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On some optimal control problems on networks, stratified domains, and controllability of motion in fluids.

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Contents

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Introduction

This thesis deals with various problems arising in deterministic control, jumping processes and control for locomotion in fluids.
It is divided in three parts. In the first part we study possible approximations for some optimal control problems on networks and stratified domains. In the second part we present a decentralized routing problem over a network and give conditions for the convergence to consensus and in the last part we investigate different strategies to overcome the so-called scallop paradox concerning periodic locomotion in fluid. We may see as possible links between the three parts the following ones:

- I-II: control, networks, Hamilton-Jacobi equations (discontinuous - multi-agent),
- I-III: optimal control, controllability, inclusion in the model of a delayed relay although in different environment,
- I-II-III: delayed relay and Play operator as important examples of hysteresis mathematical model which is also dedicated a section (see Sect. 3.1).

Before explaining singularly the content of the three parts, we point out that in the first two chapters of the thesis we review some standard results on Hamilton-Jacobi equations, later focusing on the discontinuous ones and on several related works. Chapter 3 is devoted to the definition of the concept of hysteresis and related operators.
Now we give a quick overview of the three parts.

Part I- Hybrid Thermostatic approximations of junctions.

In this part of the thesis we focus on some optimal control problems on network with junctions (see Bagagiolo-Maggistro [14]) where each edge has its own controlled dynamics and cost. In the case of optimal control and, in particular, of dynamic programming and Hamilton-Jacobi-Bellman (HJB) equation, the presence of junctions is a problem. Indeed, due to the discontinuous feature of HJB, the corresponding uniqueness of the value function as solution of HJB is not in general guaranteed. The usual double-variable technique for proving the comparison result between sub and super-viscosity solutions cannot work in these cases because the points of minimum and of maximum, even if very close, may belong to different arcs (the edge of the network) for which dynamics and costs are absolutely non-comparable (the junction, indeed). Anyway, we point out that in this thesis the discontinuity of the Hamiltonian is always given by discontinuities through some hypersurfaces (a point in the one dimensional case) of the state space. That is, the Hamiltonians are anyway continuous one some connected components. Some authors have recently studied optimal control and HJB on networks, see for instance Achdou-Camilli-Cutrì-Tchou [2], Camilli-Marchi [36], Camilli-Marchi-Schieborn [37], Schieborn-Camilli [91], Imbert-Monneau-Zidani [68], Achdou-Oudet-Tchou [4], Achdou-Tchou [5] (see also Chapter 2 for more details). A mean field games problem on network is studied for example in Camilli-Carlini-Marchi [35]. We propose a different approach that permits to overcome the above mentioned difficulty and consists in replacing the junctions with a suitable combinations of delayed thermostats (from now on we will use the word thermostat and relay as synonyms), each of them characterized by a threshold parameter $\varepsilon > 0$. We start considering a twofold junction problem, namely the simple situation where two half-lines (the edges) are separated by one point (the junction) (see Figure 1). We replace the junction, which represents a unique threshold for passing both from one edge to the other one and vice-versa, by a delayed thermostat consisting in two different thresholds for passing separately from one edge to the other one and vice-versa (see Figure 1). The problem is then transformed

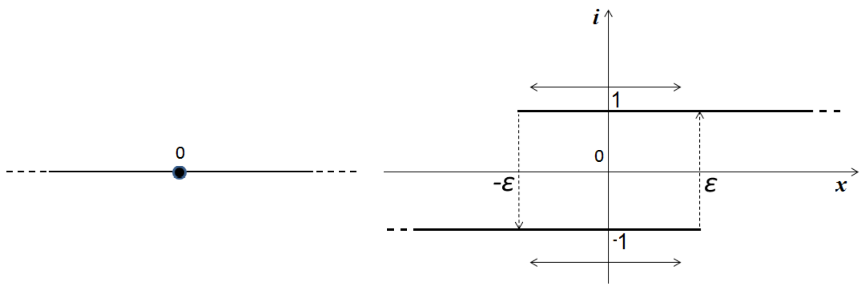


Figure 1: The two-fold junction and its thermostatic approximation.

in a so-called hybrid problem (continuous/discrete evolution, see for example Goebel-Sanfelice-Teel [61]) for which the discontinuity of HJB is replaced by some suitable mutually exchanged boundary conditions on the extreme points of the two branches. This allows to obtain a uniqueness result for HJB for this kind of thermostatic problem. Then, using as starting point the results in Bagagiolo [8] (see also Bagagiolo-Danieli [12]), (where it is proved that the value function V_ε of this (and others) switching thermostatic problem is the unique viscosity solution of a coupled exit-time problem (see Theorem 4.1.5)), we study the limit of the value functions V_ε and of the HJB problem when the threshold distance ε tends to zero, and hence we recover the twofold junction situation. In Barles-Briani-Chasseigne [19], among others, a one-dimensional twofold junction problem as the one described here above is also studied and some possible approximations are given. Introducing a different kind of approximation (the thermostatic one), we recover, with a different kind of proof, similar results: we characterize the limit problem and we get that the limit of V_ε is the corresponding maximal viscosity subsolution. Before doing this, we prove, under suitable assumptions, that the limit function $\tilde{V}$ is the unique viscosity solution of (see Theorem 4.3.1)

$$\begin{cases} \lambda V + H_1(x, V') = 0 \quad \text{for } x > 0, \\ \lambda V + H_{-1}(x, V') = 0 \quad \text{for } x < 0, \\ V(0) = \min\left\{u_0(0), V_{sc(-1)}(0), V_{sc(1)}(0)\right\}. \end{cases} \tag{1}$$

where on the junction point ($x = 0$), some further dynamics are considered: the ones given by the suitable convexification of “outward pointing” dynamics and somehow corresponding to stable equilibria on the junction point (stable equilibria of dynamics interpreted as forces). Note that in this simple one-dimensional case the condition in $x = 0$ of the system (1) coincides with the (regular) one in Barels-Briani-Chasseigne [19]. Hence $\tilde{V} = U^+$, where U^+ is the value function of the so-called regular problem in [19]. Next, we are able to give different proof of such an equality where, using the thermostatic approximation, we show that $\tilde{V}$ is a viscosity solution and the maximal subsolution (as in [19]) of the following Hamilton-Jacobi problem

$$\begin{cases} \lambda V + H_1(x, \nabla V) = 0 \ \text{ in } \{x > 0\}, \\ \lambda V + H_{-1}(x, \nabla V) = 0 \ \text{ in } \{x < 0\}, \\ \min\{\lambda V + H_1, \lambda V + H_{-1}\} \le 0 \text{ on } x = 0, \\ \max\{\lambda V + H_1, \lambda V + H_{-1}\} \ge 0 \text{ on } x = 0. \end{cases} \tag{2}$$

Afterwards, we extend the results to a threefold junction problem, namely a junction given by three half-lines entering the same point (see Figure 2). As in [8] we consider a thermostatic approximation and prove that the value function of the approximating problem $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is the unique viscosity solution of a suitable exit time problem (see Proposition 4.4.4). Differently from the twofold junction situation, the convexification of dynamics on the junction point seems to be not more applicable (the physical interpretation as forces equilibrium is also failing). However, inspired by the previous thermostatic approximation, we introduce a special kind of “convexification parameters” that somehow corresponds to the length of the time intervals that the trajectory spends on every single branch of a “threefold” thermostatic approximation. Here, we consider more than one way for passing to the limit

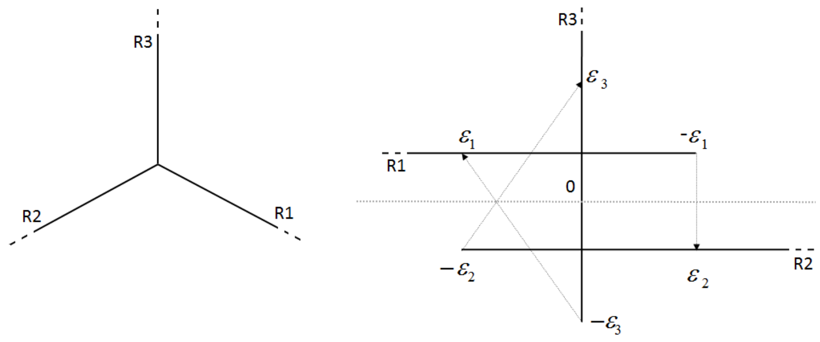


Figure 2: The threefold junctions and its thermostatic-type approximation.

in the thermostatic approximation due to the choice of either uniform or non-uniform switching thresholds.
In the first case, where we take as thresholds $(\varepsilon, \varepsilon, \varepsilon)$, we characterize the value function V of the limit problem as the unique viscosity solution of a Dirichlet problem with a particular boundary data in the junction (see Theorem 4.4.7). This characterization permit us to prove that V is the uniform limit of $V_{\varepsilon,\varepsilon,\varepsilon}$, the value function of the thermostatic problem (see Theorem 4.4.8). Next we introduce a special class of test functions and give a definition of viscosity solution for the associated Hamilton-Jacobi equations on the network. Suitably extending them to the thermostatic approximation (see Figure 2 right), being V the limit of $V_{\varepsilon,\varepsilon,\varepsilon}$ we prove that it is a viscosity solution of the Hamilton-Jacobi-Bellman problem (Theorem 4.4.11)

$$\begin{cases} \lambda V + H_1(x, \nabla V) = 0 \;\; \text{in } int(R_1), \\ \lambda V + H_2(x, \nabla V) = 0 \;\; \text{in } int(R_2), \\ \lambda V + H_3(x, \nabla V) = 0 \;\; \text{in } int(R_3), \\ \min\{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \leq 0 \text{ on } x = 0, \\ \max\{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \geq 0 \text{ on } x = 0. \end{cases} \tag{3}$$

Moreover, we provide the corresponding comparison result showing that V is the maximal subsolution of (3).
After, we take into account the case with non uniform switching thresholds $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. As before we characterize the value function V^* of the limit problem and show that (Theorem 4.4.14)

$$V^*(x,i) = \liminf_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)\to(0^+,0^+,0^+)} V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x,i) \;\forall\; (x,i) \in R_i, \;\; i = 1,2,3. \tag{4}$$

Also in this case we prove, under suitable assumptions, that V^* is a viscosity solution and the maximal subsolution of the HJB problem (3) (see Theorem 4.4.18). Anyway we point out that the definition of viscosity solution in this case is different from the previous one. Indeed, denoting by φ the test function, in this last case for $x = 0$ (the junction) the local maximum/minimum point of $V - \varphi$ may be consider with respect to two of the three branches only, while in the case of uniform switching thresholds it must be considered with respect to all the three branches. We now observe that the proven results for the threefold junction problem may also be extended to the case of manifold junction problem and, as we show in this thesis, also to a problem with two threefold junction points given by a segment and four half-lines (see Figure 4.4). Hence, in Sect. 4.5 we give a result of viscosity solution and maximal subsolution for the latter problem.

The problem with junctions is also related to n-dimensional optimal control problems on multi-domains, where the dynamics and costs incur in discontinuities when crossing some fixed hypersurfaces. These kind of problems, whose study started with Bressan-Hong [33], has been studied, in connection with HJB, in Barles-Briani-Chasseigne [19], Barnard-Wolenski [23], Rao-Zidani [89], Barles-Briani-Chasseigne [20], Rao-Siconolfi-Zidani [88], Barles-Chasseigne [21], Imbert-Monneau [67] (see Chapter 2 for more details). In this framework we study a multi-domain problem on $\mathbb{R}^2$ and, in line with what has been done in the one-dimensional case, we propose a thermostatic approximation transforming the problem in a hybrid problem (see Figure 3). In this way the discontinuity over the line $\Gamma = \{x = 0\}$ is replaced by proper

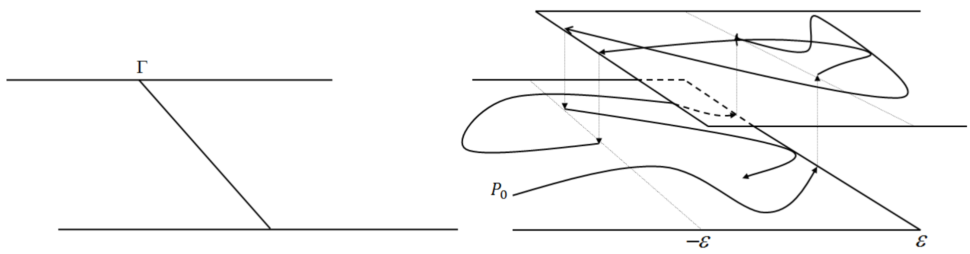


Figure 3: The two-domain problem and its thermostatic approximation.

mutually exchanged boundary conditions on the extreme "border lines" of the two extended planes and it is proved that the value function V_ε of the switching thermostatic problem is the unique viscosity solution of a pair of exit time problems (see Proposition 5.19). Then, as before we study the limit of V_ε for $\varepsilon \to 0$ and we get (see Theorem 5.2.7) that it uniformly converges to a function $\tilde{V}$ which, under proper assumptions is a viscosity solution of

$$\begin{cases} \lambda u + H_1 (x, y, \nabla u) = 0 & \text{in } \mathcal{H}_1, \\ \lambda u + H_{-1} (x, y, \nabla u) = 0 & \text{in } \mathcal{H}_{-1}, \\ \min\{\lambda u + H_1 (x, y, \nabla u), \lambda u + H_{-1} (x, y, \nabla u)\} \leq 0 & \text{on } \Gamma, \\ \max\{\lambda u + H_1 (x, y, \nabla u), \lambda u + H_{-1} (x, y, \nabla u)\} \geq 0 & \text{on } \Gamma. \end{cases} \tag{5}$$

as in Barles-Briani-Chasseigne [19]. Then, building the dynamics on Γ by the convexification of the outward pointing dynamics, we define a new Hamiltonian on Γ that we indicate with H_T, which under suitable assumptions is satisfied by any subsolution of (5). A result of maximal viscosity subsolution will be the subject of a future work.
On the theory of viscosity solutions we refer to works of Crandall, Lions [48], Crandall-Evans-Lions [50], Crandall-Ishii-Lions [51].

Part II- Decentralized feedback control in Markov Jump Processes.

In the second part of the thesis we investigate a routing problem, that involves a population of individuals referred to as players, defined over a network (see Bagagiolo-Bauso-Maggistro-Zoppello [13]). A similar problem is studied in Bauso-Zhang-Papachristodoulou [26], in which the authors consider a centralized control and a density flow for each edge dependent on the density of the whole population. This implies that each player minimizes a common cost functional which depends on the whole population's density distribution. Differently from [26], we describe a problem of jumping between nodes instead of flowing in the edges. Moreover we consider a decentralized control (as in Bauso et al. [25], Como et al. [46, 47]), in which the density of each node is locally controlled. In particular each agent ignores both controls of the far agents and the network topology. There are three distinct approaches related to routing/jump problems. The first one consists in controlling the probability to jump from a node to another one (or to flow along the edges), see for example [26]. The second one consists in controlling the transition rate from nodes (or edges), see for example Kelly-Maulloo-Tan [73], and the last one in assigning the product among the probability and the relative transition rate. As in Basna-Hilbert-Kolokoltsov [24], we use the last approach, in particular we control the product between the probability to jump from one node to an adjacent one and the relative transition rate. We formulate the problem as follows: from a microscopic point of view, each player jumps from a node to an adjacent one according to a continuous-time Markov process

$$\{X(t), t \geq 0\}, \qquad q_{ij}(u_{ij}) = \begin{cases} u_{ij}, & j \in N(i), j \neq i, \\ -\sum_{k \in N(i), k \neq i} u_{ik}, & i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $i \in V$ is the player's initial state where V is the set of nodes of the graph $L(G) = (V, E)$, $N(i)$ the set of neighbor nodes of i, q_{ij} is the microscopic dynamics from i to j, and u the decentralized routing policy described by the matrix-value function $u(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}_+^{n \times n}$, $t \longmapsto u(t)$. From a macroscopic point of view, each node is characterized by a dynamics describing the time-evolution of the density. Such dynamics depends on a decentralized control

$$\begin{cases} \dot{\rho}(t) = \rho(t)A(u), \\ \rho(0) = \rho_0, \end{cases} \tag{7}$$

where ρ is a row vector such that the sum of his components is equal to one, ρ_0 is the initial condition and the matrix-valued function $A : \mathbb{R}_+^{n\times n} \to \mathbb{R}^{n\times n}$ is defined as q_{ij} (see (6.4)). The problem that we want to solve consists in designing a decentralized routing policy to minimize the output disagreement with respect to a possible consensus/equilibrium. In particular each player, regarding its microscopic evolution, solves the minimization problem

$$\begin{cases} \inf_{u(\cdot)} J(i, u(\cdot), \rho[\cdot](\cdot), \cdot), \\ J(\cdot) = \int_t^T \ell(X(\tau), \rho(\tau), u(\tau))d\tau + g(X(T), \rho(T)), \\ \{X(t), t \geq 0\} \text{ as in (6)}, \\ X(t) = i, \end{cases} \tag{8}$$

where ℓ and g are the running cost and the exit cost respectively defined as

$$\ell(i, \rho, u) = \sum_{j\in N(i), j\neq i} \frac{u_{ij}^2}{2} (\gamma_{ij}(\rho))^+ , \; g(i, \rho) = dist(\rho, \hat{M}_i), \tag{9}$$

where γ_{ij} is a suitable coefficient yet to be designed, $(\cdot)^+$ is the positive part and $\hat{M}_i$ is the local consensus manifold/local Wardrop equilibrium set for the player i defined as $\hat{M}_i = \{\xi \in \mathbb{R}^n : \xi_j = \xi_i \; \forall \; j \in N(i)\}$. At first we rearrange the problem as a mean-field game (see Lemma 6.1.1) where the macroscopic evolution of ρ is indeed the mean field entering the cost (8). The mean-field game theory was developed in the work of Huang-Malhamé-Caines [65, 66] and independently in that of Lasry-Lions [77, 78], where the new standard terminology of Mean Field Games (MFG) was introduced. This theory includes methods and techniques to study differential games with a large population of rational players, and it is based on the assumption that the population influences the individuals' strategies through mean-field parameters. In addition to this theory, the notion of Oblivious Equilibria for large population dynamical game was introduced by Weintraub-Benkard-Van Roy [98] in the framework of Markov Decision Processes. Several application domains, such as economic, physics, biology and network engineering accommodate mean-field game theoretical models (see Lasry-Lions [78], Guent-Lasry-Lions [64], Lachapelle-Salomon-Turinici [76], Achdou-Camilli-Capuzzo Dolcetta [1]). Decision problems with mean-field coupling terms have also been formalized and studied in Bauso-Zhu-Basar [27], and application to power grid management are recently provided in Bagagiolo-Bauso [11]. The literature provides explicit solutions in the case of linear quadratic structure. In most cases, a variety of solution schemes have been recently proposed, based on discretization and/or numerical approximations (see Achdou-Capuzzo

Dolcetta [3], Achdou-Camilli-Capuzzo Dolcetta [1]).
Computing an explicit solution in the nonlinear case is difficult, and therefore we recast the problem in the framework of optimal control using a state space extension approach. This approach consists to review the density ρ as an additional state variable and no longer as an unknown variable. Then, from (8) the resulting problem is

$$\begin{aligned} \inf_{u(\cdot)} \quad & J(i, u(\cdot), \rho[\cdot](\cdot), \cdot), \\ \text{subject to} \quad & \{X(t), t \geq 0\} \text{ as in (6.2)}, \\ & \dot{\rho}(t) = \rho(t) A(u). \end{aligned}$$

In this way the mean-field system of Lemma 6.1.1 is reduced to an HJB equation plus a boundary condition

$$\begin{cases} \partial_t \tilde{V}(i, \rho, t) + \tilde{H}(i, \rho, \Delta(\tilde{V}), \partial_\rho \tilde{V}, t) = 0 \text{ in } V \times [0,1]^n \times [0, T[, \\ \tilde{V}(i, \rho, T) = g(i, \rho(T)), \end{cases} \tag{10}$$

(see Lemma 6.2.1).
The state space extension procedure is reminiscent of the McKaen-Vlasov control problem, in which the statistical distribution is encoded by our density. Similarities and differences between the McKean-Vlasov and the Mean-Field framework are analyzed in Carmona-Delarue-Lachapelle [39]. In spirit with Casti [40], Bagagiolo-Bauso [10], Chitour-Jean-Mason [44], we reformulate the problem as an inverse optimal control one, providing an explicit expression of the running cost function ℓ in order to obtain a suitable a-priori fixed decentralized feedback control u^*_{ij} (see (6.15) in Chapter 6) as the optimal feedback control of the problem. Then, using this control we prove some convergence results. In particular we obtain a uniform distribution of the density ρ (consensus/equilibrium) at first on a neighborhood of a node and then throughout the whole network. Indeed, given a suitable attainability assumption (see (6.22)) which we prove to be satisfied by our optimal control u^*_{ij} through numerical simulations, we get a first converge result to a local Wardrop equilibrium (see Theorem 6.3.1). As corollary we prove a similar convergence result for the global equilibrium, characterized by a uniform distribution of the density over all nodes (see Corollary 6.3.2). Stability of the macroscopic dynamic (7) is treated also when u^*_{ij} is affected by a hysteresis phenomena modeled by a scalar play operator (see Sect. 3.1.2). A similar problem with hysteresis was already discussed in Ceragioli-De Persis-Frasca [41]. The authors make a rigorous treatment of continuous-time average consensus dynamics with uniform quantization in communications. The consensus is reached by quantized measurements which are transmitted using a delay thermostat. In contrast to this, we use the play operator, that can be considered as a concatenation of delayed thermostats and this results in a continuous nonlinear dynamics.

The use of hysteresis captures a scenario where the players have distorted information on the density distribution in neighbor nodes. This implies that the problem has multiple equilibrium points. Indeed, considering a simple case of a network with four nodes (see Sect. 6.5), we define different types of equilibrium points according to the choice of the value of the play parameter $\varepsilon > 0$. Then, considering the global equilibrium manifold $\overline{M}$ (see (6.35)) and assuming another attainability assumption (see (6.36)), analogous to the previous one but in presence of hysteresis, we show in Theorem 6.5.3 that the solution of the hysteretic macroscopic system converge asymptotically to $\overline{M}$. This means that it is asymptotically stable and we have a global equilibrium.

Part III- Controllability results for a planar scallop swimmer.

The third part of this thesis is concerned with the study of the controllability of a swimmer immersed in a fluid (see Bagagiolo-Maggistro-Zoppello [15]). In particular we focus on the system describing the motion of a scallop for which it is well known from Purcell [86] that the *scallop theorem/paradox* holds (see also Alouges-DeSimone [7], Munnier-Chambrion [83]). This means that it is not capable to achieve any net motion performing cyclical shape changes, either in a viscous or in an inviscid fluid. The study of locomotion strategies in fluids is attracting increasing interest in recent literature, especially for its connection with the realization of artificial devices that can self-propel in fluids. Theories of swimming generally uses either low Reynolds number approximation ($Re \to 0$, hence a viscous fluid), or the assumption of inviscid ideal fluid dynamics (high Reynolds number, $Re \to +\infty$) where the *Reynolds number* (that arises from the adimesionalization of the Navier-Stokes equations) is defined by

$$Re = \frac{VL\rho}{\eta} = \frac{VL}{\nu}$$

where V is the characteristic velocity of the body immersed in the fluid, L its characteristic length, ρ the density of the fluid, η its viscosity and $\nu = \frac{\eta}{\rho}$ is the kinematic viscosity. The above approximations are useful, together with suitable assumptions on body's composition of the swimmer, to determine the state evolution of the scallop both in viscous and ideal fluid (see (7.6) and (7.13) respectively). Coming back to the scallop paradox, some authors tried to overcome it changing the geometry of the swimmer, for example adding a degree of freedom, introducing the Purcell swimmer [86], or the three sphere swimmer, as Golestanian-Ajdari [62]. Others, instead, supposed the scallop immersed in a non Newtonian fluid, in which the viscosity is not constant, ending up with a non reversible dynamics, see Cheng-DeMont [42] , Qiu et al. [87]. Inspired by this last approach, our aim is to propose some strategies which maintain the swimmer geometry and exploit instead a change in the dynamics. For example in Cicconofri-Desimone [45] the

authors modulate the speed of actuation to steer the swimmer along curved trajectories. Differently from us they never leave the Stokes regime. Our idea is based on switching dynamics depending on the angular velocity $\dot{\theta}$ of opening and closing of the scallop's valves. More precisely we analyze two cases:
1) we suppose that if the modulus of the angular velocity $|\dot{\theta}|$ is high, the fluid regime can be approximated by the ideal one, instead if this modulus is low the fluid can be considered as completely viscous. This situation is well represented by a switching system in which the change of the dynamics is determined by the modulus of the angular velocity. Using this representation we provide a first result in which we overcome the Scallop paradox, and hence obtain a partial controllability of the mechanical system in the state x (see Theorem 7.2.1).
2) The fluid reacts in a different way between the opening and closing of the valves: it facilitates the opening, so that it can be considered ideal, and resists the closing, like a viscous fluid. These last approximations model a fluid in which the viscosity η changes with the sign of the angular velocity $(sign(\dot{\theta}))$. More precisely we use two constant viscosities: one high (resp. one very small) if the angular velocity is negative (resp. positive). Also this situation can be presented by a switching system (different from the previous one) but we get a more weaker result then the one in case 1) because we are able to achieve a net displacement but only forward (see Theorem 7.2.3). Inspired by [87], where the scallop's opening and closing is actuated by an external magnetic field, in this last case we introduce an hysteresis mechanism through a thermostat to model a delay in the change of fluid's regime. In such a way, assuming to be able to prescribe the angular velocity and using it as a control parameter, we prove that the system is controllable, i.e. the scallop is able to move both forward and backward using cyclical deformations (Theorem 7.2.4). Furthermore, we also show that it is always possible to move between two fixed configurations in which we prescribe both the initial and final positions and angles. We then get a result of global controllability (Theorem 7.2.7). In conclusion we carry out some numerical simulations to support our theoretical predictions.

Chapter 1

Overview of results on Hamilton-Jacobi equations

In this chapter we presents some classical results on deterministic optimal control and the associated Hamilton-Jacobi approach. Using the dynamic programming techniques the study of optimal control problems can be linked to the resolution of a particular class of nonlinear partial differential equations: the Hamilton-Jacobi-Bellman equations. In many cases the existence of classical solutions of such equations is not guaranteed. The suitable context to obtain several existence and uniqueness results turned out to be the viscosity solution framework, introduce by Crandall, Lions [48], Crandall-Evans-Lions [50], Crandall-Ishii-Lions [51].
The chapter is organized as follows. In Sect. 1.1 the notion of viscosity solution for the more general class of Hamilton-Jacobi equations is given. We state some comparison and uniqueness theorems and conclude recalling some important stability results. In Sect. 1.2 a general formulation of the deterministic optimal control with finite or infinite horizon is given. The Dynamic Programming Principle is stated and its differential version, the Hamilton-Jacobi-Bellman equation, provided. The exit-time problem is analysed in order to introduce the Dirichlet problem with boundary condition in the viscosity sense (see Barles-Perthame [22], Ishii [71] for a complete treatment). For a comprehensive account of the theory see for example Barles [17] and Bardi-Capuzzo Dolcetta [16].

1.1 Viscosity solutions

We consider the Hamilton-Jacobi equation

$$\lambda u + H(x, Du(x)) = 0, \quad x \in \Omega \tag{1.1}$$

where Ω is an open domain of $\mathbb{R}^n$, $\lambda > 0$ is a constant strictly positive, u is a real valued function in Ω, Du is the gradient of u and the Hamiltonian

$H = H(x,p)$ is a continuous real valued function on $\Omega \times \mathbb{R}^n$. It is well know that for the equation (1.1) no classical solution (that is of class $C^1(\Omega) \cap C(\overline{\Omega})$) may exist. Even for a very simple 1-dimensional eikonal equation with a Dirichlet boundary condition

$$\begin{cases} |u'(x)| = 1, & x \in (-1,1) \\ u(x) = 0, & x = \pm 1 \end{cases} \tag{1.2}$$

we can find infinite solutions almost everywhere, but not classic (see Figure 1.1). The theory of viscosity solutions was developed in order to overcome these problems. The basic idea of this theory is to use a class of regular test functions in order to replace the partial derivatives in (1.1) that may not exist. We give here two equivalent definitions of viscosity solution.

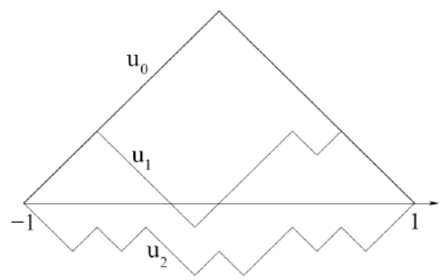


Figure 1.1: Multiple solutions of the eikonal equation (1.2).

Definition 1.1.1. (I version). *A function $u \in C(\Omega)$ is a viscosity solution of the equation* (1.1) *if the following conditions are satisfied:*

- $\lambda u(x) + H(x,p) \leq 0$ *for all $x \in \Omega$, $p \in D^+u(x)$ (viscosity subsolution),*

- $\lambda u(x) + H(x,q) \geq 0$ *for all $x \in \Omega$, $q \in D^-u(x)$ (viscosity supersolution).*

where D^+, D^- are the super and sub-differential of u in the point x, i.e.

$$D^+u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y-x)}{|y-x|} \leq 0 \right\},$$
$$D^-u(x) = \left\{ q \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - q \cdot (y-x)}{|y-x|} \geq 0 \right\}.$$

Definition 1.1.2. (II version). *A function $u \in C(\Omega)$ is a viscosity solution of the equation* (1.1) *if the following conditions are satisfied:*

- *if for any $x_0 \in \Omega$, any $\varphi \in C^1(\Omega)$ such that $u - \varphi$ has a local maximum point at x_0 with respect to Ω, then*

$$\lambda u(x_0) + H(x_0, D\varphi(x_0)) \leq 0 \quad (\textit{viscosity subsolution}),$$

- *if for any $x_0 \in \Omega$, any $\varphi \in C^1(\Omega)$ such that $u - \varphi$ has a local minimum point at x_0 with respect to Ω, then*

$$\lambda u(x_0) + H(x_0, D\varphi(x_0)) \geq 0 \quad (\textit{viscosity supersolution}).$$

The motivation for the terminology *viscosity solutions* is that we can see this kind of solution as the limit function $u = \lim_{\varepsilon \to 0^+} u^\varepsilon$, where $u^\varepsilon \in C^2(\Omega)$ is the classical solution of the regularized problem

$$-\varepsilon \Delta u^\varepsilon + \lambda u^\varepsilon + H(x, Du^\varepsilon) = 0, \quad x \in \Omega \tag{1.3}$$

in the case u^ε exists and converges locally uniformly to some continuous function u. This method is named *vanishing viscosity*, and it is the original idea behind the notion of viscosity solution presented by Crandall and Lions in [48].
In the following we state some comparison results between viscosity sub- and supersolution in the cases Ω bounded and $\Omega = \mathbb{R}^n$. As a simple corollary, each comparison result produces a uniqueness theorem for the associated Dirichlet problem. Moreover we present some important stability properties. Before to do this we recall some properties satisfied by the Hamiltonian H that we will take into account in the following results.

$$\begin{aligned} |H(x,p) - H(y,p)| &\leq \omega_1(|x-y|(1+|p|)), \\ |H(x,p) - H(x,q)| &\leq \omega_2(|p-q|), \end{aligned} \tag{1.4}$$

for $x, y \in \Omega$, $p, q \in \mathbb{R}^n$, and ω_i, $i = 1, 2$ is a modulus of continuity, that is $\omega_i : [0, +\infty) \to [0, +\infty)$ is continuous non decreasing with $\omega_i(0) = 0$.

Theorem 1.1.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that $u_1, u_2 \in C(\overline{\Omega})$ are, respectively, viscosity sub- and supersolution of*

$$\lambda u(x) + H(x, Du(x)) = 0, \quad x \in \Omega \tag{1.5}$$

and

$$u_1 \leq u_2 \quad \textit{on } \partial\Omega. \tag{1.6}$$

Then $u_1 \leq u_2$ in $\overline{\Omega}$.

Remark 1.1.4. *If u_1, u_2 are both viscosity solutions of* (1.5) *with $u_1 = u_2$ on $\partial\Omega$, from Theorem* (1.1.3) *it follows that $u_1 = u_2$ in $\overline{\Omega}$.*

We consider now the case $\Omega = \mathbb{R}^n$ and state a comparison result in the space $BC(\mathbb{R}^n)$ of bounded continuous function on $\mathbb{R}^n$.

Theorem 1.1.5. *Assume that* u_1, $u_2 \in BC(\mathbb{R}^n)$ *are, respectively, viscosity sub- and supersolution of*

$$\lambda u(x) + H(x, Du(x)) = 0, \quad x \in \mathbb{R}^n. \tag{1.7}$$

Then $u_1 \leq u_2$ *in* $\mathbb{R}^n$.

Remark 1.1.6. *Theorem* (1.1.5) *can be generalized to cover the case of a general unbounded open set* $\Omega \in \mathbb{R}^n$. *Furthermore, the assumption* $u_1, u_2 \in BC(\mathbb{R}^n)$ *can be replaced by* $u_1, u_2 \in UC(\mathbb{R}^n)$, *that is the space of uniformly continuous functions on* $\mathbb{R}^n$.

Stability Results

The first stability result is the one relative to the lattice operations in $C(\Omega)$:

$$(u \vee v)(x) = \max\{u(x), v(x)\},$$
$$(u \wedge v)(x) = \min\{u(x), v(x)\}.$$

Proposition 1.1.7.

1. *Let* $u, v \in C(\Omega)$ *be a viscosity subsolution of* (1.1)*; then* $u \vee v$ *is a viscosity subsolution of* (1.1).

2. *Let* $u, v \in C(\Omega)$ *be a viscosity supersolution of* (1.1)*; then* $u \wedge v$ *is a viscosity supersolution of* (1.1).

3. *Let* $u \in C(\Omega)$ *be a viscosity subsolution of* (1.1) *such that* $u \geq v$ *for any viscosity subsolution* $v \in C(\Omega)$ *of* (1.1)*; then* u *is a viscosity supersolution and therefore a viscosity solution of* (1.1).

Proposition 1.1.8. *Let* $u_n \in C(\Omega)$ $(n \in \mathbb{N})$ *be a viscosity solution of*

$$\lambda u_n(x) + H_n(x, Du_n(x)) = 0 \quad \textit{in } \Omega. \tag{1.8}$$

Assume that

$$u_n \longrightarrow u \textit{ locally uniformly in } \Omega,$$
$$H_n \longrightarrow H \textit{ locally uniformly in } \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

Then u *is a viscosity solution of* (1.1) *in* Ω.

1.2 Optimal control problem

We now recall some classical results on optimal control problem and the associated Hamilton-Jacobi approach.
Let us consider the controlled nonlinear dynamical system

$$\begin{cases} \dot{y}(t) = f(y(t), \alpha(t)), \quad t > 0 \\ y(0) = x. \end{cases} \tag{1.9}$$

where $y(t)$ is the state of the system, $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is the dynamics, $\alpha : [0, +\infty[\to A$ is a measurable control, $A \subset \mathbb{R}^m$.
We consider the following assumption on A and f:

$$A \subset \mathbb{R}^m \text{ is a compact set;} \tag{1.10}$$

$$\begin{cases} f \text{ is continuous on } \mathbb{R}^n \times A; \\ \exists M > 0 \text{ such that } |f(x, a)| \leq M \ \forall x \in \mathbb{R}^n, \forall a \in A; \\ \exists L > 0 \text{ such that } |f(x, a) - f(z, a)| \leq L|x - z| \ \forall x, z \in \mathbb{R}^n \ \forall a \in A. \end{cases} \tag{1.11}$$

We denote with $\mathcal{A}$ the set of admissible controls defined as

$$\mathcal{A} := \{\alpha : [0, +\infty[\to A, \text{measurable}\}$$

and we call A the set of constant controls.
It is well know that under the above assumptions, for all initial state $x \in \mathbb{R}^n$ and for all measurable control $\alpha \in \mathcal{A}$ there exists a unique solution (trajectory) of (1.9). We define this solution, denoting it by $y_x(\cdot, \alpha)$, as the unique function such that

$$y_x(t, \alpha) = x + \int_0^t f(y_x(s, \alpha), \alpha(s))ds \quad \forall t \geq 0. \tag{1.12}$$

In particular, $y_x(\cdot, \alpha)$ is absolutely continuous on compact interval of $[0, +\infty[$ and it solves (1.12) almost everywhere.

The final goal is to find an optimal control α^* such that the corresponding trajectory $y_x(t; \alpha^*)$ is the "most convenient" one with respect to some given criterion, typically minimizing a cost functional, between all possible trajectories starting from x.

Before to do this we recall a basic estimate on $y_x(t, \alpha)$ that we will use later:

$$|y_x(t, \alpha) - y_z(t, \alpha)| \leq e^{Lt}|x - z| \quad \text{for all } \alpha \in \mathcal{A} \text{ and } t > 0. \tag{1.13}$$

1.2.1 The infinite horizon problem

In the infinite horizon problem the *cost functional* J, associated to every trajectory, which has to be minimized is

$$J(x,\alpha) := \int_0^\infty e^{-\lambda s}\ell(y_x(s,\alpha),\alpha(s))ds,$$

where the running cost $\ell : \mathbb{R}^n \times A \to \mathbb{R}$ and the discount factor λ satisfy

$$\begin{cases} \ell \text{ is continuos on } \mathbb{R}^n \times A; \\ \exists M > 0 \text{ and a modulus of continuity } \omega_\ell \text{ such that} \\ |\ell(x,a)| \le M \ \forall x \in \mathbb{R}^n, \forall a \in A; \\ |\ell(x,a) - \ell(z,a)| \le \omega_\ell(|x-z|) \ \ \forall x,z \in \mathbb{R}^n \ \forall a \in A; \\ \lambda > 0. \end{cases} \tag{1.14}$$

The associated *value function* is

$$V(x) := \inf_{\alpha \in \mathcal{A}} J(x,\alpha). \tag{1.15}$$

We recall some basic results:

Proposition 1.2.1. *Under the assumptions* (1.11) *and* (1.14) *the value function V is bounded and uniformly continuous on $\mathbb{R}^n$.*

Proposition 1.2.2. (Dynamic Programming Principle). *Assume* (1.10), (1.11) *and* (1.14). *Then for all $x \in \mathbb{R}^n$ and $t > 0$*

$$V(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t e^{-\lambda s}\ell(y_x(s,\alpha),\alpha(s))ds + e^{-\lambda t}V(y_x(t,\alpha)) \right\} \tag{1.16}$$

Remark 1.2.3. If there is an optimal control α^* for x, that is, $V(x) = J(x,\alpha^*)$, then the infimum in (1.16) is attained at α^* for all $t > 0$:

$$V(x) = \int_0^t e^{-\lambda s}\ell(y_x(s,\alpha^*),\alpha^*(s))ds + e^{-\lambda t}V(y_x(t,\alpha^*)).$$

Furthermore we have that

$$V(y_x(t,\alpha^*)) = J(y_x(t,\alpha^*),\alpha^*(\cdot + t))$$

namely a control α^* is optimal for each point of the corresponding trajectory, provided it is appropriately shifted in time. In other words, that could be roughly interpreted as "pieces of optimal trajectories are optimal". That is the Bellman's original formulation of the principle (Bellman [29]): " *An optimal policy has the property that whatever the initial data and choice are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*".

The main consequence of the Dynamic Programming Principle is the following

Proposition 1.2.4. *Assume* (1.10), (1.11) *and* (1.14). *Then the value function V is a viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$\lambda V(x) + \sup_{a \in A}\{-f(x,a) \cdot \nabla V(x) - \ell(x,a)\} = 0, \quad x \in \mathbb{R}^n. \tag{1.17}$$

The following result characterizes the value function as the unique viscosity solution of (1.17).

Theorem 1.2.5. *Assume* (1.10),(1.11) *and* (1.14). *Then the value function V is the unique bounded and uniformly continuous function which is viscosity solution of* (1.17).

1.2.2 The finite horizon problem

The *cost functional* J defined on $\mathbb{R}^n \times [0, +\infty[\times \mathcal{A}$ which has to be minimized is

$$J(x,t,\alpha) = \int_0^t e^{-\lambda s} \ell(y_x(s,\alpha), \alpha(s))ds + e^{-\lambda t} g(y_x(t,\alpha)), \tag{1.18}$$

where the running cost ℓ and the discount factor λ are such that

$$\ell \text{ satisfies the assumptions in (1.14)}, \ \lambda \geq 0, \tag{1.19}$$

and the terminal cost $g : \mathbb{R}^n \to \mathbb{R}$ is bounded and uniformly continuous.
The value function V in this case is defined as

$$V(x,t) := \inf_{\alpha \in \mathcal{A}} J(x,t,\alpha). \tag{1.20}$$

Proposition 1.2.6. *Assume* (1.10), (1.11), (1.19) *and the hypothesis on g introduce above. Then the value function V is bounded and continuous in $\mathbb{R}^n \times [0,T]$, for all $T > 0$.*

Proposition 1.2.7. (Dynamic Programming Principle).
Assume (1.10),(1.11), (1.19) *and the hypothesis on g. Then for all $x \in \mathbb{R}^n$ and $0 < \tau \leq t$*

$$V(x,t) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^\tau e^{-\lambda s} \ell(y_x(s,\alpha), \alpha(s))ds + e^{-\lambda \tau} V(y_x(\tau,\alpha), t-\tau) \right\}. \tag{1.21}$$

Proposition 1.2.8. *With the same assumptions as before, the value function* (1.20) *is the unique viscosity solution of*

$$\begin{cases} V_t + \lambda V + \sup_{a \in A}\{-f(x,a) \cdot \nabla V(\cdot,t) - \ell(x,a)\} = 0 & \text{on } \mathbb{R}^n \times]0, +\infty[, \\ V(x,0) = g(x) & x \in \mathbb{R}^n. \end{cases} \tag{1.22}$$

where $\nabla V(\cdot,t)$ denotes the gradient of the value function with respect to the n spatial variables.

Another common way to study the finite horizon problem is to minimize the following cost functional defined on $\mathbb{R}^n \times]-\infty, T[\times \mathcal{A}$

$$I(x,t,\alpha) = \int_t^T e^{\lambda(t-s)} \ell(y_x(s,\alpha), \alpha(s))ds + e^{\lambda(t-T)} g(y_x(T,\alpha)), \tag{1.23}$$

where the running cost ℓ and the discount factor λ satisfy (1.19), and the terminal cost $g : \mathbb{R}^n \to \mathbb{R}$ is bounded and uniformly continuous as before. The value function W in this case is defined as

$$W(x,t) := \inf_{\alpha \in \mathcal{A}} I(x,t,\alpha). \tag{1.24}$$

and it is related to the function V (1.20) by the formula

$$V(x,t) = W(x, T-t).$$

Proposition 1.2.9. (Dynamic Programming Principle).
Assume (1.10),(1.11), (1.19) *and the hypothesis on* g. *Then for all* $x \in \mathbb{R}^n$ *and* $t < \tau \leq T$

$$W(x,t) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_t^\tau e^{\lambda(t-s)} \ell(y_x(s,\alpha), \alpha(s))ds + e^{\lambda(t-\tau)} W(y_x(\tau,\alpha), \tau) \right\}. \tag{1.25}$$

Proposition 1.2.10. *With the same assumptions as before, the value function* (1.24) *is the unique viscosity solution of*

$$\begin{cases} -W_t + \lambda W + \sup\limits_{a \in A}\{-f(x,a) \cdot \nabla W(\cdot,t) - \ell(x,a)\} = 0 & \textit{on } \mathbb{R}^n \times]-\infty, T[, \\ W(x,T) = g(x) & x \in \mathbb{R}^n. \end{cases} \tag{1.26}$$

where $\nabla W(\cdot,t)$ *denotes the gradient of the value function with respect to the* n *spatial variables.*

1.2.3 The exit-time problem

Consider a *target set* $\mathcal{T} \subset \mathbb{R}^n$ such that

$$\mathcal{T} \text{ is closed with compact boundary } \partial\mathcal{T}. \tag{1.27}$$

We study a problem with initial state x in $\mathcal{T}^c := \mathbb{R}^n \setminus \mathcal{T}$ whose dynamics is stopped and the payoff computed when the system either 1) reaches $\mathcal{T}$ or 2) its interior $int(\mathcal{T})$.
We start considering the case 1) for which the cost functional J to minimize is

$$J(x,\alpha) := \begin{cases} \int_0^{t_x(\alpha)} e^{-\lambda s} \ell(y_x(s,\alpha), \alpha(s))ds + e^{\lambda t_x(\alpha)} g(y_x(t_x(\alpha)) & \text{if } t_x(\alpha) < +\infty, \\ \int_0^{+\infty} e^{-\lambda t} \ell(y_x(t,\alpha), \alpha(t))dt & \text{if } t_x(\alpha) = +\infty, \end{cases} \tag{1.28}$$

where $t_x(\alpha)$ is the exit-time from $\mathcal{T}^c$ (or the entry time in $\mathcal{T}$) defined as

$$t_x(\alpha) := \begin{cases} +\infty & \text{if } \{t : y_x(t,\alpha) \in \mathcal{T}\} = \emptyset, \\ \min\{t : y_x(t,\alpha) \in \mathcal{T}\} & \text{otherwise.} \end{cases} \tag{1.29}$$

The running cost ℓ and the discount factor λ satisfy (1.14), while the *final cost* $g : \partial\mathcal{T} \to [0, +\infty[$ is such that

$$g \in BC(\partial\mathcal{T}). \tag{1.30}$$

The associated value function V is

$$V(x) = \inf_{\alpha \in \mathcal{A}} J(x,\alpha), \tag{1.31}$$

and the following holds

Proposition 1.2.11. *(Dynamic Programming Principle). Assume* (1.10),(1.11) *and*(1.14)*. Then*

$$V(x) = \inf_{\alpha \in \mathcal{A}} \int_0^{t_x(\alpha)\wedge t} e^{-\lambda s}\ell(y_x(s),\alpha(s)) + e^{-\lambda(t_x(\alpha)\wedge t)} V(y_x(t_x(\alpha)\wedge t)). \tag{1.32}$$

In case 2) instead, the cost functional $\hat{J}$ to minimize is

$$\hat{J}(x,\alpha) := \begin{cases} \int_0^{\hat{t}_x(\alpha)} e^{-\lambda s}\ell(y_x(s,\alpha),\alpha(s))ds + e^{\lambda \hat{t}_x(\alpha)} g(y_x(\hat{t}_x(\alpha)) & \text{if } \hat{t}_x(\alpha) < +\infty, \\ \int_0^{+\infty} e^{-\lambda t}\ell(y_x(t,\alpha),\alpha(t))dt & \text{if } \hat{t}_x(\alpha) = +\infty, \end{cases} \tag{1.33}$$

where $\hat{t}_x(\alpha)$ is the exit-time from $\overline{\mathcal{T}^c}$ (ot the entry time in $int(\mathcal{T})$) defined as

$$\hat{t}_x(\alpha) := \begin{cases} +\infty & \text{if } \{t : y_x(t,\alpha) \in int(\mathcal{T})\} = \emptyset, \\ \inf\{t : y_x(t,\alpha) \in int(\mathcal{T})\} & \text{otherwise.} \end{cases} \tag{1.34}$$

The conditions on ℓ, λ and g are the same of case 1) above. The associated value function $\hat{V}$ is

$$\hat{V}(x) = \inf_{\alpha \in \mathcal{A}} \hat{J}(x,\alpha), \tag{1.35}$$

that satisfies

Proposition 1.2.12. *(Dynamic Programming Principle). Assume* (1.10),(1.11) *and*(1.14)*. Then*

$$\hat{V}(x) = \inf_{\alpha \in \mathcal{A}} \int_0^{\hat{t}_x(\alpha)\wedge t} e^{-\lambda s}\ell(y_x(s),\alpha(s)) + e^{-\lambda(\hat{t}_x(\alpha)\wedge t)} V(y_x(\hat{t}_x(\alpha)\wedge t)). \tag{1.36}$$

We now characterize the function V (1.31) as viscosity solution of the HJB equation with the natural boundary conditions. By subsolution (respectively, supersolution) of the Dirichlet condition

$$V = g \quad \text{on } \partial\mathcal{T}, \tag{1.37}$$

we mean a function $\leq g$ (respectively, $\geq g$) at each point of $\partial\mathcal{T}$.

Proposition 1.2.13. *Assume* (1.10), (1.11), (1.14), (1.27), (1.30) *and suitable conditions of controllability. Then $V(x)$ is a bounded and uniformly continuous viscosity solution of*

$$\lambda V(x) + H(x, DV(x)) = 0 \quad in \quad \mathcal{T}^c, \tag{1.38}$$

with the boundary condition:

$$V = g \quad on \quad \partial\mathcal{T}.$$

Proposition 1.2.14. *Assume* (1.10), (1.11), (1.14), (1.27) *and* (1.30). *If $u \in C(\overline{\mathcal{T}^c})$ is a subsolution of* (1.38) *such that $u \leq g$ on $\partial\mathcal{T}$ and u is bounded above, then $u \leq V$ in $\overline{\mathcal{T}^c}$.*

In order to give similar results for the exit time problem from $\overline{\mathcal{T}^c}$ we need to use the definition of boundary conditions in viscosity sense (see the follow section). This because the HJB equation (1.38) may be satisfied also on $\partial\mathcal{T}^c$.

1.2.4 Boundary conditions in the viscosity sense

In this section we deal with Dirichlet boundary value problem

$$\begin{cases} \lambda u(x) + H(x, Du(x)) = 0 & \text{in } \ \mathcal{T}^c := \Omega, \\ u(x) = g(x) & \text{on } \ \partial\Omega, \end{cases} \tag{1.39}$$

in cases where a solution does not verify the boundary condition in classical sense. The following holds:

Definition 1.2.15. *A function $u \in C(\overline{\Omega})$ is a viscosity subsolution (respectively, supersolution) of*

$$u = g \quad or \quad \lambda u + H(x, Du) = 0 \quad on\ \partial\Omega \tag{1.40}$$

if, for any $\varphi \in C^1(\overline{\Omega})$ and $x \in \partial\Omega$ such that $u - \varphi$ has a local maximum (respectively, minimum) at x with respect to $\overline{\Omega}$,

$$(u-g)(x) \leq 0 \;\; (resp., \geq 0) \quad or \quad \lambda u(x) + H(x, D\varphi(x)) \leq 0 \;\; (resp., \geq 0). \tag{1.41}$$

Equivalently, we say that subsolutions (respectively, supersolutions) of (1.40) *satisfy*

$$(u-g) \wedge (\lambda u + H(x, Du)) \leq 0 \quad \textit{on } \partial\Omega$$

(respectively,

$$(u-g) \vee (\lambda u + H(x, Du)) \geq 0 \quad \textit{on } \partial\Omega)$$

in the viscosity sense.

Proposition 1.2.16. *Assume* (1.10), (1.11), (1.14)*, suitable conditions of controllability and $g \in BC(\partial\Omega)$. Then the value function $\hat{V}(x)$* (1.35) *is a bounded and uniformly continuous viscosity solution of* (1.39) *with boundary condition in viscosity sense.*

Also for (1.39) there are comparison results. We start to consider the case with Ω bounded and the following assumptions on H and on $\partial\Omega$. We require that H satisfies (1.4) and with H as

$$H(x,p) = \sup_{a \in A}\{-f(x,a) \cdot p - \ell(x,a)\}. \tag{1.42}$$

The assumption on $\partial\Omega$ is to be a Lipschitz surface, namely

$$\begin{aligned} &\text{there are } c > 0 \text{ and } \eta : \overline{\Omega} \to \mathbb{R}^n \text{ continuous such that} \\ &B(x + t\eta(x), ct) \subseteq \Omega \text{ for all } x \in \overline{\Omega}, 0 < t \leq c. \end{aligned} \tag{1.43}$$

Theorem 1.2.17. *Assume Ω bounded and the above conditions on H and $\partial\Omega$. Suppose $u_1, u_2 \in C(\overline{\Omega})$ are, respectively, a subsolution and a supersolution of*

$$\lambda u + H(x, Du) = 0 \quad \textit{in } \Omega, \tag{1.44}$$

and u_2 satisfies

$$(u_2 - u_1) \vee (u_2 + H(x, Du_2)) \geq 0 \quad \textit{on } \partial\Omega \tag{1.45}$$

in the viscosity sense. Then $u_1 \leq u_2$ in $\overline{\Omega}$. The same if u_1 instead of u_2 satisfies

$$(u_1 - u_2) \wedge (u_1 + H(x, Du_1)) \leq 0 \quad \textit{on } \partial\Omega \tag{1.46}$$

If we take Ω unbounded the previous comparison theorem continues to be valid provided that

1. (1.43) holds with η bounded and uniformly continuous;
2. u_1 and u_2 are bounded;
3. H is given by (1.42) with $f : \overline{\Omega} \times A \to \mathbb{R}^n$ and $\ell : \overline{\Omega} \times A \to \mathbb{R}$ continuous and Lipschitz continuous for all $x, y \in \overline{\Omega}$, $a \in A$, A compact, f bounded in $\partial\Omega \times A$.

Theorem 1.2.18. *Assume the above hypotheses 1. and 3, and suppose there exists a solution $u \in BC(\partial\Omega)$ of* (1.39)*. Then $u \in C(\overline{\Omega})$ and it is the unique solution of* (1.39)*.*

Chapter 2

Discontinuous Hamilton-Jacobi equations

In this chapter, we give an overview on the first order Hamilton-Jacobi equations with discontinuous Hamiltonians with respect to the state variable. In particular, in Sect. 2.1 we focus on new problems arising by this discontinuity and recall in Sect. 2.2 the results obtained by various authors. In Sect. 2.3 we restrict to consider discontinuous HJ equations on networks and discuss about existing results in this topic. Anyway, we point out that in this thesis the discontinuity of the Hamiltonian is always given by discontinuities through some hypersurfaces of the state space. That is, the Hamiltonians are anyway continuous one some connected components.

2.1 An overview

Discontinuous HJ equations present new difficulties from the viewpoint of the viscosity solutions theory. In addition to the problem of existence of solutions, the main difficulty lies in the proof of the uniqueness or more precisely of a comparison principle. Indeed, when using the classical double-variable method for proving comparison results between sub- and superviscosity solutions (see Crandall-Lions [49], Lions [80], Crandall-Lions [48], Crandall-Evans-Lions [50], Barles [17], Bardi-Capuzzo Dolcetta [16], Barles [18]) we cannot in general conclude as in the standard way because the points of minimum and of maximum, even if very close, may belong to different regions for which dynamics and costs are absolutely non-comparable. We consider, for example, a very simple equation for which we can not directly apply the double-variable method:

$$\lambda u(x) + H(x, Du(x)) = 0, \quad x \in \mathbb{R}, \tag{2.1}$$

where λ is a strictly positive constant and the Hamiltonian $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$H(x,p) = \begin{cases} H_R(x,p) & \text{if } x \geq 0, \\ H_L(x,p) & \text{if } x < 0, \end{cases} \tag{2.2}$$

with H_L and H_R convex and coercive and with the Hamiltonian H discontinuous on $x = 0$. An idea, explained by Barles-Briani-Chasseigne [19, 20], is to use the Ishii's definition of viscosity solution for the discontinuous Hamiltonians (see Ishii [69]). More precisely, an upper semi-continuous function (usc) $u : \mathbb{R} \to \mathbb{R}$ is a subsolution of (2.1) if for any $x \in \mathbb{R}$, any $\varphi \in C^1(\mathbb{R})$ such that $u - \varphi$ has a local maximum point at x, then

$$\begin{cases} \lambda u(x) + H_L(x, \varphi'(x)) \leq 0, & \text{if } x < 0, \\ \lambda u(x) + H_R(x, \varphi'(x)) \leq 0, & \text{if } x > 0, \\ \min\{\lambda u(x) + H_L(x, \varphi'(x)), \lambda u(x) + H_R(x, \varphi'(x))\} \leq 0, & \text{if } x = 0. \end{cases}$$

In the same way, a lower semi-continuous function (lsc) $u : \mathbb{R} \to \mathbb{R}$ is a supersolution of (2.1) if for any $x \in \mathbb{R}$, any $\varphi \in C^1(\mathbb{R})$ such that $u - \varphi$ has a local minimum point at x, then

$$\begin{cases} \lambda u(x) + H_L(x, \varphi'(x)) \geq 0, & \text{if } x < 0, \\ \lambda u(x) + H_R(x, \varphi'(x)) \geq 0, & \text{if } x > 0, \\ \max\{\lambda u(x) + H_L(x, \varphi'(x)), \lambda u(x) + H_R(x, \varphi'(x))\} \geq 0, & \text{if } x = 0. \end{cases}$$

As said before the doubling-variables technique is not sufficient in this framework. To show this, we try to prove that $u - v \leq 0$ on $\mathbb{R}$, where u is a bounded subsolution and v a bounded supersolution. We introduce the test function

$$\phi_\varepsilon(x,y) = u(x) - v(y) - \frac{|x-y|^2}{\varepsilon}. \tag{2.3}$$

One supposes that it admits $(x_\varepsilon, y_\varepsilon) \in \mathbb{R}^2$ as maximum point. By classical results one proves that both x_ε and y_ε converge to the same point $\tilde{x}$ for $\varepsilon \to 0$. If $\tilde{x} > 0$ (the case $\tilde{x} < 0$ is analogous) and for ε small enough we get by standard arguments that

$$\lambda(u(x_\varepsilon) - v(y_\varepsilon)) + H_R\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) - H_R\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) \leq 0,$$

hence $u \leq v$. If $\tilde{x} = 0$, assuming that $x_\varepsilon < 0 < y_\varepsilon$ we obtain

$$\lambda(u(x_\varepsilon) - v(y_\varepsilon)) + H_L\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) - H_R\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) \leq 0.$$

Since the Hamiltonians H_R and H_L are totally independent we do not have that

$$H_L\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) - H_R\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) \to 0 \quad \text{for} \quad \varepsilon \to 0,$$

and hence we cannot conclude.
We realize that the problem not only refers to the above example but arises when the Hamiltonians are discontinuous with respect to the state variable. Then, to characterize the solution, it is necessary to define further conditions on the interface (namely the surface of discontinuity), called transmission conditions. Once this is done, one will propose a new definition of viscosity solution and develop a new strategies in order to obtain the uniqueness of the solution.

2.2 Related literature

The study of discontinuous Hamilton-Jacobi equations is the subject of several works in which the techniques used come from both the theory of partial differential equations and the optimal control theory.
In the particular case of discontinuous Hamilton-Jacobi-Bellman equations we start by recalling the work of Ishii [69]. Afterward we can cite, for example, the works of Soravia [94, 95] and Garavello-Soravia [59, 60] published among 2002 and 2006. In [59] for example, the authors study the HJB equations resulting from optimal control problems with a no bounded control set and a discontinuous running cost with respect to the state variable. In particular the running cost is of type $\ell(x, a) = h(x, a) + g(x)$ with continuous h and Borel measurable g. The value function of this problem is not the unique solution of the associated HJB equation. The sub- and super-optimality principles for sub- and supersolutions are provided in order to get some necessary and sufficient conditions which give the uniqueness of the solution. When there is no uniqueness, the sub- and super-optimality principles allow to characterize the minimal and maximal solutions.
In the articles mentioned above, the Hamiltonians have a specific structure, they are Borel measurable in the state variables and there are no precise information on the set of discontinuity points. We will see that among the various studies on discontinuous Hamilton-Jacobi equations, many of them refer to classes of problems for which we have information on the set of discontinuity points of the Hamiltonian. This is the case of the work of Bressan and Hong [33], in which the authors provide a rather complete study of deterministic control problems in stratified domains, i.e. control problems for which the dynamics and the running costs may have discontinuities on the submanifolds of $\mathbb{R}^n$. In particular, they show that the value function satisfies some HJB inequalities (in the viscosity sense) and prove, under certain conditions, a comparison result between sub and supersolutions of these HJB equations, ensuring that the value function is the unique solution of these equations. This comparison principle is obtained by arguments from control theory, which imply that any subsolution (resp. supersolution) is

lower (resp. greater) than the value function.

The articles of Barles, Briani and Chasseigne [19, 20] greatly influenced the results contained in this thesis; in particular, using different tools, we recover similar results. This is the reason for which we explain in detail the methods developed in these articles. In [19], the authors consider an infinite horizon optimal control problem in $\mathbb{R}^n$ whose dynamics and running costs present discontinuities along an interface Γ. They consider the standard regularity assumptions on the dynamics $f_i : \overline{\Omega}_i \times A_i \to \mathbb{R}^n$ $(i = -1, 1)$ and the running costs $\ell_i : \overline{\Omega}_i \times A_i \to \mathbb{R}$ $(i = -1, 1)$ in each half-space Ω_i of $\mathbb{R}^n$, and a strong controllability hypothesis on the whole space $\mathbb{R}^n$. In order to properly define the problem, they introduce a specific dynamics and running cost over the interface. To do so, they use the trajectory/solutions concept of Filippov [54] and use the approach through differential inclusion. As a consequence, they show the existence of trajectories which stay on Γ at least for a while.
The discontinuous viscosity solutions theory developed by Ishii leads to consider the following problem:

$$\begin{cases} \lambda u(x) + H_1(x, Du) = 0 & \text{in } \Omega_1, \\ \lambda u(x) + H_{-1}(x, Du) = 0 & \text{in } \Omega_{-1}, \\ \min \left\{ \lambda u(x) + H_1(x, Du), \lambda u(x) + H_{-1}(x, Du) \right\} \leq 0 & \text{on } \Gamma, \\ \max \left\{ \lambda u(x) + H_1(x, Du), \lambda u(x) + H_{-1}(x, Du) \right\} \geq 0 & \text{on } \Gamma, \end{cases} \tag{2.4}$$

where for $i = -1, 1$, $H_i : \overline{\Omega}_i \times \mathbb{R}^n \to \mathbb{R}$ is the classical Hamiltonians

$$H_i(x, p) := \sup_{\alpha_i \in A_i} \left\{ -f_i(x, \alpha_i) \cdot p - l_i(x, \alpha_i) \right\}. \tag{2.5}$$

One proves that the problem (2.4) has not a unique solution. Nevertheless, the authors identify the functions U^+ and U^- as viscosity solutions of (2.4). Moreover, they are both value functions of a suitable optimal control problems which differ in the strategies used on the interface Γ. The authors call *singular* strategies those which are obtained by using on Γ any convex combination of the dynamics and costs in Ω_1 and Ω_{-1}. While, the *regular* strategies are those for which only push-push convex combinations are allowed. The solution U^- is achieved when taking into account both kind of controlled strategies, while U^+ when only the regular strategies are considered. Afterwards the authors show that U^- also satisfies the tangential inequality

$$\lambda u(x) + H_T(x, D_\Gamma u(x)) \leq 0 \quad \text{on } \Gamma, \tag{2.6}$$

where $H_T : \Gamma \times \mathbb{R}^{n-1} \to \mathbb{R}$ is the tangential Hamiltonian. Then, adding (2.6) to the problem (2.4), not only U^- becomes the unique solution of (2.4)-(2.6) but one obtains a strong comparison result for this new problem (i.e., a comparison result between the subsolutions of (2.4)-(2.6) and the supersolutions of (2.4)). One of the key results to prove the comparison principle is the one according to which for any $\bar{x} \in \Gamma$, a supersolution of (2.4) is both a supersolution of the tangential Hamiltonian

$$\lambda u(x) + H_T(x, D_\Gamma u(x)) \geq 0 \quad \text{on } \Gamma,$$

and super-optimal along certain trajectories located near $\bar{x}$. A direct consequence of the comparison principle is that U^- is the minimal supersolution (and solution) of (2.4). At last Barles et al. demonstrate that U^+ is the maximal subsolution (and solution) of (2.4).

In the work [20] Barles et al. generalize the results obtained in [19]. The methods provided in [19] for the infinite horizon optimal control problems are re-adapt for the finite horizon optimal control problems. Furthermore, the authors extend the results to a more general class of domains assuming a $W^{2,\infty}$-regularity on the interface Γ. At the end, they weaken the global controllability assumption considering only the normal one. These more general settings imply new difficulties. In particular, due to the weakening of the controllability assumption, the value functions U^- and U^+ (achieved as in [19]]) are not a priori continuous. This involves to deal with discontinuous viscosity sub- and supersolutions, which entail more technicalities. The comparison principle is analogue to the one in [19] and implies the continuity and the uniqueness of the value function U^- as solution of the system which appears in the proof of the principle. As before U^- is the continuous minimal supersolution (and solution) of the equivalent equation to (2.4) and U^+ is the continuous maximal subsolution (and solution) of the same equation. In conclusion the author address the question of the stability for the problem associated to U^- and U^+.

The results in [19, 20] concerning the value function U^- can be review in the general framework of stratified domains. This is what Barles and Chasseigne do in [21]. More precisely the aim of the work [21] is to generalize the results of Bressan-Hong [33]. First, contrarily to [33], they use a general approach through differential inclusion and do not start from dynamics and costs defined on the disjoint embedded submanifolds of the whole space $\mathbb{R}^n$. This has the advantage that the global Lipschitz assumption on the dynamics can be reduced to a locally Lipschitz one. A normal controllability assumption similar to the one in [20] is given and a value function U, which can be seen as analogous to the value function U^- in [19, 20] is introduced. The authors prove a general comparison principle so that the function U is well defined. For the proof of comparison, they assume neither the Holder-continuity as in [33] for the value function nor that its restriction to each submanifold is differentiable almost everywhere. Furthermore, the comparison principle holds for upper semi-continuous sub and lower semi-continuous supersolutions whereas in [33] for Lipschitz subsolutions and continuous supersolutions. Finally they provide a stability result allowing the stratification depends on a parameter ε.

We also recall the papers of Rao-Zidani [89] and Rao-Siconolfi-Zidani [88] in the framework of the stratified domains. In [89], the authors study a system of Hamilton-Jacobi-Bellman equations associated to some finite horizon optimal control problem on multi-domains. A multi-domain is a stratified domain for which one distinguishes the submanifolds Ω_i (with dimension n) and the interfaces Γ_j (which are also open embedded manifolds with dimensions strictly smaller than n). The authors are interested in the problems where there are different HJB equations for each Ω_i without specifying the equations at junction points. The above prescribed equations on Ω_i are the following

$$\begin{cases} -\partial_t u(t,x) + H_i(x, Du(t,x)) = 0 & \text{on} \quad (0,T)\times\Omega_i, \\ u(T,x) = \varphi(x) & \text{on} \quad \Omega_i, \end{cases} \tag{2.7}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a given function and $H_i : \overline{\Omega}_i \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$H_i(x,p) = \sup_{q\in F_i(x)} \{-q\cdot p\} \tag{2.8}$$

with $F_i : \overline{\Omega}_i \rightsquigarrow \mathbb{R}^n$ a Lipschitz continuous multifunction (with respect to the Hausdorff metric) with non-empty, convex, and compact values. Taking the Hamiltonians as in (2.8), with dynamics given by the multifunction F_i and running costs zero, permits the authors to deal with a more general class of multi-domains. Z. Rao and H. Zidani address the question to understand which condition should be considered on the interfaces Γ_j in order to get the existence and uniqueness of solution of (2.7). Their approach is completely different from the one used in [19, 20] and seems to be easy to generalize for two or multi-domains problems. They propose a junction condition involving an Hamiltonian H^E, says Essential Hamiltonian, built from a set of essential dynamics F^E which contains the dynamics from each adjacent region to Ω_i pointing inside Ω_i. The concept of Essential Hamiltonian was introduced for the first time by Barnard-Wolenski in [23]. In conclusion the authors consider the following problem

$$\begin{cases} \partial_t u(t,x) + H_i(x, Du(t,x)) = 0 & \text{in } (0,T)\times\Omega_i, \\ \partial_t u(t,x) + H^E(x, Du(t,x)) \geq 0 & \text{in } (0,T)\times\Gamma_j, \\ \partial_t u(t,x) + H^E(x, Du(t,x)) \leq 0 & \text{in } (0,T)\times\Gamma_j, \\ u(T,x) = \varphi(x) & \text{in } \Omega_i. \end{cases} \tag{2.9}$$

They prove that, if the function φ is Lipschitz continuous, then the above system has a unique viscosity solution in the sense of the new definition that they introduce. The definitions of sub- and supersolution proposed here are more different. Indeed, the supersolution inequality involves test function $\phi \in C^1(\mathbb{R}^n)$, while the subsolution inequality uses test functions $\phi \in C^1(\Gamma_j)$. The uniqueness result follows by a comparison principle, valid for continuous sub and supersolutions.

The paper by Rao-Siconolfi-Zidani [88] is devoted to the analysis of an infinite horizon problem on $\mathbb{R}^n$, where the latter is partitioned in two disjoint open sets Ω_{-1}, Ω_1 plus their common boundary, the interface, denoted by $\Gamma \in C^2$. As in Dupuis [52] and [19, 20] the authors define the dynamics and the running cost on Γ through the convex combinations of dynamics and running costs of the adjacent regions. Moreover, in spite of [89], the running cost is not zero and the controllability assumptions are weakened. Indeed, in [89], the authors assume a strong controllability assumption in order to ensure the Lipschitz continuity of the value function. Here the controllability assumptions are located in the points of the interface and they are just of tangential type. This implies that all the subsolutions of HJB are Lipschitz continuous when restricted to the interface. The equation proposed to characterize the value function is the following

$$\begin{cases} \lambda u(x) + H_i(x, Du(x)) = 0 & \text{in } \Omega_i, \\ \lambda u(x) + \max(H_{-1}(x, Du(x)), H_1(x, Du(x))) \geq 0 & \text{in } \Gamma_j, \\ \lambda u(x) + H_\Gamma^T(x, Du(x)) \leq 0 & \text{in } \Gamma_j, \end{cases} \tag{2.10}$$

where H_i is the classical Hamiltonian (2.5) and H_Γ^T is built by selecting the convex combinations of dynamics and costs which allow to stay in Γ. In (2.10) the authors consider the usual Hamiltonian, indicated by Ishii's theory, for the supersolution part on Γ . On the other hand, the sub-solution inequality is less standard because only test functions $\phi \in C^1(\Gamma)$ are taken into account. One proves that the value function of the optimal control problem studied here is the unique solution of (2.10). The uniqueness result follows by a comparison principle obtained as in [89] using the tools of the optimal control. But this principle is more general than the one in [89] because it applies to continuous subsolutions and lower semi-continuous supersolutions.

2.3 Hamilton-Jacobi equations on networks

Several phenomena in physics, chemistry and biology, described by interaction of different media, can be translated into mathematical problems involving differential equations which are defined on so-called ramified spaces. A ramified space is a connected and closed subset of $\mathbb{R}^n$, obtained as a finite union of embedded sub-manifolds whose dimensions are strictly smaller than n. The simplest example of ramified space is the network, i.e. a set of points, called vertices or nodes, that are connected by (one-dimensional) lines called arcs or edges. The definition of ramified space is reminiscent of the one of stratified domain introduced by Bressan and Hong in [33] and also used in the articles [19, 20, 21, 89, 88] recalled in Sect. 2.2. Remember that we speak about stratification of $\mathbb{R}^n$, when $\mathbb{R}^n$ can be decomposed into a finite union of its sub-manifolds. Thus, the ramified spaces are constructed as stratified domains, but not considering the sub-manifolds of dimension n. An immediate consequence of this construction is that a ramified space has empty interior in the $\mathbb{R}^n$ topology and presents generally geometric singularities. In the recent years there was an increasing interest in the investigation of dynamical systems and differential equations on network, for example in connection with problems of data transmission, traffic flows and consensus problems (see for example Garavello-Piccoli [58], Engel et al [53] and Ren-Beard [90]). In the framework of HJB equations on networks if one supposes to take different HJB equations in each edge then passing form an arc to another one through a node the system experiences a discontinuity. Hence one recovers the same issue of the problems with discontinuous Hamiltonians with respect to the state variable: finding a good notion of viscosity solution on the interface that allows to ensure the existence and uniqueness of solution and the stability.

2.3.1 Related literature

There exist several papers concerning control problems with state constrained in closures of open sets (see Soner [92, 93], Capuzzo Dolcetta-Lions [38], Ishii-Koike[72]) and also with empty interior (Frankowska, Plaskacz [55, 56]); while there is to our knowledge much fewer literature on problems on networks: we recall the paper of Camilli and Schieborn [91] devoted to the eikonal equations on topological networks, i.e. graphs embedded in Euclidean space. More precisely, the authors consider the equations of the form

$$H(x, Du(x)) = 0 \quad \text{on } G$$

where G is a network and $H : G \times \mathbb{R} \to \mathbb{R}$ is the Hamiltonian. They suppose that H is continuous with respect to the state variable. Accordingly, one considers test functions that are continuous in G and whose restrictions to each edges is C^1. Using the latter, the authors extend the concept of viscosity solutions to the class of first order Hamilton-Jacobi equations of eikonal type and prove an existence and uniqueness result. The uniqueness relies on a comparison principle inspired by Ishii's classical argument for eikonal equations [70].

In [2] Y. Achdou, F. Camilli, A. Cutrì and N. Tchou characterize the value function of an infinite horizon optimal control problem whose dynamics in $\mathbb{R}^2$ are constrained to a network. Due this constraint, the set of admissible controls depends on the state of the system and drastically changes from a point in the interior of the edges, where only one direction is admissible, to a vertex where the admissible directions are given by all the edges connected to it. The authors, indeed, restrict themselves to a star-shaped network, namely a network given by the union of a finite number of half-line of the plane with a single common point (the junction point). The Hamiltonian is discontinuous at the vertex and suitable assumptions are made to ensure both the continuity with respect to the state variable of the value function and the existence of admissible controls at any point of the network. The authors select continuous test functions in the network whose restrictions to each edge is C^1. Then they give a definition of viscosity solution of the associated Hamilton-Jacobi equation on the network. Via the dynamic programming principle one establishes that the value function is a viscosity solution of the above equation. The uniqueness derives by a comparison principle in which proof the classical doubling technique is still used, but choosing as penalty term a not symmetric generalization of the geodetic distance.

Also in [68] C. Imbert, R. Monneau and H. Zidani restrict themselves to a star-shaped network as in [2]. A particular class of Hamiltonians, independent of the state variable and possibly discontinuous at the junction, is studied. The test functions' space, taken into account to define the viscosity solution, is the no-stationary version of the one in [2]. Existence and uniqueness of the viscosity solution is obtained considering the equivalence between the viscosity supersolution and the super-optimality principle, and by using representation formulas "*à* la Hopf-Lax" for the viscosity subsolutions.

In conclusion Camilli and Marchi have proved in [36] the equivalence among the notions of viscosity solution proposed in [2] and [68], and in the eikonal context also with the one given in Schieborn-Camilli [91].

A general comparison result has finally been obtained in Imbert-Monneau [67] and then in Achdou-Oudet-Tchou[4] through different methods.
In [67] the main assumption is that the Hamiltonian in each arc, say $H_i(x,p)$, is bimonotone, namely non increasing (resp. non decreasing) for p smaller (resp. large) than a given threshold $p_i^0(x)$. Moreover, the authors consider more general transmission conditions than in [2, 68] with an addiction running cost at the junctions. The proof of the comparison principle is based on arguments from the theory of partial differential equations: in particular, when all the Hamiltonians related to the arcs are strictly convex and reach their minimum at $p = 0$, the doubling-variable technique is applied with a suitable test function. Then, in the general case, perturbation arguments are used in order to apply the results proved in the former case.
In [4], the authors are interest to infinite horizon optimal control problems on network in which dynamics and running costs are different on each edge,

and each Hamiltonian is a priori completely independent from each other, as in [67]. The main result is the proof of a general comparison principle only using arguments from the theory of control which were first introduced in Barles-Briani-Chasseigne [19, 20]. The problems studied in [19, 20] have in common with [4] that the data are discontinuous across a low dimensional subregion.

Chapter 3

Hysteresis operators: Delay relay and Play

In this chapter, after defining the concept of hysteresis and its particular properties, we introduce two hysteresis operators, the delay relay in Sect. 3.1.1 and the play operator in Sect. 3.1.2, which will be very useful tools in the next treatment. In particular we will use the delay relay to overcome the problem of the discontinuity along the interface of a network and of a multi-domain (see Chapter 4 and 5 respectively), and to obtain controllability results for a planar scallop swimmer (see Chapter 7). While, the play operator will be applied to the study of stability of a control system affected by a hysteresis phenomena (see Chapter 6, Sect. 6.5).

What is Hysteresis?

We state that an input/output relationship between time-dependent quantities is affected by hysteresis if: (*memory*) the output's value at time t does not only depend on the input's value at the same instant, but also on the whole previous evolution of the input; (*rate-independence*) the value of the output at time t does not depend on the velocity of the input's evolution but only on the sequence of values reached by the input. There are several physical and natural phenomena in which hysteresis occurs; surely the most known is the relationship between magnetic field and magnetization of ferromagnetic material. Others examples are the filtration through porous media, phase transition, superconductivity, shape memory alloys and behavior of thermostats (see Visintin [96] for further details). Another interesting phenomenon, modeled by hysteresis, is the communication delay as in Ceragioli-De Persis-Frasca [41]. One way to represent hysteresis effects is the use of the hysteresis operators. The concept of hysteretic operator is due to Krasnoselskii and Pokrovskii [74] which perform a systematic analysis of the mathematical properties of these operators. Other references on the mathematical aspects of hysteresis, in particular in connection with PDEs and applicative problems are Mayergoyz [82], Visintin [96], Brokate-Sprekels [34], Krejci [75].

3.1 Hysteresis operators

Let $[0,T]$ be a time interval, B a Banach space and A a suitable set of functions depending on time. An operator

$$\mathcal{F} : D \subseteq C^0([0,T]) \times B \to A \qquad (u, w_0) \mapsto \mathcal{F}[u, w_0](\cdot)$$

is said to be an hysteresis operator if the following two properties hold
Casuality: $\forall\ (u, w_0), (v, w_0) \in D$, $\forall t \in [0,T]$

$$\text{If} \quad u = v \text{ in } [0,t] \quad \text{then} \quad \mathcal{F}[u, w_0](t) = \mathcal{F}[v, w_0](t); \tag{3.1}$$

Rate independence: For every $(u, w_0) \in D$ and for every continuous nondecreasing function $\varphi : [0,T] \to [0,T]$

$$(u \circ \varphi, w_0) \in D, \quad \mathcal{F}[u \circ \varphi, w_0] = \mathcal{F}[u, w_0] \circ \varphi. \tag{3.2}$$

Moreover, it is also natural to require the following
Semigroup property: for every $(u, w_0) \in D$, for every $[t_1, t_2] \in]0,T]$, setting $w(t_1) := \mathcal{F}[u, w_0](t_1)$, then

$$\mathcal{F}[u, w_0](t_2) = \mathcal{F}[u(t_1 + \cdot), w(t_1)](t_2 - t_1). \tag{3.3}$$

Now, if the set $A = C^0([0,T])$ then we talk about continuous hysteresis, otherwise discontinuous one. In what follows we study a discontinuous hysteresis operator (the delayed relay) and a continuous one (the play operator). Both satisfy the previous properties, but they differ in the domain of the output w.

3.1.1 The delayed relay

We denote by $BV(0,T)$ the Banach space of functions $[0,T] \to \mathbb{R}$ having finite total variation. For a fixed threshold parameter $\varepsilon > 0$ we introduce the delayed relay operator

$$h_\varepsilon : C^0([0,T]) \times \{-1, 1\} \to BV(0,T) \cap L^\infty(0,T).$$

To better understand the behavior of the delayed relay operator with its switching rules we can observe the figure below.

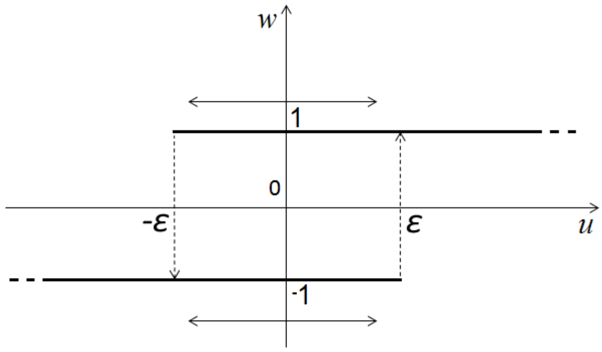


Figure 3.1: Delayed relay operator.

For every input function $u \in C^0([0,T])$ and for every initial state $w_0 \in \{-1,1\}$, the function $w(\cdot) = h_\varepsilon[u,w_0] : [0,T] \to \{-1,1\}$ is defined as follows:

$$w(0) := \begin{cases} -1 & \text{if} \quad u(0) \leq -\varepsilon, \\ w_0 & \text{if} \quad -\varepsilon < u(0) < \varepsilon, \\ 1 & \text{if} \quad u(0) \geq \varepsilon, \end{cases} \tag{3.4}$$

while the definition of $w(t)$ for $t \in]0,T]$ depends on the past evolution of u, on the initial value $w(0)$ and on the switching rule that we take into account. Anyway $w(\cdot)$ exists and it is uniquely defined in $[0,T]$ (see Visintin [96]).
We will consider two different switching rules: the *immediate switching rule* and the *getting over rule*, both satisfying the condition

$$u(t) > \varepsilon \Rightarrow w(t) = 1, \qquad u(t) < -\varepsilon \Rightarrow w(t) = -1. \tag{3.5}$$

We start to define the getting over rule using $w(0)$ as in (3.4). We give the rule for switching from -1 to 1, the other one being analogous. Let us suppose $u(t) \leq \varepsilon$ and $w(t) = -1$.

Getting over rule.
We define the switching time $\tilde{t} := \sup\{\tau \geq t : x(\tau) \leq \varepsilon\} \in [t, +\infty]$. Therefore, at least for a suitable $\delta > 0$, we have that $w(\tau) = -1\ \forall \tau \in [t, \tilde{t}]$, while $w(\tau) = 1\ \forall \tau \in]\tilde{t}, \tilde{t} + \delta]$.
In this case, the switching occurs just after the time when u is on the threshold and it is going to get over. The state $(-\varepsilon, 1)$ and $(\varepsilon, -1)$ are admissible and contained in Figure 3.1.
We now introduce the immediate switching rule.

Immediate switching rule.
If we take $w(0)$ as

$$w(0) := \begin{cases} -1 & \text{if} \quad u(0) < -\varepsilon, \\ w_0 & \text{if} \quad -\varepsilon \leq u(0) \leq \varepsilon, \\ 1 & \text{if} \quad u(0) > \varepsilon, \end{cases} \tag{3.6}$$

then the relay switches from 1 (resp. -1) to -1 (resp. 1) exactly when u reaches the threshold $-\varepsilon$ (resp. ε).
Therefore, for instance, if $u(t) = -\varepsilon$ then $w(t) = -1$ and it will remain constant until u possibly reach ε and in that exact moment w will switch to 1. In this case, the two point $(-\varepsilon, 1)$ and $(\varepsilon, -1)$ are not contained in the graph of Figure 3.1.

Note that, being the input u continuous in every compact interval of the time, the number of switchings is finite (possibly null). Moreover, the use of the getting over rule is linked to the problem with exit time from a closed set as well the immediate switching rule to the exit time problem from an open set.

3.1.2 The play operator

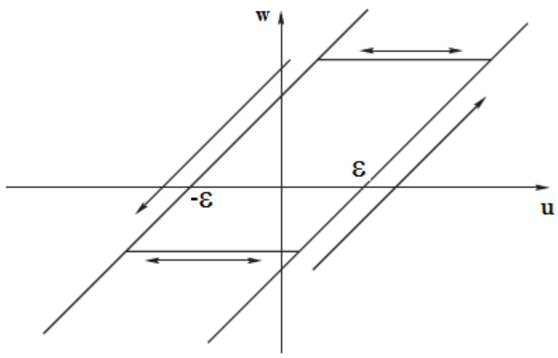


Figure 3.2: Hysteresis play operator

Let $\varepsilon > 0$ be a parameter which characterizes the play operator and define

$$\Omega_\varepsilon := \left\{ (u, w) \in \mathbb{R}^2 \; |u - \varepsilon < w < u + \varepsilon \right\}.$$

We define the play operator by

$$P : D \subseteq C^0([0,T]) \times \mathbb{R} \to C^0([0,T]) \qquad (u, w_0) \mapsto P[u, w_0](\cdot)$$

where $D = \{(u, w_0) \in C^0([0,T]) \times \mathbb{R} | (u(0), w_0) \in \overline{\Omega}_\varepsilon\}$. The behavior of the play operator $w(\cdot) := P[u, w_0](\cdot)$, with its typical hysteresis loops, can be described using Figure 3.2. For instance, supposing that u is piecewise monotone, if $(u(t), w(t)) \in \Omega_\varepsilon$ then w is constant in a neighborhood of t; if $w(t) = u(t) - \varepsilon$ and u is non increasing in $[t, t+\tau]$ (with small τ) then w stays constant in $[t, t+\tau]$; if $w(t) = u(t) - \varepsilon$ and u is non decreasing in $[t, t+\tau]$ then $w = u(t) - \varepsilon$ in $[t, t+\tau]$. A similar argument holds when replacing $u(t) - \varepsilon$ by $u(t) + \varepsilon$. Moreover we have $w(0) = w_0$.
The play operator behavior just explained can be extended to continuous inputs by density (of the piecewise monotone inputs in the continuous ones) and by continuity of the Play operator (see Krasnoselskii [74], Visintin [96]).

Before to conclude the chapter we consider the following result about the existence and uniqueness of a solution for a Cauchy problem with a hysteresis operator.

Consider the following Cauchy problem

$$\begin{cases} x'(t) = f(x(t), w(t)), \\ w(t) = \mathcal{F}[x, w_0](t) \\ x(0) = x_0, \quad w(0) = w_0 \end{cases} \tag{3.7}$$

with $(x_0, w_0) \in \mathbb{R} \times \mathbb{R}$ fixed ($B = \mathbb{R}$), $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a suitably regular bounded function and $\mathcal{F}$ an hysteresis operator. One proves that (3.7) has a unique "mild" solution, namely a solution in the integral sense.
In particular if we identify $\mathcal{F}$ with a delayed relay then:

- for the immediate switching rule it is easy to prove the existence and uniqueness of the solution $(x(\cdot), w(\cdot)) \in C^0([0,T[\times C^0_r([0,T[)$, where $C^0_r([0,T[)$ is the linear space of functions which are continuous on the right in $[0,T[$;

- for the getting over rule we get existence and uniqueness considering the particular definition of solution of (3.7) given in Bagagiolo [8] (a sort of gluing by hand of pieces of trajectory with constant output w) and the semigroup property of the delayed relay.

If instead $\mathcal{F}$ is the Play then the result follows by the Lipschitz continuity of $\mathcal{F}$ as operator from $C^0([0,T]) \to C^0([0,T])$ (see Visintin [96]) being w_0 fixed.

Part I

Hybrid thermostatic approximations of junctions for some optimal control problems on networks and stratified domains

Chapter 4

Optimal control on networks

In this chapter we study a possible approximation of some optimal control problems on networks with junctions. Then, starting from the results in Bagagiolo [8] explained in Sect. 4.1, and after to clarify the basic assumptions on the junction problems (Sect. 4.2), we introduce in Sect. 4.3 the thermostatic approximation of a twofold junction problem. Our approach consists in replacing the junctions with a suitable combinations of delayed thermostats, each of them characterized by a threshold parameter $\varepsilon > 0$ and to study the passage to the limit when ε goes to zero. We characterize the limit function as viscosity solution and maximal subsolution of a suitable Hamilton-Jacobi problem. In Sect. 4.4 the case of a one dimensional threefold junction problem is discussed. Considering still the thermostatic approximation we admits more than a way for passing to the limit, due to the choice of either uniform or non-uniform switching thresholds. In both case we define the value function of the associated limit problem as the unique viscosity solution of a suitable HJ system. Moreover, we provide the corresponding uniqueness results in the sense of maximal subsolution. This study can be found also in Bagagiolo-Maggistro [14].
Finally in Sect. 4.5 we extend the results proven in Sect. 4.4 to a two threefold junctions problem.

4.1 Preliminaries

In Bagagiolo [8], the author studies the dynamic programming method and the corresponding HJB problem for optimal control problems whose dynamics has a thermostatic behavior. This means that the dynamics f (as well as the cost ℓ) besides the control, depends on the state variable $x \in \mathbb{R}$, which evolves with continuity via the equation $x' = f$, and also depends on a discrete variable $i \in \{-1, 1\}$ whose evolution is governed by a delayed thermostatic rule, subject to the evolution of the state x (see Chapter 3, Figure 3.1 for more details). In particular, let us define

$$\begin{aligned} &\mathcal{O}^\varepsilon := (]-\infty,\varepsilon]\times\{-1\})\cup([-\varepsilon,+\infty[\times\{1\}) =: \mathcal{O}^\varepsilon_{-1}\cup\mathcal{O}^\varepsilon_1,\\ &\Omega^\varepsilon_1 = \{x\in\mathbb{R}|x\geq-\varepsilon\}\quad \Omega^\varepsilon_{-1} = \{x\in\mathbb{R}|x\leq\varepsilon\}. \end{aligned} \tag{4.1}$$

Let A be a compact set and let us consider the dynamics $f:\mathcal{O}^\varepsilon\times A\to\mathbb{R}$ and the running cost $\ell:\mathcal{O}^\varepsilon\times A\to[0,+\infty[$ which are continuous, bounded and satisfy the following properties:
$\exists L>0$ such that, for any $(x_j,i,a)\in\mathcal{O}^\varepsilon\times A,\ j=1,2$

$$|f(x_1,i,a)-f(x_2,i,a)|\leq L\,|x_1-x_2|\,; \tag{4.2}$$

there exist a modulus of continuity $\omega_\ell:[0,+\infty[\to[0,+\infty[$ (i.e. continuous, increasing and $\omega_\ell(0)=0$), such that for any $(x_j,i,a)\in\mathcal{O}^\varepsilon\times A,\ j=1,2$

$$|\ell(x_1,i,a)-\ell(x_2,i,a)|\leq\omega_\ell\,|x_1-x_2|\,. \tag{4.3}$$

Finally let $\mathcal{A}$ be the set of measurable controls $\alpha:[0,+\infty[\to A$. The controlled evolution is then given by

$$\begin{cases} x'(t)=f(x(t),i(t),\alpha(t)),\\ i(t)=h_\varepsilon[x](t)\\ x(0)=x_0,\quad i(0)=i_0 \end{cases} \tag{4.4}$$

where $h_\varepsilon[\cdot]$ represents the thermostatic delayed relationship between the input x and the output i. Note that the initial value $i_0\in\{-1,1\}$ must be coherent with the thermostatic relation and that, fixed the thresholds $-\varepsilon,\varepsilon$, for each continuous scalar input $t\mapsto x(t)$, and for each initial output $i_0\in\{-1,1\}$ coherent with $x(0)$, there exist a unique output $t\mapsto i(t)$ satisfying $i(0)=i_0$. The following proposition holds

Proposition 4.1.1. *For every $(x_0,i_0)\in\mathcal{O}^\varepsilon$ and for every $\alpha\in\mathcal{A}$, there exists a unique solution $(x(\cdot),i(\cdot))$ of the system* (4.4) *and $(x(t),i(t))\in\mathcal{O}^\varepsilon$ for all $t\in[0,+\infty[$. In particular $x(\cdot)\in C^0([0,+\infty[)$ and $i(\cdot)\in C_l([0,+\infty[)$, the set of left-continuous functions.*

Proof. See Bagagiolo [8]. □

The infinite horizon optimal control problem is then, given a running cost ℓ and a discount factor $\lambda>0$, the minimization over all measurable control of the cost functional

$$J(x_0,i_0,\alpha)=\int_0^\infty e^{-\lambda t}\ell(x(t),i(t),\alpha(t))dt, \tag{4.5}$$

whose associated value function is

$$V_\varepsilon(x_0,i_0)=\inf_{\mathcal{A}}\int_0^\infty e^{-\lambda t}\ell(x(t),i(t),\alpha(t))dt. \tag{4.6}$$

The uniqueness of the solution of (4.4) implies that the proof of the following result is standard.

Proposition 4.1.2. *(Dynamical Programming Principle). Assume* (4.2) *and* (4.3)*. Then for all $t \geq 0$ and for all $(x_0, i_0) \in \mathcal{O}_\varepsilon$*

$$V_\varepsilon(x_0, i_0) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t e^{-\lambda s} \ell(x(s), i(s), \alpha(s)) ds + e^{-\lambda t} V_\varepsilon(x(t), i(t)) \right\}. \quad (4.7)$$

For every initial point $(x_0, i_0) \in \mathcal{O}^\varepsilon$ and for every $\alpha \in \mathcal{A}$, let us consider the first switching time

$$t_{(x_0,i_0)}(\alpha) := \inf \{t \geq 0 | i(t) \neq i_0\}, \quad (4.8)$$

with the convention $t_{(x_0,i_0)}(\alpha) = +\infty$ if the set in the right-hand side of (4.8) is empty. In the framework of optimal control problems, we can regard (4.8) as the *first exit time* from the connected components of $\mathcal{O}^\varepsilon$ containing the initial point (x_0, i_0).

Proposition 4.1.3. *For every $(x_0, i_0) \in \mathcal{O}^\varepsilon$ let $\varepsilon^{(i_0)}$ be the threshold of the branch of $\mathcal{O}^\varepsilon$ containing i_0 (i.e. $\varepsilon^{(i_0)}$ is $-\varepsilon$ if $i_0 = 1$ and ε if $i_0 = -1$). Then we have*

$$V_\varepsilon(x_0, i_0) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{t_{(x_0,i_0)}} e^{-\lambda s} \ell(x(s), i_0, \alpha(s)) ds + e^{-\lambda t_{(x_0,i_0)}} V_\varepsilon(\varepsilon^{(i_0)}, -i_0) \right\}, \quad (4.9)$$

where if $t_{(x_0,i_0)}(\alpha) = +\infty$, then the second addendum in the right-hand side is zero.

Proof. See [8]. □

Proposition 4.1.4. *The value function V_ε is bounded and uniformly continuous on each of the two connected components of $\mathcal{O}^\varepsilon = \mathcal{O}^\varepsilon_{-1} \cup \mathcal{O}^\varepsilon_1$.*

Proof. See [8]. □

Using the last two Proposition, the authors rewrite the optimal control problem as a coupling of two exit time optimal control problems which mutually exchange their exit-costs. In $\mathcal{O}^\varepsilon_1$ (resp. $\mathcal{O}^\varepsilon_{-1}$), the function $x \mapsto V_\varepsilon(x, 1)$ (resp. $x \mapsto V_\varepsilon(x, -1)$) coincides with the value function of the exit-time optimal control problem on Ω^ε_1 (resp. Ω^ε_{-1}), where the exit-cost on $-\varepsilon$ (resp. on ε) is given by $V_\varepsilon(-\varepsilon, -1)$ (resp. $V_\varepsilon(\varepsilon, 1)$). In other words, under the assumptions (4.2) and (4.3), is proved the following Theorem.

Theorem 4.1.5. *The value function V_ε in* (4.6) *is the unique bounded, continuous function on $\mathcal{O}^\varepsilon$ which solves the following coupled Dirichlet problem in the viscosity sense, where the boundary conditions (the two exit-costs) are* *also in the viscosity sense.*

$$\begin{cases} \lambda V_\varepsilon(x,1) + \sup_{a\in A}\{-f(x,1,a)V_\varepsilon'(x,1) - \ell(x,1,a)\} = 0 \ \textit{in } int(\mathcal{O}_1^\varepsilon) \\ V_\varepsilon(-\varepsilon,1) = V_\varepsilon(-\varepsilon,-1) \\ \lambda V_\varepsilon(x,-1) + \sup_{a\in A}\{-f(x,-1,a)V_\varepsilon'(x,-1) - \ell(x,-1,a)\} = 0 \ \textit{in } int(\mathcal{O}_{-1}^\varepsilon) \\ V_\varepsilon(\varepsilon,-1) = V_\varepsilon(\varepsilon,1) \end{cases} \tag{4.10}$$

where V' denotes the derivative with respect to x.

Proof. See [8]. □

Starting from the last Theorem we study the limit of the value function V_ε and of HJB problem when the thresholds distance ε tends to zero, and hence recovering the junction situation. Before to do this we introduce some basic assumption on the problem with junction.

4.2 Basic assumption on the junction problem

Let the junction be given by a finite number of co-planar half-lines R_i, $i = 1,\dots,n$, originating from the same point O, and we consider the half-lines as closed, that is the point $O \in R_i$ for every i. On each branch R_i we consider a one-dimensional coordinate $x \geq 0$ such that $x(O) = 0$. The state position may be then encoded by the pair (x, i).

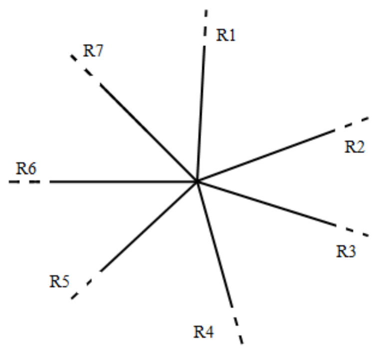


Figure 4.1: A star-shaped network.

We consider a controlled evolution on a star-shaped network (see Figure 4.1), given by the following dynamics. On R_i the system is driven by a continuous and bounded dynamics $f_i : \mathbb{R} \times A \to \mathbb{R}$, where A is compact, with the standard Lipschitz assumption

$\exists L > 0$ such that, for any $x, y \in \mathbb{R}$ and $a \in A$

$$|f_i(x,a) - f_i(y,a)| \leq L|x-y|. \tag{4.11}$$

Controllability: For every i,

$$\exists\ a_i^-, a_i^+ \ \in A \ \text{ s.t. } \quad f_i(0,a_i^-) < 0 < f_i(0,a_i^+) \tag{4.12}$$

The controlled system on the star-shaped network is then

$$\begin{cases} y'(t) = f_j(y(t),\alpha(t)) & \text{for } t>0 \text{ and } y(t)\in R_j \\ y(0) = x \\ x \in R_i \end{cases} \tag{4.13}$$

where $\alpha \in \mathcal{A}$.

To this controlled systems we associate an infinite horizon optimal control problem. For every branch R_i we consider a running cost $\ell_i : \mathbb{R}\times A \to [0,+\infty[$, and the problem is given by the minimization, over all measurable controls $\alpha \in \mathcal{A}$, of the cost functional

$$J(x,i,\alpha) = \int_0^{+\infty} e^{-\lambda t}\ell_j(y(t),\alpha(t))dt. \tag{4.14}$$

In (4.14), $\lambda > 0$ is a fixed discount factor, the trajectory $y(\cdot)$ is the solution of (4.13), and the index j is given by $y(t) \in R_j$. Moreover, for every i, the function $\ell_i : \mathbb{R}\times A \to \mathbb{R}$ is continuous and bounded, and there exists a modulus of continuity ω_ℓ as in (4.3), such that, for any $x, y \in \mathbb{R}$ and $a \in A$ and for any i

$$|\ell_i(x,a) - \ell_i(y,a)| \le \omega_\ell\left(|x-y|\right). \tag{4.15}$$

We finally consider the value function

$$V(x,i) = \inf_{\alpha\in\mathcal{A}} J(x,i,\alpha).$$

Of course, the concept of solution (or trajectory) for the system (4.13) is not a-priori well-posed. When we are on the junction point O, we can choose the index i we prefer, but the existence of the trajectory is not guaranteed, due to possible fast oscillations of the index i (behavior which is linked to the pointing versus of $f_i(O,a)$). Then, considering the delay relay operator (see Section 3.1.1) we give an approximation and the corresponding passage to the limit for such possible oscillating behavior in the context of optimal control.

4.3 A twofold junction problem

In this section we study the simple situation in which the "junction" is given by two closed half-lines entering in the same point $x = 0$ (see Figure 4.2). We define the dynamics $f : \mathbb{R}\times A \to \mathbb{R}$ such that it is $f(x,\cdot) = f_1(x,\cdot)$ (resp. $f(x,\cdot) = f_{-1}(x,\cdot)$) if $x > 0$ (resp. if $x < 0$) , where $f_1 : [0,+\infty[\times A \to \mathbb{R}$, $f_{-1} :]-\infty,0]\times A \to \mathbb{R}$.

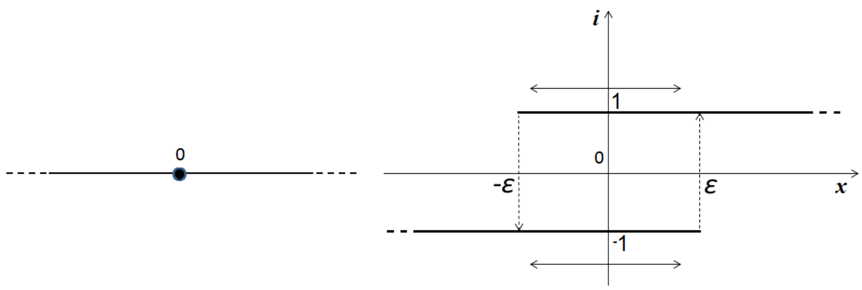


Figure 4.2: The two-fold junction and its thermostatic approximation.

For $\varepsilon > 0$ we approximate the junction problem by a delayed thermostatic problem (see Figure 4.2) with output $i \in \{-1, 1\}$. Still denoting by f_1, f_{-1} two extensions by constancy in the space variable x of the dynamics to $[-\varepsilon, +\infty[\times A$ and to $]-\infty, \varepsilon] \times A$ respectively, we may consider the controlled system

$$\begin{cases} x'(t) = f_{i(t)}(x(t), \alpha(t)), \\ i(t) = h_\varepsilon [x](t) \\ x(0) = x_0, \quad i(0) = i_0 \end{cases} \tag{4.16}$$

Similarly, we extend a running cost ℓ which is given in two different ways, ℓ_1, ℓ_{-1} in the two half-lines.

Let V_ε be the value function of the thermostatic optimal control problem (4.10) with dynamics given by (4.16) and corresponding costs. We define the function

$$\tilde{V}_\varepsilon : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \ \ \tilde{V}_\varepsilon(x) = \begin{cases} V_\varepsilon(x, 1) & x > 0 \\ V_\varepsilon(x, -1) & x < 0. \end{cases}$$

Note that, in general, $V_\varepsilon(0, -1) \neq V_\varepsilon(0, 1)$.

Theorem 4.3.1. *As $\varepsilon \to 0^+$, the sequence of functions $\tilde{V}_\varepsilon$ uniformly converges on $\mathbb{R} \setminus \{0\}$ to a continuous function $\tilde{V}_1, \tilde{V}_{-1}$ respectively. If (4.11) and (4.15) hold, then $\tilde{V}$ continuously extends to $x = 0$ and, if* (4.12) *also holds, it is a viscosity solution of*

$$\begin{cases} \lambda V + H_1(x, V') = 0 \quad \text{for } x > 0 \\ \lambda V + H_{-1}(x, V') = 0 \quad \text{for } x < 0 \\ V(0) = \min\left\{u_0(0), V_{sc(-1)}(0), V_{sc(1)}(0)\right\} \end{cases} \tag{4.17}$$

where H_1, H_{-1} are the Hamiltonians in (4.10), $u_0(0)$ is the convexification

$$u_0(0) = \frac{1}{\lambda} \min_{A_0} \{\mu \ell_{-1}(0, a_{-1}) + (1 - \mu)\ell_1(0, a_1)\} \tag{4.18}$$

with

$$A_0 = \big\{(\mu, a_{-1}, a_1) \in [0,1] \times A \times A : \\ \mu f_{-1}(0, a_{-1}) + (1-\mu) f_1(0, a_1) = 0, f_1(0, a_1) \le 0, f_{-1}(0, a_{-1}) \ge 0\big\} \quad (4.19)$$

and $V_{sc(i)}(0)$ is the value function at $x = 0$ of the state-constraint optimal control problem on the branch i.

Proof. We are going to use the notation in (4.1). We also recall that the state-constraint problem in branch i is the optimal control problem restricted to the branch i and such that, when we are on the point $x = 0$ we can only use controls that make us to not leave the branch. We first prove that V_ε uniformly converges to a continuous function on $\mathbb{R} \setminus \{0\}$. We have some cases and we illustrate some of them.

i) $f_{-1}(0, a) \le 0$ for all $a \in A$. Hence, when starting from a point of $\mathcal{O}^\varepsilon_{-1}$, it is impossible to switch on the other branch $\mathcal{O}^\varepsilon_1$. Hence, $x \to V_\varepsilon(x, -1)$ is the value function of the optimal control problem with dynamics f_{-1} and cost ℓ_{-1} and state-constraint in $\mathcal{O}^\varepsilon_{-1}$, which uniformly converges on $]-\infty, 0]$ to the value function with same dynamics and cost and with state constraints in $]-\infty, 0]$ (note that dynamics and costs are bounded), that is to $V_{sc(-1)}$. On the other branch, being $V_\varepsilon(-\varepsilon, -1)$ convergent to $V_{sc(-1)}(0)$, by standard stability results (applied to the first two lines of (4.10)), we also get the uniform convergence of $V_\varepsilon(\cdot, 1)$ to the unique solution in $[0, +\infty[$ with viscosity boundary datum $V_{sc(-1)}(0)$.

ii) $\exists\ a_{-1}, a_1 \in A$ such that $f_1(0, a_1) < 0 < f_{-1}(0, a_{-1})$. In this case, when ε is sufficiently small, starting from $(\varepsilon, 1)$ (resp. from $(-\varepsilon, -1)$) we can always switch on the other branch, and we can reach $(-\varepsilon, -1)$ (resp. $(\varepsilon, 1)$) in a time interval whose length is infinitesimal as ε. It is then easy to check that the difference $|V_\varepsilon(\varepsilon, 1) - V_\varepsilon(-\varepsilon, -1)|$ (as well as $|V_\varepsilon(0, 1) - V_\varepsilon(0, -1)|$) is also infinitesimal as ε. Moreover, for every pair $(\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 > 0$, $\|V_{\varepsilon_1} - V_{\varepsilon_2}\|$ is also infinitesimal as $\max\{\varepsilon_1, \varepsilon_2\}$. Hence, again from stability results for (4.10), V_ε uniformly converges on $\mathbb{R} \setminus \{0\}$, and also the continuous limit function continuously extends to $x = 0$.
We now define

$$\tilde{V}(x) = \begin{cases} \tilde{V}_1(x) & \text{if } x \ge 0 \\ \tilde{V}_{-1}(x) & \text{if } x \le 0 \end{cases}$$

Now, assuming (4.12) (which of course implies the conditions in ii)), we prove that the continuous limit function $\tilde{V}$ satisfies the third equation of (4.17). Again, we proceed illustrating some cases.

a) $V_{sc(-1)}(0)$ strictly realizes the minimum in (4.17). Then, there exists a measurable control α such that the corresponding trajectory starting from $x = 0$ with dynamics f_{-1} does not exit from $]-\infty, 0]$, and the corresponding cost, with running cost ℓ_{-1}, is strictly less than $u_0(0)$ and $V_{sc(1)}(0)$. Note that, such a control α has exactly the same cost for the thermostatic problem

with initial point $(0,-1)$ (no switchings may occur).
Now, we observe that, for every $(\mu, a_{-1}, a_1) \in A_0$ with $f_{-1}(0,a_{-1}), f_1(0,a_1) \neq 0$, the alternation of the constant controls a_{-1}, a_1 correspondingly to every switching, gives a cost for the thermostatic problem in $(0,-1)$ as well as in $(0,1)$, which, when ε goes to zero, converges to $(\mu\ell_{-1}(0,a_{-1}) + (1-\mu)\ell_1(0,a_1))/\lambda$. Indeed, the condition $\mu f_{-1}(0,a_{-1}) + (1-\mu)f_1(0,a_1) = 0$ implies that $f_{-1}(0,a_{-1}), f_1(0,a_1)$ are in the same (inverse) proportion as μ and $1-\mu$, and the corresponding time-durations for covering the distance 2ε (from one threshold to the other one) are in the same (direct) proportion as μ and $1-\mu$. Hence, the required convergence holds.
From this we get that $V_{sc(-1)}(0) = V_\varepsilon(0,-1)$ and so $\tilde{V}(0) = V_{sc(-1)}(0)$.

b) $u_0(0)$ strictly realizes the minimum. Then, let $(\mu, a_{-1}, a_1) \in A_0$ be such that $\mu\ell_{-1}(0,a_{-1}) + (1-\mu)\ell_1(0,a_1)$ is the minimum in the definition of $u_0(0)$. Again, as in the previous point, we get that a switching trajectory using controls a_{-1} and a_1 is near optimal for V_ε, and then the conclusion. □

Remark 4.3.2. *In this one-dimensional case, Theorem 4.3.1 also proves that $\tilde{V} = U^+$, where U^+ is the value function of the so-called regular problem in Barles-Briani-Chasseigne [19]. In the sequel we are also given a different proof of such an equality where, using the thermostatic approximation, we show that $\tilde{V}$ is the maximal subsolution of a suitable Hamilton-Jacobi problem as in [19], namely next problem (4.20).*

Theorem 4.3.3. *Assume* (4.11), (4.12) *and* (4.15). *The function $\tilde{V}$ is a viscosity solution of the Hamilton-Jacobi-Bellman problem*

$$\begin{cases} \lambda V + H_1(x,\nabla V) = 0 \quad \textit{in } \{x>0\} =: \Omega_1 \\ \lambda V + H_{-1}(x,\nabla V) = 0 \quad \textit{in } \{x<0\} =: \Omega_{-1} \\ \min\{\lambda V + H_1, \lambda V + H_{-1}\} \le 0 \ \textit{on } x=0 \\ \max\{\lambda V + H_1, \lambda V + H_{-1}\} \ge 0 \ \textit{on } x=0. \end{cases} \tag{4.20}$$

We recall that here we mean that $\tilde{V}$ is a subsolution of the first three equations and a supersolution of the first two together with the fourth one according the Definition (1.1.2).

Proof. From Theorem 4.1.5 and Theorem 4.3.1, and by a classical convergent result, we have

$$\lambda\tilde{V}(x) + \sup_{\alpha\in A}\left\{-f_1(x,\alpha)\tilde{V}'(x) - \ell_1(x,\alpha)\right\} = 0 \text{ in } \Omega_1,$$

and similarly for the equation in Ω_{-1}.

We now prove the third equation in (4.20). Let $\varphi \in C^1(\mathbb{R})$ be a test function such that $\tilde{V} - \varphi$ has a strict relative maximum at $x=0$. By uniform convergence, there exists a sequence $x_\varepsilon \in \Omega_1^\varepsilon$ of points of relative maxima for

$V_\varepsilon(\cdot,1)-\varphi$ which converge to $x=0$. We may have two cases: 1) at least for a subsequence, at x_ε the HJB equation satisfied by $V_\varepsilon(\cdot,1)$ has the right sign "$\leq$", 2) it is definitely true that the boundary point $x_\varepsilon=-\varepsilon$ is a strict maximum point and the HJB equation has the wrong sign "$>$". Note that, when x_ε is in the interior of Ω_1^ε, case 1) always occurs. Also note that the boundary of Ω_1^ε, i.e. $x=-\varepsilon$, is also converging to $x=0$.

Case 1). Sending $\varepsilon\to 0$, we get $\lambda\tilde{V}+H_1\leq 0$ in $x=0$ and so the third equation in (4.20).

Case 2). Since the boundary conditions in (4.10) are in the viscosity sense and by virtue of the controllability condition (4.12), we have

$$V_\varepsilon(-\varepsilon,1)=V_\varepsilon(-\varepsilon,-1) \tag{4.21}$$

Now, the same argumentations and cases also hold for the branches Ω_{-1}^ε. If the corresponding case 1) holds, then we get the conclusion as before. Otherwise we have

$$V_\varepsilon(\varepsilon,-1)=V_\varepsilon(\varepsilon,1) \tag{4.22}$$

We now prove that case 2) cannot simultaneously holds in both branches. Indeed, let us observe that $(-\varepsilon,-1)$ is in the interior of Ω_{-1}^ε and $(\varepsilon,1)$ is in the interior of Ω_1^ε, therefore, using (4.22), (4.21), we get the following contradiction which concludes the proof

$$\begin{aligned}&V_\varepsilon(-\varepsilon,-1)-\varphi(-\varepsilon)<V_\varepsilon(\varepsilon,-1)-\varphi(-\varepsilon)=V_\varepsilon(\varepsilon,1)-\varphi(\varepsilon)\\&<V_\varepsilon(-\varepsilon,1)-\varphi(\varepsilon)=V_\varepsilon(-\varepsilon,-1)-\varphi(-\varepsilon)\end{aligned}$$

In order to prove the fourth equation in (4.20), we argue in the same way. □

We now want to prove that $\tilde{V}$ is the maximal subsolution of (4.20). We first prove the following lemma.

Lemma 4.3.4. *Let us assume that $\forall\,\varepsilon>0$ small enough, the optimal strategy for the approximating problem ε, starting by any $(x,1),(x,-1)$ with $x\in[-\varepsilon,\varepsilon]$, is to run through infinitely many switches between the two branches (i.e. no state-constraint behavior is optimal). Let then $(\bar{\mu},\bar{a}_{-1},\bar{a}_1)\in A_0$ be such that $f_{-1}(0,\bar{a}_{-1})>0, f_1(0,\bar{a}_1)<0$, and that*

$$\tilde{V}(0)=u_0(0)=\frac{1}{\lambda}\{\bar{\mu}\ell_{-1}(0,\bar{a}_{-1})+(1-\bar{\mu})\ell_1(0,\bar{a}_1)\}. \tag{4.23}$$

For every $x\in[-\varepsilon,\varepsilon]$, we consider the two switching trajectories (compare with (4.16)*)*

$$\begin{cases}y'(t)=f_{i(t)}(0,\bar{a}_{i(t)}),\\ i(t)=h_\varepsilon\left[y\right](t)\\ y(0)=x,\quad i(0)=1\end{cases},\qquad\begin{cases}y'(t)=f_{i(t)}(0,\bar{a}_{i(t)}),\\ i(t)=h_\varepsilon\left[y\right](t)\\ y(0)=x,\quad i(0)=-1.\end{cases}$$

Note that the two trajectories on every branch move with constant velocity ($f_1(0,\bar a_1)$ towards left and respectively towards right $f_{-1}(0,\bar a_{-1})$), and they switch infinitely many times. We then consider the corresponding two functions

$$\begin{aligned}\bar V_\varepsilon(x,1) &= \int_0^\infty e^{-\lambda t}\ell_{i(t)}(0,\bar a_{i(t)})dt \quad \text{with} \quad i(0)=1,\\ \bar V_\varepsilon(x,-1) &= \int_0^\infty e^{-\lambda t}\ell_{i(t)}(0,\bar a_{i(t)})dt \quad \text{with} \quad i(0)=-1.\end{aligned} \tag{4.24}$$

Then $\bar V_\varepsilon(\cdot,1)$ and $\bar V_\varepsilon(\cdot,-1)$ are differentiable in $[-\varepsilon,\varepsilon]$ and

$$\sup_{x\in[-\varepsilon,\varepsilon]} |\bar V_\varepsilon'(x,1)-\bar V_\varepsilon'(x,-1)| \to 0 \quad \text{for} \quad \varepsilon\to 0. \tag{4.25}$$

Proof. The derivability comes form the constancy of dynamics and costs. Note that we can rewrite the two functions in (4.24) as

$$\begin{aligned}\bar V_\varepsilon(x,1) &= \int_0^{\frac{x+\varepsilon}{|f_1(0,\bar a_1)|}} e^{-\lambda t}\ell_1(0,\bar a_1)dt + e^{\frac{-\lambda(x+\varepsilon)}{|f_1(0,\bar a_1)|}}\bar V_\varepsilon(-\varepsilon,1),\\ \bar V_\varepsilon(x,-1) &= \int_0^{\frac{\varepsilon-x}{f_{-1}(0,\bar a_{-1})}} e^{-\lambda t}\ell_{-1}(0,\bar a_{-1})dt + e^{\frac{-\lambda(\varepsilon-x)}{f_{-1}(0,\bar a_{-1})}}\bar V_\varepsilon(\varepsilon,-1),\end{aligned} \tag{4.26}$$

where the upper extremal of the integration is the reaching time of the threshold in the corresponding initial branch. Then we have

$$\bar V_\varepsilon(-\varepsilon,1)=\bar V_\varepsilon(-\varepsilon,-1) \quad \text{and} \quad \bar V_\varepsilon(\varepsilon,-1)=\bar V_\varepsilon(\varepsilon,1), \tag{4.27}$$

and by (4.23) for any i, $\lim_{\varepsilon\to0}\bar V_\varepsilon(x,i)=\tilde V(0)=u_0(0)$ uniformly in $x\in[-\varepsilon,\varepsilon]$. A direct calculation gives

$$\begin{aligned}\bar V_\varepsilon'(x,1) &= \frac{1}{|f_1(0,\bar a_1)|}e^{\frac{-\lambda(x+\varepsilon)}{|f_1(0,\bar a_1)|}}\ell_1(0,\bar a_1) - \frac{\lambda e^{\frac{-\lambda(x+\varepsilon)}{|f_1(0,\bar a_1)|}}}{|f_1(0,\bar a_1)|}\bar V_\varepsilon(-\varepsilon,1),\\ \bar V_\varepsilon'(x,-1) &= -\frac{1}{f_{-1}(0,\bar a_{-1})}e^{\frac{-\lambda(\varepsilon-x)}{f_{-1}(0,\bar a_{-1})}}\ell_{-1}(0,\bar a_{-1}) + \frac{\lambda e^{\frac{-\lambda(\varepsilon-x)}{f_{-1}(0,\bar a_{-1})}}}{f_{-1}(0,\bar a_{-1})}\bar V_\varepsilon(\varepsilon,-1).\end{aligned}$$

and then for $\varepsilon\to0$

$$\begin{aligned}\bar V_\varepsilon'(x,1) &\longrightarrow \frac{\bar\mu\big(\ell_1(0,\bar a_1)-\ell_{-1}(0,\bar a_{-1})\big)}{|f_1(0,\bar a_1)|},\\ \bar V_\varepsilon'(x,-1) &\longrightarrow \frac{(\bar\mu-1)\big(\ell_{-1}(0,\bar a_{-1})-\ell_1(0,\bar a_1)\big)}{f_{-1}(0,\bar a_{-1})}.\end{aligned}$$

Recalling the definition of A_0 (4.19), calculating $\overline{\mu}$ we get

$$\begin{aligned}\bar V_\varepsilon'(x,1) &\longrightarrow \frac{\ell_1(0,\bar a_1)-\ell_{-1}(0,\bar a_{-1})}{f_{-1}(0,\bar a_{-1})-f_1(0,\bar a_1)},\\ \bar V_\varepsilon'(x,-1) &\longrightarrow \frac{\ell_1(0,\bar a_1)-\ell_{-1}(0,\bar a_{-1})}{f_{-1}(0,\bar a_{-1})-f_1(0,\bar a_1)}\end{aligned}$$

and then (4.25). □

Comparison result

Theorem 4.3.5. *Let u be a bounded and continuous subsolution of* (4.20)*. Then $u \le \tilde{V}$ in $\mathbb{R}$.*

Proof. We first note that we can assume to be in the situation as in Lemma 4.3.4. Indeed, otherwise in at least one branch $\tilde{V}$ coincides with the corresponding state-constraint value function which, see for example Soner [92], is greater than any subsolution (note that, in general the state-constraint value functions do not satisfy the third line of (4.20)). We then also get $u \le \tilde{V}$ on the other branch.
We assume by contradiction that $\sup_{x\in\mathbb{R}}(u-\tilde{V})(x) > \delta > 0$. If

$$\exists r > 0 | \forall \delta' > 0 \; \exists \; \overline{x} \in [r, +\infty[: \sup_{x\in\mathbb{R}} \big((u-\tilde{V})(x) - (u-\tilde{V})(\overline{x}) \big) \le \delta,$$

then, by Theorem 4.3.3 and known comparison techniques we get a contradiction because, in $]r, +\infty[$, $\tilde{V}$ is a supersolution and u is a subsolution of the same HJB. Similarly for the opposite case $]-\infty, -r[$. Hence we may restrict to the case where $u-\tilde{V}$ has the maximum with respect to r in $x=0$. Since $\bar{V}_\varepsilon(x,i)$ converges to $\tilde{V}(0)$, with $\bar{V}_\varepsilon$ defined in (4.26), then for small ε,

$$u(z^i) - \bar{V}_\varepsilon(z^i, i) = \max_{[-\varepsilon,\varepsilon]} (u(\cdot) - \bar{V}_\varepsilon(\cdot, i)) > \frac{\delta}{2} > 0, \tag{4.28}$$

with $z^i \in [-\varepsilon, \varepsilon]$. If for example $\max(u(\cdot) - \bar{V}_\varepsilon(\cdot, 1))$ is reached in $x = -\varepsilon$ and $\max(u(\cdot) - \bar{V}_\varepsilon(\cdot, -1))$ is reached only in ε, then using (4.27) we get the following contradiction

$$\begin{aligned} & u(-\varepsilon) - \bar{V}_\varepsilon(-\varepsilon, 1) = u(-\varepsilon) - \bar{V}_\varepsilon(-\varepsilon, -1) \\ & < u(\varepsilon) - \bar{V}_\varepsilon(\varepsilon, -1) = u(\varepsilon) - \bar{V}_\varepsilon(\varepsilon, 1). \end{aligned} \tag{4.29}$$

This implies that in at least one branch we can assume z^i not equal to corresponding switching threshold. Then let us assume $z^{-1} \in [-\varepsilon, \varepsilon[$.

We are now comparing u and $\overline{V}_\varepsilon$. We first note that by (4.26) we have for every $i = -1, 1$

$$\lambda \bar{V}_\varepsilon(x,i) - f_i(x, \bar{a}_i) \bar{V}_\varepsilon'(x,i) - \ell_i(x, \bar{a}_i) \ge -O(\varepsilon),$$

in $x \in [-\varepsilon, \varepsilon[$ or in $]-\varepsilon, \varepsilon]$ respectively, where, here and in the sequel $O(\varepsilon)$ is a suitable positive infinitesimal quantity as $\varepsilon \to 0$. Recalling that $\bar{V}_\varepsilon(\cdot, i)$ is derivable in $[-\varepsilon, \varepsilon]$ and recalling the sign of $f_i(0, \bar{a}_i)$ we then get for every $i = -1, 1$

$$\lambda \bar{V}_\varepsilon(x,i) + H_i(x,p) \ge -O(\varepsilon), \tag{4.30}$$

for every $x \in [-\varepsilon, \varepsilon[$ and for every p subgradient in x with respect to $[-\varepsilon, \varepsilon]$ of $\bar{V}_\varepsilon(\cdot, -1)$ (respectively for any $x \in]-\varepsilon, \varepsilon]$ and p subgradient of $\bar{V}_\varepsilon(\cdot, 1)$).

Let $\eta : [-\varepsilon, \varepsilon] \to \mathbb{R}$ be continuous and $c > 0$ such that

$$]x + \xi\eta(x) - \xi c, x + \xi\eta(x) + \xi c[\subseteq]-\varepsilon, \varepsilon[\ \forall x \in [-\varepsilon, \varepsilon], 0 < \xi \leq c. \quad (4.31)$$

For any $0 < \xi \leq c$, we define the function in $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$:

$$\Phi_\xi(x, y) = u(x) - \bar{V}_\varepsilon(y, -1) - \left| \frac{x - y}{\xi} - \eta(z^{-1}) \right|^2 - \left| y - z^{-1} \right|^2 .$$

Let (x_ξ^{-1}, y_ξ^{-1}) be a point of maximum for $\Phi_\xi \in [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$. Recalling $z^{-1} \in [-\varepsilon, \varepsilon[$ by standard estimates (see Soner [92] or Bardi-Capuzzo Dolcetta[16] p. 271) for small ξ we get $x_\xi^{-1} \in]-\varepsilon, \varepsilon[$, $y_\xi^{-1} \in [-\varepsilon, \varepsilon[$ and

$$\frac{x_\xi^{-1} - y_\xi^{-1}}{\xi} \to \eta(z^{-1}) \quad \text{and} \quad x_\xi^{-1}, y_\xi^{-1} \to z^{-1} \quad \text{as} \quad \xi \to 0. \quad (4.32)$$

We have the following possible cases, for a subsequence $\xi \to 0$: (i) $(x_\xi^{-1}, y_\xi^{-1}) \in]-\varepsilon, 0[\times[-\varepsilon, \varepsilon[$; (ii) $x_\xi^{-1} = 0$ and $y_\xi^{-1} \in]-\varepsilon, \varepsilon[$; (iii) $(x_\xi^{-1}, y_\xi^{-1}) \in]0, \varepsilon[\times]-\varepsilon, \varepsilon[$.

Case (i). We get for any small ξ

$$\lambda u(x_\xi^{-1}) + H_{-1}\left(x_\xi^{-1}, \frac{2}{\xi}\left(\frac{x_\xi^{-1} - y_\xi^{-1}}{\xi} - \eta(z^{-1}) \right) \right) \leq 0, \quad (4.33)$$

$$\lambda \bar{V}_\varepsilon(y_\xi^{-1}, -1) + H_{-1}\left(y_\xi^{-1}, \frac{2}{\xi}\left(\frac{x_\xi^{-1} - y_\xi^{-1}}{\xi} - \eta(z^{-1}) \right) + 2(z^{-1} - y_\xi^{-1}) \right) \geq -O(\varepsilon). \quad (4.34)$$

and we conclude in the standard way getting the contradiction to (4.28) first sending $\xi \to 0$ and then $\varepsilon \to 0$.

Case (ii). By $x_\xi^{-1} = 0$ we have that

$$\min\left\{ \lambda u(0) + H_1\left(0, \frac{2}{\xi}\left(\frac{-y_\xi^{-1}}{\xi} - \eta(z^{-1}) \right) \right), \lambda u(0) + H_{-1}\left(0, \frac{2}{\xi}\left(\frac{-y_\xi^{-1}}{\xi} - \eta(z^{-1}) \right) \right) \right\} \leq 0. \quad (4.35)$$

If $\lambda u(0) + H_{-1}\left(0, \frac{2}{\xi}\left(\frac{-y_\xi^{-1}}{\xi} - \eta(z^{-1}) \right) \right) \leq 0$ for a subsequence ξ tends to 0 we conclude as in case (i). Otherwise, we have

$$\lambda u(0) + H_1\left(0, \frac{2}{\xi}\left(\frac{-y_\xi^{-1}}{\xi} - \eta(z^{-1}) \right) \right) \leq 0. \quad (4.36)$$

The inequality (4.34) and (4.36) cannot be compared because they have different Hamiltonians. However, noting that $y_\xi^{-1} \in]-\varepsilon, \varepsilon[$, we have

$$\left(\bar{V}_\varepsilon(y_\xi^{-1}, -1) \right)' = \frac{2}{\xi}\left(\frac{x_\xi^{-1} - y_\xi^{-1}}{\xi} - \eta(z^{-1}) \right) + 2(z^{-1} - y_\xi^{-1}).$$

By (4.25), we have

$$\begin{aligned}\bar V_\varepsilon(y_\xi^{-1},1) &= \bar V_\varepsilon(y_\xi^{-1},-1) + O(\varepsilon),\\ \Big(\bar V_\varepsilon(y_\xi^{-1},1)\Big)' &= \Big(\bar V_\varepsilon(y_\xi^{-1},-1)\Big)' + O(\varepsilon),\end{aligned}$$

and using (4.30) in y_ξ^{-1} for $i=1$, we get

$$\lambda \bar V_\varepsilon(y_\xi^{-1},1) + H_1\left(y_\xi^{-1}, \frac{2}{\xi}\left(\frac{-y_\xi^{-1}}{\xi} - \eta(z^{-1})\right) + 2(z^{-1} - y_\xi^{-1})\right) \geq -O(\varepsilon). \quad (4.37)$$

Then using(4.36) and (4.37) we may conclude in standard way obtaining a contradiction as in the case (i).

Case (iii). For $x_\xi^{-1} \in]0,\varepsilon[$ we have

$$\lambda u(x_\xi^{-1}) + H_1\left(x_\xi^{-1}, \frac{2}{\xi}\left(\frac{x_\xi^{-1} - y_\xi^{-1}}{\xi} - \eta(z^{-1})\right)\right) \leq 0 \qquad (4.38)$$

that cannot be compared with (4.34). Being also in this case $y_\xi^{-1} \in [-\varepsilon,\varepsilon[$, we conclude as in the case before. $\square$

4.4 A threefold junction problem

In the previous section we have considered the case where the junction is given by a line divided into two half-line by a point. Here we consider a junction given by three half-lines entering the same point (see Figure 4.3). In this case we have three labels $\{1,2,3\}$, one for every half-line R_1, R_2, R_3, that we identify with the labeled half-line $R_i = [0,+\infty[\times\{i\}$. We also consider the controlled dynamics $f_i : R_i \times A \to \mathbb{R}$ and the running costs $\ell_i : R_i \times A \to [0,+\infty[$. We approximate these triple discontinuity by a thermostatic-type combination in the following way. We extend f_i and ℓ_i to $[-\varepsilon_i,+\infty[\times\{i\}\times A$, where all thresholds ε_i are greater than 0 and not necessarily the same for every i.

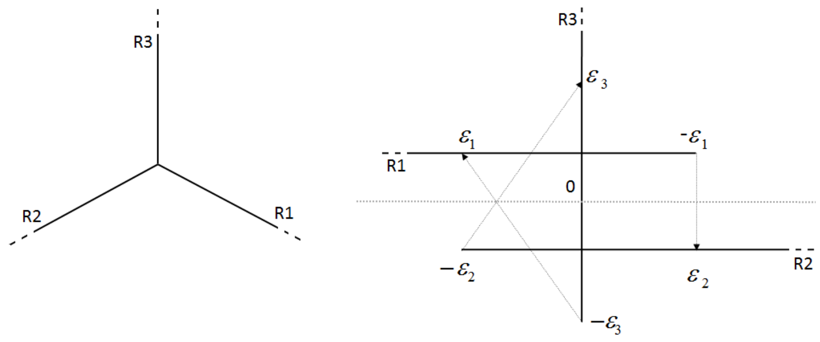


Figure 4.3: The threefold junctions and its thermostatic-type approximation.

The thermostatic controlled dynamics is given by

$$\begin{cases} x'(t) = f_{i(t)}(x(t), \alpha(t)), \\ i(t) = \tilde{h}[x](t), \\ i(0) = i_0 \in \{1, 2, 3\}, \\ x(0) = x_0 \in \ [-\varepsilon_{i(0)}, +\infty[, \end{cases} \tag{4.39}$$

where $\tilde{h}[x](t)$ is the delayed thermostatic rules as shown in Figure 4.3. In this thermostatic representation, denoting by $R_{\varepsilon_i} := [-\varepsilon_i, +\infty[\times\{i\}$ (and by $int(R_{\varepsilon_i}) =]-\varepsilon_i, +\infty[\times\{i\}$), we can only switch from R_{ε_1} to R_{ε_2}, from R_{ε_2} to R_{ε_3} and from R_{ε_3} to R_{ε_1}. This is an arbitrary choice, because when we are in the junction-point, we can in general switch in anyone of the other branches. However, we will recover this kind of behavior in the limit procedure. Moreover, in the switching rule given by $\tilde{h}$, also the variable x is subject to a discontinuity at the switching instant unlike the twofold case in previous section (see Figure 4.3 and also note in the thermostat, the branch R_{ε_1} is oriented in the opposite way with respect to the standard one). For every $i_0 \in \{1, 2, 3\}$ and $\forall x_0 \in \ [-\varepsilon_{i(0)}, +\infty[$ we consider the value function

$$V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_0, i_0) = \inf_{\alpha\in\mathcal{A}} \int_0^\infty e^{-\lambda t} \ell_{i(t)}(x(t), \alpha(t))dt, \tag{4.40}$$

and we also have for every $i = 1, 2, 3$ the Hamiltonians

$$H_i(x, p) = \sup_{a\in A} \{-f_i(x, a) \cdot p - \ell_i(x, a)\}. \tag{4.41}$$

where we drop the index i in the entries of f_i, ℓ_i and consequently in H_i. We will sometimes use this simplification of the notation in the sequel too, without remember it.

The continuity of the value function (4.40) comes from the controllability (4.12), regularity ((4.11) and (4.15)) and by the same procedures used in

Bagagiolo [9]. Moreover, as in Proposition 4.1.1 the system (4.39) admits a unique solution $(x(\cdot), i(\cdot)) \in \mathcal{O} = R_{\varepsilon_1} \cup R_{\varepsilon_2} \cup R_{\varepsilon_3}$ and the function $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ (4.40) satisfies the Dynamic Programming Principle (DPP).

Proposition 4.4.1. *For all $t \geq 0$ and for all $(x_0, i_0) \in \mathcal{O}$*

$$V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_0, i_0) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t e^{-\lambda s} \ell_{i(s)}(x(s), \alpha(s)) ds + e^{-\lambda t} V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x(t), i(t)) \right\}.$$

Proof. The proof is standard and uses the semigroup property of the trajectories and the fact that translating a measurable control you still get a measurable control. □

Considering the first switching time as in (4.8) and the fact that it can be regarded as the first exit time from the connected components of $\mathcal{O}$, a similar result to Proposition 4.1.3 holds

Proposition 4.4.2. *For every $(x_0, i_0) \in \mathcal{O}$ let ε_{i_0} be the threshold of the branch of $\mathcal{O}$ containing i_0 (i.e. ε_{i_0} is $-\varepsilon_1$ if $i_0 = 1$, $-\varepsilon_2$ if $i_0 = 2$ and $-\varepsilon_3$ if $i_0 = 3$). Then we have*

$$V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_0, i_0) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{t_{(x_0,i_0)}(\alpha)} e^{-\lambda s} \ell_{i_0}(x(s), \alpha(s)) ds \right. \\ \left. + e^{-\lambda t_{(x_0,i_0)}(\alpha)} V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(-\varepsilon_{i_{0_s}}, i_{0_s}) \right\} \tag{4.42}$$

where i_{0_s} is the next value to the output i_0 and $\varepsilon_{i_{0_s}}$ the relative threshold.

Proof. Let us suppose for instance $i_0 = 1$ and hence $x_0 \in [-\varepsilon_1, +\infty[$. If $t_{(x_0,i_0)}(\alpha) = +\infty$ then (4.42) is just the definition of $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$. If instead $t_{(x_0,i_0)}(\alpha) < +\infty$, by (4.12) there exists $\overline{a} \in A$ such that $f_1(-\varepsilon_1, \overline{a}) < 0$ and since still by (4.12) follows that $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(-\varepsilon_1, 1) \leq V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(\varepsilon_2, 2)$, applying DPP (Proposition 4.4.1), we have the "$\leq$" inequality in (4.42).
To prove the other inequality, let us take $\rho > 0$ and $\widetilde{\alpha}$ such that $J(x_0, 1, \widetilde{\alpha}) - \rho \leq V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_0, 1)$. We can suppose $\bar{t} := t_{(x_0,1)}(\widetilde{\alpha}) < +\infty$ (the other case is easier). Take $\delta > 0$ small enough such that every trajectory starting from $(\varepsilon_2, 2)$ does not switch in the time interval $[0, \delta]$. Let $(x_{0_\delta}, 2)$ be the point reached after the time $\bar{t} + \delta$ and note that $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_{0_\delta}, 2) \to V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(\varepsilon_2, 2)$ as $\delta \to 0$. Let us denote the integral in (4.42) by I_1. We have

$$\begin{aligned} & V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_0, 1) \geq J(x_0, 1, \widetilde{\alpha}) - \rho \geq \\ & I_1 + e^{-\lambda(\bar{t}+\delta)} J(x_{0_\delta}, 2, \widetilde{\alpha}(\cdot + \bar{t} + \delta)) - \rho \geq \\ & I_1 + e^{-\lambda(\bar{t}+\delta)} V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_{0_\delta}, 2) - \rho. \end{aligned} \tag{4.43}$$

We pass to the limit as $\delta \to 0$ in (4.43) and conclude by the arbitrariness of ρ. □

Proposition 4.4.3. *The value function $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is bounded and uniformly continuous on each of the three connected components of $\mathcal{O} = R_{\varepsilon_1} \cup R_{\varepsilon_2} \cup R_{\varepsilon_3}$.*

Proof. From Proposition 4.4.2, in each connected component R_{ε_i}, $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is the value function of the exit time problem from R_{ε_i} with exit cost $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(-\varepsilon_{i_{0_s}}, i_{0_s})$. Since by controllability (4.12) $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(-\varepsilon_{i_{0_s}}, i_{0_s})$ is bounded and continuous on the boundary points, we conclude applying the result of Bardi-Capuzzo Dolcetta [16], Chapter IV, Section 3. $\square$

Proposition 4.4.4. *For any choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ the value function $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is a bounded and uniformly continuous solution of the following problem for Hamilton-Jacobi equations*

$$\begin{cases} \text{for every } i_0 \in \{1,2,3\}, V_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \text{ is the unique bounded} \\ \text{and uniformly continuous viscosity solution} \\ \text{of the following Dirichlet problem in } R_{\varepsilon_i}, \\ \text{with boundary conditions in viscosity sense} \\ \begin{cases} \lambda V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_0, i_0) + H_i\Big(x, V'_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x_0, i_0)\Big) = 0 \quad \text{in } int(R_{\varepsilon_i}), \\ V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(\varepsilon_{i_0}, i_0) = V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(-\varepsilon_{i_{0_s}}, i_{0_s}). \end{cases} \end{cases} \tag{4.44}$$

Proof. The boundedness and the uniformly continuity come from Proposition 4.4.2 and Proposition 4.4.3. Moreover $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is a solution of (4.44) because in each branch, by virtue of (4.42), is the value function of the exit time problem with f_{i_0}, ℓ_{i_0} and exit cost $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(\cdot, i_{0_s})$. Being the latter bounded and continuous on the boundary point, by Proposition 1.2.16 follows that $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is solution of the system in (4.44) on each branch. The uniqueness come from the results of Chapter V of Bardi-Capuzzo Dolcetta [16] (see in particular Theorem 4.20). $\square$

In the sequel, when $i_0 \in \{1,2,3\}$ is fixed, we will denote the Dirichlet problem in (4.44) by $(4.44)_{i_0}$.

Proposition 4.4.5. *For any choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ the value function $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ of the switching three-thermostatic optimal control problem is the only bounded and continuous function on $\mathcal{O}$ which is solution of* (4.44).

Proof. The boundedness and the continuity come from Proposition 4.4.3. The fact that $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is a viscosity solution of (4.44) follows by Proposition 4.4.4. Regarding the uniqueness we prove that every solution of (4.44) is a fixed point of a contraction map $\mathcal{G} : BC(\mathcal{O}) \to BC(\mathcal{O})$, where $BC(\mathcal{O})$ is the space of the real bounded and continuous function on $\mathcal{O}$. By the completeness, we provide the uniqueness.

Considering $BC(\mathcal{O}) = BC(R_{\varepsilon_1}) \times BC(R_{\varepsilon_2}) \times BC(R_{\varepsilon_3})$, for every $c \geq 0$ and for every $i_0 \in \{1,2,3\}$, let $z_c^{(i_0)}$ be the solution of the correspondent

Hamilton-Jacobi equation $(4.44)_{i_0}$ with boundary datum c. Hence, for each $(\xi,\eta,\sigma)\in BC(\mathcal{O})$ we define

$$\mathcal{G}(\xi,\eta,\sigma) := \left(z^{(1)}_{\left(z^{(2)}_{\xi(-\varepsilon_2)}(\varepsilon_2)\right)}(\cdot), z^{(2)}_{\left(z^{(3)}_{\eta(-\varepsilon_3)}(\varepsilon_3)\right)}(\cdot), z^{(3)}_{\left(z^{(1)}_{\sigma(-\varepsilon_1)}(\varepsilon_1)\right)}(\cdot)\right).$$

This means that, for instance, the first component of $\mathcal{G}(\xi,\eta,\sigma)$ is the solution on the branch R_{ε_1} with boundary datum equal to the value on ε_2 of the solution on the branch R_{ε_2} with boundary datum equal to $\xi(-\varepsilon_2)$. By the uniqueness of the solution in any branch, it is easy to see that every solution of (4.44) is a fixed point of $\mathcal{G}$. Then for every $(\xi,\eta,\sigma),(\widehat{\xi},\widehat{\eta},\widehat{\sigma})\in BC(\mathcal{O})$, for the first component of $\mathcal{G}$ we have

$$\begin{aligned}&\|(\mathcal{G}(\xi,\eta,\sigma))_1-(\mathcal{G}(\widehat{\xi},\widehat{\eta},\widehat{\sigma}))_1\|_\infty \le |z^{(2)}_{\xi(-\varepsilon_2)}(\varepsilon_2)-z^{(2)}_{\widehat{\xi}(-\varepsilon_2)}(\varepsilon_2)|\\ &\le e^{\frac{-\lambda(2\varepsilon_2)}{M}}|\xi(-\varepsilon_2)-\widehat{\xi}(-\varepsilon_2)| \le e^{\frac{-\lambda(2\varepsilon_2)}{M}}\|\xi-\widehat{\xi}\|_\infty,\end{aligned}$$

with M the bound of the running cost ℓ_i. A similar inequality holds for the others components of $\mathcal{G}$. Since $\lambda>0$ we get the conclusion. □

We now study two different thermostatic problem: the first in which we take into account uniform switching thresholds and the second with non-uniform switching thresholds.

4.4.1 Uniform switching thresholds

In this subsection we consider the thresholds $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(\varepsilon,\varepsilon,\varepsilon)$.

Looking to the twofold junction it is easy to see that the convexification parameters $\mu, 1-\mu$ are given by the ratio between the time spent using $f_i(0,a_i)$ to go from a threshold to the other one (namely $2\varepsilon/f_i(0,a_i)$)and the total time to perform a complete switching. Coherently to that, when for $f_1,f_2,f_3<0$ (here we drop the entries in the dynamics), namely when we perform the whole cycle, the right convex parameters to be considered are

$$\mu_1=\frac{f_2f_3}{f_2f_3+f_1f_3+f_1f_2},\ \mu_2=\frac{f_1f_3}{f_2f_3+f_1f_3+f_1f_2},\ \mu_3=\frac{f_1f_2}{f_2f_3+f_1f_3+f_1f_2}. \tag{4.45}$$

Moreover $(\mu_1,\mu_2,\mu_3)\in[0,1]^3$ and $\sum_{i=1}^3\mu_i=1$. Observe that now we have not anymore the interpretation as balance of forces, indeed in general $\sum_{i=1}^3\mu_if_i(0,a_i)\neq 0$, regardless to our choice of the signs of the branches R_i and dynamics f_i. We also note that (4.45) is meaningful with the same interpretation when at most one f_i is null, in which case we definitely remain in the corresponding branch. In order to identify the limit optimal control problem when $\varepsilon\to 0$ we start to define its controlled dynamics. In particular, calling $TR=R_1\cup R_2\cup R_3$, we have that if $(x,i)\in TR$, with $x\neq 0$ then the dynamics is the usual $f_i(x,a_i)$ with $a_i\in A$. If instead $x=0$, being $(0,i)=(0,j)$ for $i,j\in\{1,2,3\}, i\neq j$, we can either choose any dynamics makes us to stay inside a single branch R_i or we may rest at zero "formally" using any combination $\sum_{i=1}^{3} \mu_i f_i(0, a_i)$ with $f_i(0, a_i)$ and μ_i as before. More specifically, we define the set of controls in the junction point

$$A(0) = A_0 \cup \widetilde{A}$$

with (note that in $\widetilde{A}$ the index i is also at disposal)

$$\begin{aligned} A_0 &= \{(a_1, a_2, a_3) \in A^3 |\ f_i(0, a_i) \leq 0 \text{ with at most one equal to } 0\}, \\ \widetilde{A} &= \{(a, i) \in A \times \{1, 2, 3\} |\ f_i(0, a) \geq 0\ \}. \end{aligned}$$

Then, calling $\hat{a}$ the generic element of $A(0)$ we define

$$f_0(0, \hat{a}) = \begin{cases} f_i(0, a) & \text{if } \hat{a} \in \widetilde{A}, \\ 0 & \text{if } \hat{a} \in A_0. \end{cases}$$

With the same arguments, if $(x, i) \in TR$ and $x \neq 0$ then the running cost is $\ell_i(x, a_i)$ with $a_i \in A$, otherwise we define

$$\ell_0(0, \hat{a}) = \begin{cases} \ell_i(0, a) & \text{if } \hat{a} \in \widetilde{A}, \\ \mu_1 \ell_1(0, a_1) + \mu_2 \ell_2(0, a_2) + \mu_3 \ell_3(0, a_3) & \text{if } \hat{a} \in A_0. \end{cases}$$

The quadruples $f = (f_1, f_2, f_3, f_0)$ and $\ell = (\ell_1, \ell_2, \ell_3, \ell_0)$ then define the threefold junction optimal control problem. In particular given an initial state $(x_0, i_0) \in TR$ and a measurable control $\alpha(t) \in A \cup A(0)$ we consider a possible admissible trajectory in TR whose evolution, denoted by $(x(t), i(t))$, is such that $i(t)$ remains constant whenever $x(t) > 0$ and $x(t)$ evolves with dynamics described above. Let us note that given an initial state, the set of measurable controls for which there exists a unique admissible trajectory is not empty and we denote it by $\mathcal{A}_{(x_0, i_0)}$. We then consider an infinite horizon problem with a discount factor $\lambda > 0$ given by

$$J(x_0, i_0, \alpha) = \int_0^{+\infty} e^{-\lambda t} \ell(x(t), i(t), \alpha(t)) dt,$$

where ℓ is the running cost described above and the corresponding value function is

$$V(x_0, i_0) = \inf_{\alpha \in \mathcal{A}_{(x_0, i_0)}} J(x_0, i_0, \alpha). \tag{4.46}$$

In the sequel when $x = 0$ we will drop the index i. Note that if we remain in $x = 0$ for all the time using controls in A_0 the best cost is given by

$$u_{1,2,3}(0) = \frac{1}{\lambda} \inf_{A_0} \{\mu_1 \ell_1(0, a_1) + \mu_2 \ell_2(0, a_2) + \mu_3 \ell_3(0, a_3)\}. \tag{4.47}$$

Remark 4.4.6. *Note that in general A_0 is not compact. However, if $(a_1^k, a_2^k, a_3^k) \in A_0$ is a minimizing sequence for $u_{1,2,3}(0)$ converging to $(\bar{a}_1, \bar{a}_2, \bar{a}_3) \notin A_0$ we have that the quantity inside the brachet in (4.47) loses meaning but we still have the inequality*

$$\lim_{k\to\infty} \left\{\mu_1^k \ell_1(0, a_1^k) + \mu_2^k \ell_2(0, a_2^k) + \mu_3^k \ell_3(0, a_3^k)\right\} \geq \min\{\ell_i(0, \bar{a}_i) | f_i(0, \bar{a}_i) = 0\}.$$

and hence we can always detect an optimal behavior among the ones making as remaining at $x = 0$.

We now have all the ingredients to characterize the value function (4.46).

Theorem 4.4.7. *Assume (4.11), (4.12) and (4.15). Then, V is continuous on TR. Moreover when $x = 0$,*

$$V(0) = \min\left\{u_{1,2,3}(0), V_{sc(1)}(0), V_{sc(2)}(0), V_{sc(3)}(0)\right\}, \tag{4.48}$$

where $V_{sc(i)}(0)$ is the value function at $x = 0$ of the state- constraint optimal control problem on R_i. Therefore
i) if $V(0) = u_{1,2,3}(0)$, then V is the unique bounded and continuous solution of the three problems (one for every $i \in \{1, 2, 3\}$)

$$\begin{cases} \lambda u + H_i(x, u') = 0 & \text{in } int(R_i) \\ u(0) = u_{1,2,3}(0) \end{cases} \tag{4.49}$$

ii) if $V(0) = V_{sc(i)}(0)$, for some $i = 1, 2, 3$, then V satisfies: $V = V_{sc(i)}$ in R_i, and uniquely solves (for every $j \in \{1, 2, 3\} \setminus \{i\}$)

$$\begin{cases} \lambda u + H_j(x, u') = 0 & \text{in } int(R_j) \\ u(0) = V_{sc(i)}(0). \end{cases} \tag{4.50}$$

Proof. The continuity of V come from by controllability (4.12) and regularity (4.11) and (4.15) in a standard way. Moreover, (4.48) comes from (4.46) because the four terms in the minimum are exactly the only allowed behaviors (see also Remark 4.4.6). Finally (4.49) and (4.50) naturally follow from the standard properties of the corresponding Dirichlet problems in the viscosity sense. □

The Theorem 4.4.7 plays a key role in the following result in which we characterize the uniform limit of the value function $V_{\varepsilon,\varepsilon,\varepsilon}$ of the approximating thermostatic problem.

Theorem 4.4.8. *Assume (4.11), (4.12) and (4.15). The value function V (4.46) (also characterizes by Theorem 4.4.7) satisfies*

$$V(x, i) = \lim_{\varepsilon\to 0} V_{\varepsilon,\varepsilon,\varepsilon}(x, i) \quad \forall\ (x, i) \in R_i,\ i = 1, 2, 3. \tag{4.51}$$

where $V_{\varepsilon,\varepsilon,\varepsilon}$ is the value function of the approximating thermostatic problem (4.44) with uniform thresholds $(\varepsilon, \varepsilon, \varepsilon)$, and the convergence is uniform. Moreover, when $x = 0$ the limit is independent from $i = 1, 2, 3$.

Proof. We first prove that (4.51) holds for $x = 0$ (the junction point). The fact that the limit (4.51), whenever it exists, is independent from i when $x = 0$ comes from the controllability hypothesis (4.12) because $|V_{\varepsilon,\varepsilon,\varepsilon}(0, i) - V_{\varepsilon,\varepsilon,\varepsilon}(0, j)|$ is infinitesimal as ε. In the sequel, we drop the symbol i in the expression $V_{\varepsilon,\varepsilon,\varepsilon}(0, i)$.
We prove (4.51) at $x = 0$ for a convergent subsequence still denoted by $(\varepsilon, \varepsilon, \varepsilon)$ which exists because $V_{\varepsilon,\varepsilon,\varepsilon}$ are equi-bounded. The uniqueness of the limit will give the whole (4.51). By contradiction, let us suppose that $V(0) < \lim V_{\varepsilon,\varepsilon,\varepsilon}(0)$. By (4.12), for every $\varepsilon > 0$, we have $V_{\varepsilon,\varepsilon,\varepsilon}(0) \leq V_{sc(i)}(0)$ for every $i = 1, 2, 3$. Hence, the absurd hypothesis implies $V(0) = u_{1,2,3}(0)$ by (4.48). Let us suppose that $(a_1, a_2, a_3) \in A_0$ realizes the minimum in the definition of $u_{1,2,3}(0)$. We then analyze some possible cases, the other ones being similar.

1) $f_1(0, a_1), f_2(0, a_2), f_3(0, a_3) < 0$. Hence, using a suitably switching control between those constant controls, we get $V_{\varepsilon,\varepsilon,\varepsilon}(0)$ is not larger then $u_{1,2,3}(0)$ plus an infinitesimal quantity as $\varepsilon \to 0$, which is a contradiction.

2) $f_1(0, a_1) = 0, f_2(0, a_2), f_3(0, a_3) < 0$. In this case we arrive at R_1 and we stop with $f_1(0, a_1)$ in $x = 0$. Hence, $u_{1,2,3}(0) = \frac{1}{\lambda}\ell_1(0, a_1)$ cannot be lower than $V_{sc(1)}(0)$ which is a contradiction.

If instead not exists the minimizing sequence for (a_1, a_2, a_3) (see Remark 4.4.6) then $u_{1,2,3}(0)$ cannot be better than a state constraints. Then as before, we have again a contradiction.
Now assume $\lim V_{\varepsilon,\varepsilon,\varepsilon}(0) < V(0)$. Let $\delta > 0$ be such that, for ε small enough, it is $V_{\varepsilon,\varepsilon,\varepsilon}(0) + \delta < V(0)$. A measurable control α which almost realizes the optimum (less than $\beta > 0$) for $V_{\varepsilon,\varepsilon,\varepsilon}(0)$ must be such that there are infinitely many switching between all branches R_i^ε (i.e for every i, $f_i(x, \alpha_i) < 0 \ \forall \ x$). Indeed, if it is not the case, then, for at least one branch R_i^ε, the trajectory definitely remains inside it. Hence, for small ε, $V_{\varepsilon,\varepsilon,\varepsilon}(0)$ is almost equal to $V_{sc(i)}(0)$, which is a contradiction. Note that we can limit to consider a piecewise constant control that we call again α since $V_{\varepsilon,\varepsilon,\varepsilon}$ defined both with measurable controls and with piecewise constant controls, satisfies the same problem (4.44) which admits a unique solution. Then, in order to obtain the optimum, on each branch R_i^ε let $x_1^i, \ldots, x_{n^i}^i$ be the points corresponding to the discontinuity instants $t_1^i, \ldots, t_{n^i}^i$ of the control α and let a_j^i be the constant controls $\forall i = 1, 2, 3$, $\forall \ j = 1, \ldots, n^i - 1$. On the assumption that $f_i(0, a_j^i) < 0 \ \forall \ i, j$ we consider the dynamics $f_i(0, a_j^i)$ and the running cost $\ell_i(0, a_j^i)$ on every spatial interval $[x_j^i, x_{j+1}^i]$. Now, for every i we consider

$$\inf_{a \in A} \left\{ \frac{\ell_i(0, a)}{|f_i(0, a)|} \ |f_i(0, a) < 0 \right\}. \tag{4.52}$$

If (4.52) is a minimum for every i obtained in $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ then in each R_i^ε we use constant dynamics $f_i(0, \bar{a}_i)$ and constant running cost $\ell_i(0, \bar{a}_i)$. Therefore

$|J(\cdot,i,\alpha) - J(\cdot,i,\bar{a}_i)| \leq O(\varepsilon)$ and we get

$$\begin{aligned} V_{\varepsilon,\varepsilon,\varepsilon}(0) &\geq J(\cdot,i,\alpha) - \beta \geq J(\cdot,i,\bar{a}_i) - O(\varepsilon) - \beta \\ &\geq u_{1,2,3}(0) - O(\varepsilon) - \beta \geq V(0) - O(\varepsilon) - \beta, \end{aligned} \tag{4.53}$$

that is a contradiction. If, for some i, (4.52) is not a minimum then we can consider the minimizing sequence a_i^k that realizes the infimum less than $O(\frac{1}{k})$. In particular $a_i^k \to \tilde{a}_i \in A$ for $k \to +\infty$ and $f_i(0,a_i^k) \to f_i(0,\tilde{a}_i) = 0$ being $f_i(0,a_i^k) < 0$. However, since the optimal strategy is to switch among the branches, we cannot stop in the branch R_i^ε with dynamics $f_i(0,\tilde{a}_i)$ paying the cost $\ell_i(0,\tilde{a}_i)$. Then, always taking into account that $f_i(0,a_j^i) < 0$ we have

$$\begin{aligned} V_{\varepsilon,\varepsilon,\varepsilon}(0) &\geq J(\cdot,i,\alpha) - \beta \geq J(\cdot,i,a_i^k) - O\left(\frac{1}{k}\right) - O(\varepsilon) - \beta \\ &\geq u_{1,2,3}(0) - O\left(\frac{1}{k}\right) - O(\varepsilon) - \beta \geq V(0) - O\left(\frac{1}{k}\right) - O(\varepsilon) - \beta, \end{aligned} \tag{4.54}$$

which is again a contradiction. Therefore at the end, $V_{\varepsilon,\varepsilon,\varepsilon}(0)$ cannot be less than $V(0) - \delta$ by the definition of $V(0)$. This is a contradiction. Hence we have $\lim V_{\varepsilon,\varepsilon,\varepsilon}(0) = V(0)$. Now note that the equations solved by $V_{\varepsilon,\varepsilon,\varepsilon}$ and by V ((4.44) and (4.49), (4.50) respectively) are the same for all $(x,i) \in int(R_i)$ and the boundary datum converges to $V(0)$. Hence, representing the solutions as the value functions of the corresponding optimal control problems, we get (4.51) and the uniform convergence. □

Now we want to show that V (4.46) is a viscosity solution of a suitable Hamilton-Jacobi problem, namely next problem (4.63). In order to do this, we introduce the test functions for the differential equations on the branches and give the definition of viscosity subsolution and supersolution of (4.63).

From now on we consider the control α piecewise constant as explained in Therem 4.4.8 and we indicate it with α *p.c.*.

Definition 4.4.9. *Let $\varphi : TR \to \mathbb{R}$ be a function such that*

$$\begin{aligned} &\varphi|_{R_i} := \varphi_i : R_i \longrightarrow \mathbb{R} \\ &(x,i) \longmapsto \varphi_i(x,i) \quad \text{if } x \neq 0, \forall i \in \{1,2,3\} \\ &(0,i) \longmapsto \varphi_i(0,i) = \varphi_j(0,j) \ \forall j \in \{1,2,3\} \setminus \{i\}, \end{aligned} \tag{4.55}$$

with $\varphi \in C^0(TR)$ and $\varphi_i \in C^1(R_i)$.

Definition 4.4.10. *A continuous function $u : TR \to \mathbb{R}$ is a viscosity subsolution of (4.63) if for any $(x,i) \in TR$, any φ as in (4.55) such that $u - \varphi$ has a local maximum at (x,i) with respect to TR, then*

$$\begin{aligned} &\lambda u(x,i) + H_i(x,\varphi_i'(x,i)) \leq 0 && (x,i) \in int(R_i), \\ &\min\left\{\lambda u(0,i) + H_i(0,\varphi_i'(0,i)),\ i = 1,2,3\right\} \leq 0 && x = 0. \end{aligned} \tag{4.56}$$

A continuous function $u : TR \to \mathbb{R}$ is a viscosity supersolution of (4.63) *if for any $(x,i) \in TR$, any φ as in* (4.55) *such that $u - \varphi$ has a local minimum at (x,i) with respect to TR, then*

$$\begin{aligned} &\lambda u(x,i) + H_i(x, \varphi_i'(x,i)) \geq 0 && x \in int(R_i), \\ &\max\left\{\lambda u(0,i) + H_i(0, \varphi_i'(0,i)),\ i = 1,2,3\right\} \geq 0 && x = 0. \end{aligned} \tag{4.57}$$

In particular note that if $x = 0$ then the local maximum/minimum $(0,i)$ is with respect to all the three branches and $\varphi_i'(0,i)$ is the right derivative on the branch i, $(\varphi_i')^+$. Since $(0,i) = (0,j)$ for $i,j \in \{1,2,3\}$, $i \neq j$, in the sequel we drop the index i in the pair $(0,i)$.

We will prove the following Theorem using the thermostatic approximation, namely considering the approximating value function $V_{\varepsilon,\varepsilon,\varepsilon}$. Differently from the twofold junction problem in which the index that identifies the branch is included in the sign of x and the test function $\varphi \in C^1(\mathbb{R})$, here, we need to extend the test function φ_i in (4.55) from R_i to R_i^ε. To do that we distinguish the case in which $V - \varphi$ has a local maximum point at $x = 0$ from that where $x = 0$ is a local minimum point, both respect to all three branches.
If $V - \varphi$ has a local maximum point at $x = 0$ then we suppose that

$$\varphi_1'(0)^+ \leq \varphi_2'(0)^+ \leq \varphi_3'(0)^+. \tag{4.58}$$

Our switching sequence is $1 \to 2 \to 3 \to 1$ which is coherent with such an order. If the order is different, then we consider a different switching sequence in the approximating thermostatic ε-problem, still coherent with the order. This is always possible because the limit function V is independent from the switching order of the chosen approximating problem. Then we define

$$\begin{aligned} &\tilde{\varphi}_i : [-\varepsilon, +\infty[\times\{i\} \longrightarrow \mathbb{R}, \\ &\tilde{\varphi}_i = \begin{cases} \varphi_i(x,i) & x \geq 0 \\ \varphi_{i_s}(-x, i_s) & x < 0 \end{cases} \end{aligned} \tag{4.59}$$

for $i = 1,2$ and with i_s the next transition to i. If $i = 3$ we construct $\tilde{\varphi}_3$ in two different way:

- if $\varphi_1'(0)^+ = \varphi_3'(0)^+$ then

$$\tilde{\varphi}_3 = \begin{cases} \varphi_3(x,3) & x \geq 0, \\ \varphi_1(-x,1) & x < 0. \end{cases} \tag{4.60}$$

- if $\varphi_1'(0)^+ < \varphi_3'(0)^+$ then

$$\tilde{\varphi}_3 = \begin{cases} \varphi_3(x,3) & x \geq 0, \\ \varphi_3(-x,3) & x < 0. \end{cases} \tag{4.61}$$

Note that, by the assumption (4.58), the first case gives, $\varphi_1'(0)^+ = \varphi_2'(0)^+ = \varphi_3'(0)^+$, and that the second case gives, at least for small ε, $\tilde{\varphi}_1(\varepsilon, 1) = \varphi_1(\varepsilon, 1) \leq \varphi_3(\varepsilon, 3) = \tilde{\varphi}_3(-\varepsilon, 3)$. Finally note that in both cases we then have $\tilde{\varphi}_1(\varepsilon, 1) \leq \tilde{\varphi}_3(-\varepsilon, 3)$.
If instead $V - \varphi$ has a local minimum point at $x = 0$ then we suppose that

$$\varphi_1'(0)^+ \geq \varphi_2'(0)^+ \geq \varphi_3'(0)^+, \tag{4.62}$$

and that the switching order is the coherent one, as above. In this case we construct $\tilde{\varphi}_3$ as in (4.59), (4.60), (4.61) (with the only difference of the case $\varphi_1'(0)^+ < \varphi_3'(0)^+$ replaced by $\varphi_1'(0)^+ > \varphi_3'(0)^+$). In this case, for at least small ε it is $\tilde{\varphi}_1(\varepsilon, 1) \geq \tilde{\varphi}_3(-\varepsilon, 3)$.

Note that the function $\tilde{\varphi}_i$ is not differentiable in $x = 0$, hence we cannot write a unique HJB equation for the function $V_{\varepsilon,\varepsilon,\varepsilon}$ in the branch R_i^ε. In order to overcome the problem of discontinuity of $\tilde{\varphi}_i'$ in $x = 0$ we interpret the behavior of the dynamic $f_i(x, a_i) < 0$ for $x \in]-\varepsilon, 0[$ as entering in the next branch of the switching rule. More precisely, considering for example the branches R_1^ε and R_2^ε, we define the function $V_{\varepsilon,\varepsilon,\varepsilon}(x, 1) =: \widetilde{V}_{\varepsilon,\varepsilon,\varepsilon}(-x, 2)$, the dynamics $-f_1(x, a) =: \tilde{f}_2(-x, a)$ and the relative running costs $\ell_1(x, a) =: \tilde{\ell}_2(-x, a)$ for $x \in]-\varepsilon, 0[$. In this way, for any $x \in]-\varepsilon, 0[$ a local maximum point of $V_{\varepsilon,\varepsilon,\varepsilon}(\cdot, 1) - \tilde{\varphi}_1(\cdot, 1)$, we get that $V_{\varepsilon,\varepsilon,\varepsilon}(\cdot, 1)$ satisfies

$$\lambda \widetilde{V}_{\varepsilon,\varepsilon,\varepsilon}(-x, 2) + \sup_{a \in A} \left\{ -\tilde{f}_2(-x, a) \varphi_2(-x, 2)' - \tilde{\ell}_2(-x, a) \right\} \leq 0.$$

which is equivalent, for the considerations before, to

$$\lambda V_{\varepsilon,\varepsilon,\varepsilon}(x, 1) + \sup_{a \in A} \left\{ -f_1(x, a) \tilde{\varphi}_1(x, 1)' - \ell_1(x, a) \right\} \leq 0,$$

The same ideas can be applied to the other pairs of branches R_2^ε with R_3^ε and R_3^ε with R_1^ε.

Theorem 4.4.11. *Assume* (4.11), (4.12) *and* (4.15). *The value function* V (4.46) *is a viscosity solution of the Hamilton-Jacobi-Bellman problem*

$$\begin{cases} \lambda V + H_1(x, \nabla V) = 0 & \text{in } int(R_1) \\ \lambda V + H_2(x, \nabla V) = 0 & \text{in } int(R_2) \\ \lambda V + H_3(x, \nabla V) = 0 & \text{in } int(R_3) \\ \min \{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \leq 0 & \text{on } x = 0 \\ \max \{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \geq 0 & \text{on } x = 0 \end{cases} \tag{4.63}$$

Proof. From Proposition 4.4.4, Theorem 4.4.8 and by classical convergence result, we get

$$\lambda V(x, 1) + \sup_{a \in A} \left\{ -f_1(x, a)(V(x, 1))' - \ell_1(x, a) \right\} = 0 \text{ in } int(R_1),$$

and similarly for the equations in $int(R_2)$ and $int(R_3)$.
We now prove the fourth equation in (4.63). Let φ as given in (4.55) such that $V-\varphi$ has a strict relative maximum at $x=0$ with respect all the three branches and consider the assumption (4.58). Let us note that, for every i it is

$$\lambda V(0)+\sup_{a\in A, f_i(0,a)\geq 0}\{-f_i(0,a)\varphi_i'^{+}(0)-\ell_i(0,a)\}\leq 0. \tag{4.64}$$

Indeed, for every $\varepsilon>0$, and for every $t>0$, we have ($V_{\varepsilon,\varepsilon,\varepsilon}$ solves DPP, see Proposition 4.4.1)

$$V_{\varepsilon,\varepsilon,\varepsilon}(0,i)\leq \inf_{\alpha\ p.c.,\ f_i(0,\alpha)\geq 0}\left(\int_0^t e^{-\lambda s}\ell_i(x(s),\alpha(s))ds+e^{-\lambda t}V_{\varepsilon,\varepsilon,\varepsilon}(x(t),i)\right),$$

hence, passing to the limit $\varepsilon\to 0^+$,

$$V(0)\leq \inf_{\alpha\ p.c.,\ f_i(0,\alpha)\geq 0}\left(\int_0^t e^{-\lambda s}\ell_i(x(s),\alpha(s))ds+e^{-\lambda t}V(x(t),i)\right),$$

and finally we get the desired inequality (4.64), being $x=0$ a local maximum for $V-\varphi_i$ with respect to R_i.

Hence, we only need to prove that, with our hypotheses, for at least one i, we get

$$\lambda V(0)+\sup_{a\in A, f_i(0,a)\leq 0}\{-f_i(0,a)\varphi_i'(0)^{+}-\ell_i(0,a)\}\leq 0. \tag{4.65}$$

For each i let (x_ε^i,i) be a sequence of local maximum points for $V_{\varepsilon,\varepsilon,\varepsilon}-\tilde{\varphi}_i$ with respect to R_i^ε convergent to $(0,i)$, with $\tilde{\varphi}_i$ as in (4.59). For each ε, for at least one branch i we may assume $x_\varepsilon^i\neq-\varepsilon$. Indeed, if it is not the case, recalling that by controllability follows that $V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,i)\leq V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,i_s)$, we get the contradiction

$$\begin{aligned}&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,1)-\tilde{\varphi}_1(\varepsilon,1)<V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,1)-\tilde{\varphi}_1(-\varepsilon,1)\leq\\&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,2)-\tilde{\varphi}_2(\varepsilon,2)<V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,2)-\tilde{\varphi}_2(-\varepsilon,2)\leq\\&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,3)-\tilde{\varphi}_3(\varepsilon,3)<V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,3)-\tilde{\varphi}_3(-\varepsilon,3)\leq\\&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,1)-\tilde{\varphi}_1(\varepsilon,1).\end{aligned}$$

Now, let i be such that the correspondent $x_\varepsilon^i\neq-\varepsilon$ for every ε (or at least for a subsequence). If $x_\varepsilon^i>0$ for all ε, in the limit we get

$$\lambda V(0)+\sup_{a\in A}\left\{-f_i(0,a)\varphi_i'(0)^{+}-\ell_i(0,a)\right\}\leq 0,$$

and we get the conclusion.

If $x_\varepsilon^i\in]-\varepsilon,0[$, in the limit we get

$$\lambda V(0)+\sup_{a\in A}\left\{-f_i(0,a)\tilde{\varphi}_i'(0)^{-}-\ell_i(0,a)\right\}\leq 0,$$

and in particular

$$\lambda V(0) + \sup_{a\in A, f_i(0,a)\le 0} \left\{-f_i(0,a)\tilde{\varphi}_i'(0)^- - \ell_i(0,a)\right\} \le 0,$$

where $\tilde{\varphi}_i'(0)^-$ is the left derivative of $\tilde{\varphi}_i$ at $x=0$. Now, if we are in the first case (all the right derivatives coincide) then we have $\tilde{\varphi}_i'(0)^- = \varphi_{i_s}'(0)^+ = \varphi_i'(0)^+$, and hence we get (4.65). If instead $\varphi_1'(0)^+ < \varphi_3'(0)^+$ then if $i=1$ then $i_s = 2$, hence, by our hypotheses, in the inequality above it is

$$-f_i(0,a)\tilde{\varphi}_i'(0)^- = -f_i(0,a)\varphi_{i_s}'(0)^+ \ge -f_i(0,a)\varphi_i'(0)^+,$$

and we conclude. Same arguments if $i=2$ and $i_s=3$. If instead $i=3$, then $\tilde{\varphi}_3(0)^- = \varphi_3'(0)^+$ and we conclude.

Finally, if $x_\varepsilon^i = 0$, then we still get

$$\lambda V(0) + \sup_{a\in A, f_i(0,a)\le 0} \left\{-f_i(0,a)\tilde{\varphi}_i'(0)^- - \ell_i(0,a)\right\} \le 0,$$

and we conclude as before, i.e. studying the two cases as above.

Now we suppose $V - \varphi$ have a local minimum with respect to TR at $(0,i)$ and consider (4.62). We have to prove that, for at least one i, we have

$$\lambda V(0) + \sup_{a\in A}\{-f_i(0,a)\varphi_i'(0)^+ - \ell_i(0,a)\} \ge 0. \tag{4.66}$$

If for some i and for $\varepsilon \to 0^+$, $V_{\varepsilon,\varepsilon,\varepsilon}(\cdot,i)$ coincides with the state-constraint value function on R_i^ε, then $V_{\varepsilon,\varepsilon,\varepsilon}(\cdot,i)$ and $V(\cdot,i)$ coincides on R_i and hence V satisfies the same HJB equation as $V_{\varepsilon,\varepsilon,\varepsilon}$, which is (4.66).

Hence we suppose that no $V_{\varepsilon,\varepsilon,\varepsilon}(\cdot,i)$ coincide with the corresponding state-constraint value function.

For each i let (x_ε^i, i) be a sequence of local minimum points for $V_{\varepsilon,\varepsilon,\varepsilon} - \tilde{\varphi}_i$ with respect to R_ε^i, which converges to $x=0$. In this case we may assume that, for a fixed i, the sequence is such that either $x_\varepsilon^i \ne -\varepsilon$ or $x_\varepsilon^i = -\varepsilon$ but the HJB equation satisfied by $V_{\varepsilon,\varepsilon,\varepsilon}$ has the right sign (≥ 0). Indeed, if it is not the case (i.e. $x_\varepsilon^i = -\varepsilon$ and HJB has the wrong sign), we must have $V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,i) = V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,i_s)$, and hence we get the following contradiction

$$\begin{aligned}
&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,1) - \tilde{\varphi}_1(\varepsilon,1) > V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,1) - \tilde{\varphi}_1(-\varepsilon,1) = \\
&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,2) - \tilde{\varphi}_2(\varepsilon,2) > V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,2) - \tilde{\varphi}_2(-\varepsilon,2) = \\
&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,3) - \tilde{\varphi}_3(\varepsilon,3) > V_{\varepsilon,\varepsilon,\varepsilon}(-\varepsilon,3) - \tilde{\varphi}_3(-\varepsilon,3) \ge \\
&V_{\varepsilon,\varepsilon,\varepsilon}(\varepsilon,1) - \tilde{\varphi}_1(\varepsilon,1).
\end{aligned}$$

If $x_\varepsilon^i > 0$, in the limit we exactly get

$$\lambda V(0) + \sup_{a\in A}\{-f_i(0,a)\varphi_i'(0)^+ - \ell_i(0,a)\} \ge 0.$$

If $x_\varepsilon^i \in [-\varepsilon, 0[$ in the limit we get

$$\lambda V(0) + \sup_{a\in A}\{-f_i(0,a)\tilde{\varphi}_i'(0)^- - \ell_i(0,a)\} \geq 0.$$

If all the right derivatives at $x = 0$ of $\varphi_i(0)$ coincide, then we conclude because $\tilde{\varphi}_i'(0)^- = \varphi_{i_s}'(0)^+ = \varphi_i'(0)^+$. Otherwise, if $i = 3$ then we have $\tilde{\varphi}_i'(0)^- = \varphi_3'(0)^+$ and we conclude; if $i = 1$ or $i = 2$, by the hypotheses on $V_{\varepsilon,\varepsilon,\varepsilon}(\cdot, i)$ not coincident with the state-constraint value function, we get that the supremum above is approximated by controls such that $f_i(0,a) \leq 0$, which means

$$-f_i(0,a)\tilde{\varphi}_i'(0)^- = -f_i(0,a)\tilde{\varphi}_{i_s}(0)^+ \leq -f_i(0,a)\varphi_i(0)^+$$

and we conclude.

If $x_\varepsilon^i = 0$, then in the limit we get (still recalling that $V_{\varepsilon,\varepsilon,\varepsilon}(\cdot, i)$ is not the state-constraint value function)

$$\lambda V(0) + \sup_{a\in A, f_i(0,a)\leq 0}\{-f_i(0,a)\tilde{\varphi}_i'(0)^- - \ell_i(0,a)\} \geq 0$$

and we conclude as before. □

As we have obtain a result of maximal viscosity subsolution for the twofold-junction, now we want to prove that V (4.46) is the maximal subsolution of (4.63).

Let us assume that $\forall\ \varepsilon > 0$ small enough, the optimal strategy for the approximating problem ε, starting by any (x, i) with $x \in [-\varepsilon, \varepsilon]$, is to run through infinitely many switches between the three branches (i.e. no state-constraint behavior is optimal). Let then μ_1, μ_2, μ_3 be as in (4.45) and $(a_1, a_2, a_3) \in A_0$ that realize the minimum in (4.47) such that

$$V(0) = u_{1,2,3}(0) = \frac{1}{\lambda}\{\mu_1\ell_1(0,a_1) + \mu_2\ell_2(0,a_2) + \mu_3\ell_3(0,a_3)\}. \tag{4.67}$$

For every $x \in [0, \varepsilon]$, we define the following functions

$$\begin{aligned}
\bar{V}^\varepsilon(x,1) &= \int_0^{\frac{x}{|f_1(0,a_1)|}} e^{-\lambda t}\ell_1(0,a_1)dt + e^{\frac{-\lambda x}{|f_1(0,a_1)|}}u_{1,2,3}(0),\\
\bar{V}^\varepsilon(x,2) &= \int_0^{\frac{x}{|f_2(0,a_2)|}} e^{-\lambda t}\ell_2(0,a_2)dt + e^{\frac{-\lambda x}{|f_2(0,a_2)|}}u_{1,2,3}(0)\\
\bar{V}^\varepsilon(x,3) &= \int_0^{\frac{x}{|f_3(0,a_3)|}} e^{-\lambda t}\ell_3(0,a_3)dt + e^{\frac{-\lambda x}{|f_3(0,a_3)|}}u_{1,2,3}(0),
\end{aligned} \tag{4.68}$$

where the upper extremal of the integration is the reaching time of the point 0 in the corresponding branch starting from $x \in [0, \varepsilon]$. Note that, for $x \in [0, \varepsilon]$, $V_{\varepsilon,\varepsilon,\varepsilon,}(x, i)$ is not larger then $\bar{V}^\varepsilon(x, i)$ plus an infinitesimal quantity as $\varepsilon \to 0$. The functions in (4.68) are differentiable in $[0, \varepsilon]$ by the constancy of dynamics and running costs. A direct computation gives

$$\begin{aligned}
\bar{V}^{\varepsilon}(x,1)' &= \frac{\ell_1(0,a_1)}{|f_1(0,a_1)|} e^{\frac{-\lambda x}{|f_1(0,a_1)|}} - \frac{\lambda e^{\frac{-\lambda x}{|f_1(0,a_1)|}}}{|f_1(0,a_1)|} u_{1,2,3}(0),\\
\bar{V}^{\varepsilon}(x,2)' &= \frac{\ell_2(0,a_2)}{|f_2(0,a_2)|} e^{\frac{-\lambda x}{|f_2(0,a_2)|}} - \frac{\lambda e^{\frac{-\lambda x}{|f_2(0,a_2)|}}}{|f_2(0,a_2)|} u_{1,2,3}(0),\\
\bar{V}^{\varepsilon}(x,3)' &= \frac{\ell_3(0,a_3)}{|f_3(0,a_3)|} e^{\frac{-\lambda x}{|f_3(0,a_3)|}} - \frac{\lambda e^{\frac{-\lambda x}{|f_3(0,a_3)|}}}{|f_3(0,a_3)|} u_{1,2,3}(0),
\end{aligned}$$

and then for $\varepsilon \to 0$

$$\begin{aligned}
\bar{V}^{\varepsilon}(x,1)' &\longrightarrow \frac{(1-\mu_1)\ell_1(0,a_1) - \mu_2\ell_2(0,a_2) - \mu_3\ell_3(0,a_3)}{|f_1(0,a_1)|},\\
\bar{V}^{\varepsilon}(x,2)' &\longrightarrow \frac{-\mu_1\ell_1(0,a_1) + (1-\mu_2)\ell_2(0,a_2) - \mu_3\ell_3(0,a_3)}{|f_2(0,a_2)|},\\
\bar{V}^{\varepsilon}(x,3)' &\longrightarrow \frac{-\mu_1\ell_1(0,a_1) - \mu_2\ell_2(0,a_2) + (1-\mu_3)\ell_3(0,a_3)}{|f_3(0,a_3)|}.
\end{aligned}$$

Moreover by (4.68) we have for every $i = 1, 2, 3$

$$\lambda \bar{V}^{\varepsilon}(x,i) - f_i(x,a_i)\bar{V}^{\varepsilon}(x,i)' - \ell_i(x,a_i) \geq -O(\varepsilon), \tag{4.69}$$

in $x \in [0, \varepsilon]$. Note that in (4.69) when $x = 0$ we use the right derivative of $\bar{V}^{\varepsilon}(x,i)$ and $\bar{V}^{\varepsilon}(0,i) = u_{1,2,3}(0)$ for every i. Furthermore, by differentiability of $\bar{V}^{\varepsilon}(x,i)$ and recalling the sign of $f_i(0,a_i)$ we then get for every i

$$\lambda \bar{V}^{\varepsilon}(x,i) + H_i(x,q) \geq -O(\varepsilon),$$

for every $x \in [0, \varepsilon]$ and for every q subgradient in x of $\bar{V}^{\varepsilon}(x,i)$.
We now define on $TR \cap \left(\cup_{i=1}^{3}[0,\varepsilon] \times \{i\}\right)$ the function

$$\bar{V}(x) = \begin{cases} \bar{V}^{\varepsilon}(x,i) & \text{if } (x,i) \in int(R_i),\\ u_{1,2,3}(x) & \text{if } x = 0. \end{cases} \tag{4.70}$$

which is in $C^1([0,\varepsilon])$ and that we extend to whole TR maintaining its differentiability. This function will be useful in the comparison result (Theorem 4.4.13) as well as the following

Theorem 4.4.12. *Let $u : [0,1] \to \mathbb{R}$ be a continuous function and exists $\delta > 0$ such that for every $x \in [0,1]$ in which the superdifferential (see Definition 1.1.1) $D^+u(x) \neq \emptyset$ one has that $D^+u(x) \subseteq]-\infty, -\delta]$. Then u is decreasing.*

Proof. Fix $\varepsilon > 0$ such that $|u(0) - u(x)| \leq \frac{\delta}{2}$ for every $x \in [0, \varepsilon]$. We consider the following dynamical system

$$\begin{cases} x'(t) = -1 & t > 0, \\ x(0) = x \in [0, \varepsilon] \end{cases} \tag{4.71}$$

and associate to this exit time optimal control problem. Given $A = [-1, 1]$ the set of constant control we take a running cost $\ell(x, a) = u(0)$ and an exit cost $g(x) = u(0)$. The problem is given by the minimization, over all measurable controls $\alpha \in \mathcal{A}$ of the cost functional

$$J(x, \alpha) = \int_0^{t_x(\alpha)} e^{-\lambda t} u(0) + e^{-\lambda t_x(\alpha)} u(0) \tag{4.72}$$

where $t_x(\alpha)$ is the reaching time of the point $x = 0$. As we can see the cost J is independent from the control α hence taking $\lambda = 1$ we have that (4.72) is equal to $u(0)$. This implies that the associated value function $V(x)$ is equal to $u(0)$ for every $x \in [0, \varepsilon]$. Furthermore $V(x)$ is a solution of

$$V(x) + V'(x) - u(0) = 0 \tag{4.73}$$

and also a supersolution of

$$V(x) + H(x, q) \geq 0,$$

(with $H(x, q) = q - u(0)$), for every $x \in]0, \varepsilon]$ and for every q subgradient in x with respect to $[0, \varepsilon]$ of $V'(x)$. Moreover, for every $p \in D^+u(x) \neq \emptyset$ we have that $u(x) + p \leq 0$ for all $x \in [0, \varepsilon]$ and that u is a subsolution of $u(x) + H(x, p) \leq 0$ for every $x \in]0, \varepsilon[$. These considerations imply that $V \geq u$ in $[0, \varepsilon]$ and that $u(0) \geq u(x)$ $\forall x \in [0, \varepsilon]$. This result holds for every $[a, b] \subseteq [0, 1]$ small enough. □

Comparison result

Theorem 4.4.13. *Let u be a bounded and continuous subsolution of* (4.63). *Then $u \leq V$ in TR.*

Proof. We can assume to be in the settings above for which (4.67) holds. Indeed, otherwise in at least one branch V coincides with the corresponding state-constraint value function which is greater than any subsolution (see Soner [92]). We then also have $u \leq V$ on the other branches. By contradiction we suppose that $\sup_{(x,i) \in TR}(u - V)(x, i) > \delta > 0$. If

$$\exists r > 0 | \forall \delta' > 0 \, \exists \, (\overline{x}, i) \in [r, +\infty[\times \{i\} : \sup_{(x,i) \in TR} \big((u - V)(x, i) - (u - V)(\overline{x}, i) \big) \leq \delta,$$

then, by Theorem 4.4.11 and known comparison techniques we get a contradiction because, in $]r, +\infty[\times \{i\}$, V is a supersolution and u is a subsolution of the same HJB. Hence we may restrict to the case where $u - V$ has the maximum with respect to r in $x = 0$. Since $\bar{V}^\varepsilon(x,i)$ converges to $V(0)$, with $\bar{V}^\varepsilon$ defined in (4.68), then for small ε,

$$u(z^i, i) - \bar{V}^\varepsilon(z^i, i) = \max_{[0,\varepsilon]\times\{i\}} (u - \bar{V}^\varepsilon) > \frac{\delta}{2} > 0. \tag{4.74}$$

Since $u(x,i)$ is a continuous subsolution of (4.63) then satisfies

$$\lambda u(x,i) - f_i(x,a_i) \cdot p - \ell_i(x,a_i) \le 0 \quad \forall p \in D^+u(x,i) \neq \emptyset, \tag{4.75}$$

where $D^+u(x,i)$ is the set of super-differentials of u at a point (x,i). Now, taking into account (4.69) and (4.74) we have that

$$p - \bar{V}^\varepsilon(x,i)' \le \frac{-\lambda\delta}{2|f_i(x,a_i)|} + O(\varepsilon), \tag{4.76}$$

whence, for $\varepsilon < \frac{1}{2}\left|\frac{\lambda\delta}{2|f_i(x,a_i)|}\right|$, we get that $p - \bar{V}^\varepsilon(x,i)' \le -\bar{\delta}$, for a suitable $\bar{\delta} > 0$ regardless to x. Hence $u(x,i) - \bar{V}^\varepsilon(x,i)$ is decreasing and, taking ε as above, has maximum point in $x = 0$. By the previous consideration we get that $\bar{V}(x)$ (4.70) is an admissible test function and that $u - \bar{V}$ has a local maximum point in $x = 0$ for suitable small $\varepsilon > 0$. Hence, being u a subsolution, exists $\bar{i} \in \{1,2,3\}$ such that

$$\lambda u(0) + H_{\bar{i}}\left(0, \left(\bar{V}^\varepsilon(0,\bar{i})'\right)\right) \le 0. \tag{4.77}$$

Moreover, by(4.68), we have

$$\lambda \bar{V}^\varepsilon(0,\bar{i}) + H_{\bar{i}}\left(0, \left(\bar{V}^\varepsilon(0,\bar{i})'\right)\right) \ge -O(\varepsilon). \tag{4.78}$$

Subtracting (4.78) to (4.77) we get the contradiction to (4.74) and then, for $\varepsilon \to 0$, $u \le V$ in TR. □

4.4.2 Non-uniform switching thresholds

In this section we suppose that the three thresholds of the three-thermostatic optimal control problem are not the same for all R_{ε_i}. This imply that the time spent in a single branch R_{ε_i} to reach the relative threshold depends on the value of ε_i. Accordingly to this, the convexification parameters $\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3$ are such that if at limit for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \to (0^+, 0^+, 0^+)$ the optimal behavior is to switch only between two branch, R_i and R_j for $i, j \in \{1,2,3\}, i \neq j$, then $\bar{\mu}_i + \bar{\mu}_j = 1$. If instead the optimal behavior is to switch among all three branches R_i then $\bar{\mu}_i = \mu_i$ as in (4.45). To identify the limit optimal control problem when $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \to (0^+, 0^+, 0^+)$ we start to the define the controlled dynamics. Using the same notation of the last section, if $(x,i) \in TR$ with $x \neq 0$ then the dynamics is the usual $f_i(x,a_i)$ with $a_i \in A$. If instead

$x = 0$, being $(0, i) = (0, j)$ for $i, j \in \{1, 2, 3\}, i \neq j$, we can either choose any dynamics makes us to stay inside a single branch R_i or we may rest at zero using any combination $\sum_{i=1}^{3} \bar{\mu}_i f_i(0, a_i)$ with $f_i(0, a_i)$ and $\bar{\mu}_i$ as before. In detail, the set of controls in the junction point is

$$A(0) = \overline{A} \cup \widetilde{A}$$

with

$$\begin{aligned}
\overline{A} = \{ & (a_1, a_2, a_3, \sigma, \bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3) \in A^3 \times \{12, 13, 23, 123\} \times [0, 1]^3 | \\
& \sigma = ij \Rightarrow \bar{\mu}_i + \bar{\mu}_j = 1, f_i(0, a_i) \leq 0; \\
& \sigma = 123 \Rightarrow \bar{\mu}_i = \mu_i, f_i(0, a_i) \leq 0 \text{ with at most one equal to } 0 \}, \\
\widetilde{A} = \{ & (a, i) \in A \times \{1, 2, 3\} |\ f_i(0, a) \geq 0 \} .
\end{aligned}$$

Note that in $\widetilde{A}$ the index i is at disposal, while in $\overline{A}$, the notation ij means that the switching is only between R_i and R_j (as well as 123 means that the switching performs among all the three branches).
Then, as in the last section, calling $\hat{a}$ the generic element of $A(0)$ we define

$$f_0(0, \hat{a}) = \begin{cases} f_i(0, a) & \text{if } \hat{a} \in \widetilde{A}, \\ 0 & \text{if } \hat{a} \in \overline{A}. \end{cases}$$

With the same arguments, if $(x, i) \in TR$ and $x \neq 0$ then the running cost is $\ell_i(x, a_i)$ with $a_i \in A$, otherwise we define

$$\ell_0(0, \hat{a}) = \begin{cases} \ell_i(0, a) & \text{if } \hat{a} \in \widetilde{A}, \\ \bar{\mu}_1 \ell_1(0, a_1) + \bar{\mu}_2 \ell_2(0, a_2) & \text{if } \sigma = 12 \quad \text{and } \hat{a} \in \overline{A}, \\ \bar{\mu}_1 \ell_1(0, a_1) + \bar{\mu}_3 \ell_3(0, a_3) & \text{if } \sigma = 13 \quad \text{and } \hat{a} \in \overline{A}, \\ \bar{\mu}_2 \ell_2(0, a_2) + \bar{\mu}_3 \ell_3(0, a_3) & \text{if } \sigma = 23 \quad \text{and } \hat{a} \in \overline{A} \\ \mu_1 \ell_1(0, a_1) + \mu_2 \ell_2(0, a_2) + \mu_3 \ell_3(0, a_3) & \text{if } \sigma = 123 \quad \text{and } \hat{a} \in \overline{A}. \end{cases}$$

The quadruples $f = (f_1, f_2, f_3, f_0)$ and $\ell = (\ell_1, \ell_2, \ell_3, \ell_0)$ then define the threefold junction optimal control problem. The same considerations of the case with uniform switching thresholds hold, hence, denoting by $\mathcal{A}_{(x_0, i_0)}$ the set of measurable controls for which there exist a unique admissible trajectory, we consider an infinite horizon problem with a discount factor $\lambda > 0$ given by

$$J(x_0, i_0, \alpha) = \int_0^{+\infty} e^{-\lambda t} \ell(x(t), i(t), \alpha(t)) dt$$

where ℓ is the running cost described above. The corresponding value function is

$$V^*(x_0, i_0) = \inf_{\alpha \in \mathcal{A}_{(x_0, i_0)}} J(x_0, i_0, \alpha). \tag{4.79}$$

Observe that if we stay in $x = 0$ for all time using controls in $\overline{A}$ the cost is

$$u_0(0) = \frac{1}{\lambda} \min_{\overline{A}} \sum_{i=1}^{3} \bar{\mu}_i \ell_i(0, a_i) = \frac{1}{\lambda} \min \{u_{1,2}(0), u_{1,3}(0), u_{2,3}(0), u_{1,2,3}(0)\}$$

where $u_{1,2}(0)$ is the minimum over $\overline{A}$ of the cost ℓ_0 when $\sigma = 12$, $u_{1,3}(0)$ is the minimum over $\overline{A}$ of the cost ℓ_0 when $\sigma = 13$ and similarly the others.

Theorem 4.4.14. *Assume* (4.11), (4.12) *and* (4.15). *The value function* V^* (4.79) *characterized as in Theorem* (4.4.7), *but with* $u_0(0)$ *in place of* $u_{1,2,3}(0)$, *namely*

$$V^*(0) = \min \left\{u_0(0), V_{sc(1)}(0), V_{sc(2)}(0), V_{sc(3)}(0)\right\}, \tag{4.80}$$

satisfies

$$V^*(x, i) = \liminf_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)\to(0^+,0^+,0^+)} V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x, i) \ \forall \ (x, i) \in R_i, \ \ i = 1, 2, 3, \tag{4.81}$$

where $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ *is the value function of the approximating thermostatic problem* (4.44), *with non uniform thresholds* $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, *and the convergence is uniform. Moreover, when* $x = 0$, *the limit is independent from* $i = 1, 2, 3$.

Proof. We start to prove that (4.81) holds for $x = 0$. The independence from i of the inferior limit (4.81) comes from the controllability (4.12) because $|V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0, i) - V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0, j)|$ is infinitesimal as $\max \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. In the sequel, we omit the symbol i in the expression $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0, i)$.
By contradiction, let us suppose that $V^*(0) < \liminf V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0)$. By (4.12), for every $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, we have $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0) \leq V_{sc(i)}(0)$ for every $i = 1, 2, 3$. Hence, it implies $V^*(0) = u_0(0)$. Let $(a_1, a_2, a_3, \sigma, \bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3) \in \overline{A}$ realize the minimum in the definition of $u_0(0)$.
We analyze some possible cases, the other ones being similar.

1) $f_1(0, a_1), f_2(0, a_2), f_3(0, a_3) < 0$ and $\sigma = 123$. Hence, considering the triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon, \varepsilon, \varepsilon)$ and using a suitably switching control between those constants controls, we get $V_{\varepsilon,\varepsilon,\varepsilon}(0)$ is not larger than $u_{1,2,3}(0)$ plus an infinitesimal quantity as $\varepsilon \to 0$, which is a contradiction.

2) $f_1(0, a_1), f_2(0, a_2), f_3(0, a_3) < 0$ and $\sigma = 23$. Here, taking the triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon^2, \varepsilon, \varepsilon)$, we get $V_{\varepsilon,\varepsilon,\varepsilon}(0)$ is not larger than $u_{2,3}(0)$ plus an infinitesimal quantity as $\varepsilon \to 0$, which is a contradiction.

3) $f_1(0, a_1) = 0, f_2(0, a_2), f_3(0, a_3) < 0$. In this setting we can study two sub-cases according to the value of σ.

3.1) If $\sigma = 123$ we consider the triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon, \varepsilon, \varepsilon)$, we arrive in R_1 and we stop there. Therefore, $u_{1,2,3}(0) = \frac{1}{\lambda}\ell_1(0, a_1)$ cannot be lower than $V_{sc(1)}(0)$ that is a contradiction.

3.2) If $\sigma = 23$ we consider the triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon^2, \varepsilon, \varepsilon)$ and argue as in the case 2).

We remark that if $\sigma = 12$ or 13 then considering $(\varepsilon, \varepsilon, \varepsilon^2)$ and $(\varepsilon, \varepsilon^2, \varepsilon)$ respectively we can conclude as in 3.1).

4) $f_1(0, a_1), f_2(0, a_2) = 0, f_3(0, a_3) < 0$. Also in this case we have different sub-cases according to the value of σ.

4.1) If $\sigma = 123$ we take the triple $(\varepsilon, \varepsilon, \varepsilon)$ and conclude using Remark 4.4.6 since $u_{1,2,3}(0)$ cannot be lower than a state constraints. Then as before we have again a contradiction.

4.2) If $\sigma = 23$ taking the triple $(\varepsilon^2, \varepsilon, \varepsilon)$ we get $u_{2,3}(0) = \frac{1}{\lambda}\ell_2(0, a_2)$ that is no lower than$V_{sc(2)}(0)$, that is a contradiction.

Now we assume $\liminf V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0) < V^*(0)$. Let $\delta > 0$ be such that, for arbitrarily small suitably chosen $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, it is $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0) + \delta < V^*(0)$. A piecewise constant control α which almost realizes the optimum (less then β) for $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0)$ must be such that there are infinitely many switching between all branches $R_{\varepsilon_1}, R_{\varepsilon_2}, R_{\varepsilon_3}$. Indeed, if it is not the case, then, for a least one branch, say R_{ε_i}, the trajectory definitely remains inside it. Hence, for small $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0)$ is almost equal to $V_{sc(i)}(0)$, which is a contradiction. Then, in order to prove that $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0)$ cannot be less than $V^*(0) - \delta$ we proceed as in the Theorem 4.4.8 considering $O(\max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\})$ and $u_0(0)$ instead of $O(\varepsilon)$ and $u_{1,2,3}(0)$ respectively.
In conclusion we have $\liminf V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(0) = V^*(0)$. Note that the equations solved by $V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ and by $V^*(0)$((4.44) and (4.49), (4.50) suitably modified) are the same in the interior of R_i and the boundary datum converges to $V^*(0)$. Then, representing the solutions as the value functions of the corresponding optimal control problems, we get (4.81) and the uniform convergence. □

Remark 4.4.15. *As we show in the proof of Theorem 4.4.14, we can restrict us to consider as thresholds:*

$$(\varepsilon, \varepsilon, \varepsilon), \quad (\varepsilon, \varepsilon, \varepsilon^2), \quad (\varepsilon, \varepsilon^2, \varepsilon), \quad (\varepsilon^2, \varepsilon, \varepsilon); \tag{4.82}$$

hence, given the dynamics f_1, f_2, f_3 and the running costs ℓ_1, ℓ_2, ℓ_3 satisfying the controllability assumptions there exists a unique choice of (4.82) *such that*

$$V^*(x, i) = \liminf_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)\to(0,0,0)} V_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(x, i) = \lim_{(\cdot,\cdot,\cdot)\to(0,0,0)} V_{(\cdot,\cdot,\cdot)}(x, i) \;\forall\; (x, i) \in R_i.$$

Note that for this problem we do not consider, for example, triples of thresholds of the kind $(c_1\varepsilon, c_2\varepsilon, c_3\varepsilon)$, $c_1, c_2, c_3 \in \mathbb{R}$ because they do not bring new possible optimal behaviors. Moreover, we have no take into account the triple of thresholds $(\varepsilon^2, \varepsilon^2, \varepsilon)$ and its permutations because at the limit this would means to stay in $x = 0$ without using the balance of the dynamics, which is physically meaningless.

Remark 4.4.16. *When the optimal strategy is to switch among all branches we have that $\ell_i(0, a_i) = \ell_j(0, j)$ $\forall i, j \in \{1, 2, 3\}, i \neq j$ and $V^*(x, i) = V(x, i)$, where V is the value function* (4.46) *of the threefold junction problem with uniform switching thresholds.*

Now, we suppose that the optimal strategy is to switch only between two branches, for example R_1 and R_2. In order to prove the following result we introduce a test function $\psi : TR \to \mathbb{R}$ such that $\psi \in C^1(TR)$ and on each branch $\psi_i : R_i \to \mathbb{R}$ is such that $\psi_i(x,i) = \psi_j(x,j)$ for every $i,j \in \{1,2,3\}, i \neq j$ when $x = 0$. Moreover, we give the following definition.

Definition 4.4.17. *A continuous function $u : TR \to \mathbb{R}$ is a viscosity subsolution of* (4.85) *if for any $(x,i) \in TR$, any $\psi \in C^1(TR)$ such that $u - \psi$ has a local maximum at (x,i), then*

$$\begin{aligned} &\lambda u(x,i) + H_i(x, \psi_i'(x,i)) \leq 0 && (x,i) \in int(R_i), \\ &\min\left\{\lambda u(0,i) + H_i(0, \psi_i'(0,i)),\ i = 1,2,3\right\} \leq 0 && x = 0; \end{aligned} \tag{4.83}$$

A continuous function $u : TR \to \mathbb{R}$ is a viscosity supersolution of (4.85) *if for any $(x,i) \in TR$, any $\psi \in C^1(TR)$ such that $u - \psi$ has a local minimum at (x,i), then*

$$\begin{aligned} &\lambda u(x,i) + H_i(x, \psi_i'(x,i)) \geq 0 && (x,i) \in int(R_i), \\ &\max\left\{\lambda u(0,i) + H_i(0, \psi_i'(0,i)),\ i = 1,2,3\right\} \geq 0 && x = 0. \end{aligned} \tag{4.84}$$

In particular if $x = 0$ then the local maximum/minimum may be considered with respect to two of the three branches only.

We point out the difference with Definition (4.4.10) where, for $x = 0$, the maximum/minimum must be with respect to all three branches.

Theorem 4.4.18. *Assume* (4.11), (4.12) *and* (4.15). *The function V^* is a viscosity solution and the maximal subsolution of the HJB problem*

$$\begin{cases} \lambda V + H_1(x, \nabla V) = 0 \quad \text{in } int(R_1), \\ \lambda V + H_2(x, \nabla V) = 0 \quad \text{in } int(R_2), \\ \lambda V + H_3(x, \nabla V) = 0 \quad \text{in } int(R_3), \\ \min\{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \leq 0 \text{ on } x = 0, \\ \max\{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \geq 0 \text{ on } x = 0. \end{cases} \tag{4.85}$$

Proof. By Proposition 4.4.4 and Theorem 4.4.14 we have that V^* satisfies the first three equations of (4.85). Since we are considering the function $V^* = \lim_{(\varepsilon,\varepsilon,\varepsilon^2)\to(0,0,0)} V_{\varepsilon,\varepsilon,\varepsilon^2} = V_{1,2}$, if $V^* - \psi$ assumes his maximum or minimum in $x = 0$ with respect to $R_1 \cup R_2$, then by the twofold junction problem we have that V^* is a viscosity solution and the maximal subsolution of

$$\begin{cases} \lambda V + H_1(x, \nabla V) = 0 \ \text{ in } \ int(R_1), \\ \lambda V + H_2(x, \nabla V) = 0 \ \text{ in } \ int(R_2), \\ \min\{\lambda V + H_1, \lambda V + H_2\} \leq 0 \text{ on } x = 0, \\ \max\{\lambda V + H_1, \lambda V + H_2\} \geq 0 \text{ on } x = 0. \end{cases}$$

If instead $V^* - \psi$ has a maximum point at $x = 0$ with respect to $R_1 \cup R_3$ we prove that the min $\{\lambda V + H_1, \lambda V + H_2\}$ is still lower or equal to zero. We can consider two cases:
1) If the optimal behavior consists to reach R_2 and stay there, namely $V^*(x,2) = V_{sc(2)}(x)$, and supposing that the cost to pay in R_1 to reach the junction is lower than the one in R_3, we have that $V^* = V_{1,2}$ on $R_1 \cup R_2$. Now, since (by assumption) that $V^* - \psi$ has maximum point at $x = 0$ locally with respect to the branch R_3, we have that $\psi_3(x,3) \geq V^*(x,3)$ for x near to zero. The optimality of $V_{sc(2)}$ implies that $V^*(\cdot,3) \geq V_{sc(2)}(\cdot) = V^*(\cdot,2)$ and hence $\psi_3(\cdot,3) \geq V^*(\cdot,2)$. Then gluing ψ_3 over R_2 we obtain that $V^* - \psi_3$ has a maximum point in $x = 0$ locally with respect to R_2. Hence, $\min\{\lambda V + H_1, \lambda V + H_2\} \leq 0$.
2) If the optimal strategy is to switch between R_1 and R_2 and the maximum point at $x = 0$ is still with respect to $R_1 \cup R_3$ we conclude as before because $\psi_3(\cdot,3) \geq V^*(\cdot,2)$.
If $V^* = V_{1,2}$ and $V^* - \psi$ has a maximum point at $x = 0$ with respect to $R_2 \cup R_3$, with similar argument as before we conclude that $\min\{\lambda V + H_1, \lambda V + H_2\} \leq 0$.
In conclusion we have shown that the following condition hold: exists a couple of indexes $(\bar{i}, \bar{j})$, fixed a priori, such that $V^* = V_{\bar{i},\bar{j}}$ on $R_{\bar{i}} \cup R_{\bar{j}}$ and that for all $\psi \in C^1(TR)$ such that $V^* - \psi$ has the maximum point at $x = 0$ with respect to any couple of edges, $\min\{\lambda V + H_{\bar{i}}, \lambda V + H_{\bar{j}}\} \leq 0$. From the latter condition follows that $\min\{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \leq 0$. Proceeding as before also for the fifth equation of (4.85) we have that V^* is a viscosity solution of (4.85). Now, let u be a continuous subsolution of (4.85) satisfying the above condition with the same couple of indexes $(\bar{i}, \bar{j})$ that we suppose to be $(1,2)$. Then

$$V^* \geq u \quad \text{on} \quad R_1 \cup R_2 \implies V^*(0) \geq u(0). \tag{4.86}$$

Furthermore V^* is a supersolution of the third equation of (4.85), u is a subsolution of the same equation and hence, by (4.86), follows $V^* \geq u$ on R_3. We can conclude that $V^* \geq u$ on TR and hence it is the maximal subsolution of (4.85). □
A similar result holds also considering $V^* = \lim_{(\varepsilon^2,\varepsilon,\varepsilon)\to(0,0,0)} V_{\varepsilon^2,\varepsilon,\varepsilon} = V_{2,3}$ or $V^* = \lim_{(\varepsilon,\varepsilon^2,\varepsilon)\to(0,0,0)} V_{\varepsilon,\varepsilon^2,\varepsilon} = V_{1,3}$.

4.5 A more general problem

In this section we consider a problem with two threefold junction points given by a segment BC and four half-lines (see Figure 4.4).

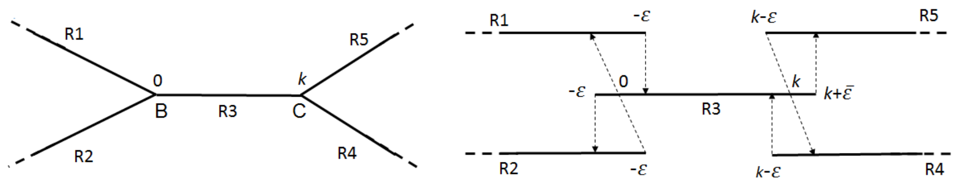


Figure 4.4: The threefold junctions and its thermostatic-type approximation.

We have five labels $\{1,2,3,4,5\}$, one for every half line $R_1,\dots,R_5$ that we identify with

$$R_i = \begin{cases} [0,+\infty[\times\{i\} & \text{if } i=1,2,\\ [0,k]\times\{i\} & \text{if } i=3,\\ [k,+\infty[\times\{i\} & \text{if } i=4,5. \end{cases}$$

We also consider the controlled dynamics $f_i : R_i \times A \to \mathbb{R}$, the running costs $\ell_i : R_i \times A \to [0,+\infty[$ and the following condition
Controllability: For every $i \in \{1,2,3\}$

$$\exists\ a_i^-, a_i^+ \ \in A \quad \text{s.t.} \quad f_i(0,a_i^-) < 0 < f_i(0,a_i^+), \tag{4.87}$$

and for every $j \in \{3,4,5\}$

$$\exists\ \bar{a}_j^-, \bar{a}_j^+ \ \in A \quad \text{s.t.} \quad f_j(k,\bar{a}_j^-) < 0 < f_j(k,\bar{a}_j^+). \tag{4.88}$$

We approximate these two triple discontinuities in the geometric points B e C with coordinates $x=0$ and $x=k$, $k>0$ respectively, by a thermostatic approximation with uniform switching thresholds in such a way: we extend f_i and ℓ_i to $[-\varepsilon,+\infty[\times\{i\}\times A$ for $i=1,2$, to $[-\varepsilon,k+\bar{\varepsilon}]\times\{i\}\times A$ for $i=3$ and to $[k-\varepsilon,+\infty[\times\{i\}\times A$ for $i=4,5$.
Then, denoting by

$$R_i^\varepsilon = \begin{cases} [-\varepsilon,+\infty[\times\{i\} & \text{if } i=1,2,\\ [-\varepsilon,k+\varepsilon]\times\{i\} & \text{if } i=3,\\ [k-\varepsilon,+\infty[\times\{i\} & \text{if } i=4,5, \end{cases}$$

the thermostatic controlled system is given by

$$\begin{cases} x'(t) = f_{i(t)}(x(t),\alpha(t))\\ i(t) = \tilde{h}[x](t)\\ (x_0,i_0) \in R_{i_0}^\varepsilon, \end{cases} \tag{4.89}$$

where $\tilde{h}[x](t)$ is the delayed thermostatic rule as in Figure 4.4. A shown in Figure 4.4 we can only switch from R_1^ε to R_3^ε, then, depending on the used control, we can either reach the threshold $-\varepsilon$ and hence switch from R_3^ε to R_2^ε and from R_2^ε to R_1^ε, or reach the threshold $k+\varepsilon$ and hence switch from R_3^ε to R_5^ε, from R_5^ε to R_4^ε and from R_4^ε to R_3^ε. Note that the variable x is subject to a discontinuity at the switching instant (see Figure 4.4 and also note in the thermostat, the branches R_1^ε and R_2^ε are oriented in the opposite way respect to the standard one). Now, for every $(x_0, i_0) \in R_{i_0}^\varepsilon$ we consider the value function

$$V_\varepsilon(x_0, i_0) = \inf_{\alpha \in \mathcal{A}} \int_0^{+\infty} e^{-\lambda t} \ell_{i(t)}(x(t), \alpha(t)) dt, \tag{4.90}$$

and we also have for every $i = 1, \dots, 5$ the Hamiltonians

$$H_i(x, p) = \sup_{a \in A} \left\{ -f_i(x, a) \cdot p - \ell_i(x, a) \right\}, \tag{4.91}$$

where we drop the index i in the entries $(x, i) \in R_i^\varepsilon$ of f_i, ℓ_i and consequently in H_i. As for the threefold junction problem, the continuity of the value function (4.90) comes from the controllability, regularity and by the same procedures used in Bagagiolo [9] and moreover the system (4.89) admits a unique solution $(x(\cdot), i(\cdot)) \in \mathcal{O} = \cup_{i=1}^5 R_{\varepsilon_i}$ for each initial state (x_0, i_0) and for any control α. The following holds

Proposition 4.5.1. *For any $\varepsilon > 0$ the value function V_ε of the switching multi-thermostatic optimal control problem is the unique bounded and continuous function on $\mathcal{O}$ which satisfies, in the viscosity sense*

$$\begin{cases} \lambda V_\varepsilon(x,1) + H_1\Big(x, V_\varepsilon'(x,1)\Big) = 0 & \text{in } int(R_1^\varepsilon), \\ V_\varepsilon(-\varepsilon, 1) = V_\varepsilon(\varepsilon, 3); & \\ \lambda V_\varepsilon(x,2) + H_2\Big(x, V_\varepsilon'(x,2)\Big) = 0 & \text{in } int(R_2^\varepsilon), \\ V_\varepsilon(-\varepsilon, 2) = V_\varepsilon(\varepsilon, 1); & \\ \lambda V_\varepsilon(x,3) + H_3\Big(x, V_\varepsilon'(x,3)\Big) = 0 & \text{in } int(R_3^\varepsilon), \\ V_\varepsilon(-\varepsilon, 3) = V_\varepsilon(\varepsilon, 2), \quad V_\varepsilon(k+\varepsilon, 3) = V_\varepsilon(k+\varepsilon, 5); & \\ \lambda V_\varepsilon(x,4) + H_4\Big(x, V_\varepsilon'(x,4)\Big) = 0 & \text{in } int(R_4^\varepsilon), \\ V_\varepsilon(k-\varepsilon, 4) = V_\varepsilon(k-\varepsilon, 3); & \\ \lambda V_\varepsilon(x,5) + H_5\Big(x, V_\varepsilon'(x,5)\Big) = 0 & \text{in } int(R_5^\varepsilon), \\ V_\varepsilon(k-\varepsilon, 5) = V_\varepsilon(k+\varepsilon, 4). & \end{cases} \tag{4.92}$$

Proof. We give here only the proof of the uniqueness of solution of (4.92) because the boundedness and continuity of V_ε come from analogous results to ones proved for the threefold junction problem. Say that, we show that every solution of (4.92) is a fixed point of a contraction mapping $G : BC(\mathcal{O}) \to BC(\mathcal{O})$, where $BC(\mathcal{O})$ is the space of real bounded and continuous function on $\mathcal{O}$, whose elements we represent by a five-uple of functions, one for ever branch . By the completeness, we will get the uniqueness. We define G in the following way: for every $c \geq 0$ and for every $i \in \{1,2,3,4,5\}$, let $z_c^{(i)}$ be the solution of the Hamilton-Jacobi equation with fixed i and with boundary datum c. Hence for every $(\xi,\eta,\sigma,\tau,\beta) \in BC(\mathcal{O})$ we define

$$G(\xi,\eta,\sigma,\tau,\beta) = \Big(z^{(1)}_{\left(z^{(3)}_{\xi(-\varepsilon)}(\varepsilon)\right)}(\cdot), z^{(2)}_{\left(z^{(1)}_{\eta(-\varepsilon)}(\varepsilon)\right)}(\cdot),$$
$$z^{(3)}_{\left(z^{(2)}_{\sigma(-\varepsilon)}(\varepsilon), z^{(5)}_{\sigma(k-\varepsilon)}(k+\varepsilon)\right)}(\cdot), z^{(4)}_{\left(z^{(3)}_{\tau(k+\varepsilon)}(k-\varepsilon)\right)}(\cdot), z^{(5)}_{\left(z^{(4)}_{\beta(k-\varepsilon)}(k+\varepsilon)\right)}(\cdot)\Big)$$

This means that, for instance, the first component of $(G(\xi,\eta,\sigma,\tau,\beta))$ is the solution on the branch R_{ε_1} with boundary datum equal to the value on ε_3 of the solution on the branch R_{ε_3} with boundary datum equal to $\xi(-\varepsilon_3)$. By the uniqueness of the solution in any branch, this is a good definition. It is easy o see that every solution of (4.92) is a fixed point of G. Then for every $(\xi,\eta,\sigma,\tau,\beta), (\widehat{\xi},\widehat{\eta},\widehat{\sigma},\widehat{\tau},\widehat{\beta}) \in BC(\Omega)$, for the first component of G we have

$$\begin{aligned}
&\|(G(\xi,\eta,\sigma,\tau,\beta))_1 - (G(\widehat{\xi},\widehat{\eta},\widehat{\sigma},\widehat{\tau},\widehat{\beta}))_1\|_\infty \leq |z^{(3)}_{\xi(-\varepsilon)}(\varepsilon) - z^{(3)}_{\widehat{\xi}(-\varepsilon)}(\varepsilon)| \\
&\leq e^{\frac{-\lambda(2\varepsilon)}{M}} |\xi(-\varepsilon) - \widehat{\xi}(-\varepsilon)| \leq e^{\frac{-\lambda(2\varepsilon)}{M}} \|\xi - \widehat{\xi}\|_\infty,
\end{aligned}$$

where M is the bound of ℓ_i. A similar inequality holds for the second, the fourth and the fifth component of G. We study apart the third component:

$$\begin{aligned}
&\|(G(\xi,\eta,\sigma,\tau,\beta))_3 - (G(\widehat{\xi},\widehat{\eta},\widehat{\sigma},\widehat{\tau},\widehat{\beta}))_3\|_\infty \\
&\leq \max\Big(|z^{(2)}_{\sigma(-\varepsilon)}(\varepsilon) - z^{(2)}_{\widehat{\sigma}(-\varepsilon)}(\varepsilon)|, |z^{(5)}_{\sigma(k-\varepsilon)}(k+\varepsilon) - z^{(5)}_{\widehat{\sigma}(k-\varepsilon)}(k+\varepsilon)| \Big) \\
&\leq e^{\frac{-\lambda(2\varepsilon)}{M}} \max(|\sigma(-\varepsilon) - \widehat{\sigma}(-\varepsilon)|, |\sigma(k-\varepsilon) - \widehat{\sigma}(k-\varepsilon)|) \\
&\leq e^{\frac{-\lambda(2\varepsilon)}{M}} \|\sigma - \widehat{\sigma}\|_\infty
\end{aligned}$$

Since $\lambda > 0$, we get the conclusion. □

We now study both the loop concerning the branches R_1, R_2, R_3 and the one relative to R_3, R_4, R_5. In particular, when $f_i(0, a_i) < 0$ for $i = 1,2,3$, namely when we perform the whole cycle, the right convex parameters to be considered are

$$\mu_1 = \frac{f_2 f_3}{f_2 f_3 + f_1 f_3 + f_1 f_2}, \ \mu_2 = \frac{f_1 f_3}{f_2 f_3 + f_1 f_3 + f_1 f_2}, \ \mu_3 = \frac{f_1 f_2}{f_2 f_3 + f_1 f_3 + f_1 f_2}, \tag{4.93}$$

where we drop the entries in the dynamics. Analogously, when $f_3(k,\bar a_3) > 0$ and $f_4(k,\bar a_4), f_5(k,\bar a_5) < 0$ the convex parameters are

$$\bar\mu_3 = \frac{f_4 f_5}{f_4 f_5 + f_3 f_4 + f_3 f_5},\ \bar\mu_4 = \frac{f_3 f_5}{f_4 f_5 + f_3 f_4 + f_3 f_5},\ \bar\mu_5 = \frac{f_3 f_4}{f_4 f_5 + f_3 f_4 + f_3 f_5}. \tag{4.94}$$

For the above parameters ((4.93) and (4.94)) the same considerations as the threefold case hold; both (4.93) and (4.94) are meaningful when at most one $f_i(0,a_i)$, $i \in \{1,2,3\}$, and one $f_j(k,\bar a_j)$, $j \in \{3,4,5\}$ respectively is null. In that case we definitely remain in the corresponding branch. To identify the limit optimal control problem when $\varepsilon \to 0$ we define its controlled dynamics. Calling $TR5 = \cup_{i=1}^5 R_i$, if $(x,i) \in TR5$ with $x \notin \{0,k\}$ then the dynamics is the usual $f_i(x,a_i)$, $a_i \in A$. If $x = 0$ being $(0,i) = (0,i')$ for $i,i' \in \{1,2,3\}$, $i \neq i'$ we can either choose any dynamics make us to stay inside a single branch R_i and possibly for the branch R_3 exit from k, or we may rest at zero using any combination $\sum_{i=1}^3 \mu_i f_i(0,a_i)$ with $f_i(0,a_i)$ and μ_i as before. The set of controls in the junction point $x=0$ is

$$A(0) = A_0 \cup \widetilde A$$

with (note that in $\widetilde A$ the index i is also at disposal)

$$\begin{aligned} A_0 &= \{(a_1,a_2,a_3) \in A^3 |\ f_i(0,a_i) \le 0 \text{ with at most one equal to } 0\},\\ \widetilde A &= \{(a,i) \in A \times \{1,2,3\} |\ f_i(0,a) \ge 0\}\,. \end{aligned}$$

Then, calling $\hat a$ the generic element of $A(0)$ we define

$$f_0(0,\hat a) = \begin{cases} f_i(0,a) & \text{if } \hat a \in \widetilde A,\\ 0 & \text{if } \hat a \in A_0. \end{cases}$$

With the same arguments, if $(x,i) \in TR5$ and $x \notin \{0,k\}$ then the running cost is $\ell_i(x,a_i)$ with $a_i \in A$, otherwise we define

$$\ell_0(0,\hat a) = \begin{cases} \ell_i(0,a) & \text{if } \hat a \in \widetilde A,\\ \mu_1 \ell_1(0,a_1) + \mu_2 \ell_2(0,a_2) + \mu_3 \ell_3(0,a_3) & \text{if } \hat a \in A_0. \end{cases}$$

If $x = k$, with similar arguments as before we define the set of controls

$$A(k) = A_k \cup \widetilde A_k$$

with

$$\begin{aligned} A_k &= \{(\bar a_3,\bar a_4,\bar a_5) \in A^3 |\ f_3(k,\bar a_3) \ge 0, f_4(k,\bar a_4), f_5(k,\bar a_5) \le 0 \text{ with at most one equal to } 0\},\\ \widetilde A_k &= \{(a,i) \in A \times \{3,4,5\} |\ f_i(k,a) \ge 0, i = 4,5 \text{ and } f_3(k,a) \le 0\}\,. \end{aligned}$$

Then, calling $\ddot{a}$ the generic element of $A(k)$ we define

$$f_k(k,\ddot{a}) = \begin{cases} f_i(k,a) & \text{if } \ddot{a} \in \widetilde{A}_k, \\ 0 & \text{if } \ddot{a} \in A_k. \end{cases}$$

and

$$\ell_k(k,\ddot{a}) = \begin{cases} \ell_i(k,a) & \text{if } \ddot{a} \in \widetilde{A}_k, \\ \bar{\mu}_3\ell_3(k,\bar{a}_3) + \bar{\mu}_4\ell_4(k,\bar{a}_4) + \bar{\mu}_5\ell_5(k,a_5) & \text{if } \ddot{a} \in A_k. \end{cases}$$

The 7-uples $f = (f_1, f_2, f_3, f_4, f_5, f_0, f_k)$ and $\ell = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_0, \ell_k)$ then define the multiple threefold junction optimal control problem. In particular given an initial state $(x_0, i_0) \in TR5$ and a measurable control $\alpha(t) \in A \cup A(0) \cup A(k)$ we consider a possible admissible trajectory in $TR5$ whose evolution, denoted by $(x(t), i(t))$, is such that $i(t)$ remains constant whenever $x(t) \notin \{0, k\}$ and $x(t)$ evolves with dynamics described above. Let us note that given an initial state, the set of measurable controls for which there exists a unique admissible trajectory is not empty and we denote it by $\mathcal{A}_{(x_0,i_0)}$. We then consider an infinite horizon problem with a discount factor $\lambda > 0$ given by

$$J(x_0, i_0, \alpha) = \int_0^{+\infty} e^{-\lambda t}\ell(x(t), i(t), \alpha(t))dt,$$

where ℓ is the running cost described above and the corresponding value function is

$$V(x_0, i_0) = \inf_{\alpha \in \mathcal{A}_{(x_0,i_0)}} J(x_0, i_0, \alpha). \tag{4.95}$$

In the sequel when $x = 0$ or $x = k$ we will drop the index i. Note that if we remain in $x = 0$ for all the time using controls in A_0 the best cost is given by

$$u_{1,2,3}(0) = \frac{1}{\lambda} \inf_{A_0} \{\mu_1\ell_1(0, a_1) + \mu_2\ell_2(0, a_2) + \mu_3\ell_3(0, a_3)\}. \tag{4.96}$$

While, if we remain in $x = k$ for all the time using controls in A_k the best cost is given by

$$u_{3,4,5}(k) = \frac{1}{\lambda} \inf_{A_k} \{\bar{\mu}_3\ell_3(k, \bar{a}_3) + \bar{\mu}_4\ell_4(k, \bar{a}_4) + \bar{\mu}_5\ell_5(k, \bar{a}_5)\}. \tag{4.97}$$

Remark 4.5.2. *Note that in general both A_0 and A_k are not compact. However, if $(a_1^n, a_2^n, a_3^n) \in A_0$ is a minimizing sequence for $u_{1,2,3}(0)$ converging to $(\dot{a}_1, \dot{a}_2, \dot{a}_3) \notin A_0$ we have that the quantity inside the bracket in* (4.96) *loses meaning but we still have the inequality*

$$\lim_{n\to\infty} \{\mu_1^n\ell_1(0, a_1^n) + \mu_2^n\ell_2(0, a_2^n) + \mu_3^n\ell_3(0, a_3^n)\} \geq \min\{\ell_i(0, \dot{a}_i) | f_i(0, \dot{a}_i) = 0\}.$$

and hence we can always detect an optimal behavior among the ones making as remaining at $x = 0$.

In the same way if $(\bar{a}_3^n, \bar{a}_4^n, \bar{a}_5^n) \in A_k$ is a minimizing sequence for $u_{3,4,5}(0)$ converging to $(\tilde{a}_3, \tilde{a}_4, \tilde{a}_5) \notin A_k$ we have that the quantity inside the bracket in (4.97) loses meaning but we still have the inequality

$$\lim_{n\to\infty} \{\bar{\mu}_3^n \ell_3(k, \bar{a}_3^n) + \bar{\mu}_4^n \ell_4(k, \bar{a}_4^n) + \bar{\mu}_5^n \ell_5(k, \bar{a}_5^n)\} \geq \min\{\ell_i(k, \tilde{a}_i) | f_i(k, \tilde{a}_i) = 0\}.$$

and hence we can always detect an optimal behavior among the ones making as remaining at $x = k$.

In order to characterize the value function (4.95) we define the value functions of the semi-state constraint optimal control problems, namely the functions that express the possibility to exit from the branch R_3. In particular, $V_{ssc(3-k)}(x)$ (resp. $V_{ssc(3-0)}(x)$) is the value function of the control problem on R_3, constrained to not exit from 0 (resp. from k) and with exit cost in k (resp. in 0). More specifically, $V_{ssc(3-k)}(x)$ satisfies

$$\begin{cases} \lambda v + H_3(x, v') \leq 0 \quad \text{in }]0, k[\\ \lambda v + H_3(x, v') \geq 0 \quad \text{in } [0, k[\\ V_{ssc(3-k)}(k) = \min\{u_{3,4,5}(k), V_{sc(3)}(k), V_{sc(4)}(k), V_{sc(5)}(k)\} =: u_*(k) \end{cases} \tag{4.98}$$

while $V_{ssc(3-0)}(x)$

$$\begin{cases} \lambda v + H_3(x, v') \leq 0 \quad \text{in }]0, k[\\ \lambda v + H_3(x, v') \geq 0 \quad \text{in }]0, k] \\ V_{ssc(3-0)}(0) = \min\{u_{1,2,3}(0), V_{sc(1)}(0), V_{sc(2)}(0), V_{sc(3)}(0)\} =: u_*(0). \end{cases} \tag{4.99}$$

Theorem 4.5.3. *Assume (4.11), (4.15), (4.87) and (4.88). Then V is continuous on $TR5$. Moreover when $x = 0$ and $x = k$*

$$V(0) = \min\left\{u_{1,2,3}(0), V_{sc(1)}(0), V_{sc(2)}(0), V_{sc(3)}(0), V_{ssc(3-k)}(0)\right\}, \tag{4.100}$$

$$V(k) = \min\left\{u_{3,4,5}(k), V_{sc(3)}(k), V_{sc(4)}(k), V_{sc(5)}(k), V_{ssc(3-0)}(k)\right\}. \tag{4.101}$$

Therefore
i) if $V(0) = u_{1,2,3}(0)$, then V is the unique solution of the three problems (one for every $i \in \{1, 2, 3\}$)

$$\begin{cases} \lambda v + H_i(x, v') = 0 \quad \textit{in } int(R_i), \\ v(0) = u_{1,2,3}(0), \end{cases} \tag{4.102}$$

ii) if $V(0) = V_{sc(i)}(0)$, for some $i = 1, 2, 3$, then V satisfies: $V = V_{sc(i)}$ in R_i, and uniquely solves (for every $i' \in \{1, 2, 3\} \setminus \{i\}$)

$$\begin{cases} \lambda v + H_{i'}(x, v') = 0 \quad \textit{in } int(R_{i'}), \\ v(0) = V_{sc(i)}(0), \end{cases} \tag{4.103}$$

iii) if $V(0) = V_{ssc(3-k)}(0)$ *then* $V = V_{ssc(3-k)}$ *in* R_3 *and uniquely solves*

$$\begin{cases} \lambda v + H_3(x, v') \le 0 & in \ ]0, k[\\ \lambda v + H_3(x, v') \ge 0 & in \ [0, k[\\ v(k) = u_*(k). \end{cases} \tag{4.104}$$

i') If $V(k) = u_{3,4,5}(k)$*, then* V *is the unique solution of the three problems (one for every* $j \in \{3, 4, 5\}$*)*

$$\begin{cases} \lambda v + H_j(x, v^{'}) = 0 & in \ int(R_j), \\ v(k) = u_{3,4,5}(k), \end{cases} \tag{4.105}$$

ii') if $V(k) = V_{sc(j)}(k)$*, for same* $j = 3, 4, 5$*, then* V *satisfies:* $V = V_{sc(j)}$ *in* R_j*, and uniquely solves (for every* $j' \in \{3, 4, 5\} \setminus \{j\}$*)*

$$\begin{cases} \lambda v + H_{j'}(x, v^{'}) = 0 & in \ int(R_{j'}), \\ v(k) = V_{sc(j)}(k), \end{cases} \tag{4.106}$$

iii') if $V(k) = V_{ssc(3-k)}(k)$ *then* $V = V_{ssc(3-k)}$ *in* R_3 *and uniquely solves*

$$\begin{cases} \lambda v + H_3(x, v') \le 0 & in \ ]0, k[, \\ \lambda v + H_3(x, v') \ge 0 & in \ ]0, k], \\ v(0) = u_*(0). \end{cases} \tag{4.107}$$

Proof. The continuity of V comes from by controllability (4.87), (4.88) and regularity (4.11), (4.15) in a standard way. Both (4.100) and (4.101) come from (4.95) because the five terms in the corresponding minimum are exactly the only allowed behaviors (see Remark 4.5.2). To prove the second part of the Theorem we solve separately the Dirichlet problem in R_1, R_2 and R_3, putting the value of the minimum as boundary condition at $x = 0$. We also solve the problem (4.104) using the results of comparison and uniqueness at the end of the section. We get a solution $\bar{w}$ in $R_1 \cup R_2 \cup R_3$ which satisfies $\bar{w}(0) = V(0)$. Hence, by uniqueness for the Dirichlet problem in each R_i and by the uniqueness of solution of (4.104) we end up with $\bar{w} = V$. In the same way for the problem evaluate in k we obtain a solution $\widetilde{w}$ in $R_3 \cup R_4 \cup R_5$ which satisfies $\widetilde{w}(k) = V(k)$ and such that $\widetilde{w} = V$. Finally, $\bar{w} = V = \widetilde{w}$ hence V is a unique viscosity solution of the limit problem. □

Theorem 4.5.4. *Assume* (4.11)*,* (4.15)*,* (4.87) *and* (4.88)*. The value function* V (4.95) *(also characterizes by Theorem 4.5.3) satisfies*

$$V(x, i) = \lim_{\varepsilon \to 0} V_\varepsilon(x, i) \quad \forall \ (x, i) \in R_i, \ i = 1, \ldots, 5. \tag{4.108}$$

where V_ε *is the value function of the approximating thermostatic problem* (4.92) *with uniform thresholds and the convergence is uniform. Moreover, when* $x = 0$ *and* $x = k$ *the limit is independent on* i*.*

Proof. We start to prove that (4.108) holds for $x = 0$. The fact that the limit, whenever it exists, is independent from i comes from the controllability conditions (4.87) because $|V_\varepsilon(0,i) - V_\varepsilon(0,i')|$ is infinitesimal as ε. In the sequel, we drop the symbol i in the expression $V_\varepsilon(0,i)$. We prove (4.108) at $x = 0$ for a convergent subsequence still denoted by ε which exists as V_ε are equi-bounded. By contradiction, let us suppose that $V(0) < \lim V_\varepsilon(0)$. By (4.87), for every $\varepsilon > 0$, we have

$$V_\varepsilon(0) \leq \min\left\{V_{sc(1)}(0), V_{sc(2)}(0), V_{sc(3)}(0), V_{ssc(3-k)}(0)\right\}$$

hence, the absurd hypothesis implies $V(0) = u_{1,2,3}(0)$ by (4.100). Let us suppose that $(a_1, a_2, a_3) \in A_0$ realizes the minimum in the definition of $u_{1,2,3}(0)$. We analyze some possible cases, the other ones being similar.

1) $f_1(0,a_1), f_2(0,a_2), f_3(0,a_3) < 0$. Hence, using a suitably switching control between those constant controls, we get $V_\varepsilon(0)$ is not larger then $u_{1,2,3}(0)$ plus an infinitesimal quantity as $\varepsilon \to 0$, which is a contradiction.

2) $f_1(0,a_1) = 0, f_2(0,a_2), f_3(0,a_3) < 0$. In this case we arrive in R_1 and we stop with $f_1(0,a_1)$ in $x = 0$. Hence $u_{1,2,3}(0) = \frac{1}{\lambda}\ell_1(0,a_1)$ cannot be lower than $V_{sc(1)}(0)$ which is a contradiction. If instead not exists the minimizing sequence for (a_1,a_2,a_3) (see Remark 4.5.2) then $u_{1,2,3}(0)$ cannot be better than a state constraints neither than a semi-state constraints. Then as before, we have again a contradiction.

We can repeat the above steps for $x = k$ assuming that $V(k) < \lim V_\varepsilon(k)$. We obtain still a contradiction.

Now assume $\lim V_\varepsilon(0) < V(0)$. Let $\delta > 0$ be such that, for ε small enough, it is $V_\varepsilon(0) + \delta < V(0)$. A measurable control α which almost realizes the optimum (less than $\beta > 0$) for $V_\varepsilon(0)$ must be such that there are infinitely many switching between branches R_i^ε (i.e for every $i \in \{1,2,3\}$, $f_i(x,\alpha_i) < 0 \ \forall\ x$). Indeed, if it is not the case, then, for at least one branch R_i^ε, either the trajectory definitely remains inside it or in the case of branch R_3^ε can also exits from it. Hence, for small ε, $V_\varepsilon(0)$ is almost equal to $V_{sc(i)}(0)$ or to $V_{ssc(3-k)}(0)$, which is a contradiction. Note that we can limit to consider a piecewise constant control that we call again α since V_ε defined both with measurable controls and with piecewise constant controls, satisfies the same problem (4.92) which admits a unique solution. Then, in order to obtain the optimum, on each branch R_i^ε let $x_1^i, \dots, x_{n^i}^i$ be the points corresponding to the discontinuity instants $t_1^i, \dots, t_{n^i}^i$ of the control α and let a_j^i be the constant controls $\forall i = 1,2,3$, $\forall\ j = 1,\dots,n^i - 1$. On the assumption that $f_i(0,a_j^i) < 0 \ \forall\ i,j$ we consider the dynamics $f_i(0,a_j^i)$ and the running cost $\ell_i(0,a_j^i)$ on every spatial interval $[x_j^i, x_{j+1}^i]$. Now, for every i we consider

$$\inf_{a \in A}\left\{\frac{\ell_i(0,a)}{|f_i(0,a)|}\ |f_i(0,a) < 0\right\}. \tag{4.109}$$

If (4.109) is a minimum for every $i \in \{1,2,3\}$ obtained in $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ then in each R_i^ε we use constant dynamics $f_i(0,\tilde{a}_i)$ and constant running cost

$\ell_i(0, \tilde{a}_i)$. Therefore $|J(\cdot, i, \alpha) - J(\cdot, i, \tilde{a}_i)| \leq O(\varepsilon)$ and we get

$$\begin{aligned} V_\varepsilon(0) &\geq J(\cdot, i, \alpha) - \beta \geq J(\cdot, i, \tilde{a}_i) - O(\varepsilon) - \beta \\ &\geq u_{1,2,3}(0) - O(\varepsilon) - \beta \geq V(0) - O(\varepsilon) - \beta, \end{aligned} \tag{4.110}$$

that is a contradiction. If, for some i, (4.109) is not a minimum then we can consider the minimizing sequence a_i^n that realizes the infimum less than $O(\frac{1}{n})$. In particular $a_i^n \to \dot{a}_i \in A$ for $n \to +\infty$ and $f_i(0, a_i^n) \to f_i(0, \dot{a}_i) = 0$ being $f_i(0, a_i^n) < 0$. However, since the optimal strategy is to switch among the branches, we cannot stop in the branch R_i^ε with dynamics $f_i(0, \dot{a}_i)$ paying the cost $\ell_i(0, \dot{a}_i)$. Then, always taking into account that $f_i(0, a_j^i) < 0$ we have

$$\begin{aligned} V_\varepsilon &\geq J(\cdot, i, \alpha) - \beta \geq J(\cdot, i, a_i^n) - O\left(\frac{1}{n}\right) - O(\varepsilon) - \beta \\ &\geq u_{1,2,3}(0) - O\left(\frac{1}{n}\right) - O(\varepsilon) - \beta \geq V(0) - O\left(\frac{1}{n}\right) - O(\varepsilon) - \beta, \end{aligned} \tag{4.111}$$

which is again a contradiction.Therefore at the end, $V_\varepsilon(0)$ cannot be less than $V(0) - \delta$ by the definition of $V(0)$. This is a contradiction. Hence we have $\lim V_\varepsilon(0) = V(0)$. Now assuming that $\lim V_\varepsilon(k) < V(k)$ and proceeding as before we have again a contradiction, hence in conclusion $\lim V_\varepsilon(k) = V(k)$. Now note that the equations solved by V_ε and by V ((4.92) and (4.102)-(4.104), (4.105)-(4.107) respectively) are the same for all $(x, i) \in int(R_i), i = 1, \ldots, 5$ and the boundary datum converges, depending on the cycle that we are taking into account, to $V(0)$ or $V(k)$. Hence, representing the solutions as the value functions of the corresponding optimal control problems, we get (4.108) and the uniform convergence. □

We now show how is possible to extend the results of viscosity solution and maximal subsolution, proved for the threefold junction, also to this double threefold junction problem. We introduce the test functions and give the definition of viscosity subsolution and supersolution of (4.115).

Definition 4.5.5. *Let $\varphi : TR5 \to \mathbb{R}$ be a function such that*

$$\begin{aligned} &\varphi|_{R_i} := \varphi_i : R_i \longrightarrow \mathbb{R} \\ &(x, i) \longmapsto \varphi_i(x, i) \quad \text{if } x \notin \{0, k\}, \forall i \in \{1, 2, 3, 4, 5\}, \\ &(0, i) \longmapsto \varphi_i(0, i) = \varphi_j(0, j) \ \forall j \in \{1, 2, 3\} \setminus \{i\}, \\ &(k, i) \longmapsto \varphi_i(k, i) = \varphi_j(k, j) \ \forall j \in \{3, 4, 5\} \setminus \{i\}, \end{aligned} \tag{4.112}$$

with $\varphi \in C^0(TR5)$ and $\varphi_i \in C^1(R_i)$.

Definition 4.5.6. *A continuous function $u : TR5 \to \mathbb{R}$ is a viscosity subsolution of* (4.115) *if for any $(x, i) \in TR5$, any φ as in* (4.112) *such that*

$u - \varphi$ has a local maximum at (x,i) with respect to $TR5$, then

$$\begin{aligned}
&\lambda u(x,i) + H_i(x, \varphi_i'(x,i)) \le 0 && \text{if } \; x \in int(R_i),\\
&\min\left\{\lambda u(0,i) + H_i(0, \varphi_i'(0,i)),\ i = 1,2,3\right\} \le 0 && \text{if } \; x = 0,\\
&\min\left\{\lambda u(k,j) + H_i(k, \varphi_j'(k,j)),\ j = 3,4,5\right\} \le 0 && \text{if } \; x = k,\ i \in \{1,2,3\}.
\end{aligned} \tag{4.113}$$

A continuous function $u : TR5 \to \mathbb{R}$ is a viscosity supersolution of (4.115) *if for any $(x,i) \in TR5$, any φ as in* (4.112) *such that $u - \varphi$ has a local minimum at (x,i) with respect to $TR5$, then*

$$\begin{aligned}
&\lambda u(x,i) + H_i(x, \varphi_i'(x,i)) \ge 0 && \text{if } \; x \in int(R_i),\\
&\max\left\{\lambda u(0,i) + H_i(0, \varphi_i'(0,i)),\ i = 1,2,3\right\} \ge 0 && \text{if } \; x = 0,\\
&\max\left\{\lambda u(k,j) + H_i(k, \varphi_j'(k,j)),\ j = 3,4,5\right\} \ge 0 && \text{if } \; x = k,\ i \in \{1,2,3\}.
\end{aligned} \tag{4.114}$$

In particular note that if $x = 0$ then the local maximum/minimum is with respect to the branches R_1, R_2, R_3, and $\varphi_i'(0,i)$, $i \in \{1,2,3\}$, is the right derivative on the branch i.
If instead $x = k$ and $i \in \{3,4,5\}$ then the local maximum/minimum is with respect to the branches R_3, R_4, R_5 and $\varphi_j'(k,j)$, $j \in \{4,5\}$, is the right derivative on the branch j, while $\varphi_3'(k,3)$ is the left derivative in the branch R_3.

Theorem 4.5.7. *Assume* (4.11), (4.15), (4.87), (4.88). *The value function V* (4.95) *is a viscosity solution of the Hamilton-Jacobi-Bellman problem*

$$\begin{cases}
\lambda V + H_1(x, \nabla V) = 0 \;\; in\ int(R_1),\\
\lambda V + H_2(x, \nabla V) = 0 \;\; in\ int(R_2),\\
\lambda V + H_3(x, \nabla V) = 0 \;\; in\ int(R_3),\\
\lambda V + H_4(x, \nabla V) = 0 \;\; in\ int(R_4),\\
\lambda V + H_5(x, \nabla V) = 0 \;\; in\ int(R_5),\\
\min\{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \le 0 \;\; on\ x = 0,\\
\max\{\lambda V + H_1, \lambda V + H_2, \lambda V + H_3\} \ge 0 \;\; on\ x = 0,\\
\min\{\lambda V + H_3, \lambda V + H_4, \lambda V + H_5\} \le 0 \;\; on\ x = k,\\
\max\{\lambda V + H_3, \lambda V + H_4, \lambda V + H_5\} \ge 0 \;\; on\ x = k.
\end{cases} \tag{4.115}$$

Proof. From Proposition 4.5.1, Theorem 4.5.4 and by classical convergence result,

$$\lambda V(x,i) + \sup_{a \in A}\left\{-f_i(x,a)(V(x,i))' - \ell_i(x,a)\right\} = 0 \text{ in } int(R_i),\ i = 1, \dots, 5.$$

Now, taken φ as in (4.112), if $V - \varphi$ has a local maximum at B with respect to the three branches then, similarly as in Theorem 4.4.11, we prove that V satisfies the condition of maximal subsolution, namely the sixth equation

in (4.115). If we suppose that $V-\varphi$ has a local minimum at $x=0$, still following Theorem 4.4.11 we get that V satisfies the seventh equation of (4.115).
It is evident that if the above arguments hold for the point B then they also hold for the point C. Anyway, if we suppose that $V-\varphi$, with φ as in (4.112), has a local maximum point at C, then consider the following change of coordinate

$$y=\begin{cases} x+k & \text{on} \quad R_i, i=1,2, \\ k-x & \text{on} \quad R_3, \\ x-k & \text{on} \quad R_i, i=4,5. \end{cases} \tag{4.116}$$

Using (4.116), we define the branches $\tilde{R}_i$

$$\tilde{R}_i=\begin{cases} [k,+\infty[\times\{i\} & \text{if } i=1,2, \\ [0,k]\times\{i\} & \text{if } i=3, \\ [0,+\infty[\times\{i\} & \text{if } i=4,5, \end{cases}$$

and the new dynamics and running costs on $TR5$

$$\tilde{f}_i(y,a)=\begin{cases} f_i(y-k,a) & \text{for } i=1,2, \\ -f_i(k-y,a) & \text{for } i=3, \\ f_i(y+k,a) & \text{for } i=4,5 \end{cases}, \quad \tilde{\ell}_i(y,a)=\begin{cases} \ell_i(y-k,a) & \text{for } i=1,2, \\ \ell_i(k-y,a) & \text{for } i=3, \\ \ell_i(y+k,a) & \text{for } i=4,5. \end{cases}$$

We also introduce the new value function $\tilde{V}$ and the test functions $\tilde{\varphi}$

$$\tilde{V}(y,i)=\begin{cases} V(y-k,i) & \text{for } i=1,2, \\ V(k-y,a) & \text{for } i=3, \\ V(y+k,a) & \text{for } i=4,5 \end{cases}, \quad \tilde{\varphi}_i(y,a)=\begin{cases} \varphi_i(y-k,i) & \text{for } i=1,2, \\ \varphi_i(k-y,i) & \text{for } i=3, \\ \varphi_i(y+k,a) & \text{for } i=4,5. \end{cases} \tag{4.117}$$

Using $\tilde{f}_i$, $\tilde{\ell}_i, \tilde{V}$ and $\tilde{\varphi}$ we can define the limit problem $\tilde{P}$ and, though a thermostatic approximation as before, the approximating problem $\tilde{P}_\varepsilon$ with the value function $\tilde{V}_\varepsilon$ which converges to $\tilde{V}$ (as in Theorem 4.5.4). Now the node C has coordinate $y=0$, the branches $\tilde{R}_3$, $\tilde{R}_4$, $\tilde{R}_5$ are all positively oriented (in the outgoing direction) and $\tilde{V}-\tilde{\varphi}$ has local maximum point at $y=0$. In this setting, as in Theorem 4.4.11, $\tilde{V}$ satisfies the corresponding condition of maximal subsolution in C $(y=0)$. By the previous definitions of $\tilde{f}_i$, $\tilde{\ell}_i$ and $\tilde{\varphi}$ follow that

$$\begin{aligned} \tilde{f}_i(y,a)=f_i(x,a), \quad \tilde{\ell}_i(y,a)=\ell_i(x,a), \\ \tilde{\varphi}_i(y,i)=\varphi_i(x,i), \quad \tilde{\varphi}_i'(y,i)=\varphi_i'(x,i), \end{aligned}$$

for $i=4,5$. Instead for $i=3$ we have that

$$\begin{aligned} \tilde{f}_3(y,a)=-f_3(x,a), \quad \tilde{\ell}_3(y,a)=\ell_3(x,a), \\ \tilde{\varphi}_3(y,3)=\varphi_3(x,3), \quad \tilde{\varphi}_3'(y,3)=-\varphi_3'(x,3). \end{aligned} \tag{4.118}$$

Hence,

$$-\tilde{f}_3(y,a)\tilde{\varphi}_3'(y,3) = -f_3(x,a)\varphi_3'(x,3), \tag{4.119}$$

namely, although we used a change of coordinate, we maintain the sign of the product dynamics-test function in the Hamiltonian H_3 and then we conclude that V verify the Ishii's condition of maximal subsolution on C $(x=k)$(the eighth equation in (4.115)) with f_i, ℓ_i and φ_i, for $i=3,4,5$. With the same arguments if we suppose that $V-\varphi$ as a local minimum point at $x=k$ we get that V satisfies the last equation in (4.115), then the proof is complete. □

Theorem 4.5.8. *Let u be a bounded and continuous subsolution of* (4.115). *Then $u \le V$ in $TR5$.*

Proof. We focus on the junction point B involving R_1, R_2, R_3 because for the junction point C the proof is similar. Suppose by contradiction that there exists $(z,i) \in \cup_{i=1}^3 R_i$ such that

$$(u-V)(z,i) = \max_{(x,i)\in\cup_{i=1}^3 R_i} (u-V)(x,i) > 0. \tag{4.120}$$

If $(z,i) \in int(R_i)$ for some $i \in \{1,2,3\}$ then we get a contradiction because by Theorem 4.5.7 and know comparison techniques V is a supersolution and u is a subsolution of the same HJB equation. Then we may restrict to the case in which $u-V$ has maximum at $z=0$ or at $z=k$. A said at the beginning of the proof we study the case $z=0$ (B), because the other one is analogous. If the optimal strategy at B is to stay inside a single branch without exiting, namely $V(0) = V_{sc(i)}(0)$, then we get the contradiction $u(0) \le V(0)$ (see Soner [92]). If the optimal behavior is to exit form k, that is $V(0) = V_{ssc(3-k)}(0)$ then V is a supersolution up to $z=0$, hence $u(0) \le V(0)$. Therefore the optimal strategy at B is to switch among the three branches. But also in this case, as in Theorem 4.4.13, we get that $u(0) \le V(0)$. Hence, $u \le V$ in $TR5$. □

Comparison and uniqueness results

Theorem 4.5.9. *(Comparison result for mixed problem)*
Let $u, v : [0,k] \to \mathbb{R}$ be a continuous subsolution and supersolution, respectively of

$$\begin{cases} \lambda\bar{u} + H_3(x,\bar{u}') \le 0 & in\ ]0,k[\\ \lambda\bar{u} + H_3(x,\bar{u}') \ge 0 & in\ [0,k[\\ \bar{u}(k) = u_*(k). \end{cases} \tag{4.121}$$

Moreover, there are $r>0$ and $\eta : [0,k] \to \mathbb{R}$ such that $x \mapsto -x + \frac{k}{2}$ with

$$]x + t\frac{k}{2} - rt, x + t\frac{k}{2} + rt[\subseteq]0,k[\ \forall\ x \in\ [0,k],\ 0 < t \le r.$$

Then $u \le v$ in $[0,k]$.

Proof. We assume by contradiction that exists $z \in \ [0,k] : (u-v)(z) > 0$. If $z \in \ [0,k[,\ \forall\ 0 < \varepsilon < r$ we define the following function in $[0,k] \times [0,k]$:

$$\Phi_\varepsilon(x,y) = u(x) - v(y) - \left|\frac{x-y}{\varepsilon} - \eta(z)\right|^2 - |y-z|^2.$$

We observe that Φ_ε is continuous in $[0,k]\times[0,k]$ and consider $(x_\varepsilon, y_\varepsilon)$ his point of maximum. By the definition of η, $z + \varepsilon\eta(z) \in\ ]0,k[$, so $\Phi_\varepsilon(z+\varepsilon\eta(z), z) \leq \Phi(x_\varepsilon, y_\varepsilon)$ and we obtain

$$\left|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right|^2 - |y_\varepsilon - z|^2 \leq u(x_\varepsilon) - v(y_\varepsilon) - (u-v)(z) + \omega(\varepsilon) \quad (4.122)$$

where ω is a modulus. By standard estimates (see Soner [92]) for small ε we get $x_\varepsilon \in]0,k[, y_\varepsilon \in [0,k[$ and

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \to \eta(z) \quad \text{and} \quad x_\varepsilon, y_\varepsilon \to z \quad \text{as} \quad \varepsilon \to 0. \quad (4.123)$$

Then, for any small ε, we have

$$u(x_\varepsilon) + H_3\left(x_\varepsilon, \frac{2}{\varepsilon}\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right)\right) \leq 0, \quad (4.124)$$

$$v(y_\varepsilon) + H_3\left(y_\varepsilon, \frac{2}{\varepsilon}\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \eta(z)\right) + 2(y_\varepsilon - z)\right) \geq 0, \quad (4.125)$$

and we conclude in the standard way getting $(u-v)(z) \leq 0$ for $\varepsilon \to 0$. Therefore $u \leq v$ in $[0,k[$.
If $z = k$, by the same arguments as before we get $x_\varepsilon \in]0,k[$, $y_\varepsilon \in [0,k]$ and

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \to \eta(k) \quad \text{and} \quad x_\varepsilon, y_\varepsilon \to k \quad \text{as} \quad \varepsilon \to 0. \quad (4.126)$$

Now, if $y_\varepsilon \in [0,k[$ then v satisfies (4.125) which compared with (4.124) gives a contradiction. If instead $y_\varepsilon = k$ then v satisfies

$$\min\{v(k) - u_*(k), \lambda v(k) + H_3(k, v')\} \geq 0. \quad (4.127)$$

By (4.122) follows that

$$v(y_\varepsilon) \leq v(k) + u(x_\varepsilon) - u(k) + \omega(\varepsilon),$$

so we use $\lim_{\varepsilon\to 0} x_\varepsilon \to k$ and the continuity of u to get

$$\lim_{\varepsilon \to 0} v(y_\varepsilon) \leq v(k). \quad (4.128)$$

Then by (4.126) and the continuity of the boundary datum we get

$$\lim_{\varepsilon \to 0} (v - u_*)(y_\varepsilon) = (v - u_*)(k) < 0. \quad (4.129)$$

Thus, for ε small enough, the boundary condition (4.127) gives (4.125) with $y_\varepsilon = k$ and as before we have a contradiction. Then $u \leq v$ in $z = k$.
Finally, combining the above results we get that $u \leq v$ in $[0,k]$. □

Theorem 4.5.10. *(Uniqueness)*
There exists a unique continuous function $v : [0, k] \to \mathbb{R}$ *which satisfies* (4.121)*, in viscosity sense.*

Proof. Let u_1 and u_2 be two viscosity solutions of (4.121). Considering u_1 as subsolution and u_2 as supersolution, by Theorem (4.5.9) we have that $u_1 \leq u_2$ in $[0, k]$. Changing the role of u_1 and u_2 we obtain equality $u_1 = u_2$ in $[0, k]$.
With the same arguments we obtain uniqueness also for (4.107). □

Chapter 5

Optimal control on a multi-domain

In this chapter we study an optimal control problem in two complementary domains of the space $\mathbb{R}^n$ as in Barles-Briani-Chasseigne [19]. After the basic assumptions of the problem on $\mathbb{R}^n$ given in Sect. 5.1, in Sect. 5.2 we focus on a multi-domain problem in $\mathbb{R}^2$. We approximate the problem by a delayed thermostat with threshold parameter $\varepsilon > 0$, and study the limit when ε goes to zero. We get that the limit function is a viscosity solution of a suitable Hamilton-Jacobi problem. In Sect. 5.3 we give some idea about a future research direction.

5.1 Basic assumptions

In this section we want to study the infinite horizon optimal control problem which has different dynamics and running costs in the two complementary domains of $\mathbb{R}^n$, namely

$$\mathcal{H}_1 = \{z \in \mathbb{R}^n : z \cdot S > 0\}, \quad \mathcal{H}_{-1} = \{z \in \mathbb{R}^n : z \cdot S < 0\}, \tag{5.1}$$

where S is a fixed unit vector of $\mathbb{R}^n$. Of course, the difficulty is to understand how to define the problem in

$$\Gamma = \overline{\mathcal{H}}_1 \cap \overline{\mathcal{H}}_{-1} = \{z \in \mathbb{R}^n : z \cdot S = 0\}, \tag{5.2}$$

where $\overline{\mathcal{H}}_i$ is the closure of $\mathcal{H}_i$. We start considering a controlled evolution on such a multi-domain of $\mathbb{R}^n$. On $\overline{\mathcal{H}}_i$ $(i = -1, 1)$ the system is driven by a continuous and bounded dynamics $f_i : \mathbb{R}^n \times A \to \mathbb{R}^n$, where A is compact, and

$$\begin{aligned} &\exists\ L > 0 \text{ such that } \forall z_1, z_2 \in \mathbb{R}^n, \forall a \in A : \\ &|f_i(z_1, a) - f_i(z_2, a)| \le L|z_1 - z_2|. \end{aligned} \tag{5.3}$$

Moreover the following condition holds

Controllability: $\exists \beta > 0$ such that for any $i = -1, 1$ and $z \in \Gamma$

$$\exists\ a_i^{'}, a_i^{''} \in A : f_i(z, a_i^{'}) \cdot S > \beta \quad \text{and} \quad f_i(z, a_i^{''}) \cdot S < -\beta. \tag{5.4}$$

The controlled system on the multi-domain is then

$$\begin{cases} z'(t) = f_i(z(t), \alpha(t)) & t > 0, \text{ if } z(t) \in \overline{\mathcal{H}}_i, \\ z(0) = z_0, \end{cases} \tag{5.5}$$

where $\alpha(\cdot)$ belongs to $\mathcal{A}$, the set of measurable controls. To this controlled system we associate an infinite horizon optimal control problem. For every $\overline{\mathcal{H}}_i$ we consider a running cost $\ell_i : \mathbb{R}^n \times A \to [0, +\infty[$, and the problem is given by the minimization, over all $\alpha \in \mathcal{A}$, of the cost functional

$$J(z_0, \alpha) = \int_0^{+\infty} e^{-\lambda t} \ell_i(z(t), \alpha(t)) dt. \tag{5.6}$$

In (5.6) the trajectory $z(t)$ is the solution of (5.5), $\lambda > 0$ is a fixed discount factor and the index i is given by $z(t) \in \overline{\mathcal{H}}_i$. Furthermore, for every i, the running cost ℓ_i is continuous, bounded and there exist $\tilde{L} > 0$ such that, for any $z_1, z_2 \in \mathbb{R}^n$ and for any i

$$|\ell_i(z_1, a) - \ell_i(z_2, a)| \leq \tilde{L}|z_1 - z_2|. \tag{5.7}$$

Finally, we consider the value function

$$V(z_0) = \inf_{\alpha \in \mathcal{A}} J(z_0, \alpha).$$

Of course, the concept of solution for the system (5.5) is not a-priori well-posed. When we are on Γ we can choose the index i how we prefer, but the existence of solution (trajectory) is not guaranteed, due to possible fast oscillations of the index i. Then we are going to use the delay relay operator and moreover we will restrict to the problem in $\mathbb{R}^2$. This is only a simplification although all what follows can be generalized for the problem in $\mathbb{R}^n$.

5.2 A multi-domain optimal control problem in $\mathbb{R}^2$

In this section we focus on the problem of multi-domain in $\mathbb{R}^2$, where $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ are two half-plane and Γ is the line $\{x = 0\}$ because we consider the unit vector $S = e_1 = (1, 0)$. We then consider a controlled dynamics which suddenly changes when passing from a plane to the other one. That is, denoting by $f : \mathbb{R}^2 \times A \to \mathbb{R}^2$, it is $f(z, \cdot) = f_1(z, \cdot)$ (resp. $f(z, \cdot) = f_{-1}(z, \cdot)$)

if $z \cdot e_1 > 0$ (resp. if $z \cdot e_1 < 0$), where $f_1 : [0, +\infty[\times \mathbb{R} \times A \to \mathbb{R}^2$ and $f_{-1} :]-\infty, 0] \times \mathbb{R} \times A \to \mathbb{R}^2$. The assumption (5.3) and (5.7) are still valid for the above dynamic. Concerning the controllability condition we rewrite it in this way:

Controllability: there is a $\beta > 0$ such that for any $i = -1, 1$ and $z \in \Gamma$

$$\exists\ a_i^{'}, a_i^{''} \in A : \Big(f_i(z, a_i^{'})\Big)_1 > \beta \quad \text{and} \quad \Big(f_i(z, a_i^{''})\Big)_1 < -\beta, \tag{5.8}$$

where $(f_i)_1 = f_i \cdot e_1$. For $\varepsilon > 0$ we approximate the problem by a delayed thermostatic problem (see Figure 5.1). Still denoting by f_1, f_{-1} two extensions by

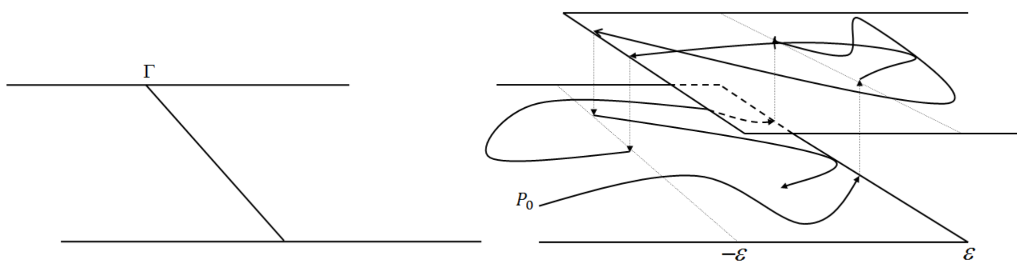


Figure 5.1: The two-domain problem and its thermostatic approximation.

constancy in the space variable $z \cdot e_1$ of the dynamics to $[-\varepsilon, +\infty[\times \mathbb{R} \times A$ and to $]-\infty, \varepsilon] \times \mathbb{R} \times A$ respectively, we may consider the controlled thermostatic system

$$\begin{cases} z'(t) = f_{i}(t)(z(t), \alpha(t)) & t > 0, \\ i(t) = h_\varepsilon[z(\cdot) \cdot e_1](t) & t \geq 0, \\ z(0) = z_0, \quad i(0) = i_0, \end{cases} \tag{5.9}$$

where $h_\varepsilon[\cdot]$ represent the thermostatic delayed relationship between the input $z(\cdot) \cdot e_1$ and the output i. The initial state (z_0, i_0) is admissible if $i_0 = 1$ and $z_0 \cdot e_1 \geq -\varepsilon$, or $i_0 = -1$ and $z_0 \cdot e_1 \leq \varepsilon$. Moreover, fixed the thresholds $-\varepsilon, \varepsilon$, for each continuous input $t \to z(t)$, and for each initial output $i_0 \in \{-1, 1\}$ coherent with $z(0)$, there exists a unique output $t \to i(t) =: h_\varepsilon[z(\cdot) \cdot e_1](t) \in \{-1, 1\}$ satisfying $i(0) = i_0$. Now, since the two extended function f_1 and f_{-1} are bounded, continuous and Lipschitz continuous, and for a coherent initial state (z_0, i_0) there exists a unique solution $(z(t), i(t))$ of the system (5.9) (as in Proposition 4.1.1). For a rigorous as well as natural definition of the trajectory we refer to Bagagiolo [8].
The solution $(z(t), i(t))$ of (5.9) can be seen as a trajectory starting from (z_0, i_0) and evolving in the subset of $\mathbb{R}^2 \times \mathbb{R}$

$$\overline{\mathcal{H}}^\varepsilon := \overline{\mathcal{H}}_1^\varepsilon \cup \overline{\mathcal{H}}_{-1}^\varepsilon, \tag{5.10}$$

where

$$\begin{aligned}\overline{\mathcal{H}}_1^\varepsilon &:= \{(z,1)\in\mathbb{R}^2\times\{1\} : z\cdot e_1 \geq -\varepsilon\},\\ \overline{\mathcal{H}}_{-1}^\varepsilon &:= \{(z,-1)\in\mathbb{R}^2\times\{-1\} : z\cdot e_1 \leq \varepsilon\},\end{aligned} \tag{5.11}$$

with the switching rule from one connected component $\overline{\mathcal{H}}_{i_0}^\varepsilon$ to the other one, given by the delayed relay. In the sequel, for every $i_0 \in \{-1,1\}$, we will use respectively the notations: $\partial\mathcal{H}_1^\varepsilon := \{(z,1)\in\mathbb{R}^2\times\{1\} : z\cdot e_1 = -\varepsilon\}$ and $\partial\mathcal{H}_{-1}^\varepsilon := \{(z,-1)\in\mathbb{R}^2\times\{-1\} : z\cdot e_1 = \varepsilon\}$. With the latter ones the closure of $\mathcal{H}_{i_0}^\varepsilon$ is given by $\overline{\mathcal{H}}_{i_0}^\varepsilon = \mathcal{H}_{i_0}^\varepsilon \cup \partial\mathcal{H}_{i_0}^\varepsilon$. To the controlled system (5.9) we associate an infinite horizon optimal control problem. Then we consider the minimization, over all measurable controls $\alpha\in\mathcal{A}$, of the cost functional

$$J(z_0,i_0,\alpha) := \int_0^{+\infty} e^{-\lambda s}\ell_{i(s)}(z(s),\alpha(s))ds. \tag{5.12}$$

where $\lambda>0$ is a fixed discount factor and the function ℓ satisfies the regularity assumptions (5.7). The value function is defined by

$$V_\varepsilon(z_0,i_0) = \inf_{\alpha\in\mathcal{A}} J(z_0,i_0,\alpha), \tag{5.13}$$

and we also have for every i the Hamiltonians

$$H_i(z,p) = \sup_{a\in A}\{-f_i(z,a)\cdot p - \ell_i(z,a)\}. \tag{5.14}$$

where we drop the index i in the entries of f_i, ℓ_i and consequently in H_i.

The continuity of the value function (5.13) comes from by controllability (5.8), regularity ((5.3) and (5.7)) and by the same procedures used in Bagagiolo [9]. For more explicit steps see the proof of Theorem 5.2.7. The uniqueness of the solution of (5.9) and the semigroup property of the delayed relay (see (3.3) in Sect. 3.1) imply that the following result is standard.

Proposition 5.2.1. *The Dynamical Programming Principle holds: for all $t\geq 0$ and for all $(z_0,i_0)\in\overline{\mathcal{H}}^\varepsilon$*

$$V_\varepsilon(z_0,i_0) = \inf_{\alpha\in\mathcal{A}}\left\{\int_0^{+\infty} e^{-\lambda s}\ell_{i(s)}(z(s),\alpha(s))ds + e^{-\lambda t}V_\varepsilon(z(t),i(t))\right\}. \tag{5.15}$$

Let us define the *first switching time*

$$t_{(z_0,i_0)}(\alpha) := \inf\{t\geq 0 : i(t)\neq i_0\} \quad \forall\ (z_0,i_0)\in\overline{\mathcal{H}}^\varepsilon, \forall\alpha\in\mathcal{A}, \tag{5.16}$$

with $t_{(z_0,i_0)}(\alpha) = +\infty$ if the set in the right-hand side of (5.16) is empty.

Proposition 5.2.2. *For every $(z,i) \in \overline{\mathcal{H}}^{\varepsilon}$ let $\rho^{(i)}$ be the point of the plane of $\overline{\mathcal{H}}^{\varepsilon}$ in which the eventual switching occurs and containing i (i.e $\rho^{(i)}$ is $(-\varepsilon, y(t_{(z,i)}(\alpha)))$ if $i = 1$ and $(\varepsilon, y(t_{(z,i)}(\alpha)))$ if $i = -1$). Then we have*

$$V_{\varepsilon}(z,i) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{t_{(z,i)}(\alpha)} e^{-\lambda s} \ell_i(z(s), \alpha(s)) ds + e^{-\lambda t_{(z,i)}(\alpha)} V_{\varepsilon}(\rho^{(i)}, -i) \right\}. \tag{5.17}$$

Proof. Let us suppose for instance $i = 1$ and hence $z \in \overline{\mathcal{H}}_1^{\varepsilon}$. If $t_{(z,i)}(\alpha) = +\infty$ then (5.17) is just the definition of V_{ε}. If instead $t_{(z,i)}(\alpha) < +\infty$, by (5.8) there exists $\overline{a} \in A$ such that $\left(f_1(\rho^{(1)}, \overline{a})\right)_1 < -\beta$ and since still by controllability (5.8) follows that $V_{\varepsilon}(\rho^{(1)}, 1) \le V_{\varepsilon}(\rho^{(1)}, -1)$, applying (DPP) (Proposition 5.2.1), we have the "$\le$" inequality in (5.17).
To prove the other inequality, let us take $\beta > 0$ and $\widetilde{\alpha}$ such that $J(z,1,\widetilde{\alpha}) - \beta \le V_{\varepsilon}(z,1)$. We can suppose $\bar{t} := t_{(z,1)}(\widetilde{\alpha}) < +\infty$ (the other case is easier). Take $\zeta > 0$ small enough such that every trajectory starting from $(\rho^{(1)}, -1)$ does not switch in the time interval $[0, \zeta]$. Let $(z_{\zeta}, -1)$ be the point reached after the time $\bar{t} + \zeta$ and note that $V_{\varepsilon}(z_{\zeta}, -1) \to V_{\varepsilon}(\rho^{(1)}, -1)$ as $\zeta \to 0$. Let us denote the integral in (5.17) by I_1. We have

$$\begin{aligned} &V_{\varepsilon}(z,1) \ge J(z,1,\widetilde{\alpha}) - \beta \ge \\ &I_1 + e^{-\lambda(\bar{t}+\zeta)} J(z_{\zeta}, -1, \widetilde{\alpha}(\cdot + \bar{t} + \zeta)) - \beta \ge \\ &I_1 + e^{-\lambda(\bar{t}+\zeta)} V_{\varepsilon}(z_{\zeta}, -1) - \beta. \end{aligned} \tag{5.18}$$

We pass to the limit as $\zeta \to 0$ in (5.18) and conclude by the arbitrariness of β. □

Proposition 5.2.3. *The value function V_{ε} (5.13) is bounded and uniformly continuous on each of the two connected components of $\overline{\mathcal{H}}^{\varepsilon}$.*

Proof. From Proposition 5.2.2, in each connected component $\overline{\mathcal{H}}_i^{\varepsilon}$, $i \in \{-1, 1\}$, V_{ε} is the value function of the exit time problem from $\overline{\mathcal{H}}_i^{\varepsilon}$ with exit cost $V_{\varepsilon}(\rho^{(i)}, -i)$. Since by (5.8) $V_{\varepsilon}(\rho^{(i)}, -i)$ is bounded and continuous on $\partial\mathcal{H}_i^{\varepsilon}$ we can conclude applying the result of Bardi-Capuzzo Dolcetta [16], Chapter IV, Section 3. □

Proposition 5.2.4. *The value function V_{ε} (5.13) is a bounded and uniformly continuous solution of the following problem for Hamilton-Jacobi equations*

$$\begin{cases} \forall i \in \{-1,1\}, V_{\varepsilon} \text{ is the unique bounded and} \\ \text{uniformly continuous viscosity solution} \\ \text{of the following Dirichlet problem in } \overline{\mathcal{H}}_i^{\varepsilon}, \\ \text{with boundary conditions in the viscosity sense} \\ \begin{cases} \lambda V_{\varepsilon}(z,i) + H_i\left(z, \nabla V_{\varepsilon}(z,i)\right) = 0 & \text{in } \mathcal{H}_i^{\varepsilon}, \\ V_{\varepsilon}(\cdot, i) = V_{\varepsilon}(\cdot, -i) & \text{on } \partial\mathcal{H}_i^{\varepsilon}. \end{cases} \end{cases} \tag{5.19}$$

Proof. The boundedness and the uniformly continuity come from Proposition 5.2.2 and Proposition 5.2.3. Moreover V_ε is a solution of (5.19) because in each plane, by virtue of (5.17), is the value function of the exit time problem with f_i, ℓ_i and exit cost $V_\varepsilon(\cdot, -i)$. Being $V_\varepsilon(\cdot, -i)$ bounded and continuous on $\partial\mathcal{H}_i^\varepsilon$, by Proposition 1.2.16 follows that V_ε is solution of the system in (5.19) on each plane. The uniqueness come from the results of Chapter V of Bardi-Capuzzo Dolcetta [16] (see in particular Theorem 4.20). □

Proposition 5.2.5. *The value function V_ε (5.13) is the only bounded and continuous function on $\overline{\mathcal{H}}^\varepsilon$ which is solution of (5.19).*

Proof. The proof follows the same arguments of ones given in [8] (see Proposition 3.6) and in Proposition 4.4.5 for the threefold junction problem. □

Before proving the following Theorem 5.2.7 we have to clarify what happens when we are on $\Gamma = \{x = 0\}$. To do so, as done for the twofold junction problem Sect. 4.3, we follow the pioneering work of Filippov [54] and we get that there exist trajectories which stay on Γ at least for a while. Such trajectories are built through a dynamics of the form

$$f_\Gamma(z, (\mu, a_1, a_{-1})) := \mu f_1(z, a_1) + (1-\mu) f_{-1}(z, a_{-1}), \tag{5.20}$$

for any $z \in \Gamma$ and with $(\mu, a_1, a_{-1}) \in A_0(z)$ so defined

$$\begin{aligned} A_0(z) = &\{a = (\mu, a_1, a_{-1}) \in [0,1] \times A \times A : \\ &f_\Gamma(z,a) \cdot e_1 = 0, f_1(z, a_1) \cdot e_1 \leq 0, f_{-1}(z, a_{-1}) \cdot e_1 \geq 0\}. \end{aligned} \tag{5.21}$$

The associated running cost is

$$\ell_\Gamma(z, a) = \ell_\Gamma(z, (\mu, a_1, a_{-1})) := \mu \ell_1(z, a_1) + (1-\mu)\ell_{-1}(z, a_{-1}). \tag{5.22}$$

Using (5.20) and (5.22) we define the following Hamiltonian on Γ

$$H_T(z, \nabla_\Gamma u) = \sup_{A_0(z)} \{-f_\Gamma(z,a) \cdot \nabla_\Gamma u - \ell_\Gamma(z,a)\}, \tag{5.23}$$

where $\nabla_\Gamma u$ is the gradient of u with respect to the Γ-variable y, i.e. $\partial u/\partial y$
Now, given V_ε (5.13), we define the function

$$\tilde{V}_\varepsilon : \mathbb{R}^2 \setminus \{x = 0\} \to \mathbb{R}, \quad \tilde{V}_\varepsilon(x,y) = \begin{cases} V_\varepsilon(x,y,1) & x > 0, \ y \in \mathbb{R}, \\ V_\varepsilon(x,y,-1) & x < 0, \ y \in \mathbb{R}. \end{cases} \tag{5.24}$$

Note that in the above expression we consider the two components (x, y) of the variable $z \in \mathbb{R}^2$ in order to make clearer the following Theorem 5.2.7 whose proof requires the following lemma.

Lemma 5.2.6. *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that, for some $\varphi \in C^1$, the function $w - \varphi$ has a strict local minimum (a strict local maximum) at a point $\bar{z} \in \mathbb{R}^2$. If $w_\varepsilon \to w$ uniformly, then there exists a sequence of points $z_\varepsilon \to \bar{z}$ with $w_\varepsilon(z_\varepsilon) \to w(\bar{z})$ and such that $w_\varepsilon - \varphi$ has a local minimum (a local maximum) at z_ε.*

Proof. See for example Bressan [31] □

Theorem 5.2.7. *As $\varepsilon \to 0^+$, the sequence of functions $\tilde{V}_\varepsilon$ (5.24) uniformly converges on $\mathbb{R}^2 \setminus \{x = 0\}$ to a continuous function $\tilde{V}_1$, $\tilde{V}_{-1}$ respectively. If* (5.3) *and* (5.7) *hold, then $\tilde{V}$ uniquely continuously extends to the line $\{x = 0\}$ and if* (5.8) *also hold, it satisfies in viscosity sense*

$$\begin{cases} \lambda u + H_1(x, y, \nabla u) = 0 & \text{in } \mathcal{H}_1, \\ \lambda u + H_{-1}(x, y, \nabla u) = 0 & \text{in } \mathcal{H}_{-1}, \\ \min\{\lambda u + H_1(x, y, \nabla u), \lambda u + H_{-1}(x, y, \nabla u)\} \le 0 & \text{on } \Gamma, \\ \max\{\lambda u + H_1(x, y, \nabla u), \lambda u + H_{-1}(x, y, \nabla u)\} \ge 0 & \text{on } \Gamma. \end{cases} \tag{5.25}$$

Proof. We use Ascoli-Arzelà theorem to prove the uniform convergence starting from the function V_ε. The boundedness of V_ε follows by Proposition 5.2.3. To prove the uniform equicontinuity we consider any two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that $x_1, x_2 \ge 0$, a constant $\mu > 0$ and a control α_μ which is "μ-optimal" for $V_\varepsilon(x_2, y_2, 1)$. Then

$$V_\varepsilon(x_2, y_2, 1) \ge \int_0^\infty \ell_{i_2(t)}(x(t, (x_2, y_2, 1), \alpha_\mu), y(t, (x_2, y_2, 1), \alpha_\mu), \alpha_\mu(t))e^{-\lambda t}dt - \mu.$$

Computing the difference between the value functions in the two given points, we get

$$\begin{aligned} & V_\varepsilon(x_1, y_1, 1) - V_\varepsilon(x_2, y_2, 1) \le \\ & \int_0^\infty \ell_{i_1(t)}(x(t; (x_1, y_1, 1), \alpha_\mu), y(t; (x_1, y_1, 1), \alpha_\mu), \alpha_\mu(t))e^{-\lambda t}dt - \\ & \int_0^\infty \ell_{i_2(t)}(x(t; (x_2, y_2, 1), \alpha_\mu), y(t; (x_2, y_2, 1), \alpha_\mu), \alpha_\mu(t))e^{-\lambda t}dt + \mu \le \\ & \int_0^T \ell_{i_1(t)}(x(t; (x_1, y_1, 1), \alpha_\mu), y(t, (x_1, y_1, 1), \alpha_\mu), \alpha_\mu(t))e^{-\lambda t}dt - \\ & \int_0^T \ell_{i_2(t)}(x(t; (x_2, y_2, 1), \alpha_\mu), y(t, (x_2, y_2, 1), \alpha_\mu), \alpha_\mu(t))e^{-\lambda t}dt + \\ & 2\int_T^\infty Me^{-\lambda t}dt + \mu. \end{aligned} \tag{5.26}$$

Moreover, for T large enough, we can bound from above the last integral of (5.26) with μ and, considering the modules, we get

$$\begin{aligned}
&\leq \int_0^T |\ell_{i_1(t)}(\cdot,\cdot,\cdot) - \ell_{i_2(t)}(\cdot,\cdot,\cdot)|dt + 3\mu \leq \\
&\int_0^{t_1} |\ell_1(\cdot,\cdot,\cdot) - \ell_1(\cdot,\cdot,\cdot)|dt + \int_{t_1}^T |\ell_{i_1(t)}(\cdot,\cdot,\cdot) - \ell_{i_2(t)}(\cdot,\cdot,\cdot)|dt + 3\mu,
\end{aligned} \tag{5.27}$$

with t_1 the first switching time for the trajectory starting from (x_2, y_2), assuming that this is the first one that switches. Then, using the Gronwall inequality, we have that (5.27) is

$$\begin{aligned}
&\leq \tilde{L}t_1 e^{Lt_1} \parallel (x_1, y_1) - (x_2, y_2) \parallel + \int_{t_1}^{t^1} |\ell_1(\cdot,\cdot,\cdot) - \ell_{-1}(\cdot,\cdot,\cdot)|dt + \\
&\int_{t^1}^T |\ell_{i_1(t)}(\cdot,\cdot,\cdot) - \ell_{i_2(t)}(\cdot,\cdot,\cdot)|dt + 3\mu,
\end{aligned} \tag{5.28}$$

where t^1 is the first switching time for the trajectory starting from (x_1, y_1). Note that if $t_1 = t^1$, then the two trajectories reach the boundary $\partial\mathcal{H}_1^\varepsilon$ at same instant and the corresponding integral in (5.28) is null. Suppose that $t_1 \neq t^1$ and since in t^1 the trajectory starting from (x_1, y_1) is on $\partial\mathcal{H}_1^\varepsilon$ by controllability (5.8) we can use a suitable control so that the trajectory switches, hence we can bound from above the first integral of (5.28) with

$$2M(t^1 - t_1) \leq 2M\tilde{L}Te^{LT} \parallel (x_1, y_1) - (x_2, y_2) \parallel . \tag{5.29}$$

Continuing to break the last integral in (5.28) on N-intervals, where N is finite (in $[0, T]$) because the switching are delayed and the dynamics are bounded, we get

$$\begin{aligned}
&V_\varepsilon(x_1, y_1, 1) - V_\varepsilon(x_2, y_2, 1) \leq \\
&C(T) \parallel (x_1, y_1) - (x_2, y_2) \parallel + \\
&\sum_{i=1}^{N_1} \int_{I_i} |\ell_{i(\cdot)}(\cdot,\cdot,\cdot) - \ell_{-i(\cdot)}(\cdot,\cdot,\cdot)|dt + \\
&\sum_{i=1}^{N_2} \int_{I_i} |\ell_{i(\cdot)}(\cdot,\cdot,\bar{\alpha}_\mu(\cdot)) - \ell_{i(\cdot)}(\cdot,\cdot,\alpha_\mu(\cdot))|dt + 3\mu,
\end{aligned} \tag{5.30}$$

where: the first addendum of the right-hand side of (5.30) represents the estimate relative to the sum of the integrals on time intervals in which the two trajectories do not switch, and $C(T) > 0$ is the constant wherewith we identify the sum of all constants given by each integral. The second addendum is the sum of the integrals on the time intervals I_i, in which the trajectory starting from (x_2, y_2) jumps while the one starting from (x_1, y_1)

does not jump. To each of this N_1-integral (N_1 is finite) we can apply the same argument of (5.29). While the last addendum is the finite sum of integrals where the trajectory starting from (x_2, y_2) does not jump using the control α_μ. Hence we impose that also the trajectory starting from (x_1, y_1) does not jump exploiting the result in Soner [92]. It states that for a suitable ball B around (x_2, y_2), and suitable constants $\bar{C}, C^*$ (that depend on the instant of no switching for the trajectory starting from (x_2, y_2)), for every point $(x_1, y_1) \in B \cap \overline{\mathcal{H}}_1^\varepsilon$, there exists a control $\bar{\alpha}_\mu$ such that the corresponding trajectory does not switch and the following hold:

$$\begin{aligned}&\left|\left(x_{(x_1,y_1,1)}(\cdot\,;\bar{\alpha}_\mu), y_{(x_1,y_1,1)}(\cdot\,;\bar{\alpha}_\mu)\right) - \left(x_{(x_2,y_2,1)}(\cdot\,;\alpha_\mu), y_{(x_2,y_2,1)}(\cdot\,;\alpha_\mu)\right)\right| \\ &\leq \bar{C}(\cdot) \parallel (x_1, y_1) - (x_2, y_2) \parallel,\end{aligned}$$

$$\int_{I_i} |\ell_{i(\cdot)}(\cdot,\cdot,\bar{\alpha}_\mu(\cdot)) - \ell_{i(\cdot)}(\cdot,\cdot,\alpha_\mu(\cdot))| dt \leq C^*(\cdot) \parallel (x_1, y_1) - (x_2, y_2) \parallel .$$

With the above considerations over all integrals of (5.30) we can conclude that

$$V_\varepsilon(x_1, y_1, 1) - V_\varepsilon(x_2, y_2, 1) \leq \mathcal{C}(T) \parallel (x_1, y_1) - (x_2, y_2) \parallel + 3\mu \quad \forall \varepsilon, \tag{5.31}$$

where $\mathcal{C}(T)$ is the sum of all constants given by the estimations of all integrals in (5.30).

Similarly, if we consider $V_\varepsilon(\cdot,\cdot,-1)$ and any two point $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that $x_1, x_2 \leq 0$ we have

$$V_\varepsilon(x_1, y_1, -1) - V_\varepsilon(x_2, y_2, -1) \leq \mathcal{C}'(T) \parallel (x_1, y_1) - (x_2, y_2) \parallel + 3\mu \quad \forall \varepsilon, \tag{5.32}$$

where $\mathcal{C}'(T)$ as $\mathcal{C}(T)$ is the sum of all constants.

Now, to prove the uniform equicontinuity of $V_\varepsilon(\cdot,\cdot,i)$ we have to show that:

$$\forall \eta > 0\ \exists \delta > 0 : \parallel (x_1, y_1) - (x_2, y_2) \parallel \leq \delta \Rightarrow | V_\varepsilon(x_1, y_1, i) - V_\varepsilon(x_2, y_2, i) | \leq \eta,$$

for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $i = -1, 1$. Then fix $\eta > 0$ and $\mu \in]0, \frac{\eta}{6}]$. If $i = 1$ by (5.31) we get $\delta \leq \eta/2\mathcal{C}(T)$. If instead $i = -1$ we obtain that $\delta \leq \eta/2\mathcal{C}'(T)$.

Hence $\tilde{V}_\varepsilon$ is equicontinuous and by Ascoli-Arzelà theorem there exists a subsequence of $\tilde{V}_\varepsilon$ that uniformly converges in the compact subsets of $\mathcal{H}_i$. Denoting this subsequence with V_ε we have:

$$V_\varepsilon(x, y, 1) \rightrightarrows \tilde{V}_1(x, y) \quad \text{for } x > 0, y \in \mathbb{R} \text{ on compact sets of } \mathcal{H}_1,$$

$$V_\varepsilon(x, y, -1) \rightrightarrows \tilde{V}_{-1}(x, y) \quad \text{for } x < 0, y \in \mathbb{R} \text{ on compact sets of } \mathcal{H}_{-1},$$

where $\rightrightarrows$ denotes the uniform convergence.

Now, in order to prove that $\tilde{V}$ continuously extends to the line $\{x = 0\}$

we show that for ε small enough the difference $|V_\varepsilon(0,y,1) - V_\varepsilon(0,y,-1)|$ is infinitesimal as ε. Let σ_ε be the time used to switch from a threshold to another one and such that goes to 0 as $\varepsilon \to 0$. We then consider a measurable control $\bar{\alpha}$ that allows to switch after σ_ε and applying Proposition (5.2.1) we get

$$\begin{aligned} V_\varepsilon(0,y,1) \leq & \int_0^\infty e^{-\lambda t} \ell_{i(t)}(x(t), y(t), \bar{\alpha}(t))dt \leq \\ & \int_0^{\sigma_\varepsilon} e^{-\lambda t} \ell_1(x(t), y(t), \bar{\alpha}(t))dt + e^{-\lambda \sigma_\varepsilon} V_\varepsilon(0, y(\tau_\varepsilon), -1), \end{aligned}$$

therefore

$$V_\varepsilon(0,y,1) - V_\varepsilon(0,y(\tau_\varepsilon),-1)e^{-\lambda\sigma_\varepsilon} \leq \int_0^{\sigma_\varepsilon} e^{-\lambda t} \ell_1(x(t), y(t), \bar{\alpha}(t))dt,$$

whence as $\varepsilon \to 0$, $\sigma_\varepsilon \to 0$ we obtain

$$\tilde{V}_1(0,y) - \tilde{V}_{-1}(0,y) \leq 0. \tag{5.33}$$

By the same arguments as above starting from $V_\varepsilon(0,y,-1)$ we have

$$\tilde{V}_{-1}(0,y) - \tilde{V}_1(0,y) \leq 0. \tag{5.34}$$

By (5.33) and (5.34) follows that $\tilde{V}_1(0,y) = \tilde{V}_{-1}(0,y)$ $\forall y \in \Gamma$ and hence $|V_\varepsilon(0,y,1) - V_\varepsilon(0,y,-1)| = O(\varepsilon)$. Then we define

$$\tilde{V}(x,y) = \begin{cases} \tilde{V}_1(x,y) & \text{if } x \geq 0,\ y \in \mathbb{R}, \\ \tilde{V}_{-1}(x,y) & \text{if } x \leq 0,\ y \in \mathbb{R}, \end{cases}$$

which is continuous as consists by functions that are uniform limits of the corresponding V_ε continuous themselves.
Afterward to provide the uniform convergence of $V_\varepsilon(x,y,i)$ on $\overline{\mathcal{H}}_i$ we take a pair of thresholds $(\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 > 0$ and both converging to zero, and show that

$$\| V_{\varepsilon_1}(x,y,i) - V_{\varepsilon_2}(x,y,i) \|_{C^0(\overline{\mathcal{H}}_i, \mathbb{R})} \leq \mathcal{K} \max\{\varepsilon_1, \varepsilon_2\} \ \forall\ (x,y) \in \overline{\mathcal{H}}_i, \mathcal{K} > 0.$$

This inequality holds since, by controllability, the optimal behavior for the problem with ε_2 (assuming to be the smaller) is also optimal for the problem with ε_1 less then $O(\varepsilon_1 - \varepsilon_2)$. Then V_ε is a Cauchy sequence and hence uniformly convergences on all $\overline{\mathcal{H}}_i$.
Now, we prove that $\tilde{V}(x,y)$ satisfies (5.25). From Proposition 5.2.4 and by classical convergence result we have

$$\lambda \tilde{V}_1(x,y) + \sup_{a \in A} \left\{ -f_1(x,y,a) \nabla \tilde{V}_1(x,y) - \ell_1(x,y,a) \right\} = 0 \text{ in } \mathcal{H}_1, \tag{5.35}$$

in the viscosity sense. Similarly for the equation in $\mathcal{H}_{-1}$.
We now prove the third equation in (5.25). Let $\varphi \in C^1(\mathbb{R}^2)$ be a test function such that $\tilde{V} - \varphi$ has a strictly maximum at $(0, \bar{y})$. Let $\{\varepsilon\}$ be a sequence such that $\varepsilon \to 0$, and by Proposition 5.2.8 we have that, for ε small enough, the maximum point of $V_\varepsilon(\cdot, \cdot, 1) - \varphi(\cdot, \cdot)$ does not belongs to

$$\partial(B((0, \bar{y}), r)) \setminus \left\{(x, y) \in \mathbb{R}^2 \mid x \leq -\varepsilon\right\} = \partial\vartheta_\varepsilon,$$

and that the maximum point of $V_\varepsilon(\cdot, \cdot, -1) - \varphi(\cdot, \cdot)$ does not belong to

$$\partial(B((0, \bar{y}), r)) \setminus \left\{(x, y) \in \mathbb{R}^2 : x \geq \varepsilon\right\}.$$

Then, there exists a sequence $(x_\varepsilon, y_\varepsilon) \in int(B((0, \bar{y}), r)) \cap \overline{\mathcal{H}}_1^\varepsilon$ of local maximum for $V_\varepsilon(\cdot, \cdot, 1) - \varphi$ which converges to $(0, \bar{y})$. We may have two case: 1) at least for a subsequence, at $(x_\varepsilon, y_\varepsilon)$ the HJB equation satisfied by $V_\varepsilon(\cdot, \cdot, 1)$ has the right sign "$\leq$", 2) it is definitely true that the boundary point $(x_\varepsilon, y_\varepsilon) = (-\varepsilon, y_\varepsilon)$ is a strict maximum point and the HJB equation has the wrong sign "$>$".

Case 1). Sending $\varepsilon \to 0$ we get $\lambda\tilde{V} + H_1 \leq 0$ in $(0, \bar{y})$ and hence the fourth equation in (5.25).

Case 2). Since the boundary conditions in (5.19) are in the viscosity sense and by the controllability (5.8), we have

$$V_\varepsilon(-\varepsilon, y_\varepsilon, 1) = V_\varepsilon(-\varepsilon, y_\varepsilon, -1). \tag{5.36}$$

Now, the same reasoning and cases also hold in $int(B((0, \bar{y}), r)) \cap \overline{\mathcal{H}}_{-1}^\varepsilon$. If the corresponding Case 1) holds, then we obtain the conclusion as before. Otherwise we get

$$V_\varepsilon(\varepsilon, \hat{y}_\varepsilon, -1) = V_\varepsilon(\varepsilon, \hat{y}_\varepsilon, 1). \tag{5.37}$$

We observe that $(-\varepsilon, y_\varepsilon, -1) \in int(B((0, \bar{y}), r)) \cap \overline{\mathcal{H}}_{-1}^\varepsilon$ and $(\varepsilon, \hat{y}_\varepsilon, 1) \in int(B((0, \bar{y}), r)) \cap \overline{\mathcal{H}}_1^\varepsilon$, therefore, using (5.36) and (5.37), we get the following contradiction which concludes the proof

$$\begin{aligned}
&V_\varepsilon(-\varepsilon, y_\varepsilon, -1) - \varphi(-\varepsilon, y_\varepsilon) < V_\varepsilon(\varepsilon, \hat{y}_\varepsilon, -1) - \varphi(\varepsilon, \hat{y}_\varepsilon) = \\
&V_\varepsilon(\varepsilon, \hat{y}_\varepsilon, 1) - \varphi(\varepsilon, \hat{y}_\varepsilon) < V_\varepsilon(-\varepsilon, y_\varepsilon, 1) - \varphi(-\varepsilon, y_\varepsilon) = \\
&V_\varepsilon(-\varepsilon, y_\varepsilon, -1) - \varphi(-\varepsilon, y_\varepsilon).
\end{aligned}$$

In order to prove the fourth equation in (5.25), we proceed in the same way. □

Proposition 5.2.8. *Let $(0, \bar{y}) \in \mathbb{R}^2$ and $r > 0$ such that $\tilde{V} - \varphi$ has a strict maximum point in $(0, \bar{y})$ with respect to $B((0, \bar{y}), r)$. For every $\varepsilon > 0$ let $(x_\varepsilon, y_\varepsilon)$ be a maximum point for $V_\varepsilon - \varphi$ with respect to $\bar{\vartheta}_\varepsilon$, with*

$$\bar{\vartheta}_\varepsilon = B((0, \bar{y}), r) \setminus \left\{(x, y) \in \mathbb{R}^2 \mid x \leq -\varepsilon\right\}.$$

Then $(x_\varepsilon, y_\varepsilon) \to (0, \bar{y})$ for $\varepsilon \to 0$.

Proof. Let $(x_{\varepsilon_k}, y_{\varepsilon_k})$, $k \in \mathbb{N}$, be any convergent sub sequence of $(x_\varepsilon, y_\varepsilon)$ to $(\tilde{x}, \tilde{y}) \in \bar{\vartheta}_0 = B((0,\bar{y}), r) \setminus \{(x,y) \in \mathbb{R}^2 : x \le 0\}$. Still denoting this sub sequence with $(x_\varepsilon, y_\varepsilon)$, by the uniform convergence of V_ε to $\tilde{V}$ we have

$$V_\varepsilon(x_\varepsilon, y_\varepsilon, 1) \to \tilde{V}(\tilde{x}, \tilde{y}) \quad \text{as } \varepsilon \to 0. \tag{5.38}$$

By the choice of such convergent sub sequence and by (5.38) follows that

$$\tilde{V}(\tilde{x}, \tilde{y}) - \varphi(\tilde{x}, \tilde{y}) \ge \tilde{V}(x, y) - \varphi(x, y) \quad \forall\ (x, y) \in \bar{\vartheta}_0.$$

Moreover, $(0, \bar{y}) \in \bar{\vartheta}_0$ hence

$$\tilde{V}(\tilde{x}, \tilde{y}) - \varphi(\tilde{x}, \tilde{y}) \ge \tilde{V}(0, \bar{y}) - \varphi(0, \bar{y}).$$

Since $(0, \bar{y})$ is a strict maximum point with respect to $B((0, \bar{y}), r)$, we have that $(\tilde{x}, \tilde{y}) = (0, \bar{y})$ and $(x_\varepsilon, y_\varepsilon) \to (0, \bar{y})$. This ensures that the maximum point of $V_\varepsilon - \varphi$ does not belong to $\partial\vartheta_\varepsilon$. □

We now come back to the Hamiltonian H_T (5.23) and consider the follow HJB equation

$$\lambda u + H_T\,(x, y, \nabla_\Gamma u) \le 0 \text{ on } \Gamma. \tag{5.39}$$

In Barles-Briani-Chasseigne [19], considering further assumptions one proves that if u is a subsolution of (5.25) then it is a subsolution of (5.39). Hence, in our case, since we prove that $\tilde{V}$ is a a viscosity solution of (5.25) then it is a subsolution of (5.39). This means that $\forall \varphi \in C^1(\mathbb{R}), \forall \bar{y} \in \mathbb{R}$ such that $\tilde{V}(0, \cdot) - \varphi(\cdot)$ has a local maximum at $\bar{y}$ with respect to Γ, then

$$\lambda \tilde{V}(0, \bar{y}) + H_T((0, \bar{y}), \varphi'(\bar{y})) \le 0. \tag{5.40}$$

We now give some ideas how to prove (5.40) using the thermostatic approximation.
To do so, we first remark that, changing $\varphi(y)$ in $\varphi(y) - |y - \bar{y}|^2$, we can assume that $\bar{y}$ is a strict local maximum point of $\tilde{V} - \varphi$. Then by Lemma 5.2.6 it follows that $V_\varepsilon - \varphi$ has a local maximum point in y_ε. Moreover we take a_1, a_{-1} as in (5.20) such that $f_1(0, \bar{y}, a_1) \cdot e_1 < 0$ and $f_{-1}(0, \bar{y}, a_{-1}) \cdot e_1 > 0$. Note that these conditions on the dynamics can be extended to a neighborhood of $\bar{y}$ of radius ε for ε small enough, and hence to y_ε. Then, calling t^ε the time using for a complete switching starting from y_ε and using $f_1(\cdot, \cdot, a_1), f_{-1}(\cdot, \cdot, a_{-1})$ we compute

$$\lim_{\varepsilon \to 0} \frac{y_\varepsilon(t^\varepsilon) - y_\varepsilon}{t^\varepsilon},$$

where $y_\varepsilon(t^\varepsilon)$ is the point on Γ obtained from y_ε after a complete switching and that still belongs to the neighborhood of $\bar{y}$. Then

$$\lim_{\varepsilon\to 0}\frac{y_\varepsilon(t^\varepsilon)-y_\varepsilon}{t^\varepsilon}=\lim_{\varepsilon\to 0}\frac{1}{t^\varepsilon}\Big(\int_0^{t_1^\varepsilon}(f_1(0,y_\varepsilon(s),a_1))_2\,ds+$$
$$\int_{t_1^\varepsilon}^{t_2^\varepsilon}(f_{-1}(0,y_\varepsilon(s),a_{-1}))_2\,ds+\int_{t_2^\varepsilon}^{t^\varepsilon}(f_1(0,y_\varepsilon(s),a_1))_2\,ds\Big)=$$
$$\lim_{\varepsilon\to 0}\Big(\frac{t_1^\varepsilon}{t^\varepsilon}\frac{1}{t_1^\varepsilon}\int_0^{t_1^\varepsilon}(f_1(0,y_\varepsilon(s),a_1))_2\,ds+\frac{t_2^\varepsilon-t_1^\varepsilon}{t^\varepsilon}\frac{1}{t_2^\varepsilon-t_1^\varepsilon}\int_{t_1^\varepsilon}^{t_2^\varepsilon}(f_{-1}(0,y_\varepsilon(s),a_{-1}))_2\,ds$$
$$+\frac{t^\varepsilon-t_2^\varepsilon}{t^\varepsilon}\frac{1}{t^\varepsilon-t_2^\varepsilon}\int_{t_2^\varepsilon}^{t^\varepsilon}(f_1(0,y_\varepsilon(s),a_1))_2\,ds\Big)=$$
$$\left(\lim_{\varepsilon\to 0}\frac{t_1^\varepsilon+t^\varepsilon-t_2^\varepsilon}{t^\varepsilon}\right)(f_1(0,\bar{y},a_1))_2+\left(\lim_{\varepsilon\to 0}\frac{t_2^\varepsilon-t_1^\varepsilon}{t^\varepsilon}\right)(f_{-1}(0,\bar{y},a_{-1}))_2, \tag{5.41}$$

where t_1^ε and t_2^ε are the time to reach $\partial\mathcal{H}_1^\varepsilon$ starting from Γ and the one to reach $\partial\mathcal{H}_{-1}^\varepsilon$ starting from $\partial\mathcal{H}_1^\varepsilon$, respectively. Moreover, since we are considering dynamics of the type (5.20), we will show that

$$\lim_{\varepsilon\to 0}\frac{t_1^\varepsilon+t^\varepsilon-t_2^\varepsilon}{t^\varepsilon}=\mu \quad\text{and}\quad \lim_{\varepsilon\to 0}\frac{t_2^\varepsilon-t_1^\varepsilon}{t^\varepsilon}=1-\mu. \tag{5.42}$$

The first ratio in (5.42) is the one between the time spent using $(f_1)_1$ to go from a threshold to the other one and the total time to perform a complete switching. Then

$$\frac{t_1^\varepsilon+t^\varepsilon-t_2^\varepsilon}{t^\varepsilon}=\frac{\frac{2\varepsilon}{-(f_1)_1}}{\frac{2\varepsilon}{-(f_1)_1}+\frac{2\varepsilon}{(f_{-1})_1}}=\frac{(f_{-1})_1}{(f_{-1})_1-(f_1)_1}. \tag{5.43}$$

While the second ratio in (5.42) is the one between the time spent using $(f_{-1})_1$ to go from a threshold to the other one and the total time to perform a complete switching, hence

$$\frac{t_2^\varepsilon-t_1^\varepsilon}{t^\varepsilon}=\frac{\frac{2\varepsilon}{(f_{-1})_1}}{\frac{2\varepsilon}{-(f_1)_1}+\frac{2\varepsilon}{(f_{-1})_1}}=\frac{-(f_1)_1}{(f_{-1})_1-(f_1)_1}. \tag{5.44}$$

With the times defined as in (5.43) and (5.44) we have that

$$\frac{t_1^\varepsilon+t^\varepsilon-t_2^\varepsilon}{t^\varepsilon}(f_1)_1+\frac{t_2^\varepsilon-t_1^\varepsilon}{t^\varepsilon}(f_{-1})_1=0,$$

therefore

$$\frac{t_1^\varepsilon+t^\varepsilon-t_2^\varepsilon}{t^\varepsilon}=\mu \quad\text{and}\quad \frac{t_2^\varepsilon-t_1^\varepsilon}{t^\varepsilon}=1-\mu.$$

Being y_ε a maximum point of $V_\varepsilon-\varphi$, by Lemma 5.2.6 follows that

$$V_\varepsilon(0,y_\varepsilon,\cdot)-\varphi(y_\varepsilon)\geq V_\varepsilon(0,y_\varepsilon(t^\varepsilon),\cdot)-\varphi(y_\varepsilon(t^\varepsilon)),$$

namely,

$$V_\varepsilon(0, y_\varepsilon(t^\varepsilon), \cdot) - V_\varepsilon(0, y_\varepsilon, \cdot) + \varphi(y_\varepsilon) - \varphi(y_\varepsilon(t^\varepsilon)) \leq 0. \tag{5.45}$$

Dividing by t^ε the above inequality we get

$$\frac{V_\varepsilon(0, y_\varepsilon(t^\varepsilon), \cdot) - V_\varepsilon(0, y_\varepsilon, \cdot)}{t^\varepsilon} + \frac{\varphi(y_\varepsilon) - \varphi(y_\varepsilon(t^\varepsilon))}{t^\varepsilon} \leq 0. \tag{5.46}$$

We call I the first addendum of the left-hand side of (5.46) and II the second ones. We can rewrite II as

$$II = \frac{-\varphi^{'}(y_\varepsilon)(y_\varepsilon(t^\varepsilon) - y_\varepsilon) + o(y_\varepsilon(t^\varepsilon) - y_\varepsilon)}{t^\varepsilon},$$

and by using (DPP) (Proposition (5.2.1)) in I we have

$$I = I \pm \frac{e^{-\lambda t^\varepsilon} V_\varepsilon(0, y_\varepsilon(t^\varepsilon), \cdot)}{t^\varepsilon} \geq$$
$$\frac{1 - e^{-\lambda t^\varepsilon}}{t^\varepsilon} V_\varepsilon(0, y_\varepsilon(t^\varepsilon), \cdot) - \frac{1}{t^\varepsilon} \int_0^{t^\varepsilon} e^{-\lambda s} \ell_{i(s)}(x_{(0,y_\varepsilon,\cdot)}(s), y_{(0,y_\varepsilon,\cdot)}(s), \alpha(s)) ds. \tag{5.47}$$

Observing that the computation (5.41) for the dynamics f_i is also valid for the running costs ℓ_i, from (5.46) and (5.47) as $\varepsilon \to 0$

$$\lambda \tilde{V}(0, \bar{y}) + \sup_{A_0(y)} \Big\{ -(\mu (f_1(0, \bar{y}, a_1))_2 + (1 - \mu)(f_{-1}(0, \bar{y}, a_{-1}))_2) \varphi^{'}(\bar{y}) -$$
$$(\mu \ell_1(0, \bar{y}, a_1) + (1 - \mu) \ell_{-1}(0, \bar{y}, a_{-1})) \Big\} \leq 0, \tag{5.48}$$

i.e. (5.40) holds.

We now assume that $f_i(0, \bar{y}, a_i) \cdot e_1 = 0$ for $i = -1, 1$, and as before we extend this condition in a neighborhood of $\bar{y}$, hence it holds in y_ε.
We show that the trajectory starting form $(0, y_\varepsilon)$ with dynamics $f_{-1}(\cdot, \cdot, a_{-1})$ does not reach $\partial \mathcal{H}^\varepsilon_{-1}$ using times s_ε with the same infinitesimal order of ε (i.e. may not occur that $\left|\frac{s_\varepsilon}{\varepsilon}\right| \leq M$ for any $M > 0$ and for every ε). (The same reasoning with suitable changes also holds for the trajectory starting form $(0, y_\varepsilon)$ with dynamics $f_1(\cdot, \cdot, a_1)$).
By contradiction, we suppose that the previous condition holds, hence $0 \leq s_\varepsilon \leq M\varepsilon$. Then, starting from $(0, y_\varepsilon)$ using any dynamics $f_{-1}(\cdot, \cdot, a)$ in the time interval $[0, M\varepsilon]$, we do not turn away from y_ε more then a quantity c_ε that goes to 0 as $\varepsilon \to 0$. And hence, we do not turn away from $\bar{y}$ more then $\tilde{c}_\varepsilon$ that goes to 0 as $\varepsilon \to 0$. Then, in every point of the trajectory with dynamics $f_{-1}(\cdot, \cdot, a_{-1})$ and $s \in [0, M\varepsilon]$ we have

$$|(f_{-1}(x(s), y(s), a_{-1}))_1 - (f_{-1}(0, \bar{y}, a_{-1}))_1| \leq L\tilde{c}_\varepsilon \tag{5.49}$$

by the Lipschitz continuity of the dynamics. Furthermore, by the absurd hypothesis we have that

$$\int_0^{s_\varepsilon} (f_{-1}(x(s), y(s), a_{-1}))_1 \, ds = \varepsilon$$

whence, using (5.49) follows

$$0 < \varepsilon \leq \int_0^{s_\varepsilon} L\tilde{c}_\varepsilon ds = L\tilde{c}_\varepsilon s_\varepsilon.$$

Then passing to the limit for $\varepsilon \to 0$ in the expression above we get a contradiction, hence if s_ε is the time to reach $\partial\mathcal{H}_{-1}^\varepsilon$ then $\varepsilon = o(s_\varepsilon)$.
With this in mind, and considering any $\mu \in [0,1]$, let $\mu \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}$ (resp. $(1-\mu)\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}$) be the time to go from Γ toward $\partial\mathcal{H}_1^\varepsilon$ (resp. $\partial\mathcal{H}_{-1}^\varepsilon$) without reaching it, using the dynamics f_1 (resp. f_{-1}), while η_ε is the time used to go from the point in which we stop with $\mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}$ (resp. $(1-\mu)\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}$) to Γ. We then define $\eta_\varepsilon = O(\varepsilon) + \tau_\varepsilon$ where τ_ε the time to go from $\partial\mathcal{H}_i^\varepsilon$ to Γ. The time t^ε for a complete switching starting from y_ε is then such that $t^\varepsilon = 2\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}$. We then prove that

$$\lim_{\varepsilon\to 0} \frac{y_\varepsilon(t^\varepsilon) - y_\varepsilon}{t^\varepsilon} = \mu(f_1)_2 + (1-\mu)(f_{-1})_2.$$

Then,

$$\lim_{\varepsilon\to 0} \frac{y_\varepsilon(t^\varepsilon) - y_\varepsilon}{t^\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{t^\varepsilon}\Bigg(\int_0^{\mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} (f_1(0, y_\varepsilon(s), a_1))_2 \, ds +$$
$$\int_{\mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}^{\eta_\varepsilon + \mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} (f_{i(s)}(0, y_\varepsilon(s), \bar{a}))_2 \, ds + \int_{\eta_\varepsilon + \mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}^{\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} (f_{-1}(0, y_\varepsilon(s), a_{-1}))_2 \, ds +$$
$$\int_{\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}^{t_\varepsilon} (f_{i(s)}(0, y_\varepsilon(s), \hat{a}))_2 \, ds \Bigg) =$$
$$\lim_{\varepsilon\to 0} \Bigg(\frac{\mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{t^\varepsilon} \frac{1}{\mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} \int_0^{\mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} (f_1(0, y_\varepsilon(s), a_1))_2 \, ds +$$
$$\frac{\eta_\varepsilon}{t^\varepsilon}\frac{1}{\eta_\varepsilon} \int_{\mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}^{\eta_\varepsilon + \mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} (f_{i(s)}(0, y_\varepsilon(s), \bar{a}))_2 \, ds +$$
$$\frac{(1-\mu)\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{t^\varepsilon} \frac{1}{(1-\mu)\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} \int_{\eta_\varepsilon + \mu\min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}^{\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} (f_{-1}(0, y_\varepsilon(s), a_{-1}))_2 \, ds$$
$$+ \frac{\eta_\varepsilon}{t^\varepsilon}\frac{1}{\eta_\varepsilon} \int_{\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}^{t^\varepsilon} (f_{i(s)}(0, y_\varepsilon(s), \hat{a}))_2 \, ds \Bigg),$$

where $\bar{a}$ is a constant control wherewith the trajectory goes from $\mathcal{H}_1^\varepsilon$ to Γ and $\hat{a}$ a constant one wherewith the trajectory goes from $\mathcal{H}_{-1}^\varepsilon$ to Γ. Note that both controls $\bar{a}$ and $\hat{a}$ are such that the corresponding trajectories point right to Γ, hence $\tau_\varepsilon = o(s_\varepsilon)$ and it has the same infinitesimal order of ε. This implies that also η_ε goes to zero as ε. Now applying a classical result in the above equality we get that it is equal to

$$\left(\lim_{\varepsilon\to 0} \frac{\mu \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{t^\varepsilon}\right)(f_1(0,\bar{y},a_1))_2 + \left(\lim_{\varepsilon\to 0}\frac{\eta_\varepsilon}{t^\varepsilon}\right)(f_i(0,\bar{y},\bar{a}))_2 +$$
$$\left(\lim_{\varepsilon\to 0} \frac{(1-\mu) \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{t^\varepsilon}\right)(f_{-1}(0,\bar{y},a_{-1}))_2 + \left(\lim_{\varepsilon\to 0}\frac{\eta_\varepsilon}{t^\varepsilon}\right)(f_i(0,y_\varepsilon(s),\hat{a}))_2.$$

Moreover when $\varepsilon \to 0$ we have

$$\frac{\mu \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{t^\varepsilon} = \frac{\mu \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{2\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} \to \mu, \quad \frac{\eta_\varepsilon}{t^\varepsilon} = \frac{\eta_\varepsilon}{2\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} \to 0,$$
$$\frac{(1-\mu) \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{t^\varepsilon} = \frac{(1-\mu) \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}}{2\eta_\varepsilon + \min\{\sqrt{\tau_\varepsilon}, s_\varepsilon\}} \to 1-\mu.$$

Again using Lemma 5.2.6 we have that

$$V_\varepsilon(0, y_\varepsilon, \cdot) - \varphi(y_\varepsilon) \geq V_\varepsilon(0, y_\varepsilon(t^\varepsilon), \cdot) - \varphi(y_\varepsilon(t^\varepsilon)),$$

from which, using the same argument as before we get (5.48) and hence (5.40).

5.3 On Comparison result

A comparison result showing that $\tilde{V}$ is the maximal subsolution of (5.25) will be the subject of a future work. In [20] a comparison result for a finite horizon problem with normal controllability conditions (as ours) is proved. Here the authors, using a local comparison result, get a global comparison for the problem in which both regular and singular controlled dynamics are allowed.
Our aim is to develop strategies which permit to prove a global comparison result for an infinite horizon problem considering only regular dynamics, through the thermostatic approximation (as we have done for the problems on network in Chapter 4). A first difficult respect to the one-dimensional case is that here we cannot control the minimum point of the supersolution because it slides along the hypersurface Γ.

Part II

Game Theoretic Decentralized Feedback Controls in Markov Jump Processes

Chapter 6

A consensus problem on network

In this chapter we study a decentralized routing problem over a network, using the paradigm of mean-field games with large number of players (see Sect. 6.1). In Sect. 6.2, introducing a state-space extension solution approach, that is an alternative method to the classical fixed point one, we turn the problem into an optimal control one for each single player and exhibits the optimal decentralized feedback control under a suitable assumption. In Sect. 6.3 we study the convergence to and the stability of a local Wardrop equilibrium and then its extension to a global equilibrium. In Sect. 6.4 we carry out numerical studies. Finally, in Sect. 6.5 applying the play operator on the control function we study both the global equilibrium and the stability of the density equation subject to this operator.
This study can be found also in Bagagiolo-Bauso-Maggistro-Zoppello [13].

6.1 Model and Problem Set-up

In this section, we provide a model of a pedestrian density flow over a network with dynamics defined on each node and using a line graph as topology. Let G be a graph with h nodes, e edges, and vertex degree d_i for $i = 1, \dots, h$. We define the line graph $L(G) = (V, E)$ to be the graph with $n = e$ nodes and $m = \frac{1}{2}\sum_{i=1}^{h} d_i^2$ edges. In particular the graph is obtained by associating a vertex to each edge of the original graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. Hence instead of considering a flux on the edges from now on we will consider jumps between vertices. Now, let a connected line graph $L(G) = (V, E)$ be given, where $V = \{1, \dots, n\}$ is the set of vertices and $E = \{1, \dots, m\}$ is the set of edges. For each node $i \in V$, let us denote by $N(i)$ the set of neighbor nodes of i:

$$N(i) = \{j \in V \,|\, \{i, j\} \in E\}\,.$$

We consider a large population of players and each of them is characterized by a time-varying state $X(t) \in V$ at time $t \in [0,T]$, where $[0,T]$ is the time horizon window. Players represent pedestrians and jump across the nodes of the graph according to a decentralized routing policy described by the matrix-valued function

$$u(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}_+^{n\times n}, \quad t \longmapsto u(t). \tag{6.1}$$

Note that u takes value in $\mathbb{R}_+^{n\times n}$ because each component u_{ij} is the product between the probability to jump from one node to an adjacent one and the relative transition rate.

Let $i \in V$ be the player's initial state. The state evolution of a single player is then captured by the following continuous-time Markov process:

$$\begin{aligned} &\{X(t), t \geq 0\} \\ &q_{ij}(u_{ij}) = \begin{cases} u_{ij}, & j \in N(i), j \neq i, \\ -\sum_{k\in N(i),k\neq i} u_{ik}, & i = j, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{6.2}$$

where q_{ij} is the microscopic dynamics from i to j.
Denote by ρ the vector whose components are the densities on vertices. This implies that the sum of the components is equal to one. Thus we have

$$\rho \in D := \{\hat{\rho} \in [0,1]^n : \sum_{i\in V} \hat{\rho}_i = 1\}.$$

The density evolution can be described by the following forward Kolmogorov Ordinary Differential Equation (ODE)

$$\begin{cases} \dot{\rho}(t) = \rho(t)A(u), \\ \rho(0) = \rho_0, \end{cases} \tag{6.3}$$

where ρ is a row vector, ρ_0 is the initial condition and the matrix-valued function $A : \mathbb{R}_+^{n\times n} \to \mathbb{R}^{n\times n}$ is given by

$$A_{ij}(u) = \begin{cases} u_{ij} & \text{if } j \in N(i), j \neq i, \\ -\sum_{j\in N(i),j\neq i} u_{ij} & \text{if } i = j, \\ 0 & \text{if } j \notin N(i). \end{cases} \tag{6.4}$$

Equation (6.3) establishes that the density variation on each node balancing densities on neighbor nodes.
It is well known that the uniform distribution of the density on a graph corresponds to a Wardrop equilibrium (Wardrop [97]) . Since we are considering a line graph, our aim is to achieve a uniform distribution of the

density over all nodes. Indeed in traffic network the Wardrop equilibrium corresponds to equidistribution of agents along edges. Therefore on its line graph we view the equilibrium as uniform distribution on nodes. We start by proving convergence to a local equilibrium, i.e. a uniform density on the nodes adjacent to i.
For each player, consider a running cost $\ell(\cdot) : V \times [0,1]^n \times \mathbb{R}_+^{n\times n} \to [0,+\infty[$, and an exit cost $g(\cdot) : V \times [0,1]^n \to [0,+\infty[$ of the form given below

$$\ell(i,\rho,u) = \sum_{j\in N(i), j\neq i} \frac{u_{ij}^2}{2} \left(\gamma_{ij}(\rho)\right)^+, \tag{6.5}$$

$$g(i,\rho) = dist(\rho, \hat{M}_i). \tag{6.6}$$

where γ_{ij} is a suitable coefficient yet to be designed and $(\cdot)^+$ is the positive part.

In (6.6) the $dist(\rho, \hat{M}_i)$ denotes the distance of the vector ρ from the manifold $\hat{M}_i$, where $\hat{M}_i$ is the local consensus manifold/local Wardrop equilibrium set for the player i defined as

$$\hat{M}_i = \{\xi \in \mathbb{R}^n | \xi_j = \xi_i \ \forall\ j \in N(i)\}. \tag{6.7}$$

Therefore, the choice of the exit cost $g(i,\rho)$ describes the difference between the number of agents in the node i and the local equidistribution of agents among the adjacent nodes.
The problem in its general form is then the following:

Problem 1: Design a decentralized routing policy to minimize the output disagreement, i.e., each player solves the following problem:

$$\begin{cases} \inf_{u(\cdot)} J(i, u(\cdot), \rho[\cdot](\cdot), \cdot), \\ J(\cdot) = \int_t^T \ell(X(\tau), \rho(\tau), u(\tau)) d\tau + g(X(T), \rho(T)), \\ \{X(t), t \geq 0\} \text{ as in (6.2)}, \\ X(t) = i, \end{cases} \tag{6.8}$$

where u is the control (6.1) taking value in $\mathbb{R}_+^{n\times n}$ for any $t \in [0,T]$ and ρ evolves as in (6.3). Note that every player minimizes a cost functional which depends on the density of his neighbours. Thus, the microscopic (6.2) and macroscopic (6.3) representations of the system are strongly intertwined which makes the problem different from classical optimal control.

Mean-field formulation

This subsection presents a mean-field formulation of problem (6.8). Let $v(i,t)$ be the value function of the optimization problem (6.8) starting from time t in state i. We can establish the following preliminary result.

Lemma 6.1.1. *The mean-field system for the decentralized routing problem in Problem 1 takes the form:*

$$\begin{cases} \dot{v}(i,t) + H(i, \Delta(v), t) = 0 \text{ in } V \times [0, T[, \\ v(i,T) = g(i, \rho(T)), \forall x \in V, \\ \dot{\rho}(t) = \rho(t) A(u^*), \\ \rho(0) = \rho_0, \end{cases} \tag{6.9}$$

where

$$H(i, \Delta(v), t) = \inf_u \Big\{ \sum_{j \in N(i)} q_{ij}(v(j,t) - v(i,t)) + \ell(i, \rho, u) \Big\}, \tag{6.10}$$

and g as in (6.6)*.*
In the expression above, $\Delta(v)$ denotes the difference of the value function computed in two successive vertices, q_{ij} is given in (6.2) *and $\ell(i,\rho,u)$ as in* (6.5)*. The optimal time-varying control $u^*(i,t)$ is given by*

$$u^*(i,t) \in \arg\min_u \Big\{ \sum_{j \in V} q_{ij}(v(j,t) - v(i,t)) + \ell(i, \rho, u) \Big\}. \tag{6.11}$$

Proof. To prove the first equation of (6.9) we know from dynamic programming that

$$\dot{v}(i,t) + \inf_u \big\{ \sum_{j \in N(i)} q_{ij}(v(j,t) - v(i,t)) + \ell(i, \rho, u) \big\} = 0 \text{ in } V \times [0, T[.$$

We obtain the first equation, by introducing the Hamiltonian in (6.10). Since (6.2) depends on the routing policy u, then the latter is obtained minimizing the Hamiltonian as expressed by (6.11). The second equation is the boundary condition on the terminal cost. The third and fourth equation are the forward Kolmogorov equation and the corresponding initial condition. □

The mean-field game (6.9) appears in the form of two coupled ODEs linked in a forward-backward way. The first equation in (6.9) is the *Hamilton-Jacobi-Bellman* (HJB) equation with variable $v(i,t)$ and parametrized in $\rho(\cdot)$. Given the boundary condition on final state and assuming a given population density behaviour captured by $\rho(\cdot)$, the HJB equation is solved backwards and returns the value function and the optimal control (6.11). The Kolmogorov equation is defined on variable $\rho(\cdot)$ and parametrized in $u^*(i,t)$. Given the initial condition $\rho(0) = \rho_0$ and assuming a given individual behaviour described by u^*, the density equation is solved forward and returns the population time evolution $\rho(t)$.

6.2 State space extension

We solve Problem 1 and the related mean-field game (6.9) through state space extension in spirit with Bauso-Zhang-Papachristodoulou [26]; namely we review ρ as an additional state variable. Then the resulting problem is of the form

$$\begin{aligned} \inf_{u(\cdot)} \quad & J(i, u(\cdot), \rho[\cdot](\cdot), \cdot), \\ \text{subject to} \quad & \{X(t), t \leq 0\} \text{ as in (6.2)}, \\ & \dot{\rho}(t) = \rho(t) A(u). \end{aligned}$$

We are looking for a value function $\tilde{V}(i, \rho, t)$ which depends on i and on the density vector ρ as a state variable, rather than as a parameter as in (6.8). The problem can be rewritten as follow.

Lemma 6.2.1. *The mean-field system for the decentralized routing problem in Problem 1 takes the form:*

$$\begin{cases} \partial_t \tilde{V}(i, \rho, t) + \tilde{H}(i, \rho, \Delta(\tilde{V}), \partial_\rho \tilde{V}, t) = 0 \ \text{in } V \times [0,1]^n \times [0, T[, \\ \tilde{V}(i, \rho, T) = g(i, \rho(T)), \end{cases} \tag{6.12}$$

where for the Hamiltonian we have

$$\tilde{H}(i, \rho, \Delta(\tilde{V}), \partial_\rho \tilde{V}, t) = \inf_u \left\{ \sum_{j \in N(i)} q_{ij} (\tilde{V}(j, \rho, t) - \tilde{V}(i, \rho, t)) + \partial_\rho \tilde{V}(i, \rho, t)(\rho A(u))^T + \ell(i, \rho, u) \right\}, \tag{6.13}$$

and the optimal time-varying control $u^*(i, \rho, t)$ is given by

$$u^*(i, \rho, t) \in \arg\min_u \left\{ \sum_{j \in N(i)} q_{ij} (\tilde{V}(j, \rho, t) - \tilde{V}(i, \rho, t)) + \partial_\rho \tilde{V}(i, \rho, t)(\rho A(u))^T + \ell(i, \rho, u) \right\}. \tag{6.14}$$

Proof. From dynamic programming we obtain

$$\partial_t \tilde{V}(i, \rho, t) + \inf_u \left\{ \sum_{j \in V} q_{ij} (\tilde{V}(j, \rho, t) - \tilde{V}(i, \rho, t)) + \partial_\rho \tilde{V}(i, \rho, t)(\rho A(u))^T + \ell(i, \rho, u) \right\} = 0.$$

By introducing the Hamiltonian $\tilde{H}(i, \rho, \Delta(\tilde{V}), \partial_\rho \tilde{V}, t)$ given in (6.13), the first equation is proven. To prove (6.14), observe that the optimal control is the minimizer in the computation of the extended Hamiltonian. Finally, the second equation in (6.12) is the boundary condition. □

Remark 6.2.2. *The use of the state space extension approach reduces our initial problem to an optimal control problem. Therefore from now on we will no longer consider the mean field formulation.*

Now, our aim is to review the optimal control problem as an inverse problem. Our aim is to find a suitable γ_{ij} (see (6.5)) such that the optimal control u^*_{ij}, which is the *argmin* of the extended Hamiltonian, is

$$u^*_{ij} = \begin{cases} \rho_i(t) - \rho_j(t) & \rho_i(t) > \rho_j(t),\ j \in N(i), \\ 0 & \text{otherwise.} \end{cases} \tag{6.15}$$

In [26] for the infinite horizon problem, the authors take the value functions as $V(\rho) = dist(\rho, M)$, where M is the global equilibrium manifold. Therefore in our finite horizon problem we assume that

$$V(i, \rho) = dist(\rho, M_i) = \sqrt{\sum_{j \in N(i)} \left(\rho_j - \frac{\sum\limits_{k \in N(i)} \rho_k}{\# N(i)} \right)^2}. \tag{6.16}$$

Note that the above satisfies the boundary condition in (6.12), according to our choice of the exit cost g (see (6.6)).
We can write (6.3) for the generic component i as

$$\dot{\rho}_i(t) = \sum_{j \in N(i), j \neq i} \rho_j(t) u_{ji} - \sum_{j \in N(i), j \neq i} \rho_i(t) u_{ij}.$$

Starting from the Hamiltonian (6.13) (see also (6.5)) we assume that if $\rho_i \neq \rho_j$, γ_{ij} is

$$\gamma_{ij}(\rho) = \left(\frac{\rho_i^2 - \rho_i\rho_j - dist(\rho, \hat{M}_j) dist(\rho, \hat{M}_i) + dist(\rho, \hat{M}_i)^2}{(\rho_i - \rho_j) dist(\rho, \hat{M}_i)} \right). \tag{6.17}$$

We want to prove that, using (6.17), the correspondent running cost (6.5) is such that our control (6.15) is the optimal one. We have the following cases:

a) $\boxed{\gamma_{ij} > 0}$
The Hamiltonian (6.13) is strictly convex in u_{ij}. Therefore the optimal control u_{ij} is the solution of

$$\frac{\partial \tilde{H}}{\partial u_{ij}} = u_{ij}\gamma_{ij}(\rho) + \frac{\rho_i\rho_j - \rho_i^2}{dist(\rho, \hat{M}_i)} + dist(\rho, \hat{M}_j) - dist(\rho, \hat{M}_i) = 0. \tag{6.18}$$

Namely if $\rho_i > \rho_j$, $u_{ij} = \rho_i - \rho_j$, instead if $\rho_i < \rho_j$ since we are supposing that $u_{ij} \in \mathbb{R}^+$ we have that the optimal control is $u_{ij} = 0$.

b) $\boxed{\gamma_{ij} \leq 0}$
The Hamiltonian (6.13) is linear in u_{ij} and is increasing or decreasing depending on the sign of

$$\beta_{ij} = \frac{\rho_i\rho_j - \rho_i^2}{dist(\rho, \hat{M}_i)} + dist(\rho, \hat{M}_j) - dist(\rho, \hat{M}_i) = -\gamma_{ij}(\rho)(\rho_i - \rho_j)$$

– if $\rho_i > \rho_j$ the Hamiltonian is increasing in u_{ij}, hence it admits minimum at $u_{ij} = 0$,
– if $\rho_i < \rho_j$ the Hamiltonian is decreasing in u_{ij}, therefore it takes smaller and smaller values as $u_{ij} \to +\infty$.

Now note that, if the densities are converging in time to the same value, which is the case if we use the control u^*_{ij}, the function (6.17) is never negative and thus case b) before cannot occur. Simulations will show this phenomenon and also suggest that

$$\lim_{\rho_i \to \rho_j, j \in N(i)\setminus\{i\}} \gamma_{ij} = +\infty. \tag{6.19}$$

which is coherent with the constraint $u^*_{ij} = 0$. Therefore, using the function (6.17), the correspondent running cost given by

$$\ell(i,\rho,u) = \sum_{j\in N(i),\, j\neq i\, \rho_i>\rho_j} \frac{u_{ij}^2}{2} \underbrace{\left(\frac{\rho_i^2 - \rho_i\rho_j - dist(\rho,\hat{M}_j)dist(\rho,\hat{M}_i) + dist(\rho,\hat{M}_i)^2}{(\rho_i-\rho_j)dist(\rho,\hat{M}_i)} \right)^+}_{\gamma_{ij}(\rho)}, \tag{6.20}$$

leads the optimal feedback control to be exactly (6.15). Moreover, when using control (6.15), the Hamiltonian (6.13) also converges to zero as t tends to infinity. Hence, the function $V(i,\rho)$ as defined in (6.16), is almost a solution of the Hamilton-Jacobi-Bellman problem (6.12). Such a consideration leads to the fact that, at least when time becomes large, the control (6.15) is optimal. The fact that the Hamiltonian (6.13) converges to zero comes from (6.18) where the second addendum of the right-hand side is bounded (the distance from the manifold is larger than $|\rho_i - \rho_j|$ up to a multiplicative constant). This boundedness leads to $(\rho_i - \rho_j)^2\gamma_{ij}(\rho) \to 0$ and hence the conclusion, because inside the Hamiltonian we almost have (6.18) multiplied by $(\rho_i - \rho_j)$.
Now with the control (6.15), we can rewrite the evolution of ρ as

$$\dot{\rho}_i(t) = \sum_{j\neq i, j\in N(i):\rho_j>\rho_i} \rho_j(t)(\rho_j(t)-\rho_i(t)) - \sum_{j\in N(i):\rho_i>\rho_j} \rho_i(t)(\rho_i(t)-\rho_j(t)) \ \ \forall i \tag{6.21}$$

Now our aim is to study the stability properties of the dynamical system (6.21). In other words if using the optimal control u^*_{ij} the system converges to an equilibrium.

6.3 Wardrop equilibrium

In this section we will show how to obtain a uniform distribution of the density ρ, at first on a neighborhood of a node and then throughout the

graph.
The right-hand side of equation (6.21) is zero only when $\rho_i = \rho_j$ $\forall i \in V$ and $j \in N(i)$, which leads to a uniform density over the nodes.
The following assumption establishes that for a given feasible target manifold, there always exists a decentralized routing policy $u(t)$ which drives the density ρ toward the relative manifold $\hat{M}_i$ (see (6.7)).
This assumption will be used later on to prove the convergence to a local Wardop equilibrium.

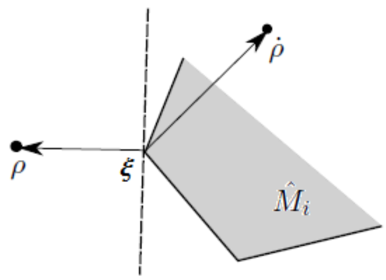


Figure 6.1: Geometric illustration of the Attainability condition.

Assumption 1 (Attainability condition)
Let $\hat{M}_i$ be given by (6.7), $r > 0$ and $S_i = \{\rho : dist(\rho, \hat{M}_i) < r\}$. For all $\rho \in S_i \setminus \hat{M}_i$ there exists an element in the projection, $\xi(i,\rho) \in \Pi_{\hat{M}_i}\rho$, such that the value $val[\lambda_i]$ is negative for every $\lambda_i = (\rho(t) - \xi(i,\rho))$, namely

$$\forall i,\ val[\lambda_i] = \inf_u \{\lambda_i \cdot [(I - \partial_\rho(\xi(i,\rho)))\dot{\rho}^T + \sum_{j \in N(i)} (\xi(j,\rho) - \xi(i,\rho))q_{ij}]\} < 0, \tag{6.22}$$

where $\partial_\rho \xi(i,\rho)$ is a constant matrix since $\xi(i,\rho)$ is a linear function of ρ.

We point out that, as we will show in Section 6.4 (see (6.26)), assumption (6.22) is satisfied by our optimal control u^*_{ij} (6.15).
Assumption (6.22) represents the trend of the agents in node i to be influenced by the choices of neighbor agents. Agents can act in order to reach the same density as in the adjacent nodes.
In the proof of the next theorem, we review the value function of (6.12) as a Lyapunov function.

Theorem 6.3.1. *Let Assumption 1 hold true. Then, $\rho(t)$ converges asymptotically to $\hat{M}_i$, i.e.*

$$\lim_{t\to\infty} dist(\rho, \hat{M}_i) = 0. \tag{6.23}$$

Proof. Let ρ be a solution of (6.3) with initial value $\rho(0) \in S_i \setminus \hat{M}_i$. Set $\tau = \{\inf t > 0 : \rho(t) \in \hat{M}_i\} \leq \infty$ and let $V(i(t), \rho(t)) = dist(\rho(t), \hat{M}_i)$. For all $t \in [0, \tau]$ and $\xi \in \Pi_{\hat{M}_i}(\rho(t))$. We wish to compute $\dot{V}(i(t), \rho(t))$ as the limit of the incremental ratio, thus at first we write its numerator, where $X(t)$ is the Markov process giving the evolution of the index $i(t)$, that is:

$$
\begin{aligned}
&V(i(t), \rho(t+dt)) - V(i(t), \rho(t)) + V(i(t+dt), \rho(t)) - V(i(t), \rho(t)) = \\
&\|\rho(t+dt) - \xi(\rho(t+dt), X(t))\| - \|\rho(t) - \xi(\rho(t), X(t))\| + \\
&\|\rho(t) - \xi(\rho(t), X(t+dt))\| - \|\rho(t) - \xi(\rho(t), X(t))\| = \\
&\|\rho(t) + \dot{\rho}(t)dt - \xi(\rho(t), X(t)) - \nabla_\rho \xi(\rho(t), X(t))\dot{\rho}(t)dt\| - \\
&\|\rho(t) - \xi(\rho(t), X(t))\| + |dt|\varepsilon(dt) + \\
&\|\rho(t) - \xi(\rho(t), X(t)) - \partial_X \xi(\rho(t), X(t))\dot{X}(t)dt + o(dt)\| - \|\rho(t) - \xi(\rho(t), X(t))\|
\end{aligned}
$$

where $\lim_{dt\to 0} \varepsilon(dt) = 0$ and $\lim_{dt\to 0} o(dt) = 0$. Hence

$$
\begin{aligned}
&\dot{V}(i(t), \rho(t)) = \\
&\lim_{dt\to 0} \frac{1}{dt}\left(\frac{\|\rho(t) + \dot{\rho}(t)dt - \xi(\rho(t), X(t)) - \nabla_\rho \xi(\rho(t), X(t))\dot{\rho}(t)dt\|^2}{\|\rho(t) + \dot{\rho}(t)dt - \xi(\rho(t), X(t)) - \nabla_\rho \xi(\rho(t), X(t))\dot{\rho}(t)dt\|} - \right. \\
&\frac{\|\rho(t) - \xi(\rho(t), X(t))\|^2}{\|\rho(t) - \xi(\rho(t), X(t))\|} + |dt|\varepsilon(dt) + \\
&\frac{\|\rho(t) - \xi(\rho(t), X(t)) - \partial_X \xi(\rho(t), X(t))\dot{X}(t)dt\|^2}{\|\rho(t) - \xi(\rho(t), X(t)) - \partial_X \xi(\rho(t), X(t))\dot{X}(t)dt\|} - \\
&\left. \frac{\|\rho(t) - \xi(\rho(t), X(t))\|^2}{\|\rho(t) - \xi(\rho(t), X(t))\|} + o(dt) \right) = \\
&\lim_{dt\to 0} \frac{1}{dt}\left(\frac{\|\rho(t) + \dot{\rho}(t)dt - \xi(\rho(t), X(t)) - \nabla_\rho \xi(\rho(t), X(t))\dot{\rho}(t)dt\|^2}{\|\rho(t) - \xi(\rho(t), X(t))\| + O(\sqrt{dt})} - \right. \\
&\frac{\|\rho(t) - \xi(\rho(t), X(t))\|^2}{\|\rho(t) - \xi(\rho(t), X(t))\|} + |dt|\varepsilon(dt) + \\
&\frac{\|\rho(t) - \xi(X(t)) - \partial_X \xi(\rho(t), X(t))\dot{X}(t)dt\|^2}{\|\rho(t) - \xi(\rho(t), X(t))\| + O(\sqrt{dt})} - \\
&\left. \frac{\|\rho(t) - \xi(\rho(t), X(t))\|^2}{\|\rho(t) - \xi(\rho(t), X(t))\|} + o(dt) \right) = \\
&\frac{1}{\|\rho(t) - \xi(\rho(t), X(t))\|} \frac{d}{dt}\left(\|\rho(t) - \xi(\rho(t), X(t))\|^2 \right) \leq \\
&\frac{2}{\|\rho(t) - \xi(i, \rho)\|} \left[(\rho(t) - \xi(i, \rho)) \cdot \right. \\
&\left. \left((I - \nabla_\rho(\xi(i, \rho)))\dot{\rho}(t)^T + \sum_{j\in N(i)} (\xi(j, \rho) - \xi(i, \rho))q_{ij} \right) \right].
\end{aligned}
$$

Using now Assumption 1 we have that the second factor of the last product is strictly negative, hence $\dot{V}(i(t), \rho(t)) < 0$. This proves not only that a

Wardrop equilibrium but also that the solution ρ of the dynamics (6.3) is locally asymptotically stable for the Lyapunov theorem. □

The next step is to prove the asymptotic convergence of ρ, solution of (6.3), to the global consensus manifold M defined as follows

$$M = \{\rho \in D : \rho = \mathbf{1}\frac{1}{n}\}, \tag{6.24}$$

where n is the number of nodes.

Corollary 6.3.2. *Let Assumption 1 hold true, then*

$$\lim_{t \to +\infty} d(\rho(t), M) = 0.$$

Proof. We are in the hypothesis of Theorem (6.3.1), then

$$\lim_{t \to \infty} dist(\rho, \hat{M}_i) = 0.$$

It follows that for any sequence $(t_m)_{m \in \mathbb{N}}$ such that $t_m \to +\infty$ we have that

$$\begin{aligned} &\rho_i \to \beta \\ &\rho_j \to \beta \ \forall\ j \in N(i) \\ &\rho_k \to \beta \ \forall\ k \in N(j) \text{ s.t } j \in N(i) \\ &\vdots \end{aligned} \tag{6.25}$$

By doing this, since the graph is connected, we can conclude that

$$\rho_i(t_m) \to \beta = \frac{1}{n} \quad \forall i \in V.$$

Then, there exists a subsequence $(t_{m_\ell})_{\ell \in \mathbb{N}}$ such that

$$\rho_i(t_{m_\ell}) \to \frac{1}{n} \quad \forall i \in V.$$

This proves that $\rho(t) \to \frac{1}{n}$ for $t \to +\infty$ and thus $\lim_{t \to +\infty} d(\rho(t), M) = 0$. □

6.4 Numerical example

In this section, numerical simulation show that on a graph with seven nodes, the provided distributed routing policy (6.15) provides convergence to the equilibrium.
Consider the following network consisting of 7 nodes and 8 edges.

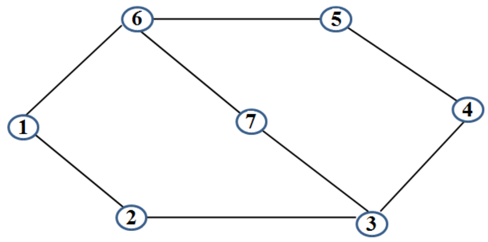


Figure 6.2: Network system with seven nodes.

Solving the Kolmogorov equation (6.21) with the following initial conditions

$$\rho_1(0) = 0.15, \quad \rho_2(0) = 0.2, \quad \rho_3(0) = 0.1, \quad \rho_4(0) = 0.3,$$
$$\rho_5(0) = 0.1, \quad \rho_6(0) = 0.15, \quad \rho_7(0) = 0,$$

we obtain the density evolution as shown in Figure 6.3

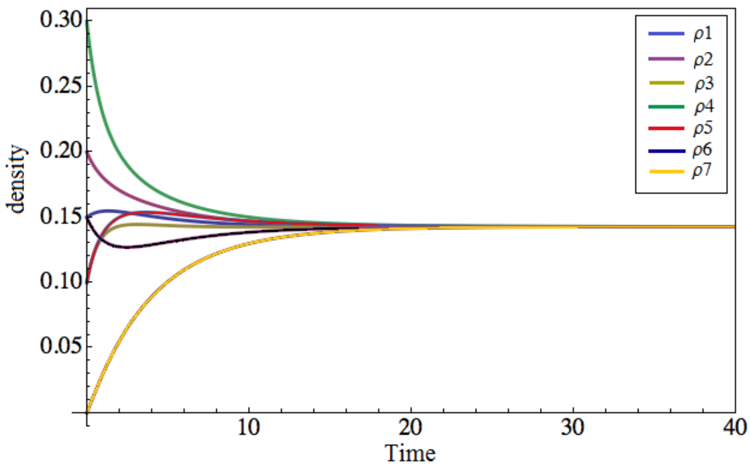


Figure 6.3: Simulation of the density.

As expected the density converges to the global equilibrium in which all the ρ_i are equal.
In Figure 6.4 we can see that the function γ_{ij} (6.17) is positive, in accordance with our statements in Section 6.2.

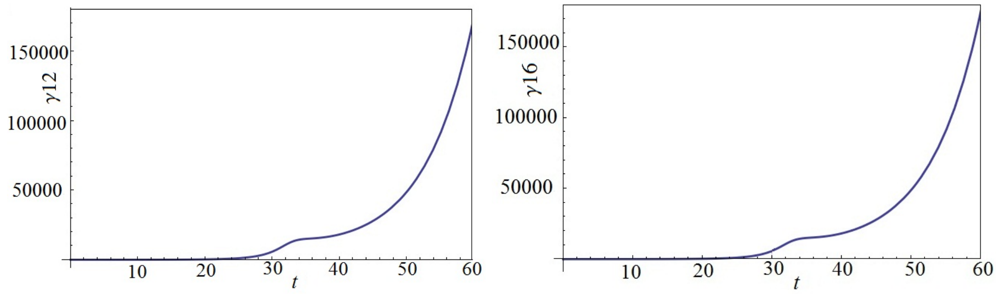


Figure 6.4: Evolution of γ_{12} and γ_{16} along the trajectories obtained using control u^*_{ij} (6.15).

Note that the optimal control $u^*_{ij} = (\rho_i - \rho_j)^+$ satisfies Assumption 1 as by defining

$$\alpha_i = \lambda_i \cdot [(I - \partial_\rho(\xi(i,\rho)))\dot{\rho}(t)^T + \sum_{j \in N(i)} (\xi(j,\rho) - \xi(i,\rho))q_{ij}], \quad \forall i = 1, \cdots, 7,$$

$$\begin{aligned} &\max_\rho\{\alpha_1\} = -6.1489 \cdot 10^{-7} \quad && \max_\rho\{\alpha_2\} = -2.1462 \cdot 10^{-6} \\ &\max_\rho\{\alpha_3\} = -3.1123 \cdot 10^{-9} \quad && \max_\rho\{\alpha_4\} = -6.7065 \cdot 10^{-7} \\ &\max_\rho\{\alpha_5\} = -8.0771 \cdot 10^{-7} \quad && \max_\rho\{\alpha_6\} = -2.1169 \cdot 10^{-6} \\ &\max_\rho\{\alpha_7\} = -7.4670 \cdot 10^{-7}. \end{aligned} \tag{6.26}$$

Then, function α_i is negative for all i, for our choice of the control. According to Theorem (6.3.1), in Figure 6.5 we show that the distance of ρ_i from the relative $\hat{M}_i$, $\forall i = 1, \ldots, 7$, converges to zero.

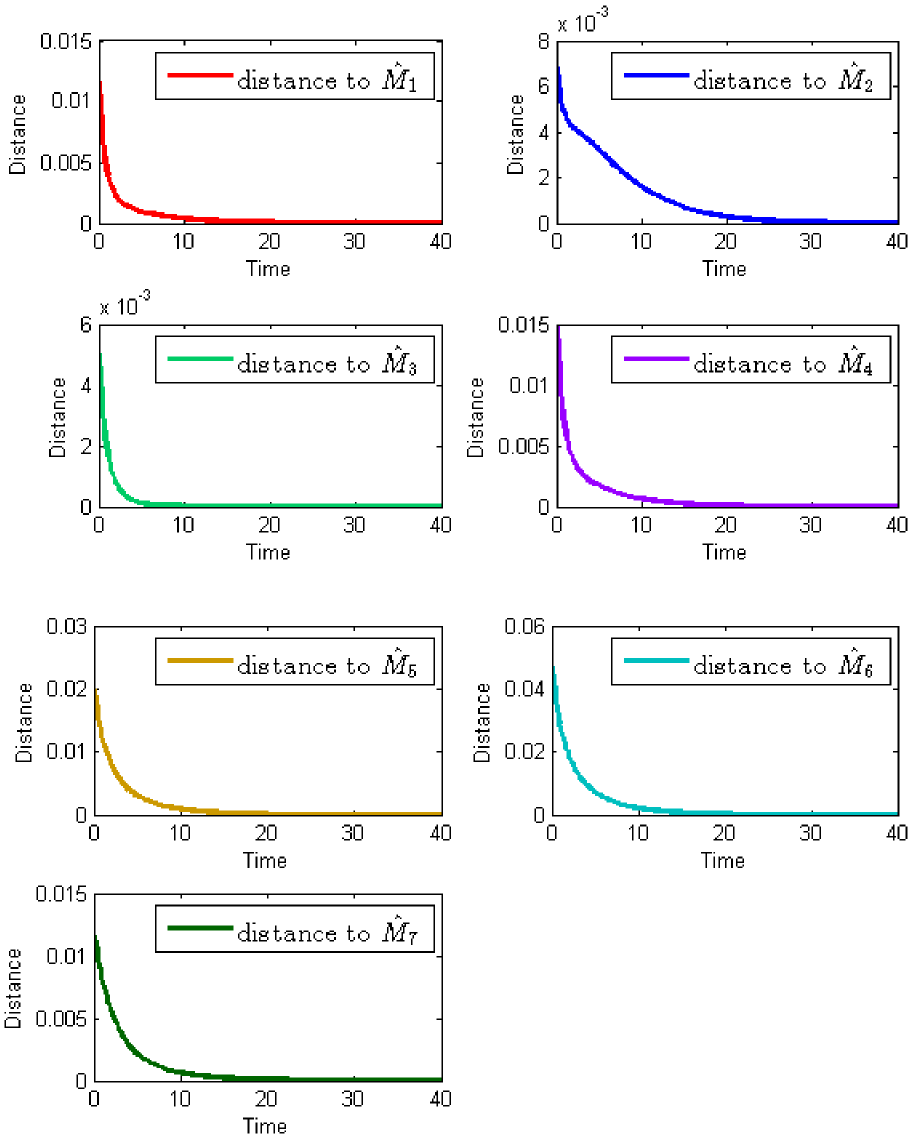


Figure 6.5: Distances to the consensus manifolds.

6.5 Stability with hysteresis

In this section we study stability of the macroscopic dynamics of the vector ρ when the optimal decentralized feedback control (6.15) is affected by a hysteresis phenomena modeled by a scalar play operator (see Chapter 3, Sect. 3.1.2 for the details on the operator). We study how the evolution of the macroscopic equation changes when we apply the play operator to the control u^* obtained from (6.15). Furthermore, we characterize the set of equilibrium points as union of several manifolds. Finally, we provide convergence condition for the resulting dynamics.

The controlled dynamical system is

$$\begin{cases} \dot{\rho}(t) = \rho(t)A(w), \\ w(t) = P[u^*]^+(t), \\ \rho(0) = \rho_0, \\ w(0) = w_0, \end{cases} \tag{6.27}$$

where $P[\cdot](\cdot)$ is the play operator whose behavior is explained in the following subsection and ^+ is the positive part. With reference to our system (6.27), we consider as input of the matrix A the positive part of the play operator, applied to the control $u^*_{ij} = (\rho_i - \rho_j)^+$, i.e. $w_{ij}(t) = P[(\rho_i - \rho_j)]^+(t)$.

Remark 6.5.1. *Note that, since $(\rho_i(0) - \rho_j(0)) = -(\rho_j(0) - \rho_i(0))$, then $(\rho_i(t) - \rho_j(t)) = -(\rho_j(t) - \rho_i(t))$ $\forall t$. Thus it is not a restriction to suppose that also $P[(\rho_i - \rho_j)](0) = -P[(\rho_j - \rho_i)](0)$, therefore*
$P[(\rho_i - \rho_j)](t) = -P[(\rho_j - \rho_i)](t)$ $\forall t$.
Moreover since we are taking the positive part of the play, we will have that if $w_{ij} > 0$ then $w_{ji} = 0$.

Equilibria

We are looking for the equilibrium points of the first equation of (6.27) considering the simple case of a network with four nodes as the one depicted in Figure 6.6

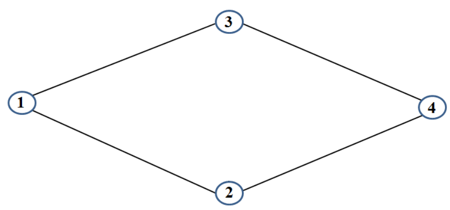


Figure 6.6: Network system with four nodes.

The evolution of the vector ρ is given by

$$\begin{cases} \dot{\rho}_1(t) = & -\big(w_{12} + w_{13}\big)\rho_1(t) + w_{21}\rho_2(t) + w_{31}\rho_3(t), \\ \dot{\rho}_2(t) = & w_{12}\rho_1(t) - \big(w_{21} + w_{24}\big)\rho_2(t) + w_{42}\rho_4(t), \\ \dot{\rho}_3(t) = & w_{13}\rho_1(t) - \big(w_{31} + w_{34}\big)\rho_3(t) + w_{43}\rho_4(t), \\ \dot{\rho}_4(t) = & w_{24}\rho_2(t) + w_{34}\rho_3(t) - \big(w_{42} + w_{43}\big)\rho_4(t). \end{cases} \tag{6.28}$$

Case 1

Assume that $w_{12} > 0, w_{31} > 0, w_{24} > 0, w_{43} > 0$. If

$$|\varepsilon| > \max\{\rho_4\big(\frac{w_{43}}{w_{12}} - \frac{w_{43}}{w_{24}}\big), \rho_4\big(\frac{w_{43}}{w_{24}} - 1\big), \rho_4\big(\frac{w_{43}}{w_{31}} - \frac{w_{43}}{w_{12}}\big), \rho_4\big(1 - \frac{w_{43}}{w_{31}}\big)\},$$

then the system to solve is

$$\begin{cases} \dot\rho_1(t) = & -w_{12}\rho_1(t) + w_{31}\rho_3(t), \\ \dot\rho_2(t) = & w_{12}\rho_1(t) - w_{24}\rho_2(t), \\ \dot\rho_3(t) = & -w_{31}\rho_3(t) + w_{43}\rho_4(t), \\ \dot\rho_4(t) = & w_{24}\rho_2(t) - w_{43}\rho_4(t), \end{cases} \tag{6.29}$$

that is zero in

$$\left(\rho_4\frac{w_{43}}{w_{12}}, \rho_4\frac{w_{43}}{w_{24}}, \rho_4\frac{w_{43}}{w_{31}}, \rho_4, w_{12}, w_{24}, w_{31}, w_{43}\right). \tag{6.30}$$

We write only the values of $w_{12}, w_{24}, w_{31}, w_{43}$ because their symmetric $w_{21}, w_{42}, w_{13}, w_{34}$ are always zero according to Remark 6.5.1. We will use this convention from now on.

Case 2

Assume that $w_{12} > 0, w_{31} > 0, w_{24} > 0, w_{43} = 0$. If $|\varepsilon| > 1$, then the system is zero in

$$\big(0, 0, 0, 1, w_{12}, w_{31}w_{24}, 0\big). \tag{6.31}$$

Case 3

For $w_{12} > 0, w_{31} > 0, w_{24} = 0, w_{43} = 0$. If $|\varepsilon| > \max\{\rho_4, 1 - \rho_4\}$, then the system is zero in

$$\big(0, 1 - \rho_4, 0, \rho_4, w_{12}, w_{31}, 0, 0\big). \tag{6.32}$$

Case 4

For $w_{12} > 0, w_{31} = 0, w_{24} = 0, w_{43} = 0$. if $|\varepsilon| > \max\{\rho_4,\ \rho_3, 1 - \rho_4 - \rho_3\}$ then the system is zero in

$$\big(0, 1 - \rho_4 - \rho_3, \rho_3, \rho_4, w_{12}\big). \tag{6.33}$$

Case 5

Assume that all $w_{ij} = 0\ \forall j \in N(i)$. If

$$|\varepsilon| > \max\{\rho_1 - \rho_2, \rho_2 - \rho_4, \rho_3 - \rho_1, \rho_4 - \rho_3\},$$

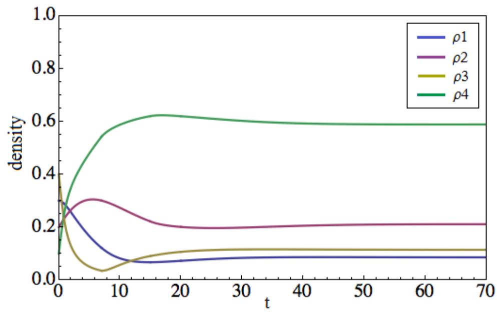
then the equilibrium point of the system is

$$(1 - \rho_2 - \rho_3 - \rho_4, \rho_2, \rho_3, \rho_4, \mathbf{w}) = (\rho_1, \rho_2, \rho_3, \rho_4, \mathbf{0}), \tag{6.34}$$

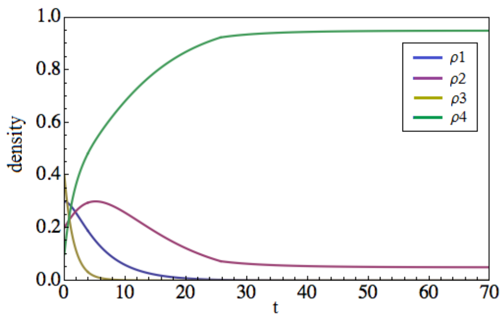
where $\mathbf{w}$ denotes the vector of all eight w_{ij}.

Remark 6.5.2. *Note that the equilibria in cases* 2, 3, 4, 5 *can be obtained as limits of the equilibrium in case* 1*. Indeed if we let* $w_{43} \to 0$ *we end up with equilibrium* (6.31) *and since* $\sum_{i=1}^{4} \rho_i = 1$*,* $\rho_4 = 1$*. If* $w_{43} \to 0$ *and* $w_{24} \to 0$ *we obtain equilibrium* (6.32) *where we denoted by* $1 - \rho_4$ *the indeterminate form* $\rho_4 \frac{w_{43}}{w_{24}}$*, taking into account the conservation of mass. Furthermore if* $w_{31} \to 0$*,* $w_{24} \to 0$ *and* $w_{43} \to 0$ *we get equilibrium* (6.33)*, where we call the indeterminate forms* $\rho_4 \frac{w_{43}}{w_{31}}$ *and* $\rho_4 \frac{w_{43}}{w_{24}}$ *respectively* ρ_3 *and* $1 - \rho_4 - \rho_3$ *for the same reason as before.*
Finally letting all $w_{ij} \to 0$ *we end up with equilibrium* (6.34)*, in which* ρ_2*,* ρ_3*, and* $1 - \rho_2 - \rho_3 - \rho_4$ *denote the indeterminate forms* $\rho_4 \frac{w_{43}}{w_{24}}$*,* $\rho_4 \frac{w_{43}}{w_{31}}$*, and* $\rho_4 \frac{w_{43}}{w_{12}}$ *that respect the conservation of mass.*
Moreover, our choice of taking $w_{12} > 0, w_{31} > 0, w_{24} > 0, w_{43} > 0$ *and not other* w_{ij} *is completely arbitrary, indeed taking any* 4 *non symmetric* $w_{ij} > 0$ *we will end up with an equilibrium of the same type of* (6.30).

In the following numerical simulations we show the behavior of the system for two different choices of the parameter ε



(a)



(b)

Figure 6.7: Numerical simulations of the system converging to the equilibria in case 1 (Figure 6.7(a)) and case 3 (Figure 6.7(b))

In Figure 6.7(a) we take $\varepsilon = 0.5$. We can see that the densities converge to the equilibrium (6.30). Instead in the Figure 6.7(b), using $\varepsilon = 0.95$, the system converges to equilibrium (6.32).

Stability

In the following subsection we show that also in the presence of the play operator we converge to the equilibrium for $t \to \infty$. Before doing this we make a further assumption for the manifold as defined next.
The global equilibrium manifold $\overline{M}$ in this case is the union of different equilibrium manifolds

$$\overline{M} = \bigcup_{z=1}^{5} \overline{M}_z, \tag{6.35}$$

where $\bar{M}_z$ denotes the manifold whose points are equilibria relative to the z-th case.

Assumption 2
Let $\overline{M}$ *be given as in* (6.35), $s > 0$ *and* $S = \{\overline{\rho} : dist(\overline{\rho}, \overline{M}) < s\}$. *For all* $\overline{\rho} = (\rho, w) \in S \setminus \overline{M}$, *there exists* $\overline{\xi} \in \Pi_{\overline{M}}\overline{\rho}$ *such that the value* $val[\lambda]$ *is negative for every* $\lambda = (\overline{\rho} - \overline{\xi})$, *namely*

$$val[\lambda] = \inf_u \{\lambda \cdot (I - \partial_{\overline{\rho}}\overline{\xi}(\overline{\rho}(t)))\dot{\overline{\rho}}(t)^T\} < 0. \tag{6.36}$$

This assumption is analogous to the attainability (6.22) in the presence of hysteresis. Note that here the term involving q_{ij} in (6.22) is not present, since depending on whether we are in the node i or in the node j the projection on the global manifold $\overline{\xi}$ is the same. Moreover, at the end of this section we will stress the fact that (6.36) is satisfied under control $w_{ij} = P[(\rho_i - \rho_j)]^+$.

Theorem 6.5.3. *Let Assumption 2 hold true. Then* $\overline{\rho}(t)$ *converges asymptomatically to* $\overline{M}$, *namely*

$$\lim_{t \to +\infty} dist(\overline{\rho}, \overline{M}) = 0. \tag{6.37}$$

Proof.: Let $\overline{\rho}$ a solution of (6.27) with initial value $\overline{\rho}(0) \in S \setminus \overline{M}$. Set $\tau = \{\inf t > 0 : \overline{\rho}(t) \in \overline{M}\} \leq \infty$ and let $V(\overline{\rho}(t)) = dist(\overline{\rho}, \overline{M})$. We compute:

$$\begin{aligned}\dot{V}(\overline{\rho}(t)) =& \frac{d}{dt}\Big(\|\overline{\rho}(t) - \overline{\xi}(\overline{\rho}(t))\|\Big) = \\ & \frac{1}{\|\overline{\rho}(t) - \overline{\xi}(\overline{\rho}(t))\|}\Big[\big(\overline{\rho}(t) - \overline{\xi}(\overline{\rho}(t))\big)\big(I - \partial_{\overline{\rho}}\overline{\xi}(\overline{\rho}(t))\big)\dot{\overline{\rho}}(t)^T\Big] < 0\end{aligned}$$

by (6.36). Then the solution $\overline{\rho}$ of (6.27) is asymptotically stable and we have a global equilibrium. □

In the following we deal with some examples of convergence to the equilibria in different $\overline{M}_z$ using the decentralized control $u^*_{ij} = (\rho_i - \rho_j)^+$. At first we suppose that $\varepsilon > 1$ thus for all t, $w(t)$ satisfies the conditions in case 2. The system to study is

$$\begin{cases} \dot\rho_1(t) = -w_{12}(t)\rho_1(t) + w_{31}(t)\rho_3(t), \\ \dot\rho_2(t) = w_{12}(t)\rho_1(t) + w_{24}(t)\rho_2(t), \\ \dot\rho_3(t) = -w_{31}(t)\rho_3(t), \\ \dot\rho_4(t) = w_{24}(t)\rho_2(t). \end{cases} \tag{6.38}$$

From the assumption on the w_{ij} we have

$$\exists c > 0 : w_{ij}(t) > c \ \forall t \geq 0.$$

Then considering the third equation of (6.38) we have that $\rho_3(t) \leq e^{-ct}\rho_3(0) \to 0$ for $t \to +\infty$. By contradiction, we suppose that $\rho_1(t) \to \bar\rho_1$ with $\bar\rho_1 > 0$. Thus,

$$\lim_{t\to+\infty} \dot\rho_1(t) = \lim_{t\to+\infty} -w_{12}(t)\bar\rho_1 + \lim_{t\to+\infty} w_{31}(t)\rho_3(t) \neq 0. \tag{6.39}$$

This is a contradiction as the left hand side should be equal to zero. Hence $\lim_{t\to+\infty}\rho_1(t) = 0$. With similar argument also $\lim_{t\to+\infty}\rho_2(t) = 0$. For the mass conservation $\rho_4(t) \to 1$ for $t \to +\infty$ hence we obtain the equilibrium point (6.31).

Assuming now that $\varepsilon > \max\{\rho_4(0), 1-\rho_4(0)\}$ and $w(0)$ satisfies the conditions in case 3, the system becomes

$$\begin{cases} \dot\rho_1(t) = -w_{12}(t)\rho_1(t) + w_{31}(t)\rho_3(t), \\ \dot\rho_2(t) = w_{12}(t)\rho_1(t), \\ \dot\rho_3(t) = -w_{31}(t)\rho_3(t), \\ \dot\rho_4(t) = 0, \end{cases} \tag{6.40}$$

for all $t \in [0, \bar t[$ where

$$\bar t = \sup\{t \geq 0 : u^*_{12} + \varepsilon > w_{12}(t) \equiv w_{12}(0) > 0, u^*_{31} + \varepsilon > w_{31}(t) \equiv w_{31}(0) > 0, \\ w_{24} \equiv 0, w_{43} \equiv 0\}.$$

We will now prove that $\bar t = +\infty$.

Let us suppose by contradiction that $\bar t < +\infty$. Obviously $\rho_4(t) \equiv \rho_4(0)$ in $[0, \bar t[$. Using the hypothesis over w_{ij} we have that $\rho_3(t) = e^{-w_{31}(0)t}\rho_3(0)$ in $[0, \bar t[$ and thus ρ_3 decreases. Moreover ρ_2 is increasing.

Let us now focus on the differences among the densities. Since ρ_4 is constant and $\rho_3 \searrow$ then $\rho_4 - \rho_3 \nearrow$. This difference is always less than or equal to

ρ_4 and thus it is less than ε. By the continuity of ρ, $\lim\limits_{t \to \bar{t}} (\rho_4(t) - \rho_3(t)) < \varepsilon$. Therefore w_{43} does not change and remains equal to 0 in $[0, \bar{t}]$.
Let us now consider $\rho_2 - \rho_4$. By (6.40), in $[0, \bar{t}[$ $\rho_2 \nearrow$, thus $\rho_2 - \rho_4$ increases and is less than $1 - \rho_4 < \varepsilon$. By the previous continuity argument $w_{24} \equiv 0$ in $[0, \bar{t}]$. From the last two results we can conclude that $\rho_4(t) \equiv \rho_4(0)$ in $[0, \bar{t}]$.
Considering $\rho_3 - \rho_1$ we have that, if $\rho_3 - \rho_1 \searrow$ in $[0, \bar{t}[$, the last difference is greater than $-\rho_1 = \rho_4 - 1 + \rho_3 + \rho_2 > \rho_4 - 1 > -\varepsilon$. This implies $\varepsilon > \rho_1$ and thus using the continuity argument $w_{31}(t) = w_{31}(0) > 0$ in $[0, \bar{t}]$. Instead if $\rho_3 - \rho_1 \nearrow$ it is always less that $\rho_3 < 1 - \rho_4 < \varepsilon$. Then as before $w_{31}(t) = w_{31}(0) > 0$ in $[0, \bar{t}]$. From the last one and $w_{43} \equiv 0$ we conclude $\rho_3(t) = \rho_3(0)e^{-w_{31}(0)t}$ in $[0, \bar{t}]$.

Again if $\rho_1 - \rho_2 \searrow$ it is greater than $-\rho_2 > \rho_4 - 1 > -\varepsilon$. Proceeding as before we conclude that $w_{12}(t) = w_{12}(0) > 0$ in $[0, \bar{t}]$. Instead if $\rho_1 - \rho_2 \nearrow$ reasoning as before we reach the same conclusion, i.e, $w_{12}(t) = w_{12}(0) > 0$ in $[0, \bar{t}]$.
Hence we have proven that in $\bar{t}$, the same conditions valid in the interval $[0, \bar{t}[$, hold. Therefore there exists $\delta > 0$ such that in $[0, \bar{t} + \delta]$, $w_{ij}(t)$ are the same as in $t = 0$. This is a contradiction as $\bar{t}$ is a supremum, thus we conclude $\bar{t} = +\infty$.

We will now prove that the system converges to equilibrium (6.32). From the assumption on the w_{ij} we have $\rho_3(t) = e^{-w_{31}(0)t}\rho_3(0) \to 0$ for $t \to +\infty$. By contradiction, we suppose that $\rho_1(t) \to \bar{\rho}_1$ with $\bar{\rho}_1 > 0$. Thus,

$$\lim_{t \to +\infty} \dot{\rho}_1(t) = \lim_{t \to +\infty} -w_{12}(t)\bar{\rho}_1 \neq 0. \tag{6.41}$$

This is a contradiction as it should be equal to zero. Hence $\lim_{t \to +\infty} \rho_1(t) = 0$. Regarding ρ_4 and ρ_2, the first is constant and $\lim_{t \to +\infty} \rho_2(t) = \bar{\rho}_2 > 0$. From the mass conservation $\bar{\rho}_2 = 1 - \bar{\rho}_4$ hence we obtain an equilibrium point as in (6.32).
Using similar arguments, if ε is like in case 4 and 5 we will converge to equilibria (6.33) and (6.34) respectively.
The above procedure can be extended to the case where ε is such that for all t we have four non symmetric $w_{ij} > 0$ like in case 1.

Note also that the decentralized control $u^*_{ij} = (\rho_i - \rho_j)^+$ satisfies Assumption 2, indeed the function $V(\overline{\rho}(t))$ is strictly decreasing along the trajectories (see Figure 6.8).
As a consequence, the distance of $\overline{\rho}$ from the manifold $\overline{M}$ is a Lyapunov function and thus Theorem 6.5.3 holds true.
The Figure 6.8 displays the distance of $\overline{\rho}$ from the manifold $\overline{M}_1$ in function of time. It is visually clear that the time plot is decreasing in accordance to our expectations.

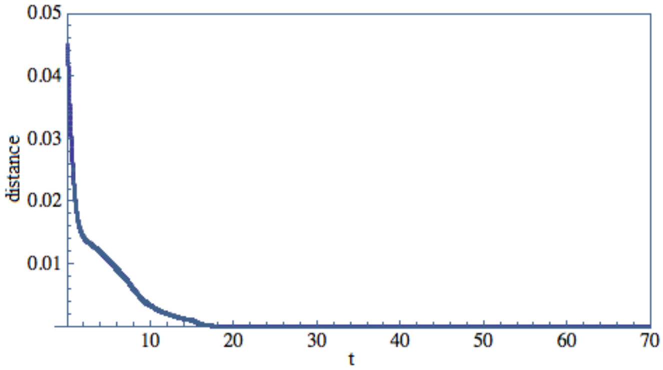


Figure 6.8: The distance of $\overline{\rho}$ from the manifold M_1.

Part III

Controllability results for a planar scallop swimmer

Chapter 7

Swimming by switching

In this chapter we investigate different strategies to overcome the scallop theorem. We will show how to obtain a net motion exploiting the fluid's type change during a periodic deformation. We are interested in two different models: in the first one that change is linked to the magnitude of the opening and closing velocity. Instead, in the second one it is related to the sign of the above velocity. An interesting feature of the latter model is the use of a switching rule through a delay relay. The latter is fundamental in order to get both forward and backward motion. The plan of the chapter is the following. In Sect. 7.1 we present the swimmer model and derive its equation of motion both in the viscous and in the ideal approximation, proving the scallop theorem. In Sect. 7.2, through the introduction of the switching strategies, we obtain both a result of partial controllability on the state, and a global controllability result of the scallop system. Finally in Sect. 7.3 we present some numerical simulations showing different kind of controls that can be used.
This study can be found also in Bagagiolo-Maggistro-Zoppello [15]

7.1 The Scallop swimmer

In this section we are interested in analyzing the motion of an articulated rigid body immersed in a fluid that changes its configuration. In order to determine completely its state we need the position of its center of mass and its orientation. Their temporal evolution is obtained solving the Newton's equations coupled with the Navier-Stokes equations relative to the surrounding fluid. We will face this problem considering the body as immersed in two kinds of different fluids: one viscous at low Reynolds number in which we neglect the effects of inertia, and another one ideal inviscid and irrotational, in which we neglect the viscous forces in the Navier-Stokes equations. First of all we recall that in both cases a swimmer that tries to moves like a scallop, opening and closing periodically its valves, does not

move at the end of a cycle. This situation is well known as scallop theorem (or paradox) (Purcell [86], Alouges-DeSimone-Lefebvre [7]).
In what follows we will consider a planar body composed by two rigid valves of elliptical shape, joined in order that they can be opened and closed. Moreover this body is constrained to move along one of the cartesian axes (the e_x-axis) and is symmetric with respect to it. Finally we will neglect the interaction between the two valves. The configuration of the system is easily described by the position x of the juncture point along the e_x-axis and by the angle θ that each valve forms with the axis.

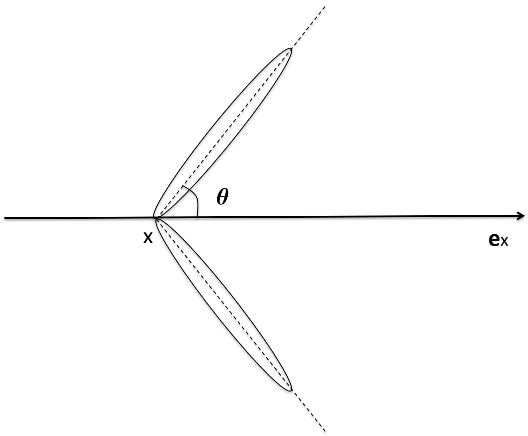


Figure 7.1: The scallop configuration

The possible translation of the system is determined by the consecutive opening and closing of the valves. Our aim is to determine the net translation of the body, given the function of time describing the angular velocity $\dot{\theta}$.

Viscous fluid

Here we focus on the case in which the scallop is immersed in a viscous fluid. In this regime the viscous forces dominate the inertial ones that can be neglected, so the equations governing the dynamics of the fluid are the Stokes ones:

$$\Delta v - \nabla p = 0,$$

together with the incompressibility condition div $v = 0$. Let us consider that the ellipses have major axis $2a$ and minor axis $2b$ with $b << a$, moreover let us suppose that $\theta \in (0, \frac{\pi}{2})$ so that it remains acute. One of the main difficulties in computing explicitly the equation of motion is the complexity of the hydrodynamic forces exerted by the fluid on the swimmer as a reaction to its shape changes. Since in our assumptions the minor axis of the ellipse is very small with respect to the major one, i.e. $b << a$, we can consider the swimmer as one-dimensional, composed essentially by two links of length $2a$ (see Fig 7.1). In the case of slender swimmers, Resistive Force Theory (RFT) (Gray-Hancock [63]) provides a simple and concise way to compute a local approximation of such forces, and it has been successfully used in several recent studies, see for example Becker-Koehler-Stone [28], Friedrich-Riedel et al. [57]. From now on we use this approach as well, in order to obtain the forces acting on the swimmer, neglecting the interaction between the valves. Since the scallop's density can be assumed to be comparable to that of the fluid and since the scallop is immersed in a viscous fluid the inertial forces are negligible with respect to the viscous ones, then the dynamics of the swimmer follows from Newton laws in which both the inertia of the fluid and of the scallop vanish:

$$F = 0, \tag{7.1}$$

where F is the total force exerted on the swimmer by the fluid. As already said we want to couple the fluid and the swimmer, using the local drag approximation of Resistive Force Theory. We denote by s the arc length coordinate on the i-th link ($0 \leq s \leq 2a$) measured from the juncture point and by $v_i(s)$ the velocity of the corresponding point. We also introduce the unit vectors

$$\begin{aligned} e_1 &= \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad e_1^{\perp} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}, \\ e_2 &= \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}, \quad e_2^{\perp} = \begin{pmatrix} -\sin(\theta) \\ -\cos(\theta) \end{pmatrix} \end{aligned}$$

in the directions parallel and perpendicular to each link and write the position of the point at arc length s as $x_i(s) = \begin{pmatrix} x \\ 0 \end{pmatrix} + s e_i$ where x is the coordinate of the joint between the two valves. By differentiation, we obtain,

$$v_i(s) = \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} + s\dot{\theta}_i e_i^{\perp}. \tag{7.2}$$

The density of the force f_i acting on the i-th segment is assumed to depend linearly on the velocity. It is defined by

$$f_i(s) := -\xi \left(v_i(s) \cdot e_i\right) e_i - \eta \left(v_i(s) \cdot e_i^{\perp}\right) e_i^{\perp}, \tag{7.3}$$

where ξ and η are respectively the drag coefficients in the directions of e_i and $e_i^{\perp}$ measured in $N\,s\,m^{-2}$. We thus obtain

$$F = \int_0^{2a} f_1(s)\,ds + \int_0^{2a} f_2(s)\,ds = 0. \tag{7.4}$$

Using (7.2) and (7.3) and since we are neglecting inertia we have

$$\begin{cases} F_x = -4a\xi\dot{x}\cos^2(\theta) - 4a\eta\dot{x}\sin^2(\theta) + 4a^2\eta\dot{\theta}\sin(\theta) = 0, \\ F_y = 0. \end{cases} \tag{7.5}$$

Observe that F_y vanishes since the scallop is symmetric with respect to the e_x axis. From (7.5) is now easy to determine the evolution of x

$$\dot{x} = V_1(\theta)\dot{\theta} = \frac{a\eta\sin(\theta)}{\xi\cos^2(\theta) + \eta\sin^2(\theta)}\dot{\theta}. \tag{7.6}$$

Ideal Fluid

While in the previous subsection we faced the problem of the self-propulsion of the scallop immersed in a viscous fluid, here we focus on the case in which it is immersed in an ideal inviscid and irrotational fluid. Let us make the same assumptions on the parameters a and b that have been done in the previous section, moreover let us denote by Ω the region of the plane occupied by the swimmer in a reference configuration.
Assigning (x, θ) as functions of time let us call

$$\begin{aligned} f^{(x,\theta)} &: \Omega \to \mathbb{R}^2, \\ &\zeta \mapsto f^{(x,\theta)}(\zeta), \end{aligned}$$

the function which maps each point of the swimmer $\zeta \in \Omega$ in $f^{(x,\theta)}(\zeta)$ that is its position in the plane at time t. Supposing that θ can be assigned and that there are not other external forces, our aim is to find equations that describe the motion of x. To this end we call v the velocity of the fluid, its motion is given by the Euler equations for ideal fluids

$$v_t + v \cdot \nabla v = -\nabla p, \tag{7.7}$$

with the incompressibility condition div $v = 0$. Moreover we impose a Neumann boundary condition, that is that the normal component of the velocity of the fluid has to be equal to the normal component of the velocity of the body:

$$\left\langle v(f^{(x,\theta)}) - \left(\frac{\partial f^{(x,\theta)}}{\partial x}\dot{x} + \frac{\partial f^{(x,\theta)}}{\partial \theta}\dot{\theta}\right), n^{(x,\theta)} \right\rangle = 0,$$

where $\left\langle \cdot \right\rangle$ denotes the scalar product, $n^{(x,\theta)}$ is the external normal to the set $f^{(x,\theta)}(\Omega)$. To find the evolution of x we should solve the Lagrange equation

$$\frac{d}{dt}\frac{\partial T^b}{\partial \dot{x}} = \frac{\partial T^b}{\partial x} + F, \tag{7.8}$$

where T^b is the kinetic energy of the body and F the external pressure force acting on the boundary of the swimmer. As already done in Mason-Burdick [81], Bressan [32], Munnier-Chambrion [84] this force F can be reinterpreted as a kinetic term, precisely thanks to the fact that we are in an ideal fluid. Therefore the system body + fluid is geodetic with Lagrangian given by the sum of the kinetic energy of the body (T^b) and the one of the fluid (T^f):

$$T^{tot} = T^b + T^f.$$

The kinetic energy of the body is the sum of the kinetic energy of the two ellipses, that reads

$$T^b = m\big(\dot{x}^2 + a^2\dot{\theta}^2 - 2a\dot{x}\dot{\theta}\sin\theta\big) + I\dot{\theta}^2, \tag{7.9}$$

with m the mass of the scallop and I is the moment of inertia.
Since we are dealing with an ideal fluid and thus inertial forces dominates over the viscous ones, in order to derive the kinetic energy of the fluid we will make use of the concept of *added mass*. In fluid mechanics, added mass or virtual mass is the inertia added to a system because an accelerating or decelerating body must move (or deflect) some volume of surrounding fluid as it moves through it. Added mass is a common issue because the object and surrounding fluid cannot occupy the same physical space simultaneously (Bessel [30]). For simplicity this can be modeled as some volume of fluid moving with the object, though in reality "all" the fluid will be accelerated, to various degrees.
Therefore the kinetic energy of the fluid will be given by the sum of the kinetic energy of the added masses of the two ellipses:

$$T^f = \frac{1}{2}v_1^T M_{1_{add}} v_1 + \frac{1}{2}v_2^T M_{2_{add}} v_2, \tag{7.10}$$

where $M_{i_{add}}$ are the added mass matrices relative to each ellipse which are diagonal, and v_i the velocities of their centre of mass, expressed in the frame solidal to each ellipse with axes parallel and perpendicular to the major axis.
Finally we can compute the total kinetic energy of the coupled system body+ fluid that is

$$\begin{aligned} T^{tot} &= m\big(\dot{x}^2 + a^2\dot{\theta}^2 - 2a\dot{x}\dot{\theta}\sin\theta\big) + I\dot{\theta}^2 + \\ &+ m_{11}\dot{x}^2\cos^2\theta + m_{22}\big(\dot{x}^2\sin^2\theta + a^2\dot{\theta}^2 - 2a\dot{x}\dot{\theta}\sin\theta\big) + m_{33}\dot{\theta}^2 \end{aligned} \tag{7.11}$$

where m_{ii}, $i = 1\dots3$, are the diagonal elements of the mass matrices.
Following a procedure introduced by Alberto Bressan in [32], in order to end up with a control system we perform a partial legendre transformation on the kinetic energy defining

$$p = \frac{\partial T^{tot}}{\partial \dot{x}} = 2\dot{x}\big(m + m_{11}\cos^2\theta + m_{22}\sin^2\theta\big) - 2a\dot{\theta}\sin\theta(m + m_{22}),$$

from which we derive

$$\dot{x} = \frac{p + 2a\dot{\theta}\sin\theta(m + m_{22})}{2(m + m_{11}\cos^2\theta + m_{22}\sin^2\theta)}. \tag{7.12}$$

There is a wide spread literature regarding the computation of added masses of planar contours moving in an ideal unlimited fluid. We will use in the rest of the chapter the added mass coefficients for the ellipse computed in Newman [85]: the added mass in the direction of the major axis is $m_{11} = \rho\pi b^2$, the one along the minor axis is $m_{22} = \rho\pi a^2$. Notice now that writing the Hamilton equation relative to p, and recalling (7.11)

$$\dot{p} = \frac{\partial T^{tot}}{\partial x} = 0,$$

thus, if we start with $p(0) = 0$, p remains null for all times and the evolution of x becomes

$$\dot{x} = V_2(\theta)\dot{\theta} = \frac{a\sin\theta(m + \rho\pi a^2)}{m + \rho\pi b^2\cos^2\theta + \rho\pi a^2\sin^2\theta}\dot{\theta}. \tag{7.13}$$

Theorem 7.1.1 (Scallop Theorem)**.** *Consider a swimmer dynamics of the type*

$$\dot{x} = V(\theta)\dot{\theta}. \tag{7.14}$$

Then for every T-periodic deformation (i.e. stroke) one has

$$\Delta x = \int_0^T \dot{x}(t)\,dt = 0, \tag{7.15}$$

that is, the final total translation is null.

Proof. Define the primitive of V by

$$F(\theta) = \int_0^\theta V(\sigma)\,d\sigma. \tag{7.16}$$

Then using (7.14)

$$\Delta x = \int_0^T V(\theta(t))\dot{\theta}(t)\,dt = \int_0^T \frac{d}{dt}F(\theta(t))\,dt = F(\theta(T)) - F(\theta(0)) = 0,$$

by the periodicity of $t \to \theta(t)$. □

Note that the dynamics (7.6) and (7.13) are of the type (7.14), therefore the scallop theorem is valid either in the viscous and in the ideal case.

7.2 Controllability

In this section we will give two different strategies to overcome the scallop theorem, both based on a switching mechanism. In particular we produce some partial and global controllability results for this switching systems.

7.2.1 Partial controllability in x

We have previously seen that if our scallop is immersed either in an ideal fluid or in a viscous one, if it experiences periodical shape changes it is not able to move after one cycle. Here we would like to find a way to overcome this problem. The main idea is to be able to change the dynamics during one periodical stroke and see if in this way we obtain a net motion and in particular some controllability. In order to do this we have to introduce the *Reynolds number*, a number which characterizes the fluid regime. It arises from the adimesionalization of the Navier-Stokes equations and it is defined by

$$Re = \frac{VL\rho}{\eta} = \frac{VL}{\nu}, \tag{7.17}$$

where V is the characteristic velocity of the body immersed in the fluid, L its characteristic length, ρ the density of the fluid, η its viscosity and $\nu = \frac{\eta}{\rho}$ is the kinematic viscosity. The Reynolds number quantifies the relative importance of inertial versus viscous effects.

- $\boxed{\eta = \eta(|\dot{\theta}|)}$

Let us recall that if $v(t, x)$ is a solution of the Navier Stokes equations, the function $u(t, x) = v(ct, x)$, $c > 0$ is still a solution of the Navier Stokes equations but with a different viscosity. Now assume that the absolute value of the speed $\dot{\theta}$ is very high, this means that rescaling the time of the solution of the Navier Stokes equations, we end up with a viscosity η that is very small and therefore the Reynolds number is large. In this case the inertial forces dominates over the viscous ones, so we can consider the scallop immersed in an ideal fluid and thus use the dynamics (7.13). Then we suppose that at a certain point of the cycle the absolute value of the angular velocity is very small. In this case we have a solution of the Navier Stokes equations with a very high viscosity η. Thus we can suppose that the scallop is immersed in a Stokes fluid, since the viscous effects dominates the inertial ones and use the dynamics (7.6). This situation is well represented by a switching system in which the change of the dynamics is determined by the modulus of the angular velocity $\dot{\theta}$: if it is big (i.e $|\dot{\theta}| > M$ with $M > 0$) we use the ideal approximation and the corresponding dynamics; if it is small (i.e $|\dot{\theta}| < M$ with $M > 0$) we use instead the viscous approximation and the relative dynamics.The switching rule in Fig 7.2 should also consider what happens

when $|\dot{\theta}| = M$. However in the sequel we are going to exhibit a function $\dot{\theta}$ which stays in M or $-M$ for only a set of times of null measure.
Our aim is to prove that using this kind of switching we are able to have a net displacement, both forward or backward, using periodic continuous functions $\dot{\theta}$

According to what said before we can prescribe the angular velocity $\dot{\theta}$ and thus use it as a control function u. Therefore we write the system as a control system that is

$$\begin{cases} \dot{x}(t) = V_{w(t)}(\theta(t))u(t), \\ \dot{\theta}(t) = u(t), \\ w(t) = h[u](t), \\ x(0) = x_0, \ \theta(0) = \theta_0 \ \ w(0) = w_0, \end{cases}$$

where u is continuous and

$$h[u] = \begin{cases} 2 & \text{if } |u| > M, \\ 1 & \text{if } |u| < M. \end{cases}$$

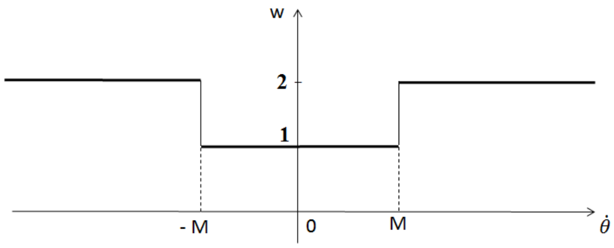


Figure 7.2: The rule of the classical switching

Moreover let us call F_i the primitives of the functions V_i, for $i = 1, 2$. They are :

$$F_1 = \frac{a\eta \operatorname{arctan} h(\sqrt{\frac{\eta-\xi}{\eta}} \cos\theta)}{\sqrt{\eta(\eta-\xi)}},$$

$$F_2 = \frac{-a\sqrt{m + a^2\rho\pi} \operatorname{arctan} h(\frac{\sqrt{(a^2-b^2)\rho\pi} \cos\theta}{\sqrt{m+a^2\rho\pi}})}{\sqrt{\rho\pi(a^2-b^2)}}.$$

Theorem 7.2.1. *With the previous switching scheme we are able to overcome the Scallop paradox, thus to move both forward and backward. More precisely there are $r > 0$ small enough (see remark 7.2.2), a final time $T > 0$ and a continuous T-periodic control function $u(t)$, which make the system move between two fixed points along the x axis, x_0 and $x_f \in]x_0 - r, x_0 + r[$, in the time T.*

Proof. **First case:** $u(0) > M$
In this case we start with the ideal approximation (i.e $w_0 = 2$)

$$V_{w(t)}(\theta(t)) = \begin{cases} V_2(\theta(t)) & 0 < t < t_1, \\ V_1(\theta(t)) & t_1 < t < t_2, \\ V_2(\theta(t)) & t_2 < t < t_3, \\ V_1(\theta(t)) & t_3 < t < t_4, \\ V_2(\theta(t)) & t_4 < t < T, \end{cases} \tag{7.18}$$

with

$$\begin{aligned} &t_1 := inf\{T > t > 0 \,|\, u(t) = M\},\ t_2 := inf\{T > t > t_1 \,|\, u(t) = -M\} \\ &t_3 := inf\{T > t > t_2 \,|\, u(t) = -M\},\ t_4 := inf\{T > t > t_3 \,|\, u(t) = M\}, \end{aligned}$$

assuming that $inf\,(\emptyset) = +\infty$. The net motion is then calculated as

$$\begin{aligned} \Delta x =&(F_2 - F_1)(\theta(t_1)) + (F_2 - F_1)(\theta(t_3)) \\ &- (F_2 - F_1)(\theta(t_2)) - (F_2 - F_1)(\theta(t_4)), \end{aligned} \tag{7.19}$$

taking into account that $\theta(0) = \theta(T)$ and that $(F_2 - F_1)(\theta(t_i))$ does not appear in the equation if $t_i = +\infty$.
We want to prove that we are able to move choosing a suitable periodic evolution for our control function $\dot{\theta} = u$. Let us call the unknowns $\theta_i := \theta(t_i)$, for $i = 1, \dots 4$. First of all we show that Δx as function of $(\theta_1, \theta_2, \theta_3, \theta_4)$ is surjective in $]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[$.
We are going to prove that

$$\nabla(\Delta x) = \begin{pmatrix} -(V_2 - V_1)(\theta_1) \\ (V_2 - V_1)(\theta_2) \\ -(V_2 - V_1)(\theta_3) \\ (V_2 - V_1)(\theta_4) \end{pmatrix} \neq 0$$
$$\text{in} \quad (\theta_1, \theta_2, \theta_3, \theta_4) \in]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[,$$

so that (7.19) is a submersion and surjective as required.
Recall that the function $(F_2 - F_1)(\cdot)$ is always increasing indeed

$$\begin{aligned} &\frac{(F_2 - F_1)(\theta)}{\partial\theta} = \\ &\left(- \frac{a\eta}{\xi\cos^2\theta + \eta\sin^2\theta} + \frac{ma + \rho\pi a^2}{m + \rho\pi b^2 cos^2\theta + \rho\pi a^2\sin^2\theta} \right) \sin\theta \\ &= \frac{\sin\theta\cos^2\theta\big(ma(\eta - \xi) + \rho\pi(\xi a^2 - \eta b^2)\big)}{(m + \rho\pi b^2 cos^2\theta + \rho\pi a^2\sin^2\theta)(\xi\cos^2\theta + \eta\sin^2\theta)} > 0 \end{aligned} \tag{7.20}$$

$$\text{for} \quad \theta \in]0, \frac{\pi}{2}[\quad \text{and } b << a.$$

From this immediately follows that $\nabla(\Delta x) \neq 0$.
The surjectivity ensures us that for any fixed Δx in a neighborhood of zero we are always able to find a $(\theta_1, \theta_2, \theta_3, \theta_4)$ which realize the desired displacement. Moreover, thanks to the symmetry properties of the function defining the displacement, also each of the 4-uplets $(\theta_1, \theta_4, \theta_3, \theta_2)$, $(\theta_3, \theta_4, \theta_1, \theta_2)$ and $(\theta_3, \theta_2, \theta_1, \theta_4)$ realizes the same displacement. Supposing $\Delta x > 0$ and recalling that the function $(F_2 - F_1)(\cdot)$ is increasing, then the angles $(\theta_1, \theta_2, \theta_3, \theta_4)$ will have a suitable order that can or not be coherent with the switching rule and the periodicity of $\dot{\theta}$. If their sorting is appropriate we will choose a control $\dot{\theta} = u$ such that $\theta(t_i) = \theta_i$. Otherwise at least one of the 4-uplets above will be right. Thus defining $(\theta_1^{'}, \theta_2^{'}, \theta_3^{'}, \theta_4^{'})$ this latter uple, we take a control u such that $\theta(t_i) = \theta_i^{'}$. This choice of the control will lead us to obtain the desired positive displacement.
For example suppose that the uplet $(\theta_1, \theta_2, \theta_3, \theta_4)$ which realizes the desired positive displacement, satisfy $\theta_3 > \theta_4 > \theta_1 > \theta_2$. Indeed

$$\left.\begin{array}{l}(F_2 - F_1)(\theta_1) - (F_2 - F_1)(\theta_2) > 0 \\ \\ (F_2 - F_1)(\theta_4) - (F_2 - F_1)(\theta_3) < 0\end{array}\right. \qquad \Longrightarrow \Delta x > 0$$

To respect the switching scheme in the time interval (t_2, t_3) the function $\dot{\theta}$ should decrease and thus $\theta_2 > \theta_3$. The latter is not satisfied by $(\theta_1, \theta_2, \theta_3, \theta_4)$, but taking $(\theta_1^{'}, \theta_2^{'}, \theta_3^{'}, \theta_4^{'}) = (\theta_3, \theta_4, \theta_1, \theta_2)$, we have the same Δx and the switching scheme is now respected. Analogous arguments can be used if $\Delta x < 0$.

Second case: $-M < u(0) < M$

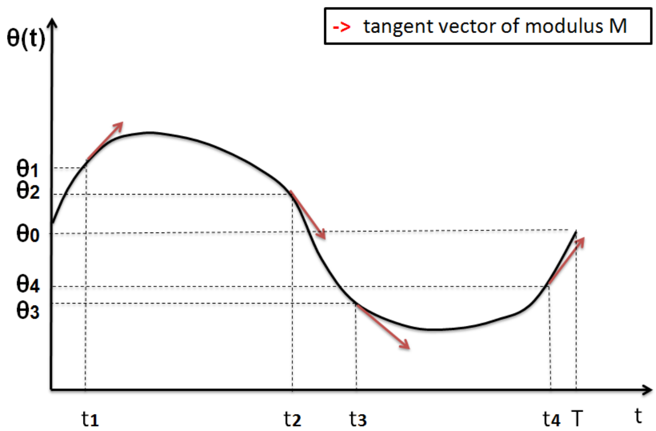


Figure 7.3: This figure shows a possible choice of $\theta(t)$ which realizes a positive displacement and respects the switching scheme

In this case we start form the viscous approximation (i.e $w_0 = 1$). Using arguments similar to the ones used before to compute Δx and to prove its surjectivity, and redefining accordingly the times t_i for $i = 1 \dots 4$, we have that

$$\begin{aligned}\Delta x =&(F_2 - F_1)(\theta(t_2)) + (F_2 - F_1)(\theta(t_4)) \\ &- (F_2 - F_1)(\theta(t_1)) - (F_2 - F_1)(\theta(t_3)).\end{aligned} \tag{7.21}$$

Going on as before, exploiting the surjetivity and the symmetry of the last function, we are able to find a control u that realizes the desired displacement.

Third case: $u(0) < -M$

This case is analogous to the first one.

In conclusion we have proved that wherever we start on the switching diagram we are able to achieve a net displacement either positive or negative and then we have the controllability. □

Remark 7.2.2. *Note that the value of r in the the last theorem is the maximal value that the function $|\Delta x(\theta_1, \theta_2, \theta_3, \theta_4)|$ can assume in $]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[$. Thus the constant r is independent from x and θ. To cover distances $|\Delta x| \geq r$ we should divide the spatial interval in N subintervals of length less than r, each one realized by a u of period $\frac{T}{N}$. Repeating N times this control u we are able to reach the desired displacement.*

- $\boxed{\eta = \eta(sign(\dot{\theta}))}$

While in the previous subsection we supposed that the change in the fluid regime was linked to the magnitude of the modulus of the angular velocity, here we would like to link the two fluids approximations to the sign of $\dot{\theta}$. This can be represented by a switching scheme as in Fig. 7.4. If the valves are opening ($\dot{\theta} > 0$) we suppose that the scallop is immersed in an ideal fluid; instead when the valves are closing ($\dot{\theta} < 0$) we assume the scallop immersed in a viscous fluid. This idea is inspired by Cheng-DeMont [42] where the fluid has a pseudoelastic nature that helps the valve opening but resist the valve closing. While [42] shows that only the moment of the scallop is subject to this pseudo-elastic dependence, we suppose instead that the forces are affected by this dependence, since we are not considering the moment equation. Despite the less physical evidences then [42], we conjecture that the model describes a different response of the fluid to the opening and closure of the scallop' valves, since it is not difficult to immagine the following situation. We suppose that during the valves closure the fluid opposes a greater resistance due to its compression. Instead when the valves are opening, the fluid assists this movement, and thus we have a small resistance. Therefore, according to our assumption the viscosity of the fluid changes between the opening and the closing of the valves, switching from one constant value to another one. This model is also mathematically interesting because it is an important

example of discontinuous hybrid switching system.
The system can be written as a control system, in which the control function $u(t)$ is the angular velocity $\dot{\theta}$:

$$\begin{cases} \dot{x}(t) = V_{w(t)}(\theta(t))u(t), \\ \dot{\theta}(t) = u(t), \\ w(t) = h[u](t) \\ x(0) = x_0, \ \theta(0) = \theta_0, \quad w(0) = w_0, \end{cases}$$

where the control u is continuous and now

$$h[u] = \begin{cases} 2 & \text{if } u > 0 \\ 1 & \text{if } u < 0 \end{cases}$$

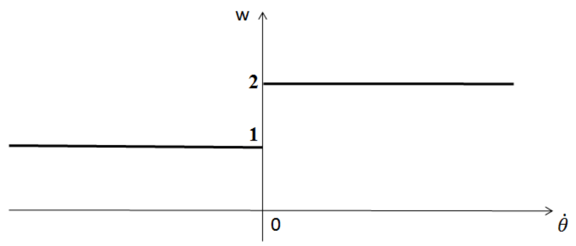


Figure 7.4: The rule of the classical switching

Theorem 7.2.3. *With the classical switching scheme (see Fig 7.4) we are able to overcome the scallop theorem but moving only forward. That is, there are $r > 0$ small enough, a time $T > 0$ and a continuous T-periodic control function, which make the system move between two fixed configurations x_0 and x_f with $x_f \in [x_0, x_0 + r[$, in the time T.*

Proof. Let us suppose to start with the ideal approximation, so that we are opening the valves

$$u(0) > 0 \quad \text{and} \quad w_0 = 2,$$

$$V_{w(t)}(\theta(t)) = \begin{cases} V_2(\theta(t)) & 0 < t < t_1, \\ V_1(\theta(t)) & t_1 < t < t_2, \\ V_2(\theta(t)) & t_2 < t < T, \end{cases} \tag{7.22}$$

with

$$t_1 := inf\{T > t > 0 \,|\, u(t) = 0\} \quad \text{and} \quad t_2 := inf\{T > t > t_1 \,|\, u(t) = 0\},$$

with $inf(\emptyset) = +\infty$. The net motion can be computed as

$$\Delta x = F_2(\theta(t_1)) + F_1(\theta(t_2)) - F_1(\theta(t_1)) - F_2(\theta(t_2)), \tag{7.23}$$

recalling as before that $\theta(0) = \theta(T)$. We want to prove that we are able to move choosing a suitable periodic evolution for our control function $\dot{\theta} = u$. Let us call $\theta_1 := \theta(t_1)$ and $\theta_2 := \theta(t_2)$, first of all we show that Δx as function of (θ_1, θ_2) is surjective in $]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[$.
Like before we prove that

$$\nabla(\Delta x) = \begin{pmatrix} (V_2 - V_1)(\theta_1) \\ (V_1 - V_2)(\theta_2) \end{pmatrix} \neq 0 \quad \text{in} \quad (\theta_1, \theta_2) \in]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[,$$

hence (7.23) is a submersion and surjective as required. Notice that

$$\Delta x = (F_2 - F_1)(\theta_1) - (F_2 - F_1)(\theta_2).$$

If we chose a control such that $\theta_1 > \theta_2$ then Δx will be positive, while if $\theta_1 < \theta_2$ then Δx will be negative. But since we need to respect the switching rule the last case could not be achieved because after t_1 $\dot{\theta} = u < 0$ and thus we are closing the valves therefore $\theta(t_2) = \theta_2$ will be necessarily less than $\theta(t_1) = \theta_1$.
The case where $u(0) < 0$ is analogous to the previous one.
In conclusion we have proved that for every choice of w_0 we are able to achieve a net displacement but only forward. □

Thermostatic-like case

In this section we introduce a mathematical variant of the previous switching in order to be able to move both forward and backward and therefore have a result of partial controllability in x. Our approach is to relate the variation of $u = \dot{\theta} \in \mathbb{R}$ to the fluid regime, by a delayed thermostat, an operator with memory, introduced rigorously in Visintin [96], consisting of two different thresholds for passing separately from one edge to the another one and vice-versa. This idea was inspired by Qiu et al. [87] in which the Scallop opening and closing is actuated by an external magnetic field, and thus a delay mechanism is reasonable. More precisely we consider (like in [87]) that an external magnetic field (H) is able to determine the opening and the closure of the valves, therefore the sign of $\dot{\theta}$ as in Figure 7.5 a). Composing it with the switching of Figure 7.4 we obtain the graph of Figure 7.5 b), which is the same of Figure 7.4 if $sign(H) = sign(\dot{\theta})$, and this is a natural assumption for example in the linear relation $\dot{\theta} = cH$ with $c > 0$. Then it is natural to consider an hysteresis phenomena between H and $sgn(\dot{\theta})$ as in Figure 7.5 c). Composing this relation with the dynamics switching introduced before (Figure 7.4), we get Figure 7.5 d). Subsequently, still supposing $\dot{\theta} = cH$ or more generally $\dot{\theta} = f(H)$ with f increasing and $f(0) = 0$, we will end up with a delay phenomena on the type of fluid regime (see Figure 7.6).

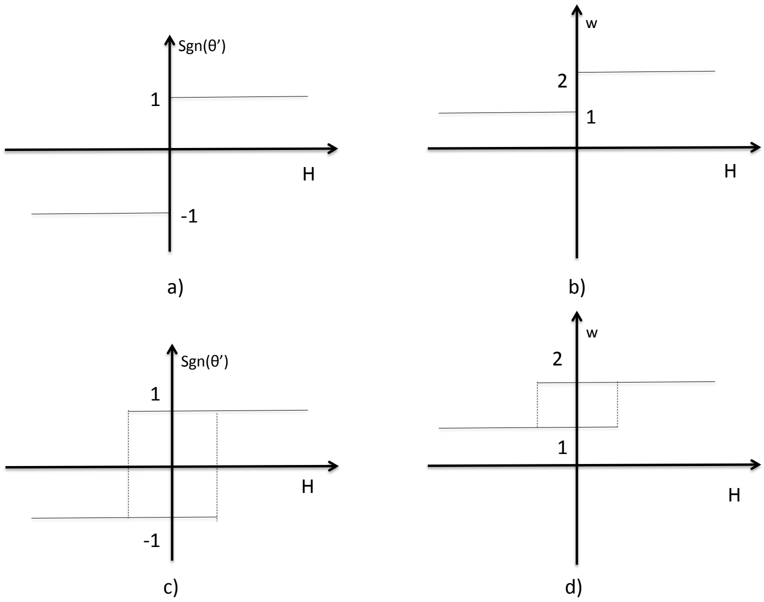


Figure 7.5: Relation between the magnetic field and the angular velocity

We suppose that the dynamics V depends on the angle $\theta \in]0, \frac{\pi}{2}[$, and also depends on a discrete variable $w \in \{1, 2\}$, whose evolution is governed by a delayed thermostatic rule, subject to the evolution of the control u. In Fig. 7.6 the behavior of such a rule is explained, correspondingly to the choice of a fixed threshold parameter $\varepsilon > 0$. See Sect. 3.1.1 for more details.

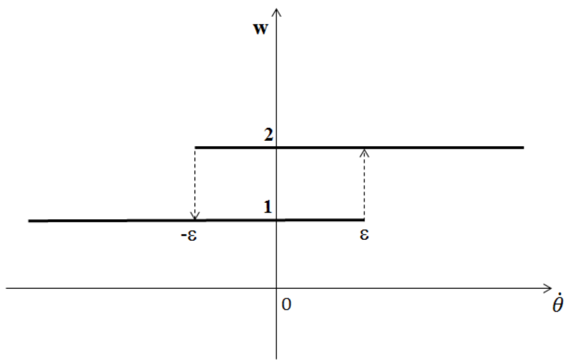


Figure 7.6: The thermostatic approximation

The controlled evolution is then given by

$$\begin{cases} \dot{x}(t) = V_{w(t)}(\theta(t))u(t), \\ \dot{\theta}(t) = u(t), \\ w(t) = h_\varepsilon[u](t), \\ x(0) = x_0, \quad \theta(0) = \theta_0, \quad w(0) = w_0, \end{cases} \tag{7.24}$$

where $h_\varepsilon[\cdot]$ represents the thermostatic delayed relationship between the input u and the output w. Note that the initial value $w_0 \in \{1, 2\}$ must be coherent with the thermostatic relation: $w_0 = 2$ (resp. $w_0 = 1$) whenever $\dot{\theta}_0 > \varepsilon$ (resp. $\dot{\theta}_0 < -\varepsilon$).
We start now to analyse the value of the displacement Δx depending of the value of u proving the following result:

Theorem 7.2.4. *Let $x_f \in]x_0 - r, x_0 + r[$ with $r > 0$ small enough. Then, there always exits a time $T > 0$ and a continuous T-periodic control function $\dot{\theta} = u$ (hence a periodic θ) such that one can move from x_0 to x_f in time T when the delayed thermostat is taken into account. In other words the system* (7.24) *is partially controllable in x.*

Proof. **First case**

$$-\varepsilon < u(0) < \varepsilon \quad \text{and} \quad w_0 = 1,$$

then we have

$$V_{w(t)}(\theta(t)) = \begin{cases} V_1(\theta(t)) & 0 < t < t_1 \\ V_2(\theta(t)) & t_1 < t < T. \end{cases} \tag{7.25}$$

where t_1 is the first time for which u goes through ε, i.e.

$$t_1 := inf\{T > t > 0 \,|\, u(t) = \varepsilon\}$$

and T is the final time. The displacement is then

$$\Delta x = F_1(\theta(t_1)) - F_1(\theta(0)) + F_2(\theta(0)) - F_2(\theta(t_1)). \tag{7.26}$$

recalling as before that $\theta(0) = \theta(T)$. We call $\theta(t_1) = \theta_1$ and we want to prove that we able to obtain $\Delta x = c, \forall\ |c| < r$ using a suitable periodic control function. In order to do this we show that $\Delta x(\theta_1)$ is surjective in a neighborhood of zero. First of all we compute the derivative and show that it is different from 0 and negative.

$$\frac{\partial \Delta x}{\partial \theta_1} = V_1(\theta_1) - V_2(\theta_1) =$$

$$\left(\frac{a\eta \sin\theta_1}{\xi\cos^2\theta_1 + \eta\sin^2\theta_1} - \frac{(ma+\rho\pi a^2)\sin\theta_1}{m+\rho\pi b^2 cos^2\theta_1 + \rho\pi a^2\sin^2\theta_1} \right)$$

$$= \frac{\sin\theta_1\cos^2\theta_1\big(-ma(\eta-\xi) - \rho\pi(\xi a^2 - \eta b^2)\big)}{(m+\rho\pi b^2 cos^2\theta_1 + \rho\pi a^2\sin^2\theta_1)(\xi\cos^2\theta_1 + \eta\sin^2\theta_1)} \neq 0$$

for $\theta_1 \in]0, \frac{\pi}{2}[$

Notice also that since in our assumptions b is negligible with respect to a, i.e $b << a$, we have that $\eta b^2 << \xi a^2$ and thus the derivative is always negative and consequently the Δx is decreesing. We are interested in $\theta_1 \in]0, \frac{\pi}{2}[$. Since the derivative of the function defining the displacement is different from 0 in $]0, \frac{\pi}{2}[$, (7.26) is locally invertible. Thus, since the inverse image of 0 is θ_0 then the inverse image of a neighborhood of 0 is a neighborhood of θ_0. Finally, recalling that Δx is decreasing, we can conclude that (7.26) can be positive or negative i.e. if we chose a control such that $\theta_1 < \theta_0$ the displacement will be positive instead if $\theta_1 > \theta_0$ it will be negative. In both cases the switching rule is respected thanks to the presence of the thermostat.

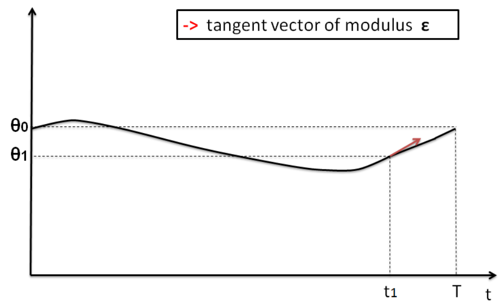


Figure 7.7: A possible choice of $\theta(t)$ starting from $0 < u(0) < \varepsilon$ which realizes a positive displacement.

Second case

$$-\varepsilon < u(0) < \varepsilon \quad \text{and} \quad w_0 = 2,$$

then we have

$$V_{w(t)}(\theta(t)) = \begin{cases} V_2(\theta(t)) & 0 < t < t_1, \\ V_1(\theta(t)) & t_1 < t < T. \end{cases} \tag{7.27}$$

where t_1 is the first time for which u goes through $-\varepsilon$

$$t_1 := inf\{T > t > 0 \,|\, u(t) = -\varepsilon\}$$

and T the final time. The displacement is

$$\Delta x = F_2(\theta(t_1)) - F_2(\theta(0)) + F_1(\theta(T)) - F_1(\theta(t_1)). \tag{7.28}$$

Calling again $\theta(t_1) := \theta_1$ also in the case we verify the surjectivity showing that the derivative of the displacement is different from zero.

Hence (7.28) is locally invertible and the inverse image of a neighborhood of 0 is a neighborhood of θ_0. We can conclude as in the previous case that (7.28) can be either positive or negative choosing a suitable control.

Third case

$$u(0) > \varepsilon \quad \text{and} \quad w_0 = 2,$$

$$V_{w(t)}(\theta(t)) = \begin{cases} V_2(\theta(t)) & 0 < t < t_1, \\ V_1(\theta(t)) & t_1 < t < t_2, \\ V_2(\theta(t)) & t_2 < t < T. \end{cases} \tag{7.29}$$

with

$$t_1 := inf\{T > t > 0 \,|\, u(t) = -\varepsilon\} \quad \text{and} \quad t_2 := inf\{T > t > t_1 \,|\, u(t) = \varepsilon\}.$$

The net motion is

$$\Delta x = F_2(\theta(t_1)) + F_1(\theta(t_2)) - F_1(\theta(t_1)) - F_2(\theta(t_2)). \tag{7.30}$$

recalling that $\theta(0) = \theta(T)$.
Also in this case we want to prove that we are able to move both forward or backward. Therefore we show that Δx is surjective in $]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[$ as in the non hysteretic case. We compute the gradient and show that it is never null

$$\nabla(\Delta x) = \begin{pmatrix} (V_2 - V_1)(\theta_1) \\ (V_1 - V_2)(\theta_2) \end{pmatrix} \neq 0 \text{ in } (\theta_1, \theta_2) \in]0, \frac{\pi}{2}[\times]0, \frac{\pi}{2}[$$

hence (7.30) is a submersion and surjective as required. Notice that

$$\Delta x = (F_2 - F1)(\theta_1) - (F_2 - F_1)(\theta_2)$$

and recall that the function $(F_2 - F1)(\cdot)$ is always increasing. Hence, if we use a control such that $\theta_1 > \theta_2$ the Δx will be positive, while if $\theta_1 < \theta_2$ then Δx will be negative. Also in this case both the alternatives can be achieved respecting the switching rule. Therefore we are able to obtain the desired displacement.

Fourth case
The case where $u(0) < -\varepsilon$ is analogous to the previous one.

In conclusion we have proved that for every choice of w_0 we are always able to find a periodic and continuous control $\dot{\theta} = u$ that allows us to obtain the desired displacement. The system (7.24) is then partially controllable in x. $\square$

Remark 7.2.5. *The introduction of the thermostat is essential because allows us to achieve displacements of every sign and thus the controllability result in* x*. This fact is strictly linked to the presence of the thresholds, indeed we are allowed to move between them without changing dynamics and therefore obtain values* $\theta_1 < \theta_2$ *either* $\theta_1 > \theta_2$*, and thus move both forward and backward.*

Remark 7.2.6. *Note that the maximal value of* r *in the last theorem is* $|\Delta x(\frac{\pi}{2})|$ *if* $-\varepsilon < u(0) < \varepsilon$*, and* $|\Delta x(\frac{\pi}{2}, 0)|$ *if* $-\varepsilon < u(0)$ *or* $u(0) > \varepsilon$*. Thus it is always independent from* x *and* θ*.*
To cover distances $|\Delta x| \geq r$ *we should divide the spatial interval in* N *subintervals of length less than* r*, each one realized by a* u *of period* $\frac{T}{N}$*. Repeating* N *times this control* u *we are able to reach the desired displacement.*

7.2.2 Global controllability result

In this subsection we are interested in studying whether it is feasible for the system of the scallop to move between two fixed configurations $((x_0, \theta_0)$ and $(x_f, \theta_f))$. This part add something to the previous one, since we are prescribing both the initial and final positions and angles. The following holds:

Theorem 7.2.7. *Let* A *and* B *be two fixed positions along the* x*-axis and* θ_0*,* θ_f *two fixed angles. Then, we are always able to find a suitable control function* $u(t)$ *such that the scallop system moves between* A *and* B *passing from* $\theta(0) = \theta_0$ *to* $\theta(T) = \theta_f$*, where* T *is a suitable big enough final time. Moreover such function* $u(t)$ *respects the switching rules modeling the dependence of the viscosity* η *from* $|\dot{\theta}|$ *(Fig.7.2) and from* $sign(\dot{\theta})$ *with the thermostat, (Fig.7.6). In other words the system* (7.24) *is controllable.*

Proof. Let $u(t)$ the periodic function that makes the system move between A and B with final angle θ_0 during a time $t^{'}$. We have proved the existence of such a function with both switching rules, in the previous subsection. Now whatever $w(t^{'})$ we open or close the valves respecting the switching rule in Fig.7.2 or Fig. 7.6 respectively until we reach the desired angle θ_f. We call $t^{''}$ the time in which we have θ_f and C the point in which we are arrived. Now starting from C with $w(t^{''})$ we move to B using another periodic $u(t)$ (hence $\theta(t)$ periodic), whose existence is ensured from Theorem 7.2.4 □

7.3 Numerical examples

In this section we will show, through numerical simulations, that our theoretical pretictions on the controllability of the Scallop along x are good. Moreover we will also describe how it is possible to obtain the same results

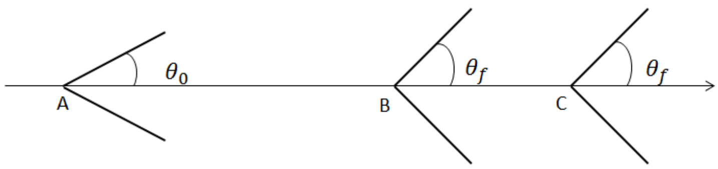


Figure 7.8: This figure represents one of the cases considered in the proof of Theorem 7.2.7

removing the continuity hypothesis on $\dot{\theta}$. In what follows the pictures are all relative to the controllability result which follows the thermostatic switching scheme (see Fig. 7.6) that is the most interesting one. Similar results can be obtained analogously using the other switching described in Fig. 7.2.
Let us suppose to start with $w(0) = 2$ which means $\dot{\theta}(0) > \varepsilon$, Figure 7.9 shows a possible choice of the control $\dot{\theta}$ to obtain a displacement $\Delta x = 1\,cm$, using the following parameters: $a = 2\,cm$, $b = 0.1\,cm$, $\eta = 2\,Nsm^{-2}$, $\xi = 1\,Nsm^{-2}$ $m = 1\,g$ and $\rho = 1\,gcm^{-3}$. More precisely in these simulations we decided to use a periodic polynomial control $\theta(t)$ that can be uniquely determined imposing the following constraints.

$$\begin{array}{llll} \dot{\theta}(0) = \dot{\theta}_0 & \dot{\theta}(t_1) = -\varepsilon & \dot{\theta}(t_2) = \varepsilon & \dot{\theta}(T) = \dot{\theta}_0 \\ \theta(0) = \theta_0 & \theta(t_1) = \theta_1 & \theta(t_2) = \theta_2 & \theta(T) = \theta_0 \end{array} \tag{7.31}$$

where θ_1 and θ_2 are determined by the numerical inversion of the function Δx (7.26) and we chose $t_1 = 2\,s$, $t_2 = 6\,s$ and $T = 7\,s$. It is easy to see that (since we want a positive displacement $\theta_1 > \theta_2$) $\dot{\theta}$ respects the thermostatic switching rule and that after a time $T = 7\,s$ we have gained the desired displacement of $1\,cm$.
Starting from the simulations in Figure 7.9 we want to build a piecewise constant control, instead of a continuous one, to obtain the same displacement. We note that in the case of delayed thermostat a discontinuous input is in general not allowed due to the presence of memory. The main difficulty of using a discontinuous control is to chose the switching times. Having in mind the previous simulations (Fig. 7.9) we can take the switching times of the continuous control and build a piecewise constant control which satisfies the constraints (7.31). See Figure 7.10. These simulations actually prove that the displacement does not depend on the whole control trajectory but only on the values that the angle θ and its derivative $\dot{\theta}$ assume in the switching times.

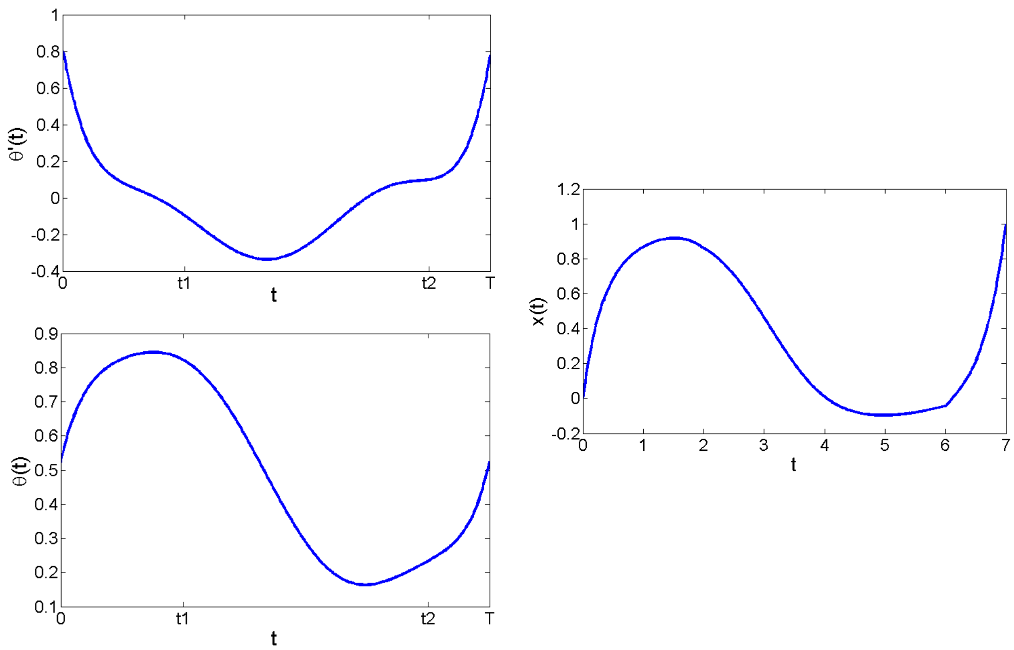


Figure 7.9: The polynomial control $\dot{\theta}(t)$, the resulting periodic angle $\theta(t)$ and the corresponding x displacement in function of time.

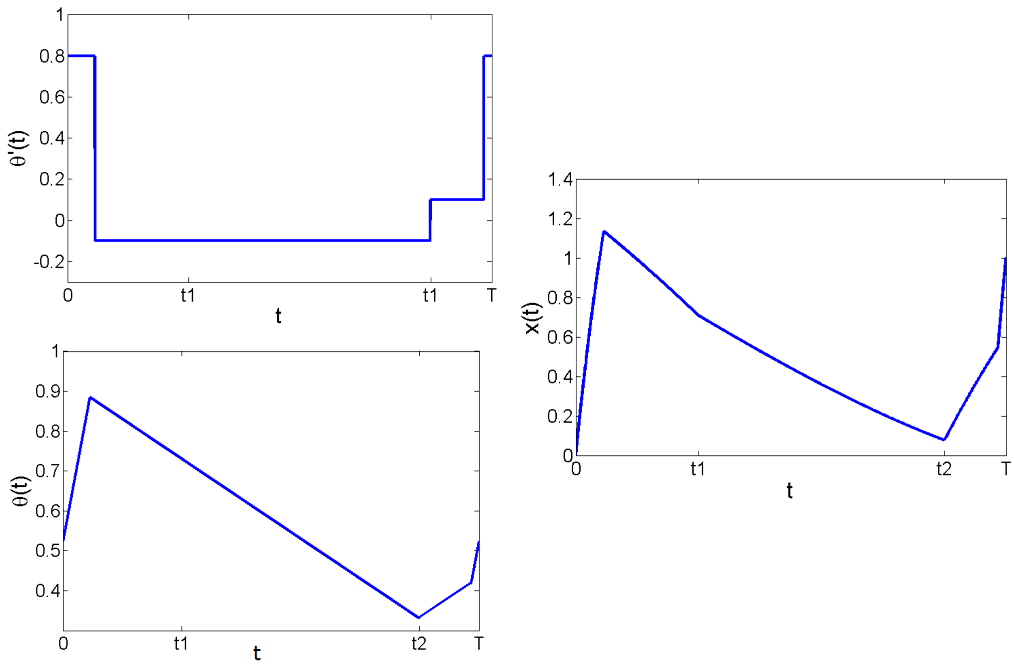


Figure 7.10: The piecewise constant control $\dot{\theta}(t)$, the resulting angle θ and the corresponding x displacement in function of time.

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