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Unilateral Commitments

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Introduction

Game Theory is a mathematical theory which deals with conflict and cooperation situations between (at least two) intelligent and rational decision-makers and provides mathematical models of them. The subjects of study for game theorists are not merely play activities as the term " game" erroneously suggests. "Conflict analysis" or "interactive decision theory " might be names more suitable to describe this theory.

A conflict or cooperation situation (*game*) is a strategic interaction between two [1] or more individuals (*players*), which jointly determine the outcome. Each player partially controls the game, but usually no player has full control.

It is convenient to be more precise about the assumptions of intelligence and rationality. With the first hypothesis, we assume that the decision makers have unlimited capacities of deduction, computation, and analysis of the situation. With the second hypothesis, we assume that the decision maker is able to make a choice between various available options. The rationality of the decision maker lies in having preferences on the consequences of his choice, in the consistency with these preferences, and in choosing the available action whose consequence he prefers. The keyword "rationality" is a term inherited from Neoclassic Economics and Decision Theory from which Game Theory derives.

A game is not only made up of players and of choices available to the players, but also of preferences of the players over the set of possible outcomes of the games. Thus, each player strives to obtain the most profitable outcome for him. Often it is assumed that these preferences are described by a von Neumann-Morgenstern utility function, hence with each player there is associated a numerical function whose expected value he tries to maximize.

[1] The case of one player usually falls under the heading of "Decision Theory".

So Game Theory is not only an empiric theory, but also a normative theory, since it prescribes what action each player should choose in a game in order to promote his interests optimally, that is, which strategy each player should play to obtain the best benefits with only his partial influence on the situation.

The aim of Game Theory, as stated in Von Neumann and Morgenstern (1944), is to find the mathematically complete and perfectly general principles which define "rational behavior" for the players in a game and to derive from them the characteristics of that behavior. While the principles ought to be perfectly general, that is, valid in all situations, the solution can be found only in some special characteristic cases and it varies with the change of conditions.

Game Theory is a relatively recent science. Its beginnig dates back to the 20th century with the works of Zermelo (1913), Borel (1921), von Neumann (1928) [2]. But the theory was considered only after the publication of the book by von Neumann and Morgenstern (1944), followed by many articles addressed to developments of this theory. We remember, for example, John F.Nash Jr. (1950), in which the author introduced the Equilibria-bargaining threat.

Many studies of Game Theory were completed during World War II at Princeton, in the same cultural circle where many theoretical physicists were also working (see Morgenstern (1976)). According to the opinion of Myerson (1991), this propinquity does not seem coincidental, however the purpose of the two groups proved different. The physicists have developed the nuclear studies which have threatened the world peace, the game theorists have created social systems for moderating human behavior in conflict. Thus, it might be desiderable that the improvements of social systems were able to study the situation and to outguess the consequences of the physical science. This convinction has moved mathematicians and social scientists to work in Game Theory during the past few years, although there is lack of collaboration between the various disciplines. For example, the production of bio-diesel, extracted from sunflower oil or colza oil, has been a positive environmental impact, but it has created problems of famine to poor countries whose economy was based on primary sectors.

Game Theory actually has proved to be versatile since used in many fields. It has been applied in Military Strategies (Cold War, Gulf War), in Economics (Oligopolies, Monopolies), in Marketing (Coca-Cola), in Finance

[2] von Neumann (1928) introduced the Minimax theorem.

(Firm's Control), in Politics (Electoral Systems), in Club Games (Bridge, Poker, Chess), in Sports (Attack-Defence Strategies), in Sociology (Migration), in Medicine (Neurons), Genetics/Biomedicine (Microarray Games), in Psychology (Prisoner's Dilemma), in Biology (Evolution), in Environment (Pollution).

The importance of developments achieved in Game Theory is supported and affirmed by the assignment of Nobel prizes in Economics. In fact, recently, eight game theorists shared the Nobel prize in economics. They were: in 1994 John F.Nash Jr., John Harsanyi, and Reinhard Selten; in 2005 Y.Robert J. Aumann and Thomas C. Schelling; in 2007 Roger Myerson, Leonid Hurwicz, and Eric Maskin. Nash defined the notion of a noncooperative (or Nash) equilibrium, and proved its existence in mixed strategies. Selten refined this notion to the recursive notion of subgame perfect equilibrium and the closely related notion of trembling hand perfection. Harsanyi defined the notions of a game with incomplete information and of a Bayesian equilibrium, in which players' lack of information about the game they are playing is encapsulated in a player's "type." These ideas have been influential in the study of games by economists during the 1980s. Four major areas in which this impact has been felt are in the study of bargaining, reputation and repeated games, signalling, and mechanism design.

Traditionally, the mathematical models of strategic interactions are divided into two classes: cooperative games and non-cooperative games. A cooperative game is a game in which the players can subscribe binding agreements. A non-cooperative game is one in which there are no possibilities for communication, correlation or (pre)commitment, except for those that are explicitly allowed by the game rules. Hence, all relevant aspects should be captured by the rules of the game. In this work, we restrict ourselves to noncooperative games.

A *solution* for a non-cooperative game is a set of recommendations, which tell each player how to behave in every situation that may arise. We request that solution be *consistent*, i.e. no player should have an incentive to deviate from the prescriptions of solution. Hence, a solution must be *self-enforcing*. A possible reading of this term is the following. It is as if, by pre-play communication, the players have agreed, in a not binding way, to play the prescriptions of this solution and no player has incentive to unilaterally deviate from strategy combination, since his reward will not increase with such an action, on the understanding that the other players do not deviate. In game theoristic terminology, this means that the solution should be a Nash equilibrium (Nash (1950), (1951)).

The concept of Nash equilibrium is a concept relevant and increased in value solution, not only for the idea of stability of non-binding agreements, but also since it plays on the assumptions of intelligence and rationality of players and it does not request the players to communicate to each other before strategic interaction. However, it has its drawbacks. Given a game, various problems can arise, from problems of existence of equilibria to problems of choice of an equilibrium that brings to an efficient outcome. For example, a game so simple as Matching Pennies is without Nash equilibria. Otherwise, we might have two or more possible choices for the players: the Coordination game has two Nash equilibria with the same payoffs, while the Battle of the Sexes has two Nash equilibria, each of them is preferred only by one player. Or, again, there are games with inefficient Nash equilibria, as it happens for Prisoner's Dilemma.

The intervention of Game Theory lies in providing the players with different kinds of solutions of the game. For example the study of refinements of Nash equilibria is one of the knottiest problems in Game Theory. Among the proposed refinements, we can mention trembling hand perfect equilibrium (Selten (1975)), proper equilibrium (Myerson (1978)), sequential equilibrium (Kreps and Wilson (1982)), stable equilibrium (Kohlberg and Mertens (1986)), and virtual subgame perfect equilibrium (García-Jurado and González-Díaz (2006)).

As stated at the beginning, Game Theory makes models and studies hypothetical examples in order to understand conflict and cooperation interactions. On one side, the symplicity of the model, obtained by ignoring the details of reality, allows to analyze the essence and the nature of these strategic interactions. On the other hand, it provides examples, sometimes unrealistic, because they do not correspond to the initial real situation. This is not strictly negative, since in this way we can identify which details, at first sight trifling, are the X-factor of the model. For example, game theory has represented the tragedy of the commons via the strategic actions of players (individuals or countries) which, in order to protect a common, decide whether to cooperate or not. The game was been represented by a Prisoner Dilemma, which provides, for the equilibrium, that both players defect: so, no agreement can be signed by all the countries. The latest literature, inspired by concrete facts, has proved that model is unsuitable for represent it, since some agreements are indeed reached. The inadequacy of the Prisoner Dilemma equilibrium can be interpreted either as a critique of the choice of the model or as a sign that the complete-information model omits some important features. In repeated games, where the reputation effects are significant, small amounts of certain kinds of incomplete information lead to cooperative equilibria. In the example of the Tragedy of the Commons, despite the pessimistic theoretical predictions about cooperation, a lot of partial agreements are signed by a subset of countries trading. Thus, it emerged that man is a social being, which spontaneously cooperates. But the cooperation is partial, not global. The final goal becomes to find how the global cooperation is reached.

Again, let us consider the case of public goods. If a public good is local, then in order to conserve it from over-exploitation a local authority is sufficient. But in case of international public goods, this is not possible, then there are international negotiations and contracts signed by a part the of countries interested in cooperation. Comparing the theoretical results with concrete facts, there is a paradox. That is, there is not always the over-exploitation of public good, as predicted by theory. In fact, more than 120 international environmental agreements are signed. At this point, the game theory intervenes in order to emerge cooperation between all the countries or among a greater and greater number of individuals. The environment is, for example, one of the international public goods. In the global environmental problems a target to reach is the reduction of greenhouse gas emissions. All the countries are actively and passively concerned in this problem, in fact they regulate the threshold of their own gas emissions and they suffer the consequence also from the economic point of view of greenhouse effect: melt of glaciers and rising water level cause damage to agriproducts. World emergency requires an international coordination between countries aiming at signing agreements to reduce gas emissions, but, in practice, the signers are a small number. Even Game Theory confirms this behaviour. Carraro and Marchiori (2003), for example, show a model where, at the equilibrium, the signatories are fewer in number and the grand coalition is not achieved. We prove the same results in Section 3.6. The seriousness of the problem needs all the countries to sign agreements and reduce their gas emissions.

The Thesis consists of four parts: the first part contains opening notes, while the others refer to three different problems: the analysis of a binary symmetric game, a modified version of Unilateral Commitments Game, and the essentializing of different equilibrium concepts.

The first part of the Thesis is made up of Chapters 1 and 2 , and contains definitions and notations. It would like to be a ushering into Game Theory of the readers not familiar with mainly standard game theoretic notions. In Chapter 1 we introduce the basic concepts of Game Theory and the solution concepts used in this thesis. The content of Chapter develops through the three forms, in which we can represent a game: the strategic form, the extensive form, and the coalitional form. First, we introduce games in strategic form, the dominated strategies , the concept of Nash equilibrium, and some games quoted in this work, such as Matching Pennies, Coordination Game, Battle of the Sexes, Prisoner's Dilemma, and Rock Paper Scissors. Then, we present games in extensive form with perfect recall, followed by games in coalitional form and, in particular we recall the definition of $TUIC$-games, since the model analyzed in Chapter 3 derives from them. In the conclusive Section of Chapter 1, we present those refinements of Nash equilibria, which we have essentialized in Chapter 5 , not based on beliefs, like Subgame Perfect Equilibrium (SPE), and Perfect Equilibrium (PE), and based on beliefs, like Sequential Rationality (SR), Sequential Equilibrium (SE), and Weak Perfect Bayesian Equilibrium (WPBE).

In Chapter 2, we present the concept of potential, and its relations with symmetric games. The Section 2.5 is the core of Chapter 2 and provides our results. The first result establishes a symmetric game with only two strategies is a potential game and then it has a pure Nash Equilibrium. The originality of the result lies in beying such a game a potential game, since Cheng and other (2004) has already showed that a symmetric game with only two strategies has a pure Nash equilibrium. The second one provides how all the NE of a symmetric, binary game are deduced from its potential function.

The second part of the Thesis is Chapter 3, where we present two models of binary games. We consider the problem of sharing the cost of facilities among the possible users. An easy way to divide costs is to divide them evenly among all of the players. But this way violates fairness, and, it would be reasonable to take into account whether a member uses or not a given facility. In order to be able to enforce the payment, we assume to have a way to make verifiable to a third part who are the users, but, to make it verifiable, an additional cost is to be added. We propose two ways, in which players reach such a decision. In the first model, the naming game, each player names the machine for which he asks for verifing. In the second model, the majority decision game, the checking is made only if the given quorum is reached. Each model has been examined first in the case with only one machine and then with different ones. Since a binary symmetric game is a potential game and, then, it has a pure Nash Equilibrium, we can model a special case of environmental game via naming game, that is via a potential or congestion game, result processed in Section 3.6.

The third part of the Thesis (Chapter 4), analyzes the model of Quality Unilateral Commitments, i.e. unilateral commitments where, in the first stage, each player declares that he will pay a penalty if he will not play, in the second stage, the restricted strategies. Adding a self-punishing scheme to this simple sequential game structure makes self-enforcing the Nash equilibria of the constituent game, if any. In particular, we embed a two player game into a two stage game, in which players can restrict their strategy spaces in the first stage. In the second stage, if player chooses a strategy from his restricted strategy space, he obtains the same payoff as in the basic game, otherwise he pays a penalty dependent on the square of distance from his restricted strategy space. Since a commitment is a binding of an individual to the others, it measures an attitude to the sense of altruism, compliance, identification and loyalty towards the group. It is no accident we have called our model as quality commitments instead of penalty commitments. In this years, the Corporate Social Responsability (briefly, CSR) is developping. A CSR is an enterprise, which not only produces wealth, but also is dealing with business within the competence of State, Church, civil society, and family. In order to favour the cooperation in a game, usually we implement an efficient disciplinary system, extern to the parts involved in the game. Instead, the CSR develops on civic virtue: the virtue cannot be negotiated, like penalty is settled by contract, but it is a product of free will. The same it happens in our model of QUC, since the sanctions is self declared by the players, then it is internal to the game.

The fourth part of the Thesis (Chapter 5) is an annotated rewrite of the paper *Essentializing Equilibrium Concepts*, together with González-Díaz , García-Jurado, and Patrone (see González-Díaz et al. (2009)). The essentializing process is a tool to identify what information about a game may be neglected, in order to check whether a specific profile correspond to an equilibrium outcome or not. Given an extensive game, an equilibrium concept selects a set of strategy profiles (empty too) satisfying well-defined conditions testing on the all game tree We characterize the essential collections for the most used equilibrium concepts, based or not on beliefs, such as SR, WPBE, SE or NE, SPE, PE. The possible applications of our analysis is to check the robustness of a concept, to analize a partially-specified game, and, finally, to study the concept of Virtual Equilibrium.

The three problems studied in the Thesis are not entirely unrelated. In Chapter 3, we analize the equilibria set of a simple non cooperative model using two different criteria ('Naming game' and 'Majority decision'), Since the found equilibria are unfair, we give fair nature to solutions without recourse to cooperative games, which tools for building equity criteria. So we look for normative prescriptions by emphasizing and focusing on the ethical behaviour of players. A possible solution is provided by Quality

Unilateral Commitments (Chapter 4). This game is a one-shot game, since we try to obtain a cooperative and efficient outcome without resorting to repeated games but using only anticipative declarations of good behaviour. Since, with the unilateral commitment amplification, the extensive form of the game has grown exponentially, it is necessary to locate which parts of the game are relevant to check if the outcome of the strategy profile is an equilibrium outcome: from here the Essentializing Equilibrium Concepts (Chapter 5).

Contents

Chapter 1

Preliminaries

In this Chapter we define the basic concepts of Game Theory, fundamental for this work, and set up standard terminology and notations. First, we introduce games in strategic form (Section 1.1), dominated strategies (Subsection 1.1.2), the concept of Nash equilibrium (Subsection 1.1.3), and some games quoted in this work (Subsection 1.1.4), then games in extensive form (Section 1.2), followed by games in coalitional form (Section 1.3, and, in particular we recall the definition of $TUIC$-games (Subsection 1.3.1), since we quote them in Chapter 3. In the conclusive Section 1.4, we present some refinements of Nash equilibria, not based on beliefs (Subsection 1.4.1), like Subgame Perfect Equilibrium (SPE), and Perfect Equilibrium (PE), and based on beliefs (Subsection 1.4.2), like Sequential Rationality (SR), Sequential Equilibrium (SE), and Weak Perfect Bayesian Equilibrium (WPBE).

As we have reminded in the Introduction, Game Theory copes with strategic interaction between at least two decisioners, called players, and makes mathematical models of it. The players (or else sets of players), intelligent and rational, interact with each other in situations of conflict and cooperation. Each player masters partially the end result of the game [1] through his actions. This way, we have identified the constituents of a game.

Definition 1. *A game G is composed of at least two players, the choices at disposal of players, and the preferences of players compared to game outcomes.*

We assume that the players are rational and intelligent, and the model

[1] A game is made up of players (at least two), of choices at disposal of the players, and preferences of the players for outcomes of the games.

is common knowledge (Lewis (1969). We say *rational* the player able to make a choice between various available options, and *intelligent* the decision maker with unlimited capacities of deduction, calculus, and analysis of the situation. The structure of the game is *common knowledge* when we assume that all players know the structure of the strategic form, and know that their opponents know it, and know that their opponents know that they know, and so on ad infinitum.

The games are divided into *cooperative games*, if players can sign binding agreements, and *non-cooperative games*, otherwise.

A game can be described in several ways, the principal forms are three: the strategic form, the extensive form, and the coalitional form. The first two classes belong to non cooperative games theory, the third class to cooperative games theory. A game in *strategic form* is represented by listing all the strategies (complete plan of action) available to each player, together with the payoffs associated with the various strategy combinations. The strategic form, or s.f., is recomended for games with simultaneous and independent actions. A game in *extensive form* is given by the rules of the game indicating the choices available to each player, the information of a player when it is his turn to move, and the payoffs each player receives at the end of the game. The extensive form, or e.f., is suitable for games with alternate moves. A game in *coalitional form* is described by the utility that each set of players can gain if they form a coalition, excluding the other players. The characteristic form, or c.f., is used for cooperative games. A game in extensive form can be transformed into strategic form (von Neumann (1928)). The possibility of reducing to strategic form also a game with non-simultaneous moves makes the strategic form very important, even if some essential information of extensive form is lost during the change to strategic form.

1.1 Games in Strategic Form and Dominance

1.1.1 Strategic Game

Definition 2. *A game form* Γ *in strategic form is*

$$\langle X_1, \ldots, X_n, E, \phi \rangle$$

where $N = \{1, \ldots, n\}$ *is a finite set of players,* X_i *is the non-empty pure strategy set of player* $i \in N$*,* E *is the set of possible final outcomes, and*

$\phi : \prod_{k \in N} X_k \longrightarrow E$ maps the set of pure strategies into the corresponding outcome.

Definition 3. *A game G in strategic form is*

$$\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$$

where $N = \{1, \ldots, n\}$ is a finite set of players, X_i is the non-empty pure strategy set of player $i \in N$, $u_i : \prod_{k \in N} X_k \longrightarrow \mathbb{R}$ is the payoff function for player i.

The payoff function u_i gives von Neumann-Morgenstern utility $u_i(x_1, \ldots, x_n)$ of player i for each strategy profile $(x_1, \ldots, x_n)$.

Definition 4. *A game $\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ in strategic form is a finite game if X_i is a finite set for all $i \in N$.*

Definition 5. *A game is a binary choice game if each player has only two pure strategies.*

Definition 6. *Given a game $\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$, the game*

$$\langle X_1, \ldots, X_n, c_1, \ldots, c_n \rangle,$$

where $c_i = -u_i$ for all $i \in \{1, \ldots, n\}$, is a cost game. c_i is called cost function.

For $S \subseteq A$, we denote $-S$ the set $A \backslash S$ and X_S the product set $\prod_{i \in S} X_i$. As a particular case, with abuse of notation, we denote X_{-i} the product set $\prod_{k \neq i} X_k$. Then x_{-i} indicates an element of X_{-i}, and (y, x_{-i}) the element of $\prod_{k \in N} X_k$ obtained from $(x_1, \ldots, x_n)$ by replacing the i-th strategy x_i by y, that is $(y, x_{-i}) = (x_1, \ldots, x_{i-1}, y, x_{i+1}, x_n)$.

Let $\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ be a finite game, where $m_i \doteq |X_i|$, for each $i \in N$. A mixed strategy p_i of player i is a probability distribution on X_i, which asssigns to the pure strategy x_{ij} of player i the probability p_{ij}.

Definition 7. *A mixed strategy of player i is $p_i \in \Delta(X_i)$, where*

$$\Delta(X_i) = \{p_i = (p_{i1}, \ldots, p_{im_i}) \in \mathbb{R}^{m_i} : p_{ij} \geqslant 0, \sum_{j=1}^{m_i} p_{ij} = 1\}$$

is the probability simplex on X_i.

Really, a mixed strategy is

$$\sum_{j=1}^{m_i} p_{ij} x_{ij},$$

where $(p_{ij})_j \in \Delta(X_i)$ and $x_{ij} \in X_i$ for each $j = 1, \dots, m_i$ are the pure strategies of player i, but we represent with $p = (p_{ij})_{j=1}^{m_i}$, that is, an element $p = (p_{ij})_{j=1}^{m_i} \in \Delta(X_i)$ corresponds to the strategy " play strategy x_{ij} with probability p_{ij}, for each $j = 1, \dots, m_i$".

1.1.2 Dominance

Definition 8. *Given a strategic game $(X_1, \dots, X_n, u_1, \dots, u_n)$, the strategy $x_i \in X_i$ strongly dominates the strategy $y_i \in X_i$ for player $i \in N$, if $\forall x_{-i} \in X_{-i}$*

$$u_i(x_i, x_{-i}) > u_i(y_i, x_{-i}).$$

The strategy $x_i \in X_i$ weakly dominates the strategy $y_i \in X_i$ for player i, if $\forall x_{-i} \in X_{-i}$

$$u_i(x_i, x_{-i}) \geqslant u_i(y_i, x_{-i}).$$

The strategy $x_i \in X_i$ strictly dominates the strategy $y_i \in X_i$ for player i, if x_i weakly dominates $y_i \in X_i$ and $\exists \bar{x}_{-i} \in X_{-i}$ such that

$$u_i(x_i, \bar{x}_{-i}) > u_i(y_i, \bar{x}_{-i}).$$

The strategy $x_i \in X_i$ is a strongly dominant strategy for player i, if x_i strongly dominates every other strategy $y_i \in X_i$ with $x_i \neq y_i$, while the strategy $x_i \in X_i$ is a strongly dominated strategy if there exists a strategy y_i which strongly dominates it [2].

Obviously, all the dominance relations are reversed for a cost game.

1.1.3 Nash Equilibrium

The Nash equilibrium is the most important equilibrium concept in Game Theory. It was introduced by Nash ((1950), (1951)). A Nash equilibrium is a profile of strategies such that the strategy of each player is the optimal response to the strategies of the opponents. Nash equilibria are consistent

[2] To avoid misunderstandings, the terminology we use about dominances is not aligned with the 'dominant' usage in Game Theory literature.

predictions of how the game will be played, in the sense that if all players predict that a particular Nash equilibrium occurs, then no player has an incentive to play differently.

Definition 9. *A strategy profile is* $(x_1, \dots, x_n) \in \prod_{i \in N} X_i$.

Definition 10. *A Nash equilibrium is a strategy profile* $(x_1, \dots, x_n)$ *such that* $\forall i \in N$ *and* $\forall y_i \in X_i$

$$u_i(x_i, x_{-i}) \geqslant u_i(y_i, x_{-i}).$$

Hence, a strategy profile $(x_1, \dots, x_n)$ is a Nash equilibrium (briefly: NE) if no player has an incentive to unilaterally deviate from $(x_1, \dots, x_n)$, since with a NE each player maximizes his payoff if the strategies of the others are held fixed. In this sense, the strategy of each player is said optimal against those of the opponents.

Remark 1. When we assume that the strategy sets are subset of an Euclidean space and the payoff function are continuous, the criterion in Definition 10 for a NE can be expressed by equating n pairs of continuous functions on the space of n-uples. Then the NE obviously form a closed subset of this space. This subset is composed of a number of pieces of algebraic varieties, cut out by other algebraic varieties.

1.1.4 Example of Games in Strategic Form

Not all games have NE in pure strategies, like it happens in Matching Pennies Games. Sometimes there are games with multiply NE: two well known examples are the Coordination Games, where the NE are the same for each player, and the Battle of the Sexes, where each NE is preferred only by one player. Therefore, the following problem arises: given a game with more than one NE and without possibility to make binding agreements, which one of these NE should be chosen as the solution of the game? Again, some NE are better qualified to be chosen as the solution than others, and not every NE has the property to be self-enforcing. The tool of eliminating the equilibria not self-enforcing (or unreasonable or non-sensible) is called refinement of NE.

Matching Pennies

A simple example of non-existence of Nash equilibria is the Matching Pennies Games, in Figure 1.1. In Matching Pennies, two players simultaneously announce heads (H) or tails (T). If the announces match, player I wins and player II looses, otherwise player I looses and player II wins. Neither of the pure strategy profile constitute an equilibrium. The unique equilibrium of MP is in mixed strategies, when each player randomizes between his two pure strategies, assigning equal probability to each.

$I \backslash II$	H		T	
H	1	0	0	1
T	0	1	1	0

Figure 1.1: Matching Pennies Game.

Coordination Game

An easy example of a game with multiple equilibria is the Coordination Game[3], illustrated by Figure 1.2. Each player receives 1 when the players

$I \backslash II$	L		R	
T	1	1	0	0
B	0	0	1	1

Figure 1.2: Coordination Game.

choose the same strategies and 0 otherwise. The game has two Nash equilibria in pure strategies, and a third in mixed strategies, when each player randomizes between his two pure strategies, assigning equal probability to each. The problems derive from the fact that there are two optimal choices for the players and the strategies are choosen simultaneously, so the players cannot effectively coordinate themselves.

Battle of the Sexes

One well-known example of a game with multiple equilibria is the Battle of the Sexes, illustrated by Figure 1.3. Two players wish to go to an event together, but disagree about whether to go to a football game or to the

[3]The Coordination Game is an example of a game described by Definition 41.

$I \backslash II$	F		B	
F	2	1	0	0
B	0	0	1	2

Figure 1.3: Battle of the Sexes.

ballet. Each player gets a utility of 2 if both go to his (or to her) preferred event, a utility of 1 if both go the other's preferred event, and a utility of 0 if the two are unable to agree and stay at home or go out individually. The game has three equilibria: two in pure strategies, (F, F) and (B, B), and one in mixed: player I plays F with probability $\frac{2}{3}$ (and B with probability $\frac{1}{3}$), and player II plays F with probability $\frac{1}{3}$ (and B with probability $\frac{2}{3}$).

If two players have not played the battle of sexes before, there is no obvious way for the players to coordinate their expectations. However, the theory of focal points of Schelling (1960) suggests that in some real-life situations, players may be able to coordinate on a particular equilibrium using information abstracted away by the strategic form.

Prisoner's Dilemma

In the Prisoner's Dilemma game, two suspects of a crime are put into separate cells. If both confess (strategy B and R, respectively) each will be sentenced to 2. If only one of them confesses, he will be freed and used as a witness against the other person, who will be sentenced to 3 years in prison. If both do not confess (strategy T and L, respectively), they will both be punished for a minor offense and spend 1 year in jail. Payoffs are represented by 3 minus the number of years spent in prison.

$I \backslash II$	L		R	
T	2	2	0	3
B	3	0	1	1

Figure 1.4: Prisoner's Dilemma.

Changing the perspective, we define as Prisoner's Dilemma each symmetric binary game with two players such that each player has a dominant strategy and the NE is not efficient. From here the interest reserved to Prisoner's Dilemma follows, since a reader might expect an efficient outcome, on account of rationality assumption of players. The Prisoner's Dilemma repetion allows to draw up the paradox. As the game perpetuates, the players are urged to cooperate (see Fudenberg and Tirole (1991) or Kreps et al. (1982).

Rock Paper Scissors

Rock Paper Scissors, depicted in the Figure 1.5 is a two player game. Each player has three strategies: rock, paper, and scissors. Rock breaks scissors, paper folds rock, and scissors cut paper. None of the pure strategy profiles constitute an equilibrium. The game has a unique symmetric equilibrium in mixed strategies: player I plays R with probability $\frac{1}{3}$, S with probability $\frac{1}{3}$, and P with probability $\frac{1}{3}$, and player II the same mixed strategy.

$I \backslash II$	R		P		S	
R	0	0	−1	1	1	−1
P	1	−1	0	0	−1	1
S	−1	1	1	−1	0	0

Figure 1.5: Rock Paper Scissors.

RSP is a three player game with no pure strategy equilibria.

1.2 Games in Extensive Form

The extensive form is a fundamental concept in Game Theory. In this work, the words extensive game will always refer to a finite game in extensive form. We follow the representation of an extensive game given in Fudenberg and Tirole (1991a), representation equivalent to the classic one given by Kuhn (1953) and further developed in Selten (1975) and Kreps and Wilson (1982). The extensive form is a more detailed description of a game. It tells exactly which players should move, when, what are the choice, the outcomes, the information of the players at every stage, and so on. The order of moves (i.e., who moves when) is represented by a game tree (a non-oriented graph, connected, and without simple cycles) finite and with root. The probability distributions over any exogenous events is represented by moves of Nature, eventually. In the following Sub-section we transfer, for completeness, the formal definition, some details of which are not essential for the rest of the work.

1.2.1 Extensive Game with Perfect Recall

We now formally define a finite game form in extensive form.

Definition 11. *(Kuhn (1953)) A finite extensive game form is*

$$\Gamma = (V, D, r, N, \mathcal{P}, \mathcal{U}, \mathcal{M}, E, \phi, (\preccurlyeq_k)_{k \in N}),$$

where:

1. (V, D, r) *is a finite tree*[4] (V, D) *with root* r.
   V *denotes the tree node (or vertex) set,* D *the tree branch set,* Z *the terminal node set, and e* $X = V \setminus Z$ *the decisional node set.*

2. $N = \{0, 1, \ldots, n\}$ *is the finite player set.* 0 *represents Nature. We assume that the random player can move only in* r.

3. $\mathcal{P} = (P_k)_{k \in N^*}$ *is a subdivision in disjoint subsets of* X*, also empty.*
   P_k *are the set of pertinent nodes to player* k*, that is the nodes in which* k *has to move.*

4. $\mathcal{U} = (U_{k,j})_{k \in N^*, j \in J_k}$ *is, for each player* k*, a partition of* P_k *in a family of sets* $U_{k,j}$*,* J_k *is a set of indices.*
   $U_{k,j}$ *are the nodes pertinent to player* k*, such that, when the player is one of them, who is not able to distinguish in which node he is.*

5. $\mathcal{A}$ *is, for* $k \neq 0$*, a family of sets* $A_{k,j}$*, one for each of* $U_{k,j}$.
   In correspondence with a node of an information set $U_{k,j}$*, player* k *has to choose an action between those contained in* $A_{k,j}$.

6. E *is the set of possible final outcomes of the game.*

7. $\phi : Z \longrightarrow E$ *associates to each terminal node an outcome.*

8. $(\preccurlyeq_k)_{k \in N}$ *represents a family of total preorder on* E *which represent the preferences for final outcomes of the game.*

[4]A tree is a non oriented, connected graph made of simplex cycles.

From here onwards, Γ denotes a game form, $U(\Gamma)$ a partition of $X(\Gamma)$, i.e. each terminal node is also an information set, $A_i(\Gamma)$ the actions available to player i, $A(u)$ the action available to him in information set u, that is in $A(u) \subseteq A_i$, $U_i(\Gamma)$ the information sets belonging to a player $i \in N$, (Γ, h) a game in extensive form, and $\mathcal{G}(\Gamma)$ the set of games with game form Γ.

Definition 12. *An information set u is a class of pertinence nodes of a player such that*

- *all nodes in u have the same number of outgoing branches, and there is a given one-to-one correspondence between the sets of outgoing branches of different nodes in u;*
- *every directed path in the tree from the root to a terminal node can cross each u at most once.*

Grafically the dashed line connects the nodes belonging to the same information set.

Definition 13. *A game is of perfect information if all the information sets are singletons.*

In a game of perfect information, there are no simultaneous moves, and at each decision point the player knows which choice has previously been made. The Figure 1.6 depicts a game with perfect information and a game without perfect information.

In Figure 1.7 is depicted a game in extensive form where two players are involved: player I and player II. The game starts at the root of the tree, depicted in the figure by $\circ$, where player I has to move. He can choose between T or B. If player I has choosen T, then player II has to move. After he has observed the move of his opponent, he can decide to go left L or right R. If player I has choosen B, the player II has not to move. We have hung the end results on the endpoints of the tree. The upper number is the payoff to player I, the lower to player II. So, for example, if player I chooses B, then player I receives 1 and player II receives 2. The game is played just once. The game has two NE, (T, L) and (B, R), with payoff $(2, 1)$ and $(1, 2)$, respectively. The strategy profile (B, R) is NE since, given the choice of player II, it is optimal for the player I choosing B at the beginning of the game (if he chooses T, then, given the choice R of player II, he gets 0 instead of 1), and, given the choice of player I, it is optimal for the player II to choose L, since his choice is indifferent to the outcome. Similarly, the strategy profile (T, L) is a NE.

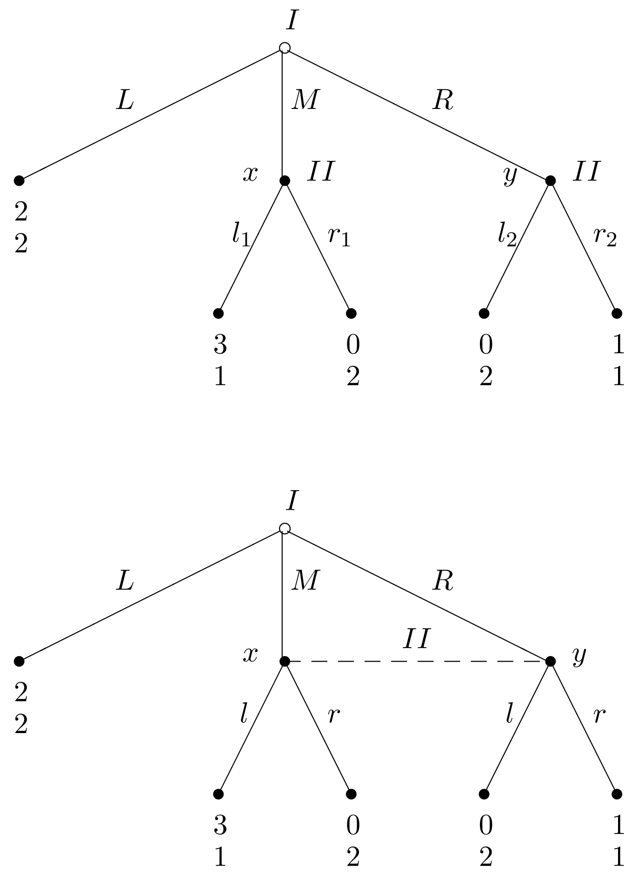


Figure 1.6: Games in extensive form with and without perfect information.

Almost all games in economics literature are games of perfect recall.

Definition 14. *A game is of perfect recall if no player ever forgets any information he once knew, and all players know the actions they have chosen previously.*

In the Figure 1.8 the player I moves first, but when he has to move again, he has forgotten his previous choice. The player is not able to distinguish between the sequence of action (L, r), (R, l), and (R, r). That is, player I does not remember whether he has chosen L or R. While, it is reasonable that he cannot distinguish between r and l since this move is choosen by player II and is not revealed to player I.

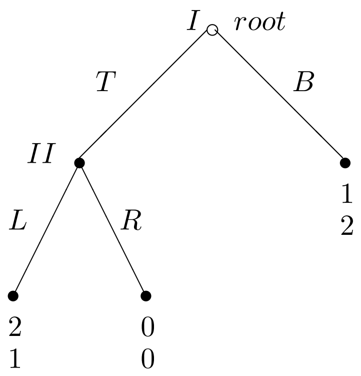


Figure 1.7: Game in extensive form.

In the Figure 1.9, player I makes his choice knowing the initial node, chosen by Nature (When a game involves Nature, the exogenous probabilities are displayed in brackets). If player I chooses d and player II chooses D, player I has to move again in the information set $\{x, y\}$, but he has forgotten the Natura choice, information he had at his disposal.

1.2.2 Behavior strategy profile

A pure strategy of a player is a complete plan for his choices in all possible contingencies in the game, that is at all his information sets. A mixed strategy means that the player chooses, before the beginning of the game, one such comprehensive plan at random, according to a certain probability distribution. An alternative method of randomization for the player is to make an independent random choice at each one of his information sets. That is, rather than selecting, for each information set, one definitive choice, as in a pure strategy, he specifies instead a probability distribution over the set of choices there. Moreover, the choices at different information sets are (stochastically) independent. These randomization procedures are called behavior strategies.

Definition 15. *A behavioral strategy profile is $b = (b_i)_{i \in N}$, where the behavioral strategy b_i for the player $i \in N$ prescribes, for each information set $u \in U_i(\Gamma)$, a probability distribution on the actions available to him in*

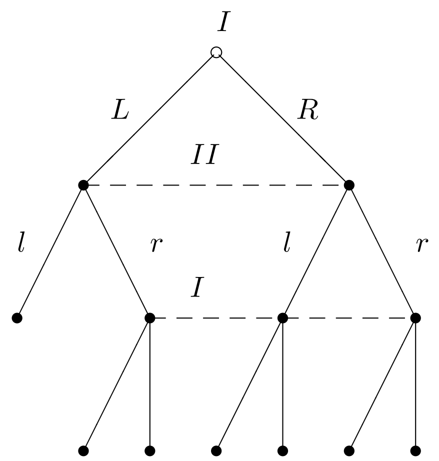


Figure 1.8: Game of non-perfect recall.

information set u, that is in $A(u) \subseteq A_i$

$$b_i : A_i \longrightarrow [0,1] \quad s.t. \quad \sum_{a \in A(u)} b_i(a) = 1, \quad u \in U_i$$

.

We denote $B(\Gamma) = \prod_{i=1}^{n} B_i(\Gamma)$ the set of behavior strategy profiles of a game G or a game form Γ, and, with a slight abuse of notation, $h_i(b)$ the (expected) payoff to player i when $b \in B(\Gamma)$ is played.

Definition 16. *A behavioral strategy profile* $b \in B(\Gamma)$ *is completely mixed if at each information set all the choices are taken with positive probability.*

Thus the beliefs associated with a completely mixed strategy profile are completely determined by Bayes rule (see Section 1.4.3).

1.3 Games in Coalitional Form

For a finite set N of players, we denote its power set, i.e. the collection of all its subsets, by 2^N and its number of elements by $|N|$. A subset $S \subseteq N$ is called a coalition.

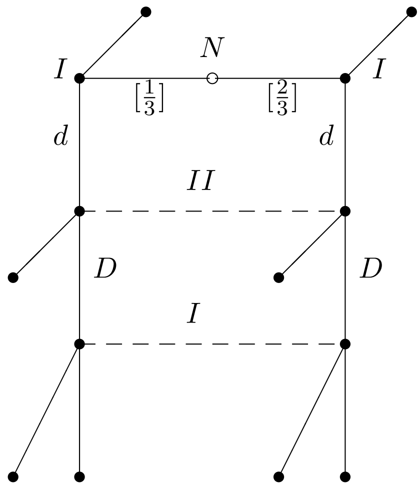


Figure 1.9: Game of non-perfect recall.

Definition 17. *A cooperative n-person game in coalitional form is an ordered pair*

$$\langle N, v \rangle$$

where $N = \{1, \ldots, n\}$ *is the set of players, and* $v : 2^N \longrightarrow \mathbb{R}$ *is a map, which assigns to each coalition* $S \in 2^N$ *a real number, such that* $v(\emptyset) = 0$. v *is called the characteristic function of the game and* $v(S)$ *the value or the worth of coalition* S.

A game in coalitional form (or characteristic function) may represent very different situations, for example it can model a simple voting game where v associates to a winning coalition the value 1 and to a losing coalition the value 0, or an economic market that generates a cooperative game.

Example 1. *(Glove game)* Let $N = \{1, \ldots, n\}$ be divided into two disjunct subsets L and R. Members of L possess a left hand glove, members of R a right hand glove. A single glove is worth nothing, a right-left pair of gloves 1. This situation can be modeled by an n-person game $\langle N, v \rangle$, where, for each $S \in 2^N$,

$$v(S) \doteq \min\{|L \cap S|, |R \cap S|\}.$$

There are two special classes of games:

- Transferable utility games (TU) (also called Games with Side Payments). The members of a coalition S can arbitrarily divide among themselves the amount $v(S)$ which S can get. So a TU-game is of the form
$$v(S) = \{(x_i)_{i \in S} \, such\, that \sum_{i \in S} x_i \leqslant v(S)\}.$$
- Non-transferable utility games (NTU), the games without transferable utility.

But these questions are not within our terms of references.

For completeness, we recall the Pure Bargaining games (PB). In these games only the grand coalition matters. Here, for all $S \neq N$,

$$v(S) = \{(x_i)_{i \in S} \, such\, that \, x_i \leqslant 0 \, , \, \forall i \in S\}.$$

But these questions are not within our terms of references.

1.3.1 TUIC games

The TUIC games represent a simple model which allows embedding a cooperative game of cost allocation in a richer structure, so that it is possible to take in account that cost information is expensive to get. In this structure, we can discuss how to balance on one hand the costs imposed by information requirements, on the other the loss of fairness when one tries to reduce these costs to the minimum. A *TUIC*-game is a family of *TU*-game, indexed by a parameter $t \in T$, with information costs χ_t, and ordered by a transitive and irreflexive relation $\prec$ on T. In addition to the function c_t of *TU*-game G_t, there is an extra cost χ_t bringing the necessary information on the cost to get c_t. For example χ_t is the additional cost to pass from a model $t_1 \in T$ to another $t_2 \in T$ or to choose the function c_t. Moreover $t_1 \prec t_2$ means the model t_2 has more information w.r.t. model t_1 and c_{t_2} approaches better the cost function than c_{t_1}.

Definition 18. *A TUIC game is*

$$\langle N, T, (c_t)_{t \in T}, (\chi_t)_{t \in T}, \prec \rangle,$$

where N is a finite set of players, T is a set of parameters (models), whose elements provide the needed information to have a TU game, $c_t : \mathcal{P}(N) \longrightarrow \mathbb{R}$ is a (cost) TU game, $\chi_t \in [0, +\infty)$ is the cost to get c_t, $\prec$ is a transitive and irreflexive relation on T.

1.4 Special classes of equilibria

1.4.1 Non-belief-based equilibria

The classic equilibrium concepts not based on beliefs are the Nash equilibrium and some of its refinements, such as the subgame perfect equilibrium (Selten (1965)), the perfect equilibrium (Selten (1975)), the proper equilibrium (Myerson (1978)), the persistent equilibrium (Kalai and Samet (1984)), the essential equilibrium (Wu Wen-Ts*ü*n and Jiahg Jia-He (1962)), and the regular equilibrium (Harsanyi (1973)). We have also defined the NE in Subsection 1.1.3. Here, we introduce only the subgame perfect equilibrium in Subsubsection 1.4.1 and the perfect equilibrium in Subsubsection 1.4.1, since the others wander off the matter of this thesis.

Selten (1965), in order to discard those NE, possible if some players give credit to irrational (that is, non-maximizing) plan of the others, introduced the subgame perfect equilibrium, that is a NE which induces a NE in each subgame. But a subgame perfect equilibrium may also be non sensible, in the sense that it prescribes a choice non-maximizing the expected payoff. Selten (1975), to eliminate unreasonable subgame perfect equilibria, assumes that there is always a small probability that a player will take a choice by mistake, with the consequence that every choice will be taken with a positive probability. Therefore, in an extensive game with mistakes (a so called perturbed game), every information set will be reached with a positive probability. Then, an equilibrium of this game will prescribe rational behavior at every information set. Assuming that mistakes occur only with a very small probability leads to define a perfect equilibrium, that is an equilibrium obtained as a limit point of a sequence of disturbed games in which the mistake probabilities go to zero. Hence, an equilibrium is perfect if the equilibrium strategy of each player is not only optimal against the equilibrium strategies of his opponent, but if it is also optimal against some slight perturbations of these strategies.

Subgame Perfect Equilibria

The subgame perfect equilibrium (Selten (1965)), or SPE, is the most important equilibrium concept within the class of extensive games. The subgame perfect equilibrium discards those NE which are only possible if some players give credit to irrational plans of others. That is, a SPE is a Nash equilibrium which induces a Nash equilibrium in every subgame.

We consider the game due to Selten (1975) in Figure 1.10. It is an

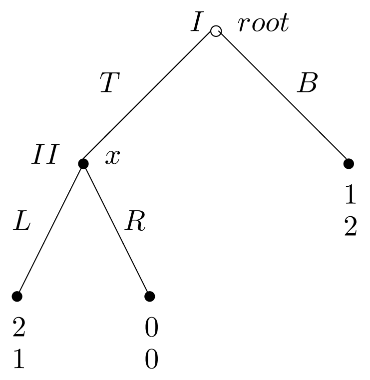


Figure 1.10: Selten game.

extensive game with perfect information. In order to identify the Nash equilibria, it is more convenient to analize the Selten game in strategic form, see Figure 1.11. The game has two NE (T, L) and (B, R), with payoff $(2, 1)$

$I \backslash II$	L	R
T	$(2, 1)$	$(0, 0)$
B	$(1, 2)$	$(1, 2)$

Figure 1.11: Selten game in strategic form.

and $(1, 2)$, respectively. The equilibrium (B, R) is not self-enforcing, while (T, L) is enforcing. Let us make ourselves clear. Let suppose the players have agreed to play (B, R). If player I considers that player II will fulfil the agreement, then it is optimal for him to play B. But I cannot expect that II will fulfil the agreement. In fact, if the node x is reached, the strategy L gives to II a higher payoff than R. So, the II will play L, if he has to move. Therefore, it is better for player I to play T, and so he will also violate the agreement. From here, the Nash equilibrium (B, R) is not enforcing. The equilibrium (B, R) can be interpreted as a *threat equilibrium* on part of player II. Player II threats player I that he will punish him by playing R, if he does not play B (which gives II the best possible result). This way, II will punish also himself: the choice R is not optimal for II, which gets a better payoff playing L. Why does a Nash equilbrium predict for a player a sub-optimal choice? The equilibrium (B, R) does not predict that II plays R, since the choice B concludes the game and II has not to move. In general, a Nash equilbrium can predict non optimal choices on part of players in nodes of the tree not reached, if the equibrium profile is played. Again, the threat is not credible since if I disregards the threat and plays T, then II will play L, following his rationality. So, using the extensive form, we have shown that not all the Nash equilibria are the same. This leads to the definition of subgame perfect equilibrium or SPE by Selten (1965).

The argument used to exclude the equilibrium (B, R) in the Selten game in Figure 1.10 generalizes to all games with perfect information. Since in a non-cooperative game there are no possibilities for commitment, once the decision point x is reached, the part of the game tree which does not come after x has become strategically irrelevant and, therefore, the decision at x should be based only on the part of the tree which comes after x. This implies that for games with perfect information only those equilibria which can be found by inductively working backwards in the game tree, are sensible, i.e. self-enforcing. Using the *backward induction* principle, we get all the SPE of an extensive game.

Sequential rationality and subgame-perfectness are backward induction principles for the analysis of games in extensive form, because they require that any predictions that can be made about the behavior of players at the end of a game are supposed to be anticipated by the players earlier in the game.

Perfect Equilibria

The Perfect Equilibrium or Trembling-Hand Perfect equilibrium is a refinement of Nash equilibrium, introduced in Selten (1975). It is very closely related to the concept of sequential equilibrium. The basic idea behind the perfectness concept is that each player makes mistakes with a small probability, therefore every pure strategy is chosen with a positive (although possibly small) probability. Mathematically, this idea is modelled via a perturbed game, that is a game in which each player is only allowed to use completely mixed strategies. The distinction with sequential equilibrium is thus that strategies must be in equilibrium along the converging subsequence and not only in the limit. In the definition, Selten requires that a strategy profile b be the limit of a sequence of totally mixed profiles b^ϵ and that b_i^ϵ be a best response to the opponents' perturbed strategies b_{-i}^ϵ.

Definition 19. *Let $G = (\Gamma, h)$ be an extensive game. Let ε be a function*

$$\varepsilon : A_i \longrightarrow (0, 1]$$

which assigns to every choice a in G a positive number $\varepsilon(a)$ such that, for every information set $u \in U_i$,

$$\sum_{a \in A(u)} \varepsilon(a) < 1.$$

The perturbed game $G^{\varepsilon} \in \mathcal{G}(\Gamma)$ is the extensive game G in which every player $i \in N$ is only allowed to use behavior strategies b_i which satisfy

$$(b_i)_u \geqslant \varepsilon(a),$$

for all $u \in U_i$ and $a \in A(u)$.

Let G^{ε} be a perturbed game and let B^{ε} be the set of admissible strategy profiles in G^{ε}. An equilibrium of G^{ε} is an admissible strategy profile $b \in B^{\varepsilon}$ which prescribes a best reply at every information set, i.e.

$$h_i(b_u) = \max_{b_i' \in B_i^{\varepsilon}} h_i(b_{-i}, b_i')_u,$$

for each $i \in N$ and each $u \in U_i$. An equilibrium of G^{ε} is perfect if it is still sensible to play this equilibrium if slight mistakes are taken into account.

Definition 20. *Let G be a game in extensive form. A behavioral strategy profile b is a perfect equilibrium of G if*

$$b^{\epsilon} \longrightarrow b, \; as \, \epsilon \to 0,$$

that is, b is a limit point of a sequence of equilibria of perturbed game G^{ε}.

In the game of Figure 1.12, only the equilibrium (R, r) is perfect. In a perturbed game associated with this game, player I will take the choices M and R with a positive probability (if only by mistake) and, therefore, the information set of player II will actually be reached, which forces player II to play r.

Theorem 1. *(Selten (1975)) Every finite game possesses at least one perfect equilibrium.*

Theorem 2. *(Kreps and Wilson (1982)) Every perfect equilibrium is sequential, but the converse is not true.*

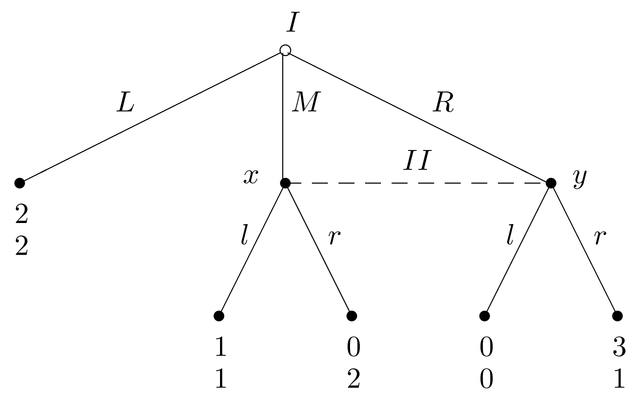


Figure 1.12: G.

To illustrate the difference between the two concepts, let us consider the game G', in Figure 1.13, obtained with a slight modification of game G in Figure 1.12. As before, player II has to play r. For player I, both R and L are the best replies against r. Therefore, in a sequential equilibrium, player I can play any combination of R and L. The only perfect equilibrium, however, is (L, r), since if player I plays L, he is sure of getting 3, whereas if he plays R, he can expect only slightly less than 3 since player 2 with a small probability will make a mistake and play l.

1.4.2 Belief-based equilibria

In this subsection, we extend the notion of subgame perfect equilibrium to extensive game with imperfect information. We focus on the concept of Sequential Rationality, and some of its refinements, such as Sequential Equilibrium and Weak Perfect Bayesian Equilibrim.

We recall that a Subgame Perfect Equilibrium of an extensive game with perfect information is a strategy profile for which the strategy of each player, given the strategies of the others, is optimal at any contingency in which it is his turn to take an action, also in tree nodes not reached by game. The natural extension of this idea to extensive games with imperfect information leads to the following requirement.

($\star$) The strategy of each player, given the strategies of the others, is optimal at each of his information sets (reached or not by the game).

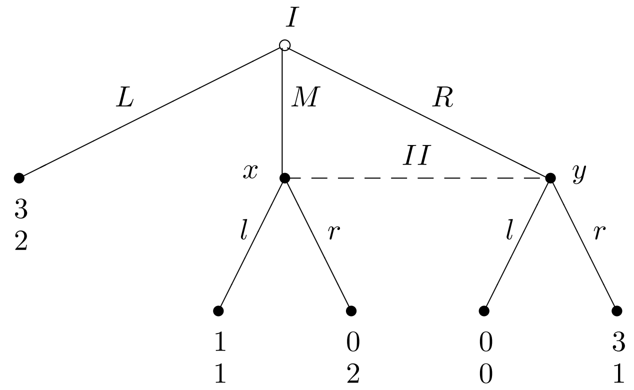


Figure 1.13: G'.

In the extensive game G with imperfect information in Figure 1.14, the requirement that each player's strategy be optimal at every information set eliminates a NE.

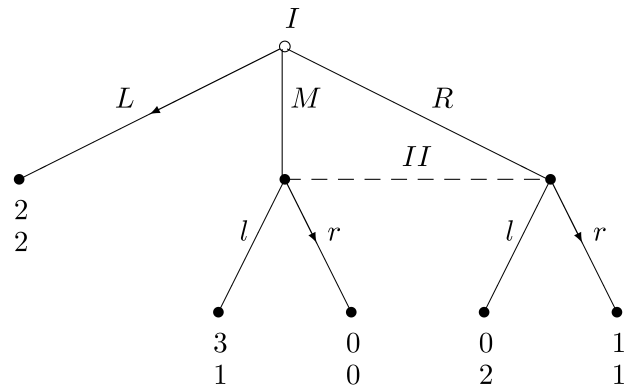


Figure 1.14: G.

This simple example is due to Selten (1975). In this game player I has tree strategies: L, M, and R. If he plays L, the game ends with payoff $(2, 2)$. If plays M or R, then player II must choose between l or r, but he does not know what action has been choosen by I. If player I chooses M and player II l, the payoffs are $(0, 0)$, and so on. The game G has two NE: (L, r) and (M, l), both of which are subgame perfect. If player I adheres to equilibrium (L, r), then the information set of player II is not reached. However, if it is reached (player I chooses M or L), the strategy r is dominated by strategy l. For any specification of player II's beliefs about the probability of M and R when player I deviates and does not play L, the optimal strategy of player II is to play r. Then (L, r) does not satisfy the condition of the extension, while the equilibrium (M, l) does. The extensive game with imperfect information G' in Figure 1.16 has a NE (L, r) that is not ruled out by an implementation $(\star)$, since optimal strategy of player II in the event that his information set is reached depends on his beliefs about the history that has occurred. The strategy r is optimal if II assigns probability of at most $\frac{1}{2}$ to the history M, while l is optimal if he assigns probability at most $\frac{1}{2}$ to this history. His belief cannot be derived from the equilibrium strategy, since (L, r) assigns probability zero to his information set being reached.

The solution for the extensive games studied in this section consists of two components: a strategy profile and a belief system.

Definition 21. *A system of beliefs μ over $X(\Gamma)\backslash Z(\Gamma)$ is a function*

$$\mu : X(\Gamma)\backslash Z(\Gamma) \longrightarrow [0, 1]$$

such that, for each $u \in U(\Gamma)$,

$$\sum_{x \in u} \mu(x) = 1.$$

That is, a belief system consists of a collection of probability measures, one for each information set of the game.

Definition 22. *An assessment in an extensive game is a pair*

$$(b, \mu),$$

where $b = (b_i)_{i \in N}$ is a behavioral strategy profile, and μ a system of beliefs .

1.4.3 Bayes Rule

We first recall some concepts of probability. Let Ω be a finite sample space. We call *events* the subsets of sample space, that is $E, F \subseteq \Omega$, so the set of events is the set of the parts 2^{Ω}. If m is the cardinality of Ω, then the cardinality of 2^{Ω} is 2^m.

Definition 23. *A probability measure on* Ω *is a function*

$$P : 2^{\Omega} \longrightarrow [0, 1],$$

such that

$$i)\, P(\emptyset) = 0, , \qquad ii)\, P(\Omega) = 1\,, \; and$$

$$iii)\, for\, each\, E, F \in \Omega\, s.t.\, E \cap F = \emptyset\,, \quad P(E \cup F) = P(E) + P(F).$$

Definition 24. *The conditioned probability of the event* E *given the event* F *is*

$$P(E|F) \doteq \frac{P(E \cap F)}{P(F)}.$$

In plain words, we restrict the sample space to F and then we calculate the probability of event E.

Since $P(F) \in [0, 1]$,

$$P(E|F) > P(E \cap F).$$

Theorem 3. *Let* E, F *be two events. Then,*

$$P(E \cup F) + P(E \cap F) = P(E) + P(F).$$

Theorem 4. *Let* $F_1, \ldots, F_m$ *be mutually disjoint and complementary events, that is,* $F_i \cap F_j = \emptyset$ *for each* $i, j = 1, \ldots, m$ *with* $i \neq j$*, and* $F_1 \cup \ldots \cup F_m = \Omega$*. Then, for each event* E*,*

$$P(E) = P(E|F_1)P(F_1) + \ldots P(E|F_m)P(F_m).$$

Theorem 5 (Bayes theorem)**.** *Let* E, F *be events, then*

$$P(E|F)P(F) = P(F|E)P(E).$$

Corollary 1 (Bayes rule)**.** *Let* $F_1, \ldots, F_n$ *be mutually exclusive and exaustive events and let* E *be an arbitrary events of sample space such that* $P(E) \neq 0$*, then*

$$P(F_1|E) = \frac{P(F_1)P(E|F_1)}{P(F_1)P(E|F_1) + \ldots + P(F_m)P(E|F_m)}.$$

We consider the game in Figure 1.15 and we assume player I chooses T with probability $\frac{1}{3}$ and B with $\frac{2}{3}$, and player II chooses t with probability $\frac{1}{4}$, and d with $\frac{3}{4}$. During the game, the node x of information set of player III is reached with probability a priori $\frac{2}{3}$, while the node y with probability a priori $\frac{1}{4}$. A priori $1 - P(Tt) = 1 - \frac{1}{3}\frac{1}{4} = 1 - \frac{1}{12} = \frac{11}{12}$ is the probability which III has to move and $\frac{1}{12}$ the probability with which a priori that the game finishes. Using Bayes rule, the belief of III, when he has to move, gives to him the probability of $(\frac{2}{3})/(\frac{11}{12}) = \frac{8}{11}$ to the possibility of being in node x and $(\frac{1}{4})/(\frac{11}{12}) = \frac{3}{11}$ to the possibility of being in node y.

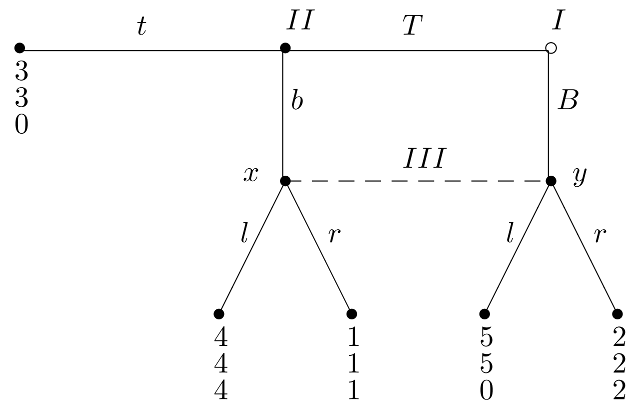


Figure 1.15: Bayes rule application.

Sequential Rationality

Definition 25. *An assessment (b, μ) is consistent if*

$$(b, \mu) = \lim_{n \longrightarrow \infty} (b^n, \mu^n),$$

that is, it is the limit of a sequence of assessments $(b^n, \mu^n)_{n \in \mathbb{N}}$ such that each b^n is completely mixed, each μ^n results from b^n using Bayes rule.

The idea for consistent condition is that the probability of the events, conditioned on events with probability zero, approximates probabilities raised by strategies which assign positive probability to each actions.

Definition 26. *An assessment (b, μ) is sequentially rational if, for each player $i \in N$ and each information set $u \in U_i(\Gamma)$ the strategy b_i of the player i who has to move is the best replay, assegned his beliefs and the strategies of his opponents.*

Sequential Equilibrium

Definition 27. *An assessment (b, μ) is a sequential equilibrium of a finite game in extensive form with perfect recall if it is sequentially rational and consistent.*

Let consider the game G in Figure 1.17.

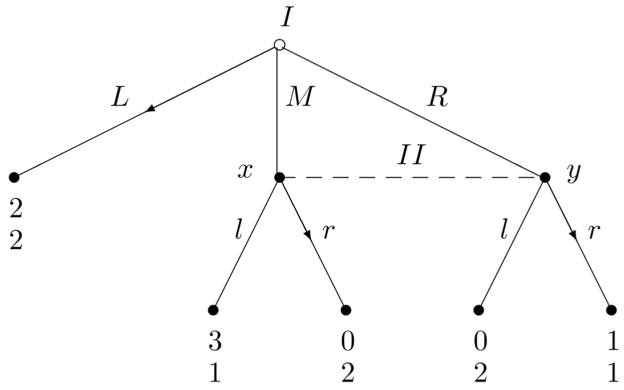


Figure 1.16: G'.

The assessment (b,μ) where

$$b=(b_1,b_2)\,,\; b_1=L\,,\; b_2=r\,,\; \mu(x)=\alpha\,,\; \mu(y)=1-\alpha\,,\; \forall\alpha\in(0,1)$$

is consistent since

$$(b,\mu)=\lim_{n\longrightarrow\infty}(b^n,\mu^n),$$

where

$$b_1^n=(1-\frac{1}{n},\alpha\frac{1}{n},(1-\alpha)\frac{1}{n})\,,\; b_2^n=(\frac{1}{n},1-\frac{1}{n})\,,\; \mu(x)=\alpha\,,\; \mu(y)=1-\alpha\,,\; \forall n.$$

If $\alpha\geq\frac{1}{2}$, then (b,μ) is sequentially rational, since $2\alpha+1(1-\alpha)\geq\alpha+2(1-\alpha)$. So (b,μ) is a sequential equilibrium.

Proposition 1. *Each finite extensive game, with perfect recall, has a sequential equilibrium.*

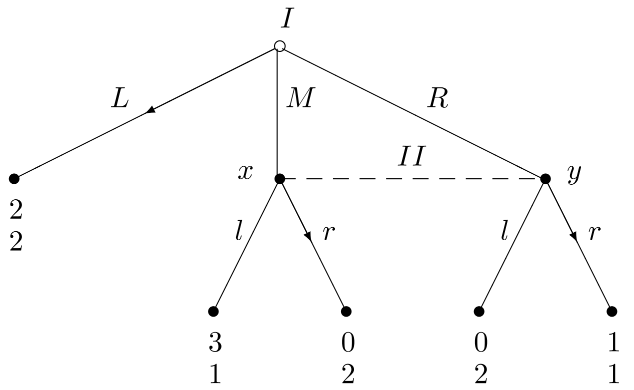
Proposition 2. *If (b,μ) is a sequential equilibrium, then b is a Nash equilibrium.*

Proposition 3. *In an extensive game with perfect recall, (b,μ) is a sequential equilibrium if and only if b is a subgame perfect equilibrium.*

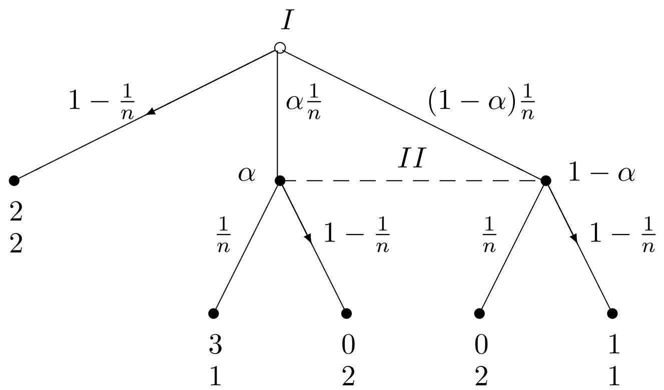
Weak Perfect Bayesian Equilibrium

Definition 28. *Let $G=(\Gamma,h)$ be an extensive game. An assessment (b,μ) is weakly consistent with Bayes rule if μ is derived using Bayesian updating in the path of b.*

Definition 29. *Let* $G = (\Gamma, h)$ *be an extensive game. A Weak Perfect Bayesian Equilibrium (or, briefly,* $WPBE$*) is an assessment* (b, μ) *sequentially rational and weakly consistent with Bayes rule.*



G and (b, μ)



G and (b_n, μ_n)

Figure 1.17: The assesment (b, μ) is the limit of the sequence (b_n, μ_n).

Chapter 2

Potential game

2.1 Introduction

Monderer and Shapley (1996) introduced, for games in strategic form, three nested classes of potential games: the ordinal potential games, the weighted potential games, and the exact potential games (or in short potential games). The basic point of these classes is the existence of a real-valued function P, called *potential function*, on the strategy space, which measures alone the incentive of each player to deviate from a strategy. In the case of ordinal potential games, P gives only indications whether the deviation increases or decreases the payoff, while P for a weighted potential games values the weighted gap of the deviation, and P for an exact potential game measures the exact gap of the deviation. In this work, we will mainly deal with last class, that is with exact potential games (or in short potential games).

The potential function is not only an useful tool to analyze equilibrium properties of potential games, since the incentives of all players are mapped into only one function, but also P provides the necessary information for the determination of the Nash equilibria: a strategy profile is a NE if every unilateral deviation from it decreases the value of the potential function. We consider, for example, the Prisoner Dilemma G and the function P depicted in Figure 2.1. The payoff change of a player, which unilaterally deviates, exactly matches the change in function P. For example, if player II deviates from (T, L) to (T, R), his payoff increases by $3 - 2 = 1$ as well as P increases by $1 - 0 = 1$. For this reason P is called *an* exact potential of the game G. We underline that P is not the unique potential of G. Another potential for G is P', illustrated in Figure 2.2. In fact, the exact potential games enjoy

$I \setminus II$	L		R	
T	2	2	0	3
B	3	0	1	1

G

	L	R
T	0	1
B	1	2

P

Figure 2.1: Game and Potential.

	L	R
T	2	3
B	3	4

P'

Figure 2.2: P'.

the property that two potentials are different in a constant. Again, (B, R) is the NE since every unilateral deviation from this strategy profile decreases the value of the potential function. Thus, the information concerning pure NE accrues to a potential function.

Moreover, if strategy spaces are finite, the potential game has at least an equilibrium in pure strategies. A point of maximum for P, which exists since the product of finite space strategies is finite, is also a point of equilibrium for G.

There are various analogies with the physical concept of potential not only in the term and in the possibility of replacing n payoff functions (a vector field) with one potential function (a scalar field), but also in the fact that, if strategy spaces are finite, the "discret" circulation is always zero. If the strategy spaces are, instead, intervals of real numbers and each payoff function is twice continuously differentiable, then the Schwarz theorem applies to potential P and moreover, P is expressed by the integral of the partial derivatives of each payoff (see Monderer and Shapley (1996)).

Historically, the first to use potential functions for strategic games was Rosenthal (1973). Rosenthal introduced the class of congestion games and proved, by explicitly constructing a potential function, that every congestion game has a pure Nash equilibrium. A congestion game is a game where players have to choose their strategy from a finite set of alternatives and their payoff depends on the number of players choosing the same alternative. Moreover, Monderer and Shapley (1996) showed that the class of congestion games coincides, up to an isomorphism, with the class of finite potential games.

This Chapter is organized as follows. In Section 2.2, we introduce potential functions, potential games, and the relative properties. In Section 2.4 we study the characterization of exact potential games by splitting them into coordination games and dummy games. In Section 2.3, we intorduce congestion game in order to have the formula of potential function. Finally, in Section 2.5, we investigate symmetric games and we present our results. The first result establishes a symmetric game with only two strategies is a potential game and then it has a pure Nash Equilibrium. The originality lies in being a potential game, since Cheng and other (2004) has already showed that a symmetric game with only two strategies has a pure Nash equilibrium. The second one provides how all the NE of a symmetric, binary game are deduced from its potential function.

2.2 Potential game and potential function

This section defines ordinal potential game, exact potential games, surveys some simple results, and provides two characterizations of exact potential games.

Let $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ be a n-person strategy game.

Definition 30. *[Monderer and Shapley (1996)] An ordinal potential for G is a function $P : \prod_{i=1}^{n} X_i \to \mathbb{R}$ such that, for all $i \in N$, $x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$,*

$$sgn\left(u_i(x_i, x_{-i}) - u_i(y_i, x_{-i})\right) = sgn\left(P(x_i, x_{-i}) - P(y_i, x_{-i})\right).$$

$sgn(x)$ denotes the sign of x, namely $+1, -1$ or 0.

Definition 31. *An exact potential (or, briefly, a potential) for G is a function $P : \prod_{i=1}^{n} X_i \to \mathbb{R}$ such that, for all $i \in \{1, \dots, n\}$, $x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$,*

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}).$$

The gap in the payoff for a unilaterally deviating player is exactly equal to the corresponding gap in the value of P.

Definition 32. *A game admitting an ordinal or an exact potential function is called an ordinal or an exact potential game respectively (or, briefly, a potential game).*

It is clear that the class of exact potential games is a proper subset of the class of ordinal potential games.

Again, a function may be a potential function or an ordinal potential one. For example, P is a potential for the Prisoner's Dilemma G and at the same time it is an ordinal potential for the game G' described in Figure 2.3.

I \ II	L	R
T	2	3
B	3	4

P

I \ II	L		R	
T	2	2	0	3
B	3	0	1	1

G

I \ II	L		R	
T	2	2	0	3
B	3	0	1	1

G'

Figure 2.3: P is a potential for G and an ordinal potential for G'.

The next lemma characterizes the equilibrium set of an ordinal potential game.

Lemma 1. *Let P be an ordinal potential for $G = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$. Then the equilibrium set of G coincides with the equilibrium set of the game $\langle X_1, \ldots, X_n, P, \ldots, P \rangle$. That is,*

$$(x_1, \ldots, x_n) \text{ is a NE for G} \Leftrightarrow \mathrm{P}(\mathrm{x_i}, \mathrm{x_{-i}}) \geqslant \mathrm{P}(\mathrm{y}, \mathrm{x_{-i}})\ \forall \mathrm{i} \in \mathrm{N},\ \forall \mathrm{y} \in \mathrm{X_i}.$$

Consequently, if P admits a maximal value in $\prod_{i=1}^{n} X_i$, then G has a pure strategy equilibrium.

Corollary 2. *Every finite ordinal potential game G has at least one pure Nash equilibrium.*

Each ordinal potential game has many ordinal potentials, instead all the exact potentials of the same game differ in a constant.

Lemma 2. *Let P_1 and P_2 be potentials for the game G. Then there is a real constant c such that*

$$P_1(x_1, \dots, x_n) - P_2(x_1, \dots, x_n) = c \quad \forall (x_1, \dots, x_n) \in \prod_{i=1}^{n} X_i.$$

The following theorem 6 characterizes potential game via a physical approach, cyclicity on a simple closed path of length 4.

Definition 33. *A path in $\prod_{i=1}^{n} X_i$ is a sequence of strategy profiles*

$$(x_1^k, \dots, x_n^k)_{k \in \mathbb{N}},$$

such that, for every $k = 1, 2, \dots$, the strategies $(x_1^k, \dots, x_n^k)_k$ and $(x_1^{k-1}, \dots, x_n^{k-1})_k$, differ in exactly one, say the i^{th}, coordinate, i.e., there is a unique player $i \in N$ such that

$$(x_i^k, x_{-i}^k) = (y, x_{-i}^{k-1}) \, \mathrm{for\, some\, y \in X_i \setminus \{x_i^k\}}.$$

$(x_1^0, \dots, x_n^0)$ is called the initial point of path, and, if the sequence is finite, its last element $(x_1^l, \dots, x_n^l)$ is called the terminal point of path, and the path is called finite.

Definition 34. *A finite path is closed if $(x_1^0, \dots, x_n^0) = (x_1^l, \dots, x_n^l)$.*

Definition 35. *A closed path is simple if, in addition, $(x_1^j, \dots, x_n^j) \neq (x_1^k, \dots, x_n^k)$, for every $0 \leq j \neq k \leq l-1$, that is, the strategy profiles are all distinct.*

Definition 36. *The length of a simple closed path is the number of distinct vertices in it.*

That is, the length of the simple closed path $(x_1^k, \dots, x_n^k)_{k=1}^l$ is l.

Definition 37. *Let $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ be a game and $\pi = (x_1^k, \dots, x_n^k)_{k \in \mathbb{N}}$ be a finite path. We set*

$$I(\pi, u_1, \dots, u_n) \doteq \sum_{k \in N} u_{i_k}(x_1^k, \dots, x_n^k) - u_{i_k}(x_1^{k-1}, \dots, x_n^{k-1}),$$

where i_k is the unique deviator at step k, that is $x_{i_k}^k \neq x_{i_k}^{k-1}$.

Theorem 6. *Let $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ be a strategic game. Then, the following claims are equivalent:*

i) G *is a potential game.*

ii) $I(\pi, u_1, \dots, u_n) = 0$*, for every finite closed path* π*.*

iii) $I(\pi, u_1, \dots, u_n) = 0$*, for every finite simple closed path* π*.*

iv) $I(\pi, u_1, \dots, u_n) = 0$*, for every finite simple closed path* π *of length* 4*.*

Corollary 3. $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ *is a potential game if and only if*

$$u_i(x_i, x_j, x_{-\{i,j\}}) - u_i(y_i, x_j, x_{-\{i,j\}}) + u_j(y_i, x_j, x_{-\{i,j\}}) +$$
$$-u_j(y_i, y_j, x_{-\{i,j\}}) + u_i(y_i, y_j, x_{-\{i,j\}}) - u_i(x_i, y_j, x_{-\{i,j\}}) +$$
$$+u_j(x_i, y_j, x_{-\{i,j\}}) - u_j(x_i, x_j, x_{-\{i,j\}}) = 0,$$

where i, j *are the active players,* $x_{-\{i,j\}} \in X_{-\{i,j\}}$ *is a fixed strategy profile of the other players,* $x_i, y_j \in X_i$ *and* $x_j, y_j \in X_j$*.*

A typical simple closed path of length 4 is described in Figure 2.4.

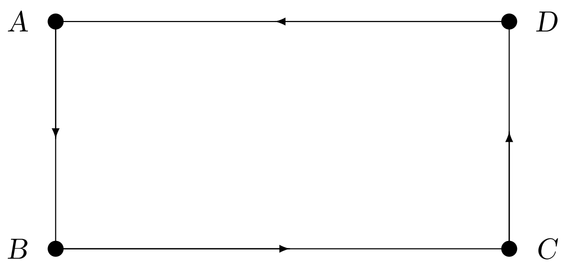


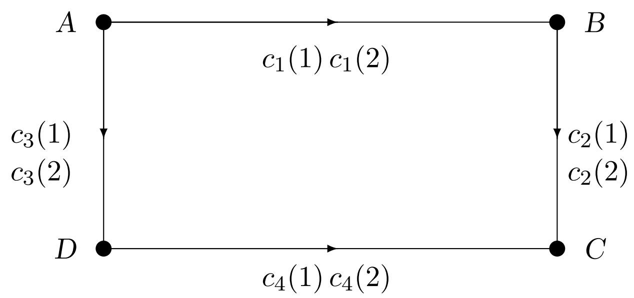
Figure 2.4: Simple closed path of length 4.

2.3 Congestion games

In this section we present the congestion mode, since we may extract the expression of potential from the construction (Rosenthal (1973)) of an exact potential function for a congestion game.

In a congestion model, players use a set of *facilities* (also called *machines*) from a common group. The costs or the benefits of a player, derived from the use of a facility, depend only on the number of players choosing the same facility. The payoff to a player is the sum of the costs or benefits associated with each facility in his strategy choice, given the choice of the other players. By constructing a potential function for such congestion game, Rosenthal proved the existence of pure-strategy Nash equilibria. Moreover, Monderer and Shapley (1996) showed every finite potential games is isomorphic to a congestion game.

Before formalizing the definitions, we introduce a very simple example where two players I and II are involved.



Player I has to go from point A to point C and player II has to go from point B to point D. Player I can travel via B or via D and player II via A or via C. The cost of using a segment depends on the number of users. We call 1 the segment AB, 2 the segment BC, 3 the segment AD and 4 the segment DC, $c_j(1)$ denotes the cost of segment j for a single user, and $c_j(2)$ the cost of j for each users if both use segment j, where $j \in \{1, \dots, 4\}$. The associated congestion game is given by

$I \backslash II$	L	R
T	$(c_1(2) + c_2(1)) \quad (c_1(2) + c_3(1))$	$(c_2(2) + c_1(1)) \quad (c_2(2) + c_4(1))$
B	$(c_3(2) + c_4(1)) \quad (c_3(2) + c_1(1))$	$(c_4(2) + c_3(1)) \quad (c_4(2) + c_2(1))$

It is straightforward to see that this symmetric game is a potential game and so it has a Nash equilibrium in pure strategies. A potential is given by:

$I \backslash II$	L	R
T	$c_1(1) + c_1(2) + c_2(1) + c_3(1)$	$c_2(1) + c_2(2) + c_1(1) + c_4(1)$
B	$c_3(1) + c_3(2) + c_4(1) + c_4(1)$	$c_4(1) + c_4(2) + c_3(1) + c_2(1)$

Definition 38. *A congestion model is described as a 4-tuple*

$$\langle N, F, (X_i)_{i\in N}, (c_f)_{f\in F}\rangle$$

where $N = \{1, 2, \dots, n\}$ *is the set of players,* F *is the set of facilities* $\{1, 2, \dots, f\}$ *involved,* $X_i \in \mathcal{P}(F)$ *is the set of pure strategies of player* i, $X_i \neq \emptyset$, $c_f : N \to \mathbb{R}$ *is the cost function of facility* f *so defined: for each* $k \in N$, $c_f(k)$ *denotes the costs to each user of facility* f *with precisely* k *users.*

Definition 39. *The congestion game corresponding to the congestion model is the cost game in strategic form* $\langle X_1, \dots, X_n, C_1, \dots, C_n\rangle$ *where the cost for player* i *is*

$$C_i(x_1, \dots, x_n) = \sum_{f\in x_i} c_f(|\{r \in N \quad s.t. \quad f \in x_r\}|).$$

The following theorem is the main result of Rosenthal (1973).

Theorem 7. *(Rosenthal (1973)) Every congestion game is a potential game.*

The potential function $P : \prod_{i\in N} X_i \longrightarrow \mathbb{R}$ is defined by

$$P(x_1, \dots, x_n) = \sum_{i\in N}(\sum_{f\in x_i} c_f(|\{r \in N \quad s.t. \quad f \in x_r\}|).$$

Monderer and Shapley (1996) showed that the class of finite potential games coincides, up to an isomorphism, with the class of congestion games.

Definition 40. *Let* $G_1 = \langle (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ *and* $G_2 = \langle (Y_i)_{i\in N}, (v_i)_{i\in N}\rangle$ *be two game with the same set of players* N. G_1 *and* G_2 *are isomophic if there exist bijections* $\phi_i : X_i \longrightarrow Y_i$, $i \in \{1, \dots, n\}$ *such that for every* $i \in N$ *and for every* $(x_1, \dots, x_n) \in \prod_{i\in N} X_i$,

$$u_i(x_1, \dots, x_n) = v_i(\phi_1(x_1), \dots, \phi_N(x_n)).$$

Proposition 4. *(Monderer and Shapley (1996)) Every finite potential game is isomorphic to a congestion game.*

2.4 Decomposition of exact potential games

We conclude the treatment of (exact) potential games with a characterization of such games.

I \ II	L		R	
T	0	0	1	1
B	1	1	2	2

I \ II	L		R	
T	2	2	−1	2
B	2	−1	−1	−1

Figure 2.5: PD is sum of coordination game and dummy game.

We can decompose the Prisoner's Dilemma in Figure 1.4 into the sum of two games showed in Figure 2.5. It is immediate to see that, in the first game, the players have the same payoffs, while, in the second game, the payoffs of a player depend not on his choice, but on the strategy of his opponent. Formally, we present the following definitions.

Definition 41. *A game $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ is a coordination game if there is a function $u : \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ such that $u_i = u$ for each $i \in N$.*

In a coordination game, players pursue the same goal, reflected by the identical payoff functions. It is a game of *pure externality*, in the sense that the strategy chosen by a player has effect only on the contestant.

Definition 42. *A game $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ is a dummy game if, for each $i \in N$ and for each $x_{-i} \in X_{-i}$, there is $k \in \mathbb{R}$ such that $u_i(x_i, x_{-i}) = k$ for each $x_i \in X_{-i}$.*

In a dummy game, each player has no reason to choose a strategy instead of another, since his payoff does not depend on his own strategy.

Facchini et *al.* (1997) provide this characterization of exact potential games by splitting them into coordination games and dummy games.

Theorem 8. *Let $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ be a strategic game. G is a potential game if and only if, for each $i = 1, \dots, n$, there exist functions $c_i : \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ and $d_i : \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ such that*

i) $u_i = c_i + d_i$, *for each* $i = 1, \dots, n$,

ii) $\langle X_1, \dots, X_n, c_1, \dots, c_n \rangle$ *is a coordination game, and*

iii) $\langle X_1, \dots, X_n, d_1, \dots, d_n \rangle$ *is a dummy game.*

Proof. The part 'if' is obvious. The payoff function of the coordination game is an exact potential function of G. Let us consider the assertion 'only if'. Let P be an exact potential for G. For all $i \in N$, we have $u_i = P + (u_i - P)$. Clearly, $\langle X_1, \dots, X_n, P, \dots, P \rangle$ is a coordination game. Let $i \in N$, $x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$. Then $u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i})$ implies $u_i(x_i, x_{-i}) - P(x_i, x_{-i}) = u_i(y_i, x_{-i}) - P(y_i, x_{-i})$. Consequently, $\langle X_1, \dots, X_n, u_1 - P, \dots, u_n - P \rangle$ is a dummy game. □

2.5 Symmetric Game

Let $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ be a game in strategic form with player set $N = \{1, \dots, n\}$. For convenience, in this section we use the notation

$$(x, y, x_{-\{i,j\}}) \in X_i \times X_j \times X_{-\{i,j\}},$$

where $i, j \in N$ are the active players, $x \in X_i$ is the strategy of player i, $y \in X_j$ is the strategy of player j, and $x_{-\{i,j\}} \in X_{-\{i,j\}}$ is a fixed strategy profile of the other players.

Definition 43. *G is a symmetric game if $X_1 = X_2 = \ldots = X_n$ and*

$$u_i(x, y, x_{-\{i,j\}}) = u_j(y, x, x_{-\{i,j\}}),$$

for $x, y \in X_1$ and $x_{-\{i,j\}} \in X_{-\{i,j\}}$, $\forall i, j \in N$.

Definition 44. *A symmetric strategy profile is a profile with all players playing the same strategy. If such a profile is a Nash equilibrium, it is a symmetric equilibrium.*

Theorem 9. *A symmetric binary game is a potential game.*

Proof. For Corollary 3, the existence of an exact potential P is equivalent to the following condition:

$$u_i(x_i, x_j, x_{-\{i,j\}}) - u_i(y_i, x_j, x_{-\{i,j\}}) + u_j(y_i, x_j, x_{-\{i,j\}}) +$$
$$-u_j(y_i, y_j, x_{-\{i,j\}}) + u_i(y_i, y_j, x_{-\{i,j\}}) - u_i(x_i, y_j, x_{-\{i,j\}}) +$$
$$+u_j(x_i, y_j, x_{-\{i,j\}}) - u_j(x_i, x_j, x_{-\{i,j\}}) = 0$$

Using symmetry,

$$u_i(x, y, x_{-\{i,j\}}) = u_j(y, x, x_{-\{i,j\}}) \quad \forall i, j \in N,$$

we get:

$$u_i(x,x,x_{-\{i,j\}}) - u_i(y,x,x_{-\{i,j\}}) + u_j(y,x,x_{-\{i,j\}}) - u_j(y,y,x_{-\{i,j\}}) +$$
$$+ u_i(y,y,x_{-\{i,j\}}) - u_i(x,y,x_{-\{i,j\}}) + u_j(x,y,x_{-\{i,j\}}) - u_j(x,x,x_{-\{i,j\}}) =$$
$$= u_i(x,x,x_{-\{i,j\}}) - u_i(y,x,x_{-\{i,j\}}) + u_i(x,y,x_{-\{i,j\}}) - u_i(y,y,x_{-\{i,j\}}) +$$
$$+ u_i(y,y,x_{-\{i,j\}}) - u_i(x,y,x_{-\{i,j\}}) + u_i(y,x,x_{-\{i,j\}}) - u_i(x,x,x_{-\{i,j\}}) = 0.$$

□

The conditions of theorem 9 are sufficient to guarantee the existence of pure equilibria. In fact, Rock-Paper-Scissors, described in 1.1.4, is a 3-player symmetric game with no pure strategy equilibria and Matching Pennies, described in 1.1.4, is a 2-player asymmetric game with no pure strategy equilibria.

Given a symmetric environment, we would expect symmetric equilibria, but the conditions of Theorem 9 alone are not sufficient to ensure symmetric profiles. In fact, the Anti-coordination game [1] is a symmetric game with no symmetric pure equilibria.

Cheng *et al.*(2004) prove the existence of symmetric equilibria for infinite symmetric games. The sufficient conditions are the following. The strategy set X_i of each player i is a nonempty, convex, compact subset of an Euclidean space, for example $\mathbb{R}$, and his payoff function is continuous in all its arguments and strictly quasi-concave in $x_i \in X_i$. The statement with its proof are quoted in Chapter 4 (Corollary 4). We remind that Nash (1951) proved the following theorem.

Theorem 10. *Every finite game has a " symmetric" equilibrium.*

Nash define as " symmetric" the profile invariant under every automorphism of the game. This is equivalent to a profile in which all the symmetric players (if any) are playing the same mixed strategies. In the case of a symmetric game this notion coincides with our definition, as Cheng *et al.*(2004) have already remarked. Then, we conclude saying that a finite symmetric game has a symmetric mixed-strategies equilibrium.

Now, we are going to deduce from potential function all the NE of a game. Let

$$G = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$$

be a symmetric binary game, with $X_1 = \{0,1\}$. Given $(x_1, \ldots, x_n) \in \prod_{i \in N} X_i$, let ν be the number of players voting 1, formally

$$\nu \doteq |\{i \in N : x_i = 1\}|.$$

[1] Each player receives 1 when the players choose different strategies and 0 otherwise.

Let $c_1(k)$ denote the payoff/cost to a player voting 1 when exactly k players vote 1, and let $c_0(k)$ denote the payoff/cost to a player voting 0 when exactly k players vote 0. Since players are identical and the game is binary, the payoff/cost to be paid depends only on the number ν of players voting 1 and not on their identities. So we identify a strategy profile $(x_1, \ldots, x_n)$ with ν The potential function for a symmetric binary game $P = P(x_1, \ldots, x_n)$ is a discrete function of variable ν:

$$P[\nu] = \sum_{k=1}^{\nu} c_1(k) + \sum_{k=1}^{n-\nu} c_0(k).$$

This expression is derived from congestion cost, since a symmetric binary game $G = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ is a congestion game whose facilities set is exactly the strategy set $X_1 = \ldots = X_n$.

Let G be a potential game and let P be a its potential. The set of all strategy profiles that maximize P is a subset of the equilibria set of the game G. The following theorem characterizes the NE of a symmetric binary game, searching the local optima of P.

Theorem 11. *Let $G = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ be a symmetric binary game with potential function $P = P[\nu]$. Then $(x_1, \ldots, x_n)$ is a NE of G if and only if the corresponding ν is such that*

$$\begin{array}{ll} P[\nu] \geqslant P[\nu+1] \text{ and } P[\nu] \geqslant P[\nu-1] & \text{if } 1 \leqslant \nu \leqslant n, \text{ or} \\ P[\nu] \geqslant P[\nu+1] & \text{if } \nu = 0, \text{ or} \\ P[\nu] \geqslant P[\nu-1] & \text{if } \nu = n. \end{array}$$

Proof. By Definition 10 $(x_1, \ldots, x_n)$ is a NE of G if for each $i \in N$ and for each $y_i \in X_i$ we have $u_i(x_i, x_{-i}) \geqslant u_i(y_i, x_{-i})$. Since G is a potential game, then $(x_1, \ldots, x_n)$ is a NE if for each $i \in N$ and for each $y_i \in X_i$ we have $P(x_i, x_{-i}) \geqslant P(y_i, x_{-i})$. By symmetry of G, P is function of the number of players choosing the same strategies and $X_1 = \ldots = X_n$. But G is also binary, then $X_1 = \{0, 1\}$, P is function only of the discrete variable ν (which represents the number of players choosing the strategy 1), and the only unilaterally deviations happen when ν increases by one and decreases by one. From this, the claim follows. $\square$

Chapter 3

Naming Games

3.1 Introduction

Inspiration for this chapter came from the TUIC games. This extension of the standard definition of TU game was given in Moretti-Patrone (2004), as a way to take into account the costs incurred when one needs to obtain the values of the various $v(S)$ (or $c(S)$, for a cost game). In that paper, particular emphasis was given to a special case, that of shared facilities, which we shall re-analize in this chapter.

Consider, for example, the problem of sharing the cost of printers, copiers, faxes among the members of a Department. An easy way to divide costs is to divide them evenly among all the Department members. But this way violates fairness, and, at the same time, seems to lack good incentive properties. It would be reasonable to take into account whether a member uses or not a given facility (e.g. a printer), and also some measure of intensity of its use. Here we shall concentrate on the case in which the 'fixed costs' of a facility are the only ones significant, that is we neglect the intensity of use. In such a case, a natural approach could be that of dividing the cost of a facility among the users (Young 1994).

Even if it is common knowledge who the users are, to be able to enforce the payment one perhaps needs to have a way to make verifiable to a third part who the users are (possible solutions could be: locked rooms, cards, or some ad hoc software to detect the users of a given facility): so, to make it verifiable an additional cost is to be added. If a group of people finds that the costs generated by these operations are less than the imputed costs for the use of facilities that they do not use, then they will have an incentive to ask that costs are made verifiable. The question is whether and how a fair allocation can be achieved. A relevant aspect is the institutional setting, that is the way in which players reach such a decision.

A rule could be that if there is someone who asks for the verification, then it must be done, and these additional costs will be imputed to the set of people that ask for the verification. These costs will be divided evenly, while costs for the use of facilities, will be divided among the users (evenly). This rule allows us to consider a non cooperative game in strategic form, in which each player has just two alternatives, to ask/not to ask that the verification is made. It turns out that, by deleting dominated strategies, the game can be reduced to one with a potential, which guarantees that it has a Nash equilibrium in pure strategies (this result can be achieved also by exploiting the symmetry structure of the game). Notice that this procedure could be applied to more than one facility. It is fairly obvious that, in such a case, the problem simply amounts to treating each facility independently.

Another rule that we shall investigate is the decision by voting (in particular, the absolute majority rule will be considered). In such a case, it becomes relevant whether there will be a set of independent votations (one for each facility) or a unique one. In the latter case, linkage effects can be obviously present.

If we introduce the possibility of abstention from voting, the effect consists in increasing the number of equilibria. That is, by adding a third strategy to the binary game, the new game has the same outcomes as the original one.

The paper is structured as follows. In Section 3.2 we introduce the notations and assumptions. In Section 3.3 we present the first model, the naming game, in which each player names the machine for which he asks for verification. We examine first the case with only one machine (Subsection 3.3.2) and then with different ones (Subsection 3.3.3). Since a symmetric game with only two strategies is a potential game and then it has a pure Nash Equilibrium, we can model a special case of environmental game via naming game, that is via a potential or congestion game, a result processed in Section 3.6, whose proofs are in Section 3.9. In Section 3.4 we investigate the correlated equilibria and games with contracts applied to the model of Naming game with one facility. In Section 3.5 we analyze the printer game on the condition of a majority decision with only one facility (Subsection 3.5.1) or with more machines (Subsection 3.5.2). In Section 3.7 we introduce the possibility of abstention from voting. In Section 3.8 we present the obtained results in this Chapter.

3.2 Notations and Assumptions

We shall assume that there is a finite set of players or agents

$$A = \{1, \ldots, a\},$$

which can use a set of facilities (or cost components)

$$M = \{1, \ldots, m\}.$$

Let

$$M_i \subseteq M \text{ , with } \mathrm{m_i} = |\mathrm{M_i}|,$$

be the set of the machines *not* used by player $i \in A$ while, dually, let

$$N^j \subseteq A \text{ , with } \mathrm{n^j} = |\mathrm{N^j}|,$$

be the set of the players *not* using the facility $j \in M$. Assuming that $N^j \neq A$ it is non restrictive. Then,

$$N \doteq \bigcap_{j \in M} N^j \text{ , with } \mathrm{n} = |\mathrm{N}|,$$

is the set of non users, and

$$U \doteq A \setminus N \text{ , with } \mathrm{u} = |\mathrm{U}|,$$

the set of machine users. Lastly, let

$$c^j > 0$$

be the cost associated to $j \in M$: we shall assume that this cost is given and independent of the (non empty) set of players using the facility j and that, to make the set of users for a game facility verifiable an additional cost

$$\chi^j > 0$$

is needed. When we look at only one machine, we will call the facility m, the machine cost c and the "checking" cost χ.

We shall use the index to indicate sets or objects linked with a specific player and the apex to indicate those with a specific machine.

The situation as described leads in a natural way to a TUIC game (Moretti and Patrone, (2004)). In our case, N is A, T is $\mathcal{P}(M)$ and $\prec$ is the relation of strict inclusion in $\mathcal{P}(M)$. The characteristic function associated with the parameter $t \in T$ is

$$c_t : \mathcal{P}(A) \longrightarrow \mathbb{R},$$

with

$$c_t(S) = \sum_{j \in t | S \cap (A \setminus N^j) \neq \emptyset} C_j + \sum_{j \notin t} C_j$$

and with

$$\chi_t = \sum_{j \in t} \chi^j.$$

3.3 Naming Games

3.3.1 An example

We begin introducing a very simple example, to show the kind of issues we shall consider. "*To name*" here means *asking for checking.*

We have three players I, II, III and one facility m, used only by player III. The associated strategic (cost) game is $G = \langle X_1, X_2, X_3, C_1, C_2, C_3 \rangle$, where $X_i = \{0, 1\}$ is the strategy set for player i (*1* stands for *naming*, *0* for *not*), in Figure 3.1 below:

$I \backslash II$	0			1		
0	$\frac{c}{3}$	$\frac{c}{3}$	$\frac{c}{3}$	0	χ	c
1	χ	0	c	$\frac{\chi}{2}$	$\frac{\chi}{2}$	c

0

$I \backslash II$	0			1		
0	0	0	$\chi + c$	0	$\frac{\chi}{2}$	$c + \frac{\chi}{2}$
1	$\frac{\chi}{2}$	0	$\frac{\chi}{2} + c$	$\frac{\chi}{3}$	$\frac{\chi}{3}$	$\frac{\chi}{3} + c$

1

III

Figure 3.1: Cost game $G = \langle X_1, X_2, X_3, C_1, C_2, C_3 \rangle$.

Since player III has a strongly dominated strategy, we work with the reduced game $\overline{G} = \langle X_1, X_2, C_1, C_2 \rangle$ with cost matrix:
Let us consider the function $P : X_1 \times X_2 \to \mathbb{R}$ given by: It follows that $P(0, x_{-i}) - P(1, x_{-i}) = C_i(0, x_{-i}) - C_i(1, x_{-i})\ \forall i \in \{1, 2\}$, i.e., the function P measures the gap of game costs of an unilaterally deviating player. So this symmetric game is a (cost) potential game, and it has two pure Nash equilibria $(0, 1)$ and $(1, 0)$, obtained by minimizing the function P. Since it

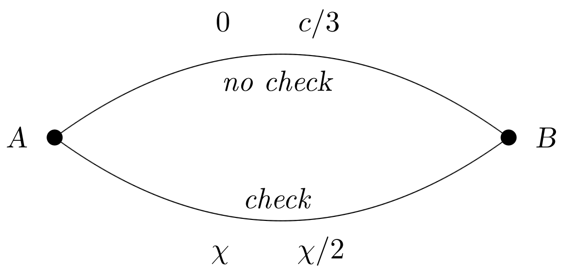
I \ II	0		1	
0	$\frac{c}{3}$	$\frac{c}{3}$	0	χ
1	χ	0	$\frac{\chi}{2}$	$\frac{\chi}{2}$

Figure 3.2: Reduced game $\overline{G} = \langle X_1, X_2, C_1, C_2 \rangle$.

I \ II	0	1
0	$\frac{c}{3}$	χ
1	χ	$\frac{3}{2}\chi$

Figure 3.3: Potential P.

has a potential, the game can be interpreted as a congestion game [1], with two roads from A to B, which we shall identify as *no check* and *check*.



The cost for using the road *no check* is 0 if one player uses it and $\frac{c}{3}$ if both players use it; the cost of using the road *check* is χ if one player asks for checking and $\frac{\chi}{2}$ if both ask.

All of these results generalize to the case of any number of players and facilities. This will be the subject of the following subsections. Hereafter, given a strategy profile $(x_1, \ldots, x_n)$, in our model ν will always denote the number of players asking for check.

[1] Given a finite set of facilities, whose costs depend on the number of users, a *congestion game* (Rosenthal (1973) or Monderer and Shapley (1996) or Voorneveld (1999)) is a cost game where each player pays the cost of all facilities he uses. Every congestion game is a potential game and every finite potential game is isomorphic to a congestion game.

3.3.2 The one facility case

We can now formalize the case with one facility. Let $G = \langle (X_i)_{i\in A}, (C_i)_{i\in A}\rangle$ be the *elementary* game, that is the game with only one machine, where $X_i = \{0,1\}$ is the strategy set for player i and $C_i : \prod_{i\in A} X_i \to \mathbb{R}$ is the cost for player i.

The strategy *check* of all users $i \in A\backslash N$ is strongly dominated by the strategy *no check*, since $C_i(1, x_{-i}) = \frac{c}{a-n} + \frac{\chi}{\nu} > C_i(0, x_{-i})$, where $C_i(0, x_{-i}) = \frac{c}{a}$ if $x_{-i} = (0, \ldots, 0)$ and $C_i(0, x_{-i}) = \frac{c}{a-n}$ otherwise. After removing strongly dominated strategies, we get the reduced game $\overline{G} = \langle (X_i)_{i\in N}, (C_i)_{i\in N}\rangle$. It is a symmetric binary game, so, from Theorem 9, it is a potential game and, for Corollary 2, it has at least a pure Nash equilibrium. The potential function is $P : \prod_{i\in N} X_i \to \mathbb{R}$, given by

$$P(x_1, \ldots, x_n) = P[\nu] = \begin{cases} \frac{c}{a} & \text{if } \nu = 0 \\ \chi + \frac{\chi}{2} + \ldots \frac{\chi}{\nu} & \text{otherwise.} \end{cases}$$

In particular we have the following results.

Theorem 12. *Let $\overline{G} = \langle (X_i)_{i\in N}, (C_i)_{i\in N}\rangle$ be the game reduced from elementary game G.*

i. If $\chi < \frac{c}{a}$, then $\overline{G}$ has n pure NE, with $\nu = 1$.

ii. If $\chi > \frac{c}{a}$, then $\overline{G}$ has only one pure NE, with $\nu = 0$.

iii. If $\chi = \frac{c}{a}$, then $\overline{G}$ has $n+1$ pure NE, with $\nu = 0$ and $\nu = 1$.

Proof. For $1 < \nu \leqslant n$, we have $\chi < \sum_{k=1}^{\nu} \frac{\chi}{k}$. Then the discrete variable function P is a strictly increasing function when $\nu > 1$and the strategy profiles corresponding to ν with $\nu > 1$ are not NE. So we search the equilibrium profiles between $\nu = 0$ and $\nu = 1$.

If $\chi < \frac{c}{a}$, the NE are $(x_1, \ldots, x_n)$ corresponding to $\nu = 1$. The symmetry assumption implies that there is not a single NE in the game, but n, i.e. the number of players of the reduced game.

If $\chi > \frac{c}{a}$, the only NE is $(0, \ldots, 0)$, that is $\nu = 0$.

If $\chi = \frac{c}{a}$, the NE are $(x_1, \ldots, x_n)$ corresponding to $\nu = 0$ and $\nu = 1$. □

3.3.3 The general case

We can generalize to the case of m machines. We will show that a game with m facilities is the sum of m elementary potential games, so it is a potential game itself. Furthermore, the potential function is the sum of elementary potential functions.

Let $G^j = \langle (X_i^j)_{i\in A}, (C_i^j)_{i\in A}\rangle$ be the elementary game of machine $j \in M$, where A is the set of players, $X_i^j = \{0,1\}$ is the strategy set for player $i \in A$ w.r.t. machine $j \in M$ and

$$C_i^j : \prod_{i\in A} X_i^j \to \mathbb{R}$$

is the cost for player $i \in A$ w.r.t. machine $j \in M$.

Let $G = \langle (X_i)_{i\in A}, (C_i)_{i\in A}\rangle$ be the global game, where A is the set of players, $X_i = \prod\limits_{j\in M} X_i^j$ is the strategy set for player $i \in A$ and

$$C_i : \prod_{i\in A}\prod_{j\in M} X_i^j \to \mathbb{R}$$

is the cost for player $i \in A$ so defined

$$C_i((x_1^1,\ldots,x_1^m),(x_2^1,\ldots,x_2^m),\ldots,(x_a^1,\ldots,x_a^m)) \doteq$$

$$C_i^1(x_1^1,x_2^1,\ldots,x_a^1) + C_i^2(x_1^2,x_2^2,\ldots,x_a^2) + \ldots + C_i^m(x_1^m,x_2^m,\ldots,x_a^m).$$

We shall briefly refer to this fact saying (somehow improperly) that $G = G^1 + \ldots + G^m$. So, we focus the attention on one machine j and we look at the elementary game G^j. We eliminate the strongly dominated strategies and work on $\overline{G}^j = \langle (X_i^j)_{i\in N^j}, (C_i^j)_{i\in N^j}\rangle$, a game reduced to players in N^j. Since the reduced game is a symmetric binary game, for theorem 9 it has potential P^j. So the overall game $\overline{G}$ [2] - sum of games $\overline{G}^j$ - has potential P - sum of potential P^j of games $\overline{G}^j$. Then the game $\overline{G}$ has at least one pure Nash equilibrium. We formulate the following general result.

Proposition 5. *Let $\Gamma = \Gamma^1 + \ldots + \Gamma^m$ be a game defined for players in $\bigcup\limits_{j=1}^m N^j$ with m facilities such that each game $\Gamma^j = \langle (X_i^j)_{i\in N^j}, (u_i^j)_{i\in N^j}\rangle$ is a*

[2] The player set of game $\overline{G}$ is $\bigcup\limits_{j=1}^m N^j$. Some players of $\overline{G}^j$ can use facility k, that is $N^j \neq N^k$ with $j,k \in M$. So, in the game $\overline{G}$, the strategy of player $i \notin N^k$ is $x_i^k = 0$, being i a user of facility k.

potential game with potential $P^j : \prod_{i\in N^j} X_i^j \to \mathbb{R}$, $\forall j = 1,\ldots,m$. *Then the function*

$$P : \prod_{i\in \bigcup_{j=1}^{m} N^j} \prod_{j\in M_i} X_i^j \to \mathbb{R}$$

so defined

$$P((x_i^j)_{j\in M_i})_{i\in \bigcup_{j=1}^{m} N^j} \doteq \sum_{j\in \bigcup_{i=1}^{m} M_i} P^j((x_i^j)_{i\in N^j})$$

is a potential of Γ.

Proof. For each $i \in \bigcup_{j=1}^{m} N^j$, each $x_{-i} = (x_{-i}^j)_{j\in M_i} \in \prod_{k\in \bigcup_{j=1}^{a} N^j, k\neq i} \prod_{j\in M_k} X_k^j$, and all $x_i = (x_i^j)_{j\in M_i}, y_i = (y_i^j)_{j\in M_i} \in \prod_{j\in M_i} X_i^j$, we have

$$P(x_{-i},y_i) - P(x_{-i},x_i) = P((x_{-i}^j, y_i^j)_{j\in M_i}) - P((x_{-i}^j, x_i^j)_{j\in M_i}) =$$
$$= \sum_{j=1}^{m} P^j(x_{-i}^j, y_i^j) - \sum_{j=1}^{m} P^j(x_{-i}^j, x_i^j) = \sum_{j=1}^{m} u_i^j(x_{-i}^j, y_i^j) - \sum_{j=1}^{m} u_i^j(x_{-i}^j, x_i^j) =$$
$$= u_i((x_{-i}^j, y_i^j)_{j\in M_i}) - u_i((x_{-i}^j, x_i^j)_{j\in M_i}) = u(x_{-i},y_i) - u(x_{-i},x_i).$$ □

We apply the previous result to the sum of reduced games $\overline{G} = \sum_{j\in \bigcup_{j=1}^{a} M_i} \overline{G}^j$ with elementary potential $P^j : \prod_{i\in N^j} X_i^j \to \mathbb{R}$, $\forall j = 1,\ldots,m$. Then the function $P : \prod_{i\in \bigcup_{j=1}^{m} N^j} \prod_{j\in M_i} X_i^j \to \mathbb{R}$ is a potential of $\overline{G}$.

3.4 Cutting down on paying or paying fairly

Here we make some comments on the reduced naming game $\overline{G}$ with one facility. First, we search for lower costs and then how to remove the problem of many equilibria.

If $\chi > \frac{c}{a}$ the strategy 1 for each player $i \in N$ is strongly dominated. Eliminating it, the game becomes trivial and we cannot improve the outcome cost.

If $\chi < \frac{c}{a}$ we have found n non symmetric pure NE, but a special case of Nash's theorem [3] ensures the existence of at least a symmetric mixed NE. The reduced game of the example of the Subsection 3.3.1 for $\chi < \frac{c}{3}$ has, in addition to the two pure equilibria, a mixed NE consisting, for each player, in playing 0 with probability $\frac{\chi}{2}/(\frac{c}{3} - \frac{\chi}{2})$ and with payoff $\frac{\chi}{2}\frac{c}{3}/(\frac{c}{3} - \frac{\chi}{2})$.

Is it possible to find a solution yielding better outcomes without having to turn to binding agreements? The correlated equilibria (Aumann (1974)) add nothing from a cost point of view. In the example of the Subsection 3.3.1 the expected payoff is $(\frac{\chi}{2}, \frac{\chi}{2})$ playing correlated equilibria μ, where μ allocates probability $(0, \alpha, \alpha, 1 - 2\alpha)$ with $0 \leqslant \alpha \leqslant 1$, to the strategy pairs $((0,0),(0,1),(1,0),(1,1))$. The correlated equilibrium is more efficient than a mixed equilibrium, since $\frac{\chi}{2} < \frac{\chi}{2}\frac{c}{3}/(\frac{c}{3} - \frac{\chi}{2})$, even if it induces the same payoff as the one when players vote unanimously for check.

How do we solve the problem of more equilibria? Moreover the ex-post situation is unworthy, while the ex-ante phase was fair. Any NE is a priori fair, but in our example only one player in equilibrium behaves as the scapegoat of the situation, taking all the checking costs upon himself. The situation is better understood with more players. If all the *non*-users are ex-ante identical, who sacrifices oneself ex-post? How to remedy to social injustice? We examine two possible ways which provide no new insight but at the same time orient towards the outcome of the great coalition of a cooperative game.

A possible solution to reduce coordination problems is to play on the symmetry of the problem and to force all the players to choose the same strategy. The equilibrium profile is realized by total coordination of players. We can still resort to correlated equilibrium or to games with contracts (Myerson (1991)). The game $\bar{G}$ assumes the form:

$I \backslash II$	0		1		c	
0	$\frac{c}{3}$	$\frac{c}{3}$	0	χ	$\frac{c}{3}$	$\frac{c}{3}$
1	χ	0	$\frac{\chi}{2}$	$\frac{\chi}{2}$	χ	0
c	$\frac{c}{3}$	$\frac{c}{3}$	0	χ	$\frac{\chi}{2}$	$\frac{\chi}{2}$

where c is the contract-signing strategy for each player [4]. The tran-

[3] *Every finite symmetric game has a symmetric Nash equilibrium*, in the sense that all the players are playing the same mixed strategy. (Nash (1951))

[4] If this contract is signed by both players, they promise to choose 1 and the payoff is $\frac{\chi}{2}$ for both undersigned, while if it is signed by only one player, then he will choose 0.

sformed game has, in addition, the equilibrium (c, c), with cost allocation $(\frac{\chi}{2}, \frac{\chi}{2})$.

Another proposal is in the direction of a partial cooperative agreement (Mallozzi and Tijs (2006)). Let $G = \langle X_1, \dots, X_n, u_1, \dots, u_n \rangle$ be a game, with the same strategy set $X_1 = \dots = X_n$ for each player in $N = \{1, \dots, n\}$. We suppose that a group $\{k+1, \dots, n\}$ of players participate in an agreement, the remaining players acting, as singletons, in a non-cooperative way. k is the level of non cooperation. The game is a two-stage game: first signatories announce their joint strategy, then non-signatories react by playing a non-cooperative game.

Definition 45. *[Mallozzi and Tijs (2006)] Given k, a partial cooperative equilibrium is a strategy profile* $(x_1, \dots, x_n) \in \prod_{i \in N} X_i$*, such that the signatories* $\{k+1, \dots, n\}$ *choose the same strategy* $x_{k+1} = \dots x_n = y \in X_1$ *and the non signatories* $\{1, \dots, k\}$ *play a NE of the partial game*

$$\langle X_1, \dots, X_k, \{y\}, \dots, \{y\}, u_1, \dots, u_n \rangle.$$

We apply this concept to a naming game with one facility. The cooperative players choose 1 and we can read $\nu = n - k$ as the level of cooperation. Since our game is binary, each partial game, fixed ν, has trivially a unique NE, so the partial cooperative equilibrium is assured. Comparing the cost of a single signatory player, we choose the more opportune ν. The result is again the total cooperation. Once more we have interiorized the externalities: each player incurs expenses from which he will benefit like the other players.

We remark, lastly, that, in order to induce the cooperative solution, we can also impose an efficient disciplinary system, which is extern to the parts involved in the cooperation.

3.5 The Decision by Majority Rule: Voting Game

Now we set another rule depending on which group of players is able to get their decision implemented. Checking is made if the minimum number of votes, called *quorum q*, is reached. This is what usually happens in several decision-making situations. We re-analyze our model on the condition of a majority decision. There are two types of NE, the indifferent NE and the allure NE. We define *indifferent* a NE if the payoff/cost to unilaterally deviating player is invariant, and *allure* a NE if it appreciably changes. The latter equilibrium is more interesting than the former.

At first we will work with only one machine to simplify matters and afterwards we will examine the cases with more facilities. In Voting Game with one facility two cases occur. Let n be the number of non users. i) If $n \leqslant q$, there are only indifferent NE, since, not executing the checking, all players pay $\frac{c}{a}$. The equilibria are obtained when ν players ask for check with $0 \leqslant \nu < q-1$ or when $\nu = q$ users do not ask. ii) If $n > q$, in addition to indifferent NE there are allure NE when exactly q non users ask for check. We will study the latter case, because it is more interesting.

Since in the naming game only one vote was sufficient to make the check, the naming model can be read as a majority decision model with $q = 1$.

3.5.1 The one facility case

Let $G = \langle (X_i)_{i\in A}, (C_i)_{i\in A}\rangle$ be the elementary game as described in 3.3.2 with quorum q. The strategy *check* for all users $i \in A\backslash N$ is strictly dominated by strategy *no check*. In fact, for $0 < \nu < q$, $C_i(1, x_{-i}) = \frac{c}{a} = C_i(0, x_{-i})$ and, for $q \leqslant \nu \leqslant a$, $C_i(1, x_{-i}) = \frac{\chi}{\nu} + \frac{c}{a-n} > \frac{c}{a-n} = C_i(0, x_{-i})$. Removing strictly dominated strategies, the reduced game $\overline{G} = \langle (X_i)_{i\in N}, (C_i)_{i\in N}\rangle$ is a binary symmetric game and, for theorem 9, it is a potential game with potential

$$P(x_1, \ldots, x_n) = P[\nu] = \begin{cases} \frac{c}{a} & \text{for } 0 \leqslant \nu \leqslant q-1 \\ \frac{\chi}{q} + \ldots + \frac{\chi}{\nu} & \text{for } q \leqslant \nu \leqslant n. \end{cases}$$

We have the following result.

Theorem 13. *Let $\overline{G} = \langle (X_i)_{i\in N}, (C_i)_{i\in N}\rangle$ be the game reduced from the elementary game G.*

i. If $\frac{\chi}{q} > \frac{c}{a}$, then $\overline{G}$ has only indifferent NE, for $0 \leqslant \nu \leqslant q-1$.

ii. If $\frac{\chi}{q} < \frac{c}{a}$, then $\overline{G}$ has both indifferent NE, for $0 \leqslant \nu < q-1$, and allure NE for $\nu = q$.

iii. If $\frac{\chi}{q} = \frac{c}{a}$, then $\overline{G}$ has both indifferent NE, for $0 \leqslant \nu \leqslant q-1$. and allure NE, for $\nu = q$.

The number of NE is $\sum\limits_{\nu=0}^{q-1} \binom{n}{\nu}$ in case (i),

$$\sum_{\nu=0}^{q-2} \binom{n}{\nu} + \binom{n}{q} \text{ in case } (ii), \text{ and } \sum_{\nu=0}^{q} \binom{n}{\nu} \text{ in case } (iii).$$

Proof. The potential $P = P[\nu]$ is a discrete variable function, constant for $\nu \in \{0, \ldots, q-1\}$, and strictly increasing for $\nu \in \{q, \ldots, n\}$. Then, each $\nu \in \{0, \ldots, q-2\}$ corresponds to indifferent NE. For $\nu \in \{q, \ldots, n\}$, P is strictly increasing, then the strategy profiles corresponding to $\nu > q$ are not NE. So we compare $v = q-1$ and $\nu = q$.

If $P[q-1] = \frac{c}{a} < P[q] = \frac{\chi}{q}$, then $q-1$ corresponds to indifferent NE.

If $P[q-1] = \frac{c}{a} > P[q] = \frac{\chi}{q}$, then q corresponds to allure NE.

If $P[q-1] = \frac{c}{a} = P[q] = \frac{\chi}{q}$, then $v = q-1$ and $\nu = q$ correspond to indifferent NE. □

A simple numerical example can help understanding the above analysis. Let us suppose to have seven players, only one user of one facility, and $q = 4$. The actions of players in N and their costs are summarized in the following table.

ν	*action*						*individual cost*						*NE*
0	0	0	0	0	0	0	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	indifferent
1	1	0	0	0	0	0	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	indifferent
2	1	1	0	0	0	0	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	indifferent
3	1	1	1	0	0	0	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	$\frac{c}{7}$	
4	1	1	1	1	0	0	$\frac{\chi}{4}$	$\frac{\chi}{4}$	$\frac{\chi}{4}$	$\frac{\chi}{4}$	0	0	allure
5	1	1	1	1	1	0	$\frac{\chi}{5}$	$\frac{\chi}{5}$	$\frac{\chi}{5}$	$\frac{\chi}{5}$	$\frac{\chi}{5}$	0	
6	1	1	1	1	1	0	$\frac{\chi}{6}$	$\frac{\chi}{6}$	$\frac{\chi}{6}$	$\frac{\chi}{6}$	$\frac{\chi}{6}$	$\frac{\chi}{6}$	

Remark 1. Also with a majority decision, the correlated equilibria, supposing $\chi < \frac{c}{N}$, improve the outcomes of each player because the total cost is constant in each equilibrium outcome.

3.5.2 Overall Game: m machines - Majority Decision

Let us review a game with m machines. We can choose between various rules of the game. By instituting *disjoint votations* each player votes for each machine so we have m independent ballots, then the overall game is split into the sum of m elementary games, whose equilibria are obtained by juxtapposing the equilibria of each elementary game. Hence we are back to the situation of a single machine. Using a *unique votation* instead, each player votes for all the machines. The case is more ticklish since the outcomes are due to the combination of different factors such as the amount of users and the cost of each individual printer.

We remind that $M \setminus M_i$ is the set of machines used by player i, $A \setminus N^j$ is the set of players using machine j with $a - n^j$ its cardinality, and $U \doteq A \setminus N$ the set of facility users.

We sort out the players into three sets.

$$\breve{A} := \{i \in A \mid \frac{\chi}{q} + \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j} > \frac{1}{a} \sum_{j \in M} c^j\},$$

$$\bar{A} := \{i \in A \mid \frac{\chi}{q} + \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j} = \frac{1}{a} \sum_{j \in M} c^j\},$$

and

$$\hat{A} := \{i \in A \mid \frac{\chi}{q} + \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j} < \frac{1}{a} \sum_{j \in M} c^j\},$$

with the convention that we set $\sum_{j \in M \setminus M_i} \frac{c^j}{a-n^j} = 0$, when $i \in N$, that is when i is a non-user, or equivalently when the set $M \setminus M_i$ is empty. If $i \in \breve{A}$, the strategy *check* is strictly dominated by the strategy *no check*. So we eliminate it, passing to the reduced game $\hat{G} = \langle (X_i)_{i \in \hat{A} \cup \bar{A}}, (C_i)_{i \in \hat{A} \cup \bar{A}} \rangle$. In this way, we are supposing that it is convenient for some users to ask for check. The game $\hat{G}$ is binary, but non symmetric, since the players are not identical. Further more $\hat{G}$ is a non potential game, since for $i \in \hat{A} \setminus N$ and $k \in N$:

$$C_i^q(1, x_{-i}) - C_i^{q-1}(0, x_{-i}) + C_k^{q+1}(1, x_{-k}) - C_k^q(0, x_{-k}) + C_i^q(0, x_{-i}) - C_i^{q+1}(1, x_{-i}) +$$

$$+ C_k^{q-1}(0, x_{-k}) - C_k^{q-1}(1, x_{-k}) = +(\frac{\chi}{q} + \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j}) - \frac{1}{a} \sum_{j \in M} c^j + \frac{\chi}{q+1} - 0 +$$

$$+ \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j} - (\frac{\chi}{q+1} + \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j})) + \frac{1}{a} \sum_{j \in M} c^j - \frac{\chi}{q} = \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j}.$$

The apex over cost function informs about the number of checking requests. The condition of Corollary 3 is dealt only in the simple closed path of length four. Obviously, if $\hat{A} = N$ and $\bar{A} = \emptyset$, then the game G is reduced to only the players in N and $\hat{G} = \langle (X_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is a potential game.

The game has indifferent NE for $0 \leqslant \nu < q - 1$ and allure NE for $\nu = q$, the profiles for $q < \nu \leqslant n$ are not NE since for each player that asks for check, it is convenient to deviate and to not ask for it. Also for $\nu = q-1$ the strategy profile is not NE, since if $i \in \hat{A} \setminus N$, $C_i^{q-1}(0, x_{-i}) = \frac{1}{a} \sum\limits_{j \in M} c^j > \frac{\chi}{q} + \sum\limits_{j \in M \setminus M_i} \frac{c^j}{a-n^j} = C_i^q(1, x_{-i})$. We get down to examining the possible deviations for allure outcomes since for the indifferent equilibria the proof is obvious, as the outcome has the same costs. Let $i \in \hat{A} \cup \bar{A}$ be a player of reduced game. Then $C_i^q(1, x_{-i}) = \frac{\chi}{q} + \sum\limits_{j \in M \setminus M_i} \frac{c^j}{a-n^j} \leqslant \frac{1}{a} \sum\limits_{j \in M} c^j = C_i^{q-1}(0, x_{-i})$[5], and $C_i^q(0, x_{-i}) = \sum\limits_{j \in M \setminus M_i} \frac{c^j}{a-n^j} < \frac{\chi}{q+1} + \sum\limits_{j \in M \setminus M_i} \frac{c^j}{a-n^j} = C_i^{q-1}(1, x_{-i})$. Between the non symmetric allure NE, neither is Pareto efficient.

3.6 Environmental Game

The model presented in this paper is suitable for concrete situations where at most one coalition can be formed. The economic applications are, for example, joint ventures, cartels, or enviromental protocols. We pause over the last ones, considering in particular the model studied by Carraro and Marchiori (2003) about policy coordination on greenhouse gases emission, since it corresponds to our model. The standard environmental game is a two-stage game whose players are a finite set of countries. In the first stage (*coalition game*), countries decide non-cooperatively and simultaneously whether or not to sign the agreement, (i.e. to join a coalition) thus accepting the burden sharing rule of the coalition. In the second stage (*emission game*) the countries in each coalition decide the level of gases emission in order to maximize the welfare of the coalition. Formally, the two stage game in its normal form is

$$\langle Y_1, \ldots, Y_n, v_1, \ldots, v_n \rangle,$$

where $N = \{1, \ldots, n\}$ is a finite set of countries, Y_i is the strategy set for player $i \in N$, and the utility function v_i allocates the coalition gain shared according to the coalition burden-sharing rule, if i is a signatory in the first game, or his payoff, if i is a singleton. In order to particularize the model, Carraro and Marchiori (2003) make the following assumptions. *a1) Uniquss.* The emission game has a unique NE for each coalition. *a2) PANE.* Inside each coalition, in the emission game, players act cooperatively in order to maximize the joint payoff, whereas coalitions and singletons compete in a non-cooperative way. *a3) Symmetry.* All players are ex-ante identical. *a4)*

[5]The sign of equality holds only if $i \in \bar{A}$.

Single coalition. Only one coalition can be formed, the non-signatoring countries play as singletons.

For the assumption $a1$), the second stage of the game can be reduced to a partition function obtained (for assumption $a2$)) as a NE payoff of the coalition game. Then, the study of the two-stage game is reduced to the analysis of the first-stage game. The symmetry assumption $a3$) means that each player has the same strategy space in the emission game and receives the same payoff as the other members of his coalition. So, each coalition can be identified with its size c. For the assumption $a4$), the strategies of the coalition game consist in a binary choice. The environmental game becomes a binary choice game:

$$\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle,$$

where X_i has two options (sign/not sign) and u_i divides in equal proportions the gain of coalition among the signatories while allocating the singleton his payoff. The structure of this game is the same as our model, even if our model is a cost game. Whereas we work on the game in a normal form, Carraro and Marchiori (2003) study the game in a partition function form, in particular they analize the per-member partition function $p(c, \pi)$ and the non-member one $p(1, \pi)$ which represent respectively the payoff u_i of a player i belonging or not to the coalition π of size c. By symmetry $a3$), these functions depend on the coalition size c, as in our model the cost to be paid depends only on the number ν of players asking for check.

In their work, Carraro and Marchiori determine the equilibrium coalition and how it changes modifying the rules of the game. The equilibrium coalition structure depends on important features: the membership rules, the order of moves, the players' conjectures, the shape of profitability function, the type of free-riders. Specifying these features, it is possible to match our model with theirs. It is enough to assume that the players are free to join or to leave the coalition without the consensus of the other coalition members (*open membership*): the game is with *Nash conjectures* (each player takes his decision given the decisions of the others, which do not change as a consequence of the decision of the first player) and the payoff function is *humped-shaped.* In particular assuming orthogonal free-riding [6] their model corresponds to our naming game, while assuming non-orthogonal free-riding [7] corresponds to our voting game. The Appendix features the proof of our

[6]The free-riders are orthogonal if they benefit from the cooperative abatement of the coalition, but have no incentive to damage it: in international environmental games, there is no leakage (Carraro and Marchiori (2001) page163)

[7]The free-riders are non orthogonal if they benefit from countries' cooperation and

claim. The research of Carraro and Marchiori and ours have been tackled in different ways. They worked to identify the size of the unique Nash stable coalition, or rather internally and externally stable coalition. By contrast, we fixed the quorum to make the checking and morover we used the condition of Nash equilibrium. They assume the payoff functions u_i are twice continuously differentiable functions, while we work with discrete functions c_i. Again, they formulate the game with at least three players and require at least two players to make a coalition, instead we suppose the players of the game to be at least two (otherwise there is no game) and the coalition to be composed of at least one player. The little difference between their size of an equilibrium coalition and our quorum is thus explained . Lastly, we note that environmental game turns out to be a potential game, being a symmetric two-strategy game. This guarantees the existence of pure Nash equilibria.

3.7 Abstention from Voting

Once again let us perturb the rules of the game. We introduce the possibility of abstention from voting, adding a third strategy Ab for each player. We shall analyze two cases: the abstentionists do not pay the checking cost or they pay a part of the quota.

When the abstentionists do not pay the cost of a possible checking, the strategy *abstention* has the same effects as strategy *no check*, since $C_i(Ab, x_{-i}) = C_i(0, x_{-i})$, for $i \in A$. We can thus think in terms of only two strategies: *check* and *no check*, and, at the end, we have duplicated the equilibria (substituting Ab with 0 in the strategy profiles).

Let us analyze the situation when the checking cost is divided into equal parts among the abstentionists and the players who vote 1, since we can refer to it also the case when the abstentionist pays a share of cheching cost (that is $\alpha\chi$, with $\alpha \in (0, 1]$) and the players who vote 1 the remaining quota.

First, we consider the naming game with one facility. The strategy *check* for a player in U is strongly dominated by the strategy *no check*, so we eliminate it. There are various possibilities. i) If $\frac{c}{a} < \frac{\chi}{a}$, that is $c < \chi$, the strategy *check* for a player in N is strongly dominated by the strategy *no check*. By eliminating it, the checking is never made since the only strategies

damage the coalition, for example, in environmental games, by increasing emission whenever cooperating countries reduce their own. (Carraro and Marchiori (2001) page 164) of players, users or not, are *no check* and *abstention*, so the game has only one outcome. *ii*) If $\frac{c}{a} > \chi$ the strategy *abstention* for a player in U is strictly dominated by strategy *check*. By removing it, we lose no equilibrium. So let us consider the game limited to the players in N. Also for them the strategy *abstention* is strictly dominated by the strategy *check*, so the reduced game is a binary symmetric game. That is, by adding a strategy to the game, the equilibria are the same as the original game with two strategies. *iii*) In the intermediate cases, $\frac{\chi}{a} \leqslant \frac{c}{a} \leqslant \chi$, the equilibria are indifferent since the checking is not made.

Now, let us consider the majority game with one facility. The strategy *check* for a user is strictly dominated by the strategy *no check*, so we eliminate it. The strategy *Ab* for an user is again strictly dominated by the strategy *no check*, so we eliminate it and we study the symmetric game reduced to players in N. But the strategy *Ab* for a non user is strictly dominated by the strategy *no check*, so, by eliminating it, the reduced symmetric game becomes a binary game. Then, also in this case, adding a strategy, the equilibria are the same with two strategies, but larger in number since the strategy *Ab* has the same rule as strategy 0.

3.8 Conclusions

This Chapter presents a study of two games from the same model, the naming game and the majority decision game, both framed in two subcases (one and more facilities) working on a game reduced to only non user players. In the naming game with one facility the NE is reached when only one player asks for checking. The game with M facilities is shared in sum of M elementary games. In the majority decision game with one facility, with quorum q, the allure NE is reached when q players ask for checking. With M facilities, beyond non users, the players of the reduced game are players such that

$$\{i \in U \mid \frac{\chi}{q} + \sum_{j \in M \setminus M_i} \frac{c^j}{a - n^j} < \frac{1}{a} \sum_{j \in M} c^j\}.$$

Also in this case we have an allure NE with q votes for checking. It is immediately noticeable that the naming game is a special majority decision game with quorum $q = 1$. We can generalize claiming that, setting q the quorum with $1 \leqslant q \leqslant a$, we have the allure equilibria exactly when q players ask for check, while we have indifferent equilibria with $1 \leqslant \nu < q - 1$. The existence of pure NE is guaranteed only for first three cases examinated by the original result of this paper: given a symmetric game, a sufficient condition to have a potential game consists in having only two strategies. This theorem cannot be applied to the fourth case since it is not a symmetric game.

A possible improvement of this work is in the area of cooperative transferable utility games which measure the power of influence among players. It is assumed that players have to make a *yes-no* decision and each player has an inclination to say *yes* or *no*. But, due to influence of others, the decision of the player can be different from his inclination. A careful study of *influence indices* and *influential functions*, also with abstention, has been performed by Grabisch and Rusinowska (2008) and (2009).

3.9 Appendix

In this Appendix we compare the model of Carraro and Marchioni (2003) with ours. Obviously, the two models are not identical since we have constructed a discrete game, while Carraro and Marchioni (2003) made a continuous game applied to a discrete example. In their paper (2003), Carraro and Marchiori introduce three functions relating to a standard (payoff) game: the *profitability function*

$$P = P(c) \doteq p(c, \pi) - p(1, \pi^S),$$

the *free-riding function*

$$Q = Q(c) \doteq p(1, \pi) - p(1, \pi^S),$$

and the *stability function*

$$L = L(c) \doteq p(c, \pi) - p(1, \pi^{'}) = P(c) - Q(c-1),$$

where π is a coalition of size c, $\pi^{'}$ is a coalition of size $c-1$, and π^S is the singleton structure, that is all the players are singletons. P measures the profitability and L is useful to identify the size of a Nash stable coalition. In fact,

$$c^* \text{ is stable} \Leftrightarrow c^* = \max\{[x] | L(x) = 0 \,\mathrm{and}\, \mathrm{L}'(\mathrm{x}) < 0\}.$$

For convenience, let $c^* = 1$ when $L(c) < 0$ for all $c \in (1, n]$. Moreover, they define

$$c^\sharp = \max\{[x] | L(c) = 0 \,\mathrm{and}\, \mathrm{L}'(\mathrm{c}) > 0\}.$$

Again, it turns out that: *i*) if Q or the partition function of a player outside the coalition $p(1,\pi)$ is increasing in c, then the game is with positive spillovers, *ii*) if $P \geqslant 0$ and Q is convex, then the free riders are orthogonal, and *iii*) if P is negative for $c < c^m$, null in c^m, positive for $c > c^m$ and if Q is concave, then the free riders are non orthogonal. Hence, the definition of c^m follows.

The conversion of their continuous model into our discrete model is the following: c corresponds to ν, c^m corresponds to q, $p(c,\pi) = u_i(1, x_{-i})$ when $\nu \neq 0$, $p(1,\pi) = u_i(0, x_{-i})$ when $\nu \neq 0$, $p(1,\pi^S) = u_i(0, x_{-i})$ when $\nu = 0$, and π^S corresponds to $\nu = 0$. Since our model is a cost game, the cost profitability and the cost free-riding functions are defined in the opposite way, that is $P(c) \doteq p(1,\pi^S) - p(c,\pi)$ and $Q(c) \doteq p(1,\pi^S) - p(1,\pi)$. We analyze first the naming game with one facility under assumption $\chi < \frac{c}{a}$. $p(1,\pi) = u_i(0, x_{-i}) = \frac{c}{a}$ if $\nu = 0$ and $p(1,\pi) = u_i(0, x_{-i}) = 0$ if $\nu \neq 0$, so the game is with positive spillovers. $p(c,\pi) = u_i(1, x_{-i}) = \frac{\chi}{\nu}$ if $1 \leqslant \nu \leqslant n$. $p(1,\pi^S) = u_i(0, x_{-i}) = \frac{c}{a}$ with $\nu = 0$, $P[\nu] \doteq p(1,\pi^S) - p(c,\pi) = \frac{c}{a} - \frac{\chi}{\nu} > 0$. P is increasing in ν and c^m that is q is 1. Also the per-member partition function is monotonic since it differs from P in a constant. $p(1,\pi) = u_i(0, x_{-i}) = 0$ if $\nu \neq 0$, then $Q[\nu] \doteq p(1,\pi^S) - p(1,\pi) = \frac{c}{a} > 0$ and constant. Since $P[\nu] > 0$ and $Q[\nu]$ is convex, the game is with orthogonal free-riding. $L = L[\nu] \doteq P[\nu] - Q[\nu - 1] = -\frac{\chi}{\nu}$. We have: $L[\nu] < 0$, so by convention $\nu^* = 1$ that corresponds to our NE. The naming game with one facility under assumption $\chi < \frac{c}{a}$ satisfies the conditions of Proposition 6 of Carraro and Marchiori (2003).

Proposition 6 (Carraro and Siniscalco (1993), Barret (1994))**.** *In a game with Nash conjectures and open membership, in which the reaction functions are orthogonal, the stable coalition structures are π^* of size c^* when $1 < c^* < n$ and the grand coalition structure π^n of size n when $c^* > n$, that is $L(c) > 0$ for all $c \in [2, n]$, both when the per-member partition function is monotonic and when it is humped-shaped.*

Then we analyze the majority game with one facility assuming $\chi < \frac{c}{a}$. We have: $p(1,\pi) = u_i(0, x_{-i}) = \frac{c}{a}$ if $0 \leqslant \nu \leqslant q-1$ and $p(1,\pi) = u_i(0, x_{-i}) = 0$ if $q \leqslant \nu \leqslant n$. Here, the partition function of a non member is constant, so the game is with positive spillovers. $p(c,\pi) = u_i(1, x_{-i}) = \frac{c}{a}$ if $0 \leqslant \nu \leqslant q-1$ and $p(c,\pi) = u_i(1, x_{-i}) = \frac{\chi}{\nu}$ if $q \leqslant \nu \leqslant n$. $p(1,\pi^S) = u_i(0, x_{-i}) = \frac{c}{a}$ with $\nu = 0$, $P[\nu] \doteq p(1,\pi^S) - p(c,\pi) = 0$ if $0 \leqslant \nu \leqslant q-1$ and $P[\nu] = \frac{c}{a} - \frac{\chi}{\nu}$ if $q \leqslant \nu \leqslant n$. We have $P[\nu] > 0$ if $q \leqslant \nu \leqslant n$, c^m is just our quorum q and $P[\nu]$ is humped shaped, that is it is locally concave. $p(1,\pi) = u_i(0, x_{-i}) = \frac{c}{a}$ if $1 \leqslant \nu \leqslant q-1$ and $p(1,\pi) = u_i(0, x_{-i}) = 0$ if $q \leqslant \nu \leqslant n$, then $Q[\nu] \doteq p(1,\pi^S) - p(1,\pi) = 0$

if $1 \leqslant \nu \leqslant q-1$ and $Q[\nu] = \frac{c}{a} > 0$ if $q \leqslant \nu \leqslant n$. Since $P[\nu]$ is null for $\nu < q$ and positive for $\nu \geqslant q$, and since $Q[\nu]$ is concave, the game is with non orthogonal free-riding. $L = L[\nu] \doteq P[\nu] - Q[\nu-1] = 0$ if $1 \leqslant \nu \leqslant q-1$, $L[\nu] = \frac{c}{a}$ if $\nu = q$, and $L[\nu] = -\frac{\chi}{\nu}$ if $q+1 \leqslant \nu \leqslant n$. We have: $DL[\nu] = 0$ for $1 \leqslant \nu \leqslant q-1$, $DL[q] = \frac{c}{a}$, $DL[q+1] = -\frac{\chi}{q+1} - \frac{c}{a}$, and $DL[\nu] = \frac{\chi}{\nu(\nu-1)}$ for $q+2 \leqslant \nu \leqslant n$, where DL is the discrete derivative of L. Since $L[q] = \frac{c}{a} > 0$ and $L[q+1] = -\frac{\chi}{\nu} < 0$,and $DL[q+1] = \frac{L[q+1]-L[q]}{1} = -\frac{q+1}{\nu} - \frac{c}{a} < 0$, we have $\nu^* = q$, which corresponds to our allure NE. Our indifferent NE are given by searching ν such that $L[\nu] = 0$ and $DL[\nu] = 0$, that is $1 \leqslant \nu \leqslant q-1$. The majority decision game with one facility under assumption $\chi < \frac{c}{a}$ satisfies the conditions of Proposition 7 of Carraro and Marchiori (2003).

Proposition 7 (Carraro and Marchiori (2003))**.** *In a game with Nash conjectures and open membership, in which the profitability function is positive for $c \geqslant c^m$ and humped-shaped, and the free-riding is non orthogonal, the stable coalition structures are π^* of size c^* when $c^m \leqslant c^\sharp \leqslant c^* \leqslant n$ and the singleton structure π^S of size 1 when $c^\sharp > n$.*

Chapter 4

Quality Unilateral Commitments

4.1 Introduction

The words *Unilateral Commitments* assume a lightly different meaning, depending on its context. In Economics, for example, the term Unilateral Commitment indicates the commitment, assumed in an independent way, by an individual towards an organization, to undertake a single action. For example, in Pereau and Tazdait (1999), a unilateral commitment occurs when some countries decide to reduce their emission in an individual manner, without coordination, if bargaining with the scope to solve global environmental problems fail. That is, they provide for a new strategy, next to the strategies to cooperate or not with the other countries. In Game Theory, instead, the term Unilateral Commitment means the commitment of an action before the other players can move, that is the player chooses to commit a single strategy or to keep all of their strategies for a later decision. We will mean the second acceptation, in a more advanced version. For us, Unilateral Commitments means to assume binding commitments, taken simultaneously and unilaterally by all the players before the beginning of the game. This model is very close to *delegation* models, as showed in García-Jurado and González-Díaz (2006).

A Unilateral Commitments Game is a game, in which players can make unilateral commitments regarding their set of strategies. Formally, we add to a game, called *component game* or *basic game*, an initial round in which each player simultaneously, unilaterally, and publicly declare, in an enforceable way, that he will play only certain strategies of the basic game. We refer to these choices as *unilateral commitments* (briefly UC). In the second stage every player is committed to use strategies only from the subset choosen in the preliminary round. We underline that *i*) the unilateral commitments become common knowledge before the second stage starts, when they are publicly announced, *ii*) if the commitments are not made simultaneously and unilaterally, the players could influence each other, the new game could be played cooperatively, and then the players could achieve in equilibrium the cooperative payoffs of the game, and *iii*) the behavior can be considered as a signal used by the players to convey information about their type.

The tool of UC is suitable for repeated games especially, but in this chapter we consider only unilateral commitments in one-shot game for two reasons. First, we want to make a Nash equilibrium self-enforcing, without maving to recur to the finitely repeated games tool. Second, if we repeat a game G a finite number of times m, G^m is nothing else but another game. This justifies to pay attention to UC added to a generic strategic game, that is, to a game which is not necessarily a finite repetition of some constituent game.

There are many strategic situations, in real life, in which people disregard some of their possible strategies. For example, a firm announces a limited edition of a certain product or picks its capacity constraints or subscribes to quality programs in which it commits to follow certain rules, a department makes the commitment of not hiring its own graduate students, a party announces, during the election campaign, that it is not going to make certain alliances regardless of the final result of the election, or a candidate promises not to raise taxes by more than $x\%$,

The benefits of using Unilateral Commitments for a finite component game has already been studied and sounded by Faíña-Medín et al. (1998), García-Jurado et al. (2000), and García-Jurado et al. (2006). Faíña-Medín et al. (1998) proved that, if a preliminary round is included in a Prisoners' Dilemma repeated a large enough number of times, then not only the players have incentives to commit themselves to a limited strategy set, but there is also a symmetric subgame perfect equilibrium in which both players act cooperatively through the post-commitment stages of the game. Instead, García-Jurado et al. (2000) worked in a more general framework, considering finite repetitions of a n-person strategic game form, and they obtain a Nash folk theorem for finitely repeated games with UC. They proved that, if a n-strategic game is repeated a number of times large enough and players restrict their strategy sets in a preliminary round of the game, then every outcome, strictly preferred to the minimax outcome by every player, can be supported by a Nash equilibrium in pure strategies. Again, adding unilateral commitments to a constituent game without Nash equilibria generates Nash equilibria when the game is sufficiently repeated and can generate Nash equilibria even in the one-shot constituent game. García-Jurado et al. (2006) deepen the influence of UC in the assumption needed for the folk theorems for repeated games. They show that, when UC are possible, it is easier to find both Nash and subgame perfect equilibria supporting the cooperative payoff of the original game, so no condition on component game is needed for the Nash folk theorem to hold, when UC are considered. Again, they introduce a new equilibrium concept for extensive games: the virtual subgame perfect equilium, which we will present in Section 5.12 in Chapter 5.

Instead, Bade et al. (2006) consider the Unilateral Commitments of a infinite component game, where the players can choose a subset of their strategy set. They start from a paper of Schelling (1960) in which players can commit to a single action before the other players move or keep all of their actions for a later decision. Not much is gained, since the only outcomes of this commitment game are the original NE and some equilibrium outcomes of the sequential version of the game. So, they allow more flexibility in term of possible commitments assuming the players keep any non-empty, compact, convex subset of their original strategy space for their choice in the second stage. Allowing players to commit on sets of strategies affects the set of equilibrium outcomes. They prove that a strategy profile is *implementable by a commitment* [1] if and only if it is implementable by a *simple commitment*, that is the commitment where a player chooses a single strategy and the other a set depending on the best reply to it. Again they prove that all strategy profiles implementable by a commitment can be described as the equilibrium outcome of a generalized sequential version of the UC game. For example the lead-follow equilibrium outcomes is implementable in the following way. First we modify G such that one player is moving first and the other follows suit (Stakelberg duopoly). To implement such outcomes it suffices that the leader commits to a single strategy (his strategy in the lead-follow profile) and the other player does not restrict his strategy space at all.

The previous literature has analyzed how allowing players to commit on (finite or infinite) sets of strategies affects the equilibrium outcome and

[1] A strategy profile implementable by a commitment is a strategy profile of component game such that, when played in the second stage, is a subgame perfect equilibrium of the UC game.

which strategy profiles can be sustained as equilibrium outcomes. In this chapter, we are interested in a slightly different question. We ask whether, in order to obtain an efficient equilibrium, the commitment are sufficient or an other *escamotage* is needed? The answer we propose is the model of Quality Unilateral Commitments, i.e. unilateral commitments where, in the first stage, each player declares that he will pay a penalty if he will not play, in the second stage, the restricted strategies. This simple sequential game structure, assuming a self-punishing scheme, makes self-enforcing the Nash equilibria of the constituent game, if any. We analyze the case of a two player game G, in which strategy sets and permissible restrictions of them are compact subintervals of the real line and in which players have strictly quasi-concave payoff functions. We embed G into a two stage game, in which players can restrict their strategy spaces in the first stage. In the second stage, if a player chooses a strategy from his restricted strategy space, he obtains the same payoff as in the basic game G, otherwise he pays a penalty dependent on the square of the distance from his restricted strategy space.

The choice of each player to make quality commitments is not only in order to build a reputation, but also because, when the relations climate has been so positive and harmonious, the players will behave properly and make commitments, as it is proved in Snape and Redman (2006). Since a commitment is a binding of an individual to the others, it measures an attitude to the sense of altruism, compliance, identification and loyalty towards the group.

Our outlook has been inspired by the declaration of high-quality of very different products, from mineral water to jeans. For example, a well-known mineral water bottling factory guarantees the purity of mineral water in a newpaper promotion and, in order to give the highest guarantee of security, the firm endows itself with ISO certification, while a famous firm, producing brand-name jeans, hides away facing a declaration of intent, which guarantees the handmade cure with which they have been realized.

It is no accident we have called our model as quality commitments instead of penalty commitments. The Corporate Social Responsability (briefly, CSR) is developping and will develop in the following years. CSR is an enterprise, which not only produces wealth, makes products on the cheap, but also is dealing with business within the competence of State, Church, civil society, and family. The CSR, according to the opinion of Bruni ((2008a), (2008b)), is an oxymoron between capitalist enterprise and civil economy. The modern civil economy is based on market, symbol of freedom and equality. The capitalist enterprise has inherited the hierarchic structure by feudal society, like Churches, military structure, and family. The oxymoron is also in the way they are realized. The civil economy and market are based on contract, that is an exchange between ugual agents. In order to favour the cooperation, there is an efficient disciplinary system, extern to the parts involved in the contract. The hierarchic mechanism has asymmetric structure: there are a principal and one or more agents. The CSR develops on public *fides* or faith and on civic virtue, in order to overcome the dualism between interprise and market. Since the enterprice is the more efficient, the more it is similar to market, then CRS extends the contract, tipical of market, to the enterprise. It imports into the enterprise the pact, which is contract plus fides. It is no accident that, one year after the publication of *Dei delitti e delle pene* of Beccaria, Dragonetti (1768) publishes *Delle virtú e dei premi*, where he write the virtue cannot be negotiated (like penalty is settled by contract), but it is a product of free will. The same it happens in our model of QUC, since the sanctions is self declared by the players, then it is internal to the game. In the 2005 a famous food firm told the press to launch on the market a fair-solidarity product. This move has been choosen by other producers, while the importers have made a trademark on producing instead of product. The die of fair-solidarity market gives a new responsability to market, since it gives value to values. The rewards are beginning always more qualitative, and the users more sensible to ethical values, to environmental respect and then they pretend corporate social responsability, otherwise they punish the enterprises.

This chapter is organised as follows. In Section 4.2, we introduce the Unilateral Commitments of a game. In Section 4.3, we present the conditions that assure the existence of NE for infinite games. In Section 4.4, we introduce the Penalty Function Method, while in Section 4.5 we apply it to our model. Finally, in Section 4.6 we study the Quality Unilateral Commitments of Cournot Duopoly.

4.2 Unilateral Commitments

Remark 2. For ease of definition of $UC(G)$, the games G and $UC(G)$ are expressed in strategic form. At the same time, $UC(G)$ is a two-stage game, grafically it is represented in extensive form, and hence the subgame perfect equilibrium is the concept of solution more suitable for our analysis.

Given a game $G = \langle (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, the corresponding game with unilateral commitments consists of embedding the component game G into a two-stage game. In the first stage, each player, independently and simultaneously, chooses a nonempty subset S_i of his strategy set X_i, i.e. he commits to play only the choosen strategies in S_i. After this preliminary stage, the choices made by all of the players are publicly known, and the game G, restricted to $S_1, \dots, S_n$, is played. The commitments are assumed to be perfectly binding, in the sense that if player i restricts his strategy set to S_i, any strategy choosen in the second stage must belong to S_i. We note that a commitment does not necessarily prescribe the choice of a strategy. Ex-post, the players are free to choose from the set of strategies specified in the commitment.

Now, we formalize the UC-extension of a game G.

Definition 46. *Let $G = \langle (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic form game. The UC-extension of G is the game*

$$UC(G) = \langle (X_i^U)_{i \in N}, (u_i^U)_{i \in N} \rangle,$$

where:

i. the set of players N remains the same of G,

ii. the set X_i^U of strategies available to player $i \in N$ is the set of all the couples (S_i, ϕ_i) such that

$$j)\, S_i \subseteq X_i, \quad S_i \neq \emptyset, \quad \text{and} \quad jj)\, \phi_i : \prod_{k \in N} 2^{X_k} \longrightarrow X_i, \quad \text{with}$$

$$\phi_i(T_1, \dots, T_n) \in S_i, \quad \text{for each}\, (T_1, \dots, T_n) \in \prod_{k \in N} 2^{X_k},$$

iii. the payoff function of player i is $u_i^U : \prod_{i \in N} X_i^U \longrightarrow \mathbb{R}$ defined by

$$u_i^U((S_1, \phi_1) \dots, (S_n, \phi_n)) = u_i(\phi_1(S_1, \dots, S_n), \dots, \phi_n(S_1, \dots, S_n)).$$

Obviously, the condition $j)$ refers to the first stage, and the $jj)$ to the second stage. That is, a strategy for player i prescribes a choice of a restriction S_i (first-stage strategy) and of a strategy ϕ_i (second-stage strategy), for each possible choice of a restriction for all players in the first stage. Again the condition $\phi_i(T_1, \dots, T_n) \in S_i$ imposes that the commitments are binding.

The outcome of a strategy profile

$$((S_1, \phi_1) \dots, (S_n, \phi_n))$$

is the vector

$$((S_1, \dots, S_n), (s_1, \dots, s_n)),$$

where, for each $i \in N$,

$$\phi_i(S_i) = s_i.$$

The payoffs over outcomes $((S_1, \dots, S_n), (s_1, \dots, s_n))$ are assumed to depend only on the strategy profiles choosen in the second stage of the game and are given by the payoffs of the $UC(G)$. That is, the utility $u_i(s_1, \dots, s_n)$ of player i derives from outcome $((S_1, \dots, S_n), (s_1, \dots, s_n))$. The result of $((S_1, \phi_1) \dots, (S_n, \phi_n))$ is

$$(s_1, \dots, s_n),$$

where $((S_1, \dots, S_n), (s_1, \dots, s_n))$ is the outcome of strategy profile $((S_1, \phi_1) \dots, (S_n, \phi_n))$.

We illustrate the model with a simple example.

Example 2. In the following Figure 4.1, it is depicted the game form Γ and its Unilateral Commitments extension. To make more readable the drawing, some labels are omitted.

Let us recall briefly the following properties due to García-Jurado et al. (2000).

i) If the original game G has a NE $(\bar{x}_1, \dots, \bar{x}_n)$, then $UC(G)$ has a NE with the same outcome. Let us just consider $((S_1, \phi_1), \dots, , (S_n, \phi_n))$, where the commitment $(S_1, \dots, S_n)$, is defined by $S_i \doteq x_i$, for each $i \in N$, and the strategy profile $(\phi_1, \dots, \phi_n)$ as follows, for each $i \in N$: $\phi_i(S_1, \dots, S_n) \doteq x_i$, $\phi_i(T_j, S_{-j}) \doteq x_i$ for all $i \in N \backslash \{j\}$ (i.e. if a unique player j deviates from the commitment $(S_1, \dots, S_n)$, the others stick to their NE strategies), and ϕ_i is defined ad libitum otherwise. (To avoid cumbersome notations, the definition is given in a colloquial style). Checking that, the couple $((S_1, \phi_1), \dots, (S_n, \phi_n))$ is a NE for $UC(G)$, is straightforward.

ii) If the original game G does not have a NE, it cannot be guaranteed that $UC(G)$ has a NE: an example in pure strategies is Matching Pennies.

iii) It can happen that a game G does not have a NE, while $UC(G)$ has, as shown in the following example (see Figure 4.2). The game G does not have NE, but $((S_1, \phi_1), (S_2, \phi_2))$ defined by $(S_1, S_2) = (\{t\}, \{l, r\})$, $\phi_1(T_1, T_2) = t$, $\phi_2(S_1, T_2) = t$ and $\phi_2(T_1, S_2) = r$ if $T_1 \neq \{t\}$, for each $T_1 \in 2^{X_1}$ and $T_2 \in 2^{X_2}$, do provide a NE for $UC(G)$.

For this reason, in the model of Quality Unilateral Commitment, we consider a component game with non-empty compact real intervals as strategic spaces, and continuous and strictly quasi-concave payoff functions. These assumptions, met by many economic models, ensure that the equilibria set of a component game is a non-empty set.

4.3 Existence and Properties of Nash Equilibria

In this section we now tackle the question of the existence of Nash equilibria.

In the case of finite games, the theorem of Nash (1950) guarantees the existence of a mixed-strategy equilibrium.

Theorem 14. *(Nash (1950)) Every finite strategic-form game has a mixed-strategy equilibrium.*

In the case of infinite games with continuous payoffs, the existence of Nash equilibria in pure strategies is ensured by the theorem of Debreau, Glicksberg, and Fan (1952). In its formulation, we adopt the concept of *nice game* introduced in $\mathbb{R}^n$ by Bade, Haeringer and Renou (2005), and we give the definition in the context of Euclidean spaces.

Before stating the theorem, some preliminaries are needed. We recall that a real *Euclidean space* of finite dimension is a linear vector space of finite dimension on $\mathbb{R}$ with a scalar product. So, it is equipped with a norm, a metric, and a Hausdorff topology, admitting convex bases, where vector operations of addition and scalar multiplication are continuous.

Definition 47. *A subset X of a Euclidean space is compact*[2] *if every sequence in X has a subsequence that converges to a limit point in X.*

Definition 48. *A subset X of a linear vector space is convex if, for any $x, y \in X$ and any $\alpha \in [0,1]$,*

$$\alpha x + (1-\alpha)y \in X.$$

Definition 49. *Let X be a convex subset of an Euclidean space. The function $f : X \longrightarrow \mathbb{R}$ is concave if for any $x, y \in X$ and for any $\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \geqslant \alpha f(x) + (1-\alpha)f(y).$$

[2]The notion of compactness for more general topological spaces uses the notion of *cover*, which is a collection of open sets whose union includes the set X. X is compact if any cover has a finite subcover.

The function f is strictly concave if for any $x, y \in X$ and for any $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y).$$

Definition 50. *Let X be a convex subset of an Euclidean space and let $f : X \longrightarrow \mathbb{R}$. The function $f : X \longrightarrow \mathbb{R}$ is quasi-concave if, for each $t \in \mathbb{R}$, the set*

$$\{x \in X : f(x) \geqslant t\}$$

is convex. The function $f : X \longrightarrow \mathbb{R}$ is strictly quasi-concave if, for each $t \in \mathbb{R}$, the set

$$\{x \in X : f(x) > t\}$$

is convex.

Or, equivalently,

Definition 51. *The function f is a quasi-concave function if for any $x, y \in X$ and for any $\alpha \in [0, 1]$ we have*

$$f(\alpha x + (1-\alpha)y) \geqslant \min(f(x), f(y)).$$

The function f is a strictly quasi-concave function if for any $x, y \in X$ and for any $\alpha \in [0, 1]$ we have

$$f(\alpha x + (1-\alpha)y) > \min(f(x), f(y)).$$

Remark 3. If f is a strictly concave function, it is also quasi concave. The converse is false. For example, $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x^3$.

Definition 52. *A game $\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ in strategic form is a nice game if for each player $i \in N$, X_i is a non-empty compact, convex subset of an Euclidean space, and the payoff function u_i is continuous in all its arguments, and strictly quasi-concave in x_i.*

The continuity is with respect to the topology induced by the metric.

Example 3. A non-empty closed real interval is a compact, convex subset of the real line.

Definition 53. *Let $F : X \longrightarrow X$ be a function (or a multifunction). $\bar{x} \in X$ is a fixed point for F if $\bar{x} \in F(X)$.*

Definition 54. *Let X, Y be topological spaces, $F : X \rightrightarrows Y$ be a multifunction, and $\mathrm{gph}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ be its graph. F is said to have closed graph if $\mathrm{gph}(F)$ is a closed of $X \times Y$.*

If $X, Y \subseteq \mathbb{R}^n$, the definition 54 is formulable in a sequential way: F has a closed reduced graph if the following condition holds. If $(x_k, y_k) \in X \times Y$ such that $(x_k, y_k) \longrightarrow (x, y)$, as $k \to +\infty$, then $(x, y) \in X \times Y$.

Or, equivalently.

If $x_k \longrightarrow x$, $y_k \longrightarrow y$, as $k \to +\infty$, and $y_k \in F(x_k)$, then $y \in F(x)$.

Definition 55. *Let X be a topological space. A function $f : X \longrightarrow \mathbb{R}$ is called upper hemicontinuous if and only if the set $\{x \in X : f(x) > t\}$ is open for each t.*

Upper hemi-continuity property requires that, for any x_0 and for any open set V that contains $f(x_0)$, there exists a neighborhood U of x_0 such that $f(x) \subseteq V$ if $x \in U$. In general, this differs from the closed-graph notion, but the two concepts coincide if the range of f is compact and $f(x)$ is closed for each x, conditions which are generally satisfied when applying fixed-point theorems.

Theorem 15. *(Kakutani[3]) Let $K \subseteq \mathbb{R}^k$ be a non-empty, compact, convex set. If $F : K \overrightarrow{\rightarrow} K$ is a non-empty, convex-valued multifunction with closed graph, then F has a fixed point.*

If $F(x)$ is a singleton for each $x \in X$, then we come across again the Browner fixed point theorem, focal theorem in differential and algebraic topology.

Theorem 16. *(Berge) Let X, Y be metric spaces, and $f : X \times Y \longrightarrow \mathbb{R}$ be a continuous function, then the multifunction $F : X \overrightarrow{\rightarrow} Y$, so defined*

$$F(x) = \operatorname{argmax}_{y \in Y} f(x, y),$$

has a closed graph.

Theorem 17. *(Debreau (1952), Glicksberg (1952), and Fan (1952)) Every nice game $G = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ has a Nash equilibrium in pure strategies.*

Proof. The idea of the proof is to apply the theorem of fixed-point of Kakutani to the best reply correspondences. The set of (pure) *best replies* of player i to $x_{-i} \in X_{-i}$, for each player $i \in N$, is defined as follows:

$$BR_i(x_{-i}) \doteq \operatorname{argmax}_{x_i \in X_i} u_i(x_i, x_{-i})$$

[3] With the intention of simplifying the original proof of von Neumann, Kakutani (1941) extended the classical Brouwer's theorem to set valued maps and derived the minimax theorem as an easy corollary.

With variation of $x_{-i} \in X_{-i}$, we define the *best replay* multifunction for the player i

$$BR_i : X_{-i} \rightrightarrows X_i,$$

which associates, with each element of X_{-i}, only one subset of X_i, empty too. But, for hypothesis, X_i is a compact, u_i is continuous, and

$$BR_i(x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) \geqslant u_i(y_i, x_{-i}) \, for \, all \, y_i \in X_i\},$$

then BR_i is a continuous, non-empty, single-valued function in any nice game. We define the correspondence BR as the Cartesian product of the best reply BR_i:

$$BR : X_1 \times \ldots \times X_n \longrightarrow X_1 \times \ldots \times X_n$$

$$BR(x_1, \ldots, x_n) \doteq \prod_{i \in N} BR_i(x_{-i})$$

A fixed point of BR is a strategy profile $(\bar{x}_1, \ldots, \bar{x}_n)$ such that

$$BR(\bar{x}_1, \ldots, \bar{x}_n) = (\bar{x}_1, \ldots, \bar{x}_n),$$

that is such that

$$BR_i(\bar{x}_{-i}) = \bar{x}_i, \quad \text{for each i} \in \text{N},$$

thus, a fixed point of BR is a Nash equilibrium of G. In order to prove our statement, we show that BR satisfies the hypotesis of Kakutani's theorem to have a fixed point.

i) $(X_1 \times \ldots \times X_n)$ is a compact, convex, non-empty subset of a finite-dimensional Euclidean space.

ii) $BR(x_1, \ldots, x_n)$ is non-empty for all $(x_1, \ldots, x_n)$.

iii) $BR(x_1, \ldots, x_n)$ is convex for all $(x_1, \ldots, x_n)$.

iv) $BR(\cdot)$ has a closed graph or is upper hemi-continuous.

Condition *i*) is easy to verify. Since the Cartesian product of non-empty sets or compact sets or convex sets is a non-empty set or compact set or convex set, respectively, then $X_1 \times \ldots \times X_n$ satisfies immediately the condition *i*). From the definition of BR as Cartesian product, it is sufficient to check that each BR_i satisfies conditions *ii*) and *iii*). Now, BR_i has non empty values for Weierstrass theorem (in fact R_i is a continuos function defined on a compact set). Let us prove that R_i is convex-valued. By definition,

$$R_i(x_{-i}) \doteq \text{argmax}_{y_i \in X_i} u_i(y_i, x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) \geqslant \max_{y_i \in X_i} u_i(y_i, x_{-i})\},$$

where the last set is convex since u_i is a quasi-concave function. If we choose $t = \max_{y_i \in X_i} u_i(y_i, x_{-i})$, then R_i is convex-valued. At least, for each $i \in N$ the payoff function $u_i : X_i \times X_{-i} \longrightarrow \mathbb{R}$ is a continuous function on metric space payoffs, so Berge's Theorem 16 implies closed-graph best reply $BR_i(x_{-i}) \doteq \text{argmax}_{x_i \in X_i} u_i(x_i, x_{-i})$. Since the sufficient conditions of Kakutani's Theorem 15 are satisfied, BR has a fixed point. □

By using Theorem 17 for a symmetric game, we have the following existence results of a pure symmetric equilibrium.

Corollary 4. *A symmetric nice game $\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ has at least one symmetric pure NE.*

Proof. Theorem 17 states that there exist a strategy $(\bar{x}_1, \ldots, \bar{x}_n) \in X_1 \times \ldots \times X_n$ such that

$$BR(\bar{x}_1, \ldots, \bar{x}_n) = (\bar{x}_1, \ldots, \bar{x}_n),$$

that is such that

$$BR_i(\bar{x}_{-i}) = \bar{x}_i, \quad \text{for each i} \in \text{N}.$$

But G is a symmetric game, so, for each $i, j \in N$,

$$\bar{x}_i = BR_i(x_{-i}) = \text{argmax}_{x_i \in X_i} u_i(x_i, x_{-i}) = \text{argmax}_{x_j \in X_j} u_j(x_j, x_{-j}) =$$

$$= BR_j(x_{-j}) = \bar{x}_j.$$

□

Corollary 5. *Let $\langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ be a symmetric potential game, where $X_1 = \ldots = X_n$ is a closed real interval, P is a continuous function in all its arguments and quasi-concave in x_i for all $i \in N = \{1, \ldots, n\}$, there exists at least one pure symmetric NE.*

Remark 4. Let $G = \langle (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a nice game. When the second-stage of $UC(G)$ is played, we can consider the sub-game G limited to $(S_1, \ldots, S_n)$, denoted with $G_{S_1, \ldots, S_n}$, as a restriction of G. So we can apply the results of Bade, Haeringer and Renou (2005), concerning the relations between the equilibria of a nice game[4] and its restriction. The first property is an obvious property of a restricted game: any equilibrium of G, which belongs to the restricted set of strategies $(S_1, \ldots, S_n)$, is also an equilibrium of $G_{S_1, \ldots, S_n}$. That is, if there exists a NE $(\bar{t}_1, \ldots, \bar{t}_n)$ such that

[4]If the game G is finite, the results do not hold.

$(\bar{t}_1, \ldots, \bar{t}_n) \in (S_1, \ldots, S_n)$, then $(\bar{t}_1, \ldots, \bar{t}_n)$ is a NE of $G_{S_1,\ldots,S_n}$. The converse is obviously not true. However, Bade, Haeringer and Renou (2005) prove that any interior equilibrium of $G_{S_1,\ldots,S_n}$ is also an equilibrium of G, where a strategy profile is interior if $(\bar{t}_1, \ldots, \bar{t}_n) \in int \prod_{i\in N} S_i$. It follows that the equilibria of $G_{S_1,\ldots,S_n}$, which are not equilibria of G, are on the boundary of $\prod_{i\in N} S_i$. In particular, if the players are two, the equilibria of the G_{S_1,S_2}, which are not equilibria of G, are in number less than or equal to four.

4.4 Penalty Function Method

The statement of the problem of QUC recurs to the Penalty Method, which is one of the techniques of Constrained Optimization. Sometimes, a problem of optimization with equality constraints and/or inequality constraints can be transformed into an equivalent unconstrained problem. The transformations can be made on the variables or on the objective function. For example, with linear constraints, it is to be preferred to treat the constraints directly and to transform the variables. Instead, when the constraints are nonlinear, it is particularly useful to use a method, such as the penalty, which acts on the objective function.

When solving a nonlinear programming problem, in which the constraints cannot easily be eliminated, it is necessary to balance the aims of reducing the objective function and staying inside or close to the feasible region, in order to induce global convergence, that is convergence to a local solution from any initial approximation. This inevitably leads to the idea of a penalty function, which is some combination of the objective function and the constraint function. Early penalty functions are smooth so as to enable to use efficient techniques for smooth unconstrained optimization.

The penalty method approximates a given constrained problem with unconstrained problems, penalizing each deviation from acceptable values.

Let us consider the following problem in $\mathbb{R}^n$:

$$(P) \qquad \max\{f(x_1, \ldots, x_n) \,|\, (x_1, \ldots, x_n) \in K\}.$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, with $n \in \mathbb{N}^*$, is the *objective* function and $K \subseteq \mathbb{R}^n$ is the *feasible region*, usually defined by

$$K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, g_j(x_1, \ldots, x_n) \geqslant 0,\; j = 1, \ldots, m\},$$

with assigned *constraint* functions $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, for all $j = 1, \ldots, m$.

We underline the general formulation of the problem (P): we can switch the maximum research to minimum, only changing the sign of f.

The penalty method associates with a constrained problem (P) a sequence of unconstrained problems $(P_k)_{k\in\mathbb{N}}$, where

$$(P_k) \quad \max\{p_k(f(x_1,\ldots,x_n), g_j(x_1,\ldots,x_n)) \,|\, j = 1,\ldots,m,\ (x_1,\ldots,x_n) \in \mathbb{R}^n\}.$$

The solution of each problem (P_k) is a point $(x_{k_1},\ldots,x_{k_n})$, which maximizes the relative unconstrained function $p_k = p_k(f, g_1,\ldots,g_m)$. These functions are constructed so that, as $k \to +\infty$, a convergent sequence of unconstrained maximizer approaches a constrained maximizer of problem (P).

Let us assume f defined, continuous on $\mathbb{R}^n$, and strictly concave. We assume K is a closed convex set defined by only one concave function g on $\mathbb{R}^n$. Again, for simplicity we assume $n = 2$.

There is a lot of offers for penalty function p. The earliest penalty function (Courant (1943)) is

$$p_n(x,y) = f(x,y) - \frac{1}{2}ng^2(x,y).$$

The penalty is formed from a difference of squares of constraint violations and the parameter n determines the amount of penalty.

Remark 5. We suppose (x_n, y_n) is a point of maximum for p_n in $\mathbb{R}^2$. If $g(x_n, y_n)$, as $n \to +\infty$, does not converge to zero, the term $-\frac{1}{2}ng^2(x,y)$ diverges to $-\infty$ as $n \to +\infty$. So, for a great value of n, the values $p_n(x_n, y_n)$ become very small, if f is not upper limited, and (x_n, y_n) cannot maximize p_n, as we have assumed. That is, the sequence of free problems defined by p_n is equivalent to penalize the point (x,y) if it does not belong to the constraint $g(x,y) = 0$. Multiplying $g^2(x,y)$ by $-\frac{1}{2}n$ implies that we subtract from f a quantity which diverges, so $-\frac{1}{2}ng^2(x,y)$ has very small values and then is not competitive in order to calculate the maximum of p_n. We say that p_n is obtained penalizing the constraint.

Obviously, $g(x,y) = 0$ if and only if $(x,y) \in K$, and, since $p_n(x,y) \leq f(x,y)$ and stricly convex, only one $(x_n, y_n) \in \mathbb{R}^2$ exists such that

$$p_n(x_n, y_n) = \max_{(x,y)\in\mathbb{R}^2} f_n(x,y).$$

Then, we have the following theorem.

Theorem 18. *Under the previous assumption on f, g, and p_n, we have*

$$(x_n, y_n) \to (\bar{x}, \bar{y}), \quad as\, n \to +\infty,$$

where $(\bar{x}, \bar{y})$ is the maximum point of f on K.

Proof. We have:

$$f(x_n, y_n) \geqslant p_n(x_n, y_n) \geqslant \max_{(x,y)\in K} p_n(x, y) \geqslant \max_{(x,y)\in K} f(x, y) = f(\bar{x}, \bar{y}).$$

From which, (x_n, y_n) is in a limited set of $\mathbb{R}^2$. We extract a subsequence in $\mathbb{R}^2$ $(x_{n_k}, y_{n_k}) \to (x^*, y^*)$, as $k \to +\infty$. From 4.4 follows

$$f(x^*, y^*) \geqslant f(\bar{x}, \bar{y}).$$

Again, from 4.4, we have

$$\frac{1}{2} k g^2(x_{n_k}, y_{n_k}) \geqslant f(\bar{x}, \bar{y}) - f(x_{n_k}, y_{n_k}),$$

and then

$$g^2(x^*, y^*) = \lim_{k\to+\infty} g^2(x_{n_k}, y_{n_k}) = 0.$$

From which, $(x^*, y^*) \in K$ which, with 4.4, implies $(x^*, y^*) = (\bar{x}, \bar{y})$.

□

It is interesting to observe that this result is obtained in absence of differentiability or Karush-Kuhn-Tucker regularity assumptions. However, not only the method suffers from necessity of a sequence of maximizations, but also these maximizations become more and more difficult numerically, as k grows.

It is also possible to get asymptotic estimates of the rate of convergence. These estimates can be used to terminate the penalty function iteration and also to provide better initial approximations when maximizing $p_k = p_k(f, g)$. Aubin (1969) proves that, under suitable assumptions, there is a valuation of maximum points $\underline{x}_k = (x_{k_1}, \ldots, x_{k_n})$ of problem (P_k) to maximum point $\underline{x} = (x_, \ldots, x_n)$ of problem (P) like $|\underline{x} - \underline{x}_k| \leq C\frac{1}{k}$, with C appropriate constant. Lucchetti and Patrone (1977) notice that this result cannot improve, that is, a convergence like $|\underline{x} - \underline{x}_k| \leq C(\frac{1}{k})^\alpha$, with $\alpha > 1$, is not possible.

The penalty method is an *exterior point* algorithm, so called since it generates a sequence of maxima exterior to the constraints, forcing convergence to a feasible point in the limit as $n \to +\infty$. Equality constraints problems can only be handled by using exterior point algorithms. Instead, problems with inequality constraints can be handled by exterior point algorithms or *interior point* algorithms, where the sequence of maxima is interior to the constraints set.

4.5 The model

We model the choice of quality unilateral commitments applied to a one-shot game, looking at the reputation, trust and loyalty effect between the players. The structure of the QUC model is traced out from UC model, but here the commitments are assumed to be not binding, since in the second stage a player can play a strategy not belonging to his restricted set declared in the first stage. The role, played in UC by binding commitments, is in the QUC performed by penalty. In fact, we must interpret in a positive way the penalty of a player, as guarantee towards the others of his perfect, unobjectable and fair behavior, and not in a negative way, as a punishment for his (possible) deviation. Again, the QUC want to make self-enforcing any NE of the reduced game $G_{S_1,\ldots,S_n}$.

Given a game $G = \langle (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, the corresponding game with quality unilateral commitments consists in embedding the component game G into a two-stage game. In the first stage, each player, independently and simultaneously, chooses a nonempty subset S_i of his strategy set X_i, i.e. he commits to play only the choosen strategies in S_i. After this preliminary stage, the choices made by all players are publicly known, and the game G is played. If player i adheres to his declaration and plays a strategy in S_i, he obtains the payoff u_i of G, otherwise his payoff is cut down.

In order to make self-enforcing a NE of the reduced game $G_{S_1,\ldots,S_n}$, we approximate to the problem of maximizing each u_i, subject to the constraints $S_1,\ldots,S_n$, by a maximization problem without constraints, in which the function to be maximized is modified by the subtraction of a penalty term, equal to the square of the distance between the strategy x_i and the declaration set S_i, multiplied by a large penalization factor. To be precise, we take a natural k, which will be made to tend to infty, and we associate with it the problem of finding, for each $i \in N$,

$$\text{argmax}\{u_i(x_1,\ldots,x_n) - k[dist(x_i,S_i)]^2 : (x_1,\ldots,x_n) \in X_1 \times \ldots \times X_n\},$$

where the distance between a strategy profile x_i and a set S_i is so defined

$$dist(x_i,S_i) \doteq \inf\{d(x_i,y) : y \in S_i\},$$

with d the usual distance of the metric space $\mathbb{R}$.

As reminded earlier, the existence of NE for the component game G is not a guarantee of existence of equilibria for its extension with UC. So, we assume for G and for its extension $QUC(G)$ the same hypothesis as Theorem 17. In particular, we assume that each player i has, as strategy set of G, a non-empty closed real interval (i.e. a compact, convex subset of the real line), and that in the preliminary stage of $QUC(G)$, i chooses a non-empty closed interval of his available strategies [5]. We denote the compact real interval S_i of player i by

$$[\underline{x}_i, \overline{x}_i] \subseteq X_i,$$

where $\underline{x}_i$ is the minimum of S_i and $\overline{x}_i$ its maximum. Player i can also commit to a singleton, when $\underline{x}_i = \overline{x}_i$. Again, we assume that the payoff functions of both games is continuous in all its arguments, and strictly quasi-concave in the respective variables.

We limited the analysis of the model, for simplycity, to the case of two players, but it extends immediately up to the case of n players. We suppose that the players are symmetric, for example two identical firms producing the same good, and we propose the same type of penalty for each player.

Let $G = \langle X, Y, f, g \rangle$ be a nice game, that is the strategy set X, Y are non-empty compact, convex subset of the real line, and the payoff functions f, g are continuous in (x, y), and strictly quasi-concave in x and in y, respectively[6]. Then Theorem 17 assures that the set of NE of G is non-empty.

Now, we can formalize the QUC-extention of a game G.

Definition 56. *Let $G = \langle X, Y, f, g \rangle$ be a nice game. The QUC-extension of G, briefly QUC(G), is the sequence of games*

$$G_n = \langle [\underline{x}, \overline{x}], [\underline{y}, \overline{y}], p_n, q_n \rangle,$$

where, for each $n \in \mathbb{N}$,

i. the set of players $N = \{I, II\}$ remains the same of G,

[5] Without loss of generality, we have assumed convex the restricted strategy space S_i. Imposing some Lipschitz conditions is sufficient to ensure that the game played in the second-stage, has a NE. We also note that imposing a convex strategy space is a common assumption in Economics literature.

[6] The model developed in this section cannot be applied to mixed extensions of finite games. In fact, payoff functions are not strictly quasi-concave, and mixed strategy space is not a subset of the real line, except for a binary game.

ii. $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$ *are respectively the set of strategies declared by player* I *and* II*,*

iii. the payoff function p_n *of player* i *is given by*

$$p_n(x, y) = \begin{cases} f(x, y) - n(\underline{x} - x)^2 & \text{if } x < \underline{x} \\ f(x, y) & \text{if } x \in [\underline{x}, \overline{x}] \\ f(x, y) - n(x - \overline{x})^2 & \text{if } x > \overline{x} \end{cases}$$

and the payoff function q_n *of player* i *is given by*

$$q_n(x, y) = \begin{cases} g(x, y) - n(\underline{y} - y)^2 & \text{if } y < \underline{y} \\ g(x, y) & \text{if } y \in [\underline{y}, \overline{y}] \\ g(x, y) - n(y - \overline{y})^2 & \text{if } y > \overline{y} \end{cases}$$

It is immediately noted that the penalty functions are strictly quasi-concave, since payoff functions of G are strictly quasi-concave and penalty is concave since it is proportional to square of the deviation.

Remark 6. The model is very flexible and foresees many enhancements. For example we could investigate what happens when the penalty functions are different,one for each player, or what happens when the players do not commit simultaneously but they play hierarchically, for example a monopolist and its consumers.

4.6 An Example: QUC of Cournot Duopoly

4.6.1 Cournot Duopoly

Let us consider a simplified version of Cournot Duopoly. Two firms I and II produce a homogeneous good, for example mineral water, and simultaneously choose their respective output level. If producer I brings on the market an amount $x \in [0, +\infty)$, and II a quantity $y \in [0, +\infty)$, then the price p of mineral water depends on the total amount $q = x + y$ brought on the market. We assume a linear demand $p(q) = a - q$[7], where $a > 0$ is a

[7] Our choice for $p(q)$ is due only to technical reasons. It simplifies the scenarios when we apply QUC to Cournot Duopoly. Obviously, our choice for $p(q)$ is less realistic than the classical assumption $p(q) = \max(0, a - q)$ of Cournot Duopoly, but it is irrelevant for the purpose of QUC. With classical assumptions, the total profit has linear and quadratic expression, but after application of QUC, the penalty function becomes quadratic, like in our model.

costant, and a symmetric linear output cost cx and cy for each corresponding firm, where $c > 0$ is the constant unit price. The firm total profits are then, respectively,

$$f(x, y) = xp(q) - cx = -x^2 + (a - c - y)x,$$

and

$$g(x, y) = yp(q) - cy = -y^2 + (a - c - x)y.$$

We assume that $c < a$, otherwise the market price for good unit, being less than or equal to a, does not exceed the output price.

The duopoly situation can be modeled as a game in strategic form where, the players are the two firms, the strategies are quantities $x, y \in [0, +\infty)$ of produced good, the payoff functions are total profits f, g. Formally,

$$G = \langle X, Y, f, g \rangle,$$

where $X = Y = [0, +\infty)$ and, for all strategy profiles $(x, y) \in X \times Y$,

$$f(x, y) = xp(q) - cx = -x^2 + (a - c - y)x,$$

$$g(x, y) = yp(q) - cy = -y^2 + (a - c - x)y.$$

It follows immediately that the Cournot game G is a nice game. The intersection (if any exists) of the two reaction functions $BR_I : Y \longrightarrow X$ and $BR_{II} : X \longrightarrow Y$ are the Nash equilibria of the Cournot game: neither firm can gain by a change in output, given the output level of its opponent. The best reply of firm I to strategy y of firm II is

$$BR_I(y) = \text{argmax}\{-x^2 + (a - c - y)x : y \in [0, +\infty)\} =$$

$$= \begin{cases} \frac{a-c-y}{2} & \text{if } 0 \leqslant y < a - c \\ 0 & \text{if } y \geqslant a - c \end{cases}$$

By simmetry, the best reply of firm II to strategy x of firm I is

$$BR_{II}(x) = \text{argmax}\{-y^2 + (a - c - x)y : x \in [0, +\infty)\} =$$

$$= \begin{cases} \frac{a-c-x}{2} & \text{if } 0 \leqslant x < a - c \\ 0 & \text{if } x \geqslant a - c \end{cases}$$

The NE must satisfy

$$\begin{cases} x = BR_I(y) \\ y = BR_{II}(x) \end{cases}$$

then, there is only one NE given by

$$(\frac{a-c}{3}, \frac{a-c}{3}).$$

At the equilibrium, each firm gains

$$f(\frac{a-c}{3}, \frac{a-c}{3}) = g(\frac{a-c}{3}, \frac{a-c}{3}) = \frac{(a-c)^2}{9}.$$

The sale price is

$$p(q) = p(2\frac{a-c}{3}) = \frac{a+2c}{3}.$$

Since, for hypothesis, $c < a$, the sale price is greater than unit price, so each firm gets a profit.

We assume firm I has a monopoly on mineral water, so II produces $y = 0$, without profit. Then, the best strategy for I is the *quantity of monopoly*

$$x = \frac{a-c}{2},$$

which gives to I a profit

$$f(\frac{a-c}{2}, 0) = \frac{(a-c)^2}{4}.$$

The sale price is

$$p(q) = p(\frac{a-c}{2}),$$

which is greater than sale price of duopoly. So the monopolist makes a profit.

If the firms agree on producing half of monopoly quantity and dividing the profit in half, their payoff is

$$f(\frac{a-c}{4}, \frac{a-c}{4}) = g(\frac{a-c}{4}, \frac{a-c}{4}) = \frac{(a-c)^2}{8},$$

which is greater then payoff of duopoly. But the strategy profile $(\frac{a-c}{4}, \frac{a-c}{4})$ is not a NE, since $\frac{a-c}{4} \neq BR_I(\frac{a-c}{4})$ and $\frac{a-c}{4} \neq BR_{II}(\frac{a-c}{4})$, so the NE $(\frac{a-c}{3}, \frac{a-c}{3})$ is inefficient.

4.6.2 QUC of Cournot Duopoly

The QUC-extention of the Cournot game G, $QUC(G)$, is the sequence of games

$$G_n = \langle [\underline{x}, \overline{x}], [\underline{y}, \overline{y}], p_n, q_n \rangle,$$

where, $[\underline{x}, \overline{x}] \subseteq [0, +\infty)$ and $[\underline{y}, \overline{y}] \subseteq [0, +\infty)$, and the payoff functions are given by

$$p_n(x,y) = \begin{cases} -x^2 + (a-c-y)x - n(x-\underline{x})^2 & \text{if } x \in [0, \underline{x}] \\ -x^2 + (a-c-y)x & \text{if } x \in [\underline{x}, \overline{x}] \\ -x^2 + (a-c-y)x - n(x-\overline{x})^2 & \text{if } x \in [\overline{x}, +\infty) \end{cases}$$

$$= \begin{cases} -(1+n)x^2 + (a-c-y+2\underline{x}n)x - n\underline{x}^2 & \text{if } x \in [0, \underline{x}] \\ -x^2 + (a-c-y)x & \text{if } x \in [\underline{x}, \overline{x}] \\ -(1+n)x^2 + (a-c-y+2\overline{x}n)x - n\overline{x}^2 & \text{if } x \in [\overline{x}, +\infty) \end{cases}$$

and

$$q_n(x,y) = \begin{cases} -y^2 + (a-c-x)y - n(y-\underline{y})^2 & \text{if } y \in [0, \underline{y}] \\ -y^2 + (a-c-x)y & \text{if } x \in [\underline{y}, \overline{y}] \\ -y^2 + (a-c-x)y - n(y-\overline{y})^2 & \text{if } y \in [\overline{y}, +\infty) \end{cases}$$

$$= \begin{cases} -(1+n)y^2 + (a-c-x+2\underline{y}n)y - n\underline{y}^2 & \text{if } y \in [0, \underline{y}] \\ -y^2 + (a-c-x)y & \text{if } x \in [\underline{y}, \overline{y}] \\ -(1+n)y^2 + (a-c-x+2\overline{y}n)y - n\overline{y}^2 & \text{if } y \in [\overline{y}, +\infty) \end{cases}$$

We focus on $p_n(x,y)$ to study the best reply function. p_n is composed of three parables, then the best reply of firm I to strategy y of firm II is the quantity corresponding to one of the vertices of parables, which are respectively,

$$\underline{V} = (\frac{a-c-y+2n\underline{x}}{2(1+n)}, \frac{(a-c-y)^2 + 4n\underline{x}(a-c-y-\underline{x})}{4(1+n)}),$$

$$V = (\frac{a-c-y}{2}, \frac{(a-c-y)^2}{4}),$$

and

$$\overline{V} = (\frac{a-c-y+2n\overline{x}}{2(1+n)}, \frac{(a-c-y)^2 + 4n\overline{x}(a-c-y-\overline{x})}{4(1+n)}).$$

Set

$$x_{\underline{V}} \doteq \frac{a-c-y+2n\underline{x}}{2(1+n)}, \; x_V \doteq \frac{a-c-y}{2}, \; and \; x_{\overline{V}} \doteq \frac{a-c-y+2n\overline{x}}{2(1+n)},$$

we have:

$$j) \quad x_{\underline{V}} < x_V \quad \Leftrightarrow \quad \underline{x} < x_V,$$

in fact, $\frac{a-c-y+2n\underline{x}}{2(1+n)} < \frac{a-c-y}{2} \Leftrightarrow a-c-y+2n\underline{x} < a-c-y+n(a-c-y) \Leftrightarrow \underline{x} < \frac{a-c-y}{2}$.

We have, again,

$$jj) \quad x_{\underline{V}} < \underline{x} \quad \Leftrightarrow \quad x_V < \underline{x},$$

in fact, $\frac{a-c-y+2n\underline{x}}{2(1+n)} < \underline{x} \Leftrightarrow a-c-y+2n\underline{x} < 2\underline{x}+2n\underline{x} \Leftrightarrow \frac{a-c-y}{2} < \underline{x}$.

If $x_V \leqslant 0$, that is $y \geqslant a-c$, it is immediate to verify that, for each $x > 0$,

$$p_n(0,y) = 0 > p_n(x,y),$$

so $BR_I^n(y) = 0$.

If $x_V > 0$, that is $y < a - c$, then, there are three scenarios, corresponding to

$$i)\quad \underline{x} < \overline{x} < x_V, \qquad ii)\quad \underline{x} < x_V < \overline{x} \qquad \textit{and} \qquad iii)\quad x_V < \underline{x} < \overline{x}.$$

Case $i)$ $\underline{x} < \overline{x} < x_V$. Then,

$$\begin{cases} x_{\underline{V}} > \underline{x} \\ x_{\underline{V}} < x_V \end{cases} \textit{and} \begin{cases} x_{\overline{V}} > \overline{x} \\ x_{\overline{V}} < x_V \end{cases}$$

So, $BR_I^n(y) = x_{\overline{V}}$.

Case $ii)$ $\underline{x} < x_V < \overline{x}$. Then,

$$\begin{cases} x_{\underline{V}} > \underline{x} \\ x_{\underline{V}} < x_V \end{cases} \textit{and} \begin{cases} x_{\overline{V}} < \overline{x} \\ x_{\overline{V}} > x_V \end{cases}$$

So, $BR_I^n(y) = x_V$.

Case $iii)$ $x_V < \underline{x} < \overline{x}$. Then,

$$\begin{cases} x_{\underline{V}} < \underline{x} \\ x_{\underline{V}} > x_V \end{cases} \textit{and} \begin{cases} x_{\overline{V}} < \overline{x} \\ x_{\overline{V}} > x_V \end{cases}$$

So, $BR_I^n(y) = x_{\underline{V}}$. The best reply of firm I to strategy y of firm II is

$$BR_I^n(y) = \text{argmax}\{p_n(x, y : x \in [0, +\infty)\} =$$

$$= \begin{cases} \frac{a-c-y+2n\overline{x}}{2(1+n)} & \text{if } 0 \leqslant y \leqslant a - c - 2\overline{x} \\ \frac{a-c-y}{2} & \text{if } a - c - 2\overline{x} < y < a - c - 2\underline{x} \\ \frac{a-c-y+2n\underline{x}}{2(1+n)} & \text{if } a - c - 2\underline{x} \leqslant y < a - c \\ 0 & \text{if } y \geqslant a - c \end{cases}$$

Simmetrically, the best reply of firm II to strategy x of firm I is

$$BR_{II}^n(x) = \text{argmax}\{q_n(x, y) : y \in [0, +\infty)\} =$$

$$= \begin{cases} \frac{a-c-x+2n\overline{y}}{2(1+n)} & \text{if } 0 \leqslant x \leqslant a - c - 2\overline{y} \\ \frac{a-c-x}{2} & \text{if } a - c - 2\overline{y} < x < a - c - 2\underline{y} \\ \frac{a-c-x+2n\underline{y}}{2(1+n)} & \text{if } a - c - 2\underline{y} \leqslant x < a - c \\ 0 & \text{if } x \geqslant a - c \end{cases}$$

Since the reaction functions are linear, there is only one NE, given by the intersection of the two reaction functions. It depends on the overall sixteen scenarios.

4.6.3 A particular case

Let's consider a particular situation: both firms commit to produce only one quantity, that is the restricted action space is a singleton, the same for both players:

$$\underline{x} = \overline{x} = \underline{y} = \overline{y}.$$

The penalty function for firm I becomes, for each $x \in [0, +\infty)$,

$$p_n(x, y) = -(1+n)x^2 + (a - c - y + 2\underline{x}n)x - n\underline{x}^2.$$

If $x_{\underline{V}} \leqslant 0$, that is $y \geqslant a - c + 2n\underline{x}$, we have, for each $x > 0$,

$$p_n(0, y) = -n\underline{x}^2 > p_n(x, y),$$

so $BR_I^n = 0$. If $x_{\underline{V}} > 0$, that is $0 \leqslant y < a - c + 2n\underline{x}$, the best reply of firm I to strategy y of firm II is the quantity corresponding to the vertex of the parable p_n, that is

$$BR_I^n(y) = \frac{a - c - y + 2n\underline{x}}{2(1+n)}.$$

Then, the best reply of firm I to strategy y of firm II is

$$BR_I^n(y) = \begin{cases} \frac{a-c-y+2n\underline{x}}{2(1+n)} & \text{if } 0 \leqslant y \leqslant a - c + 2n\underline{x} \\ 0 & \text{if } y \geqslant a - c + 2n\underline{x} \end{cases}$$

Simmetrically, the best reply of firm II to strategy x of firm I is

$$BR_{II}^n(x) \begin{cases} \frac{a-c-x+2n\underline{x}}{2(1+n)} & \text{if } 0 \leqslant x \leqslant a - c + 2n\underline{x} \\ 0 & \text{if } x \geqslant a - c + 2n\underline{x} \end{cases}$$

Then, there is only one NE given by the intersection of the two reaction function graphics in the plane xy:

$$(\frac{a - c + 2n\underline{x}}{2n+3}, \frac{a - c + 2n\underline{x}}{2n+3}).$$

As $n \to +\infty$, the NE converges to $(\underline{x}, \underline{x})$, that is the wished result. (We recall that, the Cournot duopoly equilibrium is not Pareto efficient, but if each player commits to play half the monopoly quantity, then we obtain a NE for QUC game.). At the equilibrium, each firm gains

$$p_n(\frac{a - c + 2n\underline{x}}{2n+3}, \frac{a - c + 2n\underline{x}}{2n+3}) = q_n(\frac{a - c + 2n\underline{x}}{2n+3}, \frac{a - c + 2n\underline{x}}{2n+3}) =$$

$$(n+1)(\frac{a - c + 2n\underline{x}}{2n+3})^2 - n\underline{x}^2.$$

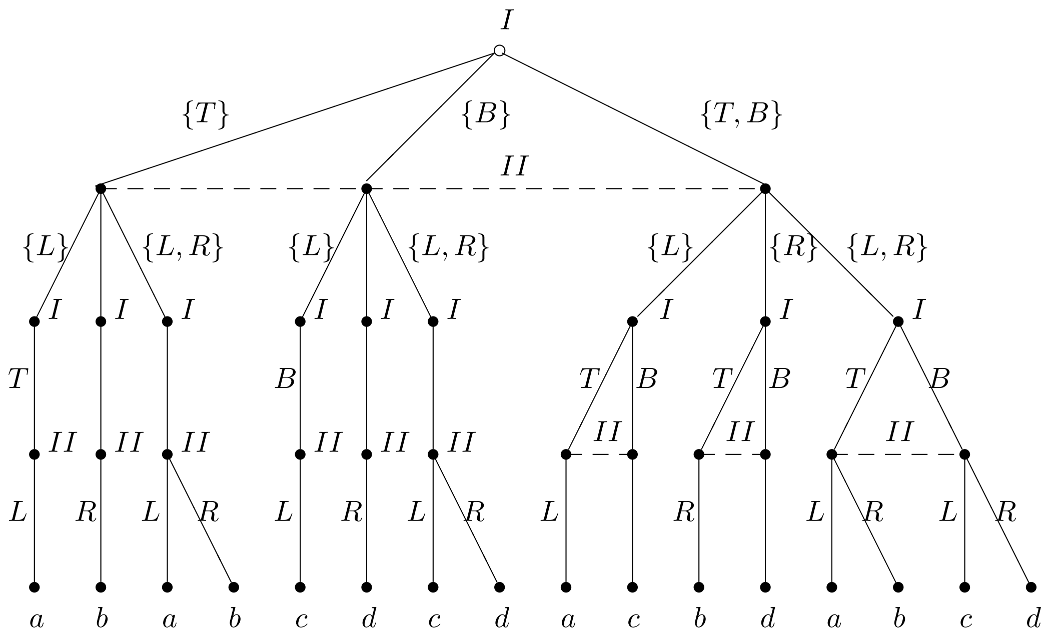
The sale price is

$$p(q) = p(2\frac{a - c + 2n\underline{x}}{2n + 3}) = \frac{(2n + 1)a + 2c - 4n\underline{x}}{2n + 3}.$$

In particular, if $\underline{x}$ is the NE duopoly quantity, that is $\underline{x} = \frac{a-c}{3}$, the sale price is the same sale price as for NE duopoly quantity $p(2\frac{a-c+2n\underline{x}}{2n+3}) = \frac{a+2c}{3}$. Instead, if $\underline{x}$ is half of the monopoly quantity, that is $\underline{x} = \frac{a-c}{4}$, the sale price is $p(2\frac{a-c+2n\underline{x}}{2n+3}) = \frac{(n+1)a+(n+2)c}{2n+3}$.

I \ II	L	R
T	a	b
B	c	d

The game form Γ.



The Unilateral Commitments of game form Γ.

Figure 4.1: Extension of the game form Γ.

I \ II	l		r	
t	5	6	1	5
b	6	0	0	1

Figure 4.2: G without NE, but UC(G) with NE.

Chapter 5

Essentializing Equilibrium Concepts

5.1 Introduction

We live in a society in which information, the cost of information, and the quality of products are very important. The efficient use of information is one of the important matters of Game Theory. In this Chapter we present a framework to identify what information about a game may be neglected, in order to check whether a specific profile corresponds to an equilibrium outcome or not.

Given a game, the goal is usually to find all equilibrium profiles. In this work, we reverse the perspective. We select a strategy profile and check whether it is an equilibrium of the game. Why this different approach? The starting point of this research is a question raised in García-Jurado and González-Díaz (2006). Given an extensive game, an equilibrium concept selects a set of strategy profiles (even empty) satisfying well-defined conditions testing on all game trees [1]. Dealing with unilateral commitments, we face extremely large game trees, with many subgames, some of which correspond to senseless commitments. From here, the demands of "pruning" the tree springs, i.e. locating the irrelevant parts of the game tree which need not be considered. So, in the preliminary stage of the unilateral commitment game, we have to choose the commitment corresponding an efficient outcome, and then to check whether the profile is an equilibrium. To *essentialize an EC for a fixed strategy profile* means to locate the parts of the game tree needed

[1] Hereafter we will shorten EC for equilibrium concept.

to check whether the profile satisfies the EC conditions.

5.1.1 An example

Let us begin by introducing a simple example to understand the essentializing of an equilibrium concept. We consider the following extensive game G in Figure 5.1. The NE are (L, l) and (R, r). To check whether the strategy

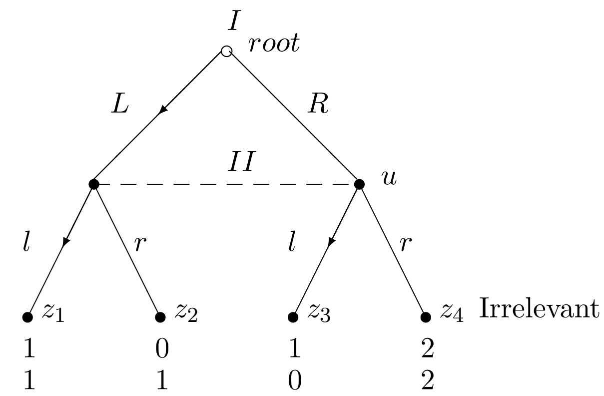


Figure 5.1: G.

profile $b = (L, l)$ is a subgame perfect equilibrium [2], the payoff $(2, 2)$ is irrelevant, since the outcome z_4 can never be reached after unilateral deviations.

Let G' be the game in Figure 5.2, obtained by G replacing the terminal node z_4 with a matching pennies game. We consider $b' = ((L, L_1), (l, l_1))$, which restricted to G coincides with b. To check if $b' = ((L, L_1), (l, l_1))$ is a SPE for G', in this case the behavior in the subgame matters. Indeed, b' is not a SPE of G', because the matching pennies subgame has not a Nash equilibrium. The problem becomes to search for an equivalent profile. That is, we search any SPE of G' realization equivalent to b', in the sense that it coincides with b' in the part of the game relevant for b'. To answer this question, the payoffs and the behavior in the matching pennies game are completely irrelevant since this subgame cannot be reached via unilateral deviations from b'.

If we replace again the matching pennies payoffs with all zero, b' is a

[2] Hereafter we will shorten SPE for subgame perfect equilibrium.

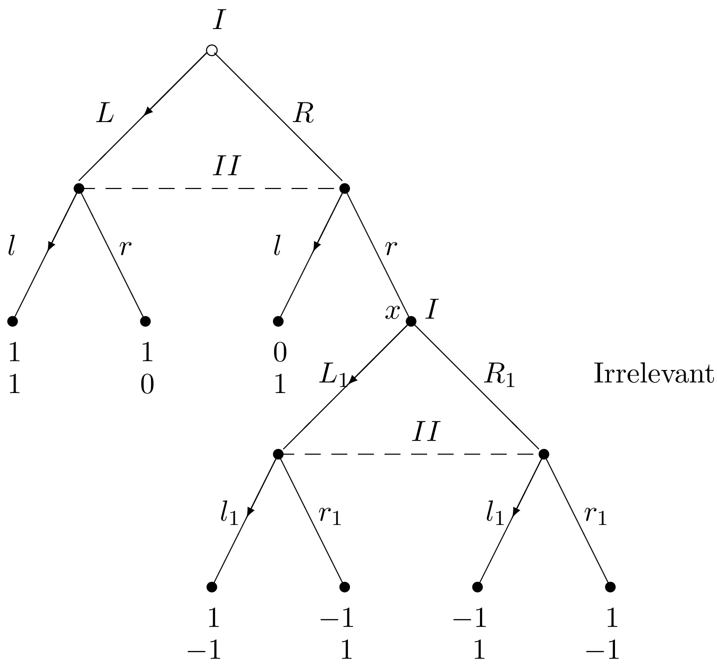


Figure 5.2: G'.

SPE of the new game, but the matching pennies subgame is not relevant for b', since it cannot be reached via unilateral deviations from b'. For this reason we abstract the game form Γ from the game G and we focus only on the game form. Throughout this chapter, a game in extensive form G is a game form together with a payoff function $G = (\Gamma, h)$. Then, for every game having the same game form as G, the payoffs and behavior in the proper subgame of the game are irrelevant to know if the outcome of b is a SPE outcome.

The above discussion suggests that we may determine whether the outcome of a given strategy profile is an equilibrium outcome, without taking into account all the parts of the game tree. So we identify the relevant part of the tree and disregard the information about the irrelevant part of the tree. Formally we study the following problem:

> *"Given an equilibrium concept EC, an (extensive) game form Γ, and a (behavior) strategy profile b, identify W, a minimal collection of information sets of Γ, with the following property:*
> *If a game G has game form Γ and b is an equilibrium of G, then, whatever changes are made in the payoffs and strategies outside W, the outcome of b will be an equilibrium outcome in the resulting game."*

The concept of inessentiality is very intuitive. Not being in W, i.e. being irrelevant,it is related to the fact of not being reachable through unilateral deviations or sequences of unilateral deviations. The difficulty lies in finding an appropriate mathematical formulation of the above problem that is operative for all equilibrium concepts. Under our approach, given an equilibrium concept, a game form, and a strategy profile, there is a unique minimal collection of information sets satisfying the above property. We refer to it as the *essential* collection for EC, Γ, and b. Then, we characterize the essential collections of different equilibrium concepts, such as Nash equilibrium, subgame perfect equilibrium, perfect equilibrium, sequential rationality, weak perfect Bayesinan equilibrium, and sequential equilibrium. The contribution of this chapter is to provide a definition of essential collection useful in the various applications, as described below.

Applications

The possible applications of essentializing equilibrium concepts concern the reduced games, the structural robustness, and the virtual equilibrium concepts.

Given an equilibrium concept, a game G, and a strategy profile b of G, the reduced version of G is a game sufficient to check whether b is an equilibrium for G. That is, if b restricted to the reduced game is an equilibrium of the reduced game, then b is an equilibrium of the original game G. Since the reduced game might be smaller than the original game, the above verification might be easier. A similar approach to the concept of reduced game has already been used for SPE in Osborne (1993) in a model of political competition. In Section 5.5 we use the model of Osborne to illustrate the different implications and applications of the essentializing of an EC and in Section 5.11 we treat the concepts of reduced game.

The results obtained for the reduced game also allow us to understand the structural robustness of the equilibrium concepts to modifications in the game, such as changes in the sets of strategies, in the players of the game, in the information available to the players, and also in the payoffs. The checks of robustness for Nash equilibrium have already been made in Kalai (2005) and in Kalai (2006) in the so-called *large games* Moreover, we show that sometimes we may determine whether a given outcome is an equilibrium outcome of a game only partially-specified.

The *virtual equilibrium concept* is the other main application of our results. For each equilibrium concept we define its virtual version by narrowing the checking conditions on the relevant parts of the game tree. We show that, given an equilibrium concept, if the original game has at least an equilibrium, the sets of equilibrium outcomes and virtual equilibrium outcomes coincide. In general, there are games without equilibria but with virtual equilibria, the virtual equilibria being still sensible in the spirit of their non-virtual counterpart. The concept of virtual equilibrium for SPE has been introduced by García-Jurado and González-Díaz (2006) to get a folk theorem for a class of repeated games in which the existence of subgame perfect equilibria is not guaranteed. The equilibrium notion used in Osborne (1993) is also very close to the virtual SPE, as well as the approach taken in Groenert (2007) in order to introduce the idea of *trimmed equilibrium* and apply it to subgame perfect equilibrium and weak perfect Bayesian equilibrium. In Section 5.12 we define the virtual equilibria of various equilibrium concepts according to the analysis developed for SPE in García-Jurado and González-Díaz (2006).

This chapter is organized as follows. In Section 5.2, we introduce the basic notations and, in particular, the concept of W-combination of games, profiles, and beliefs (Subsection 5.2.3). In Section 5.3 we define the main concepts of essential collection which we will analyze. In Section 5.4, we present an overview of the main results and, in Section 5.5, we build upon the model in Osborne (1993) to illustrate some implications and applications. In Sections 5.6 and 5.8, we characterize the essential collections for the most used equilibrium concepts (NE, SPE, PE, and SR, WPBE, and SE). In the remaining sections we present some applications of our analysis. In particular, in Sections 5.9 and 5.11, we introduce the decomposition (with respect to a collection) of a game into two games, the reduced and the complementary. In Subsection 5.11.1, we analyze the robustness of different equilibrium concepts with respect to changes in the games, and what happens when a game is partially-specified game. Finally, in Section 5.12 we study the concept of Virtual Equilibrium.

5.2 Notations

5.2.1 Game and Game Form

We develop our analysis for finite extensive games with perfect recall, although it can be extended to extensive games of infinite length. We follow the representation of an extensive game given in Fudenberg and Tirole (1991a), representation equivalent to the classic one given by Kuhn (1953) and further developed in Selten (1975) and Kreps and Wilson (1982).

Definition 57. *A game form Γ in extensive form is characterized by i) a finite game tree with root $r(\Gamma)$, ii) a finite set of players $N = \{1, \ldots, n\}$, iii) the sets of nodes $X(\Gamma)$, terminal nodes $Z(\Gamma)$, and information sets $U(\Gamma)$, iv) the probabilities of natural choice, if any.*

Under this representation, nature moves only once at $r(\Gamma)$ and $U(\Gamma)$ is a partition of $X(\Gamma)$, i.e. each terminal node is also an information set. Let $U_i(\Gamma)$ denote the information sets belonging to a player $i \in N$.

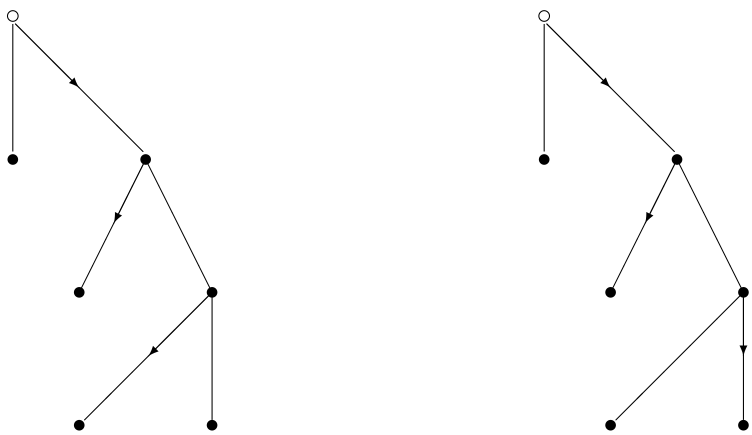
Definition 58. *A game G in extensive form is a pair $\langle \Gamma, h \rangle$, where Γ is a game form and $h\colon Z(\Gamma) \to \mathbb{R}^n$ is the* payoff function*, such that $h(z) = (h_1(z), \ldots, h_n(z))$. $h_i(z)$ is the payoff of player $i \in N$ if z is occurred.*

$\mathcal{G}(\Gamma)$ denotes the set of games with game form Γ, $B(\Gamma) = \prod_{i=1}^{n} B_i(\Gamma)$ the set of behavior strategy profiles of a game G or a game form Γ, and, with slight abuse of notation, $h_i(b)$ the (expected) payoff to player i when $b \in B(\Gamma)$ is played. Given $G \in \mathcal{G}(\Gamma)$, let $M_G := \max_{i \in N, z \in Z(\Gamma)} |h_i(z)| + 1$.

Definition 59. *Let Γ be a game form, and $b, \bar{b} \in B(\Gamma)$. b and $\bar{b}$ are realization equivalent if all the nodes of Γ are reached with the same probabilities under b and $\bar{b}$.*

Being b and $\bar{b}$ realization equivalent does not mean that $b \equiv \bar{b}$, as proved by Figure 5.3.

We will show that b and $\bar{b}$ are equivalent realization if the profiles are the same when restricted to an essential collection W. In fact, by Definition 70 of essential collection, W contains $\pi(b)$, which in turn contains the nodes reached with positive probability. Out of W, the information sets are reached with null probability. Then, if two behavior strategy profiles coincide in W, they coincide in the information sets we can reach with positive probability.



b $\bar{b}$

Figure 5.3: b and $\bar{b}$ are equivalent realization.

Given $b \in B(\Gamma)$, $\pi(b)$ denotes the collection of information sets that are reached with positive probability when b is played, i.e., $\pi(b)$ can be seen as the union of all paths of play that might be realized when b is played. Hence, with slight abuse of language, we refer to $\pi(b)$ itself as the *path* of b. Only if the strategies are pure, we can characterize two equivalent profile in the following way.

The profiles $b, b' \in B(\Gamma)$ *are equivalent realization if and only if* $\pi(b) = \pi'(b)$.

In fact, with mixed strategies, there might be two strategy profiles which reach the same nodes but with different probability.

The concept of equivalence which we apply to profiles is similar to the one introduced for strategies by Aumann and Hart (1992). The authors define equivalent two strategies b_i and $b_i^{'}$ of player i if they yield the same payoffs for any strategies b_{-i} of the other players. They show that, for each terminal node z, the probabilities that z is reached under (b_i, b_{-i}) and under $(b_i^{'}, b_{-i})$ are the same for any b_{-i}. The two concepts are different, not only since we apply to strategy profiles instead of strategies, but also because our case equivalent realization requires that the probability that each node is reached is the same for the two strategy profiles.

Now we define some binary relations for the sets $X(\Gamma)$ and $U(\Gamma)$.

Definition 60. *The node $x \in X(\Gamma)$ is a* predecessor *of $y \in X(\Gamma)$, denoted by $x \prec y$, if $x \neq y$ and x is in the path from the root to y; $x \preccurlyeq y$ means that either $x \prec y$ or $x = y$. If $x \preccurlyeq y$, then the* path of nodes from x to y *is the sequence formed by x, y, and the nodes between x and y.*

The precedence relation is transitive (if $x \prec y$ and $y \prec z$, then $x \prec z$) and asymmetric (if $x \prec y$ then not $y \prec x$), hence it is a partial order. It is not a complete order since two nodes may not be comparable: in figure 5.1, neither $z_1 \prec z_3$ nor $z_3 \prec z_1$.

Similarly,

Definition 61. *The information set $u \in U(\Gamma)$ is a* predecessor *of $v \in U(\Gamma)$, denoted by $u \prec v$, if $u \neq v$ and there are $x \in u$ and $y \in v$ such that $x \prec y$; $u \preccurlyeq v$ means that either $u \prec v$ or $u = v$.* [3] *If $x \preccurlyeq y$, then the* path of information sets from x to y *is the sequence formed by u_x, u_y, and the information sets containing nodes in between x and y.*

Whenever we represent a path of nodes or information sets as a sequence $\{x^1, \ldots, x^k\}$ it is implicitly assumed that $x^1 \prec x^2 \prec \ldots \prec x^k$. Also, given $x \in X(\Gamma)$ and $u \in U(\Gamma)$, $x \prec u$ and $u \prec x$ are defined in the obvious manner.

5.2.2 Collections

Definition 62. *A collection W of information sets is a subset of $U(\Gamma)$.*

We denote W_i the information sets belonging to player $i \in N$.

Definition 63. *A collection $W \subset U(\Gamma)$ is closed (under $\preccurlyeq$) if, for each $v \in W$ and each $u \in U(\Gamma)$,*

$$u \prec v \quad \Rightarrow \quad u \in W.$$

It is immediate to prove that arbitrary unions and intersections of closed collections lead to closed collections. Then, given an arbitrary family of

[3] Note that it is possible to have both $u \prec v$ and $v \prec u$.

closed collections containing a collection W, it makes sense to introduce the smallest closed collection containing W, since this set is the intersection of all elements of the family. According to the previous considerations, $\langle W \rangle$ denotes the smallest closed collection containing a collection W. Then, we can define a closed collection.

Definition 64. *A collection $W \subset U(\Gamma)$ is closed (under $\preccurlyeq$), if*

$$\langle W \rangle = W.$$

Remark 7. In the definition of Essential Collection, we will require closedness, not only in order to facilitate the analysis (and the proofs) of the results in Sections 5.6 and 5.8, and applications in Section 5.11, but also since we want, from a game form,to build a structure which is as close as possible to a game form. The Example in Figure 5.8 shows a collection sufficient but not closed.

Definition 65. *A collection $W \subset U(\Gamma)$ is terminal if, for each $u \in W$ and each $x \in u$, there is $z \in W \cap Z(\Gamma)$ such that $x \preccurlyeq z$.*

It is immediate to prove that arbitrary unions of terminal collections lead to terminal collections.

The next lemma is useful for the proofs of the main results in this chapter. We show that a closed and terminal collection is different from a terminal collection in terminal nodes only.

Lemma 3. *Let Γ be a game form. Let W and $\bar{W}$ be two collections in $U(\Gamma)$ closed under $\preccurlyeq$. If $\bar{W}$ is terminal and $\bar{W} \backslash W \neq \emptyset$, then $(\bar{W} \backslash W) \cap Z(\Gamma) \neq \emptyset$.*

Proof. Let $u \in \bar{W} \backslash W$. Since $\bar{W}$ is terminal, there is $z \in \bar{W} \cap Z(\Gamma)$ such that $u \preccurlyeq z$. Now, since W is closed under $\preccurlyeq$, $u \notin W$, and $u \preccurlyeq z$, we have that $z \notin W$. □

5.2.3 W-combination

Given a collection W relating to a given game form Γ, we define an operation $\otimes_W$ which allows to combine games, payoffs, strategy profiles, and beliefs.

Given $b \in B(\Gamma)$ and $W \subset U(\Gamma)$, b_W denotes the restriction of b to the information sets in W. Similarly, b_{-W} denotes the restriction of b to the information sets outside W.

Definition 66. *Given a collection $W \subset U(\Gamma)$, the W-combination of two games $G = (\Gamma, h), \bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ is the game of $\mathcal{G}(\Gamma)$*

$$G \otimes_W \bar{G} := (\Gamma, h \otimes_W \bar{h}),$$

where

$$h \otimes_W \bar{h}(z) = \begin{cases} h(z) & \text{if } z \in Z(\Gamma) \cap W \text{ and} \\ \bar{h}(z) & \text{if } z \in Z(\Gamma) \backslash W. \end{cases}$$

The Figures 5.4, 5.5, and 5.6 show the two games with their combination and that $\otimes_W$ is not commutative.

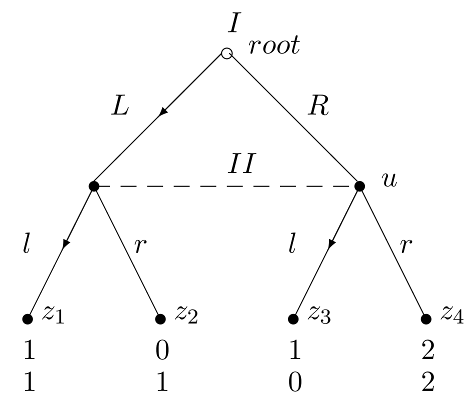


Figure 5.4: First factor game: $G \quad b = (L, l)$.

Similarly,

Definition 67. *Given a collection $W \subset U(\Gamma)$, the W-combination of two profiles $b, \bar{b} \in B(\Gamma)$ is the profile of $B(\Gamma)$*

$$b \otimes_W \bar{b} := (b_W, \bar{b}_{-W}),$$

i.e., the profile that consists of playing according to b in W and to $\bar{b}$ elsewhere.

Clearly, the payoffs associated with b and $b \otimes_W \bar{b}$ coincide, since they define the same path.

Let $\mathcal{M}(\Gamma)$ denote the set of all beliefs that can be defined on Γ.

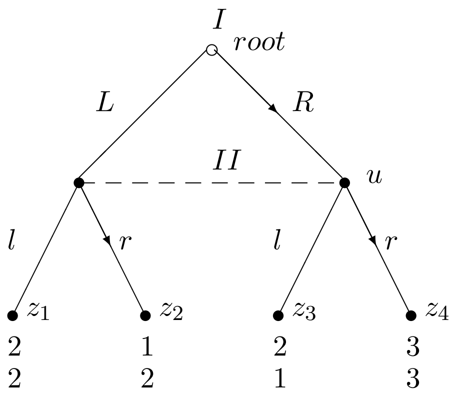


Figure 5.5: Second factor game: $\bar{G}$ $\bar{b} = (R, r)$.

Definition 68. *Given a collection $W \subset U(\Gamma)$, the W-combination of two beliefs $\mu, \bar{\mu} \in \mathcal{M}(\Gamma)$ is the belief of $\mathcal{M}(\Gamma)$*

$$\mu \otimes_W \bar{\mu} := (\mu_W, \bar{\mu}_{-W}).$$

For sake of notation, when no confusion arises, we use the abbreviated notations $G^{\otimes}$, $h^{\otimes}$, $b^{\otimes}$, and $\mu^{\otimes}$.

5.3 Essential collections

5.3.1 Essential collections

Definition 69. *Fixed an equilibrium concept* EC*, let Γ be a game form and $b \in B(\Gamma)$. A collection $W \subset U(\Gamma)$ is sufficient for* EC*, Γ, and b if it satisfies the following properties:*

i. $\pi(b) \subset W$, *i.e., W contains the path of b.*

ii. If $G, \bar{G} \in \mathcal{G}(\Gamma)$ are such that $b \in \mathrm{EC}(G)$ and $\mathrm{EC}(\bar{G}) \neq \emptyset$, then, there is $\hat{b} \in \mathrm{EC}(G \otimes_W \bar{G})$ such that b and $\hat{b}$ coincide in W.

Note that the conditions *i)* and *ii)* together imply that b and $\hat{b}$ are realization equivalent. If $\mathrm{EC}(\bar{G}) = \emptyset$, the definition is without consequences. The property ii) contains the gist of being a sufficient collection and the

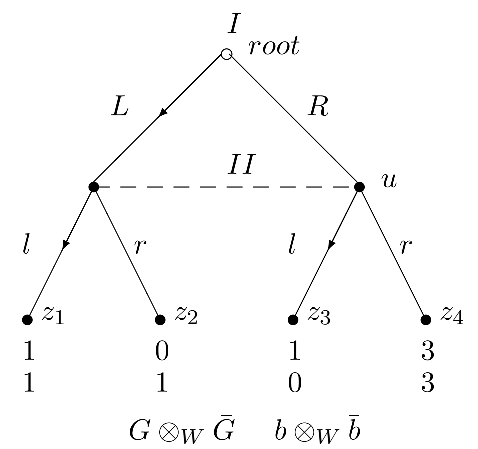


Figure 5.6: The combination of games.

property *i*) imposes some minimality requirement, since an empty collection always satisfies *ii*). Thus, the path of b is a natural candidate since we then ensure that $(b_W, \hat{b}_{-W})$ is realization equivalent to b, which was an important element in the motivation section.

In place of condition *ii*), the most natural thing to do would be to request that:

iii. $b \in \mathrm{EC}(G)$ and $\bar{b} \in \mathrm{EC}(\bar{G}) \Rightarrow b \otimes_W \bar{b} \in \mathrm{EC}(G \otimes_W \bar{G})$.

This natural condition is suitable for non belief-based equilibria (see Section 5.6), but it is too restrictive for belief-based equilibria. We will debate about it in Section 5.8, where we will show that, for SE as equilibrium concept, condition *iii*) is false.

The idea behind the sufficient collections is the following. Taken a collection W sufficient (for EC, Γ, and b) and taken $G \in \mathcal{G}(\Gamma)$ for which b is an equilibrium, if we change the payoffs outside W, provided that the new game has some equilibrium, then there will be one that is realization equivalent to b. Consider again the game G' introduced in section 5.1.1, with SPE as the equilibrium concept. There will always be a SPE of the game that is realization equivalent to b', whatever payoffs we put instead of those of the matching pennies subgame, since the collection W, left after removing the matching pennies subgame, is sufficient for SPE, Γ, and b'. Indeed to answer the question "does any equilibrium of G' coincide with b' in W?", the behavior outside W does not matter and if the answer is positive, then it remains positive for the new game, regardless of how we change the payoffs outside W. In particular, the outcome of b' will be an equilibrium outcome in the new game.

The property of being a sufficient collection only depends on the equilibrium concept at hand and on the given game form and strategy profile, while it does not depend on the possible payoffs associated with the game form.

Lemma 4. *If W is sufficient for EC, Γ and b, then it is also sufficient for any other $\bar{b}$ such that $b_W = \bar{b}_W$.*

Proof. Straightforward. □

Lemma 5. *The finite intersection of sufficient collections is a sufficient collection.*

Proof. Fixed an equilibrium concept EC, let Γ be a game form, $b \in B(\Gamma)$, and W,$\bar{W}$ be two sufficient collections (for EC, Γ, and b). First, the definition of sufficient collection implies $W \cap \bar{W}$ contains $\pi(b)$. Then, let G and $\bar{G} \in \mathcal{G}(\Gamma)$ be such that $b \in \mathrm{EC}(G)$ and $\mathrm{EC}(\bar{G}) \neq \emptyset$. We want to find $\hat{b} \in \mathrm{EC}(G \otimes_{W \cap \bar{W}} \bar{G})$ such that b and $\hat{b}$ coincide in $W \cap \bar{W}$. Since W is a sufficient collection, there is $\tilde{b} \in \mathrm{EC}(G \otimes_W \bar{G})$ that coincides with b in W. Let $\tilde{G} = G \otimes_W \bar{G}$. Since $\bar{W}$ is a sufficient collection, there is $\hat{b} \in \mathrm{EC}(\tilde{G} \otimes_{\bar{W}} \bar{G})$ that coincides with $\tilde{b}$ in $\bar{W}$. Now, by definition, $\hat{b}$ coincides with b in $W \cap \bar{W}$ and $\tilde{G} \otimes_{\bar{W}} \bar{G} = G \otimes_{W \cap \bar{W}} \bar{G}$. □

Corollary 6. *Fixed an equilibrium concept* EC*, let Γ be a game form and $b \in B(\Gamma)$. Then, there is a unique minimal collection that is sufficient for* EC*, Γ, and b. Moreover, there is a unique minimal collection that is closed and sufficient for* EC*, Γ, and b.*

Proof. We consider the intersection of all the sufficient collections for EC, Γ, and b. Since Γ is always a sufficient collection and all the sufficient collections contain $\pi(b)$, non-emptiness is guaranteed. The above intersection is contained in all the sufficient collections and its sufficiency follows from Lemma 5. The proof of the second statement is analogous, since Γ is a closed collection and the intersection of closed collections is a closed collection. □

Remark 8. If W and $\bar{W}$ are two collections such that $W \subset \bar{W}$ and W is sufficient (for some EC, Γ, and b), then $\bar{W}$ need not be also sufficient. In fact the condition that b and $\hat{b}$ coincide in $\bar{W}$ (see Definition 69) can be much more demanding than the corresponding condition for W.

Definition 70. *Fixed an equilibrium concept* EC*, let* Γ *be a game form and* $b \in B(\Gamma)$*. The essential collection for* EC*,* Γ*, and* b*, denoted by* $W_{\text{EC}}(\Gamma, b)$*, is the unique minimal collection closed under* $\preccurlyeq$ *and sufficient for* EC*,* Γ*, and* b*.*

Definition 71. *To essentialize an equilibrium concept* EC *is to find the map* W_{EC} *that assigns, to each pair* (Γ, b)*, the essential collection* $W_{\text{EC}}(\Gamma, b)$*.*

The closing request for an essential collection is quite natural. For example, in the case of belief-based equilibrium concept, let $u \in U_i(\Gamma)$ be an information set in the essential collection, that is player i's behavior at u is relevant for EC, Γ, and b. The closedness under $\preccurlyeq$ assures that what b prescribes for information sets that precede u should also be relevant, as it might affect the beliefs and behavior of i at u. The closedness provides to a collection a structure similar to a game form. Again, if this requirement is removed, then some unnatural essential collections might appear.

5.4 Discussion of the contribution

The definition of *essentializing an equilibrium concept* is so general that it is implementable to all the classical equilibrium concepts. But the analysis is cumbersome already for Nash equilibrium. Before the formal characterization of the essential collections associated with different equilibrium concepts, we present in this section the main results of the chapter.

We divide the equilibrium concepts into two groups: non-belief-based equilibrium concepts (NE, SPE, and PE) and belief-based-equilibrium concepts (SR, WPBE, SE, and a whole family of intermediate equilibrium concepts). *i*) The characterization for the first group is quite intuitive and provides no new insights into the nature of those equilibria. Let Γ be a game form and $b \in B(\Gamma)$, then we have:

Nash equilibrium: the essential collection consists of all the information sets that can be reached after an unilateral deviation from b.

Subgame perfect equilibrium: the essential collection is constructed iteratively. At each step, we add to the essential collection those information sets that can be reached after an unilateral deviation from b, deviation narrowed to subgames reached in the previous step.

Perfect equilibrium: every information set belongs to the essential collection.

In particular, in a game with perfect information, since a subgame begins at every node, the essential collection for SPE contains all the nodes of the game, that is it coincides with the essential collection for PE. This is not the case for NE. For the above characterizations, the more demanding an equilibrium concept is, the larger its corresponding essential collection. *ii*) This natural result does not hold for belief-based equilibrium concepts, rather the opposite one does. More specifically, let Γ be a game form and (b, μ) an assessment, where $b \in B(\Gamma)$, and $\mu \in M(\Gamma)$, then

Sequential rationality: every information set belongs to the essential collection.

Weak perfect Bayesian equilibrium: μ is calculated using Bayes rule in the path of b and the essential collection is constructed iteratively. At each step, we add to the essential collection those information sets that can be reached after an unilateral deviation from b, deviation narrowed to the continuation games reached with positive probability, according to μ, in the previous step.

Sequential equilibrium: μ is consistent with b and the essential collection is constructed iteratively. At each step we add to the essential collection those information sets that can be reached after an unilateral deviation from b narrowed to the continuation games reached with positive probability, according to μ, in the previous step.

We can note the parallelism between the characterization of essential collection for WPBE and for SE. When we use the same approach to SR, we require that μ is a system of beliefs and we apply the iterative construction for the essential collection. With no restrictions on the beliefs, every node can always be reached after a series of unilateral deviations. From the characterization for belief-based-equilibrium concepts, the more demanding an equilibrium concept is, the smaller its corresponding essential collection. For instance, for every game form and every strategy profile, the essential information sets for SE are a subset of those for WPBE or, equivalently, if an information set is irrelevant for WPBE, then it is irrelevant for SE as well. We show in the example below that the converse is not true in general. The definition of essential collection for belief-based equilibrium concepts, compared to one for non-belief-based equilibrium concepts, requires a condition on the belief set to be considered. The less restrictive equilibrium concepts allow for more beliefs, hence more parts of the game tree can be reached after a sequence of unilateral deviations, then the essential collections became larger.

We present now an example to illustrate some implications of the above characterizations and also some applications of results in the paper. We mainly concentrate our discussion on the essential collections for WPBE and SE.

5.5 A candidate positioning game (Osborne (1993))

In this Section, we read the model of Osborne (1993) in the light of the arguments introduced in the Chapter. We omit some elements not needed to our approach. We consider a game with three players, which represent the three potential candidates in an election. We denote $\Gamma(3)$ the three player game. The voters are a continuum set, each of whom has a most preferred or ideal policy. The ideal policies of voters are given by the continuous distribution function F, whose support is the $[0, 1]$ interval. Voters vote sincerely, i.e. each voter endorses the candidate whose position is closest to his ideal, if indifferent, he decides randomly. The candidate, who obtains a majority (plurality rule), wins. At each period $t \in \{1, \ldots, T\}$, with $T > 2$, candidates simultaneously decide whether to wait, which is denoted by w, or to enter the competition announcing a policy $p_i \in [0, 1]$. Policies are decided once and for all. Hence, at each period, a player who has already announced a policy cannot take any further action and, otherwise, he can either announce a policy, i.e. a number in $[0, 1]$, or decide to wait, i.e. w. Candidates can only use pure strategies[4]. The player who plays w in every period has decided to stay out of the election. Once reached period T, the election is held and the candidate with more votes wins. The notion of essential collection is very close to the idea behind the observation of Osborne: "in $\Gamma(3)$, as in other sequential games in which some choices are made simultaneously, the spirit of subgame perfect equilibrium is captured by a notion that requires only a partial specification of the player's strategies".

Suppose we want to study the strategy profile b where the player I and II enter in period 1 with policies p_1 and p_2 respectively, whereas player III chooses w in each period. Again, following Osborne (1993): to fully describe b, for player I we must "specify an action in period 2 for every first-period profile of action (w, s_2, s_3), where s_2 and s_3 are members of $[0, 1] \cup \{w\}$. However, there is just one relevant subgame in which player I has to take an action: the one that follows the first-period action profile (w, s_2, w)".

[4]Osborne argues that, in this setting, "the problem of finding equilibria in mixed strategies seems intractable" and, moreover, "voters may have an aversion to candidates who choose their positions randomly..."

Essential collection for SPE. Let W_{SPE} be the essential collection for SPE, Γ, and b. Then, following the informal characterization above, the only information set of the form (w, x_2, w) that would belong to W_{SPE} would indeed be (w, s_2, w), since all the others involve a multilateral deviation at period 1. That shows an important advantage of the essentializing approach. In order to study whether an outcome of the game is an equilibrium outcome or not, we need to check the incentive only in some subgames of the game. *The reduced game.* In Section 5.11, given a game G, we associate a reduced game G_W with each closed collection of information sets W. The idea is to remove from G all the information sets that are not in W, in such a way that what is left still forms a game. For instance, when studying the strategy profile b, none of subgames starting at information sets of the form (w, s_2, s_3) would be the root of a subgame in the reduced game, except for (w, x_2, w). Now, for Proposition 15, if the restriction of b to the reduced game is a SPE of the reduced game, then b is a SPE of the game $\Gamma(3)$, provided that $\Gamma(3)$ indeed has at least one SPE. *Structural robustness.* The reduced game is also applied to the study of the structural robustness of the different equilibrium concept. For example, let $b \in SPE(\Gamma(3))$. How robust would this equilibrium be to the structural changes in the game? We suppose that the following rule is valid, in order to encourage early positioning of candidates. If no candidate has entered the competition after period 2, then the election is suspended. Would b still be an equilibrium of the new game? Since no subgame at which the election is suspended belongs to the reduced game associated with b (they cannot be reached after unilateral deviations from b, where two candidates enter already in period 1), the above change in the rules of the game would have no impact for the profile b. That is, whether b is an equilibrium outcome or not is independent (robust) from those changes in the rules of $\Gamma(3)$ that only affect information sets outside the reduced game associated with b. *Partial-specifications of the game.* This issue is related to the one above. The idea is that essential collections may provide some information about the equilibrium outcomes of games that are not completely specified. We suppose that in $\Gamma(3)$ we have no idea about how the game unfolds if no player has entered the competition after period T. Even in this case, we know (by Corollary 9) that, no matter how the game is defined from that point onwards, the outcome of b is going to be an SPE outcome. Hence, essential collections help to identify what misspecifications in the game are irrelevant for different strategies and equilibrium concepts. *Virtual equilibrium concepts.* We suppose that there are some subgames of game $\Gamma(3)$ for which we do not even know whether a Nash equilibrium exists or not. Then, it might be that the game $\Gamma(3)$ has no SPE. The concept of virtual equilibrium intervenes in studying a game with subgames for which we do not even know whether a Nash equilibrium exists or not. A strategy profile b is a virtual SPE, if it is a SPE of the reduced game associated with its collection essential for SPE and the game form at hand. The virtual version of any other equilibrium concept is defined analogously. Hence, for the strategy b to be a virtual SPE it is needed that all the subgames of the corresponding reduced game have a Nash equilibrium, but we do not care about this for subgames outside the essential collection associated with b. Given a virtual equilibrium, we can always replace the non-equilibrium behavior outside the essential collection by equilibrium behavior (if this exists) to get an equilibrium in the classic sense. Then, by Proposition 15, if the set of SPE of the original game is nonempty, the set of SPE outcomes and virtual SPE outcomes coincides: which justify the name virtual. The equilibrium notion introduced in Osborne (1993) is extremely close to the virtual version of SPE. Indeed, Osborne wrote "the advantage of working with this notion of equilibrium in the game $\Gamma(3)$ is that it is not necessary ... to worry about the existence of an equilibrium, in *irrelevant subgames*" and " the relation between an equilibrum in this sense and a subgame perfect equilbrium is close: a subgame perfect equilibrium is an equilibrium and if every subgame has a subgame perfect equilibrium then an equilibrium is associated with at least one subgame perfect equilibrium", which is anologous to what we said above for virtual equilibrium concepts: every EC is a VEC and, if an EC exists, for each VEC we can find an EC with the same outcome.

5.6 Essentializing non-belief-based equilibrium concepts

The classic equilibrium concepts not based on beliefs are the Nash equilibrium, the subgame perfect equilibrium, and the perfect equilibrium. In this section we present their essentializing, even if the characterizations add nothing more to common knowledge of each equilibrium concept. We report these intuitive results not only for completeness, but also to support the adequacy of definitions and to make the readers familiar with the approach and with the tecniques of the proofs.

We introduce the stronger concept of sufficiency that will be quite useful to prove the characterization results.

Definition 72. *Fix an equilibrium concept* EC*. Let* Γ *be a game form and* $b \in B(\Gamma)$*. A collection* $W \subset U(\Gamma)$ *is strongly sufficient for* EC*,* Γ*, and* b *if it has the following properties:*

i) $\pi(b) \subset W$*, i.e.* W *contains the path of* b*.*

ii) Let $\bar{b} \in B(\Gamma)$ *and* $G, \bar{G} \in \mathcal{G}(\Gamma)$ *be such that* $b \in \mathrm{EC}(G)$ *and* $\bar{b} \in \mathrm{EC}(\bar{G})$*. Then* $b \otimes_W \bar{b} \in \mathrm{EC}(G \otimes_W \bar{G})$ *.*

Obviously a strongly sufficient collection is also a sufficient collection.

5.6.1 Nash equilibrium

Let Γ be a game form and $b \in B(\Gamma)$. First, we consider the collection V of information sets reached after at most one unilateral deviation from b.

$$V := \{u \in U(\Gamma) \,:\, \exists i \in N, b_i' \in B_i(\Gamma) \text{ such that } u \in \pi(b_{-i}, b_i')\}.$$

Then, we consider its closure under $\preccurlyeq$:

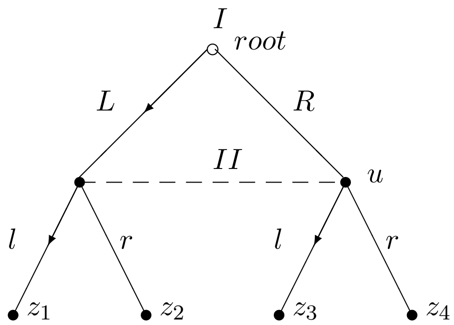
$$W_{NE}^b := \langle V \rangle.$$

We note that $W_{NE}^b \supset \pi(b)$ taking $b_i' = b_i$ and W_{NE}^b is a terminal collection. Figure 5.7 illustrates the definition of W_{NE}^b. The collection W_{NE}^b suffices to essentialize NE.

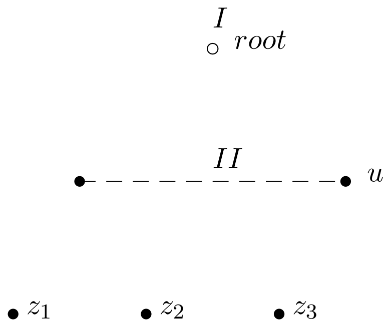
Proposition 8. W_{NE}^b *is the essential collection for NE,* Γ*, and* b*.*

Proof. First we prove that W_{NE}^b is strongly sufficient for NE, Γ, and b. By definition, choosing $b_i' = b$, we have $W_{NE}^b \supset \pi(b)$. Let $\bar{b} \in B(\Gamma)$ and $G = (\Gamma, h), \bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ be such that $b \in NE(G)$ and $\bar{b} \in NE(\bar{G})$. We suppose, by contradiction, $b^{\otimes} \notin NE(G^{\otimes})$. Then, there are a player $i \in N$ and $b_i^{'} \in B_i(\Gamma)$ such that $h_i^{\otimes}(b_{-i}^{\otimes}, b_i^{'}) > h_i^{\otimes}(b^{\otimes})$. Since $W_{NE}^b \supset \pi(b)$, $h_i^{\otimes}(b^{\otimes}) = h_i(b)$. By definition of W_{NE}^b, $\pi(b_{-i}^{\otimes}) \subset W_{NE}^b$. Hence, $h_i^{\otimes}(b_{-i}^{\otimes}, b_i^{'}) = h_i(b_{-i}^{\otimes}, b_i^{'})$. Moreover, since$(b_{-i}, b_i^{'})$ and $(b_{-i}^{\otimes}, b_i^{'})$ coincide in W_{NE}^b, $\pi(b_{-i}, b_i^{'}) = \pi(b_{-i}^{\otimes}, b_i^{'})$. Hence, $h_i(b_{-i}, b_i^{'}) = h_i(b_{-i}^{\otimes}, b_i^{'}) > h_i(b^{\otimes}) = h_i(b)$, contradicting the fact that $b \in NE(G)$.

Second, we show that W_{NE}^b is a minimal closed and sufficient collection and thus, essential. By definition, $W_{NE}^b = \langle W_{NE}^b \rangle$. Let W be a sufficient and closed collection for NE, Γ, and b that does not contain W_{NE}^b. By Lemma 3, since W_{NE}^b is terminal, there is $\bar{z} \in (W_{NE}^b \backslash W) \cap Z(\Gamma)$. Let $i \in N$ and $b_i' \in B(\Gamma)$ such that $\bar{z} \in \pi(b_{-i}, b_i')$. Consider the path of information sets from the root to $\bar{z}$, $\{u^1, \ldots, u^k\}$, *i.e.*, $u^1 = r(\Gamma)$ and $u^k = \bar{z}$. Since W_{NE}^b is closed, $\{u^1, \ldots, u^k\} \subset W_{NE}^b$. Since W is closed, $u^1 \in W$ and $u^k \notin W$, there is a unique $\bar{k}$ such that $u^{\bar{k}-1} \in W$ and $u^{\bar{k}} \notin W$. Let $G = (\Gamma, h)$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, $h_i(z) = 0$. Let $\bar{G} = (\Gamma, \bar{h})$



$\Gamma \quad b = (L, l)$



$W_{NE}(b)$

Figure 5.7: The collection W_{NE}^b.

be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, if $u^{\bar{k}} \preccurlyeq z$, $\bar{h}_i(z) := 1$ and $\bar{h}_i(z) := 0$ otherwise. Note that, since W is closed, $\bar{h}_i(z) := 1$ implies that $z \notin W$. Note that $b \in NE(G)$ and $G \otimes_W \bar{G} = \bar{G}$. Since $\pi(b) \subset W$, in game $\bar{G}$, all the payoffs in $\pi(b)$ are 0. Take now $\hat{b} \in B(\Gamma)$ such that it coincides with b in W. Then, for each $i \in N$, $h_i(\hat{b}) = 0$. By construction, there is $z \in Z(\Gamma)$ such that $u^{\bar{k}} \preccurlyeq z$ and $z \in \pi(\hat{b}_{-i}, \bar{b}_i)$. Hence, $h_i(\hat{b}_{-i}, \bar{b}_i) > 0 = h_i(\hat{b})$, $\hat{b} \notin NE(G \otimes_W \bar{G})$, contradicting the sufficiency of W. □

In general, $W_{NE}^b \neq V$, as depicted in Figure 5.8. The node x cannot be reached by unilateral deviations, but x belongs to W_{NE}^b. The closedness is a natural request to an essential collection since it gives to the collection a structure similar to a game form. In fact, given any game in $\mathcal{G}(\Gamma)$, x is not relevant to know if there is a NE that is realization equivalent to b. This example cannot be trivially adapted, for instance, to SE, since the beliefs of player III might depend on the behavior at x and hence, adding x to an essential collection might be natural.

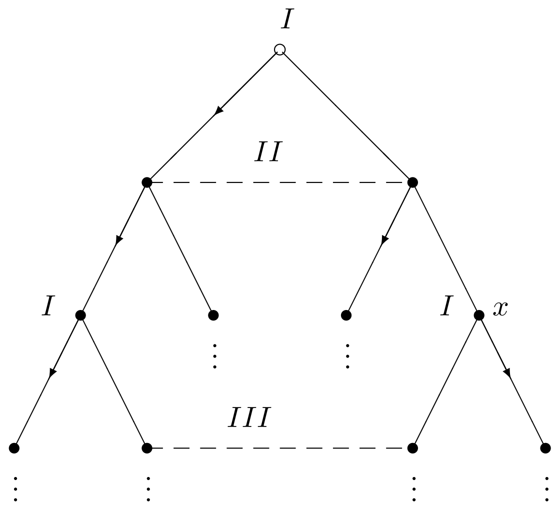


Figure 5.8: $W^b_{NE} \neq V$.

5.6.2 Essentializing NE in Strategic Form

In the previous section we have essentialized the Nash equilibrium concept for an extensive game. It is natural to query what happens whether we make a similar analysis for a strategic game. We are going to show that the two procedures reach the same conclusion, and, hence, the analysis starting from strategic form sustains the validity of the other through extensive form. Using strategic games to identify inessential elements for other equilibrium concepts like SPE or SE would become very cumbersome.

We consider the two-players strategic game $G = \langle X, Y, f, g \rangle$ in Figure 5.9. Using the definition of NE, we identify which elements of G are essential to check if a given strategy profile is a NE. In fact, (B, R) is a Nash equilibrium for G, since

$$f(B,R) \geqslant f(x,R) \quad \forall x \in X$$

and

$$g(B,R) \geqslant g(B,y) \quad \forall y \in Y.$$

$I \backslash II$	W		L		C		R		E	
N	0	2	1	0	4	1	***1***	***1***	0	3
T	2	1	4	0	3	1	***2***	***1***	1	4
M	1	2	0	0	2	1	***0***	***4***	2	1
B	***2***	***1***	***3***	***0***	***1***	***1***	***2***	***2***	***3***	***0***
S	3	0	0	2	0	0	***0***	***1***	0	4

Figure 5.9: Essentializing NE in strategic game.

Then, all the essential strategy profiles are only shaped like

$$(x, R), \forall x \in X \quad and \quad (B, y), \forall y \in Y.$$

We cannot change these strategy profiles, without the risk of losing the equilibrium, while the strategy profiles outside the cross, formed by row B and column R (see figure 5.9), are irrelevant, that is, they do not influence the fact that (B, R) is a NE.

Note that the restriction of G to the set $(\{B\} \times Y) \cup (X \times \{R\}) \subseteq X \times Y$ is not a game in strategic form. We recall that, in the essentializing process with an extensive game, we were looking for "something" very close to the structure of an extensive game. It is sufficient to work with the outcomes, corresponding to the selected strategy profiles, in order to reconstruct "something" very close to a game or to a game form. Note again that the correspondence that maps each strategy profile into its outcomes is surjective, but not injective. The following example will clarify better this statement.

Let us consider the extensive game G depicted in Figure 5.10. The game has two NE: (E, l) and (L, r). Let $b = (L, r)$, then $W^b_{NE} = \{u, v, z_2, z_4, z_5\}$. The game G in strategic form is depicted in Figure 5.11, where the outcomes corresponding to the terminal nodes in W^b_{NE} are underlined. Instead, the strategy profiles relevant are those inside the cross, formed by row L and column r (see figure 5.11). It is immediately noted that the strategy profile (E, l) appears. This happens since, in the node z_5, player II does not have to move and two strategy profiles (E, l) and (E, r) correspond to the same outcome. It is sufficient to make a quotient, in order to solve the problem. If the strategy profile b is (E, l), it adds up.

Formally, let $G = (\Gamma, h)$ be an extensive game, and $b \in B(\Gamma)$ a profile.

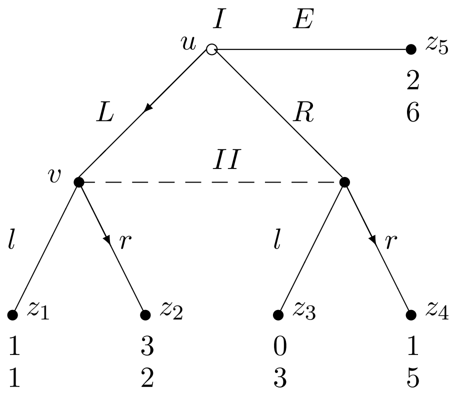


Figure 5.10: G in extensive form.

I \ II	l		r	
L	<u>*1*</u>	<u>*1*</u>	<u>*3*</u>	<u>*2*</u>
R	0	3	<u>*1*</u>	<u>*5*</u>
E	<u>*2*</u>	<u>*6*</u>	<u>*2*</u>	<u>*6*</u>

Figure 5.11: G in strategic form.

Essentializing the NE for the extensive game, we find the collection W^b_{NE}. Let, in parallel, $G = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ and $(\overline{x}_1, \ldots, \overline{x}_n)$ be the same game G and the same profile b in strategic form. Essentializing the NE for the strategic game, we find the set

$$P^{\overline{x}}_{NE} \doteq \{(x_i, \overline{x}_{-i}) : x_i \in X_i, \overline{x}_{-i} \in \prod_{j \neq i} X_j, i = 1, \ldots, n\}.$$

To each terminal node of the collection W^b_{NE}, we associate the corresponding outcome and to each outcome the class of equivalence of a strategy profile $(x_1, \ldots, x_n)$, where we define two strategy profiles *equivalent* if they have the same outcome of the game. This way we find the set $P^{\overline{x}}_{NE}$ of the essentialized strategy profiles. The equivalent strategy profiles are such that they prescribe the same actions in the decision-making nodes in W^b_{NE}, such as Proposition 15 establishes. It is no coincidence that we again come across the theme of the equivalence between two strategy profiles. In fact, we may tackle in a different way the following Essentializing problem.

> *"Given an equilibrium concept EC, an (extensive) game form Γ, and a (behavior) strategy profile b, identify W, a minimal collection of information sets of Γ, with the following property:*
> *If a game G has game form Γ and b is an equilibrium of G, then, whatever changes are made in the payoffs and strategies outside W, the outcome of b will be an equilibrium outcome in the resulting game."*

We rephrase the property in the case of non belief based equilibria.

> *"If a game G has game form Γ and b is an equilibrium of G, that is it satisfies the specifications of the concept, then, $b \otimes_W \bar{b}$ satisfies the specifications of the concept and has the same path as b, for every profile $\bar{b}$ satisfying the specification of the concept."*

Let us assume, for simplicity, the Nash equilibrium as an equilibrium concept. We essentialize NE, identifying those information sets reachable with a unilateral deviation from b, that is W^b_{NE}. We construct a partition of strategy profile relating to equivalence relation so defined.

Definition 73. *Let Γ be a game form, and $b, \bar{b} \in B(\Gamma)$. We say that b and $\bar{b}$ are realization equivalent, if $W^b_{NE} = W^{\bar{b}}_{NE}$.*

Then, the following propositions are true.

Proposition 9. *If $b \in [b]$ and $\bar{b} \in [\bar{b}]$ (where $[b]$ and $[\bar{b}]$ denote, respectively, the equivalence classes of b and $\bar{b}$, possibly coincident) then $b \otimes_{W^b_{NE}} \bar{b} \in [b]$.*

Proposition 10. *b and $\bar{b}$ are equivalent if and only if they have the same actions in the information set in W^b_{NE}.*

That is, we once more come across the definition of realization equivalence. The method through partition of profiles supports the results of the Essentializing method, since the first method, starting from the collection of information sets, determines the equivalence relation, while the second, starting from the equivalent realization, obtains the results established in Proposition 15.

5.6.3 Subgame perfect Nash equilibrium

A Subgame perfect equilibrium is a solution concept defined in recursive way: a SPE is a Nash equilibrium which induces a Nash equilibrium in every subgame. The same recursion is needed to define the essential collection $W_{SPE}^{b,\Gamma}$. To test if a fixed strategy profile b is a SPE, we restrict it to each subgame (identified through its root) or, more generally, to each continuation game.

Given $u \in U(\Gamma)$, let $W_u \doteq \{v \in U(\Gamma) : u \preccurlyeq v\}$ be the continuation game, and let b_u the restriction of b to W_u. In particular, given $x \in u \in U(\Gamma)$, b_x denotes the restriction of b to W_u.

Definition 74. *A node $x \in X(\Gamma)$ is elemental if either x is a terminal node or, for each game (Γ, h), a subgame begins at x.* [5].

In particular if x is elemental, then $u_x = x$, that is, x is a singleton, where we recall that

Definition 75. *A node $x \in X(\Gamma)$ is a singleton if its information set is degenerate.*

Singleton node and elemental node are two different notions. A singleton node is a node with no other nodes in its information set. An elemental node is a singleton which, moreover, has the property that a subgame starts at this node.

We consider the following construction of nested subsets of $U(\Gamma)$, that is, of elemental nodes, each of them identifying a subgame.

- Step 0: $X^0(b)$ coincides with the root of Γ.
- Step t: An elemental node x belongs to $X^t(b)$ if there are $i \in N$, $b'_i \in B_i(\Gamma)$, and $y \in X^{t-1}(b)$ such that x is reached by $(b_{-i}, b'_i)_y$.

Then, we consider the limit of these sequence

$$X_{SPE}(b) := \lim_{t \to +\infty} X^t(b).$$

Since the game tree is finite, $X_{SPE}(b)$ is well defined. In other words, $X_{SPE}(b)$ consists of the elemental nodes that can be reached with a series of unilateral deviations from b. Step by step, we identify each subgame and the elemental nodes which we manage to reach. Finally, we close the collection. Let

$$W_{SPE}^b := \langle X_{SPE}(b) \rangle.$$

[5]The notion of subgame we use is the standard one introduced in Selten (1975).

We note that we can exchange the order of operations: if first we close each collection $X^t(b)$ and then we take the limit, the result is invariant. We observe still that W^b_{SPE} is a terminal collection, since $\langle X^1(b)\rangle = W^b_{NE}$.

We consider, for example, the game G in Figure 5.12 and the profile $b = ((L,l),L)$. At each step of the procedure, we identify all the sub-

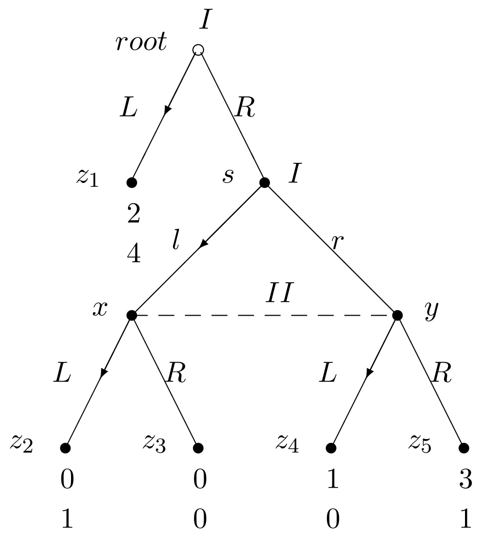


Figure 5.12: Essentializing SPE.

games of G, that in our example are only two. At step 1, we identify G, while at step 2 the proper subgame beginning at s. So $X^0(b) = \{root\}$, $X^1(b) = \{root, z_1, s, z_2, z_4\}$, $X^2(b) = \{root, z_1, s, z_2, z_4, z_3\} = X^t(b)$ for each $t \geqslant 3$. Now we close the collection, passing from a set of elemental nodes to a collection of information sets. The collection becomes $W^b_{SPE} = \{root, z_1, s, z_2, z_4, z_3, x, y\}$. $z_5 \notin W^b_{SPE}$ since in non subgame we can reach this node only with unilateral deviations. We note, further, that W^b_{SPE} do not constitute a game form, since the nodes x, y of the same information set do not have the same actions.

Proposition 11. *W^b_{SPE} is the essential collection for SPE, Γ and b.*

Proof. First, we show that W^b_{SPE} is strongly sufficient for SPE, Γ and b. Obviously $W^b_{SPE} \supset \pi(b)$. Let $\bar{b} \in B(\Gamma)$ and $G, \bar{G} \in \mathcal{G}(\Gamma)$ be such that $b \in SPE(G)$ and $\bar{b} \in SPE(\bar{G})$. We show that $b \otimes_W \bar{b} \in SPE(G \otimes_W \bar{G})$, or, in short form, that $b^{\otimes} \in SPE(G^{\otimes})$. Let $x \in X(\Gamma)$ be an elemental node. If $x \notin W^b_{SPE}$, then, since W^b_{SPE} is closed, $W_{u_x} \cap W^b_{SPE} = \emptyset$. Since $b^{\otimes}_x = \bar{b}_x$

and $\bar{b} \in SPE(\bar{G})$, $b^{\otimes}$ induces a Nash equilibrium in the subgame of $G^{\otimes}$ that begins at x. If $x \in W^b_{SPE}$, by definition of W^b_{SPE}, no elemental node outside W^b_{SPE} can be reached with unilateral deviation from b at nodes in W^b_{SPE}. Hence, since $b^{\otimes}_x = b_x$ and $b \in SPE(G)$, $b^{\otimes}$ induces a Nash equilibrium in the subgame of $G^{\otimes}$ that begins at x. Hence $b^{\otimes} \in SPE(G \otimes_W \bar{G})$.

Secondly, we show that W^b_{SPE} is a minimal closed and sufficient collection, and, thus, essential. By definition, $W^b_{SPE} = \langle W^b_{SPE} \rangle$. Let W be a closed and sufficient collection for SPE, Γ, and b that does not contain W^b_{SPE}. By Lemma 3, since W^b_{SPE} is terminal, there is $\bar{z} \in (W^b_{SPE} \backslash W) \cap Z(\Gamma)$. We consider the elemental nodes in the path from the root to $\bar{z}$, namely $\{x^1, \ldots, x^k\}$, where $x^1 = r(\Gamma)$ and $x^k = \bar{z}$. By definition $W^b_{SPE} = \langle W^b_{SPE} \rangle$, so W^b_{SPE} is closed and then $\{x^1, \ldots, x^k\} \subset W^b_{SPE}$. But also W is closed, then for Lemma 3, there is a unique $\bar{k} \geqslant 1$ such that $x^{\bar{k}-1} \in W$ and $x^{\bar{k}} \notin W$. Since W is sufficient, $\pi(b) \subset W$ and hence, $x^{\bar{k}} \in W^b_{SPE} \backslash \pi(b)$. Then, there are $i \in N$, $b'_i \in B_i(\Gamma)$, and $y \in X_{SPE}(b)$ such that $x^{\bar{k}}$ is reached by $(b_{-i}, b'_i)_y$ and not by b_y. Let $G = (\Gamma, h)$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, $h_i(z) = 0$. Let $\bar{G} = (\Gamma, \bar{h})$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, $\bar{h}_i(z) \doteq 1$ if $x^{\bar{k}} \preccurlyeq z$ and $\bar{h}_i(z) \doteq 0$ otherwise. Since W is closed, $\bar{h}_i = 1$ implies $z \notin W$. Note that $b \in SPE(G)$ and $G \otimes_W \bar{G} = \bar{G}$. We consider $\hat{b} \in B(\Gamma)$ such that it coincides with b in W. Then, for each $i \in N$, $h_i(\hat{b}) = 0$. By construction, there is $z \in Z(\Gamma)$ such that $x^{\bar{k}} \preccurlyeq z$ that is reached by $(\hat{b}_{-i}, b'_i)_y$. Hence, in the subgame of $\bar{G}$ that begins at y, payoff 1 is obtained with positive probability instead of getting 0 for sure. Therefore, $\hat{b} \notin SPE(\bar{G}) = SPE(G \otimes_W \bar{G})$, contradicting the sufficiency of W.

□

5.7 Perfect equilibrium

Given a game form Γ and a strategy profile $b \in B(\Gamma)$, the unique sufficient collection for PE, Γ, and b is $U(\Gamma)$. Therefore, $U(\Gamma)$ is the essential collection for PE, regardless of the strategy profile b.

Proposition 12. *$U(\Gamma)$ is the essential collection for PE, Γ and b.*

Proof. By definition, $U(\Gamma)$ is always a closed and sufficient collection. Hence, it is sufficient to show that $U(\Gamma)$ is a minimal closed and sufficient collection and thus essential. Let W be a closed and sufficient collection for PE, Γ and b, strictly contained in $U(\Gamma)$. By Lemma 3, since $U(\Gamma)$ is terminal, there is $\bar{z} \in (U(\Gamma)\backslash W) \cap Z(\Gamma)$ and, in particular, $\bar{z} \notin \pi(b)$. Let $G = (\Gamma, h)$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, $h_i(z) \doteq 0$. Let $\bar{G} = (\Gamma, \bar{h})$ be such that, for each $i \in N$, $h_i(\bar{z}) \doteq 1$ and $h_i(z) \doteq 0$ for each $z \in Z(\Gamma)\backslash\{\bar{z}\}$. Note that $b \in PE(G)$ and $G \otimes_W \bar{G} = \bar{G}$. Note that G has an unique perfect equilibrium in which $\bar{z}$ is reached with probability 1. Hence, if $\hat{b} \in B(\Gamma)$ coincides with b in W, since $\bar{z} \notin \pi(b)$, then $\hat{b} \notin PE(\bar{G}) = PE(G \otimes_W \bar{G})$, contradicting the sufficiency of W. □

5.8 Essentializing belief-based equilibrium concepts

In this Section, we head towards some primary concepts for extensive games with imperfect information. So the main result of this Chapter, that is the Theorem 19, applies to a wide family of belief-based equilibrium concepts.

The extensive game refinements based on beliefs are principally the sequential equilibrium, the perfect Bayesian equilibrium, the weak perfect Bayesian equilibrium or weak sequential equilibrium. They require that each player i assigns to each node x of each his information set $u \in U_i$ a probability, which represents his belief to be in x, provided he is in u. The previous equilibrium concepts impose different limitations to belief systems. In principle, they require the beliefs in the equilibrium path are derived by strategy profile using the theory of conditioned probability, the beliefs out of the equilibrium path, on the other hand, can be arbitrary, as it happens for weak perfect Bayesian equilibrium, or not, as for sequential equilibrium.

5.8.1 Belief-based equilibrium concepts. A first approach.

Let $G = (\Gamma, h)$ be an extensive game, (b, μ) an assessment, where $b \in B(\Gamma)$ is a profile and $\mu \in \mathcal{M}(\Gamma)$ a belief system. We consider the following construction of nested closed subset of $U(\Gamma)$. For the sake of exposition, we do not make explicit the dependence of U^t and V^t on b and μ.

- Step 0: $U^0 = \langle \pi(b) \rangle$.
- Step t: An information set $v \in U(\Gamma)$ belongs to V^t if there are $i \in N$, $b'_i \in B_i(\Gamma)$, and an information set $u \in U^{t-1} \cap U_i(\Gamma)$ such that v is reached with positive probability by $(b_{-i}, b'_i)_u$ when the probabilities of the nodes in u are given by μ. Let

$$U^t := \langle V^t \rangle.$$

Then, we consider the limit of these sequence

$$W^{b,\mu} := \lim_{t \to +\infty} U^t.$$

Since the game tree is finite, $W^{b,\mu}$ is well defined. Note that $W^{b,\mu}$ is a terminal collection.

We consider, for example, the game in Figure 5.13 with the assessment (b, μ), where $b = ((L, l), L)$ and $\mu(x) = 0$ $\mu(y) = 1$. Then the collection is $W^{b,\mu} = \{r, z_1, t, x, z_2, y, z_4, z_5\}$. In fact, at step 0, $U^0 = \{r, z_1\}$. At step 1, we have $V^1 = \{r, z_1; t, x, z_2\}$, since t, x, z_2 are reached with positive probability ($\mu(r) = 1$) from $((R, l), L)_{\{r\}} = ((R, l), L)$, and, closing V^1, $U^1 = \{r, z_1; t, x, z_2; y\}$. At step 2 we have $V^2 = \{r, z_1; t, x, z_2, y; z_4, z_5\}$, since z_4 is reached with positive probability ($\mu(y) = 1$) from $b_{\{x,y\}} = (l)$, and z_5 is reached with positive probability ($\mu(y) = 1$) from $((L, r), L)_{\{x,y\}} = (r)$, and closing V^2, $U^2 = V^2$. For each $t \geqslant 3$ $V^t = U^t = U^2$. In no way, z_3 can be reached from an information set in U^2 with unilateral deviations. So $W^{b,\mu} = U^2$.

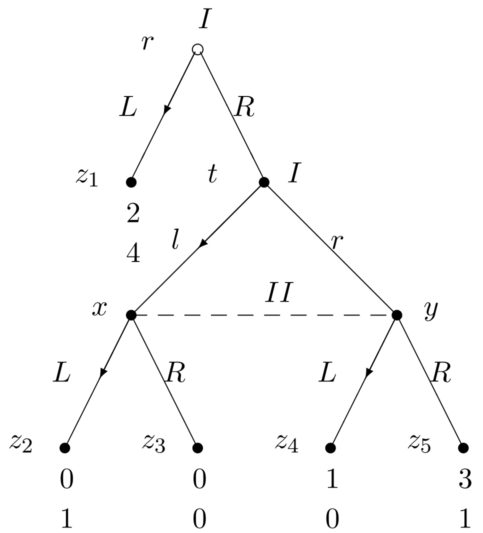


Figure 5.13: Example of $W^{b,\mu}$.

Weak perfect Bayesian equilibrium

Definition 76. *Let $G = (\Gamma, h)$ be an extensive game. An assessment (b, μ) is weakly consistent with Bayes rule if μ is derived using Bayesian updating* *in the path of* b.

Definition 77. *Let* $G = (\Gamma, h)$ *be an extensive game. A Weak Perfect Bayesian Equilibrium (or, briefly,* $WPBE$*) is an assessment* (b, μ) *sequentially rational and weakly consistent with Bayes rule.*

Let $G = (\Gamma, h)$ be an extensive game and (b, μ) be an assessment weakly consistent with Bayes rule. $W^{b,\mu}$ is a natural candidate to be a sufficient collection for $WPBE$, Γ, and b, but it is not enough. Let us consider, in Figure 5.13, the game G with the assessment (b, μ), where $b = ((L, l), L)$, $\mu(x) = 0$, and $\mu(y) = 1$, and in Figure 5.14, the same game G with the assessment $(b, \bar{\mu})$, where $b = ((L, l), L)$, $\bar{\mu}(x) = 1$, and $\bar{\mu}(x) = 0$.

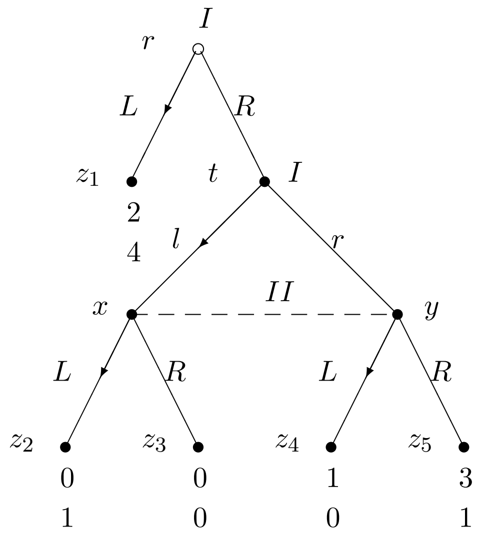


Figure 5.14: G $(b, \bar{\mu})$.

We know, beforehand, that the collection $W^{b,\mu}$ is $W^{b,\mu} = U(\Gamma)\backslash\{z_3\} = \{r, z_1, t, x, z_2, y, z_4, z_5\}$. Since the information set of player 2 is off-path, all the beliefs in this game are weakly consistent with Bayes rule. Note that $(b, \mu) \notin WPBE(G)$, and $(b, \bar{\mu}) \in WPBE(G)$. Then, in order to know that $b \in WPBE(G)$, it does not suffice to look at the payoffs in $W^{b,\mu}$. In fact, $W^{b,\mu}$ is not sufficient for $WPBE$, Γ, and b. Recall that $b \in WPBE(G)$.

Now, let us consider, in Figure 5.15, the game $\bar{G}$ with the assessment $(b, \bar{\mu})$. That is, $\bar{G}$ is identical to G, except for the fact that $\bar{h}(z_3) = (0, 2)$. Then, $\bar{b} = ((R, l), R) \in WPBE(\bar{G})$. Now, let us consider, as depicted in Figure 5.16, the combination of two games and the combination of two assessments. We have: $G \otimes_W \bar{G} = \bar{G}$, $b \otimes_W \bar{b} = b$, and, obviously, $\bar{\mu} \otimes_W \bar{\mu} = \bar{\mu}$. Again, $b \otimes_{W^{b,\mu}} \bar{b} = b \notin WPBE(G \otimes_{W^{b,\mu}} \bar{G})$, since, in $G \otimes_{W^{b,\mu}} \bar{G}$ the choice L for player II is strictly dominated and hence no beliefs make that choice sequentially rational. Similarly, by adequately rearranging the payoffs in the game G, it can be shown that $W^{b,\bar{\mu}}$ is not sufficient for $WPBE$, Γ, and b.

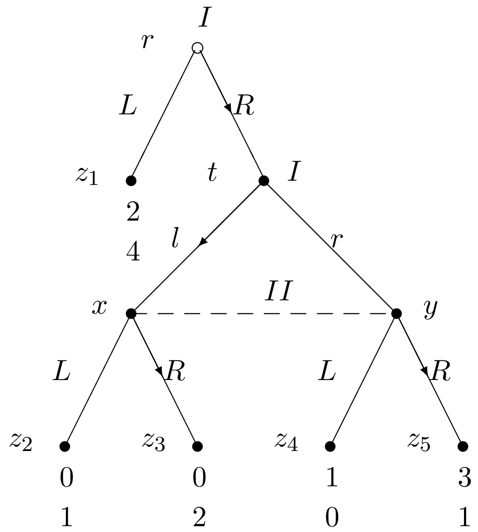


Figure 5.15: $\bar{G}$ $(\bar{b}, \bar{\mu})$.

Despite of discussion above, the collections $W^{b,\mu}$ are the key to essentialize $WPBE$. Let

$$\mathcal{M}^{wc}(b) := \{\mu \in \mathcal{M}(\Gamma) : (b, \mu) \text{ is weakly consistent with Bayes rule}\}.$$

Now, define the collection $W^b_{WPBE} := \bigcup_{\mu \in M^{wc}(b)} W^{b,\mu}$. Since the union of closed and terminal collections is a closed and terminal collection, W^b_{WPBE} is closed and terminal.

Proposition 13. *W^b_{WPBE} is the essential collection for $WPBE$, Γ and b.*

Proof. This result is a particular case of the general result in Section 5.8.2. □

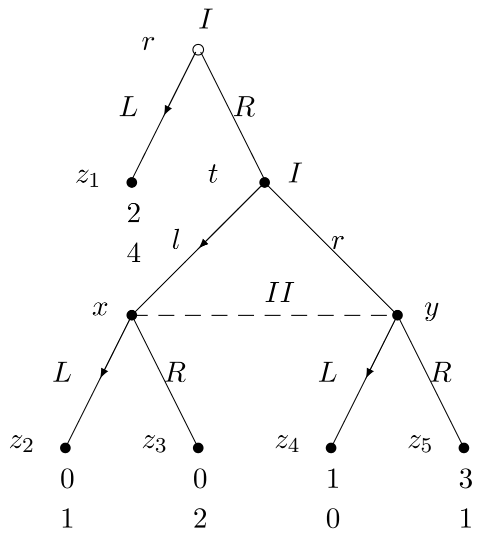


Figure 5.16: $G \otimes_W \bar{G} \quad (b \otimes_W \bar{b}, \bar{\mu} \otimes_W \bar{\mu})$.

5.8.2 Belief-based equilibrium concepts. A general result.

Let us develop a general approach that concerns several belief-based equilibrium concepts, but a sequential equilbrium that needs a separate treatment.

Let $\mathcal{F}$ be the set of all correspondences that select, for each game form Γ and each $b \in B(\Gamma)$, a subset of $\mathcal{M}(\Gamma)$ (the set of all beliefs that can be defined for Γ) [6]. Let Γ be a game form, $b \in B(\Gamma)$, and $G \in \mathcal{G}(\Gamma)$. Let $f \in \mathcal{F}$. We say that b is sequentially rational under f in game G, denoted by $b \in SR^f(G)$, if there is $\mu \in f(\Gamma, b)$ such that the assessment (b, μ) is sequentially rational. The above definition can be used to account for most belief-based solution concepts:

- Sequential rationality: $f^{SR}(\Gamma, b) := \mathcal{M}(\Gamma)$.
- WPBE: $f^{WPBE}(\Gamma, b) := \{\mu \in \mathcal{M}(\Gamma) : \mu \,\text{is derived by Bayes rule in} \pi(\text{b})\} = \mathcal{M}^{\text{wc}}(\text{b})$.
- SE: $f^{SE}(\Gamma, b) := \{\mu \in \mathcal{M}(\Gamma) : \mu \,\text{is consistent with b}\}$.
- Moreover, also the different versions of perfect Bayesian equilibrium, that have been discussed in literature, can be defined as sequentially

[6]More formally, let A denote the set of all pairs (Γ, b), where Γ is a game form and $b \in B(\Gamma)$. Then, $\mathcal{F} \doteq \{\text{functions from A to } 2^{\mathcal{M}(\Gamma)}\}$.

rational under some $f \in \mathcal{F}$.

Given $f \in \mathcal{F}$, a game form Γ, and $b \in B(\Gamma)$, define the collection

$$W_f^b := \bigcup_{\mu \in f(\Gamma,b)} W^{b,\mu}.$$

Note that, in particular,

$$W^b_{f^{WPBE}} = W^b_{WPBE}.$$

Since the union of closed and terminal collections is a closed and terminal collection, all the W_f^b collections are closed and terminal.

Lemma 6. *Let $f, f' \in \mathcal{F}$ be such that, for each Γ and each $b \in B(\Gamma)$,*

$$f(\Gamma, b) \subset f'(\Gamma, b).$$

Then, for each game G,

$$SR^f(G) \subset SR^{f'}(G).$$

Proof. Straightforward. □

The next auxiliary lemma plays an important role in the proofs of the results in this section. Let $u, v \in U(\Gamma)$.

Lemma 7. *Let $f \in \mathcal{F}$. Let Γ be a game form and $b \in B(\Gamma)$. Let $W \subset U(\Gamma)$ be a closed collection containing $\pi(b)$ such that $W_f^b \backslash W \neq \emptyset$, Then, there are $i \in N$, $\tilde{u} \in W \cap U_i(\Gamma)$, $\tilde{u} \in W_f^b \backslash W$, $\tilde{\mu} \in f(\Gamma, b)$, $x_{\tilde{u}} \in \tilde{u}$, $x_{\tilde{v}} \in \tilde{v}$, and $\tilde{b}_i \in B_i(\Gamma)$ such that*

- *i) $x_{\tilde{u}} \prec x_{\tilde{v}}$ (and hence, $\tilde{v}$) is reached with positive probability under $\tilde{\mu}$ by $(b_{-i}, \tilde{b}_i)_{\tilde{u}}$.*
- *ii) let $\{x^1 = x_{\tilde{u}}, \ldots, x^l = x_{\tilde{v}}\}$ be the path from $x_{\tilde{u}}$ to $x_{\tilde{v}}$. For each $\bar{l} < l$, $u_{x^{l}} \in W$.*

Proof. By Lemma 3, there is $z \in (W_f^b \backslash W) \cap Z(\Gamma)$. Let $\tilde{\mu} \in f(\Gamma, b)$ such that $z \in W^{b,\tilde{\mu}}$. Recall the iterative definition of $W^{b,\tilde{\mu}}$. Since $U^0 \subset \langle \pi(b) \rangle$ and $\pi(b) \subset W = \langle W \rangle$, then $U^0 \subset W$. Hence, there is $t \geqslant 1$ such that $z \in U^t \setminus U^{t-1}$. Let $u^t := z$. We now proceed backwards to identify the information sets used to reach u^t. Since $u^t \in U^t \setminus U^{t-1}$, there is $v^t \in$

$V^t \setminus U^{t-1}$ such that $u^t \preccurlyeq v^t$ (indeed, since $u^t = z \in Z(\Gamma)$, in this first step $v^t = u^t$). Since $v^t \in V^t \setminus U^{t-1}$, there are $i^t \in N$, $b^t_{i^t} \in B_{i^t}(\Gamma)$, and $u^{t-1} \in (U^{t-1} \setminus U^{t-2}) \cap U_{i^t}(\Gamma)$, [7] such that v^t is reached with positive probability by $(b^t_{i^t}, b_{-i^t})_{u^{t-1}}$. Hence, we can define a sequence $\{u^0, v^1, u^1, \dots, v^t, u^t\}$, where $u^0 \in \langle \pi(b) \rangle$. Then, $u^0 \in W$ and, since $u^t \notin W$, $W = \langle W \rangle$ and $u^t \preccurlyeq v^t$, we have that $v^t \notin W$. Similarly, for each $t' \in \{0, \dots, t\}$, if $u^{t'} \notin W$, then $v^{t'} \notin W$. Let $\bar{t} := \min_{t' \in \{0,\dots,t\}} \{t' : u^{t'-1} \in W \, \text{and} \, \text{v}^{t'} \notin \text{W}\}$. Define $i := i^{\bar{t}}$, $\tilde{u} := u^{\bar{t}-1}$, and $\tilde{b}_i := u^{\bar{t}}_{i^{\bar{t}}}$. Let $\bar{x} \in v^{\bar{t}}$ be such that $\bar{x}$ is reached with positive probability under $\tilde{\mu}$ by $(\tilde{b}_i, b_{-i})_{\tilde{u}}$. Let $x_{\tilde{u}}$ be the node in $\tilde{u}$ such that $x_{\tilde{u}} \prec \bar{x}$. Let $\{\tilde{u} = w^0, w^1, \dots, w^k = v^{\bar{t}}\}$ be the path of information sets from $x_{\tilde{u}}$ to $\bar{x}$. All the information sets in $\{\tilde{u} = w^0, w^1, \dots, w^k = v^{\bar{t}}\}$ are reached with positive probability under $\tilde{\mu}$ by $(\tilde{b}_i, b_{-i})_{\tilde{u}}$. Since $w^0 \in W$, $w^k \notin W$, and $W = \langle W \rangle$, there is a unique $\bar{k}$ such that $w^{\bar{k}-1} \in W$ and $w^{\bar{k}} \notin W$. Now let us define $\tilde{v} \doteq w^{\bar{k}}$ and let $x_{\tilde{v}}$ be the node in the path from $x_{\tilde{u}}$ to $\bar{x}$ that belongs to $\tilde{v}$. So defined, it is clear that $\bar{u} \in W \cap U_i$, $\tilde{v} \in W^{b,\tilde{\mu}}$, and hence, $\tilde{v} \in W^b_f \setminus W$. *i)* and *ii)* follow from the construction. □

For our general result, we need to restrict to a subset of $\mathcal{F}$.

Definition 78. *A function $f \in \mathcal{F}$ is regular if, given $b, \bar{b} \in B(\Gamma)$, the following properties hold.*

i) for each $\mu \in f(\Gamma, b)$ and each $\bar{\mu} \in f(\Gamma, \bar{b})$, $\mu \otimes_{W^b_f} \bar{\mu} \in f(\Gamma, b \otimes_{W^b_f} \bar{b})$, and, conversely,

ii) for each $\bar{\mu} \in f(\Gamma, b \otimes_{W^b_f} \bar{b})$, there is $\mu \in f(\Gamma, b)$ such that $\bar{\mu}$ and μ coincide in W^b_f.

In words, the beliefs inside W^b_f do not impose any restrictions in the beliefs outside W^b_f and *vice versa*. According to the above definition, f^{SE} fails to be regular (see Example in SubSection 5.8.3) and hence, sequential equilibrium needs to be studied on his own [8]. Nonetheless, sequential rationality, WPBE, and many natural refinements of the latter can be defined through regular functions [9] .

[7] If $t = 1$, then $U^{t-2} = U^{-1} := \emptyset$

[8] Moreover, also the perfect Bayesian equilibrium, as defined in Fudenberg and Tirole (1991b) for multistage games with observed actions, fails to be regular.

[9] For instance, Kreps and Wilson (1982) defined an equilibrium concept called *extended subgame perfect equilibrium*, a refinement of WPBE that imposes the use of Bayes rules off the equilibrium path, and hence, refines SPE as well. This equilibrium concept can be defined using regular functions.

Lemma 8. *Let $f \in \mathcal{F}$ be regular. If b and $\bar{b}$ coincide in W_f^b, then $W_f^b = W_f^{\bar{b}}$.*

Proof. Note that $\bar{b} = b \otimes_{W_f^b} \bar{b}$. We prove first that $W_f^b \subset W_f^{\bar{b}}$. Suppose, on the contrary, that there is $u \in W_f^b \setminus W_f^{\bar{b}}$. Take $i \in N$, $\tilde{u} \in W_f^{\bar{b}} \cap U_i$, $\tilde{v} \in W_f^b \setminus W_f^{\bar{b}}$, $\tilde{\mu} \in f(\Gamma, b)$, and $\tilde{b} \in B_i(\Gamma)$ as in Lemma 7 . Since f is regular, there is $\bar{\mu} \in f(\Gamma, \bar{b})$ that coincides with $\tilde{\mu}$ in W_f^b. Since $b_{W_f^b} = \bar{b}_{W_f^b}$ and $W_f^b = \langle W_f^b \rangle$, for each $w \in U(\Gamma)$ such that $w \prec \tilde{v}$, $b_w = \bar{b}_w$ and hence, $\tilde{v}$ is reached with positive probability under $\tilde{\mu}$ by $(\tilde{b}_i, \bar{b}_{-i})_{\tilde{u}}$. Therefore, $\tilde{v} \in W_f^{\bar{b}}$ and we have a contradiction. Hence $W_f^b \subset W_f^{\bar{b}}$.

We prove now that $W_f^{\bar{b}} \subset W_f^b$. Suppose, on the contrary, that there is $u \in W_f^{\bar{b}} \backslash W_f^b$. Take now $i \in N$, $\tilde{u} \in W_f^b \cap U_i$, $\tilde{v} \in W_f^{\bar{b}} \backslash W_f^b$, $\tilde{\mu} \in f(\Gamma, \bar{b})$, and $\tilde{b}_i \in B_i(\Gamma)$ as in Lemma 7. Since f is regular, there is $\mu \in f(\Gamma, b)$, that coincides with $\tilde{\mu}$ in W_f^b. If we had $b_{W_f^{\bar{b}}} = \bar{b}_{W_f^{\bar{b}}}$, we could continue as above. Yet, we just know that $b_{W_f^{\bar{b}}} = \bar{b}_{W_f^{\bar{b}}}$. From *ii*) in Lemma 7, all the information sets in the path from $x_{\tilde{u}}$ to $x_{\tilde{v}}$ belong to W_f^b. Hence, by *i*) in Lemma 7, if b and $\bar{b}$ coincide in W_f^b, $\tilde{v}$ is reached with positive probability under μ by $(b_{-i}, \tilde{b}_i)_{\tilde{u}}$ and we can derive the same contradiction as before. □

Theorem 19. *Let $f \in \mathcal{F}$ be regular. W_f^b is the essential collection for SR^f,Γ, and b.*

Proof. First, we show that W_f^b is a strongly sufficient collection for SR^f, Γ, and b. By definition of strongly sufficient collection, $\pi(b) \subset W_f^b$. Let $\bar{b} \in B(\Gamma)$ and $G = (\Gamma, h)$, $\bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ be such that $b \in SR^f(G)$ and $\bar{b} \in SR^f(\bar{G})$ respectively. We claim that $(b^{\otimes}, \mu^{\otimes}) \in SR^f(G^{\otimes})$. Since f is regular, $(b^{\otimes}, \mu^{\otimes}) \in f(\Gamma, b^{\otimes})$. We show now that it is sequentially rational. Let first $u \in U(\Gamma)$. There are two possibilities: $u \notin W_f^b$ or $u \in W_f^b$. Let $u \notin W_f^b$. For each $z \in Z(\Gamma)$ such that $u \prec z$, $z \notin W_f^b$ since $W_f^b = \langle W_f^b \rangle$. Hence $h^{\otimes}(z) = \bar{h}(z)$. Therefore, since $\bar{b} \in SR^f(\bar{G})$, $b^{\otimes}$ is sequentially rational at u in $G^{\otimes}$. Let now $u \in W_f^b$. By definition of W_f^b, as far as beliefs in $f(W, b)$ are considered, no terminal node outside W_f^b is reached with positive probability after unilateral deviations from b at information sets in W_f^b. Besides, by Lemma 8, $W_f^b = W_f^{b^{\otimes}}$ and hence, those terminal nodes are not reached either when the beliefs in the information sets in W_f^b are taken from $f(W, b^{\otimes})$. Hence, since $b \in SR^f(G)$, $b^{\otimes}$ is sequentially rational at u in G. Then, $b^{\otimes} \in SR^f(G^{\otimes})$.

Now let us show the second condition, that is W_f^b is a minimal closed and sufficient collection and thus essential. By definition, $W_f^b = \langle W_f^b \rangle$. Let W be a closed and sufficient collection for SR^f, Γ, and b that does not contain W_f^b. By Lemma 3, since W_f^b is terminal, there is $\bar{z} \in (W_f^b \setminus W) \cap Z(\Gamma)$. Let $\mu \in f(\Gamma, b)$ be such that $\bar{z} \in W^{b,\mu}$. Take $i \in N$, $\tilde{u} \in W \cap U_i$, $\tilde{v} \in W_f^b \setminus W$, $\tilde{\mu} \in f(\Gamma, b)$, , $\tilde{b}_i \in B_i(\Gamma)$, $x_{\tilde{v}}$, and $x_{\tilde{u}}$ as in Lemma 7. Since $x_{\tilde{v}}$ is reached with positive probability under $\tilde{\mu}$, by $(\tilde{b}_i, b_{-i})_{\tilde{u}}, \tilde{\mu}(x_{\tilde{u}}) > 0$. Let $\tilde{c}$ denote the choice at $x_{\tilde{u}}$ that is in the path to $x_{\tilde{v}}$. We distinguish two cases, regardless of whether $\tilde{u}$, and $\tilde{v}$ are singleton and whether $\tilde{v}$ is or is not a terminal node. Case *i*): $b_i(\tilde{c}) = 0$, i.e., according to b, choice c is never made. Then, $\tilde{v}$ is not reached with positive probability under $\tilde{\mu}$ by $b_{\tilde{\mu}}$. Let $G = (\Gamma, h)$ the game such that *i*) $(b, \tilde{\mu}) \in SR^f(G)$ and *ii*) given a choice $c \neq \tilde{c}$ at $\tilde{u}$, conditional on $\tilde{u}$ being reached, c is strictly dominated by $\tilde{c}$ in all nodes of $\tilde{u}$ but $x_{\tilde{u}}$. Since $\tilde{\mu}(x_{\tilde{u}}) > 0$, *i*) and *ii*) are compatible. Let $\bar{G} = (\Gamma, \bar{h})$ the game such that, for each $j \in N$ and each $z \in Z(\Gamma)$, $\bar{h}_j(z) \doteq M_G$ if $x_{\tilde{v}} \preccurlyeq z$ and $\bar{h}_j(z) \doteq h_j(z)$ otherwise. Recall that $M_G \doteq \max_{i \in N, z \in Z(\Gamma)} |h_i(z)| + 1$. Since $\tilde{v} \notin W$ and $W = \langle W \rangle$, for each $z \in Z(\Gamma)$ such that $\tilde{v} \prec z$, we have $z \notin W$. Now, $b \in SR^f(G)$ and $SR^f(\bar{G}) \neq$, just take any strategy profile with payoff M_G. Let us claim that if $\hat{b} \in B(\Gamma)$ coincides with b in W, then $\hat{b} \notin SR^f(G \otimes_W \bar{G})$, where $G \otimes_W \bar{G} = \bar{G}$. By construction, in the game $\bar{G}$, conditional on $\tilde{u}$ being reached, $\tilde{c}$ is strictly dominant at $\tilde{u}$. In fact, to play $\tilde{b}_i(\tilde{u})$ at $x_{\tilde{u}}$ leads to a payoff of M_G. Since $\tilde{u} \in W$, $\tilde{b}_i(c) = 0$ and $\tilde{b}_W = b_W$, $\tilde{b}$ is not sequentially rational at $\tilde{u}$. Case *ii*): $b_i(\tilde{c}) = 0$, i.e., $\tilde{v}$ is reached with positive probability under $\tilde{\mu}$ by $b_{\tilde{u}}$. Let $G = (\Gamma, h)$ be the game such that *i*) $(b, \tilde{\mu}) \in SR^f(G)$ and *ii*) there is a choice $c \neq \tilde{c}$ at $\bar{u}$ such that, conditional on $\tilde{u}$ being reached, c strictly dominates $\tilde{c}$ in all nodes of $\tilde{u}$ but $x_{\tilde{u}}$. Let $\bar{G} = (\Gamma, \bar{h})$ be such that for each $j \in N$ and each $z \in Z(\Gamma)$, $\bar{h}_j(z) \doteq -M_G$ if $x_{\tilde{v}} \preccurlyeq z$ and $\bar{h}_j(z) \doteq h_j(z)$ otherwise. The remaining proof is analogous to case *i*). □

Corollary 7. *$U(\Gamma)$ is the essential collection for SR, Γ, and b.*

Proof. It follows from Theorem 19 and from the fact $f(\Gamma, b) = \mathcal{M}(\Gamma)$. □

Corollary 8. *Let $f, f' \in \mathcal{F}$ be regular and let Γ and $b \in B(\Gamma)$ be such that $f(\Gamma, b) \subset f'\Gamma, b)$. Then $W_{SR^f}(\Gamma, b) \subset W_{SR^{f'}}(\Gamma, b)$.*

$U(\Gamma)$ is the essential collection for SR, Γ, and b.

The above Corollary and Lemma 6 imply that the following relation holds. Let EC^1 and EC^2 be two belief-based equilibrium concepts such that, for each game G,

$$EC^1(G) \subset EC^2(G),$$

then, for each game form Γ and each strategy profile $b \in B(\Gamma)$,

$$W_{EC^1}(\Gamma, b) \subset W_{EC^2}(\Gamma, b).$$

For example, let us just think of SR and $WPBE$.

5.8.3 Strong sufficiency and sequential equilibrium

The example below shows that f^{SE} is not regular and that W^b_{SE} needs not be a strongly sufficient collection for sequential equilibrium.

We consider the game $G \in \mathcal{G}(\Gamma)$ in Figure 5.17. Let $b = (L, L, (L, L))$,

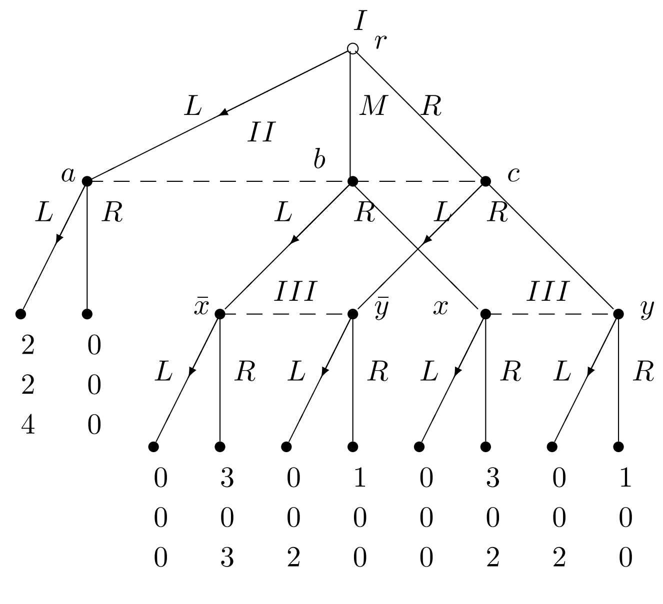


Figure 5.17: $G \quad (b, \mu)$.

then the assessment (b, μ) is consistent if and only if $\mu(a) = 1$, $\mu(b) = \mu(c) = 0$, $\mu(x) = \mu(\bar{x})$, $\mu(y) = \mu(\bar{y})$, and (b, μ) is sequentially rational if and only if $\mu(x) \leqslant \frac{1}{2}$, $\mu(\bar{x}) \leqslant \frac{2}{5}$. Now, W^b_{SE} is the collection that consists of removing the upper information set of player III and the four terminal nodes that come after it. More formally, $W^b_{SE} = U(\Gamma) \backslash W_x$, where $W_x \doteq \{v \in U(\Gamma) : u_x \preccurlyeq v\}$. Let $\mu \in \mathcal{M}^{cons}(b)$ be such that $\mu(x) = \mu(\bar{x}) = 0$. So defined, $(b, \mu) \in SE(G)$. Now, let us take the game $\bar{G}$, depicted in Figure 5.18, with the profile $\bar{b} = b$. Let $\bar{\mu} \in \mathcal{M}^{cons}(b) = \mathcal{M}^{cons}(\bar{b})$ such that $\bar{\mu}(x) = 1$ $\bar{\mu}(\bar{x}) = 1$. So defined, $(\bar{b}, \bar{\mu}) \in SE(\bar{G})$. We consider now, in Figure 5.19, the

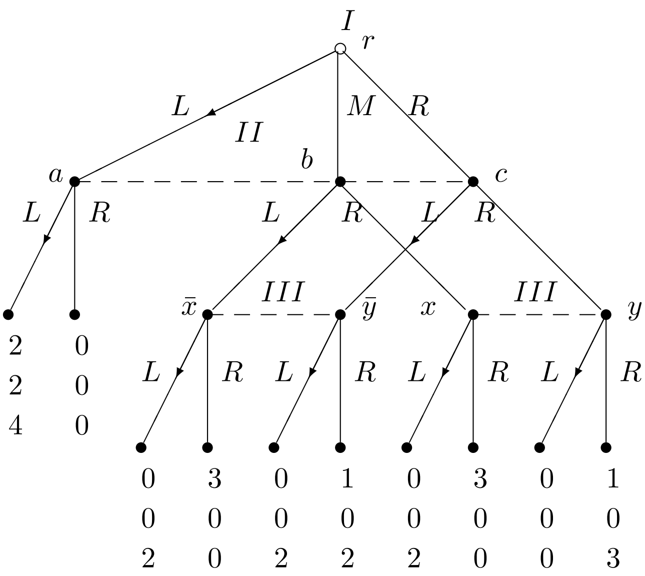


Figure 5.18: $\bar{G}$ $(\bar{b}, \bar{\mu})$.

game $G^{\otimes} = G \otimes_W \bar{G}$ with the assessment $(b^{\otimes}, \mu^{\otimes})$. Since $b^{\otimes} = b \otimes_W \bar{b} = b$, $\mathcal{M}^{cons}(b^{\otimes}) = \mathcal{M}^{cons}(b)$. Therefore, since $(\mu \otimes_W \bar{\mu})(x) = 1 \neq (\mu \otimes_W \bar{\mu})(\bar{x}) = 0$, $\mu^{\otimes}$ is not consistent with $b^{\otimes}$. Hence, $(b, \mu \otimes_W \bar{\mu}) \notin SE(G^{\otimes})$. Since $\mu \in \mathcal{M}^{cons}(b) = f^{SE}(\Gamma, b)$, $\bar{\mu} \in f^{SE}(\Gamma, \bar{b})$, and $\mu^{\otimes} \notin^{SE} (\Gamma, b^{\otimes})$, we have shown that f^{SE} is not regular.

5.9 Decomposition of a game with respect to a collection

In this Section we introduce a decomposition of a game useful to characterize the essential collections for sequential equilibrium in Section 5.10 and also for the analysis in Section 5.11.

Let Γ be a game form, $b \in B^0(\Gamma)$, $G = (\Gamma, h)$, and $W \subset U(\Gamma)$. With respect to the closed collection W, the game G is decomposable in two games, the reduced game G_W, containing the information sets in W, and its complementary game $G(-W, b)$, containing the information sets in $U(\Gamma) \backslash W$.

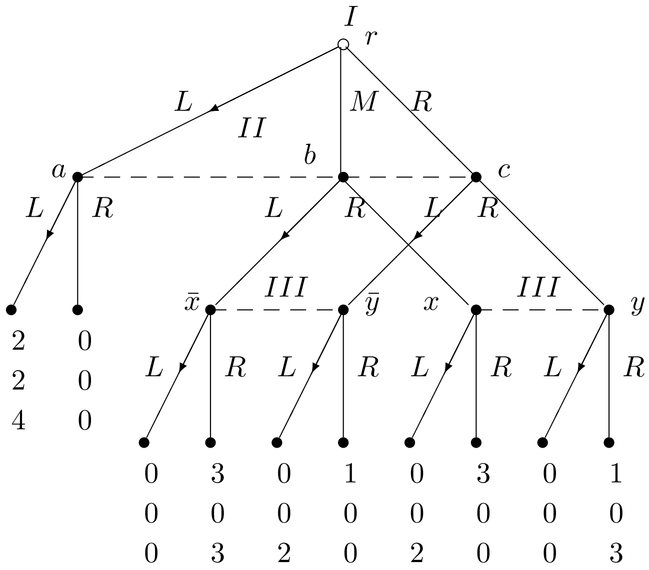


Figure 5.19: $G \otimes_W \bar{G} \quad (b \otimes_W \bar{b}, \mu \otimes_W \bar{\mu})$.

The reduced game G_W is important since to check, whether a profile is an equilibrium of original game G, it is necessary and sufficient to check only on the reduced game.

To the construction of G_W, we are interested in Γ_W, obtained by restriction of Γ to $X(W)$, with the addition of some artificial terminal nodes, necessary to ensure that Γ_W is a game form. For example, we consider the game in Figure 5.7. If we reduce Γ to the nodes in $X(W)$ without adding any extra nodes, in the information set u the number of choices available to player II would not be the same for the different nodes.

Figures 5.20 and 5.21 illustrate how decompose the game with respect to a collection.

For the sole purpose of construction, we distinguish between three sets of nodes: X(W), X(-W), and $A(W)$. $X(W)$ and $X(-W)$ are the sets of nodes whose information sets are, respectively, in and out of the essential collection W, while $A(W)$ is the set of nodes to be added, that is, $A(W)$ contains the nodes in $X(-W)$ with no predecessors in $X(-W)$.

$$X(W) \doteq \{x \in X(\Gamma) : u_x \in W\},$$

$$X(-W) \doteq X(\Gamma) \backslash X(W),$$

and

$$A(W) \doteq \{x \in X(-W) : x \prec y \Rightarrow y \in X(W)\}.$$

Definition 79. *Let Γ be a game form, $G = (\Gamma, h)$, and $W \subset U(\Gamma)$. The reduced game form Γ_W is characterized in the following way:*

i) the set of nodes is $X(\Gamma_W) \doteq X(W) \cup A(W)$,

ii) the set of information sets is $U(\Gamma_W) \doteq (U(W) \cap W) \cup A(W)$,

iii) the terminal nodes are $Z(\Gamma_W) \doteq (Z(W) \cap W) \cup A(W)$, and

iv) all the other elements are defined restricting Γ to $X(W)$ in the natural way.

Now, we can define the reduced game. Let $M \in \mathbb{R}$ be a fixed constant. We can take $M = M_G$, but the choice of the payoff for terminal nodes outside W is irrelevant for the analysis, as the choice of setting equal all the payoffs is irrelevant.

Definition 80. *Let Γ be a game form, $G = (\Gamma, h)$, and $W \subset U(\Gamma)$. The reduced game is $G_W = (\Gamma_W, h_W)$, where $h_W(z) = h(z)$ if $z \in Z(\Gamma_W) \cap W$ and $h_W(z) = (M, \ldots, M)$ if $z \in Z(\Gamma_W) \setminus W$.*

We discuss the importance of the reduced games in Section 5.11.

Now, we construct the complementary game. We need to fix a strategy profile $b \in B^0(\Gamma)$. For each $x \in A(W)$, let $p(x, b)$ denote the probability that x is reached given b and conditional on $X(-W)$ being reached. Now, we use b and the node in $X(-W)$ to define the complementary game form $\Gamma_{-W,b}$ and the complementary game $G(-W, b) = (\Gamma_{-W,b}, h_{-W,b})$, as depicted in Figures 5.20 and 5.21.

Definition 81. *Let Γ be a game form, $b \in B^0(\Gamma)$, $G = (\Gamma, h)$, and $W \subset U(\Gamma)$. The complementary game form $\Gamma_{-W,b}$ is characterized in the following way:*

i) the root is a node $r_{-W} \notin X(\Gamma)$,

ii) the set of nodes is $X(\Gamma_{-W,b}) \doteq X(-W) \cup r_{-W}$,

ii) for each $x \in A(W)$ there is an arc from r_{-W} to x, and the corresponding choice has probability $p(x, b)$, *iii) the terminal node is* $Z(\Gamma_{-W,b}) \doteq Z(\Gamma) \cap X(-W)$*, and*

iv) all the other elements are defined restricting Γ *to* $X(\Gamma_{-W,b})$ *in the natural way.*

Definition 82. *Let* Γ *be a game form,* $b \in B^0(\Gamma)$*,* $G = (\Gamma, h)$*, and* $W \subset U(\Gamma)$*. The complementary game is* $G(-W, b) = (\Gamma_{-W,b}, h_{-W,b})$*, where, for each* $z \in Z(\Gamma_{-W,b})$*,* $h_{-W,b} = h(z)$*.*

Note that, given $b, \bar{b} \in B^0(\Gamma)$, $G(-W, b)$ and $G(-W, \bar{b})$ only differ in the probabilities of nature move at the root. The games $G(W, b)$ are crucial to prove Proposition 14 below.

5.10 Sequential equilibrium

Proposition 14. W^b_{SE} *is the essential collection for* SE*,* Γ*, and* b*.*

Proof. First we show that W^b_{SE} is a sufficient collection for SE, Γ, and b. By definition $\pi(b) \subseteq W^b_{SE}$. Let $G = (\Gamma, h), \bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ be such that $b \in SE(G)$. We show that there is $\hat{b} \in SE(G^{\otimes})$ that coincides with b in W^b_{SE}. Since $b \in SE(G)$, there is $\mu \in \mathcal{M}^{cons}$ such that (b, μ) is sequentially rational. Hence there is a sequence $(b_n)_{n \in \mathbb{N}}$ of completely mixed strategies converging to b such that the associated consistent beliefs, namely $(\mu_n)_{n \in \mathbb{N}}$, converge to μ. For each $b_n \in B^0(\Gamma)$ let us consider the complementary game, so we have a sequence of games $(\bar{G}(-W^b_{SE}, b_n))_{n \in \mathbb{N}}$. Let $n \in \mathbb{N}$ and u be an information set of $\bar{G}(-W^b_{SE}, b_n)$, formed by nodes in $A(W^b_{SE})$. By definition, the beliefs induced by nature move at $r_{-W^b_{SE}}$ in u coincide with μ_n. For each $n \in \mathbb{N}$, let $(\bar{b}_n, \bar{\mu}_n)$ be a sequential equilibrium of $\bar{G}(-W^b_{SE}, b_n)$. The sequence $(\bar{b}_n, \bar{\mu}_n)_{n \in \mathbb{N}}$ has a convergent subsequence. We assume, without loss of generality, that the sequence itself converges to $(\bar{b}, \bar{\mu})$. We claim that $(b \otimes_{W^b_{SE}} \bar{b}, \mu \otimes_{W^b_{SE}} \bar{\mu})$ is a sequentially rational and consistent assessment. *Consistency.* Let Γ_n be the game form of $\bar{G}(-W^b_{SE}, b_n)$. By definition, for each $n, \bar{n} \in \mathbb{N}$, $B^0(\Gamma_n) = B^0(\Gamma_{\bar{n}})$. Let $\bar{B}^0 \doteq B^0(\Gamma_n)$. Each $\bar{b}_n$ is a sequential equilibrium of $\bar{G}(-W^b_{SE}, b_n)$. Hence, for each $n \in \mathbb{N}$, there is $\{\bar{b}_{n,k}\}_{k \in \mathbb{N}} \subset \bar{B}^0$ converging to b_n, and such that associated beliefs (satisfying Bayes rule) converge to $\bar{\mu}_n$. Hence, for each $n \in \mathbb{N}$, there is $g(n) \in \mathbb{N}$ such that $\| \bar{b}_n - \bar{b}_{n,g(n)} \| \leqslant \frac{1}{n}$. Then, $\| \bar{b}_n - \bar{b}_{n,g(n)} \| \leqslant \| \bar{b}_n - \bar{b}_{\bar{n}} \| + \| \bar{b}_{\bar{n}} - \bar{b}_{n,g(n)} \| \leqslant \| \bar{b}_n - \bar{b}_{\bar{n}} \| + \frac{1}{n}$. Hence, since $\bar{b}_n \longrightarrow \bar{b}$, $(\bar{b}_{n,g(n)})_{n \in \mathbb{N}} \longrightarrow \bar{b}$. The convergence result for the corresponding beliefs, namely $(\bar{\mu}_{n,g(n)})_{n \in \mathbb{N}}$, to $\bar{\mu}$ is analogous. Our construction ensures that, for each $n \in \mathbb{N}$ and

each $x \in A(W^b_{SE})$, $\mu_n(x) = \bar{\mu}_n(x)$ and $\mu(x) = \bar{\mu}(x)$, i.e. the beliefs match in $A(W^b_{SE})$. Hence, for each $n \in \mathbb{N}$, the beliefs associated with $b_n \otimes_{W^b_{SE}} \bar{b}_{n,g(n)} \in B^0(\Gamma)$ are $\mu_n \otimes_{W^b_{SE}} \bar{\mu}_{n,g(n)}$. Therefore, the consistency of $(b \otimes_{W^b_{SE}} \bar{b}, \mu \otimes_{W^b_{SE}} \bar{\mu})$ is obtained by considering the sequence $(b_n \otimes_{W^b_{SE}} \bar{b}_{n,g(n)})_{n\in\mathbb{N}}$.
Sequential rationality. The sequential rationality in the information sets in W^b_{SE} immediately follows from the sequential rationality of (b, μ) in G and the fact that, according to μ, no node outside W^b_{SE} can be reached with unilateral deviations from information sets W^b_{SE} and hence, the payoffs at all the terminal nodes, that can be reached by unilateral deviations from information sets in W^b_{SE}, are given by h. Similarly, only terminal nodes outside W^b_{SE} can be reached with unilateral deviations from the information sets outside W^b_{SE} and hence, the payoffs are given by $\bar{h}$. Thus, since all the $(\bar{b}_n, \bar{\mu}_n)_{n\in\mathbb{N}}$ are sequentially rational also the limit, $(\bar{b}, \bar{\mu})$, is sequentially rational.

Second, since $W^b_{SE} = W^b_{f^{SE}}$, the proof is analogous to the one for W^b_f in Theorem 19 (the regularity of f was not needed to show that W^b_f is minimally sufficient). □

It can be easily verified that

$$W_{SPE} \subset W_{SE},$$

hence, combining the result in Sections 5.6 and 5.8, we have:

$$W_{NE} \subset W_{SPE} \subset W_{SE} \subset W_{WSPE} \subset W_{SR} = W_{PE} = U.$$

5.11 Reduced Game and its Applications

In this Section we present some applications of the concepts of sufficient and essential collections. All of them are based on the reduced games defined in Section 5.9.

Proposition 15. *Let EC be an equilibrium concept, Γ a game form, and $b \in B(\Gamma)$ a strategy profile. Let $G = (\Gamma, h) \in \mathcal{G}(\Gamma)$ be such that $EC(G) \neq \emptyset$. Let W be a closed collection sufficient for EC, Γ, and b. Then,*

there is $\hat{b} \in EC(G)$ such that $\hat{b}_W = b_W$ if and only if $b_W \in EC(G_W)$.

Moreover, since $\pi(b) \subset W$,

b and $\hat{b}$ are realization equivalent.

Proof. We suppose there is $\hat{b} \in EC(G)$ such that $\hat{b}_W = b_W$ and we prove that b_W is an equilibrium concept of the reduced game G_W. We consider

$\bar{G} \in \mathcal{G}(\Gamma)$ a game with constant payoff $(M, \ldots, M)$. Since $\hat{b}_W = b_W$ and W is sufficient for EC, Γ, and b, then, for Lemma 4, W is sufficient for EC, Γ, and $\hat{b}$. Hence, there is $b^* \in EC(G \otimes_W \bar{G})$ such that $b^*_W = \hat{b}_W$. Since in game $G \otimes_W \bar{G}$, all the payoffs outside W coincide with $(M, \ldots, M)$, it is straightforward to check that $b_W \in EC(\Gamma)$. Now, we suppose that $b_W \in EC(\Gamma)$ and we prove that there is $\hat{b} \in EC(G)$ such that $\hat{b}_W = b_W$. We consider the game $G^* = (\Gamma, h^*)$, so defined: $h^*(z) \doteq h(z)$ for each $z \in Z(\Gamma) \cap W$ and $h^*_i(z) \doteq M$ for each $z \in Z(\Gamma) \backslash W$ and each $i \in N$. Since all the players are indifferent among the choices outside W and $b_W \in EC(G_W)$, we have $b \in EC(G^*)$. By definition $G^* \otimes_W G = G$. Since W is sufficient for EC, Γ, and b and $EC(G) \neq \emptyset$, there is $\hat{b} \in EC(G)$ that coincides with b in W and moreover, since $\pi(b) \subset W$, $\hat{b}$ is realization equivalent to b. □

The above result provides a first application of sufficient collections. Given a strategy profile b and a closed and sufficient collection W for b, if b_W is an equilibrium of the reduced game, then the outcome of b is an equilibrium outcome in the original game. If b_W is not an equilibrium outcome of the reduced game, then no equilibrium of the original game will coincide with b in W. In particular, the reduced game associated to the essential collection would be the simplest among the games associated with b. Quite generally, the reduced game associated with an essential collection is simpler than the original game. Recall the discussion in the motivation section and refer to the game in Figure 5.23. Beyond the immediate application described above, the reduced games can also be applied in different directions. We discuss two of them in the remainder of this section.

5.11.1 Structural robustness and partially-specified games

In the previous Section we have analyzed what happens when we reduce a game to the essential collection. In this Subsection, we change direction since we extend the reduced game. In order to measure the structural robustness of essentializing equilibrium concept process, we will compare the reduced game and the reduction of its extension.

Let $G = (\Gamma, h)$ be a game, $W \in U(\Gamma)$ a closed collection. Let $\Omega(\Gamma)$ denote the set of game forms such that if $\Lambda \in \Omega(W)$, then $W \subset U(\Lambda)$, W is closed in Λ and the nodes in W that are terminal in Γ are also terminal in Λ [10]. Now, let $\mathcal{G}(W)$ denote the set of games $\hat{G} = (\Lambda, \hat{h})$ such that $\Lambda \in \Omega(W)$ and $\hat{h}(z) = h(z)$, for each $z \in W \cap Z(\Gamma)$. We consider, for example, the

[10] In a nutshell, Λ is made attaching a game tree to the root of W.

game G_W in Figure 5.21.c. Clearly, $G_W \in \mathcal{G}(W)$ and each game in $\mathcal{G}(W)$ is defined from G_W by adding new branches to the nodes in $A(W)$. Those with payoff (M, M, M) also belong to $\mathcal{G}(W)$. The new branches can intersect each other, but cannot intersect W, otherwise W would not be a closed collection in the resulting game form. In particular, also the game G belongs to $\mathcal{G}(W)$. We refer to the elements of $\mathcal{G}(W)$ as extension of G_W.

Remark 9. Given an equilibrium concept EC, let Γ be a game form and W be a closed collection. In the previous sections we have seen how to construct the reduction Γ_w. Now we expand Γ to $\Lambda \in \Omega(W)$. From definition of $\Omega(W)$ and Γ_W it follows $\Gamma_W = \Lambda_W$, that is the corresponding reduced forms, associated with W, coincide.

In the remainder of this section EC refers only to the equilibrium concepts whose essential collections we have characterized in Sections 5.6 and 5.8, that is NE, SPE, PE, and WPBE, SR, SE.

Proposition 16. *Given an equilibrium concept EC, let Γ be a game form, $b \in B(\Gamma)$, and $\Lambda \in \Omega(W_{EC}(\Gamma, b))$. If $\bar{b} \in B(\Lambda)$ is such that b and $\bar{b}$ coincide in $W_{EC}(\Gamma, b)$, then*

$$W_{EC}(\Gamma, b) = W_{EC}(\Lambda, \bar{b}),$$

i.e., the essential collection coincide.

Proof. We can show that $W_{EC}(\Gamma, b)$ is sufficient for EC, Λ, and $\bar{b}$ using the same arguments in the sufficiency part of the proof that $W_{EC}(\Gamma, b)$ is sufficient for EC, Γ, and b, replacing clearly Γ and b with Λ and $\bar{b}$. Hence $W_{EC}(\Lambda, \bar{b}) \supset W_{EC}(\Gamma, b)$. Similarly, we can show that $W_{EC}(\Lambda, \bar{b})$ is sufficient for EC, Γ, and b. Hence $W_{EC}(\Lambda, \bar{b}) \subset W_{EC}(\Gamma, b)$. Therefore $W_{EC}(\Gamma, b) = W_{EC}(\Lambda, \bar{b})$. □

Now, let us simplify the notation, replacing W_{EC} with W. Let W be the essential collection for EC, Γ, and b. Let $\bar{G}$ be an extension of G_W and let $\bar{b}$ a strategy in $\bar{G}$ that coincides with b in W. Then, the next result shows that, an equilibrium of the reduced game is the reduction of a given game. Thus, to check whether a profile of an extension game is an equilibrium, equivalent to a fixed equilibrium profile, it is sufficient to check it in the reduced game.

Corollary 9. *Given an equilibrium concept EC, let $\mathcal{G}(W)$ be the set of the extensions of a reduced game $G_W = (\Gamma_W, h_W)$, let $b \in B(\Gamma_W)$ be such that $W_{EC}(\Gamma, b) = W$, let $\bar{G} = (\Lambda, h) \in \mathcal{G}(W)$ and $\bar{b} \in B(\Lambda)$ be such that $b = \bar{b}_W$. Then,* *i)* $W_{EC}(\Lambda, \bar{b}) = W$.

ii) If $b \in EC(G_W)$, *then there is* $\hat{b} \in EC(\bar{G})$ *such that* $\hat{b}_W = \bar{b}_W = b$.

iii) If $b \notin EC(G_W)$, *then there is no* $\hat{b} \in EC(\bar{G})$ *such that* $\hat{b}_W = \bar{b}_W = b$.

In the following Subsubsections 5.11.1 and 5.11.1 we describe two applications of Corollary 9.

Structural robustness

The first to use Structural robustness was Kalai (2005, 2006). Kalai studied the robustness of Nash equilibrium in large games when similar changes are made in the underlying games. Our approach allows us to study how robust the different equilibrium concepts are with respect to structural changes in the game. González-Díaz et al.(2009) have already provided an illustration of this fact when comparing SE and $WPBE$ in the Licensing game.

Let EC be an equilibrium concept, $G = (\Gamma, h)$ an extensive game, and $b \in EC(G)$. We assume to modify the game G by some changes in Γ or by some changes in h, without eating into the path of b. We denote $\bar{G}$ the modified game. It is natural to ask whether the outcome of b is an equilibrium outcome for EC in $\bar{G}$ or not. Essential collections are very useful here.

We suppose we have characterized the essential collections for EC. Then, if the changes in G affected neither $W_{EC}(\Gamma, b)$ nor the payoffs in its terminal nodes, b is indeed an equilibrium outcome for EC in G. In fact, in the latter case, $\bar{G} \in \mathcal{G}(W_{EC}(\Gamma, b))$, so Corollary 9 implies the desidered result.

The structural changes in a game can be of different nature, since they can: affect payoffs, change the sets of strategies, change the information available to the players, account for addiction, elimination, or merging of players, enlarge or reduce the game, and so on... If these changes do not affect the essential collection associated to a given equilibrium profile b, its outcome will be an equilibrium outcome also in the modified game. On the other hand, if the changes affected the essential collection, whether the outcome of b remains an equilibrium outcome or not will depend on the specific payoffs of the games at hand.

Therefore, if the essential collections associated with an equilibrium concept EC_1 are always smaller than the ones associated with EC_2, then EC_1 is more robust to structural changes than EC_2. The latter statement and the inclusion relations obtained for the essential collections analyzed in Sections 5.6 and 5.8, imply that SR and PE are the less robust equilibrium concepts followed, in this order by $WPBE$, SE, SPE, and SR. The Licensing Game in González-Díaz et al.(2009) exemplified this fact for SE and $WPBE$.

Remark 10. We make ourselves plain, when we say that SE is structurally more robust than $WPBE$. Let G be a game and the strategy profile b be a SE. From the above discussion it follows that, after any change in G that does not affect the essential collection for SE, Γ, and b, the outcome b remains a SE outcome in the modified game. No further calculation is needed, regardless of the actual payoffs of the modified game. On the other hand, we suppose b is just a $WPBE$, but not a SE. Then, the latter changes might have affected the corresponding essential collection for $WPBE$, which is not smaller than the one for SE, b might not be a $WPBE$ outcome anymore. Yet, our statement is mute about changes inside the essential collections. Indeed, since SE is more demanding than $WPBE$, it is natural to think that SE will be less robust to changes inside the essential collection.

Partially-specified games

As also discussed in Kalai (2005, 2006), the idea of structural robustness is very related to the possibility of dealing with partially-specified games. Let $G^p = (\Gamma, h)$ be a partially-specified game, i.e. it lacks a full description of Γ or some payoffs are unknown. Can we still say something about the equilibria of this game? Maybe. We suppose there is a possibly partially-specified strategy $b \in B(\Gamma)$ such that $W_{EC}(\Gamma, b)$ can be characterized and the corresponding reduced game is completely specified. Then, if $b_{W_{EC}(\Gamma,b)} \in EC(G^p_{W_{EC}(\Gamma,b)})$, we know that, for whatever specification of the unknown elements of G^p, there is $\hat{b} \in EC(G^p)$ that is realization equivalent to b, i.e. the outcome of b will be an equilibrium outcome of any game satisfying the partial specifications of G^p. A situation as the one described above may arise even in very simple settings. We present now a simple example.

Example 4. We consider the partially-specified game $G = (\Gamma, h)$ in Figure 5.22 below. We do not know how the game continues after node x. It may be that x is a terminal node, or we know the subgame beginning there, but that it is too complicated for its sequential equilibria to be found. And it might also be that we do not know anything at all about how the game follows once x is reached. In any case, $W = \{u, v, z_1, z_2, z_3\}$ is the essential collection for SE, any such Γ and any strategy in which players I and II play L_1 and L_2 at their initial information sets. Hence, since $b = (L_1, L_2) \in SE(G_W)$, there is a sequential equilibrium of G in which L_1 and L_2 are played, leading to the payoff vector $(1, 1)$.

5.12 Virtual Equilibrium Concepts

In the previous sections we worked on games for which the existence of analyzed equilibrium concepts was assured. In this section we widen the scope of the enquery also to games without equilibria, for example games with non-compact sets of strategies, or discontinuous payoff functions, or games in which only pure strategies are possible. In order to guarantee the existence of an equilibrium, we introduce a new concept for extensive games: the virtual equilibrium concept, shortly VEC. Once we have essentialized an equilibrium concept [11], we define *virtual equilibrium concept* the strategy profile which, restricted to essential collection, is an equilibrium concept.

Let us consider, for instance, the concept of SPE and the corresponding $VSPE$. The virtual subgame perfect equilibrium is based on the same simple idea of SPE. The subgame perfect equilibrium (Selten (1965)) discards those NE which are only possible if some players give credit to irrational plans of others. That is, a SPE is a Nash equilibrium which induces a Nash equilibrium in every subgames. In the same way, we can define the virtual subgame perfect equilibrium, as a NE which induces a NE in every subgame relevant or essential for a given strategy profile, and, more in general, the virtual equilibrium concept as a profile that is EC in the essential collection.

The $VSPE$ has the same effect as subgame perfection, but, it has the advantage, over the SPE concept, in that $VSPE$ exists in many games which do not have SPE. Hence, it is especially useful when dealing with extensive games having large trees, since there are many extensive games without SPE, but with sensible equilibria. For instance, the proper subgame of MP in Example 5 is irrelevant for the strategy profile (L_1, l_1, L_2, l_2) and it lacks Nash equilibria.

Now, we formalize our assertions. Let EC be an equilibrium concept, Γ a game form, b a strategy profile, and $W_{EC}(\Gamma, b)$ the essential collection.

Definition 83. *A strategy profile b is a virtual equilibrium concept (shortly*

[11] To *essentialize an equilibrium concept* means to assign, to each pair (Γ, b), the essential collection $W_{EC}(\Gamma, b)$.

VEC) of an extensive game $G = (\Gamma, h)$, if

$$b_{W_{EC}(\Gamma,b)} \in EC(G_{W_{EC}(\Gamma,b)}).$$

We denote $VEC(G)$ the sets of VEC of the extensive game G.

The definition implies that, for each extensive game G,

$$EC(G) \subseteq VEC(G),$$

and

$$EC(G) \neq \emptyset \Rightarrow EC(G) = VEC(G).$$

In fact, if the set of $EC(G)$ is nonempty, the Proposition 15 provides that, for each $b \in VEC(G)$, there is $\hat{b} \in EC(G)$, realization equivalent to b, which justifies the name virtual. More exactly, the following statements hold.

Proposition 17. *Let $b, \bar{b} \in B(\Gamma)$ be two strategy profile of $G = (\Gamma, h)$, and $b^{\otimes} \doteq b \otimes_{W_{EC}(\Gamma,b)} \bar{b}$. Then,*

i) the payoffs associated with b and $b^{\otimes}$ coincide, since they define the same path,

ii) if $b \in VEC(G)$, then $b^{\otimes} \in VEC(G)$, and

iii) if $b \in VEC(G)$ and $\bar{b} \in EC(G)$, then $b^{\otimes} \in EC(G)$.

The profile $b^{\otimes}$ inherits the virtual properties from b, that is to be an equilibrium only in the essential collection, and from $\bar{b}$ the property to be an equilibrium in the remaining part of game. Given a virtual equilibrium, we can always replace the non-equilibrium behavior outside the essential collection by an equilibrium behavior (if this exists) to get an equilibrium in the classic sense. Hence, if the original game has some equilibrium, the sets of equilibrium outcomes and virtual equilibrium outcomes coincide. But, that is not always true. There can be games in which the set of virtual equilibria is nonempty where there is no equilibrium, as illustrated by the following example.

Example 5. We consider the extensive game G depicted in Figure 5.23 and we restrict the attention to pure strategies. Let $b = ((L_1, l_1), (L_2, l_2))$ be the strategy profile, but we might apply the same argument to $b = ((L_1, r_1), (L_2, l_2))$, or $b = ((L_1, r_1), (L_2, r_2))$. The subgame that begins in x, after playing (R_1, R_2), is not essential for (Γ, b), since two deviations are needed to reach it. We consider the reduced game $G_{W_{SPE}(\Gamma,b)}$, with $M = M_G$, depicted in Figure 5.24. Clearly, b restricted to $G_{W_{SPE}(\Gamma,b)}$ is a SPE of $G_{W_{SPE}(\Gamma,b)}$ and, hence, b is a $VSPE(G)$. However, the game G does not have any SPE in pure strategies, but the equilibrium b is "sensible". A sensible equilibrium, in the spirit of SPE, has the following meaning. The players cannot use backwards induction to solve game G, because the proper subgame does not have any NE. Still, we suppose the players keep on with backwards induction and insist on assigning payoffs at that subgame and then go backwards in the tree. Then it does not matter what payoffs they assign to that subgame, they would find that b is a solution of the game.

The following result is an immediate consequence of the corresponding characterizations of the essential collections for NE, SR, and PE. The virtual versions of these equilibrium concepts coincide with the non-virtual versions.

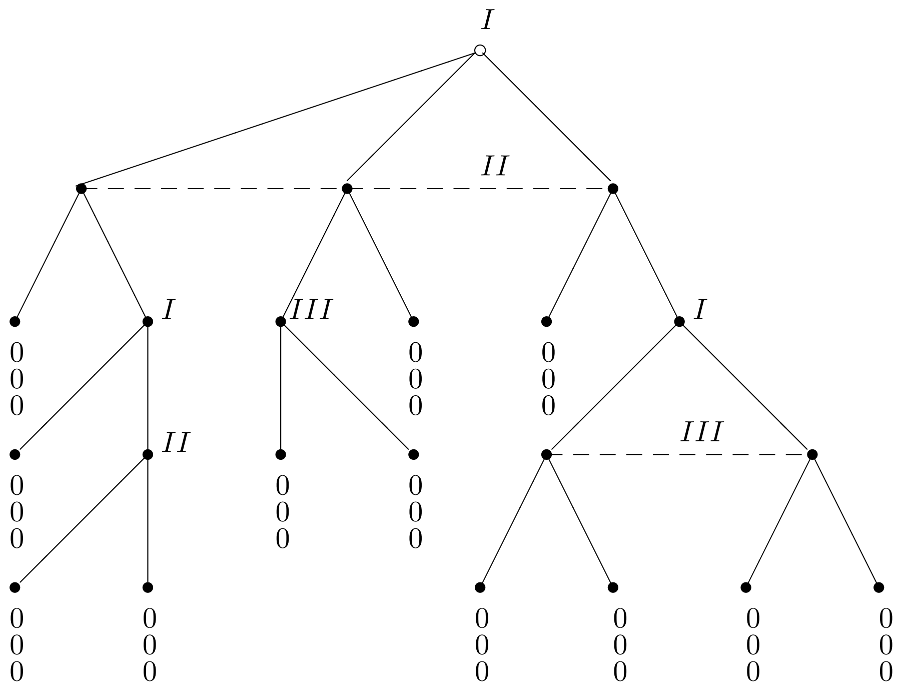
Corollary 10. *For each game form* Γ *and each game* $G \in \mathcal{G}(\Gamma)$*, we have*

$$NE(G) = VNE(G), \quad SR(G) = VSR(G), \quad \textit{and} \quad PE(G) = VPE(G).$$

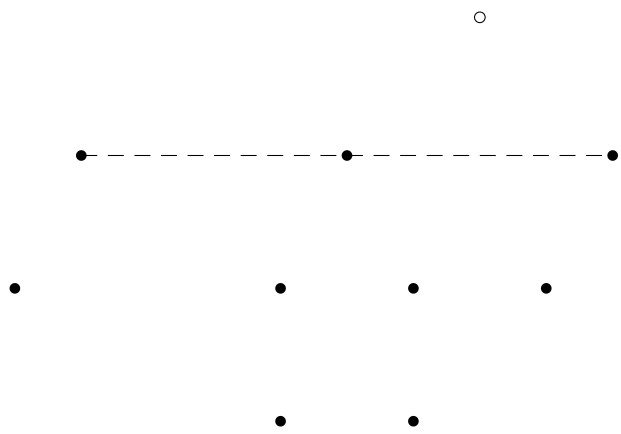
Nonetheless, for other equilibrium concepts, the virtual version can lead to reasonable equilibrium behavior in settings where the classic equilibrium concepts do not exist. See, for example, 5.23. Sometimes, the set of equilibria is empty. For example, since the payoff functions are discontinuous, as it happens in the Licensing game in González-Díaz *et al.* (2009), the payoffs are unbounded. The second version of the Licensing game, LG^m has some VSE, but SE is not even defined.

Finally, the virtual subgame perfect equilibrium is applied to derive a folk theorem in a repeated games setting in which the set of subgame perfect equilibrium may be empty. We refer the reader to García-Jurado I. and J.González-Díaz (2006) for the proof of the results.

Remark 11. Virtual equilibria remind the *trimmed equilibria* introduced in Groenert (2007). Although born from the same ideas, there are some differences between the two equilibrium concepts, since the approaches were different. First of all, the analysis in Groenert (2007) concerns the trimmed versions of subgame perfect equilibrium and weak perfect equilibrium. Further, for each assessment (b, μ) with $\mu \in M^{wc(b)}$, the author identifies those information sets that are irrelevant to check whether $(b, \mu) \in WPBE$, i.e. those outside $W^{b,\mu}$. Instead, our process of essentializing identifies, for each strategy profile b, those information sets irrelevant to check if (for whatever beliefs) $b \in WPBE$, i.e. those information sets outside $\bigcup_{\mu \in M^{wc(b)}} W^{b,\mu}$. Also, the analysis in Groenert (2007) focuses on the definition of trimmed equilibria and there is no closedness requirement involved. Hence, the differences between the final equilibrium concepts. In fact, a virtual $WPBE$ is always a trimmed $WPBE$, but there can be trimmed $WPBE$ that are not virtual $WPBE$.

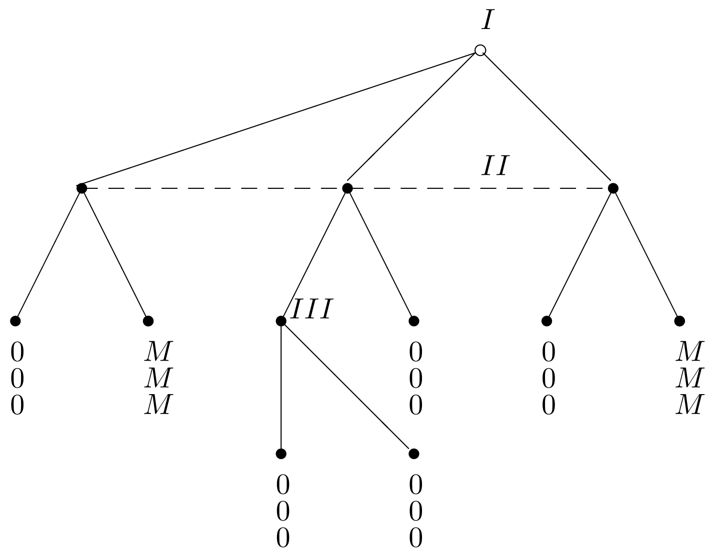


(a) *The game G.*

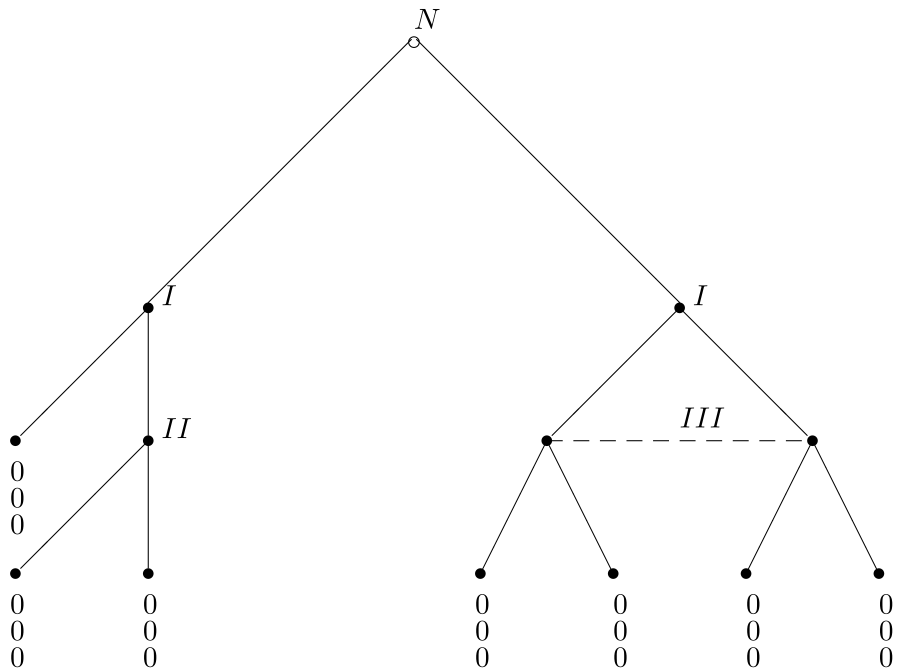


(b) *A closed collection* $W \subset U(\Gamma)$.

Figure 5.20: Decomposition of a game with respect to a closed collection. Game G and collection W.



(*c*) *The reduced game* G_W.



(*d*) *The complementary game* $G(-W)$.

Figure 5.21: Decomposition of a game with respect to a closed collection. Reduced game and Complementary game.

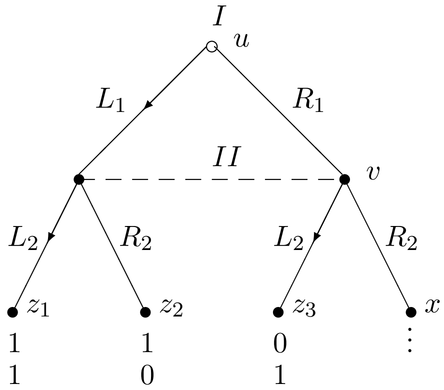


Figure 5.22: A partially-specified game.

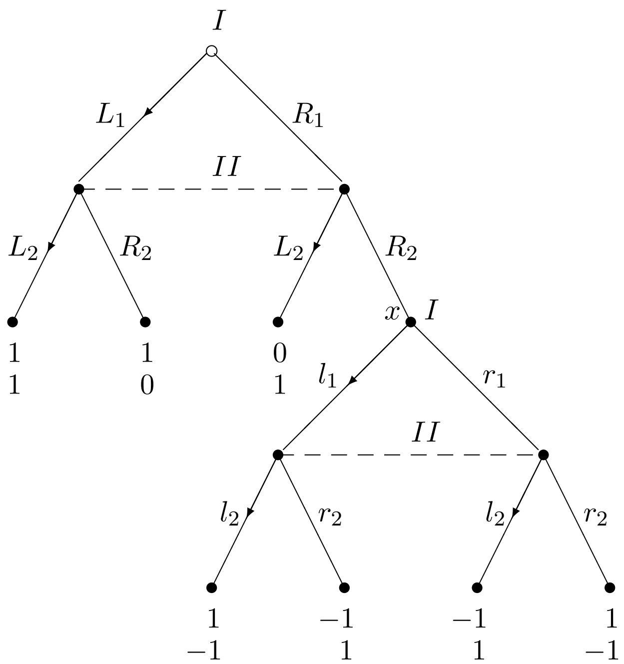


Figure 5.23: A game without SPE, with VSPE.

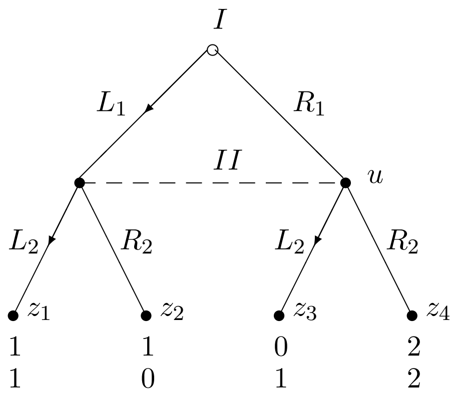


Figure 5.24: $G_{W_{SPE}(\Gamma,b)}$.

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