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On Semi-isogenous Mixed Surfaces

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The Faithful who gather at the mosque of Amr, in Cairo, are acquainted with the fact that the entire universe lies inside one of the stone pillars that ring its central court... No one, of course, can actually see it, but those who lay an ear against the surface tell that after some short while they perceive its busy hum...

- J.L.Borges, *The Aleph*

Introduction

During the first half of the 20th century, Federigo Enriques classified complex algebraic surfaces up to birational equivalence; this classification was extended to all compact complex manifold of dimension 2 in the 60's of the same century by Kunihiko Kodaira. The classification, that is widely known as the *Enriques-Kodaira classification*, divides compact complex surfaces in four families, according to their Kodaira dimension. Three of these families, namely those surfaces with Kodaira dimension $-\infty, 0$ and 1, are nowadays quite well known, while the fourth family, the one consisting of surfaces with Kodaira dimension 2, still poses numerous open problems. These latter are known as *surfaces of general type*.

Each surface of general type is associated with certain invariants; topological invariants, such as the self intersection of the canonical divisor K^2, and birational invariants, such as the *irregularity* q and the *geometric genus* p_g. These invariants determine the other classical invariants: the *(holomorphic) Euler-Poincaré characteristic* $\chi := 1 - q + p_g$ and the *topological Euler-Poincaré characteristic* $e = 12\chi - K^2$.

All of these values are related by several inequalities, that force minimal surfaces of general type to live in a bounded region of the $(\chi, K^2)-$space. Actually, it is not know if minimal surfaces of general type fill this region and this leads to the *geography problem*: once two admissible values for χ and K^2 are given, does there exist a minimal surface of general type having these invariants? One of the strategies we can adopt to solve the problem is to develop a way to construct surfaces of general type for fixed values of χ and K^2. It is important to have as many example of surfaces of general type as possible, for they often are useful to test problems and conjectures.

In [Bea83b], Beauville proposed a simple construction of a surface of general type, considering the quotient of the product of a curve C with itself with respect to the free action of a finite group G. Inspired by this construction, Catanese defined in [Cat00] *surfaces isogenous to a product*, quotients $(C_1 \times C_2)/G$ where C_i's are Riemann surfaces of genus at least two and G is a finite group acting freely on $C_1 \times C_2$. The significant aspect of these surfaces is that they are determined by combinatorial data that can be managed by an algorithm.

According to [Cat00], the action of a finite group on a product of two curves C_1 and C_2 both of genus at least two can be of two types: *mixed*, when some elements of G exchange the two factors (and then $C_1 \cong C_2$), and *unmixed* otherwise.

After Catanese, many authors studied surfaces birational to a quotient of a product of two curves, and several new surfaces of general type have been constructed in this way, mainly in the case with $\chi = 1$; see, for example, [BC04],[BCG08], [BCGP12], [BP12], [BP16], [Fra13], [FP15], [CP09], [Pol09], [MP10], [Pen11], [Zuc03]. In all of these works, the authors assume the group action to be free outside of a finite set of points. We call this case *quasi-étale*.

In the present work, we drop the quasi-étale assumption and we consider the following situation. Let C be a smooth projective curve of genus at least two and G be a finite subgroup of $\mathrm{Aut}(C \times C)$ whose action is mixed. The quotient surface $X := (C \times C)/G$ is a *mixed quotient*, and its minimal resolution of the singularities $S \to X$ is a *mixed surface*. We denote by $G^0 \triangleleft G$ the index two subgroup of G consisting of those elements that do not exchange the factors.

In general, the singularities of X are rather complicated, but if we assume the action of G^0 to be free, i.e. $(C \times C)/G^0$ to be a surface isogenous to a product, then X is smooth and we call it a *semi-isogenous mixed surface*. This work is devoted to the study of these surfaces.

Following the strategies of the above mentioned papers, our classification method combines geometry and group theory. To each semi-isogenous mixed surface we can associate the group G and a generating vector for G^0 (see Definition 1.25). The idea is that the geometry of X is encoded in this pair of algebraic data, hence the problem of constructing surfaces is translated into the problem of finding pairs (group, generating vector) subjected to certain conditions of combinatorial type.

One of the results of this work is an algorithm which, once the integers p_g, q and K^2 are given, produces all semi-isogenous mixed surfaces with those invariants. The algorithm was implemented using the computer algebra software MAGMA (see Appendix A for the commented script).

The algorithm works for arbitrary values of K^2, p_g and q; running the program for all possible positive values of K^2 and $p_g = q$ we obtained the following theorems.

Theorem A. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surfaces with $p_g(X) = q(X) = 0$ and $K_X^2 > 0$, such that $|G^0| \leq 2000$ and $|G^0| \neq 1024$. Then X belongs to one of the 15 families collected in Table 1 and it is of general type.*

Theorem B. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surfaces with $p_g(X) = q(X) = 1$ and $K_X^2 > 0$. Then X belongs to one of the 35 families collected in Table 2 and Table 3 and it is of general type.*

Theorem C. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surfaces with $p_g(X) = q(X) = 2$ and $K_X^2 > 0$. Then X belongs to one of the 9 families collected in Table 4 and it is of general type.*

For convenience, the tables can be found both on pages xi-xvii and on pages 77-83; they are explained and commented in Subsection 4.6.3. In Theorem A, the assumption $|G^0| \leq 2000$ and $|G^0| \neq 1024$ is a computational assumption; since MAGMA has some limitations (see Remark 4.46), we ask the algorithm to skip some cases. For $p_g = q \geq 3$, we already have a complete classification of the surfaces of general type (see [Bea82], [CCML98], [Pir02], [HP02]).

This classification led to the construction of new surfaces of general type; we mention, above all, one of the first examples of minimal surface of general type with $K^2 = 7$ and $p_g = q = 2$.

The last part of this thesis is dedicated to the minimality problem of semi-isogenous mixed surfaces. Let $X := (C \times C)/G$ be a semi-isogenous mixed surface and let

$$\eta \colon C \times C \to X$$

be the quotient map, ramified along R. We prove that $H^1(2K_X)$ is isomorphic to the cokernel of the G^0 invariant part of the restriction map of the global sections of the bicanonical system of $C \times C$ to R, i.e. if

$$\rho_0 \colon H^0(2K_{C\times C})^{G^0} \to H^0(2K_{C\times C}|_R)^{G^0}$$

is the restriction map, then $H^1(2K_X) \cong \operatorname{coker}(\rho_0)$.

Moreover, it is possible to displace this problem onto the curves C and R; we prove that $H^1(2K_X) \cong \operatorname{coker}(\Psi_0)$, where

$$\Psi_0 \colon (H^0(2K_C) \otimes H^0(2K_C))^{G^0} \to H^0(4K_R)^{G^0}$$

and the G^0-actions are defined in Theorem 5.16.

The thesis is organised as follows.

- Chapter 1 is devoted to the study of covering spaces and Riemann surfaces. In Section 1.1 we recall some well known results about covering spaces and lifting properties; moreover, we review the action of the fundamental group $\pi_1(X, x)$ on the fibre $p^{-1}(x)$ of a covering space $p\colon X' \to X$.

We focus on Galois covering of Riemann surfaces; in Section 1.2 and Section 1.3 we state some classical results, such as Hurwitz Formula and Riemann Existence Theorem. In Subsection 1.3.1 we see how a Galois covering $p\colon C \to C/G$ determines an algebraic datum, an *appropriate orbifold homomorphism.* In Subsection 1.3.2 we reverse this construction: we prove that once we are given an appropriate orbifold homomorphism, we can define a Galois covering $p\colon C \to C/G$.

In Section 1.4 we consider a Riemann surface C and a finite subgroup G of $\mathrm{Aut}(C)$; we show that the action of $\pi_1(C)$ on the universal cover of C extends to the action of a bigger group. This will be useful for the classification of semi-isogenous mixed surfaces.

In Section 1.6 we recall the definition of the canonical ring $R(X)$ associated with a compact complex manifold X and the definition of its Kodaira dimension $\kappa(X)$; in particular, we study the the canonical ring $R(C)$ of a Riemann surface C. Eventually, we give the classification of Riemann surfaces according to their Kodaira dimension.

- In Chapter 2 we recall some standard results concerning smooth complex surfaces. In Section 2.6 we describe the Enriques-Kodaira classification of compact complex surfaces. In Section 2.7 we direct our attention to surfaces of general type; in particular, we prove that for this class of surfaces, the first cohomology group of the bicanonical system is strictly related with the minimality problem. Then we recall some inequalities that hold for the invariants associated with a minimal surface of general type; these inequalities naturally lead to the *geography problem.* In the last section of this chapter, we illustrate the well known classification of surfaces of general type with $p_g = q \geq 3$.

- In Chapter 3 we examine group actions on a product of curves; in particular, referring to the results of Catanese contained in [Cat00], we give a description of the automorphism group of the product $C_1 \times C_2$, where both C_i's are curves of genus at least two. We see that in the case $C_1 \cong C_2$, there are two types of actions: *unmixed* and *mixed.* In Section 3.2 we give a description of *surfaces isogenous to a product.* Referring to the work of Frapporti and Pignatelli in [FP15], we briefly discuss *mixed quasi-étale surfaces.*

- Chapter 4 is dedicated to the study of *semi-isogenous mixed surfaces.* Let C be a curve of genus $g(C) \geq 2$, let G be a finite group acting on $C \times C$ and let G^0 denote the index 2 subgroup of elements that do not exchange the factors. We give a description of the locus of points in $C \times C$ having a non-trivial stabilizer, setting a bijection between the set of point-wise fixed curves and the elements of $G \setminus G^0$ of order 2.

After the definition of *semi-isogenous mixed surface*, in Section 4.2 we study the ramification and branch locus of the quotient map

$$\eta\colon C \times C \to X := (C \times C)/G,$$

while in Section 4.3 we compute the invariants associated with X. In Section 4.4 we compute the genus of the general Albanese fibre when $q(X) = 1$. In Section 4.5 we exploit the main theorem of [Arm68] in order to determine the fundamental group of X.

These results are combined to develop an algorithm that classifies all semi-isogenous mixed surfaces with fixed invariants K^2, p_g and q. This algorithm - implemented in MAGMA - and its theoretical background are described in Section 4.6, where we give some explicit bounds for the algebraic data that guarantee the finiteness of the algorithm. We run the program for $0 \leq p_g = q \leq 4$ and $1 \leq K^2 \leq 8$; results are listed and commented in Subsection 4.6.3. Because of some computational limits of MAGMA, the algorithm is forced to skip some cases, which are listed in Subsection 4.6.4. We complete the chapter extending the construction of semi-isogenous mixed surfaces to the case of low values of $g(C)$.

- In Chapter 5 we address the minimality problem of semi-isogenous mixed surfaces. In Section 5.1 we study the bicanonical system of a semi-isogenous mixed surface $X := (C \times C)/G$; we prove that this problem can be translated into a problem concerning the curve C and the ramification locus R of the quotient map $C \times C \to X$. As a byproduct, in Section 5.2 we prove that if X is of general type and the ramification locus is irreducible, then it is minimal. Exploiting Hodge Index Theorem, in Section 5.3 we give an explicit bound for $h^1(2K_X)$ when $\chi(\mathcal{O}_X) = 1$. We complete the chapter using this result to compute the minimal model of some of the surfaces constructed in the Chapter 4; in particular, we prove that all semi-isogenous mixed surfaces with $\chi = 1$ and $K^2 \geq 6$ are minimal.

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	min?
8	$D_{2,8,5} \rtimes \mathbb{Z}_2^2$	64, 92	$\mathbb{Z}_2^2 \times D_4$	32, 46	9	[0;2^5]	$\emptyset$	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$	Yes
8		256, 3679	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	[0;4^3]	$\emptyset$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
8		256, 3678	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	[0;4^3]	$\emptyset$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
8		256, 3678	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	[0;4^3]	$\emptyset$	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	Yes
8		256, 3678	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	[0;4^3]	$\emptyset$	$\mathbb{Z}_4^3$	Yes
6	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	32,43	$\mathbb{Z}_2 \times D_4$	16,11	9	[0; 2^6]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
6	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	32,27	$\mathbb{Z}_2^4$	16,14	9	[0; 2^6]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
6	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	32,27	$\mathbb{Z}_2^4$	16,14	9	[0; 2^6]	$(3,-8)$	$\mathbb{Z}_4^3$	Yes
6	$\mathbb{Z}_7 \times D_7$	98,3	$\mathbb{Z}_7^2$	49,2	15	[0; 7^3]	$(3,-8)$	$\mathbb{Z}_7^2$	Yes
6	$\mathbb{Z}_7 \times D_7$	98,3	$\mathbb{Z}_7^2$	49,2	15	[0; 7^3]	$(3,-8)$	$\mathbb{Z}_7^2$	Yes
6	$\mathbb{Z}_4^2 \rtimes D_4$	128, 734	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2$	64,211	17	[0; 2^5]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$	Yes
6	$(\mathbb{Z}_2^2 \times D_8) \rtimes \mathbb{Z}_2$	128, 750	$\mathbb{Z}_2^2 \times D_8$	64,250	17	[0; 2^5]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$	Yes
6	$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2^2$	128, 1797	$\mathbb{Z}_2^2 \times D_8$	64,250	17	[0; 2^5]	$(2,-4)^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$	Yes
2	$(\mathbb{Z}_2^3 \rtimes D_4) \rtimes \mathbb{Z}_2^2$	256, 47930	$\mathbb{Z}_2^4 \rtimes D_4$	128, 1135	33	[0; 2^5]	$(3,-8)^3$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	No
2	$(\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2) \rtimes \mathbb{Z}_2^2$	256, 45303	$\mathbb{Z}_2^4 \rtimes D_4$	128, 1135	33	[0; 2^5]	$(3,-8)^2, (2,-4)^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	No

Table 1: $p_g = q = 0,\ K^2 > 0$

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	g_{alb}	min?
8	$D_{2,8,5}$	16,6	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8,2	5	[1;2^2]	$\emptyset$	$\mathbb{Z}_4 \times \mathbb{Z}^2$	5	Yes
8	$D_{2,8,3}$	16,8	D_4	8,3	5	[1;2^2]	$\emptyset$	$\mathbb{Z}_4 \times \mathbb{Z}^2$	5	Yes
8	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	16,3	$\mathbb{Z}_2^3$	8,5	5	[1;2^2]	$\emptyset$	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$	5	Yes
7	$\mathbb{Z}_3 \rtimes D_4$	24,8	D_6	12,4	7	[1;2^2]	$(2,-4)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	5	Yes
7	$\mathbb{Z}_3 \times D_4$	24,10	$\mathbb{Z}_2 \times \mathbb{Z}_6$	12,5	7	[1;2^2]	$(2,-4)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	5	Yes
6	D_4	8,3	$\mathbb{Z}_2^2$	4,2	5	[1;2^4]	$(3,-8)$	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_3 \times S_3$	18,3	$\mathbb{Z}_3^2$	9,2	7	[1;3^2]	$(3,-8)$	$\mathbb{Z}_3 \times \mathbb{Z}^2$	4	Yes
6	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	32,43	$\mathbb{Z}_2 \times D_4$	16,11	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_2^2 \rtimes D_4$	32,28	$\mathbb{Z}_2 \times D_4$	16,11	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_2^2 \rtimes D_4$	32,28	$\mathbb{Z}_2 \times D_4$	16,11	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,11	$\mathbb{Z}_4^2$	16,2	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	Yes
6	$D_8 \rtimes \mathbb{Z}_2$	32,42	$D_4 \rtimes \mathbb{Z}_2$	16,13	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,31	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	16,3	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
6	$(\mathbb{Z}_2^2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	32,30	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	16,3	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_4 \times \mathbb{Z}^2$	3	Yes
6	$D_{2,8,5} \rtimes \mathbb{Z}_2$	32,38	$\mathbb{Z}_2 \times \mathbb{Z}_8$	16,5	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_4 \times D_4$	32,25	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	16,10	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
6	$(\mathbb{Z}_2^2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	32,30	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	16,10	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
4	$S_3 \times D_4$	48,38	$\mathbb{Z}_2^2 \times S_3$	24,14	13	[1;2^2]	$(2,-4), (4,-12)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	
4	$D_{12} \rtimes \mathbb{Z}_2$	48,37	$\mathbb{Z}_4 \times S_3$	24,5	13	[1;2^2]	$(2,-4), (4,-12)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	

Table 2: $p_g = q = 1$, $K^2 \geq 4$

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	g_{alb}	min?
2	$(\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2) \rtimes \mathbb{Z}_2$	64,153	$D_{2,8,5} \rtimes \mathbb{Z}_2$	32,7	17	$[1;2^2]$	$(3,-8),(5,-16)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$\mathbb{Z}_8 \rtimes D_4$	64,150	$D_4 \rtimes \mathbb{Z}_4$	32,9	17	$[1;2^2]$	$(3,-8),(5,-16)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$\mathbb{Z}_2^2 \rtimes D_8$	64,147	$D_4 \rtimes \mathbb{Z}_4$	32,9	17	$[1;2^2]$	$(2,-4)^2,(5,-16)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$	64,128	$\mathbb{Z}_2 \times D_8$	32,39	17	$[1;2^2]$	$(2,-4)^2,(3,-8)^2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$Q \rtimes D_4$	64,130	$\mathbb{Z}_2 \times D_{2,8,3}$	32,40	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$D_4 \rtimes D_4$	64,134	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	32,43	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$(\mathbb{Z}_2 \times D_4) \rtimes \mathbb{Z}_2^2$	64,227	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$	32,22	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_2 \times D_4) \rtimes \mathbb{Z}_2^2$	64,227	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$	32,22	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$\mathbb{Z}_4 \rtimes (D_4 \rtimes \mathbb{Z}_2)$	64,228	$(\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$	32,23	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_4 \times D_4) \rtimes \mathbb{Z}_2$	64,234	$(\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$	32,23	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_4 \times D_4) \rtimes \mathbb{Z}_2$	64,234	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,24	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_4 \rtimes Q) \rtimes \mathbb{Z}_2$	64,236	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,24	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2$	64,219	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_2^2 \rtimes D_4) \rtimes \mathbb{Z}_2$	64,221	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes D_4$	64,213	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2$	64,206	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8),(2,-4)^4$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No

Table 3: $p_g = q = 1$, $0 < K^2 < 4$

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	min?
8	$\mathbb{Z}_4$	4,1	$\mathbb{Z}_2$	2,1	3	[2;-]	$\emptyset$	$\mathbb{Z}_2 \times \mathbb{Z}^4$	Yes
7	$\mathbb{Z}_6$	6,2	$\mathbb{Z}_3$	3,1	4	[2;-]	$(2,-4)$	$\mathbb{Z}^4$	Yes
6	D_4	8,3	$\mathbb{Z}_2^2$	4,2	5	[2;-]	$(3,-8)$	$\mathbb{Z}^4$	Yes
6	D_4	8,3	$\mathbb{Z}_2^2$	4,2	5	[2;-]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}^4$	Yes
6	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8,2	$\mathbb{Z}_4$	4,1	5	[2;-]	$(2,-4)^2$	$\mathbb{Z}^4$	Yes
4	D_6	12,4	S_3	6,1	7	[2;-]	$(2,-4),(4,-12)$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 5$
2	$\mathbb{Z}_2 \times D_4$	16,11	D_4	8,3	9	[2;-]	$(2,-4)^2,(3,-8)^2$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 4$
2	$\mathbb{Z}_2 \times D_4$	16,11	D_4	8,3	9	[2;-]	$(2,-4)^2,(3,-8)^2$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 4$
2	$D_4 \rtimes \mathbb{Z}_2$	16,13	Q	8,4	9	[2;-]	$(3,-8)^3$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 4$

Table 4: $p_g = q = 2$, $K^2 > 0$

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Contents

Chapter 1

Riemann Surfaces

1.1 Covering spaces

Definition 1.1. Let X be a topological space. A *covering space* (or *étale covering*) of X is a pair $(\tilde{X}, p)$, where $\tilde{X}$ is a topological space and $p\colon \tilde{X} \to X$ that satisfies the following property: for each $x \in X$ there exists a path-connected open neighbourhood U such that each component of $p^{-1}(U)$ is mapped homeomorphically onto U by p.

Remark 1.2. For every $x \in X$ the topology induced by the topology of $\tilde{X}$ on the fibre $p^{-1}(x)$ is the discrete topology.

Once we are given a covering space $p\colon \tilde{X} \to X$ and $x_0 \in X$ we can define a $\pi_1(X, x_0)-$action on the fibre $p^{-1}(x_0)$. Let γ be a loop on X based at x_0, let $\overline{\gamma}$ be the inverse path, i.e. $\overline{\gamma}(t) := \gamma(1-t)$, and let $x \in p^{-1}(x_0)$; by the so called *path lifting property* (cf. [Hat02, page 60]) there exists a unique lift $\tilde{\gamma}$ of $\overline{\gamma}$ with starting point x, therefore we have a well defined map

$$L_\gamma\colon p^{-1}(x_0) \to p^{-1}(x_0)$$

that sends the starting point $\tilde{\gamma}(0)$ of each lift $\tilde{\gamma}$ to its ending point $\tilde{\gamma}(1)$.

The map L_γ depends only on the homotopy class of γ (cf. [Mas02, Lemma V.3.3]), then the association $\gamma \mapsto L_\gamma$ gives an homomorphism from $\pi_1(X, x_0)$ to the group of permutation of $p^{-1}(x_0)$.

This defines a left action of $\pi_1(X, x_0)$ on the fibre $p^{-1}(x_0)$, and for $\tilde{x} \in p^{-1}(x_0)$ the stabilizer of $\tilde{x}$ is the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}))$.

Definition 1.3. An *isomorphism* between two covering spaces $p_1\colon \tilde{X}_1 \to X$ and $p_2\colon \tilde{X}_2 \to X$ is a homeomorphism $\phi\colon \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ \phi$.

For further results and details concerning covering spaces we refer to [Hat02, Section 1.3].

We give the proof of the following theorem.

Theorem 1.4 (cf. [Hat02, Proposition 1.36]). *Let X be a topological space which is path-connected, locally path connected, and semilocally simply connected. Then, for every subgroup K of the fundamental group $\pi_1(X, x_0)$, there exists a covering space $p\colon X_K \to X$ such that $p_*(\pi_1(X_K, \tilde{x})) = K$ for a suitable choice of the base point $\tilde{x} \in p^{-1}(x_0)$.*

Proof. This proof consists of two steps; in the first one we construct an universal cover $\tilde{X}$ of X; in the second one we construct X_K starting from $\tilde{X}$.

Let us define

$$\tilde{X} := \{[\gamma] | \gamma \text{ is a path in } X \text{ such that } \gamma(0) = x_0\},$$

where $[\gamma]$ denotes the homotopy class of the path γ. The function

$$\begin{array}{cccc} p\colon & \tilde{X} & \to & X \\ & [\gamma] & \mapsto & \gamma(1) \end{array}$$

is well defined and it is surjective, because X is path-connected.

Now we need to define a topology on $\tilde{X}$.

Let $\mathcal{U}$ be the collection of path connected open sets $U \subseteq X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Note that if $\pi_1(U) \to \pi(X)$ is trivial for a choice of a base point, then it is trivial for every choice of a base point, because U is path connected. Moreover, if V is a path-connected open set such that $V \subset U \in \mathcal{U}$, then also $V \in \mathcal{U}$, since the composition $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ is trivial.

Suppose that there exists $U_1, U_2 \in \mathcal{U}$ such that $x \in U_1 \cap U_2$; since X is locally path connected, there exists a path connected open set $V \subset U_1 \cap U_2$ with $x \in V$, which means that $V \in \mathcal{U}$. Moreover, since X is semi-locally simply connected, for every $x \in X$ there exists $U \in \mathcal{U}$ with $x \in U$. Then $\mathcal{U}$ is a basis for the topology on X.

Given $U \in \mathcal{U}$ and a path γ in X with $\gamma(0) = x_0$ and $\gamma(1) \in U$, let us define

$$U_{[\gamma]} := \{[\gamma\eta] | \ \eta \text{ is a path in } U \text{ such that } \eta(0) = \gamma(1)\}.$$

The set $U_{[\gamma]}$ only depends on the homotopy class $[\gamma]$. The map $p|_{U_{[\gamma]}}\colon U_{[\gamma]} \to U$ is clearly surjective because U is path connected. Let us prove injectivity of $p|_{U_{[\gamma]}}$. Suppose that $\gamma\eta(1) = \gamma\eta'(1)$, then $[\gamma\eta\overline{\eta}'\overline{\gamma}] = 1$, because by definition of $\mathcal{U}$, the map $\pi_1(U) \to \pi_1(X)$ is trivial. Then $[\gamma\eta] = [\gamma\eta']$.

If $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma]} = U_{[\gamma']}$; indeed, if $\gamma' = \gamma\eta$, then the elements of $U_{[\gamma']}$ can be written as $[\gamma\eta\nu]$ and then belong to $U_{[\gamma]}$, and elements of $U_{[\gamma]}$ are $[\gamma\nu] = [\gamma\eta\overline{\eta}\nu] = [\gamma'\overline{\eta}\nu]$ and hence belong to $U_{[\gamma']}$.

The collection $\tilde{\mathcal{U}} := \{U_{[\gamma]} | \ U \in \mathcal{U}, \gamma \text{ path in } X, \gamma(0) = x_0, \gamma(1) \in U\}$ is basis for a topology on $\tilde{X}$. Let $U_{[\gamma]}$ and $V_{\gamma'}$ elements of $\tilde{\mathcal{U}}$ and let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, then $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Let $W \in \mathcal{U}$ be such that $W \subset U \cap V$ and $\gamma''(1) \in W$, then $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$.

The bijection $p|_{U_{[\gamma]}} \colon U_{[\gamma]} \to U$ is a homeomorphism. Let $V_{[\gamma']} \in \tilde{\mathcal{U}}$ such that $V_{[\gamma']} \subset U_{[\gamma]}$, then $p(V_{[\gamma']}) = V$. On the other side, for $V \in \mathcal{U}$ and $V \subset U$, $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$, where $[\gamma'] \in U_{[\gamma]}$ is a path with end point in V. This means that $p \colon \tilde{X} \to X$ is continuous.

We shall also remark that it is a covering space, since for $U \in \mathcal{U}$ the sets $U_{[\gamma]}$ forms a partition of $p^{-1}(U)$ as $[\gamma]$ varies: if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$, then $U_{[\gamma]} = U_{[\gamma']}$.

The last thing we need to prove is that $\tilde{X}$ is simply connected. Let $[\gamma] \in \tilde{X}$; we define the path γ_t in the following way:

$$\gamma_t(\tau) := \begin{cases} \gamma(\tau) & \tau \in [0, t] \\ \gamma(t) & \tau \in (t, 1] \end{cases}$$

The function $t \mapsto \gamma_t$ is a path in $\tilde{X}$ that starts at $[x_0]$ (the homotopy class of the constant path x_0) and ends at $[\gamma]$ Since $[\gamma]$ is an arbitrary point in $\tilde{X}$, we get that $\tilde{X}$ is path connected. To show that $\pi_1(\tilde{X}, [x_0])$ is trivial, it is sufficient to prove that $p_*(\pi_1(\tilde{X}, [x_0])) = 1$, being p_* injective (cf. [Hat02, Proposition 1.31]).

The elements in the image of p_* are represented by loops γ with base points in x_0 that lift to loops in $\tilde{X}$ based at $[x_0]$. The path $\tilde{\gamma} \colon t \mapsto [\gamma_t]$ lifts γ starting at $[x_0]$; as we want $\tilde{\gamma}$ to be a loop, $[\gamma_1] = [x_0]$. Since $\gamma_1 = \gamma$, this means that $[\gamma] = [x_0]$, so γ is homotopic to the constant loop and p_* is trivial.

In this way we have constructed the universal covering space $\tilde{X} \to X$.

The next step of the proof is to construct the covering space X_K. For $[\gamma], [\gamma'] \in \tilde{X}$, we define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma\overline{\gamma}'] \in K$. This is an equivalence relation since K is a subgroup: it is reflexive since $1 \in K$, it is symmetric since K is closed respect to the inverse operation, and it is transitive since K is closed under multiplication.

Let $X_K := \tilde{X}/\sim$ be the quotient space respect to this equivalence relation endowed with the quotient topology.

If $\gamma(1) = \gamma'(1)$, then $[\gamma] \sim [\gamma']$ if and only if $[\gamma\eta] \sim [\gamma'\eta]$; this means that if any two points in $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified in X_K, then the whole sets are identified. Then the natural projection $X_K \to X$ induced by $[\gamma] \mapsto \gamma(1)$ is a covering space.

Let $\tilde{x}_0 \in X_K$; it corresponds to the equivalence class of the constant path c based in $x_0 \in X$. A loop γ in X based in x_0 lifts to $\tilde{X}$ to a path with starting point $\tilde{x}_0$ and ending point $[\gamma]$, then γ lifts to a loop in X_K if and only if $[\gamma] \sim [c]$, or equivalently $[\gamma] \in K$. Therefore the image of $p_*\colon \pi_1(X_K, \tilde{x}_0) \to \pi_1(X, x_0)$ is K. □

Remark 1.5. If the group K is normal in $\pi_1(X, x_0)$, then we can define a $\pi_1(X, x_0)-$action on X_K in the following way: for $t \in \pi_1(X, x_0)$ and $[\gamma] \in X_K$, we define $t \cdot [\gamma] := [t\gamma]$.

This is equivalent to take the final point of the unique lift of the path $t\gamma$ of base point $[c] \in X_K$, where this latter denotes the homotopy class of the constant path based at x_0.

The action is well defined: if $\gamma \sim \gamma'$ then $t\gamma\overline{t\gamma'} = t(\gamma\overline{\gamma'})\bar{t}$. But by assumption $\gamma\overline{\gamma'} \in K$, which is normal, so $t(\gamma\overline{\gamma'})\bar{t} \in K$. By a straightforward computation, we see that this is a left action.

1.2 Galois Coverings

In the first part of this section we recall some basic definition and results concerning coverings of varieties, that we suppose to be algebraic, irreducible and normal. In the second part we discuss some well known results about Galois coverings of Riemann surfaces.

Definition 1.6. Let $f\colon X \to Y$ be a finite proper morphism between varieties of the same dimension. Then the inverse image of every point is a finite set of points. We say that such a map is a *branched covering.*

Definition 1.7. Let X be a variety and let G be a finite subgroup of $\mathrm{Aut}(X)$. We say that $f\colon X \to X/G$ is a *Galois covering.*

Definition 1.8. Let X and Y be varieties of the same dimension and let $f\colon X \to Y$ be a regular map such that $f(X) \subset Y$ is dense. The degree of the field extension $f^*(\mathbb{C}(Y)) \subset \mathbb{C}(X)$ is finite and is called the *degree* of f:

$$\deg(f) := [\mathbb{C}(X) : f^*(\mathbb{C}(Y))].$$

Proposition 1.9 (cf. [Sha77, Theorem 6.3.3])**.** *Let $f\colon X \to Y$ be a finite map between varieties of the same dimension. Then $|f^{-1}(y)| \leq \deg(f)$ for all $y \in Y$.*

Definition 1.10. Let $f\colon X \to Y$ be a branched covering, let $x \in X$ and $y \in f(x)$. If $|f^{-1}(y)| < \deg(f)$, then y is said to be a *branch point* and x is said to be a *ramification point.* The set of all branch points is called *branch locus.* If there are no branch points, then f is said to be *étale* (or *unbranched*).

Definition 1.11. Let $f\colon X \to Y$ be a branched covering, let $x \in X$ and $y = f(x)$. Let V be a neighbourhood of y such that the connected component U of $f^{-1}(V)$ containing x does not contain any other preimages of y. Then we define the *ramification index* of f at x as the number of the preimages in U of a generic point other than y in V. We will denote such a number as r_x.

Proposition 1.12 (cf. [Sha77, Theorem 6.3.4])**.** *Let $f\colon X \to Y$ be a branched covering. The complementary of the branch set in Y is an open set in the Zariski topology.*

Let us now focus our attention on the Riemann surfaces, complex manifolds of dimension one. Any non constant holomorphic map between two compact Riemann surfaces is a branched covering (cf. [Mir95, pages 48-49]) and for such a map it holds the well known *Hurwitz's formula.*

Theorem 1.13 (Hurwitz's formula, cf. [Mir95, Proposition II.4.16])**.** *Let $f\colon X \to Y$ be a non constant holomorphic map between compact Riemann surfaces. Then*

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{x \in X} (r_x - 1). \tag{1.1}$$

We should remark that, since the number of ramification points of a branched covering between compact curves is finite, the sum in (1.1) is finite.

Proposition 1.14 (cf. [Mir95, Proposition III.3.1, Proposition III.3.2])**.** *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. Then the stabilizer of each point is a cyclic group and the set points of X with non trivial stabilizers is discrete.*

Given a finite group acting on a Riemann surface C, then it is possible to define a complex structure on C/G (cf. Proposition III.3.3, [Mir95]).

Theorem 1.15 (cf. [Mir95, Theorem III.3.4])**.** *Let C be a Riemann surface and let G be a finite group acting on C. Then C/G has a structure of Riemann surface such that the quotient map $f\colon C \to C/G$ is holomorphic,* $\deg(f) = |G|$ *and* $r_p(f) = |\mathrm{Stab}_G(p)|$ *for any $p \in C$.*

Lemma 1.16 (cf. [Mir95, Theorem III.3.6])**.** *Let C be a compact Riemann surface and let G be a finite group acting on C. Let $f\colon C \to C/G$ be the quotient map. Then for every branch point $y \in Y$, there is an integer $r \geq 2$ such that $f^{-1}(y)$ consists of $|G|/r$ points of C, each one with ramification index r.*

Thanks to Lemma 1.16, we can rephrase Theorem 1.13 for Galois coverings in the following way.

Theorem 1.17 (cf. Theorem III.3.7, [Mir95]). *Let G be a finite group acting holomorphically and effectively on a compact Riemann surface C; let $\pi\colon C \to C/G =: Y$ be the quotient map. Suppose that there are r branch points $p_1, \dots, p_r$ in Y, and that $|\mathrm{Stab}(x)| := m_i$ for all $x \in \pi^{-1}(p_i)$ for each $i = 1, \dots, r$. Then*

$$2g(X) - 2 = |G|\left(2g(X/G) - 2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i}\right). \tag{1.2}$$

Corollary 1.18. *Let C be a Riemann surface of genus $g(C) \geq 2$ and let $\pi\colon C \to C/G \cong \mathbb{P}^1$ be a Galois covering of $\mathbb{P}^1$ with branch locus $\{p_1, \dots, p_r\}$. Then $r \geq 3$. Moreover, if $m_i := |\mathrm{Stab}(x)|$ for $x \in \pi^{-1}(p_i)$ for all $i = 1, \dots, r$, then*

$$-2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \geq \frac{1}{42}.$$

Proof. By Theorem 1.17,

$$2(g(C) - 1) = |G|\left(-2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i}\right).$$

Then $\sum_{i=1}^{r} \frac{m_i-1}{m_i} > 2$. But $\sum_{i=1}^{r} \frac{m_i-1}{m_i} < r$, then $2 < r$.

Without loss of generality, we can suppose $m_1 \leq m_2 \leq \cdots \leq m_r$. Since $m_i \geq 2$ for all $i = 1, \dots r$

$$-2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \geq -2 + \sum_{i=1}^{3} \frac{m_i - 1}{m_i} = 1 - \sum_{i=1}^{3} \frac{1}{m_i}$$

Suppose then that $r = 3$ and let us compute the minimum value of $\delta := 1 - m_1^{-1} - m_2^{-1} - m_3^{-1}$, taking into account that $\delta > 0$. Since $m_1 \geq 2$, $\delta \geq -1/2 + m_2^{-1} + m_3^{-1}$. Let us suppose $m_1 = 2$; if $m_2 = 2$, then $\delta < 0$, which is impossible. Then for $m_1 = 2$ it holds $m_2 \geq 3$, therefore $\delta \geq 1/6 - m_3^{-1}$. Suppose now $m_1 = 2$ and $m_2 = 3$; if $m_3 \leq 6$, then $\delta \leq 0$, which is impossible. Then the minimum value of δ is $1/42$ and it is reached for $(m_1, m_2, m_3) = (2, 3, 7)$. □

In the following chapters we will consider Riemann surfaces of genus $g \geq 2$; for these objects, we have two strong results concerning the automorphism group. The first one, due to Schwartz, asserts that the automorphism group of a Riemann surface C of genus ≥ 2 is finite. The second one, due to Hurwitz, gives a bound (that is sharp) for the order of such group.

Theorem 1.19 (Schwartz, [Sch90]). *Any compact Riemann surface of genus $g \geq 2$ has a finite number of automorphism, i.e. the automorphism group* $\mathrm{Aut}(C)$ *is finite.*

Theorem 1.20 (Hurwitz, cf. Theorem III.3.9, [Mir95]). *Let C be a compact Riemann surface of genus $g \geq 2$ and let G be a subgroup of* $\mathrm{Aut}(C)$*, then*

$$|G| \leq 84(g-1).$$

1.3 Riemann Existence Theorem

One of the main tools we will use in the following sections is provided by the Riemann Existence Theorem. This theorem allows to construct a Galois covering of a Riemann surface once such a surface is given along with the group associated to the covering.

Let C' be a Riemann surface, let $x_1, \dots, x_r$ be r points on C' and let $\bar{F} \colon \bar{C} \to C' \setminus \{x_1, \dots, x_r\}$ be an étale covering. Riemann Existence Theorem states that we can extend F uniquely, up to isomorphisms, to a Galois covering $F \colon C \to C'$, where C is a Riemann surface.

Proposition 1.21. *Let $f' \colon X \setminus A \to X'$ be a holomorphic map between two Riemann surfaces, where $A \subset X$ is finite. If there exists a continuous function $f \colon X \to X'$ that extends f', then f is holomorphic.*

Proof. Let $x \in A$ and let $\varphi \colon U \to \mathbb{C}$ and $\psi \colon V \to \mathbb{C}$ local charts defined in a neighbourhood of x and $f(x)$ respectively. The map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \mathbb{C}$$

is holomorphic in $\varphi(U \cap f^{-1}(V)) \setminus \varphi^{-1}(x)$ and it is bounded in a neighbourhood of $\varphi(x)$. By Riemann extension theorem (cf. [Lan03, Theorem V.3.1]), the map is holomorphic in $\varphi(x)$, hence f is holomorphic in x. □

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the unitary open disc and let $D^* := D \setminus \{0\}$ be the punctured disc.

Theorem 1.22 (cf. [GF12, Theorem 5.10]). *Let X be a Riemann surface and let $s \colon X \to D^*$ be a connected covering space of degree $m < +\infty$. There exists a biholomorphic map $\psi \colon X \to D^*$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "s"'] & & D^* \arrow[dl, "p_m"] \\ & D^* & \end{tikzcd}$$

where $p_m(z) = z^m$*.*

Theorem 1.23 (Riemann existence Theorem)**.** *Let* $\bar{C}$ *and* C' *be two Riemann surfaces and let* $A \subset C'$ *be a finite set. Let* $f\colon \bar{C} \to C' \setminus A$ *be a proper étale covering.*

Then f *can be extended to a ramified covering of* C'*, that is there exists a Riemann surface* C*, a proper holomorphic map* $F\colon C \to C'$ *and a biholomorphic map* $\varphi\colon C \setminus F^{-1}(A) \to \bar{C}$ *such that the diagram*

$$\begin{tikzcd} \bar{C} \arrow[d, "f"'] & C \setminus F^{-1}(A) \arrow[l, "\varphi"'] \arrow[r, hook] & C \arrow[d, "F"] \\ C' \setminus A \arrow[rr, hook] & & C' \end{tikzcd}$$

commutes. The couple (C, F) *is unique up to isomorphism.*

Proof. For each $x \in A$ let (U_x, ψ_x) be a chart centered in x; we can assume that $\psi_x(U_x) \cong D$ and that $U_{x_1} \cap U_{x_2} = \emptyset$ if $x_1 \neq x_2$. Let $U_x^* := U_x \setminus \{x\}$; since f is proper

$$f^{-1}(U_x^*) = V_{x,1}^* \sqcup \cdots \sqcup V_{x,N}^*,$$

where for each $i = 1, \ldots, N$ $V_{x,i}^* \to U_x^*$ is a connected covering of finite degree m_i. By Theorem 1.22, for each $i = 1, \ldots, N$ there exists a biholomorphic map $h_i\colon V_{x,i}^* \to D^*$ such that, defining $p_{m_i}\colon D^* \to D^*$ as $p_{m_i}(z) = z^{m_i}$, the diagram

$$\begin{tikzcd} V_{x,i}^* \arrow[r, "h_i"] \arrow[d, "f"'] & D^* \arrow[d, "p_{m_i}"] \\ U_x^* \arrow[r, "\psi_x"] & D^* \end{tikzcd}$$

commutes. As we want to add a point $y_{x,i}$ to each $V_{x,i}^*$, we define $V_{x,i} := V_{x,i}^* \cup \{y_{x,i}\}$ and the topology we consider is the one that makes the natural extension of h_i to $V_{x,i} \to D$ that sends $y_{x,i}$ to 0 an homeomorphism. We define

$$C := \bar{C} \cup \{y_{x,i}, i = 1, \ldots, N\}_{x \in A}.$$

There exists a unique topology on C such that the inclusion $i\colon \bar{C} \hookrightarrow C$ is continuous and for every open neighbourhood W of x the set $\{y_{x,i}\} \cup (f^{-1}(W) \cap V_{x,i}^*)$ is an open neighbourhood of $y_{x,i}$. This topology is Hausdorff.

Finally, we define $F\colon C \to C'$ as $F(z) = f(z)$ if $z \in \bar{C}$ and $F(y_{x,i}) = x$. The map F is proper. The charts $(V_{x,i}, h_i)$ are compatible with the charts of $\bar{C}$ and define a complex structure on C. The covering $f\colon \bar{C} \to C' \setminus A$ extends to a continuous map $F\colon C \to C'$ that is holomorphic by Proposition 1.21. Since $F^{-1}(A) = \{y_{x,i}\}$, its complement is by construction identified with $\bar{C}$. Indeed, the map $i^{-1}\colon C \backslash F^{-1}(A) \to \bar{C}$ is a biholomorphic map.

Suppose that there exists another Riemann surface C_1 and an holomorphic map $F_1\colon C_1 \to C'$ satisfying the statement. What we want to prove is that there exists an isomorphism $\Phi\colon C \to C_1$ such that $F = F_1 \circ \Phi$.

By hypothesis, there exists a biholorphism $\varphi_1\colon C_1 \setminus F_1^{-1}(A) \to C'$. Let us consider then the biholomorphism $\hat{\Phi} := \varphi_1 \circ \varphi^{-1}\colon C \backslash F^{-1}(A) \to C_1 \backslash F_1^{-1}(A)$; we want to extend it to a continuous function $\Phi\colon C \to C_1$.

By construction, $F_1^{-1}(U_x^*) = \hat{\Phi}(F^{-1}(U_x^*))$, so, since F_1 is proper, $F_1^{-1}(x)$ contains at least a point for each connected component of $\hat{\Phi}(F^{-1}(U_x^*))$. If $F_1^{-1}(x)$ contains some extra points, then these would be isolated, so C_1 would not be a Riemann surface in a neighbourhood of such points. Since C was defined such that for each $x \in A$, $F^{-1}(x)$ has many elements as the connected components of $F^{-1}(U_x^*)$, we can extend the map $\hat{\Phi}$ to a bijective continuous map $\Phi\colon C \to C_1$ sending each point $y_{x,i}$ to the unique accumulation point of $\hat{\Phi}(V_{x,i}^*)$. By Proposition 1.21, this map is holomorphic and then it is an isomorphism. □

1.3.1 From a Galois covering to an appropriate orbifold homomorphism

In order to understand the important tool that Riemann existence Theorem provides for the study of Galois coverings of a Riemann surface, we will need some notions of group theory.

Given integers $g \geq 0$ and and $m_1, \ldots, m_r > 1$ the *orbifold surface group of signature (or type)* $(g; m_1, \ldots, m_r)$ is defined as

$$\mathbb{T}(g; m_1, \ldots, m_r) := \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r | c_1^{m_1}, \ldots, c_r^{m_r}, \prod_{i=1}^{g} [a_i, b_i] \cdot c_1 \cdots c_r \rangle.$$

Remark 1.24. For $r = 0$, $\mathbb{T}(g)$ is the fundamental group of a Riemann surface of genus g.

Let H be a finite group; we say that an homomorphism

$$\psi\colon \mathbb{T}(g; m_1, \ldots, m_r) \to H$$

is an *appropriate orbifold homomorphism* if it is surjective and $\psi(c_i)$ has order m_i for each $i = 1, \ldots, r$.

Definition 1.25. Let H be a finite group and let $g, m_1, \ldots, m_r$ as above. A *generating vector* for H of type $(g; m_1, \ldots, m_r)$ is a $(2g+r)-$tuple of elements of H:

$$V := (d_1, e_1, \ldots, d_g, e_g, h_1, \ldots, h_r)$$

such that V generates H, $\prod_{i=1}^{g}[d_i, e_i] \cdot h_1 \cdots h_r = 1$ and there exists a permutation $\sigma \in \mathfrak{S}_r$ such that $\mathrm{ord}(h_i) = m_{\sigma(i)}$ for $i = 1, \ldots, r$. If such V exists, we will say that H is $(g; m_1, \ldots, m_r)-$generated.

Remark 1.26. Giving a generating vector of type $(g; m_1, \ldots, m_r)$ of a group H is equivalent to giving an appropriate orbifold homomorphism

$$\psi \colon \mathbb{T}(g; m_1, \ldots, m_r) \to H.$$

The last part of this section is dedicated to the proof of the following fact: once we are given a Galois covering $C \to C/G$ of a Riemann surface, we can define a generating vector of the group G.

Let C be a compact Riemann surface with genus $g(C) \geq 2$ and let G be a subgroup of $\mathrm{Aut}(C)$; let $C' := C/G$ be the quotient curve and let us denote by $g' := g(C')$ its genus. Let $B := \{p_1, \ldots, p_r\}$ be the branch locus of the quotient map $f \colon C \to C'$. For the details of the results contained in this section and their proofs we refer to [Mir95, pages 84-92].

Let us fix an element of B, say p_1 and let us consider its fibre $f^{-1}(p_1) := \{q_1, \ldots, q_t\}$. Then $H := \mathrm{Stab}(q_1) \cong \mathbb{Z}_n$ for some integer $n \geq 2$ (cf. [Mir95, Proposition III.3.1]). By construction, for each $i = 1, \ldots, t$ there exists $g_i \in G$ such that $g_i q_1 = q_i$. It is straightforward to prove the following lemma.

Lemma 1.27. $g_i H g_i^{-1} \cong \mathrm{Stab}_G(q_i)$.

This means that the stabilizers of q_i's are isomorphic and they all have the same cardinality $n = |G|/t$.

Let $X := C' \setminus B$ and let $p \in X$. Then there exists a set of loops

$$\{\alpha_1, \ldots, \alpha_{g'}, \beta_1, \ldots, \beta_{g'}, \gamma_1, \ldots, \gamma_r\} \subset \pi_1(X, p),$$

where

- $\pi_1(C', p) = \langle \alpha_1, \ldots, \alpha_{g'}, \beta_1, \ldots, \beta_{g'} \,|\, \prod_{i=1}^{g'} [\alpha_i, \beta_i] \rangle$;
- for each $i = 1, \ldots, r$ γ_i is a loop travelling once around q_i and no other point in B

such that

$$\pi_1(X,p) = \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \gamma_1, \dots, \gamma_r | \prod_{i=1}^{g'} [\alpha_i, \beta_i] \cdot \gamma_1 \cdots \gamma_r \rangle. \tag{1.3}$$

If we define $C^0 := C \setminus f^{-1}(B)$, then $f|_{C^0} \colon C^0 \to X$ is a regular covering (cf. [Hat02, pages 70-71]) and there is a surjective map

$$\theta \colon \pi_1(X,p) \to G.$$

The map θ is the so called *monodromy map* associated with the Galois covering $f|_{C^0}$. The map is defined as follows: let us fix $q \in f^{-1}(p)$, if γ is a loop based at p, $L_\gamma(q) = gq$ for some $g \in G$ (cf. Section 1.1), then $\theta([\gamma]) := g$. Monodromy is well defined up to conjugacy in G.

Given the presentation (1.3), for $i = 1, \dots, r$ let $h_i := \theta(\gamma_i)$ and let m_i be the order of the stabilizer of the points in $f^{-1}(p_i)$. For $j = 1, \dots, g'$ let $a_j := \theta(\alpha_j)$ and $b_j := \theta(\beta_j)$.

Lemma 1.28. *Using the notation defined above $\{a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r\}$ is a generating vector for G of type $(g'; m_1, \dots, m_r)$.*

Remark 1.29. One may ask what happens if we have a Galois covering $f \colon C \to C' := C/G$ where C consists in multiple connected components. Using the same notation as above, let $\theta \colon \pi_1(X,p) \to G$ be the monodromy map associated with $f|_{C^0}$.

Let us fix $x_0 \in f^{-1}(p)$; the fibre $f^{-1}(p) := \{gx_0 : g \in G\}$ is in bijection with G. Let $y_0, y_1 \in f^{-1}(p)$; they are in the same connected component of C if and only if there exists a path $\gamma \colon I \to F$ such that $\gamma(0) = y_0$ and $\gamma(1) = y_1$. This holds if and only if there exists a path $\eta \colon I \to F$ such that $\eta(0) = x_0$ and $\eta(1) = g_0^{-1} g_1 x_0$. This means that $g_0^{-1} g_1 \in \mathrm{Im}(\theta)$, that is $g_1 \in g_0 \mathrm{Im}(\theta)$.

In this way we have proved that two points in the fibre of p are in the same connected component of C if and only if the corresponding elements of G differ by an element in the image of the monodromy map. Moreover, the number of connected components of C is equal to $|G : \mathrm{Im}(\theta)|$.

1.3.2 From an appropriate orbifold homomorphism to a Galois covering

What we have seen so far shows that every Galois covering $C \to C/G \cong C'$ induces an appropriate orbifold homomorphism

$$\psi \colon \mathbb{T}(g(C'); m_1, \dots, m_r) \to G,$$

or equivalently a generating vector of type $(g'; m_1, \dots, m_r)$ for G.

Riemann existence Theorem is the key result that allows to reverse this construction: given a compact Riemann surface C' and a generating vector of type $(g'; m_1, \dots, m_r)$ for the finite group G, we construct a compact Riemann surface C such that $C' \cong C/G$.

Let $g' := g(C')$ be the genus of C', $p \in C'$ and

$$\pi_1(C') = \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{g'} [\alpha_i, \beta_i] \rangle$$

the fundamental group of C'. Let $(a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r)$ be a generating vector of type $(g'; m_1, \dots, m_r)$ for G. Fix $B := \{p_1, \dots, p_r\} \subset C'$ and $p \in X := C' \backslash B$. For each $j = 1, \dots, r$, let γ_j be a loop travelling around p_j such that $\prod_{i=1}^{g'} [\alpha_i, \beta_i] \cdot \prod_{j=1}^{r} \gamma_j = 1$, therefore

$$\pi_1(X) = \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \gamma_1, \dots, \gamma_r | \prod_{i=1}^{g'} [\alpha_i, \beta_i] \cdot \prod_{j=1}^{r} \gamma_j \rangle. \tag{1.4}$$

The generating vector $(a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r)$ induces a surjective homomorphism

$$\begin{array}{cccc} \theta\colon & \pi_1(X,p) & \longrightarrow & G \\ & \alpha_j & \longmapsto & a_j \\ & \beta_j & \longmapsto & b_j \\ & \gamma_i & \longmapsto & h_i \end{array}$$

let $K := \ker(\theta)$. By Theorem 1.4, we can associate to the normal subgroup K a Galois covering $f : X_K \to X$ such that $\pi_1(X_K, y) \cong K$.

Let $t \in \pi_1(X, p)$ and $[\gamma] \in X_K$; if we define $t \cdot [\gamma] := [t\gamma]$, we get a left action of $\pi_1(X, p)$ on X_K (cf. Remark 1.5). We can also define a $G-$action on X_K: let $h \in G$, then $h \cdot [\gamma] := [\delta\gamma]$, where $\delta \in \theta^{-1}(h)$. First of all, let us prove that this action is well defined: suppose $\delta_1, \delta_2 \in \theta^{-1}(h)$, then $\delta_1 = k\delta_2$ for some $k \in K$, therefore $[\delta_1\gamma] = [k\delta_2\gamma] = [\delta_2\gamma]$. For $h \in G$ and $[\gamma] \in X_K$ we will write $[\theta^{-1}(h)\gamma] := h[\gamma]$.

The $G-$action on X_K is faithful:

$$\begin{aligned} [\gamma] = h[\gamma] = [\theta^{-1}(h)\gamma] &\Leftrightarrow \theta^{-1}(h)\gamma\overline{\gamma} \in K \\ &\Leftrightarrow \theta^{-1}(h) \in K \\ &\Leftrightarrow h = 1_G. \end{aligned}$$

By Theorem 1.23, we can extend the étale covering $f\colon X_K \to X = C' \setminus B$ to a Galois covering $F\colon C \to C'$.

The following results underline the close link between the algebraic data and the geometry of the covering space.

Let us fix $p_1 \in B$ and let W be a small open neighbourhood of p_1 in C' such that $W \setminus \{p_1\}$ is isomorphic to a punctured disc. Let $f^{-1}(W \setminus \{p_1\}) = D_1^* \sqcup \dots D_s^*$.

Let τ be a loop in W with base point p' and travelling around p_1 once; let γ be a path on X starting at p and ending at p' such that γ_1 is homotopic to $\gamma\tau\overline{\gamma}$.

Proposition 1.30. *$f^{-1}(W \setminus \{p_1\}) = D_1^* \sqcup \dots D_s^*$ has $s = |G : \langle h_1 \rangle|$ connected components.*

Proof. By definition of X_K,$[\gamma] \in f^{-1}(p')$; let $[\gamma']$ be another point in $f^{-1}(p')$. The two points $[\gamma]$ and $[\gamma']$ belong to the same connected component D_i^* if and only if there exists a path $\delta \colon I \to X_K$ with $\delta(0) = [\gamma]$, $\delta(1) = [\gamma']$ and such that $\eta := f \circ \delta$ is contained in W. This means that η is a loop in W , hence there exists a k such that $\eta = \tau^k$ and $[\gamma'] = [\gamma\eta]$. Since $\theta(\gamma\eta\overline{\gamma}) = h_1^k$, to each point of $f^{-1}(p') \cap D_i^*$ corresponds an element of $S := \langle h_1 \rangle$. Therefore $s \leq |G : \langle h_1 \rangle|$

Conversely, to each $\gamma_1^k \in \langle \gamma_1 \rangle$ we associate the point $[\gamma_1^k\gamma] \in D_i^*$. These points are exactly $m_1 = \mathrm{ord}(h_1)$:

$$\begin{aligned} [\gamma_1^a\gamma] = [\gamma_1^b\gamma] \iff \gamma_1^{a-b} \in K \iff h_1^{a-b} = \theta(\gamma_1^{a-b}) = 1_G \\ \iff a \cong b \mod m_1. \end{aligned}$$

□

Proof of Proposition 1.30 implies that $[\gamma]$ and $[\gamma\tau^k]$ belong to the same connected component D_i^* for each $k \in \mathbb{Z}$.

Let us define $S := \langle h_1 \rangle$, then $\{[\gamma\tau^k]\}_k = S \cdot [\gamma]$.

Proposition 1.31. *The correspondence $h \mapsto h \cdot [\gamma]$ is a bijection between G and $f^{-1}(p')$.*

Proof. We first prove injectivity. Let $h, h' \in G$ such that $h[\gamma] = h'[\gamma]$, that is $\theta^{-1}(h)\gamma\overline{\gamma}\theta^{-1}(h'^{-1}) = \theta^{-1}(hh'^{-1}) \in K$, hence $hh'^{-1} = 1_K$.

In order to prove surjectivity let $[\eta] \in f^{-1}(p)$; if we define $h := \theta(\eta\overline{\gamma})$ we get $h \cdot [\gamma] = [\theta^{-1}(h)\gamma] = [\eta]$. □

Proposition 1.32. *Being in the same connected component D_i^* is equivalent to belonging to the same left coset.*

Proof. By Proposition 1.31, to each $h \in G$ is associated a unique element of the fibre $f^{-1}(p')$: $[\theta^{-1}(h)\gamma]$. Let $h, h' \in G$; then $hS = h'S$ if and only if $h\theta(\gamma\tau^k\overline{\gamma}) = h'$ for some k, that is $h\theta(\gamma\tau^k\overline{\gamma})[\gamma] = h'[\gamma]$. That is $[\theta^{-1}(h')\gamma] = [\theta^{-1}(h)\gamma\tau^k\overline{\gamma}\gamma] = [\theta^{-1}(h)\gamma\tau^k]$, that is equivalent to be in the same D_i^* by the argument of Proposition 1.30. □

Proposition 1.30, Proposition 1.31 and Proposition 1.32 imply the following result.

Lemma 1.33. *There is a bijection*

$$\begin{array}{ccc} F^{-1}(p_i) & \longleftrightarrow & \{kS\} \\ y_j & \longleftrightarrow & k_jS \end{array}$$

where $S := \langle h_i \rangle$.

A straightforward computation leads to the following results.

Lemma 1.34. $\mathrm{Stab}_G(y_j) = k_j S k_j^{-1}$.

The construction we have given so far proves to the following Proposition.

Proposition 1.35. *Given the following algebraic data:*

- *a finite group* G*;*
- *a curve* C'*;*
- *points* $p_1, \dots, p_r \in C'$ *and* $\alpha_i, \beta_j, \gamma_k \in \pi_1(C' \setminus \{p_1, \dots, p_r\})$ *as in (1.4);*
- *integers* $m_1, \dots, m_r > 1$*;*
- *a generating vector* $V = (a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r)$ *for* G^0 *of type* $(g(C'); m_1, \dots, m_r)$

there exists a Galois covering $c\colon C \to C/G^0 \cong C'$ *branched over* $\{p_1, \dots, p_r\}$ *with ramification index equal to* m_i *over* p_i.

The set of elements of G^0 *with non empty fixed locus is the set*

$$\Sigma_V := \bigcup_{g \in G^0} \bigcup_{i=r}^{r} \bigcup_{j=1}^{m_i} \{g \cdot h_i^j \cdot g^{-1}\}. \tag{1.5}$$

1.4 Group actions on the universal covering

Let C' be a Riemann surface of genus g', let $\{p_1, \dots, p_r\} \subset C'$ and p a point in $X := C' \setminus \{p_1, \dots, p_r\}$ and let

$$\theta\colon \mathbb{T}(g'; m_1, \dots, m_r) \to G$$

be an appropriate orbifold homomorphism. Let us take a representation for $\pi_1(X, p)$ as the one in (1.4). As we showed in Section 1.3.2, θ induce a Galois covering $\theta\colon C \to C'$.

Let $u\colon \Delta \to C$ the universal cover of C; as seen in the proof of Theorem 1.4, the points of Δ corresponds to the homotopy classes of paths in C with base point $y = [p]$, the constant path in X of base point p.

By Remark 1.5, the group $\pi_1(C, y)$ acts on Δ as follows: for $\eta \in \pi_1(C, y)$ and $[\delta] \in \Delta$, $\eta \cdot [\delta] = [\eta\delta]$, that is the final point of the unique lift of $\eta\delta$ with starting point $[x_0]$, the class of the constant path in C with base point y.

The path $\gamma_i^{m_i} \in \pi_1(X, p)$, travelling m_i times around p_i, lifts to a path $c_i \in \pi_1(C_0, y)$ that travels once around a point of the fibre of p_i. For all $g \in G$ let us fix a path $\alpha_g\colon [0, 1] \to C_0$ with $\alpha_g(0) = y$ and $\alpha_g(1) = gy$.

The normal subgroup

$$H := \langle\langle \alpha_g \cdot g_* c_i \cdot \overline{\alpha_g} : g \in G, i = 1, \dots r\rangle\rangle$$

is a subgroup of $\pi_1(C, y) \cong \langle a_1, b_1, \dots, a_g, b_g | \prod_{i=1}^g [a_i, b_i]\rangle$, where with $\langle\langle S\rangle\rangle$ we denote the subgroup normally generated by S.

We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & H \arrow[r, "c_i \mapsto \gamma_i^{m_i}"] \arrow[d] & \langle\langle \gamma_i \rangle\rangle \arrow[r] \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \pi_1(C_0, y) \arrow[r, "(f_0)_*"] \arrow[d] & \pi_1(X, p) \arrow[r, "\theta"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(C, y) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

By construction, it follows that

$$\begin{aligned} F &= \pi_1(X, p)/\langle\langle \gamma_i^{m_i} \rangle\rangle \\ &= \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \gamma_1, \dots, \gamma_r | \prod [\alpha_i, \beta_i] \cdot \gamma_1 \cdots \gamma_r, \gamma_j^{m_j} \rangle =: \mathbb{T}. \end{aligned} \tag{1.6}$$

In this way we have proved the following

Lemma 1.36. *The sequence*

$$1 \longrightarrow \pi_1(C, y) \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow 1$$

is exact.

The following results will be useful to compute the fundamental group of the surfaces we are going to study in the next sections.

Lemma 1.37. *The action of* $\pi_1(C, y)$ *on* Δ *extends to an action of* $\mathbb{T}$ *on* Δ.

Proof. Let $[\delta] \in \Delta$, $w := u([\delta]) \in C$ and $z := f(w) \in C'$; let $t \in \mathbb{T}$. Suppose that $w = [\gamma]$ is the homotopy class of the path $\gamma \subset C'$ with starting point p.

Recall that, by (1.6), t is a loop on X based at p; we have defined $t \cdot w = t \cdot [\gamma] = [t\gamma] =: w'$, which is the final point of the unique lift η of $t\gamma$ with base point $y = [p]$, where $[p]$ is the homotopy class of the constant path based at p.

We lift η to the unique lift with base point $[x_0]$, the homotopy class of the constant path in C with base point y; we define $t \cdot [\delta]$ as the final point of this lifting. Exploiting the uniqueness of the lift, this is a well defined action that coincides on $\pi_1(C, y)$ with the usual action. $\square$

Given the $\mathbb{T}-$action on Δ, we want to compute the fixed locus of an element of $\mathbb{T}$.

Lemma 1.38. *Let* $[\delta] \in \Delta$ *then*

$$\mathrm{Stab}_{\mathbb{T}}([\delta]) = \begin{cases} \{1\} & \text{if } f(u([\delta])) \notin \{p_1, \ldots, p_r\} \\ \alpha\langle\gamma_i\rangle\alpha^{-1} & \text{if } f(u([\delta])) = p_i, \text{ for some } \alpha \in \mathbb{T} \end{cases}$$

Proof. Let $[\delta] \in \Delta$, $w := u([\delta])$ and $z := f(w) = [\gamma]$; let $t \in \mathbb{T}$. If $t \cdot [\delta] = [\delta]$, then by definition it holds $t \cdot [\gamma] = [\gamma]$. Either $z \notin \{p_1, \ldots, p_r\}$ or $z = p_i$ for some $i = 1, \ldots, r$.

If $z \notin \{p_1, \ldots, p_r\}$, then w is not a ramification point for f, so $\mathbb{T}$ acts as $\pi_1(X, p)$, whose action on C_0 is free, then $\mathrm{Stab}(w) = \{1\}$, that is $t = 1$.

If $z = p_i$ for some i, then w is a ramification point for the quotient map f, then by Lemma 1.34, $\mathrm{Stab}_G(w) = kSk^{-1}$ where $S = \langle h_i\rangle$ and $k \in G$, but $g[\gamma] = [\theta^{-1}(g)\gamma] = (\alpha\gamma_i^d\alpha^{-1})[\gamma]$ for some $\alpha \in \mathbb{T}$ and $d \in \{1, \ldots, m_i - 1\}$, so $\mathrm{Stab}_{\mathbb{T}}(w) = \alpha\langle\gamma_i\rangle\alpha^{-1}$. $\square$

1.5 Riemann-Roch Theorem for Curves

In this section we give some basic definitions on divisors and invertible sheaves on a Riemann surface. We will state, without giving any proof, some well known and fundamental classical results.

For further details we refer to [Mir95, Chapter VI] and [Har77, Section IV.1].

Let C be a Riemann surface. A *divisor* on C is an element of the free abelian group generated by the set of points of C. We write a divisor as

$$D = \sum_i n_i \cdot P_i \qquad \text{with } n_i \in \mathbb{Z}.$$

Its degree is defined as $\deg(D) := \sum n_i$. Each divisor is associated with an invertible sheaf on C (cf. [Har77, page 144]) that we denote by $L(D)$.

Theorem 1.39 (cf. [Mir95, Theorem VI.3.11]). *Let D be a divisor on an algebraic curve C of genus g. Then*

$$h^0(L(D)) - h^1(L(D)) = \deg(D) - g + 1.$$

Corollary 1.40 (cf. [Mir95, Corollary VI.3.12]). *Let D be a divisor of degree at least $2g-1$ on an algebraic curve C of genus g. Then $h^1(L(D)) = 0$ and $h^0(L(D)) = \deg(D) - g + 1$.*

From now on, for any divisor D on a curve we will write $h^i(D)$ instead of $h^i(L(D))$.

1.6 The Canonical Ring of a complex manifold

Let X be a compact complex manifold of dimension n, and let $\mathcal{L}$ be a line bundle on X; we define the graded ring

$$R(X, \mathcal{L}) := \bigoplus_{m \geq 0} H^0(\mathcal{L}^{\otimes m}).$$

If we take $\mathcal{L} := \omega_X$ to be the canonical bundle of X, then $R(X) := R(X, \omega_X)$ is called the *canonical ring* of X.

This ring is commutative; let $\mathrm{tr}(R(X))$ be its degree of trascendency over $\mathbb{C}$. The number $P_m := h^0(\omega_X^{\otimes m})$ is called the *$m-th$ plurigenus* of X.

Definition 1.41. Let X be a compact complex manifold. We define the *Kodaira dimension* of X $\kappa(X)$ as follows:

$$\kappa(X) := \begin{cases} -\infty & \text{if } R(X) \cong \mathbb{C} \\ \mathrm{tr}(R(X)) - 1 & \text{otherwise} \end{cases}$$

For a compact complex manifold X, its Kodaira dimension $\kappa(X)$ can assume the values: $-\infty, 0, \ldots, \dim X$.

Remark 1.42 (cf. [Har77, page 421]). Let X be a smooth compact complex manifold, let K be a canonical divisor of X and let ϕ_{mK} be the rational map from X to the projective space associated with the linear system $|mK|$. The Kodaira dimension of X is equal to the maximal dimension of the images $\phi_{mK}(X)$ for $m \geq 1$.

Definition 1.43. A variety X is said to be *of general type* if its Kodaira dimension is maximal; that is $\kappa(X) = \dim X$.

Theorem 1.44 (cf. [BHPV04, Theorem I.7.2])**.** *Let X be a smooth compact complex variety. Then*

- $\kappa(X) = -\infty$ *if and only if* $P_m(X) = 0$ *for all* $m \geq 1$.
- $\kappa(X) = 0$ *if and only if* $P_m(X) = 0$ *or* 1 *for* $m \geq 1$*, but not always* 0.
- $\kappa(X) = k$ *for* $1 \leq k \leq \dim X$ *if and only if there are real constants* $\alpha > 0$ *and* $\beta > 0$ *such that* $\alpha m^k < P_m(X) < \beta m^k$ *for* m *large enough.*

Corollary 1.45. *Let X be a smooth compact complex variety of dimension k. Then X is of general type if and only if*

$$\limsup_{m\to\infty} \frac{P_m(X)}{m^k} > 0.$$

Lemma 1.46 (cf. [Bea83b, Exercise VII.7.2])**.** *If V, W are two smooth projective varieties, then $\kappa(V \times W) = \kappa(V) + \kappa(W)$.*

Lemma 1.47 (cf. [Bea83b, Exercise VII.7.3])**.** *Let $f : V \to W$ be a surjective morphism of smooth projective varieties. Then $\kappa(W) \leq \kappa(V)$, with equality if f is étale.*

1.6.1 The Canonical Ring of a Riemann surface

In this section we see some well known results about canonical ring and Kodaira dimension of a Riemann surface. The first Theorem we will state, without giving the proof, is a classic result due to Max Noether.

Theorem 1.48 (Max Noether's Theorem, cf. [ACGH13, page 117])**.** *If C is a non-hyperelliptic curve, then the homomorphisms*

$$\mathrm{Sym}^n H^0(C, K_C) \to H^0(C, nK_C)$$

are surjective for $n \geq 1$.

Theorem 1.48 implies that for a non-hyperelliptic curve, the canonical ring $R(C)$ is generated in degree 1.

An analogous result holds for hyperelliptic curves as well.

Proposition 1.49. *Let C be a hyperelliptic Riemann surface of genus $g(C) \geq 3$, then the canonical ring $R(C)$ is generated in degree 2.*

Proposition 1.49 is a well known result; we can prove it and give a precise description of $R(C)$ as well exploiting the following result.

Lemma 1.50 (cf. [Rei06, Section 4.4]). *Let C be a hyperelliptic Riemann surface, then*

$$R(C, g^1_2) = \mathbb{C}[t_1, t_2, w]/\langle w^2 = f(t_1, t_2)\rangle,$$

where t_i's are of degree 1, w of degree $g+1$ and $f \in \mathbb{C}[t_1, t_2]$ has degree $2g+2$.

Proof of Proposition 1.49. Using the same notation of Lemma 1.50, it holds $\omega_C = (g-1)g^1_2$, therefore $R(C, \omega_C) = R(C, g^1_2)^{[g-1]}$, where the latter denotes the subring of $R(C, g^1_2)$ of elements whose degree is multiple of $(g-1)$.

But since $g \geq 3$, $R(C, g^1_2)^{[g-1]} = \langle \mathrm{Sym}^{g-1}(t_1, t_2),\ \mathrm{Sym}^{g-3}(t_1, t_2)w\rangle$, hence we get the thesis. □

As we see in Definition 1.41, $\kappa(C)$ can assume the values $-\infty, 0$ and 1. The following Theorem gives a characterization for the Kodaira dimension of a Riemann surface C.

Theorem 1.51 (cf. [Bea83b, Example VII.2]). *Let C be a smooth curve of genus g. Then:*

$$\begin{aligned} \kappa(C) = -\infty &\quad\Leftrightarrow\quad g = 0 \\ \kappa(C) = 0 &\quad\Leftrightarrow\quad g = 1 \\ \kappa(C) = 1 &\quad\Leftrightarrow\quad g \geq 2. \end{aligned}$$

Chapter 2

Surfaces

In this section we recall some notions about surfaces, where by *surface* we mean compact complex manifold of dimension 2. Namely, we will briefly illustrate the main definitions and the main results concerning intersection theory for surfaces and the birational transformations.

Eventually, we will recall the Enriques-Kodaira classification of surfaces.

Throughout this chapter, by *curve on a surface* we mean *effective divisor*.

We refer to [Bea83b], [Har77] and [BHPV04] for the proofs and the details of the subjects contained in this chapter.

2.1 Invertible sheaves on a surface and intersection theory

Let S be a smooth variety of dimension n. The Picard group of S, denoted by $\mathrm{Pic}(S)$, is the group of isomorphism classes of invertible sheaves (or line bundles) on S (cf. [Har77, page 143]). To every divisor D on S corresponds an invertible sheaf $\mathcal{O}_S(D)$ and a meromorphic global section s unique up to a scalar multiplication such that $\mathrm{div}(s) = D$.

The map $D \mapsto \mathcal{O}_S(D)$ identifies $\mathrm{Pic}(S)$ with the group of linear equivalence classes of divisors on S (for further details see [Har77, Section II.6])

Let Ω^k_S the sheaf of holomorphic $k-$forms; for $k = n$, the canonical bundle $\omega_S := \Omega^n_S$ is a line bundle. A *canonical divisor* is a divisor K_S such that $\mathcal{O}_S(K_S) = \omega_S$.

Let X be another smooth variety and $f\colon S \to X$ a morphism. The inverse image of an invertible sheaf with respect to f defines a homomorphism $f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(S)$. Let us suppose now that f is a morphism of surfaces, which is generically finite of degree d. Let C be an irreducible curve contained in S, then we define f_*C the *direct* *image* of C as

$$f_*C := \begin{cases} 0 & \text{if } f(C) \text{ is a point} \\ r\Gamma & \begin{array}{l}\text{if } \Gamma := f(C) \text{ is a curve and the morphism} \\ f|_C \colon C \to \Gamma \text{ is finite of degree } r\end{array} \end{cases}$$

We define f_*D for all divisors D on S by linearity. From the definition,

$$f_*f^*D = dD \quad \text{for all divisors } D \text{ on } S.$$

Let us recall, eventually, the *Projection formula.* Though it holds under more general hypotheses, we state it in a way more convenient for our purposes.

Theorem 2.1 (Projection Formula cf. [Har77, Exercise II.5.1])**.** *Let X and Y be two smooth surfaces, let $f\colon X \to Y$ be a finite morphism and $\mathcal{F}$ and $\mathcal{E}$ two line bundles on X and Y respectively. Then*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} \tag{2.1}$$

Definition 2.2. Let C, C' be two distinct irreducible curves on a surface S, let x in $C \cap C'$ and $\mathcal{O}_x$ be the local ring of S at x (cf. [Har77, page 16]). Let f (respectively g) be an equation of C (resp. C') in $\mathcal{O}_x$, then the *intersection multiplicity* of C and C' ar x is defined to be

$$m_x(C \cap C') := \dim_{\mathbb{C}} \mathcal{O}_x/\langle f, g\rangle.$$

By the Nullstellensatz the ring $\mathcal{O}_x/\langle f, g\rangle$ is a finite-dimensional vector space over $\mathbb{C}$.

Definition 2.3. Let C, C' be two distinct irreducible curves on S, the intersection number $C.C'$ is defined as

$$(C.C') := \sum_{x \in C \cap C'} m_x(C \cap C').$$

The definition we have given corresponds to the intuitive idea of evaluating *properly* the intersection of two curves; for instance, if C and C' are two irreducible curves then $C.C' = 1$ if and only if they meet in a single point x which is smooth for both curves with different tangent directions, in this case the curves are said to be *transversal* at x.

We extend Definition 2.3 to divisors by linearity.

Definition 2.4. Let S be a surface and let L be a sheaf on S, then the *Euler-Poincaré characteristic* of L is the integer

$$\chi(L) := \sum_i (-1)^i h^i(S, L).$$

Theorem 2.5 (cf. [Bea83b, Theorem I.4]). *For $L, L' \in \mathrm{Pic}(S)$, define*

$$L.L' := \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}). \tag{2.2}$$

Then (.) is a symmetric bilinear form on $\mathrm{Pic}(S)$ *such that if C and C' are two distinct irreducible curves on S then*

$$\mathcal{O}_S(C).\mathcal{O}_S(C') = C.C'$$

Definition 2.6 (cf. [Bea83b, Lemma I.6]). Two divisors D_1, D_2 on a surface S are said to be *numerically equivalent*, written $D_1 \equiv_{num} D_2$ if $D_1.C = D_2.C$ for every irreducible curve $C \subset X$.

Lemma 2.7 (cf. [Bea83b, Lemma I.6]). *Let C be a non-singular irreducible curve on S. For all $L \in \mathrm{Pic}(S)$ we have*

$$\mathcal{O}_S(C).L = \deg(L|_C).$$

Definition 2.8 (cf. [BHPV04, page 28]). Let S be a smooth surface and $L \in \mathrm{Pic}(S)$; we say that L is a *nef* line bundle if

$$\mathcal{O}_S(C).L \geq 0$$

for any curve $C \subset S$. A divisor D on S is said to be *nef* if the associated line bundle $\mathcal{O}_S(D)$ is nef.

Definition 2.9. Let D be a divisor on the surface S, we say that $D^2 := D.D = \mathcal{O}_S(D).\mathcal{O}_S(D)$ is the *self-intersection* of D.

Proposition 2.10 (cf. [Bea83b, Proposition I.8]). *Let S and S' be two smooth surfaces and $g\colon S \to S'$ a generically finite morphism of degree d, D and D' divisors on S'. Then $g^*D.g^*D' = d(D.D')$,*

Definition 2.11 (cf. [Bea83b, page 120, page 153]). Let S be a surface. We say that $L \in \mathrm{Pic}(S)$ is *very ample* if there exists an embedding $i\colon S \to \mathbb{P}^n$ such that $L \cong i^*(\mathcal{O}_{\mathbb{P}^n}(1))$. $L \in \mathrm{Pic}(S)$ is said to be *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

A divisor D on S is said to be *very ample* (respectively *ample*) if $\mathcal{O}_S(D)$ is very ample (resp. ample).

Remark 2.12. Let D be an ample divisor on S, then D is a nef divisor, whereas the converse does not hold.

Theorem 2.13 (Nakai-Moishezon Criterion, cf. [Har77, Theorem V.1.10]). *Let S be a smooth projective surface and let D be a divisor on S. Then D is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in S.*

Definition 2.14 (cf. [Laz07, Corollary 2.2.7]). Let D be a divisor on a projective surface S. Then D is said to be *big* if there exist an ample divisor A, a positive integer $m > 0$ and an effective divisor N such that $mD \equiv_{num} A + N$.

A very useful way to prove the bigness of a divisor is given by the following result.

Theorem 2.15 (cf. [Laz07, Theorem 2.2.7]). *Let D be a nef divisor on projective surface. Then D is big if and only $D^2 > 0$.*

Theorem 2.16 (Mumford vanishing Theorem, cf. [BHPV04, Theorem IV.12.1]). *Let D be a big and nef divisor on a smooth projective surface X. Then $H^1(X, -D) = 0$.*

Theorem 2.17 (Kawamata-Viehweg vanishing Theorem, cf. [Laz07, Theorem 4.3.1]). *Let D be a big and nef divisor on a smooth projective surface X. Then*

$$H^i(X, K_X + D) = 0 \qquad \text{for } i > 0.$$

2.2 Riemann-Roch Theorem for Surfaces

In this section we state, without any proofs, some well known results concerning line bundles on a surface. Riemann-Roch Theorem, in particular, is a powerful tool to compute the Euler-Poincaré characteristic of a line bundle.

Theorem 2.18 (Serre's duality, cf. [Har77, Section II.7]). *Let X be a compact, connected complex manifold of dimension n and L be a line bundle on X. Then for each $0 \leq j \leq n$ the vector spaces*

$$H^i(M, L) \qquad \text{and} \qquad H^{n-i}(M, \omega_X \otimes L^{-1}),$$

are dual. In particular

$$\chi(L) = (-1)^n \chi(\omega_X \otimes L^{-1}).$$

Theorem 2.19 (Riemann-Roch Theorem, cf. [Bea83b, Theorem I.12]). *Let S be a smooth projective surface, for every $L \in \mathrm{Pic}(S)$*

$$\chi(\mathcal{O}(L)) = \chi(\mathcal{O}_S) + \frac{L^2 - L.K_S}{2} \tag{2.3}$$

Proposition 2.20 (Genus formula, cf. [Bea83b, Proposition I.15])**.** *Let C be an irreducible curve on a surface S of genus $g(C)$. Then*

$$g(C) = 1 + \frac{C^2 + C.K_S}{2}. \tag{2.4}$$

Proposition 2.21 (Adjunction formula, cf. [GH78, page 147])**.** *Let M be a compact complex manifold, let $V \subset M$ be a smooth analytic hypersurface. Then*

$$K_V = (K_M + V)|_V.$$

2.3 Birational transformations and minimal models

In this section we briefly recall the definition of blow-up of a surface at a point and its properties, in order to justify the classification of surfaces up to birational equivalence.

For further details we refer to [Bea83b, Section II].

Definition 2.22. Let S be a smooth surface and $p \in S$. Then there exist a smooth surface $\hat{S}$, called the *blow-up* of S at p, and a morphism $\epsilon\colon \hat{S} \to S$ such that

i. the restriction of ϵ to the set $\epsilon^{-1}(S \setminus \{p\})$ is an isomorphism onto $S \setminus \{p\}$;

ii. $E := \epsilon^{-1}(p)$ is isomorphic to $\mathbb{P}^1$. The set E is called *exceptional divisor* of ϵ.

To describe explicitly the behaviour of the blow-up in a neighbourhood of p, it is sufficient to describe the case of $S = \mathbb{C}^2$ and $p = (0, 0)$; for any surface we can reproduce this construction using local coordinates.

Let us denote the coordinate on $\mathbb{C}^2$ by (x, y) and those on $\mathbb{P}^1$ by $(t_0 : t_1)$. We define

$$\hat{\mathbb{C}}^2 := \{xt_1 = yt_0\} \subset \mathbb{C}^2 \times \mathbb{P}^1$$

and

$$\begin{array}{rccc} \epsilon\colon & \hat{\mathbb{C}}^2 & \longrightarrow & \mathbb{C}^2 \\ & ((x, y), (t_0 : t_1)) & \longmapsto & (x, y) \end{array}$$

First of all $\epsilon^{-1}(0) = \{0\} \times \mathbb{P}^1$. Let us prove that $\hat{\mathbb{C}}^2$ is smooth.

Let $q \in \hat{\mathbb{C}}^2$. If $t_1(q) \neq 0$, then let us consider the open set $V_1 := \mathbb{C}^2 \times \{t_1 \neq 0\} \subset \mathbb{C}^2 \times \mathbb{P}^1$ with coordinates $x, y, t := t_0/t_1$; $U_1 := \hat{\mathbb{C}}^2 \cap V_1$ is the zero locus of $f(x, y, t) := x - yt$. Since $\partial f/\partial x(q) \neq 0$, by the local diffeomorphism theorem (cf. [Mir95, Theorem 2.2.1]) y and t are local coordinates for U_1 in a neighbourhood of q. Hence $\hat{\mathbb{C}}^2$ is smooth in q. We shall remark that in this open set E is the divisor of the regular function y.

If $t_0(q) \neq 0$, then in $V_0 := \mathbb{C}^2 \times \{t_0 \neq 0\}$ we shall use as local coordinates $x, y, u := t_1/t_0$; in this open set $\hat{\mathbb{C}}^2$ is the zero locus of $g(x, y, u) = y - xu$ and x, u are local coordinates for $\hat{\mathbb{C}}^2$ in a neighbourhood of q. In this open set E is the divisor of the regular function x.

Moreover, $\epsilon|_{\hat{\mathbb{C}}^2 \setminus E}$ is invertible:

$$\begin{array}{rccc} \epsilon|_{\hat{\mathbb{C}}^2\setminus E}^{-1}\colon & \mathbb{C}^2 \setminus \{p\} & \longrightarrow & \hat{\mathbb{C}}^2 \setminus E \\ & (x, y) & \longmapsto & ((x, y), (x : y)) \end{array}$$

Let $\epsilon\colon \hat{S} \to S$ be the blow-up of S in $p \in S$ and let C be an irreducible curve C on S passing through p with multiplicity m. The closure of $\epsilon^{-1}(C \setminus \{p\})$ in $\hat{S}$ is an irreducible curve $\hat{C}$ which is called the *strict transform* of C.

Lemma 2.23 (cf. [Bea83b, Lemma II.2]). *Let $\epsilon\colon \hat{S} \to S$ be the blow-up of S in p. Let C be an irreducible curve on S passing through p with multiplicity m, then*

$$\epsilon^* C = \hat{C} + mE.$$

Proposition 2.24 (cf. [Bea83b, Proposition II.3]). *Let $\epsilon\colon \hat{S} \to S$ be the blow-up of S at $p \in S$. Let E be the exceptional curve, then*

i. The map $Pic(S) \oplus \mathbb{Z} \to \mathrm{Pic}(\hat{S})$ defined by $(D, n) \mapsto \epsilon^* D + nE$ is an isomorphism;

ii. Let D be a divisor on S. Then $\epsilon^* D.E = 0$ and $E^2 = -1$.

Lemma 2.25 (cf. [Bea83b, Proposition II.3]). *Let $\epsilon\colon \hat{S} \to S$ be the blow-up of S at $p \in S$. The canonical divisor of $\hat{S}$ is given by $\epsilon^* K_S + E$ and $K^2_{\hat{S}} = K^2_S - 1$.*

Proof. By definition of the map ϵ, the canonical sheaf on $\hat{S} \setminus E$ and $S \setminus \{p\}$ are isomorphic via ϵ^*, thus $K_{\hat{S}} = \epsilon^* K_S + nE$ for some integer n. By Proposition 2.21

$$-2 = 2g(E) - 2 = (K_{\hat{S}} + E).E \quad\Longrightarrow\quad K_{\hat{S}}.E = -1.$$

Then $-1 = K_{\hat{S}}.E = \epsilon^* K_S.E + nE^2 = 0 - n$, where last equality holds by Proposition 2.24, so $n = 1$. The formula for $K^2_{\hat{S}}$ follows by Proposition 2.24 and Proposition 2.10. □

Let us recall some well known results about blow-ups and rational maps.

Theorem 2.26 (Elimination of indeterminacy, cf. [Bea83b, Theorem II.7]). *Let S be a surface, X a projective variety and let $\Phi\colon S \dashrightarrow X$ be a rational map. Then there* *exists a surface S', a morphism $\eta\colon S' \to S$ which is the composition of a finite number of blow-ups, and a morphism $f\colon S' \to X$ such that the diagram*

$$\begin{tikzcd} & S' \arrow[dl, "\eta"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "\Phi"] & & X \end{tikzcd}$$

is commutative.

Theorem 2.27 (Universal property of blowing-up, cf. [Bea83b, Proposition II.8])**.** *Let $f\colon S \to X$ be a birational morphism of surfaces, and suppose that the rational map f^{-1} is not defined at the point $p \in X$. Then f factorizes as*

$$f : S \xrightarrow{g} \hat{X} \xrightarrow{\epsilon} X$$

where g is a birational morphism and ϵ is the blow-up at p.

Theorem 2.28 (cf. [Bea83b, Theorem II.11])**.** *Let $f\colon S \to S_0$ be a birational morphism of surfaces. Then there is a sequence of blow-ups $\epsilon_k\colon S_k \to S_{k-1}$ $(k = 1, \ldots, n)$ and an isomorphism $u\colon S \to S_n$ such that $f = \epsilon_1 \circ \cdots \circ \epsilon_n \circ u$.*

Corollary 2.29 (cf. [Bea83b, Corollary II.12])**.** *Let $\varphi\colon S' \dashrightarrow S$ be a biraional map of surfaces. Then there is a surface $\hat{S}$ and a commutative diagram*

$$\begin{tikzcd} & \hat{S} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S' \arrow[rr, dashed, "\varphi"] & & S \end{tikzcd}$$

where the morphisms f, g are compositions of blow-ups and isomorphisms.

Definition 2.30. Let S_1 and S_2 be two surfaces, we say that S_1 *birationally dominates* S_2 if there exists a birational morphism $S_1 \to S_2$.

A smooth surface S is said to be *minimal* if every birational morphism $S \to S'$ is an isomorphism.

Proposition 2.31 (cf. [Bea83b, Proposition II.16])**.** *Every smooth surface birationally dominates a minimal surface.*

Definition 2.32. Let $S' \to S$ be a birational morphism between smooth surfaces. If S is minimal, we say that S is a *minimal model* of S'.

As we will see in Section 2.6, apart from a particular class of surfaces, every surface has a unique minimal model and in this case we will talk about *the* minimal model of a surface. For this reason, one way to classify non ruled surfaces consists in the classification of minimal surfaces.

Remark 2.33. By Theorem 2.28, a surface is minimal if and only if it contains no exceptional curve.

If $E \subset S$ is en exceptional curve, then by definition $E \cong \mathbb{P}^1$ and by Proposition 2.24 $E^2 = -1$. Actually, this is a characterization of exceptional curves by the following important result.

Theorem 2.34 (Castelnuovo's contractibility criterion, cf. [Bea83b, Theorem II.17])**.** *Let S be a surface and let $E \subset S$ be a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then E is an exceptional curve on S.*

Proposition 2.35 (cf. [BHPV04, Proposition III.2.2])**.** *An irreducible curve $C \subset S$ is an exceptional curve if and only if*

$$C^2 < 0 \quad \textit{and} \quad K_S.C < 0.$$

2.4 Birational invariants

Definition 2.36. Let S be a smooth surface. We define the following integers:

$$\begin{aligned} q(S) &:= h^1(S, \mathcal{O}_S) \\ p_g(S) &:= h^0(S, \mathcal{O}_S(K_S)) = h^2(S, \mathcal{O}_S) \text{ (by Serre duality)} \\ P_n(S) &:= h^0(S, \mathcal{O}_S(nK_S)) \text{ for } n \geq 1. \end{aligned}$$

The integer $q(S)$ is called the *irregularity* of S; $p_g(S)$ is the *geometric genus* and P_n is called the $n-$*th plurigenus* of S.

The *(holomorphic) Euler-Poincaré characteristic* of S is defined as the Euler-Poincaré characteristic of the structure sheaf $\mathcal{O}_S$, that is (cf. Definition 2.4)

$$\chi(S) := \chi(\mathcal{O}_S) = 1 - q(S) + p_g(S).$$

Proposition 2.37 (cf. [Bea83b, Proposition III.20])**.** *The integers q, p_g and P_n are birational invariants.*

Definition 2.38. Let S be a smooth surface. We define the following integers:

$$b_i := \dim_{\mathbb{C}} H^i(S, \mathbb{C}) \qquad e(S) := \sum_{i=1}^{4} (-1)^i b_i,$$

where b_i is the *$i-th$ Betti number* and $e(S)$ is the *topological Euler-Poicaré characteristic* of S.

Betti numbers are topological invariants, moreover we have $b_0 = b_4 = 1$ and by Poincaré duality $b_3 = b_1$, so $e(S) = 2 - 2b_1 + b_2$.

Remark 2.39. Topological Euler-Poincaré characteristic is not a birational invariant: indeed if $\epsilon\colon S' \to S$ is the blow up of S at p, then $e(S') = e(S) + 1$, since we replace a point $(e(p) = 1)$ with a rational curve E $(e(E) = 2)$.

Theorem 2.40 (Noether's formula, cf. [BHPV04, Theorem I.5.5]). *Let S be a smooth projective surface. Then*

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + e(S)). \tag{2.5}$$

Lemma 2.41 (cf. [Bea83b, Lemma VI.3]). *Let $\pi\colon S \to S'$ be an étale map of surfaces of degree n. Then $K_{S'}^2 = nK_S^2$, $e(S') = ne(S)$, $\chi(\mathcal{O}'_S) = n\chi(\mathcal{O}_S)$.*

2.5 Albanese map

In this section we give some definitions and results concerning the Albanese variety and the Albanese map associated with a surface.

For further details, we address, for example, to [Bea83b, pag 60-64] or [BHPV04, pag 46-48].

Definition 2.42. A *complex torus* is a manifold T obtained as a quotient $T = V/\Gamma$, where V is a complex vector space and Γ a lattice in V. If there exists an embedding of T into a projective space, then T is said to be an *Abelian variety.*

Remark 2.43. A complex torus is a compact manifold equipped with the structure of an abelian group.

The following Theorem, known as the universal property of the Albanese variety, will be stated without proof.

Theorem 2.44 (cf. [Bea83b, Theorem V.13]). *Let X be a smooth projective variety. There exists a unique abelian variety $A := \mathrm{Alb}(X)$ and a morphism $\alpha\colon X \to \mathrm{Alb}(X)$ such that for every complex torus T and $f\colon X \to T$ morphism, there exists a unique* *morphism* $\tilde{f}\colon A \to T$ *such that the diagram*

$$\begin{tikzcd} X \arrow[d, "\alpha"] \arrow[dr, "f"] & \\ A \arrow[r, "\tilde{f}"'] & T \end{tikzcd}$$

commutes. The abelian variety $A = \mathrm{Alb}(X)$ *is called the* Albanese variety *of* X *and* α *is called* Albanese map. *Moreover the morphism* α *induces an isomorphism*

$$\alpha^*\colon H^0(\Omega^1_A) \to H^0(\Omega^1_X).$$

Remark 2.45 (cf. [Bea83b, Remark V.14]). Let X be a smooth projective variety, then the following properties hold:

i. $\dim \mathrm{Alb}(X) = \dim H^0(X, \Omega^1_X)$;

ii. The Abelian variety $\mathrm{Alb}(X)$ is generated as a group by $\alpha(X)$, in particular $\dim \alpha(X) = 0$ if and only if $\dim \mathrm{Alb}(X) = 0$;

iii. If X is a curve, $\mathrm{Alb}(X)$ is equal to the Jacobian $J(C)$.

2.6 Enriques-Kodaira classification of surfaces

In Section 1.6, we gave the definition of the Kodaira dimension $\kappa(X)$ of a complex manifold X of dimension n; for a surface S, $\kappa(S) \in \{-\infty, 0, 1, 2\}$, and by Corollary 1.45 and Proposition 2.37, it is straightforward to see that it is a birational invariant.

In order to give a classification of surfaces up to birationality, this invariant assumes a central role. Its importance is underlined by the following fundamental result, which is known as the *Enriques-Kodaira classification*.

Theorem 2.46 (cf. [BHPV04, Theorem VI.1.1]). *Every surface has a minimal model* S *in exactly one of the classes listed in Table 2.1.*

Remark 2.47. Surfaces belonging to classes (3) and (6) are not algebraic (cf. [Kod64, Theorem 25] and [BHPV04, Theorem VI.1.1]).

A *rational surface* is a surface birational to $\mathbb{P}^2$. The only minimal surfaces of this type are $\mathbb{P}^2$ and the Hirzebruch surfaces $\Sigma_n := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathbb{O}_{\mathbb{P}^1}(n))$ with $n = 0, 2, 3, \ldots$ ($\mathbb{P}^1 \times \mathbb{P}^1 = \Sigma_0$).

<table>
<tr><th>$\kappa(S)$</th><th>Class of S</th><th>K_S^2</th><th>e(S)</th><th>p_g</th><th>q</th></tr>
<tr><td rowspan="3">$-\infty$</td><td>(1) Minimal rational surfaces</td><td>8 or 9</td><td>3 or 4</td><td>0</td><td>0</td></tr>
<tr><td>(2) Ruled surfaces of genus $g \geq 1$</td><td>$8(1-g)$</td><td>$4(1-g)$</td><td>0</td><td>g</td></tr>
<tr><td>(3) Minimal surfaces of class VII</td><td>≤ 0</td><td>≥ 0</td><td>0</td><td>1</td></tr>
<tr><td rowspan="7">0</td><td>(4) Enriques surfaces</td><td rowspan="7">0</td><td>12</td><td>0</td><td>0</td></tr>
<tr><td>(5) Bi-elliptic surfaces</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>(6) Kodaira surfaces</td><td></td><td></td><td></td></tr>
<tr><td>(a) Primary</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>(b) Secondary</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>(7) K3 surfaces</td><td>24</td><td>1</td><td>0</td></tr>
<tr><td>(8) Tori</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>1</td><td>(9) Minimal properly elliptic surfaces</td><td>0</td><td>≥ 0</td><td></td><td></td></tr>
<tr><td>2</td><td>(10) Minimal surfaces of general type</td><td>> 0</td><td>> 0</td><td></td><td></td></tr>
</table>

Table 2.1: Enriques-Kodaira classification of surfaces

Theorem 2.48 (Castelnuovo's Rationality Criterion, cf. [BHPV04, Corollary VI.3.4])**.** *An algebraic surface X is rational if and only if $q(S) = P_2(S) = 0$.*

Ruled surfaces of genus g have smooth morphism to a curve of genus g whose fibres are $\mathbb{P}^1$.

A *surface of class VII* is a surface S with $\kappa(S) = -\infty$ and $b_1 = 1$, moreover $q = 1$. As stated in Remark 2.47, these surfaces are neither algebraic nor Kähler. Examples of this type of surfaces are Hopf surfaces ([Hop48]) and Inoue surfaces ([Ino74]).

Theorem 2.49 (Enriques, cf. [BHPV04, Theorem IV.12.1])**.** *Let S be a smooth projective complex surface, then the following are equivalent:*

- *S is ruled;*
- *$P_n = 0$ for all n;*
- *$P_{12} = 0$.*

An *Enriques surface* S is a surface with $q(S) = 0$, non-trivial canonical bundle (i.e. $\omega_S \ncong \mathcal{O}_S$) and $\omega_S^{\otimes 2} \cong \mathcal{O}_S$.

A *bi-elliptic surface* (or *hyperelliptic surface*) is a surface S with $q(S) = 1$ and an elliptic fibration over an elliptic curve. Surfaces of this type are quotient of a product of two elliptic curves by a finite abelian group (cf. [BHPV04, Section V.5]).

Kodaira surfaces are usually divided into two subtypes: the primary Kodaira surfaces with $b_1 = 3$ and an elliptic fibration over an elliptic curve; the secondary Kodaira surfaces are surfaces which admit a primary Kodaira surface as étale covering of degree ≥ 2. As stated in Remark 2.47, these surfaces are not algebraic.

A *K3 surface* is a surface S with $q(S) = 0$ and trivial canonical bundle.

A *torus* is a surface isomorphic to the quotient of $\mathbb{C}^2$ by a lattice of real rank 4.

A *properly elliptic surface* is a surface S admitting an elliptic fibration with $\kappa(S) = 1$. A very simple example is provided by the product of two curves, one elliptic and the other of genus ≥ 2.

The following results hold for surfaces with non negative Kodaira dimension.

Theorem 2.50 (cf. [BHPV04, Proposition III.4.6]). *If S is a non singular compact connected surface with $\kappa(S) \geq 0$, then all minimal models of S are isomorphic.*

Proposition 2.51 (cf. [Laz07, Proposition 2.2.2]). *Let S be a smooth compact connected surface with $\kappa(S) \geq 0$ and let D be an effective divisor on S such that $D.K_S < 0$. Then D contains an exceptional curve.*

Proof. It is sufficient to prove that if D is an irreducible curve with $K_X.D < 0$, then D is an exceptional curve. By Theorem 1.44 for some $n \geq 1$ there is a non negative $n-$canonical divisor $nK_S \cong \tilde{K} = \sum c_i C_i$ with $c_i \geq 0$. Since $K_S.D < 0$, $\tilde{K}.D = nK_S.D < 0$ therefore the curve D must be one of the C_i's, say $D = C_0$. Hence $D(\hat{K} - c_0 D) \geq 0$ and $D^2 < 0$. By Proposition 2.35 we get the thesis. □

Lemma 2.52. *Let S be a surface. If $\kappa(S) \geq 0$ then one of the following occurs:*

1. $h^2(2K_S) = 0$;
2. $K_S = 0$ *and therefore* $\kappa(S) = 0$.

Proof. Since $\kappa(S) \geq 0$, there exists $n > 0$ such that $h^0(nK_S) > 0$. We can choose then an effective divisor $A \in |nK_S|$. Suppose $h^2(2K_S) > 0$; by Serre duality $h^0(-K_S) > 0$, let then $B \in |-K_S|$: $A + nB$ is a principal effective divisor on S, that is $A + nB = 0$, but both A and B are effective, then $A = B = 0$. Thus $-K_S = 0$. □

2.7 Surfaces of general type and the geography problem

Enriques-Kodaira classification gives a good description for surfaces with Kodaira dimension $\kappa \leq 1$, but surfaces belonging to the remaining class, namely the surfaces with Kodaira dimension $\kappa = 2$, are far from being completely classified.

Definition 2.53. A surface S is said to be of *general type* if $\kappa(S) = 2$.

Remark 2.54 (cf. [BHPV04, Corollary IV.6.5]). Every smooth surface of general type is projective.

Proposition 2.55 (cf. [Laz07, Example 2.2.2]). *Let S be a surface of general type, then K_S is a big divisor.*

Proposition 2.56. *Let S be a surface of general type. Then S is minimal if and only if K_S is nef.*

Proof. It follows immediately by Proposition 2.51 and Proposition 2.55. □

Lemma 2.57. *Let S be a minimal surface of general type, then $h^0(2K_S) = \chi(\mathcal{O}_S)+K_S^2$.*

Proof. By Riemann-Roch Theorem 2.19, $\chi(\mathcal{O}_S(2K_S)) = \chi(\mathcal{O}_S) + K_S^2$. Since S is minimal, K_S is big and nef, then by Theorem 2.17 $h^1(2K_X) = 0$. By Lemma 2.52 $h^2(2K_X) = 0$. Then $\chi(\mathcal{O}_S(2K_S)) = h^0(2K_S)$, whence the statement. □

Proposition 2.58. *Let S be a surface of general type, and let $\hat{S}$ be its minimal model. Then $h^1(2K_S) = K_{\hat{S}}^2 - K_S^2$.*

Proof. By Riemann-Roch Theorem and Proposition 2.52

$$\begin{aligned} h^0(2K_S) - h^1(2K_S) &= h^0(2K_S) - h^1(2K_S) + h^2(2K_S) \\ &= \chi(\mathcal{O}_S(2K_S)) \\ &= \chi(\mathcal{O}_S) + K_S^2. \end{aligned}$$

By Lemma 2.57, $h^0(2K_{\hat{S}}) = \chi(\mathcal{O}_{\hat{S}}) + K_{\hat{S}}^2$. Since Euler characteristic and $m-$th plurigenus are birational invariants, $\chi(\mathcal{O}_{\hat{S}}) = \chi(\mathcal{O}_S)$ and $h^0(2K_{\hat{S}}) = h^0(2K_S)$, whence the thesis. □

Theorem 2.59 (cf. [BHPV04, Theorem VII.2.2]). *If S is a minimal surface of general type, then $K_S^2 > 0$.*

Theorem 2.60 (cf. [Bea83b, Theorem X.4]). *Let S be a surface of general type, then $e(S) \geq 0$ and $\chi(\mathcal{O}_S) \geq 1$.*

By Noether's formula, the condition $e(S) \geq 0$ is equivalent to $K_S^2 \leq 12\chi(\mathcal{O}_S)$. For a surface of general type Bogomolov and Miyaoka, and independently Yau proved the stronger inequality that is named after them.

Theorem 2.61 (cf. [BHPV04, Theorem VII.4.1])**.** *Let S be a smooth surface of general type. Then*

$$K_S^2 \leq 9\chi(\mathcal{O}_S). \tag{BMY}$$

In literature there are other well-known inequalities involving the invariants of minimal surfaces of general type:

Theorem 2.62 (cf. [BHPV04, Theorem VII.3.1])**.** *Let S be a minimal surface of general type. Then*

$$K_S^2 \geq 2p_g(S) - 4 \tag{N}$$

$$\textit{if } q > 0 \Rightarrow K_S^2 \geq 2p_g(S) \tag{D}$$

The inequality (N) is due to Noether, while (D) is due to Debarre.

Inequalities we have listed so far outline a region in the $(\chi, K^2)-$plane.

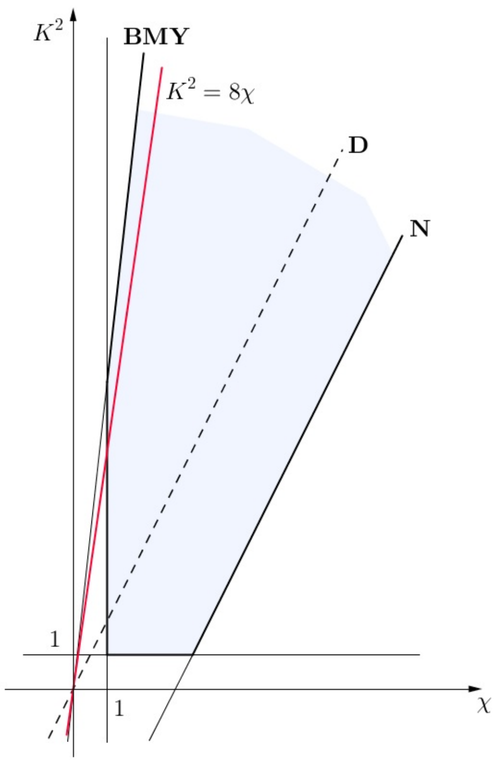


Figure 2.1: The geography of minimal surfaces of general type.

A minimal surface of general type corresponds to a point with integral coordinates in the coloured convex region of Figure 2.1. By Debarre's inequality (D), if a minimal surface of general type S has irregularity $q(S) > 0$ then it corresponds to a point lying in the region above the line D.

We will understand the importance of line with equation $K^2 = 8\chi$ displayed in Figure 2.1 in following chapters.

The question that naturally arises looking to Figure 2.1 is: for any point $(K^2, \chi) \in \mathbb{Z}^2$ lying in the bounded region, does there exist a minimal surface of general type with such invariants? This is the so called *geography problem* and is still far from being solved, so one can try to develop a way to construct explicitly surfaces for fixed values of K^2 and χ.

2.8 The classification of surfaces of general type with $p_g = q \geq 3$

In our work, we focused on the classification of surfaces of general type with $p_g = q$. These surfaces lie on the boundary of the region represented in Figure 2.1 and are far from being fully classified, nevertheless there are some important results. In particular, surfaces with $p_g = q \geq 3$ are fully classified.

Theorem 2.63 ([Bea82, Theorem IV.12.1])**.** *If S is a minimal surface of general type, then $p_g \geq 2q - 4$. Moreover, if $p_g = 2q - 4$, then S is a product of a curve of genus 2 and a curve of genus $q - 2$.*

Corollary 2.64. *Let S be a surface of general type with $p_g(S) = q(S)$ (i.e. $\chi(\mathcal{O}_S) = 1$), then $p_g = q \leq 4$. Moreover, minimal surfaces of general type with $p_g = q = 4$ are exactly the products of two genus 2 curves.*

Surfaces of with $p_g = q = 3$ have been studied in [CCML98], [Pir02] and [HP02] and they are completely classified.

Theorem 2.65. *Let S be a minimal surface of general type with $p_g = q = 3$ then one of the following cases occurs:*

- *$K_S^2 = 6$ and S is the symmetric product of a genus 3 curve;*
- *$K_S^2 = 8$ and $S = (C_2 \times C_3)/\tau$, where C_g is a curve of genus g and τ is an involution, acting on C_2 as an elliptic involution and on C_3 as a fixed point free involution.*

Chapter 3

Group Actions on a Product of two Curves

In his seminal paper [Cat00], Catanese laid the foundations for the study of the so called *isogenous surfaces*, surfaces that are obtained as the quotient of the product of two curves with respect to the free action of a finite group. In the last years, these objects have been studied in many works; we mention, among others, [BC04], [BCG08], [CP09], [Pen11], [Pol08]. In the first part of the present chapter we will see some of the results contained in these works, since they will play an important role in the next sections.

Despite being a very rich source of new examples of surfaces of general type, isogenous surfaces are not sufficient to give an answer to the geography problem, for they all lie along the red line in the (χ, K^2) plane in Figure 2.1. For this reason the construction has been generalised, allowing the group to act freely outside of a finite set of points; a surface constructed under these hypotheses is called *quasi-étale quotient* and the minimal resolution of its singularities is a *quasi-étale surface.* These surfaces has been widely studied in the last years; we mention among others [BP12], [BCGP12], [Fra13], [FP15], [MP10], [Pen11], [Pol09], [Zuc03]. The results contained in Section 3.3 show how quasi-étale surfaces allow us to fill, in principle, an area on the $(\chi, K^2)-$plane which is bigger than the one we can reach with the isogenous surfaces.

For a complete list of the surfaces classified in the works we mentioned, we refer to [Pig15].

3.1 Mixed and unmixed actions on a product of two curves

In [Bea83b], Beauville proposed a very simple construction to get a surface of general type: considering the Fermat quintic plane curve C and a free action of $\mathbb{Z}_5^2$ on $C \times C$, the quotient surface $(C \times C)/\mathbb{Z}_5^2$ is a minimal surface of general type with $p_g = q = 0$ and $K^2 = 8$.

Example 3.1 (cf. [Bea83b, Exercise X.13.4]). *Let $C \subset \mathbb{P}^2$ be the plane quintic $X^2 + Y^2 + Z^5 = 0$, let $\zeta := \exp(2\pi i/5)$ be a $5-th$ root of unity and let us define an action of $G := (\mathbb{Z}_5)^2$ on C as*

$$(a,b)[X : Y : Z] = [\zeta^a X : \zeta^b Y : Z].$$

Then there are fifteen points on C with non trivial stabilizer

$$\begin{array}{lll} p_i := [0 : -\zeta^k : 1] & k = 0, \dots, 4 & \mathrm{Stab}(p_i) = \langle (1,0) \rangle \\ q_i := [-\zeta^k : 0 : 1] & k = 0, \dots, 4 & \mathrm{Stab}(q_i) = \langle (0,1) \rangle \\ r_i := [1 : -\zeta^k : 0] & k = 0, \dots, 4 & \mathrm{Stab}(r_i) = \langle (1,1) \rangle \end{array}$$

By Riemann-Hurwitz Formula (1.2), $g(C/G) = 0$.

Let us define $\phi \in \mathrm{Aut}(G)$ as $\phi(a,b) := (a + 2b, 3a - b)$ and a $G-$action on $C \times C$ as $g(p,q) = (gp, \phi(g)q)$.

By definition of ϕ, this latter action is free. Let us consider the surface $X := (C \times C)/G$ and the corresponding quotient map $\eta \colon C \times C \to X$.

Being the action free, the map η is étale and X is smooth. By Theorem 1.51 and Lemma 1.47, X is of general type. Moreover, using some results that we are going to state and prove in the next sections under more general hypotheses, we have $p_g(X) = q(X) = 0$ and $K_X^2 = 8$.

The idea underlying this example turned out to be very fruitful in the classification of surfaces of general type and led to the detailed study of the group actions on product of curves. Most of the following results hold for a finite product of curves $C_1 \times \cdots \times C_n$ (cf. [Gle16]), but we will focus on the surface case, that is $n = 2$.

Our first aim is to understand the structure of the automorphism group $\mathrm{Aut}(C_1 \times C_2)$ of the product of two Riemann surfaces of genus $g(C_i) \geq 2$. The following result provides a simple description of this group.

Lemma 3.2 (Rigidity Lemma, cf. [Cat00, Lemma 3.8]). *Let $f \colon C_1 \times C_2 \to B_1 \times B_2$ be a surjective holomorphic map between products of curves. Assume that both B_1, B_2 have genus at least 2. Then, after possibly exchanging B_1 with B_2, there are holomorphic maps $f_i : C_i \to B_i$ such that $f(x,y) = (f_1(x), f_2(y))$.*

Corollary 3.3 (cf. [Cat00, Corollary 3.9]). *Let C_1, C_2 two Riemann surfaces of genus $g(C_i) \geq 2$ of $i = 1, 2$. Then one of the following occurs:*

- $C_1 \not\cong C_2$ *and* $\mathrm{Aut}(C_1 \times C_2) = \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$*;*
- $C_1 \cong C_2 \cong C$ *and* $\mathrm{Aut}(C \times C) = \mathrm{Aut}(C)^2 \rtimes \mathbb{Z}_2$*, where the group $\mathbb{Z}_2$ is given by the involution exchanging the two coordinates.*

Remark 3.4. Given a group G acting on the product of two curves $C_1 \times C_2$, let K be the kernel of the map $G \to \mathrm{Aut}(C_1 \times C_2)$. The group $G' := G/K$ acts faithfully on $C_1 \times C_2$ and $(C_1 \times C_2)/G \cong (C_1 \times C_2)/G'$. For this reason, we will consider only faithful action, and we will omit this hypothesis for the sake of brevity.

Remark 3.5. If $g(C_i) \geq 2$ for $i = 1, 2$ then by Theorem 1.20 $\mathrm{Aut}(C_i) \leq 84(g(C_i) - 1)$, hence by Corollary 3.3 $\mathrm{Aut}(C_1 \times C_2)$ is finite. Therefore, taking into account Remark 3.4, if G is a group acting on $C_1 \times C_2$ we can always suppose it to be finite without loss of generality.

Definition 3.6. Let C_1, C_2 be two Riemann surfaces of genus at least two. Let G be a group acting on $C_1 \times C_2$ and let us denote $G^0 := G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2))$.

Let us consider the two projection maps $p_i \colon G^0 \to \mathrm{Aut}(C_i)$ for $i = 1, 2$. We can define a G^0-action on C_i for $i = 1, 2$ in the following way: let $x \in C_i$ and $g \in G^0$, then $g(x) := p_i(g)(x)$. Analogously, we can define a G^0-action on $C_1 \times C_2$: for $g \in G^0$ and $(x, y) \in C_1 \times C_2$

$$g(x, y) := (p_1(g)x, p_2(g)y). \tag{3.1}$$

We say that G^0 acts *diagonally* on $C_1 \times C_2$.

If G^0 acts faithfully on both factor, we say that the action of G is minimal, and we say that $X := (C_1 \times C_2)/G$ is a *minimal realization* of X.

Remark 3.7 (cf. [Cat00, Remark 3.10]). Using the same notation of Definition 3.6, let $K_i := \ker(p_i)$. Then K_i acts trivially on C_i and the action of $G' := G/\langle K_1, K_2\rangle$ is minimal on $(C_1/p_1(K_2)) \times (C_2/p_2(K_1))$. The surface $(C_1/p_1(K_2) \times C_2/p_2(K_1))/G'$ is a minimal realization of X.

By the previous remark, if we are given a surface $X = (C_1 \times C_2)/G$, obtained as the quotient of the product of two curves by a group action, then it is always possible to construct a minimal realization of X. Hence, from now on we will only consider minimal action of a group G on a product of curves $C_1 \times C_2$.

Definition 3.8 (cf. [Cat00, Proposition 3.15]). Let C be a Riemann surface of genus $g(C) \geq 2$ and let $G \subset \mathrm{Aut}(C \times C)$. Let $G^0 := G \cap \mathrm{Aut}(C)^2$ (cf. Definition 3.6), then we say that the action of G is *unmixed* if $G = G^0$ and *mixed* otherwise.

Definition 3.9. Let C_1, C_2 be two Riemann surfaces both of genus at least two, and let G be a group acting on $C_1 \times C_2$. Then $(C_1 \times C_2)/G$ is said to be an *unmixed quotient* (respectively a *mixed quotient*) if the action of G is unmixed (resp. mixed).

Remark 3.10. Definition 3.9 can be extended to the case $g(C_i) \geq 0$ for $i = 1, 2$, requiring G to be finite.

Since we are interested in surfaces of general type, assuming both genus to be at least two does not effect our investigation.

Proposition 3.11. *Let $X = (C_1 \times C_2)/G$ be the quotient of the product of two curves respect to the action of a finite group G. If X is of general type, then both C_1 and C_2 have genus at least two.*

Proof. Since the quotient map $\eta : C_1 \times C_2 \to X$ is surjective, by Lemma 1.47 $\kappa(X) \leq \kappa(C_1 \times C_2)$. By Lemma 1.46, $\kappa(C_1 \times C_2) = \kappa(C_1) + \kappa(C_2)$, therefore $2 \leq \kappa(C_1) + \kappa(C_2)$. This means that $\kappa(C_i) = 1$ for $i = 1, 2$, thus, by Theorem 1.51 $g(C_i) \geq 2$ for $i = 1, 2$. □

If the action of G on $C \times C$ is unmixed, then $G = G^0$ and the action may be assumed without loss of generality to be the diagonal one, described by (3.1).

The mixed case is well described in the following result.

Theorem 3.12 (cf. [Cat00, Proposition 3.16])**.** *Let C be a Riemann surface of genus $g(C) \geq 2$ and let G be a subgroup of $\mathrm{Aut}(C)^2 \rtimes \mathbb{Z}_2$ whose action on $C \times C$ is minimal and of mixed type. Fix $\tau' \in G \setminus G^0$: it determines and element $\tau := \tau'^2 \in G^0$ and $\varphi \in \mathrm{Aut}(G^0)$ defined by $\varphi(h) := \tau' h \tau'^{-1}$. Then, up to a coordinate change, G acts as follows:*

$$\begin{aligned} g(x,y) &= (p(g)x, p(\varphi(g))y) \\ \tau' g(x,y) &= (p(\varphi(g))y, p(\tau g)x) \end{aligned} \qquad \text{for } g \in G^0, \tag{3.2}$$

where $p\colon G^0 \to \mathrm{Aut}(C)$ is the action map.

Conversely, for every finite subgroup $G^0 < \mathrm{Aut}(C)$ and G degree 2 extension of G^0, once $\tau' \in G \setminus G^0$ is fixed and once $\tau \in G^0$ and $\varphi \in \mathrm{Aut}(G^0)$ are defined as above, (3.2) defines a minimal mixed action on $C \times C$.

Since there is no risk of ambiguity, we will always omit the action map $p : G^0 \to \mathrm{Aut}(C)$.

Remark 3.13 (cf. [Cat00, Proposition 3.16]). Using the same notation of Theorem 3.12, the quotient $(C \times C)/G$ does not depend on the choice of the element $\tau' \in G \setminus G^0$.

Suppose that we fix another $\hat{\tau}' \in G \setminus G^0$, then $\hat{\tau}' = \tau' g$ for some $g \in G^0$; according to Theorem 3.12, $\hat{\tau}'$ induces an automorphism $\hat{\varphi} \in \mathrm{Aut}(G^0)$ defined as $\hat{\varphi}(h) := \hat{\tau}' h \hat{\tau}'^{-1}$, an element $\hat{\tau} := (\hat{\tau}')^2$ and a $G-$action on $C \times C$ defined as

$$\begin{aligned} h(x,y) &= (hx, \hat{\varphi}(h)y) \\ \hat{\tau}' h(x,y) &= (\hat{\varphi}(h)y, \hat{\tau}hx) \end{aligned} \qquad \text{for } h \in G^0.$$

Let us denote the $G-$action induced by τ' and $\hat{\tau}'$ with $G_{\tau'}$ and $G_{\hat{\tau}'}$ respectively. Let Φ be the isomorphism

$$\begin{array}{rccc} \Phi: & C \times C & \longrightarrow & C \times C \\ & (x,y) & \longmapsto & (x, \varphi(g)y) \end{array}$$

Let $h \in G^0$, then

$$\begin{aligned} \Phi(h_{\tau'}(x,y)) = \Phi(hx, \varphi(h)y) &= (hx, \varphi(g)\varphi(h)y) = \\ &= (hx, \hat{\varphi}(h)\varphi(g)y) = h_{\hat{\tau}'}(x, \varphi(g)y) = h_{\hat{\tau}'}\Phi(x,y). \end{aligned}$$

Analogously, we can prove the equality for elements in $G \setminus G^0$. This means that Φ induces an isomorphism between $(C \times C)/G_{\tau'}$ and $(C \times C)/G_{\hat{\tau}'}$.

When the action of a group G on $C \times C$ is mixed, the quotient map $\eta \colon C \times C \to X := (C \times C)/G$ can be factorized in the following way:

$$C \times C \xrightarrow{\sigma} Y := (C \times C)/G^0 \xrightarrow{\pi} X, \tag{3.3}$$

where σ is the quotient respect to the diagonal action of the subgroup G^0 on $C \times C$, and π is the double covering determined by the involution $\iota \colon Y \to Y$ such that $\iota[(x,y)] = [(y, \tau x)]$.

Remark 3.14. By Theorem 3.12 and Lemma 1.28, a mixed action G on a product of curve $C \times C$ with $g(C) \geq 2$ determines a subgroup G^0 of G with $|G : G^0| = 2$, a curve $C' := C/G^0$, a set of points $\{p_1, \ldots, p_r\} \subset C'$ corresponding to the branch locus of the quotient map $C \to C'$ and, for every choice of $\alpha_i, \beta_j, \gamma_k \in \pi_1(C' \setminus \{p_1, \ldots, p_r\})$ as in (1.3), a generating vector V for G^0.

Vice versa, once we are given a set of algebraic data as the one listed in 1.35 and a degree two extension G of the group G^0, by Theorem 3.12 we get a mixed action on $C \times C$ and the set of elements of G^0 fixing some points on C is given by (1.5).

3.2 Surfaces isogenous to a product

The first case we address is the one in which the group action on the product of two curves is free. The surfaces we obtain in this case have been studied in many works, such as [Cat00], [BCG08], [CP09], [MP10], [Pen11],[Pol08], [Pol10].

Definition 3.15 (cf. [Cat00, Definition 3.1])**.** A surface is said to be *isogenous to a product* if it is the quotient of a product of two curves both of genus at least 2 by a free action of a group.

Remark 3.16. In literature (cf. [Cat00]) a surface is said to be *isogenous to a product* if it is the quotient of a product of two curves by a free action of a group; if both curves are of genus at least two, then the surface is said to be *isogenous to a higher product.* As stated in Remark 3.10, since we are interested in surfaces of general type, we will always consider product of curves of genus at least two, then, by sake of brevity, we give Definition 3.15 omitting *higher.*

Proposition 3.17 (cf. [Cat00, Proposition 3.13])**.** *If X is a surface isogenous to a product, then a minimal realization of X is unique.*

Proposition 3.18 (cf. [Har77, Exercise II.8.3])**.** *Let C_1, C_2 be two Riemann surfaces, then denoting by $\pi_i\colon C_1 \times C_2 \to C_i$ the projection map onto C_i for $i = 1, 2$*

$$\omega_{C_1 \times C_2} \cong \pi_1^* \omega_{C_1} \otimes \pi_2^* \omega_{C_2}.$$

Lemma 3.19. *Let C_1 and C_2 be two smooth curves of genus $g_1 \geq 2$ and $g_2 \geq 2$ respectively and let $S := C_1 \times C_2$, then S is a smooth minimal surface of general type.*

Proof. Since both C_i's are smooth, $C_1 \times C_2$ is smooth.

Let $\pi_i\colon C_1 \times C_2 \to C_i$ be the projection on the $i-$th coordinate, and denote by F_i a general fibre of π_i, then $K_{C_1 \times C_2} \cong \pi_1^* K_{C_1} + \pi_2^* K_{C_2} \equiv 2(g_1 - 1)F_1 + 2(g_2 - 1)F_2$ (cf. Proposition 3.18).

Let $D \subset C_1 \times C_2$ be an irreducible curve. If $D = F_1$, then $K_{C_1 \times C_2}.F_1 = 2(g_2 - 1)$; analogously $K_{C_1 \times C_2}.F_2 = 2(g_1 - 1)$. If D is not a fibre of π_i, then $K_{C_1 \times C_2}.D \geq 2(g_1 - 1) + 2(g_2 - 1)$. The self-intersection of the canonical divisor of $C_1 \times C_2$ is $K^2_{C_1 \times C_2} = 8(g_1 - 1)(g_2 - 1) > 0$.

By Nakai-Moishezon criterion 2.13 $K_{C_1 \times C_2}$ is ample, in particular it is big and nef. Being $C_1 \times C_2$ of general type (cf. proof of Proposition 3.11), by Proposition 2.56 the surface is minimal. □

Theorem 3.20. *Let $X := (C_1 \times C_2)/G$ be a surface isogenous to a product. Then X is smooth and minimal surface of general type.*

Proof. The action of G on $C_1 \times C_2$ is free, hence the quotient map $\eta\colon C_1 \times C_2 \to X$ is a finite étale morphism of degree $|G|$: being $C_1 \times C_2$ smooth by Lemma 3.19, X is smooth as well. Moreover, by Lemma 1.47 X is of general type.

Let D be an irreducible curve in X, then by Proposition 2.10 $K_X.D = (\eta^* K_X.\eta^* D)/|G|$; $\eta^* K_X = K_{C_1\times C_2}$ by [Bea83b, Lemma VI.3] and $\eta^* D$ is effective, then $K_X.D > 0$. By Nakai-Moishezon criterion 2.13 K_X is ample, then by Proposition 2.56 X is minimal. □

The values of the invariants of a surface isogenous to a product are given by the following results.

Proposition 3.21. *Let $X := (C_1 \times C_2)/G$ be a surface isogenous to a product, then*

$$e(X) = \frac{4(g(C_1)-1)(g(C_2)-1)}{|G|}$$

$$K_X^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|}$$

$$\chi(X) = \frac{(g(C_1)-1)(g(C_2)-1)}{|G|}$$

Proof. Let $\eta\colon C_1 \times C_2 \to X$ be the projection; since the action of G is free, it is an étale covering of X.

The topological Euler characteristic is multiplicative, hence $e(C_1 \times C_2) = e(C_1)e(C_2) = 4(1-g(C_1))(1-g(C_2))$. By Lemma 2.41, $e(C_1 \times C_2) = |G|e(X)$. As $K_{C_1\times C_2} \cong 2(g(C_1)-1)F_1 + 2(g(C_2)-1)F_2$, where F_i denotes a general fibre of the projection $\pi_i\colon C_1 \times C_2 \to C_i$ onto the i−th coordinate, $K^2_{C_1\times C_2} = 8(g(C_1)-1)(g(C_2)-1)$. By Lemma 2.41 $K^2_{C_1\times C_2} = |G|K_X^2$.

By Noether's formula (2.5) the third equality follows. □

Remark 3.22. For a surface X isogenous to a product it holds $K_X^2 = 8\chi(\mathcal{O}_X)$.

Corollary 3.23. *Let $X = (C_1 \times C_2)/G$ be a surface isogenous to a product. Then*

$$h^0(2K_X) = \frac{9(g(C_1)-1)(g(C_2)-1)}{|G|}.$$

Proof. By Proposition 3.21

$$K_X^2 + \chi(\mathcal{O}_X) = \frac{9(g(C_1)-1)(g(C_2)-1)}{|G|};$$

by Theorem 3.20, X is a minimal surface of general type, therefore by Lemma 2.57 $h^0(2K_X) = \chi(\mathcal{O}_X) + K_X^2$, whence the thesis. □

Proposition 3.24 (cf. [Cat00, Theorem C])**.** *Let $X = (C_1 \times C_2)/G$ be a surface isogenous to a product. Then the fundamental group of X sits in an exact sequence*

$$1 \longrightarrow \pi_1(C_1) \times \pi_1(C_2) \longrightarrow \pi_1(X) \longrightarrow G \longrightarrow 1.$$

3.3 Quasi-étale quotients

In the previous section, we saw that if the group G acts freely on a product of curves $C_1 \times C_2$, the quotient $X = (C_1 \times C_2)/G$ is a smooth, minimal surface such that $K_X^2 = 8\chi(\mathcal{O}_X)$. This means that all surfaces isogenous to a product lie along the red line in Figure 2.1; we cannot possibly find any example that answers to questions not concerning that really limited geography region. The idea now is to drop the hypothesis for which the action of G is free.

In this case, the quotient $X := (C_1 \times C_2)/G$ may be singular, so we will consider the minimal resolution $\rho : S \to X$ of its singularities.

In this section we will study the case in which only a finite number of points has non trivial stabilizer, the so called *quasi étale* case. For further details we address to [BP12], [BCGP12], [Fra13], [FP15], [Pol09] and [Pig15].

Definition 3.25 (cf. [Fra13, Definition 2.8]). Let C_1, C_2 be two Riemann surfaces of genus at least two. Let G be a subgroup of $\mathrm{Aut}(C_1 \times C_2)$ whose action is free outside a finite subset of points. We will say that the quotient surface $X = (C_1 \times C_2)/G$ is a *quasi étale quotient*. We will denote by $\rho\colon S \to C_1 \times C_2$ the minimal resolution of the singularities of X, and we say that S is a *quasi étale surface*.

Remark 3.26. If the action of G on $C_1 \times C_2$ is unmixed, than the action of G is free outside a finite set of points. Let $(x, y) \in C_1 \times C_2$; since the action of G is diagonal, $\mathrm{Stab}(x, y) = \mathrm{Stab}(x) \cap \mathrm{Stab}(y)$. By Proposition 1.14 a non trivial automorphism of a curve fixes a finite number of points, hence only a finite number of points in $C_1 \times C_2$ has a non trivial stabilizer.

Lemma 3.27 (cf. [Fra13, Theorem 2.7]). *Let C be a Riemann surface of genus at least two and let G be a group whose action on $C \times C$ is mixed. Then the quotient $X = (C \times C)/G$ is quasi étale if and only if the exact sequence*

$$1 \longrightarrow G \longrightarrow G^0 \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{3.4}$$

does not split.

Proof. Using the notation of Theorem 3.12, suppose there exists $\tau' g \in G \setminus G^0$ such that $(\tau' g)^2 = \varphi(g)\tau g = 1$, then

$$\tau' g(x, \tau g x) = (\varphi(g)\tau g x, \tau g x) = (x, \tau g x),$$

which means that there exists a non trivial element in G that fixes a curve. Hence X is not quasi étale.

Vice versa, let X be quasi étale and let us consider the factorization given by (3.3). As we saw in Remark 3.26, $\sigma\colon C \times C \to Y$ has a finite number of branch points $r_1, \dots, r_t$. Suppose now that there exists a branch curve for π, that is a curve $D \subset X$ such that $|\pi^{-1}(q)| = 1$ for all $q \in D$. Let $q \in D$ such that $\pi^{-1}(q) = r' \notin \{r_1, \dots, r_t\}$. The degree of the map σ is $n := |G^0|$, then $\sigma^{-1}(r') = \{p_1, \dots, p_n\}$. This means that $|\eta^{-1}(q)| = |(\pi \circ \sigma)^{-1}(q)| = n$. Therefore $|\mathrm{Stab}(p_1)| = 2$, which means that $\mathrm{Stab}(p_1) \cong \mathbb{Z}_2$ is generated by an element not in G^0. Then the sequence (3.4) splits. □

Remark 3.28. (cf. [BP16], [MP10], [Pol10]) Let $X = (C_1 \times C_2)/G$ be a quasi étale quotient. Then the singularities of X are the images of all the points in $C_1 \times C_2$ with non trivial stabilizer in G.

The first problem we address is the relation between the singularities of the quotient and the action of G on $C_1 \times C_2$.

Definition 3.29. A surface S has a *cyclic quotient singularity* in $p \in S$ if there exists a neighbourhood U of p such that $U \cong \mathbb{C}^2/H$, with

$$H = \left\langle \begin{pmatrix} e^{\frac{2\pi i p}{r}} & 0 \\ 0 & e^{\frac{2\pi i q}{r}} \end{pmatrix} \right\rangle,$$

for some $p, q, r \in \mathbb{Z}$. In this case we say that $\frac{1}{r}(p, q)$ is the *type* of the cyclic quotient singularity in p.

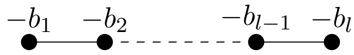
Lemma 3.30 (cf. [BHPV04, pages 104-105])**.** *Every cyclic quotient singularity of type $\frac{1}{r}(p, q)$ is isomorphic to a cyclic quotient singularity of type $\frac{1}{n}(1, a)$ with $1 \le a \le n$ and* $\gcd(a, n) = 1$.

Definition 3.31. Let $1 \le a \le n$ and $\gcd(a, n) = 1$. We denote a cyclic quotient singularity of type $\frac{1}{n}(1, a)$ by $C_{n,a}$.

Definition 3.32. Let $n, a \in \mathbb{Z}$ such that $0 < a < n$ and $\gcd(n, a) = 1$ The *continued fraction* of n/a is the finite expression

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}} := [b_1, \dots, b_l].$$

The resolution of a cyclic quotient singularity of type $\frac{1}{n}(1, a)$ is well known and it is described in [BHPV04, Section III.5]: the exceptional divisor of the minimal resolution of such a singularity is given by $E = \sum_{i=1}^{l} E_i$, where each E_i is a rational smooth curve and $E_i^2 = -b_i$, $E_i.E_{i+1} = 1$ and $E_i.E_j = 0$ if $|i - j| > 1$. The coefficient b_i's are given by the continued fraction $n/a = [b_1, \dots, b_l]$. Such a configuration is called *Hirzebruch-Jung string of type* (n, a) and its dual graph is

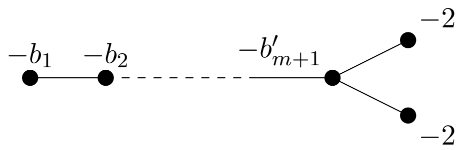


Remark 3.33. Arguing as in Remark 3.26, we see that if the action of G on $C_1 \times C_2$ is unmixed, then the singular points in $X = (C_1 \times C_2)/G$ are all cyclic quotient, as the stabilizer of each point in $C_1 \times C_2$ is a cyclic group.

Lemma 3.34 (cf. [FP15]). *Let $X := (C \times C)/G$ be a mixed quasi étale quotient, then* $\mathrm{Sing}(X) = \pi(\mathrm{Sing}(Y))$ *. Moreover let $p \in Y$ be a singular point of type $C_{n,a}$, with $n/a = [b_1, \dots, b_l]$; then one of the following occurs:*

i. $\iota(p) \neq p$ and $\pi(p)$ is a singular point of type $C_{n,a}$;

ii. $\iota(p) = p$; then $l = 2m+1$ is odd and b_{m+1} is even, $\pi(p)$ is a singular point whose minimal resolution has exceptional divisor with the following dual graph



where $b'_{m+1} = 1 + b_{m+1}/2$. Such a singularity is said to be of type $D_{n,a}$.

Proposition 3.35 (cf. [FP15, Proposition 2.18]). *Let $X = (C_1 \times C_2)/G$ be a quasi étale quotient. For a singular point of type $C_{n,a}$, we will denote by a' the only integer $0 < a' < n$ with $aa' \equiv 1 \bmod n$ and $n/a = [b_1, \dots, b_l]$.*

Then for each singular point $x \in \mathrm{Sing}(X)$ of type $C_{n,a}$ we define the two integers:

$$k_x := -2 + \frac{2+a+a'}{n} + \sum_{i=1}^{l}(b_i - 2) \qquad B_x := \frac{a+a'}{n} + \sum_{i=1}^{l} b_i.$$

For each singular point $x \in \mathrm{Sing}(X)$ of type $D_{n,a}$ we define

$$k_x := -1 + \frac{2+a+a'}{2n} + \sum_{i=1}^{l}\frac{b_i - 2}{2} \qquad B_x := 6 + \frac{a+a'}{2n} + \sum_{i=1}^{l}\frac{b_i}{2}.$$

Then, if $\rho\colon S \to X$ is the minimal resolution of the singularities of X

$$K_S^2 = \frac{8(g(C_1)-2)(g(C_2)-1)}{|G|} - \sum_{x \in \mathrm{Sing}(X)} k_x$$

$$\chi(\mathcal{O}_S) = \frac{K_S^2}{8} + \frac{\sum_{x \in \mathrm{Sing}(X)} B_x}{24}.$$

The last proposition shows that if we allow the points of $C_1 \times C_2$ to have non trivial stabilizer, we obtain a surface S with $K_S^2 \leq 8\chi$, reaching the area below the red line in Figure 2.1.

Chapter 4

Semi-isogenous Mixed Surfaces

So far the classification of quotients of products of two curves only concerned quasi étale actions of groups. In this chapter we drop the quasi étale hypothesis and allow a product $C_1 \times C_2$ to contain curves with non trivial stabilizer.

By Remark 3.26, if the action of a group G on a product $C_1 \times C_2$ is unmixed, then the quotient map $\eta \colon C_1 \times C_2 \to X := (C_1 \times C_2)/G$ is quasi étale. Therefore we will study only mixed action. The notation we will use from now on is the one defined in Theorem 3.12 and in (3.3).

4.1 Fixed points of mixed actions

As stated in the previous chapter, since we consider C of genus at least 2 (cf. Remark 3.10), a group $G \subset \mathrm{Aut}(C \times C) \cong \mathrm{Aut}(C) \rtimes \mathbb{Z}_2$ is finite.

Definition 4.1. Let C be a Riemann surface of genus at least 2 and let G be a group whose action on $C \times C$ is mixed. Then we define the set

$$O_2 := \{g \in G \setminus G^0 : g^2 = 1\}.$$

Remark 4.2. By Lemma 3.27, the quotient X is quasi-étale if and only O_2 is empty.

Let G be a group whose action on $C \times C$ is mixed. Since we can assume the action of G to be minimal (cf. Remark 3.7), Theorem 3.12 allows us to identify the normal subgroup $G^0 \triangleleft G$ acting diagonally on $C \times C$ with its image via $p : G^0 \to \mathrm{Aut}(C)$. Let Σ be the subset of elements of G^0 having some fixed point on the curve C.

Lemma 4.3. *Let G be a group whose action on $C \times C$ is mixed. Then the following hold:*

i. Let $g \in G^0$*, then* $\mathrm{Fix}(g) \neq \emptyset$ *if and only if* $g \in \Sigma \cap \varphi(\Sigma)$*;*

ii. Let $g \in G \setminus G^0$*, then* $\mathrm{Fix}(g) \neq \emptyset$ *if and only if* $g^2 \in \Sigma$*.*

Proof. i. Let $(x, y) \in C \times C$, then $g(x, y) = (gx, \varphi(g)y) = (x, y)$ if and only if $g \in \Sigma \cap \varphi^{-1}(\Sigma)$. The set Σ is $\mathrm{Inn}(G^0)-$invariant and $\varphi^2 \in \mathrm{Inn}(G^0)$, then $\varphi^{-1}(\Sigma) = \varphi(\Sigma)$.

ii. Once we have fixed τ', there exists a unique $h \in G^0$ such that $g = \tau' h$. Let $(x, y) \in C \times C$, then

$$\tau' h(x, y) = (\varphi(h)y, \tau hx) = (x, y) \Leftrightarrow \begin{cases} y = \tau hx \\ x = \varphi(h)y \end{cases} \Leftrightarrow \begin{cases} y = \tau hx \\ x = \varphi(h)\tau hx \end{cases}$$

So $\mathrm{Fix}(g) \neq \emptyset$ if and only if $\varphi(h)\tau h = (\tau' h \tau'^{-1})\tau h = (\tau' h)^2 = g^2 \in \Sigma$.

□

Definition 4.4. Let G be a group whose action on $C \times C$ is mixed. For each $g \in O_2$ we define $R_g := \mathrm{Fix}(g)$.

Remark 4.5. Each R_g is a smooth irreducible curve isomorphic to C; indeed, once we have fixed τ', the curve R_g is the graph of the automorphism $\tau' g \in \mathrm{Aut}(C)$, i.e. $R_g = \{(x, \tau' gx) : x \in C\}$.

Proposition 4.6. *Let* G *be a group whose action on* $C \times C$ *is mixed. Let* D *be an irreducible curve contained in the ramification locus of the quotient map* $\eta : C \times C \to X$*. Then there exists* $g \in O_2$ *such that* $D = R_g$*.*

Proof. Let $P \subset C \times C$ be the finite set of points fixed by a non trivial element of G^0. Each point in $D \setminus P$ has stabilizer of order 2 generated by an element in $G \setminus G^0$, otherwise the point would be stabilized by a non trivial element of G^0; therefore each point in $D \setminus P$ belongs to one of the R_g's. Noting that if $D \neq R_g$ then $D \cap R_g$ is a finite set, we get the thesis. □

Remark 4.7. By the previous proposition, if the action of G on $C \times C$ is mixed, then we have a bijection between O_2 and the set of ramification curves of the quotient map $\eta : C \times C \to X := (C \times C)/G$.

Proposition 4.8. *Let* $X = (C \times C)/G$ *be a mixed quotient. Let* $\pi \colon Y = (C \times C)/G^0 \to X$ *the double covering induced by the involution* $\iota \colon Y \to Y$*. Then* $\mathrm{Sing}(X) \subset \pi(\mathrm{Sing}(Y))$*.*

Proof. Let $u := \sigma(p,q) \in Y$ be a smooth point and let $z := \iota(u) \in Y$. By Remark 3.28, $\{1\} = \mathrm{Stab}_{G^0}(p,q) = \mathrm{Stab}_G(p,q) \cap G^0$. If $u \neq z$, then $\pi(u)$ is clearly a smooth point. If $u = z$, then $\mathrm{Stab}_G(p,q) = \langle g \rangle \cong \mathbb{Z}_2$ for some $g \in O_2$. Therefore $\dim \mathrm{Fix}(g) = \dim R_g = 1$ (cf. Remark 4.5) and $\mathrm{Stab}_G(p,q)$ is generated by a pseudo reflection. This means that there exists coordinates (x,y) in a neighbourhood of (p,q) such that locally $g(x,y) = (x,-y)$. In a neighbourhood of $\pi(u)$, X is isomorphic to $\mathbb{C}^2/\langle g \rangle$; the variety structure in such a neighbourhood is given by the ring $\mathbb{C}[x,y]^{\langle g \rangle} \cong \mathbb{C}[x,y^2]$, hence $\pi(u)$ is smooth. □

Since, by Remark 3.28, the singular points of Y correspond to the G^0-orbits of the points of $C \times C$ with non trivial stabilizer, the following corollary follows.

Corollary 4.9. *Let $X = (C \times C)/G$ be a mixed quotient and suppose the action of G^0 on $C \times C$ to be free. Then X is smooth.*

The objects we want to investigate are the surfaces we obtain when the diagonal subroup G^0 acts freely on the product $C \times C$; the quotient $X = (C \times C)/G$ will then be the quotient of the surface isogenous to a product $Y = (C \times C)/G^0$ respect to the involution ι. This justify the following definition.

Definition 4.10. Let $X := (C \times C)/G$ be a mixed quotient and let $Y := (C \times C)/G^0$. If Y is a surface isogenous to a product, then X is said to be a *semi-isogenous mixed surface.*

Remark 4.11. By Theorem 3.12 and Lemma 4.3, in order to construct a semi-isogenous mixed surface, one has to provide the algebraic data listed in Remark 3.14 such that for the set Σ_V defined in Proposition 1.35 it holds $\Sigma_V \cap \varphi(\Sigma_V) = \{1\}$. Moreover, we shall remark that $\varphi(\Sigma_V) = \Sigma_{\varphi(V)}$.

Proposition 4.12. *Let $X = (C \times C)/G$ be a semi-isogenous surface of general type. Then $g(C) \geq 3$.*

Proof. Suppose $g(C) = 2$, then we factorize the quotient map $\eta \colon C \times C \to X$ in the following way:

$$C \times C \to C^{(2)} \to X,$$

where the first map is the quotient determined by the action of $\mathbb{Z}_2 \cong \langle h \rangle$ on $C \times C$ with $h \in O_2$. It is well known that $C^{(2)}$ is birational to an abelian surface isomorphic to the Jacobian variety $J(C)$. Hence, by Lemma 1.47, $\kappa(X) \leq \kappa(C^{(2)}) = 0$, which is a contradiction. □

4.2 The branch and the ramification locus

In this section we give a description of the ramification and branch locus of the quotient map $\eta\colon C\times C\to X$; these results underline the connection between the geometry of semi-isogenous mixed surfaces and the algebraic properties of the groups involved in the construction in a simple yet clear way.

Proposition 4.13. *Let $X := (C\times C)/G$ be a semi-isogenous mixed surface. Then the ramification locus of the quotient map $\eta\colon C\times C\to X$ is the disjoint union*

$$\bigsqcup_{g\in O_2} R_g.$$

Proof. Let $(x,y)\in C\times C$ be a point with non trivial stabilizer $g\in \mathrm{Stab}_G(x,y)$. Since $g^2\in G^0$ and G^0 acts freely on $C\times C$, $g^2=1$, therefore $(x,y)\in R_g$. Let $g,h\in O_2$ such that there exists $(x,y)\in R_g\cap R_h$; then $g^{-1}h(x,y)=(x,y)$, but $g^{-1}h\in G^0$ and fixes a point, whence $g=h$. □

Proposition 4.14. *Let $X := (C\times C)/G$ be a semi-isogenous mixed surface and let $g,h\in O_2$. Then $h=\gamma g\gamma^{-1}$ if and only if $\gamma R_g = R_h$. In particular, R_g is $\gamma-$invariant if and only if γ belongs to $Z(g)$, the centralizer of g.*

Proof. For any $\gamma\in G$, the curve γR_g is fixed pointwise by $\gamma g\gamma^{-1}=h$, hence $\gamma R_g = R_h$. Vice versa, if $\gamma R_g=R_h$, then $\gamma g\gamma^{-1}$ fixes R_h pointwise, then $\gamma g\gamma^{-1}=h$. □

Remark 4.15. Let $X=(C\times C)/G$ be a semi-isogenous mixed surface and let $g,h\in O_2$. Then there exists $\alpha\in G^0$ such that $\alpha R_g=R_h$ if and only if there exists $\beta\in G\setminus G^0$ such that $\beta R_g = R_h$.

Proof. Suppose there exists $\alpha\in G^0$ such that $\alpha R_g=R_h$, then $\alpha g R_g=\alpha R_g = R_h$ and $\beta := \alpha g\in G\backslash G^0$. The same argument with $\alpha \in G\backslash G^0$ proves the other implication. □

Definition 4.16. Let us say $O_2 := \{g_1,\dots,g_N\}$, then by Proposition 4.14, for $h\in G$ $hR_{g_i}=R_{g_j}$ if and only if $g_j = hg_ih^{-1}$. We can then define an homomorphism between G and the permutation group of N elements

$$\begin{array}{cccc}\varrho\colon & G & \to & \mathfrak{S}_N\\ & h & \mapsto & \varrho(h)\end{array}$$

where, if $hR_{g_i}=R_{g_j}$, $\varrho(h)(i)=j$.

Let G be a group whose action on $C\times C$ is mixed. We denote by $\mathrm{Cl}(g)$ the conjugacy class of $g\in G$ and we define $\mathrm{Cl}(O_2) := \{\mathrm{Cl}(g) : g\in O_2\}$.

Proposition 4.17. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface. Let $\mathrm{Cl}(O_2) = \{\mathrm{Cl}_1, \dots, \mathrm{Cl}_M\}$ and for each $i = 1, \dots, M$ let us fix $g_i \in \mathrm{Cl}_i$. Then the branch locus B of the quotient map $\eta\colon C \times C \to X$ is the disjoint union $B = B_1 \sqcup \cdots \sqcup B_M$, where $B_i := \eta(R_{g_i})$.*

Moreover, for each $g \in O_2$ the map $\eta|_{R_g}\colon R_g \to B_g := \eta(R_g)$ is an étale covering of degree $|Z(g)|/2$ and

$$g(B_g) = \frac{2(g(C) - 1)}{|Z(g)|} + 1. \tag{4.1}$$

Proof. The first claim follows from Proposition 4.13 and Proposition 4.14.

By Proposition 4.14, for each $g \in O_2$ the subgroup $Z(g)$, i.e. the centralizer of g, is the stabilizer group of R_g. This group does not act faithfully on R_g, since $g \in Z(g)$ fixes the curve pointwise. Therefore $Z(g)/\langle g \rangle \cong Z(g) \cap G^0 =: Z^0(g)$ acts freely on R_g.

This means that the map $\eta|_{R_g}\colon R_g \to B_g$ is unbranched, and its degree is $\deg(\eta|_{R_g}) = |Z(g)|/|\langle g \rangle| = |Z(g)|/2$. Since $R_g \cong C$, equation (4.1) follows from Hurwitz's formula (1.2). □

Remark 4.18. Proposition 4.13 and Proposition 4.17 imply that we have a bijection between the set of ramification curves (the set of branch curves) and O_2 ($\mathrm{Cl}(O_2)$ respectively).

4.3 Birational invariants

The following result, which is proved in [Bea83b] for varieties of arbitrary dimension, will be used in the case of surfaces.

Lemma 4.19 (cf. [Bea83b, Lemma VI.11])**.** *Let X be a smooth variety and G a finite group acting on X. Let $\pi : X \to Y := X/G$ be the quotient map and assume Y to be smooth. Then the $G-$invariant $k-$fold $p-$forms $\alpha \in H^0(X, (\Omega^p_X)^{\otimes k})$ are the forms $\pi^*\omega$, where ω is a $k-$fold rational $p-$form on Y such that $\pi^*\omega$ is regular on X.*

In particular, if the group G acts freely (or equivalently, if π is étale) then the map $\pi^*\colon H^0(Y, (\Omega^p_Y)^{\otimes k}) \to H^0(X, (\Omega^p_X)^{\otimes k})^G$ is an isomorphism.

Remark 4.20. We should stress that Lemma 4.19 holds for X eventually non connected.

Proposition 4.21. *Let $X = (C \times C)/G$ be a semi-isogenous mixed surface, then $q(X)$ is equal to the genus of the quotient curve $C' := (C/G^0)$.*

Proof. First of all, we want to prove that $H^0(\Omega^1_{C\times C})^G \cong H^0(\Omega^1_X)$ via pull-back η^*; by Lemma 4.19, it suffices to prove that if a $1-$form on X α has poles, then $\eta^*\alpha$ has poles as well.

This is obvious if the poles of α are not in the branch locus of η.

Suppose then that $(\alpha) = A - kB$, where A and B have no common components, B is a branch curve for η and $k \geq 1$.

Let q be a generic point of B and $p \in C \times C$ such that $\eta(p) = q$. Let (x, y) and (u, v) be local coordinates centered at p and q respectively such that locally $\eta(x, y) = (x, y^2) = (u, v)$. Then $B = \{v = 0\}$ and locally

$$\alpha = \frac{1}{v^k}(f(u, v)du + g(u, v)dv),$$

where $\alpha' = f(u, v)du + g(u, v)dv$ is a holomorphic $1-$form that does not vanish in a neighbourhood of q.

Therefore

$$\eta^*\alpha = \frac{1}{y^{2k}}(f(x, y^2)dx + 2yg(x, y^2)dy)$$

in a neighbourhood of p. The $1-$form $f(x, y^2)dx + 2yg(x, y^2)dy$ either does not vanish or vanishes with multiplicity 1 along $\{y = 0\}$. Therefore $\eta^*\alpha$ has a pole along $\{y = 0\}$ of order at least $2k - 1 > 0$. We conclude that $H^0(\Omega^1_X) \cong H^0(\Omega^1_{C\times C})^G$.

Now, arguing as in [Cat00, Proposition 3.15]:

$$\begin{aligned} H^0(\Omega^1_X) &\cong (H^0(\Omega^1_{C\times C}))^G = (H^0(\Omega^1_C) \oplus H^0(\Omega^1_C))^G \\ &= (H^0(\Omega^1_C)^{G^0} \oplus H^0(\Omega^1_C)^{G^0})^{G/G^0} \\ &= (H^0(\Omega^1_{C'}) \oplus H^0(\Omega^1_{C'}))^{G/G^0}, \end{aligned}$$

where last equality is proved with computations which are analogous to the ones done in the first part of the proof (cf. [Bea83b, Examples VI.12.2]). Since X is a mixed quotient $G/G^0 \cong \mathbb{Z}_2$ exchanges the last two summands, hence $q(X) = h^0(\Omega^1_X) = h^0(\Omega^1_{C'}) = g(C')$. □

Let $X := (C \times C)/G$ be a semi-isogenous mixed surface. We denote by $B := B_1 + \cdots + B_M$ the branch divisor of the quotient map $\eta\colon C \times C \to X$ and we define the integer

$$\delta(B) := \sum_{j=1}^{M} (g(B_j) - 1).$$

Remark 4.22. By Proposition 4.17, the branch curves are pairwise disjoint, hence $\delta(B) = p_a(B) - 1$, where $p_a(B)$ denotes the arithmetic genus of B.

Moreover, by Proposition 4.17, it is immediate to see that

$$\delta(B) = \frac{2(g(C) - 1)}{|G|} \cdot |O_2|. \tag{4.2}$$

Let $X = (C \times C)/G$ be a semi-isogenous mixed surface; the map $\pi\colon Y = (C \times C)/G^0 \to X$ is a double covering - therefore a cyclic covering - branched along the smooth divisor B. Let us define $T := \pi^{-1}(B)$. Then the following lemma holds:

Lemma 4.23 (cf. [BHPV04, Lemma 17.2])**.** *There exists a line bundle $\mathcal{L}$ on X such that $\pi_*(\mathcal{O}_Y) = \mathcal{O}_X \oplus \mathcal{L}^*$. For such $\mathcal{L}$ the following hold:*

- $\mathcal{O}_Y(T) = \pi^*(\mathcal{L})$;
- $K_Y = \pi^*(K_X \otimes \mathcal{L})$;
- $\mathcal{O}_Y(B) = \mathcal{L}^{\otimes 2}$.

Proposition 4.24. *Let $X = (C \times C)/G$ be a semi-isogenous mixed surface, then*

$$e(X) = \frac{2(g(C)-1))}{|G|} \cdot (2(g(C)-1) - |O_2|) = \frac{4(g(C)-1)^2}{|G|} - \delta(B), \tag{4.3}$$

and

$$K_X^2 = \frac{2(g(C)-1)}{|G|} \cdot (4(g(C)-1) - 5|O_2|) = \frac{8(g(C)-1)^2}{|G|} - 5\delta(B). \tag{4.4}$$

Proof. For $i = 1, \dots, M$ let $g_i \in O_2$ such that $\eta(R_{g_i}) = B_i$. By Proposition 4.17

$$e(B_i) = \frac{-4(g(C)-1)}{|Z(g_i)|} = -4\frac{N_i}{|G|}(g(C)-1),$$

where $N_i = |G|/|Z(g_i)|$ is the cardinality of the conjugacy class of g_i. Note that $\sum_{i=1}^M N_i = |O_2|$. Since the ramification locus $R := \eta^{-1}(B)$ is the disjoint union of $|O_2|$ smooth curves, each one isomorphic to C, it holds

$$e(C \times C \setminus R) = e(C \times C) - e(R) = 4(g(C)-1)^2 + 2|O_2|(g(C)-1).$$

It follows

$$\begin{aligned}
e(X) = e(X \setminus B) + e(B) &= \frac{e(C \times C \setminus R)}{|G|} + \sum_{i=1}^M e(B_i) \\
&= \frac{4(g(C)-1)^2}{|G|} + \frac{2|O_2|}{|G|}(g(C)-1) - \frac{4(g(C)-1)}{|G|}\sum_{i=1}^M N_i \\
&= \frac{2(g(C)-1)}{|G|}(2(g(C)-1) + |O_2| - 2|O_2|) \\
&= \frac{2(g(C)-1)}{|G|}(2(g(C)-1) - |O_2|)
\end{aligned}$$

The second equality in (4.3) follows by (4.2).

The map $\sigma\colon C \times C \to Y := (C \times C)/G^0$ is an unramified covering of degree $|G^0| = |G|/2$ and the canonical divisor $K_{C\times C}$ is numerically equivalent to the divisor $2(g(C)-1)F_1 + 2(g(C)-1)F_2$, were F_1, F_2 denote a general fibre of the projections on the first and on the second coordinate respectively. Then, by Proposition 2.10

$$K_Y^2 = \frac{K_{C\times C}^2}{\deg(\sigma)} = \frac{16(g(C)-1)^2}{|G|}.$$

On the other side, by Lemma 4.23, $2K_Y = \pi^*(2K_X + B)$, therefore, by Proposition 2.10 $4K_Y^2 = 2(4K_X^2 + B^2 + 4K_X.B)$, that is

$$K_Y^2 = 2\left(K_X^2 + \frac{B^2}{4} + K_X.B\right).$$

By Proposition 4.17, $B_i.B_j = 0$ if $i \neq j$, then $B^2 = B_1^2 + \cdots + B_M^2$.

For all $i = 1, \ldots, M$, $\eta^*(B_i) = 2R_{i,1} + \cdots + 2R_{i,N_i}$; these curves are pairwise disjoint by Proposition 4.13 and of genus $g(C)$. Applying the genus formula (2.4) to $R_{i,j} \subset C\times C$ we get

$$\begin{aligned} R_{i,j}^2 &= 2(g(C)-1) - K_{C\times C}.R_{i,j} \\ &= 2(g(C)-1) - [2(g(C)-1)F_1.R_{i,j} + 2(g(C)-1)F_2.R_{i,j}] \\ &= -2(g(C)-1). \end{aligned}$$

According to Lemma 2.41

$$\begin{aligned} B_i^2 &= \frac{1}{|G|}\eta^*(B_i).\eta^*(B_i) \\ &= \frac{1}{|G|}\left(\sum_{j=1}^{N_i} 2R_{i,j}\right).\left(\sum_{j=1}^{N_i} 2R_{i,j}\right) \\ &= \frac{4}{|G|}\sum_{j=1}^{N_i} R_{i,j}^2 = -\frac{8N_i(g(C)-1)}{|G|}, \end{aligned}$$

where third equality holds because $R_{i,j}.R_{i,k} = 0$ whenever $j \neq k$.

By genus formula (2.4)

$$K_X.B_i = 2(g(B_i)-1) - B_i^2 = \frac{4N_i(g(C)-1)}{|G|} + \frac{8N_i(g(C)-1)}{|G|}.$$

Then

$$\begin{aligned} K_X^2 &= \frac{K_Y^2}{2} - K_X.B - \frac{B^2}{4} \\ &= \frac{8(g(C)-1)^2}{|G|} - \frac{12(g(C)-1)}{|G|}\sum_{i=1}^{M} N_i + \frac{2(g(C)-1)}{|G|}\sum_{i=1}^{M} N_i \\ &= \frac{2(g(C)-1)}{|G|}(4(g(C)-1) - 6|O_2| + |O_2|), \end{aligned}$$

whence the thesis. □

By Proposition 4.24 and Noether's formula, we immediately get the following.

Corollary 4.25. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surfaces. Then*

$$\chi(\mathcal{O}_X) = \frac{g(C)-1}{|G|} \cdot (g(C) - 1 - |O_2|) = \frac{(g(C)-1)^2}{|G|} - \frac{1}{2}\delta(B). \tag{4.5}$$

Remark 4.26. By Proposition 4.24 and Corollary 4.25 we get

$$8\chi(\mathcal{O}_X) - K_X^2 = \frac{2(g(C)-1)}{|G|} \cdot |O_2| = \delta(B). \tag{4.6}$$

Remark 4.27. By the proof of Proposition 4.24, for every branch curve B_i we get the following equalities:

$$K_X.B_i = 6(g(B_i) - 1), \qquad B_i^2 = -4(g(B_i) - 1),$$

therefore, being the branch curves pairwise disjoint

$$K_X.B = 6\delta(B), \qquad B^2 = -4\delta(B).$$

Moreover

$$10\chi(\mathcal{O}_X) - K_X^2 = \frac{2(g(C)-1)^2}{|G|} = \frac{(g(C)-1)^2}{|G^0|} = \chi(\mathcal{O}_Y),$$

where last equality holds for by Proposition 3.21.

4.4 Albanese fibre of a Semi-isogenous Mixed Surface with $q = 1$

By Remark 2.45, the Albanese map of a surface X with irregularity $q(X) = 1$ is a fibration onto the elliptic curve $\mathrm{Alb}(X)$ and the genus g_{alb} of the general Albanese fibre is an important deformation invariant. In this section we explain how to compute g_{alb} for a semi-isogenous mixed surface.

The argument is analogous to the one described in [FP15, Section 3] for the quasi étale surfaces.

Let $X = (C \times C)/G$ be a semi-isogenous mixed surface with $q(X) = 1$. By Proposition 4.21, C/G^0 is an elliptic curve, which will be denoted by E, and the Galois covering $c\colon C \to E$ has branch locus $\{p_1, \dots, p_r\}$. Up to translation, we may assume that $0 \in E$ is not in B and that $-p_i \notin B$ for each $i \in \{1, \dots, r\}$.

Let us define

$$\begin{array}{rccc} Q\colon & C \times C & \longrightarrow & E \times E \\ & (p,q) & \longmapsto & (c(p), c(q)) \end{array}$$

and let

$$\begin{array}{rccc} \tilde{\alpha}\colon & E^{(2)} & \longrightarrow & E \\ & \{x,y\} & \longmapsto & x+y \end{array}$$

be the Abel-Jacobi map.

Let us define the morphism $\alpha\colon X \to E$ defined as $\alpha([p,q]) := c(p) + c(q)$. First, let us remark that it is well defined: let $(p', q') \in [(p,q)]$, then either one of the following occurs:

- $(p', q') = g(p,q)$ for some $g \in G^0$; then $p' = gp$ and $q' = \varphi(g)q$, therefore $c(p') = c(p)$ and $c(q') = c(q)$;
- $(p', q') = \tau' g(p,q)$ for some $g \in G^0$; then $p' = \varphi(g)q$ and $q' = \tau gp$, therefore $c(p') = c(q)$ and $c(q') = c(p)$.

In both cases $c(p') + c(q') = c(p) + c(q)$. By Theorem 2.44, the morphism factorizes through the Albanese map: $\alpha = \psi \circ f$.

What we get by these definitions is the following commutative diagram:

$$\begin{tikzcd} C \times C \arrow[r, "Q"] \arrow[d, "\eta"'] & E \times E \arrow[d, "\epsilon"] \\ X \arrow[r] \arrow[d, "f"'] \arrow[dr, "\alpha"] & E^{(2)} \arrow[d, "\tilde{\alpha}"] \\ \mathrm{Alb}(X) \arrow[r, "\psi"'] & E \end{tikzcd}$$

Let $E' := \epsilon^*(\tilde{\alpha}^*(0)) = \{(u, -u) : u \in E\}$, consider $F^* := Q^*(E')$ and $F := \alpha^*(0)$.

Lemma 4.28. *F^* is smooth.*

Proof. For every $p \in C$ let $m_p := |\mathrm{Stab}_{G^0}(p)|$; for any chart $\phi_2 \colon U_2 \to \mathbb{C}$ on E centered at $c(p)$, there exists a chart $\phi_1 \colon U_1 \to \mathbb{C}$ centered at p such that $\phi_2(c(\phi_1^{-1}(z))) = z^{m_p}$ (cf. [Mir95, Proposition II.4.1]).

Analogously, let $(p, q) \in C \times C$; then there exist local coordinates for C z_1 and z_2 centered respectively in p and q such that - with a slight abuse of notation - the $(G^0 \times G^0)-$cover Q assumes the form $(z_1, z_2) \mapsto (z_1^{m_p}, z_2^{m_q})$ in an open neighbourhood $U \subset C \times C$ of (p, q).

Suppose that $Q(p, q) \in E'$ and let $V := Q(U)$, then, being E' a smooth curve, up to a restriction of the open set V, there exist local coordinates y_1, y_2 on V such that $E' \cap V$ is given by $ay_1 + by_2 + O(2) = 0$ with $ab \neq 0$. Then a local equation for F^* in $U \cap F^*$ is given by $az_1^{m_p} + bz_2^{m_q} + O(2 \min\{m_p, m_q\}) = 0$. Since by hypothesis $m_p = 1$ or $m_q = 1$, F^* is smooth in (p, q). □

Remark 4.29. $F^* = Q^*(\epsilon^*(\tilde{\alpha}^*(0))) = (\tilde{\alpha} \circ \epsilon \circ Q)^*(0) = (\alpha \circ \eta)^*(0) = \eta^*(\alpha^*(0)) = \eta^*(F)$, then $\eta(F^*) = F$.

The group G acts freely on F^*: suppose that $g(x, y) = (x, y)$ for some $g \in G^0$ and $(x, y) \in F^*$, then $gx = x$ and $\varphi(g)y = y$. Therefore $x \in c^{-1}(p_i)$ and $y \in c^{-1}(p_j)$ for some $i, j = 1, \dots, r$, but then $Q(x, y) = (c(x), c(y)) = (p_i, p_j) \in E'$; this means that $p_j = -p_i$, which contradicts $-p_k \notin B$ for any $p_k \in B$. Analogously, we can prove that $\tau' g(x, y) \neq (x, y)$ for $(x, y) \in F^*$ and $g \in G^0$.

Being F^* smooth and $\eta|_{F^*} \colon F^* \to F$ an étale cover, F is smooth.

Let us define the points of $E \times E$ $q_i := (p_i, -p_i)$ and $q_i' := (-p_i, p_i)$ and the set $B' := \{q_i, q_i' : i = 1, \dots, r\}$; we observe that $0' := (0, 0) \in E' \setminus B'$.

As we saw in Subsection 1.3.1, given a suitable choice of loops $\alpha, \beta, \gamma_1, \dots, \gamma_r \in \pi_1(E \setminus B, 0)$, the Galois covering $c \colon C \to E$ is determined by a generating vector of type $(1; m_1, \dots, m_r)$ of G^0 that represents the monodromy map $\mu_c \colon \pi_1(E \setminus B, 0) \to G^0$ of c.

Q induces by restriction the $(G^0 \times G^0)-$cover $F^* \to E'$, whose branch locus is B'. In order to describe the monodromy map of this cover, we define a set of generators for $\pi_1(E' \setminus B', 0')$ in the following way:

- $\delta := (\alpha, -\alpha)$;
- $\theta := (\beta, -\beta)$;
- $\gamma_i' := (\gamma_i, -\gamma_i)$ is a loop travelling around q_i, for $i = 1, \dots, r$;

- $\gamma_i'' := (-\gamma_i, \gamma_i)$ is a loop travelling around q_i', for $i = 1, \dots, r$.

We need to pay some attention to choosing α, β and γ_i in order that δ, θ, γ_i' and γ_i'' do not meet B'. Moreover,the classes of δ, θ, γ_i' and γ_i'' depend on α, β, and γ_i itself, not only on their classes in $\pi_1(E \setminus B, 0)$.

The classes of δ, θ, γ_i' and γ_i'' generate $\pi_1(E' \setminus B', 0')$ and the monodromy map of $Q|_{F^*} : F^* \to E'$ is the unique homomorphism $\mu' : \pi_1(E' \setminus B', 0') \to G^0 \times G^0$ such that

$$\begin{aligned} \delta &\stackrel{\mu'}{\longmapsto} (a, a^{-1}) & \qquad \theta &\stackrel{\mu'}{\longmapsto} (b, b^{-1}) \\ \gamma_i' &\stackrel{\mu'}{\longmapsto} (h_i, 1) & \qquad \gamma_i'' &\stackrel{\mu'}{\longmapsto} (1, h_i) \end{aligned}$$

Remark 4.30. By Remark 1.29, the number of connected components of F^* is equal to the index of $\mathrm{Im}(\mu')$ in $G^0 \times G^0$.

Let us fix $\tau' \in G \setminus G^0$, let $\tau := (\tau')^2 \in G^0$ and $\varphi \in \mathrm{Aut}(G^0)$ defined by $\varphi(h) := \tau' h \tau'^{-1}$. We define a $G-$action on $G^0 \times G^0$ in the following way:

$$\begin{aligned} g(h_1, h_2) &= (gh_1, \varphi(g)h_2) \\ \tau' g(h_1, h_2) &= (\varphi(g)h_2, \tau g h_1) \end{aligned} \qquad \text{for } g \in G^0.$$

Let us define

$$l := \left| \bigcup_{g \in G} g \ \mathrm{Im}(\mu') \right|.$$

Lemma 4.31 (cf. [FP15, Lemma 3.2]). *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface with $q(X) = 1$, then* $\deg \psi = |G^0|^2/l$.

Proof. Let $u \in E' \setminus B'$. The action of $G^0 \times G^0$ on $Q^{-1}(u)$ induces a bijection between $G^0 \times G^0$ and $Q^{-1}(u)$; arguing as in Remark 1.29, two points $p, p' \in Q^{-1}(u)$ belong to the same connected component of F^* if and only if $gp = p'$ where $g \in \mathrm{Im}(\mu')$.

Moreover, two points $p, p' \in F^*$ are mapped by η onto the same point of X if and only if there exists $g \in G$ such that $gp = p'$. So, exactly l points of $Q^{-1}(u)$ are mapped into each connected component of $F = \eta(F^*)$.

Since $\deg \psi$ equals the number of connected components of F, we get the thesis. $\square$

Proposition 4.32. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface with $q(X) = 1$. Then*

$$g_{alb} = 1 + l \cdot \frac{g(C) - 1 - |O_2|}{|G^0|^2}.$$

Proof. Since C is constructed via a generating vector of type $(1; m_1, \dots, m_r)$ for G^0, $e(C) = -|G^0| \sum_{i=1}^r \frac{m_i - 1}{m_i}$. The $(G^0 \times G^0)-$covering Q is branched along the union of r *horizontal* copies of E and r *vertical* copies of E; moreover, for each i there are one horizontal copy and one vertical copy with branching index m_i. Since E' is an elliptic curve that intersects each of these copies of E transversally in one point, by Hurwitz's Formula 1.17 applied to $Q|_{F^*} : F \to E'$, we get

$$e(F^*) = -|G^0|^2 \sum_{i=1}^r 2 \left(\frac{m_i - 1}{m_i} \right) = e(C) \cdot |G|.$$

Let us now consider the map $\eta|_{F^*} : F^* \to F$. This map has degree $|G| = 2|G^0|$ and by Proposition 4.13 is ramified in $F^* \cap \left(\bigsqcup_{g \in O_2} R_g \right)$, and

$$\left| F^* \cap \bigsqcup_{g \in O_2} R_g \right| = \sum_{g \in O_2} |F^* \cap R_g|.$$

On one side $R_g = \{(x, \tau' g x) : x \in C\}$, hence $Q(R_g) = \{(u,u) : u \in E\}$; on the other side $Q(F^*) = \{(u, -u) : u \in E\}$. Therefore, a point in $F^* \cap R_g$ is mapped to a point $(u_0, u_0) \in E'$ with $2u_0 = 0$. There are four such points and by assumption none of them lies on the branch curves of Q. Then, for each choice of such a u_0, there are exactly $|G^0|^2$ points in $F^* Q^{-1}(u_0, u_0)$, but only $|G^0|$ of them lie on R_g, because once we have fixed the first coordinate x, the second is forced to be $(\tau' g)x$.

We get that $\eta|_{F^*}$ is ramified in $\sum_{g \in O_2} |F^* \cap R_g| = |O_2| \cdot 4 \cdot |G^0|$ points, and each one of them has ramification index 2, thus we have $4 \cdot |O_2|$ branching points on F. Finally, by Hurwitz's Formula (Theorem 1.17),

$$e(C) \cdot |G| = e(F^*) = |G| \cdot \left(e(F) - \left(\frac{4 \cdot |O_2|)}{2} \right) \right) = |G| \cdot (e(F) - 2 \cdot |O_2|).$$

By Lemma 4.31, F is the disjoint union of $|G^0|^2/l$ curves of genus g_{alb}, therefore

$$2 - 2g(C) + 2 \cdot |O_2| = e(F) = \frac{|G^0|^2}{l}(2 - 2g_{alb}),$$

whence the thesis. □

4.5 The fundamental group of a Semi-isogenous Mixed Surface

In this section we show how to compute the fundamental group of a semi-isogenous mixed surface. The strategy we follow is the one described in [Fra13] and it holds for general mixed surfaces.

The key result we use for this computation is the following.

Theorem 4.33 (cf. [Arm68, Theorem]). *Let G be a discontinuous group of homeomorphisms of a path connected, simply connected, locally compact metric space X, and let H be the normal subgroup of the elements of G having some fixed points. Then $\pi_1(X/G) \cong G/H$.*

Let $X := (C \times C)/G$ be a semi-isogenous mixed surface and let $\psi\colon \mathbb{T}(g'; m_1, \dots, m_r) \to G$ be the appropriate orbifold homomorphism associated to the G^0-covering $C \to C/G^0$. According to Lemma 1.36, $\ker(\psi) \cong \pi_1(C)$ and, as we showed in Lemma 1.37, the action of $\pi_1(C)$ on the universal covering Δ of C can be extended to a faithful discontinuous action of $\mathbb{T} := \mathbb{T}(g'; m_1, \dots, m_r)$.

The covering map $u\colon \Delta \to C$ is $\psi-$equivariant and $C/G^0 \cong \Delta/\mathbb{T}$.

Fix $\tau' \in G \setminus G^0$, let $(\tau) := (\tau')^2 \in G^0$ and $\varphi \in \mathrm{Aut}(G^0)$ defined by $\varphi(h) := \tau' h \tau'^{-1}$.

Let $\mathbb{H} := \{(t_1, t_2) \in \mathbb{T} \times \mathbb{T} | \psi(t_1) = \varphi^{-1}\} < \mathrm{Aut}(\Delta \times \Delta)$. Since ψ is surjective and $\varphi(\tau) = \tau$, there exists $t \in \mathbb{T}$ such that $\tilde{\tau} := (t, t) \in \mathbb{H}$. We define the automorphism

$$\begin{array}{rccc} \tilde{\tau}'\colon & \Delta \times \Delta & \longrightarrow & \Delta \times \Delta \\ & (x, y) & \longmapsto & (y, t \cdot x) \end{array}$$

This map satisfies $(\tilde{\tau}')^2 = \tilde{\tau}$. We also define the map $\tilde{\varphi}\colon \mathbb{H} \to \mathbb{H}$ as the conjugation by $\tilde{\tau}'$: $\tilde{\varphi}(t_1, t_2) = (t_2, t \cdot t_1 \cdot t^{-1})$.

Let $\mathbb{H} = \langle \mathrm{gen}(\mathbb{H}) | \mathrm{rel}(\mathbb{H}) \rangle$ be a presentation of H; we define

$$\mathrm{REL} := \{\tilde{\varphi}(h) \tilde{\tau}' h^{-1} \tilde{\tau}'^{-1} | h \in \mathrm{gen}(\mathbb{H})\}.$$

Let $\mathbb{G}$ the subgroup of $\mathrm{Aut}(\Delta \times \Delta)$ generated by $\mathbb{H}$ and $\tilde{\tau}'$; a presentation of $\mathbb{G}$ is given by

$$\mathbb{G} := \langle \mathrm{gen}(\mathbb{H}), \tilde{\tau}' | \mathrm{rel}(\mathbb{H}), (\tilde{\tau}')^2 \tilde{\tau}^{-1}, \mathrm{REL} \rangle.$$

We note that $\mathbb{H}$ is an index 2 subgroup of $\mathbb{G}$, and there is a natural $\mathbb{G}-$action on $\Delta \times \Delta$ given by

$$\begin{aligned} (h_1, h_2) \cdot (x, y) &= (h_1 \cdot x, h_2 \cdot y) \\ \tilde{\tau}'(h_1, h_2)(x, y) &= (h_2 \cdot y, (t \cdot h_1) \cdot x) \end{aligned} \qquad \text{for } (h_1, h_2) \in \mathbb{H}.$$

We define the homomorphism $\vartheta\colon \mathbb{G} \to G$ as

$$\begin{aligned} \vartheta(h_1, h_2) &= \psi(h_1) = \varphi^{-1}(\psi(h_2)) \\ \vartheta(\tilde{\tau}'(h_1, h_2)) &= \tau' \psi(h_1) = \tau' \varphi^{-1}(\psi(h_2)) \end{aligned} \qquad \text{for } (h_1, h_2) \in \mathbb{H}.$$

Let $\mathcal{U} := (u, u)\colon \Delta \times \Delta \to C \times C$; then for all $p \in \Delta \times \Delta$ and for all $h \in \mathbb{G}$, $\mathcal{U}(h \cdot p) = \vartheta(h) \cdot \mathcal{U}(p)$, hence

$$\frac{\Delta \times \Delta}{\mathbb{G}} \cong \frac{C \times C}{G}.$$

Remark 4.34. The $\pi_1(C \times C)-$ action on $\Delta \times \Delta$ is free, hence for all $x \in \Delta \times \Delta$ $\pi_1(C \times C) \cap \mathrm{Stab}_{\mathbb{G}}(x) = \{1\}$. This means that, for $x \in \Delta \times \Delta$, the restriction of ϑ to $\mathrm{Stab}_{\mathbb{G}}(x)$ gives an isomorphism with $\mathrm{Stab}_G(\mathcal{U}(x))$.

Lemma 4.35 (cf. [Fra13, Lemma 4.3])**.** *The $\mathbb{G}-$action on $\Delta \times \Delta$ is discontinuous, which means:*

i. The stabilizer of each point is finite;

ii. For all $x \in \Delta \times \Delta$ there exists a neighbourhood U of x such that for all $\tilde{g} \in \mathbb{G} \setminus \mathrm{Stab}(x)$, $\tilde{g}(U) \cap U = \emptyset$.

Proof. i. By Remark 4.34, the restriction of ϑ to the stabilizer of $x \in \Delta \times \Delta$ gives an isomorphism with $\mathrm{Stab}(\mathcal{U}(x))$, which is finite because G is finite.

ii. Let $x \in \Delta \times \Delta$ and let $y := \mathcal{U}(x) \in C \times C$; since G is finite and $C \times C$ is Hausdorff, there exists a neighbourhood U' of y such that $g(U') \cap U' = \emptyset$ for all $g \in G \setminus \mathrm{Stab}_G(y)$. Let V' be the connected component of $\mathcal{U}^{-1}(U')$ containing x. There exists a connected neighbourhood $V \subseteq V'$ of x such that V is mapped isomorphically by $\mathcal{U}$ onto its image, $\mathcal{U}(V) =: U \subset U'$ is $\mathrm{Stab}_G(y)-$invariant and V is $\mathrm{Stab}_{\mathbb{G}}(x)-$invariant. Let $\bar{g} \in \mathbb{G} \setminus \mathrm{Stab}(x)$, then we claim that $\bar{g}(V) \cap V = \emptyset$:

$$\mathcal{U}(\bar{g}(V) \cap V) \subseteq \mathcal{U}(\bar{g}(V)) \cap \mathcal{U}(V) = \vartheta(\bar{g})(U) \cap U,$$

then either $\bar{g}(V) \cap V = \emptyset$ of $\vartheta(\bar{g}) \in \mathrm{Stab}_G(y)$. In this latter case, by Remark 4.34, there exists a unique $\bar{g}' \in \mathrm{Stab}(x)$ such that $\vartheta(\bar{g}') = \vartheta(\bar{g})$, so $\bar{g} = k\bar{g}'$ with $k \in \pi_1(C \times C) \setminus \{1\}$, thus:

$$\bar{g}(V) \cap V = k\bar{g}'(V) \cap V = k(V) \cap V = \emptyset.$$

□

By Lemma 4.35 the hypothesis of Theorem 4.33 hold, hence we have the following.

Theorem 4.36. *Let $X = (C \times C)/G$ be a semi-isogenous mixed surface. Then*

$$\pi_1(X) \cong \frac{\mathbb{G}}{\mathbb{G}'},$$

where $\mathbb{G}'$ is the normal subgroup of $\mathbb{G}$ having some fixed points on $\Delta \times \Delta$.

Theorem 4.36 can be stated in another form, which will be more useful under a *computational* point of view.

Lemma 4.37 (cf. [Fra13, Lemma 4.5]). *Under the hypothesis of Theorem 4.36, the normal subgroup* $\mathbb{G}'$ *is exactly the subgroup* $\mathrm{Tors}(\mathbb{G})$*, that is the normal subgroup of* $\mathbb{G}$ *of those elements with finite order.*

Proof. We need to show that every element $g \in \mathbb{G}$ that fixes some points in $\Delta \times \Delta$ has finite order, and vice versa. Let us distinguish two cases:

i. Let $g = (h_1, h_2) \in \mathbb{H}$ and let $(x, y) \in \Delta \times \Delta$, then

$$(h_1, h_2)(x, y) = (x, y) \Leftrightarrow \begin{cases} h_1 = \alpha c_i^{m_i} \alpha^{-1} \\ h_2 = \beta c_j^{m_j} \beta^{-1} \end{cases} \Leftrightarrow (h_1, h_2) \text{ has finite order;}$$

the first equivalence follows by the proof of Theorem 1.23, while for the second equivalence we refer to [Bea83a, Theorem 10.3.2].

ii. Let $g = \tilde{\tau}'(h_1, h_2) \in \mathbb{G} \setminus \mathbb{H}$. If g fixes $(x, y) \in \Delta \times \Delta$, then $g^2 \in \mathbb{H}$ fixes (x, y) as well, hence, by point [i.], g has finite order. Conversely, if g has finite order, then $g^2 \in \mathbb{H}$ has finite order, thus by point [i.] $g^2(x, y) = (x, y)$ for some $(x, y) \in \Delta \times \Delta$. Therefore $g(x, (h_2)^{-1}x) = (x, (h_2)^{-1}x)$.

□

The last result we see in this section will be useful for the development of the classification algorithm, since it provides a recipe to find a finite set of generators of $\mathbb{G}'$ for a semi-isogenous mixed surface.

Proposition 4.38. *Let* $X = (C \times C)/G$ *be a semi-isogenous mixed surface. Then* $\mathbb{G}'$ *is normally generated by the finite set* $\mathcal{N}$ *defined as follows: for each element* $h \in O_2$*, we choose an element* $h_1 \in \psi^{-1}(h)$ *and we include in* $\mathcal{N}$ *the element* $\tilde{\tau}'(h_1, (t \cdot h_1)^{-1}) \in \mathbb{G}$.

Proof. Let $(h_1, h_2) \in \mathbb{H}$ and assume that it fixed the point $(x, y) \in \Delta \times \Delta$, i.e. $(h_1, h_2)(x, y) = (x, y)$. The map $\mathcal{U}$ is $\vartheta-$equivariant, hence $\theta(h_1, h_2) \in G^0$ fixed the point $\mathcal{U}(x, y) \in C \times C$, but G^0 acts freely, so $(h_1, h_2) \in \ker \vartheta = \pi_1(C \times C)$. The group $\pi_1(C \times C)$ acts freely on $\Delta \times \Delta$, hence (h_1, h_2) is trivial; in other words, $\mathbb{H}$ acts freely on $\Delta \times \Delta$.

Let $g := \tilde{\tau}'(h_1, h_2) \in \mathbb{G} \setminus \mathbb{H}$ and assume that it fixes the point $(x, y) \in \Delta \times \Delta$. Then $g^2 \in \mathbb{H}$ fixes (x, y) as well, hence g has order 2 (i.e. $h_2 = (t \cdot h_1)^{-1}$). Conversely each element $\tilde{\tau}'(h_1, (t \cdot h_1)^{-1}) \in \mathbb{G} \setminus \mathbb{H}$ fixes point-wise the curve $\{(x, (t \cdot h_1)x) : x \in \Delta\}$.

Each element $g \in \mathbb{G} \setminus \mathbb{H}$ of order 2 maps, via ϑ, to an element of order 2 in $G \setminus G^0$. If two such elements map via ϑ to the same element, then they are conjugated in $\mathbb{G}$. Indeed, let $\tilde{\tau}'(h_1, (t \cdot h_1)^{-1})$ and $\tilde{\tau}'(s_1, (t \cdot s_1)^{-1})$ be such that $\psi(h_1) = \psi(s_1)$, then there exists $k \in \ker \psi = \pi_1(C)$ such that $h_1 = s_1 \cdot k$. We have the following equalities:

$$\begin{aligned} \tilde{\tau}'(h_1, (t \cdot h_1)^{-1}) &= \tilde{\tau}'(s_1 \cdot k, k^{-1} s_1^{-1} t^{-1}) \\ &= \tilde{\tau}'(1, k^{-1}) \cdot (s_1, (t \cdot s_1)^{-1}) \cdot (k, 1) \\ &= \tilde{\varphi}(1, k^{-1}) \cdot \tilde{\tau}'(s_1, (t \cdot s_1)^{-1}) \cdot (k, 1) \\ &= (k, 1)^{-1} \cdot \tilde{\tau}'(s_1, (t \cdot s_1)^{-1}) \cdot (k, 1) \end{aligned}$$

and we are done, since $(k, 1) \in \mathbb{H}$. $\square$

4.6 The classification of Semi-isogenous Mixed Surfaces with $\chi = 1$

In this section we give an algorithm to classify semi-isogenous mixed surfaces $X = (C \times C)/G$ with $g(C) \geq 2$ and fixed values of the invariants K_X^2, $p_g(X)$ and $q(X)$.

4.6.1 Finiteness of the classification

First of all, in this section we will prove that for fixed values of K_X^2, $p_g(X)$ and $q(X)$ the classification problem is finite.

Let $X = (C \times C)/G$ be a semi-isogenous mixed surface with $g(C) \geq 2$ and let $(q; m_1, \ldots, m_r)$ be the type of an induced generating vector for G^0. We define the following rational numbers:

$$\Theta := 2q(X) - 2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i}, \qquad \beta := \frac{2(10\chi(\mathcal{O}_X) - K_X^2)}{\Theta}.$$

By Remark 4.26 and Remark 4.27 we get

$$|G^0| = \frac{(g(C) - 1)^2}{10\chi(\mathcal{O}_X) - K_X^2}, \qquad |O_2| = \frac{8\chi(\mathcal{O}_X) - K_X^2}{10\chi(\mathcal{O}_X) - K_X^2}(g(C) - 1). \tag{4.7}$$

Proposition 4.39. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface and let $(q; m_1, \ldots, m_r)$ be the type of an induced generating vector for G^0. Then*

(a) $\Theta > 0$ *and* $\beta = g(C) - 1$;

(b) $r \leq \dfrac{4(10\chi(\mathcal{O}_X) - K_X^2)}{\beta} + 4(1 - q)$;

(c) each m_i divides β;

(d) $m_i \leq \dfrac{1+2(10\chi(\mathcal{O}_X)-K_X^2)}{M}$, *where* $M := \max\left\{\dfrac{1}{6}, \dfrac{r-3+4q}{2}\right\}$.

Proof. (a) Since, by Proposition 4.21, $q(X) = g(C/G^0)$, by Riemann-Hurwitz formula (1.2):

$$2(g(C)-1) = |G^0| \cdot \Theta,$$

hence $\Theta = \frac{2(g(C)-1)}{|G^0|} > 0$ since $g(C) \geq 2$. The equation $\beta = g(C) - 1$ follows now from equation 4.7.

(b) By definition, $\Theta \geq 2q - 2 + r/2$, whence $r \leq 2\Theta + 4(1-q)$.

(c) By Theorem 1.23, there exists an element $h \in G^0$ of order m_i such that $h \cdot x = x$ for some $x \in C$. Since $h \in G^0$ does not fix any point in $C \times C$, it holds $\varphi(h) \cdot y \neq y$ for all $y \in C$. Thus the map $C \to C/\langle\varphi(g)\rangle := \tilde{C}$ is étale of degree $\operatorname{ord}(\varphi(h)) = m_i$. By Riemann-Hurwitz formula (1.2) $2(g(C)-1) = 2m_i(g(\tilde{C})-1)$.

(d) (cf. [FP15, Proposition 5.4.e]) We first prove that

$$\Theta + \frac{1}{m_i} \geq M := \max\left\{\frac{1}{6}, \frac{r-3+4q}{2}\right\}.$$

Since $\Theta = 2q - 2 + r - \sum_{i=1}^{r} \dfrac{1}{m_j}$, we get

$$\Theta + \frac{1}{m_i} = 2q - 2 + r - \sum_{i \neq j} \frac{1}{m_j} \geq 2q - 2 + r - \frac{r-1}{2} = \frac{r-3+4q}{2}.$$

Since $\Theta > 0$, $\frac{r-3+4q}{2} \geq \frac{1}{2} \geq \frac{1}{6}$ unless $r = 3$ and $q = 0$. In this case $\Theta > 0$ implies that at most one m_i can be equal to 2. Hence also in this case $\Theta + \frac{1}{m_i} \geq 0 - 2 + 3 - \sum_{j \neq i} \frac{1}{m_j} \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Therefore $M \leq \Theta + \frac{1}{m_i}$.

By part (c) we have $\Theta m_i \leq \Theta\beta$, then by definition of β we get

$$M \cdot m_i \leq 1 + \Theta \cdot m_i \leq 1 + 2(10\chi(\mathcal{O}_X) - K_X^2).$$

□

Remark 4.40. We can bound the value of Θ from below:

$$\Theta_{min} = \begin{cases} 1/42 & \text{for } q = 0 \\ 1/2 & \text{for } q = 1 \\ 2q-2 & \text{for } q \geq 2 \end{cases}$$

The first follows from Corollary 1.18.

Suppose $q = 1$, then

$$\Theta = \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \geq \frac{m_1 - 1}{m_1} \geq \frac{1}{2}.$$

Finally, if $q \geq 2$, an for $G^0 = \{1\}$ we get $\Theta_{min} = 2q - 2$.

Therefore

$$\beta = \frac{2(10\chi(\mathcal{O}_X) - K_X^2)}{\Theta} \leq \frac{2(10\chi(\mathcal{O}_X) - K_X^2)}{\Theta_{min}}.$$

The value β is by definition at least 1; but, if we are interested only in surfaces of general type, by Proposition 4.39 and Proposition 4.12, we can assume $\beta \geq 2$.

Moreover, r corresponds to the number of branch points of the quotient map $C \to C/G^0$; by Hurwitz's formula

$$r \geq R_{min} = \begin{cases} 3 & \text{for } q = 0 \\ 1 & \text{for } q = 1 \\ 0 & \text{for } q \geq 2 \end{cases}$$

As Remark 4.11 states, in order to construct a semi-isogenous mixed surface, it is sufficient to provide a set of algebraic data.

Once we fix the three values K^2, p_g and q, the first steps that the classification algorithm performs are the following:

- By Remark 4.40, we get Θ_{min} and R_{min};
- Still by Remark 4.40, we compute the maximum value for β, β_{max};
- For every $\beta = \beta_{max}, \ldots, 1$, by point ($b$) of Proposition 4.39 we obtain R_{max} the maximum possible value for r, and by (4.7) we compute the order of the group G^0;
- For every $r = R_{min}, \ldots, R_{max}$, we get an upper bound for every m_i with $i = 1, \ldots, r$ according to point (d) of Proposition 4.39.

This shows that once we fix K^2, p_g and q, there exists only a finite number of integers such that fulfil Proposition 4.39 and Remark 4.40.

In order to make our algorithm more efficient, we can prove the following result, giving some restrictions for the m_i's.

Proposition 4.41. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface, let $(q; m_1, \dots, m_r)$ be the type of an induced generating vector for G^0 and suppose that $|O_2| > 0$. Then $m_i \leq |G^0|/|O_2|$ and m_i divides $|O_2|$ for all $i = 1, \dots, r$.*

Proof. Let $x_0 \in C$ and let $k := |\mathrm{Orb}(x_0)|$ be the cardinality of its orbit for the G^0-action on C. Let L be the curve $L := \{(x_0, y) : y \in C\}$, the $G-$invariant set $\hat{L} := \bigcup_{g\in G} gL$ is the union of $2k$ irreducible components, each one isomorphic to C: k disjoint *horizontal* copies of C and k disjoint *vertical* copies of C. Since the action of G^0 is free and the elements of $G\backslash G^0$ switch horizontal and vertical components of $\hat{L}$, a ramification points of $\eta|_{\hat{L}} \colon \hat{L} \to \eta(\hat{L})$ belongs to the set $S := \{(h_1 \cdot x_0, h_2 \cdot x_0) : h_1, h_2 \in G^0\}$ which has cardinality k^2. On the other side, the ramification locus of $\eta|_{\hat{L}}$ is

$$\hat{R} := \hat{L} \cap \left(\bigcup_{g \in O_2} R_g \right) = \{(h \cdot x_0, (\tau' g) \cdot h \cdot x_0) : h \in G^0,\ g \in O_2\},$$

and it has cardinality $k \cdot |O_2|$: once $h \cdot x_0$ is fixed (k choices), there are $|O_2|$ possibilities for g and each one gives a different point in $\hat{R}$ by Proposition 4.13. Thus $k \cdot |O_2| \leq k^2$.

By Proposition 1.35, for each $i = 1, \dots, r$ there exists $h_i \in G^0$ and $x_i \in C$ such that $\mathrm{Stab}(x_i) = \langle h_i \rangle$ and $\mathrm{ord}(h_i) = m_i$, for $k = |\mathrm{Orb}(x_i)| = |G^0|/m_i$. We get $|O_2| \leq |G^0|/m_i$.

Let $V := \{(x_i, (\tau' g) \cdot x_i) : g \in O_2\}$ be the set of ramification points of η lying on the vertical line $\{(x_i, y) : y \in C\}$. The group $\langle h_i \rangle$ acts faithfully and freely on V, indeed

$$h_i^{\alpha}(x_i, (\tau' g) \cdot x_i) = (x_i, \varphi(h_i^{\alpha}(\tau' g) \cdot x_i)) \neq (x_i, (\tau' g) \cdot x_i), \quad \text{for } \alpha \in \{1, \dots, m_i - 1\},$$

since G^0 acts freely on $C \times C$; whence m_i divides $|V| = |O_2|$. □

Remark 4.42. We shall remark that Proposition 4.41 shows more: $m_i \leq N_j := |\mathrm{Cl}(g_j)|$ for $i = 1, \dots, r$ and $g_j \in O_2$. Indeed, the points in V belong to m_i ramification curves and N_j is the number of ramification curves mapped onto the same branch curve.

4.6.2 Hurwitz moves

Let C' be a curve of genus g'; given two different generating vectors V_1 and V_2 of the same type $(g'; m_1, \dots, m_r)$ for the groups G_1^0 and G_2^0, index two subgroups of the group G, one may ask if they define two Galois coverings which are isomorphic.

The answer to this question is studied in detail in [Pen15]: the Galois coverings $C_1 \to C'$ and $C_2 \to C'$ induced respectively by V_1 and V_2 are equivalent if the two generating vectors belong to the same *Hurwitz equivalence class*.

In this section we will first see what are the *Hurwitz moves* and how they act on a generating vector, then we will define the Hurwitz equivalence classes. We refer to [BCG08, Sections 1-2] and [Pen15] for further details.

Let $T := (a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r)$ be a $(2g' + r)-$tuple of elements of G. Hurwitz moves are different according to the type of generating vector and they define a group that we will denote by $M_{g',[r]}$; we need to distinguish three different cases.

Definition 4.43. If $g' = 0$, then for $i = 1, \dots, r-1$ we define

$$\sigma_i := \begin{cases} h_i & \mapsto h_{i+1} \\ h_{i+1} & \mapsto h_{i+1}^{-1} h_i h_{i+1} \\ h_j & \mapsto h_j \quad \text{for } j \neq i, i+1 \end{cases}$$

and $M_{0,[r]} = \langle \sigma_i : i = 1, \dots, r-1 \rangle$.

Definition 4.44 (cf. [Pol08, Proposition 1.10])**.** If $g' = 1$ and $r = 1$, then we define

$$t_{\tilde{\delta}} := \begin{cases} a_1 & \mapsto a_1 \\ b_1 & \mapsto b_1 a_1 \\ h_1 & \mapsto h_1 \end{cases}, \qquad t_\delta := \begin{cases} a_1 & \mapsto a_1 b_1^{-1} \\ b_1 & \mapsto b_1 \\ h_1 & \mapsto h_1 \end{cases}$$

and $M_{1,[1]} := \langle t_{\tilde{\delta}}, t_\delta \rangle$.

Definition 4.45. If $g' \neq 0$ and $g' > 1$ or $r > 1$. We define the following maps:

$$t_{\delta_j} := \begin{cases} a_j & \mapsto a_j b_j^{-1} \\ a_i & \mapsto a_i \quad \text{for } i \neq j \\ b_i & \mapsto b_i \quad \text{for all } i \\ h_i & \mapsto h_i \quad \text{for all } i \end{cases} \qquad t_{\tilde{\delta}_j} := \begin{cases} a_i & \mapsto a_i \quad \text{for all } i \\ b_j & \mapsto b_j a_j \\ b_i & \mapsto b_i \quad \text{for } i \neq j \\ h_i & \mapsto h_i \quad \text{for all } i \end{cases}$$

$$t_{\sigma_l} := \begin{cases} a_i & \mapsto a_i \quad \text{for all } i \\ b_i & \mapsto b_i \quad \text{for all } i \\ h_l & \mapsto h_{l+1} \\ h_{l+1} & \mapsto h_{l+1}^{-1} h_l h_{l+1} \\ h_i & \mapsto h_i \quad \text{for } i \neq l, l+1 \end{cases} \qquad t_{\tau_k} := \begin{cases} a_k & \mapsto a_k \eta_k^{-1} \\ b_k & \mapsto \eta_k b_k \eta_k^{-1} \\ a_{k+1} & \mapsto \eta_k a_{k+1} \\ a_i & \mapsto a_i \quad \text{for } i \neq k, k+1 \\ b_i & \mapsto b_i \quad \text{for } i \neq k \\ h_i & \mapsto h_i \quad \text{for all } i \end{cases}$$

$$t_{\xi^1_{j,d}} := \begin{cases} a_j & \mapsto \chi_{j,d} a_j \\ a_i & \mapsto a_i \quad \text{for } i \neq j \\ b_i & \mapsto b_i \quad \text{for all } i \\ h_d & \mapsto \epsilon_{j,d} h_d \epsilon_{j,d}^{-1} \\ h_i & \mapsto h_i \quad \text{for } i \neq d \end{cases} \qquad t_{\xi^2_{j,d}} := \begin{cases} a_i & \mapsto a_i \quad \text{for all } i \\ b_j & \mapsto a_j^{-1} \chi_{j,d} a_j b_j \\ b_i & \mapsto b_i \quad \text{for } i \neq j \\ h_d & \mapsto \epsilon'_{j,d} h_d {\epsilon'}_{j,d}^{-1} \\ h_i & \mapsto h_i \quad \text{for } i \neq d \end{cases}$$

for $1 \leq j \leq g'$,$1 \leq k \leq (g'-1)$, $1 \leq l \leq (r-1)$ and $1 \leq d \leq r$, where we set

- $\eta_k := b_k^{-1} a_{k+1} b_{k+1} a_{k+1}^{-1}$;
- $\chi_{j,d} := \left(\prod_{k=1}^{j-1} [a_k, b_k]\right)^{-1} h_d \prod_{k=1}^{j-1} [a_k, b_k]$;
- $\epsilon_{j,d} := h_d \left(\prod_{k=1}^{j-1} [a_k, b_k]\right) a_j b_j a_j^{-1} \left(\prod_{k=1}^{j-1} [a_k, b_k]\right)^{-1}$;
- $\epsilon'_{j,d} := h_d \left(\prod_{k=1}^{j} [a_k, b_k]\right) a_j^{-1} \left(\prod_{k=1}^{j-1} [a_k, b_k]\right)^{-1}$.

Finally, let $M_{g',[r]} := \langle t_{\delta_j}, t_{\tilde{\delta}_j}, t_{\sigma_l}, t_{\tau_k}, t_{\xi^1_{j,d}}, t_{\xi^2_{j,d}} \rangle$.

A direct computation shows that if T is a generating vector of type $(g'; m_1, \ldots, m_r)$, then its image under the maps defined in previous definition is still a generating vector of the same type. Also the automorphism group $\mathrm{Aut}(G)$ of G acts on the set of generating vector by simultaneous application of an automorphism to each one of its elements.

Given $(\gamma, \eta) \in M_{g',[r]} \times \mathrm{Aut}(G)$ and $T = (a_1, b_1, \ldots, a_{g'}, b_{g'}, h_1, \ldots, h_r)$ a generating vector of type $(g'; m_1, \ldots, m_r)$ for G^0 subgroup of index 2 of G, we define

$$(\gamma, \eta) \cdot (G^0, T) := (\eta(G^0), \eta(\gamma(T))).$$

The orbits of this action are called *Hurwitz equivalence classes* of generating vectors, and each one of them correspond to a deformation class of the surfaces we are studying.

4.6.3 Classification of Semi-isogenous Mixed Surfaces with $\chi = 1$

As done in [BP12], [Fra13], [FP15], we developed a MAGMA [BCP97] algorithm which computes the semi-isogenous mixed surfaces with fixed values of invariants p_g, q, and K^2.

The step the algorithm performs are the following.

Once the values K^2, p_g and q are fixed, by Proposition 4.39 and Remark 4.40, we have only finitely many possible types for the generating vector V associated with the G^0 on the curve C. Then we produce the finite list of all types $(q; m_1, \ldots, m_r)$ respecting the conditions in Proposition 4.39 and Proposition 4.41.

For each type, the orders of G and G^0 are computed by $|G| = 2|G^0| = 4(10\chi - K^2)/\Theta^2$.

Then the script searches among the finitely many groups of order $|G^0|$, the group having a disjoint pair of generating vector of the prescribed type. For these groups, the script checks their degree 2 extensions and discards the ones that have the wrong number of elements of order 2 and/or do not satisfy the condition of Proposition 4.41.

We get a list of quadruples (type, G^0, generating vector, extension G) and each quadruple gives a family of mixed quotients, as explained in Remark 4.11, and all semi-isogenous mixed surfaces with the prescribed invariants are here. Anyway, in the list there might be surfaces whose branch locus does not correspond to the expected one, then the script discards them.

Moreover, as stated in Subsection 4.6.2, different generating vectors give deformation equivalent surfaces if they differ by some Hurwitz moves. The script computes this action on the generating vectors, and returns only a representative for each orbit. Finally, the script computes the fundamental groups of the resulting surfaces, and, if $q = 1$, the genus of the Albanese fibre too.

Remark 4.46. The algorithm works for arbitrary values of the invariants K^2, p_g and q, but the implemented MAGMA version has some technical limitations. To perform the algorithm, we have to run over all groups of a given order. Here we have to use the database of Small Groups, which contains:

- all groups of order up to 2000, excluding groups of order 1024;
- the groups whose order is a product of at most 3 primes;
- the groups of order dividing p^6 for p prime;
- the groups of order $p^n q$, where p^n is a prime-power dividing 2^8, 3^6, 5^5 or 7^4 and q is a prime different from p.

In the other cases we cannot run among the groups of prescribed order and the script returns the list of skipped cases, which have to be studied separately.

Let $X = (C \times C)/G$ be a semi-isogenous mixed surface with $\chi(\mathcal{O}_X) = 1$ and $K_X^2 > 0$, then by Remark 4.26, $K_X^2 \leq 8\chi(\mathcal{O}_X)$ so the possible values of K_X^2 are in $\{1, \ldots, 8\}$.

We ran the program for $1 \leq K_X^2 \leq 8$ and $0 \leq p_g = q \leq 4$. As expected by the classification results we mentioned in Section 2.8, for $p_g = q = 4$ the output is empty, while for $p_g = q = 3$ we have only one family: it has $K^2 = 6$ and it is the family of the symmetric products of curves of genus 3, which forms an irreducible connected component of dimension 6 of the moduli space of minimal surfaces of general type.

For $0 \leq p_g = q \leq 2$ we get the following theorems.

Theorem 4.47. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface with $p_g(X) = q(X) = 0$ and $K_X^2 > 0$, such that $|G^0| \leq 2000$ and $|G^0| \neq 1024$. Then X belongs to one of the 15 families collected in Table 4.2 (page 77) and it is of general type.*

Theorem 4.48. *The semi-isogenous mixed surfaces $X := (C \times C)/G$ with $p_g = q = 1$ and $K_X^2 > 0$ form the 35 families collected in Table 4.3 and Table 4.4 (pages 79, 81). In all cases X is of general type.*

Theorem 4.49. *The semi-isogenous mixed surfaces $X := (C \times C)/G$ with $p_g = q = 2$ and $K_X^2 > 0$ form the 9 families collected in Table 4.5 (page 83). In all cases X is of general type.*

According to Table 2.1, a surface with $K^2 > 0$ is either of general type or rational, therefore regular and simply connected: an inspection of Table 4.2, Table 4.3, Table 4.4 and Table 4.5 shows that this latter case does not occur, so all the surfaces listed are of general type.

In Tables 4.2, 4.3, 4.4 and 4.5, every row corresponds to a family and we use the following notation: columns $Id(G)$ and $Id(G^0)$ report the MAGMA identifier of the groups G and G^0: the pair (a, b) denotes the b^{th} group of order a in the database of Small Groups.

In columns G and G^0 we denote by $\mathbb{Z}_n$ the cyclic group of order n, by S_n the symmetric group on n letters, by Q the group of quaternions, by D_n the dihedral group of order $2n$, and by $D_{p,q,r} := \langle x, y \mid x^p = y^q = 1, xyx^{-1} = y^r \rangle$. The groups $(258, 3678)$ and $(258, 3679)$ do not have a representation as semidirect product of non trivial groups of smaller order, so we leave the relative spots blank.

The column Type gives the type of the generating vector for G^0 is a short way, e.g. $[0; 2^5]$ stands for $(0; 2, 2, 2, 2, 2)$. The Branch Locus B of $\eta \colon C \times C \to X$ is also given in a short way, e.g. $(3, -8)^2, (2, -4)^2$ means that B consists of 4 curves, two of genus 2 and self-intersection -4 and two of genus 3 and self-intersection -8. The last column "min?" report (if known) whether the surface X is minimal or not. These results will be proved in the following chapter. In Table 4.3 and Table 4.4, we also report the genus g_{alb} of a general fibre of the Albanese map.

The semi-isogenous mixed surfaces with $K_X^2 = 8\chi(X)$ are those for which the action is free; indeed all the examples with $K^2 = 8$ in Tables 4.2, 4.3, 4.4 and 4.5 appeared in the papers already cited in previous chapter (see, for example, [BCG08], [CP09], [Pen11]).

In Table 4.2, there is a surface with $K^2 = 6$ and $G = (32, 43)$. It realizes a new topological type of surface of general type with $p_g = 0$, indeed its fundamental group is different from those present in literature (see [BCGP12, BCF15, Ino94, Kul04]). To the best of our knowledge the surfaces with $K^2 = 6$ and $H_1 = \mathbb{Z}_7^2$ or $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_4^2$ provide the first examples of minimal regular surfaces of general type with such invariants, and so realize at least other two new topological types.

Also the examples in Table 4.3 with $K^2 = 6, 7$ may be, to the best of our knowledge, new, although other surfaces with these invariants have been already constructed (see [BCF15, Pol09, MP10, Rit07, Rit10a, Rit10b, Rit15]).

Finally, we mention an example of a minimal surfaces of general type with $K^2 = 7$ and $p_g = q = 2$. The first example of a minimal surface of general type with these invariants appeared very recently ([Rit15]). By the recent paper ([PP16]) of Pignatelli and Polizzi, this surface is indeed different from Rito's one.

4.6.4 Skipped cases

As mentioned, the surfaces returned by the program may not be all semi-isogenous mixed surfaces with the required invariants, since the program is forced to skip some types, giving rise to groups of large order. The program returns the list of these skipped cases; the triples (type of generating vector, $|G^0|$, K_X^2) of these cases are listed in Table 4.1. For the cases $p_g = q \neq 0$ this list is empty.

	type	$\lvert G^0\rvert$	K_S^2
	2, 3, 7	21168	7
	2, 3, 8	6912	7
$\star$	2, 4, 5	4800	7
	2, 3, 9	3888	7
	2, 3, 10	2700	7
	2, 3, 7	28224	6
$\star$	2, 3, 8	9216	6
$\star$	2, 4, 5	6400	6
	2, 3, 9	5184	6
$\star$	2, 3, 10	3600	6
	2, 3, 12	2304	6
	2, 4, 6	2304	6
	3, 3, 4	2304	6
	2, 4, 8	1024	6
	2, 3, 7	35280	5
$\star$	2, 3, 8	11520	5
$\star$	2, 4, 5	8000	5
	2, 3, 9	6480	5
$\star$	2, 3, 10	4500	5
	2, 3, 12	2880	5
	2, 4, 6	2880	5
	3, 3, 4	2880	5

	type	$\lvert G^0\rvert$	K_S^2
	2, 3, 7	42336	4
	2, 3, 8	13824	4
	2, 4, 5	9600	4
$\star$	2, 3, 9	7776	4
	2, 3, 10	5400	4
	2, 3, 12	3456	4
	2, 4, 6	3456	4
	3, 3, 4	3456	4
	2, 3, 14	2646	4
	2, 5, 5	2400	4
	2, 3, 7	49392	3
	2, 3, 8	16128	3
$\star$	2, 4, 5	11200	3
$\star$	2, 3, 9	9072	3
$\star$	2, 3, 10	6300	3
	2, 3, 12	4032	3
	2, 4, 6	4032	3
	3, 3, 4	4032	3
$\star$	2, 5, 5	2800	3
	2, 3, 18	2268	3

	type	$\lvert G^0\rvert$	K_S^2
	2, 3, 7	56448	2
	2, 3, 8	18432	2
	2, 4, 5	12800	2
	2, 3, 9	10368	2
	2, 3, 10	7200	2
	2, 3, 12	4608	2
	2, 4, 6	4608	2
	3, 3, 4	4608	2
	2, 3, 14	3528	2
	2, 5, 5	3200	2
	2, 3, 18	2592	2
	2, 4, 8	2048	2
	2, 3, 7	63504	1
	2, 3, 8	20736	1
	2, 4, 5	14400	1
	2, 3, 9	11664	1
$\star$	2, 3, 10	8100	1
	2, 3, 12	5184	1
	2, 4, 6	5184	1
	3, 3, 4	5184	1
	2, 5, 5	3600	1
	2, 3, 18	2916	1
	2, 4, 8	2304	1
	3, 3, 5	2025	1

Table 4.1: The skipped cases for $p_g = q = 0$ and $K^2 > 0$

According to Conder, that listed in [Con12] the largest order of a group of automorphisms of a compact Riemann surface of given genus g for $2 \leq g \leq 301$, cases marked by $\star$ in Table 4.1 cannot occur.

4.6.5 Semi-isogenous Mixed Surfaces with $g(C) \leq 2$: a non general type case

By Proposition 4.12, if $X := (C \times C)/G$ is a semi-isogenous mixed surface of general type, then $g(C) \geq 3$. One may ask what surfaces arise when we consider $g(C) \leq 2$. In this section we consider a curve C and a finite group G defining an action on $C \times C$ described by (3.2), such that the subgroup G^0 acts freely on $C \times C$. We shall remark that under these hypotheses all the results contained in Section 4.3 hold, since they do not depend on $g(C) \geq 2$, but on the action itself.

- Suppose $g(C) = 0$, that is $C \cong \mathbb{P}^1$. Since every automorphism of $\mathbb{P}^1$ has non empty fixed locus, G^0 has to be trivial and $G \cong \mathbb{Z}_2$ is generated by the involution that exchange the two coordinates. Hence X is the symmetric product $(\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2$.

- Suppose $g(C) = 1$. By Proposition 4.24, the surface $X = (C \times C)/G$ has invariants $K_X^2 = e(X) = \chi(X) = 0$. By Proposition 4.21, $q(X) = g(C/G^0) \leq 1$; if $q(X) = 0$, then $p_g = \chi(X) - 1 = -1$, therefore $g(C/G^0) = q(X) = 1$. The quotient map $C \times C \to Y := (C \times C)/G^0$ is étale, hence, arguing as in Proposition 3.21, $e(Y) = K_Y^2 = \chi(Y) = 0$. By Lemma 1.47, $\kappa(Y) = \kappa(C \times C) = 0$ and $q(Y) = 2g(C/G^0) = 2$. Since Y is projective, according to Enriques-Kodaira classification (cf. Table 2.1) and by Remark 2.47, Y is an abelian surface. Looking at the fixed locus of the involution $\iota\colon Y \to Y$ we distinguish two cases:

  (a) The fixed locus is empty, then $\pi\colon Y \to X$ is étale and, according to Table 2.1, X is a bi-elliptic surface.

  (b) The fixed locus is non empty, then (cf. [Kat87, Lemma 2.3, Lemma 2.6]) the ramification locus of $\pi\colon Y \to X$ is the union of non-singular elliptic curves and $\kappa(X) = -\infty$, whence, since X is projective, according to table 2.1 and Remark 2.47, X is a ruled surface of genus 1.

  Moreover, both cases (a) and (b) occur.

- Suppose $g(C) = 2$. Then, by Corollary 4.25

$$2\chi = \frac{1}{|G^0|} - \delta.$$

  By definition $\delta \in \mathbb{N}$ and $\chi \in \mathbb{Z}$, therefore $|G^0| = 1$. This means that $G \cong \mathbb{Z}_2 = \langle \tau' \rangle$ and $\tau'(x, y) = (y, x)$, hence $X \cong C^{(2)}$, which, as we said in the proof of Proposition 4.12, is the blow up of $J(C)$ in one point, so X is an abelian surface blown up in a point.

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	min?
8	$D_{2,8,5} \rtimes \mathbb{Z}_2^2$	64, 92	$\mathbb{Z}_2^2 \times D_4$	32, 46	9	$[0;2^5]$	$\emptyset$	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$	Yes
8		256, 3679	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	$[0;4^3]$	$\emptyset$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
8		256, 3678	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	$[0;4^3]$	$\emptyset$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
8		256, 3678	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	$[0;4^3]$	$\emptyset$	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	Yes
8		256, 3678	$(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	128, 36	17	$[0;4^3]$	$\emptyset$	$\mathbb{Z}_4^3$	Yes
6	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	32,43	$\mathbb{Z}_2 \times D_4$	16,11	9	$[0; 2^6]$	$(3, -8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
6	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	32,27	$\mathbb{Z}_2^4$	16,14	9	$[0; 2^6]$	$(3, -8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	Yes
6	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	32,27	$\mathbb{Z}_2^4$	16,14	9	$[0; 2^6]$	$(3, -8)$	$\mathbb{Z}_4^3$	Yes
6	$\mathbb{Z}_7 \times D_7$	98,3	$\mathbb{Z}_7^2$	49,2	15	$[0; 7^3]$	$(3, -8)$	$\mathbb{Z}_7^2$	Yes
6	$\mathbb{Z}_7 \times D_7$	98,3	$\mathbb{Z}_7^2$	49,2	15	$[0; 7^3]$	$(3, -8)$	$\mathbb{Z}_7^2$	Yes
6	$\mathbb{Z}_4^2 \rtimes D_4$	128, 734	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2$	64,211	17	$[0; 2^5]$	$(3, -8)$	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$	Yes
6	$(\mathbb{Z}_2^2 \times D_8) \rtimes \mathbb{Z}_2$	128, 750	$\mathbb{Z}_2^2 \times D_8$	64,250	17	$[0; 2^5]$	$(3, -8)$	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$	Yes
6	$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2^2$	128, 1797	$\mathbb{Z}_2^2 \times D_8$	64,250	17	$[0; 2^5]$	$(2, -4)^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$	Yes
2	$(\mathbb{Z}_2^3 \rtimes D_4) \rtimes \mathbb{Z}_2^2$	256, 47930	$\mathbb{Z}_2^4 \rtimes D_4$	128, 1135	33	$[0; 2^5]$	$(3, -8)^3$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	No
2	$(\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2) \rtimes \mathbb{Z}_2^2$	256, 45303	$\mathbb{Z}_2^4 \rtimes D_4$	128, 1135	33	$[0; 2^5]$	$(3, -8)^2, (2, -4)^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	No

Table 4.2: $p_g = q = 0,\ K^2 > 0$

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	g_{alb}	min?
8	$D_{2,8,5}$	16,6	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8,2	5	[1;2^2]	$\emptyset$	$\mathbb{Z}_4 \times \mathbb{Z}^2$	5	Yes
8	$D_{2,8,3}$	16,8	D_4	8,3	5	[1;2^2]	$\emptyset$	$\mathbb{Z}_4 \times \mathbb{Z}^2$	5	Yes
8	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	16,3	$\mathbb{Z}_2^3$	8,5	5	[1;2^2]	$\emptyset$	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$	5	Yes
7	$\mathbb{Z}_3 \rtimes D_4$	24,8	D_6	12,4	7	[1;2^2]	$(2,-4)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	5	Yes
7	$\mathbb{Z}_3 \times D_4$	24,10	$\mathbb{Z}_2 \times \mathbb{Z}_6$	12,5	7	[1;2^2]	$(2,-4)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	5	Yes
6	D_4	8,3	$\mathbb{Z}_2^2$	4,2	5	[1;2^4]	$(3,-8)$	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_3 \times S_3$	18,3	$\mathbb{Z}_3^2$	9,2	7	[1;3^2]	$(3,-8)$	$\mathbb{Z}_3 \times \mathbb{Z}^2$	4	Yes
6	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	32,43	$\mathbb{Z}_2 \times D_4$	16,11	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_2^2 \rtimes D_4$	32,28	$\mathbb{Z}_2 \times D_4$	16,11	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_2^2 \rtimes D_4$	32,28	$\mathbb{Z}_2 \times D_4$	16,11	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^3 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,11	$\mathbb{Z}_4^2$	16,2	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	Yes
6	$D_8 \rtimes \mathbb{Z}_2$	32,42	$D_4 \rtimes \mathbb{Z}_2$	16,13	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,31	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	16,3	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
6	$(\mathbb{Z}_2^2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	32,30	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	16,3	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_4 \times \mathbb{Z}^2$	3	Yes
6	$D_{2,8,5} \rtimes \mathbb{Z}_2$	32,38	$\mathbb{Z}_2 \times \mathbb{Z}_8$	16,5	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	Yes
6	$\mathbb{Z}_4 \times D_4$	32,25	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	16,10	9	[1;2^2]	$(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
6	$(\mathbb{Z}_2^2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$	32,30	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	16,10	9	[1;2^2]	$(3,-8)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	Yes
4	$S_3 \times D_4$	48,38	$\mathbb{Z}_2^2 \times S_3$	24,14	13	[1;2^2]	$(2,-4),(4,-12)$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	3	
4	$D_{12} \rtimes \mathbb{Z}_2$	48,37	$\mathbb{Z}_4 \times S_3$	24,5	13	[1;2^2]	$(2,-4),(4,-12)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	

Table 4.3: $p_g = q = 1,\ K^2 \geq 4$

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	g_{alb}	min?
2	$(\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2) \rtimes \mathbb{Z}_2$	64,153	$D_{2,8,5} \rtimes \mathbb{Z}_2$	32,7	17	$[1;2^2]$	$(3,-8),(5,-16)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$\mathbb{Z}_8 \rtimes D_4$	64,150	$D_4 \rtimes \mathbb{Z}_4$	32,9	17	$[1;2^2]$	$(3,-8),(5,-16)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$\mathbb{Z}_2^2 \rtimes D_8$	64,147	$D_4 \rtimes \mathbb{Z}_4$	32,9	17	$[1;2^2]$	$(2,-4)^2,(5,-16)$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$	64,128	$\mathbb{Z}_2 \times D_8$	32,39	17	$[1;2^2]$	$(2,-4)^2,(3,-8)^2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$Q \rtimes D_4$	64,130	$\mathbb{Z}_2 \times D_{2,8,3}$	32,40	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$D_4 \rtimes D_4$	64,134	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	32,43	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	3	No
2	$(\mathbb{Z}_2 \times D_4) \rtimes \mathbb{Z}_2^2$	64,227	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$	32,22	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_2 \times D_4) \rtimes \mathbb{Z}_2^2$	64,227	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$	32,22	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$\mathbb{Z}_4 \rtimes (D_4 \rtimes \mathbb{Z}_2)$	64,228	$(\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$	32,23	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_4 \times D_4) \rtimes \mathbb{Z}_2$	64,234	$(\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$	32,23	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_4 \times D_4) \rtimes \mathbb{Z}_2$	64,234	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,24	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_4 \rtimes Q) \rtimes \mathbb{Z}_2$	64,236	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	32,24	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2$	64,219	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_2^2 \rtimes D_4) \rtimes \mathbb{Z}_2$	64,221	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8)^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes D_4$	64,213	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8)^2,(2,-4)^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No
2	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2^2$	64,206	$\mathbb{Z}_4 \times D_4$	32,25	17	$[1;2^2]$	$(3,-8),(2,-4)^4$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	2	No

Table 4.4: $p_g = q = 1$, $0 < K^2 < 4$

K_X^2	G	$Id(G)$	G^0	$Id(G^0)$	$g(C)$	Type	Branch Locus B	$H_1(X,\mathbb{Z})$	min?
8	$\mathbb{Z}_4$	4,1	$\mathbb{Z}_2$	2,1	3	[2;-]	$\emptyset$	$\mathbb{Z}_2 \times \mathbb{Z}^4$	Yes
7	$\mathbb{Z}_6$	6,2	$\mathbb{Z}_3$	3,1	4	[2;-]	$(2,-4)$	$\mathbb{Z}^4$	Yes
6	D_4	8,3	$\mathbb{Z}_2^2$	4,2	5	[2;-]	$(3,-8)$	$\mathbb{Z}^4$	Yes
6	D_4	8,3	$\mathbb{Z}_2^2$	4,2	5	[2;-]	$(3,-8)$	$\mathbb{Z}_2 \times \mathbb{Z}^4$	Yes
6	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8,2	$\mathbb{Z}_4$	4,1	5	[2;-]	$(2,-4)^2$	$\mathbb{Z}^4$	Yes
4	D_6	12,4	S_3	6,1	7	[2;-]	$(2,-4),(4,-12)$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 5$
2	$\mathbb{Z}_2 \times D_4$	16,11	D_4	8,3	9	[2;-]	$(2,-4)^2,(3,-8)^2$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 4$
2	$\mathbb{Z}_2 \times D_4$	16,11	D_4	8,3	9	[2;-]	$(2,-4)^2,(3,-8)^2$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 4$
2	$D_4 \rtimes \mathbb{Z}_2$	16,13	Q	8,4	9	[2;-]	$(3,-8)^3$	$\mathbb{Z}^4 = \pi_1(X)$	No, $K^2_{X_{min}} = 4$

Table 4.5: $p_g = q = 2,\ K^2 > 0$

Chapter 5

Minimality of Semi-isogenous Mixed Surfaces

In this chapter we will study minimality of semi-isogenous mixed surfaces. We will address this problem in two different ways.

In the first section we will study the bicanonical system of a semi-isogenous mixed surface. As we saw in Section 2.7, this is strictly related to the minimality problem of a surface: if we have a surface of general type, the number of contractions of exceptional curves we need to perform to get the minimal model is equal to the dimension of the first cohomology group of its bicanonical system.

In the second section, thanks to the Hodge's Index Theorem, we will give a bound to $h^1(2K_X)$ for semi-isogenous mixed surfaces with $p_g = q$.

5.1 The bicanonical system of a Semi-isogenous Mixed Surface

In this section we will study the bicanonical system of a semi-isogenous mixed surface $X = (C \times C)/G$; namely we will prove that $H^1(2K_X)$ is isomorphic to the cokernel of the G^0 invariant part of the restriction map of the global sections of the bicanonical system of $C \times C$ to R. We will prove this result in three steps:

i. we find a relation between the bicanonical system of the semi-isogenous mixed surface X and the one of the isogenous surface $Y = (C \times C)/G^0$;

ii. thanks to a result by Beauville, we prove that we can relate the bicanonical system of Y to the one of $C \times C$;

iii. we prove some results that will allow us to displace the problem onto the curves C and R.

5.1.1 Displacing the problem onto Y

Lemma 5.1. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface, and let $\sigma \colon C \times C \to Y := (C \times C)/G^0$ be the étale covering determined by the diagonal action. Let $T := \sigma(R)$, then $h^2(2K_Y - T) = 0$ and*

$$\chi(\mathcal{O}_Y(2K_Y - T)) = \frac{9(g-1)^2}{|G^0|} - 7\delta(B).$$

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_Y(2K_Y - T) \to \mathcal{O}_Y(2K_Y) \to \mathcal{O}_T(2K_Y) \to 0$$

and the long exact sequence in cohomology it induces

$$\begin{array}{cccccccc} 0 & \to & H^0(2K_Y - T) & \to & H^0(2K_Y) & \to & H^0(2K_Y|_T) & \to \\ & \to & H^1(2K_Y - T) & \to & H^1(2K_Y) & \to & H^1(2K_Y|_T) & \to \\ & \to & H^2(2K_Y - T) & \to & H^2(2K_Y) & \to & 0 & . \end{array} \tag{5.1}$$

Y is surface isogenous to a product, then by Theorem 3.20 it is a minimal surface of general type, thus $H^1(2K_Y) = H^2(2K_Y) = 0$.

We claim that $H^1(2K_Y|_T) = 0$. Let us define $T_i := \sigma(R_i)$, then, if we denote by $B_i := \eta(R_i)$, by Proposition 4.17

$$g(T_i) = g(B_i) = \frac{2(g-1)}{|Z(\tau' g_i)|} + 1.$$

Let us compute $\deg(2K_Y|_{T_i}) = 2K_Y.T_i$: the map $\sigma : C \times C \to Y$ is étale and $\sigma^*(T_i) = R_{i,1} + \cdots + R_{i,N_i}$, with $N_i = |G^0|/|Z(\tau' g_i)|$, then

$$\begin{aligned} 2K_Y.T_i &= \frac{1}{|G^0|} \cdot 2K_{C\times C}.(R_{i,1} + \cdots + R_{i,N_i}) \\ &= \frac{2N_i}{|G^0|} \cdot 2K_{C\times C}.R_{i,1} \\ &= \frac{2N_i}{|G^0|} \cdot 8(g-1) = \frac{16(g-1)}{|Z(\tau' g_i)|} = 8(g(T_i) - 1). \end{aligned} \tag{5.2}$$

Since $\deg(2K_Y|_{T_i}) > 2(g(T_i)-1)$, $h^1(2K_Y|_{T_i}) = 0$, hence $h^1(2K_Y|_T) = \sum h^1(2K_Y|_{T_i}) = 0$. By exactness of the sequence (5.1), $H^2(2K_Y - T) = 0$.

By Theorem 2.19

$$\chi(\mathcal{O}_Y(2K_Y - T)) = \chi(\mathcal{O}_Y) + \frac{(2K_Y - T).(K_Y - T)}{2}$$
$$= \chi(\mathcal{O}_Y) + K_Y^2 + \frac{T^2 - 3K_Y.T}{2}. \tag{5.3}$$

Since T_i's are pairwise disjoint

$$T^2 - 3T.K_Y = \sum_{i=1}^{M} (T_i^2 - 3T_i.K_Y).$$

By Proposition 2.20 $T_i^2 + T_i.K_Y = 2(g(T_i) - 1)$, then by (5.2)

$$T_i^2 - 3T_i.K_Y = -14(g(T_i) - 1).$$

This means that $T^2 - 3T.K_Y = -14\delta(B)$. Plugging this value in (5.3) and exploiting Proposition 3.21, we get

$$\chi(\mathcal{O}_Y(2K_Y - T)) = \frac{(g-1)^2}{|G^0|} + \frac{8(g-1)^2}{|G^0|} - \frac{14\delta(B)}{2},$$

whence the thesis. □

Lemma 5.2. *Let X be a semi-isogenous mixed surface. Then $K_X + B$ is a big and nef divisor on X.*

Proof. Let us compute the self intersection of $K_X + B$.

$$(K_X + B)^2 = K_X^2 + 2K_X.B + B^2$$
$$= K_X^2 + 12\delta(B) - 4\delta(B)$$
$$= 64\chi(\mathcal{O}_X) - 7K_X^2,$$

where second and third equality hold by Remark 4.27 and Remark 4.26 respectively. Again, by Remark 4.27, $64\chi(\mathcal{O}_X) - 7K_X^2 = 4\chi(\mathcal{O}_Y) + 3\delta(B)$, which is strictly positive because $\delta(B) \geq 0$ and $\chi(\mathcal{O}_Y) > 0$ being Y of general type. Hence $(K_X + B)^2 > 0$.

Let $D \subset X$ be an irreducible curve. First, suppose $D \subset B$, then $D = B_i$ for some $i = 1, \ldots, M$. Then by the Genus Formula (2.4) $(K_X + B).D = (K_X + B_i).B_i = 2g(B_i) - 2 \geq 0$, where last inequality holds by Proposition 4.17. Now suppose that D is not contained in the branch locus, then by Proposition 2.10

$$(K_X + B).D = 1/2\pi^*(K_X + B).\pi^*(D)$$
$$= 1/2(K_Y + T).\pi^*(D).$$

$K_Y.\pi^*(D) \geq 0$ because Y is a minimal surface. Since D is irreducible and it is not contained in the branch locus of π, $\pi^*(D)$ has no component in T, thus $T.\pi^*(D) \geq 0$. By Theorem 2.15, $K_X + B$ is a big and nef divisor. □

Lemma 5.3. *Let X be a semi-isogenous mixed surface. Then*

$$h^0(2K_X + B) = \chi(\mathcal{O}_X(2K_X + B)) = \chi(\mathcal{O}_X) + K_X^2 + 7\delta(B).$$

Proof. By Lemma 5.2 and Theorem 2.17 $h^1(2K_X + B) = h^2(2K_X + B) = 0$, hence the first equality of the statement holds. By Theorem 2.19

$$\begin{aligned}\chi(\mathcal{O}_X(2K_X + B)) &= \chi(\mathcal{O}_X) + \frac{(2K_X + B).(K_X + B)}{2}\\ &= \chi(\mathcal{O}_X) + K_X^2 + \frac{3K_X.B + B^2}{2}\\ &= \chi(\mathcal{O}_X) + K_X^2 + 7\delta(B),\end{aligned}$$

where last equality holds for Remark 4.27. □

Proposition 5.4. *Under the hypotheses of Lemma 5.1*

$$h^1(2K_X) = h^1(2K_Y - T) \quad \textit{and} \quad h^2(2K_X) = 0.$$

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_X(2K_X) \to \mathcal{O}_X(2K_X + B) \to \mathcal{O}_B(2K_X + B) \to 0$$

and the exact sequence in cohomology it induces

$$\begin{array}{cccccccc} 0 & \to & H^0(2K_X) & \to & H^0(2K_X + B) & \to & H^0(2K_X + B|_B) & \to \\ & \to & H^1(2K_X) & \to & H^1(2K_X + B) & \to & H^1(2K_X + B|_B) & \to \\ & \to & H^2(2K_X) & \to & H^2(2K_X + B) & \to & 0 & . \end{array} \tag{5.4}$$

By Lemma 5.2 and Theorem 2.17, $H^1(2K_X + B) = H^2(2K_X + B) = 0$.

Since by Proposition 4.17, B_i's are pairwise disjoint, $h^1(2K_X+B|_B) = \sum_i h^1(2K_X + B|B_i)$. By Remark 4.27, we get

$$(2K_X + B).B_i = (2K_X + B_i).B_i = 8(g(B_i) - 1) > 2(g(B_i) - 1),$$

where last inequality since $g(B_i) > 1$ by Proposition 4.17. Therefore by Corollary 1.40 $h^1(2K_X + B|_{B_i}) = 0$, hence $h^1(2K_X + B|_B) = 0$. By exactness of (5.4), this implies that $H^2(2K_X) = 0$.

By Lemma 4.23, there exists $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\pi_*(\mathcal{O}_Y) = \mathcal{O}_X \oplus \mathcal{L}^*$ and $2K_Y = \pi^*(\omega_X^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}) = \pi^*(2K_X + B)$, then by Projection Formula (2.1)

$$\begin{aligned}\pi_*(\omega_Y^{\otimes 2}) &= \pi_*(\pi^*(\omega_X^{\otimes 2} \otimes \mathcal{O}_X(B))) \\ &= \pi_*(\pi^*(\omega_X^{\otimes 2} \otimes \mathcal{O}_X(B)) \otimes \mathcal{O}_Y)) \\ &= (\omega_X^{\otimes 2} \otimes \mathcal{O}_X(B)) \otimes (\mathcal{O}_X \oplus \mathcal{L}^*) \\ &= (\omega_X^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}) \oplus (\omega_X^{\otimes 2} \otimes \mathcal{L}).\end{aligned}$$

Then $H^0(2K_Y) \cong H^0(2K_X + B) \oplus H^0(\omega_X^{\otimes 2} \otimes \mathcal{L})$. By Proposition 3.23

$$h^0(2K_Y) = \frac{9(g(C)-1)^2}{|G^0|},$$

then by Lemma 5.3

$$h^0(\Omega_X^{\otimes 2} \otimes \mathcal{L}) = \frac{9(g(C)-1)^2}{|G^0|} - (\chi(\mathcal{O}_X + K_X^2 + 7\delta(B)).$$

Moreover, by Lemma 4.23, $\mathcal{O}_Y(T) = \pi^*(\mathcal{L})$, then

$$\begin{aligned}\omega_Y^{\otimes 2} \otimes \mathcal{O}_Y(-T) &= \pi^*(\omega_X^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}) \otimes \pi^*(\mathcal{L}^*) \\ &= \pi^*(\omega_X^{\otimes 2} \otimes \mathcal{L}),\end{aligned}$$

then

$$\begin{aligned}\pi_*(\omega_Y^{\otimes 2} \otimes \mathcal{O}_Y(-T)) &= \pi_*(\pi^*(\omega_X^{\otimes 2} \otimes \mathcal{L}) \otimes \mathcal{O}_Y) \\ &= (\omega_X^{\otimes 2} \otimes \mathcal{L}) \otimes (\mathcal{O}_X \oplus \mathcal{L}^*) \\ &= (\omega_X^{\otimes 2} \otimes \mathcal{L}) \oplus (\omega_X^{\otimes 2}).\end{aligned}$$

Thus $h^0(2K_Y - T) = h^0(\omega_X^{\otimes 2} \otimes \mathcal{L}) + h^0(2K_X)$, that is

$$h^0(2K_X) = h^0(2K_Y - T) - \frac{9(g(C)-1)^2}{|G^0|} + \chi(\mathcal{O}_X) + K_X^2 + 7\delta(B),$$

or, equivalently, by Lemma 5.1

$$h^0(2K_X) = h^1(2K_Y - T) + K_X^2 + \chi(\mathcal{O}_X).$$

By Theorem 2.19 $\chi(2K_X) = \chi(\mathcal{O}_X) + K_X^2$, and since $h^2(2K_X) = 0$ we get the thesis. $\square$

Remark 5.5. If X is a semi-isogenous mixed surface, then by Proposition 5.4 $h^2(2K_X) = 0$. By Serre's duality (cf. Theorem 2.18) $h^0(-K_X) = 0$, i.e. the anticanonical system of X is empty; this means that X cannot be neither a Del Pezzo surface, nor a minimal K3 surface, nor a minimal abelian surface.

Remark 5.6. By Proposition 5.4, the exact sequence

$$0 \longrightarrow H^0(2K_Y - T) \longrightarrow H^0(2K_Y) \xrightarrow{\hat{\rho}} H^0(2K_Y|_T) \longrightarrow H^1(2K_Y - T) \longrightarrow 0;$$

tells us that $h^1(2K_X) = \dim \operatorname{coker}(\hat{\rho})$.

5.1.2 Displacing the problem onto $C \times C$

What we have proved so far is that, in order to understand the dimension of the space $H^1(2K_X)$, we can focus on the isogenous surface $Y = (C \times C)/G^0$. What we want to do now is to simplify further the problem displacing it onto the product of curves $C \times C$.

Let us first consider the exact sequence of sheaves on $C \times C$

$$0 \longrightarrow \mathcal{I}_R \omega_{C\times C}^{\otimes 2} \longrightarrow \omega_{C\times C}^{\otimes 2} \longrightarrow \omega_{C\times C}^{\otimes 2}|_R \longrightarrow 0,$$

where $\mathcal{I}_R$ denotes the ideal defining the ramification locus R and with $\omega_{C\times C}^{\otimes 2}|_R$ we denote the quotient sheaf $\omega_{C\times C}^{\otimes 2}/\mathcal{I}_R\omega_{C\times C}^{\otimes 2} \cong \omega_{C\times C}^{\otimes 2} \otimes_{\mathcal{O}_{C\times C}} \mathcal{O}_R$.

It induces a long exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow H^0(2K_{C\times C} - R) \xrightarrow{\iota} H^0(2K_{C\times C}) \xrightarrow{\rho} \\ \xrightarrow{\rho} H^0(2K_{C\times C}|_R) \longrightarrow H^1(2K_{C\times C} - R) \longrightarrow 0. \end{aligned} \tag{5.5}$$

We have a natural G^0-action on the sheaf $\omega_{C\times C}^{\otimes 2}$ given by pull-back: for $g \in G^0$

$$\begin{array}{cccc} g\colon & H^0(U, \omega_{C\times C}^{\otimes 2}) & \longrightarrow & H^0(g^{-1}(U), \omega_{C\times C}^{\otimes 2}) \\ & \omega & \longmapsto & g^*\omega \end{array}$$

Remark 5.7. The action defined above is a *right* action: if $g, h \in G^0$ and $\omega \in H^0(U, \omega_{C\times C}^{\otimes 2})$, then $(gh)\omega = (gh)^*\omega = h^*(g^*\omega) = h(g\omega)$.

Being R G^0-invariant, the G^0-action on $\omega_{C\times C}^{\otimes 2}$ restricts to a G^0-action on the sheaf $\mathcal{I}_R\omega_{C\times C}^{\otimes 2}$. Therefore, G acts on the quotient $\omega_{C\times C}^{\otimes 2}|_R$ as well.

All of these actions naturally descend to G^0-actions on the spaces of global sections with respect to which the maps ι and ρ displayed in (5.5) are G^0-equivariant. This means that from (5.5) we get an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(2K_{C\times C} - R)^{G^0} \xrightarrow{\iota_0} H^0(2K_{C\times C})^{G^0} \xrightarrow{\rho_0} \\ \xrightarrow{\rho_0} H^0(2K_{C\times C}|_R)^{G^0} \longrightarrow \operatorname{coker}(\rho_0) \longrightarrow 0, \end{aligned} \tag{5.6}$$

where ι_0 and ρ_0 denotes the restriction of the maps ι and ρ respectively to the G^0-invariant subspaces of their domains.

The key result that now allow us to displace the problem from the isogenous surface Y onto the surface $C \times C$ is Lemma 4.19.

Remark 5.8. The following commutative diagram

$$\begin{tikzcd} Y & T \arrow[l, hook', "j"'] \\ C \times C \arrow[u, "\sigma"] & R \arrow[l, hook', "i"'] \arrow[u, "\sigma|_R"'] \end{tikzcd}$$

induces the following commutative diagram

$$\begin{tikzcd} H^0(2K_Y) \arrow[r, "j^* \equiv \hat{\rho}"] \arrow[d, "\sigma^*"'] & H^0(2K_Y|_T) \arrow[d, "\sigma|_R^*"] \\ H^0(2K_{C\times C})^{G^0} \arrow[r, "i^* \equiv \rho_0"] & H^0(2K_{C\times C}|_R)^{G^0} \end{tikzcd} \tag{5.7}$$

By Lemma 4.19, σ^* is an isomorphism.

We will now prove some results regarding the sheaves $\omega_{C\times C}^{\otimes 2}$ and $\omega_{C\times C}^{\otimes 2}|_R$; this will allow us to translate the problem of computing $h^1(2K_X)$ into a problem concerning he product $C \times C$.

As we already said before, by Proposition 3.18, $\omega_{C\times C} \cong \pi_1^*\omega_C \otimes \pi_2^*\omega_C$. Therefore

Fact 5.9. $\omega_{C\times C}^{\otimes 2} \cong \pi_1^*\omega_C^{\otimes 2} \otimes \pi_2^*\omega_C^{\otimes 2}$.

Remark 5.10. The isomorphism

$$\begin{array}{cccc} \Phi\colon & H^0(2K_C) \otimes H^0(2K_C) & \to & H^0(2K_{C\times C}) \\ & \alpha \otimes \beta & \mapsto & \pi_1^*(\alpha) \otimes \pi_2^*(\beta) \end{array}$$

induces a G^0-action on $H^0(2K_C) \otimes H^0(2K_C)$. Let $\alpha \otimes \beta \in H^0(2K_C) \otimes H^0(2K_C)$ and $h \in G^0$, then

$$\begin{aligned} h\Phi(\alpha \otimes \beta) &= h(\pi_1^*\alpha \otimes \pi_2^*\beta) = (h\pi_1^*\alpha) \otimes (h\pi_2^*\beta) \\ &= (h^*\pi_1^*\alpha) \otimes (h^*\pi_2^*\beta) \\ &= (\pi_1 \circ h)^*\alpha \otimes (\pi_2 \circ h)^*\beta \\ &= (h \circ \pi_1)^*\alpha \otimes (\varphi(h) \circ \pi_2)^*\beta \\ &= \pi_1^*(h\alpha) \otimes \pi_2^*(\varphi(h)\beta). \end{aligned}$$

Therefore the G^0-action on $H^0(2K_C) \otimes H^0(2K_C)$ is defined as

$$h(\alpha \otimes \beta) = (ha) \otimes (\varphi(h)\beta) \qquad \text{for } \alpha \otimes \beta \in H^0(2K_C) \otimes H^0(2K_C),\ h \in G^0.$$

Proposition 5.11. *For each ramification curve $R_i \subset C \times C$*

$$\omega_{C\times C}|_{R_i} \cong \omega_{R_i}^{\otimes 2}.$$

Proof. Let $\pi_j|_{R_i} \colon R_i \to C$ for $j = 1, 2$ be the restriction of the projection on the first (respectively second) coordinate on the curve R_i. In both cases $\pi_j|_{R_i}$ is an isomorphism and the sheaves $(\pi_j|_{R_i})^*(\omega_C)$ and ω_{R_i} are isomorphic. Therefore

$$\begin{aligned}\omega_{C\times C}|_{R_i} &\cong (\pi_1^*\omega_C \otimes \pi_2^*\omega_C)|_{R_i} = \\ &= (\pi_1|_{R_i})^*\omega_C \otimes (\pi_2|_{R_i})^*\omega_C \cong \omega_{R_i} \otimes \omega_{R_i} = \omega_{R_i}^{\otimes 2}.\end{aligned}$$

□

Corollary 5.12. *There exists an isomorphism of sheaves*

$$Q \colon \omega_{C\times C}^{\otimes 2}|_R \xrightarrow{\cong} \omega_R^{\otimes 4}.$$

Remark 5.13. On the sheaf $\omega_R^{\otimes 4}$ we have the natural G^0-action given by pull-back; isomorphism Q in Corollary 5.12 is G^0-equivariant, i.e. for $\omega \in H^0(U, \omega_{C\times C}^{\otimes 2}|_R)$ and $g \in G^0$ it holds $g \cdot Q(\omega) := Q(g \cdot \omega)$.

In particular

$$Q^0 \colon H^0(2K_{C\times C}|_R)^{G^0} \to H^0(4K_R)^{G^0}$$

is an isomorphism.

Let us see in detail how this G^0-action works on the space of global section $H^0(4K_R)$. Let $\omega = (\omega_i), \eta = (\eta_i) \in H^0(4K_R)$, where ω_i (respectively η_i) is the component of ω (resp. η) along the subspace $H^0(4K_{R_i})$, let $h \in G^0$ and suppose that $h\omega = \eta$. Then $\eta_i = h^*(\omega_{\varrho(h)(i)})$, where $\varrho \colon G \to \mathfrak{S}_N$ is the homomorphism defined in Definition 4.16.

Lemma 5.14. *The map $\sigma|_R^* \colon H^0(2K_Y|_T) \to H^0(2K_{C\times C}|_R)^{G^0}$ is an isomorphism.*

Proof. The map $\sigma|_R^*$ is injective, then it is sufficient to prove that $h^0(2K_Y|_T) = \dim(H^0(2K_{C\times C}|_R)^{G^0})$ to get the thesis.

Let us first compute $h^0(2K_Y|_T) = \sum h^0(2K_Y|_{T_i})$. Using the same arguments we used in Lemma 5.1 and by Theorem 1.39

$$h^0(2K_Y|_{T_i}) = \chi(2K_Y|_{T_i}) = 8(g(T_i) - 1) - (g(T_i) - 1) = 7(g(T_i) - 1),$$

therefore $h^0(2K_Y|_T) = 7\delta(B)$.

By Lemma 4.19 $\sigma|_R^*\colon H^0(4K_T) \to H^0(4K_R)^{G^0}$ is an isomorphism. Again, since T_i's are pairwise disjoint, $h^0(4K_T) = \sum h^0(4K_{T_i})$;

$$\deg(4K_{T_i}) = 8(g(T_i) - 1) > 2(g(T_i) - 1),$$

therefore by Corollary 1.40, $h^1(4K_{T_i}) = 0$. Hence, by Theorem 1.39

$$h^0(4K_{T_i}) = 8(g(T_i) - 1) - (g(T_i) - 1) = 7(g(T_i) - 1),$$

finally by Remark 5.13

$$\dim(H^0(2K_{C\times C}|_R)^{G^0}) = \dim(H^0(4K_R)^{G^0}) = h^0(4K_T) = 7\delta(B).$$

$\square$

Theorem 5.15. *Let $\rho_0\colon H^0(2K_{C\times C})^{G^0} \to H^0(2K_{C\times C}|_R)^{G^0}$ be the restriction map described in (5.6), then $h^1(2K_X) = \dim\operatorname{coker}(\rho_0)$.*

Proof. By Remark 5.8 and Lemma 5.14, we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & H^0(2K_Y) \arrow[r, "\hat\rho"] \arrow[d, "\sigma^*"'] & H^0(2K_Y|_T) \arrow[d, "\sigma|_R^*"] \\
 & H^0(2K_{C\times C})^{G^0} \arrow[r, "\rho_0"] \arrow[d] & H^0(2K_{C\times C}|_R)^{G^0} \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

Let $a \in \operatorname{Im}(\rho_0)$; being $\sigma|_R^*$ an isomorphism there exists a unique $b \in H^0(2K_Y|_T)$ such that $\sigma|_R^*(b) = a$. Let $a' \in H^0(2K_{C\times C})^{G^0}$ such that $\rho_0(a') = a$; since σ^* is an isomorphism there exists a unique $b' \in H^0(2K_Y)$ such that $\sigma^*(b') = a'$. By commutativity of the diagram, and uniqueness of b and b', $\hat\rho(b') = b$. This means that $\operatorname{Im}(\hat\rho)$ and $\operatorname{Im}(\rho_0)$ are isomorphic via $\sigma|_R^*$, therefore $\operatorname{coker}(\hat\rho) \cong \operatorname{coker}(\rho_0)$. By Remark 5.6 we get the thesis. $\square$

5.1.3 Displacing the problem onto C

The last move is to displace the problem onto C, using Remark 5.10 and Corollary 5.12. Composing the map Φ, the restriction map $\rho\colon H^0(2K_{C\times C}) \to H^0(2K_{C\times C}|_R)$ and Q we get the map

$$\Psi\colon H^0(2K_C) \otimes H^0(2K_C) \longrightarrow H^0(4K_R),$$

which is defined in the following way: let $\alpha\otimes\beta \in H^0(2K_C)\otimes H^0(2K_C)$, then $\Psi(\alpha\otimes\beta)_i$, that is the component of $\Psi(\alpha\otimes\beta)$ along the subspace $H^0(4K_{R_i})$ is

$$\begin{aligned}\Psi(\alpha\otimes\beta)_i &= (\pi_1|_{R_i})^*\alpha\cdot(\pi_2|_{R_i})^*\beta\\ &= (\pi_1|_{R_i})^*\alpha\cdot(\tau' g_i\circ\pi_1|_{R_i})^*\beta\\ &= (\pi_1|_{R_i})^*\alpha\cdot(\pi_1|_{R_i})^*(\tau' g_i\beta)\\ &= (\pi_1|_{R_i})^*(\alpha\cdot(\tau' g_i\beta)),\end{aligned} \tag{5.8}$$

where $\cdot$ denotes the multiplication in the canonical ring.

Using the properties of Φ, ρ and Q, it is clear that Ψ is G^0-equivariant, therefore we can restrict Ψ to the G^0-invariant subspace, eventually getting

$$\Psi_0\colon (H^0(2K_C)\otimes H^0(2K_C))^{G^0}\to H^0(4K_R)^{G^0}.$$

Being Ψ and Q two isomorphisms, $\operatorname{coker}(\rho_0)\cong\operatorname{coker}(\Psi_0)$.

We can summarise all of these results in the following theorem, which exploits Theorem 5.15 too.

Theorem 5.16. *Let $\Psi\colon H^0(2K_C)\otimes H^0(2K_C)\to H^0(4K_R)$ be the map defined on the decomposable tensors as*

$$\Psi(\alpha\otimes\beta)_i := (\pi_1|_{R_i})^*(\alpha\cdot(\tau' g_i\beta)),$$

where $\Psi(\alpha\otimes\beta)_i$ denotes the component of $\Psi(\alpha\otimes\beta)$ along $H^0(4K_{R_i})$.

Let $g\in G^0$ and $\alpha\otimes\beta\in H^0(2K_C)\otimes H^0(2K_C)$; the map

$$\alpha\otimes\beta\mapsto(g\alpha)\otimes(\varphi(g)\beta)$$

defines a G^0-action on $H^0(2K_C)\otimes H^0(2K_C)$.

Let $h\in G^0$ and $\omega := \{\omega_i\}\in H^0(4K_R)$, where ω_i denotes the component of ω along $H^0(4K_{R_i})$; the map

$$\{\omega_i\}\mapsto\{(h^*(\omega_{\varrho(h)(j)}))_j\},$$

where ϱ is the homomorphism defined in Definition 4.16, gives a G^0-action on $H^0(4K_R)$.

Respect to these actions the map Ψ is G^0-equivariant, and restricts to a map

$$\Psi_0\colon (H^0(2K_C)\otimes H^0(2K_C))^{G^0}\to H^0(4K_R)^{G^0}.$$

Moreover, $h^1(2K_X)=\dim\operatorname{coker}(\Psi_0)$.

Remark 5.17. Since Ψ is G^0-equivariant, it respects the decompositions in irreducible characters both of domain and codomain. Therefore, if for any $\chi \in (G^0)^*$ irreducible character of G^0, we consider the restriction of Ψ to the subspace corresponding to χ, i.e.

$$\Psi_\chi \colon (H^0(2K_C) \otimes H^0(2K_C))^\chi \to H^0(4K_R)^\chi,$$

it holds

$$\operatorname{coker}(\Psi) \cong \bigoplus_{\chi \in (G^0)^*} \operatorname{coker}(\Psi_\chi).$$

Therefore, by Proposition 5.16

$$h^1(2K_X) \leq \dim \operatorname{coker}(\Psi), \tag{5.9}$$

this means that surjectivity of Ψ is a sufficient condition for the vanishing of $h^1(2K_X)$.

5.2 Minimality of Semi-isogenous Mixed Surfaces with irreducible ramification locus

The results proved in previous sections gives a very concrete way to prove minimality of a semi-isogenous mixed surface. In this section we see an application of these results; we will prove that if $X := (C \times C)/G$ is a semi-isogenous mixed surface of general type and the ramification locus R consists in only one curve, then X is minimal.

Proposition 5.18. *Let $X = (C \times C)/G$ be a semi-isogenous mixed surface. If the ramification locus R of the quotient map $\eta \colon C \times C \to X$ is irreducible, then*

- *if $g(C) \geq 3$ then $h^1(2K_X) = 0$;*
- *if X is of general type, then it is minimal.*

Proof. By Proposition 4.13, the ramification locus R is irreducible if and only if O_2 has only one element. Since X is a semi-isogenous mixed surface, in Theorem 3.12 we can take $\tau' \in O_2$, then $R = \Delta := \{(x, x) : x \in C\}$.

According to Proposition 5.16, the cohomology group $H^1(2K_X)$ is trivial if and only if the map $\Psi_0 \colon (H^0(2K_C)^{\otimes 2})^{G^0} \to H^0(4K_R)^{G^0}$ is surjective.

Let us denote by $p \colon \Delta \to C$ the projection on the first coordinate. Let $\alpha \otimes \beta$ in $H^0(2K_C) \otimes H^0(2K_C)$, then by definition (cf. Theorem (5.16))

$$\Psi(\alpha \otimes \beta) = p^*(\alpha \cdot \beta). \tag{5.10}$$

We will prove that Ψ is surjective; as stated in Remark 5.17, this implies surjectivity of Ψ_0.

Suppose $g(C) \geq 3$, then by Theorem 1.48 and Proposition 1.49, the canonical ring $R(C)$ is generated either in degree 1 if C is not hyperelliptic or in degree 2 if C is hyperelliptic; in both cases multiplication map $\cdot : H^0(2K_C) \otimes H^0(2K_C) \to H^0(4K_C)$ is surjective. Since the map $p^* \colon H^0(4K_C) \to H^0(4K_R)$ is an isomorphism, surjectivity of Ψ follows.

By inequality (5.9), $h^1(2K_X) = 0$.

If X is of general type, X is minimal by Proposition 2.58. $\square$

Proposition 5.18 gives a new proof for the following well known result.

Corollary 5.19. *Let C be a curve of genus $g := g(C) \geq 3$, then the symmetric product $C^{(2)}$ is a minimal surface of general type.*

Proof. By definition $C^{(2)} := (C \times C)/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle \tau' \rangle$ is generated by the switch of the two coordinates. By Proposition 4.24

$$K^2_{C^{(2)}} = (g-1)(4(g-1)-5).$$

For $g \geq 4$ $K^2_{C^{(2)}} > 9$, hence by Theorem 2.46 $C^{(2)}$ is of general type.

The only case we miss is $g = 3$; in this case $K^2_{C^{(2)}} = 6$, then it is either rational or of general type. Being G^0 trivial, by Proposition 4.21 $q(C^{(2)}) = g(C/G^0) = g(C) = 3$, therefore it is of general type.

The ramification locus of the quotient map $C \times C \to C^{(2)}$ consists in the diagonal $\Delta := \{(x,x) : x \in C\}$, therefore, by Proposition 5.18, $C^{(2)}$ is minimal. $\square$

Remark 5.20. By Proposition 5.18, the only semi-isogenous mixed surface with $K^2 = 7$ and $p_g = q = 2$ listed in Table 4.5 is minimal, since $|O_2| = 1$.

5.3 Minimality of Semi-isogenous Mixed Surfaces with $\chi = 1$

In this section we compute a bound for $h^1(2K_X)$ of a semi-isogenous mixed surface X with $\chi(\mathcal{O}_X) = 1$. First of all we will determine a bound for the number of points in the intersection between the branch locus B of the quotient map $\eta : C \times C \to X$ and the possible exceptional curves on X. Then, using the Hodge's Index Theorem, we will see how this result provides a first answer to the minimality problem of semi-isogenous mixed surfaces with $\chi(\mathcal{O}_X) = 1$ and $K^2_X > 0$.

Remark 5.21. Let D be a (possibly singular) irreducible curve on Y. Let $\tilde{D}$ be the normalization of D: there exists a proper map $\nu : \tilde{Y} \to Y$ consisting of a finite number of blow-ups such that the strict transform $\tilde{D}$ of D is smooth (see [BHPV04, Theorem II.7.1]). We have the following commutative diagram

$$\begin{array}{ccc} C \times C & \xleftarrow{\gamma_1} & (C \times C) \times_Y \tilde{Y} \\ \sigma \downarrow & & \downarrow \gamma_2 \\ Y & \xleftarrow{\nu} & \tilde{Y} \end{array} \tag{5.11}$$

where $(C \times C) \times_Y \tilde{Y}$ denotes the fibre product and γ_1 and γ_2 the natural projections; the map γ_2 is étale because σ is étale. Let D' be an irreducible component of $\gamma_2^{-1}(\tilde{D})$, its image $\gamma_1(D')$ is a curve in $C \times C$, and therefore surjects onto C, whence $g(D') \geq g(C) \geq 2$. Since γ_2 is étale, and D' and $\tilde{D}$ are both smooth, we deduce that $g(\tilde{D}) \geq 2$.

Let E be a smooth rational curve on X. If $E \cap B = \emptyset$, then there exists a rational curve in $\pi^{-1}(E) \subset Y$, contradicting Remark 5.21. By Proposition 4.17, B is finite union of disjoint curves of genus strictly greater than 1, hence E and B meet in a finite number of points. We split these points in two sets accordingly to the parity of their intersection multiplicity:

$$\begin{aligned} A_0 &:= \{p \in E \cap B : m_p(E \cap B) \text{ is even}\}, \\ A_1 &:= \{p \in E \cap B : m_p(E \cap B) \text{ is odd}\}, \end{aligned}$$

where $m_p(E \cap B)$ denotes the intersection multiplicity of E and B in p. We define $\mu_0 := |A_0|, \mu_1 := |A_1|$ and $\mu := \mu_0 + \mu_1 = |E \cap B|$.

Lemma 5.22. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface and let $E \subset X$ be a smooth rational curve. Then μ_1 is even and $\mu_1 \geq 6$.*

Proof. Let $D := \pi^{-1}(E) \subset Y$. Being π a double cover, by Riemann-Hurwitz's formula (1.2) $e(D) = 2e(E) - \mu = 4 - \mu$. The map $\pi|_D : D \to E$ is finite of degree 2. In particular, if D is reducible, $D = D_1 + D_2$ with $\pi|_{D_i} : D_i \to E$ biregular, contradicting Remark 5.21. So D is irreducible.

Let $T := \pi^{-1}(R) \subset Y$; since π is a local isomorphism out of T, the singularities of D lie on T. Let us fix $p \in E \cap B$, and suppose $m_p(E \cap B) = k$, then we can take local coordinates (x, y) centred in p such that $B = \{x = 0\}$ and $E = \{x = y^k\}$. Denoting by (z, w) local coordinates centred in $p' = \pi^{-1}(p)$, the local expression of the map $\pi \colon Y \to X$ is $(z, w) \mapsto (z^2, w)$: $T = \{z = 0\}$ and $D = \{z^2 = w^k\}$. This means that p' is a singular point if and only if $k \geq 2$.

As we saw in Section 2.3, the blow-up of Y in p' is given on a chart (say V_1) by $(x_1, y_1) \mapsto (x_1y_1, y_1)$, and on the other chart (say W_1) by $(u_1, v_1) \mapsto (u_1, u_1v_1)$ and the glueing $V_1 \to W_1$ is given by $(x_1, y_1) \mapsto (x_1y_1, x_1^{-1})$. So the strict transform D_1 of D on V_1 is $\{x_1^2 = y_1^{k-2}\}$, while on W_1 is given by $\{u_1^{k-2}v_1^k = 1\}$.

If $k - 2 \geq 2$ then D_1 has a singular point and we blow up again, otherwise D_1 is smooth and we stop. According to the parity of k, we get eventually either $x_n^2 = 1$ or $x_n^2 = y_n$; thus, if k is even there are two points on the strict transform D_n lying over p', otherwise there is only one point over p'. Repeating this process for each singular points of D, we get that for the normalization $\tilde{D}$ of D it holds $e(\tilde{D}) = e(D) + \mu_0 = 4 - \mu_1$, whence $2 - 2g(\tilde{D}) = 4 - \mu_1$, i.e. $\mu_1 = 2g(\tilde{D}) + 2 \geq 6$, where the inequality follows by Remark 5.21. □

Proposition 5.23. *Let $X := (C \times C)/G$ be a semi-isogenous mixed surface and $E \subset X$ be a smooth rational curve. Then $E.B$ is even and $E.B \geq 6$.*

Proof. Being

$$E.B = \sum_{p \in E \cap B} m_p(E \cap B) = \sum_{p \in A_0} m_p(E \cap B) + \sum_{p \in A_1} m_p(E \cap B),$$

by Lemma 5.22, we get the claim. □

We recall the following.

Theorem 5.24 (Hodge Index Theorem, cf. [BHPV04, Corollary IV.2.16])**.** *Let S be a smooth surface, $NS(S)$ be its Neron-Severi group and consider $NS(S) \otimes_{\mathbb{Z}} \mathbb{R}$ endowed with the quadratic form induced by the intersection product. Let D be a divisor on S with $D^2 > 0$. Then the intersection product is negative definite on the orthogonal complement $D^{\perp}$ of D in $NS(S) \otimes_{\mathbb{Z}} \mathbb{R}$.*

For a divisor D on S, we denote by $[D]$ its class in the vector space $NS(S) \otimes_{\mathbb{Z}} \mathbb{R}$.

Lemma 5.25 (cf. [BP12, Remark 4.3])**.** *On a smooth surface S of general type every irreducible curve C with $K_S.C \leq 0$ is smooth and rational.*

Theorem 5.26. *Let X be a semi-isogenous mixed surface of general type and suppose that $\chi(\mathcal{O}_X) = 1$. If $K_X^2 \geq 6$, then X is minimal. If $1 \leq K_X^2 \leq 5$, then*

$$h^1(2K_X) \leq \left\lfloor \frac{7 - K_X^2}{2} \right\rfloor.$$

Proof. Let us denote by X_{min} the minimal model of X, then by Proposition 2.58, $h^1(2K_X) = K^2_{X_{min}} - K^2_X$. Let us then compute the number of blow downs we have to perform in order to obtain the minimal model of X.

Let $E_1 \subset X$ be a (-1)-curve. By Lemma 5.25, E_1 is smooth, hence by Proposition 5.23 $n_1 := E_1.B \geq 6$. Let W be the subspace of $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by $[K_X], [B], [E_1]$. The matrix of the quadratic form induced by the intersection product on W, by Remark 4.26 and Remark 4.27, is

$$M_1 := \begin{pmatrix} K_X^2 & K_X.B & K_X.E_1 \\ K_X.B & B^2 & E_1.B \\ K_X.E_1 & E_1.B & E_1^2 \end{pmatrix} = \begin{pmatrix} K_X^2 & 6(8-K_X^2) & -1 \\ 6(8-K_X^2) & -4(8-K_X^2) & n_1 \\ -1 & n_1 & -1 \end{pmatrix}$$

it has determinant $\det M_1 = -K_X^2 n_1^2 - 12(8-K_X^2)n_1 + 4(8-K_X^2)(73-8K_X^2)$. As quadratic polynomial in n_1, $\det M_1$ has roots ($x_1 \leq x_2$):

$$x_{1,2} = \frac{-6(8-K_X^2) \pm \sqrt{36(8-K_X^2)^2 + K_X^2(8-K_X^2)(73-8K_X^2)}}{K_X^2}.$$

and it is easy to see that $x_1 \leq 0 \leq x_2$. Since $K_X^2 > 0$, by Theorem 5.24 we have $\det M_1 \geq 0$, and the leading coefficient $-K_X^2$ is negative hence $6 \leq n_1 \leq \lfloor x_2 \rfloor$. For $K_X^2 > 0$, the round-down of x_2 is:

K_X^2	8	7	6	5	4	3	2	1
$\lfloor x_2 \rfloor$	0	2	4	6	8	10	13	17

(5.12)

For $K_X^2 \in \{6,7,8\}$, it holds $\lfloor x_2 \rfloor < 6$, a contradiction, whence there are no (-1)-curves on X.

Assume now $K_X^2 < 6$ and assume there exists another rational curve $E_2 \subset X$ such that either $E_2^2 = -2$ and $E_1.E_2 = 1$ or $E_2^2 = -1$ and $E_1.E_2 = 0$. Up to change E_2 with $E_1 + E_2$, the matrix of the intersection form for $[K], [B], [E_1], [E_2]$ is

$$M_2 := \begin{pmatrix} K_X^2 & K_X.B & K_X.E_1 & K_X.E_2 \\ K_X.B & B^2 & E_1.B & E_2.B \\ K_X.E_1 & E_1.B & E_1^2 & E_1.E_2 \\ K_X.E_2 & E_2.B & E_1.E_2 & E_2^2 \end{pmatrix} = \begin{pmatrix} K_X^2 & 6(8-K_X^2) & -1 & -1 \\ 6(8-K_X^2) & -4(8-K_X^2) & n_1 & n_2 \\ -1 & n_1 & -1 & 0 \\ -1 & n_2 & 0 & -1 \end{pmatrix},$$

where $n_2 := E_2.B \geq 6$.
It has $\det M_2 = n_2^2(1+K_X^2) + n_2(12(8-K_X^2) - 2n_1) + n_1^2(1+K_X^2) + 12n_1(8-K_X^2) + 8(8-K_X^2)(4K_X^2 - 37)$.
As quadratic polynomial in n_2, $\det M_2$ has roots $y_1 \leq y_2$.

Since $K_X^2 > 0$, by Theorem 5.24 we have $\det M_2 \leq 0$, and the leading coefficient $1 + K_X^2$ is positive hence $6 \leq n_2 \leq \lfloor y_2 \rfloor$. For $6 > K_X^2 > 0$, and $n_1 \geq 6$, even (Lemma 5.25) and bounded from above by the value in (5.12), y_1 is negative and the round-down of y_2 is:

K_X^2	5	4		3			2				1					
n_1	6	6	8	6	8	10	6	8	10	12	6	8	10	12	14	16
$\lfloor y_2 \rfloor$	-2	3	-2	6	4	0	9	7	5	1	12	11	9	7	4	-1

For $K_X^2 \in \{4, 5\}$, it holds $\lfloor y_2 \rfloor < 6$, a contradiction, whence there is at most one (-1)-curves on X.

Arguing in the same way, one proves the statements in the remaining cases: $K_X^2 \in \{1, 2, 3\}$. □

Thanks to Theorem 5.26 and some *ad hoc* arguments, we are able to determine for most of the surfaces listed in Table 4.2, Table 4.3, Table 4.4 and Table 4.5 whether they are minimal or not.

5.3.1 The cases $p_g(X) = q(X) = 2$ and $K_X^2 = 2$

Let X be a semi-isogenous mixed surface of general type with $K_X^2 = 2$ and $p_g(X) = q(X) = 2$. By Debarre's inequality (cf. Theorem 2.62), for a minimal irregular surface of general type S it holds $K_S^2 \geq 2p_g(S)$, thus X is not minimal and $K_{X_{min}}^2 \geq 4$, i.e. we need to contract at least two (-1)-curves.

On the other side, by Theorem 5.26 we can contract at most two (-1)-curves. Therefore the minimal model X_{min} of X has $K_{X_{min}}^2 = 4$.

5.3.2 The case $p_g(X) = q(X) = 2$ and $K_X^2 = 4$

Let $X := (C \times C)/G$ be a semi-isogenous mixed surface with $K_X^2 = 4$ and $p_g(X) = q(X) = 2$. The surface X is of general type and, according to Table 4.5, C is a curve of genus 7, $G^0 \cong S_3$ and $G \cong S_3 \times \mathbb{Z}_2$: $\mathbb{Z}_2$ acts on $C \times C$ exchanging the factors.

By Theorem 5.26, X contains at most one (-1)-curve. We explicitly construct a (-1)-curve on X, thus X_{min} has $K_{X_{min}}^2 = 5$.

By Proposition 4.21, $C' := C/G^0$ is a curve of genus 2: it is hyperelliptic. Let $f' \colon C' \to C'$ be the hyperelliptic involution and $c \colon C \to C'$ the projection. According to [Acc94, Corollary 2], f' lifts to an automorphism $f \in \mathrm{Aut}(C)$, i.e. f satisfies $c(f(p)) = f'(c(p))$ for all $p \in C$.

By the uniqueness of the lift, $faf \in S_3$ for all $a \in S_3$; in particular $f^2 \in S_3$ and

$S_3 \triangleleft H := \langle S_3, f \rangle < \mathrm{Aut}(C)$ with H of order 12. The map $C \to C/H \cong \mathbb{P}^1$ ramifies in 36 points, each one of them with stabilizer generated by an element of order 2 in $H \setminus S_3$. Let $T := \{p_i\}_{i=1,\dots,36}$ be the ramification locus of $C \to C/H$.

Since $\mathrm{Aut}(S_3) = \mathrm{Inn}(S_3)$, there exists a unique $\bar{f} \in H \setminus S_3$ such that $\bar{\varphi}(g) := \bar{f} g \bar{f}^{-1} = \varphi(g)$ for all $g \in S_3$. Let $\Gamma := \{(x, \bar{f}x) : x \in C\} \subset C \times C$ be the graph of $\bar{f}$. A direct computation shows that the curve Γ is G-invariant. Let $\tilde{\Gamma} := \eta(\Gamma) \subset X$; the ramification locus of the map $\eta|_\Gamma : \Gamma \to \tilde{\Gamma}$ is $\{(\bar{f}p, \bar{f}^2 p) : p \in T\}$, and each ramification point has stabilizer of order 2 generated by an element of $G \setminus G^0$, then, by Riemann-Hurwitz's formula (1.2),

$$12 = 2g(\Gamma) - 2 = 12(2g(\tilde{\Gamma}) - 2) + 36,$$

that is $g(\tilde{\Gamma}) = 0$. The canonical divisor $K_{C\times C}$ is numerically equivalent to

$$2(g(C) - 1)F_1 + 2(g(C) - 1)F_2 = 12F_1 + 12F_2\,,$$

where F_1, F_2 denote a general fibre of the projections on the first and on the second coordinate respectively, then $K_{C\times C}.\Gamma = 24$. By the adjunction formula

$$\Gamma^2 = 2g(\Gamma) - 2 - K_{C\times C}.\Gamma = -12\,.$$

Finally, since $\eta^*(\tilde{\Gamma}) = \Gamma$, by Proposition 2.10 $\Gamma^2 = \eta^*(\tilde{\Gamma})^2 = \deg \eta \cdot \tilde{\Gamma}^2$ then $\tilde{\Gamma}^2 = -1$.

5.3.3 The cases $p_g(X) = q(X) = 0$ and $K_X^2 = 2$

Let S be a *numerical Campedelli surface*, that is a minimal surface of general type with $K^2 = 2$ and $p_g = 0$; it is known (cf. [Rei]) that its algebraic fundamental group $\pi_1^{alg}(S)$ is a finite group of order ≤ 9. As remarked in [BCP11, Section 2.1] (see also [PPS13]), if $H_1(S, \mathbb{Z})$ is finite, it is isomorphic to the abelianization of $\pi_1^{alg}(S)$ and so it has order ≤ 9.

Let X be one of the surfaces in Table 4.2 with $K_X^2 = 2$. The order of $H_1(X, \mathbb{Z})$ is 32; by the above argument, X is not minimal.

5.3.4 The cases $p_g(X) = q(X) = 1$ and $K_X^2 = 2$

By [Cat81], the minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 2$ form a connected component in the moduli space: the Albanese map of these surfaces is a genus 2 fibration, and their fundamental group is isomorphic to $\mathbb{Z}^2$ (cf. [FP15]). We conclude that the surfaces in Table 4.4 with $K^2 = 2$ are not minimal.

Appendix A

The Classification Algorithm

```
// Input: K^2, p_g and q
//
// Step 1: the types
//
// Once we fix K^2, p_g and q there are finitely many possible
// signatures satisfying all the condition of the Proposition 5.1.
//
// We will represent a signature as the multiset of
// positive integers {* m_i *}.

// We define the invariants Theta and Beta:

Theta:=function(q, sig)
a:=2*q-2;
for m in sig do a+:=(1-1/m); end for;
return a;
end function;

Beta:=func<K, chi,  T | 2*(10*chi-K)/T>; // it coincides with g(C)-1

// These two transform a multiset, resp. a tuple  into a sequence

MsetToSeq:=function(mset)
seq:=[ ];
```

```
while #mset ne 0 do Append(~seq, Minimum(mset));
Exclude(~mset, Minimum(mset)); end while;
return seq;
end function;

TupleToSeq:=function(tuple)
seq:=[ ];
for el in tuple  do  Append(~seq,el); end for;
return seq;
end function;

// The input of the next program are 3 numbers, Length, HB and n
//and its output are all types with #type=Length such that
//  each m_i is smaller than HB and it divides n;

CandTypes:=function(Length,HB,n)
D:={x: x in Divisors(n) |  x in {2..HB}};
//divisors of n smaller than HB and different from 1
Types:=Multisets(D,  Length);
return Types;
end function;

// The function ListTypes calculate all the types giving the
// expected value of beta (=2*(10*chi-Ksquare)/Theta).

ListOfTypes:=function(Ksquare,pg, q, beta)
list:={};    chi:=1+pg-q;

N:=IntegerRing()!(beta*(8*chi-Ksquare)/(10*chi-Ksquare)); // N is |O_2|
ordG0:=IntegerRing()!( (beta^2)/(10*chi-Ksquare));

if q eq 0 then Rmin:=3;
elif q eq 1 then Rmin:=1;
else Rmin:=0;
end if;
```

```
Rmax:=Floor((4*(10*chi-Ksquare)/beta)+4*(1-q));
for R in [Rmin..Rmax] do

M:=Max(1/6,(R-3+4*q)/2);
HB:= Min({beta, Floor( (1+2*(10*chi-Ksquare) )/M )     });

if N gt 0
then HB:=Min({beta, N, Floor(ordG0/N), Floor((1+2*(8*chi-Ksquare))/M)});
end if;

for cand in CandTypes(R,HB,GCD({N, beta, ordG0}) ) do

T:=Theta(q,cand);
if T eq 2*(10*chi-Ksquare)/beta then
Include(~list, MsetToSeq(cand) cat [ordG0] );
end if;
end for; end for;

return list;
end function;

// ListTypes returns, for given K^2, p_g and q,
// all  the possible types (using ListOfTypes) (see Proposition 5.1)

ListTypes:=function(Ksquare, pg,q)

List:=[ ]; chi:=1-q+pg;

if q eq 0 then Tmin:=1/42;
elif q eq 1 then Tmin:=1/2;
else  Tmin:=2*q-2;
end if;

BetaMax:=Floor(Beta(Ksquare,chi,Tmin));

for beta in [BetaMax.. 1 by -1] do
```

```
N:=beta*(8*chi-Ksquare)/(10*chi-Ksquare);
ordG0:= (beta^2)/(10*chi-Ksquare);
if IsIntegral(N) and IsIntegral(ordG0)  and (N le ordG0) then
for T in  ListOfTypes(Ksquare, pg,q, beta) do
Append(~List,T );
end for;
end if; end for;

return List;
end function;

//
// Step 2: admissible groups G^0
//
// Fixed K^2, p_g, q, and the type, we can compute the order
//of the group G^0. We search among the
//group of order |G^0|  which groups
// have a  generating vector of the prescribed type.

ElsOfOrd:=func<group, order | {g: g in group| Order(g) eq order}>;

// TuplesOfGivenOrder creates a sequence of length equal to
// the length of the input sequence type plus 2*q,
//whose entries are subsets of the group in
// the input, and precisely the subsets of elements of order
// the corresponding entry of type and 2*q copies of the whole group.

TuplesOfGivenOrders:=function(group,q,type)
SEQ:=[ ];

for i in [1..2*q] do Append(~SEQ,Set(group)); end for;
for i in [1..#type] do
if IsEmpty(ElsOfOrd(group,type[i])) then return [ ];
else Append(~SEQ,ElsOfOrd(group,type[i]));
end if;end for;
```

```
return SEQ;
end function;

/// This script returns all the generating vector
//  for a group isomorphic to H < G of prescribed type.

VectGens:=function(G, H, q, type: OnlyExistence:= false)
Vect:={}; SetCands:=TuplesOfGivenOrders(G,q,type);

if not IsEmpty(SetCands) then
if #type ne 0 then
Prune(~SetCands);
else type:=[1];
end if;
cands:= CartesianProduct(SetCands);
for cand in cands do m:=Id(G);
for i in [1..q] do     m:=m*(cand[2*i-1]^-1, cand[2*i]^-1); end for;
for i in [1..#type-1] do m:=m*cand[2*q+i]; end for;
if Order(m) eq type[#type] then
S:=sub<G|TupleToSeq(cand)>;
if #S eq #H then
if #H eq #G then
Include(~Vect, Append(TupleToSeq(cand),(&*cand)^-1));
if OnlyExistence  then return true; end if;
elif CanIdentifyGroup(#H) then
if IdentifyGroup(S) eq IdentifyGroup(H) then
Include(~Vect, Append(TupleToSeq(cand),(&*cand)^-1));
if OnlyExistence  then return true;end if;
end if;
else
if IsIsomorphic(S, H) then
Include(~Vect,  Append(TupleToSeq(cand),(&*cand)^-1));
if OnlyExistence  then return true;end if;
end if;
end if; end if; end if; end for; end if;
```

```
if OnlyExistence  then
return false;
end if;

return Vect;
end function;

// The following function are use to construct the "Hurwitz moves".
// These moves are described in [Pen15].

// The next script takes a sequence of elements of a group
// and a further element g and conjugates each element
// of the sequence with g.

Conjug:=function(seq,el)
output:=[];
for h in seq do Append(~output,h^el); end for;
return output;
end function;

HurwitzMove0:= func<seq,idx|Insert(Remove(seq,idx),idx+1,
                                   seq[idx]^seq[idx+1])>;

// HurwitzOrbit0, starting from a sequence of elements of a group,
// creates all sequences of elements which are equivalent to
// the given one for the equivalence relation generated
// by the Hurwitz moves of type (0; m_1,..., m_r),
// and return (to spare memory) only the ones whose entries have never
// decreasing order.

HurwitzOrbit0:=function(seq)
orb:={ }; shortorb:={  }; Trash:={ seq };
repeat
ExtractRep(~Trash,~gens); Include(~orb, gens);
for k in [1..#seq-1] do
newgens:=HurwitzMove0(gens,k);
```

```
if newgens notin orb then
Include(~Trash, newgens);
end if; end for;
until IsEmpty(Trash);
for gens in orb do  test:=true;
for k in [1..#seq-1] do
if Order(gens[k]) gt Order(gens[k+1]) then test:=false; break k;
end if; end for;
if test then Include(~shortorb, gens); end if;
end for;
return shortorb;
end function;

// The next one create the Hurwitz move of type (1; m)

HurwitzMove1N:=function(seq)
moves:={};
t1:=[seq[1],seq[2]*seq[1], seq[3]];
Include(~moves,t1);
t2:=[seq[1]*seq[2]^-1,seq[2], seq[3]];
Include(~moves,t2);
return moves;
end function;

// This one, starting from a sequence of elements of a group,
// creates all sequences of elements which are equivalent
// to the given one for the equivalence relation generated
// by the Hurwitz moves of type (1; m).

HurwitzOrbit1N:=function(seq)
orb:={};  Trash:={ seq };
repeat
ExtractRep(~Trash,~gens); Include(~orb, gens);
for newgens in HurwitzMove1N(gens) do
if newgens notin orb then Include(~Trash, newgens);
end if; end for;
```

```
until IsEmpty(Trash);
return orb;
end function;

// The next one create the Hurwitz move of type
//(g'; m_1, ..., m_r), with g'>1 or g'>0 and r>1.

HurwitzMoveGen:=function(h,seq)
moves:={};
fund:=[seq[i]: i in [1..2*h]];
ram:=[seq[i]:  i in [2*h+1.. #seq] ];

//Type 1-2, t_j, t_(\delta_j)

for j in [1.. h] do
t1:=seq; t2:=seq;
t1[2*j-1]:=seq[2*j-1]*(seq[2*j]^-1);
t2[2*j]:=seq[2*j]*seq[2*j-1];
Include(~moves, t1);Include(~moves, t2);
end for;

//Type 3, t_(\sigma_h)

for h in [1.. #ram -1] do
t3:=fund cat (HurwitzMove0(ram, h) );
Include(~moves, t3);
end for;


//Type 4, t_(\tau_k)

for k in [1.. h-1] do
x:=(seq[2*k]^-1) * seq[2*k+1] * seq[2*k+2] *(seq[2*k+1]^-1) ;
t4:=seq;
t4[2*k-1]:=seq[2*k-1]*(x^-1);
t4[2*k]:=x*seq[2*k]*(x^-1);
```

```
t4[2*k+1]:=x*seq[2*k+1];
Include(~moves, t4);
end for;

//Type 5-6, t_(\xi^1_j,d),  t_(\xi^2_j,d)

for j in [1..h]  do
for d in [1..#ram] do

u:=seq[1]*(seq[1]^-1);
for  k in [1..j-1] do
u:= u* (seq[2*k-1]*seq[2*k]*(seq[2*k-1]^-1)*(seq[2*k]^-1)  );
end for;

s:=seq[1]*(seq[1]^-1);
for z in [2*h+d+1..#seq] do s:=s*seq[z]; end for;
c:=(u^-1)*(seq[#fund+ d])*u;

e:= u *seq[2*j-1]*seq[2*j]*(seq[2*j-1]^-1) *(u^-1);

f:=u*(seq[2*j-1]*seq[2*j]*(seq[2*j-1]^-1)*
    (seq[2*j]^-1))*(seq[2*j-1]^-1)*(u^-1);

t5:=seq; t6:=seq;

t5[2*j-1]:= c*seq[2*j-1];
t5[2*h+d]:=  seq[2*h+d]*s*e*seq[2*h+d]*(e^-1)*(seq[2*h+d]^-1)*(s^-1);

t6[2*j]:= (seq[2*j-1]^-1)*c*seq[2*j-1]*seq[2*j];
t6[2*h+d]:= seq[2*h+d]*s*f* seq[2*h+d]*(f^-1)*( seq[2*h+d]^-1)*(s^-1);

Include(~moves, t5);Include(~moves, t6);

end for;end for;

return moves;
```

```
end function;

// This one, starting from a sequence of elements of a group,
// creates all sequences of elements which are equivalent
// to the given one for the equivalence relation
// generated by the Hurwitz moves of type (g'; m_1, ..., m_r),
// with g'>1 or g'>0 and r>1.

HurwitzOrbitGen:=function(h,seq)
orb:={};  Trash:={ seq };
repeat
ExtractRep(~Trash,~gens); Include(~orb, gens);
for newgens in HurwitzMoveGen(h,gens) do
if newgens notin orb then Include(~Trash, newgens);
end if; end for;
until IsEmpty(Trash);
return orb;
end function;

HurwitzOrbit:=function(h,type,seq)
orb:={};
// type (0; m_1,..., m_r)
if h eq 0 then
orb:=HurwitzOrbit0(seq);
// type (1; m)
elif h eq 1 and #type eq 1 then
orb:=HurwitzOrbit1N(seq);
// type  (g'; m_1, ..., m_r),
else
orb:=HurwitzOrbitGen(h,seq);
end if;

return orb;
end function;

// End of Hurwitz moves
```

```
// ClassVectGenscomputes all possible generating vectors
//  for a group isomorphic to H < G of prescribed type and
// return only one representative for  each orbit
// of the equivalence relation generated by the Hurwitz moves

ClassVectGens:=function(G,H,h, type)
Repres:={};   Vects:=VectGens(G,H,h, type);

while not IsEmpty(Vects) do
v:=Rep(Vects);
Include(~Repres,v);
orb:=HurwitzOrbit(h,type,v);
for v1 in orb do
for g in G do  w1:=Conjug(v1,g);
Exclude(~Vects, w1);
end for;end for;
end while;

return Repres;
end function;

// If a group has a generating vector of the right
// type before to look for an  extension, we check
// if the group has a disjoint pair of generating vectors.
// If this is not the case surely a generating vector
// and its conjugation by tau' are not disjoint and Y is not smooth.

DisjGV:=function(H,q,gens1,gens2)
test:=true;
for i in [2*q+1..#gens1] do gen1:=gens1[i];
for j in [2*q+1..#gens2] do gen2:=gens2[j];
for d1 in [1..Order(gen1)-1] do
for d2 in [1..Order(gen2)-1] do
if IsConjugate(H,gen1^d1,gen2^d2) then
test:=false; break i;
```

```
end if;
end for; end for; end for; end for;
return test;
end function;

// These function checks if a group has a disjoint pair of
// generating vectors

ExistDisjointVectGens:=function(groupH,q,type)

s:=SetToSequence(ClassVectGens(groupH,groupH,q,type));
c:=1; test:= false;

for i in [1..#s] do   gens1:=s[i];
for j in [c..#s] do   gens2:=s[j];
if DisjGV(groupH,q, gens1, gens2) then
test:=true; break i;
end if;  end for; c+:=1; end for;

return test;
end function;

// Orbifold builds the orbifold surface group
// of the signature given by q and seq

Orbifold:=function(q,seq)
F:=FreeGroup(2*q+#seq);      G:=Id(F); Rel:={};
for i in [1..q] do  G:=G*(F.(2*i-1)^-1,F.(2*i)^-1);
end for;
for i in [1..#seq] do G:=G*F.(2*q+i); Include(~Rel,F.(2*q+i)^(seq[i]));
end for;
Include(~Rel, G);
return quo<F|Rel>;
end function;

// The group G^0 is a quotient of orbifold surface group
```

```
// T of type (q;type) and so (G^0)^{ab} is
// a quotient of T^{ab}.
// Next script makes this test (only for q=0) before checking
// wheter the group has a generating vector of type (q;type).

AbCheck:=function(q, type, H, group)
test:=false;
AQH:=AbelianQuotient(H);
if q ne 0 then test:=VectGens(H, H, q, type: OnlyExistence:= true);
else  for g in Subgroups(group) do
h:=group/(g`subgroup);
if Order(h) eq Order(AQH) then
if IsIsomorphic(h, AQH) then
test:=VectGens(H, H, q, type: OnlyExistence:= true);  break g;
end if;  end if; end for;end if;

return test;
end function;


//ListGroupsG0 searches among the groups of order ordG0
// for groups with a disjoint pair of generating vectors of type (q;type)

ListGroupsG0:=function(q,type,ordG0)
badorders:={ 512, 768,1152,1280,1536,1920};

set:={}; i:=1;

if  q ne 0 then group:=SmallGroup(1,1);
else
group:=AbelianQuotient(Orbifold(q,type));
end if;

if ordG0 in badorders then
P:= SmallGroupProcess(ordG0);
repeat
```

```
H := Current(P); i;
if AbCheck(q,type,H,group) then
if ExistDisjointVectGens(H,q, type) then
Include(~set, i);
end if; end if; i:=i+1; Advance(~P);
until IsEmpty(P);
else
for H in SmallGroups(ordG0) do i;
if AbCheck(q,type,H,group) then
if ExistDisjointVectGens(H,q, type) then
Include(~set, i);
end if; end if; i:=i+1;
end for;
end if;
return set;
end function;

//
// Step 3: the extensions
//
// We check if the given group "groupH"=G^0
// has some extension of degree 2 with the expected number of
// elements of order 2 and which satisfies
// the condition of Remark 5.4.

Ind:=function(el2G ,G, type)

set:={};
for x in el2G do
cent:=Centralizer(G, x); n:=#G/#cent;
if forall{m: m in type | m le n} then
Include(~set, x);
end if;end for;

return  set;
end function;
```

```
// If 8*chi is not equal to Ksquare, the map CxC-> X is not
// q.e. and so G is a semidirect product of the normal subgroup
// G^0 and the cyclic group Z_2= <z>, that is, we have to give
// morphisms f:Z_2-> Aut(G^0). But if for two morphisms f_1, f_2,
// the images f_1(Z_2) and f_2(Z_2) are conjugated then the
// semidirect products are isomorph, hence we have
// to consider all the images of z in Aut(G^0) up to conjugation.

AutGr:= function(Aut)
A:={ Aut!1 };
repeat
for g1 in Generators(Aut) do
for g2 in A do
Include (~A,g1*g2);
end for; end for;
until  #A eq #Aut;
return A;
end function;

ConjugCl:=function(group, order)
Set:={}; Rep:=[];
list:=[x: x in group | Order(x) eq order];
for el in list do
if el notin Set then
for a in group do
Include(~Set, el^a);
end for;  Append(~Rep, el);
end if; end for;
return Rep;
end function;

// Extension checks if the given group "groupH"=G^0
// has some extension of degree 2 with the expected number
// of elements of order 2 and which satisfies the
```

```
// condition of Remark 5.4, and returns all of them.
//
// If it is possible it constructs the extension as
// semidirect products, otherwise it runs over the groups
// of order |G|.

Extension:=function(idH,q,type,N)
H:=SmallGroup(idH[1],idH[2]);

ext:={};

Aut:=AutomorphismGroup(H);
#Aut;
if ( N ne 0)   and (#Aut le 5000) and CanIdentifyGroup(2*idH[1]) then

A:=AutGr(Aut);
AutOfOrd2:=ConjugCl(A,2) cat [ Aut!1]; //inclusa l'identit
"Aut2!\n";

C2:=SmallGroup(2,1);

for i in [1..#AutOfOrd2] do
map:=hom<C2->Aut| AutOfOrd2[i]>;
G, em :=SemidirectProduct(H,C2,map);
el2G:={x: x in G | Order(x) eq 2 and x notin em(H)};
if #el2G eq N and #Ind(el2G,G,Set(type)) eq N  then
//each m_i is smaller than N_j's
Include(~ext, IdentifyGroup(G));
end if; end for;

else
"ExplicG.\n";
el2H:={x: x in H | Order(x) eq 2};
ordG:= 2*Order(H);
NoG:=NumberOfGroups(SmallGroupDatabase(), ordG);
```

```
for i in [1..NoG] do G:=SmallGroup(ordG, i);
el2G:={x: x in G | Order(x) eq 2};
if (#el2G-#el2H) eq N then
if (N eq 0) or (N le #Ind(el2G,G,Set(type)) ) then
if VectGens(G,H,q,type: OnlyExistence:= true)  then
Include(~ext, <ordG, i>);
end if; end if; end if;
end for;

end if;

return ext;
end function;

// Step 4a: the fundamental group
//
// Next scripts allow us to calculate the topological
// fundamental group of the surfaces we constructed.

// Orbi constructs the orbifold surface group and the
// appropriate orbifold homomorphism.

Orbi:=function(seq, gr, q)
F:=FreeGroup(#seq);    G:=Id(F); Rel:={};
for i in [1..q] do  G:=G*(F.(2*i-1)^-1,F.(2*i)^-1); end for;
for i in [2*q+1..#seq] do G:=G*F.i; Include(~Rel,(F.i)^(Order(seq[i])));
end for;
Include(~Rel, G);
P:=quo<F|Rel>;
return P, hom<P->gr|seq>;
end function;

// MapProd computes given two maps f,g:A->B the map product
// induced by the product on B

MapProd:=function(map1,map2)
```

```
seq:=[];
A:=Domain(map1); B:=Codomain(map1);
if Category(A) eq GrpPC then n:=NPCgens(A);
else n:=NumberOfGenerators(A); end if;
for i in [1..n] do Append(~seq, map1(A.i)*map2(A.i)); end for;
return hom<A->B|seq>;
end function;

// Pi1 uses a generating vector for G^0
// inside G to construct the corresponding orbifold surface group
// and the group HH that acts on the universal cover of CxC.
// Then it constructs the degree 2 extension GG.
// Finally it takes the quotient by Tors(GG).

Pi1:=function(seq,G,q)

H:=sub<G|seq>;
//el:=[g: g in G | g notin H][1];
el:=Random({g: g in G | g notin H});

phi1:=hom<H->H| x:-> (el^-1)*x*(el)>;

T,f1:=Orbi(seq,H,q);  t:=(el^2)@@f1;

TxT,inT,proT:=DirectProduct([T,T]);
HxH,inH:=DirectProduct([H,H]);

Diag:=MapProd(inH[1],inH[2])(H);
f:=MapProd(proT[1]*f1*inH[1],proT[2]*f1*phi1*inH[2]);

bigH:=Rewrite(TxT,Diag@@f);
tt:=inT[1](t)*inT[2](t);
PHI:=hom<bigH->bigH| x:-> inT[1](proT[2](x))*inT[2](t*proT[1](x)*(t^-1))>;

genH:=SetToSequence(Generators(bigH)); relH:=Relations(bigH);
```

```
REL:=[]; im:=[];
F:=FreeProduct(bigH,FreeGroup(1));

for i in [1..#genH] do Append(~im,F.i);  end for;
map:=hom<bigH->F|im>; tau:=map(tt);
ul:=F.(#Generators(F)); Append(~REL, ul^2*(tau^-1));
for i in [1..#genH] do
Append(~REL, map(PHI(genH[i]))* ul * map(genH[i]^-1 )*(ul^-1));
end for;

bigG,pr:=quo<F|REL>;

TG:={};
N:={h: h in H | Order(el*h) eq 2};

for h in N do
x:=h@@f1;   y:=ul*( map( inT[1](x)*inT[2]((t*x)^-1) ) );
Include(~TG, pr(y) );
end for;

return Simplify(quo<bigG|TG>);
end function;

// Step 4b: the Albanese fibre

// GenAlb uses the generating vector seq for G^0
// inside G (giving a surface X with q=1)
// to compute the genus of the Albanese fibre of X.

GenAlb:=function(G, seq);

H,f:=sub<G|seq>;
el:=[x: x in G | x notin H][1];
N:={h: h in H | Order(el*h) eq 2};
HxH,inj, proj:=DirectProduct([H,H]);
r:=[];
```

```
r[1]:= inj[1](seq[1])*inj[2](seq[1]^-1);
r[2]:= inj[1](seq[2])*inj[2](seq[2]^-1);
for i in {3..#seq} do
r[i]:=inj[1](seq[i]);
r[i-2+#seq]:=inj[2](seq[i]);
end for;
Im:=sub<HxH|r>;

set:={};
for g in Im do  for h in H do
Include(~set,inj[1](h* proj[1](g))*inj[2](el*h*el^-1* proj[2](g)));
Include(~set, inj[1](el*h*el^-1*proj[2](g))*inj[2](el^2*h* proj[1](g)));
end for; end for;

comp:=(#H)^2/#set;

T:=0;   for i in {3..#seq} do T+:=(1-1/Order(seq[i])); end for;
eul:= (#H*T)/2-#N;
genus:=1+ (eul/comp);
return genus;
end function;

//
// Step 5:to find all the surfaces
//

// There is a surface for each pair (group, generating vector)
// which passes previous  tests, but they are often
// deformation equivalent. More precisely,
// it happens if the generating vectors
// are equivalent for the equivalence relation generated
// by Hurwitz moves and the automorhisms of the group.
// We need to construct orbits for this equivalence relation.

OrbitsVectGens:=function(G,H,q, type)
Orbits:={};   Vects:=VectGens(G,H,q, type);
```

```
Aut:=AutomorphismGroup(G);
A:=AutGr(Aut);
while not IsEmpty(Vects) do
v:=Rep(Vects);
Include(~Orbits,v);
orb:=HurwitzOrbit(q,type,v);
for v1 in orb do
for phi in A do
Exclude(~Vects,phi(v1));
if IsEmpty(Vects) then break v1;    end if;
end for;
end for;
end while;

return Orbits;
end function;

// BranchCurves chechs if a generating vector for G^0
// and the extension G give a smooth surface
// Y and  a quotient map CxC-> X with the expected branch locus

BranchCurves:=function(groupG, q, gens, expN, type)

curves:=[**]; N:=0; test:=true; gens2:=[];
groupH:= sub<groupG|gens>; F:=( Theta(q,type)/2 ); //F=beta/#G0
tp:=[g: g in groupG | g notin groupH][1]; //tau'

for i in [1..#gens] do Append(~gens2, gens[i]^tp); end for;

if DisjGV(groupH,q,gens,gens2) then

RamCurves:={x: x in groupG| (not x in groupH) and (Order(x) eq 2)};
while not IsEmpty(RamCurves) do
c:=Rep(RamCurves);
cent:=Centralizer(groupG, c); nj:=#groupG/#cent;
```

```
// index of Z(tau' k) = number of ram. curves mapped onto
// the same branch curve, i.e. cardinality of the conj. class
if IsIntegral(nj*F) then
N:=N+nj; Append(~curves, [ 1+nj*F,-4*nj*F]);
for x in groupG do Exclude(~RamCurves, c^x); end for;
// we remove the elements conjugated to c.
else
test:=false; return test, [**];
end if;
end while;

else test:=false;

end if;

return (test and (expN eq N)),  curves;
end function;

// For each triple (G,G^0, type) in the output of Step 3,
// we check which pairs (G, gen.vect),
// give a semi-isogenous mixed surface with the expected
// invariants and we print them in the file F.

FindSurfaces:=function(idG,idH,type,Ksquare,pg,q, New,F)
chi:=1+pg-q; d:=0;

G:=SmallGroup(idG[1],idG[2]);
H:=SmallGroup(idH[1],idH[2]);

beta:=Beta(Ksquare, chi,Theta(q,type));
N:=IntegerRing()!(beta*(8*chi-Ksquare)/(10*chi-Ksquare) );

Orbs:=OrbitsVectGens(G,H, q, type);

for gens in Orbs do
t, curves:=BranchCurves(G, q, gens, N, type) ;
```

```
if t then d:=d+1;
fprintf F, "Surface number %o \n", New+d;
fprintf F, "G:%o\n", idG;
fprintf F, "G^0: %o\n",  idH;
fprintf F, "genus of C: %o\n",  1+Beta(Ksquare, 1+pg-q, Theta(q,type));
fprintf F, "\n";
fprintf F, "Type of generating vector: %o\n",type;
fprintf F, "Generating Vector: %o \n", gens;
fprintf F, "\n";
fprintf F, "Branch curves: %o \n", curves;
fprintf F, "\n";
fprintf F, "H1(S,ZZ): %o \n", AbelianQuotient(Pi1(gens, G,q));
if q eq 1 then
fprintf F, "\n";
fprintf F, "genus Albanese Fiber: %o \n", GenAlb(G, gens);
end if;
fprintf F, "\n\n\n";
end if; end for;

return d;
end function;

// Output is the main function of the scripts.
// It calls the previous function and manages the outputs.

Output:=function(Ksquare, pg,q)

RT:=Realtime();

chi:=1-q+pg;

F:= "Semi_isog_mix_surf_pg" cat IntegerToString(pg)
cat "_q" cat IntegerToString(q) cat  "_Ks" cat
IntegerToString(Ksquare) cat ".txt";

fprintf F, "K^2=%o, pg=%o and q=%o\n\n\n", Ksquare,pg,q;
```

```
Types:=ListTypes(Ksquare, pg,q);

"Number of types", #Types;

ListG0s:=[**];
for T in Types do T;
ordG0:=T[#T];
if (not IsInSmallGroupDatabase(ordG0))  then
fprintf F, " TO CHECK : %o\n", T;
else AdmisG0:=ListGroupsG0(q,Prune(T), ordG0);
if #AdmisG0 ne 0 then  Append(~ListG0s, [*ordG0, AdmisG0, Prune(T)*]);
end if; end if; end for;

"Number of types with a G0:", #ListG0s;

fprintf F,"Types with a G0:%o\n\n\n", ListG0s;

ListG:=[**];
for triple in ListG0s do
ordG:=2*triple[1];
if  (not IsInSmallGroupDatabase(ordG))  then
fprintf F, "TO CHECK: %o\n", triple;
else for x in triple[2] do
printf "Searching extension\n";
N:=IntegerRing()!(2*(8*chi-Ksquare)/Theta(q,triple[3]));
Extens:=Extension([triple[1],x],q,triple[3],N);
if #Extens ne 0 then Append(~ListG,[*Extens,<triple[1],x>,triple[3]*]);
end if; end for; end if;
end for ;

"Number of candidate groups G:", #ListG;
fprintf F,"Number of candidate groups G:%o\n\n\n", ListG;

New:=0;
```

```
for trip in ListG do
Ext:=trip[1]; idH:=trip[2];  type:=trip[3]; printf "Checking Groups\n";
trip;
for idG in Ext do idG;
newsurf:=FindSurfaces(idG, idH, type, Ksquare, pg, q, New, F);
New:=New+newsurf;
end for; end for;

fprintf F,"Time: %o\n", Realtime(RT);
printf "Time: %o\n", Realtime(RT);
return "Found ",New," families of surfaces";
end function;
```

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