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A digital simulation model of out-of-equilibrium market behaviour:

a Keynesian-Sraffian approach

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Abstract

Keynesian economics has devoted particular attention to out of full employment equilibrium phenomena, but the lack of an analytical framework for describing how economic agents interact and organize their production and consumption decisions has made it difficult to establish whether the economic system is self-adjusting at an out-of-full employment equilibrium.

One of the purposes of Sraffa's book *Production of Commodities by Means of Commodities* was to understand the conditions that allow the system to reproduce itself. But how trade takes place remains an open question. For Sraffian economics, the lack of an analytical framework for describing how economic agents interact and organize their production and consumption decisions, has made it difficult to consider out-of-equilibrium behaviour.

The present thesis is a first attempt to model Keynes's principle of effective demand with the aid of Sraffian schemes. The notion of effective demand requires the possibility for buyers and sellers to buy and sell thanks to the existence of newly generated or previously accumulated financial purchasing power. Stated differently, the very notion of out-of-equilibrium exchanges may require the emergence of new credit and debt relations.

Sraffa's original production schemes have been enriched with algorithmic behavioural functions that describe agent decisions and exchanges of property rights.

In the resulting algorithmic model of an economic system (with labour and goods market), decisions of production and consumption are made by a population of algorithmic rational agents (ARAs), divided between producers and workers. The ARAs are characterized by behavioural functions, specific trading rules, and are connected in a network. Exchanges are made by signing virtual contracts that involve the use of financial means. The creation of new financial means of exchange, credit and debt, is endogenous. Production is heterogeneous and conceived as a circular process.

The fundamental conclusion is that this *digital economic laboratory* has generated virtual economies able to converge towards production prices, uniform wage rates but non-uniform profit rates with an unequal distribution. The virtual economies result to be stable and efficient from a technological point of view despite economic policy can improve the performance of the whole economy.

The *digital economic laboratory* is a powerful instrument able to answer different research questions. It represents an answer to the need to develop models able to explore the complexity of out-of-equilibrium behaviour, grounded on bookkeeping principles and computable methods.

keywords: Keynesian economics, Sraffian economics, computability, algorithm, simulation approach.

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1 Introduction

1.1 The context

According to Walrasian general equilibrium theory, under perfect competition, marginal productivities are at equilibrium equal to the prices of the factors of production. This theory implies that producers decide the production level that would maximize profits under given budget constraints and consumers would maximize utility subjected to their budget constraints and they all are, producers and consumers, price takers. Given the standard assumptions related to free market conditions and well behaved production and utility functions (see Debreu, 1959; Arrow and Hahn, 1971) it is claimed that equilibrium prices where market clearing is realized exist and are also implicitly assumed to be actual market prices. From such a setting it could be possible to claim that the distribution of the produced surplus is fair and equal because it is determined by the "laws of nature" embedded in the so-called "well-behaved" production functions, utility functions and free market conditions. In modern dynamic (equilibrium) macroeconomics the Walrasian general equilibrium is assumed to hold at each point in time (or interval). The sources of oscillations are due only to exogenous shocks that lead agents to modify their decisions and with markets "adjusting" instantaneously so that there is always instantaneous market clearing (e.g., the Real Business Cycle (RBC) model of Kydland and Prescott, 1982), but are not to be explained as attempts by the system to adjust with respect to out-of-equilibrium not market clearing prices.

The Keynesian Cambridge school from the Keynes's *General Theory of Employment, Interest and Money* (Keynes, 1936) to the Sraffa's *Production of Commodities by Means of Commodities* (Sraffa, 1960)[1], has challenged mainstream neoclassical theory[2] and questioned the validity of most of its assumptions. Keynesian economics dominated economic theory for decades after the publication of Keynes's *General Theory*. Nevertheless, it fell out of favour with the emergence of the Great Stagflation of the 1970s, a phenomenon that Keynesian economics was not able to explain because the Phillips curve excluded the simultaneous presence of high inflation and high unemployment. Milton Friedman's alternative explanation of stagflation (based on the natural rate of unemployment developed also by Edmund Phelps) and the introduction of the rational expectation hypothesis (Lucas,

[1] Hereafter *PCMC*.

[2] It is worth remembering that Keynes's aim, inside the *General Theory*, was to criticize *Classical economists*. With this term Keynes was referring to Ricardo's *followers* such as Mill, Marshall, Edgeworth and Pigou (Keynes, 1936, p.3). Nevertheless, Keynes's critique to *the postulates of the classical economics* (Keynes, 1936, ch.2) holds perfectly also to the subsequent neoclassical interpretation of unemployment, which is based on the same postulates.

1976) induced a rethinking of the whole Keynesian economics. As a consequence, the 1980s witnessed the dominance of monetarism and new classical macroeconomics and the emergence of new Keynesian economics[3]. As a result, much of current research has followed the main guiding principles of the new neoclassical synthesis, and currently focuses mostly on Dynamic Stochastic General Equilibrium (DSGE) models in which new Keynesian features may be introduced (e.g., Woodford, 2003). Also the new research field of the Computable General Equilibrium (CGE) theory (i.e., models on the Herbert Scarf tradition) or the models based on *agent based computational methods* (see for a summary Tesfatsion and Judd, 2006) do not represent attempts to displace the pre-eminence of neoclassical theory but a questionable attempt to put realism in the model, a "complement not a substitute" of modelling approaches (Tesfatsion and Judd, 2006, p.864).

The emergence of the financial crisis in 2007, however, has dramatically underlined the fragility of this framework. Even the Nobel laureate economist Robert Lucas had to admit that the crisis was unpredictable because, according to the modern economic theories (such as the efficient-market hypothesis), such events are not considered (see Lucas, 2009). In addition, one of the principal colleagues of Lucas, Thomas Sargent, confirmed the importance of the "rational expectation revolution" by claiming not only that remarks about mathematics are "foolish and intellectually lazy" but also that most criticisms after the crisis "reflect[ed] either woeful ignorance or intentional disregard for what much of modern macroeconomics is about and what it has accomplished" (Rolnick, 2010, p.28). With reference to the macro Seminars at Princeton, in 2009, Thomas Sargent argued that:

> There were interesting discussions of many aspects of the financial crisis. But the sense was surely not that modern macro needed to be reconstructed. On the contrary, seminar participants were in the business of using the tools of modern macro, especially rational expectations theorizing, to shed light on the financial crisis (Rolnick, 2010, p.28).

But for workers, businesspeople, and policymakers, things are not so easy. The pointlessness of these mathematical models has been underlined by the former President of the European Central Bank, Jean-Claude Trichet, who found "the available models of limited help" and "felt abandoned by conventional tools" (Trichet, 2010, p.18). The financial crisis of 2007 has rapidly become the *Great Recession* through the sovereign debt crisis. This

[3]New Keynesian economics represents a response to the critique of new classical macroeconomics. Its objective is to provide a Walrasian microfoundation to Keynesian economics. It accepts rational expectations but consider the possibility that market failures (wage and price stickiness) and imperfect competition could bring to under-full employment equilibria.

development has implied unpredictable social and political costs that seem to confirm how inappropriate the unconditional trust in *modern macro* has been. Paul Krugman observes that:

> What is almost certain is that economists will have to learn to live with messiness. That is, they will have to acknowledge the importance of irrational and often unpredictable behaviour, face up to the often idiosyncratic imperfections of markets and accept that an elegant economic "theory of everything" is a long way off (Krugman, 2009).

As a consequence, it seems a matter of common sense to reconsider seriously the Cambridge school's criticisms in order to comprehend current difficulties, such as distributional inequality and unemployment, and to develop more responsible economic policies (i.e., alternatives to the debatable austerity measures supported by mainstream economics).

1.2 The problem

Despite the wide recognition of Keynes's relevance, during the current crisis, the revival of Keynes has been superficial, restricted only to the so-called *Keynesian economic policies* (Pasinetti, 2012a, p.1435). Nevertheless "those who seek to develop an alternative to the conventional neoclassical theory are well advised to take into account both the findings of Keynes and those of Sraffa" (Kurz, 1995, p.84). As claimed by Roncaglia, among others:

> The integration of Sraffa's and Keynes's analyses could constitute the core of *non-neoclassical economics*. However, this integration requires that Sraffa's analysis of relative prices and their relationship with income distribution should not contradict basic elements of the Keynesian paradigm [...] Proceeding along this path we will discover, I believe, that a solid stream of non-neoclassical economics is *already available*, integrating not only Keynes's and Sraffa's analyses, but also the contributions of a wide group of economists (Roncaglia, 1995, p.120, emphasis added).

Despite this, there is widespread agreement that Keynesian Revolution remains incomplete (Pasinetti, 2007, p.49). The reasons behind this *state of the art* are twofold. Indeed, Keynes's ambiguity on the real consequences of his *General Theory* for the orthodox scheme (Robinson, 1963, p.78) created a sort of *doctrinal fog* (Blaug, 1980, p.221) that promoted the controversial neoclassical synthesis. This ambiguity is determined by the impossibility for Keynes to escape from the marginalist theory of value and the Marshallian perspectives at the base of his analysis. Moreover, the alleged lack of microfoundations in Keynes's theoretical construction has reinforced mainstream economists's arguments (Lucas and Sargent, 1979).

Sraffa, who instead was coping for a "foundation" of economic theory starting from the classical theory, was surely a great candidate for fill in the blank inside Keynesian theory. Unfortunately, Sraffa's *objectivism* (probably) prevented his analysis from considering money, contracts, debts, dynamics, and behavioural or psychological features. This because Sraffa's purpose was primarily to derive the "production prices" (i.e., the equilibrium prices) by solving a simultaneous equation system in order to explain the concept of value. As a consequence, he did not consider, at that stage, the possibility of a lack of coordination as a result of an interacting process between producers and workers. Moreover, a widespread interpretation of Sraffa, that interprets his production prices as long-run equilibrium positions towards which the economy should gravitate, puts doubts on the possibility to merge the Keynesian short run approach characterized by instability with the supposed long-run equilibrium approach of Sraffa.

1.3 The Solution

Even though it is true that Keynes and Sraffa gave rise to parallel Revolutions and that they worked on different topics with different methodologies, it should be underlined that some observations about their apparent lack of compatibility are not exact. One fundamental point of connection between Keynes and Sraffa that should be emphasised here is their reliance on bookkeeping principles and constructive mathematics. This topic will be developed in the next chapter. **The purpose of the present thesis is to contribute to fill this gap by merging Keynes's *effective demand* with Sraffa's *production schemes* through the construction of an algorithmic model based on bookkeeping principles and able to consider market out-of-equilibrium behaviour.** The final result is the construction of a coherent set of computer code so as to allow for the construction of a virtual market, in which decisions of production and consumption are made by a population of algorithmic rational agents (ARA producers and workers) which interact exchanging quantities and property rights on the base of a double-entry bookkeeping coherency. The importance of the construction of the algorithmic model is due to the fact that the combination of some important elements of Keynesian theory with Sraffian schemes could be attempted only if the "constructive mathematics" at the base of the Sraffian schemes is respected (Velupillai, 2008b). As a consequence, Sraffian schemes should be considered as algorithms that can be improved on the basis of the computable principles, in order to sustain only logical propositions and conduct simulations. In details, the algorithms have been developed to include micro units (workers and producers) which have been programmed as algorithmic rational agents interacting, exchanging, and producing goods. This implies that *market prices*, instead of *produc-*

tion prices, have been derived and the market out-of-equilibrium behaviour has been considered. This novelty has the power to free Sraffian Schemes from their role of pure (supposed) long-run equilibrium positions introducing the key role of debt and credit relations fundamental conjunction ring with the Keynesian conception of the economy as a *real-world entrepreneurial economy*. Indeed, this simulation approach has generated an accounting out-of-equilibrium dynamics in which monetary credit and debt relations have emerged. This model represents a *digital economic laboratory*.

1.4 Innovative Aspects

The idea to construct *thought experiments* able to capture the mechanism of exchange is not a complete novelty. Among many, at the beginning of last century, Irving Fisher has struggled with the thought experiment of thinking how to capture the elements describing individual exchanges, occurring at different prices, and made by a huge number of agents into an aggregate equation: the quantity theory of money (see Irving Fisher's *Purchasing Power of Money* (1911)). Irving Fisher did also construct an analogue machine with the aim of "computing" aggregate values. In fact, the very scope of Walras's work was that of modelling a system where there is an immense amount of exchanges. Despite this, the difference between previous thought experiments of this type and those of the present thesis is the care put on the accounting coherence, the presence of heterogeneous and multiple production (coherently with Sraffian schemes) and the consistency with the mathematics of digital computers.

In Physics experiments, after the famous Fermi-Pasta-Ulam (FPU) 1955 digital experiments, are not any longer conducted exclusively on real laboratories, but also on digital or analogical ones (Weissert, 1997; Galavotti, 2008; Zambelli, 2015). The FPU problem and the way computer simulations were effectively utilized to understand a theoretical problem is instructive for economists, in particular, for the tailoring *laboratory experiments*. It is here instructive to quote Ulam:

> Fermi became interested in the development and the potentialities of electronic computing machines [...] We decided to try a selection of problems for heuristic work where in absence of closed analytical solutions experimental work on computing machines would perhaps contribute to the understanding of properties of solutions. This could be particularly fruitful for problems involving the asymptotic-long time or "in the large" behaviour of non-linear physical system. *In addition, such experiments on computing machines would have at least the virtue of having the postulates clearly stated. This is not always the case in an actual physical object or model where the assumptions are not perhaps explicitly recognized* (Fermi et al., 1955, p.977, emphasis added).

The relevance of the digital computer for the economic theory is evident. Indeed, it is undoubtful that the "enormous developments in the theoretical and practical technology of the computer have made a tremendous impact on *economic methodology* in general, but also in economic theory in particular [and] it must be emphasised that these references are to the *digital computer*." (Velupillai and Zambelli, 2011, p.260, emphasis added).

Surely one effect of this impact is the even stronger necessity to be aware of the difference between the mathematics of the digital computer (i.e., recursion theory and constructive mathematics) with respect to the real analysis at the core of the mathematics of the general equilibrium theory (i.e., set theory plus axiom of choice) (Velupillai and Zambelli, 2011, p.260). This difference is important. Indeed, even though theoretical abstraction cannot match reality, they have to be meaningful, which implies that they have to "produce an effective algorithm for the solution of a problem" (Zambelli, 2010, p.34). According to Velupillai (2012), most neoclassical economic results (such as the standard choice theory, the Walrasian equilibrium, and Debreu's theory of value) are meaningless from a constructive prospective (i.e., they are uncomputable, unsolvable or undecidable) because of the "non-algorithmic content of mathematical economics" (Velupillai, 2012, p.23)[4].

[4] This distinction could help not to confuse the *digital economic laboratory* of the present thesis with models inspired by the Computable General Equilibrium (CGE) theory (i.e., models on the Herbert Scarf tradition) or by the modern *agent based computational methods*. Indeed, the models inspired by this filed of research have not as purpose a critical investigation of the validity of the mainstream's postulates. Tesfatsion explains clearly this point with reference in particular to Agent-based Computational Economics (ACE): "ACE modeling is surely a complement, not a substitute, for analytical and statistical modeling approaches. As seen in the work by Sargent (1993), ACE models can be used to evaluate economic theories developed using these more standard tools. Can agents indeed learn to coordinate on the types of equilibria identified in these theories and, if so, how? If there are multiple possible equilibria, which equilibrium (if any) will turn out to be the dominant attractor, and why? ACE models can also be used to evaluate the robustness of these theories to relaxations of their assumptions, such as common knowledge, rational expectations, and perfect capital markets" (Tesfatsion and Judd, 2006, p.864). The fundamental problem is that CGE theory has not developed a clear definition of computability and constructiveness. Indeed, despite the importance of constructiveness is sometimes claimed (see for example Tesfatsion and Judd, 2006, p.865), there is not a clear definition of constructiveness, computability, neither a serious study of the pioneering works of Turing, von Neumann or Ulam fundamental in order to develop agent based computational methods (Velupillai, 2011, p.16). Indeed, starting from these pioneers, the consequence would have been the recognition that Computable General Equilibria are neither computable nor constructive so that, as a consequence, one should have to admit that "Agent-Based Economic Modelling have no foundations in any kind of rigorous algorithmic formalism and, hence, epistemologically vacuous" (Velupillai, 2011, p.4). Briefly, the reason is that the foundations of CGE theory lie in Uzawa's Equivalence Theorem. This theorem demonstrates the equivalence of two existence theorems: the existence of a Walrasian general equilibrium in an economy with a continuous excess demand function and the Brouwer's fixed-point theorem. But the problem is that Brouwer's fixed-point theorem is not valid (it is non-constructive and uncomputable) because it invokes in its

The *digital economic laboratory* of the present thesis is coherent with the mathematics of the digital computer. All the solutions are algorithmically grounded and based on effectively computable methods (Velupillai and Zambelli, 2011, p.279). The *digital economic laboratory* is coherent with the principle of Computable Economics, - i.e., "every concept must have an operationally relevant content so as to be understood and implemented by the virtual economic agents" (Zambelli, 2010, p.38). In other words, all the model is represented by mathematical equations that can be computed also *by hand* without using stochastic elements or theorems. Only on this computable basis the *digital economic laboratory* will be able to represent an environment to be used to run *thought experiments*; where by *thought experiments* we mean computations where theoretical economic questions are posed to the computer and the computer provides answers to these questions.

The construction of a *digital economic laboratory* can represent a concrete answer to the aims of Herbert Simon's *empirical microeconomics*. Indeed, Herbert Simon stated clearly that empirical research is fundamental for a real progress of microeconomics, and that simulation models can play a crucial role for a deep understanding of decision processes and economic behaviour (see Simon, 1997, pp.79-81). It is interesting to report what Herbert Simon said:

> Alternatives to neoclassical economics, including the theory of bounded rationality, have often been accused of attacking the existing theory without making concrete proposals for an alternative. In these essays I have tried to show that such a criticism is not warranted. First, the difficulties of the neoclassical theory are real, and ignoring them will not remove them. The theory is grossly inadequate for describing what goes on inside the business firm [...] and for either understanding the events that occur in the whole economy or providing a basis for macroeconomic policy. [...] The alternatives to neoclassical theory do not depend any less on empirical facts. Their advantages are twofold: first, they embody much weaker assumptions about human rationality; assumptions whose validity has considerable empirical support. Yet they are able to reach most of the empirically validated conclusions reached by neoclassical theory on the basis of these weaker assumptions: dispensing with the need for a "principle of unreality", and meeting the test of Occam's Razor. (Simon, 1997, pp.89-90).

In particular, Herbert Simon has been a pioneer in the development of a constructive theoretical alternative to the *homo oeconomicus* based on bounded and procedural rationality (i.e., Classical Behavioural Economics, as explained by Kao and Velupillai, 2012), for the construction of models in

proof the Bolzano-Weierstrass theorem that can be proven to be undecidable (see for more details on this point Velupillai, 2011, pp.4-7).

which the behaviour of agents is constructive and comparable with empirical evidence. Another content that can be developed thanks to the *digital economic laboratory* is the concept of *near-decomposability* as described by Simon (1962). *Near-decomposability* is a property of *hierarchic system* i.e., systems composed by interrelated subsystems (Simon, 1962, p.468). In a nearly decomposable system "the short-run behaviour of each of the component subsystems is approximately independent of the short-run behaviour of the other components [while] in the long run, the behaviour of any one of the components depends in only an aggregate way on the behaviour of the other components" (Simon, 1962, p.474). The *digital economic laboratory* is able to map interrelation starting from elementary parts, and to construct hierarchical systems in which the property of *near-decomposability* can be concretely experimented. The fact that "there is widespread agreement that it is necessary to introduce into economics both dynamical relations and general interdependence" (Goodwin, 1947, p.181) is another way to sustain that economic system should be recognized as *complex systems*. A definition of *complex system* can be found in the words of Herbert Simon:

> Roughly, by a complex system I mean one made up of a large number of parts that interact in a not simple way. In such systems, the whole is more than the sum of the parts, not in an ultimate, metaphysical sense, but in the important pragmatic sense that, given the properties of the parts and the laws of their interaction, it is not a trivial matter to infer the properties of the whole (Simon, 1962, p.468).

The complexity of the economic system makes of the *digital economic laboratory* a precious tool for economic theory. The fundamental units object of this interaction able to generate complexity (the *parts* that make up the *whole*) are the algorithmic rational agents (ARAs). The notion of Algorithmic Rational Agent is inspired by the work of Kumaraswamy Velupillai. The exact phrase *algorithmically rational agent* is to be found in Velupillai (1991, p.32) and represents a conception of agent distinct with respect to the perfectly rational optimizer, the *homo economicus*. Indeed, the concept of ARA is coherent with Herbert Simon's economic actor characterized by bounded and procedural rationality:

> A theory of rational behavior must be quite as much concerned with the characteristics of the rational actors - the means they use to cope with uncertainty and cognitive complexity - as with the characteristics of the objective environment in which they make their decisions. In such a world, we must give an account not only of *substantive rationality* - the extent to which appropriate courses of action are chosen - but also *procedural rationality* - the effectiveness, in light of human cognitive powers and limitations, of the *procedures* used to choose actions. As economics moves out toward situations of increasing cognitive complexity, it becomes increasingly concerned with the ability of actors to

> cope with the complexity, and hence with the procedural aspects of rationality (Simon, 1978, pp.8-9).

The transition from the theories of *substantive rationality* to the theories of *procedural rationality* requires a methodological jump:

> The shift from theories of substantive rationality to theories of procedural rationality requires a basic shift in style, from an emphasis on deductive reasoning from a tight system of axioms to an emphasis on detailed *empirical exploration of complex* ***algorithms*** *of thought* (Simon, 1976, p.85, emphasis added).

One important instrument for the "exploration of complex *algorithms* of thought" is the digital computer. Simon expresses explicitly this concept:

> Like a modern digital computer's, Man's equipment for thinking is basically serial in organization. That is to say, one step in thought follows another, and solving a problem requires the execution of a large number of steps in sequence. The speed of his elementary processes, especially arithmetic processes, is much slower, of course, than those of a computer, but there is much reason to think that the basic repertoire of processes in the two systems is quite similar. Man and computer can both recognize symbols (patterns), store symbols, copy symbols, compare symbols for identity, and output symbols. These processes seem to be the fundamental components of thinking as they are of computation. (Simon, 1976, p.71, emphasis added).

These words of Simon demonstrate the importance of the digital computer for the development of algorithms able to investigate the complexity of the human mind and its decision-making processes. The lesson that economic theory should learn from these words of Simon is, above all, the recognition of the importance of the algorithm with respect to the axiom, the empirical exploration with respect to the deductive reasoning for the construction of a coherent macroeconomic and microeconomic theory:

> Economists have been relatively uninterested in descriptive microeconomics - understanding the behavior of individual economic agents - except as this is necessary to provide a foundation for macroeconomics. The normative microeconomist "obviously" doesn't need a theory of human behavior: he wants to know how people *ought* to behave, not how they do behave. On the other hand, the macroeconomist's lack of concern with individual behavior stems from different considerations. First, he assumes that the economic actor is rational, and hence he makes strong predictions about human behavior without performing the hard work of observing people. Second, he often assumes competition, which carries with it the implication that only the rational survive. Thus, the classical economic theory of markets with perfect

> competition and rational agents is deductive theory that requires almost no contact with empirical data once its assumptions are accepted. Undoubtedly there is an area of human behavior that fits these assumptions to a reasonable approximation, where the classical theory with its assumptions of rationality is a powerful and useful tool. Without denying the existence of this area, or its importance, *I may observe that it fails to include some of the* ***central problems*** *of conflict and dynamics with which economics has become more and more concerned.* (Simon, 1959, p.254, emphasis added).

We interpret these words of Simon as an encouragement to cope to solve these *central problems* in economic theory through the development of new instruments, able to recognise the importance of the algorithm and of the digital computer. The *digital economic laboratory* developed in this thesis represents an attempt of development of one of these “new instruments” for a coherent investigation of the complex microeconomic and macroeconomic problems.

1.5 Structure of the Thesis

The structure of the thesis is divided into the following chapters:

2 **Literature review**: in this chapter the theoretical background that supports the relevance of the thesis is analysed: the connection between Sraffa and Keynes and the importance of their integration.

3 **The digital economic laboratory**: here a summary of the research questions inside the *laboratory*[5] is presented. Then, the building blocks of the laboratory and the fundamental equations are presented. Finally, a detailed description of the mathematical structure of the *laboratory* is reported.

4 **The setting of the thought experiments**: here a detailed description of the experimental setting and behavioural functions that have been developed in order to run the thought experiments is presented.

5 **Experiment 1**: simulation of a virtual market economy with the set of methods used by Sraffa in his example given at page 19 of *PCMC*. A study of the convergence properties and the relationships between preferences, surplus and prices follows.

6 **Experiment 2**: simulation of different virtual economies with different sets of methods. A study of the convergence properties follows.

7 **Experiment 3**: with respect to experiment 2, this chapter considers the possibility for producers to choose the best method among a set of different methods. The convergence towards the *wage-profit frontier* has been examined. Moreover, the role of innovation has been investigated.

8 **Experiment 4**: with respect to experiment 3, in this chapter the effect of the introduction of an economic policy (exogenous and endogenous interest rate) has been considered.

9 **Conclusions**: analysis of the results, considerations about the potentialities of the *digital economic laboratory* and further investigations.

10 **Appendix**: a numerical example, details about technology and methods, notes about *machine learning* and *artificial neural network* in the *digital economic laboratory* and, finally, the table of contest have been added in the Appendix.

[5]We will use the terms *laboratory* and *digital economic laboratory* as synonyms.

2 Literature Review

2.1 Keynes and Sraffa

Even though it is still controversial whether the passage from classical to neoclassical theory represented a natural evolution or a strong discontinuity, it is evident that the marginal revolution has entailed deep methodological and theoretical changes. For classical economists, distribution "is the principal problem in Political Economy" (Ricardo, 1817, p.1). They interpreted economics as a sociological science and focused on the elaboration of a labour theory of value.

In the 1870s, the Marginal Revolution transformed economics into a pure mathematical science that can be hardly recognized as political economy (Lunghini, 2003, p.71) and changed also the object and the foundation of the economic investigations. Indeed the central concepts were no more producibility of commodities and surplus but scarcity and marginal utility (Lunghini, 1971, p.11). The final destination of this evolution has been, not without jumps, the modern neoclassical theory that can be defined as "the science which studies human behaviour as a relationship between ends and scarce means which have alternative uses" (Robbins, 1935, p.16).

The Cambridge school, which includes the group of economists, pupils, and scholars that surrounded Keynes, such as Joan Robinson, Piero Sraffa, Nicholas Kaldor, and Richard Goodwin (Pasinetti, 2007), could be considered the principal adversary of marginalism. Neoclassical theory has been questioned by them in different ways, but surely a key role has been played by Keynes and Sraffa. Even though they were at Cambridge, they worked apparently on complete different topics, giving rise to different revolutions (Skidelsky, 1986, p.73). In 1936, after the Great Depression, Keynes's *General Theory* claimed the centrality of underemployment and disequilibrium revolutionizing macroeconomics and political economy. In 1960, Sraffa wrote a short and cryptic book, *Production of Commodities by Means of Commodities* (*PCMC*), which became the milestone of the *Cambridge capital controversy.* The debate challenged the neoclassical interpretation of production and distribution at a strictly logical and mathematical level[6]. One of the most important results of this debate has been the proof of the fallacy of the aggregation procedure of capital goods that is the basis of the Cobb-Douglas production function. As a consequence, neither distributive justice nor equilibrium can be ensured. Even though the value of this criticism has been confirmed by Samuelson (1966), nowadays the controversy is simply considered as an episode, or a *curiosum* (Lunghini, 1975, p.xiii). Cambridge school scholars have tried to question marginalism on other occasions (Pasinetti,

[6]The debate was between some economists at the University of Cambridge (such as Joan Robinson and Piero Sraffa) and other economists at MIT (such as Paul Samuelson and Robert Solow).

2012a, p.1433) without significant influence on the mainstream.

This also because despite *Cambridge capital controversy* has been recognized to be destructive for Wicksellian capital theory, the original capital theory of Walras (in particular the Arrow-Debreu version of the general equilibrium theory that is considered the development of the general equilibrium approach in the intertemporal dimension by neoclassical economists) has been claimed not to be touched by the controversy (see Bliss, 1975; Malinvaud, 1953; Mandler, 2002; Bidard, 1990)[7]. Indeed, while Walras developed a model of general economic equilibrium based on the assumption of heterogeneous capital goods available in arbitrary given initial quantities; Wicksell built on the notion of aggregate capital measured in value terms in order to overcome the logical contradictions in the Walrasian construction once capital is introduced (Tosato, 2013, p.106). As said, Sraffa's analysis has been recognized by the neoclassical side to be destructive for Wicksellian capital theory but not for the Arrow-Debreu version of the general equilibrium theory. One criticism to this line of defence has been that in the Arrow-Debreu version "the issues concerning the production of new capital goods are, to some extent, 'concealed' inside an extremely general formulation of the production sets which sidesteps the distinction between fixed capital goods and other factors of production and of consumers' choices, which obscures the aspects concerning the saving decision" (Tosato, 2013, p.106). Garegnani tried to demonstrate that Sraffian critique of neoclassical capital theory applies as much to general equilibrium as to aggregative models. First of all, Garegnani accused the modern neoclassical models to solve Walras's contradictions simply by abandoning Walras method, in other words, by abandoning the traditional concept of equilibrium in favour of short-run equilibrium models that allow an arbitrary initial physical condition for the capital stock (see Garegnani, 1976). But aside this, Garegnani claims that anyway the concept of capital as a single quantity reappears necessarily in the market of savings and investments inside intertemporal equilibrium:

> Intertemporal equilibrium *does not* avoid the dependence on the notion of the capital as a single magnitude. Though it no longer occupies its highly visible position as a *fund* among the factor endowments, *the homogeneous commodity 'future income'* demanded by savers, can be shown to emerge as a flow, with the respective demand and supply functions and the corresponding market. They are respectively what, after Keynes, we are used to call (gross) savings *supply*, (gross) investment demand, and saving-investment market. The implications of the inconsistency of that notion of capital - the same implications which

[7]For example Bidard claims that "the aggregate version à la Clark or the Austrian version à la Böhm-Bawerk of the marginalist theory are faulty from a logical standpoint [...]. The main stream of modern economic thought has basically ignored this discussion because it does not affect its core, the Arrow-Debreu version of the general equilibrium theory inherited from Walras" (Bidard, 1990, p.129).

enforced the abandonment of the traditional analysis in pure theory - are still there to be faced (Garegnani, 2012, p.1431).

This debate remains open but without significant effects on the mainstream.

2.2 The school of thought after Keynes and Sraffa

While Keynesian economics has been developed mainly by new Keynesians and post Keynesians (even though deep differences exist among them[8]), Sraffian analysis has not generated a real Sraffian School (Aspromourgos, 2004, p.181), but rather contradictory interpretations (Blankenburg et al., 2012). Sraffa's *PCMC* is an "amazingly concise little book" (Pasinetti, 2012b, p.1303), based on strict logical reasoning and dense "mathematical and methodological elixirs" (Velupillai, 2006, p.2), with profound consequences for economic theory but lacking detailed explanations (Newman, 1972). *PCMC* had the fundamental aim of criticising the marginal method, indeed its subtitle is *Prelude to a critique of economic theory*, and rehabilitated the forgotten classical theory through the resolution of the Ricardian problem of an *invariable measure of value* (i.e., the *Standard commodity*). Sraffa interpreted the economy as a circular flow, demonstrating how it is possible to determine prices, wages, and profits. The fundamental result is that distribution, independently from value theory, depends not only on production conditions, but also on sociological patterns, realising the conflict between wages and profits.

A first point of debate is the grade of compatibility of Sraffa's analysis with neoclassical theory. Some researchers have tried to interpret *PCMC* as a *special case* inside neoclassical theory (for example Hahn, 1975, 1982)[9] or as a non-destructive critique to neoclassical theory (for example Sen,

[8]Indeed, new Keynesians, starting from the *Lucas Critique*, try to reconcile Keynesian economics inside the mainstream orthodox framework. Gregory Mankiw and David Romer can be recognized the pioneers of these approach that accepts the neoclassical microfoundations, the rational expectation hypothesis, but supposes the presence of market failures and stickiness of prices and wages that require a fiscal and monetary policy intervention (see Mankiw and Romer, 1991). The extreme development of this school of thought is the construction of DSGE models with Keynesian features (for an introduction to this evolution see Galí, 2008, pp.4-6). The post Keynesians decisively disapprove the interpretations of Keynes's thought of the neoclassical synthesis and of the new Keynesians and try to demonstrate the total incompatibility between the *real* Keynes and the neoclassical approach. Starting in particular from Keynes's *General Theory*, the post Keynesians attempt to develop a Keynesian framework that reject methodological individualism and focus upon the role of money and effective demand (see Fontana and Realfonzo, 2005, p.9-10). The most important economists considered the pioneers of this school of thought are Joan Robinson, Michal Kalecki, Nicholas Kaldor, Hyman Minsky, Paul Davidson, Augusto Graziani, Jan Kregel and Alfred Eichner. Important contributions came also from Robert Clower and Axel Leijonhufvud.

[9]According to Hahn "there is no correct neo-Ricardian proposition which is not con-

2003)[10]. It is worth underlining how these positions are questionable especially because Pasinetti (1988) has claimed that it could be demonstrated "that Sraffa's analysis is entirely incompatible with marginal economic theory" (Pasinetti, 2012a, p.1436). A position confirmed in particular by Garegnani (2003) who engage in a detailed demonstration of the fallacy, in particular, of Hanh's positions in Hahn (1975, 1982)[11]. More interesting debates about *PCMC* have been raised because Sraffa's model is not closed; its simultaneous equation system leaves one degree of freedom - i.e., one distributional parameter has to be exogenously defined. Sraffa's reference to a closure of the model inside financial markets, through the definition of "money rates of interest" (Sraffa, 1960, p.33), is a foundation "of a theory of income distribution" (Pasinetti, 1988, p.135); open to different closures (see Blankenburg et al., 2012). While some researchers are convinced that Sraffa's price system is the result of a gravitational process through free competition (Caminati, 1990), others focus instead upon the interdependence between societal configurations and the "surplus product" (Garegnani, 1979). Another debate focuses upon the relationship between Sraffa and Marx (Steedman, 1977). The claim that Sraffa has solved the *transformation problem* invalidating at the same time Marx's analysis should consider that while Marx's analysis is an historical-sociological-philosophical interpretation of capitalism's dynamics, Sraffa's *PCMC* is a logical-mathematical analysis of a production system. Few results have come from attempts to insert the typical features of Marxian (and Keynesian) economics inside Sraffian schemes, such as money[12], institutions, technology, or dynamics. Pasinetti noted that researchers have been unable to elaborate satisfactory dynamic production models (Pasinetti, 1988, pp.245-246). The only result has been the elaboration of *quasi-dynamic* models, with population as the only dynamic variable (i.e., no disequilibrium or sociological-behavioural patterns can be considered).

An interesting evolution has been experimented by the so called *monetary theory of production* who seems to find a possible connection between Keynes and Sraffa. The idea of a *monetary theory of production* is traced

tained in the set of propositions which can be generated by orthodoxy" (Hahn, 1982, p.353), he emphasised the need to distinguish between Sraffa and his followers and the fact that (according to Hahn) it can be demonstrated that Sraffa should be considered as a *special neoclassical case.*

[10]Amartya Sen, after a short description of Sraffa's results in *PCMC*, claims: "This is a powerful technical result. We can ask: what difference does it make? Aggregative neoclassical models with capital as a factor of production are irreparably damaged. But neoclassical economic theory need not be expounded in an aggregative form" (Sen, 2003, p.1246).

[11]The arguments of Garegnani on this point make reference to what has been already explained in section 2.1. See in particular the answer of Garegnani to Hahn in Garegnani (2003).

[12]For example, Hodgson (1981) introduced money as a good.

back to Keynes (1973a)[13]. The fundamental idea is that money affects motives and decisions and plays an essential role in transforming a *real-wage economy* into a *money-wage or entrepreneur economy* (Keynes, 1979, p.81). According to Keynes, fluctuations in effective demand are the mayor factor explaining the difference between the two types of economies and are strictly connected to the fact that money plays a fundamental role in the entrepreneur economies (Keynes, 1979, p.76). Following these intuitions, especially after 1984, started a strong debate between heterodox economists around the *monetary theory of production* (see Graziani, 2003). Whereas most of these heterodox economists share a critical view of Walrasian approach to money, they have not been able to share a common view for the construction of an alternative theory of money; for these reasons there are different school of thought (i.e., the circuitist school and the post Keynesian school). Despite the differences a common body of doctrine has been built up: all reject methodological individualism and feel the need of a theory of endogenous money, recognize the role of effective demand and uncertainty and follow a Kaleckian-Keynesian analysis of income distribution (see Fontana and Realfonzo, 2005, pp.9-10). Inside the theory of the monetary circuit (developed in particular by Augusto Graziani, Edward Nell and Alain Parguez), the credit nature of money and the role of credit and money for the circulation of commodities has been emphasised. In particular Edward Nell found some interesting connections between the theory of the monetary circuit and the classical theory of production as revived by Sraffa (see Nell, 2004, 2005). These connections seem important for the construction of a *monetary theory of production* based on Keynes's and Sraffa's contributions (see for example Febrero, 2006).

2.3 Keynes and Sraffa: is an integration possible?

One interesting point of debate is the possibility to integrate Keynes's and Sraffa's findings. The answer to this topic is not trivial. The difficulty of using production theory and Keynes's economics as an alternative microeconomic foundation for constructing satisfactory macro-dynamic models is a matter of fact (Pasinetti, 1975). The first issue is the explanation of why it has been so difficult to create a synthesis between Sraffa and Keynes.

The strictness of the Sraffian methodology could be one of the reasons for the difficulties in finding a synthesis with Keynes's approach. Sraffa's approach was based on the "reliance on observable measurable quantities alone, to the exclusion of all 'subjectivist' concepts" (D3/12/7: 46[14], quoted

[13]Despite some researchers say that the *monetary theory of production* has a longer tradition; see for example Fontana and Realfonzo (2005).

[14]This notation make reference to Sraffa's unpublished manuscripts of the Sraffa Archive kept in the Wren Library at Trinity College, Cambridge. The notation follows the catalogue prepared by the archivist Jonathan Smith.

in Davis, 2012, p.1342). For Sraffa, the concept of social surplus was fundamental in explaining value, in opposition to the Marginalist method, which Sraffa considered "basically ideological in nature" (Davis, 2012, p.1343). For Sraffa, value was a sort of "physical or chemical quality" (D3/12/12: 7, quoted in Kurz, 2012, p.1548), and he elaborated his equation like a chemist who presents "chemical reactions first as a balance sheet and then as an algebraic equation" (Kurz, 2012, p.1547), following the principle that "for every effect there must be sufficient cause" (D3/12/7: 161(3) quoted in Kurz, 2012, p.1547).

This approach allowed him to surpass the concept of value used by classical economists. According to Sraffa, their error has been to "regard 'labour' as a quantity, to be measured in ... [terms] of human energy, and thus commensurate to value" (D3/12/11: 36 quoted in Kurz and Salvadori, 2005, p.418). What Sraffa suggested, instead, was to restart from Quesnay's *Economic Tables* in order to obtain a measure of value consistent with the physiocrats's concept of real cost. Indeed, the combination between Quesnay's circular flow model and the simultaneous equation solution method would provide, according to Sraffa, "the value of a set of commodities in terms of how those commodities were used up in production" (Davis, 2012, p.1344).

On the other hand, Keynes's ambiguity on the real consequences of his *General Theory* for the orthodox scheme (Robinson, 1963, p.78) created a sort of *doctrinal fog* (Blaug, 1980, p.221) that promoted the controversial neoclassical synthesis. Moreover, the alleged lack of microfoundations in Keynes's theoretical construction has reinforced mainstream economists's arguments (Lucas and Sargent, 1979). Surely this ambiguity is also linked to a concrete problem inside the Keynesian scheme, that is, the impossibility for Keynes to escape from the marginalist theory of value and the Marshallian perspectives at the base of his analysis, oriented to the short run. Even those economists that found possible some connection between Keynes and Sraffa have found difficult to imagine how to merge Keynes's unstable short run approach with the widespread vision of Sraffian schemes as long run point of equilibrium convergence (see for example the approach of Kurz and Salvadori, 2005).

As a consequence, instead of a merge between Keynes and Sraffa, the final results has been an incorporation of Keynes into neoclassical theory. A result that does not make justice, surely, of the real value of Keynes's contribution. As noted by Pasinetti:

> It was quite natural to expect the theorists to be eager to transcribe Keynes's arguments into simple, possibly algebraic and diagrammatic terms, suited for didactical purposes. [...] This situation, not surprisingly, led to encouraging the use of the traditional analytical tools and to try to insert the Keynesian innovations into the prevailing paradigm.

> The most successful of all devices in this direction was the IS-LM model of J.R.Hicks [...]. Hicks more recently (1980-81) stated quite plainly that there was no attempt on his part to convert himself to Keynes [...] as he writes, "the IS-LM was in fact a translation of Keynes' nonflex-price model into my terms [...] the idea of the IS-LM diagram came to me as a result of the work I had been doing on three-way exchange, conceived in a Walrasian manner" (Pasinetti, 2007, pp.141-142).

This situation had dramatic consequences for the interpretation of Keynes's *General Theory*:

> Franco Modigliani (1944), starting from Hicks's model, proceeded to a well-known formalization of Keynes's theory that had the explicit purpose as he put it of 'digesting' Keynes's hard tools or 'difficult' concepts into traditional economic analysis. This whole process went on, without much opposition from the 'genuine' Keynesians. It was later crystallized by the expression, attributed to Paul Samuelson, of a 'grand neoclassical synthesis'. (Pasinetti, 2007, p.30).

In summary, Sraffian and Keynesian economics did not achieve full integration. On one side, the strictness of the Sraffian methodology, which analysis has no money, no contracts, no debts, no dynamics, and no behavioural or psychological features; i.e., the typical Keynesian elements; on the other side, the *doctrinal fog* (Blaug, 1980, p.221) around the Keynesian economics that promoted the controversial neoclassical synthesis. If this is the situation the questions is: **how is it possible to reconcile Sraffa and Keynes?** The answer is, in our opinion, that the two theories are not to be reconciled but completed through their integration. In fact there is no evidence to say that the two approaches are incompatible or conflicting, but there are elements which demonstrate their complementarity: the Sraffian theory can represent the theory of production that is lacking in the Keynesian theory; the Keynesian theory provides the instruments for the investigation of the disequilibrium dynamics which is lacking in the Sraffian theory. One should not forget that: "both Keynes and Sraffa rejected Say's Law, although for different reasons" (Kurz, 2013, p.9). In the next section we will demonstrate the total incompatibility between the Keynesian thought and neoclassical theory. Keynes, as Sraffa, was convinced that the complexity of the economic phenomenon should be investigated on the basis of robust accounting principles. This means that both the Keynesian theory that the Sraffian theory require the use of a constructive mathematics incompatible with the neoclassical theory. At this point, one may be tempted to doubt the compatibility between the Sraffian rigor and the typical Keynesian approach oriented to the study of the influence of the uncertainty and of the *animal spirits*. Sraffa's unpublished manuscripts seem to show that the rigor demonstrated by Sraffa should not be interpreted as an absolute refusal to consider the complexity of the economic phenomenon[15]. Indeed, as it will be demonstrated in the next section, Sraffa recognized the importance of institutions, human behaviour and society for the determination of the surplus. Moreover Sraffa never claimed that demand plays no role in his analysis and demand is crucial for Keynes who based his *General Theory* on the controversial concept of effective demand. Sraffa was hoping that his fundamental contribution could represent the foundation for the creation of an alternative to the marginal theory: "if the foundation holds, the critique may be attempt later, either by the writer or by someone younger or better equipped for the task" (Sraffa, 1960, p.vi). It is hard to believe that this *critique* could be carried out regardless of the study of the dynamics of disequilibrium. It is also hard to believe that behind Keynes's ambiguities and *silences* the hope that his precious and fundamental contributions could once break free from the neoclassical theory was not concealed. Several well-known economists who advocate an integration between Keynesian and Sraffian economics probably share this interpretation (see for example Roncaglia (1995, p.120) cited in the *Introduction* of the present thesis). It cannot be excluded that the lack of integration has been determined mainly by the contrasts between the successors of Keynes and Sraffa. A contrast that certainly contributed to the solidification of the mainstream economics. But one should not forget that:

> despite claims to the contrary, there is a strong bond uniting post-Keynesians of various brands and Sraffa: it is their opposition to the marginalist or neoclassical theory (Kurz, 2013, p.1).
> Those who seek to develop an alternative to the conventional neoclassical theory are well advised to take into account both the findings of Keynes and those of Sraffa (Kurz, 1995, p.84).

As a consequence the problem is not (or should not be) the compatibility between the two theories, but **how to integrate them**. The present thesis represents one step towards a possible answer to this question.

2.4 The reasons and conditions for a new integration

Before addressing the issue of a possible integration between Keynes and Sraffa it is important to underline the incompatibility between Keynes and the mainstream. Indeed, a fundamental misunderstanding of Keynes's thought has contributed to this development; in particular, the debatable simplification of Keynes's *General Theory* by the neoclassical synthesis (e.g., the IS-LM model in Hicks, 1937) have reinforced the habits (inherited by most new Keynesians) of considering Keynes's contribution a part of macroeconomics to be derived from Walrasian microeconomics.

[15] Although it is undeniable that it is not an easy task to combine the analysis of the economic phenomenon in its complexity with the Sraffian rigor.

Keynes was convinced that his theory was revolutionary with respect to the neoclassical one and he was not oriented to find compromises through simplifications. It appears clear in this Keynes's oft-quoted letter to Harrod:

> The general effect of your reaction [...] is to make me feel that my assault on the classical school ought to be intensified rather than abated [...]. I am frightfully afraid of the tendency, of which I see some signs in you to appear, to accept my constructive part and to find some accommodation between this and deeply cherished views which would in fact only be possible if my constructive part has been partially misunderstood (Letter dated 27 August 1935 from Keynes to Harrod cited in Pasinetti (2007, pp.31-32)).

The theoretical gaps in modern microeconomics, demonstrated by the empirical evidence of the actual crisis, underline the impossibility to restrict Keynes's theoretical framework within modern microeconomics. According to Clower the compromise between Keynes and the mainstream was implausible (Clower, 1965, pp.278-279): if Keynesian economics added something new to economics, one has to recognize its incompatibility with Walras's law and the mainstream theory of household behaviour. Indeed, Keynes's novelty could be understood only by admitting that he "made tacit use of a more general theory" (Clower, 1965, p.279), another microfoundation never explicitly explained by Keynes himself, but that probably was "at the back of his mind" (Clower, 1965, p.290). Even though there is no direct evidence of this inside Keynes's writings, it is a matter of fact that the most distinctive elements of Keynesian economics (e.g., *animal spirits* or *beauty contests*) can hardly find a place inside neoclassical theory.

Inside Neo-Walrasian theory, there are no markets, no endogenous institutions, no announcements by agents (the only active agent is the auctioneer), no competition, no money and no trading: it is concerned solely with demands and supplies based on "trading dispositions", as Walras called them, that could be interpreted as hypothetical mental states (Clower, 1994, pp.808-809). Keynes's approach, on the other hand, has always focused on the dynamic behaviour of monetary real-world economies, characterized by decentralized, self-organizing and never "clear" markets, where prices are "made" by real agents without perfect knowledge (Clower, 1994, pp.807-808). It is worth remembering that the discontents with Neo-Walrasian theory "concern not its lack of 'realism' but its scientific vacuousness" (Clower, 1994, p.810) in theorizing observable events. It has been already explained the nature of this *vacuousness* in the previous chapter, when the importance of the introduction of constructive mathematics inside economics has been emphasised. Keynesian microeconomics should be not only real and close to Keynes's *psychological economic theory* but also coherent with the principle of computable economics, i.e., "every concept must have an operationally relevant content so as to be understood and implemented by the virtual economic agents" (Zambelli, 2010, p.38).

At this point a possible connection with Sraffa results fundamental. Indeed, it is worth remembering that Sraffa's schemes are mathematically constructive[16] and, probably, they are a good candidate to be added to the Keynesian framework so as to develop a root to be used for the analysis of the behaviour of the economic system as a whole.

Sraffa's unpublished manuscripts, which have been available since the opening in 1993 of the Sraffa Archive kept in the Wren Library at Trinity College (Cambridge)[17] seem precious for "revisiting and placing *PCMC* in the wider context of ongoing debates on economic philosophy, economic theory and economic policy" (Blankenburg et al., 2012, p.1267), especially because "Sraffa is reported to have called his notes and papers an *iceberg*, the tip of which is his published work" (Kurz, 2012, p.1539).

Thanks to the unpublished manuscripts, which have become the object of a growing literature, it has been emphasised how Sraffa realised, at some point of his intellectual journey (Kurz and Salvadori, 2005, p.431), that surplus could not be explained only in terms of real costs, but depends also on the behaviour of capitalists and institutions. Indeed, as Sraffa observes

> When we have defined our "economic field", there are still outside causes which operate in it; and its effects go beyond the boundary [...]. The surplus may be the effect of the outside causes; and the effects of the distribution of the surplus may lie outside (D3/12/7: 161(3-5) cited in Davis, 2012, p.1348).

Sraffa never explored these "outside causes", nor other potential subjective factors. Nevertheless, Sraffa admitted the importance of "systemic inducement", a concept not far from Smithian "forces" or the Marxian "coercive law of competition" (Kurz, 2012, p.1559). Moreover, Sraffa's concern with the consequences of the relationship between policy issues and social conflict for distribution is confirmed by his writings on banking systems and monetary policy (Blankenburg et al., 2012, p.1276). His relationship with Gramsci permitted him to develop a "social point of view" that marginalism denied. As claimed by Sraffa:

> [The] chief objection to utility is that it makes of value an *individual* conception: it implies that problems of Rob[inson] Crusoe and those of an economic man living in the City are exactly the same. Now, value is a *social* phenomenon: it would not exist outside society: *all* our utilities are derived from social conventions and therefore dependent upon social conditions and standard (D1/16: 1, Sraffa's emphasis. Cited in Signorino, 2001, p.758).

[16]See on the constructiveness of Sraffa's schemes Velupillai (1989, 2008a) and Zambelli (2010, p.26).

[17]See Smith (1998) and Kurz (1998) for a summary of the contents of the manuscripts.

If it can be sustained that there is no reason to conclude that Keynes's and Sraffa's approaches are incompatible, some strong connections between them should also be emphasised.

Firstly, both are based on strong bookkeeping principles, fundamental for computation. Indeed, Sraffa's equation systems are based on the equality between costs and revenues, while Keynes's *General Theory* "set the stage for national income accounting...[and] his recognition of the equality of saving and investment provided the basis for an accounting approach" (Ruggles and Ruggles, 1999, p.135).

Secondly, both of them were aware of the complexity of real economies. Indeed, Keynes's claim that the causal relationship runs from investment to saving starts from the recognition that it is micro-level behaviour which generates the non-isomorphic macro-level aggregate (i.e., the *fallacy of composition*). Each agent's behaviour is subject to uncertainty and it is influenced by other people (i.e., the *beauty contest*). The failure of coordination is intrinsic in the mechanism of economic system, because different agents take different decisions according to their social role in a highly unstable, complex, and interconnected environment. Also Sraffa was aware, as just previously described, of this complexity that led him to recognize that "the surplus may be the effect of the outside causes" (D3/12/7: 161(3-5) cited in Davis, 2012, p.1348).

It is worth remembering that for Keynes investments determine savings, because saving is the decision to abstain from consumption an earned income (i.e., income, saving and investment are flow and not stocks or endowments) and income depends on the level of employment and production which depend on investments. This is the core mechanism at the base of Keynes's definition of *effective demand*[18].

It is well-known that the role of demand inside Sraffian Schemes is debated. Anyway "Sraffa never stated that prices can be conceived independently from demand" (Bellino, 2008, p.24) so it is not possible to declare that demand plays no role in Sraffa's opinion. Famous and instructive is Sraffa's letter to Arun Bose in which he states:

> [...] Your opening sentence is for me an obstacle which I am unable to get over. You write: 'It is a basic proposition of the Sraffa theory that prices are determined exclusively by the physical requirements of production and the social wage-profit division, with consumers' demand playing a purely passive role.' Never have I said this: certainly not in the two places to which you refer in your note 2. Nothing, in my view, could be more suicidal than to make such a statement. You are asking me to put my head on the block so that the first fool who comes along can cut it off neatly (C32/3 the letter is reproduced integrally in Bellino, 2008, p.39).

[18]In section 2.5.1, a detailed disccusion about Keynes's principle of *effective demand* is reported.

It could be claimed that the produced surplus of the Sraffian scheme is itself "effective demand at production prices" and the meaning of this sentence can be understood through a simulation approach (as developed in the present thesis) that consider out-of-equilibrium behaviour inside Sraffian schemes, so that the role of effective demand appears clear and the nature of the eventual convergence towards production prices can be studied.

2.5 The integration in practice: the Keynesian side of the laboratory

In the previous sections, the reasons and conditions for a possible integration between some elements of Keynesian and Sraffian economics have been considered. In this section, the Keynesian elements that will be effectively integrated in the *digital economic laboratory* will be described. It has been said that the present thesis is an attempt to model Keynes's principle of effective demand with the aid of Sraffian schemes. At this point, a careful consideration of the meaning of the controversial *principle of effective demand* inside the *digital economic laboratory* is needed. This would help on one side to clarify our position about this important topic and on the other side it will allow to explain in more details the elements of Keynesian economics effectively present in the *digital economic laboratory*.

2.5.1 Keynes's principle of effective demand: some notes

Keynes defines effective demand in the third chapter of his *General Theory*, titled *The Principle of Effective Demand*. In that chapter Keynes claims:

> It follows that in a given situation of technique, resources and factor cost per unit of employment, the amount of employment, both in each individual firm and industry and in the aggregate, depends on the amount of the proceeds which the entrepreneurs expect to receive from the corresponding output [...]. Thus the volume of employment is given by the point of intersection between the aggregate demand function and the aggregate supply function; for it is at this point that the entrepreneurs' expectation of profits will be maximised. [...] the point of the aggregate demand function, where it is intersected by the aggregate supply function, will be called the *effective demand* [...] this is the substance of the General Theory of Employment [...] (Keynes, 1936, pp.24-25).

The concept of effective demand and the aggregate supply and demand functions introduced by Keynes in order to determine effective demand have been object of debates, criticisms and different interpretations. Different economists have given different explanations of the troubles around one of the most important concept introduced by Keynes in the *General Theory*[19]. For example, Pasinetti claims that:

> After appearing as the title of the chapter, the principle of effective demand is not stated. The only time the term itself is mentioned in the whole of Chapter 3 is on page 31, in an incidental sentence, where it is taken for granted that the reader knows what it is. It is not mentioned again in the whole book. It is not even mentioned anywhere else in Keynes's writings (as far as I have been able to discover) with two exceptions: in a perfunctory sentence in the centenary allocution on Malthus (C W. X: 107) and in a letter to Sraffa (to be mentioned below, p. 100) (Pasinetti, 1997, p.93).

Other economists, such as for example Asimakopulos, underlines other difficulties that go beyond Keynes's silence on a clear explanation of the principle of effective demand:

> The aggregate supply and demand functions presented by Keynes in Chapter 3 of The General Theory of Employment, Interest and Money in order to define effective demand, have continued to draw comment, criticism and suggested reinterpretation. A definitive interpretation of, and commentary on, these concepts is *made difficult by inconsistencies and errors in the way Keynes presented them.* (Asimakopulos, 1982, p.18, emphasis added).

Before of whatever interpretation of the *real* Keynes, the problem seems to be related to the fact that Keynes defines but not states in an unequivocal way the principle of effective demand. Moreover, the way in which Keynes introduces the concept of effective demand has been found, at least, ambiguous. Keynes's inconsistencies and silences have been probably determined also by the fact that "the principle of effective demand belongs to a more fundamental level of investigation, towards which Keynes was able to go only partially. Perhaps, he never had the time or the necessary calm or the appropriate analytical tools to go back and face it explicitly" (Pasinetti, 1997, p.93). As a consequence, the commentators of Keynes have had the hard task to fill the silences of Keynes by overcoming his ambiguities and contradictions in order to find *what really Keynes wanted to say*[20]. The final purpose was the construction of a solid theory of effective demand to use as foundation for the construction of a real alternative to the neoclassical theory demonstrating that Keynes's theory was really revolutionary. A really hard task, that led to unavoidable debates, as recognized by Clower:

> The history of post-General Theory macroeconomics is a story of repeated attempts to extract from Keynes's classic more than it actually

[19] One should not forget that, according to some researchers, *effective demand* has been anticipated by Kalecki. On this point see Arestis (1996, p.112).

[20] In particular with the support of Keynes's Collected Writings.

> contains: formally valid arguments that would simultaneously substantiate its disputed revolutionary claims and rationalize its undisputed revolutionary impact. After some sixty years of exegesis and debate, all that appears to have been established is that Keynes's General Theory was 'revolution-making' but not 'revolutionary' (I am here paraphrasing T.S. Kuhn 1953: 135). No one disputes the audacity of Keynes's aggregative approach to the economy as a whole, but many have questioned the conventional comparative-statics method that Keynes adopted (Clower, 1997, p.47).

Despite everything, lot of economists have tried to develop a clear theory of effective demand but "the ambiguities in his exposition [of Keynes] were to be reflected, and often magnified, in the subsequent debates" (King, 1994, pp.4-5). A detailed description of these debates are not object of the present thesis. Despite this, a brief summary of the relevant literature and an introduction to some interesting points of discussion will be briefly commented. Firstly, different economists have tried to summarise the concept of effective demand in quite different ways. Garegnani defines effective demand as "the principle that aggregate demand may be insufficient to absorb the output produced from normal use of existing capacity" (Kregel, 1983, p.69); Amedeo, instead, enunciated Keynes's principle of effective demand as follows: "given a change in investment demand (or any autonomous component of aggregate demand), the level of income (that is, the levels of price and output) will change in such a way that, in equilibrium, the corresponding change in saving will be equal to the initial change in investment" (Amadeo, 1989, p.1). Beyond these different alternative definition (themselves subject to dispute), a first object of debate has been the role of aggregate demand and aggregate supply in determining employment. Despite "not all of the original propagators of the General Theory were convinced of the significance of aggregate supply and demand analysis" (King, 1994, pp.4-5), this topic started to become object of discussion. Economists such as Dennis Robertson, Ralph Hawtrey and F.J. de Jong emphasized the role of aggregate demand as the major innovation of Keynes, in determining employment, considering aggregate supply function "familiar and largely uncontroversial" (King, 1994, pp.4-5). Others, such as Asimakopulos (1982), have criticized this unilateral vision by emphasizing both the role of aggregate supply and of aggregate demand in determining effective demand and finally employment. Some economists have emphasized the importance of considering effective demand as the fundamental feature of an entrepreneur monetary economy. Robert W. Clower, with Axel Leijonhufvud[21], has been surely one of the most prominent economists of this strand: "Clower's original venture into the uncomfortable no-man's land between Neoclassicism

[21] Axel Leijonhufvud, among other things, focused on effective demand failures. See for example Leijonhufvud (1973).

and Keynesianism sought to provide a microtheoretical foundation for the core concept of Keynesian theory - Effective Demand" (Leijonhufvud, 1973, p.32). Bharadwaj summarizes very well the importance of the contribution of Clower and Leijonhufvud:

> Their arguments are narrowly directed to show that Keynes's underemployment state can be explained independently of the "liquidity trap", "money illusion", "wage rigidity", "elasticity pessimism" or "imperfections" in the system. [...] The central question that Keynes faced appears to be the same that troubled general equilibrium theorists. [...] To Clower and Leijonhufvud, it is the trading that actually occurs at non-market-clearing prices and its consequences that is at the root of the Keynesian results. [...] Clower emphasizes the distinctive character of the money economy in the possibility it generates of violating Walras's Law (Bharadwaj, 1983, pp.5-7).

Despite their desire to demonstrate the true revolutionary nature of the thought of Keynes must have seemed laudable to all supporters of Keynes[22], their interpretation of Kyenes's thought has been object of criticism. Indeed "while a description of the disequilibrium, non-tâtonnement process, amplifying disturbances and thus leading to unemployment, emerges from these attempts at providing microfoundations to Keynesian results, they have left a number of "false trails" and landed the theory into a state of theoretical uncertainty" (Bharadwaj, 1983, p.9). Brendan Sheehan claims that "Clower's use of the term effective demand has caused considerable pedagogic confusion in Keynesian circles [because] Clower's definition of effective demand is very different from that of Keynes" (Sheehan, 2009, p.287). Despite these criticisms it is undeniable that Keynes's objective is to understand the functioning of an entrepreneurial monetary economy. Therefore, monetary and financial factors cannot be ousted from the analysis. Hyman Minsky has been one of the most prominent economist that has considered the importance of these factors also with reference to the theory of effective demand:

> Any understanding of the generation of effective demand and the possibility of effective demand failures in our economies in our time must take the specific nature of our big government capitalism into account. [...] Under capitalist conditions effective demand is financed demand. [...] One great advance that I have always associated with Keynes is that he held that we cannot dichotomize the financial and the "real" when it comes to understanding capitalism (Minsky, 1983, pp.46-49).

If the importance of money and financial factors for Keynes is incontestable, their particular role inside the short-run and long-run theory of effective demand cannot be defined trivially. The debate between Kregel

[22]With this term we consider those economists that actually hope to build a totally independent theory with respect to neoclassical theory.

and Garegnani is a good example of the importance of this issue. According to Kregel: “in Keynes’s theory the real and monetary sectors are integrated both in and out of equilibrium; his investigation of the transition process led to a redefinition of the determinants of the natural position in terms of both money and real factors. This is Keynes’s theory of effective demand” (Kregel, 1983, p.62). Kregel conviction is that “Keynes certainly considered the analysis of effective demand to have been an integral part of the Classical discussion of economic problems” (Kregel, 1983, p.50). This seems coherent with this Keynes’s observation:

> To me, the most extraordinary thing regarded historically, is the disappearance of the theory of demand and supply for output as a whole, i.e. the theory of employment, after it had been for a quarter of a century the most discussed thing in economics (Keynes, *Collected Writings*, XIV, 1973, p.85; reported in Kregel (1983, p.50)).

Garegnani, instead, holds a different view: “Keynes’s liquidity preference is not necessary to establish the principle of effective demand in the short or in the long period” (Garegnani, 1983, p.78). Garegnani is convinced that most problems in considering a long run theory of effective demand is related to Keynes’s big troubles with orthodox theory, a position somehow shared also by Pasinetti and most Sraffian economists. In details Garegnani claims that:

> Keynes also believed in the premises of orthodox theory and did not challenge the notion of substitution between labour and capital on which, as we saw, the orthodox theory based its conclusions concerning the tendency towards the full employment of labour. The problem therefore is: how did Keynes attempt to reconcile these two contradictory strata of his thought? We know how he did it. [...] he relied basically on the joint outcome of two elements: (i) liquidity preference, [...] and (ii) the effect of expectations concerning the future profitability of production making investment fluctuate. This second, Keynesian route to effective demand, with its heavy reliance upon expectations, was (at least until very recently) quite successful in getting the effective demand principle established for the short period policies which were Keynes’s immediate concern. The same route has however turned out to be much less successful in getting the principle of effective demand accepted for long-run theory as Keynes clearly thought it should. [...] Keynes’s acceptance of the traditional premises also helps to explain, a second factor which underlies his failure to get the principle of effective demand established in long-period theory. Except for brief unsystematic statements we do not find in his work any alternative to the orthodox long-period theory of the level of aggregate output (Garegnani, 1983, pp.75-76).

Keynes acceptance of some premises of the orthodox theory is surely at the core of that *doctrinal fog* (Blaug, 1980, p.221) that promoted the neoclassical synthesis and the abandonment of the concept of effective demand within it. Indeed, in this compromise, there was no place for a fundamental and revolutionary concept such as the effective demand. As a consequence, and not surprisingly, "those who have tried to re-absorb [Keynes's] analysis into traditional theory have been careful not to mention the principle [of effective demand] at all". (Pasinetti, 1997, p.93). In new Keynesian economics effective demand disappeared. Indeed, this concept is at the end one of the most important points of disagreement between new Keynesian economics and post Keynesian economics. Effective demand in post Keynesian analysis implies that it is scarcity of demand rather than scarcity of resources that is to be confronted in modern economics, so that output is ordinarily limited by effective demand, although it is recognised that supply constraints are present in modern capitalist economies (Arestis, 1996, p.112). The situation of the debate around the theory of effective demand nowadays is well summarised by Hayes:

> King (1994, p.28) notes a decline in post Keynesian interest in Keynes's principle of effective demand since 1989, which he attributes to its neoclassical flavour. Consistent with this view, the recent literature on effective demand has largely divided into two strands, one tending to follow Kalecki rather than Keynes (see Arestis, 1996), the other addressing Keynes's methodology (see Chick and Dow, 2001). Orthodoxy has long discounted the suggestion that Keynes offers a serious alternative to Walrasian general equilibrium analysis, and has followed Lucas (1981) in its negative assessment of Keynes's theoretical contribution (Hayes, 2007, p.56).

2.5.2 Keynes in the digital economic laboratory

The perception is that this long and important debate around the *principle of effective demand* is fated to remain open. On this *state of the art* a comment of Minsky (in a contribution titled *Notes on effective demand*) seems particularly pertinent:

> *Perhaps we pay too much attention to the literature of our discipline - not only to what our great predecessors said but also to what our contemporaries are saying - and not enough to what happens in the economies we study.* [...] Instead of trying to say general things about economies, we should concentrate on understanding the range of behaviour which economies exhibit as institutions as well as the relations among variables change (Minsky, 1983, p.49, emphasis added).

Following the wisdom of the words of Minsky, it can be concluded that economists should start to focalize to what really happens in the economic system, beyond the theoretical debates. This seems to suggest, coherently also with Herbert Simon[23], that an economist should not rely only on the abstraction of the traditional modelling but should be also open to use instrument able to investigate "what happens in the economies we study" such as a simulation approach. This type of investigation can be used as an instrument in order to understand how effective demand emerge from the mechanisms of entrepreneurial monetary economy. This is exactly the approach at the base of the construction of the *digital economic laboratory.* Its purpose is to see how effective demand emerges and aggregate demand and supply function behave in a dynamical framework where out-of-equilibrium behaviour is possible thanks to the presence of some fundamental elements typical of Keynesian economics that can be summarized as follows:

- Producers produce in order to earn profits.
- Producers produce and assume workers according to expected demand: without a profit expectation based on expected demand producer can also not assume workers so that unemployment can emerge.
- ARAs (workers and producers) buy their commodities according to their preferences and income at disposal.
- Debt and credit relations can emerge in order to carry on production and consumption. Debt and credit accumulate in time and represent the financial *wealth* associated to each ARA.
- There can be cases of excess demand or excess supply. In the first case ARAs can be rationed, in the second case some commodities are unsold.

According to these premises the *digital economic laboratory* is able to experience effective demand, and with this term we mean the phenomenon as described by Keynes:

> In a given situation of technique, resources and factor cost per unit of employment, the amount of employment [...] depends on the amount of the proceeds which the entrepreneurs expect to receive from the corresponding output [...]. Thus the volume of employment is given by the point of intersection between the aggregate demand function and the aggregate supply function [...] called the *effective demand* (Keynes, 1936, pp.24-25).

Monetary and financial aspects in the digital economic laboratory: some important remarks

It can be sustained that some fundamental elements of Keynesian economics that characterize a *monetary entrepreneurial economy* are absent or, at least,

[23] See on this point the previous chapter section 1.4.

not so clearly defined: monetary system, financial system and banking system. First of all, it is important to clarify the role of money inside the *digital economic laboratory*. Inside the *laboratory*, each exchange is associated to the underwriting of an IOU (*I owe you*) denominated in a socially accepted unit of account[24]. It is assumed that each ARA accepts to underwrite an IOU when needed[25]. This means that a type of money is present inside the *laboratory*. Indeed, this mechanism respects the modern definition of money according to which "money today is a type of IOU, but one that is special because everyone in the economy trusts that it will be accepted by other people in exchange for goods and services" (McLeay et al., 2014, p.1). Moreover, we should not forget that:

> In principle, there might be no need for a special financial asset such as money to keep track of who is owed goods and services. Everyone in the economy could instead create their own financial assets and liabilities by giving out IOUs every time they wanted to purchase something, and then mark down in a ledger whether they were in debt or credit in IOUs overall. Indeed, in medieval Europe merchants would often deal with one another by issuing IOUs. And merchant houses would periodically settle their claims on one another at fairs, largely by cancelling out debts. But such systems rely on everyone being confident that everyone else is completely trustworthy. [...] Money is a *social institution* that provides a solution to the problem of a lack of trust (McLeay et al., 2014, p.4).

The possibility inside the *laboratory* to experience the emergence of credit and debt relations through the underwriting of IOUs as previously defined; the fact that each ARA takes decisions considering the values of the commodities with the purpose of maximizing his income (for producers expected profits); the fact that workers are hired according to expected demand and the fact that imbalances between supply and demand are possible are sufficient elements to consider the economy simulated inside the *laboratory* a *monetary entrepreneurial economy* in Keynesian terms. The presence of a banking system or other complex financial institutions that allows, for example, to exchange IOUs versus IOUs are important features of a real economy but not necessary elements for defining an essential and simple version of a *monetary entrepreneurial economy* in Keynesian terms. Indeed, it can be sustained that the only element that is necessary to have in order to talk about a *monetary entrepreneurial economy* in Keynesian

[24]An IOU is a contract, a deferred payment of an exchange which is taking place now. For more details see section 3.3.4.

[25]In this way *trust problems* are not considered because it is assumed that each ARA accepts to underwrite an IOU when needed according to a shared rule. This hypothesis could be considered too strong. Despite this one should not forget that this hypothesis should be considered as a first step in order to understand more clearly the role of a successive introduction of the banking system.

terms is *effective demand*. This concept, not so clear inside the *General Theory*, has been considered with more care by Keynes in the attempt to introduce a *monetary theory of production* (see Keynes's Collected Writings; volume XXIX: *The General Theory and After: a supplement*), where Keynes's speaks at length also about *effective demand* and defines it:

> *Effective demand* may be defined by reference to the expected excess of sale proceeds over variable cost (what is included in variable cost depending on the length of the period in view). Effective demand fluctuates if this excess fluctuates, being deficient if it falls short of some normal figure (not yet defined) and excessive if it exceeds it (Keynes, 1979, p.80).

Then Keynes explains the difference between a *real-wage cooperative economy*, a *neutral entrepreneur economy* and a *money-wage entrepreneur economy*. According to Keynes the first type of economy is a barter economy in which factors of production are "rewarded by allocating in agreed proportions the actual outcome of their cooperative effort" (Fontana and Realfonzo, 2005, p.2). In the *neutral entrepreneur economy* money is a simple mean of exchange that do not change the 'barter nature' of this economy in which "sale proceeds exceed variable cost by a determinate amount" (Keynes, 1979, p.80). Finally, the *money-wage entrepreneur economy* is, first of all, an economy in which, following Marx's intuition, the purpose of production is to obtain even more money:

> The distinction between a co-operative economy and an entrepreneur economy bears some relation to a pregnant observation made by Karl Mark, - though the subsequent use to which he put this observation was highly illogical. He pointed out that the nature of production in the actual world is not, as economists seem often to suppose, a case of C-M-C', i.e., of exchanging commodity (or effort) for money in order to obtain another commodity (or effort). That may be the standpoint of the private consumer. But it is not the attitude of *business*, which is a case of M-C-M', i.e. of parting with money for commodity (or effort) in order to obtain more money. (Keynes, 1979, p.81).

Keynes develops this point and arrives to explain clearly the difference between a *real-wage* or *co-operative economy* and a *money-wage entrepreneur economy* as follows:

> The first type of society we will call a *real-wage* or *co-operative economy*. The second type, in which the factors are hired by entrepreneurs for money but there is a mechanism of some kind to ensure that the exchange value of the money incomes of the factors is always equal in the aggregate to the proportion of current output which would have been the factors' share in a co-operative economy, we will call a *neutral entrepreneur economy*, or a *neutral economy* for short. The third type,

> of which the second is a limiting case, in which the entrepreneurs hire the factors for money but without such a mechanism above, we will call a *money wage* or *entrepreneur economy.* (Keynes, 1979, pp.77-78).

The real difference between the two types of economy can be traced back, ultimately, to the *fluctuation of effective demand*:

> In a co-operative or in a neutral economy, in which sale proceeds exceed variable costs by a determinate amount, effective demand cannot fluctuate [...]. But in an entrepreneur economy the fluctuations of effective demand may be the dominating factor in determining the volume of employment. (Keynes, 1979, p.80).

It is interesting to note that Keynes asks himself explicitly "could then such an entrepreneur economy exist without money?" (Keynes, 1979, p.85). To this question Keynes answers as follows:

> What [...] is the answer to our original question? Money is *par excellence* the means of remuneration in an entrepreneur economy which lends itself to fluctuations in effective demand. But if employers were to remunerate their workers in terms of plots of land or obsolete postage stamps, the same difficulties could arise. Perhaps anything in terms of which the factors of production contract to be remunerated, which is not and cannot be a part of current output and it is capable of being used otherwise that to purchase current output, is, in a sense, money. If so, but not otherwise, the use of money is a necessary condition for fluctuations in effective demand. (Keynes, 1979, p.86).

In the *laboratory* the fundamental elements that allow to explain the origin of a *money-wage entrepreneur economy* (as described by Keynes) are present. It would be a theoretical error to argue that other elements (such as a banking system) have been considered *necessary* by Keynes for defining a *monetary entrepreneurial economy*[26]. It is obvious that the economic system presented in the *laboratory* has been created with the purpose to be implemented by important institutions such as the banking system or a complex financial system[27]. Despite this, from a methodological point of view, it is desirable to study the dynamics of a *money-wage entrepreneur economy* starting from the simplest version. This is exactly the first step attempt by the present thesis.

[26]Remember that Keynes overlooks the banking system and bank money inside the *General Theory*. See section 8.3 for more details on this topic.

[27]These points are object of further research that go beyond this thesis. See section 9.2 for more details.

2.6 The integration in practice: the Sraffian side of the laboratory

In the previous section, the Keynesian elements that will be effectively integrated in the *digital economic laboratory* have been described in details. It has been said that the present thesis is an attempt to model Keynes's principle of effective demand with the aid of the Sraffian schemes. In this section the Sraffian elements that will be present in the *laboratory* will be clearly described. The introduction of the Sraffian elements inside the *laboratory* as complement of the other Keynesian elements represents an attempt to "take into account both the findings of Keynes and those of Sraffa" in order "to develop an alternative to the conventional neoclassical theory" (Kurz, 1995, p.84). In the *laboratory* the elements and the concepts of the Sraffian tradition can be summarised in the following list:

- Production is conceived as a circular process.
- Production is heterogeneous with methods of the fixed proportion type.
- Production of commodities by means of commodities: in this version of the *laboratory* joint production or fixed capital are excluded.
- This conception of production allows to compute the wage rate and the profit rate, to observe how the surplus is divided between producers and workers and to construct the wage-profit curve.
- In some *thought experiments* the possibility to choose the best method of production among a set of different methods for the production of the same commodity will be introduced. This allows to use the different wage-profit curves for the construction of the *wage-profit frontier* as a technological benchmark.

The introduction of Sraffian schemes in a dynamic environment where the exchanges can take place at market prices by signing IOUs is one of the key innovations of this thesis, compared to the existing literature on this subject. The fact that production is conceived as a circular process on the basis of the Sraffian schemes allows to analyse the pattern of distribution, to verify the convergence towards uniform profit rates and uniform wage rates and to use the *wage-profit curves* (and the *frontier*) as technological benchmarks. In the next chapters, this integration between Sraffian and Keynesian elements will be clarified through a detailed description (also mathematical) of the *laboratory*. The potentialities of this first attempt towards a Sraffian-Keynesian approach will be demonstrated through the thought experiments.

3 The digital economic laboratory

3.1 Introduction

The algorithmic *digital economic laboratory* is characterized by the presence of a virtual market formed by a population of algorithmic rational agents (ARAs)[28], divided between workers and producers. ARAs, located inside a lattice, have a deterministic network of relations. ARAs are characterized by behavioural functions describing decisions with respect to the producing, consuming, buying and selling of economic magnitudes. Exchanges between ARAs (deterministic, bilateral and in property rights), are registered through a double-entry bookkeeping system. ARAs producers sell their products in local markets which are connected spatially. Production is heterogeneous and conceived as a circular process; the methods of production available to the ARAs producers are of the fixed proportions type. Each ARA producer will choose (if different alternative methods are present) the method that allows him to maximize his expected revenues. Each local market represents a local unit of the whole economy and the presence of more interconnected local markets, with more ARAs producers producing the same commodity in different local markets, allows to make a distinction between local and global dynamics.

This algorithmic model has been written in the form of a coherent set of computer codes. We aim at running simulation experiments to be used for the clarification of theoretical as well as applied economic issues. The algorithmic model is a *digital economic laboratory*, a benchmark for all the experiments. Indeed, behavioural functions are going to be designed according to the particular *thought experiment* in order to generate a high number of virtual economies and collect statistics.

The experiments generated by this *digital economic laboratory* have the properties of the digital experiments conducted on computing machines. After Fermi-Pasta-Ulam experiment in 1955 the importance of digital experiments for simulation has been recognized (Weissert, 1997). Indeed, the experiments on computing machines present “the virtue of having the postulates clearly stated” (Fermi et al., 1955, p.977) and allows to construct simulations able to offer unexpected results, as happened in the case of the Fermi-Pasta-Ulam experiment. These characteristics allows to identify digital experiment as a versatile instrument useful not only in physics but also in other research fields such as economic dynamics. For an application of the Fermi-Pasta-Ulam method to economics see Zambelli (2015).

The chapter is organized as follows: in section 2 the research topics in the *digital economic laboratory* are introduced, in section 3 the building

[28]The notion of an Algorithmic Rational Agent is inspired by the work of Kumaraswamy Velupillai. The exact phrase *algorithmically rational agent* is to be found in Velupillai (1991, p.32).

blocks of the *laboratory* are briefly presented; in section 4 the fundamental equations of the *digital economic laboratory* are presented while in section 5 a detailed description of the mathematical structure of the *laboratory* is reported.

3.2 The research topics in the laboratory

At this point it is possible to clarify what research questions can be formulated in the *digital economic laboratory*. It is worth remembering that the answers to these *research questions* through the *digital economic laboratory* do not represent the final word on the problem studied, but simply a step forward towards the clarifications of some aspects of the economic problem studied. To repeat, we endorse here Ulam's appreciation of digital experiments where "postulates [are] clearly stated" (Fermi et al., 1955, p.977). The "answer" to the question will be catch in terms of frequencies of the results with respect to the changes of some details of the experiments.

3.2.1 The relevant theoretical and applied research questions

Here is a list of the possible *research questions* for the *digital economic laboratory*. These research questions will be object of the *thought experiments* of the present thesis, but other research questions can be considered inside the *digital economic laboratory*. Please note that the tailoring of different *thought experiments* and research questions may be based on the results of the answers to other research questions.

1. **Research questions without economic policy**
   (a) **Self-adjusting economic systems**
      i. **Production prices and uniform wage rates and profit rates**: will the variables in the experiments converge towards production prices and uniform wages and profits? Will some local markets exhibit local behaviours different from global ones?
      ii. **Choice of methods**: will the methods chosen be the ones most efficient? Will the different producers belonging to the same industry adopt the same method of production? Will some local markets exhibit local behaviours different from global ones?
      iii. **Financial magnitudes. Credit and debt**: will emerge persistent credit and debt relations between workers and producers? How will evolve in time these credit and debt relations?

iv. **Distribution issues**: will the system be able to ensure an equal distribution of income between workers and producers?

(b) **Introduction of innovations**: study of the conditions that can make a producer - the innovator - to introduce a new production method (i.e., study of possible *lock-in phenomena*).

2. **Research questions with economic policy**

(a) **Introduction of an interest rate**: economic policy can be introduced to check the alternative dynamics. There are different types of economic policies, for example policies that concentrate on the revenues and costs of the financial holdings (the IOUs) through the introduction of an interest rate.

3.3 The building blocks of the laboratory

The algorithmic model is a *digital economic laboratory* in which *thought experiments* can be run in order to answer economic theoretical questions. The model can be changed according to the experimental setting, although some fundamental features remain fixed. These features are the building blocks of the *digital economic laboratory.*

3.3.1 The algorithmic rational agent (ARA)

An ARA is a set of algorithms and a collection of property rights. They are characterized by a determined location and have a specific network of relations. ARAs can be producers, producers and workers[29] or only workers. In any case they are consumers. Workers of the same enterprise are organized into trade unions whose unique task is to announce the wage. ARAs are described by behavioural functions and they buy, sell, consume, produce, work and migrate inside the system. Each individual ARA is represented by a combination of features in terms of preference-propensity-technical parameters. Each ARA takes decisions according to his characteristics and on the basis of the information set at his disposal. It is worth remembering that in the *digital economic laboratory* it is supposed that each ARA has the capability to exploit his information set. Each ARA, according to his characteristics, his information set, his location, his financial wealth, his endowment and his knowledge of the production methods at disposal (in the case of producers) is able to make decisions with respect to the producing, consuming, buying and selling of economic magnitudes. The decision process (represented by algorithms) transforms input into output. Some decision processes (such as demand functions, wage/price functions and error correction mechanisms) are behavioural functions designed according to the particular *thought experiment.* Other decisional processes (such as the choice of the method) are part instead of the *digital economic laboratory.*

It is important to emphasize that by *wealth* we mean only the accumulation of debt and credit relations. The value of the endowment is not *wealth* (despite it is for sure a source of value for the ARA) because, according to the terminology used in the *digital economic laboratory*, the *wealth* represents only the financial wealth as previously defined.

[29] In this case the producer can work only in his enterprise. This implies that producers cannot migrate.

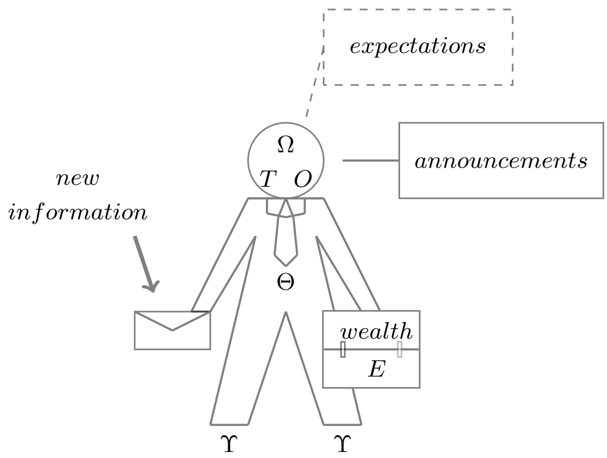


Figure 3.1: **The ARA producer**: in this picture the main characteristics of the ARA producer are summarised. What is written on his head represents what an ARA has *in his mind*, his information set (Ω), the knowledge of the methods of production (T) and his access to these methods for the production of his commodity (O). In the body, there are the preference-propensity-technical parameters of the ARA that are fixed (Θ). Under the feet each ARA is identified by his position (Υ). The dashed rectangle represents the expectations of the ARA while the not-dashed rectangle represents the effective announcements of the ARA (about consumption, labour supply, prices, wages etc.). The bag represents the financial *wealth* and endowment (E) the ARA has at disposal. The letter represents the information coming from the local market (for example, the changes in prices and wages).

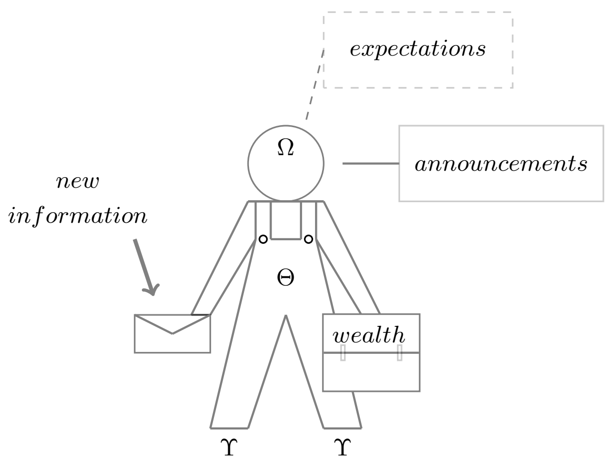


Figure 3.2: **The ARA worker**: the main characteristics of the ARA worker have been represented with the same logic of the producer. Workers do not organize production and for this reason E, T and O are not present.

3.3.2 The topology of the model

Inside the virtual market 3 different commodities are produced. Each producer produces only one commodity and more producers can produce the same commodity. Each producer needs the other commodities and labour of his workers in order to produce his commodity. Considering that an algorithmic model needs to register the time, price, quantities and the agents that are engaged in the exchange, exchanges have to be deterministic between ARAs characterized by an univocal location. The environment is represented by a set of cells positioned inside a grid. Each ARA is assigned to a particular cell. The action of the ARA located inside a particular cell depends on the state of the neighbouring cells. In details, the topology of the model has been designed through the use of the *cellular automaton* and in particular the *ring one-dimensional cellular automata* in Wolfram-von Neumann's terms (see Wolfram, 2002). The ring lattice is necessary in order to allow each ARA producer to have as neighbours two producers of a different commodity (see figure 3.3). Each ARA worker is hired by one producer. Each producer interacts with the other producers (according to the nearest neighbourhood rule), with the workers of his enterprise, and the workers hired by the neighbouring producers. At the end of each production period workers can decide to migrate to one of the neighbouring enterprises. In this way the location of each worker can change in time. Each worker interacts with the producer of the enterprise in which is working and with the other workers of the same enterprise only for the *consumer exchange phase* (see sections 3.3.7 and 3.5.10 for more details).

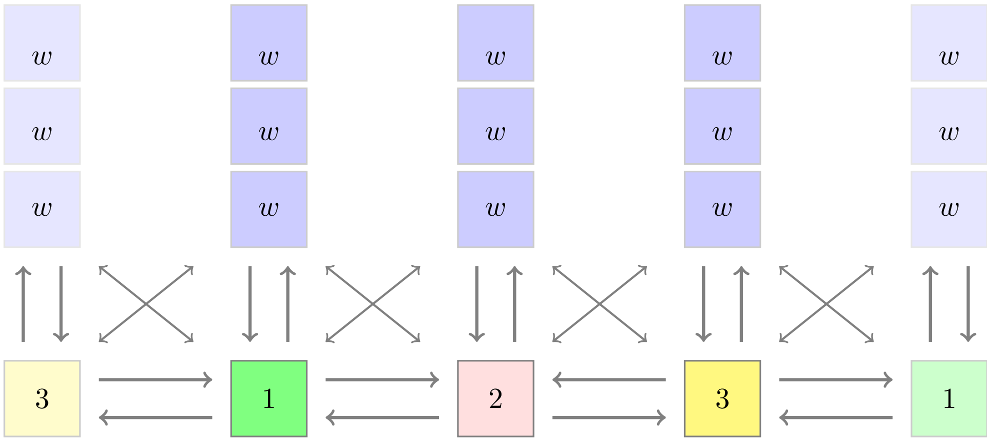


Figure 3.3: **The ARAs in a ring one-dimensional lattice**: in this example 3 producers are considered: producer 1,2 and 3. The colour of the cell identifies the commodity produced: commodity 1 is green (wheat), commodity 2 is pink (coal) commodity 3 is yellow (iron). As a consequence, producer 1 produces wheat, producer 2 produces coal and producer 3 produces iron. Each producer hires workers for the production of his commodity (cells w). In this example each producer hires 3 workers. Each producer interacts with all the other neighbouring producers, with his group of workers (which sells labour to him and buy commodities inside his local market) and the workers of the other enterprises (which represent simply potential consumers of his commodity). In order to give an idea of the *ring-lattice*, producer 1 and 3, present also on the extremes of the figure, have been represented in light colours.

3.3.3 Local markets

This fundamental structure based on 3 producers and a defined number of enrolled workers represents a fundamental local unit that can be called *local market*. It is a small, local, and interconnected economic system localized in a specific position inside the *ring-lattice* and described by its map of spatial relationships. The local market can be defined as a local system in which each producer can reproduce his commodity through labour and the commodities of the other neighbouring producers. It is possible to enlarge the model by adding more local markets producing the same 3 initial commodities. This passage from a low to a high dimension has a fundamental economic meaning. Indeed, it allows to study the effects of the aggregation of interconnected local economies producing the same commodities. Consider for example the case of 9 producers producing 3 commodities (see figure 3.4). The total number of local markets is 9, with all producers of the same commodity representing one industry.

	9	1	2	3	4	5	6	7	8	9	1
local market 1	9	1	2								
local market 2		1	2	3							
local market 3			2	3	4						
local market 4				3	4	5					
local market 5					4	5	6				
local market 6						5	6	7			
local market 7							6	7	8		
local market 8								7	8	9	
local market 9									8	9	1

Figure 3.4: **The local markets**: for a total number of producers equal to 9 producing 3 commodities.

3.3.4 Exchanges

The exchanges are bilateral and deterministic. The topology of the exchanges inside the lattice is displayed in figure 3.5. Any exchange is associated to the underwriting of an IOU denominated in a socially accepted unit of account. An IOU (*I owe you*) is a contract, a deferred payment of an exchange which is taking place now. In our framework only two types of exchange are allowed[30]: commodities versus financial contracts (IOUs) and labour employment contracts versus financial contracts (IOUs); exchanges of one type of IOUs against other types of IOUs has been excluded, because it would incorporate non-essential features of the functioning of a virtual market. An exchange between two ARAs can take place only if one has a commodity or labour to sell and the other has the necessary quantities of IOUs. Obviously, in the case in which the two ARAs do not have commodities or labour at their disposal the exchange does not take place. If instead one of the two does not have the necessary purchasing power, i.e. does not have IOUs, the exchange may still take place if and only if the two ARAs agree to underwrite a new IOU. In this case new debt and credit is generated and the commodity or labour is exchanged. In the *digital economic laboratory* it is assumed that ARAs always agree to underwrite IOUs if needed.

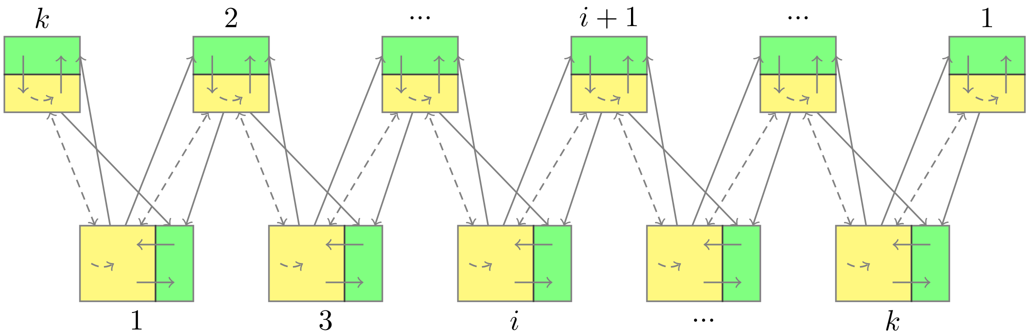


Figure 3.5: **The structure of the exchanges**: each producer i (for i=1,...,k) is represented by a yellow box while the group of workers of each enterprise is represented by a green box. Each enterprise is represented by the rectangle green/yellow. The arrows display the structure of exchanges. The dashed arrows are exchanges between producers while the line arrows are exchanges with workers. In order to produce, each producer has to exchange his commodity with the commodities of the other producers of his local market. Each producer has also to pay the workers in order to organize production. Each producer sells his commodity to the producers and workers of his local market. Each producer exchanges only with the ARAs of his local market.

[30]In this section the *consumer exchange phase* is not considered. For more details see sections 3.3.7 and 3.5.10.

3.3.5 Production, consumption and markets

When the act of consumption takes place, commodities are destroyed. When the act of production takes place, commodities and labour are transformed into new commodities. Producers can produce if they have at their disposal the necessary means of production and the necessary labour force. Technology consists of alternative methods of production of the fixed proportions type. The production is conceived as a circular process and does not consider the presence of capital or inventories i.e., simple production of commodities by means of commodities. In details, each producer has to choose the best method of production according to the prices and the wage in his local market. The method that maximizes his expected profit is chosen and the desired investment and labour demand can be computed. At the same time all ARAs workers calculate their desired labour supply. After that, the exchanges with the other producers and workers take place (production goods market and labour force market). In this phase, ARA can be rationed. The new level of IOUs can be computed. At this point all ARAs know their level of income so that they can compute their desired consumption while, and at the same time, producers transform the bought resources into new commodities (i.e., production). At this point producers decide how much of the new production to keep for the production of the next period and how much to offer inside the consumption goods market of the present production period. After the new exchanges in the consumption goods market and the new computation of the IOUs, ARAs of the same enterprise (workers and producers) exchange their consumption goods in order to reach their desired consumption basket (if not reached before). It is worth remembering that this phase does not imply the creation of new IOUs (i.e., it is a barter exchange). Finally, ARA workers decide if and where to migrate among the neighbouring enterprises.

3.3.6 Double-entry bookkeeping system and national accounting

Any exchange is associated with a registration in the economic accounts and in the balance sheets of the two ARAs involved. The nature of the generations of IOUs allows to identify the sum of IOUs as the total aggregated debt which is equal to the total aggregated credit. The sum of the values of the quantities sold and of the values of the quantities bought are by definition equal. This represents the foundations of the definition and measurement of *standard national accounting magnitudes* (that in this model are generated from the micro level data) based on double-entry accounting system. It is worth remembering that the rules of double-entry bookkeeping system are always followed and that actual budget constraints are never broken.

3.3.7 Dynamics

Each production period (that we can consider as a budget or accounting period from t to $t+1$) is subdivided into different logical phases (τ), in which ARAs take a decision with respect to a particular step in the production process. The fundamental logical steps are displayed in figure 3.6. At the beginning of the production period t, ARAs know the history of their local market and can compute the new levels of physical endowment and financial wealth.

- τ_1**: Price and wage announcement**. According to their information sets, producers (trade unions) announce their prices (wages).
- τ_2**: Exchange 1: production goods and labour**. After the computation of the desired labour supply, labour demand, and investment level, exchanges between ARAs take place on the production goods market and labour force market. The exchange implies a new computation of the total amount of IOUs.
- τ_3**: Production**. At this point ARAs produce and decide how much of the new production to devote to the consumption goods market and how much to keep for the production of the next period. All ARAs know their revenues and can calculate their consumption demands.
- τ_4**: Exchange 2: consumption goods**. ARAs announce their desired level of consumption. Exchanges between ARAs take place on the consumption goods market. The exchanges imply a new computation of the total amount of IOUs.
- τ_5**: Consumer exchange**. Given their budget constraints, ARAs of the same enterprise (workers and producer) can exchange their consumption goods in order to reach their desired consumption basket. This exchange does not imply a new computation of the total amount of IOUs because it is a barter exchange at the relative prices.
- τ_6**: Migration**. Workers decide if and in which enterprise to migrate inside their local market.

At the end of the production period, the information set of each ARA is updated: the investment and the new amount of IOUs become respectively the new level of physical endowment and the new level of financial wealth of the new production period.

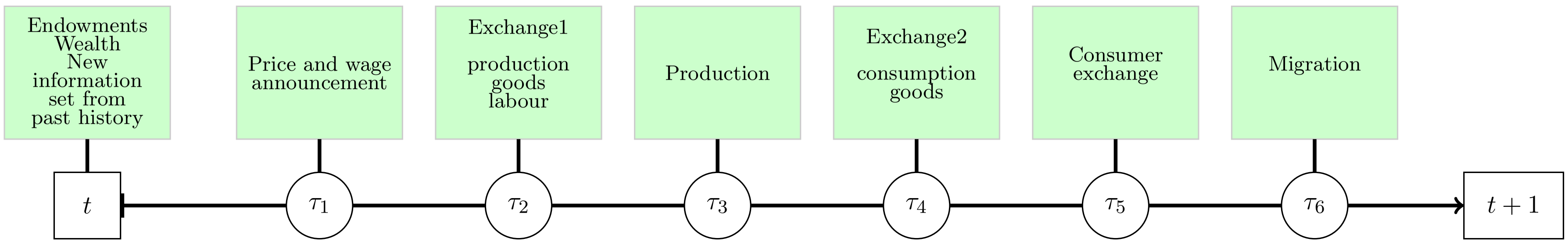


Figure 3.6: **The dynamics of the model**: the fundamental logical steps.

3.4 The structure of the digital economic laboratory: the fundamental equations

This section presents the fundamental equations[31] that characterize the *digital economic laboratory* and that are associated to each phase τ of the period t. The *digital economic laboratory* considers the presence of producers (i=1,...,k) and workers (j=1,...,K). Each producer produces only one commodity (g=1,...,n) and hires a particular number of workers. Workers can migrate in time in the neighbouring enterprises. Each producer exchanges his commodity with the other two neighbouring producers. This triple of producers represents a *local market* (s=1,...,k). For the present section, with *local market* s, we will make reference to the producer $s = i$ and the other two neighbouring producers $i-1$ and $i+1$ in growing order according to the commodity produced. Inside each local market the producer is indicated with $\bar{i}$ ($\bar{i}$=1,...,n)[32]. For example, with the *local market* s=5 we make reference to the producers 4,5,6 (in this order) and with the *local market* s=7 we make reference to the producers 7,8,6 (in this order).

3.4.1 Price and wage announcement

According to his information set, each producer announces his price p_i and each trade union announces his wage w_i that will be uniform among all the workers enrolled in the same enterprise. The vector of prices and wages of each *local market* $\boldsymbol{p}_s$ and $\boldsymbol{w}_s$ compose the matrix of the prices and wages of the whole economy $\boldsymbol{P}_t$ and $\boldsymbol{W}_t$:

$$\boldsymbol{P}_t = [\boldsymbol{p}_s]_t \ \ for \ \ s = 1, ..., k \tag{3.1}$$

$$\boldsymbol{W}_t = [\boldsymbol{w}_s]_t \ \ for \ \ s = 1, ..., k \tag{3.2}$$

3.4.2 Exchange 1: production goods and labour

After the price and wage announcement each producer i exchanges his physical endowment $\alpha_{i,t-1}b_{i,t-1}$ ($\alpha_{i,t-1}$ is the share of his commodity $b_{i,t-1}$ produced in the previous production period and devoted to the exchange in the production goods market[33]) in order to buy the other commodities and labour force that allows him to organize production. In the same way each worker j would enter the labour market at the beginning of period t endowed with his labour force L^s_j. During the period t each worker j will sell

[31] For a detailed mathematical description of these phases and all the equations see next section. See in the Appendix section 10.6 the tables 10.6, 10.7 and 10.8 for the description of the rules used for the matrix notation.

[32] For the whole system producers are, as already said, k (with i=1,...,k).

[33] For this section the hypothesis is that $\eta_{i,t-1}$=0. See for more details section 3.5.5 and section 3.5.8 about investment decision.

a fraction of his endowment in terms of labour force, so as to be able to consume during period t. The computation of the desired quantities of each producer i[34] depends on the methods of production at disposal.

Each commodity g is produced with one method of production linking the *means of production* with the *output*. Considering that each producer can choose among a set of possible alternative methods for the production of his commodity g, the method z_g (for z_g=1,...,m_g) is given by:

$$\mathbb{a}_{\bar{i}1}^{z_g}, \mathbb{a}_{\bar{i}2}^{z_g}, \ldots, \mathbb{a}_{\bar{i}g}^{z_g}, \ldots \mathbb{a}_{\bar{i},n-1}^{z_g}, \mathbb{a}_{\bar{i}n}^{z_g}, \mathbb{l}_{\bar{i}}^{z_g} \xrightarrow{\textit{production}} \mathbb{b}_{\bar{i}}^{z_g} \tag{3.3}$$

where $\mathbb{l}_{\bar{i}}$ is the necessary labour, $\mathbb{a}_{\bar{i}g}$ is the quantity of the commodity g means of production of the enterprise $\bar{i}$ and $\mathbb{b}_{\bar{i}}$ is the the quantity produced by the enterprise $\bar{i}$. Each producer $\bar{i}$ will choose the method z_g that allows him to maximize his expected profit given the market prices and the wage in his *local market*.

After the exchange in the production goods market and the labour force market (where rationing is possible) each producer has at disposal the quantities of the commodities q and the labour force q^L that allows him to organize the production of his commodity g:

$$\alpha_{i,t-1}b_{i,t-1} \xrightarrow{\textit{exchange}} [q_{i1}, q_{i2}, \ldots, q_{ig}, \ldots q_{i,n-1}, q_{in}, {q^L}_i]_t \tag{3.4}$$

Obviously, the quantity q_{it}^L bought by the producer i corresponds to the sum of the labour force sold by each workers j (q_{jt}^L) enrolled in the enterprise i.

3.4.3 Production

According to the purchased quantities, each producer can organize the production of his commodity g. The quantities effectively used and produced correspond to[35]:

$$[a_{i1}, a_{i2}, \ldots, a_{ig}, \ldots a_{i,n-1}, a_{in}, l_i]_t \xrightarrow{\textit{production}} b_{it} \tag{3.5}$$

Each producer i has to choose the share of production to devote to the exchange in the consumption goods market $\beta_{it}b_{it}$.

[34]Demand of commodities in the production goods market and demand of labour force in the labour force market.

[35]Remember that it is possible to use for production not only the purchased commodities, but also a share of the commodity produced in the previous production period $\eta_{i,t-1}b_{i,t-1}$, called $\tilde{e}_{it}$, stored as physical endowment for production and not devoted to exchange in the production goods market. For more details, see equation 3.26 and subsection 3.5.8 about investment decision.

3.4.4 Exchange 2: consumption goods

Each producer decides the share β_{it} of the production b_{it} to devote to exchange in the consumption goods market. The share of production that will not be exchanged in the consumption goods market α_{it} is the physical endowment for the production in the next period $t+1$. Each ARA (worker and producer) arrives to the consumption goods market with his demand of consumption goods (the vector $\boldsymbol{c}^*=[c_g]^*$). Each ARA can be rationed in the consumption goods market so that the effective consumption is equal to $\boldsymbol{c}=[c_g]$. Producers will consume also the part of commodities bought but not used for production $[q_{ig}-a_{ig}]_t$ for g=1,...,n.

3.4.5 Revenues, expenditures and financial balances

The consumption vectors are updated according to the barter exchanges in phase τ_5. Workers decide to migrate and can change their location in phase τ_6. At the end of the period t the *national accounting magnitudes* can be computed. The difference between the revenues and the expenditures of each ARA represents the variation in his financial position. The *local market* in which producer i is present is denoted by $s(i)$, the *local market* in which worker j is present is denoted by $s(j)$[36].

$$revenues_{it} = \alpha_{i,t-1}b_{i,t-1}p_{it} + \beta_{it}b_{it}p_{it} \tag{3.6}$$

$$expenditures_{it} = \bar{\boldsymbol{q}}_{it}\boldsymbol{p}_{s(i)t} + q^L_{it}w_{it} + \bar{\boldsymbol{c}}_{it}\boldsymbol{p}_{s(i)t} \tag{3.7}$$

$$F_{it} = revenues_{it} - expenditures_{it} \tag{3.8}$$

$$revenues_{jt} = q^L_{jt}w_{it} \tag{3.9}$$

$$expenditures_{jt} = \bar{\boldsymbol{c}}_{jt}\boldsymbol{p}_{s(j)t} \tag{3.10}$$

$$F_{jt} = revenues_{jt} - expenditures_{jt} \tag{3.11}$$

At this point the financial wealth (the sum in time of the financial positions) is updated. Producers and workers that did not have sufficient purchasing power have underwritten new IOUs that represent negative values of F:

$$wealth_{it} = wealth_{i,t-1} + F_{it} \tag{3.12}$$

[36] With reference to the equation of workers's revenues, the quantity of labour sold by worker j is multiplied for the wage of the enterprise i in which the worker j is enrolled. Remember that it is possible to imagine that also producers work inside their enterprise. In this case the revenues from work should be added in eq.3.6 and considered in the computation of q^L_{it}. In these fundamental equations, unosold commodities have not been considered. See eq.3.91 for more details.

$$wealth_{jt} = wealth_{j,t-1} + F_{jt} \tag{3.13}$$

The level of profits depends on the difference between revenues and expenditures of each enterprise. In the case in which expenditures have been greater than revenues, the loss corresponds to a negative variation in the financial position of the enterprise[37].

$$revenues_{it}^{ent} = \alpha_{i,t-1} b_{i,t-1} p_{it} + \beta_{it} b_{it} p_{it} \tag{3.14}$$

$$expenditures_{it}^{ent} = \bar{\boldsymbol{q}}_{it} \boldsymbol{p}_{s(i)t} + q_{it}^{L} w_{it} \tag{3.15}$$

$$F_{it}^{ent} = revenues_{it}^{ent} - expenditures_{it}^{ent} = profit_{it} \tag{3.16}$$

[37]For the enterprise the superscript is *ent*.

3.5 A detailed description of the mathematical structure of the laboratory

In this section, a detailed description of the mathematical structure and of the equations that are at the basis of the *digital economic laboratory* is presented[38]. For the comprehension of the *digital economic laboratory* the equations presented in the previous section are sufficient. For those not interested in the mathematical details it is possible to skip this section.

3.5.1 The ARAs, the lattice and the local markets

The system considers the presence of producers and workers. Both of them must be recognized by an identification number and located on a lattice. Each producer i (for i=1,...,k) produces only one commodity (g=1,...,n) and represents one enterprise. Each worker j (for j=1,...,K) is hired by one producer but can migrate to other enterprises in time. Each producer hires a variable number of workers $\tilde{k}$, so that a number of hired workers $\tilde{k}_i$ is associated to each producer i. Obviously each producer can hire a maximum number of workers equal to K and it must be true that $\sum_{i=1}^{k} \tilde{k}_i = K$.

3.5.2 The ARAs workers and migration inside the lattice

Consider as an example the case of 3 producers and 9 workers with 3 workers enrolled for each enterprise at time t (k=3 and K=9). At time t it is possible to have a particular allocation of workers among the enterprises. After one production period, workers can migrate so that their location change as in figure 3.7. This graphical representation can be summarised into the matrix $\boldsymbol{M}$ where each worker j (first row of the matrix $\boldsymbol{M}$) is associated to the enterprise i where he is hired (second row of the matrix $\boldsymbol{M}$). The matrix $\boldsymbol{M}$(1:2,1:K) corresponds[39], for the example of the figure 3.7, for time t and time t+1:

$$\boldsymbol{M}(1:2,1:K)_t = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \end{bmatrix}$$

$$\boldsymbol{M}(1:2,1:K)_{t+1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 3 & 2 & 1 & 1 & 1 & 3 & 3 \end{bmatrix}$$

[38]See in the Appendix section 10.6 the tables 10.6, 10.7 and 10.8 for the description of the rules used for the matrix notation.

[39]We have followed the typical notation present in MATLAB®: $\boldsymbol{M}$(1:2,1:K) means that the we are considering the structure of matrix $\boldsymbol{M}$ where the rows of the matrix $\boldsymbol{M}$ are from 1 to 2 and the columns are from 1 to K.

For the case with 9 producers and 81 workers (k=9 and K=81, figure 3.8), with 9 workers enrolled for each enterprise at time t, the matrices are:

$$\boldsymbol{M}_t = \begin{bmatrix} 1 & 2 & ... & 9 & 10 & ... & 21 & ... & 45 & 46 & ... & 81 \\ 1 & 1 & ... & 1 & 2 & ... & 3 & ... & 5 & 6 & ... & 9 \end{bmatrix}$$

$$\boldsymbol{M}_{t+1} = \begin{bmatrix} 1 & ... & 17 & 18 & ... & 45 & ... & 62 & 63 & ... & 80 & 81 \\ 1 & ... & 2 & 3 & ... & 6 & ... & 7 & 6 & ... & 9 & 1 \end{bmatrix}$$

In general, the matrix is (denoting by i^* the enterprise where the worker decides to migrate):

$$\boldsymbol{M}(1,j)_t = j \quad for \quad j = 1, ..., K \tag{3.17}$$

$$\boldsymbol{M}(2,j)_t = i^* \quad for \quad j = 1, ..., K \quad and \quad i^* \in [1,k] \tag{3.18}$$

The matrix $\boldsymbol{M}$ represents the description of the position of each worker in each period in time. It is possible also to compute for each time t the matrix $\boldsymbol{\chi}_{it}$ of the workers hired in each enterprise i. The matrix $\boldsymbol{\chi}_{it}(1:\tilde{k}_{it},1)$ corresponds, for the case of the previous example in figure 3.7, for time t and time t+1:

$$\boldsymbol{\chi}_{1t} = [1 \;\; 2 \;\; 3]^T \quad \boldsymbol{\chi}_{2t} = [4 \;\; 5 \;\; 6]^T \quad \boldsymbol{\chi}_{3t} = [7 \;\; 8 \;\; 9]^T \quad with \quad \tilde{k}_{1t,2t,3t} = 3$$

$$\boldsymbol{\chi}_{1,t+1} = [2 \;\; 5 \;\; 6 \;\; 7]^T \quad \boldsymbol{\chi}_{2,t+1} = [1 \;\; 4]^T \quad \boldsymbol{\chi}_{3,t+1} = [3 \;\; 8 \;\; 9]^T$$
$$with \quad \tilde{k}_{1,t+1} = 4 \qquad \tilde{k}_{2,t+1} = 2 \qquad \tilde{k}_{3,t+1} = 3$$

In general the matrix is:

$$\boldsymbol{\chi}_{it}(1:\tilde{k}_{it},1) = [\; j \;]_{it} \quad \forall\, j \mid \boldsymbol{M}(2,j)_t = i \tag{3.19}$$

In the *thought experiment* it is possible to consider the case in which workers can migrate by constructing a behavioural function that describes how matrix $\boldsymbol{M}_t$ changes in time. It is possible also to consider the case in which workers do not migrate simply imposing $\boldsymbol{M}_{t+1}$=$\boldsymbol{M}_t$. In each case, matrix $\boldsymbol{M}_t$ identifies each worker's location inside the lattice.

It is worth remembering that workers can migrate only in the neighbouring enterprises. This means that in time workers can arrive to work also in an enterprise which is located very far from the original enterprise simply by moving through neighbouring enterprises through time.

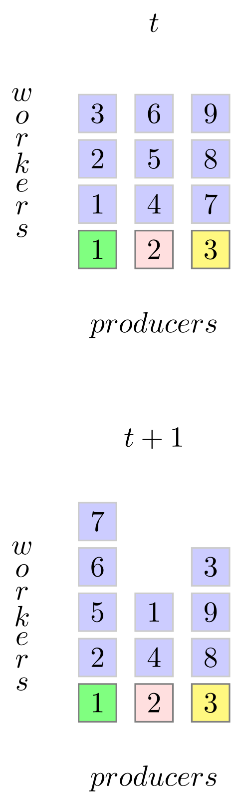


Figure 3.7: **The lattice and migration**: in the figure there is an example of the lattice and the migration of workers. The 3 coloured squares represent the producers (k=3; producer 1 is pink, producer 2 is yellow, producer 3 is green) while the violet squares represent the workers. Each producer can hire a maximum number of workers equal to K (K=9). At time t each enterprise hired 3 workers. Workers 1,2,3 are hired by producer 1, workers 4,5,6 are hired by producer 2 and workers 7,8,9 are hired by producer 3. At time t+1 workers migrate so that workers 2,5,6,7 are hired by producer 1; workers 1 and 4 are hired by producer 2 while workers 8,9,3 are hired by producer 3. Producer 1 hires 4 workers, producer 2 hires 2 workers while producer 2 hires 3 workers. Workers can migrate in the neighbouring enterprises. Thanks to the ring structure of the lattice workers enrolled in enterprise 1 (or 3) can migrate to the enterprise 3 (or 1).

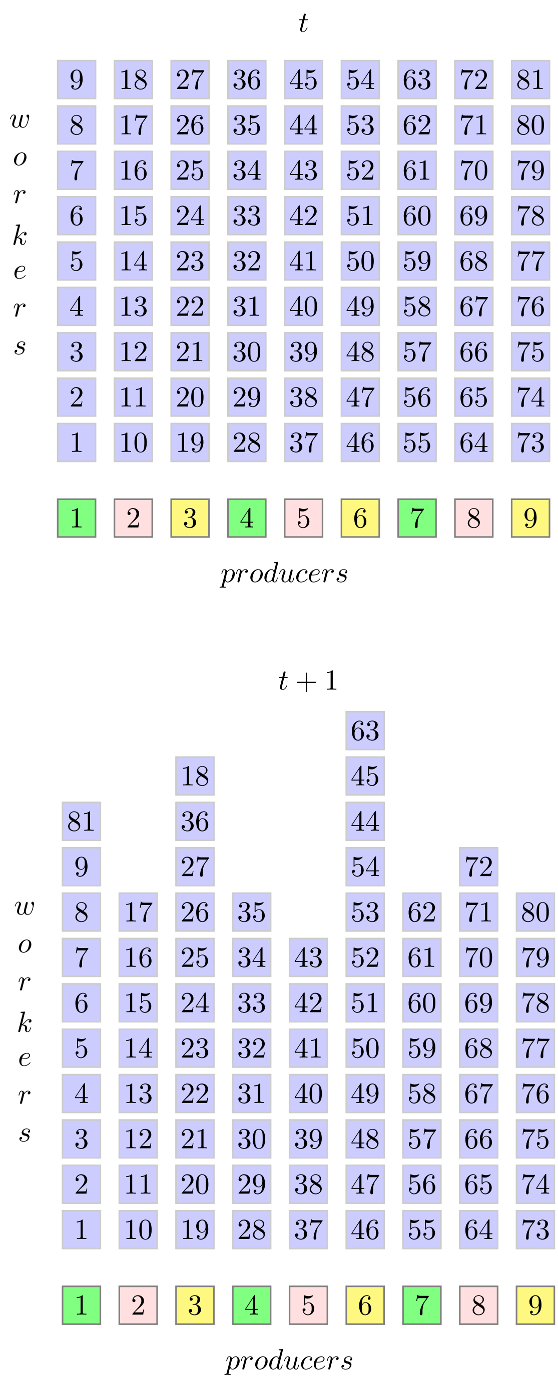


Figure 3.8: **The migration for the case with 9 producers**: in the figure there is an example of the lattice and the migration of workers for the case of 9 producers and 81 workers. The rules are the same of the previous figure. Thanks to the ring structure of the lattice workers enrolled in enterprise 1 (or 9) can migrate to the enterprise 9 (or 1).

3.5.3 The ARAs producers and the local markets

In the *digital economic laboratory* it is supposed that each producer i produces only one commodity g. In order to know which commodity is produced by each producer simply calculate $\iota(i)=mod(i/n)$, for $\iota(i)=1,...,n$ considering that if $i=\mu n$ for $\mu \in \mathbb{N}_0$ (natural number different from zero), $\iota(i)=n$. For example, producer 7 produces commodity 1 ($i=7$: $\iota(7)=mod(7/3)=1$); while producer 27 produces commodity 3 ($\iota(27)=3$ because in this case $i=\mu n$, 27 is a multiple of 3)[40].

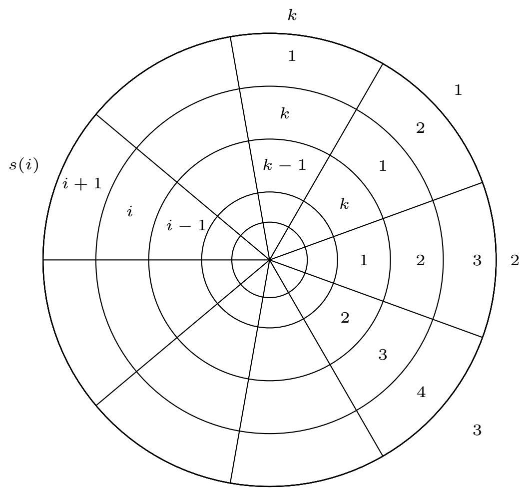
The local market is the fundamental local unit of the *digital economic laboratory*. In each local market are present n producers $\bar{i}$ (for $\bar{i}=1,...,n$) with their enrolled workers. The number of local markets is equal to the number of producers. The local market s (for $s=1,...,k$) is defined as a triple of producers where the producer $i=s$ produces commodity $\iota(i)$ and exchange his commodity with the other two neighbouring producers of his local market. The local market s of producer i can be indicated also as $s(i)$ so that is always verified that $s(i)=i$[41]. Local markets result, as a consequence, interconnected because each producer i is present in n local markets. Indeed, he sells his commodity in his local market but, as buyer of other commodities, he is present also in the local market of the other two neighbouring producers[42].

For a summary consider the figure 3.9, which represents a general graphical representation of the organization of the local markets inside the *ring one-dimensional lattice* and the relationships between the producers (for $i=1,...,k$ and $g=1,...,n$ with $n=3$).

[40] Remember that *mod* is the abbreviation of modulo operation. The result of the operation a *mod* b is the remainder of the Euclidean division of a by b. In other words, it finds the remainder after division of one number a by another number b.

[41] For example, the local market of producer $i=5$ is the local market $s=5$ because $s(5)=5$. This local market is composed by producer 5 that exchanges his commodity with the neighbouring producers 4 and 6.

[42] For example, producer $i=5$ is also present in the local markets $s=4$ and $s=6$ where he is a buyer of commodity 1 and commodity 3 respectively.



$$s(i) = i$$

$$\iota(i) = mod(i/n)$$

$$\tilde{\mathbf{\Psi}}_s = [\tilde{\psi}]_s = [i-1, \;\; i, \;\; i+1]$$

$$\mathbf{\Psi}_s = [\psi]_s \quad \psi_{gs} = \tilde{\psi}_{[\iota(\tilde{\psi}_{\tilde{g}s})=g]s}$$

$$for \;\; g = 1, ..., n \quad \tilde{g} = 1, ..., n$$
$$for \;\; i = 1, ..., k$$

$i=k$	$i=1$	$i=2$	$\cdots$	$i=1$
$\iota(k)$	$\iota(1)$	$\iota(2)$	$\iota(...)$	$\iota(1)$

$$for \;\; i = 1, ..., k$$
$$for \;\; g = 1, ..., n$$

Figure 3.9: **The local markets**: the general structure of local markets for ARAs positioned inside a *ring one-dimensional lattice - nearest neighbour rule* - for the production of n commodities with k producers. In this figure it has been considered the case of the production of 3 commodities (n=3), which is the case considered in the *digital economic laboratory.*

According to figure 3.9, $\tilde{\mathbf{\Psi}}$ is the matrix of the components of all local markets. $\tilde{\mathbf{\Psi}}_{\boldsymbol{s}}$ is the vector inside the matrix $\tilde{\mathbf{\Psi}}$ of the producers of the local market s (the vector $\mathbf{\Psi}_s$ is in order according to the commodity produced).

$$\tilde{\mathbf{\Psi}} = [\tilde{\mathbf{\Psi}}_{\boldsymbol{s}}] \;\; for \;\; s = 1, ..., k \tag{3.20}$$

$$\tilde{\mathbf{\Psi}}_{\boldsymbol{s}} = [\tilde{\psi_{\bar{i}}}]_s \;\; for \;\; \bar{i} = 1, ..., n \tag{3.21}$$

For example, for the case s=9 (9 local markets) the matrix $\tilde{\mathbf{\Psi}} = [\tilde{\psi_{\bar{i}s}}]$ corresponds to:

$$\tilde{\mathbf{\Psi}} = \left[\; \tilde{\mathbf{\Psi}}_1 \quad \tilde{\mathbf{\Psi}}_2 \quad \tilde{\mathbf{\Psi}}_3 \quad \tilde{\mathbf{\Psi}}_4 \quad \tilde{\mathbf{\Psi}}_5 \quad \tilde{\mathbf{\Psi}}_6 \quad \tilde{\mathbf{\Psi}}_7 \quad \tilde{\mathbf{\Psi}}_8 \quad \tilde{\mathbf{\Psi}}_9 \; \right]$$

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix} 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{bmatrix}$$

The matrix $\mathbf{\Psi} = [\psi_{\bar{i}s}]$ corresponds to:

$$\mathbf{\Psi} = \left[\; \mathbf{\Psi}_1 \quad \mathbf{\Psi}_2 \quad \mathbf{\Psi}_3 \quad \mathbf{\Psi}_4 \quad \mathbf{\Psi}_5 \quad \mathbf{\Psi}_6 \quad \mathbf{\Psi}_7 \quad \mathbf{\Psi}_8 \quad \mathbf{\Psi}_9 \; \right]$$

$$\mathbf{\Psi} = \begin{bmatrix} 1 & 1 & 4 & 4 & 4 & 7 & 7 & 7 & 1 \\ 2 & 2 & 2 & 5 & 5 & 5 & 8 & 8 & 8 \\ 9 & 3 & 3 & 3 & 6 & 6 & 6 & 9 & 9 \end{bmatrix}$$

Also the prices could make reference to the single producer i (p_i) or to the local market s ($\boldsymbol{p}_s$). Following the rules explained above, with the column vector $\boldsymbol{p}_s$=[p_{Ψ_s}] are considered the prices of the producers inside the local market s. For an example of the topology structure in the case of 9 producers (k=9), see in the Appendix section 10.1.

3.5.4 The description of the ARAs

An ARA is a set of algorithms. The ARA is the virtual agent that takes decisions in the *digital economic laboratory.* Each ARA has his particular characterization, which influences the way in which he takes decisions. This characterization is represented by a combination of features in terms of preference-propensity-technical parameters. These parameters are collected in the matrix $\mathbf{\Theta}$, for each ARA producer i and worker j.

In the *digital economic laboratory* the particular *thought experiment* determines the parameters of the matrix $\mathbf{\Theta}$. Each ARA takes decisions according to his characteristics and on the basis of the information set at his disposal. It is worth remembering that in the *digital economic laboratory* it is supposed that each ARA has the capability to exploit this information.

Each ARA producer and worker has his information set, which will be respectively $\mathbf{\Omega}_{it}$ and $\mathbf{\Omega}_{jt}$ (for i=1,...,k and j=1,...,K) in a particular period t. These information sets are represented by the multiple dynamic array $\boldsymbol{H}_{s(i,1:t-1)}$ ($\boldsymbol{H}_{s(j,1:t-1)}$ for workers[43]): a collection of matrices with all the past values of the variables that make reference to the local market s in which the ARA is present.

$$\mathbf{\Omega_{it}} = [\boldsymbol{H}_{s(i,1:t-1)}]_{it} \tag{3.22}$$

$$\mathbf{\Omega_{jt}} = [\boldsymbol{H}_{s(j,1:t-1)}]_{jt} \tag{3.23}$$

Each ARA has resources at disposal: $wealth_{it}$ ($wealth_{jt}$ for workers) is the financial wealth and e_{it} represents the physical endowment. ARAs have also at their disposal an endowment in terms of potential labour supply: L^{sp} for producers and L^{sw} for workers. $\mathbf{\Upsilon}_{it}$ ($\mathbf{\Upsilon}_{jt}$ for workers) represents the location of the ARA. $\boldsymbol{T}_{it}$ is the technological set (it is supposed in general to be equal for all producers) while $\boldsymbol{O}_{it}$ is a matrix that accounts for which

[43]It is important to note that the information set of a particular trade union corresponds to the information sets of all the workers enrolled in the relative enterprise in that moment. Remember that with the notation 1:t-1 we mean from time t=1 to time t-1.

methods of production (present inside the technological set) are accessible to the producer $i$at time t[44]. For the producer, the matrix $\boldsymbol{\Upsilon}_{it} = [\boldsymbol{\Psi}, \boldsymbol{\chi}]_{s(i)t}$ contains two vectors: $\boldsymbol{\Psi}_{s(i)}$ is the vector of the producers present in the local market of the producer i, the vector $\boldsymbol{\chi}_{s(i)t}$ indicates the workers enrolled in his enterprise (see section 3.5.2). For workers, $\boldsymbol{\Upsilon}_{jt}$ contains the location of the worker j, $\boldsymbol{\Upsilon}_{jt} = [\boldsymbol{M}(:, j)]_{jt}$.

Each ARA, according to his characteristics, his information set, his location, his endowment, his financial wealth and his knowledge of the production methods at disposal (in the case of producers) is able to make decisions with respect to the producing, consuming, buying and selling of economic magnitudes. These decisions have as output the updating of his information set with the new variables that have been modified by his decisions during the period t ($\boldsymbol{H}_{it}$ or $\boldsymbol{H}_{jt}$):

$$\boldsymbol{H}_{it} = \mathfrak{F}_{ARA_{it}}(\boldsymbol{\Omega}_{it},\ \ \boldsymbol{\Theta}_{it},\ \ wealth_{it},\ \ e_{it},\ \ L^{sp},\ \ \boldsymbol{\Upsilon}_{it},\ \ \boldsymbol{T}_{it},\ \ \boldsymbol{O}_{it}) \tag{3.24}$$

$$\boldsymbol{H}_{jt} = \mathfrak{F}_{ARA_{jt}}(\boldsymbol{\Omega}_{jt},\ \ \boldsymbol{\Theta}_{jt},\ \ wealth_{jt},\ \ L^{sw},\ \ \boldsymbol{\Upsilon}_{jt}) \tag{3.25}$$

$\boldsymbol{H}_{it}$ (or $\boldsymbol{H}_{jt}$ for workers) is a multiple dynamic array containing all the variables that the ARA influences with his decisions (or that influence him) in a particular period t. These variables are the results of the decision process. The decision process (represented by algorithms) transforms input into output and it is represented by the function $\mathfrak{F}_{ARA}$. Some decisional processes (such as demand function, wage/price function, expectations and error correction mechanisms) are behavioural functions designed according to the particular *thought experiment*. Other decisional processes (such as the choice of methods) is part of the *digital economic laboratory*.

[44]For a detailed description of the structure of the technological set and the access to it see section 3.5.6.

3.5.5 The initial conditions

Endowments of ARA producers: the commodities

At the beginning of each period t, each producer i has at disposal an initial endowment e_{it}. In the *digital economic laboratory* it is supposed that each ARA producer produces only one commodity and has as endowment only his commodity. This endowment is defined in real terms and represents the share $\gamma_{i,t-1}$ of the previous production $b_{i,t-1}$ of his commodity $\iota(i)$ devoted to the new production so that e_{it}=$\gamma_{i,t-1}b_{i,t-1}$. Despite all the endowment is devoted to production, not all this quantity is devoted to the exchanges on the production goods market. Indeed, each producer keeps part of his endowment $\tilde{e}_{it}$=$\eta_{i,t-1}b_{i,t-1}$ (always for production) and exchanges the rest s_{it}^{pgm}=$\alpha_{i,t-1}b_{i,t-1}$ in order to have the other commodities necessary for the reproduction of his commodity. As a consequence, the supply of the commodity devoted to exchange in the production goods market for each producer correspond to s_{it}^{pgm}[45]. The endowment not devoted to exchange corresponds to $\tilde{e}_{it}$ for each producer. Remember that γ_{it}=η_{it}+α_{it} for each production period.

$$\boldsymbol{s}_t^{pgm} = [s_i^{pgm}]_t \quad s_{it}^{pgm} = [e - \tilde{e}]_{it} \tag{3.26}$$

Endowments of ARA workers: potential labour supply

Workers (and producers if they decide to work) have at their disposal an endowment in terms of potential labour supply. It is possible to imagine this endowment as a parameter, which correspond to the maximum number of hours to devote to work each day L^{sw} (for workers) and L^{sp} (for producers). Each ARA will be able to calculate his desired labour supply (L_{it}^{sp*} for each producer i=1,...,k that decides to work and L_{jt}^{sw*} for each worker j=1,...,K) that should be necessarily equal or inferior to L^{sp} and L^{sw} (respectively)[46].

These quantities are considered given in the mathematical description of the *digital economic laboratory* because they will be designed according to the particular *thought experiment* and are determined by the behavioural function.

[45] The superscript *pgm* stands for *production goods market*. It is important in order to distinguish between production goods market and consumption goods market, whose superscript will be *cgm*.

[46] As it is possible to note from table 10.6 the general rule used is to indicate single values with small letters. The use of the capital letter has been preferred for the case of aggregate labour demand and labour supply (inside the labour force market) in order to make clear the distinction with respect to the labour used by the producers.

Wealth of ARA producers and workers

Each producer and worker has also an initial financial wealth at his disposal: respectively $wealth_{it}$ and $wealth_{jt}$. Also the enterprise considers his wealth as $wealth_{it}^{ent}$. The level of wealth depends on past history[47]. The total amount of wealth of enterprises, producers and workers is equal to:

$$\sum_{i=1}^{k} wealth_{it}^{ent} = wealth_{t}^{ent} \quad \sum_{i=1}^{k} wealth_{it} = wealth_{t}^{p} \tag{3.27}$$
$$\sum_{j=1}^{K} wealth_{jt} = wealth_{t}^{w}$$

For the sake of accounting consistency, the sum of the total wealth of producers[48] and workers is always equal to zero. The wealth represents the accumulation of debt and credit positions (i.e., the negative or positive difference between revenues and expenditures for each production period). Remember that by *wealth* we mean only the accumulation of debt and credit relations. The value of the endowment is not *wealth* (despite it is for sure a source of value for the ARA) because, according to the terminology used in the *digital economic laboratory*, the *wealth* represents only the financial wealth as previously defined.

3.5.6 The announcement of prices and wages: phase τ_1

According to the information sets, each producer will announce his price p_{it}. This price cannot be changed after announcement and it is the same for the whole production period[49]. Also wages are announced in the same way by trade unions and cannot be changed in the production period. The wage w_{it} is the same for all the workers enrolled in the same enterprise. Trade unions represent an aggregation of all the workers enrolled by the same enterprise. This means that in the *digital economic laboratory* there is a number of trade unions equal to the number of enterprises. It is worth remembering that the only action of trade unions is the wage announcement. This decision is taken on the basis of the aggregation of the information set $\mathbf{\Omega}_{jt}$ of the workers of the same trade union. The prices and wages announced by each producer and trade union are represented by the vectors:

$$\mathbf{p}_t = [p_i]_t \;\; and \;\; \mathbf{w}_t = [w_i]_t \;\; for \;\; i = 1, ..., k \tag{3.28}$$

[47] At the beginning of the first production period the financial wealth of each ARA is equal to zero.

[48] That incorporates the total wealth of enterprises.

[49] It is worth remembering that the price announced by each producer is unique for all the other producers.

The triple of prices and wages considered in each local market are represented by the vectors $\mathbf{p}_s$ and $\mathbf{w}_s$ while for the whole economy we have to consider the matrix $\mathbf{P}_t$ and $\mathbf{W}_t$:

$$\mathbf{P}_t = [\boldsymbol{p}_s]_t \ \ and \ \ \mathbf{W}_t = [\boldsymbol{w}_s]_t \ \ for \ \ s = 1, ..., k \tag{3.29}$$

$$\mathbf{p}_{st} = [p_{\Psi_s}]_t \ \ and \ \ \mathbf{w}_{st} = [w_{\Psi_s}]_t \ \ for \ \ s = 1, ..., k \tag{3.30}$$

These matrices $\boldsymbol{P}_t$ and $\boldsymbol{W}_t$ are considered given in the mathematical description of the *digital economic laboratory* because they will be designed according to the particular *thought experiment* and depend on the behavioural function.

Computation of the desired labour supply

After the announcement of prices and wages, ARAs workers and ARAs producers (if they decide to work) can decide their levels of labour supply L^{sw*}_{jt} (for j=1,...,K) and L^{sp*}_{it} (for i=1,...,k) respectively. These quantities will be announced in the labour force market. These values are considered given in the mathematical description of the *digital economic laboratory* because they will be designed according to the particular *thought experiment* and depend on the behavioural function.

Technology and the mechanism of the choice of the method

After the announcement of prices and wages ARAs producers can compute their desired quantity of commodities and labour. The exchanges happen inside each local market. As already said, each local market is a local unit in which each producer $\bar{i}$ ($\bar{i}$=1,...,n)[50] produces only one commodity g (g=1,...,n) through the use of the labour force bought in the labour force market, his endowment, and the commodities bought from the neighbouring producers (i.e., production of commodities by means of commodities without inventories or fixed capital). In each local market, the technological input requirements in terms of commodities and labour are represented by the matrices $\mathbb{A}$ and $\mathbb{L}$, while the relative level of the possible output is represented by the matrix $\mathbb{B}$. In particular, consider as in Sraffa (1960)[51], $\mathbb{A}$ and $\mathbb{B}$ square matrices of order n: $\mathbb{A}$=[$\mathbb{a}_{\bar{i}g}$] and $\mathbb{B}$=[$\mathbb{b}_{\bar{i}g}$], with $\bar{i}$,g=1,...,n. $\mathbb{A}$=[$\mathbb{a}_{\bar{i}g}$] is the matrix of the means (methods) of production (semi-positive and indecomposable), where $\mathbb{a}_{\bar{i}g}$ is the quantity of the commodity g means

[50] It is worth remembering that the producer $\bar{i}$ for $\bar{i}$=1,...,n is the producer of a particular local market s. For the whole system producers are, as already said, k (with i=1,...,k). See figure 3.9.

[51] See also Zambelli (2004) for the matrix notation.

of production of the enterprise $\bar{i}$. $\mathbb{B}$=[$\mathbb{b}_{\bar{i}g}$] is the production matrix (diagonal and semipositive definite), where $\mathbb{b}_{\bar{i}g}$ is the quantity of the commodity g produced by the enterprise $\bar{i}$. Each producer (enterprise) $\bar{i}$ produces only one commodity g, mean of production for the other producers. $\mathbb{L}$=[$\mathbb{l}_{\bar{i}}$] is the labour vector and $\mathbb{l}_{\bar{i}}$ indicates the amount of labour required by the producer $\bar{i}$.

$$\mathbb{A}(1:n,1:n) = \begin{bmatrix} \mathbb{a}_{11} & \dots & \mathbb{a}_{1n} \\ \mathbb{a}_{21} & \dots & \mathbb{a}_{2n} \\ \vdots & \vdots & \vdots \\ \mathbb{a}_{n1} & \dots & \mathbb{a}_{nn} \end{bmatrix} \tag{3.31}$$

$$\mathbb{L}(n,1) = \begin{bmatrix} \mathbb{l}_1 \\ \mathbb{l}_2 \\ \vdots \\ \mathbb{l}_n \end{bmatrix} \tag{3.32}$$

$$\mathbb{B}(1:n,1:n) = \begin{bmatrix} \mathbb{b}_{11} & \dots & \mathbb{b}_{1n} \\ \mathbb{b}_{21} & \dots & \mathbb{b}_{2n} \\ \vdots & \vdots & \vdots \\ \mathbb{b}_{n1} & \dots & \mathbb{b}_{nn} \end{bmatrix} \tag{3.33}$$

It is supposed that each ARA producer produces only one commodity[52] and has different alternative methods z_g at his disposal for the production of his commodity (z_g=1,...,m_g). This implies that the producers of each local market will choose a particular combination of methods that will compose their matrices $\mathbb{A}$, $\mathbb{L}$ and $\mathbb{B}$. Remember that if the number of the alternative methods is m_g, m_g^3 possible alternative configurations of the matrices $\mathbb{A}$, $\mathbb{L}$ and $\mathbb{B}$ exist.

These alternative methods, for all the commodities, are organized in an input-output multidimensional matrix $\boldsymbol{T}$, which is supposed to be known by all the producers of the whole virtual economy and constant in time[53]:

[52]The Sraffian case of joint production is not considered so that $b_{\bar{i}g} = 0$ for each $g{\neq}\iota(\bar{i})$ and the diagonal matrix of $\mathbb{B}$ can be considered in the subsequent equations.

[53]Remember that it is possible to imagine the presence of some special methods that allow to produce one commodity with the use of only one other commodity and labour, only two commodities and labour or only labour (with n=3). These methods collected in the matrix $\tilde{\boldsymbol{T}}$ (if all combinations are considered these mew methods are in total 7 with n=3 and are embedded in matrix $\boldsymbol{T}$) will be obviously more expensive than the other methods and can became useful for the reuse of residuals from production. For more details, see section 3.5.8 and in the Appendix figure 10.2.

$$
\boldsymbol{T}(1:n, 1:(n+2), 1:m_g) = \begin{bmatrix} \mathbb{a}_{11}^{z_g} & \dots & \mathbb{a}_{1n}^{z_g} & \mathbb{l}_1^{z_g} & \mathbb{b}_1^{z_g} \\ \mathbb{a}_{21}^{z_g} & \dots & \mathbb{a}_{2n}^{z_g} & \mathbb{l}_2^{z_g} & \mathbb{b}_2^{z_g} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbb{a}_{n1}^{z_g} & \dots & \mathbb{a}_{nn}^{z_g} & \mathbb{l}_n^{z_g} & \mathbb{b}_n^{z_g} \end{bmatrix} \tag{3.34}
$$

The collection of the methods of the matrix $\boldsymbol{T}$ that allows to produce the same commodity is organized inside the input-output multidimensional matrix $\boldsymbol{\Phi}$. For the production of his commodity, each producer i has to choose the method $\boldsymbol{\phi}_i^*$ which allows him to maximize the expected profit given the prices and the wage inside his local market. In details the structure of the matrix $\boldsymbol{\Phi}$ is the subsequent[54]:

$$
\boldsymbol{\Phi}(1:m_g, 1:(n+2), g) = \begin{bmatrix} \mathbb{a}_{g1}^{1} & \dots & \mathbb{a}_{gn}^{1} & \mathbb{l}_g^{1} & \mathbb{b}_g^{1} \\ \mathbb{a}_{g1}^{2} & \dots & \mathbb{a}_{gn}^{2} & \mathbb{l}_g^{2} & \mathbb{b}_g^{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbb{a}_{g1}^{m_g} & \dots & \mathbb{a}_{gn}^{m_g} & \mathbb{l}_g^{m_g} & \mathbb{b}_g^{m_g} \end{bmatrix} \tag{3.35}
$$

In which:

$$
\boldsymbol{\phi}(z_g, :, g) = [\mathbb{a}_{g1}^{z_g}, \mathbb{a}_{g2}^{z_g}, \dots, \mathbb{a}_{gn}^{z_g}, \mathbb{l}_g^{z_g}, \mathbb{b}_g^{z_g}] \tag{3.36}
$$

Considering the whole economy, different producers of the same commodity g can use different methods of the matrix $\boldsymbol{\Phi}(:, :, g)$ or converge to the same method between the m_g at disposal for each commodity. Each producer knows which commodity he can produce. This information is collected inside the multidimensional[55] matrix $\boldsymbol{O}$:

$$
\boldsymbol{O}(1:n, 1:m_g, i) = [o_{gz_g}]_{it} \;\; for \;\; o_{gz_g}^{it} = 0, 1 \;\; and \;\; i = 1, ..., k \tag{3.37}
$$

Consider that o_{gz_g}=1 if g=$\iota(i)$ and o_{gz_g}=0 in the other cases[56]. For the choice of the method the producer will make reference to the prices and the wage announced in his local market. Given this information firstly he will calculate the matrix $\boldsymbol{\Sigma}_{it}$, which represents the cost of production of

[54]It will be followed mainly the notation of Zambelli and Fredholm (2010).

[55]In this section it is hypothesised that the multidimensional matrix $\boldsymbol{O}$ is constant in time.

[56]For example, in the case each producer has full access to all the m_g alternative methods the matrix $\boldsymbol{O}$ would became for producer 2 which produces commodity 2 with at disposal

the commodity g for each possible alternative method z_g (for g=1,...,n and z_g=1,...,m_g)[57].

$$\boldsymbol{\Sigma}_{it}(1:n,1:m_g) = [\sigma_{gz_g}]_{it} \quad \sigma^{it}_{gz_g} = \boldsymbol{T}(g,1:(n+1),z_g) * [\bar{\boldsymbol{p}}_{s(i)}\ w_i]^T \quad (3.38)$$

Secondly, the producer will calculate the matrix of the value of production $\boldsymbol{V}_{it}$ (for g=1,...,n and z_g=1,...,m_g given s=i) given the prices and the wage inside his local market.

$$\boldsymbol{V}_{it}(1:n,1:m_g) = [v_{gz_g}]_{it} \quad v^{it}_{gz_g} = T(g,(n+2),z_g) * p_{\psi_{gs}} \quad (3.39)$$

At this point each producer can find the method z_g that maximize his expected profit as the difference between the possible value of production and the correspondent cost of production. In order to find it, it is necessary to consider which commodities the producer can produce[58]. As already said, this information is embedded into his matrix $\boldsymbol{O}$. Consider the matrix $\boldsymbol{\Pi}_{it}$ which has the same dimension of the matrix $\boldsymbol{V}_{it}$ and represents the expected profit for each method at disposal:

$$\boldsymbol{\Pi}_{it}(1:n,1:m_g) = [\pi_{gz_g}]_{it} \quad \pi^{it}_{gz_g} = (v^{it}_{gz_g} - \sigma^{it}_{gz_g}) * o^{it}_{gz_g} \quad (3.40)$$

Considering that each producer produces only one commodity so that, as said, $o^{it}_{gz_g}$=1 if g=$\iota(i)$ and $o^{it}_{gz_g}$=0 in the other cases, $\pi^{it}_{gz_g} \neq 0$ only if g=$\iota(i)$. At this point the producer can select the maximum value for each row g

5 alternative methods for the production of his commodity (n=3, m_g=5, $\iota(2)$=2):

$$\boldsymbol{O}(1:n,1:m_g,2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is important to underline that it is possible to change this hypothesis in order to test different access to methods for different producers in time. See for example the experiment in section 7.8 where the hypothesis of access to a new method (innovation) for a single producer is tested.

[57] The cost for the production of each commodity is computed, because producers have access to the whole input-output multidimensional matrix $\boldsymbol{T}$, but the matrix of the access to methods $\boldsymbol{O}$ will determine which commodity can be effectively produced. As already explained, in the *digital economic laboratory* it is supposed that each producer can produce only one commodity.

[58] It is worth remembering that each producer i can calculate his matrix $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ because he has access to the whole matrix $\boldsymbol{T}$ but he can use effectively only the methods coherent with his matrix $\boldsymbol{O}$.

inside the matrix $\mathbf{\Pi}_{it}$ so that max($\mathbf{\Pi}_{it}(g,:)$)=${\pi^*_{gz^*_g}}^{it}$ with z^*_g corresponding to the *optimal* method for the production of each commodity.

$$[z^*_g]_{it} = [z_g]_{it} \mid [\pi_{gz_g}]_{it} = max(\mathbf{\Pi}_{it}(g,:)) \tag{3.41}$$

In this way, for each producer, the vector of selected methods $\boldsymbol{z}^*_{it}$ is a column vector in which each element is the method chosen for the production of each commodity $\boldsymbol{z}^*_{it}$=$[z^*_g]_{it}$. If each producer produces only one commodity (as supposed in the *digital economic laboratory*) it will be posed $[z^*_g]_{it}$=0 for each g$\neq\iota(i)$. As a consequence, the selected method for the production of the unique commodity will be for each producer (considering g=$\iota(i)$ for time t):

$$\boldsymbol{\phi}^*_{it} = \mathbf{\Phi}(z_g^{it*}, :, g) = [\mathbb{a}_{g1}^{z^*_g}, \mathbb{a}_{g2}^{z^*_g}, \dots, \mathbb{a}_{gn}^{z^*_g}, \mathbb{l}_{g}^{z^*_g}, \mathbb{b}_{g}^{z^*_g}]_{it} \tag{3.42}$$

In which it is possible to distinguish (considering g=$\iota(i)$):

$$\boldsymbol{\phi}^{\mathbb{A}^*}_{it} = \mathbf{\Phi}(z_g^{it*}, 1:n, g) = [\mathbb{a}_{g1}^{z^*_g}, \mathbb{a}_{g2}^{z^*_g}, \dots, \mathbb{a}_{gn}^{z^*_g}]_{it} \tag{3.43}$$

$$\boldsymbol{\phi}^{\mathbb{AL}^*}_{it} = \mathbf{\Phi}(z_g^{it*}, 1:(n+1), g) = [\mathbb{a}_{g1}^{z^*_g}, \mathbb{a}_{g2}^{z^*_g}, \dots, \mathbb{a}_{gn}^{z^*_g}, \mathbb{l}_{g}^{z^*_g}]_{it} \tag{3.44}$$

$$\phi^{\mathbb{L}^*}_{it} = \mathbf{\Phi}(z_g^{it*}, (n+1), g) = [\mathbb{l}_{g}^{z^*_g}]_{it} \tag{3.45}$$

$$\phi^{\mathbb{B}^*}_{it} = \mathbf{\Phi}(z_g^{it*}, (n+2), g) = [\mathbb{b}_{g}^{z^*_g}]_{it} \tag{3.46}$$

As a consequence each local market will compose his matrices $\mathbb{A}^*$, $\mathbb{L}^*$ and $\mathbb{B}^*$, while for the whole system, in each period, the matrix of the methods chosen by all the producers will be:

$$\mathbf{\Phi}^*_{\boldsymbol{t}} = [\boldsymbol{\phi}^*_i]_t \quad for \;\; i = 1, ..., k \tag{3.47}$$

The determinants of aggregate demand and supply for the production goods market and the labour force market

The desired quantities of commodities to be bought on the production goods market by the producer i depend on the volume of desired production of his commodity vp_{it}. This quantity is determined by the particular behavioural function[59]. If producer i produces only one commodity (as supposed), each producer will compute[60]:

$$\boldsymbol{\phi}_{it}^{\mathbb{AL}^*} * \delta_{it} = [\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{l}_{n+1}]_{it} \tag{3.48}$$

$$\delta_{it} = vp_{it}/\phi_{it}^{\mathbb{B}^*} \tag{3.49}$$

Remember that $\tilde{a}_g^{it}$=0 for g=$\iota(i)$, because each producer has already at disposal his commodity for production[61]. Each producer can calculate his desired demand of commodities so that the matrix $\tilde{\boldsymbol{A}}_{it}$ becomes for the whole economy:

$$\tilde{\boldsymbol{A}}_t(1:k, 1:n, t) = \begin{bmatrix} \tilde{a}_{11} & \ldots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \ldots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots \\ \tilde{a}_{k1} & \ldots & \tilde{a}_{kn} \end{bmatrix}_t \tag{3.50}$$

$$\bar{\tilde{\boldsymbol{a}}}_{it}(i, 1:n, t) = \begin{bmatrix} \tilde{a}_{i1} & \ldots & \tilde{a}_{in} \end{bmatrix}_t \tag{3.51}$$

While his desired demand of labour becomes:

$$\tilde{\boldsymbol{l}}_{it}(1:k, 1, t) = \begin{bmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \vdots \\ \tilde{l}_k \end{bmatrix}_t \tag{3.52}$$

[59]For example, producers could decide the volume of desired production according to the expected demand d_{it}^e so that vp_{it}=d_{it}^e.

[60]In equation 3.49 it is supposed that the desired quantities are computed on the basis of the method that maximises the expected profits. It is possible inside the *thought experiment* to imagine situations in which production depends not only by the optimal methods but depends also by the method (or combination of methods) that in the past allowed to minimize residuals. For simplicity, in eq.3.49, it has been assumed the base case.

[61]See equation 3.26 for more details on this point.

Now it is possible to calculate the level of the desired demand for production goods inside each local market and as a consequence in the whole economy[62] as d_{st}^{pgm} for s=1,...,k:

$$\boldsymbol{d}_t^{pgm} = [d_s^{pgm}]_t \ \ with \ \ d_s^{pgm} = [\sum_{\bar{i}=1}^{n} \tilde{a}_{\tilde{\psi}_{\bar{i}s},\iota(s)}]_t \tag{3.53}$$

At the same time workers (and producers if they decide to work) will announce their desired labour supply.

$$L_{it}^{sp*} \ \ for \ \ i = 1, ..., k \tag{3.54}$$

$$L_{jt}^{sw*} \ \ for \ \ j = 1, ..., K \tag{3.55}$$

The labour supply of the workers of the same enterprise can be aggregated as:

$$L_{it}^{sw*} = \sum_{j \in \boldsymbol{\chi}_{it}} L_{jt}^{sw*} \ \ for \ \ i = 1, ..., k \tag{3.56}$$

The matrices of labour demand and supply for the whole economy correspond to:

$$\boldsymbol{L}_t^d = [L_{it}^d] \ \ with \ \ L_{it}^d = \tilde{l}_{it} \tag{3.57}$$

$$\boldsymbol{L}_t^{s*} = [L_{it}^{s*}] \ \ with \ \ L_{it}^{s*} = L_{it}^{sp*} + L_{it}^{sw*} \tag{3.58}$$

The labour demand and supply and the demand and supply of commodities has been computed. In the next phase, the exchanges between producers and producers and workers in the production goods market and in the labour force market respectively will take place.

[62]The superscript *pgm* stands for *production goods market*. It is important in order to distinguish between demand in the production goods market and demand in the consumption goods market, which superscript will be *cgm*.

3.5.7 Exchanges, rationing and purchased quantities in the labour force market and the production goods market: phase τ_2

At the moment of exchange, in production goods and labour force market, if the demand is greater than the supply ARAs are rationed. The levels of rationing inside the whole virtual economy are indicated respectively for the production goods and labour force market[63] with:

$$\boldsymbol{rat}_t^{pgm} = \boldsymbol{s}_t^{pgm} \oslash \boldsymbol{d}_t^{pgm} \tag{3.59}$$

$$\boldsymbol{rat}_t^{L} = \boldsymbol{L}_t^{s*} \oslash \boldsymbol{L}_t^{d} \tag{3.60}$$

Each producer (worker) is potentially rationed by the suppliers of his local market. This means that the commodities and labour effectively bought by each producer are equal to his desired quantities corrected for the rationing computed in the matrix $\boldsymbol{rat}_t^{pgm}$ and $\boldsymbol{rat}_t^{L}$. The quantities effectively bought by each producer for production after rationing is represented for the whole economy by the matrix $\boldsymbol{Q}_t$ (commodities) and $\boldsymbol{q}_t^L$ (labour force):

$$\mathbf{Q}_t(1:k,1:n,t) = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ q_{21} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots \\ q_{k1} & \cdots & q_{kn} \end{bmatrix}_t \quad \mathbf{q}_t^L(1:k,1,t) = \begin{bmatrix} q_{11}^L \\ q_{21}^L \\ \vdots \\ q_{k1}^L \end{bmatrix}_t \tag{3.61}$$

$$\bar{\boldsymbol{q}}_{it}(i,1:n,t) = \begin{bmatrix} q_{i1} & \ldots & q_{in} \end{bmatrix}_t \tag{3.62}$$

We can compute the values of the matrix as:

$$\bar{\boldsymbol{q}}_{it} = \bar{\tilde{\boldsymbol{a}}}_{it} \circledast \mathbf{rat}_{\boldsymbol{\Psi}_{s(i)}t}^{pgm} \tag{3.63}$$

$$q_{it}^L = \tilde{l}_{it} * rat_{it}^L \tag{3.64}$$

Remembering that[64]:

$$\mathbf{rat}_{\boldsymbol{\Psi}_{s(i)}t}^{pgm} = [rat_{\psi_{\bar{i}s(i)}}^{pgm}]_t^T \quad for \quad \bar{i} = 1, ..., n \tag{3.65}$$

[63] Remember that if supply is superior with respect to demand there is no rationing and the corresponding value inside the matrix $\boldsymbol{rat}_t^{pgm}$ is posed equal to 1. As explained in the Appendix, the symbol $\oslash$ is referred to the division between two matrices element by element. It will be used also $\circledast$ as symbol for a multiplication between matrices element by element. This operation is referred to the Hadamard product. For the normal products and divisions between matrices we use * and /.

[64] Remember that we are considering this different notation: i=1,...,k for all the producers of the system, $\bar{i}$=1,...,n for the restricted number of the producers inside the local market.

It is worth remembering that q^L_{it} corresponds to the total amount of labour force bought by the ARA producer i under the hypothesis that the producer produces only his commodity $\iota(i)$. By multiplication of the rows of these matrices $\boldsymbol{Q}$ and $\boldsymbol{q}^L$ for the prices and wages of the respective local markets, it is possible to compute the values of the purchased quantities. The total amount of labour force bought by the ARA producer i corresponds to the sum of the total amount of labour effectively sold[65] by all the workers enrolled in his enterprise i[66]:

$$q^L_{it} = L^s_{it} \tag{3.66}$$

In details:

$$L^s_{it} = [L^{sp} + L^{sw}]_{it} \tag{3.67}$$

Where:

$$L^{sp}_{it} = L^d_{it} * rat^L_{it} * shl^p_{it} \qquad with \qquad shl^p_{it} = L^{sp*}_{it} / L^{s*}_{it} \tag{3.68}$$

$$L^{sw}_{it} = L^d_{it} * rat^L_{it} * shl^w_{it} \qquad with \qquad shl^w_{it} = L^{sw*}_{it} / L^{s*}_{it} \tag{3.69}$$

$$L^{sw}_{it} = \sum_{j \in \boldsymbol{\chi}_{it}} L^{sw}_{jt} \;\; for \;\; i = 1, ..., k \tag{3.70}$$

It is possible to compute the level of the unsold commodities as the difference between what has been offered in the local market s of producer i (s=i) and what has been effectively sold to the neighbouring producers:

$$uns^{pgm}_{it} = s^{pgm}_{it} - [\sum_{\bar{i}=1}^{n} q_{\tilde{\psi}_{\bar{i}s(i)}, \iota(i)}]_t \tag{3.71}$$

[65]Remember that L^s represents the quantity of labour effectively sold while L^{s*} represents the desired labour supply.

[66]The workers enrolled in the same enterprise i are represented by the vector $\boldsymbol{\chi}_{it}$. See section 3.5.2 for a detailed description of migration rules and the relative notation.

3.5.8 Production: phase τ_3

Given the quantities bought in the production goods market and in the labour force market, it is possible to compute the resources for production at disposal of each producer. In order to obtain this, it is necessary to add to the commodities bought also the part of endowment not exchanged ($\tilde{e}_{it}$) plus the unsold commodity in the production goods market (uns_{it}^{pgm}). The effective quantities at disposal of each ARA producer i for production correspond to the vector $\boldsymbol{qprod}$:

$$\boldsymbol{qprod}_{it} = [q_{i1}, \ldots, q_{in}, q_{i,n+1}^{L}]_t \tag{3.72}$$

Where:

$$[q_{i,\iota(i)}]_t = [\tilde{e} + uns^{pgm}]_{it} \tag{3.73}$$

These commodities are used by each producer i for the production of the commodity $\iota(i)$[67]. This value is called b_{it} and represents the level of real production of the commodity $\iota(i)$ of each producer i considering the effective use of the resources devoted to production $\boldsymbol{qused}$:

$$\boldsymbol{qused}_{it} = [a_{i1}, \ldots, a_{in}, l_{i,n+1}]_t \tag{3.74}$$

$$[a_{i1}, \ldots, a_{in}, l_{i,n+1}]_t \xrightarrow{production} b_{it} \tag{3.75}$$

The production process may generate residuals out of commodities that are not used up. This happen frequently because the method chosen previously, in most cases, is not able to consume all the resources with the same intensity and while some commodities will be exhausted, other resources will not. In this case the producer can try to exploit residuals for further production through the use of the last set of methods (part of the matrix $\boldsymbol{T}$) that has been called previously $\tilde{\boldsymbol{T}}$ (they are 7 with n=3) and that allows to use combination of commodities (or only labour) also when one, two or three of them are missing (labour is always necessary). This set of methods is more expensive with respect to the original one (i.e., these methods are not efficient) but allows to reduce the volume of the residuals. The producer can use these methods and carry on production until residuals are minimized. It is possible that labour is missing. In this case it is possible to imagine that the producer devotes his free time for labour in order not to waste the

[67] Before to start production, because now producer knows exactly the amount of resources at disposal for production, the chosen method can be updated. The updating principle is to select the method that allows to maximize production (i.e., minimize residuals) given the resources at disposal. At this point it is possible, for each producer, to correct the level of resources devoted to production for the effective use of the resources according to the method chosen after the updating z_g^{**}. This process can be introduced in the setting of the particular *thought experiment*.

purchased commodities. The final residuals arise out of this process. For a graphical representation of this process see in the Appendix the figure 10.2. Remember that this residual mechanism process can be excluded or modified by the *thought experiment.* In each case at the end of this process it is possible to obtain the effective levels of production and the effective amount of labour and commodities used in production. The commodities of the other producers bought but not used (neither through this eventual residual mechanism process) are forced consumption because commodities are perishable and cannot be used in the next production period. Residual of producer's commodity become instead part of the final production.

$$\boldsymbol{residual}_{it} = \boldsymbol{qprod}_{it} - \boldsymbol{qused}_{it} \tag{3.76}$$

The investment decision

Once production has been completed, producers have at disposal a new amount of their commodity. At this point they have to decide how much of this amount to offer on the consumption goods market and how much to keep for next production. At the same time, they have to decide how much of this quantity to devote to the exchange in the production goods market of next production period ($s^{pgm}_{i,t+1}$) and how much to keep directly for next production ($\tilde{e}_{i,t+1}$). This decision depends on the behavioural function of the *thought experiment* that will define each production period the value of 3 variables:

- η_{it}: percentage of production to keep for next production.
- α_{it}: percentage of production to keep for exchange in the next production goods market.
- β_{it}: percentage of actual production to devote to consumption goods market of the present production period.

The sum of these 3 percentages must sum 1 (η_{it}+α_{it}+β_{it}=1). Remember that γ_{it}=η_{it}+α_{it}, where γ_{it} represents the percentage of production not devoted to consumption so that $e_{i,t+1}$=$\gamma_{it}b_{it}$ is the total endowment for production; $s^{pgm}_{i,t+1}$=$\alpha_{it}b_{it}$ is the endowment that will be exchanged in the production goods market; $\tilde{e}_{i,t+1}$=$\eta_{it}b_{it}$ is the endowment for production that will not be exchanged in the production goods market and finally s^{cgm}_{it}=$\beta_{it}b_{it}$ is the share of production devoted to exchange in the consumption goods market.

The determinants of aggregate demand and supply in the consumption goods market

At this point producers know how much to offer on the consumption goods market. This quantity would be equal to[68]:

$$\boldsymbol{s}_t^{cgm} = [s_i^{cgm}]_t \quad s_{it}^{cgm} = \beta_{it} b_{it} \tag{3.77}$$

At the same time also workers have earned their income from labour. As a consequence, they can decide how much of their financial resources to devote to consumption. This decision is taken according to the particular behavioural function of the *thought experiment* (e.g., an utility function). For each ARA producer and worker, the column vector of desired consumption is equal to:

$$\boldsymbol{c}_{it}^{*} = [c_g]_{it}^{*} \ \ for \ \ g = 1, ..., n \ \ for \ \ i = 1, ..., k \tag{3.78}$$

$$\boldsymbol{c}_{jt}^{*} = [c_g]_{jt}^{*} \ \ for \ \ g = 1, ..., n \ \ for \ \ j = 1, ..., K \tag{3.79}$$

For the whole economy it is possible to obtain the matrices:

$$\boldsymbol{C}_t^{p*} = [\boldsymbol{c}_i]_t^{p*} \ \ for \ \ i = 1...k \tag{3.80}$$

$$\boldsymbol{C}_t^{w*} = [\boldsymbol{c}_i]_t^{w*} \ \ for \ \ i = 1...k \tag{3.81}$$

Considering the aggregation of the workers enrolled in each enterprise:

$$\boldsymbol{c}_{it}^{w*} = \sum_{j \in \boldsymbol{\chi}_{it}} \boldsymbol{c}_{jt}^{*} \ \ \forall \, j \ \in \ \boldsymbol{\chi}_{it} \tag{3.82}$$

The aggregate demand for consumption goods inside each local market and for the whole economy can be computed as (for g=1,...,n and s=1,...,k):

$$\boldsymbol{d}_t^{cgm} = [d_s^{cgm}]_t \ \ with \ \ d_s^{cgm} = \sum_{\bar{i}=1}^{n} ([c_{\iota(s)}]_{\tilde{\psi}_{\bar{i}s}}^{p*} + [c_{\iota(s)}]_{\tilde{\psi}_{\bar{i}s}}^{w*})_t \tag{3.83}$$

[68]The superscript *cgm* stands for *consumption goods market*. It is important in order to distinguish between demand in the production goods market and consumption goods market. The superscript *pgm* stands for *production goods market*.

3.5.9 Exchanges, rationing and purchased quantities in the consumption goods market: phase τ_4

At the moment of exchange, in consumption goods market, if the demand is greater that the supply ARAs are rationed. The levels of rationing inside the whole system is indicated[69] with:

$$\boldsymbol{rat}_t^{cgm} = \boldsymbol{s}_t^{cgm} \oslash \boldsymbol{d}_t^{cgm} \tag{3.84}$$

Each producer (worker) is rationed by the suppliers of his local market. This means that:

- $\boldsymbol{c}_{it}$ and $\boldsymbol{c}_{jt}$: are the quantities *effectively* bought respectively by each producer i and worker j.
- $\boldsymbol{c}_{it}^*$ and $\boldsymbol{c}_{jt}^*$: are the quantities *desired* respectively by each producer i and worker j.

The quantity effectively bought by each ARA for consumption after rationing is represented for the producers and workers by the vectors:

$$\boldsymbol{c}_{it} = [c_g]_{it} \;\; for \;\; g = 1, ..., n \quad i = 1, ..., k \tag{3.85}$$

$$\boldsymbol{c}_{jt} = [c_g]_{jt} \;\; for \;\; g = 1, ..., n \quad j = 1, ..., K \tag{3.86}$$

These vectors have the same dimensions of the vectors $\boldsymbol{c}_{it}^*$ and $\boldsymbol{c}_{jt}^*$. Each element of the vector is computed as:

$$\boldsymbol{c}_{it} = \mathbf{c}_{it}^* \circledast \mathbf{rat}_{\boldsymbol{\Psi}_{s(i)}t}^{cgm} \tag{3.87}$$

$$\boldsymbol{c}_{jt} = \mathbf{c}_{jt}^* \circledast \mathbf{rat}_{\boldsymbol{\Psi}_{s(i)}t}^{cgm} \quad with \;\; j \in \boldsymbol{\chi}_{it} \tag{3.88}$$

$$\mathbf{rat}_{\boldsymbol{\Psi}_{s(i)}t}^{cgm} = [rat_{\psi_{\bar{i}s(i)}}^{cgm}]_t^T \quad for \;\; \bar{i} = 1, ..., n \tag{3.89}$$

The value of each basket of commodities bought by each ARA is computed through the multiplication of the quantities bought for the prices of his local market. It is possible to compute also for this market the unsold quantities of commodities for each producer:

[69]If $rat_{it}^{cgm} \geq 1$, $rat_{it}^{cgm}=1$ is imposed. As explained in the Appendix, the symbol $\oslash$ is related to the division between two matrices element by element. It will be used also $\circledast$ as symbol for a multiplication between matrices element by element. This operation is related to the Hadamard product. For the normal products and divisions between matrices we use * and / as usual.

$$uns_{it}^{cgm} = s_{it}^{cgm} - [\sum_{\bar{i}=1}^{n}(c_{\iota(i)}^{p} + c_{\iota(i)}^{w})_{\tilde{\psi}_{\bar{i}s(i)}}]_t \tag{3.90}$$

What is not sold in the consumption goods market (but it has been produced in the present production period) will be devoted to future production.

3.5.10 Consumer exchange: phase τ_5

After the exchanges in the consumption goods market, each ARA has at disposal his final consumption vector. Nevertheless, it is possible that this final vector is different with respect to his desired consumption basket. This could happen due to rationing in the consumption goods market; indeed, if ARA's consumption demand has not been fully satisfied, the final consumption vector $\boldsymbol{c}$ is necessarily different with respect to the desired consumption vector $\boldsymbol{c}^*$. In this phase the ARAs of each enterprise (producer and his workers) can exchange mutually their consumption goods at the relative prices in order to catch their desired consumption basket. Remember that in this phase these exchanges do not imply a new computation of the total amount of IOUs because it is a barter exchange. For this reason, in the figure 3.10 this type of exchange has been coloured in red.

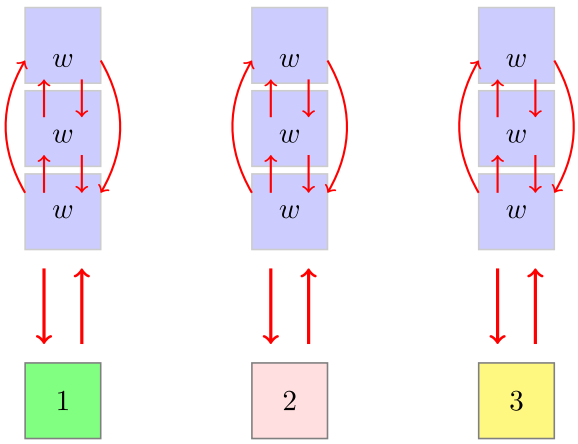


Figure 3.10: **The lattice and the consumer exchange**: this example follows the structure of figure 3.3. In this case the red lines indicates the relations valid only in the consumer exchange phase.

3.5.11 Migration: phase τ_6

At the end of the production period workers can decide to change their location and to work for another enterprise. Each worker can move only to a neighbouring enterprise and can move only one time each production period. Considering several production periods, each worker can work for more production periods for the same enterprise, return to the same enterprise or can move along all the lattice and work inside different local markets (see for an example of a possible path of migration of a single worker during 7 production periods the figure 3.11).

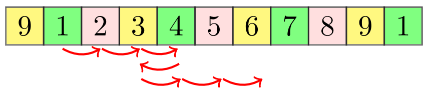


Figure 3.11: **The lattice and the migration**: the arrows indicate the possible path of migration of a particular worker considering 7 production periods. At the beginning the worker is enrolled in enterprise 1. Then he moves to enterprise 2,3,4, he returns back to 3 and then moves to 4,5 and at the end he arrives to enterprise 6.

At the end of the production period each worker decides if to move and where to move. At this point the matrix $\boldsymbol{M}_t$ has to be updated and the matrix $\boldsymbol{M}_{t+1}$ can be computed. The particular rules of migration are defined in the experimental setting relative to the particular *thought experiment*[70].

[70]See for example the rules used in the *thought experiments* considered in the present thesis and explained in section 4.4.

3.5.12 Revenues, expenditures and financial balances

At the end of the production period t it is possible to compute the revenues and the expenditures of the production period. As a consequence, it is also possible to compute the financial magnitudes. The revenues and expenditures of each producer can be computed as:

$$revenues_{it} = (s^{pgm}_{it} - uns^{pgm}_{it} + s^{cgm}_{it} - uns^{cgm}_{it})p_{it} + L^{sp}_{it} w_{it} \tag{3.91}$$

$$expenditures_{it} = \bar{\boldsymbol{q}}_{it} \boldsymbol{p}_{s(i)t} + q^{L}_{it} w_{it} + \bar{\boldsymbol{c}}_{it} \boldsymbol{p}_{s(i)t} \tag{3.92}$$

$$F_{it} = revenues_{it} - expenditures_{it} \tag{3.93}$$

The revenues and expenditures of each worker can be computed as[71]:

$$revenues_{jt} = L^{sw}_{jt} w_{it} \tag{3.94}$$

$$expenditures_{jt} = \bar{\boldsymbol{c}}_{jt} \boldsymbol{p}_{s(j)t} \tag{3.95}$$

$$F_{jt} = revenues_{jt} - expenditures_{jt} \tag{3.96}$$

The revenues and expenditures of each enterprise can be computed as:

$$revenues^{ent}_{it} = (s^{pgm}_{it} - uns^{pgm}_{it} + s^{cgm}_{it} - uns^{cgm}_{it})p_{it} \tag{3.97}$$

$$expenditures^{ent}_{it} = \bar{\boldsymbol{q}}_{it} \boldsymbol{p}_{s(i)t} + q^{L}_{it} w_{it} \tag{3.98}$$

$$F^{ent}_{it} = revenues^{ent}_{it} - expenditures^{ent}_{it} = profit_{it} \tag{3.99}$$

The wealth can be updated:

$$wealth_{it} = wealth_{i,t-1} + F_{it} \tag{3.100}$$

$$wealth^{ent}_{it} = wealth^{ent}_{i,t-1} + F^{ent}_{it} \tag{3.101}$$

$$wealth_{jt} = wealth_{j,t-1} + F_{jt} \tag{3.102}$$

[71] With reference to the equation of workers's revenues, the quantity of labour sold by worker j is multiplied for the wage of the enterprise i in which the worker j is enrolled. The same for the producer: the quantity of labour of the producer i is multiplied for the wage of his enterprise i.

By construction, if the financial accounting of all producers and workers are aggregated, it must be always true that:

$$F_t = F_t^p + F_t^w \equiv 0 \tag{3.103}$$

$$F_t^p = revenues_t^p - expenditures_t^p \tag{3.104}$$

$$F_t^w = revenues_t^w - expenditures_t^w \tag{3.105}$$

$$F_t^{ent} = revenues_t^{ent} - expenditures_t^{ent} \tag{3.106}$$

$$wealth_t = wealth_t^p + wealth_t^w \equiv 0 \tag{3.107}$$

$$wealth_{t+1}^p = wealth_t^p + F_t^p \qquad wealth_{t+1}^w = wealth_t^w + F_t^w \tag{3.108}$$

$$wealth_{t+1}^{ent} = wealth_t^{ent} + F_t^{ent} \tag{3.109}$$

It is possible also to compute other *national accounting magnitudes* such as the total production of a particular commodity g or the total value of wages and profits. All these magnitudes can be trivially computed through aggregation of the computed variable and presented previously for the single ARA worker, producer or enterprise.

3.5.13 The mathematical structure of the digital economic laboratory: a summary

In the previous section, the mathematical structure of the *digital economic laboratory* has been described in details. The equations can be associated to the decisional steps:

- **New information set from past history**: in this phase, considering the information set (eq.3.22-3.23 and eq.3.24-3.25), there is the computation of the new levels of endowments and supply of commodities for the production goods market (eq.3.26) and financial wealth (eq.3.27).

- **Price and wage announcement**: on the basis of the information sets, there is the effective announcement of prices and wages (eq.3.28-3.30). Considering as given (i.e., determined by the behavioural function which depends on the experimental setting) the desired labour supply of each ARA, each producer will choose a particular method of production (eq.3.38-3.47) from the set of available methods (eq.3.34). At this point it is possible to compute the desired demand and supply of commodities for the production goods market (eq.3.48-3.53) and the labour demand and supply in the labour force market (eq.3.54-3.58).

- **Exchange 1 in the production goods market**: in this phase, with exchanges, it is possible for ARAs to be rationed in labour and production goods markets. The quantities actually bought by each ARA can be computed (commodities eq.3.59 and eq.3.61-3.65; labour eq.3.60 and eq.3.66-3.70). Also the unsold quantities can be computed (eq.3.71).

- **Production**: after the exchanges, commodities and labour are used for production and the quantities effectively used, produced and the residuals can be computed (eq.3.72-3.76). After production, producers choose how much of the product to offer in the consumption goods market. At the same time ARA workers and producers, according to their resources (and the behavioural functions), decide their desired consumption. In this way, aggregate demand and supply in the consumption goods market are determined (eq.3.77-3.83).

- **Exchange 2 in the consumption goods market**: in this phase, with exchanges, it is possible for ARAs to be rationed in the consumption goods market. The quantities actually bought by each ARA can be computed (eq.3.84-3.89). Also in this market unsold quantities can be computed (eq.3.90).

- **Consumption**: after exchanges[72], commodities are consumed. The final amount of IOUs can be computed and the level of wealth can be updated (eq.3.91-3.109). The information set of each ARA will be updated (eq.3.24-3.25).

It is worth remembering that the rules of double-entry bookkeeping system are always followed and that actual budget constraints are never broken. Any virtual bilateral exchange is associated to a registration in the economic accounts and in the balance sheets of the two ARAs involved.

[72]Considering also the consumer exchange in phase τ_5 without IOUs. See for more details subsection 3.5.10.

4 The setting of the thought expriements

4.1 General setting of the thought experiments

The *digital economic laboratory* described in the previous chapter has been transformed into a concrete set of algorithms through the computer program MATLAB®. The use of this specific computer program is not necessary because all the model is composed by mathematical equations that are algorithmically coherent and can be computed with whatever calculator. Indeed, there are no stochastic elements in this algorithmic model and its dimension is parametric (a high number of ARAs and local markets can be considered). The use of the computer is necessary only because the computational time is high and grows with the dimension of the experiment (i.e., with the number of production periods and ARAs)[73]. It is worth remembering that the mathematical structure of the *digital economic laboratory* is described in details in chapter 3. The *digital economic laboratory* has been built with the purpose to use it to run *laboratory experiments*, which compose the *thought experiment*. The *laboratory experiments* will be different depending on the different functional forms of the behavioural functions and the different values of the parameters and initial conditions. Nevertheless, the building blocks of the *digital economic laboratory* remain fixed.

Having decided to study a specific theoretical economic problem (see the list of the research topics that will be investigated in the *digital economic laboratory* in section 3.2), it is necessary to prepare the *thought experiment* accordingly. Whatever *thought experiment* has to follow a precise sequence of steps in order to be theoretically rigorous and able to produce robust results. These steps imply the definition of the features of the *laboratory experiments*: initial conditions and parameters (that determine the structure of the virtual market and represent its initial data set), the behavioural functions, the particular rules of trading and the rules of migration and production. A lot of *laboratory experiments* can be needed in order to collect statistics and to have some form of generality. This allows to construct *thought experiments* theoretically rigorous and able to produce robust results.

In order to compute the dynamic evolution of one *laboratory experiment* inside the *digital economic laboratory* it is necessary to specify initial conditions and parameters. Changes of initial conditions and parameters will identify different *laboratory experiments*. Each *laboratory experiment* generates a virtual economy identified by a unique identity code. The identity code is associated to a unique set of initial conditions and parameters, rules and behavioural functions. In this way each *laboratory experiment* can be exactly repeated. Even though some initial conditions and parameters can

[73]It is worth remembering that the computational time of each simulation is relevant also for the computer. Indeed, each virtual economy imply a computation time in the order of hours with a high performance computer and few ARAs.

be generated randomly, they can be reproduced exactly by the identity code. The experimental setting can be described as follows:

- **Data Set**: the whole data used in the *laboratory experiment* is collected in the data set. At the beginning of the experiment some fundamental values (parameters and initial conditions) have to be defined in order to start the simulation.
  - **Parameters and initial conditions**:
    * **The structure of the lattice**: it is necessary to define the number of commodities (n), the number of the producers (k), the number of the workers (K) and the number of workers enrolled in each enterprise at the beginning of the experiment ($\tilde{k}_{i1}$). According to these parameters it is possible to construct the *ring one-dimensional lattice* and the matrices that determine the location of each ARA ($\mathbf{\Upsilon}$).
    * **Production periods**: it is necessary to define the total production periods that will be computed. This parameter $\bar{t}$ represents the *stopping rule.*
    * **Labour market**: it is necessary to define the maximum total labour supply of workers and producers (L^{sw} and L^{sp}).
    * **Methods of production and access to methods**: it is necessary to define the technological set ($\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$) and the matrix that define the access to the methods ($\boldsymbol{O}$). The methods of production (the matrix $\boldsymbol{T}$) take here the form of *book of blueprints* and characterize the existing production possibilities[74].
    * **Characteristics of ARA**: the combination of features in terms of preference-propensity-technical parameters that characterize each ARA have to be defined in matrix $\mathbf{\Theta}$. These values can be fixed according to the behavioural function that will be used.
    * **Endowments and wealth**: at the beginning of the period the wealth of each ARA and the initial endowments have to be defined. These values (and also the other parameters and initial conditions) can be computed with the purpose to start from an *equilibrium position.*

[74]This set may be formed by a huge variety of alternative methods. The structure of the set can be quite different. For example, there can be high degrees of substitutability among the different means of production, or this substitutability may be limited or inexistent. It is important here to stress that the number of alternative methods may be extremely large.

        * **Information set**: the information set is represented by the matrix $\boldsymbol{\Omega}$. The information set contains the parameters and the initial conditions that have been previously described (see for more details section 3.5.4) and will be updated each production period with the relevant variables influenced or that influence the ARA.

- **Trade, migration and production settings**: the trading will take place in the general framework described for the *digital economic laboratory* but the possibility of designing more sophisticated trading settings is not excluded. Rules that determine the way in which migration evolves ($\boldsymbol{M}$) have to de designed: for example the number of workers that can migrate each production period, the rule of selection of workers etc. Also the production sphere can be designed in order to consider, for example, the presence of a mechanism that allows to reuse residuals (i.e., matrix $\tilde{\boldsymbol{T}}$) or to update the method.

- **Algorithmic precise specification of the ARAs**: the ARAs will be characterized by behavioural equations. There are many ARAs, but each ARA has to be specified. The decision process of the ARA can take different forms. They can be assumed to be described with the limited *neoclassical* characteristics or with solid procedural rules of behaviour (as suggested, for example, in Simon (1997)). Clearly, this would have to take an algorithmic form where the information and data processing type would have to be made explicit and grounded in cognitive science. The behavioural function determines the decision of labour supply and demand of consumption goods. Moreover, also the behavioural rules that determine the investment decision and the changes in prices and wages have to be defined in this phase of the *laboratory experiment*.

    – **ARA consumer and worker**: each consumer would be described to take decisions in accordance with his objectives. The algorithmic specification would capture the characteristics of the described agents. Decisions would depend on a great variety of factors that have to be included in the algorithmic description (the information set). Regardless of the algorithmic description chosen for the particular *laboratory experiment* each ARA would have desired quantities of commodities to be bought and desired quantity of labour to be sold and would have actual quantities of commodities bought and actual quantity of labour sold. Remember that wages are announced by trade unions and not by the single worker.

    – **ARA consumer (worker) and producer**: a producer is a consumer (and worker) that has access to production possibilities. He will have similar characteristics with respect to the consumer, but his principal source of income is through production, profits. He will choose the method that allows him to maximize expected profits, he will take investment decisions and will also announce the price of his commodity.

- **Running the experiment and collection of relevant data**: production periods are computed and data are collected until the *stopping rule* is reached ($t=\bar{t}$).

- **Statistics and Results**: attempt of conclusions with respect to the original *research question* through the computation and analysis of relevant statistics.

It is necessary to run several *laboratory experiments*, changing some of the elements that are specific to the particular *laboratory experiment* (i.e., parameters, initial conditions, rule or behavioural functions) so as to be able to have some form of generality in order to answer the initial research questions and construct robust *thought experiments*.

4.2 The definition of the data set: initial conditions and parameters

For the *thought experiments* considered in the present thesis, the initial conditions have been designed so that to allow the virtual economy to start from an *equilibrium position* (i.e., no debt or credit relations and production prices). A point close to the *wage-profit frontier* has been chosen, where distribution is egalitarian (i.e., each producer and/or worker has at disposal the same share of the surplus). The initial production prices and wages have been derived starting from this initial distribution. During the experiments[75] the *numéraire* to be used to measure values is adopted and the prices computed are real prices. The shock that allows to start the dynamics of the system is represented by a change in the preferences of the ARAs. The number of production periods computed is equal to 1000 which represents the *stopping rule*: the virtual economy can converge before or after this number.

For *convergence*, we mean a situation in which the virtual economy[76] is able to reach an *equilibrium position.*

We define an *equilibrium position* as a position in which prices and wages announced do not change anymore; all markets are in equilibrium (demand equal to supply in all markets; i.e., there is no more rationing), migration stops, the system reproduce itself (i.e., same level of production and surplus without residuals) without the creation of new IOUs (no new debt and credit relations). This implies that the prices of convergence are *production prices* with zero or positive profit rates.

If the virtual system is not able to converge before time 1000 it is considered not convergent. This does not mean that it is excluded that it can converge in a production period bigger than 1000, but the study of this convergence will be part of further investigation not treated in this thesis.

[75]For simplicity, we will use the terms *experiment* and *laboratory experiment* as synonyms.

[76]Each virtual economy is identified by a unique identity code and represent a single experiment. We will use the terms *virtual economy* and *virtual system* as synonyms.

4.3 The definition of the behavioural functions

According to the particular *thought experiment*, it is necessary to define the behavioural functions able to generate the subsequent variables:

- Labour supply $\mathbf{L}^{s*}$
- Demand for consumption goods $\boldsymbol{c}^{*}$
- Production volume **vp**
- Investment decision $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\eta}$ and $\boldsymbol{\gamma}$
- Prices **p**
- Wages **w**

These matrices and vectors have been considered as given in the *laboratory* because they are determined according to the *behavioural function*. The *behavioural function* can change according to the *thought experiment* because it could be part of the research question to test a particular behavioural function instead of another one.

For all the *thought experiments* of the present thesis, each behavioural function has been constructed equally for all the *laboratory experiments*. This would allow to check how the conclusions about the research questions will change only according to different initial conditions and parameters[77].

4.3.1 Behavioural functions and learning models

The model of learning is common to all of the behavioural functions considered in the present thesis. Indeed, it is supposed that ARAs change their decisions according to the changes in their information sets. The information sets are updated according to the events inside each local market. This means that each ARA is influenced by the behaviour of the other ARAs and tries to correct past errors according to experience. This adaptive learning principle, common to all the behavioural functions, has been translated into specific algorithms that are described in detail in the next sections.
It is worth remembering that different learning techniques have been developed in order to be applied to different theoretical learning models inside economic theory. Arrow has been a pioneer in recognizing the importance of learning in the economy (see Arrow, 1962) and a large literature followed about the relevance of learning by doing (see Thompson, 2010) and the role of learning and training for firms in competitive markets and inside

[77]The influence on the results of different behavioural functions is object of further research that goes beyond this thesis. See section 9.2 for more details.

labour markets (from Becker (1962) to the more recent Acemoglu and Pischke (1999)). Inside neoclassical theory, the idea of *rational learning* has been developed in order to solve learning issues in game theory[78], when differential or incomplete information is considered, and in general equilibrium models, when the issues of expectation formation is considered (see for more details Blume and Easley, 1998). Learning has been often associated to typical biological phenomena such as mutation, selection and imitation (on the theory of learning by imitation see for example Başçı (1999) and Schlag (1998)). Artificial intelligence, psychology, artificial sciences, computer sciences and neuroscience are research fields that have allowed a strong development of our understanding of human learning capabilities. Herbert Simon is surely one of the most important pioneers for the introduction of learning in economics but on the basis of a difference conception of rationality of humans with respect to neoclassical theory:

> A comparative examination of the models of adaptive behaviour employed in psychology (e.g., learning theories), and of the models of rational behaviour employed in economics, shows that in almost all respects the latter postulate a much greater complexity in the choice mechanisms, and a much larger capacity in the organism for obtaining information and performing computations, than do the former. Moreover, in the limited range of situations where the predictions of the two theories have been compared [...], the learning theories appear to account for the observed behaviour rather better than do the theories of rational behaviour (Simon, 1956, p.129).

For Herbert Simon "learning denotes changes in the system that are adaptive in the sense that they enable the system to do the same task or tasks drawn from the same population more efficiently and more effectively the next time" (Simon, 1983, p.26) and he is sure that "intelligence cannot be separated from social interactions" (Levi and Kernbach, 2010, p.340). As a consequence, different learning algorithms should consider that biological entities, such as human beings, don't optimise but search satisfaction according to the characteristics of the environment. Between 1955 and 1956 Herbert Simon, Allen Newell and Cliff Shaw have developed a computer program, the so-called *Logic Theorist*, with the intent to mimic human decision-making processes. This program has had the merit of giving a strong contribution to the development of the future artificial intelligence field. This *top-down* approach to create intelligent machinery has

[78]Indeed, in the last decades, the focus has been even more about how people learn to play games. Evolutionary game theory relaxes the assumptions of neoclassical game theory so that to recognize that agents need not be perfectly rational and that they can have imperfect information and can adjust their strategies myopically. In these games imitation and mutation come into the scene in a dynamical process towards the selection of the equilibrium.

been flanked recently by a *bottom-up* approach to model intelligence and that starts from the neuron (i.e., neural network):

> A conventional computer will never operate as brain does, but it can be used to simulate or model human thought. In 1955, Herbert Simon and Allen Newell announced that they had invented a thinking machine. Their program, the logic theorist, dealt with problems of proving theories based on assumptions it was given. Simon and Newell later developed the general problem solver, which served as the basis of artificial intelligence (AI) systems. Simon and Newell believed that the main task of AI was figuring out the nature of the symbols and rules that the mind uses. For many years AI engineers have used the "top-down" approach to create intelligent machinery. *The top-down approach starts with the highest level of complexity, in this case thought, and breaks it down into smaller pieces to work with.* A procedure is followed step by step. AI engineers write very complex computer programs to solve problems. *Another approach to the modelling of brain functioning starts with the lowest level, the single neuron. This could be referred to as a bottom-up approach to modelling intelligence* (Chaturvedi, 2008, p.23, emphasis added).

Both approaches are nowadays object of further research. Both of them have to deal with different difficulties. New learning models surely will be developed by considering the advantages and disadvantages from the use of both these approaches.

Some notes on the learning models and techniques used in the *thought experiments*

In the *thought experiments* considered in the present thesis, the behavioural functions have been constructed considering only *adaptive learning*, in the sense that each ARA is influenced by the behaviour of the other ARAs and tries to correct past errors according to experience. The effect of imitation, for example, can be object of further research, but it is surely a good methodological habit to consider firstly the most simple case, in order to be able to understand the influence, for example, of imitation with respect to the base case[79]. It is worth remembering that in the behavioural function that describe the price and wage decisions *neural networks* have been used inside a more complex decision algorithm. This could be considered in contradiction with Herbert Simon's approach. Indeed, "neural activities are the basis of the bottom-up approach, while symbolic descriptions are the basis of the top-down approach" and Herbert Simon has been one "vigorous advocates of symbolic AI" in the debate between symbolic versus connectionist approaches to AI (Copeland, 2017). The fact that nowadays both

[79] This point is object of further research that goes beyond this thesis. See section 9.2 for more details.

approaches have to cope with different problems suggests that probably the use of one method with respect to the other will depend also on the particular task. Indeed it cannot be excluded that for the construction of a particular behavioural function one of the two methods is not feasible.

> During the 1950s and '60s the top-down and bottom-up approaches were pursued simultaneously, and both achieved noteworthy, if limited, results. During the 1970s, however, bottom-up AI was neglected, and it was not until the 1980s that this approach again became prominent. Nowadays both approaches are followed, and both are acknowledged as facing difficulties. *Symbolic techniques work in simplified realms but typically break down when confronted with the real world; meanwhile, bottom-up researchers have been unable to replicate the nervous systems of even the simplest living things* (Copeland, 2017, emphasis added).

Probably further research will require the development of hybrid models. Indeed, "the top-down/bottom-up dichotomy is somewhat simplistic" and many researchers in the field of AI "advocate hybrid approaches that combine the best aspects and opportunities of both" (Gunkel, 2012, p.82). The use of *neural networks* in the algorithm for the announcement of prices and wages is an answer to this need to develop hybrid approaches for solving the problems that arise when one tries to develop an algorithm that mimics realistic decisional processes. It is worth remembering that the announcement of prices and wages is probably for the ARA a more complex task than one can imagine. Indeed, the environment in which this decision has to be taken is extremely complex and changes continuously in time. In order to judge a particular algorithm suitable to fulfil this task, it must be able to approximate the decisional process of an ARA who tries to maximize his income by changing the price of its commodity (if producer) or the value of the wage (if a trade union) without postulating for the ARA (trade union) any particular knowledge about economic theory . Indeed, each ARA has simply to learn the best price (or wage for trade unions) to announce on the basis of the knowledge and the experience developed in the laboratory over time. As far as I know, a similar algorithm (in a similar *digital economic laboratory*) has not been developed yet and it is arguably very difficult[80] to develop it with a simple top-down approach that, we know, "typically break[s] down when confronted with the real world" (Copeland, 2017).

[80]But not impossible, in order to answer to this question we need further research that go beyond the purposes of the present thesis. See section 9.2 for more details.

4.3.2 Labour and consumption decision

ARAs have to decide, after prices and wages announcement, the amount of labour to offer in the labour force market. After this choice, between phase τ_3 and phase τ_4, when each ARA has earned his income, there is the announcement of the desired quantities of commodities for consumption. It is possible to interpret the behavioural function that determines these quantities as the resolution of an utility maximization problem under budget constraint.

Given the initial conditions, each ARA maximizes his utility and calculates his optimum level of consumption and free time[81] according to his utility function U and a standard budget constraint. The level of the parameters inside the vectors $\boldsymbol{\lambda}_i$ and $\boldsymbol{\kappa}_i$ are given and do not change in time. $\boldsymbol{x}$ is the vector containing the desired consumption levels of the n commodities (g=1,...,n with n=3) and the level of free time L^f, $\boldsymbol{p}$ is the vector of the prices in ARA's local market, *wealth* is the level of financial wealth[82], w is the wage rate and finally L^{s*} is the desired labour supply while L^s is the endowment in terms of maximum labour supply. As a consequence, the maximization problem of each ARA i (for i=1,...,K+k)[83] in time t becomes[84]:

$$\begin{gathered} maxU(\boldsymbol{x})_{it} = \boldsymbol{\lambda_i}^T \boldsymbol{x_{it}^{\kappa_i}} \quad i = 1, ..., K + k \\ s.t. : \boldsymbol{p_{it}^T c_{it}^*} = wealth_{i,t-1} + w_{it} L_{it}^{s*} \\ \boldsymbol{x}_{it} = [\boldsymbol{c}^* \ , \ L^{f*}]_{it} \\ L_{it}^{f*} = L^s - L_{it}^{s*} \\ \boldsymbol{c}_{it}^* = [c_g^*]_{it} \quad g = 1, ..., n \end{gathered} \tag{4.1}$$

This utility maximization problem represents the standard model of consumer behaviour inside the general equilibrium framework: the market clearing condition ensures the consistency of the solutions of individual choice problems. Indeed, given the equilibrium prices, equilibrium between demand and supply in all markets (labour and goods) is guarantee by construction.

[81] In this phase the desired quantities are computed and for this reason the superscript $*$ represents the latter.

[82] The wealth corresponds to the accumulation of debt and credit positions in time.

[83] In this case all the ARAs are considered. For this reason i indicates not only ARAs producers but also workers, from 1 to K+k (K workers plus k producers).

[84] The presence of IOUs implies the possibility to have negative wealth. In this case each ARA makes his choice in order to pay off debts and at the same time to maximize his utility function. In order to do that, he considers in the budget constraint only a fraction of his negative wealth. This fraction will be equal to a percentage ε of his wage, so that he will have always at disposal a positive level of resources for consumption. For the present thesis the hypothesis is that each ARA can devote only the 50% of his wage to the settlement of debt.

In the *digital economic laboratory* consumption demand and labour supply could be rationed (i.e., the market could be unable to absorb ARA's labour supply and/or satisfy his demand of consumption goods). This because in the *digital economic laboratory* the equilibrium is a possible outcome of the dynamics of the interaction process between ARAs and not a construction hypothesis based on an instantaneous market clearing condition. For this reason, in order to use the utility function, it is necessary to hypothesise that:

- Each ARA decides his labour supply and his demand of consumption goods in order to optimize the utility function subject to the budget constraint.
- When the production phase takes place, each ARA updates the original desired consumption demand according to the income effectively earned (i.e., if the ARA has been rationed in the labour force market, his consumption decision will be updated).

The coherence between markets and individual choices can be obtained through an error correction mechanism that prevents ARAs to do recurrent errors and allows them to announce quantities that are consistent with the level of rationing (i.e., with the purpose not to be rationed). Once the labour is sold and production takes place, each ARA knows exactly his level of income. If this income is different with respect to that considered in the maximization problem, the desired consumption demand should be updated considering that L_{it}^{s*} must be replaced by the effective labour supply L_{it}^{s}:

$$\begin{aligned} maxU(\boldsymbol{c}^*)_{it} &= \boldsymbol{\lambda_i}^T \boldsymbol{x_i}^{\boldsymbol{\kappa_i}} \quad i = 1, ..., K + k \\ s.t. : \boldsymbol{p_{it}^T c_{it}^*} &= wealth_{i,t-1} + w_{it} L_{it}^s \\ \boldsymbol{x}_{it} &= [\boldsymbol{c}^* \ , \ L^f]_{it} \\ L_{it}^f &= L^s - L_{it}^s \\ \boldsymbol{c}_{it}^* &= [c_g^*]_{it} \quad g = 1, ..., n \end{aligned} \tag{4.2}$$

This final computation of $\boldsymbol{c}_{it}^*$ corresponds to the final demand of consumption goods that will be announced in the consumption goods market.

4.3.3 Production decision and investment decision

The production decision of producer i depends on the volume of the desired production vp_{it}. This value can be influenced by different factors. It is hypothesized that the desired production depends only on expected demand. Expected demand is computed through an error correction mechanism that allows each ARA producer to compute an expected demand coherent with the effective demand experimented in the past.

A correct expected demand corresponds to the quantity of the commodity that will be effectively sold (for production and consumption) or used for new production. In details, this quantity, that has to be estimated today but that will be produced next period, is $b^e_{i,t+1}$ (for i=1,...,k)[85]. It corresponds to the quantity of commodity that is expected today to be produced and sold in the next production period (time t+1) in the consumption goods markets $d^{cgm,e}_{i,t+1}$ and next in the production goods markets $d^{pgm,e}_{i,t+2}$ [86] plus the endowment needed for the production of his commodity $\tilde{e}^e_{i,t+2}$ (both at time t+2).

$$b^e_{i,t+1} = d^{cgm,e}_{i,t+1} + d^{pgm,e}_{i,t+2} + \tilde{e}^e_{i,t+2} \tag{4.3}$$

At this point, each producer has to decide how much of the present production b_{it} to invest with the purpose to produce $b^e_{i,t+1}$ and how much to devote to the consumption goods market of the present production period. As a consequence, each producer has to calculate the percentage of actual production to keep for next production η_{it}, the percentage of actual production to keep for next exchange on production goods market α_{it} and finally the percentage of actual production that can be destined to the consumption goods market of the present production period β_{it} (is computed as residual). Remember that $\alpha_{it}+\beta_{it}+\eta_{it}=1$ with $\gamma_{it}=\alpha_{it}+\eta_{it}$.

If the expectation on future demand is exact, and the actual production is enough in order to organize next production[87], each producer will be able to sell, thanks to this mechanism, all his future production $b^e_{i,t+1}$. With correct expectations it should be verified for each production period $\eta_{it}b_{it}=\tilde{e}_{i,t+1}$, $\alpha_{it}b_{it}=d^{pgm}_{i,t+1}$ and $\beta_{it}b_{it}=d^{cgm}_{it}$, with uns^{pgm}_{it}=0 and uns^{cgm}_{it}=0. In this way all the production is sold without residuals.

[85] The superscript e is the expectation.

[86] The superscript *cgm* stands for *consumption goods market* while the superscript *pgm* stands for *production goods market*.

[87] It could be possible that the producer has not enough commodities for organizing next production as desired. In this case ARAs producers will devote no resources to the consumption goods market and will compute the best combination of α_{it} and η_{it} according to the resources at disposal.

4.3.4 Price decision and wage decision

At the beginning of each production period, in phase τ_1 (see figure 3.6), each ARA producer announces the price of his commodity, while trade unions (that represent the workers of the same enterprise) announce their wage. This announcement is once for each production period, because prices and wages remain the same for all the other phases of the production period.

A behavioural function determines prices and wages. At the core of this particular behavioural function there is the fundamental idea that each producer (trade union) should be able to exploit in the most efficient way his information set in order to announce the price (wage) that allows him (the trade union) to reach his objective (i.e., maximization of income). This implies that ARA producer (trade union) should be able to learn from past errors the best price (wage) to announce so that to develop the experience (i.e., understand the dynamics of the environment) that allows the ARA producer (the trade union) to reach his objective.

The complexity of the dynamics of the *laboratory*, the size of the information set of each ARA producer (trade union), and the requirement of learning capabilities implies that the behavioural function that describes price and wage decisions should be represented by an algorithm able to learn and to take decisions. For the *thought experiments* within this thesis *artificial neural networks* have been used as part of a more general algorithm constructed in order to describe behavioural functions for price and wage announcement[88].

Artificial neural network is a particular approach of *machine learning. Machine learning* is the multidisciplinary field that studies the development of algorithms that allow computers to develop the ability to learn[89] in order to perform tasks. According to a more precise definition "the field of machine learning is concerned with the question of how to construct computer programs that automatically improve with experience" (Mitchell, 1997, p.xv)[90].

There are different approaches to machine learning, such as decision tree learning, deep learning, clustering, Bayesian networks etc. One particular

[88]See section 4.3.1 for a discussion about this topic.

[89]Learning is needed in cases in which it is not possible to develop a specific computer program to solve directly the task because the task change in time, space, according to circumstances or simply when we need the computer to simulate a human attitude that we cannot translate in a specific computer program. Remember that Herbert Simon says that "learning denotes changes in the system that are adaptive in the sense that they enable the system to do the same task or tasks drawn from the same population more efficiently and more effectively the next time" (Simon, 1983, p.26).

[90]Equivalently *machine learning* could be defined as "programming computers to optimize a performance criterion using example data or past experience" (Alpaydin, 2010, p. xxxi). More formally, "a computer program is said to *learn* from experience E with respect to some class of tasks T and performance measure P, if its performance at tasks in T, as measured by P, improves with experience E" (Mitchell, 1997, p.2, emphasis added).

approach of *machine learning* that seems suitable to our task is *artificial neural network*. Indeed, the idea to construct a behavioural functions able to mimic the behaviour of producers (trade unions) when they decide the price (wage) to announce imply the construction of an algorithm able to recognize the environment, learn from errors and develop a decision. *Neural network* is probably the most suitable approach of *machine learning* for this task because other approaches such as decision trees are more suitable for classification and regressions while clustering is more suitable for statistical data analysis (see appendix 10.3 for a detailed discussion of this topic).

In details, the idea is to create an algorithm able to select the relevant information, *understand* the behaviour of the economic system through the analysis of past experience and *take a decision* that should allow to reach the final objective (i.e., maximization of income). The sequential order of the algorithm can be summarised as follows:

- **Update the information set**: the information set of the ARA producer[91] is updated with the new information collected between the beginning of the production period and the moment in which ARA producer takes the decision.

- **Select relevant information**: not all information is relevant in order to take the decision. The ARA uses a subset of his information set composed by the relevant information.

- **Create, train and test the neural network**: using relevant information a *neural network* is created, trained, validated and tested. It is a *layer recurrent network* with 2 layers and 10 neurons for the hidden layer (see for a detailed description section 10.3.5).

- **Use the neural network for simulation**: once the *neural network* is validated and tested, it can be used for simulation. In other words, in order to take a decision with respect to the price (wage) to announce, the ARA producer selects a range of possible prices (wages) to announce according to past experience and uses the neural network in order to evaluate which price (wage) should ensure the highest income.

- **Final announcement**: at the end of this process the ARA producer (trade union) evaluates the coherence of the output of the *neural network* with his experience and announces the final price (wage).

[91] Or the trade union in the case of the wage announcement.

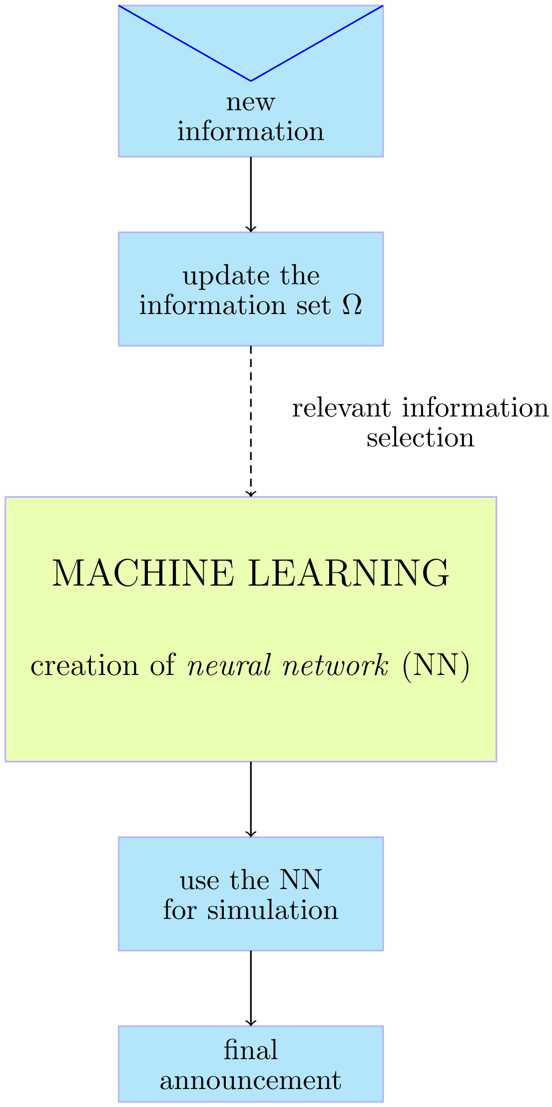


Figure 4.1: **The behavioural function for prices and wages announcement**: in this figure, the logical steps that characterize the decision process for the price and wage announcement are represented. It is based on an efficient use of information and the creation of an *artificial neural network* for the simulation of the best price/wage to announce according to the final objective of ARAs (maximization of income).

4.4 Definition of trade, migration and production settings

The trading will take place in the general framework described for the *digital economic laboratory*. Rules that determine the way in which migration evolves ($\boldsymbol{M}$) have been determined. For the present *thought experiments*, it is supposed that workers migrate where there is the highest income inside their local market. A maximum number of workers equal to 5 can migrate to each enterprise each production period. Workers are chosen according to their *preference for working*, their income and the number of times they have tried to migrate in the desired location. Each contract last 3 years, so each worker can migrate only at the end of the contract. The production sphere has been designed in order to consider the updating of the method of production after that commodities have been bought. The presence of a mechanism that allows to reuse residuals (i.e., matrix $\tilde{\boldsymbol{T}}$) has been considered only inside the last *thought experiment* (chapter 8).

4.5 The thought experiments: a summary

In this section, the *thought experiments* that will be considered in the next chapters are summarised:

- **Experiment 1**: simulation of a virtual market economy with the set of methods used by Sraffa in his example given at page 19 of *PCMC*. Two cases are considered: *case 1* is characterized by the presence 3 producers and 48 workers (the cases of homogeneous and heterogeneous ARAs will be considered); *case 2* is characterized by the presence 9 producers and 144 workers (will be considered both cases with homogeneous or heterogeneous ARAs). This last case implies the presence of 9 local markets as in figure 3.4 with 3 producers producing the same commodity. A study of the convergence properties follows. In particular, the convergence towards uniform profit rates, uniform wage rates and equal distribution of the surplus will be checked. Finally, the relationships between surplus, prices and preferences has been investigated.

- **Experiment 2**: simulation of different virtual market economies with a set of methods generated under the condition of vitality[92]. As in Sraffa, each producer has a unique method for the production of his commodity at disposal. Two cases are considered: *case 1* is characterized by the presence 3 producers and 54 workers (will be considered both cases with homogeneous or heterogeneous ARAs); *case 2* is characterized by the presence 9 producers and 162 workers (will be considered both cases with homogeneous or heterogeneous ARAs). This last

[92]The methods allow the system to reproduce itself with a positive surplus.

case implies the presence of 9 local markets as in figure 3.4 with 3 producers producing the same commodity. 20 virtual economies have been generated for *case 1* and 20 virtual economies for *case 2*. A study of the frequency of convergence[93] follows. Among all virtual economies, 10 of them (for each case) will be selected in order to analyse their convergence properties. In particular, the convergence towards uniform profit rates, uniform wage rates and equal distribution of the surplus will be checked.

- **Experiment 3**: the organization and structure of the experiment is the same of experiment 2 with the difference that, with respect to experiment 2, this experiment considers the possibility for producers to choose the best method among a set of different methods (10 methods) for the production of their commodity. Beyond the convergence properties also the convergence towards the *wage-profit frontier* has been examined and other efficiency indexes have been generated. Moreover, the role of innovation has been investigated.

- **Experiment 4**: in this chapter the effect of the introduction of an exogenous and endogenous interest rate has been considered. For this experiment the production capabilities of the virtual economies have been strengthened through the introduction of the *residual mechanism* (see figure 10.2 and chapter 3.5.8). 3 different virtual economies have been considered, with 3 producers and 54 workers for the heterogeneous case[94]. The 3 economies have been tested for a range of possible interest rates (from -5% to 5% step 1%). The relevant case has been analysed and tested also under the case of endogenous interest rate. The characteristics of convergence have been analysed and the Pareto improvements with respect to the base case with interest rate equal to zero have been computed.

[93]This number of virtual economies should be considered significant and sufficient for drawing conclusions with respect to the frequency of converge. Indeed, firstly it should be considered that the generation of each virtual economy implies a computation time in the order of hours with a high performance computer; secondly it has been preferred to generate a lower number of virtual economies with a significant value of the stopping rule $\bar{t}$=1000 instead of too many virtual economies with a lower stopping rule not necessary neither significant in order to investigate convergence properties. As a consequence, 20 economies have been considered respectively for the generation of figure 6.1, 6.2, 7.2 and 7.3.

[94]For the purposes of this experiment, 3 producers and 54 workers (heterogeneous case) have been considered sufficient in order to investigate the relative research question about the role of an economic policy.

5 Experiment 1: convergence towards uniform wage rates and non-uniform profit rates for the Sraffian case

5.1 Introduction to experiment 1

Sraffa begins Chapter 1 in *Production of Commodities by Means of Commodities* (*PCMC*) with the following statement:

> Let us consider an extremely simple society which produces [...] to maintain itself. Commodities are produced by separate industries and are exchanged for one another at a market held after the harvest (Sraffa, 1960, p.3).

There are many ways that one can read or interpret *PCMC*. Here we endorse a particular interpretation that sees *PCMC* as the study of the virtual prices that would or could allow the economic system to replicate. As explained by Sraffa, his major concern is not on how trade takes place "at a market held after the harvest" but the *ex-ante* study of the prices that could allow the system to replicate itself. In *PCMC* Sraffa shows what are the conditions for the determination of these virtual prices are. He is very careful in pointing out, that these prices are not *market prices*:

> Such classical terms as 'necessary price', 'natural price', or 'price of production' would meet the case, but value and price have been preferred as being shorter and in the present context (*which contains no reference to market prices*) no more ambiguous (Sraffa, 1960, p.9, emphasis added).

What would actually happen during the market days is not discussed. As it is well known, Sraffa makes the further assumption that these prices should be such that the profit rates are uniform. Zambelli (2016) has relaxed this assumption; he concludes that the uniform rate of profit case is a very special one: there are price vectors that would allow the system to replicate, but would imply a vector of profit rates that are not necessarily uniform. Furthermore, the case in which there is just a uniform rate of profits is, according to Zambelli (2016), a very special one.

In this chapter we make assumptions with respect to the functioning of the market by using the *digital economic laboratory* as described in chapter 3 and the behavioural functions and the setting of the *thought experiments* described in chapter 4. The process going from production, exchange and production is clearly dynamic in the sense that quantities and prices do change in time, from one period to the other. Zambelli (2016), coherently with a Sraffian approach, has shown that the prices that would allow the system to reproduce itself are not unique and profit rates may not be uniform. Hence there is a huge variety of distributions of the generated surplus among producers and workers, that would allow the economic system to reproduce. The reproduction of the system can take place also thanks to the emergence of credit and debt relations. Zambelli (2016) studies, following the Sraffian method outlined in *PCMC*, the possibility that the system may reproduce itself exactly in time in the same proportions and with the generation of the same surplus as occurred in the previous production period[95]. In Sraffa (1960) and in Zambelli (2016) behavioural and trading rules are not specified. This is also true for the great number of contributions that may be considered to follow the Sraffian tradition.

Here we will not discuss this tradition, but through the use of the *digital economic laboratory*, where virtual agents (the ARAs) trade, produce and consume, and where from one production period to the other quantities and prices do change in accordance with agents behavioural description and location, we investigate whether there would be convergence towards the production or natural prices computed in Sraffa (1960) or Zambelli (2016).

Clearly the prices observed in the *digital economic laboratory* are not necessarily the prices that would allow the system to replicate. These prices will be called *market prices* in contrast with Sraffa's prices or values. Indeed, Sraffian prices are the virtual prices computed *ex ante* before the market days begins, while *market prices* are the prices computed as the result of agents interactions and market structure. It is only when the two set of prices are the same that one can say that the system has converged towards Sraffian prices (natural or production).

In this chapter the set of methods used by Sraffa in the example given at page 19 of *PCMC* has been considered. In the next chapter more general cases have been considered. Here our main objective has been the study of whether the economic system having the same production methods as in *PCMC*, page 19, once ARAs are specified and trading rules are given, will converge towards Sraffian production prices. Given these prices we can check whether there is convergence towards the uniform wage rates and uniform profit rates.

In the subsequent sections of this chapter the fundamental equations introduced in chapter 3 have been developed according to Sraffa's periodization. Then the experimental setting has been described in details. Finally, the results have been reported and commented.

[95] Clearly, this is a theoretical starting point. In the *thought experiments* of the present thesis the assumption is that methods of production do not change in time.

5.2 The market after the harvest

5.2.1 Methods of production and divisibility

Sraffa's main goal in *PCMC* is to determine the:

> Set of exchange-values which *if adopted* by the market restores the original distribution of the products and makes it possible for the process to be *repeated*; such values spring directly from the methods of production (Sraffa, 1960, p.3, emphasis added).

Here the focus will not be on the determination of the prices and wages that, "if adopted", will "restore the original distribution of the product", but on the sequence going from a period t to the other[96].

We assume that there are k producers producing n commodities[97]. Each commodity g is produced with one method of production linking the *means of production* with the *output*. Considering that each producer i (i=1,...,k)[98] can choose among a set of possible alternative methods for the production of his commodity g (g=1,...,n), the method z_g (for z_g=1,...,m_g) for the production of commodity g is given by:

$$\mathbb{a}_{i1}^{z_g}, \mathbb{a}_{i2}^{z_g}, \ldots, \mathbb{a}_{ig}^{z_g}, \ldots \mathbb{a}_{i,n-1}^{z_g}, \mathbb{a}_{in}^{z_g}, \mathbb{l}_i^{z_g} \xrightarrow{production} \mathbb{b}_i^{z_g} \tag{5.1}$$

where $\mathbb{l}_i$ is the necessary labour, $\mathbb{a}_{ig}$ is the amount of commodity g used as physical mean of production by the producer i for the production of the quantity $\mathbb{b}_i$. We assume that production (or method of production) is divisible. This means that given any positive value x_i, also defined as activity level, the following would be feasible:

$$x_i\mathbb{a}_{i1}^{z_g}, x_i\mathbb{a}_{i2}^{z_g}, \ldots, x_i\mathbb{a}_{ig}^{z_g}, \ldots x_i\mathbb{a}_{i,n-1}^{z_g}, x_i\mathbb{a}_{in}^{z_g}, x_i\mathbb{l}_i^{z_g} \xrightarrow{production} x_i\mathbb{b}_i^{z_g} \tag{5.2}$$

[96] In a way we follow a similar method as suggested by Hicks (1939, Ch.10, pp. 121-29) in *Value and Capital*. But, as it will be clarified, accounting disequilibrium and credit and debt relations are allowed to emerge.

[97] The notation is coherent with that used in chapter 3.

[98] It has been explained in chapter 3 that the total number of producers in the whole economy is equal to k for i=1,...,k but that inside the local market the total number of producers is equal to n for $\bar{i}$=1,...,n. In this chapter we are dealing with Sraffa's case of page 19 in *PCMC* with only 3 producers. In order to make the notation easier to follow, in this chapter we will denote with i a producer inside the local market (for i=1,...,k with k=n).

5.2.2 Periodization: exchanges, production and consumption

If we try to design one possible dynamics of this *simple society* as defined by Sraffa (1960, p.3) we can imagine that at the beginning of the market day, producers enter the market endowed with the commodities produced during the previous *production and harvest period* and previously accumulated *financial means of exchange*. Producers's purpose is to exchange their endowments in order to organize next production that will be partly exchanged in the consumption goods market of the present time t and partly devoted to the restoration of production and exchanged in the production goods market during period $t + 1$[99].

Our aim is to study the dynamics, period after period, of prices and quantities produced and exchanged. Here we will consider the period t composed of three moments or time intervals:

- *Exchange 1*: exchange of the endowments (labour force and previous production) in the production goods market (*pgm*) and the labour force market (*lm*).
- *Production*: production and decision of the quantities to devote to the consumption goods market (*cgm*).
- *Exchange 2*: exchange in the consumption goods market (*cgm*). The other production is endowment for the next production period $t + 1$.

If the prices are market prices (that is not necessarily natural prices or production prices) it may not be the case that the purchasing power of the individual agents will be sufficient to buy the quantities which they desire or need to buy. In other words, there might be situations where some agents do not have enough purchasing power to buy what they would and others cannot sell what they would precisely because the potential buyers do not have the necessary purchasing power. This situation would not emerge if (*ex-post*) market prices happened to be equal to (*ex-ante*) natural or production prices. Clearly in this type of situation where market prices are not the same as natural prices the sellers could extend credit (as deferred forms of payments, such as IOUs) to the buyers, so that the desired exchanges can take place. The existence and determination of these *financial means of exchange* would have a significant impact on the decisions of the ARAs and hence on market prices, exchanges, consumption and production.

[99] This is exactly the dynamics considered in the *laboratory*. The purpose of the present section is to explain the coherence between Sraffian analysis and the *laboratory*.

Producers

In our virtual economy the producers would enter the market at the beginning of period t endowed with part of the quantities[100] produced during the period $t-1$ and their stocks of financial assets or liabilities accumulated up to period $t-1$[101].

During each period t each producer i (for i=1,...,k) producing commodity g (for g=1,...,n) will sell his physical endowment represented by the share $\alpha_{i,t-1}$ of the previous production devoted to the new production $[\alpha b]_{i,t-1}$ in the production goods market[102] and will aim at buying the necessary means of production (in terms of commodities a_g and labour force q^L) so as being able to produce during the time t the quantity b_{it} that will be partly (the share β_{it}) exchanged in the consumption goods market and partly (the share α_{it}) devoted to the exchange in period $t+1$ in the production goods market. Pay attention that in the same period t we are considering quantities produced in different production periods: indeed, the commodities sold in the production goods market have been produced in the previous production period while the commodities sold in the consumption goods market have been produced during the present period t. It is always true that $b_{it} = [\alpha b]_{it} + [\beta b]_{it}$, but the actual quantities would obviously be determined by the prices and market conditions[103].

$$\overbrace{[\alpha b]_{i,t-1} \xrightarrow{\textit{exchange pgm lm}} [q_{i1},...,q_{in},q_i^L]_t}^{\textit{exchange 1}} \overbrace{\xrightarrow{\textit{production}} [a_{i1},...,a_{in},l_i]_t \to b_{it}}^{\textit{production}} \quad ...$$

$$\overbrace{... \xrightarrow{\textit{exchange cgm}} [\beta b]_{it}}^{\textit{exchange 2}} \overbrace{\xrightarrow{\textit{consumption}} \boldsymbol{c}_{it} \xrightarrow{\textit{new endowment}} [\alpha b]_{it}}^{\textit{consumption and new endowment}}$$

The quantities $[q_{i1},...,q_{in},q^L]_t$ represent what each producer i has at disposal for production after the exchange in the production goods market pgm and the labour force market lm. The part of production devoted to consumption ($[\beta b]_{it}$) corresponds to the quantities effectively sold to producers and workers. The vector of consumption of each producer (for $i = 1,...,k$) corresponds to $\boldsymbol{c}_{it}$.

[100] The part of the previous production devoted to the next production.

[101] In the *laboratory* this stock of financial assets is called financial *wealth*.

[102] In order to be as close as possible to Sraffa's case it has been assumed in this section that $\eta_{i,t-1}$=0. For more details, see equation 3.26 and subsection 3.5.8 about investment decision.

[103] This distinction will become important when the profit rates will be computed. See section 5.2.3.

Workers

In the same way each worker j (for j=1,...,K) would enter the market at the beginning of period t endowed with its labour force L_j^{sw} and his stocks of financial assets or liabilities accumulated up to period $t-1$.

During the period t each worker j will sell a fraction of what he owns (his labour force), so as to be able to consume during next period t. There could be a difference between his desired labour supply L_j^{sw*} and what effectively he will be able to sold in the labour market q_j^L[104]. Indeed, prices and wages can determine a potential rationing in the labour market. Each worker j (for $j = 1, ..., K$) will consume his vector of consumption goods $\boldsymbol{c}_{jt}$.

$$L_{jt}^{s*} \xrightarrow{exchange} q_{jt}^{L} \xrightarrow{exchange\ \ cgm} \boldsymbol{c}_{jt}$$

5.2.3 Economic accounts: revenues and expenditures

Each producer i exchanges his commodity with the other two neighbouring producers. This triple of producers represent a *local market*. By *local market* s, the producer $s = i$ and the other two neighbouring producers $i-1$ and $i+1$ in growing order according to the commodity produced will be denoted. The *local market* in which producer i is present will be denoted by $s(i)$, the *local market* in which worker j is present will be denoted by $s(j)$. In Sraffa's example page 19 in *PCMC* there are only 3 producers but with this notation it is possible to imagine also to enlarge the number of producer in order to have many local markets. This hypothesis has been tested in the section 5.3.4 and the following equations can refer to whatever local market[105].

The economic accounts of the producers

The difference between the revenues and the expenditures of each producer i represents the variation in his financial position[106].

[104] Obviously, the quantity q_i^L bought by the producer i correspond to the sum of the labour force sold by each worker j (q_j^L) enrolled in the enterprise i.

[105] In the *digital economic laboratory*, as explained in chapter 3, a lot of local markets and different producers producing the same commodity can be considered. For example, with the *local market* s=5 we make reference to the producers 4,5,6 (in this order) and with the *local market* s=7 we make reference to the producers 7,8,6 (in this order). Remember that in the case of 9 producers, producers 1,4,7 produce the first commodity, producers 2,5,8, produce the second commodity while producers 3,6,9 produce the third one. Remember that with $\boldsymbol{p}_{s(i)}$ and i=7 we are making reference to s=7 and the prices of producers 7,8,6 in this order.

[106] In the *digital economic laboratory* and in the *thought experiments* it is hypothesized that also producers can work in their enterprise. In order to reconstruct these equations as close as possible to Sraffa's framework, this hypothesis has not been considered in this section.

$$revenues_{it} = \alpha_{i,t-1}b_{i,t-1}p_{it} + \beta_{it}b_{it}p_{it} \tag{5.3}$$

$$expenditures_{it} = \bar{\boldsymbol{q}}_{it}\boldsymbol{p}_{s(i)t} + q^L_{it}w_{it} + \bar{\boldsymbol{c}}_{it}\boldsymbol{p}_{s(i)t} \tag{5.4}$$

$$F_{it} = revenues_{it} - expenditures_{it} \tag{5.5}$$

The economic accounts of the local markets

A different accounting becomes useful or even necessary when we consider the productive units (enterprises[107]) with the aim of computing the profit rates. We have to consider what happens inside each *local market s*. The total level of profits depends on the difference between revenues and expenditures of each enterprise. In the case in which expenditures have been bigger with respect to revenues, the loss corresponds to a negative variation in the financial position of the enterprise.

$$revenues^{ent}_{it} = \alpha_{i,t-1}b_{i,t-1}p_{it} + \beta_{it}b_{it}p_{it} \tag{5.6}$$

$$expenditures^{ent}_{it} = \bar{\boldsymbol{q}}_{it}\boldsymbol{p}_{s(i)t} + q^L_{it}w_{it} \tag{5.7}$$

$$F^{ent}_{it} = revenues^{ent}_{it} - expenditures^{ent}_{it} \tag{5.8}$$

The above equations imply that with market prices it is possible to have inside each *local market s* (where are present k producers producing the n commodities with k=n)[108]:

$$\begin{aligned} a_{11}p_{1t} + a_{12}p_{2t} + ... + a_{1n}p_{nt} + l_1w_{1t} &\geq b_1\tilde{p}_{1,t+1} \\ a_{21}p_{1t} + a_{22}p_{2t} + ... + a_{2n}p_{nt} + l_2w_{2t} &\leq b_2\tilde{p}_{2,t+1} \\ ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... & \quad ... \quad ... \quad ... \quad ... \\ a_{n1}p_{1t} + a_{n2}p_{2t} + ... + a_{nn}p_{nt} + l_nw_{nt} &\geq b_n\tilde{p}_{n,t+1} \end{aligned} \tag{5.9}$$

which in matrix notation could be written as:

$$\boldsymbol{A}_{st}\boldsymbol{p}_{st} + diag(\boldsymbol{w}_{st})\boldsymbol{L}_{st} \lesseqqgtr \boldsymbol{B}_{st}\tilde{\boldsymbol{p}}_{s,t+1} \tag{5.10}$$

Remember that here the wage rate can be heterogeneous among enterprises and for this reason is no more a single value as in Sraffa. The presence of $\tilde{p}$ is necessary because part of this production has been sold at the current

[107]When we make reference to the enterprises we use *ent* as superscript.

[108]For example, for the case of k=3, we consider the producers 1,2,3 inside local market s. They can correspond, for example, to producers 7,8,6 in the local market s=7.

prices in the consumption goods market. $\tilde{p}_{i,t+1}$ is the price that allows to have[109]:

$$b_{it}\tilde{p}_{i,t+1} = [\beta b]_{it}p_{it} + [\alpha b]_{it}p_{i,t+1} \tag{5.11}$$

The profit rate can be interpreted as:

$$\begin{aligned}
(a_{11}p_{1t} + a_{12}p_{2t} + ... + a_{1n}p_{nt} + l_1 w_{1t})(1 + r_1) &= b_1\tilde{p}_{1,t+1} \\
(a_{21}p_{1t} + a_{22}p_{2t} + ... + a_{2n}p_{nt} + l_2 w_{2t})(1 + r_2) &= b_2\tilde{p}_{2,t+1} \\
... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \\
(a_{n1}p_{1t} + a_{n2}p_{2t} + ... + a_{nn}p_{nt} + l_n w_{nt})(1 + r_n) &= b_n\tilde{p}_{n,t+1}
\end{aligned} \tag{5.12}$$

Which is equivalent to:

$$\begin{aligned}
a_{11}p_{1t} + a_{12}p_{2t} + ... + a_{1n}p_{nt} + l_1 w_{1t} + F^{ent}_{1t} &= b_1\tilde{p}_{1,t+1} \\
a_{21}p_{1t} + a_{22}p_{2t} + ... + a_{2n}p_{nt} + l_2 w_{2t} + F^{ent}_{2t} &= b_2\tilde{p}_{2,t+1} \\
... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \\
a_{n1}p_{1t} + n_{n2}p_{2t} + ... + a_{nn}p_{nt} + l_n w_{nt} + F^{ent}_{nt} &= b_n\tilde{p}_{n,t+1}
\end{aligned} \tag{5.13}$$

In matrix notation:

$$(\boldsymbol{I} + \boldsymbol{R}_{st})\boldsymbol{A}_{st}\boldsymbol{p}_{st} + diag(\boldsymbol{w}_{st})\boldsymbol{L}_{st} = \boldsymbol{B}_{st}\tilde{\boldsymbol{p}}_{s,t+1} \tag{5.14}$$

$$\mathbf{A}_{st}\mathbf{p}_{st} + diag(\boldsymbol{w}_{st})\boldsymbol{L}_{st} + \mathbf{F}^{ent}_{st} = \mathbf{B}_{st}\tilde{\mathbf{p}}_{s,t+1} \tag{5.15}$$

where $\mathbf{B}_{st}$ is a diagonal matrix whose diagonal elements are the single producers' quantity produced b_{it} while $\mathbf{R}_{st}$ is a diagonal matrix whose diagonal elements are the single producers' profit rates, r_{1t}, r_{2t}, r_{3t} (vector $\mathbf{r}_{st}$) at time t in the *local market* s. When $revenues^{ent}_{it} - expenditures^{ent}_{it} \leq 0$ this implies that $r_{it} \leq 0$ and also $F^{ent}_{it} \leq 0$. The equation can be rewritten as:

$$\boldsymbol{B}_{st}\tilde{\boldsymbol{p}}_{s,t+1} = (\boldsymbol{I} + \boldsymbol{R}_{st})\boldsymbol{A}_{st}\boldsymbol{p}_{st} + diag(\boldsymbol{w}_{st})\boldsymbol{L}_{st} \tag{5.16}$$

The above equations correspond to the Sraffian schemes for the non-uniform rate of profits (Zambelli, 2016) when $\boldsymbol{R}_{st} \geq 0$ and $\tilde{\boldsymbol{p}}_{s,t+1} = \boldsymbol{p}_{st}$.

[109] It is important to pay attention to the periodization. In the *digital economic laboratory*, explained in chapter 3, the production b_{it} is partly sold in the consumption goods market at time t and partly devoted to the production goods market at time $t + 1$. In Sraffa, b_{it} represents the quantities sold in the production goods market and the consumption goods market of the production period t. In order to compute profit rates coherently with Sraffa's equation, the presence of $\tilde{p}$ is necessary. Remember that despite $\boldsymbol{w}$ is a matrix and $\boldsymbol{L}$ is a vector, they have been written in this way in order to be as close as possible to Sraffa's notation.

The economic accounts of the workers

The revenues of worker j (for j=1,...,K) depends on the labour sold in the labour force market[110]:

$$revenues_{jt} = q^L_{jt} w_{it} \tag{5.17}$$

The expenditures during the period t are given by the quantity of commodities that the worker j is able to buy $\bar{\boldsymbol{c}}_{jt}$. The worker's *expenditures* would be given by:

$$expenditures_{jt} = \bar{\boldsymbol{c}}_{jt} \boldsymbol{p}_{s(j)t} \tag{5.18}$$

The variation in the financial position of the individual workers j is given by:

$$F_{jt} = revenues_{jt} - expenditures_{jt} \tag{5.19}$$

5.2.4 Financial Balances

Clearly those industries that are in the condition for which the expenditures would be higher than the revenues would not have the purchasing power to buy the means of production necessary to replicate the production of the previous period. These enterprises are in a condition of potential financial deficit. They would be able to purchase the necessary means of production only by agreeing to a deferred payment to take place during the years to follow in favour of the enterprises in potential surplus. If the quantities have to be restored independently from any given price we must have that in general the inequalities of eq.5.10 may be *eliminated* if we allow for deferred payments which might take the form of IOUs (i.e., *I owe you*: selling of real quantities or labour now with the promise of paying back sometimes in the future). The accounting balances would require, if we accept the existence of credit and debt relations:

$$\sum_{i=1}^{k} revenues_{it} + \sum_{j=1}^{K} revenues_{jt} = \sum_{i=1}^{k} expenditures_{it} + \sum_{j=1}^{K} expenditures_{jt} \tag{5.20}$$

Which implies:

$$\sum_{i=1}^{k} F_{it} + \sum_{j=1}^{K} F_{jt} = 0 \tag{5.21}$$

These equations mean that, for accounting consistency, the sum of all the debts and credits has to be always equal to zero.

[110]The quantity of labour sold by worker j is multiplied for the wage of the enterprise i in which the worker j is enrolled.

5.3 The thought experiment 1

The experimental setting common to all the *thought experiments* of the present thesis has been already described in chapter 4. Here are reported only the particular features that characterize the present *thought experiment.*

5.3.1 Experimental setting

The experimental setting considered in the present chapter is subdivided in two main cases:

- *Case 1*: is characterized by the presence 3 producers and 48 workers (two cases, with homogeneous or heterogeneous ARAs, will be considered)[111].
- *Case 2*: is characterized by the presence 9 producers and 144 workers (two cases, with homogeneous or heterogeneous ARAs, will be considered). This last case implies the presence of 9 local markets as in figure 3.4 with 3 producers producing the same commodity.

Each producer has at disposal, at the beginning of the production periods, 16 workers, that can migrate[112]. There is only one method of production at disposal and it is exactly the one present in Sraffa's *PCMC* page 19.

5.3.2 The research questions

The first purpose of the present chapter is to verify if Sraffa's example of a three sector economy enriched with behavioural functions in a dynamical framework is able to converge towards an *equilibrium position.*

We define an *equilibrium position* as a position in which prices and wages announced do not change anymore; all markets are in equilibrium (demand equal to supply in all markets; i.e., there is no more rationing), migration stops, the system reproduces itself (i.e., same level of production and surplus without residuals) without the creation of new IOUs (no new debt and credit relations). This implies that the prices of convergence are *production prices* with zero or positive profit rates. In particular, these characteristic of the eventual equilibrium have been analysed:

[111] Homogenous ARAs imply that the matrices $\mathbf{\Theta}$ and $\mathbf{\Omega}$ are equal for all the ARAs. In other words, the ARA have the same characteristics and the same initial information set.

[112] The total number of worker is higher with respect to the original Sraffa's example because in the present experiments migration is allowed and in order to test convergence towards uniform wage rates it is necessary to consider a higher number of workers. Despite this, the method at disposal of each ARA producer is the same as in Sraffa's *PCMC* page 19.

- **Convergence towards uniform profit rates**. Given the convergence towards production prices it is possible to check convergence towards uniform profit rates.

- **Convergence towards uniform wage rates**. Given the possibility for workers to migrate where the income is higher, if the system converges towards a position of equilibrium, the wage rates should become uniform among all enterprises and the migration should stop.

- **Convergence towards an equal distribution of the surplus**. Given the initial condition of equal distribution of the physical surplus (94% of the surplus has been associated to workers given that they are 48, with respect to the 3 producers, for a total population of 51 ARAs)[113] it is interesting to test the type of distribution of the surplus associated with the convergence position.

These characteristics of the equilibrium position will be investigated also for the cases with 9 producers and 144 workers in order to test the role of local markets for the problem of convergence. Both cases with homogeneous and heterogeneous ARAs will be tested.

[113] The same proportions also for the case with local markets where workers are 144 and producers 9. Also in this case 94% of the surplus has been associated to workers. It is worth remembering that the initial production prices and wage have been computed starting from this distribution.

5.3.3 Results for the *case 1*

In the subsequent table the results for the *case 1* with 3 producers and 48 workers, with homogeneous and heterogeneous ARAs are reported.

Research questions	Homogeneous case	Heterogeneous case
Convergence towards production prices	*yes*	*yes*
Convergence towards uniform profit rates	*no*	*no*
Convergence towards uniform wage rates	*yes*	*yes*
Percentage of surplus to producers	98	98
Percentage of surplus to workers	2	2

Table 5.1: **Results of the experiments**: using the numbers of Sraffa's example of a three sector economy of *PCMC* page 19 and a total population of 51 ARAs (48 workers and 3 producers) for the two cases with homogeneous or heterogeneous ARAs. Remember that at the beginning of the production periods the 94% of the surplus was associated to workers.

According to the results of the experiments, the following answers to the initial research questions have been obtained:

- ***The system converges towards production prices.*** The production system is able to converge towards an *equilibrium position* stable in time characterized by the presence of production prices, no creation of new IOUs and positive profit rates.

- ***The system DOES NOT converge towards uniform profit rates.*** This could be surprising for those that interpret the convergence towards an uniform profit rate as the natural outcome of the competition process between producers in time. This result should be interpreted instead in the light of the results presented in Zambelli (2016). If the objective of the agents is to maximize their surplus shares, there is no reason to consider an uniform profit rate as a natural outcome of the competition process.

- ***The system converges towards uniform wage rates.*** This implies that migration stops at an equilibrium level in which there is no more incentive to migrate. This enforces the stability of the equilibrium position and the idea that migration is a competitive process between workers able to bring to convergence towards the same wage rate.

- ***The system DOES NOT converge towards an equal distribution.*** Despite a starting position characterized by an equal distribution (94% of the surplus was associated to workers) at the end of the production periods the situation is completely reversed: 98% of the surplus goes to producers and only 2% to workers, who represent the 94% of the population.

5.3.4 Results for the *case 2*: the influence of the local markets

In these further experiments, the role of local markets has been tested. 9 producers and 144 workers have been considered. In the following table the results are summarised:

Research questions	Homogeneous case	Heterogeneous case
Convergence towards production prices	*yes*	*yes*
Convergence towards uniform profit rates	*no*	*no*
Convergence towards uniform wage rates	*yes*	*yes*
Percentage of surplus to producers	98	98
Percentage of surplus to workers	2	2

Table 5.2: **Results of the experiments**: considering local markets and a total population of 153 ARAs (144 workers and 9 producers) for the two cases with homogeneous or heterogeneous ARAs. Remember that at the beginning of the production periods the 94% of the surplus was associated to workers.

The presence of local markets does not change the conclusions reached in the previous case. Producers of the same commodity converge towards the same prices and the same profit rates only in the homogeneous case. The values reached in each *local market* in the homogeneous case are the same with respect to the homogeneous case with only 3 producers. Indeed, all producers of the same commodity converge towards the same profit rate while workers converge towards an uniform wage rate. This is not surprising because the initial conditions and parameters are the same and ARAs are all homogeneous. The heterogeneous case is instead different with respect to the case with only 3 producers. This because now ARAs are heterogeneous and different workers can migrate in the neighbouring enterprises in time along all the lattice. Moreover, producers of the same commodity are now different and can take different decisions. This competition process in the labour market is the mechanism that allows to reach an uniform wage rate also in the heterogeneous case but the profit rates are now different among producers of the same commodity (i.e., inside the same industry). Also in this experiment workers represent the 94% of the population and it is interesting to note that also in these experiments the competition process brings to an unequal distribution.

5.3.5 The dynamics towards convergence

It has been said previously that when the economy converges towards an equilibrium position, this implies that debt and credit positions tend to zero and market prices are equal to production prices. This implies that profit rates converge towards positive values (or values equal to zero). This convergence is clear in the figure 5.2, where the dynamics of financial *wealth* positions of all ARAs and the dynamics of the profit rates in the last 200 periods have been reported. The convergence towards positive profit rates implies that ARA producers need no more to underwrite IOUs and the financial *wealth* accumulated can converge towards zero in time. Despite this convergence, it is interesting to note how profit rates are non-uniform.

In the figure 5.1 the wage-profit curves that correspond to all the combination of the profit rate and the wage rate for which the profit rate is maximum for each enterprise are reported. Because the unique method at disposal is the same of Sraffa's page 19 of *PCMC* the curves are equal to the ones reported in Zambelli (2016, p.20), but here we are able to add also the profit rates and wages rates of convergence. The profit rates of convergence are the same reported in figure 5.2 where instead also the path of convergence for the last 200 periods is reported. Obviously these profit rates of convergence are lower than their wage-profit curves (see figure 5.1). As already noted, profit rates are non-uniform as in the case with only 3 producers (see table 5.1). In this case, with 9 local markets and heterogeneous ARAs, it is possible to note that neither the producers of the same commodity have converged towards the same profit rate despite the wage rates are uniform for all the 9 enterprises. Wage rates are uniform thanks to the competition process promoted by the migration of workers but a similar process has not been able to bring profit rates to uniformity (neither for the case of homogenous ARA). This seems to sustain the proposition of Zambelli (2016) who has questioned the relevance of the assumption that sees the uniform profit rate as a natural long run point of convergence determined by market forces in a free competition environment. Remember that the idea according to which the uniform profit rate is a natural long run point of convergence has been developed after Sraffa (see Kurz and Salvadori, 2005) and it is questionable the idea that Sraffa considered the uniform profit rate something more that an assumption (see on this point Zambelli, 2016, pp.3-4). Finally, it is interesting to note how different is the distribution of convergence with respect to the initial equilibrium position where distribution was equal (see also the results from the table 5.2).

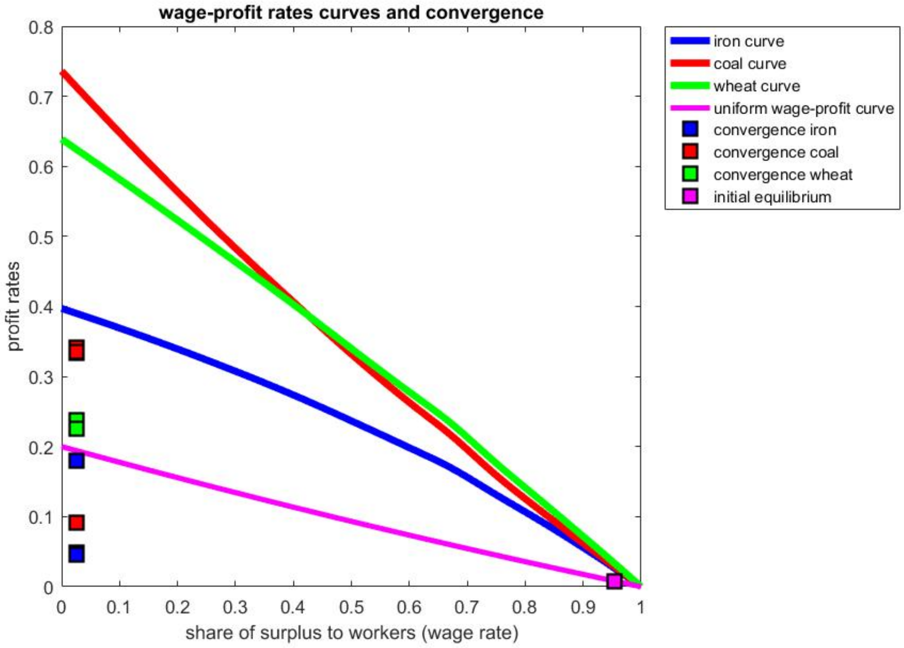


Figure 5.1: **Maximum enterprise profit rate as a function of the share of the surplus to workers. Heterogeneous ARAs case with 9 local markets**: in the figure, the wage-profit curves (in colour red (coal), green (wheat) and blue (iron)) that correspond to all the combination of the profit rate and the wage rate for which the profit rate is maximum for each enterprise are reported. When, for example, a share of the surplus equal to zero is associated to the workers of the iron enterprise, so that the profit rate of the iron enterprise is maximum, the profit rates of the wheat and coal industry are equal to zero. The other points of the curve can be uniquely determined. The lowest curve is the *wage-profit curve* with uniform profit rates. The points are the profit rates and the wage rates of convergence. The point on the *uniform wage-profit curve* is the initial equilibrium. It is interesting to note how the system converges towards uniform wage rates but non-uniform profit rates.

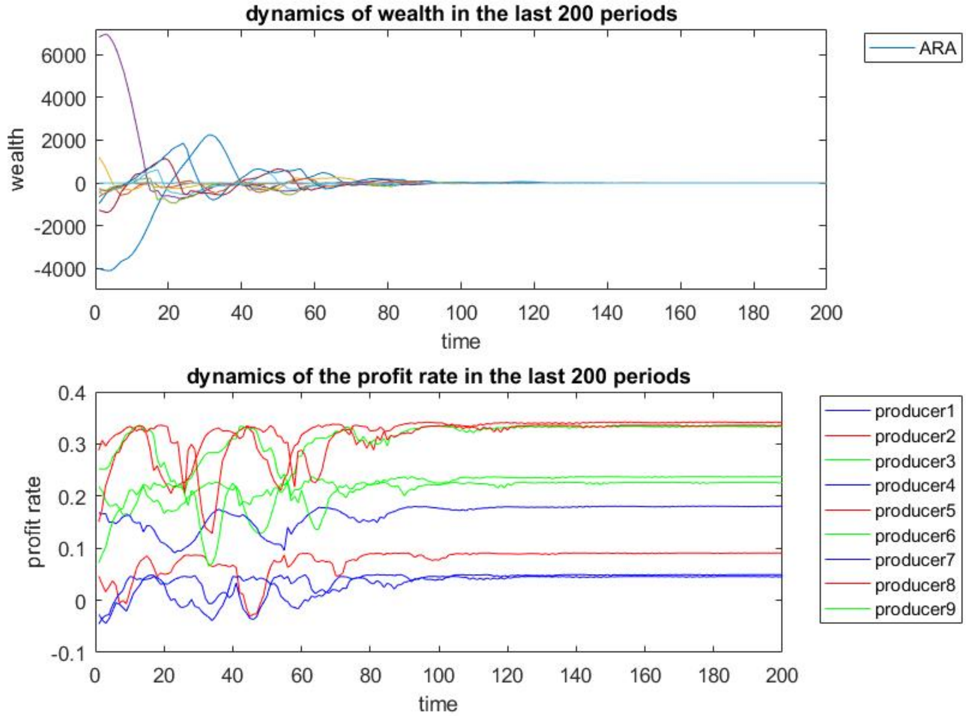


Figure 5.2: **The dynamics of wealth of each ARA and the dynamics of the profit rate for the case of heterogeneous ARAs and 9 local markets**: in the first figure the dynamics of wealth in the last 200 periods are reported. Each line represents the wealth of a single ARA (worker or producer, in total 153 ARAs). When convergence is reached wealth approaches to zero for all ARAs. This because prices and wages reach particular levels that allow all ARAs to earn a positive income. In the figure at the bottom the dynamics of the profit rates in the last 200 periods are represented. Before convergence, profit rates can be negative. The accumulation of losses explains the negative wealth of the upper figure. If convergence prices and wages do not change anymore and also profit rates converge towards positive values, market prices correspond to production prices. The colour of the line that corresponds to the profit rate of a particular producer allows to make a comparison with figure 5.1. Indeed, in the figure the profit rates of all producers of coal in red, of wheat in green and of iron in blue are reported. It can be noted that the converge points are the same of figure 5.1.

5.4 Relationship between preferences, prices, demand, supply and distribution

The construction of the *digital economic laboratory* and the subsequent *thought experiments* allow to create virtual economies in which only the characteristics of ARAs, the connected network, the trading rules and the behavioural functions have been exogenously constructed. This means that the relationship between demand, supply, prices and distributional variables have not been postulated but are endogenously determined by the dynamics of the systems. At the end of the experiment, the total quantities demanded, the total quantities supplied, the prices and the distribution among ARAs can be observed, so that the relationships between economic magnitudes can be investigated.

5.4.1 The experimental setting

These relationships have been investigated considering a further generalization of the previous experiments. Indeed, the first experiment of the *case 1* (with 3 producers and homogeneous ARAs) has been identically repeated changing only the preferences of each commodity. This means that each parameter of the utility function has been progressively augmented on a scale of 4 steps[114] given the other parameters (that remain fixed) for a total number of 15 experiments (3 commodities for 4 growing steps plus the initial states). In order to generalize the results, the experiment has been repeated 4 times for 4 different initial conditions (i.e., different initial combination of the parameters for a total number of virtual economies equal to 60).

5.4.2 The research questions

This new set of virtual economies has been analysed in order to:

- Investigate the influence of preferences on prices and surplus.
- Verify the well-known inverse relationship between prices and demand (the negative slope of the demand function) and the direct relationship between prices and supply (the positive slope of the supply function).
- Investigate the relationship between preferences and distribution.

[114] Between a minimum equal to the initial value of the parameter and a maximum close to 1.

5.4.3 Relationship between prices and quantities demanded and supplied

The well-known inverse relationship between prices and demand and the direct relationship between prices and supply have been investigated through the computation of the frequencies (i.e., percentage of the times in which the expected relation has been verified; for example, the case of the relation between prices and demand corresponds to the percentage of the times in which demand grows (declines) while price declines (grows)) and the estimation of demand and supply elasticity using *Ordinary Least Squares*. In particular, the subsequent regression model (respectively for demand d_{gt} and supply q_{gt} of each good g=1,...,n n=3 at time t) has been considered:

$$ln(d_{gt}) = \beta_0 + \beta_1 ln(p_{gt}) + v_{gt} \;\; g = 1, ..., n \tag{5.22}$$

$$ln(q_{gt}) = \beta_0 + \beta_1 ln(p_{gt}) + u_{gt} \;\; g = 1, ..., n \tag{5.23}$$

In the subsequent table the results are reported: the frequencies, the estimates (*beta*), the *t Statistics* and the values of R^2 (all average values for all commodities and all experiments of the cases in which the *t Statistic* was significant).

Relationship	frequency	β_1	tStat	R^2
demand and prices	60	-0.26	-8.77	0.14
production and prices	50	-0.24	-8.03	0.13

Table 5.3: **Results of the experiment**: computation of the frequencies (i.e., percentage of the times in which the expected relation has been verified; for example, the case of the relation between prices and demand corresponds to the percentage of the times in which demand grows (declines) while price declines (grows)) and the estimation of demand and supply elasticity using *Ordinary Least Squares*: in particular the average values of the β_1, the *t-Statistics* and the R^2 of the cases in which the β_1 was significant have been reported.

The fundamental conclusion is that:

- The inverse relation between **prices and demand** is verified for the **60%** of the virtual economies.
- The direct relation between **prices and supply** is verified for the **50%** of the virtual economies.

These results suggest a weak relation between prices and demand and in particular between prices and supply. The OLS estimation, by measuring the intensity of these relations, seems to confirm this impression. Despite the sign of the *beta* confirms a negative relation between prices and demand, the absolute value of the elasticity of demand is less than 1 thus indicating that demand is inelastic. Moreover, it is worth remembering that the sign of *beta* was negative in the 70% of the cases while in the 15% it was not and finally in the 15% of the cases the *beta* was not significant. If we consider the case of the relation between prices and supply, the results of the OLS estimation are surprising. Indeed, the sign of the *beta* is negative. This could be justified by the fact that a rise in the price of a particularly commodity implies that the other producers that use that commodity in their production processes will decrease their production with probably negative effects for the production of the commodity which price initially was grown. Despite this, the absolute value of *beta* is less than 1 indicating that the supply is substantially inelastic to price. Moreover, it is worth remembering that for this regression the sign of *beta* was positive (but on average low) in the 64% of the cases while in the 18% it was negative (and on average not so low) and finally in the 18% of the cases the *beta* was not significant. R^2 is very low in both cases suggesting that prices cannot be considered the only determinant of demand and supply.

5.4.4 The influence of preferences on prices and surplus

It has been possible to test the influence of preferences (i.e., demand) on prices and surplus. According to a well-known economic hypothesis the price of the preferred commodity should grow as its demand grows and also the production of that commodity is expected to grow. According to the results:

- The growing preference for a particular commodity implies a growing share of the **surplus** composed by that commodity in the **97%** of the cases.
- The growing preference for a particular commodity implies a growing **price** of that commodity in the **51%** of the cases.

This seems to suggest that the composition of the social surplus is a function of effective demand but the relation between the converged prices and the effective demand is not so evident.

5.4.5 The influence of preferences on distribution

A further elaboration on what determines a particular distribution, as the one reported in the figure 5.1 needs to be considered. It could be possible that the particular distribution reported in the figure 5.1 is determined by the particular preference parameters associated to each ARA. In order to verify this hypothesis, the relationship between preferences (which determines the pattern of demand) and distribution has been investigated using the new set of experiments. In detail, all the distributional patterns associated with each combination of the preference parameters considered in all the new 60 virtual economies have been considered. It is interesting to note how only a small proportion of these new virtual economies has generated a pattern of distribution that is closer to the egalitarian (less than 10%). Despite this, it is worth remembering that all these particular virtual economies, whose distributions are closer to the egalitarian, are associated to levels of production very closed to zero. Moreover, more than 90% of the virtual economies converge to a situation in which the share of surplus to workers is lower than 10%. These results suggest that distribution could be influenced not only by preferences (demand). Probably also other factors, such as the methods, can influence the distributional pattern. In this chapter, only the methods reported by Sraffa at page 19 of *PCMC* have been considered. This implies that it is better to reconsider this research question in the next chapter, where the determinants of distribution will be investigated considering also different methods of production.

5.5 Conclusion

In this chapter we have used *digital economic laboratory* in order to investigate whether a simple three sector economy with the characteristics of Sraffa's example in *PCMC* page 19 would be able to converge towards production prices. According to the results, this *Sraffian economic system* is able to converge towards production prices and uniform wage rates but this equilibrium does not imply convergence towards uniform profit rates neither an equal distribution of the surplus. In order to generalize the results, the cases with homogeneous and heterogeneous ARAs and also the case with more local markets (i.e., more producers producing the same commodity in a *ring one-dimensional lattice*) have been considered. All experiments share the same conclusions described previously.

These results sustain the proposition of Zambelli (2016) who has questioned the relevance of the assumption that sees the uniform profit rates as a natural long run point of convergence determined by market forces in a free competition environment. Indeed, the maximization of the surplus shares and not the profit rate may be seen as the final object of the producers. It is also interesting to note how a free competition environment, such as the framework of the present experiments, has as final outcome an unequal distribution.

The further experiments aimed to investigate the relation between prices and quantities seems to underline how well-known economic relations (such as the negative slope of demand function and the positive slope of supply function) represents postulate that have not been clearly confirmed by our experiments. In particular, the experiments seem to demonstrate that the composition of the social surplus is a function of effective demand while the dependence of prices on effective demand is more questionable.

6 Experiment 2: convergence towards uniform wage rates and non-uniform profit rates for general cases

6.1 Introduction to experiment 2

Sraffa's purpose in his book *Production of Commodities by Means of Commodities* (*PCMC*) was primarily the determination of the prices that allow the system to reproduce itself under the hypothesis of uniform profit rates. What would actually happen during the market day is not discussed and how trade takes place remains an open question.

In chapter 5 the main objective was to study whether a virtual economy having the same production methods as in *PCMC*, page 19, once ARAs are specified and trading rules are given, will converge towards Sraffian production prices, uniform wage rates, uniform profit rates and equal distribution. The fundamental conclusion was that the virtual system converge towards production prices, uniform wage rate but non-uniform profit rates with an unequal distribution.

In this chapter we check whether these results apply to a general case without the number of Sraffa's *PCMC* page 19. 10 different economies for the case of homogeneous and heterogeneous ARAs will be considered, given the same behavioural functions introduced in chapter 4 and used also in chapter 5.

6.2 The experimental setting

The experimental setting considered in the present chapter is the same with respect to the previous one. Each virtual economy, identified by its identity code, is associated with different characteristics of the ARAs. Moreover, one fundamental change is represented by the fact that the method is not that used by Sraffa, but different virtual economies with different sets of methods at disposal for the production of the 3 commodities have been generated (but each producer has, as in Sraffa, only one method at disposal for the production of his commodity). Moreover each producer hires at the beginning of the first production period 18 workers that can migrate. This would allow us to generalize the results obtained in the previous chapter. The experimental setting considered in the present chapter is subdivided in two main cases:

- *Case 1*: is characterized by the presence of 3 producers and 54 workers (both cases, with homogeneous or heterogeneous ARAs, will be considered).

- *Case 2*: is characterized by the presence of 9 producers and 162 workers (both cases, with homogeneous or heterogeneous ARAs, will be considered). This last case implies the presence of 9 local markets as in figure 3.4 with 3 producers producing the same commodity.

6.3 The research questions

The purpose of the present chapter is to verify if a generalization of Sraffa's example of a three sector economy enriched with behavioural functions in a dynamic framework is able to converge towards an *equilibrium position*.

As in the previous chapter, we define an *equilibrium position* as a position in which prices and wages announced do not change anymore; all markets are in equilibrium (demand equal to supply in all markets; i.e., there is no more rationing), migration stops, the system reproduces itself (i.e., same level of production and surplus without residuals) without the creation of new IOUs (no new debt and credit relations). This implies that the prices of convergence are *production prices* with zero or positive profit rates.
In particular, these characteristic of the eventual equilibrium have been analysed:

- **Convergence towards uniform profit rates**. Given the convergence towards production prices it is possible to check convergence towards uniform profit rates.

- **Convergence towards uniform wage rates**. Given the possibility for workers to migrate where the income is the highest, if the system converges towards a position of equilibrium, wage rates should become uniform among all enterprises and the migration should stop.

- **Convergence towards an equal distribution of the surplus**. Given the initial condition of equal distribution (95% of the surplus has been associated to workers given that they are 54 with respect to the 3 producers for a total population of 57 ARAs)[115] it is interesting to test the type of distribution of the surplus associated with the convergence position.

A high number of different virtual economies with homogenous (i.e., all ARAs producers and workers are equal) or heterogeneous ARAs (each ARA is different with respect to the others) has been generated, in the case with 3 and 9 local markets, in order to study the relation between the time of convergence and the experimental setting. This allowed us to

[115]The same proportions also for the case with local markets. It is worth remembering that the initial production prices and wages have been computed starting from this distribution.

collect statistics about the time of convergence and the frequency of the eventual convergence[116]. Among the virtual economies able to converge, we have selected 10 virtual economies for each experimental setting in order to study the characteristics of convergence.

6.4 Results for the *case 1*

6.4.1 Convergence and time of convergence

A high number of virtual economies has been generated (i.e., with different characteristics of the ARAs and different methods at disposal) with homogenous or heterogeneous ARAs for this *case 1* with only 3 producers. The time of convergence and the percentage of virtual economies that converged are reported in the figure 6.1.

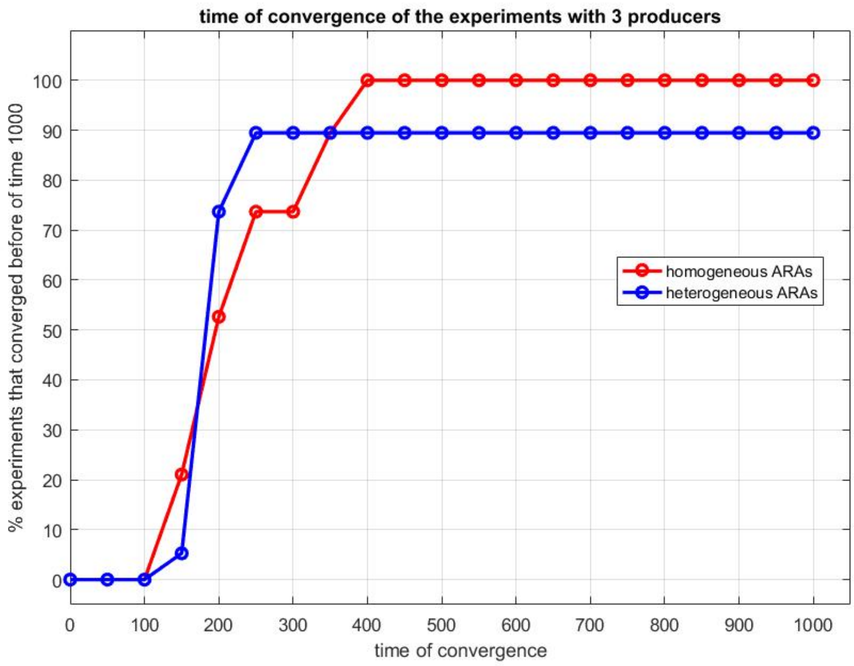


Figure 6.1: **Time of convergence and percentage of economies able to converge before time t=1000**: in the figure, in colour red (case homogeneous ARAs) and blue (case heterogeneous ARAs), the percentages of experiments able to converge to an equilibrium position for each point in time are reported. The total number of virtual economies considered for each experimental setting is equal to 20. For more details see sections 4.2 and 4.5.

[116]The study of the convergence has been conducted considering 20 virtual economies for each particular experimental setting. For more details on this point see sections 4.2 and 4.5.

As appears clear from the figure, for the case with homogenous ARAs all experiments converge towards an equilibrium position at time 400 or before. The presence of heterogeneous ARAs implies that 90% of the virtual economies are able to converge before time 1000. Probably heterogeneity represents an improvement in the complexity of the dynamics of the system. These virtual economies that do not converge before time 1000 will be part of further investigation (beyond this thesis) aimed at understanding whether convergence can be achieved for more than 1000 periods or not. In the latter case, the causes for lack of convergence should be further analysed.

6.4.2 Characteristics of the convergence

Among the experiments of the previous section, 10 virtual economies able to converge towards production prices have been selected and analysed. In the subsequent table are reported the results of the 10 virtual economies (E) with 3 producers and 54 workers, with homogeneous and heterogeneous ARAs.

Homogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	27	39	42	47	22	55	30	23	58	26
% surplus to workers	73	61	58	53	78	45	70	77	42	74

Table 6.1: **Results of the experiments**: considering 10 different virtual economies E and a total population of 57 ARAs (54 workers and 3 producers) for the case with homogeneous ARAs.

Heterogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	32	49	28	47	50	45	40	46	25	16
% surplus to workers	68	51	72	53	50	55	60	54	75	84

Table 6.2: **Results of the experiments**: considering 10 different virtual economies E and a total population of 57 ARAs (54 workers and 3 producers) for the case with heterogeneous ARAs.

According to the results, the following answers to the initial research questions have been obtained:

- ***The systems converge towards production prices.*** The virtual systems are able to converge towards an *equilibrium position* stable in time characterized by the presence of production prices, no creation of new IOUs and positive profit rates.

- ***The systems DO NOT converge towards uniform profit rates.*** This could be surprising for those that interpret the convergence towards an uniform profit rate as the natural outcome of the competition process between producers in time. This result should be interpreted instead in the light of the result presented in Zambelli (2016). If the objective of the agents is to maximize their surplus shares, there is no reason to consider uniform profit rates as a natural outcome of the competition process.

- ***The systems converge towards uniform wage rates.*** This implies that migration stops at an equilibrium level in which there is no more incentive to migrate. This enforces the stability of the equilibrium position and the idea that migration is a competitive process between workers able to bring to convergence towards the same wage rate.

- ***The systems DO NOT converge towards an egalitarian distribution.*** Despite a starting position characterized by an equal distribution (95% of the surplus was associated to workers) at the end of the production periods the situation is sharply different. Despite it is true that the distributional pattern is not completely reversed as in the previous chapter where were used Sraffa's numbers, we observe anyway a departure from the equal distribution.

It is interesting to note how the difference between homogenous or heterogeneous ARAs does not change the conclusions that are common to both cases.

6.5 Results for the *case 2*: the influence of the local markets

6.5.1 Convergence and time of convergence

In these further experiments, the role of local markets has been tested. 9 producers and 162 total workers have been considered. We have generated a high number of virtual economies (i.e., with different characteristics of the ARAs and different methods at disposal) with homogenous or heterogeneous ARAs. The time of convergence and the percentage of virtual economies that converged are reported in the figure 6.2.

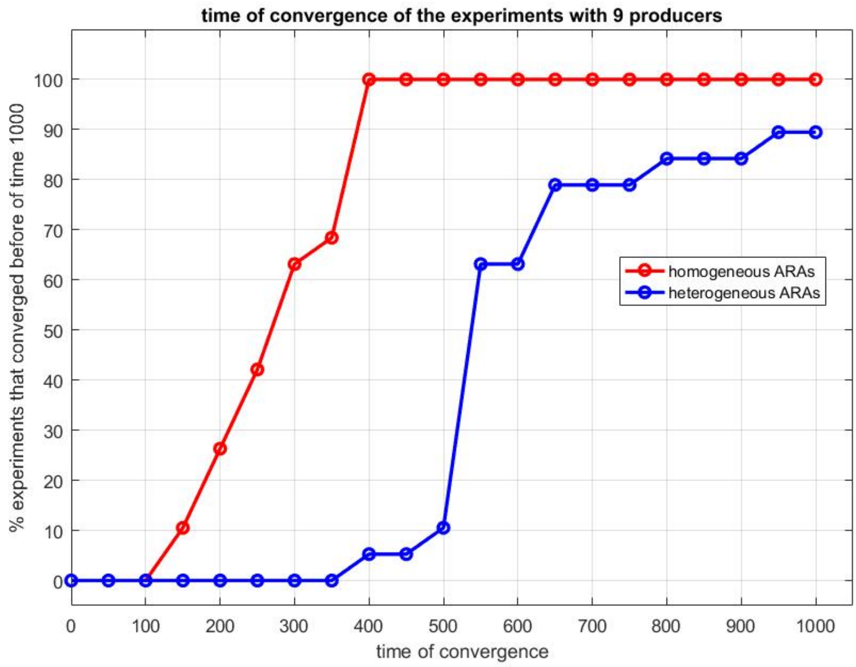


Figure 6.2: **Time of convergence and percentage of economies able to converge before time t=1000**: in the figure, in colour red (case homogeneous ARAs) and blue (case heterogeneous ARAs), the percentages of experiments able to converge to an equilibrium position for each point in time are reported. The total number of virtual economies considered for each experimental setting is equal to 20. For more details on this point see sections 4.2 and 4.5.

As the figure clearly shows, for the case with homogenous ARAs all virtual economies converge towards an equilibrium position at time 400 or before. The presence of heterogeneous ARAs implies that 90% of the experiments are able to converge before period 1000 and time of converge is even closer to 1000. These virtual economies will be part of further investigation (beyond this thesis) aimed at understanding whether convergence can be achieved for more than 1000 periods or not. In the latter case, the causes for lack of convergence should be further analysed.

6.5.2 Characteristics of the convergence

Among the experiments of the previous section, 10 virtual economies able to converge towards production prices have been selected and analysed. In the subsequent table, the results of the 10 virtual economies (E) with 9 producers and 162 workers are reported, with homogeneous and heterogeneous ARAs.

Homogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	27	39	23	13	26	27	26	37	47	19
% surplus to workers	73	61	77	87	74	73	74	63	53	81

Table 6.3: **Results of the experiments**: considering 10 different virtual economies E and a total population of 171 ARAs (162 workers and 9 producers) for the case with homogeneous ARAs.

Heterogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	37	33	21	35	32	48	36	21	39	37
% surplus to workers	63	67	79	65	68	52	64	79	61	63

Table 6.4: **Results of the experiments**: considering 10 different virtual economies E and a total population of 171 ARAs (162 workers and 9 producers) for the case with heterogeneous ARAs.

According to the results, the presence of local markets does not change the conclusions reached in the previous case. Producers of the same commodity converge towards the same prices and the same profit rates only in the homogeneous case. In particular, experiments $E1$ and $E2$ have also the same initial conditions and parameters with respect to experiments $E1$ and $E2$ with only 3 producers. It is interesting to note that they reach exactly the same results, so that the global dynamics is equal to the local dynamics in these virtual economies. For the case of heterogeneous ARAs the global and the local dynamics change. Indeed, only wages converge towards the same value in all local markets. This is because ARAs are heterogeneous and producers of the same commodity can take different decisions while workers can migrate in the neighbouring enterprises and in time this implies that workers can migrate along all the network. This competition process in the labour market is the mechanism that allows to reach a uniform wage rate while profit rates are non-uniform.

6.5.3 The dynamics towards convergence

It has been said previously that when the virtual economy converges towards an equilibrium position, this implies that debt and credit positions tends to zero and market prices are equal to production prices. This implies that profit rates converge towards positive values (or values equal to zero). This convergence is clear in the figure 6.4, where the dynamics of financial *wealth* positions of all ARAs and the dynamics of the profit rates in the last 200 periods have been reported. The convergence towards positive profit rates implies that ARA producers need no more to underwrite IOUs and the financial *wealth* accumulated can converge towards zero in time. Despite this convergence, it is interesting to note how profit rates are non-uniform.

In the figure 6.3 the wage-profit curves that correspond to all the combination of the profit rate and the wage rate for which the profit rate is maximum for each enterprise are reported. We have added also the profit rates and wages rates of convergence. The profit rates of convergence are the same reported in figure 6.4 where instead also the path of convergence for the last 200 periods is reported. Obviously these profit rates of convergence are lower than their wage-profit curves (see figure 6.3). As already noted, profit rates are non-uniform as in the case with only 3 producers (see table 6.2). In this case, with 9 local markets and heterogeneous ARAs, it is possible to note that neither the producers of the same commodity have converged towards the same profit rates despite the wage rates are uniform for all the 9 enterprises. Wage rates are uniform thanks to the competition process promoted by the migration of workers but a similar process has not been able to bring profit rates to uniformity (neither for the case of homogenous ARA). This seems to sustain the proposition of Zambelli (2016) who has questioned the relevance of the assumption that sees the uniform profit rate as a natural long run point of convergence determined by market forces in a free competition environment. Finally, it is interesting to note that the distribution of convergence is unequal (see also the results from the table 6.4). Remember that in the initial equilibrium position distribution was equal. These results confirm those obtained in the previous chapter where the methods of Sraffa's example at page 19 of *PCMC* were considered. This seems to indicate that these properties of the convergence (i.e., convergence towards uniform wage rates but non-uniform profit rates with unequal distribution) do not depend on the particular method considered (e.g., the methods used by Sraffa at page 19 of *PCMC*) but, probably, they reveal some fundamental properties of our virtual economies.

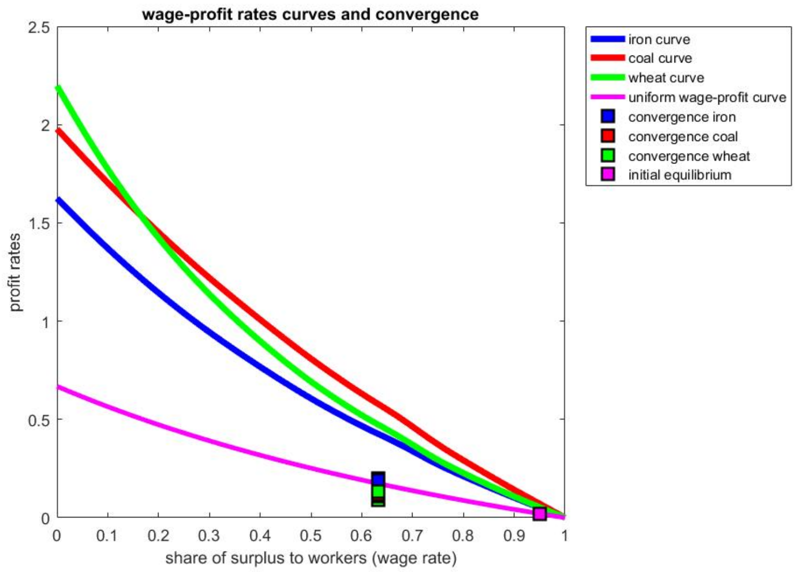


Figure 6.3: **Maximum enterprise profit rate as a function of the share of the surplus to workers. Virtual economy E10 with heterogeneous ARAs and 9 local markets**: in the figure, the wage-profit curves (in colour red (coal), green (wheat) and blue (iron)) that correspond to all the combination of the profit rate and the wage rate for which the profit rate is maximum for each enterprise are reported. When, for example, a share of the surplus equal to zero is associated to the workers of the iron enterprise, so that the profit rate of the iron enterprise is maximum, the profit rates of the wheat and coal industry are equal to zero. The other points of the curve can be uniquely determined. The lowest curve is the *wage-profit curve* with uniform profit rates. The points are the profit rates and the wage rates of convergence. The point on the *uniform wage-profit curve* is the initial equilibrium. It is interesting to note how the system converges towards uniform wage rates but non-uniform profit rates.

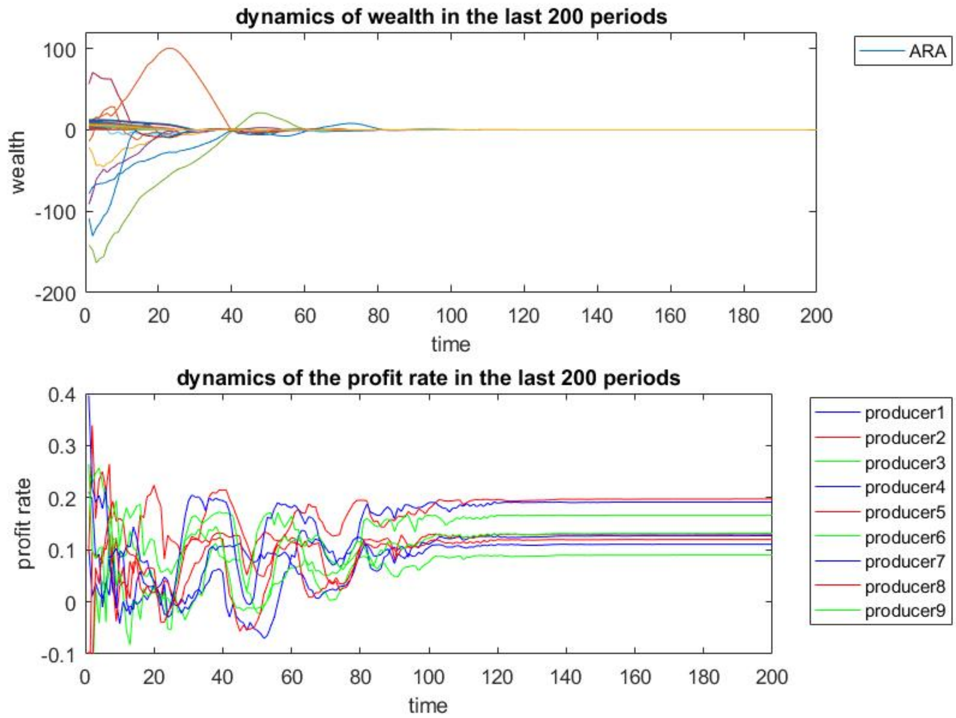


Figure 6.4: **The dynamics of wealth of each ARA and the dynamics of the profit rate for the virtual economy $E10$ with heterogeneous ARAs and 9 local markets**: in the first figure, the dynamics of wealth in the last 200 periods are reported. Each line represents the wealth of a single ARA (worker or producer, in total 171 ARAs). When convergence is reached wealth approaches to zero for all ARAs. This because prices and wages reach particular levels that allow all ARAs to earn a positive income. In the figure at the bottom, the dynamics of the profit rates in the last 200 periods is represented. Before convergence, profit rates can be negative. The accumulation of losses is at the base of the negative wealth of the upper figure. If convergence prices and wages do not change anymore and also profit rates converge towards positive values, market prices correspond to production prices. The color of the line that corresponds to the profit rate of a particular producer allows to make a comparison with figure 6.3. Indeed, in the figure the profit rates of all producers of coal in red, of wheat in green and of iron in blue are reported. It can be noted that the converge points are the same of figure 6.3.

6.6 Robustness checks: some notes

All the different virtual economies considered in this chapter are identified by an unique identity code. Each identity code is associated to different characteristics of the ARAs and a different technological sets, so that each virtual economy is associated to an unique set of initial conditions and parameters. This heterogeneity in the initial conditions and parameters represents the principal attempt to provide a robustness check to our *thought experiment*. Despite this, it is interesting to test also the effect of a particular type of perturbation, i.e., a number of workers and producers which is not a power of 3. One single experimental setting for this robustness check was considered sufficient: with 6 producers and 11 workers (that can migrate) for each enterprise at the beginning of the production periods. The experiment has been repeated for the case with homogenous and heterogeneous ARAs. The results do not change at all the answers to our initial research questions.

6.7 The role of the methods and the pattern of distribution

With the purpose to develop the interesting results obtained in the section 5.4.5 of the present thesis, here the effect of a variation in the preference parameters for two particular types of virtual economies characterized by extremely different technological sets is tested. It is worth remembering that all the virtual economies generated in these chapters have different technological sets but in this section we consider only *extreme methods*, i.e., very *labour-intensive* or *capital-intensive* methods. The different methods used in the previous experiments considered, instead, balanced combinations of labour and commodities for production.

6.7.1 Experimental setting

The experimental setting of this section is the same for both the types of virtual economies: heterogeneous ARAs with 3 producers and 54 workers. In detail, the two types of virtual economies can be described in the following way:

- *Economy1*: uses *labour-intensive* methods (high use of labour with respect to commodities).
- *Economy2*: uses *capital-intensive* methods (low use of labour with respect to commodities).

For each type of virtual economy (*Economy1* and *Economy2*), the effect on distribution of a variation in the preference parameters has been tested. Each parameter of the utility function has been progressively augmented on a scale of 4 steps: between a minimum equal to the initial value of the parameter and a maximum close to 1 (given the other parameters, that remain fixed) for a total number of 15 different experiments (3 commodities for 4 growing steps plus the initial states) for each type of economy (in total 30 different virtual economies).

6.7.2 The results

In the figures 6.5 and 6.6 all the wage-profit curves that correspond to all the combination of the profit rate and the wage rate for which the profit rate is maximum for each enterprise are plotted. We have added also the profit rates and wages rates of convergence of all the experiments (15 for each type of virtual economy). The results seem to suggest that the type of method is extremely important for the determination of the distributional pattern. Indeed, only with *labour-intensive* methods a change in preferences can play a role in the determination of the distributional pattern, while with *capital-intensive* methods seem to prevent the distribution to become equal, despite the changes in the preference parameters (demand). This is an interesting result because it seems to suggest that *capital-intensive* methods (that typically are driven by technological progress) can have very dramatic effects on distribution equality. Despite this, this conclusion and its theoretical implications have to be taken with *extreme care*. Indeed, further research, able to consider different behavioural functions, needs to be conducted in order to understand the dynamics and the causes of this phenomenon[117].

[117] See section 9.2 for more details on this point.

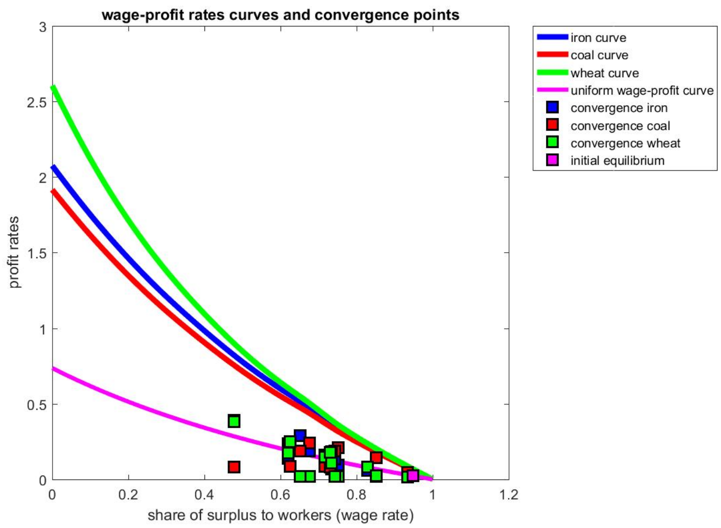


Figure 6.5: ***Economy1*: labour-intensive methods.** Maximum enterprise profit rate as a function of the share of the surplus to workers. How distribution changes according to preferences.

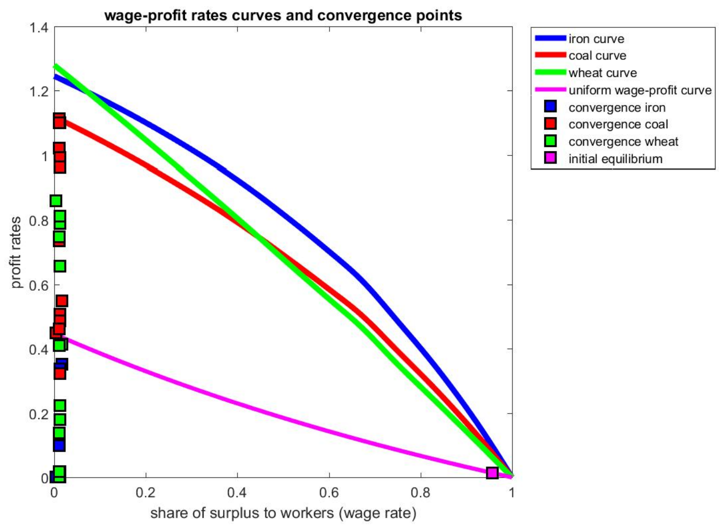


Figure 6.6: ***Economy2*: capital-intensive methods.** Maximum enterprise profit rate as a function of the share of the surplus to workers. How distribution changes according to preferences.

6.8 Conclusion

The purpose of the present chapter was to generalize the conclusion of *experiment 1* by considering the presence of 10 different virtual economies with different methods with respect to Sraffa's example in *PCMC* page 19.

The fundamental conclusion of the present chapter is that the most important findings of *experiment 1* have been confirmed also in the general case characterized by the presence of 10 different virtual economies with different characteristics of the ARA and different methods.

Indeed, according to the results, despite all the virtual economies *selected* are able to converge towards production prices and uniform wage rates, these equilibrium positions do not imply convergence towards uniform profit rates neither a convergence towards an equal distribution of the surplus. Moreover, the cases with homogeneous, heterogeneous ARAs and also the case with 9 local markets (i.e., more producers producing the same commodity in a *ring one-dimensional lattice*) have been considered. These further experiments have not changed the conclusions.

This seems to indicate that these properties of the convergence (i.e., convergence towards uniform wage rate but non uniform profit rate with unequal distribution) do not depend on the particular method considered (e.g., the methods used by Sraffa at page 19 of *PCMC*) but, probably, they reveal some fundamental properties of our virtual economies.

Other interesting results, that deserve further research, have emerged. In general not all virtual economies are able to converge before time t=1000. For the case with heterogeneous ARA (with 3 or 9 producers) only 90% of the virtual economies are able to converge. Moreover, the distributional pattern seems to be influenced by the technology (*labour-intensive* methods or *capital-intensive* methods).

7 Experiment 3: convergence towards the wage-profit frontier and the role of technological innovation

7.1 Introduction to experiment 3

The *digital economic laboratory* introduced in chapter 3 represents an attempt to consider the functioning of the market inside Sraffian schemes. In the experiments of the previous chapters were considered the set of methods used by Sraffa in the example given at page 19 of *PCMC* (chapter 5), then the result has been generalized considering 10 different virtual economies with different methods with respect to those of Sraffa's example (chapter 6). The cases with homogeneous, heterogeneous ARAs and also the case with 9 local markets have been considered in both previous chapters.

The fundamental conclusion was that the virtual economies converge towards production prices, uniform wage rates but non-uniform profit rates with an unequal distribution of the surplus.

In this chapter we check whether these results are confirmed in virtual economies where producers have the possibility to choose among a set of alternative methods for the production of their commodity. 10 different virtual economies for the case of homogeneous and heterogeneous ARAs and with 3 or 9 local markets will be considered, given the same behavioural functions of chapter 4. In addition, the efficiency of the virtual economies will also be studied; i.e., if the ARA producers are able to adopt the most efficient method. The indexes that allows to measure efficiency will be computed starting from the construction of the *wage-profit frontier*.

Moreover, the role of innovation will be considered. For the case with 9 local markets and homogeneous ARAs, once convergence has been reached, one producer posed in the centre of the lattice (producer 5 will be chosen) will start to have access to a new and more efficient production method with respect to the others. This means that the new *frontier* that can be computed with the new method dominates (in one interval or on the whole domain, both cases will be considered) the initial one. The purpose of the experiment is to check whether producer 5 will be able to adopt or not the new method (i.e., *lock-in* phenomena). Before to introduce the experimental setting of the present chapter, a brief introduction about the role of the *wage-profit frontier* is needed.

7.2 The wage-profit frontier

The presence of more methods of production allows each producer to choose the method that allows him to maximize the expected profit. The convergence towards technological efficiency (i.e., the convergence towards the most efficient method of production) is a possible outcome of the simulation. Indeed, the dynamics of prices, wages and the consumers' choices can prevent from adopting the best method at disposal. The presence of local markets, with more producers producing the same commodity, allows to make a comparison between the technological efficiency of different local markets. If the global behaviour is equal to the local behaviour, the methods adopted should be the same for all the producers producing the same commodity (i.e., all the producers of the same industry).

In order to check whether in our virtual economies the convergence towards the same method of production is realized and the chosen methods are the most efficient (in the sense of being methods belonging to the *wage-profit frontier*), some measures of technological efficiency are needed. These measures are based on the construction of the *wage-profit frontier*.

Consider for example the case of 9 producers organized in 9 different local markets (as in figure 3.4). Each producer i (i=1,...,k in this case k=9) produces only one commodity g (for g=1,...,n with n=3) through the use of the labour force and the commodities offered inside his local market (with inside each local market n producers $\bar{i}$ for $\bar{i}$=1,...,n). In each local market, the technological input requirements in terms of commodities and labour are represented by the matrices $\mathbb{A}$ and $\mathbb{L}$, while the relative level of the possible output is represented by the matrix $\mathbb{B}$. In particular[118] consider $\mathbb{A}$ and $\mathbb{B}$ square matrices of order n: $\mathbb{A}=[\mathbb{a}_{\bar{i}g}]$ and $\mathbb{B}=[\mathbb{b}_{\bar{i}g}]$, with $\bar{i}$,g=1,...,n. $\mathbb{A}=[\mathbb{a}_{\bar{i}g}]$ is the matrix of the means of production (semi-positive and indecomposable), where $\mathbb{a}_{\bar{i}g}$ is the quantity of the commodity g means of production of the producer (enterprise) $\bar{i}$. $\mathbb{B}=[\mathbb{b}_{\bar{i}g}]$ is the production matrix (diagonal and semipositive definite), where $\mathbb{b}_{\bar{i}g}$ is the quantity of the commodity g produced by the producer $\bar{i}$. Each producer $\bar{i}$ produces only one commodity g, which is means of production for the other neighbouring producers. $\mathbb{L}=[\mathbb{l}_{\bar{i}}]$ is the labour vector and $\mathbb{l}_{\bar{i}}$ indicates the amount of labour required by the producer $\bar{i}$. It is supposed that each producer has at disposal m_g different alternative methods z_g for the production of his commodity g (z_g=1,...,m_g). These alternative methods, for all the commodities, are organized in an input-output multidimensional matrix $\boldsymbol{T}$, which is supposed to be known by all the producers of the whole virtual economy and equal for each production period t:

[118] See chapter 3 for the matrix notation and for more details about technology and the mechanism of the choice of the techniques.

$$\boldsymbol{T}(1:n, 1:(n+2), 1:m_g) = \begin{bmatrix} \mathbb{a}_{11}^{z_g} & \dots & \mathbb{a}_{1n}^{z_g} & \mathbb{l}_1^{z_g} & \mathbb{b}_1^{z_g} \\ \mathbb{a}_{21}^{z_g} & \dots & \mathbb{a}_{2n}^{z_g} & \mathbb{l}_2^{z_g} & \mathbb{b}_2^{z_g} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbb{a}_{n1}^{z_g} & \dots & \mathbb{a}_{nn}^{z_g} & \mathbb{l}_n^{z_g} & \mathbb{b}_n^{z_g} \end{bmatrix} \tag{7.1}$$

The collection of the methods of the matrix $\boldsymbol{T}$ that allows to product the same commodity are organized inside the matrix $\boldsymbol{\Phi}$[119]. For the production of his commodity, each producer has to choose the method $\boldsymbol{\phi}^*$ which allows him to maximize the expected profit given the market prices and the wage inside his local market. Each production period, the producers of each local market will choose a particular combination of methods that will compose the matrix $\mathbb{A}^*$, $\mathbb{L}^*$ and $\mathbb{B}^*$ of their local market. The production of each local market can be described with the simultaneous equation system (considering also the equation of the *numéraire* $\boldsymbol{n}^{\boldsymbol{T}}\boldsymbol{p}$=1).

$$\mathbb{A}^*\boldsymbol{p}(1+r) + \mathbb{L}^*w = \mathbb{B}^*\boldsymbol{p} \tag{7.4}$$

Denoting by $\boldsymbol{p}$ the production prices that can be computed with the hypothesis of uniform profit rate r and uniform wage rate w, it is possible to exploit the information embedded into the input-output matrices through the computation of the *wage-profit curve* (i.e., all the possible combination of the distributional parameters which generates the production prices).

$$w = (\boldsymbol{n}(\mathbb{B}^* - \mathbb{A}^*(1+r)))^{-1}\mathbb{L}^*))^{-1} \tag{7.5}$$

Remember that the production prices are here considered exclusively in order to construct the *wage-profit curve.* In the virtual economy, market prices can be different from production prices and the objective of the construction of the *wage-profit curve* is exactly the comparison between the actual

[119] In detail the structure of the matrix $\boldsymbol{\Phi}$ is the following:

$$\boldsymbol{\Phi}(1:m_g, 1:(n+2), g) = \begin{bmatrix} \mathbb{a}_{g1}^{1} & \dots & \mathbb{a}_{gn}^{1} & \mathbb{l}_g^{1} & \mathbb{b}_g^{1} \\ \mathbb{a}_{g1}^{2} & \dots & \mathbb{a}_{gn}^{2} & \mathbb{l}_g^{2} & \mathbb{b}_g^{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbb{a}_{g1}^{m_g} & \dots & \mathbb{a}_{gn}^{m_g} & \mathbb{l}_g^{m_g} & \mathbb{b}_g^{m_g} \end{bmatrix} \tag{7.2}$$

In which:

$$\boldsymbol{\phi}(z_g, 1:(n+2), g) = [\mathbb{a}_{g1}^{z_g}, \mathbb{a}_{g2}^{z_g}, \dots, \mathbb{a}_{gn}^{z_g}, \mathbb{l}_g^{z_g}, \mathbb{b}_g^{z_g}] \tag{7.3}$$

market prices and the production prices that are associated to the methods of production chosen by the producers in each local market at the end of the production periods and that compose the final matrices $\mathbb{A}^*$, $\mathbb{L}^*$ and $\mathbb{B}^*$. It is worth remembering that the number of the *wage-profit curves* must be equal to the number of local markets (see for example the blue lines of figure 7.1). Obviously, if all the local markets converge to the same methods of production, all the *wage-profit curves* will be the same.

Considering that each producer has at disposal m_g different alternative methods for the production of his commodity, there are m_g^3 possible alternative configurations of the matrix $\mathbb{A}^*$, $\mathbb{L}^*$ and $\mathbb{B}^*$. In the experiments considered in this chapter, each producer has at disposal m_g=10 different technological combinations for the production of his commodity (in other words, each row of the matrix $\mathbb{A}^*$, $\mathbb{L}^*$ and $\mathbb{B}^*$ can be substituted with other 9 alternatives), so that the total number of possible *wage profit curves* is equal to 10^3, which is exactly the number of possible combinations of the methods. Considering the whole economy, different producers of the same commodity g can use different methods of the matrix $\mathbf{\Phi}(1 : m_g, 1 : (n+2), g)$ or converge to the same method between the m_g at disposal for each commodity.

Each *wage-profit curve* represents all the possible combination of the profit rate and the wage rate given a particular combination of the methods. If another combination implies for the same value of the wage rate a higher level of the profit rate (or the contrary), this combination is considered more efficient. As a consequence, if we consider all the different *wage-profit curves* relative to all the possible combinations of the methods, the outer envelope computed from all the possible *wage-profit curves* is a piecewise-nonlinear function representing the *wage-profit frontier*: the most productive and efficient system of methods (see for an example figure 7.1)[120].

The *wage-profit frontier* represents the potential technological efficiency of a particular virtual economy and it has been used as a robust benchmark against which to compare each *wage-profit curve* generated by the single local market. A brute force algorithm exists and it allows the computation of the *wage-profit frontier* but it becames computationally intractable for sets of methods with high cardinality. In this case, a more efficient algorithm can be applied, the FVZ-algorithm (presented by Zambelli et al. (2014)), which allows to reduce drastically the computational time for the exploitation of the *wage-profit frontier*. The *wage-profit frontier* and the *wage-profit curves* calculated for each local market (with k=9 the *wage-profit curves* are equal to 9 as in the figure 7.1) represent a benchmark that allows not only to check

[120]It is worth remembering that different *wage-profit curves* can be compared independently of the cardinality of their productive systems. Moreover the wage rate and the production prices are not dependent on the activity level thanks to the *non-substitution theorem*. Always thanks to this theorem *wage-profit curves* and *wage-profit frontiers* are scale independent. More details on these aspects are present in Zambelli et al. (2014). In the same paper, there is an example of a frontier computed starting from empirical data.

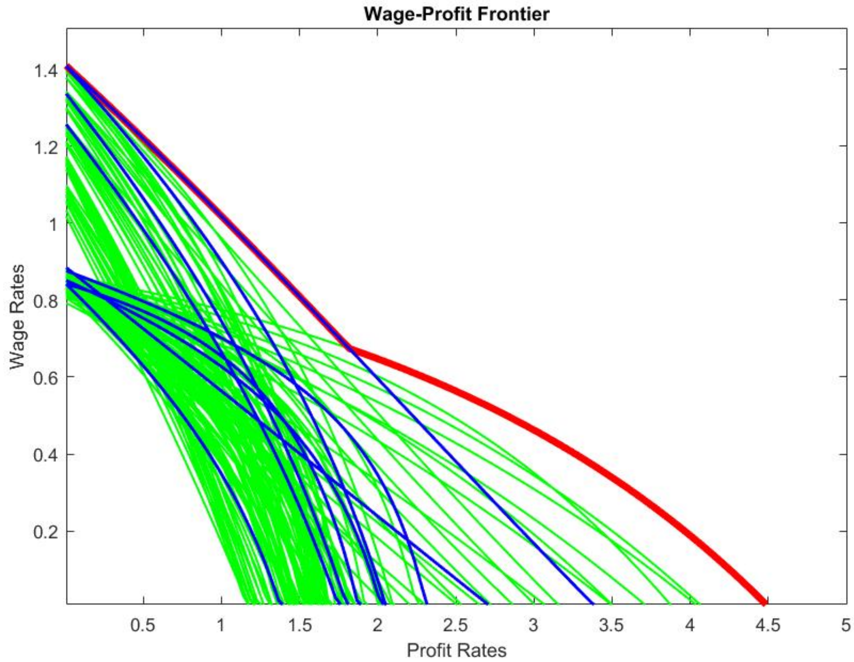


Figure 7.1: **The *wage-profit frontier* and the *wage-profit curves***: The *wage profit frontier* (red line) as the envelope of all the *wage-profit curves* (green lines). The total number of the *wage profit curves* of the virtual economy considered in this example is 5^3 (125 *wage-profit curves*). The blue lines represent the *wage-profit curves* to which the 9 local markets converge at the end of the production periods. It is possible to note that in this case there is no convergence towards the same method of production (there are 9 different blue lines) and only 1 local market has been able to choose an efficient combination of the methods of production (i.e., only one blue line is on the *wage-profit frontier*).

whether the convergence towards the same method of production is realized (this is obtained if all the *wage-profit curves* of the local markets are the same[121]) or whether the chosen methods are the most efficient (this is obtained if the *wage-profit curves* of the local markets *touch the frontier*), but allows also to check whether for each local market, the market prices have

[121] Remember that if each producer of the same commodity chooses a different method for the production of his commodity, there will be (in the case of k=9 producers) 9 different *wage-profit curves*. If all the producers of only one commodity (for example iron) in all local markets converge towards the same method of production there will be 6 different *wage-profit curves*. If all the producers of two different commodities (for example iron and wheat) in all local markets converge respectively towards the same method of production there will be 3 different *wage-profit curves*. Finally, if all the producers of all the commodities, for each commodity (for example coal, wheat and iron) in all local markets converge respectively towards the same method of production there will be one single *wage-profit curve*.

converged towards the production prices associated to the methods chosen by the producers and that compose the *wage-profit curve.* It is worth remembering that in the previous chapter, with one single method, the *wage-profit curve* coincided with the *wage-profit frontier.* The *wage-profit curve* has been used in the previous experiments in order to check if the virtual economy converged towards uniform profit rates and wage rates. Obviously, with one single method, it was not possible to investigate technological efficiency.

7.3 The experimental setting

The experimental setting considered in the present chapter coincides with the experiments of the previous chapter. Each virtual economy, identified by its identity code, is associated to different characteristics of the ARAs. The fundamental change is represented by the fact that now each producer has at disposal 10 methods for the production of his commodity. This would allow us to generalize the results obtained in the previous chapters and to investigate technological efficiency. The experimental setting considered in the present chapter is subdivided in two main cases:

- *Case 1*: is characterized by the presence of 3 producers and 54 workers (both cases, with homogeneous or heterogeneous ARAs, will be considered).
- *Case 2*: is characterized by the presence of 9 producers and 162 workers (both cases, with homogeneous or heterogeneous ARAs, will be considered). This last case implies the presence of 9 local markets as in figure 3.4 with 3 producers producing the same commodity.

7.4 Research questions

The purpose of the present chapter is to verify if a generalization of the Sraffa's example of a three sector economy enriched with behavioural functions in a dynamical framework with the possibility to choose among a set of methods is able to converge towards an *equilibrium position*, and if this equilibrium is efficient from a technological point of view.

We define an *equilibrium position* as a position in which prices and wages announced do not change anymore; all markets are in equilibrium (demand equal to supply in all markets; i.e., there is no more rationing), migration stops, the system reproduce itself (i.e., same level of production and surplus without residuals) without the creation of new IOUs (no new debt and credit relations). This implies that the prices of convergence are *production prices* with zero or positive profit rates.
In particular, these characteristic of the eventual equilibrium have been analysed:

- **Convergence towards uniform profit rates**. Given the convergence towards production prices it is possible to check convergence towards uniform profit rates.
- **Convergence towards uniform wage rates**. Given the possibility for workers to migrate where the income is the highest, if the system converges towards a position of equilibrium, wage rates should become uniform among all enterprises and the migration should stop.
- **Convergence towards an equal distribution**. Given the initial condition of equal distribution (95% of the surplus has been associated to workers given that they are 54 with respect to the 3 producers for a total population of 57 ARAs)[122] it is interesting to test the type of distribution of the surplus associated with the convergence position.

The convergence towards an equilibrium position does not imply the convergence towards the **technological efficiency**. With respect to the previous experiments, it will be considered also:

- **Convergence in methods**. This type of efficiency refers to the virtual economies with more local markets. The convergence in methods is measured by the index *CONV* and indicates whether the producers of the same commodity have chosen the same method of production.
- **Convergence towards the *wage-profit frontier***. The convergence towards the *wage-profit frontier* corresponds to the case in which all the producers of the same local market have chosen the most efficient method of production so that their *wage-profit curve* is part of the *wage-profit frontier*. It is measured by the index *EFF*.
- **Stability of the methods chosen**. The technological efficiency should be stable in time in order to talk about convergence towards the technological efficiency. Indeed, there could be cases in which the virtual economy is on the *frontier* cyclically or occasionally. These cases do not represent a convergence towards the technological efficiency. The convergence towards technological efficiency is measured by the index *TechStab*.

The meaning of these indexes has been described in details in the table 7.1.

[122] The same proportions also for the case with 9 local markets. It is worth remembering that the initial production prices and wages have been computed starting from this distribution.

CONV	This index measures for how many commodities it is verified the convergence towards the same method of production, i.e., producers of different local markets producing the same commodity converge to the same method of production. If all producers of the same commodity use different methods CONV=0. If all the producers of one commodity in all local markets converge to the same method CONV=1, if all the producers of two commodities converge respectively to the same method CONV=2. Finally, if all the producers of the 3 commodities converge respectively to the same method CONV=3.
EFF	This index measures the number of the *wage-profit curves* of the local markets on the *frontier*. This index can assume values from 0 (no one local market catchs up the *frontier*) to k (all the local markets catch up the *frontier*).
TechStab	This index measures the number of producers who have never changed the method of production in the last 10 production periods. This index indicates the stability of the choice of the methods. It can assume values from 0 (all the producer have changed their method of production at least one time in the last 10 production periods) to k (no one producer has changed his method of production in the last 10 production periods). A value of the index equal to k indicates a perfect stability in the choice of the methods.

Table 7.1: **Description of the efficiency indexes**: in the present chapter the experiments consider the presence of 3 and 9 ARA producers so that k=3 and k=9.

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7.5 Results for the *case 1*

7.5.1 Convergence and time of convergence

A high number of virtual economies has been generated (i.e., with different characteristics of the ARAs and different methods at disposal) with homogenous or heterogeneous ARAs for this *case 1* with only 3 producers. The time of convergence and the percentage of virtual economies that converged are reported in figure 7.2.

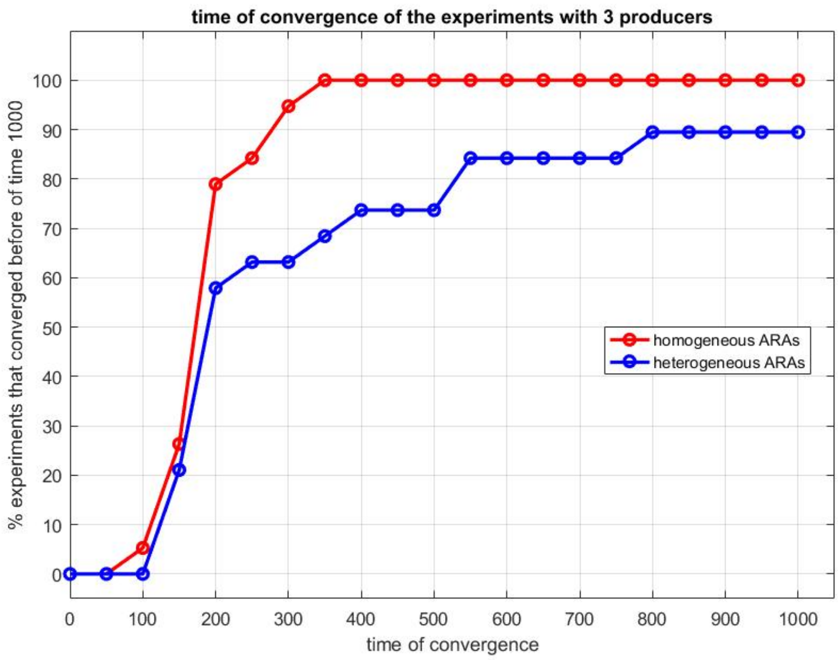


Figure 7.2: **Time of convergence and percentage of economies able to converge before time t=1000**: in colour red (case homogeneous ARAs) and blue (case heterogeneous ARAs), the percentages of experiments able to converge to an equilibrium position for each point in time are reported in the figure. The total number of virtual economies considered for each experimental setting is equal to 20. For more details on this point see sections 4.2 and 4.5.

As it clearly appears from the figure, for the case with homogenous ARAs all experiments converge towards an equilibrium position before time 400. The presence of heterogeneous ARAs implies that only 90% of the experiments are able to converge before time 1000. These virtual economies will be part of further investigations (beyond this thesis) aimed at understanding whether convergence can be achieved for more than 1000 periods or not. In the latter case, the causes for lack of convergence should be further analysed.

7.5.2 Characteristics of the convergence

Among the experiments of the previous section, 10 virtual economies able to converge towards production prices have been selected and analysed. In the following tables, the results of the 10 virtual economies (E) with 3 producers and 54 workers are reported, with homogeneous and heterogeneous ARAs.

Homogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	35	39	22	39	56	19	30	33	21	33
% surplus to workers	65	61	78	61	44	81	70	67	79	67
EFF	3	3	3	3	3	3	3	3	3	3
TechStab	3	3	3	3	3	3	3	3	3	3

Table 7.2: **Results of the experiments**: considering 10 different virtual economies E and a total population of 57 ARAs (54 workers and 3 producers) for the case with homogeneous ARAs.

Heterogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	29	21	30	42	37	43	28	35	26	31
% surplus to workers	71	79	70	58	63	57	72	65	74	69
EFF	3	3	3	3	3	3	3	3	3	3
TechStab	3	3	3	3	3	3	3	3	3	3

Table 7.3: **Results of the experiments**: considering 10 different virtual economies E and a total population of 57 ARAs (54 workers and 3 producers) for the case with heterogeneous ARAs.

According to the results of the experiments it has been obtained these answers to the initial research questions:

- ***The systems converge towards production prices.*** The virtual systems are able to converge towards an *equilibrium position* stable in time characterized by the presence of production prices, no creation of new IOUs and positive profit rates.

- ***The systems DO NOT converge towards uniform profit rates.*** This could be surprising for those that interpret the convergence towards an uniform profit rate as the natural outcome of the competition process between producers in time. This result should be interpreted instead in the light of the results presented in Zambelli (2016). If the objective of economic agents is to maximize their surplus share there is no reason to consider the uniform profit rate as a natural outcome of the competition process.

- ***The systems converge towards uniform wage rates***. This implies that migration stops at an equilibrium level in which there is no more incentive to migrate. This enforces the stability of the equilibrium position and the idea that migration is a competitive process between workers able to bring to convergence towards the same wage rate.

- ***The systems DO NOT converge towards an equal distribution.*** Despite a starting position characterized by an equal distribution (95% of the surplus was associated to workers) at the end of the production periods the convergence is towards an unequal distribution.

- ***The systems ARE EFFICIENT.*** According to the efficiency indexes the virtual systems converge towards the *wage-profit frontier*. This convergence is stable.

It is interesting to note how the difference between homogenous or heterogeneous ARAs does not change the conclusions that are common to both cases.

7.6 Results for the *case 2*: the influence of the local markets

7.6.1 Convergence and time of convergence

In these further experiments, the role of local markets has been tested. 9 producers and 162 workers have been considered. A high number of virtual economies (i.e., with different characteristics of the ARAs and different methods at disposal) with homogenous or heterogeneous ARAs (i.e., 9 local markets) has been generated. The time of convergence and the percentages of virtual economies that converged are reported in the figure 7.3.

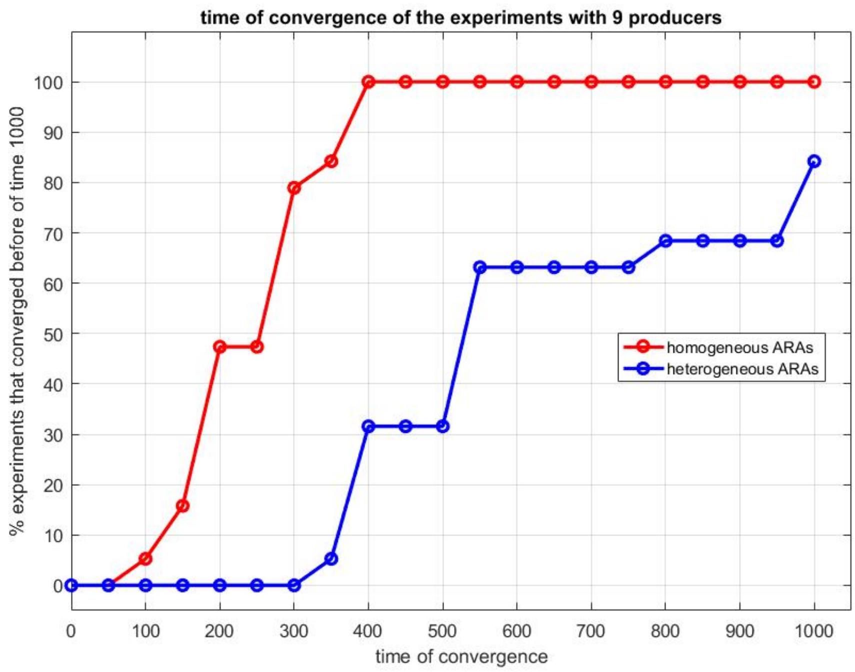


Figure 7.3: **Time of convergence and percentage of economies able to converge before time t=1000**: in the figure, in colour red (case homogeneous ARAs) and blue (case heterogeneous ARAs), the percentages of experiments able to converge to an equilibrium position for each point in time are reported. The total number of virtual economies considered for each experimental setting is equal to 20. For more details on this point see sections 4.2 and 4.5.

As the figure clearly shows, for the case with homogenous ARAs all experiments converge towards an equilibrium at time 400 or before. The presence of heterogeneous ARAs implies that less than 90% of the experiments are able to converge before time 1000. These experiments will be part of further investigations (beyond this thesis) aimed at understanding whether convergence can be achieved for more than 1000 periods or not. In the latter case, the causes for lack of convergence should be further analysed.

7.6.2 Characteristics of the convergence

Among the experiments of the previous section, 10 virtual economies able to converge towards production prices have been selected and analysed. The results of the 10 virtual economies with 9 producers and 162 workers, with homogeneous and heterogeneous ARAs, are reported in the following tables.

Homogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	35	39	40	24	40	23	24	26	29	21
% surplus to workers	65	61	60	76	60	77	76	74	71	79
CONV	3	3	3	3	3	3	3	3	3	3
EFF	9	9	9	9	9	9	9	9	9	9
TechStab	9	9	9	9	9	9	9	9	9	9

Table 7.4: **Results of the experiments**: considering 10 different virtual economies E and a total population of 171 ARAs (162 workers and 9 producers) for the case with homogeneous ARAs.

Heterogeneous ARAs	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
uniform profit rates	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*	*no*
uniform wage rates	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*
% surplus to producers	33	21	33	32	50	29	30	28	40	32
% surplus to workers	67	79	67	68	50	71	70	72	60	68
CONV	3	3	3	3	3	3	3	3	3	3
EFF	9	9	9	9	9	9	9	9	9	9
TechStab	9	9	9	9	9	9	9	9	9	9

Table 7.5: **Results of the experiments**: considering 10 different virtual economies E and a total population of 171 ARAs (162 workers and 9 producers) for the case with heterogeneous ARAs.

The presence of 9 local markets does not change the conclusions reached in previous experiments. It is interesting to note that for homogeneous ARAs all local markets are equal. In particular, experiments $E1$ and $E2$ have also the same intial condition and parameters with respect to experiments E1 and E2 with only 3 producers. It is interesting to note that they reach exactly the same results, so that the global dynamics is equal to the local dynamics. For the case of heterogeneous ARAs the global and the local dynamics change. Indeed, only wage rates converge towards the same value in all local markets. This because workers can migrate in the neighbouring enterprises and in time this implies that workers can migrate along all the network. This competition process in the labour market is the mechanism that allows to reach an uniform wage rate.

7.7 Robustness checks: some notes

All the different virtual economies considered in this chapter are identified by an unique identity code. Each identity code is associated to different characteristics of the ARAs and a different technological set, so that each virtual economy is associated to an unique set of initial conditions and parameters. This heterogeneity in the initial conditions and parameters represents the principal attempt to provide a robustness check to our *thought experiment*. Despite this, it is interesting to test the effect of a particular type of perturbation, i.e., a number of workers and producers which is not a power of 3. This robustness check has been considered also in the previous chapter in the section 6.6. It has been considered sufficient one single experimental setting for this robustness check: 6 producers and 11 workers for each enterprise at the beginning of the period. The experiment has been repeated for the case with homogenous and heterogeneous ARAs. The results do not change at all the answers to our initial research questions.

7.8 The role of innovation and the possibility of lock-in phenomena

The results of the previous experiments indicate that all virtual economies that converge towards equilibrium are able to reach technological efficiency (as defined according to the efficiency indexes). At this point it is interesting to verify if innovation, represented by the introduction of a more efficient method of production at disposal of a particular producer, would be able to be adopted or would be victim of *lock in* phenomena. *Lock in* phenomena can be defined as situations in which, despite a particular method is more efficient, some circumstances and factors inside the economic system prevent it to be adopted. So the interesting point is to investigate, as Brian Arthur claims, the "circumstances under which the economy might become locked-in by 'historical events' to the monopoly of an inferior technology" (Arthur, 1989, p.117). In order to test this, the case with 9 producers and homogeneous ARAs in which producer 5 (a producer in the centre of the lattice) has access to a method more efficient with respect to the other producers (that have access only to the old methods) has been considered. The innovation could be of different types:

- *Innovation type 1*: the new method (the innovation) allows to compute a new *wage-profit frontier* that dominates the old one on the whole domain.
- *Innovation type 2*: the new method (the innovation) allows to compute a new *wage-profit frontier* that dominates the old one only on a part of the domain (low).

- *Innovation type 3*: the new method (the innovation) allows to compute a new *wage-profit frontier* that dominates the old one only on a part of the domain (high).

The comparison between the *wage-profit frontier* before and after the innovation is displayed in the figures 7.4, 7.5 and 7.6. The presence of a more efficient method implies that the new *wage-profit frontier* dominates (on the whole domain or simply on a part of it, it depends on the method) the old *wage-profit frontier*.

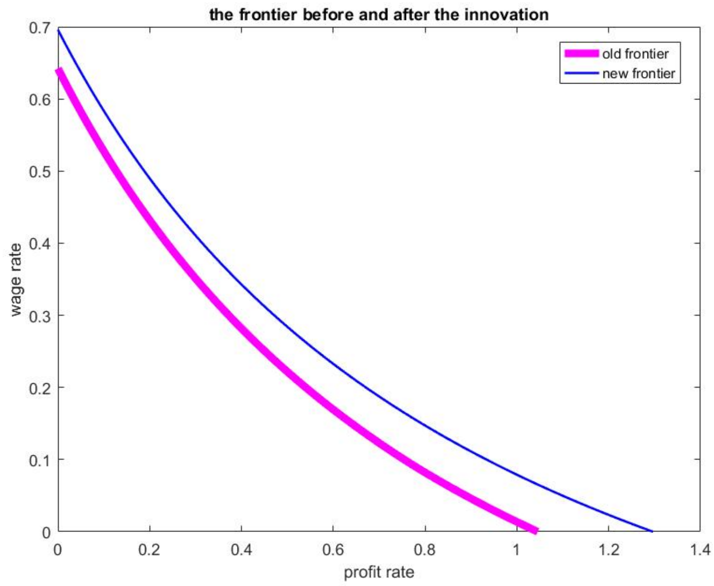


Figure 7.4: **Innovation type 1**: the new method (the innovation) allows to compute a new *wage-profit frontier* that dominates the old one on the whole domain.

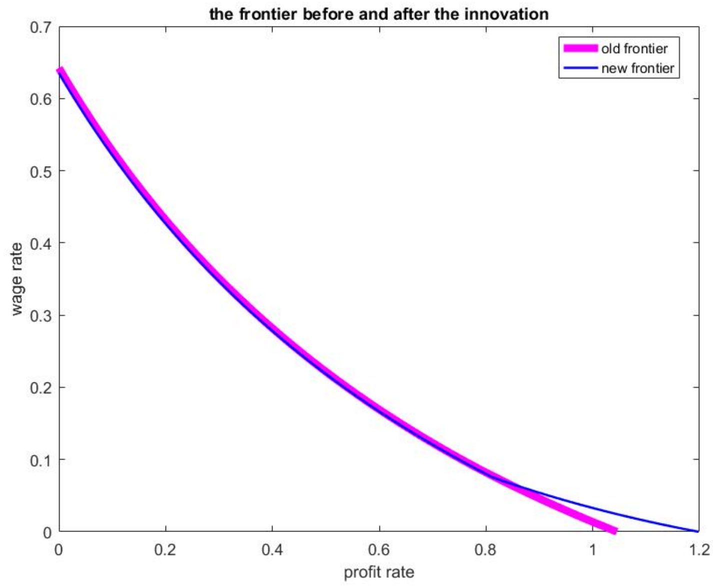


Figure 7.5: **Innovation type 2**: the new method (the innovation) allows to compute a new *wage-profit frontier* that dominates the old one only on a part of the domain (low).

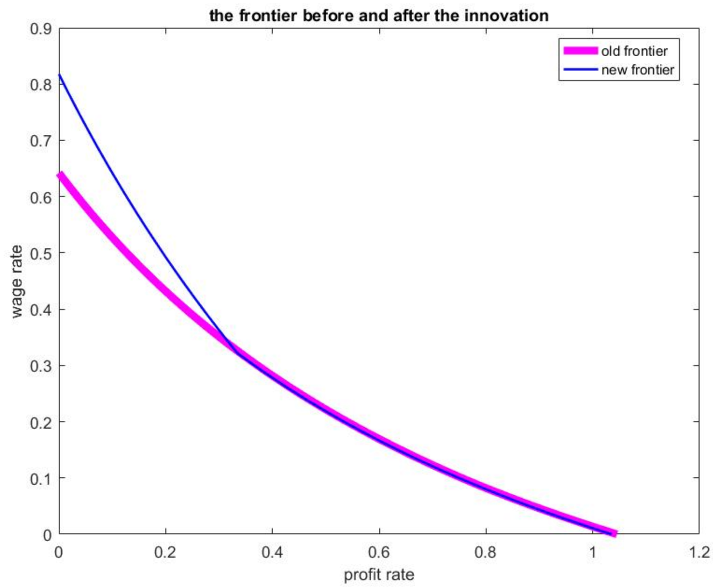


Figure 7.6: **Innovation type 3**: the new method (the innovation) allows to compute a new *wage-profit frontier* that dominates the old one only on a part of the domain (high).

7.8.1 The experimental setting

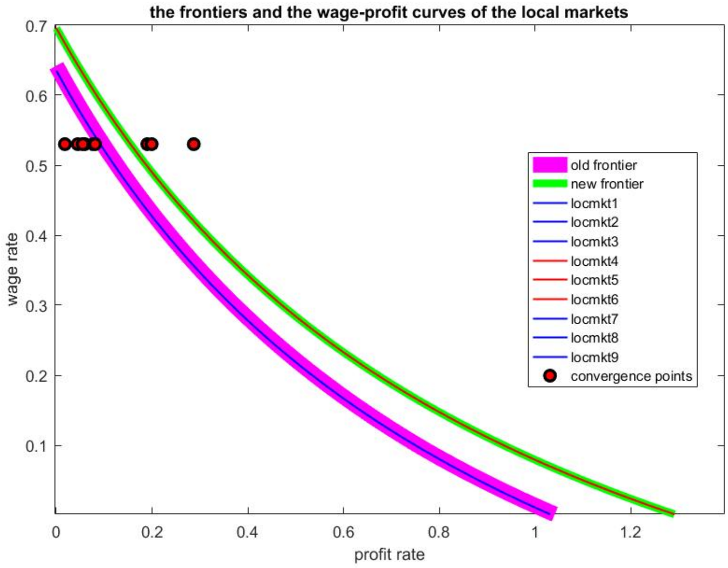
In order to test the effect of the introduction of the new method (the innovation), the case with 9 producers and 162 workers (9 local markets) with homogeneous ARAs has been considered. After convergence towards an equilibrium position, only ARA producer i=5 starts to have access to a new method for the production of his commodity. The other ARA producers have no access to this new method. ARA producer i=5 has been chosen because he is at the centre of the lattice and this allows to test the effect of this perturbation of the equilibrium on the lattice (indeed, the innovation represents a sort of shock for the equilibrium reached). The experiment has been repeated for each type of innovation considering the same experiment (i.e., the same identity code).

7.8.2 Results: the role of innovation

The profit rates and wage rates associated to the new equilibrium are displayed in the figure 7.7 where it has been also reported the new and the old *wage-profit frontier*. A first interesting fact is that the introduction of an innovation at disposal of a single producer implies that, despite ARAs are homogeneous, the dynamics of local markets become different. Indeed, as appears clear from figure 7.7, despite the ARAs are homogeneous, the producers of the same commodity converge towards different profit rates so that, instead of converging towards 3 different[123] profit rates, the virtual economy converges towards 9 different rates of profit. This means that the local and the global dynamics can diverge persistently in presence of asymmetric shocks (i.e., shocks - as in the present case - where only one producer has access to a new method of production).
According to the results of the 3 experiments ARA producer 5 is able to converge towards the new method for innovation of type 1 and 3 but not 2. This because the new method will be adopted only if market prices and wages make its adoption convenient. For innovation of type 1 and 3 market prices and wages converge towards levels that make convenient for producer 5 to adopt the new method, for innovation type 2 this is not the case. This is evident if one compares the area in which the profit rates and wage rates converge for each experiment and the area in which the new frontier dominates the old one. But this result *does not* represent an example of *lock-in* phenomena. Indeed, the experiments results demonstrate that some types of innovation (i.e., some new methods that are not able to generate a new *wage-profit frontier* that dominates the old one on the whole domain) are not adopted simply because they are not coherent with the characteristics and the distributional-technological pattern of the virtual economy. In order

[123]According to the results of the previous experiments, by now we should expect to converge towards non-uniform profit rates.

to have a real *lock-in* phenomena, we should observe a situation in which the profit rates and wage rates converge towards the region in which the new *frontier* dominates the old one and despite this the new method is not adopted. In our case this does not happen. One explanation of this phenomena could be that innovation type 2 was not suitable for the distributional and technological pattern of the virtual economy. The virtual economy is not characterized by *capital intensive methods*. An innovation that allows to produce more with a lower use of labour is a real innovation for the economy (as innovation type 1 and 3), but an innovation that allows to produce more using less capital (as innovation type 2) is not an useful innovation for a virtual economy not characterized by *capital-intensive methods*. Moreover, the virtual system is efficient and it is able to adopt an innovation (i.e., whatever new method able to improve the old *wage-profit frontier*) if the innovation is coherent with the characteristics of the virtual economy. This allows to observe an interesting point. Indeed, usually it is common to talk about innovation as an *a priori* improvement for the economy. The experiment demonstrates that a new method is a real innovation if and only if it is coherent with the features of that particular economy.


the frontiers and the wage-profit curves of the local markets
wage rate
profit rate
old frontier
new frontier
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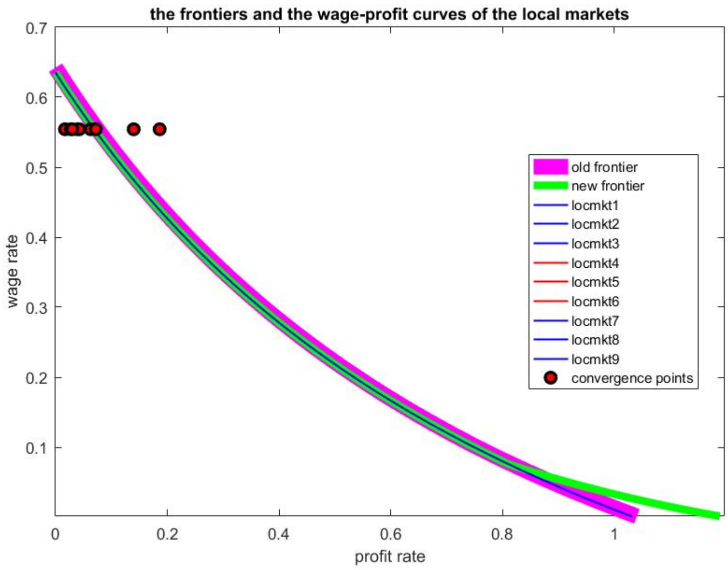
convergence points


the frontiers and the wage-profit curves of the local markets
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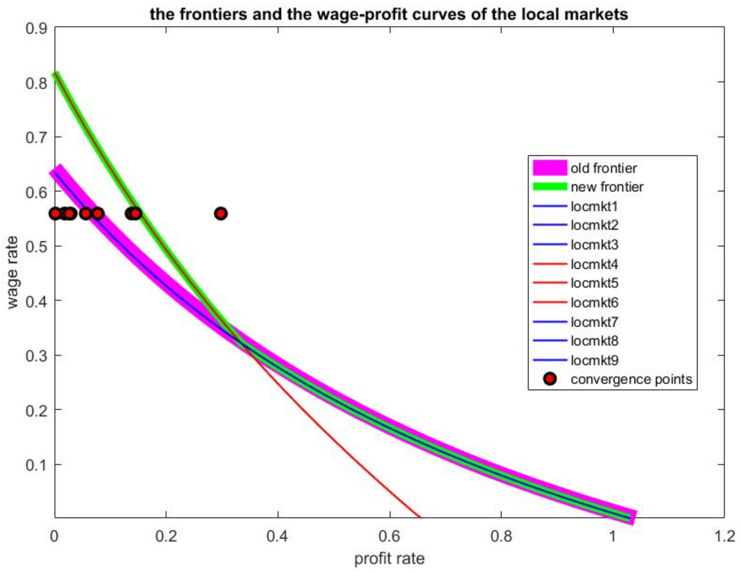


Figure 7.7: **Results of the experiments**: the figures represent the tests of innovations type 1,2 and 3 respectively. The old frontier (magenta), the new frontier (green) after the introduction of the new method and the wage-profit curves towards which each local market converges after the introduction of the new method (blue lines) have been reported. In particular, the wage-profit curves of the local markets in which producer 5 (the innovator) is present are reported in red. Indeed, only these local markets can converge towards the new wage-profit frontier. The profit rates and the wage rates associated to the new equilibrium have been reported. It is possible to note that in the new equilibrium wage rates are uniform. Only in the first and in the third figure local markets where producer 5 is present are able to converge towards the new frontier.

7.9 Conclusion

The fundamental conclusion of the present chapter is that the most important findings of the previous experiments have been confirmed also in the general case characterized by the presence of 10 different virtual economies with the possibility to choose the best method among a set of alternatives.

Indeed, according to the results, all the *Sraffian virtual economies* are able to converge towards production prices and uniform wage rates but these equilibrium positions do not imply convergence towards uniform profit rates and neither an equal distribution of the surplus. Another time, the cases with homogeneous, heterogeneous ARAs and also the case with more local markets (i.e., more producers producing the same commodity in a *ring one-dimensional lattice*) have been considered. These further experiments have not changed the conclusions. It is interesting to note how the probability of convergence in reasonable time (before the *stopping rule* $\bar{t}$=1000) declines with the complexity of the experimental setting (i.e., heterogeneous ARAs, more local markets and the possibility to choose the best method of production).

The possibility for ARA producers to choose the best method for the production of their commodity allowed to check if the virtual economies were efficient; i.e., if the ARA producers were able to adopt the most efficient method. The indexes that allowed to measure efficiency have been computed starting from the construction of the *wage-profit frontier*. According to the results, all virtual economies can be considered efficient: we have obtained convergence in methods inside each industry (i.e., all producers of the same commodity converge towards the same method) and a stable converge towards the *wage-profit frontier*.

But efficiency does not imply that an innovation would be not victim of *lock-in phenomena* (i.e., situations in which, despite a new method more efficient that the others is at disposal, some factors prevents its adoption). It has been considered 3 types of innovation. These further experiments have demonstrated that an innovation can be adopted only if the new method is coherent with the distributional-technological pattern (unless the innovation interests the whole domain of the *wage-profit frontier*; i.e., the new frontier dominates the old one on the whole domain). This *does not* represent an example of *lock-in* phenomena but anyway it is an interesting result because indicates how a new method can be considered an innovation if and only if it is coherent with the characteristics of the economy.

8 Experiment 4: the role of the rate of interest

8.1 Introduction to experiment 4

The most important findings revealed by the previous experiments are that the virtual systems converge towards production prices, uniform wage rates but non-uniform profit rates with an unequal distribution. The probability of convergence in reasonable time (i.e., before the stopping rule fixed to time $\bar{t}$=1000) declines with the growing of the complexity of the experimental setting, which is maximum with heterogeneous producers, 9 local markets and multiple methods of production at disposal. The virtual systems result to be stable and efficient from a technological point of view (i.e., all local markets reach the *wage-profit frontier* and all producers of the same industry adopt the same method of production) is adopted if coherent with the features of the economy.

In this chapter the purpose is to check whether these findings can be influenced by the presence of an authority which is able to put an interest rate on the debt and credit accumulated (the financial *wealth*). The interest rate can be exogenous or endogenous. Both experimental settings will be considered. The presence of an interest rate allows to make a comparison between the position of each ARA in time and subject to different levels of the interest rate. This allows to check if, for each single ARA, an interest rate is able to introduce Pareto improvements with respect to other scenarios. The introduction of experiments aimed at understanding the role of the interest rate for the credit and debt relations and its effects on distribution and production allows to analyse a topic that is crucial within Keynesian economics. A comparison between the role of the interest rate inside the *digital economic laboratory* and the role of the interest rates inside the Keynesian tradition will be considered.

8.2 The role of the interest rate for the debt and credit relations

In the previous experiments, in the convergence position, debt and credit relations disappear and also the financial wealth accumulated by each ARA tends towards the zero. This because after the convergence towards the production prices (that imply profit rates bigger or equal to zero) the virtual system is able to reproduce itself without any need to create new IOUs. For all the experiments it has been supposed that the IOUs are automatically underwritten when needed, without any restriction or any cost. We recall the fundamental equations related to the financial positions that are reported also in chapter 3. Consider with F a negative financial position (if $F < 0$) or a positive financial position (if $F > 0$), that is the difference between revenues and costs for each producer i (for i=1,...k) and worker j

(for j=1,...,K):

$$F_{it} = revenues_{it} - expenditures_{it} \tag{8.1}$$

$$F_{jt} = revenues_{jt} - expenditures_{jt} \tag{8.2}$$

The debt and credit accumulation in time represents the level of wealth (positive or negative) associated to each ARA producer or worker:

$$wealth_{i,t-1} = \sum_{t=1}^{t-1} F_{it} \qquad wealth_{it} = wealth_{i,t-1} + F_{it} \tag{8.3}$$

$$wealth_{j,t-1} = \sum_{t=1}^{t-1} F_{jt} \qquad wealth_{jt} = wealth_{j,t-1} + F_{jt} \tag{8.4}$$

The accumulation of wealth does not imply restrictions, costs or revenues for each ARA. Moreover, wealth growth has no limit. Despite this, most virtual economies tend to converge towards production prices so that the level of wealth of each ARA tends to zero and the creation of new IOUs is no more needed. At this point it would be interesting to investigate how the existence of an external authority (e.g., a central bank) that does not impose limits on the creation of IOUs and accumulation of wealth but fixes an interest rate that has to be computed on the *wealth* at the end of each production period t can influence the dynamics of the system. As it will clearly appear, the introduction of the interest rate represents an investigation of the role of political economy inside our virtual systems. The interest rate can be positive or negative.

- **Positive interest rate**: a positive interest rate represents a revenue for those who have a positive wealth (i.e., credits that have been accumulated) and a cost for those who have a negative wealth (i.e., debts that have been accumulated). Indeed, a positive interest rate computed on the level of wealth at the end of each production period corresponds to revenues to be added to the financial holdings if wealth is positive and corresponds to a cost that decreases even more the financial holdings if wealth is negative.

- **Negative interest rate**: a negative interest rate is instead a sort of *taxation*, a redistribution of the wealth. It represents a revenue for those with negative wealth and a cost for those with positive wealth. Indeed, a negative interest rate computed on the level of wealth at the end of each production period corresponds to a cost that decreases the value of the financial holdings if wealth is positive and corresponds to a revenue that increases the financial holdings if wealth is negative.

The interest rate can be fixed exogenously by the monetary authority and remain fixed for all the duration of the experiment or can be endogenous and change according to the behaviour of the system. Both alternatives will be considered. The presence of the interest rate δ changes the equations in the following way:

$$wealth_{i,t-1} = \sum_{t=1}^{t-1} F_{it} \qquad wealth_{it} = wealth_{i,t-1}(1+\delta) + F_{it} \tag{8.5}$$

$$wealth_{j,t-1} = \sum_{t=1}^{t-1} F_{jt} \qquad wealth_{jt} = wealth_{j,t-1}(1+\delta) + F_{jt} \tag{8.6}$$

Remember that the presence of interest rates do not break the accounting consistency:

$$\sum_{i=1}^{k} revenues_{it} + \sum_{j=1}^{K} revenues_{jt} = \sum_{i=1}^{k} expenditures_{it} + \sum_{j=1}^{K} expenditures_{jt} \tag{8.7}$$

Which implies:

$$\sum_{i=1}^{k} F_{it} + \sum_{j=1}^{K} F_{jt} = 0 \tag{8.8}$$

$$\sum_{i=1}^{k} wealth_{it} + \sum_{j=1}^{K} wealth_{jt} = 0 \tag{8.9}$$

A comparison between an experiment with interest rate equal to zero and the same experiment simulated with positive or negative interest rates allows to investigate the effect of an economic policy that concentrates on the revenues and costs of the financial holdings. The effect can be measured in terms of **Pareto improvements**. Indeed, in each experiment characterized by a different level of the interest rate, each ARA will arrive at the end of the periods with a particular level of his utility function. It is possible to investigate which level of the interest rate is able to bring each ARA (or social classes, i.e., workers and producers) to the higher level of his utility function.

8.3 The role of the interest rate: a comparison with the Keynesian tradition

The interpretation of the nature and the role of the interest rate in Keynes is completely different with respect to the marginalist approach. Keynes introduces his conception of the interest rate in chapter 17 of his *General Theory.* For Keynes interest rate is a "monetary phenomenon [...], i.e. that it equalises the advantages of holding actual cash and a deferred claim on cash" (Keynes, 1937, p.245). This is in contrast with the classical theory according to which the interest rate depends on "the interaction of the schedule of the marginal efficiency of capital with the psychological propensity to save" (Keynes, 1936, p.165). According to Keynes the *theory of interest* is fundamental for a clear understanding of the determinants of effective demand and employment:

> There is, I am convinced, a fatal flaw in that part of the orthodox reasoning which deals with the theory of what determines the level of effective demand and the volume of aggregate employment; the flaw being largely due to the failure of the classical doctrine to develop a satisfactory theory of the rate of interest (Keynes, 1973b, p.489).

In Keynes the monetary nature of the interest rate is explained through the *liquidity preference theory.* This aspect has been criticized by the supporters of the *loanable funds theory* such as Bertil Ohlin, Dennis H. Robertson and John R. Hicks. According to Ohlin: "the rate of interest is simply the price of credit, and is therefore governed by the supply of and demand for credit. The banking system - through its ability to give credit - can influence, and to some extent does affect, the interest level" (Ohlin, 1937, p.221). It is well known that Keynes, in his *General Theory*, explicitly assumes that the money supply is fully controlled by the central bank and it is a matter of fact that the *liquidity preference theory* "overlooks the presence of banks and bank money" (Bertocco, 2011, p.8). Keynes was convinced that there was no compatibility between his *liquidity preference theory* and the *loanable funds theory*:

> The liquidity-preference theory of the rate of interest which I have set forth in my *General Theory of Employment, Interest and Money* makes the rate of interest to depend on the present supply of *money* and the demand schedule for a present claim on money in terms of a deferred claim on money. This can be put briefly by saying that the rate of interest depends on the demand and supply of money; though this may be misleading, because it obscures the answer to the question, Demand for money in terms of what? The alternative theory held, I gather, by Prof. Ohlin and his group of Swedish economists, by Mr. Robertson and Mr. Hicks, and probably by many others, makes it to depend, put briefly, on the demand and supply of *credit* or, alternatively (meaning

> the same thing), of *loans*, at different rates of interest. Some of the writers (as will be seen from the quotations given below) believe that my theory is on the whole the same as theirs and mainly amounts to expressing it in a somewhat different way. Nevertheless the theories are, I believe, radically opposed to one another. (Keynes, 1937, p.241)

It is well know that Kaldor (starting from Kaldor (1939)) was critical of some aspects of Keynes's theory of money and interest rate (see also Sardoni, 2007). Also Sraffa underlined his perplexities about Keynes's *liquidity preference theory*, that he called 'Keynes's system', and Keynes's use of the concept of commodity rate of interest:

> In his debate with Hayek Sraffa introduced the concept of commodity rate of interest. This concept Keynes was eager to pick up in the *General Theory* (Keynes [1936] 1973, chap. 17, especially 223n), because he thought that it would provide him with the long-sought choice- and capital-theoretic foundation of his theory of investment behavior, both real and financial. The lack of such a foundation was a major objection Hayek had put forward against the *Treatise*. The new concept allowed Keynes, or so he thought, to drive home the main message of the *General Theory*, that it is the downward rigidity of the money rate of interest that is the source of all the trouble. This downward rigidity is in turn explained in terms of the liquidity preference of wealth owners. [...] *Sraffa was not at all happy with Keynes's use of the concept of commodity rates of interest*, and he was critical of his explanation of why liquidity preference was to prevent the [...] money rate of interest from falling sufficiently not only in the short run, but also in the long (see Kurz, 2010). *In Sraffa's view, as it is expressed both in his annotations of his personal copy of the* General Theory *and in two manuscript fragments, Keynes's argument was a mess, confused and confusing*. He argued, among other things, that *the concept of liquidity that Keynes uses is vague and ambiguous* (Kurz, 2013, pp.11-12, emphasis added).

Despite this, some Keynesians are convinced that one should make reference to Keynes's *Treatise on Money* in order to find a richer treatment of how money is created through the interaction of banks and central banks (de Carvalho, 2013, p.432). As a consequence, the importance of the endogeneity of money has been widely recognized and accepted: "many strands of Keynesian macroeconomics agree that money supply is endogenous in some sense. More particularly, practically all post-Keynesian varieties of macroeconomic theory explicitly reject the so-called 'verticalist' assumption that central banks can fully determine the money supply" (de Carvalho, 2013, p.431).

What emerges from this brief summary about the Keynesian tradition on money and interest rate, is that the original contribution of Keynes has been object of debate and different interpretations despite nowadays most

Keynesians recognize the importance of endogenous money, an aspect that has not been clearly developed by Keynes. The purpose of the present chapter is not to enter into the details of this debate. In the *digital economic laboratory*, as already explained in chapters 1 and 2, the integration between Sraffian schemes and Keynes's effective demand has been constructed considering the most simple type of society: without monetary or financial institutions with ARAs that always underwrite IOUs when needed. This simple type of society encapsulates some fundamental characteristics of a *monetary entrepreneurial economy* in Keynesian terms. In this framework a monetary interest rate is simply the cost (if negative) or the revenues (if positive) of the financial wealth (if positive, the contrary in the other case). This allows to understand the influence of this monetary phenomenon on real magnitudes such as demand, production and distribution in the most simple framework[124].

8.4 The experimental setting

The mathematical structure of the *laboratory* is described in details in chapter 3. The behavioural functions are the same of the previous chapters. A detailed description of the behavioural functions is present in chapter 4.

The experimental setting of the present chapter considers the simple case with 3 producers and 54 workers. The ARAs are heterogeneous. The methods at disposal of each producer are 10. In order to make the virtual economies as much productive as possible in this set of new experiments each ARA producer has the possibility to produce new commodities with the residuals from production through the use of a particular set of methods that allows him to use not all the commodities for the production of his commodity (see in the Appendix section 10.2 and section 4.5). 3 different virtual economies (identified by 3 different identity codes) will be tested under two different types of scenarios:

- ***Scenario 1***: an external authority fixes an interest rate. The interest rate is fixed at the beginning of the experiment and does not change in time. The interest rate can be positive or negative. A grid of interest rates (from -5% to 5% step 1%) has been considered.
- ***Scenario 2***: an external authority imposes an interest rate which level changes in time according to a trivial *rule of thumb*: if the maximum wealth is higher with respect to the value of the total production the interest rate will be decreased of the 0.5% and *vice versa*. At the beginning the level of the interest rate is equal to zero.

[124] Financial and monetary markets will be added in a second moment as further research beyond this thesis, see section 9.2.

8.5 The research questions

The two different scenarios are suitable to answer different research questions. *Scenario 1* and *scenario 2* will check if the virtual economy is able to:

- **Convergence towards production prices**: despite the presence of the interest rate, it is important to check whether the system is able to converge towards production prices. The production prices identify the *equilibrium position*. Remember that we define an *equilibrium position* as a position in which prices and wages announced do not change anymore; all markets are in equilibrium (demand equal to supply in all markets; i.e., there is no more rationing), migration stops, the system reproduces itself (i.e., same level of production and surplus without residuals) without the creation of new IOUs (no new debt and credit relations). This implies that the prices of convergence are *production prices* with zero or positive profit rates.

- **Convergence towards an equal distribution**. Given the initial condition of equal distribution (95% of the surplus has been associated to workers given that they are 54 with respect to the 3 producers for a total population of 57 ARAs)[125] it is interesting to test the type of distribution of the surplus associated with the convergence position.

- **Pareto improvements with respect to the base case with interest rate equal to zero**. For each experiment each ARA will arrive at the end of the periods with a particular level of his utility function, according to his consumption and labour supply. It is possible to check for each social class (i.e., producers and workers) which level of the interest rate is able to bring to a final solution that represents a Pareto improvement with respect to the base case with interest rate equal to zero.

- **Pareto improvements in time with respect to the base case with interest rate equal to zero**. It is possible to compute for each social class (i.e., producers and workers) the number of times in which all producers or all workers obtained Pareto improvements with respect to the base case with interest rate equal to zero.

- **Production**. It has been measured how production is changed for each commodity in percentage with respect to the base case with interest rate equal to zero.

[125] It is worth remembering that the initial production prices and wages have been computed starting from this distribution.

8.6 Results: *scenario 1*

In table 8.1, the results relative to the virtual economy tested with the *scenario 1* are reported. It has been considered the most representative case among 3 different virtual economies tested for all the values of the grid for the interest rate (from -5% to 5% step 1%). As appears clear only some levels of the interest rate are able to bring the virtual economy to convergence. Indeed, when the interest rate is negative it represents a form of taxation that facilitates the convergence. When interest rate is positive and high, this means that ARAs with a positive wealth will increase their credit positions while ARAs with negative wealth will increase their level of debt in time, thus preventing convergence.

When interest rate is negative, it represents a form of taxation of wealth able to redistribute resources among ARAs (in particular producers); this allows to improve the level of production of all commodities. Also when the interest rate is positive production increase considerably. This because workers are the ARAs that most of the times have positive wealth and for this reason they are the most favoured by the increase in the interest rate. With a growing wealth, ARA sharply increase their demand of commodities and this stimulates production. Remember that, despite ARAs (in particular producers) with a negative level of wealth will see an increase in their level of indebtedness in the presence of a positive interest rate, this has no effect on their decisions of production because they know that there is not a limit to the level of debt. Despite this, the level of indebtedness can have effects on their price decisions, consumption decisions, on their Pareto improvements and on the final distribution.

The increase in production allows some ARAs to consume more with respect to the base case with interest rate equal to zero and for this reason in each experiment some ARAs experience Pareto improvements. Which ARAs will experience these Pareto improvements (i.e., workers or producers) depends on the dynamics of prices, wages and the final distribution associated to each experiment. In general, workers benefit the most by the presence of the interest rate while producers are penalized in particular by the positive interest rate for the reasons explained before.

The fundamental message that can be derived by the experiments of this *scenario 1* is that the interest rate has the potentiality to improve production, bring to convergence and Pareto Improvements for all ARAs. This particular combination of effects has been obtained only with an interest rate equal to -5% (it is interesting to observe that with this level of the interest rate for most of the periods the majority of the ARAs improves his utility). Despite this, it is true that a positive interest rate has the potentiality to improve sharply production even more that with negative interest rates. This result is extremely interesting, because it shows that production can greatly improve without convergence. This result calls into question the idea that convergence towards a point of equilibrium where debt and credit relations disappear is a necessary requirement to obtain a better economic performance. In fact, it seems that high and increasing values of debt and credit can have a positive effect on production. This counterintuitive result certainly deserves further investigation beyond the present thesis. In order to test the potentialities of a combination of positive and negative interest rates have been tested the idea of an endogenous interest rate with *scenario 2*, considering the experimental setting of table 8.1.

Rate of interest δ	-0.05	-0.04	-0.03	-0.02	-0.01	0	0.01	0.02	0.03	0.04	0.05
production prices	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*no*	*no*	*no*	*no*
% surplus to producers	20	14	10	26	21	20	16	23	27	5	4
% surplus to workers	80	86	90	74	79	80	84	77	73	95	96
Δ% commodity 1 w.r.t δ=0	26	23	49	21	-1	0	2	93	58	85	50
Δ% commodity 2 w.r.t δ=0	37	31	59	20	14	0	6	80	56	80	56
Δ% commodity 3 w.r.t δ=0	38	29	59	3	10	0	12	93	59	104	68
% producers that improve utility w.r.t δ=0	100	37	33	64	33	0	0	67	33	0	0
% times that all producers improve utility w.r.t δ=0	65	4	2	5	2	0	4	3	2	3	3
% workers that improve utility w.r.t δ=0	100	100	100	78	65	0	0	100	100	100	100
% times that all workers improve utility w.r.t δ=0	70	75	70	2	3	0	62	86	75	84	76

Table 8.1: **Results of the experiments**: considering a representative virtual economy, results have been computed according to the level of the interest rate.

8.7 Results: *scenario 2*

The results relative to the virtual economy tested with the *scenario 2* are reported in the following table. It is the same experimental setting of the previous scenario (and the same identity code). In order to make the role of the endogenous interest rate clear, the results related to the experiment tested with zero interest rates have also been reported.

Rate of interest δ	0	*endo*
production prices	*yes*	*yes*
% surplus to producers	20	22
% surplus to workers	80	78
Δ% commodity 1 w.r.t δ=0	0	154
Δ% commodity 2 w.r.t δ=0	0	139
Δ% commodity 3 w.r.t δ=0	0	162
% producers that improve utility w.r.t δ=0	0	100
% times that all producers improve utility w.r.t δ=0	0	64
% workers that improve utility w.r.t δ=0	0	100
% times that all workers improve utility w.r.t δ=0	0	70

Table 8.2: **Results of the experiments**: considering the same representative virtual economy of the previous table, a comparison has been made, between the case with interest rate equal to zero and the case with endogenous interest rate.

An endogenous interest rate is able to exploit the positive effects of a negative interest rate (i.e., convergence towards equilibrium and Pareto improvements for all the ARAs) and those of a positive interest rate (i.e., sharp increase in production). This experiment underlines clearly how the presence of an authority able to impose a variable interest rate (i.e., the introduction of an economic policy) can redistribute resources in such a way to improve sharply the performance of the whole economy and to bring to a final equilibrium position that represents a Pareto improvement with respect to the case of a virtual economy where prices, wages and distribution are determined only by the free interaction between agents. This interesting result certainly deserves further investigations (together with the results of the *scenario 1*) in order to draw more general conclusions about the role of the interest rate. This analysis is beyond the scope of this chapter and represents one of the most important and interesting points for further research. Indeed, a deeper understanding of the implications of these results requires the testing of alternative behavioural functions and a shift to more complex trading patterns that are object of further research (see section 9.2).

8.8 Conclusion

The purpose of the present chapter was to investigate the effect of the introduction of an interest rate on the financial holdings (the accumulation of the IOUs called financial *wealth*) on the dynamics of the virtual system.

In particular, the influence of the interest rate on convergence, production levels, distribution and the utilities of the ARAs has been investigated through the computation of the Pareto improvements with respect to the base case with interest rate equal to zero.

Two different types of interest rate have been considered. In *scenario 1* an external authority fixes an interest rate. The interest rate is fixed at the beginning of the experiment and does not change in time. The interest rate can be positive or negative. A grid of interest rates (from -5% to 5% step 1%) has been considered. In *scenario 2* an external authority impose an interest rate whose level changes in time according to a trivial *rule of thumb*: if the maximum wealth is higher with respect to the value of the total production the interest rate will be decreased. The contrary in the other case. At the beginning the level of the interest rate is equal to zero.

The most important findings from *scenario 1* are that only a negative level of the interest rate is able to bring the virtual system towards equilibrium with Pareto improvements for all the ARAs; on the other hand, a positive interest rate is able to stimulate production and distributional equality. In *scenario 2* the introduction of an interest rate which level changes endogenously according to the relationship between the level of production and wealth has been considered. The most important findings from this *scenario 2* is that an endogenous interest rate is able to bring to convergence, Pareto improvements for all ARAs with very high level of production with respect to the base case. This experiment demonstrates how an endogenous interest rate is able to exploit the positive effects of a negative interest rate and those of a positive interest rate. This experiment underlines clearly how the presence of an authority (i.e., the introduction of an economic policy) able to impose a variable interest rate can redistribute resources such a way to improve sharply the performance of the whole virtual economy and to bring to a final equilibrium position that represents a Pareto improvement with respect to the case of an economy where prices, wages and distribution are determined only by the free interaction between agents.

9 Conclusions

Despite the wide recognition of the relevance of Keynes's and Sraffa's contributions, as a matter of fact, an integration between Keynesian and Sraffian economics has not been reached yet. Nevertheless, the integration of Sraffa's and Keynes's analyses could constitute the core of *non-neoclassical economics* (Roncaglia, 1995, p.120). In the literature review of the present thesis (chapter 2), the reasons and the conditions for an integration of Sraffa's and Keynes's investigations have been discussed.

The purpose of the present thesis was to contribute to fill this gap by merging Keynes's *effective demand* with Sraffa's *production schemes* through the construction of a *digital economic laboratory* where to run *thought experiments* aimed at answering different research questions about equilibrium, distribution, technological efficiency and economic policy. One of the novelty of the *digital economic laboratory* of the present thesis is its coherence with the mathematics of the digital computers. Following computable methods, the *laboratory* has been constructed in order to be algorithmically grounded. It is based on bookkeeping principles and it is able to consider market out-of-equilibrium behaviours.

The *digital economic laboratory* has been described in details in chapter 3 where the mathematical reconstruction of the model has been reported. This *digital economic laboratory* has been written in the form of a coherent set of computer codes so as to allow for the construction of a virtual market (labour and goods market), in which decisions of production and consumption are made by a population of algorithmic rational agents (producers and workers) which interact exchanging quantities and property rights coherently with double-entry bookkeeping principles. Exchanges are made by signing virtual contracts that involve the use of financial means. The creation of new financial means of exchange, credit and debt, is endogenous. Production is heterogeneous and conceived as a circular process. The ARAs are characterized by behavioural functions and specific trading rules. ARAs, located inside a lattice, have a deterministic network of relations. ARAs producers sell their products in local markets. The local markets are connected spatially. Each local market represents a local unit of the whole economy and the presence of more interconnected local markets, with more ARAs producers producing the same commodity in different local markets, allows to make comparisons between the local and the global dynamics. The *digital economic laboratory* is a benchmark for all the *thought experiments*. Indeed, behavioural functions have to be designed according to the particular *thought experiment* in order to generate different virtual economies and collect statistics.

In chapter 4 the experimental setting and particular behavioural functions have been designed and detailed described. In particular, the algorithm that described the announcement of prices and wages has been constructed using *machine learning* and in particular *artificial neural network*. This approach represents an interesting application of *machine learning* inside economics and a powerful tool for describing agent capabilities to learn and to develop decisions coherently with their environment.

9.1 The results

From chapter 5 to chapter 8 four different experiments have been tested. The results have been detailed described and commented. In the following list the main findings have been summarised:

- The probability for a virtual economy to convergence in reasonable time (i.e., before the stopping rule fixed to time $\bar{t}$=1000) declines with the growing of the complexity of the experimental setting, which is maximum with heterogeneous producers, 9 local markets and multiple methods of production at disposal.

- The virtual economies (i.e., not only the case reported by Sraffa at page 19 of *PCMC*) tend to converge towards production prices, uniform wage rates but non-uniform profit rates with an unequal distribution. The further experiments, aimed to investigate the relationship between prices and quantities, seem to underline how well-known economic relations (such as the negative slope of demand function and the positive slope of supply function) represent postulates that have not been clearly confirmed by our experiments. In particular, the experiments seem to demonstrate that the composition of the social surplus is a function of effective demand while the dependence of prices on effective demand is more questionable.

- The virtual systems result to be stable and efficient from a technological point of view (i.e., all local markets reach the *wage-profit frontier* and all producers of the same industry adopt the same method of production).

- Innovation (represented by a new method of production at disposal of a single producer) is not subject to technological *lock-in phenomena*: innovation is adopted if it is coherent with the features of the virtual economy. It is interesting to note how innovation is a shock capable of provoking a persistent divergence between local and global behaviour.

- The presence of an authority able to impose a variable interest rate (i.e., an economic policy) can redistribute resources in order to improve sharply the performance of the whole economy towards a final equilibrium position that represents a Pareto improvement with respect to the case of an economy where prices, wages and distribution are determined only by the free interaction between agents.

9.2 Further research

Throughout the thesis, we have been referring often to theoretical issues and controversial results that would have been developed in further research, beyond the present thesis. In this section we present a summary of these topics. First of all, the *digital economic laboratory* can be improved through the introduction of:

- *Different behavioural functions and learning methods*: for example learning methods that consider the role of imitation, algorithmic methods that use only top-down approaches, and behavioural functions different with respect to the utility function for the determination of the consumption demand and the labour supply.
- *Financial system and banking system*: more complex institutions can be added in order to verify their influence on production and distribution.
- *Different out-of-equilibrium starting points*: in all the experiments of the present thesis, the initial condition was an equilibrium point with production prices and equal distribution. The role of different initial conditions (e.g., non-equilibrium points) can be tested.

Some interesting results that emerged in the present thesis needs to be further developed:

- *Time of convergence*: in the previous chapters, we have seen that some virtual economies have not been able to converge before time $\bar{t}$=1000. The dynamics of these virtual economies will be part of further investigations aimed at understanding whether convergence can be achieved for more than 1000 periods or not. In the latter case, the causes for lack of convergence should be further analysed.
- *Unemployment*: unemployment phenomena can emerge in the *laboratory*. In the thought experiments considered in the present thesis, unemployment phenomena were on the background of the out-of-equilibrium dynamics. Despite this, in the present thesis, unemployment phenomena have not been analysed because the primary focus was the analysis of the convergence properties. Specific indexes will be developed in further investigations in order to study properly this phenomenon.

- *The determinants of distribution*: it seems that distribution is influenced by preferences (i.e., the demand) if the technological set of the virtual economy is not composed by *capital-intensive* methods. Indeed, with emphcapital-intensive methods, distribution is extremely unequal regardless of the preference parameters. The causes of this phenomenon should be carefully analysed considering also other behavioural functions and more experiments.

- *The role of the interest rate*: the presence of more complex institutions, such as banking system and financial system, allows to investigate in a more realistic framework the influence of a monetary interest rate on the economic magnitudes.

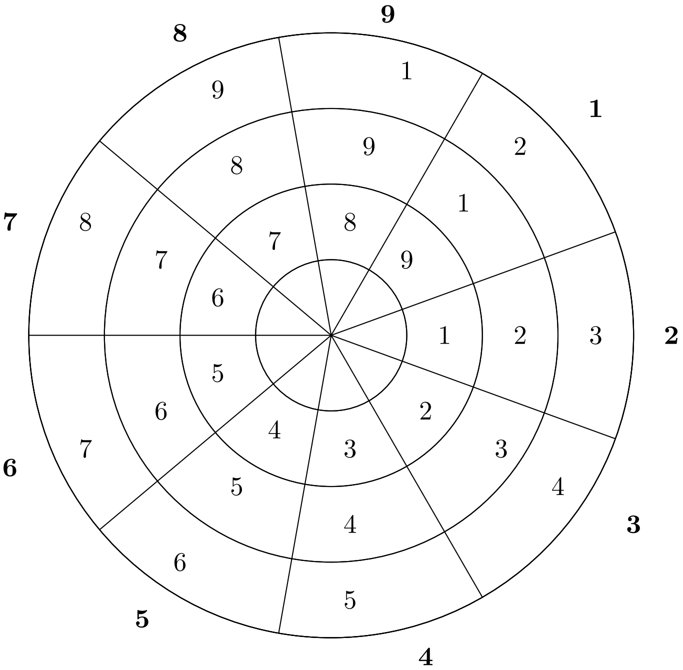
9.3 Last considerations

Obviously all these results have not to be interpreted as final answers to the initial research questions but as first steps towards a new way to face them for the development of further research questions and intuitions. In the previous section, a first list of further research topics has been presented, but the list can be enlarged with other interesting *thought experiments* and theoretical issues. Indeed, the real purpose of the present thesis was the introduction of a powerful instrument, i.e., the *digital economic laboratory*, as an answer to the need to develop models able to explore the complexity of the out-of-equilibrium behaviours, grounded on bookkeeping principles and computable methods and, mostly, coherent with the spirit and the method of Sraffa's and Keynes's investigations.

10 Appendix

10.1 Numerical example

Consider the topology of a virtual economy with 9 producers and 81 workers, which produces 3 commodities. Consider the following graphical representation of the *ring one-dimensional lattice*.



9	1	2	3	4	5	6	7	8	9	1	$for\;\; i=1,...,k$
3	1	2	3	1	2	3	1	2	3	1	$for\;\; g=1,...,n$

Figure 10.1: **The local markets: case k=9**

For each producer, it is possible to calculate (remember that if i=μn for μ=1,...,n, $\iota(i)$=n):

i=1 ι(1)=mod(1/3)=1
i=2 ι(2)=mod(2/3)=2
i=3 ι(3)=3
i=4 ι(4)=mod(4/3)=1
i=5 ι(5)=mod(5/3)=2
i=6 ι(6)=3
i=7 ι(7)=mod(7/3)=1
i=8 ι(8)=mod(8/3)=2

i=9 $\iota(9)$=3

And for the each local market:

s=1	$\iota(1)$=1	s(1)=1	$\tilde{\Psi}_1$=[9 1 2]	Ψ_1=[1 2 9]
s=2	$\iota(2)$=2	s(2)=2	$\tilde{\Psi}_2$=[1 2 3]	Ψ_2=[1 2 3]
s=3	$\iota(3)$=3	s(3)=3	$\tilde{\Psi}_3$=[2 3 4]	Ψ_3=[4 2 3]
s=4	$\iota(4)$=1	s(4)=4	$\tilde{\Psi}_4$=[3 4 5]	Ψ_4=[4 5 3]
s=5	$\iota(5)$=2	s(5)=5	$\tilde{\Psi}_5$=[4 5 6]	Ψ_5=[4 5 6]
s=6	$\iota(6)$=3	s(6)=6	$\tilde{\Psi}_6$=[5 6 7]	Ψ_6=[7 5 6]
s=7	$\iota(7)$=1	s(7)=7	$\tilde{\Psi}_7$=[6 7 8]	Ψ_7=[7 8 6]
s=8	$\iota(8)$=2	s(8)=8	$\tilde{\Psi}_8$=[7 8 9]	Ψ_8=[7 8 9]
s=9	$\iota(9)$=3	s(9)=9	$\tilde{\Psi}_9$=[8 9 1]	Ψ_9=[1 8 9]

The whole matrix $\tilde{\mathbf{\Psi}} = [\tilde{\psi}_{\bar{i}s}]$ corresponds to:

$$\tilde{\mathbf{\Psi}} = \left[\begin{array}{ccccccccc} \tilde{\mathbf{\Psi}}_1 & \tilde{\mathbf{\Psi}}_2 & \tilde{\mathbf{\Psi}}_3 & \tilde{\mathbf{\Psi}}_4 & \tilde{\mathbf{\Psi}}_5 & \tilde{\mathbf{\Psi}}_6 & \tilde{\mathbf{\Psi}}_7 & \tilde{\mathbf{\Psi}}_8 & \tilde{\mathbf{\Psi}}_9 \end{array} \right]$$

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix} 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{bmatrix}$$

The whole matrix $\mathbf{\Psi} = [\psi_{\bar{i}s}]$ corresponds to:

$$\mathbf{\Psi} = \left[\begin{array}{ccccccccc} \mathbf{\Psi}_1 & \mathbf{\Psi}_2 & \mathbf{\Psi}_3 & \mathbf{\Psi}_4 & \mathbf{\Psi}_5 & \mathbf{\Psi}_6 & \mathbf{\Psi}_7 & \mathbf{\Psi}_8 & \mathbf{\Psi}_9 \end{array} \right]$$

$$\mathbf{\Psi} = \begin{bmatrix} 1 & 1 & 4 & 4 & 4 & 7 & 7 & 7 & 1 \\ 2 & 2 & 2 & 5 & 5 & 5 & 8 & 8 & 8 \\ 9 & 3 & 3 & 3 & 6 & 6 & 6 & 9 & 9 \end{bmatrix}$$

10.2 Methods and residuals

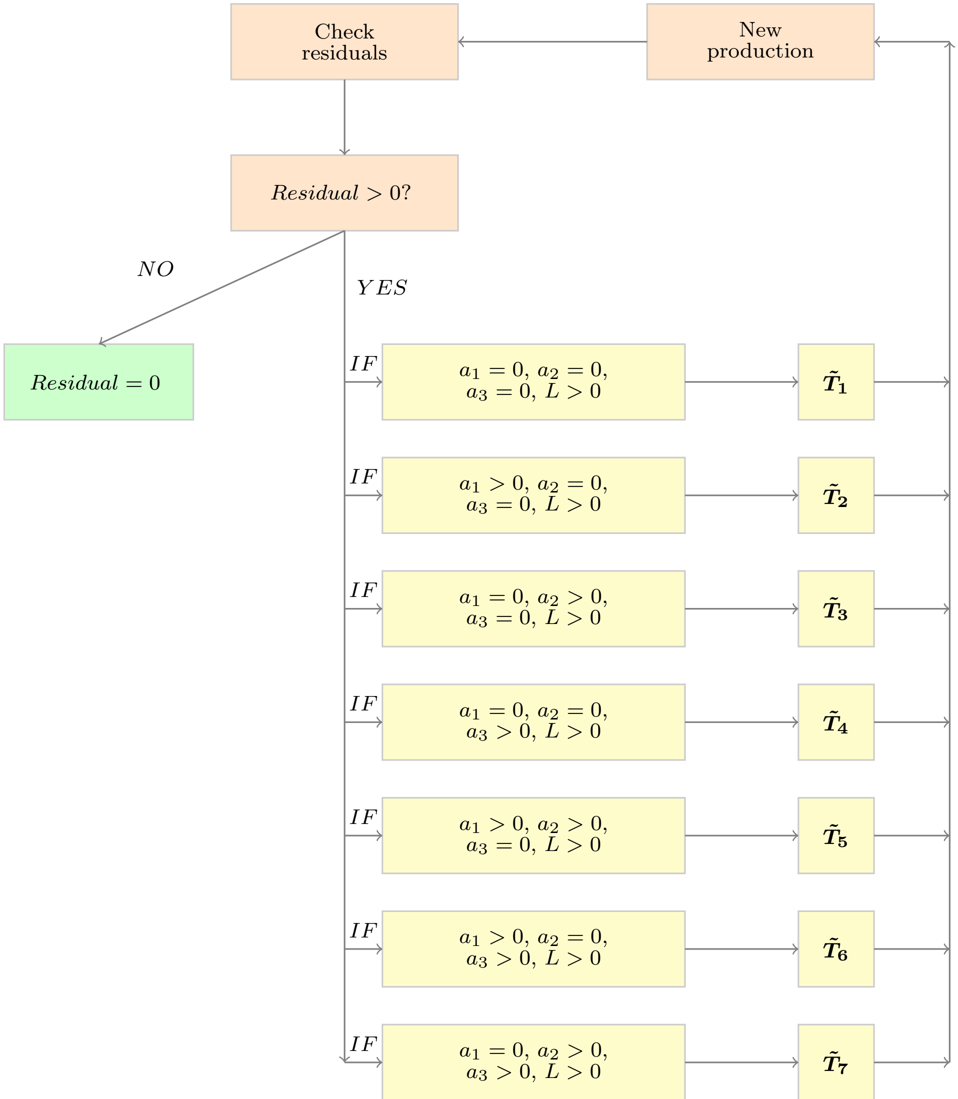


Figure 10.2: **The residual mechanism**: each ARA producer uses the residuals from production through the use of a particular set of techniques which allows him to use not all the commodities for the production of his commodity. In the figure, with $\tilde{\boldsymbol{T}}_1$, the first method among those of the matrix $\tilde{\boldsymbol{T}}$ should be considered.

10.3 Notes on machine learning in the thought experiments: the construction of the artificial neural network

Machine learning is the multidisciplinary field that studies the development of algorithms that allow computers to develop the ability to learn[126] in order to perform tasks. According to a more precise definition, "the field of machine learning is concerned with the question of how to construct computer programs that automatically improve with experience" (Mitchell, 1997, p.xv)[127].

10.3.1 A particular approach: artificial neural networks

There are different approaches to machine learning, such as decision tree learning, deep learning, clustering, Bayesian networks etc. One particular approach that seems suitable to our task is artificial neural network. Indeed, the idea to construct a behavioural function, able to mimic the behaviour of producers (trade unions) when they decide the prices (wages) to announce, implies the construction of an algorithm able to recognize the environment, learn from errors and develop a decision. Neural network is probably the most suitable approach of machine learning for this task because other approaches, such as decision trees, are more suitable for classification and regressions while clustering is more suitable for statistical data analysis.

Intuitively, the role of neural network in our case is to find an algorithm able to mimic the mechanism inside the *brain of the ARAs* when they take decisions. Indeed, neural network syntax is inspired by neurobiology and tries to reconstruct mechanisms that refer to some fundamental features of real brains (such as, neurons and their interrelations). A system of artificial and interconnected neurons is called *neural network* and can be defined as:

> An interconnected assembly of simple processing elements, *units* or *nodes*, whose functionality is loosely based on the animal neuron. The processing ability of the network is stored in the interunit connection strengths, or *weights*, obtained by a process of adaption to, or *learning* from, a set of training patterns (Gurney, 1997, p.1).

[126] Learning is needed in cases in which it is not possible to develop a specific computer program to solve directly the task because the task changes in time, space, according to circumstances or simply when we need the computer to simulate a human attitude that we cannot translate in a computer program. Herbert Simon says that "learning denotes changes in the system that are adaptive in the sense that they enable the system to do the same task or tasks drawn from the same population more efficiently and more effectively the next time" (Simon, 1983, p. 26).

[127] Equivalently machine learning could be defined as "programming computers to optimize a performance criterion using example data or past experience" (Alpaydin, 2010, p. xxxi). In particular, "a computer program is said to *learn* from experience E with respect to some class of tasks T and performance measure P, if its performance at tasks in T, as measured by P, improves with experience E" (Mitchell, 1997, p.2, emphasis added).

10.3.2 Artificial neural networks in economics

The idea to model the economic agents in such a way that they behave as real agents is not new in the field of economics. For example, Simon (1983) compared what is involved in human learning with respect to machine learning and said that:

> Anybody who is interested in machine learning because he wants to simulate human learning - because he wants to understand human learning and thinking, and perhaps improve it - it can pursue his interest in good conscience (Simon, 1983, p.26).

It is claimed that the economic agents's learning capabilities and decision processes can be modelled through 3 basic techniques inspired by biology: artificial neural networks, evolutionary algorithms and artificial economies related to artificial life (Marks and Schnabl, 1999, p.197). The purpose of artificial economics is to use computer simulation approach in order to understand socioeconomic process and how economic macro phenomena can emerge from the interaction of single economic agents (Marks and Schnabl, 1999, p.198). Evolutionary algorithms are inspired by biological evolution and processes. Genetic algorithms are surely the most popular type of evolutionary algorithms and they are used in order to solve optimization and search problem through mechanism inspired by natural selection principles such as recombination and mutation (see Mitchell (1998) for an interesting introduction to genetic algorithms). Artificial neural network is an approach of machine learning and its purpose is to develop a mechanism able to mimic the ability of real brains in learning how to solve a variety of tasks.

The 3 basic techniques inspired by biology, and in particular genetic algorithms and artificial neural networks, are sometimes considered both suitable (or used together[128]) to solve a wide variety of optimization problems also inside the economic field. Even though it is not object of the present chapter to make a detailed distinction between the two approaches, it is important to emphasize how genetic algorithms are suitable for the search of an exact solution to a well-defined optimization or search problem while artificial neural network emphasises the desire to mimic the learning capabilities of real brains in order to solve task or to recognise new patterns.

The idea of neural networks to construct models able to mimic the fundamental mechanism at the base of the functioning of real brains is fascinating and distinguishes this method from the other algorithms inspired by biology. The literature on application of artificial neural networks in economics shows how artificial neural network has been used for classification of economic agents and prediction of time series (Herbrich et al., 1999, pp.178-179). Despite this, probably a more interesting but less common application

[128] For example, genetic algorithms can be used to train a neural network.

is that related to the model of bounded rational economic agents. In this last contest:

> Neurons are interpreted as agents who update their perception of the environment according to the information they receive. Their decisions (the output of the neuron) then exert an influence on the environment which might be fed back to the agent (Herbrich et al., 1999, p.181).

Another application could be oriented to use neural network in order to mimic the learning capabilities of a single agent. In this perspective, neurons are not agents but the basic elements of the artificial brain of the economic agent. This particular application, uncommon in the existing literature, has been followed in the construction of the behavioural function for the wage and price announcement in present thesis (see chapter 4.3.4).

10.3.3 Artificial neural network: historical notes

From biological neural network to artificial neural network

An artificial neural network is based on the operation of biological neural networks. In human brains billions of neurons (i.e., nerve cells) communicate via electrical signals (i.e., *spikes* in the voltage of the cell wall called *membrane*). Each neuron receives thousands of incoming signals from other neurons (i.e., neurons are connected through electrochemical junctions called *synapses*, located on branches called *dentries*). These signals can be inhibitory (i.e., prevent firing) or excitory (i.e., promote firing). All these signals are *summed* together (i.e., in the *cell body*) and if this final signal exceeds some *threshold* the neuron will generate a response signal (i.e., firing or not firing), that is transmitted to the other neurons (the transmission is possible through a branching fibre called *axon*). This fundamental architecture based on the interconnection and processing abilities of neurons is at the base of artificial neural network constructions. This simple artificial neuron is called *threshold logical unit* (TLU)[129]. The dynamics of the neural network stands in the learning process. In real neurons, the learning capability is determined by the possibility to change the synaptic strengths of neurons according to inputs so that neurons adapt themselves according to circumstances. In artificial neural network this capability is stored in the *weight* values. The *learning rule* determines the way in which weight updates. For each pattern presented (i.e., input) the output is compared with the *target* and the weights update accordingly through an iterative process which stops (hopefully) when the output (i.e., the responds to each pattern) converges to the target (this process is based on the dynamics of the *postsynaptic potential PSP*). This process is part of the *training algorithm.* The convergence to the output means that the neural network has learnt the underlying structure and so it should be able to generalize and interpret correctly patterns never seen before.

The intuitive and informal description of neurons and networks proposed previously should now be reformulated in details through a mathematical description of the fundamental elements cited:

- input signals: x_1,x_2,x_3,...,x_n
- potential action: α,β
- weights: w_1,w_2,w_3,...,w_n
- signal production (PSP): γ
- activation: $a = \gamma(x, w)$

[129]Make reference to Gurney (1997, pp.1-2) for a more detailed introduction to biological and artificial neural networks.

- threshold value: θ
- transfer function: $\Phi(a, \theta)$
- output: z

Briefly, neurons receive input signals that can be inhibitory (value α) or excitory (value β). All these signals are *combined* together through γ so that to obtain the activation term a. The net input pass through the transfer function so that to obtain the output z. In this phase the threshold value considered inside the transfer function determines the final output z. There are 3 functional process that take place in this TLU:

- *weight function*: the inputs are multiplied for the respective weights.
- *net input function*: the weighted input can be added to a scalar bias b to form the net input a.
- *transfer function*: the net input passes through the transfer function so that to obtain the output z.

The biological neural network and the general structure of the artificial neural network are represented in the figure:

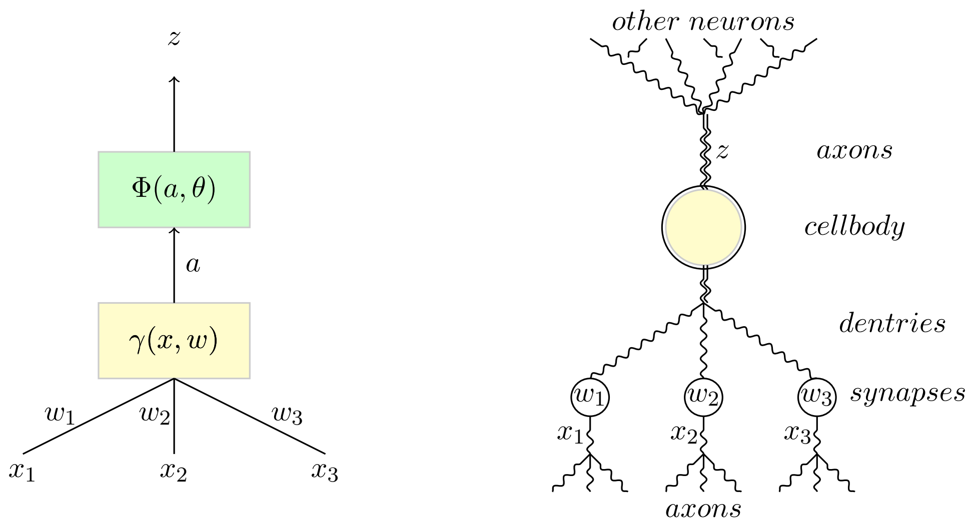


Figure 10.3: **The biological and the artificial neural network**: the general structure of the artificial neural network (on the left) and the biological neural network represented in a stylized style (on the right).

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The artificial neuron of McCulloch and Pitts

Consider now a more detailed specification of the artificial neural network in which there are:

- input signals: x_1,x_2,x_3,...,x_n
- potential action: 0,1 (binary or Boolean signal)
- weights: w_1,w_2,w_3,...,w_n
- signal production (PSP): w_1x_1,w_2x_2,w_3x_3,...,w_nx_n
- activation: $a = w_1x_1 + w_2x_2 + w_3x_3 + ... + w_nx_n$
- threshold value: θ
- transfer function: Φ=hard-limit type (Heaviside step function)
- output: z

Briefly, neurons receive input signals that can be inhibitory (value 0) or excitory (value 1). All these signals are *summed* together so that to obtain the activation term a:

$$a = \sum_{i=1}^{n} w_i x_i \tag{10.1}$$

If this activation term a exceeds or is equal to the threshold θ the final output z is equal to 1, otherwise is equal to 0. z represents the final signal response:

$$z = \begin{cases} 1 & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases} \tag{10.2}$$

In this case, we have considered as the *weight function* the simple multiplication of weights for inputs. For the *net input function*, the bias b has not been considered. The Heaviside step function has been considered as the *transfer function*. The threshold value can be considered as a particular weight and implemented inside the network. Other alternatives for the transfer function are possible such as the *linear transfer function* or the *sigmoid transfer function*. They allow to have different range of outputs (for example not only value 0 or 1 but also any other value between them). This particular artificial neural network in which are used 0 and 1 as potential actions and the hard-limit transfer function is the basic example of the

McCulloch and Pitts artificial neuron (1943). Despite it represents a fundamental prototype, the limitation of this artificial neuron is that this network has a fixed set of weights. If weights are known, the artificial neuron can solve some Boolean logical problems, such as the AND function.

The *perceptron* of Rosenblatt

The fundamental limitation of the mathematical model of neuron of McCulloch and Pitts is that the weights are given. A fundamental characteristic of a biological neural network is the possibility to modify the strength of the connections between neuron through learning process. This feature has been introduced inside the neural network world by Hebb in 1949 with the concept of *synaptic plasticity*. This concept known also as *hebbian plasticity* opened a season of neural network models able to learn. A fundamental neural network able to learn is the *perceptron*, introduced by Rosenblatt in 1957: a model of neural network able to solve some problems of pattern recognition. The elements of the *perceptron* are very similar to the McCulloch and Pitts neuron model. The fundamental difference is that the values of the weights has to be computed through a learning process.

The behaviour of a neural network able to learn is characterized by two phases:

- **training**: in this phase the network is trained with example and weight are computed given the input and output values of the examples.
- **work**: the network gives output for input never seen before, given the weights computed in the training phase.

This learning process is called *supervised learning*, because the network is trained starting from examples. There is also another type of learning called *unsupervised learning* where example are not associated to an output.

In the case of *supervised learning* the computation of weights is described by an algorithm. The perceptron learning rule is represented by a simple algorithm that adjust the network weights w to minimize the difference between the desired and the actual outputs. In detail, the algorithm is described by the subsequent logical phases:

- Give random values to the weights.
- For each training vector pair (x,z) of the examples at disposal compute the activation term a ($a = w_1x_1 + w_2x_2 + w_3x_3 + ... + w_nx_n$).
- Compare the z obtained with the z of the example (called $\bar{z}$).

- If $z \neq \bar{z}$ update the weight according to this rule: $w' = w + \gamma(\bar{z} - z)x$.
- Repeat the procedure until z=$\bar{z}$ in all the examples.

An example could help to understand this learning algorithm. Consider the desired function as the AND function in the following table.

x1	x2	$\bar{z}$
0	0	0
0	1	0
1	0	0
1	1	1

Table 10.1: **The AND function**: logical function.

.

The algorithm could be constructed with whatever computer program [130]. The objective is to train the perceptron so that to classify correctly the points. The algorithm (sometime called *delta rule*) allows to arrive to a perfect separation of the inputs in the two classes as appears clear in the figure 10.4.

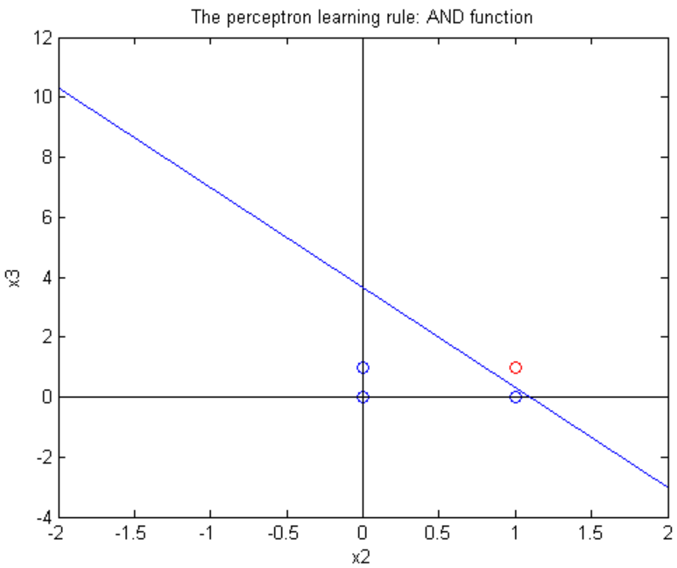


Figure 10.4: **The perceptron learning rule**: with a learning rate equal to 2.

[130]For example the computer program MATLAB®; see an example of the algorithm for the solution of the AND function developed with MATLAB® in the Appendix 10.4.

The limits of the perceptron

We have seen that perceptron is able to solve classification problems (such as the AND function). The algorithm will always converge to weights able to solve the classification problem (i.e., converge to a solution) if a solution exists. The problem is that perceptron is able to solve only *linear problems*. A perceptron with n inputs is able to represent a n-dimensional hyperplane. This means that a perceptron is able to solve only linear separable problems in which inputs can be put in two different classes dividable with a line (perceptron with 2 inputs), a plane (perceptron with 3 inputs) or a hyperplane (perceptron with 3 inputs). A typical non-linear problem is the XOR. XOR function is not linearly separable.

x1	x2	$\bar{z}$
0	0	0
0	1	1
1	0	1
1	1	0

Table 10.2: **The XOR function**: logical function.

.

Despite the initial enthusiasm for the perceptron, once, in 1959, Marvin Minsky and Seymour A. Papert demonstrated that perceptron was able to solve only linear problems, the interest for neural network declined sharply. Only after more than 10 years, the problem has been solved considering more complex architecture of neural network (multi-layers perceptron) and different algorithms for training. A pioneer in this field has been in 1974 Paul Werbos. One of the most famous training algorithms is the *error back-propagation* proposed in 1986 by Rumelhart, Hinton and Williams. Before talking about this class of training algorithms, we describe the more complex architecture of neural networks.

The architecture of neural networks

More neurons of the previously shown types can be combined in a layer. A network can contain more layers. The basic example is a single layer of more neurons as in the figure 10.5, but it is possible to have also multiple layers of neurons. Each layer has his weights, inputs and outputs. The layers which produces the output is called *output layer* while the others are called *hidden layers*. The weights can be summarised in a matrix. These complex neural networks, regardless of the number of layers or neurons, have in common the fact that the signal propagation goes forward without circles or traversal connections (i.e., arrows are straight line that goes ahead). For this reason, this class of artificial neural networks are called **feedforward**. Recurrent or feedback neural networks have instead another architecture that transform them in dynamic systems. In this last type of architecture there are circles or traversal connections.

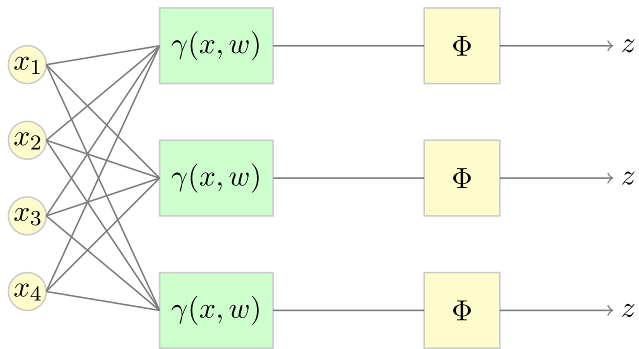


Figure 10.5: **A one-layer network**: in this case there is one layer with 4 inputs and 3 output neurons.

Beyond the perceptron: the backpropagation algorithm

The limits of the perceptron can be solved considering multi-layers-perceptron. This allows to have a more complex representation of inputs. Indeed, perceptron can separate space only in two spaces. In the XOR case a more complex division of the space is needed. With more layers and nodes, it is possible to solve even more complex classification patterns problems (see figure 10.6).

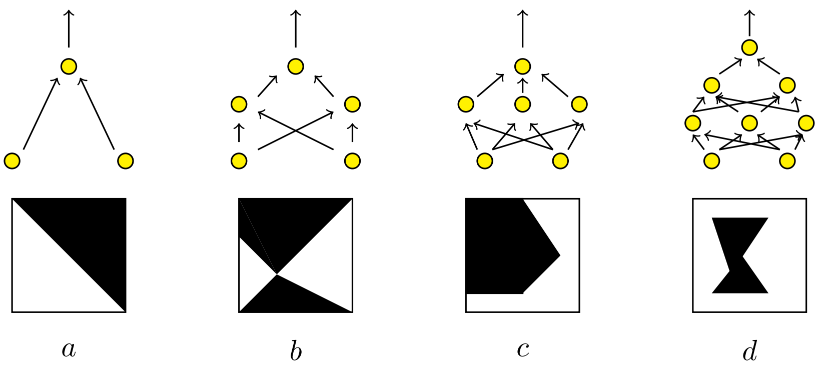


Figure 10.6: **Decision boundaries**: the perceptron is able to solve only linear separable classification problems such as the AND function (figure *a*). In order to solve more complex classification problems such as the convex forms in figure *b* (e.g., the XOR function) and figure *c*, it is sufficient to construct architecture with 2 layers. An architecture with 3 layers is sufficient for whatever classification problem (figure *d*). This type of representation of the decision boundaries has been developed by Cammarata (1990) and reported also in Macchiavello (1992, p.120).

Consider for example the XOR problem. As it clearly appears from the figure 10.7, a perceptron cannot solve this problem, because it does not exist a line able to separate the input whose output is one (red points) to the inputs which output is zero (blue points). In order to solve XOR problem (and non-linear problem in general), it is necessary to consider more complex architectures, with more neurons and layers. In the case of the XOR problem, it is sufficient to consider two layers (i.e., the so called multi-layer perceptron).

The learning process that allows to solve the XOR problem with a multi-layer perceptron is the so called *backpropagation algorithm* (also called *generalized delta rule* based on the Widrow-Hoff learning rule). The fundamental idea of this algorithm is to identify the vector of weights and biases that minimize a particular cost function (*error function*). The cost function is represented by the mean squared error. For this reason, the backpropagation represents a generalization of the LMS (Least Mean Square) algorithm.

The function error is minimized using the *gradient descendent algorithm*. The intuition at the base of this optimization algorithm is to minimize the error by changing the weights. The first order partial derivative of error with respect to weights (the gradient), scaled for a factor called learning rate, gives a direction for the weights to change in order to reduce error (Priddy and Keller, 2005, p.113). In the next section, the *backpropagation algorithm* will be presented in detail.

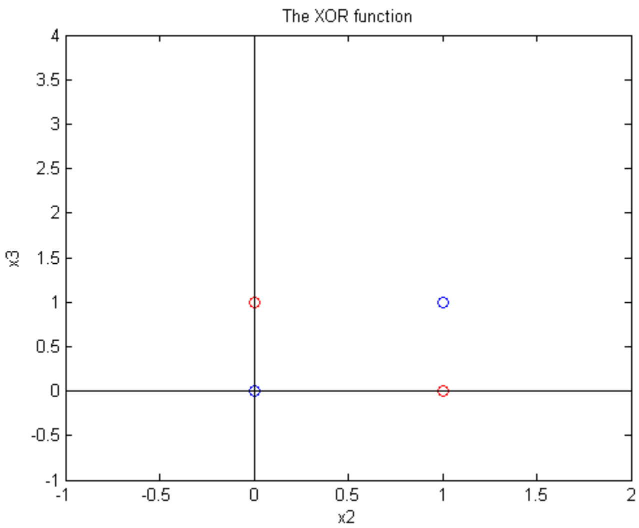
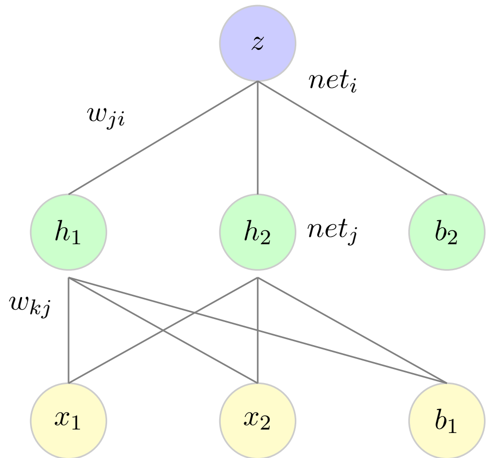


Figure 10.7: **The XOR function**: logical function.

The backpropagation algorithm: derivation

In order to understand the backpropagation algorithm consider for example the structure of a multi-layer perceptron (suitable also for solving the XOR problem):



output neuron z_i

hidden neuron h_j

input neuron x_k

Figure 10.8: **The multi-layer perceptron**

Call *input pattern* ψ each couple of x_1 and x_2 (4 couples in the XOR problem so that ψ=1,...,n and n=4). The value of the *target* corresponds to $\bar{z}$. If the initial weights are random values, the computation of the z will be different with respect to the target. The objective is to minimize this difference by adjusting the weights. The adjustment of the hidden layer is not trivial. The idea is to adjust the weight in proportion to the error.

So we define an *error function* able to measure this difference and we try to minimize it using the *gradient descendent* on the output units over all the inputs patterns. The *performance index* that characterize this *error function* is the *mean squared error* (MSE).

A generalization of the *delta rule*[131] will be used. It calculates the error for the current input example and then *backpropagate* this error from layer to layer.

In order to understand how the backpropagation algorithm works, it is useful to understand how to derive the fundamental equations of the algorithm. Consider the figure 10.8. In general, we can have a number of different output neurons z_i, a number of different hidden neurons h_j and a number of different input neurons x_k. For the XOR problem we have one output neuron, 2 hidden neurons and 2 input neurons plus the 2 biases. The

[131] A gradient descent learning rule, precursor to backpropagation, used in perceptron learning by Widrow and Hoff in the late 1950s (Priddy and Keller, 2005, p.111).

input signal for an output neuron is called net_i while the input signal for a hidden neuron is called net_j. w_{kj} is the weight between one input and one hidden neuron and w_{ji} is the weight between one hidden and one output neuron. The *error function* over all neurons and patterns can be written as:

$$E = 1/2 \sum_i \sum_\psi (\bar{z}_i^\psi - z_i^\psi)^2 \tag{10.3}$$

Now we can consider the equations for the input signals and the activations of the hidden and output neurons as function of their inputs:

$$net_j^\psi = \sum_k w_{kj} x_k^\psi + b_j \tag{10.4}$$

$$h_j^\psi = f(net_j^\psi) = f(\sum_k w_{kj} x_k^\psi + b_j) \tag{10.5}$$

$$net_i^\psi = \sum_j w_{ji} h_j^\psi + b_i \tag{10.6}$$

$$z_i^\psi = f(net_i^\psi) = f(\sum_j w_{ji} h_j^\psi + b_i) \tag{10.7}$$

Substitute these equations in the *error function*:

$$E = 1/2 \sum_i \sum_\psi (\bar{z}_i^\psi - z_i^\psi)^2$$

$$E = 1/2 \sum_i \sum_\psi (\bar{z}_i^\psi - f(\sum_j w_{ji} h_j^\psi + b_i))^2 \tag{10.8}$$

At this point we apply the iterative *steepest gradient descent optimization algorithm* that allows to update the weights so that to minimize the *error function.* Briefly, if we imagine the error function as the error surface, the algorithm changes the weights in order to reach a minimum. The steps are proportional to the negative of the gradient called *learning rate* η:

$$\Delta w_{ji} = -\eta \frac{\partial E}{\partial w_{ji}} = \eta \sum_\psi (\bar{z}_i^\psi - z_i^\psi) f'(net_i^\psi) h_j^\psi = \eta \sum_\psi \delta_i^\psi h_j^\psi \tag{10.9}$$

$$\delta_i^\psi = (\bar{z}_i^\psi - z_i^\psi) f'(net_i^\psi) \tag{10.10}$$

Considering that δ_i^ψ is the *error signal* for the output neuron i, for hidden neurons the computation of the derivatives uses the *chain rule*:

$$\Delta w_{kj} = -\eta \frac{\partial E}{\partial w_{kj}} = -\eta \frac{\partial E}{\partial h_j^\psi} \frac{\partial h_j^\psi}{\partial w_{kj}} = \tag{10.11}$$

$$= \eta \sum_{\psi,i} (\bar{z}_i^\psi - z_i^\psi) f'(net_i^\psi) w_{ji} f'(net_j^\psi) x_k^\psi =$$

$$= \eta \sum_{\psi,i} \delta_i^\psi w_{ji} f'(net_j^\psi) x_k^\psi = \eta \sum_\psi \delta_j^\psi x_k^\psi$$

$$\delta_j^\psi = f'(net_j^\psi) \sum_i w_{ji} \delta_i^\psi \tag{10.12}$$

At this point we can update the weights for the *input pattern* ψ:

$$w_{ji}^{new} = w_{ji} + \Delta w_{ji} \tag{10.13}$$

$$w_{kj}^{new} = w_{kj} + \Delta w_{kj} \tag{10.14}$$

In order to compute the updated values of the weights, we need to define the function of the inputs *net*. If we use the *sigmond transfer function* we can write them as:

$$f(net_i^\psi) = \frac{1}{1 + e^{-net_i^\psi}} \tag{10.15}$$

$$f(net_j^\psi) = \frac{1}{1 + e^{-net_j^\psi}} \tag{10.16}$$

We can compute also the *first derivatives* as:

$$f'(net_i^\psi) = \frac{\partial z_i^\psi}{\partial net_i^\psi} = \frac{\partial \frac{1}{1+e^{-net_i^\psi}}}{\partial net_i^\psi} = \frac{e^{-net_i^\psi}}{(1 + e^{-net_i^\psi})^2} = z_i^\psi (1 - z_i^\psi) \tag{10.17}$$

$$f'(net_j^\psi) = \frac{\partial h_j^\psi}{\partial net_j^\psi} = \frac{\partial \frac{1}{1+e^{-net_j^\psi}}}{\partial net_j^\psi} = \frac{e^{-net_j^\psi}}{(1 + e^{-net_j^\psi})^2} = h_j^\psi (1 - h_j^\psi) \tag{10.18}$$

The computation of the *first derivatives* allows to rewrite:

$$\delta_i^\psi = (\bar{z}_i^\psi - z_i^\psi) f'(net_i^\psi) \tag{10.19}$$

$$\delta_j^\psi = f'(net_j^\psi) \sum_i w_{ji} \delta_i^\psi \tag{10.20}$$

in a more suitable form for computation:

$$\delta_i^\psi = (\bar{z}_i^\psi - z_i^\psi) z_i^\psi (1 - z_i^\psi) \tag{10.21}$$

$$\delta_j^\psi = h_j^\psi (1 - h_j^\psi) \sum_i w_{ji} \delta_i^\psi \tag{10.22}$$

The most important shortcomings of this method are connected to the fact that it is not excluded that the algorithm could converge towards a local minimum or it could converge too slowly to the solution. Despite this, the backpropagation algorithm has some universal properties:

- "Any continuous function can be uniformly approximated by a continuous neural network having only one internal, hidden layer and with an arbitrary continuous sigmoidal nonlinearity. [...] Arbitrary decision functions can be arbitrarily well approximated by a neural network with one internal layer and a continuous sigmoidal nonlinearity" (Cybenko, 1989, p.312). This implies that it is sufficient a single hidden layer in order to represent every boolean function.

- According to a result of Lippmann (1987) and Cybenko (1988) any function can be approximated by a network with two hidden layers.

The backpropagation algorithm: a summary

This derivation brings to the fundamental equations of the backpropagation algorithm that represents the steps that that have to be followed for each *input pattern* ψ. The algorithm can be summarised in the steps reported in the subsequent box. The algorithm has to be repeated until a solution is reached. The solution is reached when the value of the *error function* E is lower with respect to an exogenously fixed threshold θ. The number of time that are needed in order to have $\mathrm{E} < \theta$ represent the number of *epochs* that have to be computed for solving the problem. In the appendix is present an example of solution of the XOR problem using MATLAB® (see appendix 10.5).

1. Consider the first *input patters* ψ (the relative values of x_1,x_2 and z) and the initial values of the weights.

   1.1 Compute the value of the input signal *net* and the neurons of the hidden layer h according to the formula:

   $$net_j^{\psi} = \sum_k w_{kj} x_k^{\psi} + b_j \qquad h_j^{\psi} = \frac{1}{1+e^{-net_j^{\psi}}}$$

   1.2 Compute the value of the input signal *net* and the neurons of the output layer z according to the formula:

   $$net_i^{\psi} = \sum_j w_{ji} h_j^{\psi} + b_i \qquad z_i^{\psi} = \frac{1}{1+e^{-net_i^{\psi}}}$$

   1.3 Compute the *error signal* for the output neuron i and the change in the weights w_{ji}:

   $$\delta_i^{\psi} = (\bar{z}_i^{\psi} - z_i^{\psi}) z_i^{\psi} (1 - z_i^{\psi}) \qquad \Delta w_{ji} = \eta \delta_i^{\psi} h_j^{\psi}$$

   1.4 Compute the *error signal* for the hidden neuron j and the change in the weights w_{kj}:

   $$\delta_j^{\psi} = h_j^{\psi}(1 - h_j^{\psi}) \sum_i w_{ji} \delta_i^{\psi} \qquad \Delta w_{kj} = \eta \delta_i^{\psi} x_k^{\psi}$$

   1.5 Compute the new values of the weights:

   $$w_{ji}^{new} = w_{ji} + \Delta w_{ji} \qquad w_{kj}^{new} = w_{kj} + \Delta w_{kj}$$

2. Consider the second *input patter* ψ and the initial values of the weights. Repeat the sequence 1.1-1.5. Repeat this sequence also for all the other *input patterns*.

3. Compute for all *input patterns* the values of the output neurons using the formula of points 1.1-1.2. Compute the new value of the *error function*:

   $$E = 1/2 \sum_i \sum_{\psi} (\bar{z}_i^{\psi} - z_i^{\psi})^2$$

   if E is bigger with respect to a determinate threshold θ repeat the algorithm from step 1 considering the new values of the weight. If $E < \theta$ the algorithm has reached a solution.

10.3.4 Problems and limits of the neural network

Neural networks can represent a powerful tool for the resolution of complex problems. In order to start to work with neural networks, it is necessary to have a clear understanding of the problem to solve. Indeed, not all problems are suitable to be solved through neural networks. Once it established that the neural network is the right tool for the solution of the problem, there are some fundamental steps to be followed in order to build properly the neural network:

- Choose the type of neural network: first of all, it is necessary to choose the type of architecture. This depends on the type of problem to solve (e.g., feedforward or recurrent). After this, it is necessary to define the architecture properties, such as the number of layers, neurons and the type of transfer function.

- Choose the learning algorithm: for example, the backpropagation algorithm. The learning algorithm is at the base of the computation of the weights of the neural network.

- Collection and organization of the data: a proper collection and organization of the data is fundamental in order train, test and validate properly the neural network.

- Train the neural network: use part of the data to fit the neural network. In this phase are computed the *optimal* weights with the learning algorithm.

- Validate the neural network: use part of the data in order to check the fitness of the architecture of the neural network. In this phase it is possible to estimate the *optimal* number of hidden neurons and to compute an *early stopping* for the learning algorithm.

- Verification of the neural network: test the performance of a fully trained neural network.

- Use the neural network: the *optimal* weights of the fully trained neural network can be used for simulation.

As previously said, the developer of a neural network has to face some possible shortcomings in the construction and use of the neural network. However, there are some suggestions in order to solve these problems:

- The neural network may not find a good solution: the learning algorithm could be trapped in a local minimum or it could converge too slowly to the global minimum (see Nawi et al., 2013, p.1-2). Usually a good tailoring of the architecture properties is able to minimize the effects of this problem.

- The neural network is sensible to the value of the learning rate: an excessively low learning rate implies slow convergence, an excessively high learning rate implies instability and poor performance. Usually in order to solve slow convergence the suggestion is to increase the learning rate or use momentum by modifying the learning rule.

- The performance is sensitive to initial conditions. Initial conditions have a great influence in the computation of the weights. A solution could be the computation of the weights starting with different initial conditions. Usually it is a good method to normalize inputs with mean equal to zero and variance equal to 1 (*z-score*). This avoids saturation.

- The neural network is sensible to the number of hidden layers and neurons: despite most problems can be solved with two hidden layers, the number of neurons present some trade-offs: too many neurons can provoke overfitting/overtraining and too few underfitting. As heuristic to start consider one hidden layer with a minimum number of neurons equal to 5, or equal to (inputs+output)/2.

It has been said that collection and organization of the data is a fundamental step for a good training, test and validation of the neural network. A good suggestion (Looney, 1996, p.216) is to divide the data set into 3 subsets with these proportions:

- *Training set* (65%): used for training the neural network (randomly selected).

- *Cross-validation set* (10%): used for validating the neural network (randomly selected).

- *Verification testing set* (25%): used for the verification of the neural network (the remained data).

Others suggest also a division of the type 50% 25% and 25% respectively (Friedman et al., 2001, p.196). These sets help the developer of the neural network to avoid overtraining. Indeed, during training, after a certain number of epochs, the error on the training set tends asymptotically towards the zero but the error of the validation set (that at the beginning decreases), after a period, tends to increase. When this happens the developer has to stop the training (*early stopping*) because at this point the error on the validation set reach the minimum and the weights computed at this point of the training represents the best estimation.

10.3.5 The neural network considered in the thought experiments

The brief introduction to the world of neural network is fundamental in order to understand the one used inside the algorithm that determines the announcement of prices and wages inside the *thought experiments* considered in the present thesis.

As said, theoretically any function can be approximated by a network with two hidden layers but it is possible to consider more suitable structures for our task. Indeed, the algorithm that determines the announcement of prices and wages inside the *thought experiments* has to elaborate data that evolves in time.

For this reason the neural network used in the *thought experiments* is a *layer recurrent neural network*. Layer recurrent neural networks are similar to feedforward networks, except that each layer has a recurrent connection: signals can travel in both directions and loops are created inside the network. In order to train a *layer recurrent neural network* a more power training algorithm is needed. We have already seen that backpropagation algorithm suffers from 2 particular drawbacks that can be mitigated but not completely solved: low and slow convergence and instability due to the possibility to be trapped in a local minimum (Nawi et al., 2013, p.1). For better performance second order learning algorithms have been introduced and among them the so called *Levenberg Marquardt algorithm* is considered one of the most effective and particularly suitable for training *layer recurrent neural network* and overcome the problems of the backpropagation algorithm (Nawi et al., 2013, p.2). The architecture of the neural network constructed has the following characteristics[132]:

- 2 layers and 10 neurons for the hidden layer.
- *Training function*: backpropagation algorithm (*Levenberg Marquardt algorithm*).
- *Transfer function*: hyperbolic tangent sigmoid transfer function.
- Linear transfer function for the output layer.
- Epochs: 5.

Once the neural network is validated and tested, it can be used for simulation using the final weights computed. In other words, in order to take a decision with respect to the price (wage) to announce, the ARA select a range of possible prices (wages) to announce according to past experience

[132]In order to develop the neural network in MATLAB® the Neural Network Toolbox has been used.

and use the neural network in order to predict which price (wage) should correspond to the highest income. The data set is composed in each period by the information set of the ARA. This means that the experience of the ARA grows in time, as his information set grows. It is worth remembering that neural networks need a big dataset in order to develop good predicting capacities and also the initial random numbers used for initialize the algorithm plays a role in the determination of the final weights. In order to face these shortcomings, the suggestion descripted in the previous chapter have been followed. Moreover, given that at the beginning of the experiment experience is low (determined only by the starting equilibrium condition), the effective number of neural networks created for each decision is equal to 10 and also the effective number of simulation is equal to 10 for each neural network. The final decision is an average of the results of all simulations.

10.4 The perceptron learning rule

```
%% THE PERCEPTRON LEARNING RULE (example: AND function)

x=[1 0 0;1 0 1;1 1 0; 1 1 1]; zhat=[0 0 0 1]; % training set
w1(1)=0.5; w2(1)=-1; w3(1)=1.5; % random weights
theta=0; % threshold
gamma=0.5; % learning rate
[nr,nc]=size(x); counter=0; check=0; % other variables

% compute the net for each row of the training set and update
% each time the weights: if zhat==z the algorithm stops

while check==0;
    for i=1:nr;
        net(i)=w1(i)*x(i,1)+w2(i)*x(i,2)+w3(i)*x(i,3);
        if net(i)≥theta; z(i)=1; else  z(i)=0; end
        w1(i+1)=w1(i)+gamma*(zhat(i)-z(i))*x(i,1);
        w2(i+1)=w2(i)+gamma*(zhat(i)-z(i))*x(i,2);
        w3(i+1)=w3(i)+gamma*(zhat(i)-z(i))*x(i,3);
    end
    % check the error
    check=all(z==zhat);
    % keep the last values for the next cycle
    w1f=w1(1,end);
    w2f=w2(1,end);
    w3f=w3(1,end);
    clear w1 w2 w3
    w1(1)=w1f;
    w2(1)=w2f;
    w3(1)=w3f;
end

%% algorithm developed by Sara Casagrande (2016)
```

10.5 The XOR problem solved with the backpropagation algorithm

An example of solution of the XOR problem using the backpropagation algorithm is reported in this section. The algorithm has been reconstructed using the computer program MATLAB®. The structure of the multi-layer perceptron for solving the XOR problem can be represented in this way:

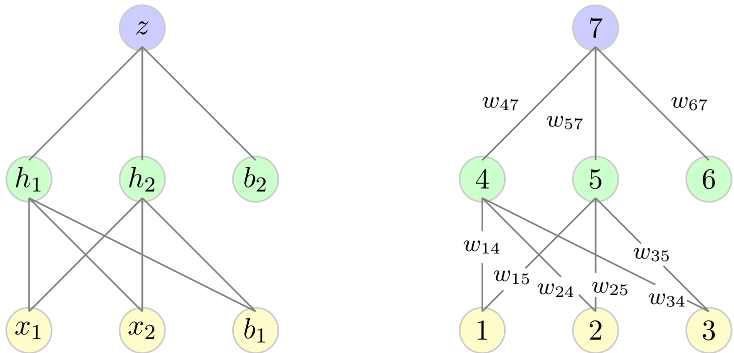


Figure 10.9: **The multi-layer perceptron for solving the XOR problem**

Consider the initial values of the weights equal to zero, a *learning rate* η equal to 0.8 and a threshold equal to 0.000001. The values of the input nodes and the value of the target for each *input pattern* are reported in the following table:

x1	x2	$\bar{z}$
0	0	0
0	1	1
1	0	1
1	1	0

Table 10.3: **The XOR function**: logical function.

.

In the next pages the algorithm code is reported.

```
%% BACKPROPAGATION ALGORITHM

%% Solve the XOR problem with the backpropagation algorithm.

% Training set: XOR problem
%x=[1 0;0 0;0 1; 1 1]; z=[1 0 1 0];
% bh=1; % error bz=1; % error
% b can be incorporated as b=x3 always with value 1:
x=[1 0 1 1;0 0 1 1;0 1 1 1; 1 1 1 1]; z=[1 0 1 0];
eta=0.8; % learning rate
threshold=0.000001; % threshold
w47=0; w57=0; w67=0; % weights with the output (z)
w14=0; w15=0; w24=0; % weights with the hidden (h)
w25=0; w34=0; w35=0;

w(1,1)=w14; w(1,2)=w24; w(1,3)=w34;
w(1,4)=w15; w(1,5)=w25; w(1,6)=w35;
w(1,7)=w47; w(1,8)=w57; w(1,9)=w67;

%v=1/(1+exp(-net)) activation function
% net=sum(w*x)

w1=w; w2=w; w3=w; w4=w; w=[w1;w2;w3;w4]; w_new=w;
checktot=0; counter=0;

while checktot==0; % until err>threshold
    counter=counter+1;
    w=w_new;
    for i=1:4; % for each input pattern
        [w_new(i,:),zhat(i)] = ...
            back_algorithm(z(i),x(i,:),w(i,:),eta);
    end
    err=0.5*sum((zhat-z).^2); % error function
    for i=1:4; % check if err<threshold
        if err<threshold; check(i)=1; else check(i)=0; end
    end
    checktot=sum(check);
end

%% algorithm developed by Sara Casagrande (2016)
```

```
function [w,zhat] = back_algorithm(z,x,w,eta)

% BACK_ALGORITHM  a function that allows the iteration of the
% backpropagation algorithm for each case: input z(1) x(1,:)

w14=w(1,1);  w24=w(1,2);  w34=w(1,3);
w15=w(1,4);  w25=w(1,5);  w35=w(1,6);
w47=w(1,7);  w57=w(1,8);  w67=w(1,9);
wh1=[w14 w24 w34];  wh2=[w15 w25 w35];  wh3=[w47 w57 w67];
neth1=x(1,1:3)*wh1'; h1=1/(1+exp(-neth1));
neth2=x(1,1:3)*wh2'; h2=1/(1+exp(-neth2));
xbis=[h1 h2 x(1,4)]; netz1=xbis*wh3'; zeff1=1/(1+exp(-netz1));

% 1) compute the change of the weights of the node with z
Δz=(z-zeff1)*zeff1*(1-zeff1);
dw47=eta*Δz*h1;
dw57=eta*Δz*h2;
dw67=eta*Δz*x(1,4);

% 2) compute the change of the weights of the node with h
Δh1=h1*(1-h1)*Δz*w47;
Δh2=h2*(1-h2)*Δz*w57;
dw14=eta*Δh1*x(1,1);
dw15=eta*Δh2*x(1,1);
dw24=eta*Δh1*x(1,2);
dw25=eta*Δh2*x(1,2);
dw34=eta*Δh1*x(1,3);
dw35=eta*Δh2*x(1,3);

% 3) compute the new weights:
w14=w14+dw14; w24=w24+dw24; w15=w15+dw15;
w25=w25+dw25; w34=w34+dw34; w35=w35+dw35;
w47=w47+dw47; w57=w57+dw57; w67=w67+dw67;

% Compute the new z
wh1=[w14 w24 w34]; wh2=[w15 w25 w35]; wh3=[w47 w57 w67];
neth1=x(1,1:3)*wh1'; h1=1/(1+exp(-neth1));
neth2=x(1,1:3)*wh2'; h2=1/(1+exp(-neth2));
xbis=[h1 h2 x(1,4)]; netz1=xbis*wh3'; zeff1=1/(1+exp(-netz1));

% output
zhat=zeff1; w=[wh1 wh2 wh3];

end

%% algorithm developed by Sara Casagrande (2016)
```

The backpropagation algorithm presented in the previous pages allowed us to solve the XOR problem (i.e., converge towards a solution so that E$< \theta$) after 472107 iterations (epochs). The final values of the output neurons z are reported in the subsequent table:

$\bar{z}$	z
1	0,9993
0	0,0007
1	0,9993
0	0,0007

Table 10.4: **The solution of the XOR function**: values from the simulation.

In general, the number of iterations needed for convergence grows as the value of the learning rate declines. The number of iterations can change also according to the threshold. With a less compelling threshold the number of iterations is in general lower as appears clear from the subsequent table:

η	θ=0.000001	θ=0.001
3	126 006	184
2	188 931	276
1.5	251 858	369
1	377 714	553
0.5	755 288	571

Table 10.5: **Number of iterations needed for convergence according to the learning rate η and the threshold θ**: the number of iterations needed for convergence change according to the level fo the learning rate and the level of the threshold.

For level of the learning rate too high the convergence becomes impossible.

10.6 Table of contents

Notation	Example	Description
Bold Capital letters	$\mathbf{E}_t = [\mathbf{E}_{it}]$	matrix (whole economy or single ARA)
small and bold letters	$\mathbf{q}$	column vector
$-$	$\bar{\boldsymbol{q}}$, $\bar{i}$	rows vector or i producer inside a local mkt
$\{n\}$	$\mathbf{q}^{\{n\}}$,$\bar{\boldsymbol{q}}^{\{n\}}$	n(th) column vector/row vector
small letters	$wealth, q$	single value of a matrix
cgm/pgm (superscript)	$\boldsymbol{S}^{cgm}$	consumption/production goods market
d (superscript)	L^d	demand
e (superscript)	p^e	expectation
ent (superscript)	F^{ent}	enterprise
f (superscript)	L^f	free time
g (super/subscript)	q_g	commodity g
i (also super/subscript)	c_i	producer i
j (also subscript)	c_j	worker j
L (superscript)	$\boldsymbol{R}^L$	labour market
p (superscript)	$wealth^p$	producer
s (subscript)	$\boldsymbol{D}_{st}$	local market s
s (superscript)	$\boldsymbol{L}_t^s$	supply
t (subscript)	c_{it}	time t
T (superscript)	$\boldsymbol{q}^T$	transpose matrix
w (superscript)	$wealth^w$	all workers enrolled in the same enterprise
$\oslash$	$\mathbf{q}\oslash\mathbf{DI}$	division element by element
$\circledast$	$\mathbf{D}\circledast\mathbf{R}$	Hadamard product (product el. by el.)
$*$	z^*	optimal/desired matrix/values
$\tilde{}$	$\tilde{\psi}_1$	variants of existing matrices or values
Blackboard Bold	$\mathbb{A}$, $\phi^{\mathbb{A}}$, $\mathbb{a}$	technological matrix or coefficient
$\mathfrak{F}$	$\mathfrak{F}_{ARA^p_{it}}$	function

Table 10.6: **Table of the notation**: The rules of notation are described in this table.

Symbol	Description
α	% of actual production to keep for next exchange
β	% of actual production devoted to consumption market
γ	% of actual production to keep for next production and exchange
δ	parameter for the calculus of ex ante investment demand
ε	fraction of wage in the budget constraint
η	% of actual production to keep for next production
Θ	matrix of the preferece-propensity-technical parameters
ι	ι(i)/ι(s): function indicating the good produced by i or inside s
κ	parameter for the utility function
λ	parameter for the utility function
μ	series of natural numbers different from zero $\mu \in \mathbb{N}_0$
Π	expected profits for each method of production
ρ	parameter for the calculus of production
Σ	cost of production for each method
σ	elements of the matrix Σ
τ	logical phases of the period t
Υ	matrix of the location of the ARA
Φ	matrix of the methods of production of a single commodity g
ϕ	single method of production of a single commodity g
$\boldsymbol{\chi}$	matrix of the enrolled workers
χ	element of the matrix of the enrolled workers
Ψ	matrix of the producers inside each local market
ψ	a particular producer of each local market
Ω	information set

Table 10.7: **Table of greek letters**: in this table, all the variables and parameters expressed in greek letters are described.

Table 10.8: **Table of letters**: variables and parameters

Letter	Description
$\mathbb{a}$	technological input requirement (commodities)
$\tilde{a}$	demand of commodities
a	quantity of commodities used in production
$\mathbb{b}$	technological output
b	quantity of commodities produced

Table 10.8: go ahead in the next page

Table 10.8: from previous page

Letter	Description
c	consumption
$\boldsymbol{d}$	aggregate demand of commodities (cgm or pgm)
e	endowment for exchange
$\tilde{e}$	endowment for production
$expenditures$	expenditures
F	financial balance
g	commodity g
$\boldsymbol{H}$	matrix of the variables of the model
i	producer i (identification number)
i^*	enterprise where worker decide to migrate
$\bar{i}$	producer inside a local market
j	worker j (identification number)
K	total number of workers (for the whole economy)
k	total number of producers/local markets
$\tilde{k}$	number of enrolled workers
$\mathbb{l}$	technological input requirement (labour)
$\tilde{l}$	demand of labour
l	quantity of labour used in production
$\boldsymbol{L}^s$	labour supply
$\boldsymbol{M}$	migration matrix
m_g	number of alternative methods of production for commodity g
n	total number of commodities
$\boldsymbol{O}$	access to techniques
$\boldsymbol{p}$	prices
$profit$	level of profit
$\boldsymbol{q}$	real quantities bought
$\boldsymbol{qprod}$	real quantities devoted to production
$\boldsymbol{qused}$	real quantities used for production
r	profit rate
$\boldsymbol{rat}$	rationing
$\boldsymbol{residual}$	residuals after production
$revenues$	revenues
$\boldsymbol{s}$	aggregate supply of commodities (cgm or pgm)
s	local market
shl	share of labour
t	period
$\boldsymbol{T}$	matrix of the methods of production
$\tilde{\boldsymbol{T}}$	matrix of all the methods of production
$\boldsymbol{uns}$	unsold commodities

Table 10.8: go ahead in the next page

Table 10.8: from previous page

Letter	Description
$\boldsymbol{V}$	value of production for each method
vp	production volume
w	wage rate
$wealth$	wealth
x	activity level or the matrix used in the utility function
z_g	alternative methods of production for commodity g

Table 10.8: conclusion from previous page

Symbol	Description
x	input signals
α,β	potential action
w	weights
w^{new}	updated weights
γ	signal production
a	activation
θ	threshold value
Φ	transfer function
z	output (neurons)
i	output neuron i
j	hidden neuron j
$\bar{z}$	target
b	bias
ϕ	input pattern
h	hidden neurons
net	input signal for hidden neurons
E	error function
η	learning rate
δ	error signal
Δ	variation symbol

Table 10.9: **Table of symbols, variables and parameters for neural network section**: in this table there are all the symbols, variables and parameters used in appendix 10.3.3 devoted to artificial neural network.

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