) and we obtain

$$\begin{aligned}
&\lambda \lim_{\epsilon\to 0}\int_{\mathbb{R}^N} \frac{\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}T(v)dv \\
&+\lim_{\epsilon\to 0}\lambda\theta(\epsilon)\int_{\mathbb{R}^N} \frac{\lambda}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}T(v)dv \qquad (2.36)\\
&=\tau\lambda\int_{\mathbb{R}^N} T(v)dv=\lambda\tau. \qquad \text{(Using that } \lim_{\epsilon\to 0}\overline{\phi}^{\epsilon}(\lambda)=1/\tau)
\end{aligned}$$

It remains to be proved that

$$I_2(\lambda,k)=\frac{1}{\theta(\epsilon)}\int_{\mathbb{R}^N} \frac{(\epsilon k\cdot v)^2}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}T(v)dv \qquad (2.37)$$

converges to $C|k|^{\alpha}$ with α well defined.

For some constant $K>0$ we split the integral in two parts:

$$I_2(\lambda,k)=I_2^1(\lambda,k)+I_2^2(\lambda,k).$$

It is immediate that

$$\begin{aligned}
I_2^1(\lambda,k)&=\frac{\epsilon^2}{\theta(\epsilon)}\int_{\{|v|\le K\}} \frac{(k\cdot v)^2}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+(\epsilon k\cdot v)^2}T(v)dv\\
&\le \frac{\epsilon^2|k|^2}{\theta(\epsilon)}\int_{\{|v|\le K\}} \frac{|v|^2}{(\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2}T(v)dv\\
&\le \frac{\tau K^2|k|^2}{(1-2\tau\eta)}\frac{\epsilon^2}{\theta(\epsilon)}\int_{\{|v|\le K\}} T(v)dv.
\end{aligned}$$

If

$$C\frac{\epsilon^2|k|^2}{\theta(\epsilon)}\to 0 \text{ when } \epsilon \text{ goes to } 0,$$

i.e., $\alpha\in(0,2)$, then

$$I_2^1(\lambda,k)\to 0. \qquad (2.38)$$

In order to control $I_2^2(\lambda,k)$ we multiply and divide for $|v|^{N+\alpha}$ the integral $I_2^2(\lambda,k)$, then

$$I_2^2(\lambda,k)=\int_{\{|v|\ge K\}} \frac{\epsilon^2\theta(\epsilon)^{-1}(k\cdot v)^2}{(\lambda\theta(\epsilon)+\overline{\phi}^{\epsilon}(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2+\epsilon^2(k\cdot v)^2}\frac{\tilde{T}(v)}{|v|^{N+\alpha}}dv,$$

where

$$\tilde{T}(v) = |v|^{N+\alpha} T(v).$$

Let $w = \epsilon v|k|$ a change of variables, then

$$v = \frac{w\epsilon^{-1}}{|k|} \quad \text{and} \quad dv = \frac{\epsilon^{-N}}{|k|^N} dw.$$

Therefore

$$\overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1} k \cdot v) = \overline{\phi}(\theta(\epsilon)\lambda + iw \cdot e) \to \overline{\phi}(iw \cdot e) \text{ for } \epsilon \to 0$$

and

$$\begin{aligned} I_2^2(\lambda, k) &= \theta(\epsilon)^{-1} \int_{\{|v| \geq K\}} \frac{\epsilon^2 (k \cdot v)^2}{(\lambda\theta(\epsilon) + \overline{\phi}^{\epsilon}(\lambda + i\epsilon\theta(\epsilon)^{-1} k \cdot v))^2 + \epsilon^2 (k \cdot v)^2} \frac{\tilde{T}(v)}{|v|^{N+\alpha}} dv \\ &= \theta(\epsilon)^{-1} \int_{\{|w| \geq \epsilon|k|K\}} \frac{(w \cdot e)^2}{(\theta(\epsilon)\lambda + \overline{\phi}^{\epsilon}(\theta(\epsilon)\lambda + iw \cdot e))^2 + (w \cdot e)^2} \frac{|k|^{N+\alpha}}{|w|^{N+\alpha}} \frac{\tilde{T}^{\epsilon}(w)}{\epsilon^{-N}\epsilon^{-\alpha}} \frac{\epsilon^{-N}}{|k|^N} dw \\ &= \frac{\theta(\epsilon)^{-1}}{\epsilon^{-\alpha}} |k|^{\alpha} \int_{\{|w| \geq \epsilon|k|K\}} \frac{(w \cdot e)^2}{(\theta(\epsilon)\lambda + \overline{\phi}^{\epsilon}(\theta(\epsilon)\lambda + iw \cdot e))^2 + (w \cdot e)^2} \frac{\tilde{T}^{\epsilon}(w)}{|w|^{N+\alpha}} dw, \end{aligned} \tag{2.39}$$

with

$$\tilde{T}^{\epsilon}(w) = T\left(\frac{w}{\epsilon|k|}\right).$$

By **(HT1)**, we have that

$$\lim_{\epsilon \to 0} \tilde{T}^{\epsilon}(w) = C_1 > 0.$$

Hence, when $\frac{\theta(\epsilon)^{-1}}{\epsilon^{-\alpha}} \to 1$, i.e. $\theta(\epsilon) \approx \epsilon^{\alpha}$, the integrand in (2.39) converges pointwise to

$$C_1 \frac{(w \cdot e)^2}{(\overline{\phi}(iw \cdot e))^2 + (w \cdot e)^2} \frac{1}{|w|^{N+\alpha}}$$

In particular we observe that

$$\left| \frac{(w \cdot e)^2}{(\theta(\epsilon)\lambda + \overline{\phi}^{\epsilon}(\theta(\epsilon)\lambda + iw \cdot e))^2 + (w \cdot e)^2} \frac{\tilde{T}^{\epsilon}(w)}{|w|^{N+\alpha}} \right| \leq \frac{\tau (w \cdot e)^2}{(1 - \tau\eta)^2 + \tau (w \cdot e)^2} \frac{\tilde{T}^{\epsilon}(w)}{|w|^{N+\alpha}} \in L^1(\mathbb{R}^N).$$

By Lesbegue's dominated convergence theorem we can pass to the limit and we obtain

$$I_2(\lambda, k) \to C_1 |k|^{\alpha} \int_{\mathbb{R}^N} \frac{(w \cdot e)^2}{(\overline{\phi}(iw \cdot e))^2 + (w \cdot e)^2} \frac{1}{|w|^{N+\alpha}} dw. \tag{2.40}$$

Therefore,

$$I_2(\lambda,k)\to C_2|k|^\alpha. \tag{2.41}$$

We look at the right side of (2.32); we use the assumption on the initial data $p_0(x,v)\in L^2_x(\mathbb{R}^N;L^1_v(\mathbb{R}^N))$, hence $\overline{p}_0(k,v)\in L^2_k(\mathbb{R}^N;L^1_v(\mathbb{R}^N))$ for q.e. $k\in\mathbb{R}^N$.

Moreover, from (2.34), we obtain again

$$\begin{aligned}\left|1-\frac{\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{\lambda\theta(\epsilon)+\epsilon ik\cdot v+\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\right| &\le \theta(\epsilon)\frac{|\lambda|}{\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\\ &+\epsilon\frac{(k\cdot v)^2}{(\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v))^2}\\ &+\epsilon\frac{|k\cdot v|}{\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)},\end{aligned} \tag{2.42}$$

Therefore, we can apply the Lebesgue's dominated convergence theorem and we pass to the limit for $\epsilon\to 0$, obtaining

$$\int_{\mathbb{R}^N}\frac{\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}{\lambda\theta(\epsilon)-\epsilon ik\cdot v+\overline{\phi}^\epsilon(\lambda+i\epsilon\theta(\epsilon)^{-1}k\cdot v)}\overline{p}_0dv\to\int_{\mathbb{R}^N}\overline{p}_0dv=\overline{\rho}_0 \text{ for a.e. } k\in\mathbb{R}^N. \tag{2.43}$$

By the results (2.33), (2.36), (2.38), (2.41), passing to the limit in the equation (2.42) we obtain that $Q_0\left[\overline{p}^\epsilon\right](\lambda,k)$ is bounded in $L^\infty(b,\infty;L^1(\mathbb{R}^N))$ for any $b>0$. Therefore there exists $\overline{\psi}$ such that, up to a subsequence,

$$Q_0\left[\overline{p}^\epsilon\right](\lambda,k)\rightharpoonup\overline{\psi} \text{ for a.e. } \lambda\in\mathbb{R}^+ \text{ and } k\in\mathbb{R}^N.$$

Then $\overline{\psi}$, passing to the limit in (2.42), satisfies the following fractional diffusion equation

$$\lambda\overline{\psi}(\lambda,k)-\overline{\rho}_0+\frac{C_2}{\tau}|k|^\alpha\overline{\psi}(\lambda,k)=0$$

for a.e. $k\in\mathbb{R}^N$ and a.e. $\lambda\in\mathbb{R}^+$.

Moreover, we observe that, by (2.26) and by the definition of $Q_0\left[\overline{p}^\epsilon\right](\lambda,k)$ in (2.30), we have

$$Q_0\left[\overline{p}^\epsilon\right](\lambda,k)-\frac{1}{\tau}\int_{\mathbb{R}^N}\overline{p}^\epsilon(t,x,v)dv\to 0 \text{ as } \epsilon\to 0,$$

i.e. up to a constant $Q_0\left[\overline{p}^\epsilon\right](\lambda,k)$ is equal to $\overline{\rho}^\epsilon=\int_{\mathbb{R}^N}\overline{p}^\epsilon(t,x,v)dv$. Then we can conclude that

$$\overline{\rho}^\epsilon(t,x)\rightharpoonup\overline{\psi} \text{ for a.e. } \lambda\in\mathbb{R}^+ \text{ and } k\in\mathbb{R}^N.$$

Analogously, from the bound in (2.28), up to a subsequence, there exists $\psi \in L^\infty(0,\infty; L^2(\mathbb{R}^N))$ such that

$$\rho^\epsilon \rightharpoonup \psi(t,x)$$

for a.e. $x \in \mathbb{R}^N$ and $t \in \mathbb{R}^+$.

Calling with ρ the solution of the problem

$$\begin{cases} \partial_t \rho + C(-\Delta)^{\frac{\alpha}{2}} \rho = 0 \text{ in } (0,\infty) \times \mathbb{R}^N \\ \rho(0.\cdot) = \rho_0 \text{ in } \mathbb{R}^N \end{cases}$$

with $\alpha < 2$ and

$$C = \frac{C_1}{\tau} \int_{\mathbb{R}^N} \frac{(w \cdot e)^2}{(\overline{\phi}(iw \cdot e))^2 + (w \cdot e)^2} \frac{1}{|w|^{N+\alpha}} dw.$$

Then $\overline{\rho}$ satisfies

$$\lambda \overline{\rho}(\lambda, k) - \overline{\rho}_0 + \frac{C_2}{\tau} |k|^\alpha \overline{\rho}(\lambda, k) = 0$$

for a.e. $k \in \mathbb{R}^N$ and a.e. $\lambda \in \mathbb{R}^+$. By the uniqueness of the solution, we can conclude that $\overline{\rho} = \overline{\psi}$, i.e. $\rho = \psi$ for a.e. $t \in \mathbb{R}^+$.

Finally we observe that the costant C not depends by k because we have the scalar product between w and the normalized vector k (we could choose any direction indistinctly because the integral is simmetric). We stress that the constant depends strictly from the exponential contribution and the function saw in the previous section.

□

Chapter 3

The generalized logistic equation driven by the s-power of the Laplacian

We consider the problem

$$\begin{cases} (-\Delta)^s u = \lambda(\beta u - g(x,u)) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_λ^β}$$

where $N \geq 2$, $s \in (0,1)$, $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{2,\alpha}$ for some $0 < \alpha \leq 1$, $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory perturbation (i.e., for all $s \in \mathbb{R}$ the map $x \mapsto g(x,s)$ is measurable and for a.e. $x \in \Omega$ the map $x \mapsto g(x,s)$ is continuous). Concerning the function β, we assume that it is a sign changing measurable weight (for a first presentation of such classes of weights, we refer to [42]).

In view of the considerations of the previous section, (P_λ^β) has a variational structure, so that u solves (P_λ^β) if and only if it is a critical point for the functional $I : X_0 \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy - \lambda\int_\Omega F(x,u)\,dx, \tag{3.1}$$

where

$$F(x,u) = \int_0^u [\beta s - g(x,s)]ds.$$

Now, we set

$$\lambda^* = \inf\left\{\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dxdy : \, u \in X_0 \text{ with } \int_\Omega \beta u^2 = 1\right\}. \tag{3.2}$$

In order to ensure that λ^* is well defined, we impose the following condition on the weight function β:
$H(\beta)$: $\beta \in L^q(\Omega)$ with $q > \frac{N}{2s}$ and $\beta^+ = \max\{\beta, 0\} \neq 0$.

Remark 3.0.3. The assumption $\beta^+ = \max\{\beta, 0\} \neq 0$ implies that the set

$$\left\{ u \in X_0 \text{ with } \int_\Omega \beta u^2 = 1 \right\}$$

is not empty, so that λ^* is well defined.

Proposition 3.0.1. *If hypothesis $H(\beta)$ holds, then $\lambda^* > 0$.*

Proof. Let $u \in X_0$ such that $\int_\Omega \beta u^2 = 1$. Since $u \in X_0$, then $u \in L^{2^\sharp}(\Omega)$, and we note that

$$\frac{1}{\frac{N}{2s}} + \frac{1}{\frac{N}{N-2s}} = \frac{2s}{N} + \frac{N-2s}{N} = 1.$$

Then, using Hölder's inequality and Proposition 1.3.5, we get

$$\begin{aligned} 1 = \int_\Omega \beta u^2 dx &\leq \left\| \beta^+ \right\|_{L^{\frac{N}{2s}}(\Omega)} \|u\|^2_{L^{2^\sharp}(\Omega)} \leq |\Omega|^{\frac{2sq-N}{Nq}} \left\| \beta^+ \right\|_{L^q(\Omega)} \|u\|^2_{2^\sharp(\Omega)} \\ &\leq C_1 \left\| \beta^+ \right\|_{L^q(\Omega)} \|u\|^2_{X_0} \end{aligned}$$

for some universal constant $C_1 = C_1(n, s, q) > 0$.
Hence,

$$\inf_{\substack{u \in X_0 \\ \int \beta u^2 = 1}} \left(\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dxdy \right) = \lambda^* \geq \left(C_1 \left\| \beta^+ \right\|_{L^q(\Omega)} \right)^{-1} > 0.$$

$\square$

We introduce the following general maximum principles.

Proposition 3.0.2. *Suppose that $H(\beta)$ holds. Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and let $u \in C^1(\overline{\Omega})$ be a solution of*

$$\begin{cases} (-\Delta)^s u(x) = \beta(x) u(x) - g(x, u) & in \quad \Omega \\ u(x) \geq 0 & in \quad \Omega \\ u(x) = 0 & on \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $g(x, 0) = 0$. Then $u > 0$ in Ω or $u \equiv 0$ in Ω.

Proof. Suppose that exists $x_0 \in \Omega$ such that $u(x_0) = \min_{x\in\Omega} u(x)$. By definition of the fractional Laplacian, we have

$$(-\Delta)^s u(x_0) = \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x-y|^{N+2s}} dy \le 0.$$

In particular if $u(x_0) = 0$, since $g(x, u(x_0)) = g(x, 0) = 0$, we have that $u(x_0) = u(y)$ for all $y \in \mathbb{R}^N$ i.e. $u(x) \equiv 0$ for all $x \in \mathbb{R}^N$. □

Remark 3.0.4. In the case of weak solutions, the previous result still holds true by [31].

3.1 The Eigenvalue Problem

Next, let us consider the eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda^* \beta u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P_{λ^*}}$$

It is clear that there are no eigenvalues (weighted by β) for $(-\Delta)^s$ smaller than λ^*; we want to show that λ^* is indeed the principal eigenvalue of $(-\Delta)^s$.

We start with the following regularity result.

Proposition 3.1.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and let $v \in X_0$ be a weak solution of (P_{λ^*}). If $H(\beta)$ holds, then $v \in C^\alpha(\overline{\Omega})$.*

Proof. It is an adaptation of the classical Moser iteration technique to the nonlocal setting.

Let $v \in X_0$ be a solution of (P_{λ^*}). For $M > 0$ we define $v_M = \min\{v^+, M\}$ and thus $v_M \in X_0 \cap L^\infty(\mathbb{R}^N)$. For any nonnegative number k we set $\phi = v_M^{2k+1}$. By definition of weak solution with $\phi = v_M^{2k+1}$ as test function, we get

$$\int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))((v_M)^{2k+1}(x) - (v_M)^{2k+1}(y))}{|x-y|^{N+2s}} dxdy = \lambda \int_\Omega \beta v v_M^{2k+1} dx,$$

which implies that

$$\int_{\mathbb{R}^{2N}} \frac{(v_M(x)-v_M(y))((v_M)^{2k+1}(x) - (v_M)^{2k+1}(y))}{|x-y|^{N+2s}} dxdy \le \lambda \int_\Omega \beta v v_M^{2k+1} dx. \tag{3.3}$$

In fact,

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy \\
&= \int_{v(x)\leq 0}\int_{v(y)\leq 0} \frac{(v_M(x)-v_M(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy \\
&+ \int_{v(x)\leq 0}\int_{0<v(y)<M} \frac{(v(x)-v_M(y))((v_M)^{2k+1}(y)-(v_M)^{2k+1}(x))}{|x-y|^{N+2s}}dxdy \\
&+ \int_{0<v(x)<M}\int_{v(y)\leq 0} \frac{(v(x)-v_M(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy \\
&+ \int_{v(x)\leq 0}\int_{0<v(y)<M} \frac{(v_M(x)-v(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy \\
&+ \int_{v(x)\geq M}\int_{v(y)\geq M} \frac{(v_M(x)-v_M(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy.
\end{aligned}
\tag{3.4}
$$

The first and the last integrals are zero, so

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy \\
&= -\int_{v(x)\leq 0}\int_{0<v(y)<M} \frac{(v(x)-v_M(y))((v_M)^{2k+1}(y)-(v_M)^{2k+1}(x))}{|x-y|^{N+2s}}dxdy \\
&+ \int_{0<v(x)<M}\int_{v(y)\leq 0} \frac{(v(x)-v_M(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy \\
&\geq \int_{\mathbb{R}^{2N}} \frac{(v_M(x)-v_M(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y))}{|x-y|^{N+2s}}dxdy.
\end{aligned}
$$

Moreover, there exists $C_k > 0$ such that

$$
(v_M(x)-v_M(y))((v_M)^{2k+1}(x)-(v_M)^{2k+1}(y)) \geq C_k|(v_M)^{k+1}(x)-(v_M)^{k+1}(y)|^2. \tag{3.5}
$$

For this pourpose let $X = v(x)$ and $Y = v(y)$ we proof that

$$
(X-Y)(X^{2k+1}-Y^{2k+1}) \geq C_k|X^{k+1}-Y^{k+1}|^2
$$

or equivalently

$$
X^{2k+2}+Y^{2k+2}-XY^{2k+1}-X^{2k+1}Y-C_k\left(X^{2k+2}+Y^{2k+2}-2X^{k+1}Y^{k+1}\right) \geq 0.
$$

Let

$$
f(Z) = (1-C_k)Z^{2k+2}+(2C_k-1)Z^{k+1}-Z+1-C_k,
$$

then

$$f'(Z) = (2k+2)(1-C_k)Z^{2k+1} + (2C_k - 1)(k+1)Z^k - 1$$

and

$$f''(Z) = (2k+2)(2k+1)(1-C_k)Z^{2k} + (2C_k - 1)k(k+1)Z^{k-1}. \tag{3.6}$$

For $\frac{1}{2} < C_k < 1$ thus $f''(Z) > 0$, so there exists a unique Z_0 such that $f'(Z_0) = 0$, i.e.

$$2(k+1)(1-C_k)Z_0^{2k+1} + (2C_k - 1)(k+1)Z_0^k = 1$$

or, equivalently,

$$2(k+1)(1-C_k)Z_0^{2k+2} = Z_0 - (2C_k - 1)(k+1)Z_0^{k+1}. \tag{3.7}$$

Using (3.7) we have

$$\begin{aligned} f(Z_0) &= (1-C_k)Z_0^{2k+2} + (2C_k - 1)Z_0^{k+1} - Z_0 + 1 - C_k \\ &= (1-C_k)Z_0^{2k+2} + Z_0 - 2(k+1)(1-C_k)Z_0^{2k+2} + 1 - C_k - Z_0 \\ &= Z_0^{2k+2}(1-C_k)(-2k-1) + 1 - C_k = (1-C_k)\left[Z_0^{2k+2}(-2k-1) + 1\right]. \end{aligned}$$

It remains to prove that

$$f(Z_0) = (1-C_k)\left[Z_0^{2k+2}(-2k-1) + 1\right] > 0$$

but, passing to the limit in (3.7),

$$(2C_k - 1)(k+1)Z_0^{k+1} - Z_0 + 2(k+1)(1-C_k)Z_0^{2k+2} = 0 \to (k+1)Z_0^{k+1} - Z_0 = 0 \text{ when } C_k \to 1$$

we get

$$Z_0(1) = \frac{1}{(k+1)^k}.$$

Evaluating f in $Z_0(1)$

$$\frac{f(Z_0(1))}{1-C_k} = (-2k-1)\left(\frac{1}{(k+1)^{\frac{2(k+1)}{k}}}\right) + 1.$$

We observe that the minimum of f is positive, infact

$$(k+1)^{\frac{2(k+1)}{k}} > k + \frac{1}{2}.$$

Then, from (3.5), we get

$$\begin{aligned} &C_k \int_{\mathbb{R}^{2N}} \frac{|(v_M)^{k+1}(x) - (v_M)^{k+1}(y)|^2}{|x-y|^{N+2s}} dxdy \\ &\le \int_{\mathbb{R}^{2N}} \frac{(v_M(x) - v_M(y))((v_M)^{2k+1}(x) - (v_M)^{2k+1}(y))}{|x-y|^{N+2s}} dxdy. \end{aligned} \tag{3.8}$$

In addition, by applying the Hölder inequality to the right-hand side of (3.3), we obtain

$$\lambda \int_\Omega \beta v v_M^{2k+1} dx \le \lambda \left\|\beta^+\right\|_{L^q(\Omega)} \left\|(v^+)^{k+1}\right\|^2_{L^{2q'}(\Omega)} \tag{3.9}$$

By applying (3.3),(3.8) and (3.9) one gets

$$C_k \int_{\mathbb{R}^{2N}} \frac{|v_M(x)^{k+1} - v_M(y)^{k+1}|^2}{|x-y|^{N+2s}} dxdy \le \lambda \left\|\beta^+\right\|_q \left\|(v^+)^{k+1}\right\|^2_{2q'}. \tag{3.10}$$

Since $\lim_{M\to\infty} v_M(x) = v^+(x)$ for a.e. $x \in \mathbb{R}^N$, applying Fatou's Lemma, we get

$$\left\|(v^+)^{k+1}\right\|_{X_0} \le \sqrt{\frac{\lambda}{C_k}} \left\|\beta^+\right\|_q^{1/2} \left\|(v^+)^{k+1}\right\|_{2q'}. \tag{3.11}$$

By the continuous embedding of X_0 in $L^{2^\sharp}(\mathbb{R}^N)$, we get the existence of $S > 0$ such that

$$\left\|(v^+)^{k+1}\right\|_{L^{2^\sharp}(\Omega)} \le S\sqrt{\frac{\lambda}{C_k}} \left\|\beta^+\right\|_q^{1/2} \left\|(v^+)^{k+1}\right\|_{L^{2q'}(\Omega)}. \tag{3.12}$$

Since $q > N$, then $2q' < 2^\sharp$. Now, define $k_0 = 0$ and by recurrence

$$2(k_n + 1)q' = (k_{n-1} + 1)2^\sharp.$$

It is easy see that $k_i > k_{i-1}$ and that $k_n \to \infty$ as $n \to \infty$. Moreover, (3.12) implies that

$$\text{if } (v^+) \in L^{2(k_{i-1}+1)q'}(\Omega) \text{ then } (v^+) \in L^{(k_{i-1}+1)2^\sharp}(\Omega) = L^{2(k_i+1)q'}(\Omega),$$

and hence $(v^+) \in L^r(\Omega)$ for every $r \ge 1$. Moreover, by (3.11) we obtain that

$$\int_{\mathbb{R}^{2N}} \frac{|(v^+)^{k_i+1}(x) - (v^+)^{k_i+1}(y)|^2}{|x-y|^{N+2s}} \le C_i \text{ for every } i \ge 0,$$

or equivalently $(v^+)^{k_i+1} \in X_0$ for every $i \ge 0$.

Analogously, choosing $v_M := \min\{M, v^-\}$, we can prove that $v^- \in L^r(\Omega)$ for every $r \geq 1$ and $(v^-)^{k_i+1} \in X_0$ for every $i \geq 0$.

Now, since $v \in L^r(\Omega)$ for all $r < \infty$, then $\beta v \in L^{\overline{q}}(\Omega)$ with $q > \overline{q} > N/2s$. We rewrite the equation in (P_{λ^*}) as

$$-\Delta\big[(-\Delta)^{s-1}v\big] = \lambda^* \beta v.$$

In this way, by the Calderón-Zygmund theory, we obtain that $(-\Delta)^{s-1}v = h \in W^{2,\overline{q}}(\Omega)$. Thus, $h \in C^{1,\alpha}(\overline{\Omega})$, and so

$$v = (-\Delta)^{1-s}h \in C^{0,\tilde{\alpha}}(\overline{\Omega}),$$

see [22, Theorem 3.13]. □

Now, let us consider

$$C_+ = \left\{u \in C^\alpha(\overline{\Omega}) \;:\; u(x) \geq 0 \;\; \forall x \in \overline{\Omega}\right\},$$

whose interior set is

$$\operatorname{int} C_+ = \left\{u \in C^\alpha(\overline{\Omega}) \;:\; u(x) > 0 \;\; \forall x \in \overline{\Omega}\right\}.$$

Let e_1 be a solution of problem (P_λ^*). We first prove that e_1 does exist. In fact, we have

Proposition 3.1.2. *If $H(\beta)$ holds, then there exists $e_1 \in X_0 \cap \operatorname{int} C_+$ which solves (P_{λ^*}), that is, e_1 is the first eigenfunction for $(-\Delta)^s$ with weight β. Moreover, e_1 is simple.*

Proof. Let us consider the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy$$

constrained on the set

$$M = \left\{u \in X_0 \,\middle|\, \int_\Omega \beta u^2 dx = 1\right\},$$

and note that J is sequentially weakly lower semicontinuous and coercive; so by the Weierstrass Theorem we can find $e_1 \in X_0$ such that

$$J(e_1) = \inf\left\{J(u) |\, u \in X_0\right\}.$$

Thus, there exists $\lambda^* \in \mathbb{R}$ such that $J'(e_1)v = \lambda^* \int_\Omega \beta e_1 v dx$ for every $v \in X_0$, that is e_1 solves the problem

$$\begin{cases} (\Delta)^s e_1 = \lambda^* \beta e_1 & \text{in } \Omega, \\ e_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover, it is clear that we can assume $e_1 \geq 0$ in Ω, and by Proposition 3.1.1 we get that $e_1 \in C_+$.

Moreover, by Remark 3.0.4, $e_1 \in \operatorname{int} C^+$. We proved that, each eigenfunctions corresponding to the first eigenvalue, are strictly positive in Ω.

Furthermore, we can check how this eigenfunctions are also simple. Let $u \in X_0$ be another eigenfunction associated to λ^*. Then, from the previous discussion, we can say that $u \in \operatorname{int} C^+$. The functional $J(u)$ is convex (see the Hidden convexity in [10]) hence, from Theorem 4.2 in [24], we have that $u = Ce_1$ for some $C \in \mathbb{R}$.

□

Remark 3.1.1. We shall also use the linear map $A : X_0 \to X_0^*$ defined by

$$\langle A(u), v\rangle_{X_0^*, X_0} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy \quad \text{for all } u, v \in X_0.$$

This map is such that, if $u_n \rightharpoonup u$ in X_0 and $\lim_{n\to\infty} \langle A(u), v\rangle_{X_0^*, X_0} = 0$ then $u_n \to u$ in X_0, as it is easy to check.

In general no other properties can be deduced for e_1 if no other hypotheses on β are given. However, we can recover the classical features of the first eigenvalue with additional assumptions. In particular, we have the following regularity result:

Proposition 3.1.3. *If $\beta \in C^1(\Omega)$, then $e_1 \in C^{2,\alpha}(\overline{\Omega})$.*

3.2 The Existence Theorem

On the absorption term g we assume that

$$g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is a Carathéodory function such that } g(x, 0) = 0 \text{ and } g(x, s)s > 0 \text{ for a.e. } x \in \Omega \text{ and for all } s \neq 0. \tag{3.13}$$

Moreover:
there exists $g_0 \in L^\infty(\Omega)$ such that:

$$\liminf_{|s|\to\infty} \frac{g(x, s)}{s} \geq g_0 \quad \text{uniformly for a.e. } x \in \Omega; \tag{3.14}$$

$$\operatorname*{ess\,inf}_{\Omega}(g_0 - \beta) := \mu > 0; \tag{3.15}$$

$$\lim_{|s|\to 0} \frac{g(x,s)}{s} = 0 \quad \text{uniformly for a.e. } x \in \Omega; \tag{3.16}$$

for every $u \in L^\infty(\Omega)$ there exists $\rho = \rho(u) \geq 0$ such that

$$|g(x, u(x))| \leq \rho \text{ for a.e. } x \in \Omega; \tag{3.17}$$

for a.e. $x \in \Omega$ the map

$$u \mapsto \frac{g(x,u)}{u} \text{ is strictly increasing in } (0, \infty). \tag{3.18}$$

Remark 3.2.1. Hypothesis (3.14) does not exclude the classical case in logistic equations, that is

$$\lim_{s\to\infty} \frac{g(x,s)}{s} = \infty,$$

as it happens, for example, if $g(x,s) = |s|^{p-2}s$, $p > 2$.

Remark 3.2.2. Hypothesis (3.17) is very general and excludes any a priori bound on the growth of the function at infinity, as it is usual in variational problems of this type.

We shall prove the following result.

Proposition 3.2.1. *If hypotheses $H(\beta)$,* (3.13)*,* (3.14)*,* (3.15)*,* (3.16) *and* (3.17) *hold, then for all $\lambda > \lambda^*$ there exists a solution $u \in \mathcal{C}^\alpha(\bar{\Omega})$ of problem (P^β_λ) such that $u(x) > 0$ for all $x \in \Omega$.*

Proof. In order to get the goal we will truncate the functional between 0 and a suitable level ψ. First, we will find a critical point for the truncated functional, then we will show that this critical point is below ψ, and so it will be solution for the original problem (P^β_λ). Let us start with the proof.

By virtue of hypothesis (3.14), for all $\epsilon \in (0, \mu)$ there exists $M = M(\epsilon) > 0$ such that for all $s > M$

$$g(x,s) \geq (g_0(x) - \epsilon)s \quad \text{for a.e.} \quad x \in \Omega. \tag{3.19}$$

Now, fix $\psi > M$ and consider the following truncation for the reaction term:

$$h(x,s) = \begin{cases} 0 & \text{if} \quad s \leq 0 \\ \beta(x)s - g(x,s) & \text{if} \quad 0 \leq s \leq \psi \\ \beta(x)\psi - g(x,\psi) & \text{if} \quad s \geq \psi. \end{cases} \tag{3.20}$$

Of course, h is still a Carathéodory function. Setting $H(x,s) = \int_0^s h(x,s)ds$, by $H(g)$ the functional $I_\psi : X_0 \longrightarrow \mathbb{R}$ defined by

$$I_\psi(v) = \frac{1}{2}\|v\|_{X_0} - \lambda \int_\Omega H(x, v(x))\, dx$$

is of class C^1 and sequentially weakly lower semicontinuous. Moreover, by the very definition of the truncation $h(x,s)$, we have that I_ψ is coercive. Therefore, by the Weierstrass Theorem we can find $\overline{v} \in X_0$ such that

$$I_\psi(\overline{v}) = \inf\Big\{ I_\psi(v) \,|\, v \in X_0 \Big\} := m \tag{3.21}$$

and m is a critical value for I_ψ.

Now we must check that $\overline{v} \neq 0$. By virtue of hypothesis (3.16), in correspondence of the previous ϵ there exists $\delta = \delta(\epsilon) > 0$ with $\delta < \min\{M, \psi\}$ such that

$$g(x,s) \le \epsilon s \quad \forall s \in [0,\delta] \text{ and a.e. } x \in \Omega,$$

so that

$$G(x,s) = \int_0^s g(x,s)\, ds \le \epsilon \frac{s^2}{2} \quad \text{for all } s \in [0,\delta] \text{ and a.e. } x \in \Omega. \tag{3.22}$$

By Proposition 3.1.2, we can find $t \in (0,1)$ such that $te_1 \in [0,\delta]$ for all $x \in \Omega$. Then, by (3.22),

$$\begin{aligned} I_\psi(te_1) &= \frac{t^2}{2}\int_{\mathbb{R}^{2N}} \frac{(e_1(x) - e_1(y))^2}{|x-y|^{N+2s}} dxdy - \lambda \int_\Omega H(x, te_1)dx \\ &= \frac{t^2}{2}\|e_1\|^2_{X_0} - \lambda\frac{t^2}{2}\int_\Omega \beta e_1^2 dx + \lambda \int_\Omega G(x, te_1)\, dx \\ &\le \frac{t^2}{2}(\lambda^* - \lambda)\int_\Omega \beta e_1^2 dx + \frac{\lambda}{2}\epsilon t^2 \|e_1\|_2^2 . \end{aligned}$$

Recalling that $\int_\Omega \beta e_1^2 dx = 1$ and $\lambda^* < \lambda$, choosing $\epsilon > 0$ sufficiently small, then we have $I_\psi(te_1) < 0$ and by (3.21) we find that

$$m = I_\psi(\overline{v}) < I_\psi(0) = 0, \tag{3.23}$$

and so $\overline{v} \neq 0$, as claimed.

Since $\overline{v}$ is a critical point for I_ψ, then $I'_\psi(\overline{v}) = 0$, that is

$$A(\overline{v}) = \lambda h(x, \overline{v}), \tag{3.24}$$

see Remark 3.1.1.

Now, we act on (3.24) with $-(\overline{v})^- \in X_0$, and recalling (3.20), we obtain

$$
\begin{aligned}
\langle A(\overline{v}), -(\overline{v})^-\rangle &= -\int_{\mathbb{R}^{2N}} \frac{(\overline{v}(x)-\overline{v}(y))((\overline{v})^-(x)-(\overline{v})^-(y))}{|x-y|^{N+2s}}dxdy \\
&= -\int_{\{\overline{v}(x)<0\}}\int_{\{\overline{v}(y)<0\}} \frac{(\overline{v}(x)-\overline{v}(y))((\overline{v})(x)-(\overline{v})(y))}{|x-y|^{N+2s}}dxdy \\
&- \int_{\{\overline{v}(x)>0\}}\int_{\{\overline{v}(y)>0\}} \frac{(\overline{v}(x)-\overline{v}(y))((\overline{v})^-(x)-(\overline{v})^-(y))}{|x-y|^{N+2s}}dxdy \\
&- \int_{\{\overline{v}(x)<0\}}\int_{\{\overline{v}(y)>0\}} \frac{(\overline{v}(x)-\overline{v}(y))\overline{v}(y)}{|x-y|^{N+2s}}dxdy \\
&- \int_{\{\overline{v}(x)>0\}}\int_{\{\overline{v}(y)<0\}} \frac{(\overline{v}(x)-\overline{v}(y))\overline{v}(x)}{|x-y|^{N+2s}}dxdy = 0.
\end{aligned}
$$

We observe that, by definition of v^-,

$$
-\int_{\{\overline{v}(x)>0\}}\int_{\{\overline{v}(y)>0\}} \frac{(\overline{v}(x)-\overline{v}(y))(\overline{v})^-(x)-(\overline{v})^-(y)}{|x-y|^{N+2s}}dxdy = 0
$$

while the other integrals are nonnegative, so that $\overline{v}^- = 0$, that is $\overline{v} \geq 0$.

Next, we act on (3.24) with $(\bar{v}-\psi)^+ \in X_0$ and by definition of h we find

$$
\begin{aligned}
\langle A(\overline{v}), (\overline{v}-\psi)^+\rangle =& \lambda \int_\Omega h(x,\overline{v})(\overline{v}-\psi)^+\, dx \\
=& \int_\Omega (\beta\psi - g(x,\psi))(\overline{v}-\psi)^+\, dx.
\end{aligned} \tag{3.25}
$$

First, let us prove that

$$
\int_\Omega (\beta\psi - g(x,\psi))(\overline{v}-\psi)^+\, dx \leq 0.
$$

Indeed, recalling that $\psi > M$ and $\epsilon < \mu$, using (3.19), by (3.15) we have

$$
\beta\psi - g(x,\psi) \leq \beta\psi - g_0(x)\psi + \epsilon\psi \ \leq (\epsilon-\mu)\psi < 0.
$$

Then, by (3.25)

$$
\begin{aligned}
0 =& \langle A(\overline{v}), (\overline{v}-\psi)^+\rangle \\
=& \int_{\mathbb{R}^{2N}} \frac{(\overline{v}(x)-\overline{v}(y))((\overline{v}-\psi)^+(x)-(\overline{v}-\psi)^+(y))}{|x-y|^{N+2s}}dxdy \\
=& \int_{\{\overline{v}(x)>\psi\}}\int_{\{\overline{v}(y)\leq\psi\}} \frac{(\overline{v}(x)-\overline{v}(y))(\overline{v}(x)-\psi)}{|x-y|^{N+2s}}dxdy \\
-& \int_{\{\overline{v}(x)\leq\psi\}}\int_{\{\overline{v}(y)>\psi\}} \frac{(\overline{v}(x)-\overline{v}(y))(\overline{v}(y)-\psi)}{|x-y|^{N+2s}}dxdy \\
+& \int_{\{\overline{v}(x)>\psi\}}\int_{\{\overline{v}(y)>\psi\}} \frac{(\overline{v}(x)-\overline{v}(y))(\overline{v}(x)-\overline{v}(y))}{|x-y|^{N+2s}}dxdy.
\end{aligned}
$$

Since the first and second integrals are nonnegative, we get

$$\int_{\{\bar{v}(x)>\psi\}}\int_{\{\bar{v}(y)>\psi\}} \frac{(\bar{v}(x)-\bar{v}(y))(\bar{v}(x)-\bar{v}(y))}{|x-y|^{N+2s}}dxdy \leq 0.$$

Then, $|\{\bar{v} > \psi\}| = 0$ or $\bar{v}$ is costant in $\{\bar{v} > \psi\}$. In any case, $\bar{v}$ is bounded. Hence, by (3.17), $g(x, \bar{v})$ is bounded and so we have that $\beta\bar{v} - g(x, \bar{v}) \in L^{\bar{q}}(\Omega)$ with $q > \bar{q} > N/2s$.

By the Calderon Zygmund theory, $(-\Delta)^{s-1}\bar{v} \in W^{2,\bar{q}}$ so $(-\Delta)^{s-1}\bar{v} \in C^{1,\sigma_0}$ for some $\sigma_0 > 0$.

We recall that

$$(-\Delta)^{1-s} : C^{1,\sigma_0} \hookrightarrow C^{0,\alpha},$$

from Theorem 3.13 in [22], and so $\bar{v} \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in [0, 1)$. □

Remark 3.2.3. Similarly, fix $\phi < -M$ and consider the following truncation for the reaction term:

$$k(x,s) = \begin{cases} \beta(x)\phi - g(x,\phi) & \text{if} \quad s \leq \phi \\ \beta(x)s - g(x,s) & \text{if} \quad \phi \leq s \leq 0 \\ 0 & \text{if} \quad s \geq 0. \end{cases} \tag{3.26}$$

Of course, k is still a Carathéodory function. Setting $K(x,s) = \int_0^s k(x,s)ds$.

By the hypothesis (3.13) on $g(x, u)$, we can introduce a new functional $I_\phi : X_0 \longrightarrow \mathbb{R}$ defined by

$$I_\phi(v) = \frac{1}{2}\|v\|_{X_0} - \lambda \int_\Omega K(x, v(x))\, dx$$

of class C^1, sequentially weakly lower semicontinuous and, like I_ψ and coercive by definition of truncated functional. Therefore, by the Weierstrass Theorem we can find $\underline{v} \in X_0$ such that

$$I_\phi(\underline{v}) = \inf\left\{I_\phi(v) \,|\, v \in X_0\right\} = q, \tag{3.27}$$

that is q is a critical value for I_ϕ. The proof is analogous to the Proposition 3.2.1 but, in this way, we will find a negative solution, that is

$$\underline{v} \in -\mathrm{int}C^+ = \left\{u \in C^\alpha(\overline{\Omega}) \;:\; u(x) < 0 \quad \text{for all } x \in \overline{\Omega}\right\}.$$

As usual, in logistic–type problems, we expect positive solutions to be unique. Indeed, we have

Proposition 3.2.2. *If hypotheses* $H(\beta)$, (3.17) *and* (3.18) *hold, then problem* (P_λ^β) *admits a unique nontrivial nonnegative solution.*

Proof. We consider two nontrivial nonnegative solutions $u, v \in C^\alpha(\overline{\Omega})$ of problem P_λ^β and we test with $\frac{u^2-v^2}{u+\epsilon}$ and $\frac{v^2-u^2}{v+\epsilon}$. Then we find

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))}{|x-y|^{N+2s}}\left(\frac{u^2(x)-v^2(x)}{u(x)+\epsilon}-\frac{u^2(y)-v^2(y)}{u(y)+\epsilon}\right)dxdy\\&=\lambda\int_\Omega\beta u\frac{u^2-v^2}{u+\epsilon}dx-\lambda\int_\Omega g(x,u)\frac{u^2-v^2}{u+\epsilon}dx\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v(x)-v(y)}{|x-y|^{N+2s}}\left(\frac{v^2(x)-u^2(x)}{v(x)+\epsilon}-\frac{v^2(y)-u^2(y)}{v(y)+\epsilon}\right)dxdy\\&=\lambda\int_\Omega\beta v\frac{v^2-u^2}{v+\epsilon}dx-\lambda\int_\Omega g(x,v)\frac{v^2-u^2}{v+\epsilon}dx.\end{aligned}\tag{3.29}$$

Now we add (3.28) and (3.29), obtaining

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}(u(x)-u(y))\left(\frac{u^2(x)-v^2(x)}{u(x)+\epsilon}-\frac{u^2(y)-v^2(y)}{u(y)+\epsilon}\right)dxdy\\&+(v(x)-v(y))\left(\frac{v^2(x)-u^2(x)}{v(x)+\epsilon}-\frac{v^2(y)-u^2(y)}{v(y)+\epsilon}\right)dxdy\\&-\lambda\int_\Omega\left[\frac{u^3}{u+\epsilon}-\frac{v^2u}{u+\epsilon}+\frac{v^3}{v+\epsilon}-\frac{u^2v}{v+\epsilon}\right]dx+\lambda\int_\Omega g(x,u)\frac{u^2-v^2}{u+\epsilon}dx\\&+\lambda\int_\Omega g(x,v)\frac{v^2-u^2}{v+\epsilon}dx=0.\end{aligned}\tag{3.30}$$

Taking into account that

$$\left|\frac{u^3}{u+\epsilon}-\frac{v^2u}{u+\epsilon}+\frac{v^3}{v+\epsilon}-\frac{u^2v}{v+\epsilon}\right|\le 2||u||_\infty^2+2||v||_\infty^2\in L^1(\Omega),$$

by the Lebesgue theorem we can pass to limit for $\epsilon\to 0$ and we find that

$$\int_\Omega\left[\frac{u^3}{u+\epsilon}-\frac{v^2u}{u+\epsilon}+\frac{v^3}{v+\epsilon}-\frac{u^2v}{v+\epsilon}\right]dx\to 0.$$

In the same way, by (3.17),

$$\left|\left(\frac{g(x,u)}{u+\epsilon}-\frac{g(x,v)}{v+\epsilon}\right)(u^2-v^2)\right|\le\left(\frac{\rho(||u||_\infty)}{||u||_\infty}+\frac{\rho(||v||_\infty)}{||v||_\infty}\right)|u^2-v^2|\in L^1(\Omega),$$

then we apply the Lebesgue theorem and we immediately find that

$$\int_\Omega \left[\frac{g(x,u)}{u+\epsilon} - \frac{g(x,v)}{v+\epsilon}\right](u^2-v^2)dx \to \int_\Omega \left[\frac{g(x,u)}{u} - \frac{g(x,v)}{v}\right](u^2-v^2)dx$$

as $\epsilon \to 0$.

Now, look at the first two integrals in (3.30)

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} (u(x)-u(y))\left(\frac{u^2(x)-v^2(x)}{u(x)+\epsilon} - \frac{u^2(y)-v^2(y)}{u(y)+\epsilon}\right)dxdy \\
&+ \int_{\mathbb{R}^{2N}} (v(x)-v(y))\left(\frac{v^2(x)-u^2(x)}{v(x)+\epsilon} - \frac{v^2(y)-u^2(y)}{v(y)+\epsilon}\right)dxdy \\
&= \int_{\mathbb{R}^{2N}} \frac{u^3(x)-v^2(x)u(x)-u(y)u^2(x)+u(y)v^2(x)}{u(x)+\epsilon}dxdy \\
&- \int_{\mathbb{R}^{2N}} \frac{u(x)u^2(y)-v^2(y)u(x)-u^3(y)+v^2(y)u(y)}{u(y)+\epsilon}dxdy \\
&+ \int_{\mathbb{R}^{2N}} \frac{v^3(x)-u^2(x)v(x)-v(y)v^2(x)+v(y)u^2(x)}{v(x)+\epsilon}dxdy \\
&- \int_{\mathbb{R}^{2N}} \frac{v(x)v^2(y)-u^2(y)v(x)-v^3(y)+u^2(y)v(y)}{v(y)+\epsilon}dxdy \\
&= \int_{\mathbb{R}^{2N}} Adxdy - \int_{\mathbb{R}^{2N}} Bdxdy + \int_{\mathbb{R}^{2N}} Cdxdy - \int_{\mathbb{R}^{2N}} Ddxdy.
\end{aligned} \tag{3.31}$$

Look at the first integrand A; for almost every $x \in \Omega$,

$$\frac{u^3(x)-v^2(x)u(x)-u(y)u^2(x)+u(y)v^2(x)}{u(x)+\epsilon} \geq -v^2(x)-u(y)u(x) \in L^1(\mathbb{R}^{2N}). \tag{3.32}$$

For the integrand B, instead, we find a upper bound, in order to find a lower bound for $-B$

$$\frac{u(x)u^2(y)-v^2(y)u(x)-u^3(y)+v^2(y)u(y)}{u(y)+\epsilon} \leq u(x)u(y)+v^2(y) \in L^1(\mathbb{R}^{2N}). \tag{3.33}$$

For the other integrands C and D we have analogous bounds.

From (3.31), using the bounds (3.32) and (3.33) we can apply Fatou's Lemma in (3.30),

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \liminf_{\epsilon\to 0}\,[A-B+C-D]\,dxdy + \liminf_{\epsilon\to 0}\int_\Omega \left[\frac{g(x,u)}{u+\epsilon} - \frac{g(x,v)}{v+\epsilon}\right](u^2-v^2)dx \\
&\leq \liminf_{\epsilon\to 0}\int_{\mathbb{R}^{2N}} [A-B+C-D]\,dxdy + \liminf_{\epsilon\to 0}\int_\Omega \left[\frac{g(x,u)}{u+\epsilon} - \frac{g(x,v)}{v+\epsilon}\right](u^2-v^2)dx = 0.
\end{aligned}$$

More explicitly,

$$\int_{\mathbb{R}^{2N}} \left[\frac{u(y)v^2(x)}{u(x)} - 2u(x)u(y) + \frac{v^2(y)u(x)}{u(y)} + \frac{u^2(x)v(y)}{v(x)} - 2v(y)v(x) + \frac{v(x)u^2(y)}{v(y)} \right] dxdy$$
$$+ \lambda \int_\Omega \left[\frac{g(x,u)}{u} - \frac{g(x,v)}{v} \right] (u^2 - v^2) dx \le 0. \tag{3.34}$$

In particular

$$\int_{\mathbb{R}^{2N}} \left[\frac{u(y)v^2(x)}{u(x)} - 2u(x)u(y) + \frac{v^2(y)u(x)}{u(y)} + \frac{u^2(x)v(y)}{v(x)} - 2v(y)v(x) + \frac{v(x)u^2(y)}{v(y)} \right] dxdy$$
$$= \int_{\mathbb{R}^{2N}} \left[u(x)\sqrt{\frac{v(y)}{v(x)}} - u(y)\sqrt{\frac{v(x)}{v(y)}} \right]^2 + \left[v(x)\sqrt{\frac{u(y)}{u(x)}} - v(y)\sqrt{\frac{u(x)}{u(y)}} \right]^2 dxdy \tag{3.35}$$

Therefore, we find that

$$\int_{\mathbb{R}^{2N}} \left[u(x)\sqrt{\frac{v(y)}{v(x)}} - u(y)\sqrt{\frac{v(x)}{v(y)}} \right]^2 + \left[v(x)\sqrt{\frac{u(y)}{u(x)}} - v(y)\sqrt{\frac{u(x)}{u(y)}} \right]^2 dxdy$$
$$+ \lambda \int_\Omega \left[\left(\frac{g(x,u)}{u} - \frac{g(x,v)}{v} \right) (u^2 - v^2) \right] dx \le 0. \tag{3.36}$$

Since both integrands are nonnegative by (3.18), we immediately get that $u = v$. $\square$

Next we prove that:

Proposition 3.2.3. *If hypotheses $H(\beta)$ and (3.13) hold, problem (P_λ^β) has no nonnegative nontrivial solutions when $\lambda \in (0, \lambda^*]$.*

Proof. Suppose that there exists a nonnegative solution $v_0 \neq 0$ for (P_λ^β), that is

$$\int_{\mathbb{R}^{2N}} \frac{(v_0(x) - v_0(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy = \lambda \int_\Omega [\beta v_0 - g(x, v_0)] v dx \quad \forall\, v \in X_0.$$

In particular

$$0 < \int_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^2}{|x-y|^{N+2s}} dxdy = \lambda \int_\Omega [\beta v_0^2 - g(x, v_0) v_0] dx < \lambda \int_\Omega \beta v_0^2 dx, \tag{3.37}$$

since $g(x, v_0) > 0$. By definition of λ^*, we have

$$\lambda^* \int_\Omega \beta v_0^2 \leq \int_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^2}{|x-y|^{N+2s}}\, dxdy,$$

and comparing with (3.37), we reach a contradiction, since $\lambda \leq \lambda^*$. □

In conclusion, we can state the following theorem:

Theorem 3.2.1. *If hypotheses $H(\beta)$, (3.13), (3.14), (3.15), (3.16), (3.17), (3.18) hold, then there exists $\lambda^* > 0$ such that*

1. *for all $\lambda > \lambda^*$ problem (P_λ^β) has a unique nonnegative solution $u \in C^\alpha(\overline{\Omega})$ such that $u > 0$ in Ω.*
2. *for all $\lambda \leq \lambda^*$ problem (P_λ^β) has no positive solutions.*

Finally, we concentrate on the behaviour of solutions near λ^*, i.e. we investigate

$$\lim_{\lambda \downarrow \lambda^{*+}} u(\lambda).$$

For this purpose,the general assumptions made so far are not enough, and some growth condition is necessary. In particular, we suppose that there exist $p \in (2, 2^\sharp)$ and $C_0 > 0$ such that

$$g(x, s) \geq C_0 |s|^{p-1} \text{ for a.e } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{3.38}$$

On the other hand, such an assumption is enough to prove a bifurcation–type result. Indeed, whenever solutions exist, we have the following result:

Proposition 3.2.4. *Assume that (3.13) holds and that $\beta^+ \in L^{\frac{p}{p-2}}(\Omega)$ with $\beta^+ \neq 0$. If hypothesis (3.38) holds, then every solution u_λ of problem (P_λ^β) converges to 0 in X_0 when $\lambda \downarrow \lambda^*$.*

Proof. Let $(\lambda_n)_n$ be a sequence which strictly decreases to λ^*, and let v_n be a corresponding solution of problem $(P_{\lambda_n}^\beta)$ (notice that under these assumptions no uniqueness is guaranteed).

Then

$$\int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}} dxdy = \lambda_n \int_\Omega [\beta v_n - g(x, v_n)] v_n\, dx \quad \forall\, n \in \mathbb{N}. \tag{3.39}$$

By the continuous inclusion of X_0 in $L^p(\Omega)$, by the Hölder inequality and (3.38), we have

$$\begin{aligned} C||v_n||_p^2 \leq \int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}} dxdy =& \lambda_n \int_\Omega \beta |v_n|^2\, dx - \lambda_n \int_\Omega g(x, v_n) v_n dx \\ \leq& \lambda_n (||\beta^+||_{(\frac{p}{2})'} ||v_n||_p^2 - C_0 ||v_n||_p^p). \end{aligned}$$

Being $p > 2$, by the integrability condition on β, we find that the sequence $(v_n)_n$ is bounded in $L^p(\Omega)$, and therefore in $L^2(\Omega)$. As a consequence, by (3.39), $(v_n)_n$ is bounded in X_0.

Thus, up to a subsequence, we have that

$$v_n \rightharpoonup v^* \text{ in } X_0 \text{ and } v_n \to v^* \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{3.40}$$

Using the definition of solution for v_n with $v_n - v^* \in X_0$ as test function, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))[(v_n - v^*)(x) - (v_n - v^*)(y)]}{|x-y|^{N+2s}} dxdy \\
=&\int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(v_n(x) - v_n(y))}{|x-y|^{N+2s}} dxdy - \int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(v^*(x) - v^*(y))}{|x-y|^{N+2s}} dxdy \\
=&\lambda_n \int_\Omega \beta v_n (v_n - v^*)\, dx - \lambda_n \int_\Omega g(x, v_n) v_n (v_n - v^*)\, dx.
\end{aligned}$$

Now, the last two integrals converge to 0 as $n \to \infty$ by (3.40), while, by the weak convergence in X_0,

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(v^*(x) - v^*(y))}{|x-y|^{N+2s}} dxdy = \|v^*\|^2.$$

In conclusion,

$$\lim_{n\to\infty} \|v_n\|^2 = \|v^*\|^2,$$

that is

$$v_n \to v^* \text{ in } X_0.$$

By Proposition 3.2.3 we conclude that $v^* = 0$. □

In the light of the previous results, we emphasize the fact that if g and β satisfy all the assumptions above, we obtain a complete bifurcation result for problem (P_λ^β), that we summarize in the following final

Theorem 3.2.2. *Assume that $\beta \in L^{\frac{p}{p-2}}(\Omega)$ with $\beta^+ \neq 0$. Moreover, assume $H(g)$ and* (3.13)–(3.18) *and* (3.38). *Then the conclusions of Theorem* 3.2.1 *and Proposition* 3.2.4 *hold true.*

3.3 Multiplicity theorem

3.3.1 Spectral properties for the eigenvalue problem

In this first part we give some spectral properties for the eigenvalue problem (P_λ), which will be useful to proof the multiplicity theorem we have in mind.

Let

$$\begin{cases} (-\Delta)^s u = \lambda \beta u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P_λ}$$

Then, by definition, λ^* is the principle eigenvalue of $(-\Delta)^s$ with weight β, with associated positive eigenfunction e_1, see Proposition 3.1.2. We now show a result similar to the usual elliptic cases, but which is also a useful characterization of eigenfunctions for $(-\Delta)^s$ with weight β.

Proposition 3.3.1. *If $H(\beta)$ holds, then every eigenvalue $\lambda \neq \lambda^*$ of problem (P_{λ^*}) has nodal (i.e. sign changing) eigenfunctions.*

Proof. Let $\lambda > \lambda^*$ be an eigenvalue and $w \in X_0$ be an eigenfunction associated to λ, and suppose that w is not nodal. Without loss of generality, using the maximum principle in Proposition 3.0.2, section 3.1, we may assume that w is strictly positive in Ω. Moreover, we take w normalized in such a way that

$$\int_\Omega \beta(x) w^2 dx = 1.$$

Let $e_{1,\epsilon} = e_1 + \epsilon$ and $w_\epsilon = w + \epsilon$ with $\epsilon << 1$ and consider the following interpolating curves:

$$\sigma_t^\epsilon(x) = (t e_{1,\epsilon}^2 + (1-t) w_\epsilon^2)^{\frac{1}{2}} \text{ with } t \in [0,1].$$

By Lemma 4.1 in [24], we know that $\sigma_t^\epsilon \in X_0$ and that the following inequality holds:

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{(\sigma_t^\epsilon(x) - \sigma_t^\epsilon(y))^2}{|x-y|^{N+2s}} dxdy \leq t \int_{\mathbb{R}^{2N}} \frac{(e_1(x) - e_1(y))^2}{|x-y|^{N+2s}} dxdy \\ + (1-t) \int_{\mathbb{R}^{2N}} \frac{(w(x) - w(y))^2}{|x-y|^{N+2s}} dxdy. \end{aligned}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{(\sigma_t^\epsilon(x) - \sigma_t^\epsilon(y))^2}{t|x-y|^{N+2s}} dxdy - \int_{\mathbb{R}^{2N}} \frac{(w(x) - w(y))^2}{t|x-y|^{N+2s}} dxdy \\ &\leq \int_{\mathbb{R}^{2N}} \frac{(e_1(x) - e_1(y))^2}{|x-y|^{N+2s}} dxdy - \int_{\mathbb{R}^{2N}} \frac{(w(x) - w(y))^2}{|x-y|^{N+2s}} dxdy = \lambda^* - \lambda \end{aligned} \tag{3.41}$$

for all $t \in (0,1]$ and $\epsilon > 0$.

For a convex function the tangent line is always below the function; then, we can use the convexity of the map $u \mapsto u^2$ to control from below the left side of (3.41) and obtain

$$2 \int_{\mathbb{R}^{2N}} \frac{(w(x) - w(y))}{|x-y|^{N+2s}} \frac{(\sigma_t^\epsilon(x) - \sigma_t^\epsilon(y) - (w(x) - w(y)))}{t} dxdy \leq \lambda^* - \lambda.$$

Recalling that w solves (P_λ), taking $v = \sigma_t^\epsilon - w_\epsilon \in X_0$ as test function, from the previous inequality and using that $w(y) - w(x) = w_\epsilon(y) - w_\epsilon(x)$, we easily get

$$2\lambda \int_\Omega \beta(x) w_\epsilon(x) \frac{\sigma_t^\epsilon(x) - w_\epsilon(x)}{t} dxdy \leq \lambda^* - \lambda. \tag{3.42}$$

Since the square root function is concave, by definition of concavity we have

$$\sigma_t^\epsilon(x) - w_\epsilon(x) \geq t(e_{1,\epsilon}(x) - w_\epsilon(x)).$$

Then, being $w_\epsilon > 0$, we get

$$\begin{aligned}
&\beta(x) w_\epsilon(x) \frac{\sigma_t^\epsilon(x) - w_\epsilon(x)}{t} \\
&\geq \beta^+(x) w_\epsilon(x)(e_1(x) - w_\epsilon(x)) - \beta^-(x) w_\epsilon(x) \frac{(\sigma_t^\epsilon(x) - w_\epsilon(x))}{t} \quad \text{for a.e. } x \in \Omega.
\end{aligned} \tag{3.43}$$

Now, the map $t \mapsto \sigma_t^\epsilon(x)$ is concave, so that $\sigma_t^\epsilon(x) - \sigma_0^\epsilon(x) \leq (\sigma_t^\epsilon(x))'_{|t=0} t$, that is

$$\sigma_t^\epsilon(x) - w_\epsilon(x) \leq \frac{e_{1,\epsilon}^2(x) - w_\epsilon^2(x)}{2 w_\epsilon(x)}.$$

As a consequence, from (3.43) we get that for a.e. $x \in \Omega$

$$\beta(x) w_\epsilon(x) \frac{\sigma_t^\epsilon(x) - w_\epsilon(x)}{t} \geq \beta^+(x) w_\epsilon(x)(e_1(x) - w_\epsilon(x)) - \beta^-(x)(e_{1,\epsilon}^2(x) - w_\epsilon^2(x)) \in L^1(\Omega).$$

Then, we can apply the Fatou Lemma and from (3.42) we obtain

$$\begin{aligned}
2\lambda \int_\Omega \beta(x) w_\epsilon(x) \liminf_{t \mapsto 0} \frac{\sigma_t^\epsilon(x) - w_\epsilon(x)}{t} dxdy &\leq 2 \liminf_{t \mapsto 0} \lambda \int_\Omega \beta(x) w_\epsilon(x) \frac{\sigma_t^\epsilon(x) - w_\epsilon(x)}{t} dxdy \\
&\leq \lambda^* - \lambda
\end{aligned}$$

for every $\epsilon << 1$. By direct calculations, the $\liminf$ in the left hand side integrand is

$$\frac{d}{dt} \sigma_t^\epsilon(x)|_{t=0} = \frac{1}{2} \frac{e_{1,\epsilon}^2(x) - w_\epsilon^2(x)}{w_\epsilon(x)}.$$

Thus,

$$\int_\Omega \beta(x) \big(e_{1,\epsilon}^2 - w_\epsilon^2\big)\, dx \leq \lambda^* - \lambda. \tag{3.44}$$

Now,

$$|\beta(e_{1,\epsilon}^2 - w_\epsilon^2)| \leq |\beta|(2e_1^2 + 2w^2 + 4) \in L^1(\Omega),$$

so that, by the Lebesgue Dominated Convergence Theorem, from (3.44) and recalling the normalization of e_1 and w, we have

$$0 = \lambda \int_\Omega \beta(x)(e_1^2(x) - w^2(x))dx \le \lambda^* - \lambda.$$

The conclusion follows recalling that λ is larger than λ^*. □

Now, set

$$V = \left\{ u \in X_0 : \int_\Omega e_1 u \, dx = 0 \right\}$$

and

$$\lambda_V = \inf \left\{ \|u\|^2_{X_0} : u \in V \text{ and } \int_\Omega \beta u^2 dx = 1 \right\}.$$

We have a characterization of the "second" eigenvalue of $(-\Delta)^s$ with weight β which is similar to the case of $-\Delta$:

Proposition 3.3.2. *If $H(\beta)$ holds, let*

$$M = \left\{ u \in X_0 : \int_\Omega \beta u^2 = 1 \right\},$$

then

$$\lambda_V = \inf_{\gamma \in \Gamma} \sup_{t \in [-1,1]} J(\gamma(t)),$$

where

$$\Gamma = \left\{ \gamma \in C([-1,1], V \cap M) : \gamma(-1) = -e_1 \text{ and } \gamma(1) = e_1 \right\}.$$

Proof. first, we prove that that J defined in Proposition 3.1.2 satisfies the PS condition on $V \cap M$. Indeed, let $\{u_n\}_n \subset V \cap M$ be such that

$$|J(u_n)| \le K \text{ for some } K > 0 \text{ and for all } n \ge 1 \tag{3.45}$$

and

$$J'(u_n)(v) \to 0 \quad \forall v \in T_{u_n}(V \cap M), \tag{3.46}$$

where $T_{u_n}(V \cap M)$ is the tangent space to $V \cap M$ at u_n, that is

$$T_{u_n}(V \cap M) = \left\{ v \in V \text{ such that } \int_\Omega \beta u_n v dx = 0 \right\}, \quad n \ge 1.$$

By (3.45) we have that

$$\{u_n\}_n \text{ is bounded in } X_0.$$

Then, we can assume that

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } u_n \to u \text{ in } L^q(\Omega) \text{ with } q < 2^\sharp. \tag{3.47}$$

For all $z \in V$,

$$y = z - \left(\int_\Omega \beta u_n z dx \right) u_n \in T_{u_n}(V \cap M). \tag{3.48}$$

Now we can rewrite (3.46) as

$$|J'(u_n)(v)| \leq \epsilon(n)||v|| \quad \text{for all } v \in T_{u_n}(V \cap M) \tag{3.49}$$

with $\epsilon(n) \to 0$ when $n \to \infty$. Testing the function (3.48) in (3.49) we get

$$|J'(u_n)(y)| = \left| J'(u_n)(z) - \left(\int_\Omega \beta u_n z dx \right) J'(u_n)(u_n) \right| \leq \epsilon(n)||y||.$$

In particular, similarly to [26], we can find $C > 0$ such that

$$|J'(u_n)(y)| = \left| J'(u_n)(z) - \left(\int_\Omega \beta u_n z dx \right) J'(u_n)(u_n) \right| \leq \epsilon(n)C||z||. \tag{3.50}$$

Using (3.50) with $z = u_n - u \in V$ (note that $u \in V$ and that $\{u_n\}_n$ is bounded in X_0),

$$\lim_{n\to\infty} J'(u_n)(u_n - u) = \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))}{|x - y|^{N+2s}} = 0$$

which easily implies that

$$u_n \to u \text{ in } X_0.$$

Now, we already know that $\pm e_1$ are local minimizer for the functional J and $J(\pm e_1) = \lambda^*$. We will prove that we can find $\rho > 0$ small enough with

$$\rho < 2||e_1||_{X_0}$$

such that,

$$J(e_1) < \inf \Big\{ J(u) \text{ such that } u \in V \cap M \text{ and } \|u - e_1\| = \rho \Big\}.$$

In fact, suppose by contradiction that for all $\rho > 0$ there exists $\{u_n\}_n \subset V \cap M \cap \partial B_\rho(e_1)$ such that $J(u_n) \to J(e_1) = \lambda^*$. By definition of J, we have that $\{u_n\}_n$ is bounded in X_0, and in particular we can assume that

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } u_n \to u \text{ in } L^q(\Omega) \text{ for every } q < 2^\sharp. \tag{3.51}$$

Hence, by the sequential weakly lower semicontinuity of the functional, we get

$$J(u) \leq \liminf_{n\to\infty} J(u_n) = J(e_1) = \lambda^*.$$

Since e_1 is simple, see Proposition 3.1.2, so we get that $u = e_1$. By the Mean Value Theorem we can find $t_n \in (0,1)$ such that

$$\begin{aligned} J(u_n) - J\left(\frac{u_n + e_1}{2}\right) &= J'\left(t_n u_n + (1-t_n)\frac{u_n + e_1}{2}\right)\left(\frac{u_n - e_1}{2}\right) \\ &= \left\langle A\left(t_n u_n + (1-t_n)\frac{u_n + e_1}{2}\right), \frac{u_n - e_1}{2}\right\rangle_{X_0^*, X_0}, \end{aligned} \tag{3.52}$$

where A is defined in Remark 3.1.1.

Assuming that $t_n \to t \in [0,1]$, by (3.51), by the convergence of the functional $J(u_n)$ to λ^* and, being $\frac{t_n+1}{2} > 0$, we obtain

$$\lim_{n\to\infty} \frac{t_n+1}{2}\left\langle A\left(t_n u_n + (1-t_n)\frac{u_n + e_1}{2}\right), u_n - e_1\right\rangle_{X_0^*, X_0} = 0$$

but

$$\frac{t_n+1}{2}(u_n - e_1) = \frac{t_n u_n - t_n e_1 + u_n - e_1}{2} = t_n u_n + (1-t_n)\frac{u_n + e_1}{2} - e_1$$

so we have

$$\lim_{n\to\infty}\left\langle A(t_n u_n + (1-t_n)\frac{u_n + e_1}{2}), t_n u_n + (1-t_n)\frac{u_n + e_1}{2} - e_1\right\rangle_{X_0^*, X_0} = 0.$$

By (3.47), $u_n \rightharpoonup e_1$ in X_0, then

$$t_n u_n + (1-t_n)\frac{u_n + e_1}{2} \rightharpoonup e_1 \text{ in } X_0$$

which implies, by Remark 3.1.1, that

$$t_n u_n + (1-t_n)\frac{u_n + e_1}{2} \to e_1 \text{ in } X_0.$$

Then

$$\left\| t_n u_n + (1-t_n)\frac{u_n + e_1}{2} - e_1 \right\| = \frac{t_n+1}{2}\left\| u_n - e_1 \right\| \to 0$$

and this gives a contradiction, since

$$0 < \rho = \left\| u_n - e_1 \right\|.$$

Similarly, we prove that we can find $\rho > 0$ small enough such that:

$$J(-e_1) < \{J(u) \text{ such that } u \in V \cap M \text{ and } \|u + e_1\| = \rho\},$$

that is $\pm e_1$ are strict local minimizer for J and every path joining $-e_1$ and e_1 crosses $\partial B_\rho(-e_1)$ and $\partial B_\rho(e_1)$. As a consequence,

$$\tilde{\lambda} = \inf_{\gamma \in \Gamma} \sup_{t \in [-1,1]} J(\gamma(t)) > J(\pm e_1),$$

where

$$\Gamma = \big\{\gamma \in C([-1,1], V \cap M) \text{ such that } \gamma(0) = -e_1 \text{ and } \gamma(1) = e_1\big\}.$$

Since $J|_M$ satisfies the PS condition, we can apply the min max result of Ghoussoub ([25], Corollary 3.6) and find that $\tilde{\lambda}$ is a critical value for J constrained on $V \cap M$ with $\tilde{\lambda} > \lambda^*$.

Now, we will show that there is no eigenvalue of problem (P_{λ^*}) contained in the open interval $(\lambda^*, \tilde{\lambda})$, and so $\tilde{\lambda} = \lambda_V$.

Suppose by contradiction that there exists $\lambda \in (\lambda^*, \tilde{\lambda})$ and $w \in X_0$ which are an eigencouple for (P_λ). By Proposition 3.3.1, w is nodal, and so $w = w^+ - w^-$, with $w^\pm \neq 0$. Moreover, we normalize w in such a way that $\int_\Omega \beta w^2 dx = 1$. We now consider the following two paths and their composition:

$$\gamma_+ : [0,1] \to M \qquad t \longmapsto \gamma_+(t) = \frac{w^+ - (1-t)w^-}{(\int_\Omega \beta(w^+(1-t)w^-)^2 dx)^{\frac{1}{2}}}$$

from w to $\frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}$, and

$$\gamma_- : [0,1] \to M \qquad t \longmapsto \gamma_-(t) = \frac{-w^- + tw^+}{(\int_\Omega \beta(-w^- + tw^+)^2 dx)^{\frac{1}{2}}}$$

from $\frac{-w^-}{(\int_\Omega \beta(x)(-w^-)^2 dx)^{\frac{1}{2}}}$ to w. Therefore

$$\gamma_1 = \gamma_- \cup \gamma_+$$

goes from $\frac{-w^-}{(\int_\Omega \beta(x)(-w^-)^2 dx)^{\frac{1}{2}}}$ to $\frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}$.

Now, we act on $A(w) = \lambda \beta w$ with $w^+, w^- \in X_0$, obtaining

$$\int_{\mathbb{R}^{2N}} \frac{(w(x) - w(y))(w^+(x) - w^+(y))}{|x-y|^{N+2s}} dxdy \qquad (\text{ using that } w^+ w^- = 0)$$
$$= \int_{\mathbb{R}^{2N}} \frac{(w^+(x) - w^+(y))^2}{|x-y|^{N+2s}} dxdy = \lambda \int_\Omega \beta(x)(w^+)^2 dx.$$

Similarly,

$$\int_{\mathbb{R}^{2N}} \frac{(w^-(x) - w^-(y))^2}{|x-y|^{N+2s}} dxdy = \lambda \int_\Omega \beta(x)(w^-)^2 dx.$$

Therefore

$$\begin{aligned} J(\gamma_+(t)) &= \frac{||w^+||_{X_0} + (1-t)^2 ||w^-||_{X_0}}{\int_\Omega \beta(x)(w^+(1-t)^2 w^-)^2 dx} \\ &= \frac{\lambda \int_\Omega \beta(x)(w^+)^2 dx + (1-t)t^2 \lambda \int_\Omega \beta(x)(w^-)^2 dx}{\int_\Omega \beta(x)(w^+(1-t)w^-)^2 dx} = \lambda \end{aligned}$$

and similarly

$$J(\gamma_-(t)) = \lambda.$$

Now, we consider the set $J^\lambda = \{u \in V \cap M \text{ such that } J(u) < \lambda\}$. If J^λ were path connected, we could consider a path γ in J^λ joining e_1 and $-e_1$, and, in this way, we would get

$$\tilde{\lambda} = \inf_{\gamma \in \Gamma} \sup_{t \in [-1,1]} J(\gamma(t)) < \max_{t \in [-1,1]} J(\gamma(t)) = \lambda,$$

which is absurd, since $\lambda < \tilde{\lambda}$. Therefore J^λ is not path connected.

Now, using Ekeland's variational principle [21] and using that $J|_{V \cap M}$ satisfies the PS-condition, we infer that every connected components of J^λ contains a critical point of $J|_{V\cap M}$. Since the only critical points of $J|_{V \cap M}$ in J^λ are $\pm e_1$ then one component of J^λ contains e_1, the other contains $-e_1$ and J^λ has exactly two path components.

Note that $\frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}$ is not a critical point of $J|_{V\cap M}$. Infact, suppose by contradiction that it was critical point, then, since $\frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}$ has constant sign and since

$$J\left(\frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}\right) = \lambda \frac{1}{\int_\Omega \beta(x)(w^+)^2 dx} \int_\Omega \beta(x)(w^+)^2 dx = \lambda$$

we would have

$$\frac{(-\Delta)^s w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}} = \lambda \frac{\beta w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}$$

i.e. $\lambda = \lambda^*$, in contradiction with the definition of λ.

So we can consider a path $s : [0,1] \to V \cap M$ such that

$$s\left(\frac{1}{2}\right) = \frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}} \text{ and } \frac{d}{dt}J(s(t)) \neq 0.$$

Therefore we can connect with s the point $\frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}$ with a generic point $y \in V \cap M$ within one connected component of J^λ. Without lost of generality, we can suppose that e_1 and y are in the same path-component of J^λ, which we will call U_1.

Now, we consider $z : [0,1] \to \overline{U}_1$ such that $z(0) = y$ and $z(1) = e_1$. Then we have that $\gamma_2 = s \cup z : [0,1] \to \overline{U}_1$ is such that

$$\gamma_2(0) = \frac{w^+}{(\int_\Omega \beta(x)(w^+)^2 dx)^{\frac{1}{2}}}$$

and $\gamma_2(1) = e_1$, with $\gamma_2(t) \in \overline{U}_1 \subset J^\lambda$ for all $t \in (0,1]$.

Similarly, we can take a path $\gamma_3 : [0,1] \to \overline{U}_2$ such that $\gamma_3(0) = -e_1$ and $\gamma_3(1) = \frac{-w^-}{(\int_\Omega \beta(x)(-w^-)^2 dx)^{\frac{1}{2}}}$. Moreover, the chain of three paths

$$\gamma^* = \bigcup_{i=1}^{3} \gamma_i : [0,1] \to V \cap M$$

connects $-e_1$ and e_1. It is immediate that

$$\tilde{\lambda} = \inf_{\gamma\in\Gamma} \sup_{t\in[-1,1]} J(\gamma(t)) \leq \sup_{t\in[-1,1]} J(\gamma^*(t)) \leq \lambda$$

so we reach a contradiction because $\tilde{\lambda} > \lambda$. In conclusion there are no eigenvalues between λ^* and $\tilde{\lambda}$, and so $\tilde{\lambda} = \lambda_V$. □

Proposition 3.3.3.

$$\lambda^* < \lambda_V.$$

Proof. Suppose by contradiction that $\lambda^* = \lambda_V$. Let M be as in Proposition 3.3.2, then we can find $(u_n)_n \subset V \cap M$ such that

$$J(u_n) = \|u_n\|^2_{X_0} \to \lambda^* = \lambda_V.$$

So by definition of J, we have that

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } u_n \to u \text{ in } L^q(\Omega).$$

with $q < 2^\sharp$. By this conclusion and by the sequential weakly lower semicontinuity of J we have

$$\lambda^* \leq J(u) \leq \liminf_{n\to\infty} J(u_n) = \lambda^* = \lambda_V.$$

This means that $u = \pm e_1$, and so we reach a contradiction since $u \in V \cap M$ and $e_1 \notin V$.

□

3.3.2 The Multiplicity result

In this section we prove the first multiplicity result for the problem

$$\begin{cases} (-\Delta)^s u = \lambda(\beta u - g(x,u)) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P_λ^β}$$

We use the same hypotheses of sections 3.1 and 3.2, where, for this problem, we proved the existence of two solutions: one in the positive cone and one in the negative cone.

Let $\overline{v}$ and $\underline{v}$ be the solutions of problem (P_λ^β) such that $\overline{v} > 0$ and $\underline{v} < 0$ in Ω, see Proposition 3.2.1 and Remark 3.2.3, and consider the following truncation for the reaction term:

$$\ell(x,s) = \begin{cases} \beta(x)\underline{v} - g(x,\underline{v}) & \text{if} \quad s \leq \underline{v}(x) \\ \beta(x)s - g(x,s) & \text{if} \quad \underline{v}(x) \leq s \leq \overline{v}(x) \\ \beta(x)\overline{v} - g(x,\overline{v}) & \text{if} \quad s \geq \overline{v}(x). \end{cases} \tag{3.53}$$

Then, ℓ is still a Carathéodory function. Setting $\mathcal{L}(x,s) = \int_0^s \ell(x,s)ds$, by $H(g)$ the functional $\tilde{I} : X_0 \longrightarrow \mathbb{R}$ defined by

$$\tilde{I}(v) = \frac{1}{2}\|v\|_{X_0} - \lambda \int_\Omega \mathcal{L}(x, v(x))\, dx \tag{3.54}$$

is still of class C^1 and sequentially weakly lower semicontinuous (like for the previous I_ϕ and I_ψ in Subsection 3.2). This functional has the following property:

Proposition 3.3.4. *If* (3.13) *and* $H(\beta)$ *hold, then* $\tilde{I}$ *satisfies the C-condition and it is coercive.*

Proof. Let $\{u_n\}_n \subseteq X_0$ be a sequence such that

$$|\tilde{I}(u_n)| \leq B$$

for some $B > 0$, and

$$(1 + \|u_n\|_n)\tilde{I}'(u_n) \to 0 \text{ in } X_0^* \text{ as } n \to \infty \tag{3.55}$$

The first condition reads

$$\left| \frac{1}{2} \|u_n\|_{X_0} - \lambda \int_\Omega \mathcal{L}(x, u_n(x))\, dx \right| \leq B;$$

in particular, by (3.53), there exists $K > 0$ such that

$$\frac{1}{2} \|u_n\|_{X_0} \leq K.$$

So

$$\{u_n\}_n \subset X_0 \text{ is bounded}$$

and we may assume that

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } u_n \to u \text{ in } L^q(\Omega) \text{ as } n \to \infty \text{ with } q < 2^{\#}. \tag{3.56}$$

Using (3.55) with $u_n - u \in X_0$ as test function, from (3.56) we easily get

$$\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0,$$

which implies that $u_n \to u$ in X_0 by Remark 3.1.1, i.e. $\tilde{I}$ satisfies the C-condition.

The second statement holds trivially by the definition of the truncation h. □

Now, we are ready to give the multiplicity result. It is clear that we can't say anything about the relation between $\overline{v}$ and the new truncated functional $\tilde{I}$, although $\overline{v}$ is a minimum for the truncated functional I_ψ. Therefore, we check that $\overline{v}$ and $\underline{v}$ are still minimum for the new functional (3.54). This allow us to find a third critical point, say v_0. This critical point will be of mountain pass type and will be such that $v_0(x) < \overline{v}(x)$ and $v_0(x) > \underline{v}(x)$ for every $x \in \Omega$.

We stress that in the following result we require the following additional hypotheses.
$\tilde{H}(g)$ g satisfies (3.13)–(3.17). In addition, $g(x, \cdot) \in C^1(\mathbb{R})$ and $g_u(x, 0) = 0$ for a.e. $x \in \Omega$.

$\tilde{H}(\beta)$ β satisfies $H(\beta)$. Moreover, $\beta^+(x) \in L^\infty(\Omega)$.

Theorem 3.3.1. *If hypotheses $\tilde{H}(g)$ and $\tilde{H}(\beta)$ hold, and $\lambda > \lambda_V$, then problem (P_λ^β) has a third non trivial solution.*

Proof. By construction, we know that $\overline{v}$ and $\underline{v}$ are solutions for the problem (P_λ^β) and in particular they are local minimizer for the functionals I_ψ and I_ϕ respectively (see Proposition 3.2.1 and Remark 3.2.3).

We want to prove that $\overline{v}$ and $\underline{v}$ are also minimizer for the functional $\tilde{I}$.

In Proposition 3.2.1 we had fixed a strictly positive ψ and proved that

$$\overline{v}(x) < \psi \text{ for every } x \in \bar{\Omega},$$

with $\overline{v}$ a local minimum for the functional I_ψ. Since $\overline{v} \in \text{int}C^+$, then we can find $\rho > 0$ such that

$$\begin{aligned} \overline{B}(\overline{v}) &= \{u \in \text{int}C^+ \text{ such that } ||u - \overline{v}||_{C^0(\overline{\Omega})} \leq \rho\} \\ &\subset [0, \psi] = \{u \in X_0 \text{ such that } u(x) \leq \psi \text{ for a. e. } x \in \Omega\}. \end{aligned}$$

Hence, we have that $\tilde{I}$ constrained on $[0, \psi]$ coincides with I_ψ constrained on the same set. Therefore, $\overline{v}$ is a local $C^0(\overline{\Omega})$-minimizer of $\tilde{I}$, as well. By Theorem 1.1 in [31] we get that $\overline{v}$ is X_0-minimizer of $\tilde{I}$.

Moreover, we can assume that it is a strict local minimizer (otherwise, we would find infinitely many solutions), which means that

$$\tilde{I}(\overline{v}) < \inf \left\{ \tilde{I}(u) \text{ such that } \|u - \overline{v}\| = \rho \right\} \text{ with } \rho > 0.$$

In analogous manner we can deduce the analogous property for $\underline{v}$, that is $\underline{v}$ is a X_0-minimizer for $\tilde{I}$ and it is a strict local minimizer.

Since every path joining $\overline{v}$ and $\underline{v}$ crosses $\partial B_\rho(\overline{v})$ and $\partial B_\rho(\underline{v})$, it follows that

$$c = \inf_{\gamma \in \tilde{\Gamma}} \sup_{t \in [0,1]} \tilde{I}(\gamma(t)) > \tilde{m}$$

with $\tilde{\Gamma} = \{\gamma : [0, 1] \to X_0 \text{ such that } \gamma(0) = \underline{v} \text{ and } \gamma(1) = \overline{v}\}$.

By Proposition 3.3.4, we can apply the Mountain Pass Theorem in [4] and infer that $c \geq \tilde{m}$ and

$$c \text{ is a critical value of } \tilde{I}.$$

Therefore, there exists $v_0 \in X_0$ such that $\tilde{I}'(v_0) = 0$ or equivalently we can write

$$A(v_0) = \lambda l(x, v_0). \tag{3.57}$$

Now, we prove that v_0 is below $\overline{v}$ for a.e. $x \in \Omega$. Since both v_0 and $\overline{v}$ are critical points for $\tilde{I}$, by definition of ℓ, we have that

$$
\begin{aligned}
0 = \tilde{I}'(v_0)((v_0-\overline{v})^+) &= \langle A(v_0),(v_0-\overline{v})^+\rangle_{X_0^*,X_0} - \lambda\int_\Omega \ell(x,v_0)(v_0-\overline{v})^+dx\\
&= \langle A(v_0),(v_0-\overline{v})^+\rangle_{X_0^*,X_0} - \lambda\int_\Omega \ell(x,\overline{v})(v_0-\overline{v})^+dx\\
&= \langle A(v_0),(v_0-\overline{v})^+\rangle_{X_0^*,X_0} - \lambda\int_\Omega \big(\beta\overline{v}-g(x,\overline{v})\big)(v_0-\overline{v})^+dx\\
&= \langle A(v_0),(v_0-\overline{v})^+\rangle_{X_0^*,X_0} + \tilde{I}'(\overline{v})((v_0-\overline{v})^+) - \langle A(\overline{v}),(v_0-\overline{v})^+\\
&= \langle A(v_0),(v_0-\overline{v})^+\rangle - \langle A(\overline{v}),(v_0-\overline{v})^+\rangle_{X_0^*,X_0}\\
&= \langle A(v_0)-A(\overline{v}),(v_0-\overline{v})^+\rangle_{X_0^*,X_0} = \langle A(v_0-\overline{v}),(v_0-\overline{v})^+\rangle_{X_0^*,X_0},
\end{aligned}
\tag{3.58}
$$

by linearity of A.

Testing (3.57) with $(v_0-\overline{v})^+$ and writing $z = v_0-\overline{v}$ we have

$$
\begin{aligned}
0 = \big\langle A(z),(z)^+\big\rangle &= \int_{\mathbb{R}^{2N}} \frac{(z(x)-z(y))((z)^+(x)-(z)^+(y))}{|x-y|^{N+2s}}dxdy\\
&= -\int_{\{z(x)\le 0\}}\int_{\{z(y)\ge 0\}} \frac{(z(x)-z(y))(z(y))}{|x-y|^{N+2s}}dxdy\\
&+ \int_{\{z(x)\ge 0\}}\int_{\{z(y)\le 0\}} \frac{(z(x)-z(y))(z(x))}{|x-y|^{N+2s}}dxdy + \int_{\{z(x)>0\}}\int_{\{z(y)>0\}} \frac{(z(x)-z(y))^2}{|x-y|^{N+2s}}dxdy
\end{aligned}
\tag{3.59}
$$

because it is obvious that

$$
\int_{\{z(x)<0\}}\int_{\{z(y)<0\}} \frac{(z(x)-z(y))((z)^+(x)-(z)^+(y))}{|x-y|^{N+2s}}dxdy = 0.
$$

From (3.58) and (3.59), we get

$$
\begin{aligned}
-\int_{\{z(x)\le 0\}}\int_{\{z(y)\ge 0\}} \frac{z(x)z(y)-z^2(y)}{|x-y|^{N+2s}}dxdy + \int_{\{z(x)\ge 0\}}\int_{\{z(y)\le 0\}} \frac{z^2(x)-z(y)z(x)}{|x-y|^{N+2s}}dxdy\\
+\int_{\{z(x)>0\}}\int_{\{z(y)>0\}} \frac{(z(x)-z(y))^2}{|x-y|^{N+2s}}dxdy = 0
\end{aligned}
$$

In the last identity we have the sum of three nonnegative integrals, hence, in particular

$$
\int_{\{z(x)>0\}}\int_{\{z(y)>0\}} \frac{(z(x)-z(y))^2}{|x-y|^{N+2s}}dxdy = 0.
$$

Then $(v_0-\overline{v})^+(x) = 0$ for almost every $x \in \Omega$. i.e. $v_0 \le \overline{v}$.

Moreover, by the maximum principle, we get that $v_0 < \overline{v}$ in Ω. In a similar fashion, we prove that $v_0 > \underline{v}$ in Ω. In this way we have produced a third critical point for $\tilde{I}$ which is different from $\overline{v}$ and $\underline{v}$. Finally, by definition of ℓ, the points v_0, $\overline{v}$ and $\underline{v}$ solve problem (P_λ^β).

Now we want to prove that the critical value c is not 0, in this way we exclude that v_0 is the trivial solution for (P_λ^β)..

Let $K_c = \left\{u \in X_0 : \tilde{I}(u) = c \text{ and } \tilde{I}'(u) = 0\right\}$ be the mountain pass critical point set. Of course, we can assume that $K_c = \{v_0\}$, otherwise we would have more then three solutions and we would have done.

Hence, by Theorem 12.31 in [2],

$$m(v_0) \leq 1$$

where m denotes the Morse index of v_0.

Now we evaluate the dimension of the maximal dimension of those subspaces where $\tilde{I}''(0)$ is negative definite. For this, let $u, v \in X_0$. Then

$$\tilde{I}'(u)(v) = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N-2s}} dxdy - \lambda \int_\Omega \beta uv dx - \int_\Omega g(x,u) v dx$$

and

$$\tilde{I}''(u)(v,v) = \int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^2}{|x-y|^{N-2s}} dxdy - \lambda \int_\Omega \beta v^2 dx - \int_\Omega g_u(x,u) v^2 dx.$$

Following the approach in [3], we introduce the Liapunov-Schmidt reduction for the problem solved by critical points of $\tilde{I}$, i.e. for the problem

$$\begin{cases} (-\Delta)^s u = \lambda \ell(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{Q_λ^β}$$

By definition of $\ell(x,u)$, we have that

$$\ell'(x,s) = \begin{cases} 0 & \text{if} \quad s \leq \underline{v}(x) \\ \beta(x) - g_s(x,s) & \text{if} \quad \underline{v}(x) \leq s \leq \overline{v}(x) \\ 0 & \text{if} \quad s \geq \overline{v}(x). \end{cases} \tag{3.60}$$

Using $\tilde{H}(g)$, we have that $g_u(x,u)$ is continuous in u and, together with $\tilde{H}(\beta)$, it is immediate that $\ell'(x,u)$ is bounded. Therefore, we can find $n \in \mathbb{N}$ such that

$$\ell'(s) < \lambda_n,$$

where λ_n is the $n-$th eigenvalue of problem (Q_λ^β).

Let $H = span\,\{e_1, \dots, e_n\}$ be the n-dimentional space generated by the first n eigenfunctions and denote by $H^\perp$ the L^2 orthogonal complement such that

$$X_0 = H \oplus H^\perp.$$

So, for any $u \in X_0$, we can write $u = v + w$ with $v \in H$ and $w \in H^\perp$ and therefore problem (P_λ) reads as

$$\begin{cases} (-\Delta)^s w = P_{H^\perp}\ell(v+w) \\ (-\Delta)^s v = P_H \ell(v+w), \end{cases} \tag{3.61}$$

where $P_{H^\perp}$ is the projection onto $H^\perp$, while P_H the projection onto H.

For every $v = \sum_{i=1}^N \alpha_i e_i$, with $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^n$, and from the bound on $\ell'(x,s)$, the first equation in (3.61) has a unique solution $w = w(\alpha)$ of class C^1 with respect to α.

Substituting w into the second equation of (3.61), we obtain on the left hand side

$$(-\Delta)^s v = (-\Delta)^s (\alpha_i e_i) = \alpha_i (-\Delta)^s e_i = \alpha_i \lambda_i e_i \quad \text{for all } i = 1, \dots, n$$

and, on the right side,

$$\ell\left(\sum_{i=0}^n \alpha_i e_i + w(\alpha)\right).$$

Therefore, we are reduced to the following system in H:

$$\alpha_i \lambda_i = \lambda_i \int_\Omega \ell\left(\sum_{i=0}^n \alpha_i e_i + w(\alpha)\right) e_i, \qquad \text{for all } i = 1, \dots, n.$$

Let $\psi : H \to \mathbb{R}$ be the functional defined as

$$\psi(\alpha) = \frac{1}{2}||w(\alpha)||_{X_0}^2 + \frac{1}{2}\sum_{i=1}^n \lambda_i \alpha_i - \lambda \int_\Omega \mathcal{L}\left(\sum_{i=0}^n \alpha_i e_i + w(\alpha)\right) dx.$$

Since that $w(\alpha)$ is solution of the first equation in (3.61), we have

$$\frac{\partial \psi}{\partial \alpha_i}(\alpha) = \lambda_i \alpha_i - \lambda \int_\Omega \ell\left(\sum_{i=0}^n \alpha_i e_i + w(\alpha)\right) e_i dx. \tag{3.62}$$

Let $\overline{\alpha} = P_H \overline{v}$ and $\underline{\alpha} = P_H \underline{v}$ be the projection in H of the positive and the negative solutions founded in Theorem 3.3.1.

Now, since $\overline{v} \in X_0$ is strictly positive in Ω, we have that $w(\overline{\alpha}) > 0$ and $\sum_{i=0}^{n} \overline{\alpha}_i e_i > 0$. Moreover, if $|\alpha - \overline{\alpha}| < \epsilon$ for $\epsilon << 1$, it is immediates that $||w(\alpha) - w(\overline{\alpha})||_{C^1} < \epsilon$, then

$$\sum_{i=0}^{n} \alpha_i e_i + w(\alpha) > 0 \text{ in } \Omega \text{ when } |\alpha - \overline{\alpha}| < \epsilon.$$

By definition, $\overline{v}$ is local minimizer for $\tilde{I}$, then

$$\psi(\overline{\alpha}) = \tilde{I}(\overline{v}) < \tilde{I}(\alpha\phi + w(\alpha)) = \psi(\alpha) \text{ for all } \alpha \text{ such that } |\alpha - \overline{\alpha}| < \epsilon$$

i.e. $\overline{\alpha}$ is a local minimizer for ψ in H.

In the same way we can prove that $\underline{\alpha}$ is local minimizer for ψ on H, too.

By $\tilde{H}(g)$, we get that $\psi \in C^2(H, \mathbb{R})$, hence differentianting (3.62) and taking $\alpha = 0$, we find

$$\begin{aligned} \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j}(0) &= \lambda_j \delta_{ij} - \lambda \int_\Omega \ell'(w(0)) e_i \left(e_j + \frac{\partial w(0)}{\partial \alpha_i} \right) \quad (\text{ since w(0)=0 and } \tilde{H}(g)) \\ &= \lambda_j \delta_{ij} - \lambda \int_\Omega \beta e_i e_j dx = (\lambda_j - \lambda)\delta_{ij}. \end{aligned} \tag{3.63}$$

Since $\psi(0) = \tilde{I}(0)$ then we evaluate $\tilde{I}''(0)$ along the direction e_1 and e_V and we obtain

$$\tilde{I}''(0)(e_1, e_1) = \lambda^* - \lambda \leq \lambda^* - \lambda_V < 0$$

and

$$\tilde{I}''(0)(e_V, e_V) = \lambda_V - \lambda < 0.$$

Moreover, if we take $v \in span\,\{e_1, e_V\}$, namely $v = \alpha^2 e_1 + \gamma^2 e_V$ with α,γ positive. Using the ortogonality of $span\,\{e_1\}$,$span\,\{e_V\}$ and that $\int_\Omega \beta e_1 e_V dx = 0$, by linearity of the operator, we can easily show that

$$\begin{aligned} \|v\|^2_{X_0} =& \alpha^2 \|e_1\|^2_{X_0} + \gamma^2 \|e_V\|^2_{X_0} = \alpha^2 \lambda^* \int_\Omega \beta e_1^2 dx + \gamma^2 \lambda_V \int_\Omega \beta e_V^2 dx < \\ &< \lambda_V \int_\Omega \beta \left[\alpha^2 e_1^2 + \gamma^2 e_V^2\right] = \lambda_V \left(\int_\Omega \beta v^2 dx \right). \end{aligned}$$

Therefore

$$\tilde{I}''(0)\,(v, v) = \|v\|^2_{X_0} - \lambda \int_\Omega \beta v^2 dx \leq (\lambda_V - \lambda) \int_\Omega \beta v^2 dx < 0$$

by the positivity of the last integral. Therefore we can conclude that

$$m(0) \geq 2.$$

□

In conclusion we have the following final Multiplicity theorem.

Theorem 3.3.2. *Assuming that $\tilde{H}(g)$ and $\tilde{H}(\beta)$ hold, then exist λ^* and λ_V such that*

- *For all $\lambda < \lambda^*$, the problem (P_λ^β) has no solutions*
- *For all $\lambda^* < \lambda < \lambda_V$, the problem (P_λ^β) has at least two solutions $\overline{v}, \underline{v} \in C^\alpha(\overline{\Omega})$ such that $\overline{v} > 0$ in Ω and $\underline{v} < 0$ in Ω.*
- *For all $\lambda > \lambda_V$, the problem (P_λ^β) has at least three non trivial solutions.*

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