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Intrinsic differentiability and Intrinsic Regular Surfaces in Carnot groups

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Ai miei genitori Tiziana e Franco,
e a mio fratello Pier Paolo,
per il loro sostegno.

A mia nonna Fidalma,
che da lassú veglia su di me.

Contents

Basic notation

$\Subset$	compactly contained
$\oplus$	direct sum of vector spaces
$\circ$	composition of functions
$\mathbb{R}^N$	N-dimensional Euclidean space
$\partial_{x_i} f$	partial derivative of the function f with respect to x_i
$\mathcal{L}^N$	Lebesgue measure in $\mathbb{R}^N$
$\mathbb{G}$	a Carnot group
$\mathbb{H}^k$	k-th Heisenberg group
$\mathfrak{g}$	Lie algebra of $\mathbb{G}$
d_{cc}	Carnot-Carathéodory distance
d	invariant distance on $\mathbb{G}$
$\lvert \cdot \rvert$	Euclidean norm in $\mathbb{R}^N$
$\lVert \cdot \rVert$	homogeneous norm in a Carnot group
χ_E	characteristic function of a measurable set $E \subset \mathbb{R}^N$
$⨍$	average integral
$\mu \llcorner \mathcal{E}$	restriction of a measure μ to a set $\mathcal{E}$
$\dot{\gamma}$	time derivative of a curve γ
$[X, Y]$	commutator of vector fields $X, Y \in \mathfrak{g}$
τ_P	left translation by an element $P \in \mathbb{G}$
δ_λ	homogeneous dilations in $\mathbb{G}$
$\mathfrak{q}$	homogeneous dimension of $\mathbb{G}$
TM, $T_P M$	tangent bundle to a manifold M and tangent space at P
HM,$H_P M$	horizontal subbundle to M and horizontal subspace at P
∇f	Euclidean gradient of f
$\nabla_{\mathbb{G}} f$	horizontal gradient of f
$\nu_S(P)$	horizontal normal to S at $P \in S$
$D^\phi \phi$	intrinsic gradient of ϕ
$d\phi_A$	intrinsic differential of ϕ at A
$\mathrm{div}(\phi)$	divergence of ϕ
$\mathrm{div}_{\mathbb{G}}(\phi)$	horizontal divergence of ϕ

$\mathrm{spt}(f)$	support of f
$\mathbb{C}^k(\Omega)$	continuously k-differentiable real functions in Ω
$\mathbb{C}^k_c(\Omega)$	functions in $\mathbb{C}^k(\Omega)$ with compact support in Ω
$\mathbb{C}^1_{\mathbb{G}}(\Omega)$	continuously $\nabla_{\mathbb{G}}$-differentiable functions in Ω
$\mathrm{BV}(\Omega)$	space of functions with bounded variation in Ω
$\mathrm{BV}_{\mathbb{G}}(\Omega)$	space of functions with bounded $\mathbb{G}$-variation in Ω
$\lvert\partial E\rvert_{\mathbb{G}}$	$\mathbb{G}$-perimeter measure of a measurable set E
$\partial^*_{\mathbb{G}}E$	reduced boundary of a measurable set E
$\mathcal{H}^t_e$	t-dimensional Hausdorff measure in $\mathbb{R}^N$ in the Euclidean metric
$\mathcal{H}^t$	t-dimensional Hausdorff measure induced by invariant metric d
$\mathcal{S}^t_e$	t-dimensional spherical Hausdorff measure in $\mathbb{R}^N$ in the Euclidean metric
$\mathcal{S}^t$	t-dimensional spherical Hausdorff measure induced by invariant metric d
$U(P,r)$	open ball associated with d, centered at P having radius r
$B(P,r)$	closed ball associated with d, centered at P having radius r
$\mathbf{M}_{k\times m}$	the set of matrices with k rows and m columns
$\mathbb{I}_m$	the unit matrix of order m
$\lVert\mathcal{B}\rVert$	operator norm of a matrix $\mathcal{B}$

Introduction

In this thesis we deal with a particular class of sub-Riemannian manifold, i.e. Carnot groups.

A sub-Riemannian manifold is defined as a manifold M of dimension N together with a distribution D of m-planes ($m \leq N$) and a Riemannian metric on D. From this structure, a distance on M is derived as follows: the length of an absolutely continuous path tangent to D is defined via the Riemannian metric on D, and the distance of two points of M is in turn defined as the infimum of the lengths of absolutely continuous paths that are tangent to D and join these two points.

Sub-Riemannian Geometry has been a research domain for many years, with motivations and ramifications in several parts of pure and applied mathematics, namely: Control Theory [18], [99]; Riemannian Geometry (of which Sub-Riemannian Geometry constitutes a natural generalization); Analysis of hypoelliptic operators [55], [91].

We should mention also here Sobolev spaces theory and its connections with Poincaré-type inequalities [20], [48], [57]; the theory of quasiconformal mappings [60], [61]; the theory of convex functions [28], [37], [88], [100]; the theory of harmonic analysis on the Heisenberg group [101], [56]. But this list of subjects is surely incomplete.

Now we recall the definition of Carnot-Carathéodory (CC) space. A CC space is an open subset $\Omega \subset \mathbb{R}^N$ (or, more generally, a manifold) endowed with a family $X = (X_1, \dots, X_m)$ of vector fields such that every two points $P, Q \in \Omega$ can be joined, for some $T > 0$, by an absolutely continuous curve $\gamma : [0, T] \to \Omega$ such that

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)).$$

We call subunit such a curve and, according to the terminology in [50] and [83], we define the Carnot-Carathéodory distance between P and Q as

$$d_{cc}(P, Q) = \inf \left\{ T \geq 0 : \text{there is a subunit curve } \gamma : [0, T] \to \mathbb{R}^N \text{ with } \gamma(0) = P, \gamma(T) = Q \right\}.$$

The problem of connecting points by means of subunit curves was studied by Rashevsky in [86] and Chow in [26]. They independently proved that a sufficient condition for connectivity is that the distribution of subspaces Lie generating the whole tangent space at every point of Ω. This condition has subsequently played a key role in several branches of Mathematics (e.g. Nonholonomic Mechanics, Subelliptic PDE's and Optimal Control Theory), under the different names of "Hörmander condition", "total nonholonomicity", "bracket generating condition", "Lie algebra rank condition" and "Chow condition".

Hence, the Chow condition ensures d_{cc} is a finite distance.

In particular, if the vector fields $X_1, \dots, X_m$ define a smooth distribution on Ω that satisfy the Chow condition the resulting CC space is a sub-Riemmaninan space.

Among CC spaces, a fundamental role is played by Carnot groups. They seem to owe their name to a paper by Carathéodory [22] (related to a mathematical model of thermodynamics) dated 1909. The same denomination was then used in the school of Gromov [49] and it is commonly used nowadays. In the literature, the name "stratified group" is also used, following the terminology of [36].

A Carnot group $\mathbb{G}$ is a connected and simply connected nilpotent Lie group. Through exponential coordinates, we can identify $\mathbb{G}$ with $\mathbb{R}^N$ endowed with the group operation $\cdot$ given by the Baker-Campbell-Hausdorff formula. Classical references to the Carnot groups are [35], [83], [104], [103], [17] and to the Baker-Campbell-Hausdorff formula are [52], [102].

If $\mathfrak{g}$ denotes the Lie algebra of all left invariant first order differential operators on $\mathbb{G}$, then $\mathfrak{g}$ admits a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_\kappa$ where κ is called the step of $\mathbb{G}$. When $\kappa = 1$, $\mathbb{G}$ is isomorphic to $(\mathbb{R}^N, +)$ and this is the only commutative Carnot group.

The stratification has the further property that the entire Lie algebra $\mathfrak{g}$ is generated by its first layer $\mathfrak{g}_1$, the so-called horizontal layer, that is

$$\begin{cases} [\mathfrak{g}_1, \mathfrak{g}_{i-1}] = \mathfrak{g}_i & \text{if } 2 \le i \le \kappa \\ [\mathfrak{g}_1, \mathfrak{g}_\kappa] = \{0\} \end{cases} \tag{1}$$

where $[\mathfrak{g}_1, \mathfrak{g}_i]$ is the subspaces of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_i$. We remark that (1) guarantees that any basis of $\mathfrak{g}_1$ satisfies the Chow condition and so $(\mathbb{G}, d_{cc})$ is a metric space.

The stratification of $\mathfrak{g}$ induces, through the exponential map, a family of non isotropic dilations δ_λ for $\lambda > 0$. These maps δ_λ, called intrinsic dilations, are one of the most important features of the group. They are compatible with the cc-metric in the sense that

$$d_{cc}(\delta_\lambda P, \delta_\lambda Q) = \lambda d_{cc}(P, Q), \qquad \text{for all } P, Q \in \mathbb{G}, \lambda > 0$$

and well behave with respect to the group operation $\delta_\lambda(P \cdot Q) = \delta_\lambda P \cdot \delta_\lambda Q$.

The intrinsic left translations of $\mathbb{G}$ are another important family of transformations of $\mathbb{G}$: for any $P \in \mathbb{G}$ the left translation $\tau_P : \mathbb{G} \to \mathbb{G}$ is defined as $\tau_P Q := P \cdot Q$, for all $Q \in \mathbb{G}$.

It is useful to consider on $\mathbb{G}$ a homogeneous norm, i.e. a nonnegative function $P \to \|P\|$ on $\mathbb{G}$ such that

1. $\|P\| = 0$ if and only if $P = 0$.

2. $\|\delta_\lambda P\| = \lambda \|P\|$ for all $P \in \mathbb{G}$ and $\lambda > 0$.

3. $\|P \cdot Q\| \leq \|P\| + \|Q\|$ for all $P, Q \in \mathbb{G}$.

Given any homogeneous norm $\| \cdot \|$, it is possible to introduce a distance in $\mathbb{G}$ given by

$$d(P, Q) = d(P^{-1}Q, 0) = \|P^{-1}Q\|$$

for all $P, Q \in \mathbb{G}$. This distance d is equivalent to d_{cc}.

The importance of Carnot groups became evident in [76], where it was proved that a suitable blow-up limit of a sub-Riemannian manifold at a generic point is a Carnot group. In other words, Carnot groups can be seen [11] as the natural "tangent spaces" to sub-Riemannian manifolds, and therefore can be considered as local models of general sub-Riemannian manifolds. Therefore there is a comparison between sub-Riemannian Geometry and Riemannian Geometry: Carnot groups are to sub-Riemannian manifolds what Euclidean spaces are to Riemannian manifolds.

All these features could remind us of the familiar Euclidean structure, but as soon as we consider non Abelian groups, we see that in many respects we are dealing with something that is closer to the fractal geometry. We stress explicitly that, in general, cc-distances are not Euclidean at any scale, and hence not Riemannian (see [93]). Indeed, there are no (even local) bilipschitz maps from a general non commutative Carnot group $\mathbb{G}$ to Euclidean spaces.

Moreover, in the non Abelian case $\kappa > 1$ the Hausdorff dimension of $\mathbb{G}$ with respect to the cc-distance is

$$\mathfrak{q} := \sum_{i=1}^{\kappa} i \dim \mathfrak{g}_i$$

and $\mathfrak{q}$ is always greater than the topological dimension of $(\mathbb{G}, d_{cc})$ which is equal to that of $\mathbb{G}$ endowed the Euclidean distance. This is a typical feature of fractal objects.

A *sub-Riemannian structure* is defined on $\mathbb{G}$ as follows: we call horizontal bundle $H\mathbb{G}$ the subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the left invariant vector fields $X_1, \ldots, X_m$ belonging to $\mathfrak{g}_1$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_P = \mathrm{span}\{X_1(P), \ldots, X_m(P)\}, \qquad P \in \mathbb{G}.$$

Then we consider $\mathbb{G}$ endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle \cdot, \cdot \rangle_P$ and a norm $| \cdot |_P$ making the basis $X_1(P), \ldots, X_m(P)$ an orthonormal basis. That is if $v = \sum_{i=1}^m (v_1)_i X_i(P)$ and $w = \sum_{i=1}^m (w_1)_i X_i(P)$ are in $H\mathbb{G}$, then $\langle v, w \rangle_P := \sum_{i=1}^m (v_1)_i (w_1)_i$ and $|v|_P^2 := \langle v, v \rangle_P$. We will write, with abuse of notation, $\langle \cdot, \cdot \rangle$ meaning $\langle \cdot, \cdot \rangle_P$ and $| \cdot |$ meaning $| \cdot |_P$.

The sections of $H\mathbb{G}$ are called *horizontal sections*, a vector of $H\mathbb{G}_P$ is an *horizontal vector* while any vector in $T\mathbb{G}_P$ that is not horizontal is a vertical vector. Each horizontal section ϕ defined on an open set $\Omega \subset \mathbb{G}$ can be written as $\phi = \sum_{i=1}^{m_1} \phi_i X_i$, where its coordinates are functions $\phi_i : \Omega \to \mathbb{R}$. When considering two such sections ϕ and ψ, we will write $\langle\psi, \phi\rangle$ for $\langle\psi(P), \phi(P)\rangle_P$.

A classical theorem due to Pansu [83] states that Lipschitz maps between Carnot groups have a differential which is a homogeneous homomorphism. In other words, Pansu extends the Rademarcher's Theorem to Carnot groups introducing a suitable notion of differentiability. Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be Carnot groups with homogeneous norm $\|\cdot\|_1, \|\cdot\|_2$ and let $\Omega \subset \mathbb{G}^{(1)}$ be an open set. Then $f : \Omega \to \mathbb{G}^{(2)}$ is P-differentiable in $P \in \Omega$ if there exists a H-linear function $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ such that

$$\|(l(P^{-1}Q))^{-1}f(P)^{-1}f(Q)\|_2 = o(\|P^{-1}Q\|_1), \quad \text{ as } \|P^{-1}Q\|_1 \to 0.$$

Here the H-linear map l is called P-differential of f in P. The P-differentiability is an intrinsic notion since it employs the group operation, dilations and the natural family of "linear maps" of the group, i.e. H-linear maps. We recall that a H-linear map is a group homomorphism that is homogeneous with respect to dilations.

We stress that for a function $f : \Omega \subset \mathbb{G} \to \mathbb{R}$ the P-differentiability simply means that f is continuous and its horizontal gradient $\nabla_{\mathbb{G}} f = (X_1 f, \dots, X_m f)$ is represented, in distributional sense, by continuous functions. In this case we write $f \in \mathbb{C}^1_{\mathbb{G}}(\Omega)$.

The differentiability on Carnot groups also provides a natural way to introduce "intrinsic regular surfaces". This concept was first introduced by Franchi, Serapioni and Serra Cassano in [44], [45], [47] in order to obtain a natural notion of rectifiability. Indeed the rectifiable sets are classically defined as contained in the countable union of $\mathbb{C}^1$ submanifolds. A general theory of rectifiable sets in Euclidean spaces has been accomplished in [34], [33], [72] while a general theory in metric spaces can be found in [6].

More precisely, we say that a subset $S \subset \mathbb{G}$ is $\mathbb{G}$-regular hypersurface (i.e. a topological codimension 1 surface) if it is locally defined as a non critical level set of $\mathbb{C}^1_{\mathbb{G}}$ function; that is if there is a continuous function $f : \mathbb{G} \to \mathbb{R}$ such that locally

$$S = \{P \in \mathbb{G} : f(P) = 0\}$$

and the horizontal gradient $\nabla_{\mathbb{G}} f$ is continuous and non vanishing on S. In a similar way, a k-codimensional $\mathbb{G}$-regular surface is locally defined as a non critical level set of a $\mathbb{C}^1_{\mathbb{G}}$ vector function $F : \mathbb{G} \to \mathbb{R}^k$.

The notion of $\mathbb{G}$-regularity can be extended to subsets of higher codimension and modeled on the geometry of another Carnot group. This is precisely explained by the notion of $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface, introduced and studied by Magnani in [69], [70]. The author distinguishes between two different classes of regular surfaces:

- $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surfaces of $\mathbb{G}^{(1)}$ which are defined as level sets of P-differentiable maps with "regular" surjective P-differential.

- $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surfaces of $\mathbb{G}^{(2)}$ which are defined as images of P-differentiable maps with "regular" injective P-differential.

Here "regular" surjective (or injective) P-differential means that we consider a special class of surjective (or injective) H-linear maps, called H-epimorphisms (or H-monomorphisms), that yield the natural splitting of $\mathbb{G}^{(1)}$ (or $\mathbb{G}^{(2)}$) as a product of complementary subgroups. We recall that $\mathbb{M}$ and $\mathbb{W}$ are complementary subgroups of $\mathbb{G}$ if they are both subgroups closed under dilations and such that $\mathbb{W} \cap \mathbb{M} = \{0\}$ and $\mathbb{G} = \mathbb{W} \cdot \mathbb{M}$ (here $\cdot$ indicates the group operation in $\mathbb{G}$ and 0 is the unit element).

To obtain a complete classification of these surfaces we need find all possible factorization of $\mathbb{G}^{(1)}$ or of $\mathbb{G}^{(2)}$. This is well known when we consider the Heisenberg groups where the only intrinsic regular surfaces are the $(\mathbb{H}^n, \mathbb{R}^k)$-regular surfaces and the $(\mathbb{R}^k, \mathbb{H}^n)$-regular surfaces for $k = 1, \dots, n$ (see [46]). But, in a general Carnot group, the understanding of the intrinsic regular surfaces is very far from being complete.

A fine characterization of $(\mathbb{H}^k, \mathbb{R})$-regular surfaces of $\mathbb{H}^k$ as suitable 1-codimensional intrinsic graphs has been established in [7]. The main purpose of this thesis is to generalize this result when we consider $(\mathbb{G}, \mathbb{R}^k)$-regular surfaces of $\mathbb{G}$. Here $(\mathbb{G}, \mathbb{R}^k)$-regular surfaces of $\mathbb{G}$ are simply called $\mathbb{G}$-regular surfaces and the intrinsic graphs are defined as follows: let $\mathbb{M}$ and $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$, then the intrinsic left graph of $\phi : \mathbb{W} \to \mathbb{M}$ is the set

$$\mathrm{graph}\,(\phi) := \{A \cdot \phi(A) \mid A \in \mathbb{W}\}.$$

Therefore the existence of intrinsic graphs depends on the possibility of splitting $\mathbb{G}$ as a product of complementary subgroups and so it depends on the structure of the algebra $\mathfrak{g}$. This concept is intrinsic because if $S = \mathrm{graph}\,(\phi)$ then, for all $\lambda > 0$ and all $Q \in \mathbb{G}, \tau_Q(S)$ and $\delta_\lambda(S)$ are also intrinsic graphs.

Differently from the Euclidean case where setting $\mathbb{C}^1$ surfaces can be locally viewed as non-critical level sets of $\mathbb{C}^1$-functions or, *equivalently*, graphs of $\mathbb{C}^1$ maps between complementary linear subspaces, in Carnot groups the corresponding notion of $\mathbb{G}$-regular surface is *not* equivalent to that of intrinsic graph any more. The objective of our research is to study the equivalence of these natural definitions in Carnot groups.

However, thanks to Implicit Function Theorem, proved in [44] for the Heisenberg group and in [45] for a general Carnot group (see also Theorem 1.3, [70]) it follows that

$$S \text{ is a } \mathbb{G}\text{-regular surface} \quad \Longrightarrow \quad S \text{ is (locally) the intrinsic graph of a map } \phi.$$

Here we say that ϕ is a parametrization of the $\mathbb{G}$-regular surface.

Consequently, our main purpose is the following fact: given an intrinsic graph of continuous map ϕ, we want to find necessary and sufficient assumptions on ϕ in order that the opposite implication is true.

We will see that these additional assumptions will be characterized in terms of an appropriate notion of differentiability, denoted *intrinsic differentiability*, for maps acting between complementary subgroups $\mathbb{W}$ and $\mathbb{M}$ of $\mathbb{G}$. More precisely, a function is intrinsic differentiable if it is well approximated by appropriate linear type functions, denoted intrinsic linear functions.

When $\mathbb{W}$ and $\mathbb{M}$ are both normal subgroup, the notion of intrinsic differentiability corresponds to that of P-differentiability.

The title of the thesis is: "Intrinsic differentiability and Intrinsic Regular Surfaces in Carnot groups". We show that the intrinsic graph of uniform intrinsic differentiable maps is (locally) a $\mathbb{G}$-regular surface. In particular, the original contributions of the author in collaboration with R. Serapioni are illustrated in Chapter 3.

Our aim is to examine the most basic properties of submanifolds in $\mathbb{G}$ from the viewpoint of Geometric Measure Theory, considering for instance perimeter measures, area formulae, parametrizations, etc.

In the last fifty years many authors have tried to develop a Geometric Measure Theory in Carnot groups or more generally in CC spaces (see [33], [34], [72], [81], [97]). The first result in this sense probably traces back to the proof of the isoperimetric inequality in the Heisenberg group [84] (see also [20], [48]). An essential item of Geometric Measure Theory such as De Giorgi's notion of perimeter [30], [31] has been extended in a natural way to CC spaces (see [20], [68], [43], [29], [16], [54]). In particular, in Carnot groups the $\mathbb{G}$-perimeter of a measurable set $E \subset \Omega$ is defined as

$$|\partial E|_{\mathbb{G}}(\Omega) := \sup \left\{ \int_E \mathrm{div}_{\mathbb{G}} \phi \, d\mathcal{L}^N : \phi \in \mathbb{C}^1_c(\Omega, H\mathbb{G}), |\phi(P)| \leq 1 \right\}$$

where $\mathrm{div}_{\mathbb{G}} \phi := \sum_{j=1}^m X_j \phi_j$.

The perimeter measure has good natural properties, such as an integral representation [79] in case of sets with smooth boundary or its $(\mathfrak{q}-1)$-homogeneity in Carnot groups setting.

More generally, it is also possible to give a good definition of functions of bounded variation [16], [20], [36], [43], which fits the one given for functions in general metric spaces [75]. The theory of minimal surfaces has been investigated [25], [19], [48], [85], and also differentiability of Lipschitz maps [83], [24], [106]; fractal geometry [10]; area and coarea formulae [43], [69], [78] and the isoperimetric problem [64], [90], [53] provided prosperous research themes. However, basic techniques of classical Euclidean Geometry do not admit any counterpart in the CC settings, like Besicovitch covering theorem [89], while many others are still open or only partially solved.

The structure of this thesis is the following. In Chapter 1 we present the basic theory of Carnot groups together with several remarks. We consider them as special classes of Lie groups equipped with a invariant distance and with the intrinsic dilations which are automorphisms of group.

In Section 1.1 we recall the definition of Carnot-Carathéodory distance and the Chow-Rashevsky theorem, and then in Section 1.2 we go to a brief analysis of the Lie groups and their Lie algebras of the left-invariant vector fields. Here we present some fundamental results proved in [107]. In particular, Theorem 1.2.4 states the exponential map is a global diffeomorphism from the Lie algebra $\mathfrak{g}$ of a Lie group $\mathbb{G}$ to $\mathbb{G}$.

Therefore in Carnot groups any point $P \in \mathbb{G}$ can be written in a unique way as $P = \exp(p_1 X_1 + \cdots + p_N X_N)$ and we identify P with $(p_1, \ldots, p_N)$ and $\mathbb{G}$ with $(\mathbb{R}^N, \cdot)$, where the group operation $\cdot$ is determined by the Baker-Campbell-Hausdorff formula. In particular, in Section 1.3.3, using well-behaved group of dilations, we equip Carnot groups with an explicit group law (see Theorem 1.3.7).

A very special emphasis is given to the examples. In Section 1.4 we introduce and discuss a wide range of explicit Carnot groups of step 2. Some of them have been known in specialized literature for several years, such as the Heisenberg groups [21], [98]; the free step 2 groups [17]; the H-type groups [58]; the H-groups in the Sense of Métivier [74]; the complexified Heisenberg group [87]. Following [17], we show that these Carnot groups are naturally given with the data on $\mathbb{R}^{m+n}$ of n suitable linearly independent and skew-symmetric matrices of order m.

In Chapter 2 we provide the definitions and some properties about the differential calculus within Carnot groups.

After a brief description of complementary subgroups of a Carnot group (Section 2.1), we analyze the notion of H-linear maps. In particular, in Section 2.2.2 we investigate the algebraic conditions under which either surjective or injective H-linear functions are respectively H-epimorphisms or H-monomorphisms. Here we present some results obtained in [70].

In Section 2.3 we give a detailed description of Pansu differentiability, with particular emphasis on $\mathbb{C}^1_{\mathbb{G}}$ functions. Lemma 2.3.7 contains an estimate on horizontal difference quotients of $\mathbb{C}^1_{\mathbb{G}}$ functions which will be crucial in the proof of Theorem 3.1.1, while the main result of this section is Whitney Extension Theorem 2.3.8: its proof was proved in [44] for Carnot groups of step 2 only, but here we give a complete one which is similar.

Then we define and characterize the notion of $\mathrm{BV}_{\mathbb{G}}$ function and of locally finite $\mathbb{G}$-perimeter set. In particular, Theorem 2.3.14 states the perimeter measure equals a constant times the spherical $(\mathfrak{q}-1)$-dimensional Hausdorff measure restricted to the so-called reduced boundary (Definition 2.3.6) when $\mathbb{G}$ is step 2 Carnot group.

In Section 2.4 we introduce one of the main objects of the book, namely $\mathbb{G}$-regular surfaces. This part is taken from a recent paper of Magnani [70]. In particular, Theorem 2.4.3 and 2.4.4 describe all regular surfaces of the Heisenberg groups and of complexified Heisenberg group, respectively.

Section 2.4.2 is devoted to a brief survey of notion of rectifiability on step 2 Carnot groups, presented and discussed in [47] and recently extended in [71]. We summarize (without proofs) the results of [47] concerning rectifiability of locally finite $\mathbb{G}$-perimeter sets. The main result in this sense is Theorem 2.4.7, whence it follows that the reduced boundary of a locally finite $\mathbb{G}$-perimeter set, up to $\mathcal{H}^{\mathfrak{q}-1}$-negligible sets, is contained in a countable union of $\mathbb{G}$-regular hypersurfaces.

In Section 2.5 we talk about intrinsic graphs theory. Implicit Function Theorem 2.5.4 shows that any regular surface locally admits a parametrization. Following the recent papers [39], [38], [94] we discuss about the intrinsic Lipschitz graphs and some their properties. We also mention the paper [42], where this concept appears for the first time when $\mathbb{G}$ is an Heisenberg group.

In Section 2.6, we give the general definition of intrinsic differentiability and then we provide the basic tools for the analysis of parametrizations of $\mathbb{G}$-regular surfaces. Namely, for any fixed continuous function $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ where $\mathbb{M}$ is horizontal subgroup (i.e. its Lie algebra is contained in horizontal layer of $\mathbb{G}$), it is possible to introduce a stronger, i.e. uniform, notion of intrinsic differentiability in a Carnot group $\mathbb{G} = \mathbb{W} \cdot \mathbb{M}$.

The main item of Chapter 3 is Theorem 3.1.1, where we prove that given a continuous map $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ where $\mathbb{M}$ is horizontal subgroup, it parametrizes a $\mathbb{G}$-regular surface if and only if ϕ is a uniform intrinsic differentiable map. Moreover Theorem 3.1.5 states that the class of uniform intrinsic differentiable functions is a large class of functions. Indeed it includes the class of $\mathbb{C}^1$ functions.

We want to stress in particular the importance of the operator D^ϕ, which appears in the proof of Theorem 3.1.1 and which seems to be the correct intrinsic replacement of Euclidean gradient for $\mathbb{C}^1$ surfaces.

In Heisenberg groups, it is known after the results in [7], [15] that the intrinsic differentiability of ϕ is equivalent to the existence and continuity of suitable 'derivatives' $D^\phi_j \phi$ of ϕ. The non linear first order differential operators D^ϕ_j were introduced by Serra Cassano et al. in the context of Heisenberg groups $\mathbb{H}^n$ (see [95] and the references therein). Following the notations in [95], the operators D^ϕ_j are denoted as *intrinsic derivatives* of ϕ and $D^\phi\phi$, the vector of the intrinsic derivatives of ϕ, is the intrinsic gradient of ϕ.

Regarding the operator D^ϕ, we also mention the papers [13], [14], [23], [80], [96] when $\mathbb{G}$ is an Heisenberg group.

In Section 3.2 we analyze the $\mathbb{G}$-regular hypersurfaces in a particular subclass of 2 step Carnot groups. More precisely, we characterize the uniform intrinsic differentiable map $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ where $\mathbb{M}$ is 1 dimensional subgroup of $\mathbb{G}$ (and consequently horizontal) in terms of suitable notions of weak solution for the non-linear first order PDEs' system

$$D^\phi\phi = w \quad \text{in } \mathcal{E}, \tag{2}$$

being w a prescribed continuous function and $\mathcal{E} \subset \mathbb{R}^{N-1}$. In particular in [7] it was introduced the concept of broad* solution of the system (2). They show that in $\mathbb{H}^1$ this notion extends the classical notion of broad solution for Burger's equation through characteristic curves provided ϕ and w are locally Lipschitz continuous. In our case ϕ and w are supposed to be only continuous then the classical theory breaks down. On the other hand broad* solution of the system (2) can be constructed with a continuous datum w.

More specifically, in Theorem 3.2.7 we prove that the intrinsic graph of continuous map ϕ is a regular surface if and only if ϕ is broad* solution of (2) and it is 1/2-little Hölder continuous. We also show that these assumptions are equivalent to the fact that ϕ and its intrinsic gradient $D^{\phi}\phi$ can be uniformly approximated by $\mathbb{C}^1$ functions.

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Chapter 1

Carnot groups

In this chapter, we introduce the main notations and the basic definitions concerning with vector fields: algebras of vector fields, exponentials of smooth vector fields, Lie brackets. Then we study Lie groups and the Lie algebra of their left invariant vector fields. Finally, we introduce the main geometric structure investigated throughout the thesis: the Carnot groups. We will take most of the material from [17], [69], [95], [105].

In Section 1.1 we provide a brief exposition of general features concerning Carnot Carathéodory spaces (see [49], [78]); we recall the definitions of subunitary curve and of Carnot Carathéodory metric, or cc-metric in short, which is an actual distance thanks to Theorem 1.1.6, so-called Chow-Rashevsky Theorem. Here Rashevsky in [86] and Chow in [26] independently proved that a sufficient condition for connectivity is the distribution of subspaces Lie generating the whole tangent space at every point.

Section 1.2 is entirely concerned with Lie groups and Lie algebras: we recall some basic facts about Lie groups, providing all the terminology and the main results about the left invariant vector fields, the homomorphisms, the exponential map, the Baker-Campbell-Hausdorff formula (see the monographs [27] and [102] for more references).

The importance of Lie algebras lies in the fact that there is a special finite dimensional Lie algebra intimately associated with each Lie group, and that properties of the Lie group are reflected in properties of its Lie algebra. For instance, the connected, simply connected Lie groups are completely determined (up to isomorphism) by their Lie algebras. Therefore the study of these Lie groups is limited in large part to a study of their Lie algebras. Theorem 1.2.1 and Theorem 1.2.4 are examples of this link.

In Section 1.3 we analyze the Carnot groups $\mathbb{G}$ with particular emphasis on their most relevant peculiarities, such as dilations and invariant metrics in $\mathbb{G}$. We recall that a Carnot group of step κ is connected, simply connected Lie group whose Lie algebra admits a step κ stratification. The Heisenberg groups $\mathbb{H}^k$ are the simplest but, at the same time, non-trivial instance of non Abelian Carnot groups. They represent a precious source of manageable examples.

When $\kappa = 1$, $\mathbb{G}$ is isomorphic to $(\mathbb{R}^N, +)$ and this is the only commutative Carnot group. Thus, when we talk about Carnot groups we always consider $\kappa \geq 2$; in this case, through the exponential map, a Carnot group $\mathbb{G}$ can be identified with $\mathbb{R}^N$, endowed with a non commutative group operation given by the Baker-Campbell-Hausdorff formula. In particular, in Section 1.3.3 using well-behaved group of dilations we equip Carnot groups with an explicit group law.

In Section 1.3.5 and 1.3.6, we study the left invariant metrics in $\mathbb{G}$, which are equivalent to Carnot-Carathéodory metric (see Proposition 1.3.6). Non commutative Carnot groups, endowed with their left invariant metric are not Riemannian manifolds, not even locally. In fact they are particular instances of so-called sub Riemannian manifolds (see [11]).

In Section 1.4 we focus our attention on a subclass of Carnot groups of step 2 shown in [17], Chapter 3: we call them *groups of class $\mathcal{B}$*. To begin with, we show that they are naturally given with the data on $\mathbb{R}^{m+n}$ of n linearly independent and skew-symmetric matrices of order m. In particular, we show that the set of examples of groups of class $\mathcal{B}$ contains the free step-two groups, H-type groups (see [58]), H-groups in the sense of Métivier (see [74]).

1.1 Carnot Carathéodory spaces

1.1.1 Vector fields on $\mathbb{R}^N$

Let $P = (p_1, \dots, p_N) \in \mathbb{R}^N$ and let $X_1, \dots, X_m$ be a family of smooth vectors fields on $\mathbb{R}^N$

$$X_j(P) = \sum_{i=1}^{N} a_{ij}(P)\partial_{p_i} \qquad \text{for } j = 1, \dots, m$$

with $a_{ij} : \mathbb{R}^N \to \mathbb{R}$ smooth functions. We also suppose that $X_1, \dots, X_m$ are linearly independent. We define $H_P\mathbb{R}^N$ the *horizontal subspaces* at the point P the subspaces of $T_P\mathbb{R}^N$ generated by $X_1(P), \dots, X_m(P)$. The collection of all horizontal fibers $H_P\mathbb{R}^N$ forms the horizontal subbundle $H\mathbb{R}^N$ of $T\mathbb{R}^N$.

Given two smooth vector fields on $\mathbb{R}^N$ X_1, X_2 we define the *commutator* (or Lie bracket) $[X_1, X_2]$ as the smooth vector field given by $X_1X_2 - X_2X_1$ (as common in literature, we tacitly identify vector fields and first order operators); if $X_1(P) = \sum_{i=1}^N a_i(P)\partial_{p_i}$ and $X_2(P) = \sum_{i=1}^N b_i(P)\partial_{p_i}$ then in coordinates $[X_1, X_2]$ is given by

$$[X_1, X_2](P) = \sum_{i,j=1}^{N} \Big(a_j(P)\partial_{p_j} b_i(P) - b_j(P)\partial_{p_j} a_i(P) \Big) \partial_{p_i}.$$

This product is antisymmetric $[X_1, X_2] = -[X_2, X_1]$ and satisfies Jacobi's identity, i.e.

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$$

For example, for the following two vector fields on $\mathbb{R}^3$ (whose points are denoted by $P = (p_1, p_2, p_3)$) $X_1(P) = \partial_{p_1} + 2p_2\,\partial_{p_3}$ and $X_2(P) = \partial_{p_2} - 2p_1\,\partial_{p_3}$, we have

$$[X_1, X_2](P) = (X_1(-2p_1) - X_2(2p_2))\partial_{p_3} = -4\partial_{p_3}.$$

We say that the vector fields on $\mathbb{R}^N$ $X_1, \dots, X_m$ satisfy the *Chow-Hörmander condition* if there is an integer $\kappa \geq 1$ such that the set of linear combinations of family of commutators of $X_1, \dots, X_m$ up to the length κ, i.e.

$$X_1, \dots, X_m,\ [X_{i_1}, X_{i_2}], [X_{i_1}, [X_{i_2}, X_{i_3}]], \dots, [X_{i_1}, [X_{i_2}, [\dots, X_{i_\kappa}] \dots], \quad 1 \leq i_\delta \leq m$$

is the tangent space to $\mathbb{R}^N$ at every point of $\mathbb{R}^N$.

Remark 1.1.1. Let $X_1, \dots, X_m$ be left invariant vector fields on $\mathbb{R}^N$. In this case the Chow-Hörmander condition is equivalent to

$$\dim(\operatorname{span}\{X_1, \dots, X_m, [X_{i_1}, X_{i_2}], \dots, [X_{i_1}, [X_{i_2}, [\dots, X_{i_\kappa}] \dots]\}(P)) = N \tag{1.1}$$

for all $P \in \mathbb{R}^N$.

1.1.2 Carnot Carathéodory distance

An absolutely continuous curve $\gamma : [0, T] \to \mathbb{R}^N$ is a *subunit curve* with respect to $X_1, \dots, X_m$ if it is an *horizontal curve*, that is if there are real measurable functions $h_1(t), \dots, h_m(t)$, $t \in [0, T]$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^{m} h_i(t) X_i(\gamma(t)), \qquad \text{for a.e. } t \in [0, T],$$

and if $\sum_{i=1}^m h_i^2 \leq 1$.

Definition 1.1.1. We define the Carnot-Carathéodory distance (cc-distance) between the points $P, Q \in \mathbb{R}^N$ as

$$d_{cc}(P, Q) = \inf \left\{ T \geq 0 : \text{there is a subunit curve } \gamma : [0, T] \to \mathbb{R}^N \text{ with } \gamma(0) = P, \gamma(T) = Q \right\}.$$

If the above set is empty we put $d_{cc}(P, Q) = \infty$.

Using the same approach followed in [78], our next task is to prove the following result:

Proposition 1.1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set. If d_{cc} is finite, i.e. $d_{cc}(P, Q) < \infty$ for all $P, Q \in \mathbb{R}^N$, then (Ω, d_{cc}) is a metric space.*

We need two lemmas. If $X_j(P) = \sum_{i=1}^N a_{ij}(P)\partial_{p_i}$ for $j = 1, \dots, m$, then we shall write the coefficients a_{ij} in the $(N \times m)$ matrix

$$\mathcal{B}(P) := \begin{pmatrix} a_{11}(P) & \dots & a_{1m}(P) \\ \vdots & \ddots & \vdots \\ a_{N1}(P) & \dots & a_{Nm}(P) \end{pmatrix}$$

and we define its norm as

$$\|\mathcal{B}\| := \sup_{h \in \mathbb{R}^m,\, |h| \le 1} |\mathcal{B}h|.$$

Notice that $\sum_{i=1}^m h_i(t) X_i(\gamma(t)) = \mathcal{B}(\gamma(t))h(t)$ for $h(t) = (h_1(t), \dots, h_m(t))$ and $t \in [0, T]$.

Lemma 1.1.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $P \in \Omega$ and $r > 0$ be such that $U(P, r) := \{Q \in \mathbb{R}^N \,|\, |Q - P| < r\} \Subset \Omega$. Moreover let $\alpha := \sup_{Q \in U(P,r)} \|\mathcal{B}(Q)\|$ and $\gamma : [0; T] \to \Omega$ be a subunit curve such that $\gamma(0) = P$. If $\alpha T < r$ then $\gamma(t) \in U(P, r)$ for all $t \in [0; T]$.*

Proof. By contradiction we suppose that

$$\bar{t} := \inf\{t \in [0, T] \,:\, \gamma(t) \notin U(P, r)\} \le T.$$

Then

$$\begin{aligned} |\gamma(\bar{t}) - P| &= \left| \int_0^{\bar{t}} \dot{\gamma}(\tau)\, d\tau \right| = \left| \int_0^{\bar{t}} \mathcal{B}(\gamma(\tau))h(\tau)\, d\tau \right| \\ &\le \int_0^{\bar{t}} |\mathcal{B}(\gamma(\tau))h(\tau)|\, d\tau \le \int_0^{\bar{t}} \|\mathcal{B}(\gamma(\tau))\| |h(\tau)|\, d\tau \\ &\le \bar{t}\alpha \le T\alpha < r. \end{aligned}$$

Consequently $\gamma(\bar{t}) \in U(P, r)$ because $U(P, r)$ is open and this is in contradiction with the definition of $\bar{t}$. □

Lemma 1.1.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $K \subset \Omega$ be a compact set. Then there is $\beta > 0$ such that*

$$d_{cc}(P, Q) \ge \beta |P - Q| \quad \textit{for all } P, Q \in K. \tag{1.2}$$

Proof. Fix $P, Q \in K$ and let $\gamma : [0, T] \to \Omega$ be a subunit curve from P to Q. We choose $\epsilon > 0$ in order that if we put $K_\epsilon := \{P \in \Omega \,:\, \min_{Q \in K} |P - Q| \le \epsilon\}$ then $K_\epsilon \Subset \Omega$. Moreover if we define $r := \min\{\epsilon, |P - Q|\}$, then

$$|\gamma(T) - \gamma(0)| = |P - Q| \ge r,$$

and by Lemma 1.1.3 we have $\alpha T \ge r$, where $\alpha := \sup_{Q \in K_\epsilon} \|\mathcal{B}(Q)\|$.

Now we consider two case: $r = \epsilon$ or $r = |P - Q|$. If $r = \epsilon$ then

$$T \geq \frac{\epsilon}{\alpha} \geq \frac{\epsilon}{\alpha\rho}|P - Q|$$

where $\rho := \sup_{P,Q \in K} |P - Q|$. If $r = |P - Q|$ then $T \geq |P - Q|/\alpha$. As a consequence by the definition of d_{cc} we get

$$d_{cc}(P, Q) \geq T \geq \min\left\{\frac{1}{\alpha}, \frac{\epsilon}{\alpha\rho}\right\} |P - Q|$$

then the thesis follows with $\beta := \min\left\{\frac{1}{\alpha}, \frac{\epsilon}{\alpha\rho}\right\}$. □

Now we are able to prove Proposition 1.1.2.

Proof. It is clear that $d_{cc}(P, P) = 0$. Moreover if $P \neq Q$ by (1.2) it follows $d_{cc}(P, Q) > 0$.

The symmetry property $d_{cc}(P, Q) = d_{cc}(Q, P)$ follows from the fact that if $\gamma : [0, T] \to \Omega$ is a subunit curve then $\bar{\gamma}(t) = \gamma(T - t)$ is a subunit curve too.

Finally, if $\gamma_1 : [0, T_1] \to \Omega$ and $\gamma_2 : [0, T_2] \to \Omega$ are subunit curves such that $\gamma_1(0) = P$, $\gamma_1(T_1) = G$, $\gamma_2(0) = G$ and $\gamma_2(T_2) = Q$ then

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, T_1] \\ \gamma_2(t - T_1) & \text{if } t \in [T_1, T_1 + T_2] \end{cases}$$

is a subunit curve from P to Q. Taking the infimum we get that

$$d_{cc}(P, Q) \leq d_{cc}(P, G) + d_{cc}(G, Q),$$

i.e. the triangle inequality is true.

□

Remark 1.1.5. Inequality (1.2) shows that the Euclidean metric is continuous with respect to the cc-metric d_{cc}. The converse is in general not true. For example, consider in $\mathbb{R}^2$ the vector fields $X_1 = \partial_{p_1}$ and $X_2 = a(p_1)\partial_{p_2}$, where $a : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map defined as $a(p_1) = 0$ if $p_1 \leq 0$ and $a(p_1) > 0$ if $p_1 > 0$. Any couple of points in $\mathbb{R}^2$ can be connected by piecewise integral curves of X_1 and X_2, which therefore induce on $\mathbb{R}^2$ a finite cc-metric d_{cc}. But if $p_1 < 0$ then

$$\lim_{p_2 \to 0} d_{cc}((p_1, p_2), (p_1, 0)) = 2|p_1| \neq 0.$$

The following theorem, called Chow-Rashevsky Theorem, gives a sufficient condition in order that d_{cc} is finite. For a proof see for instance [11], Theorem 2.4. or [77], Theorem 1.6.2

Theorem 1.1.6. *Let the vector fields $X_1, \ldots, X_m$ in $\mathbb{R}^N$ satisfy the Chow-Hörmander condition. Then any two points in $\mathbb{R}^N$ can be connected by horizontal curve and hence d_{cc} is a distance on $\mathbb{R}^N$.*

According to the terminology in [49] (see also [77]), we have the following definition:

Definition 1.1.2. The metric space $(\mathbb{R}^N, d_{cc})$ is called Carnot-Carathéodory space (CC space).

The CC spaces satisfying Chow-Hörmander's condition are also called Sub-Riemannian spaces. Classical examples of Carnot-Carathéodory spaces are the Carnot groups (see Section 1.3).

1.2 Lie groups and Lie algebras

Before stating the definition of Carnot groups, we want to briefly recall some basic facts on Lie groups and Lie algebras: a more complete description of these structures can be found in [102], [103], [104], [107].

Definition 1.2.1. A Lie group $\mathbb{G}$ is a manifold endowed with the structure of differential group, i.e. a group where the map

$$\mathbb{G} \times \mathbb{G} \ni (P, Q) \longmapsto P^{-1} \cdot Q \in \mathbb{G}$$

is of class $\mathbb{C}^\infty$.

We denote by 0 the identity of the group, while we define, for any $P \in \mathbb{G}$, the $\mathbb{C}^\infty$ map $\tau_P : \mathbb{G} \to \mathbb{G}$ as

$$Q \mapsto \tau_P(Q) := P \cdot Q.$$

We call τ_P the left translation by P.

The general notion of Lie algebra is the following

Definition 1.2.2. A Lie algebra $\mathfrak{g}$ is a vector space together with a bilinear operation

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket such that, for all $X_1, X_2, X_3 \in \mathfrak{g}$, one has

1. anti-commutativity : $[X_1, X_2] = -[X_2, X_1]$
2. Jacobi identity : $[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$

A linear subspaces $\mathfrak{a} \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ if $\mathfrak{a}$ is closed with respect to $[\cdot, \cdot]$, i.e. $[X_1, X_2] \in \mathfrak{a}$ for every $X_1, X_2 \in \mathfrak{a}$. Given two subalgebras $\mathfrak{a}, \mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ we will denote by $[\mathfrak{a}, \mathfrak{b}]$ the vector subspace generated by the elements of $\{[X, Y] : X \in \mathfrak{a}, Y \in \mathfrak{b}\}$.

An classical example of Lie algebra is $\Gamma(TM)$, the linear space of smooth sections of TM (with M a smooth manifold), with the product $[X_1, X_2] = X_1X_2 - X_2X_1$.

Definition 1.2.3. A vector field $X \in \Gamma(T\mathbb{G})$ on a Lie group $\mathbb{G}$ is left invariant if

$$X(P) = d\tau_P(X(0)), \qquad \text{for all } P \in \mathbb{G}$$

where $d\tau_P : T\mathbb{G} \to T\mathbb{G}$ denotes the differential of the left translation by P.

A vector field X is left invariant if and only if

$$(Xf)(\tau_P Q) = X(f \circ \tau_P)(Q)$$

for any $f \in \mathbb{C}^\infty(\mathbb{G})$ and $P, Q \in \mathbb{G}$. [1]

If X_1, X_2 are left invariant vector fields, then the Lie bracket $[X_1, X_2]$ is also left invariant (see Proposition 3.7 in [107]). This implies that by iterated brackets the left invariant vector fields $X_1, \dots, X_m$ generate a Lie algebra.

Precisely, we define *the Lie algebra of the Lie group* $\mathbb{G}$ to be the Lie algebra of left invariant vector fields on $\mathbb{G}$.

This Lie algebra is denoted by $\mathfrak{g}$ and, for every $P \in \mathbb{G}$, is the vector space

$$\mathrm{span}\{X_1, \dots, X_m, [X_{i_1}, X_{i_2}], [X_{i_1}, [X_{i_2}, X_{i_3}]], \dots, [X_{i_1}, [X_{i_2}, [\dots, X_{i_\kappa}] \dots]\}(P).$$

$\mathfrak{g}$ is canonically isomorphic to the tangent spaces $T_0\mathbb{G}$ at the identity via the isomorphism

$$T_0\mathbb{G} \ni v \longleftrightarrow X \in \mathfrak{g} \ \text{ such that } \ X(P) = d\tau_P(v).$$

Example 1.2.1. We denote the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$ with the group law given by

$$(p_1, p_2, p_3) \cdot (q_1, q_2, q_3) = (p_1 + q_1, p_2 + q_2, p_3 + q_3 + \frac{1}{2}(p_1 q_2 - q_1 p_2)).$$

Note that $(p_1, p_2, p_3)^{-1} = (-p_1, -p_2, -p_3)$ and the neutral element of the group is the origin.

The Heisenberg group $\mathbb{H}^1$ is an example of a Lie group, indeed the map

$$(P, Q) \mapsto Q^{-1} \cdot P = (p_1 - q_1, p_2 - q_2, p_3 - q_3 + \frac{1}{2} p_2(p_1 - q_1) - \frac{1}{2} p_1(p_2 - q_2))$$

is of class C^∞.

[1] Let $\Omega \subset \mathbb{R}^N$ be an open set. We recall that if $X = \sum_{j=1}^N a_j \partial_{p_j}$ and $f : \Omega \to \mathbb{R}$ is a differentiable function, then we denote by Xf the function on Ω defined by

$$Xf(P) = \sum_{j=1}^N a_j(P) \partial_{p_j} f(P), \qquad \text{for } P \in \Omega.$$

The class of left invariant vector fields can be identified with the tangent space $T_0\mathbb{H}^1$ to $\mathbb{H}^1$ at 0. Hence $\partial_{p_1}, \partial_{p_2}$ and ∂_{p_3} form a basis of $T_0\mathbb{H}^1$. The corresponding left invariant vector fields X_1, X_2, Y are

$$\begin{aligned} X_1(P) &= d\tau_P \partial_{p_1}(0) \\ X_2(P) &= d\tau_P \partial_{p_2}(0) \\ Y(P) &= d\tau_P \partial_{p_3}(0) \end{aligned}$$

for $P = (p_1, p_2, p_3)$. Consequently, using the fact

$$d\tau_P(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}p_2 & \frac{1}{2}p_1 & 1 \end{pmatrix}$$

we obtain

$$\begin{aligned} X_1(P) &= (1, 0, -\frac{1}{2}p_2) = \partial_{p_1} - \frac{1}{2}p_2\,\partial_{p_3} \\ X_2(P) &= (1, 0, \frac{1}{2}p_1) = \partial_{p_2} + \frac{1}{2}p_1\,\partial_{p_3} \\ Y(P) &= (0, 0, 1) = \partial_{p_3} \end{aligned}$$

The left invariant vector fields X_1, X_2, Y form a basis of left invariant vector fields for the Lie algebra of $\mathbb{H}^1$. Moreover, since $[X_1, X_2] = Y$ and since any commutator involving X_1, X_2 more than twice is identically zero, then the Lie algebra of $\mathbb{H}^1$ is $\operatorname{span}\{X_1, X_2, [X_1, X_2]\}$.

1.2.1 Homomorphisms and isomorphisms

There is a special finite dimensional Lie algebra intimately associated with each Lie group, and the properties of the Lie group are reflected in properties of its Lie algebra. Here we show their connection in terms of Lie group homomorphism and Lie algebra homomorphism (see [62], [107]).

Definition 1.2.4. Let $(\mathbb{G}, \cdot)$ and $(\mathbb{H}, \star)$ be Lie groups. A map $F : \mathbb{G} \to \mathbb{H}$ is said a Lie group homomorphism if it belongs to $\mathbb{C}^\infty$ and it is a group homomorphism, i.e.

$$F(P \cdot Q) = F(P) \star F(Q), \qquad \text{for all } P, Q \in \mathbb{G}.$$

A map F is an isomorphism of Lie groups if it is a Lie groups homomorphism and a diffeomorphism of differentiable manifolds. An isomorphism of $\mathbb{G}$ onto itself is called an automorphism of $\mathbb{G}$.

Definition 1.2.5. Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ and $(\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h})$ be Lie algebras. A map $f : \mathfrak{g} \to \mathfrak{h}$ is said a Lie algebra homomorphism if it is linear and it preserves brackets

$$f([X_1, X_2]_\mathfrak{g}) = [f(X_1), f(X_2)]_\mathfrak{h} \qquad \text{for all } X_1, X_2 \in \mathfrak{g}.$$

A map f is an isomorphism of Lie algebras if it is a bijective homomorphism of Lie algebras. An isomorphism of $\mathfrak{g}$ onto itself is called an automorphism of $\mathfrak{g}$.

The first connection between Lie groups and their Lie algebras is that each Lie group homomorphism indices a Lie algebra homomorphism: if $F : \mathbb{G} \to \mathbb{H}$ is a Lie group homomorphism, we have that $F(0) = 0$ and the differential at the identity $dF_0 : T_0\mathbb{G} \to T_0\mathbb{H}$ is a linear transformation which preserves the bracket operation.

Viceversa, we have the following results:

Theorem 1.2.1 ([107], Theorem 3.27)**.** *Let $\mathbb{G}$ and $\mathbb{H}$ be two Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Assume $\mathbb{G}$ simply connected. Let $f : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a unique Lie group homomorphism $F : \mathbb{G} \to \mathbb{H}$ such that $dF = f$.*

Theorem 1.2.2 ([107], Corollary 3.28)**.** *If simply connected Lie groups $\mathbb{G}$ and $\mathbb{H}$ have isomorphic Lie algebras, then $\mathbb{G}$ and $\mathbb{H}$ are isomorphic.*

1.2.2 Exponential map

In the theory of Lie groups we define the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ as the map from the Lie algebra $\mathfrak{g}$ of Lie group $\mathbb{G}$ to $\mathbb{G}$, given by

$$\exp(X)(P) := \gamma_P^X(1)$$

where $\gamma_P^X : \mathbb{R} \to \mathbb{G}$ is the (unique) solution of the following Cauchy problem

$$\begin{cases} \dot\gamma(t) = X(\gamma(t)) \\ \gamma(0) = P \end{cases} \tag{1.3}$$

We recall the following basic result proved in [107], Theorem 3.31:

Theorem 1.2.3. *Let $\mathbb{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. If $X \in \mathfrak{g}$ then*

1. $\exp((t+s)X) = \exp(tX) \cdot \exp(sX)$ *for all* $t, s \in \mathbb{R}$
2. $\exp(-X) = (\exp(X))^{-1}$
3. $\exp : \mathfrak{g} \to \mathbb{G}$ *is smooth and* $(d\exp)_0$ *is the identity map,*

$$(d\exp)_0 = id_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$$

so $\exp$ *gives a diffeomorphism of a neighborhood of* 0 *in* $\mathfrak{g}$ *onto a neighborhood of* 0 *in* $\mathbb{G}$.

4. the left invariant vector fields are complete. [2]

From the unique solvability of the Cauchy problem (1.3) related to smooth vector fields

1. $\exp(-tX)(P) = \exp(t(-X))(P)$
2. $\exp(-tX)\big(\exp(tX)(P)\big) = P$
3. $\exp((t+s)X)(P) = \exp(tX)\big(\exp(sX)(P)\big)$
4. $\exp((ts)X)(P) = \exp(t(sX))(P)$

for every $X \in T\mathbb{G}$, $P \in \mathbb{G}$ and $t, s \in \mathbb{R}$.

Moreover if $\mathbb{G}$ and $\mathbb{H}$ are two Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and $F : \mathbb{G} \to \mathbb{H}$ is a Lie group homomorphism, then the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow[dF]{} & \mathfrak{h} \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ \mathbb{G} & \xrightarrow{F} & \mathbb{H} \end{array}$$

Example 1.2.2 (The exponential map on $\mathbb{H}^1$)**.** Let us consider once again the Heisenberg group $\mathbb{H}^1$. In Example 1.2.1, we showed that a basis for its Lie algebra $\mathfrak{h}$ is given by X_1, X_2, Y where $X_1 = \partial_{p_1} - \frac{1}{2}p_2\partial_{p_3}$, $X_2 = \partial_{p_2} + \frac{1}{2}p_1\partial_{p_3}$ and $[X_1, X_2] = Y = \partial_{p_3}$. Let us construct the exponential map. We set, for $(q_1, q_2, q_3) \in \mathbb{R}^3$

$$Z := q_1X_1 + q_2X_2 + q_3Y = \begin{pmatrix} q_1 \\ q_2 \\ -\frac{1}{2}q_1p_2 + \frac{1}{2}q_2p_1 + q_3 \end{pmatrix}.$$

Hence for fixed $P = (p_1, p_2, p_3) \in \mathbb{H}^1$, we have

$$\exp(Z)(P) = \gamma_P^Z(1),$$

where $\gamma_P^Z(t) = \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ is the solution of the Cauchy problem (1.3) , i.e.

$$\begin{cases} \dot{\gamma}(t) = (q_1, q_2, q_3 - \frac{1}{2}q_1\gamma_2(t) + \frac{1}{2}q_2\gamma_1(t)) \\ \\ \gamma(0) = P \end{cases}$$

[2] A smooth vector field X on a Lie group $\mathbb{G}$ is complete if, for every $P \in \mathbb{G}$, the integral curve γ of X such that $\gamma(0) = P$ is defined on the whole $\mathbb{R}$ (i.e. its maximal interval of definition is $\mathbb{R}$).

Solving the above system of ODE's, one gets

$$\exp\begin{pmatrix} q_1 \\ q_2 \\ -\frac{1}{2}q_1p_2 + \frac{1}{2}q_2p_1 + q_3 \end{pmatrix}(P) = \begin{pmatrix} p_1+q_1 \\ p_2+q_2 \\ p_3+q_3+\frac{1}{2}(p_1q_2 - p_2q_1) \end{pmatrix}.$$

Example 1.2.3 (Pyramid-shaped vector fields)**.** Let us consider the Lie algebra $T\mathbb{R}^N$ of the vector fields on $\mathbb{R}^N$ equipped with the product $[X_1, X_2] = X_1X_2 - X_2X_1$.

We consider in $\mathbb{R}^N$ vector fields of the following type

$$X = \sum_{j=1}^{N} a_j(p_1, \ldots, p_{j-1})\partial_{p_j}, \tag{1.4}$$

where $a_1 =$ constant.

The function a_j only depends on the variables $p_1, \ldots, p_{j-1}$ when $j > 1$. Roughly speaking, such a vector field is pyramid-shaped,

$$X = \begin{pmatrix} a_1 \\ a_2(p_1) \\ a_3(p_1,p_2) \\ \vdots \\ a_N(p_1,\ldots,p_{N-1}) \end{pmatrix}$$

For instance, the vector fields X_1, X_2, Y in above example have this form.

For any smooth vector field X of the form (1.4), the map $(P,t) \mapsto \exp(tX)(P)$ is well defined for every $P = (p_1, \ldots, p_N) \in \mathbb{R}^N$ and $t \in \mathbb{R}$ and can be easily computed. Indeed, if $\gamma_P^X = \gamma = (\gamma_1, \ldots, \gamma_N)$ is the solution to the Cauchy problem

$$\begin{cases} \dot\gamma(t) = X(\gamma(t)) \\ \gamma(0) = P \end{cases}$$

then $\dot\gamma_1(t) = a_1$ and $\dot\gamma_j(t) = a_j(p_1, \ldots, p_{j-1})$ for $j = 2, \ldots, N$. As a consequence,

$$\gamma_1(t) = p_1 + ta_1, \quad \gamma_j(t) = p_j + \int_0^t a_j(\gamma_1(s), \ldots, \gamma_{j-1}(s))\,ds$$

and $\gamma_j(t)$ is defined for every $P \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Moreover $\gamma_j(t)$ only depends on the first components $p_1, \ldots, p_j$ of $P = (p_1, \ldots, p_N)$ for $j = 1, \ldots, N$. Let us put $A_1(t) := ta_1$ and for $j = 2, \ldots, N$

$$A_j(p_1, \ldots, p_{j-1}, t) := \int_0^t a_j(\gamma_1(s), \ldots, \gamma_{j-1}(s))\,ds.$$

Then, for all $P = (p_1, \dots, p_N) \in \mathbb{R}^N$ we conclude that

$$\exp(X)(P) = \begin{pmatrix} p_1 + A_1(1) \\ p_2 + A_2(p_1, 1) \\ \vdots \\ p_N + A_N(p_1, \dots, p_{N-1}, 1) \end{pmatrix}$$

and the map $P \mapsto \exp(X)(P)$ is a global diffeomorphism of $\mathbb{R}^N$ onto $\mathbb{R}^N$.

1.2.3 The Baker-Campbell-Hausdorff formula

The algebraic structure of $\mathfrak{g}$ determines that of $\mathbb{G}$, and precisely

$$\exp(X_1)\exp(X_2) = \exp(\mathfrak{C}(X_1, X_2)), \quad \text{for all } X_1, X_2 \in \mathfrak{g}$$

where $\mathfrak{C}(X_1, X_2)$ is given by the following formula, called Baker-Campbell-Hausdorff formula (1.5)

$$\begin{aligned}\mathfrak{C}(X_1, X_2) = X_1 + X_2 + \frac{1}{2}[X_1, X_2] + \frac{1}{12}[X_1, [X_1, X_2]] - \frac{1}{12}[X_2, [X_1, X_2]] \\ - \frac{1}{48}[X_2, [X_1, [X_1, X_2]]] - \frac{1}{48}[X_1, [X_2, [X_1, X_2]]] + \{\text{brackets of height} \geq 5\}.\end{aligned}$$

$\mathfrak{C}(X_1, X_2)$ is an infinite linear combination of X_1, X_2 and their iterated commutators. It defines a binary operation in $\mathfrak{g}$ whenever the series converges.

Example 1.2.4. Let $\mathfrak{g}$ be an abelian Lie algebra, i.e. a Lie algebra such that $[X_1, X_2] = 0$ for all $X_1, X_2 \in \mathfrak{g}$. Then

$$\mathfrak{C}(X_1, X_2) = X_1 + X_2$$

and consequently $\exp(X_1)\exp(X_2) = \exp(X_1 + X_2)$ for all $X_1, X_2 \in \mathfrak{g}$.

Example 1.2.5 (The Baker-Campbell-Hausdorff formula on $\mathbb{H}^1$)**.** Let $\xi, \eta \in \mathbb{R}^3$ and $Z := X_1 + X_2 + Y$ where X_1, X_2, Y is the basis for the Lie algebra of Heisenberg group $\mathbb{H}^1$ defined in Example 1.2.1. We explicitly write down $\mathfrak{C}(\xi Z, \eta Z)$ in $\mathbb{H}^1$, thus obtaining

$$\begin{aligned}\mathfrak{C}(\xi Z, \eta Z) &= \xi Z + \eta Z + \frac{1}{2}[\xi Z, \eta Z] + \frac{1}{12}[\xi Z, [\xi Z, \eta Z]] + \dots \\ &= \xi Z + \eta Z + \frac{1}{2}[\xi Z, \eta Z] \\ &\qquad \text{(since any commutator involving } X_1, X_2 \text{ more than twice is identically 0)} \\ &= \xi_1 X_1 + \xi_2 X_2 + \xi_3 Y + \eta_1 X_1 + \eta_2 X_2 + \eta_3 Y \\ &\quad + \frac{1}{2}[\xi_1 X_1 + \xi_2 X_2 + \xi_3 Y \,,\, \eta_1 X_1 + \eta_2 X_2 + \eta_3 Y] \\ &\qquad \text{(here we use } [X_1, X_2] = Y,\ [X_1, Y] = [X_2, Y] = 0) \\ &= (\xi_1 + \eta_1)X_1 + (\xi_2 + \eta_2)X_2 + (\xi_3 + \eta_3 + \frac{1}{2}\xi_1\eta_2 - \frac{1}{2}\xi_2\eta_1)Y\end{aligned}$$

which is the group operation in $\mathbb{H}^1$.

1.2.4 Nilpotent groups

Definition 1.2.6 ([27])**.** Let $\mathfrak{g}$ be a Lie algebra. For each $s \in \mathbb{N}$ we define by induction the following sequence of subspaces

$$\mathfrak{g}^{(s)} := \mathfrak{g}$$
$$\mathfrak{g}^{(s+1)} := [\mathfrak{g}^{(s)}, \mathfrak{g}].$$

If there exists a positive integer $\kappa \in \mathbb{N}$ such that $\mathfrak{g}_\kappa \neq \{0\}$ and $\mathfrak{g}^{(\kappa+1)} = \{0\}$ we say that $\mathfrak{g}$ is a nilpotent Lie algebra. The integer κ is called the step of $\mathfrak{g}$. A Lie group $\mathbb{G}$ is nilpotent if its Lie algebra is nilpotent..

Notice that if $\mathfrak{g}$ is nilpotent of κ step, then for all $s \in \{1, \dots, \kappa\}$ the subalgebra $\mathfrak{g}^{(s+1)}$ is strictly contained in $\mathfrak{g}^{(s)}$.

Theorem 1.2.4 ([27], Theorem 1.2.1)**.** *Let $\mathbb{G}$ be a connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{g}$. Then*

1. *The exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is an analytic diffeomorphism.*

2. *The Baker-Campbell-Hausdorff formula holds for all couple of elements of $\mathfrak{g}$.*

Definition 1.2.7. Let $\mathbb{G}$ be a connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{g}$. Let $\{X_1, \dots, X_N\}$ be a basis for $\mathfrak{g}$. A system of *exponential coordinates* associated with the basis $\{X_1, \dots, X_N\}$ of $\mathfrak{g}$ is the map $\Psi : \mathbb{R}^N \to \mathbb{G}$

$$\Psi(p_1, \dots, p_N) := \exp\left(\sum_{i=1}^{N} p_i X_i\right).$$

An important application of Theorem 1.2.4 involves coordinates on $\mathbb{G}$. Since the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism, the map Ψ is also a diffeomorphism. Consequently, any $P \in \mathbb{G}$ can be written in a unique way as $P = \exp(p_1X_1 + \cdots + p_NX_N)$ and we can identify P with the N-tuple $(p_1, \dots, p_N) \in \mathbb{R}^N$.

Using $\mathfrak{C}(\cdot, \cdot)$ the Baker-Campbell-Hausdorff formula (1.5), we define a group law on $\mathbb{R}^N$ as follows: let $(p_1, \dots, p_N), (p_1', \dots, p_N') \in \mathbb{R}^N$. If

$$\mathfrak{C}\left(\sum_{i=1}^{N} p_i X_i, \sum_{i=1}^{N} p_i' X_i\right) = \sum_{i=1}^{N} q_i X_i$$

then

$$(p_1, \dots, p_N) \cdot (p_1', \dots, p_N') := (q_1, \dots, q_N).$$

In this way, $(\mathbb{R}^N, \cdot)$ is a Lie group, whose Lie algebra is isomorphic to $\mathfrak{g}$. Moreover, thanks to Theorem 1.2.2, we can identify $\mathbb{G}$ with $(\mathbb{R}^N, \cdot)$.

Example 1.2.6. The Heisenberg group $\mathbb{H}^1$ (see Example 1.2.1) is an example of 2 step nilpotent Lie group.

Example 1.2.7 (A non-polynomial non-nilpotent Lie group on $\mathbb{R}^2$)**.** The following operation on $\mathbb{R}^2$

$$(p_1, p_2) \cdot (q_1, q_2) = (p_1 + q_1, q_2 + p_2 \mathrm{e}^{q_1})$$

defines a Lie group structure where $(p_1, p_2)^{-1} = (-p_1, -p_2 \mathrm{e}^{-p_1})$. Moreover a basis of the Lie algebra associated is

$$Y_1 = \partial_{p_1} + p_2 \partial_{p_2}, \qquad Y_2 = \partial_{p_2}$$

Hence, the Lie algebra is *not* nilpotent, for $[Y_2, Y_1] = Y_2$, so that, inductively,

$$[\dots [Y_2, \underbrace{Y_1], Y_1], Y_1] \dots Y_1]}_{k \text{ times}} = Y_2 \qquad \text{for all } k \in \mathbb{N}.$$

1.3 Carnot groups of step κ

We now enter into the core of the chapter by introducing the central definition of this thesis, i.e. Carnot groups. For a general account see e.g. [17], [35], [62], [95], [104], [103].

Definition 1.3.1. A Carnot group $\mathbb{G} = (\mathbb{G}, \cdot, \delta_\lambda)$ of step κ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e. a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_\kappa$. The stratification has the further property that the entire Lie algebra $\mathfrak{g}$ is generated by its first layer $\mathfrak{g}_1$, the so-called horizontal layer, that is

$$\begin{cases} [\mathfrak{g}_1, \mathfrak{g}_{i-1}] = \mathfrak{g}_i & \text{if } 2 \le i \le \kappa \\ [\mathfrak{g}_1, \mathfrak{g}_\kappa] = \{0\} \end{cases} \tag{1.6}$$

where $[\mathfrak{g}_1, \mathfrak{g}_i]$ is the subspaces of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_i$.

By Theorem 1.2.4, we can identify $\mathbb{G}$ with $(\mathbb{R}^N, \cdot, \delta_\lambda)$, where the group operation $\cdot$ is determined by the Baker-Campbell-Hausdorff formula (see Section 1.2.3) and the family of automorphisms $\{\delta_\lambda\}_\lambda$ of $\mathbb{G}$ is defined in Section 1.3.2.

Let $X_1, \dots, X_{m_1}$ be a base for $\mathfrak{g}_1$. The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \dots, X_{m_1}$ plays a particularly important role in the theory, it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_P = \operatorname{span}\{X_1(P), \dots, X_{m_1}(P)\}, \quad P \in \mathbb{G}.$$

A *sub Riemannian structure* is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle \cdot, \cdot \rangle_P$ and a norm $| \cdot |_P$ making the basis $X_1(P), \dots, X_{m_1}(P)$ an orthonormal basis.

If $v = \sum_{i=1}^{m_1} (v_1)_i X_i(P)$ and $w = \sum_{i=1}^{m_1} (w_1)_i X_i(P)$ are in $H\mathbb{G}$, then $\langle v, w\rangle_P := \sum_{i=1}^{m_1} (v_1)_i (w_1)_i$ and $|v|_P^2 := \langle v, v\rangle_P$. We will write, with abuse of notation, $\langle \cdot, \cdot \rangle$ meaning $\langle \cdot, \cdot \rangle_P$ and $|\cdot|$ meaning $|\cdot|_P$.

The sections of $H\mathbb{G}$ are called *horizontal sections*, a vector of $H\mathbb{G}_P$ is an *horizontal vector* while any vector in $T\mathbb{G}_P$ that is not horizontal is a vertical vector. Each horizontal section ϕ defined on an open set $\Omega \subset \mathbb{G}$ can be written as $\phi = \sum_{i=1}^{m_1} \phi_i X_i$, where its coordinates are functions $\phi_i : \Omega \to \mathbb{R}$. When considering two such sections ϕ and ψ, we will write $\langle \psi, \phi\rangle$ for $\langle \psi(P), \phi(P)\rangle_P$.

Remark 1.3.1. Let $\mathbb{G}$ be a Carnot group of step κ and let $\{X_1, \dots, X_{m_1}\}$ be a basis of the horizontal layer $\mathfrak{g}_1$. By (1.6) we know that $X_1, \dots, X_{m_1}$ generates $\mathfrak{g}$ by commutations and consequently $X_1, \dots, X_{m_1}$ satisfy Chow-Hörmander condition (1.1).

1.3.1 Uniqueness of stratifications

The stratification of a Lie algebra is unique up to isomorphism. Hence, also the structure of a Carnot group is essentially unique:

Proposition 1.3.2 ([63], Proposition 1.17)**.** *Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g}$ has two stratification, $\mathfrak{g} = \mathfrak{h}_1 \oplus \dots \oplus \mathfrak{h}_\kappa$ and $\mathfrak{g} = \mathfrak{h}'_1 \oplus \dots \oplus \mathfrak{h}'_r$.*

Then $\kappa = r$ and there is a Lie algebra automorphism $f : \mathfrak{g} \to \mathfrak{g}$ such that

$$f(\mathfrak{h}_s) = \mathfrak{h}'_s, \quad \textit{for all } s = 1, \dots, \kappa.$$

Proof. We begin observing the following simple fact (see Lemma 1.16 in [63]): if $\mathfrak{g} = \mathfrak{h}_1 \oplus \dots \oplus \mathfrak{h}_\kappa$, then $\mathfrak{g}$ is nilpotent of step κ (see Definition 1.2.6) and $\mathfrak{g}^{(s)} = \mathfrak{h}_s \oplus \dots \oplus \mathfrak{h}_\kappa$.

Consequently, $\mathfrak{g}^{(s)} = \mathfrak{h}_s \oplus \dots \oplus \mathfrak{h}_\kappa = \mathfrak{h}'_s \oplus \dots \oplus \mathfrak{h}'_r$ and so $\kappa = r$.

Now we consider the quotient mappings $\pi_s : \mathfrak{g}^{(s)} \to \mathfrak{g}^{(s)}/\mathfrak{g}^{(s+1)}$. We have that its restrictions $\pi_{s|\mathfrak{h}_s} : \mathfrak{h}_s \to \mathfrak{g}^{(s)}/\mathfrak{g}^{(s+1)}$ and $\pi_{s|\mathfrak{h}'_s} : \mathfrak{h}'_s \to \mathfrak{g}^{(s)}/\mathfrak{g}^{(s+1)}$ are linear isomorphisms.

For $X \in \mathfrak{h}_s$ and $s = 1, \dots, \kappa$, define $f_s : \mathfrak{h}_s \to \mathfrak{h}'_s$ as

$$f_s(X) := \left(\pi_{s|\mathfrak{h}'_s}\right)^{-1} \circ \left(\pi_{s|\mathfrak{h}_s}\right)(X).$$

More precisely, for $X \in \mathfrak{h}_s$ and $Y \in \mathfrak{h}'_s$

$$f_s(X) = Y \quad \Longleftrightarrow \quad X - Y \in \mathfrak{g}^{(s+1)} \tag{1.7}$$

In this way we can define a linear map $f : \mathfrak{g} \to \mathfrak{g}$. This is clearly a linear isomorphism and $f(\mathfrak{h}_s) = \mathfrak{h}'_s$ for each $s = 1, \dots, \kappa$.

Consequently, it remains to show that f preserves brackets, i.e. $f([X, Y]) = [fX, fY]$, for all $X, Y \in \mathfrak{g}$. Let $X, Y \in \mathfrak{g}$ with $X = \sum_{s=1}^{\kappa} X_s$ and $Y = \sum_{s=1}^{\kappa} Y_s$ such that $X_s, Y_s \in \mathfrak{h}_s$. Then

$$f([X, Y]) = \sum_{s,l=1}^{\kappa} f([X_s, Y_l]), \qquad [fX, fY] = \sum_{s,l=1}^{\kappa} [fX_s, fY_l].$$

Therefore it sufficient to prove that

$$f([X_s, Y_l]) = [fX_s, fY_l] \text{ for } X_s \in \mathfrak{h}_s,\ Y_l \in \mathfrak{h}'_l \tag{1.8}$$

First, notice that

$$[X_s, Y_l] - [fX_s, fY_l] = [X_s - fX_s, Y_l] - [fX_s, fY_l - Y_l] \in \mathfrak{g}^{(s+l+1)}$$

because, on the other hand, $X_s - fX_s \in \mathfrak{g}^{(s+1)}$, $Y_l \in \mathfrak{h}'_l$ and consequently $[X_s - fX_s, Y_l] \in \mathfrak{g}^{(s+l+1)}$; on the other hand, $fX_s \in \mathfrak{h}'_s$, $fY_l - Y_l \in \mathfrak{g}^{(l+1)}$ and so $[fX_s, fY_l - Y_l] \in \mathfrak{g}^{(s+l+1)}$.

Then by (1.7) and since $[X_s, Y_l] \in \mathfrak{h}_{s+l}$ and $[fX_s, fY_l] \in \mathfrak{h}'_{s+l}$

$$f([X_s, Y_l]) = [fX_s, fY_l] \quad \Longleftrightarrow \quad [X_s, Y_l] - [fX_s, fY_l] \in \mathfrak{g}^{(s+l+1)}$$

i.e. (1.8) holds and so f preserves brackets. □

Example 1.3.1. Not all nilpotent Lie algebras admit a stratification, see [51]. Indeed, consider the 7-dimensional Lie algebra $\mathfrak{g}$ generated by $X_1, \dots, X_7$ with only non trivial brackets

$$\begin{aligned}
&[X_1, X_2] = X_3, \quad [X_1, X_3] = 2X_4, \quad [X_1, X_4] = 3X_5,\\
&[X_2, X_3] = X_5, \quad [X_1, X_5] = 4X_6, \quad [X_2, X_4] = 2X_6,\\
&[X_1, X_6] = 5X_7, \ [X_2, X_5] = 3X_7, \quad [X_3, X_4] = X_7.
\end{aligned}$$

This Lie algebra $\mathfrak{g}$ is not stratifiable.

1.3.2 The dilation structure

The construction of the dilation structure deeply uses the stratification of the algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa$. We denote by $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ the family of (non isotropic) dilation defined by

$$\delta_\lambda\left(\sum_{i=1}^{\kappa} X_i\right) := \sum_{i=1}^{\kappa} \lambda^i X_i, \qquad \lambda \geq 0$$

where $X = \sum_{i=1}^{\kappa} X_i$ with $v_i \in \mathfrak{g}_i$, $1 \leq i \leq \kappa$. The dilations δ_λ are automorphisms of $\mathfrak{g}$ and are uniquely determined by the homogeneity conditions

$$\delta_\lambda X_i := \lambda^i X_i, \qquad \forall X_i \in \mathfrak{g}_i,\ 1 \leq i \leq \kappa.$$

Moreover, from the grading, it is obvious that

1. $\delta_{\lambda\eta} = \delta_\lambda \circ \delta_\eta$;
2. $\delta_\lambda([X, Y]) = [\delta_\lambda X, \delta_\lambda Y]$;

3. $\delta_\lambda(\mathfrak{C}(X,Y)) = \mathfrak{C}(\delta_\lambda X, \delta_\lambda Y)$, where $\mathfrak{C}(\cdot,\cdot)$ is given by (1.5).

By Theorem 1.2.4, in Carnot groups, the map $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism, so any element $P \in \mathbb{G}$ can be represented as $\exp(X)$ for some unique $X \in \mathfrak{g}$, and therefore it can be uniquely written in the form

$$\exp\left(\sum_{i=1}^{\kappa} X_i\right), \quad X_i \in \mathfrak{g}_i,\ 1 \le i \le \kappa.$$

Therefore we can define a one-parameter group of automorphism of $\mathbb{G}$, which we still denote with $\{\delta_\lambda\}_{\lambda \ge 0}$ via the formula

$$\delta_\lambda\left(\exp\left(\sum_{i=1}^{\kappa} X_i\right)\right) := \exp\left(\sum_{i=1}^{\kappa} \lambda^i X_i\right),$$

i.e. $\exp \circ \delta_\lambda = \delta_\lambda \circ \exp$.

Let $f : \mathbb{G} \to \mathbb{G}$, then

$$X(f \circ \delta_\lambda)(P) = (\delta_\lambda X) f(\delta_\lambda P), \quad \text{for all } P \in \mathbb{G},\ \lambda \ge 0.$$

Indeed,

$$\begin{aligned} X(f \circ \delta_\lambda)(P) &= \frac{d}{dt} f \circ \delta_\lambda(P \exp(tX))_{|t=0} \\ &= \frac{d}{dt} f(\delta_\lambda P \delta_\lambda \exp(tX))_{|t=0} \\ &= \frac{d}{dt} f(\delta_\lambda P \exp(t\delta_\lambda X))_{|t=0} \\ &= (\delta_\lambda X) f(\delta_\lambda P). \end{aligned}$$

Then the map $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is the unique group homomorphism with δ_λ as differential, whose existence is given by Theorem 1.2.1 since $\mathbb{G}$ is simply connected.

From the properties of dilations in Lie algebras we immediately deduce the associated ones for dilations of Carnot group:

1. $\delta_{\lambda\eta} = \delta_\lambda \circ \delta_\eta$, indeed

$$\begin{aligned} \delta_{\lambda\eta}(P) &= \exp(\delta_{\lambda\eta} \exp^{-1}(P)) \\ &= \exp(\delta_\lambda \delta_\eta \exp^{-1}(P)) \\ &= \exp(\delta_\lambda \exp^{-1}(\exp \delta_\eta \exp^{-1}(P))) \\ &= \exp(\delta_\lambda \exp^{-1}(\delta_\eta(P))) \\ &= \delta_\lambda \delta_\eta(P). \end{aligned}$$

2. $\delta_\lambda(P \cdot Q) = \delta_\lambda(P) \cdot \delta_\lambda(Q)$ for all $P, Q \in \mathbb{G}$. Indeed

$$\begin{aligned}\delta_\lambda(PQ) &= \exp\left(\delta_\lambda \exp^{-1}(PQ)\right)\\ &= \exp \delta_\lambda(\mathfrak{C}(\exp^{-1} P, \exp^{-1} Q))\\ &= \exp(\mathfrak{C}(\delta_\lambda \exp^{-1} P, \delta_\lambda \exp^{-1} Q))\\ &= \exp(\delta_\lambda \exp^{-1} P) \exp(\delta_\lambda \exp^{-1} Q))\\ &= \delta_\lambda(P)\delta_\lambda(Q).\end{aligned}$$

where $\mathfrak{C}(\cdot,\cdot)$ is given by (1.5).

1.3.3 The Composition Law of $\mathbb{G}$

Following [17], in this section we give a structure theorem for the group operation in a Carnot group $\mathbb{G}$ (see Theorem 1.3.7).

Let us introduce some notations. Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_\kappa$ and, for $s = 1, \dots, \kappa$, let $n_s := \dim \mathfrak{g}_s$ and $m_s - m_{s-1} := n_s$ with $m_0 = 0 < m_1 < \cdots < m_\kappa = N$.

We say that a basis $\mathcal{X}$ of $\mathfrak{g}$ is adapted to $(\mathfrak{g}_1, \dots, \mathfrak{g}_\kappa)$ if

$$\mathcal{X} = (X_1^1, \dots, X_{n_1}^1, \dots, X_1^\kappa, \dots, X_{n_\kappa}^\kappa)$$

where $n_s := \dim \mathfrak{g}_s$ and $(X_1^s, \dots, X_{n_s}^s)$ is a basis for $\mathfrak{g}_s$ for all $s = 1, \dots, \kappa$.

Definition 1.3.2. Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathfrak{g}$. A system of exponential coordinates $\Psi : \mathbb{R}^N \to \mathbb{G}$ is a system of *graded coordinates* if it is associated with an adapted basis of $\mathfrak{g}$.

Let $\Psi : \mathbb{R}^N \to \mathbb{G}$ be a system of graded coordinates. For simplicity we denote with $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ the dilations read in coordinates, so that $\delta_\lambda \circ \Psi = \Psi \circ \delta_\lambda$. Then

$$\delta_\lambda(P) = (\lambda^{\alpha_1} p_1, \dots, \lambda^{\alpha_N} p_N), \qquad \forall\, P = (p_1, \dots, p_N) \in \mathbb{R}^N \tag{1.9}$$

where $1 = \alpha_1 = \cdots = \alpha_{m_1} < \alpha_{m_1+1} = 2 \le \cdots \le \alpha_N = \kappa$.

Now we show some basic properties of homogeneous functions and homogeneous differential operators with respect to the family $\{\delta_\lambda\}_\lambda$ which is crucial for the proof of Theorem 1.3.7.

Definition 1.3.3. Let $f : \mathbb{R}^N \to \mathbb{R}$. We say that f is δ_λ-homogeneous of degree $l \in \mathbb{R}$ if f does not vanish identically and, for every $\lambda > 0$, it holds

$$f \circ \delta_\lambda = \lambda^l f.$$

Moreover a non-identically-vanishing linear differential operator X is called δ_λ-homogeneous of degree $l \in \mathbb{R}$ if, for every $g \in \mathbb{C}^\infty(\mathbb{R}^N)$ and $\lambda > 0$, it holds

$$X(g \circ \delta_\lambda) = \lambda^l (Xg) \circ \delta_\lambda.$$

Remark 1.3.3. Let f be a smooth δ_λ-homogeneous function of degree $l \in \mathbb{R}$ and X be a linear differential operator δ_λ-homogeneous of degree $h \in \mathbb{R}$. Then Xf is a δ_λ-homogeneous function of degree $l - h$ (unless $Xf \equiv 0$). Indeed for every $P \in \mathbb{G}$ and $\lambda > 0$ we have

$$\lambda^h (Xf)(\delta_\lambda(P)) = X(f(\delta_\lambda(P))) = X(\lambda^l f(P)) = \lambda^l (Xf)(P).$$

Definition 1.3.4. Let $\mathbb{G} = (\mathbb{R}^N, \cdot, \delta_\lambda)$ be a Carnot group of step κ where δ_λ is defined as (1.9). Then given a multi-index $\beta = (\beta_1, \dots, \beta_N) \in (\mathbb{N} \cup \{0\})^N$, we define the δ_λ-length (or the $\mathbb{G}$-length of β) as

$$|\beta|_\alpha := \sum_{i=1}^N \beta_i \alpha_i,$$

where α is given by (1.9). Moreover, if $\mathcal{P} : \mathbb{G} \to \mathbb{R}$ is a polynomial function (the sum below is intended to be finite), i.e.

$$\mathcal{P}(P) = \sum_\sigma c_\sigma P^\sigma, \quad c_\sigma \in \mathbb{R}, [3]$$

then we say that

$$\deg_{\mathbb{G}}(\mathcal{P}) := \max\{|\beta|_\alpha \,:\, c_\beta \neq 0\}$$

is the $\mathbb{G}$-degree or δ_λ-homogeneous degree of $\mathcal{P}$.

Notice that since $P \mapsto p_j$ is δ_λ-homogeneous of degree α_j, the function $P \mapsto P^\beta$ is δ_λ-homogeneous of degree $|\beta|_\alpha$.

Moreover, since ∂_{p_j} for $j \in \{1, \dots, N\}$ is δ_λ-homogeneous of degree α_j, the differential operator D^β is δ_λ-homogeneous of degree $|\beta|_\alpha$.

Proposition 1.3.4 (Smooth δ_λ-homogeneous functions)**.** *Let δ_λ be as in* (1.9)*. Suppose that $f \in \mathbb{C}^\infty(\mathbb{R}^N)$. Then f is δ_λ-homogeneous of degree $l \in \mathbb{R}$ if and only if f is a polynomial function of the form*

$$f(P) = \sum_{|\beta|_\alpha = l} f_\beta P^\beta \tag{1.10}$$

with some $f_\beta \in \mathbb{R} - \{0\}$. As a consequence, the set of the degrees of the smooth δ_λ-homogeneous functions is precisely the set of the nonnegative integer numbers

$$\mathcal{A} = \{|\beta|_\alpha \,:\, \beta \in (\mathbb{N} \cup \{0\})^N\},$$

with $|\beta|_\alpha = 0$ if and only if f is constant.

[3] P^σ simply means $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_N^{\alpha_N}$ for $P = (p_1, \dots, p_N)$ and $\alpha = (\alpha_1, \dots, \alpha_N)$.

Proof. It is clear that every polynomial function of the form (1.10) is δ_λ-homogeneous of degree l. Consequently it remains to prove that if f is δ_λ-homogeneous of degree $l \in \mathbb{R}$, then f is a polynomial function of the form (1.10).

If $f(Q) \neq 0$ for some $Q \in \mathbb{R}^N$, then $l \geq 0$. Indeed, from $f(\delta_\lambda Q) = \lambda^l f(Q)$ we get

$$\lim_{\lambda \to 0} \lambda^l = \lim_{\lambda \to 0} \frac{f(\delta_\lambda Q)}{f(Q)} = \frac{f(0)}{f(Q)} < \infty.$$

Moreover, the continuous and δ_λ-homogeneous of degree 0 functions are precisely the constant (non-zero) functions. Indeed,

$$f(P) = f(\delta_\lambda(P)) = \lim_{\lambda \to 0^+} f(\delta_\lambda(P)) = f(0).$$

Then we can suppose that f is a non constant map. Consequently $D^\beta f$ is not identically zero and since $D^\beta f$ is smooth and δ_λ-homogeneous of degree $l - |\beta|_\alpha$, we have that $l - |\beta|_\alpha \geq 0$. This result can be restated as follows:

$$D^\beta f \equiv 0 \quad \forall \beta \text{ such that } |\beta|_\alpha > l.$$

Thus f is a polynomial function, i.e. $f(P) = \sum_{\beta \in \mathcal{A}} f_\beta P^\beta$, where $\mathcal{A}$ is a finite set of multi-indices and $f_\beta \in \mathbb{R}$ for every $\beta \in \mathcal{A}$. Because f is δ_λ-homogeneous of degree l, we obtain

$$\sum_{\beta \in \mathcal{A}} \lambda^l f_\beta P^\beta = \lambda^l f(P) = f(\delta_\lambda(P)) = \sum_{\beta \in \mathcal{A}} \lambda^{|\beta|_\alpha} f_\beta P^\beta.$$

Hence $\lambda^l f_\beta = \lambda^{|\beta|_\alpha} f_\beta$ for every $\lambda > 0$, so that $|\beta|_\alpha = l$ if $f_\beta \neq 0$. Then f is a polynomial function of the form (1.10), as desired.

□

Now we present some elementary properties of the δ_λ-homogeneous functions.

Lemma 1.3.5. *Let δ_λ be as in (1.9). Let $\mathcal{P} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a smooth function with the following property: there is $j \in \{1, \dots, N\}$ such that for all $P = (p_1, \dots, p_N)$ and $Q = (q_1, \dots, q_N)$*

1. $\mathcal{P}(\delta_\lambda(P), \delta_\lambda(Q)) = \lambda^{\alpha_j} \mathcal{P}(P, Q)$, *for all* $\lambda > 0$
2. $\mathcal{P}(P, 0) = p_j$
3. $\mathcal{P}(0, Q) = q_j$

Then

$$\mathcal{P}(P,Q) = \begin{cases} p_j + q_j & \text{if } j \in \{1,\dots,m_1\} \\ p_j + q_j + \tilde{\mathcal{P}}_j(P,Q), & \text{if } j \in \{m_1+1,\dots,N\} \end{cases}$$

where $\tilde{\mathcal{P}}_j$ is a polynomial and is the sum of mixed monomials in $p_1,\dots,p_{j-1},q_1,\dots,q_{j-1}$. Moreover, $\tilde{\mathcal{P}}_j(\delta_\lambda(P),\delta_\lambda(Q)) = \lambda^{\alpha_j}\tilde{\mathcal{P}}_j(P,Q)$. Finally, $\mathcal{P}(P,Q)$ only depends on the p_k's and q_k's with $\alpha_k < \alpha_j$.

Proof. By Proposition 1.3.4, $\mathcal{P}$ is a polynomial function of the following type:

$$\mathcal{P}(P,Q) = \sum_{|\sigma|_\alpha + |\rho|_\alpha = \alpha_j} c_{\sigma,\rho} P^\sigma Q^\rho, \quad c_{\sigma,\rho} \in \mathbb{R}$$

and by the properties 2. and 3. it follows that

$$p_j = \mathcal{P}(P,0) = \sum_{|\sigma|_\alpha = \alpha_j} c_{\sigma,0} P^\sigma \qquad \text{and} \qquad q_j = \mathcal{P}(0,Q) = \sum_{|\rho|_\alpha = \alpha_j} c_{0,\rho} Q^\rho.$$

Then

$$\mathcal{P}(P,Q) = p_j + q_j + \sum_{\substack{|\sigma|_\alpha + |\rho|_\alpha = \alpha_j \\ \sigma,\rho \neq 0}} c_{\sigma,\rho} P^\sigma Q^\rho. \tag{1.11}$$

We can complete the proof by noticing that the condition $|\sigma|_\alpha + |\rho|_\alpha = \alpha_j, \sigma, \rho \neq 0$ is empty when $j = 1,\dots,m_1$, whereas it implies $\sigma = (\sigma_1,\dots,\sigma_{j-1},0,\dots,0)$, $\rho = (\rho_1,\dots,\rho_{j-1},0,\dots,0)$ when $j \geq m_1 + 1$.

As for the last assertion of the lemma, being $\sigma, \rho \neq 0$ in the sum in the right-hand side of (1.11), the sum itself may depend only on the σ's and ρ's with $|\sigma|_\alpha + |\rho|_\alpha < \alpha_j$, hence, on the p_k's and q_k's with $\alpha_k < \alpha_j$. □

Lemma 1.3.6. *Let δ_λ be as in* (1.9). *Let $\mathcal{P} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a smooth function. Assume that there is $l \geq 0$ such that for all $P, Q \in \mathbb{R}^N$ and for all $\lambda > 0$ such that*

$$\mathcal{P}(\delta_\lambda(P), \delta_\lambda(Q)) = \lambda^l \mathcal{P}(P,Q).$$

Then

$$P \mapsto \partial_{q_j}\mathcal{P}(P,0)$$

is δ_λ-homogeneous of degree $l - \alpha_j$ (unless it vanishes identically).

Proof. By Lemma 1.3.5, $\mathcal{P}$ is a polynomial of the following type

$$\mathcal{P}(P,Q) = \sum_{|\sigma|_\alpha + |\rho|_\alpha = l} c_{\sigma,\rho} P^\sigma Q^\rho, \quad c_{\sigma,\rho} \in \mathbb{R}.$$

Then, denoting by e_j the j-th element of the canonical basis of $\mathbb{R}^N$, we obtain

$$\partial_{q_j}\mathcal{P}(P,Q) = \sum_{|\sigma|_\alpha+|\rho|_\alpha=l} c_{\sigma,\rho}\rho_j P^\sigma Q^{\rho-e_j},$$

so that, since $|e_j|_\alpha = \alpha_j$,

$$\partial_{q_j}\mathcal{P}(P,0) = \sum_{\substack{|\sigma|_\alpha=l-\alpha_j\\ \rho=e_j}} c_{\sigma,\rho}P^\sigma$$

This ends the proof. □

Now, we are in the position to prove the previously mentioned structure theorem for the composition law of a Carnot group.

Theorem 1.3.7. *Let $\mathbb{G} = (\mathbb{R}^N, \cdot, \delta_\lambda)$ be a Carnot group of step κ. Then the group operation $\cdot$ has polynomial component functions. Furthermore, for each $P = (p_1,\dots,p_N), Q = (q_1,\dots,q_N) \in \mathbb{R}^N$ we have*

$$(PQ)_j = \begin{cases} p_j + q_j, & \text{for } j = 1,\dots,m_1 \\ p_j + q_j + \mathcal{Q}_j(P,Q) & \text{for } j = m_1+1,\dots,N \end{cases}$$

and the following facts hold:

1. *$\mathcal{Q}_j$ is a sum of mixed monomials in P, Q.*
2. *$\mathcal{Q}_j(\delta_\lambda P, \delta_\lambda Q) = \lambda^{\alpha_j}\mathcal{Q}_j(P,Q)$ for all $P,Q \in \mathbb{G}$ and $\lambda > 0$.*
3. *$\mathcal{Q}_j(P,Q)$ only depends on the p_k's and q_k's with $\alpha_k < \alpha_j$.*

More precisely,

$$\mathcal{Q}_j(P,Q) = \sum_{l,h} \mathcal{R}^j_{l,h}(P,Q)(p_l q_h - p_h q_l), \qquad \text{for } m_1 < j \le N \tag{1.12}$$

where $\mathcal{R}^j_{l,h}$ are polynomials, homogenous of degree $\alpha_j - (\alpha_l + \alpha_h)$ with respect to group dilations, and the sum is extended to all l, h such that $\alpha_l + \alpha_h \le \alpha_j$.

Proof. Fix $j = 1,\dots,N$. We define

$$\mathcal{Q}_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \qquad \mathcal{Q}_j(P,Q) = (PQ)_j.$$

Since δ_λ is an automorphism of $\mathbb{G}$, we get

$$\mathcal{Q}_j(\delta_\lambda(P), \delta_\lambda(Q)) = (\delta_\lambda(PQ))_j = \lambda^{\alpha_j}(PQ)_j = \lambda^{\alpha_j}\mathcal{Q}_j(P,Q).$$

Moreover, because $P \cdot 0 = P$, $0 \cdot Q = Q$, we obtain that

$$\mathcal{Q}_j(P, 0) = p_j, \quad \mathcal{Q}_j(0, Q) = q_j,$$

Hence we can apply Lemma 1.3.5 to $\mathcal{Q}_j$ and consequently we obtain that the conditions 1-2-3 hold for all $j = 1, \dots, N$. Finally (1.12) follows from Proposition 2.2.22 (4) in [17]. □

It is useful to know that $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \cdots \oplus \mathbb{G}^\kappa$ where $\mathbb{G}^i = \exp(\mathfrak{g}_i) = \mathbb{R}^{n_i}$ is the i^{th} layer of $\mathbb{G}$ and to write $P \in \mathbb{G}$ as $(p^1, \dots, p^\kappa)$ with $p^i \in \mathbb{G}^i$. According to this

$$P \cdot Q = (p^1 + q^1, p^2 + q^2 + \mathcal{Q}^2(p^1, q^1), \dots, p^\kappa + q^\kappa + \mathcal{Q}^\kappa((p^1, \dots, p^{\kappa-1}), (q^1, \dots, q^{\kappa-1})) \tag{1.13}$$

for every $P = (p^1, \dots, p^\kappa)$, $Q = (q^1, \dots, q^\kappa) \in \mathbb{G}$.

From (1.12) it follows that for all $P \in \mathbb{G}$

$$\mathcal{Q}^s(P, 0) = \mathcal{Q}^s(0, P) = 0 \quad \text{and} \quad \mathcal{Q}^s(P, P) = \mathcal{Q}^s(P, -P) = 0 \tag{1.14}$$

for $s = 2, \dots \kappa$. Moreover for each $\mathcal{E}$ bounded subset of $\mathbb{G}$ there exists $C_{\mathcal{E}} > 0$ such that

$$\begin{aligned} |\mathcal{Q}^s(-P, Q)| &\le C_{\mathcal{E}}(|q^1 - p^1| + \cdots + |q^{s-1} - p^{s-1}|) \\ &\le C_{\mathcal{E}}(|(P^{-1}Q)^1| + \cdots + |(P^{-1}Q)^{s-1}|) \end{aligned} \tag{1.15}$$

for all $P, Q \in \mathcal{E}$. This fact follows from

$$\begin{aligned} \mathcal{Q}_j(-P, Q) &= \sum_{l,h} \mathcal{R}^j_{l,h}(-P, Q)(-p_l q_h + p_h q_l) \\ &= \sum_{l,h} \mathcal{R}^j_{l,h}(-P, Q)(q_h(q_l - p_l) - q_l(q_h - p_h)), \end{aligned}$$

Corollary 1.3.8. *Let $\mathbb{G} = (\mathbb{R}^N, \cdot, \delta_\lambda)$ be a Carnot group of step κ. Then for all $Q \in \mathbb{G}$*

$$Q^{-1} = -Q.$$

Proof. By the explicit form of the composition in Theorem 1.3.7 we deduce that if $P \cdot Q = 0$, then

$$p_j = -q_j \quad \text{whenever } \alpha_j = 1. \tag{1.16}$$

Moreover if $\alpha_j = 2$ we have $p_j = -q_j + \mathcal{Q}_j(P, Q)$, where $\mathcal{Q}_j$ only depends on the p_k's and q_k's with $\alpha_k = 1$ and consequently using (1.14) and (1.16)

$$p_j = -q_j + \mathcal{Q}_j((-q_1, \dots, -q_{m_1}), (q_1, \dots, q_{m_1})) = -q_j \qquad \text{whenever } \alpha_j = 2.$$

This procedure can be iterated to get the thesis. □

Now we give two estimates (see [39]).

Proposition 1.3.9. *If $P, Q \in \mathbb{G}$ then*

$$P^{-1}QP = Q + \mathcal{P}(P,Q),$$

where $\mathcal{P}(P,Q) = (\mathcal{P}^1(P,Q),\dots,\mathcal{P}^\kappa(P,Q))$ with $\mathcal{P}^1(P,Q) = 0$ and for each $s = 2,\dots\kappa$, $\mathcal{P}^s(P,Q)$ are (vector valued) polynomial functions δ_λ-homogeneous of degree s. Moreover, if $\mathcal{E} \subset \mathbb{G}$ is bounded, there exists $C_\mathcal{E} = C_\mathcal{E}(\mathbb{G}) > 0$ such that for $s = 2,\dots,\kappa$

$$|\mathcal{P}^s(P,Q)| \le C_\mathcal{E}\big(|q^1| + \dots + |q^{s-1}|\big),$$

for all $P = (p^1,\dots,p^\kappa), Q = (q^1,\dots,q^\kappa) \in \mathcal{E}$.

Proof. By Theorem 1.3.7 we have

$$\begin{aligned} P^{-1}QP &= Q + \mathcal{Q}(P,Q) + \mathcal{Q}(-P, Q + P + \mathcal{Q}(P,Q)) \\ &=: Q + \mathcal{P}(P,Q), \end{aligned}$$

for all $P, Q \in \mathbb{G}$. Because $\mathcal{Q}^1(P,Q) = 0$ then $\mathcal{P}^1(P,Q) = 0$. Moreover thanks to (1.12) it follows that $\mathcal{P}^s(P,Q)$ is the sum of monomials each one containing a positive power of some q_i for $i = 1,\dots,m_{s-1}$. □

Corollary 1.3.10. *There is a vector valued polynomial function $\mathcal{P} : \mathbb{G} \times \mathbb{G} \to \mathbb{R}^N$ such that*

$$G^{-1}P^{-1}QG = Q - P + \mathcal{P}(G, P^{-1}Q) \quad \forall\, G, P, Q \in \mathbb{G}.$$

Moreover if $\mathcal{E} \subset \mathbb{G}$ is bounded, there is $C_\mathcal{E} = C_\mathcal{E}(\mathbb{G}) > 0$ such that for $s = 2,\dots,\kappa$,

$$|\mathcal{P}^s(G, P^{-1}Q)| \le C_\mathcal{E}\left(|(P^{-1}Q)^1| + \dots + |(P^{-1}Q)^{s-1}|\right) \quad \forall\, G, P, Q \in \mathcal{E}. \tag{1.17}$$

Proof. From Proposition 1.3.9 we have

$$\begin{aligned} G^{-1}P^{-1}QG &= P^{-1}Q + \mathcal{P}(G, P^{-1}Q) \\ &= Q - P + \mathcal{Q}(-P,Q) + \mathcal{P}(G, P^{-1}Q). \end{aligned}$$

We can estimate $\mathcal{Q}(-P,Q)$ with (1.15). From again Proposition 1.3.9 and (1.15) we get for $s = 2,\dots,\kappa$

$$\begin{aligned} |\mathcal{P}^s(G, P^{-1}Q)| &\le C_\mathcal{E}\left(|(P^{-1}Q)^1| + \dots + |(P^{-1}Q)^{s-1}|\right) \\ &\le C_\mathcal{E} \sum_{l=1}^{s-1} \left(|(Q-P)^l| + |\mathcal{Q}^l(-P,Q)|\right) \\ &\le \hat{C}_\mathcal{E} \sum_{l=1}^{s-1} |(Q-P)^l| \end{aligned}$$

for all $G, P, Q \in \mathcal{E}$. □

Example 1.3.2. The Euclidean group $(\mathbb{R}^N, +)$ is a Carnot group of step 1. Here, for any $\lambda > 0$, the dilation is given by

$$\delta_\lambda(A) = \lambda A, \qquad \text{for every } A \in \mathbb{R}^N.$$

The Euclidean spaces are the only Carnot groups of step 1 and the only commutative Carnot groups.

Consequently, a Carnot group is not the Euclidean group $\mathbb{R}^N$ if and only if $\mathfrak{g}_2 \neq \{0\}$. In this case, the first layer $\mathfrak{g}_1$ must be at least 2 dimensional since $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2 \neq \{0\}$ and hence the homogeneous dimension of non commutative Carnot group is larger than 4.

Example 1.3.3. The simplest example of a non-Abelian Carnot group is provided by Heisenberg group $\mathbb{H}^k = \mathbb{R}^{2k+1}$. Exhaustive introductions to Heisenberg groups can be found in [21], [98].

For $P = (p_1, \dots, p_{2k}, p_{2k+1}), Q = (q_1, \dots, q_{2k}, q_{2k+1}) \in \mathbb{H}^k$ we define the group operation given by

$$P \cdot Q = \left(p_1 + q_1, \dots, p_{2k} + q_{2k}, p_{2k+1} + q_{2k+1} + \frac{1}{2} \sum_{i=1}^{k} (p_i q_{k+i} - p_{k+i} q_i) \right)$$

and the family of (non isotropic) dilations

$$\delta_\lambda(P) := (\lambda p_1, \dots, \lambda p_{2k}, \lambda^2 p_{2k+1}), \quad \text{for all } P \in \mathbb{H}^k, \lambda > 0.$$

A basis of left invariant vector fields is given by

$$\begin{aligned} X_i &= \partial_{p_i} - \frac{1}{2} p_{k+i}\, \partial_{p_{2k+1}}, && \text{for all } i = 1, \dots, k \\ X_{k+i} &= \partial_{p_{k+i}} + \frac{1}{2} p_i\, \partial_{p_{2k+1}}, && \text{for all } i = 1, \dots, k \\ Y &= \partial_{p_{2k+1}} \end{aligned}$$

The only non trivial commutator relations being

$$[X_i, X_{k+i}] = Y \quad i = 1, \dots, k.$$

Thus the vector fields $X_1, \dots, X_{2k}$ satisfy Chow-Hörmander condition (1.1) and $\mathbb{H}^k$ is a step 2 Carnot group. Moreover the stratification of the Lie algebra $\mathfrak{h}$ of the left invariant vector fields is given by $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$,

$$\mathfrak{h}_1 = \text{span}\{X_1, \dots, X_{2k}\}, \quad \mathfrak{h}_2 = \text{span}\{Y\}.$$

1.3.4 Left invariant vector fields

A basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is given by the following N left invariant vector fields:

Proposition 1.3.11. *Let $\mathbb{G}$ be a Carnot group identified with $\mathbb{R}^N$ through graded coordinates associated with a basis $X_1, \dots, X_N$. Let $\{\partial_{p_i}\}_{i=1,\dots,N}$ be the standard basis of vectors of $\mathbb{R}^N$ and set $X_j(P) := \sum_{i=1}^N a_{ij}(P)\partial_{p_i}$. If $m_{s-1} < i \le m_s$, $1 \le s \le \kappa$, then*

$$X_i(P) = \partial_{p_i} + \sum_{j>m_s}^{N} a_{ji}(P)\partial_{p_j} \tag{1.18}$$

with

$$a_{ji}(P) = \partial_{q_i}\mathcal{Q}_j(P,Q)_{|Q=0}$$

and such that $a_{ji}(P) = a_{ji}(p_1, \dots, p_{m_{s-1}})$ and $a_{ji}(0) = 0$.

We highlighted that X_i defined as (1.18) are "pyramid"-shaped vector fields (see Example 1.2.3).

The proof of Proposition 1.3.11 relies on the following characterization of the smooth δ_λ-homogeneous vector fields (see Proposition 1.3.5 in [17]).

Proposition 1.3.12. *Let δ_λ be as in* (1.9)*. Let X be a smooth non vanishing vector field on $\mathbb{R}^N$,*

$$X(P) = \sum_{i=1}^N a_i(P)\partial_{p_i}.$$

Then X is δ_λ-homogeneous of degree $l \in \mathbb{R}$ if and only if a_i is a polynomial function δ_λ-homogeneous of degree $\alpha_i - l$ (unless $a_i = 0$). Hence, the degree of δ_λ-homogeneity of X belongs to the set of real (possibly negative) numbers

$$\mathcal{A}_i = \{\alpha_i - |\beta|_\alpha \, : \, \beta \in (\mathbb{N} \cup \{0\})^N\},$$

whenever i is such that a_i is not identically zero.

For example, the vector fields $X_1 = \partial_{p_1} - \frac{1}{2}p_2\partial_{p_3}$ and $X_2 = \partial_{p_2} - \frac{1}{2}p_1\partial_{p_3}$ on $\mathbb{R}^3$ are δ_λ-homogeneous of degree 1 with respect to the dilation

$$\delta_\lambda(p_1, p_2, p_3) = (\lambda p_1, \lambda p_2, \lambda^2 p_3).$$

Also, the vector fields $p_1^3 X_1 := p_1^3\partial_{p_1} - \frac{1}{2}p_1^3 p_2\partial_{p_3}$ and $p_2 X_2 := p_2\partial_{p_2} - \frac{1}{2}p_2 p_1\partial_{p_3}$ are respectively δ_λ-homogeneous of degrees -2 and 0 with respect to the same dilation.

Corollary 1.3.13. *Let δ_λ be as in (1.9). Let X be a smooth non vanishing vector field on $\mathbb{R}^N$. Then X is δ_λ-homogeneous of degree $l \in \mathbb{R}$ if and only if*

$$\delta_\lambda(X(P)) = \lambda^l X(\delta_\lambda(P)).$$

Proof. Let $X = \sum_{i=1}^N a_i(P)\partial_{p_i}$. By Proposition 1.3.12, X is δ_λ-homogeneous of degree l if and only if $a_i(\delta_\lambda(P)) = \lambda^{\alpha_i - l} a_i(P)$ for any $i \in \{1, \dots, N\}$. This is equivalent to

$$\begin{aligned}\delta_\lambda(X(P)) &= (\lambda^{\alpha_1} a_1(P), \dots, \lambda^{\alpha_N} a_N(P)) \\ &= \lambda^l (a_1(\delta_\lambda(P)), \dots, a_N(\delta_\lambda(P))) \\ &= \lambda^l X(\delta_\lambda(P)),\end{aligned}$$

as desired. □

Now we are able to show the proof of Proposition 1.3.11.

Proof. Let $P = (p_1, \dots, p_N), Q = (q_1, \dots, q_N) \in \mathbb{G}$. Because X_i is a left invariant vector field we have

$$\begin{aligned}X_i f(P) &= \partial_{q_i} f(PQ)\big|_{Q=0} \\ &= \partial_{p_i} f(P) + \sum_{j>m_s} \partial_{q_i} \mathcal{Q}_j(P,Q)_{|Q=0}\, \partial_{p_j} f(P) \\ &= \partial_{p_i} f(P) + \sum_{j>m_s} a_{ji}(P)\partial_{p_j} f(P)\end{aligned}$$

To conclude observe that from (1.13) in each polynomial $\mathcal{Q}_j$ there is no term depending on a single variable, hence $a_{ji}(0) = \partial_{q_i} \mathcal{Q}_j(P,Q)_{|Q=P=0} = 0$.

Finally by Lemma 1.3.6 and Proposition 1.3.12 we obtain that $a_{ji}(P)$ is δ_λ-homogeneous of degree $\alpha_j - \alpha_i$ and consequently X_i is δ_λ-homogeneous of degree α_i. □

1.3.5 Metrics on Carnot groups

Let $\mathbb{G} = (\mathbb{R}^N, \cdot, \delta_\lambda)$ be a Carnot group and let d_{cc} be the cc-metric defined in Definition 1.1.1.

By Theorem 1.1.6 and Remark 1.3.1, $(\mathbb{G}, d_{cc})$ is a Carnot-Carathéodory space. Moreover the presence of a stratification induces many "good" properties of d_{cc}, with respect to both left translations and dilations, which are collected in the following proposition.

Proposition 1.3.14. *For any $P, Q, Q' \in \mathbb{R}^N$ and $\lambda > 0$ we have*

1. $d_{cc}(\tau_P Q, \tau_P Q') = d_{cc}(Q, Q')$
2. $d_{cc}(\delta_\lambda Q, \delta_\lambda Q') = \lambda d_{cc}(Q, Q')$.

Proof. The statement 1. of the thesis follows from the fact that $\gamma : [0,T] \to \mathbb{R}^N$ is a subunit path from Q to Q' if and only if $\tilde{\gamma} = \tau_P \circ \gamma : [0,T] \to \mathbb{R}^N$ is a subunit path from PQ to PQ'. In fact, if $\dot{\gamma}(t) = \sum_{j=1}^{n_1} h_j(t) X_j(\gamma(t))$, then

$$\dot{\tilde{\gamma}}(t) = d\tau_P(\gamma(t))\,\dot{\gamma}(t) = d\tau_P(\gamma(t)) \sum_{j=1}^{m_1} h_j(t) X_j(\gamma(t)) = \sum_{j=1}^{m_1} h_j(t) X_j(\tilde{\gamma}(t)),$$

where $d\tau_P$ denotes the differential of the left translation by P.

Now we prove the statement 2. of the thesis. It will be sufficient to show that a path $\gamma : [0,T] \to \mathbb{R}^N$ from Q to Q' is subunit if and only if so is the curve $\gamma_\lambda : [0, \lambda T] \to \mathbb{R}^N$, joining $\delta_\lambda(Q)$ and $\delta_\lambda(Q')$, defined by

$$\gamma_\lambda(t) := \delta_\lambda(\gamma(t/\lambda)).$$

Indeed, we know that

$$\dot{\gamma}(t) = \sum_{j=1}^{m_1} h_j(t) X_j(\gamma(t)) = \sum_{l=1}^{N} \left(\sum_{j=1}^{m_1} h_j(t) a_{lj}(\gamma(t)) \right) \partial_{p_l}$$

and since $\alpha_j = 1$ for all $j = 1, \dots, m_1$, by Proposition 1.3.11 all the a_{lj}'s appearing in the sum are δ_λ-homogeneous of degree $(\alpha_l - 1)$ and so

$$\begin{aligned} \dot{\gamma}_\lambda(t) &= \sum_{l=1}^{N} \lambda^{\alpha_l - 1} \left(\sum_{j=1}^{m_1} h_j(t/\lambda) a_{lj}(\gamma(t/\lambda)) \right) \partial_{p_l} \\ &= \sum_{l=1}^{N} \left(\sum_{j=1}^{m_1} h_j(t/\lambda) a_{lj}(\gamma_\lambda(t)) \right) \partial_{p_l} = \sum_{j=1}^{m_1} h_j(t/\lambda) X_j(\gamma_\lambda(t)). \end{aligned}$$

□

It is useful to consider on $\mathbb{G}$ a homogeneous norm:

Definition 1.3.5. A nonnegative function $P \to \|P\|$ on $\mathbb{G}$ is said a homogeneous norm if

1. $\|P\| = 0$ if and only if $P = 0$.
2. $\|\delta_\lambda P\| = \lambda \|P\|$ for all $P \in \mathbb{G}$ and $\lambda > 0$.
3. $\|PQ\| \leq \|P\| + \|Q\|$.

Given any homogeneous norm $\|\cdot\|$, it is possible to introduce a distance in $\mathbb{G}$ given by

$$d(P,Q) = d(P^{-1}Q, 0) = \|P^{-1}Q\| \tag{1.19}$$

for all $P, Q \in \mathbb{G}$.

Both the cc-metric d_{cc} and the metric d in (1.19) are invariant, i.e.

Definition 1.3.6. Let $\mathbb{G}$ be a Carnot group. A invariant (or homogeneous) distance d on $\mathbb{G}$ is a map $d : \mathbb{G} \times \mathbb{G} \to [0, +\infty)$ with the following properties:

1. $d(\tau_P Q, \tau_P Q') = d(Q, Q')$, for all $P, Q, Q' \in \mathbb{G}$ (Left invariance)
2. $d(\delta_\lambda Q, \delta_\lambda Q') = \lambda d(Q, Q')$, for all $Q, Q' \in \mathbb{G}$ and $\lambda > 0$ (Homogeneity)

The distance d in (1.19) is equivalent to d_{cc}. More precisely

Proposition 1.3.15. *Let d_1 and d_2 be invariant distances on $\mathbb{G}$. Then they are equivalent, i.e. there exist $C_M, C_m > 0$ such that for all $P, Q \in \mathbb{G}$*

$$C_m d_1(P, Q) \leq d_2(P, Q) \leq C_M d_1(P, Q).$$

Proof. We define the sphere $S := \{P \in \mathbb{G} \,:\, d_1(P, 0) = 1\}$ and the numbers

$$C_m := \min\{d_2(0, Q) \,:\, Q \in S\}, \qquad C_M := \max\{d_2(0, Q) \,:\, Q \in S\}.$$

because $d_2(0, \cdot)$ is strictly positive and continuous on S, we deduce that C_M and C_m are positive constants. Then by homogeneity of d_1 and d_2 we get

$$C_m d_1(0, Q) \leq d_2(0, Q) \leq C_M d_1(0, Q)$$

for any $Q \in \mathbb{G}$. Now the thesis follows from the left invariant property of distances d_1 and d_2. □

For any bounded subset Ω of $\mathbb{G}$ there exist $c_1 = c_1(\Omega), c_2 = c_2(\Omega) > 0$ such that

$$c_1|P - Q| \leq d(P, Q) \leq c_2|P - Q|^{1/\kappa} \tag{1.20}$$

for every $P, Q \in \Omega$. Therefore, $id : (\mathbb{G}, d) \to (\mathbb{R}^N, |\cdot|)$ is locally Lipschitz and $id : (\mathbb{R}^N, |\cdot|) \to (\mathbb{G}, d)$ is locally $\mathbb{C}^{0,1/\kappa}$ Hölder continuous.

By (1.20), we have that the topologies defined by d and by the Euclidean distance coincide, and so the topological dimension of $(\mathbb{G}, d)$ is N; but the invariant metric d is not locally equivalent to the Euclidean distance. Indeed it was proved in [93] that there are no bi-Lipschitz maps from from a general non commutative Carnot group $\mathbb{G}$ to any Euclidean space.

Example 1.3.4. We present an example of homogeneous norm that is used in [44] and denote by $\|\cdot\|_\infty$.

Let us consider the Heisenberg group $\mathbb{H}^k$ (see Example 1.3.3). For any $P = (p^1, p^2) \in \mathbb{H}^k$ we define a homogeneous norm as

$$\|P\|_\infty := \max\{|p^1|, |p^2|^{1/2}\}.$$

Then the invariant distance associated to $\|\cdot\|_\infty$ is the continuous map $d_\infty : \mathbb{H}^k \times \mathbb{H}^k \to \mathbb{R}$ given by

$$d_\infty(P, Q) := \|P^{-1}Q\|_\infty$$

for all $P, Q \in \mathbb{H}^k$.
It is clear that $\|P\| = 0$ if and only if $P = 0$, i.e. the condition 1. is true.
Moreover observe that for any $P = (p^1, p^2) \in \mathbb{H}^k$

$$\|\delta_\lambda P\|_\infty = \max\{|\lambda p^1|, |\lambda^2 p^2|^{1/2}\} = \lambda \max\{|p^1|, |p^2|^{1/2}\} = \lambda \|P\|_\infty$$

i.e. the condition 2. holds.
Finally we show the triangle inequality. Let $P = (p^1, p^2), Q = (q^1, q^2) \in \mathbb{H}^k$ with $p^1, q^1 \in \mathbb{R}^{2k}$ and $p^2, q^2 \in \mathbb{R}$. We have that

$$PQ = (p^1 + q^1, p^2 + q^2 + \mathcal{Q}^2(p^1, q^1))$$

with $\mathcal{Q}^2(p^1, q^1) = \frac{1}{2}\sum_{i=1}^k (p_i q_{k+i} - p_{k+i} q_i)$. If $\|PQ\|_\infty = |p^1 + q^1|$, then

$$\|PQ\|_\infty \le |p^1| + |q^1| \le \|P\|_\infty + \|Q\|_\infty.$$

On the other hand, if $\|PQ\|_\infty = |p^2 + q^2 + \mathcal{Q}^2(p^1, q^1)|^{1/2}$, then

$$\|PQ\|_\infty^2 \le |p^2| + |q^2| + |p^1||q^1| \le \|P\|_\infty^2 + \|Q\|_\infty^2 + 2\|P\|_\infty \|Q\|_\infty.$$

and so the triangle inequality follows.

Example 1.3.5. (see Theorem 5.1 in [47]) Let $\mathbb{G}$ be a Carnot group of step κ then there are constants $\epsilon_1 = 1, \dots, \epsilon_\kappa$ with $\epsilon_i \in (0, 1]$, such that the map $\|\cdot\| : \mathbb{G} \to \mathbb{R}$ defined as

$$\|(p^1, \dots, p^\kappa)\| := \max_{s=1,\dots,\kappa} \left\{\epsilon_s |p^s|^{1/\alpha_s}\right\} \tag{1.21}$$

is a homogeneous norm.

1.3.6 Hausdorff measures in a metric space: Application to Carnot groups

In this section we provide the definition and basic properties of the notion of Hausdorff measure in a general metric spaces and then we consider the specific case of a Carnot group $\mathbb{G}$ equipped with an invariant distance (see Definition 1.3.6). The reader can see [33] and [95].

For $t > 0$ let

$$\alpha(t) := \frac{\pi^{t/2}}{\Gamma(\frac{t}{2} + 1)}$$

where $\Gamma : (0, +\infty) \to \mathbb{R}$ is the usual gamma function defined as $\Gamma(t) = \int_0^\infty \mathrm{e}^{-r} r^{t-1}\, dr$. We recall that

$$\mathcal{L}^N(B_e(x,r)) = \alpha(N) r^N$$

for all Euclidean balls $B(x,r) \subset (\mathbb{R}^N, |\cdot|)$.

Moreover we recall that if $\mathcal{C}$ is a subset of a separable metric space $(\mathcal{G}, d)$ then the diameter of $\mathcal{C}$ is denoted as

$$\mathrm{diam}(\mathcal{C}) := \sup\{d(A,B) \,|\, A, B \in \mathcal{C}\}.$$

Definition 1.3.7. Let $(\mathcal{G}, d)$ be a separable metric space. Let $\mathcal{U} \subset \mathcal{G}$, $0 \le t < \infty$ and $0 < \delta \le \infty$. Define

$$\mathcal{H}^t_{d,\delta}(\mathcal{U}) := \inf \left\{ \sum_{i=1}^\infty \alpha(t) \mathrm{diam}(\mathcal{C}_i)^t \,\middle|\, \mathcal{U} \subset \cup_{i=1}^\infty \mathcal{C}_i,\ \mathrm{diam}(\mathcal{C}_i) \le \delta \right\}.$$

Then

$$\mathcal{H}^t_d(\mathcal{U}) = \lim_{\delta \to 0} \mathcal{H}^t_{d,\delta}(\mathcal{U})$$

is called the t-dimensional Hausdorff measure of $\mathcal{U}$.

We define $\mathcal{S}^t_d$ the t-dimensional spherical Hausdorff measure of $\mathcal{U}$, when the infimum is taken over all possible coverings of $\mathcal{U}$ by a (finite or) countable family of balls $\{B(P_i, r_i) : i \in \mathbb{N}\}$.

It is well know that $\mathcal{H}^t_d$ and $\mathcal{S}^t_d$ are equivalent and they are Borel measures, i.e.

$$\mathcal{H}^t_d(\mathcal{U}_1 \cup \mathcal{U}_2) = \mathcal{H}^t_d(\mathcal{U}_1) + \mathcal{H}^t_d(\mathcal{U}_2), \quad \text{if } \mathrm{dist}(\mathcal{U}_1, \mathcal{U}_2) > 0.$$

Definition 1.3.8. Let $(\mathcal{G}, d)$ be a separable metric space and let $\mathcal{U} \subset \mathcal{G}$. We define the Hausdorff (or metric) dimension of $\mathcal{U}$ as the value

$$\mathrm{Hdim}(\mathcal{U}) := \inf\{t \in [0, +\infty) \,|\, \mathcal{H}^t_d(\mathcal{U}) = 0\}.$$

Remark 1.3.16. Observe that if $\mathrm{Hdim}(\mathcal{U}) = t$, then

$$\begin{aligned} \mathcal{H}^s_d(\mathcal{U}) &= 0 \quad \text{for all } s > t \\ \mathcal{H}^s_d(\mathcal{U}) &= \infty \quad \text{for all } s < t \end{aligned}$$

and $\mathcal{H}^t_d(\mathcal{U})$ may be any number between 0 and ∞, inclusive.

Because $\mathcal{H}^t_d, \mathcal{S}^t_d$ are equivalent, the notion of Hausdorff dimension can be equivalently stated by means of measure $\mathcal{S}^t_d$. Moreover, the Hausdorff dimension is stable with respect to equivalent metrics on $\mathcal{G}$ and with respect to the countable union, i.e.

$$\mathrm{Hdim}(\cup_{i=1}^\infty \mathcal{U}_i) = \sup_i \mathrm{Hdim}(\mathcal{U}_i) \quad \text{if } \mathcal{U}_i \subset \mathcal{G},\ i \in \mathbb{N}.$$

It is monotonous, i.e.

$$\mathrm{Hdim}(\mathcal{U}_1) \le \mathrm{Hdim}(\mathcal{U}_2) \quad \text{if } \mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{G}.$$

Let $\mathbb{G}$ be a Carnot group endowed with an invariant metric d. We denote by $U(P,r)$ and $B(P,r)$, respectively, the open and closed ball centered at P and with radius $r>0$, that is

$$U(P,r) = \{Q \in \mathbb{G} \,:\, d(P,Q) < r\}, \quad B(P,r) = \{Q \in \mathbb{G} \,:\, d(P,Q) \le r\}.$$

By Proposition 2.4 in [47], we know that there is $\delta > 0$ such that for all $r \in (0,\delta)$ and all $P \in \mathbb{G}$

$$\mathrm{diam}(B(P,r)) = 2r.$$

Moreover because d is a left invariant distance, all Hausdorff measures are left invariant, i.e.

$$\mathcal{H}_d^t(\tau_P(\mathcal{U})) = \mathcal{H}_d^t(\mathcal{U}), \qquad \text{for all } \mathcal{U} \subset \mathbb{G},\ P \in \mathcal{U}$$

and t homogeneous with respect to dilations δ_λ, i.e.

$$\mathcal{H}_d^t(\delta_\lambda(\mathcal{U})) = \lambda^t\,\mathcal{H}_d^t(\mathcal{U}), \qquad \text{for all } \mathcal{U} \subset \mathbb{G},\ \lambda > 0.$$

We denote by $\mathcal{H}_e^t$ the t-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^N \simeq \mathbb{G}$, by $\mathcal{H}^t$ the t-dimensional Hausdorff measure obtained from the metric d in $\mathbb{G}$. Analogously, $\mathcal{S}_e^t$ $\mathcal{S}^t$ denote the corresponding spherical Hausdorff measure.

The following theorem, proved for the first time in [76], gives an explicit formula of the metric dimension of Carnot group:

Theorem 1.3.17. *Let $\mathbb{G}$ be a Carnot group endowed with an invariant metric. If we put*

$$\mathfrak{q} := \sum_{j=1}^{N} \alpha_j = \sum_{i=1}^{\kappa} i\, dim \mathfrak{g}_i \tag{1.22}$$

then

$$Hdim(\mathbb{G}) = \mathfrak{q}.$$

We call $\mathfrak{q}$ the homogeneous dimension of $\mathbb{G}$.

Remark 1.3.18. In the non-Abelian case $\kappa > 1$, one clearly has

$$N < \mathfrak{q},$$

i.e. the topological dimension N is strictly less than the metric dimension $\mathfrak{q}$. The equality holds if and only if $\mathbb{G}$ is a step 1 Carnot group, i.e. if $\mathbb{G}$ is isomorphic to Euclidean space $\mathbb{R}^N$. For instance in the case of the Heisenberg group $\mathbb{H}^k = \mathbb{R}^{2k+1}$ we have that

$$N = 2k+1 < 2k+2 = \mathfrak{q}.$$

Proposition 1.3.19. *Let $\mathbb{G} = (\mathbb{R}^N, \cdot)$ be a Carnot group. Then the N-dimensional Lebesgue measure $\mathcal{L}^N$ is the Haar measure of the group $\mathbb{G}$. Indeed if $E \subset \mathbb{R}^N$ is measurable then $\mathcal{L}^N(\tau_P(E)) = \mathcal{L}^N(E)$ for all $P \in \mathbb{G}$. Moreover if $\lambda > 0$ then $\mathcal{L}^N(\delta_\lambda(E)) = \lambda^{\mathfrak{q}}\mathcal{L}^N(E)$ and*

$$\mathcal{L}^N(B(P,r)) = \mathcal{L}^N(\mathcal{U}(P,r)) = r^{\mathfrak{q}}\mathcal{L}^N(B(P,1)) = r^{\mathfrak{q}}\mathcal{L}^N(B(0,1)) \tag{1.23}$$

where $\mathfrak{q}$ is the homogeneous dimension of $\mathbb{G}$ defined as (1.22)*.*

1.3.7 Sum of Carnot Groups

In this section, we show how to build a Carnot group starting from two fixed groups (see Section 4.1.5 in [17]).

Let $\mathbb{G}^{(1)} = (\mathbb{R}^N, \cdot, \delta_\lambda^{(1)})$, $\mathbb{G}^{(2)} = (\mathbb{R}^M, \star, \delta_\lambda^{(2)})$ be Carnot groups with dilations

$$\begin{aligned} \delta_\lambda^{(1)}(P) &= (\lambda p^1, \dots, \lambda p^\kappa), \qquad P \in \mathbb{G}^{(1)} \\ \delta_\lambda^{(2)}(Q) &= (\lambda q^1, \dots, \lambda q^r), \qquad Q \in \mathbb{G}^{(2)} \end{aligned}$$

where

$$\begin{aligned} p^s \in \mathbb{R}^{n_s} \quad &\text{for } s = 1, \dots, \kappa, \quad n_1 + \dots + n_\kappa = N \\ q^s \in \mathbb{R}^{l_s} \quad &\text{for } s = 1, \dots, r, \quad l_1 + \dots + l_r = M. \end{aligned}$$

We define a Carnot group $\mathbb{G}$ on $\mathbb{R}^{N+M}$ as follows.
Suppose $r \le \kappa$. We denote the points of $\mathbb{G} = \mathbb{R}^{N+M}$ by

$$R(P,Q) = (p^1, q^1, \dots, p^r, q^r, p^{r+1}, \dots, p^\kappa), \qquad \text{for } (P,Q) \in \mathbb{R}^N \times \mathbb{R}^M.$$

Moreover the group law $\diamond$ and the dilation δ_λ on $\mathbb{G}$ are defined as follows: for every $\xi = R(P,Q)$, $\eta = R(P',Q') \in \mathbb{G}$

$$\begin{aligned} \xi \diamond \eta &= R(P \cdot P', Q \star Q'), \\ \delta_\lambda(\xi) &= R(\delta_\lambda^{(1)}(P), \delta_\lambda^{(2)}(Q)). \end{aligned}$$

Then $\mathbb{G} = (\mathbb{R}^{N+M}, \diamond, \delta_\lambda)$ is a Carnot group of step κ and $n_1 + l_1$ generators.

Precisely if $(\mathfrak{g}_1^{(1)}, \dots, \mathfrak{g}_\kappa^{(1)})$ is a stratification of $\mathfrak{g}^{(1)}$, the Lie algebra of $\mathbb{G}^{(1)}$, and $(\mathfrak{g}_1^{(2)}, \dots, \mathfrak{g}_r^{(2)})$ is a stratification of $\mathfrak{g}^{(2)}$, the Lie algebra of $\mathbb{G}^{(2)}$, then

$$(\mathfrak{g}_1, \dots, \mathfrak{g}_\kappa) \qquad \text{where } \mathfrak{g}_i = \begin{cases} (\mathfrak{g}_i^{(1)}, \mathfrak{g}_i^{(2)}) & \text{for } i = 1, \dots, r \\ \mathfrak{g}_i = \mathfrak{g}_i^{(1)} & \text{for } i = r+1, \dots, \kappa \end{cases}$$

is a stratification of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$.

Example 1.3.6. If $\mathbb{G}^{(1)}$ is the Euclidean group on $\mathbb{R}^2$ and $\mathbb{G}^{(2)}$ is the Heisenberg group $\mathbb{H}^1$, then the sum of $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ is the Carnot group on $\mathbb{R}^5$ (whose points are denoted $\xi = (p_1, p_2, q_1, q_2, q_3)$ for $P = (p_1, p_2) \in \mathbb{R}^2$ and $Q = (q_1, q_2, q_3) \in \mathbb{H}^1 = \mathbb{R}^3$) with the group operation

$$(p_1, p_2, q_1, q_2, q_3) \diamond (p_1', p_2', q_1', q_2', q_3') = \begin{pmatrix} p_1 + p_1' \\ p_2 + p_2' \\ q_1 + q_1' \\ q_2 + q_2' \\ q_3 + q_3' + \frac{1}{2}(q_1 q_2' - q_1' q_2) \end{pmatrix}$$

and

$$\delta_\lambda(p_1, p_2, q_1, q_2, q_3) = (\lambda p_1, \lambda p_2, \lambda q_1, \lambda q_2, \lambda^2 q_3).$$

1.4 Groups of class $\mathcal{B}$

The aim of this section is to define and to collect some results and many explicit examples of group of class $\mathcal{B}$, which is a subclass of Carnot groups of step 2 shown in [17], Chapter 3.

1.4.1 Characterization of Carnot groups of Step 2

Let $(\mathbb{R}^{m+n}, \cdot)$ with the composition law $\cdot$ defined as

$$(x, y) \cdot (x', y') := (x + x', y + y' + \frac{1}{2}\langle \mathcal{B}x, x' \rangle), \quad \text{for all } (x, y), (x', y') \in \mathbb{R}^{m+n} \tag{1.24}$$

where $\langle \mathcal{B}x, x' \rangle := (\langle \mathcal{B}^{(1)}x, x' \rangle, \dots, \langle \mathcal{B}^{(n)}x, x' \rangle)$, $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ are $m \times m$ matrices with real entries and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$.

Then $(\mathbb{R}^{m+n}, \cdot)$ is a Lie group where identity is the origin and the inverse is

$$(x, y)^{-1} = (-x, -y + \frac{1}{2}\langle \mathcal{B}x, x \rangle).$$

Notice that the inverse map is the usual $(-x, -y)$ if and only if, for every $s = 1, \dots, n$ it holds

$$\langle \mathcal{B}^{(s)}x, x \rangle = 0, \qquad \text{for all } x \in \mathbb{R}^m$$

i.e. iff the matrices $\mathcal{B}^{(s)}$'s are skew-symmetric.

In particular we have the following result:

Proposition 1.4.1 ([17], Proposition 3.2.1)**.** *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^{m+n}$ such that the dilation $\delta_\lambda : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ defined as*

$$\delta_\lambda(x, y) := (\lambda x, \lambda^2 y), \qquad \textit{for all } (x, y) \in \mathbb{R}^{m+n} \tag{1.25}$$

is an automorphism of $\mathbb{G}$ for every $\lambda > 0$.

Then $\mathbb{G}$ is equipped with the composition law defined as (1.24) *where $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ are n suitable $m \times m$ matrices. Moreover, a characterization of Carnot groups of step* 2 *and m generators is given by the above $\mathbb{G} = (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$, where the skew-symmetric parts of the $\mathcal{B}^{(s)}$'s are linearly independent.*

Note that the above arguments show that there exist 2 step Carnot groups of any dimension $m \in \mathbb{N}$ of the first layer and any dimension

$$n \leq \frac{m(m-1)}{2}$$

of the second layer: it suffices to choose n linearly independent matrices $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ in the vector space of the skew-symmetric $m \times m$ matrices (which has dimension $m(m-1)/2$) and then define the composition law as in (1.24).

Remark 1.4.2. We explicitly remark that the linear independence of the skew-symmetric parts of $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ is necessary for $\mathbb{G}$ to be a Carnot group.

Indeed if $\mathcal{B}^{(s)} = (b_{ij}^{(s)})_{i,j=1}^m$, then a basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, is given by the $m+n$ left invariant vector fields

$$X_j(x,y) = \partial_{x_j} + \frac{1}{2}\sum_{s=1}^n \left(\sum_{i=1}^m b_{ji}^{(s)} x_i \right) \partial_{y_s}, \qquad Y_s(x,y) = \partial_{y_s}, \tag{1.26}$$

for $j = 1, \dots, m$, and $s = 1, \dots, n$ and it easy to see that

$$[X_j, X_i] = \sum_{s=1}^n \frac{1}{2}\big(b_{ij}^{(s)} - b_{ji}^{(s)}\big)\partial_{y_s} =: \sum_{s=1}^n c_{ij}^{(s)} \partial_{y_s}$$

where we denote by $\mathcal{C}^{(s)} = (c_{ij}^{(s)})_{i,j=1}^m$ the skew-symmetric part of $\mathcal{B}^{(s)}$, i.e.

$$\mathcal{C}^{(s)} := \frac{1}{2}(\mathcal{B}^{(s)} - (\mathcal{B}^{(s)})^T).$$

Moreover thanks to the linear independence of $\mathcal{C}^{(s)}$ we have that the matrix $m^2 \times n$

$$\begin{pmatrix} c_{11}^{(1)} & \dots & c_{11}^{(n)} \\ c_{12}^{(1)} & \dots & c_{12}^{(n)} \\ \vdots & \dots & \vdots \\ c_{1m}^{(1)} & \dots & c_{1m}^{(n)} \\ c_{21}^{(1)} & \dots & c_{21}^{(n)} \\ \vdots & \dots & \vdots \\ \multicolumn{3}{c}{\text{(proceed analogously up to)}} \\ \vdots & \dots & \vdots \\ c_{mm}^{(1)} & \dots & c_{mm}^{(n)} \end{pmatrix}$$

has rank equal to n and consequently

$$\mathrm{span}\{[X_j, X_i] \,|\, i, j = 1, \ldots, m\} = \mathrm{span}\{Y_1, \ldots, Y_n\}.$$

Therefore $X_1, \ldots, X_m$ satisfy the Chow-Hörmander condition (1.1), i.e.

$$\dim(\mathrm{span}\{X_1, \ldots, X_m, [X_j, X_i]\}(x, y)) = m + n$$

for all $(x, y) \in \mathbb{G}$.

Example 1.4.1. We consider $\mathbb{G} = (\mathbb{R}^{3+2}, \cdot, \delta_\lambda)$ with

$$\mathcal{B}^{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}^{(2)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then the composition law on $\mathbb{G}$ as in (1.24) becomes

$$(x_1, x_2, x_3, y_1, y_2) \cdot (x_1', x_2', x_3', y_1', y_2') = \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \\ y_1 + y_1' + \frac{1}{2}\big(x_1 x_1' + x_1' x_2 - x_2' x_1\big) \\ y_2 + y_2' + \frac{1}{2}\big(x_2 x_2' - x_1' x_3 - x_3' x_1\big) \end{pmatrix}$$

and the dilation (1.25) is

$$\delta_\lambda(x_1, x_2, x_3, y_1, y_2) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda^2 y_1, \lambda^2 y_2).$$

In this case $\mathbb{G}$ is a Carnot group. Indeed the skew-symmetric parts of $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ are linearly independent

$$\frac{1}{2}(\mathcal{B}^{(2)} - (\mathcal{B}^{(1)})^T) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \frac{1}{2}(\mathcal{B}^{(2)} - (\mathcal{B}^{(2)})^T) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Moreover we can compute the first three vector fields of the basis of Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ and verify that they are Lie-generators for $\mathfrak{g}$,

$$\begin{aligned} X_1 &= \partial_{x_1} + \frac{1}{2}(x_1 + x_2)\partial_{y_1} - \frac{1}{2}x_3\partial_{y_2}, \\ X_2 &= \partial_{x_2} - \frac{1}{2}x_1\partial_{y_1} + \frac{1}{2}x_2\partial_{y_2}, \\ X_3 &= \partial_{x_3} + \frac{1}{2}x_1\partial_{y_2}, \\ [X_1, X_2] &= -\partial_{y_1}, \\ [X_1, X_3] &= \partial_{y_2}, \\ [X_2, X_3] &= \frac{1}{2}\partial_{y_2}. \end{aligned}$$

Example 1.4.2. We consider $\mathbb{G} = (\mathbb{R}^{3+2}, \cdot, \delta_\lambda)$ where $\cdot$ is given by (1.24) with

$$\mathcal{B}^{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}^{(2)} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the explicit group operation on $\mathbb{G}$ is

$$(x_1, x_2, x_3, y_1, y_2) \cdot (x_1', x_2', x_3', y_1', y_2') = \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \\ y_1 + y_1' + \frac{1}{2}\big(x_1 x_1' + x_1' x_2 - x_2' x_1\big) \\ y_2 + y_2' + \frac{1}{2}\big(x_2 x_2' - 2x_1' x_2 + 2x_2' x_1\big) \end{pmatrix}$$

and the dilation is

$$\delta_\lambda(x_1, x_2, x_3, y_1, y_2) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda^2 y_1, \lambda^2 y_2).$$

Here by Proposition 1.4.1 we have that $\mathbb{G}$ is *not* a Carnot group because the skew-symmetric parts of $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ are linearly dependent,

$$\frac{1}{2}(\mathcal{B}^{(2)} - (\mathcal{B}^{(1)})^T) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \frac{1}{2}(\mathcal{B}^{(2)} - (\mathcal{B}^{(2)})^T) = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, the first three vector fields of the related basis of Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ are not Lie-generators for $\mathfrak{g}$, since

$$\begin{aligned} X_1 &= \partial_{x_1} + \frac{1}{2}(x_1 + x_2)\partial_{y_1} - x_2 \partial_{y_2}, \\ X_2 &= \partial_{x_2} - \frac{1}{2}x_1 \partial_{y_1} + \Big(\frac{1}{2}x_2 + x_1\Big)\partial_{y_2}, \\ X_3 &= \partial_{x_3}, \\ [X_1, X_2] &= -\partial_{y_1} + 2\partial_{y_2}, \\ [X_1, X_3] &= [X_2, X_3] = 0. \end{aligned}$$

Consequently $\operatorname{span}\{[X_j, X_i] \,|\, i,j = 1,2,3\} \neq \operatorname{span}\{\partial_{y_1}, \partial_{y_2}\}$ and the Chow-Hörmander condition (1.1) is not checked.

Proposition 1.4.3 ([17], Proposition 3.5.1)**.** *Let $\mathbb{G} = (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$ be equipped with a Lie group structure by the composition law is* (1.24) *where $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ are fixed $m \times m$ matrices and the dilation is* (1.25)*.*

Then $\mathbb{G}$ is isomorphic to the Lie group $\mathbb{H} = (\mathbb{R}^{m+n}, \diamond, \delta_\lambda)$ where:

1. *δ_λ is the same non isotropic dilation as above.*
2. *$\diamond : \mathbb{R}^{m+n} \times \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is defined by*

$$(x,y)\diamond(\xi,\tau) = \left(x+\xi, y+\tau+\frac{1}{2}\langle\tilde{\mathcal{B}}x,\xi\rangle\right)$$

   where $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}^{(1)},\dots,\tilde{\mathcal{B}}^{(n)})$ and $\tilde{\mathcal{B}}^{(s)}$ is the skew-symmetric parts of $\mathcal{B}^{(s)}$ for every $s = 1,\dots,n$. Moreover the inverse map on $\mathbb{H}$ is $(-x,-y)$.

3. *the Lie group isomorphism is $\Psi : \mathbb{H} \to \mathbb{G}$ with*

$$\Psi(\xi,\tau) = \left(\xi, \tau+\frac{1}{4}\langle\tilde{\mathcal{B}}\xi,\xi\rangle\right),$$

   so that Ψ is the identity map iff all the $\mathcal{B}^{(s)}$'s are skew-symmetric.

4. *if $\mathbb{G}$ is a Carnot group, then the same is true for $\mathbb{H}$.*

Example 1.4.3. If $\mathbb{G} = (\mathbb{R}^{3+2},\cdot,\delta_\lambda)$ is as in Example 1.4.1, then $\mathbb{G}$ is isomorphic to the Carnot group $\mathbb{H} = (\mathbb{R}^{3+2},\diamond,\delta_\lambda)$, where $\diamond$ is given by

$$(x,y)\diamond(\xi,\tau) = \left(x+\xi, y+\tau+\frac{1}{2}\langle\tilde{\mathcal{B}}x,\xi\rangle\right)$$

where $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}^{(1)},\tilde{\mathcal{B}}^{(2)})$ and $\tilde{\mathcal{B}}^{(s)}$ is the skew-symmetric parts of $\mathcal{B}^{(s)}$, i.e.

$$\tilde{\mathcal{B}}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{\mathcal{B}}^{(2)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

1.4.2 Groups of class $\mathcal{B}$

Definition 1.4.1. We say that $\mathbb{G} := (\mathbb{R}^{m+n},\cdot,\delta_\lambda)$ is a Carnot group of class $\mathcal{B}$ if there are n linearly independent, skew-symmetric $m\times m$ real matrices $\mathcal{B}^{(1)},\dots,\mathcal{B}^{(n)}$ such that for all $(x,y),(x',y') \in \mathbb{R}^m\times\mathbb{R}^n$ and for all $\lambda > 0$

$$(x,y)\cdot(x',y') = (x+x', y+y'+\frac{1}{2}\langle\mathcal{B}x,x'\rangle) \tag{1.27}$$

where $\langle\mathcal{B}x,x'\rangle := (\langle\mathcal{B}^{(1)}x,x'\rangle,\dots,\langle\mathcal{B}^{(n)}x,x'\rangle)$ and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^m$ and

$$\delta_\lambda(x,y) := (\lambda x, \lambda^2 y). \tag{1.28}$$

Under these assumptions $\mathbb{G}$ is a Carnot group of step 2 with $\mathbb{R}^m$ the horizontal layer and $\mathbb{R}^n$ the vertical layer (see Chapter 3 of [17]).

Moreover, if $\mathcal{B}^{(s)} := (b_{ij}^{(s)})_{i,j=1}^m$, then a basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, is given by the $n+m$ left invariant vector fields

$$X_j(x,y) = \partial_{x_j} + \frac{1}{2}\sum_{s=1}^{n}\left(\sum_{i=1}^{m} b_{ji}^{(s)} x_i\right)\partial_{y_s}, \qquad Y_s(x,y) = \partial_{y_s}, \tag{1.29}$$

for $j = 1, \dots, m$, and $s = 1, \dots, n$. Finally, we make the following choice of the homogeneous norm in $\mathbb{G}$:

$$\|(x,y)\| := \max\{|x|, \epsilon|y|^{1/2}\} \tag{1.30}$$

for a suitable $\epsilon \in (0,1]$ (see Example 1.3.5).

Remark 1.4.4. According to (1.22), if $\mathbb{G}$ is a Carnot group of class $\mathcal{B}$ then the homogeneous dimension of $\mathbb{G}$ is

$$\mathfrak{q} = m + 2n.$$

Remark 1.4.5. As we said, the space of skew-symmetric $m \times m$ matrices has dimension $\frac{m(m-1)}{2}$. Hence in any group $\mathbb{G}$ of class $\mathcal{B}$ the dimensions of the horizontal layer and of the vertical layer are related by the inequality

$$n \leq \frac{m(m-1)}{2}.$$

1.4.3 Example: Free Step 2 Groups

We present free step 2 groups which are an example of groups of class $\mathcal{B}$.

Let $m \geq 2$ be a fixed integer. We denote by $(\mathbb{F}_{m,2}, \star)$ the Carnot group on $\mathbb{R}^m \times \mathbb{R}^{\frac{m(m-1)}{2}}$ with the composition law (1.27) defined by the matrices $\mathcal{B}^{(s)} \equiv \mathcal{B}^{(i,j)}$ where $1 \leq j < i \leq m$ and $\mathcal{B}^{(i,j)}$ has entries -1 in position (i,j), 1 in position (j,i) and 0 everywhere else.

In the sequel of this section, we shall use the following notation, different from the one used in the previous section: instead of using the notation τ for the coordinate in the vertical layer of the group, we denote the points of $\mathbb{F}_{m,2}$ by (x,τ), where the coordinates of $\tau \in \mathbb{R}^n$ are denoted by

$$\tau_{ij} \quad \text{where } (i,j) \in \mathcal{I} := \{(i,j) \, : \, 1 \leq j < i \leq m\}.$$

Here we have ordered $\mathcal{I}$ in an arbitrary (henceforth) fixed way. Then, the group operation $\star$ is given by

$$(x,\tau) \star (x',\tau') = \begin{pmatrix} x + x' \\ \tau_{lh} + \tau'_{lh} + \frac{1}{2}(x_l x'_h - x_h x'_l) \quad (l,h) \in \mathcal{I} \end{pmatrix}$$

A base of Lie algebra of $\mathbb{F}_{m,2}$ is

$$
\begin{aligned}
X_i &= \partial_{x_i} + \frac{1}{2}\sum_{1\le h<l\le m}\Big(\sum_{j=1}^{m} b_{ij}^{(l,h)} x_j\Big)\partial_{\tau_{lh}} \\
&= \begin{cases}
\partial_{x_1} + \frac{1}{2}\sum_{1<l\le m} x_l \partial_{\tau_{l1}} & \text{if } i=1 \\[2ex]
\partial_{x_i} + \frac{1}{2}\sum_{i<l\le m} x_l \partial_{\tau_{li}} - \frac{1}{2}\sum_{1\le h<m} x_h \partial_{\tau_{ih}} & \text{if } 1<i<m \\[2ex]
\partial_{x_m} + \frac{1}{2}\sum_{1\le h<m} x_h \partial_{\tau_{mh}} & \text{if } i=m
\end{cases} \\
Y_{lh} &= \partial_{\tau_{lh}}, \qquad (l,h)\in\mathcal{I}.
\end{aligned}
$$

Moreover, for every $(l,h)\in\mathcal{I}$, we have the commutator identities

$$[X_l, X_h] = \frac{1}{2}(\partial_{\tau_{lh}} - \partial_{\tau_{hl}}).$$

Remark 1.4.6. The Heisenberg group $\mathbb{H}^k$ is a free Carnot group if and only if $k=1$. Indeed, since the Lie algebra $\mathfrak{g}$ of $\mathbb{H}^k$ has step 2 and $2k$ generators, a necessary condition for $\mathbb{H}^k$ to be free is that

$$2k+1 = \dim \mathfrak{g} = 2k(2k+1)/2,$$

i.e. that $k=1$.

Example 1.4.4. We consider free step 2 group $\mathbb{F}_{3,2}$. Here $m=3$, $n=3$ and

$$\mathcal{B}^{(2,1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}^{(3,1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{B}^{(3,2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Hence the group operation $\star$ is defined as

$$(x_1,x_2,x_3,\tau_{21},\tau_{31},\tau_{32}) \star (x_1',x_2',x_3',\tau_{21}',\tau_{31}',\tau_{32}') = \begin{pmatrix} x_1+x_1' \\ x_2+x_2' \\ x_3+x_3' \\ \tau_{21}+\tau_{21}'+\frac{1}{2}\big(x_2x_1'-x_1x_2'\big) \\ \tau_{31}+\tau_{31}'+\frac{1}{2}\big(x_3x_1'-x_1x_3'\big) \\ \tau_{32}+\tau_{32}'+\frac{1}{2}\big(x_3x_2'-x_2x_3'\big) \end{pmatrix}$$

and the dilation is

$$\delta_\lambda(x_1, x_2, x_3, \tau_{21}, \tau_{31}, \tau_{32}) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda^2\tau_{21}, \lambda^2\tau_{31}, \lambda^2\tau_{32}).$$

Moreover a basis of Lie algebra of $\mathbb{F}_{3,2}$ is given by

$$\begin{aligned}
X_1 &= \partial_{x_1} + \frac{1}{2}\big(x_2\partial_{\tau_{21}} + x_3\partial_{\tau_{31}}\big),\\
X_2 &= \partial_{x_2} + \frac{1}{2}\big(x_3\partial_{\tau_{32}} - x_1\partial_{\tau_{21}}\big),\\
X_3 &= \partial_{x_3} + \frac{1}{2}\big(- x_1\partial_{\tau_{31}} - x_2\partial_{\tau_{32}}\big),\\
Y_{21} &= \partial_{\tau_{21}}, \qquad Y_{31} = \partial_{\tau_{31}}, \qquad Y_{32} = \partial_{\tau_{32}}.
\end{aligned}$$

1.4.4 Example: H-Type Groups

Another examples of groups of class $\mathcal{B}$ are H-type Groups.

The groups of H-type were introduced by A. Kaplan in [58]. Kaplan's definition of H-type groups is more abstract than the one given here. In [17], Chapter 18 the authors show that, up to an isomorphism, the two definitions are equivalent.

Definition 1.4.2. We say that a Lie group $H = (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$ is H-type group if the dilation is defined as (1.28) and the composition law is of the form (1.27) where the matrices $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ have the following properties:

1. $\mathcal{B}^{(s)}$ is an $m \times m$ skew-symmetric and orthogonal matrix for all $s = 1, \dots n$
2. $\mathcal{B}^{(s)}\mathcal{B}^{(l)} = -\mathcal{B}^{(l)}\mathcal{B}^{(s)}$ for every $s, l = 1, \dots, n$ with $s \neq l$.

Remark 1.4.7. Since conditions 1. and 2. we have that $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ are linear independent. Indeed if $\eta = (\eta_1, \dots, \eta_n) - \{0\}$, then $\frac{1}{|\eta|}\sum_{s=1}^n \eta_s\mathcal{B}^{(s)}$ is orthogonal (hence non-vanishing), as the following computation shows,

$$\begin{aligned}
\left(\frac{1}{|\eta|}\sum_{s=1}^n \eta_s\mathcal{B}^{(s)}\right)\left(\frac{1}{|\eta|}\sum_{s=1}^n \eta_s\mathcal{B}^{(s)}\right)^T &= -\frac{1}{|\eta|^2}\sum_{l,s=1}^n \eta_l\eta_s\mathcal{B}^{(l)}\mathcal{B}^{(s)}\\
&= -\frac{1}{|\eta|^2}\sum_{l\leq n}\eta_l^2(\mathcal{B}^{(l)})^2 - \frac{1}{|\eta|^2}\sum_{l,s\leq n,\, l\neq r}\eta_l\eta_s\mathcal{B}^{(l)}\mathcal{B}^{(s)}\\
&= \mathbb{I}_m
\end{aligned}$$

Here we used the following facts: $(\mathcal{B}^{(l)})^2 = -\mathbb{I}_m$, since $\mathcal{B}^{(l)}$ is skew-symmetric and orthogonal; $\sum_{l\neq r}\eta_l\eta_s\mathcal{B}^{(l)}\mathcal{B}^{(s)} = 0$ according to condition 2.

Remark 1.4.8. Because $\mathcal{B}^{(s)}$ is a $m \times m$ skew-symmetric orthogonal matrix, we have $\mathbb{I}_m = \mathcal{B}^{(s)}(\mathcal{B}^{(s)})^T = -(\mathcal{B}^{(s)})^2$, whence $1 = (-1)^m(\det\mathcal{B}^{(s)})^2$. As a consequence, the first layer of a H-type group has even dimension m.

Remark 1.4.9. If $\mathbb{G} = (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$ is H-type group, then

$$\{(0, y) \mid y \in \mathbb{R}^n\} \text{ is the center of } \mathbb{G}.^4$$

Indeed, let $(x, y) \in \mathbb{G}$ be such that

$$(x, y) \cdot (x', y') = (x', y') \cdot (x, y) \quad \text{for all } (x', y') \in \mathbb{G}.$$

This holds if and only if

$$\langle \mathcal{B}^{(s)} x, x' \rangle = \langle \mathcal{B}^{(s)} x', x \rangle$$

for all $x \in \mathbb{R}^m$ and for all $s = 1, \dots, n$. Then, since $(\mathcal{B}^{(s)})^T = -\mathcal{B}^{(s)}$, we have $\langle \mathcal{B}^{(s)} x', x \rangle = 0$ for every $x \in \mathbb{R}^m$ and $s = 1, \dots, n$. Consequently $x = 0$ because $\mathcal{B}^{(s)}$ is orthogonal (hence non singular).

Remark 1.4.10. The Heisenberg groups are H-type groups and consequently they are groups of class $\mathcal{B}$. Indeed in $\mathbb{H}^k = \mathbb{R}^{2k} \times \mathbb{R}$ the group law is of the form (1.27) with

$$\mathcal{B}^{(1)} = \begin{pmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{pmatrix}.$$

More specifically, the Heisenberg groups are the only, up to isomorphism, H-type group with one-dimensional center.

Proposition 1.4.11. *Let m, n be two positive integers. Then there exists a H-type group of dimension $m + n$ whose center has dimension n if and only if it holds $n < \rho(m)$, where $\rho : \mathbb{N} \to \mathbb{N}$ is the so-called Hurwitz-Radon function, i.e.*

$$\rho(m) := 8p + q, \qquad \textit{where } m = (\textit{odd})\, 2^{4p+q},\ 0 \le q \le 3.$$

Notice that if m is odd, then $\rho(m) = 0$, whence the first layer of any H-type group has even dimension.

Example 1.4.5. We conclude this section by presenting the *complexified Heisenberg group* $\mathbb{H}^1_2$, which is an example of H-type group. More information on this group can be found in [87].

We consider $\mathbb{H}^1_2 = (\mathbb{R}^{4+2}, \cdot, \delta_\lambda)$ where $\cdot$ is given by (1.24) with

$$\mathcal{B}^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathcal{B}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

[4] We recall that the center of $\mathbb{G}$ is $\{P \in \mathbb{G} \mid PQ = QP, \forall\, Q \in \mathbb{G}\}$.

Then the explicit group operation on $\mathbb{H}^1_2$ is

$$(x_1, x_2, x_3, x_4, y_1, y_2) \cdot (x_1', x_2', x_3', x_4', y_1', y_2') = \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \\ x_4 + x_4' \\ y_1 + y_1' + \frac{1}{2}\big(- x_2 x_1' + x_1 x_2' + x_4 x_3' - x_3 x_4'\big) \\ y_2 + y_2' + \frac{1}{2}\big(- x_4 x_1' + x_3 x_2' - x_2 x_3' + x_1 x_4'\big) \end{pmatrix}$$

and the dilation is

$$\delta_\lambda(x_1, x_2, x_3, y_1, y_2) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda^2 y_1, \lambda^2 y_2).$$

Notice that $\mathbb{H}^1_2$ is H-type group whose center has dimension 2 and the first layer has dimension 4. Moreover a basis of Lie algebra $\mathfrak{h}^1_2$ of $\mathbb{H}^1_2$ is

$$\begin{aligned} X_1 &= \partial_{x_1} - \frac{1}{2}(x_2 \partial_{y_1} + x_4 \partial_{y_2}), \\ X_2 &= \partial_{x_2} + \frac{1}{2}(x_1 \partial_{y_1} + x_3 \partial_{y_2}), \\ X_3 &= \partial_{x_3} + \frac{1}{2}(x_4 \partial_{y_1} - x_2 \partial_{y_2}), \\ X_4 &= \partial_{x_4} - \frac{1}{2}(x_3 \partial_{y_1} - x_1 \partial_{y_2}), \\ Y_1 &= \partial_{y_1}, \qquad Y_2 = \partial_{y_2} \end{aligned}$$

1.4.5 Example: H-groups in the Sense of Métivier

Following G. Métivier in [74] (see also [17]), we give the following definition.

Definition 1.4.3. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra, and let us denote by Z its center. [5] We say that $\mathfrak{g}$ is of H-type in the sense of Métivier if it admits a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \qquad \begin{cases} [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2 \\ \mathfrak{g}_2 \subseteq Z \end{cases}$$

with the following additional property: for every $\eta \in \mathfrak{g}_2^*$ (i.e. the dual space of $\mathfrak{g}_2$) the skew-symmetric bilinear form on $\mathfrak{g}_1$ defined by

$$\mathcal{B}_\eta : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}, \quad \mathcal{B}_\eta(X_1, X_2) := \eta([X_1, X_2])$$

[5] The center of $\mathfrak{g}$ is $\{Y \in \mathfrak{g} \,|\, [Y, X] = 0, \forall\, X \in \mathfrak{g}\}$.

is non degenerate [6] whenever $\eta \neq 0$.

We say that a Lie group $\mathbb{G}$ is a H-group in the sense of Métivier, or a HM-group in short, if its Lie algebra is of H-type in the sense of Métivier.

Proposition 1.4.12. *The HM-groups are Carnot groups of step* 2.

Proof. By definition we have that HM-type algebra is obviously nilpotent of step two. Moreover by $\mathfrak{g}_2 \subseteq Z$ and $\mathfrak{g}_1 \subseteq \mathfrak{g}$ we obtain

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1 + \mathfrak{g}_2, \mathfrak{g} + \mathfrak{g}_2] = [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq [\mathfrak{g}, \mathfrak{g}].$$

Consequently,

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1]. \tag{1.31}$$

Now we want to prove that

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_2. \tag{1.32}$$

From (1.31) and $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2$ we get that $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2$. Hence it remains to show that

$$\mathfrak{g}_2 \subseteq [\mathfrak{g}, \mathfrak{g}]. \tag{1.33}$$

Suppose to the contrary that there exists $Y \in \mathfrak{g}_2$ such that $Y \notin [\mathfrak{g}, \mathfrak{g}]$ (and consequently $Y \neq \{0\}$). Then there certainly exists $\eta \in \mathfrak{g}_2^*$ such that $\mathfrak{g}_2(Y) \neq 0$ (whence $\eta \neq 0$) and η vanishes identically on $[\mathfrak{g}, \mathfrak{g}]$ because $Y \in \mathfrak{g}_2 - [\mathfrak{g}, \mathfrak{g}]$. But this implies that, for every $X_1, X_2 \in \mathfrak{g}_1$, we obtain

$$\mathcal{B}_\eta(X_1, X_2) = \eta([X_1, X_2]) = 0,$$

for $[X_1, X_2] \in [\mathfrak{g}, \mathfrak{g}]$ and $\eta_{|[\mathfrak{g},\mathfrak{g}]} = 0$. This is in contradiction because $\mathcal{B}_\eta$ is non degenerate and consequently (1.33) holds.

Now putting together (1.31) and (1.32), we get that a HM-type algebra is stratified: indeed

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \qquad \text{with } [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2 \text{ and } [\mathfrak{g}_1, \mathfrak{g}_2] = \{0\},$$

and this completes the proof. □

Collecting the above result, we have proved the following proposition.

[6] A bilinear map $\mathcal{B}_\eta$ on a finite dimensional vector space $\mathfrak{g}_1$ is non degenerate if any (or equivalently, if one) of the matrices representing it with respect to a fixed basis is non singular or, equivalently, if for every $v \in \mathfrak{g}_1 - \{0\}$, there exists $w \in \mathfrak{g}_1$ such that $\mathcal{B}_\eta(v, w) \neq 0$.

Proposition 1.4.13 ([17], Proposition 3.7.3)**.** *A HM-group is a Carnot group $\mathbb{G}$ of step 2 such that if*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \qquad \textit{with } [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2, \ [\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$$

is any stratification of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, then the following property holds: For every non-vanishing linear map η from $\mathfrak{g}_2$ to $\mathbb{R}$, the (skew-symmetric) bilinear form $\mathcal{B}_\eta$ on $\mathfrak{g}_1$ defined by

$$\mathcal{B}_\eta(X_1, X_2) := \eta([X_1, X_2]), \quad \textit{for all } X_1, X_2 \in \mathfrak{g}_1$$

is non degenerate.

Now we give a characterization of HM-groups:

Proposition 1.4.14. *Let $\mathbb{G} = (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$ be a group of class $\mathcal{B}$.*

Then $\mathbb{G}$ is a HM-group if and only if every non vanishing linear combination of the matrices $\mathcal{B}^{(s)}$'s is non singular.

Proof. If $\mathcal{B}^{(s)} := (b_{ij}^{(s)})_{i,j=1}^m$, then a basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, is given by the $n+m$ left invariant vector fields defined as (1.26), i.e.

$$X_j(x,y) = \partial_{x_j} + \frac{1}{2} \sum_{s=1}^n \left(\sum_{i=1}^m b_{ji}^{(s)} x_i \right) \partial_{y_s}, \qquad\qquad Y_s(x,y) = \partial_{y_s},$$

for $j = 1, \dots, m$, and $s = 1, \dots, n$.

Moreover if $\eta : \mathfrak{g}_2 \to \mathbb{R}$ is a linear map, we know that there exist $\eta_1, \dots, \eta_n \in \mathbb{R}$ such that

$$\eta(\partial_{y_s}) = \eta_s \qquad \text{for all } s = 1, \dots, n.$$

In particular the map $\mathcal{B}_\eta : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}$, as in Proposition 1.4.13, can be explicitly written as follows

$$\text{if } Z_1 = \sum_{j=1}^m \alpha_j X_j \quad \text{and} \quad Z_2 = \sum_{j=1}^m \alpha'_j X_j$$

$$\text{then } \mathcal{B}_\eta(Z_1, Z_2) = \sum_{i,j=1}^m \left(-\sum_{s=1}^n \eta_s b_{ij}^{(s)} \right) \alpha_i \alpha'_j.$$

In other words, the matrix representing the skew-symmetric bilinear map $\mathcal{B}_\eta$ with respect to the basis $X_1, \dots, X_m$ of $\mathfrak{g}_1$ is the matrix

$$\eta_1 \mathcal{B}^{(1)} + \cdots + \eta_n \mathcal{B}^{(n)}.$$

Hence, to ask for $\mathcal{B}_\eta$ to be non degenerate (for every $\eta \neq 0$) is equivalent to ask that any linear combination of the matrices $\mathcal{B}^{(s)}$'s is non singular, unless it is the null matrix (recall that the $\mathcal{B}^{(s)}$'s are linearly independent). $\square$

Remark 1.4.15. Let $\mathbb{G} = (\mathbb{R}^N, \cdot)$ be a HM-group. It isn't necessary that $\mathcal{B}^{(s)}, \ldots, \mathcal{B}^{(n)}$ are skew-symmetric, but thanks to Proposition 1.4.3 $\mathbb{G}$ is isomorphic to $\mathbb{H} = (\mathbb{R}^N, \diamond)$ where the matrices $\tilde{\mathcal{B}}^{(1)}, \ldots, \tilde{\mathcal{B}}^{(n)}$ associated to $\diamond$ are the skew-symmetric part of $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(n)}$. Hence we may also suppose that they are.

Remark 1.4.16. If $\mathbb{G}$ is a HM-group, then the $\mathcal{B}^{(s)}$'s are all non singular $m \times m$ matrices. Moreover since the $\mathcal{B}^{(s)}$'s are also skew-symmetric, this implies that m is necessarily even.

Remark 1.4.17. Any H-type group (according to Definition 1.4.2) is a HM-group. Indeed, as it can be seen from the computations on Remark 1.4.7, in a H-type group, for every $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n - \{0\}$, we proved that $\sum_{s=1}^n \eta_s \mathcal{B}^{(s)}$ is $|\eta|$ times an orthogonal matrix, hence $\sum_{s=1}^n \eta_s \mathcal{B}^{(s)}$ is non singular.

The converse is not true. For example, consider the group on $\mathbb{R}^5 = \mathbb{R}^4 \times \mathbb{R}$ with the composition law $\cdot$ given by (1.27) where

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Then $\mathbb{G} = (\mathbb{R}^5, \cdot, \delta_\lambda)$ is obviously a HM-group because $\mathcal{B}$ is a non singular skew-symmetric matrix. But $\mathbb{G}$ is not a H-type group because $\mathcal{B}$ is not orthogonal. But more is true: $\mathbb{G}$ is not even isomorphic to any prototype H-type group (Chapter 18, [17]).

Chapter 2

Differential calculus within Carnot groups

In this chapter we provide the definitions and some properties of the differential calculus within Carnot groups.

One of the main objects of this thesis is the notion of $\mathbb{G}$-regular surfaces, that is related to the notion of rectifiability. Indeed, rectifiable sets are classically defined as contained in the countable union of $\mathbb{C}^1$ submanifolds: in Euclidean spaces one can see [33], [72] while a general theory in metric spaces can be found in [34], [5], [6]. We also mention [73] for the Heisenberg groups.

In Carnot groups, the importance of the $\mathbb{G}$-regular surfaces became evident in [44], [47] where Rectifiability Theorem on the class of 2 step Carnot groups is proved (see Theorem 2.4.7). Here the authors establish that the reduced boundary of a $\mathbb{G}$-Caccioppoli set (see Section 2.3.3) in a step two Carnot group is a countable union of $\mathbb{G}$-regular surfaces up to $\mathcal{H}^{\mathfrak{q}-1}$-negligible sets, where $\mathfrak{q}$ is the homogeneous dimension of the group. The validity of this result for groups of higher step is partially solved in [71] to the much larger class of step 2 Carnot groups. On the other hand, in [47] there is a counterexample to the De Giorgi's (classical) method when $\mathbb{G}$ is a particular Carnot group of step three called Engels group.

According to the notion of Federer, a "good" surface in a metric space should be the image of an open subset of an Euclidean space via a Lipschitz map. Unfortunately, this definition is not appropriate in Heisenberg groups, and in many other Carnot groups. Indeed, in [6] the authors show that the surfaces in Heisenberg groups would be not rectifiable in the Federer sense.

On the other hand, in the Euclidean setting $\mathbb{R}^N$, a $\mathbb{C}^1$ hypersurface (i.e. a topological codimension 1 surface) can be equivalently viewed as the level set of a function $f : \mathbb{R}^N \to \mathbb{R}$ with non vanishing gradient. Such a concept was easily transposed in [44] to the Heisenberg group and in [45] to Carnot group by means of $\mathbb{C}^1_{\mathbb{G}}$ functions (see Definition 2.3.2): we consequently define $\mathbb{G}$-regular surfaces as non critical level sets of $\mathbb{C}^1_{\mathbb{G}}$ functions, i.e. continuous real functions on $\mathbb{G}$ whose horizontal derivatives are represented, in distributional sense, by continuous functions.

In a similar way, a k-codimensional regular surface $S \subset \mathbb{G}$ is locally defined as a non critical level set of a $\mathbb{C}^1_{\mathbb{G}}$ vector function $F : \mathbb{G} \to \mathbb{R}^k$ (see Definition 2.4.5).

Another important object of this thesis is the notion of the intrinsic graph. The simple idea of intrinsic graph is the following one: let $\mathbb{M}$ and $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$, i.e. homogeneous subgroups such that $\mathbb{W} \cap \mathbb{M} = \{0\}$ and $\mathbb{G} = \mathbb{W} \cdot \mathbb{M}$, then the intrinsic left graph of $\phi : \mathbb{W} \to \mathbb{M}$ is the set

$$\operatorname{graph}(\phi) := \{A \cdot \phi(A) \mid A \in \mathbb{W}\}.$$

Hence the existence of intrinsic graphs depends on the possibility of splitting $\mathbb{G}$ as a product of complementary subgroups. Hence it depends on the structure of the algebra $\mathfrak{g}$.

Differently from the Euclidean case where the surfaces can be locally defined as non critical level sets or, equivalently, as continuously intrinsic differentiable graphs, in Carnot groups the notion of regular surfaces is *not* equivalent to the notion of intrinsic graphs any more. One of the main aim of this thesis is to find the additional assumptions in order that they are equivalent in $\mathbb{G}$.

More precisely, from Implicit Function Theorem, proved in [44] for the Heisenberg group and in [45] for a general Carnot group (see also Theorem 1.3, [70]) it follows

$$S \text{ is a } \mathbb{G}\text{-regular surface} \quad \Longrightarrow \quad S \text{ is (locally) an intrinsic graph of a map } \phi.$$

Consequently, given an intrinsic graph of ϕ, we want to find necessary and sufficient assumptions on ϕ in order that the opposite implication is true.

We will see that these additional assumptions will be characterized in terms of an appropriate notion of differentiability, denoted *uniform intrinsic differentiability*, for maps acting between complementary subgroups of $\mathbb{G}$. Quite precisely, a function is intrinsic differentiable if it is well approximated by appropriate linear type functions, denoted *intrinsic linear* functions. In Proposition 2.6.3 we prove that any intrinsic linear map acting between complementary subgroups of $\mathbb{G}$, where one of these complementary subgroups is horizontal, can be represented by a matrix denoted by $D^{\phi}\phi$. We call $D^{\phi}\phi$ intrinsic gradient of map and it will play a crucial role in the next chapter.

In Section 2.1 we focus our attention on the concept of subgroups of $\mathbb{G}$: we begin by recalling the concept of homogeneous, complementary and horizontal subgroup. Then we present some results about projection maps on a subgroup.

Section 2.2 is entirely concerned with horizontal linear maps, in short H-linear maps. We present some results given in [70] about the surjective or injective H-linear maps. Note that we will frequently use the adjective "horizontal" to indicate objects related to the horizontal subbundle and we will often use the prefix H. Here in Proposition 2.3.4 we see the link between this notion and the horizontal subbundle.

In Section 2.3 we introduce and analyze the Pansu differentiability in Carnot groups (see [83]). Moreover we give a natural definition of bounded variation functions and of finite perimeter sets. The main result of the section is Whitney's Extension Theorem: its proof was sketched in [47], here we give a complete one.

In Section 2.4, we define the $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surfaces where $\mathbb{G}^{(1)}, \mathbb{G}^{(2)}$ are Carnot groups, according to [70]. Quite precisely, the author partially extends the definition of $\mathbb{G}$-regular surfaces introduced and studied in Heisenberg groups in [44], [45], [46]. We say "partially" because there is not a complete classification of $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surfaces. We stress that not all codomains $\mathbb{G}^{(2)}$ are "good" to be considered; for instance, the family of $(\mathbb{H}^n, \mathbb{H}^k)$-regular surfaces is empty whenever $n > k$ (see Proposition 2.2.10).

In Section 2.4.2 we consider the surfaces $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surfaces when $\mathbb{G}^{(2)} = \mathbb{R}^k$. We simply call them $\mathbb{G}$-regular surfaces and they are the main object of this thesis.

Section 2.5 is entirely devoted to the intrinsic graphs with particular emphasis on Lipschitz graphs. The notion of Lipschitz graphs in Heisenberg group appeared for the first time in [44] and was studied, more diffusely, in [94], [46], [38], [39], [13], [15] and [42]. Intrinsic Lipschitz functions play the same role as Lipschitz functions in Euclidean context.

First we propose here a geometric definition: a $\mathbb{M}$-graph S is said to be an intrinsic Lipschitz $\mathbb{M}$-graph if S intersects intrinsic cones with axis $\mathbb{M}$, fixed opening and vertex on S only in the vertex (see Definition 2.5.4). Then in Proposition 2.5.10 we give an equivalent analytic form. Finally, in Section 2.5.2 we characterize the intrinsic Lipschitz functions in terms of boundedness of appropriately defined intrinsic difference quotients (see [94]).

In Section 2.6, we conclude this chapter with the definition and some important properties of intrinsic differentiability (see Definition 2.6.2).

If we consider a map $\phi : \mathbb{W} \to \mathbb{M}$ where $\mathbb{M}$ is a horizontal subgroup of $\mathbb{G}$, then we can define a quasi metric ρ_ϕ depends on ϕ, which we call "graph distance" because it is equivalent to the invariant metric d restricted to the graph of ϕ, under suitable hypothesis. Moreover using ρ_ϕ we can also introduce a stronger pointwise notion of intrinsic differentiability here denoted as uniform intrinsic differentiability (see Definition 2.6.4).

2.1 Complementary subgroups

Now we introduce the concept of complementary subgroups.

An *homogeneous subgroup* $\mathbb{W}$ of $\mathbb{G}$ is a Lie subgroup such that $\delta_\lambda A \in \mathbb{W}$ for every $A \in \mathbb{W}$ and for all $\lambda > 0$. Homogeneous subgroups are linear subspaces of $\mathbb{G}$, when $\mathbb{G}$ is identified with $\mathbb{R}^N$.

Definition 2.1.1. Let $\mathbb{W}$, $\mathbb{M}$ be homogeneous subgroups of $\mathbb{G}$. We say that $\mathbb{W}$ and $\mathbb{M}$ are complementary subgroups in $\mathbb{G}$, if $\mathbb{W} \cap \mathbb{M} = \{0\}$ and if

$$\mathbb{G} = \mathbb{W} \cdot \mathbb{M},$$

i.e. if for every $P \in \mathbb{G}$ there are $P_{\mathbb{W}} \in \mathbb{W}$ and $P_{\mathbb{M}} \in \mathbb{M}$ such that $P = P_{\mathbb{W}} P_{\mathbb{M}}$.

In particular through all this thesis we consider $\mathbb{G} = \mathbb{W} \cdot \mathbb{M}$, where as usual $\mathbb{W}$ and $\mathbb{M}$ are complementary subgroups of $\mathbb{G}$ and $\mathbb{M}$ is also a *horizontal* subgroup, i.e. the Lie algebra of $\mathbb{M}$ is contained in the first layer $\mathfrak{g}_1$ of $\mathbb{G}$. More precisely, we assume the existence of $X_1, \dots, X_k \in \mathfrak{g}_1$ such that

$$\mathbb{M} := \exp(\operatorname{span}\{X_1, \dots, X_k\}).$$

If $\mathbb{W}$ and $\mathbb{M}$ are complementary subgroups of $\mathbb{G}$ and one of them is a normal subgroup [1] then $\mathbb{G}$ is said to be the semi-direct product of $\mathbb{W}$ and $\mathbb{M}$. Moreover if both $\mathbb{W}$ and $\mathbb{M}$ are normal subgroups then $\mathbb{G}$ is said to be the direct product of $\mathbb{W}$ and $\mathbb{M}$ and in this case we will also write that $\mathbb{G} = \mathbb{W} \times \mathbb{M}$.

The elements $P_{\mathbb{W}} \in \mathbb{W}$ and $P_{\mathbb{M}} \in \mathbb{M}$ such that $P = P_{\mathbb{W}} \cdot P_{\mathbb{M}}$ are unique because of $\mathbb{W} \cap \mathbb{M} = \{0\}$ and are denoted as components of P along $\mathbb{W}$ and $\mathbb{M}$ or as projections of P on $\mathbb{W}$ and $\mathbb{M}$. We stress that each component $P_{\mathbb{W}}$ and $P_{\mathbb{M}}$ depends on both the complementary subgroups $\mathbb{W}$ and $\mathbb{M}$ and also on the order in which they are taken. The projection maps $\mathbf{P}_{\mathbb{W}} : \mathbb{G} \to \mathbb{W}$ and $\mathbf{P}_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ defined

$$\mathbf{P}_{\mathbb{W}}(P) = P_{\mathbb{W}} \quad \mathbf{P}_{\mathbb{M}}(P) = P_{\mathbb{M}}$$

are not always Lipschitz maps, when $\mathbb{W}$ and $\mathbb{M}$ are endowed with the restriction of the left invariant distance of $\mathbb{G}$ (see Example 2.1.1). In Proposition 2.2.14 in [39], the authors prove that the projection maps are $\mathbb{C}^{\infty}$, as function from $\mathbb{G} \equiv \mathbb{R}^N$ to $\mathbb{G} \equiv \mathbb{R}^N$.

Observe that, in general,

$$(P_{\mathbb{W}})^{-1} \neq (P^{-1})_{\mathbb{W}} \quad \text{and} \quad (P_{\mathbb{M}})^{-1} \neq (P^{-1})_{\mathbb{M}}.$$

Example 2.1.1. We consider the Heisenberg group $\mathbb{H}^1$ and

$$\mathbb{M} := \{P = (p_1, p_2, p_3) \in \mathbb{H}^1 \mid p_2 = p_3 = 0\} \quad \text{and} \quad \mathbb{W} := \{P = (p_1, p_2, p_3) \in \mathbb{H}^1 \mid p_1 = 0\}.$$

Here $\mathbb{M}, \mathbb{W}$ are complementary subgroup of $\mathbb{H}^1$ where $\mathbb{W}$ is normal and $\mathbb{M}$ is 1-dimensional horizontal. Moreover

$$\mathbf{P}_{\mathbb{W}}(P) = (0, p_2, p_3 - \frac{1}{2} p_1 p_2) \quad \mathbf{P}_{\mathbb{M}}(P) = (p_1, 0, 0).$$

The projection map $\mathbf{P}_{\mathbb{W}}$ is not Lipschitz. Indeed if we consider $P = (1, 1, 0)$ and $Q_h = (1 + h, 1 + h, 0)$, then

$$\mathbf{P}_{\mathbb{W}}(P) = \left(0, 1, -\frac{1}{2}\right) \quad \mathbf{P}_{\mathbb{W}}(Q_h) = \left(0, 1 + h, -\frac{1}{2}(1 + h)^2\right)$$

[1] We recall that $\mathbb{W}$ is a normal subgroup in $\mathbb{G}$ if it is a subgroup such that

$$P^{-1} A P \in \mathbb{W}, \qquad \text{for all } P \in \mathbb{G}, A \in \mathbb{W}.$$

and consequently as $h \to 0^+$

$$\|P^{-1}Q_h\| = \|(h,h,0)\| \approx h$$
$$\|\mathbf{P}_{\mathbb{W}}(P)^{-1}\mathbf{P}_{\mathbb{W}}(Q_h)\| = \|(0,h,-h-h^2/2)\| \approx \sqrt{h}$$

We recall the following inequality proved in [9], Proposition 3.2 (see also [8])

Proposition 2.1.1. *If $\mathbb{W}$ and $\mathbb{M}$ are complementary subgroups in $\mathbb{G}$ there is $c_0 = c_0(\mathbb{W},\mathbb{M})$ with $c_0 \in (0,1]$ such that for each $P_{\mathbb{W}} \in \mathbb{W}$ and $P_{\mathbb{M}} \in \mathbb{M}$*

$$c_0(\|P_{\mathbb{W}}\| + \|P_{\mathbb{M}}\|) \le \|P_{\mathbb{W}}P_{\mathbb{M}}\| \le \|P_{\mathbb{W}}\| + \|P_{\mathbb{M}}\| \tag{2.1}$$

Remark 2.1.2. By elementary fact in group theory if $\mathbb{W},\mathbb{M}$ are complementary subgroup of $\mathbb{G}$, so that $\mathbb{G} = \mathbb{W}\cdot\mathbb{M}$ then is also true that

$$\mathbb{G} = \mathbb{M}\cdot\mathbb{W},$$

that is, each $P \in \mathbb{G}$ can be written in a unique way as $P = \bar{P}_{\mathbb{M}}\bar{P}_{\mathbb{W}}$ with $\bar{P}_{\mathbb{M}} \in \mathbb{M}$ and $\bar{P}_{\mathbb{W}} \in \mathbb{W}$. Rephrased differently, if $\mathbb{W},\mathbb{M}$ are complementary subgroup of $\mathbb{G}$ also $\mathbb{M},\mathbb{W}$ are complementary subgroup of $\mathbb{G}$.

Remark 2.1.3. The stratification of $\mathbb{G}$ induces a stratifications on the complementary subgroups $\mathbb{W}$ and $\mathbb{M}$. If $\mathbb{G}$ as $\mathbb{G} = \mathbb{G}^1\oplus\cdots\oplus\mathbb{G}^\kappa$ then also $\mathbb{W} = \mathbb{W}^1\oplus\cdots\oplus\mathbb{W}^\kappa$, $\mathbb{M} = \mathbb{M}^1\oplus\cdots\oplus\mathbb{M}^\kappa$ and $\mathbb{G}^i = \mathbb{W}^i \oplus \mathbb{M}^i$. If $\mathbb{M}$ is horizontal then the complementary subgroup $\mathbb{W}$ is normal.

Remark 2.1.4. Complementary subgroups always exist in any Carnot group $\mathbb{G}$. Indeed, choose any horizontal homogeneous subgroup $\mathbb{M} = \mathbb{M}^1 \subset \mathbb{G}^1$ and subgroup $\mathbb{W} = \mathbb{W}^1 \oplus\cdots\oplus \mathbb{W}^\kappa$ such that

$$\mathbb{M} \oplus \mathbb{W}^1 = \mathbb{G}^1$$
$$\mathbb{W}^s = \mathbb{G}^s \quad \text{for all } s = 2,\dots,\kappa.$$

Then $\mathbb{M},\mathbb{W}$ are complementary subgroups in $\mathbb{G}$ and the product $\mathbb{G} = \mathbb{W}\cdot\mathbb{M}$ is semi-direct because $\mathbb{W}$ is a normal subgroup.

Example 2.1.2. Let $\mathbb{G}$ be a Carnot group of step κ and let

$$\mathbb{M} := \{P \in \mathbb{G} \,|\, p_2 = \cdots = p_N = 0\} \quad \text{and} \quad \mathbb{W} := \{P \in \mathbb{G} \,|\, p_1 = 0\}.$$

Then $\mathbb{W},\mathbb{M}$ are complementary subgroups of $\mathbb{G}$. Moreover $\mathbb{M}$ is horizontal subgroup and $\mathbb{W}$ is normal.

Example 2.1.3. There are subgroups in Carnot groups without a complementary subgroup.

For instance, in Heisenberg group $\mathbb{H}^k$ if we consider the subgroup

$$\mathbb{M} := \{P = (p_1,\dots,p_{2k+1}) \in \mathbb{H}^k \,|\, p_1 = \cdots = p_{2k} = 0\},$$

then there is not a subgroup $\mathbb{W} \subset \mathbb{H}^k$ such that $\mathbb{W}\cap\mathbb{M} = \{0\}$ and $\mathbb{H}^k = \mathbb{W}\cdot\mathbb{M}$.

Example 2.1.4. We now give an example of complementary subgroups of a Carnot group which are not normal. Indeed, recalling Section 1.3.7, if we consider $\mathbb{G}$ equals to the sum of $\mathbb{G}^{(1)} = \mathbb{H}^1$ with itself, then $\mathbb{G} = \mathbb{R}^{4+2}$ is a group of class $\mathcal{B}$ where the group operation is defined as (1.24) with

$$\mathcal{B}^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{B}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

i.e.

$$\begin{aligned} &(x_1, x_2, y_1, y_2, x_3, y_3) \cdot (x_1', x_2', y_1', y_2', x_3', y_3') = \\ &\left(x_1 + x_1', x_2 + x_2', y_1 + y_1', y_2 + y_2', x_3 + x_3' + \frac{1}{2}(x_1 x_2' - x_1' x_2), y_3 + y_3' + \frac{1}{2}(y_1 y_2' - y_1' y_2) \right). \end{aligned}$$

Moreover $\mathbb{W} := \{(0, x_2, y_1, 0, x_3, 0) \,|\, x_2, x_3, y_1 \in \mathbb{R}\}$ and $\mathbb{M} := \{(x_1, 0, 0, y_2, 0, y_3) \,|\, x_1, y_2, y_3 \in \mathbb{R}\}$ are complementary subgroups of $\mathbb{G}$ but both $\mathbb{W}$ and $\mathbb{M}$ are not normal; indeed if we consider $P = (\hat{x}_1, \hat{x}_2, 0, 0, 0, 0), Q = (0, 0, \hat{y}_1, \hat{y}_2, 0, 0) \in \mathbb{G}$ then

$$\begin{aligned} P^{-1}(x_1, 0, 0, y_2, 0, y_3)P &= (x_1, 0, y_2, \hat{x}_2 x_1, 0, y_3) \quad \notin \mathbb{M} \\ Q^{-1}(0, x_2, y_1, 0, x_3, 0)Q &= (0, x_2, y_1, 0, x_3, \hat{y}_2 y_1) \quad \notin \mathbb{W}. \end{aligned}$$

Hence the product $\mathbb{G} = \mathbb{W} \cdot \mathbb{M}$ is not semi-direct.

Now we present an estimate which we will use later:

Lemma 2.1.5. *Let $\mathbb{G}$ be a step κ Carnot group. There is $C = C(\mathbb{G}) > 0$ such that*

$$\|P^{-1}Q^{-1}PQ\| \le C \left(\|P\|^{\frac{1}{\kappa}} \|Q\|^{\frac{\kappa-1}{\kappa}} + \|P\|^{\frac{\kappa-1}{\kappa}} \|Q\|^{\frac{1}{\kappa}} \right) \quad \textit{for all } P, Q \in \mathbb{G}$$

and consequently

$$\|Q^{-1}PQ\| \le \|P\| + C \left(\|P\|^{\frac{1}{\kappa}} \|Q\|^{\frac{\kappa-1}{\kappa}} + \|P\|^{\frac{\kappa-1}{\kappa}} \|Q\|^{\frac{1}{\kappa}} \right) \quad \textit{for all } P, Q \in \mathbb{G}. \tag{2.2}$$

Proof. By direct computation it can be checked

$$\begin{aligned} &(P^{-1}Q^{-1}PQ)^1 = 0 \\ &(P^{-1}Q^{-1}PQ)^2 = \mathcal{Q}^2(-P, -Q) + \mathcal{Q}^2(P, Q) \\ &(P^{-1}Q^{-1}PQ)^s = \mathcal{Q}^s(-P, -Q) + \mathcal{Q}^s(P, Q) + \mathcal{Q}^s(-P - Q + \mathcal{Q}(-P, -Q), P + Q + \mathcal{Q}(P, Q)), \end{aligned}$$

for each $s > 2$. As a consequence, from (1.12) we have that if $s = 2$ we can estimate $|(P^{-1}Q^{-1}PQ)^2|^{\frac{1}{2}}$ by $\|P\|^{\frac{1}{2}}\|Q\|^{\frac{1}{2}}$ up to a constant factor depending only on $\mathbb{G}$.

On the other hand if $s = 3, \dots, \kappa$, since Theorem 1.3.7 we know that $\mathcal{Q}^s(-P,-Q) + \mathcal{Q}^s(P,Q) + \mathcal{Q}^s(-P-Q+\mathcal{Q}(-P,-Q), P+Q+\mathcal{Q}(P,Q))$ are δ_λ-homogeneous polynomials of degree s in the variables P and Q and by (1.14)

$$\begin{aligned}
&\mathcal{Q}^s(P,0) = 0,\\
&\mathcal{Q}^s(0,Q) = 0,\\
&\mathcal{Q}^s(-P,0) + \mathcal{Q}^s(P,0) + \mathcal{Q}^s(-P+\mathcal{Q}(-P,0), P+\mathcal{Q}(P,0)) = 0,\\
&\mathcal{Q}^s(0,-Q) + \mathcal{Q}^s(0,Q) + \mathcal{Q}^s(-Q+\mathcal{Q}(0,-Q), Q+\mathcal{Q}(0,Q)) = 0.
\end{aligned}$$

Hence $\mathcal{Q}^s(-P,-Q) + \mathcal{Q}^s(P,Q) + \mathcal{Q}^s(-P-Q+\mathcal{Q}(-P,-Q), P+Q+\mathcal{Q}(P,Q))$ contain only mixed monomials in the variables P and Q and using again (1.12) we can estimate $|(P^{-1}Q^{-1}PQ)^s|^{\frac{1}{s}}$ by $\sum_{i=1}^{s-1} \|P\|^i \|Q\|^{s-i}$ up to a constant factor. More specifically because $\|P\|^i\|Q\|^{s-i}$ is estimated by $\|P\|\|Q\|^{s-1}$ when $\|P\| \le \|Q\|$ and by $\|P\|^{s-1}\|Q\|$ when $\|P\| \ge \|Q\|$, then

$$\|P\|^i\|Q\|^{s-i} \le \|P\|\|Q\|^{s-1} + \|P\|^{s-1}\|Q\|, \qquad \text{for all } i = 1, \dots, s-1.$$

Consequently, there exists $C_s > 0$ such that

$$\begin{aligned}
&|(P^{-1}Q^{-1}PQ)^s|^{\frac{1}{s}}\\
&\le |\mathcal{Q}^s(-P,-Q)|^{\frac{1}{s}} + |\mathcal{Q}^s(P,Q)|^{\frac{1}{s}} + |\mathcal{Q}^s(-P-Q+\mathcal{Q}(-P,-Q), P+Q+\mathcal{Q}(P,Q))|^{\frac{1}{s}}\\
&\le C_s\left(\|P\|^{\frac{1}{s}}\|Q\|^{\frac{s-1}{s}} + \|P\|^{\frac{s-1}{s}}\|Q\|^{\frac{1}{s}}\right).
\end{aligned}$$

Finally (2.2) follows from $\|Q^{-1}PQ\| \le \|P\| + \|P^{-1}Q^{-1}PQ\|$. □

A direct consequence of Lemma 2.1.5 is the following corollary:

Corollary 2.1.6. *Let $\mathbb{G}$ be a Carnot group of step κ where $\mathbb{W}, \mathbb{M}$ are complementary subgroups of $\mathbb{G}$. If $A, A' \in \mathbb{W}$ and $B, B' \in \mathbb{M}$ such that*

$$AB = B'A'$$

then

$$\begin{aligned}
\|A\| &\le \frac{1}{c_0}\|A'\| + \frac{C}{c_0}\left(\|A'\|^{\frac{1}{\kappa}}\|B'\|^{\frac{\kappa-1}{\kappa}} + \|A'\|^{\frac{\kappa-1}{\kappa}}\|B'\|^{\frac{1}{\kappa}}\right),\\
\|B\| &\le \frac{1}{c_0}\|B'\| + \frac{C}{c_0}\left(\|A'\|^{\frac{1}{\kappa}}\|B'\|^{\frac{\kappa-1}{\kappa}} + \|A'\|^{\frac{\kappa-1}{\kappa}}\|B'\|^{\frac{1}{\kappa}}\right),
\end{aligned}$$

where c_0 is given by (2.1). Moreover for every $\delta > 0$ there exists $c(\delta) = c(\delta, \mathbb{W}, \mathbb{M}) > 0$ such that $\|AB\| \le \delta$ yields

$$\|A\| \le c(\delta)\|A'\|^{\frac{1}{\kappa}} \qquad \textit{and} \qquad \|B\| \le c(\delta)\|B'\|^{\frac{1}{\kappa}}.$$

Proof. With the notation
$$AB = (B'A')_{\mathbb{W}}(B'A')_{\mathbb{M}}$$
by the uniqueness of the components, we obtain that
$$A = (B'A')_{\mathbb{W}} = (B'A'B'^{-1})_{\mathbb{W}} \quad \text{and} \quad B = (B'A')_{\mathbb{M}} = (A'^{-1}B'A')_{\mathbb{M}}.$$
Moreover from (2.1) and (2.2) follows
$$\|A\| = \|(B'A'B'^{-1})_{\mathbb{W}}\| \leq \frac{1}{c_0}\|B'A'B'^{-1}\| \leq \frac{1}{c_0}\|A'\| + \frac{C}{c_0}\left(\|A'\|^{\frac{1}{\kappa}}\|B'\|^{\frac{\kappa-1}{\kappa}} + \|A'\|^{\frac{\kappa-1}{\kappa}}\|B'\|^{\frac{1}{\kappa}}\right)$$
$$\|B\| = \|(A'^{-1}B'A')_{\mathbb{M}}\| \leq \frac{1}{c_0}\|A'^{-1}B'A'\| \leq \frac{1}{c_0}\|B'\| + \frac{C}{c_0}\left(\|B'\|^{\frac{1}{\kappa}}\|A'\|^{\frac{\kappa-1}{\kappa}} + \|B'\|^{\frac{\kappa-1}{\kappa}}\|A'\|^{\frac{1}{\kappa}}\right),$$
as desired. Finally we obtain that $\|A'\|, \|B'\| \leq \delta/c_0$ because $\|AB\| \leq \delta$. □

2.2 H-linear maps

We introduce the concept of H-linear maps and we propose some properties proved in [69], [95]. These functions play the same role of linear functions in Euclidean spaces; indeed the class of H-linear maps coincides with that of linear maps when the group is an Euclidean space (i.e. Carnot group of step 1).

Definition 2.2.1. Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be Carnot groups with homogeneous norms $\|\cdot\|_1, \|\cdot\|_2$ and dilations $\delta_\lambda^{(1)}, \delta_\lambda^{(2)}$. Then $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ is said to be *horizontal linear map* or, shortly, H-linear map if

1. l is a group homomorphism from $\mathbb{G}^{(1)}$ to $\mathbb{G}^{(2)}$.
2. l is homogeneous map, i.e. $l(\delta_\lambda^{(1)}P) = \delta_\lambda^{(2)}l(P)$, for all $P \in \mathbb{G}^{(1)}$ and $\lambda > 0$.

An invertible H-linear map will be called H-isomorphism.

Following [95], we denote by $\mathcal{L}_H(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$ the set of all H-linear maps $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ endowed with the norm defined as
$$\|l\|_{\mathcal{L}_H(\mathbb{G}^{(1)},\mathbb{G}^{(2)})} = \|l\| := \sup\{\|l(P)\|_2 \,:\, \|P\|_1 \leq 1\}.$$

It has a natural structure of Lie group with respect to the following operations:
$$l \cdot h(P) := l(P)h(P) \quad \text{and} \quad (l(P))^{-1} := -l(P)$$
for any $l, h \in \mathcal{L}_H(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$. Moreover there is a natural group of dilations defined as $\delta_\lambda l(P) := \delta_\lambda(l(P))$ for $\lambda > 0$.

If $\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}$ are the Lie algebra of $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ respectively, we also define $\mathcal{L}_H(\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)})$ as the set of all maps $\tilde{l} : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(2)}$ such that

$$\exp_2 \circ \tilde{l} \circ \exp_1^{-1} \in \mathcal{L}_H(\mathbb{G}^{(1)}, \mathbb{G}^{(2)}).$$

Here $\exp_i : \mathfrak{g}^{(i)} \to \mathbb{G}^{(i)}$ for $i = 1, 2$ and $\exp^{-1}$ is the inverse map of exponential map (see Theorem 1.2.4).

Consequently any function of $\mathcal{L}_H(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$ induces uniquely a function of $\mathcal{L}_H(\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)})$ and viceversa.

Remark 2.2.1. Let $l \in \mathcal{L}_H(\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)})$. Then

1. l is linear map.

2. l preserves the bracket operation.

3. $l(\mathfrak{g}_1^{(1)}) \subset \mathfrak{g}_1^{(2)}$ where $\mathfrak{g}_1^{(i)}$ is the horizontal layer of $\mathfrak{g}^{(i)}$ for $i = 1, 2$.

The following results give a simply characterization of H-linear maps.

Proposition 2.2.2. *Let $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ be an injective H-linear map and $l(\mathbb{G}^{(1)}) = \mathbb{W}$. Then $\mathbb{W}$ is a subgroup of $\mathbb{G}^{(2)}$ and $l^{-1} : \mathbb{W} \to \mathbb{G}^{(1)}$ is H-linear.*

Proposition 2.2.3. *Let $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$, $h : \mathbb{G}^{(2)} \to \mathbb{G}^{(3)}$ be H-linear maps. Then $h \circ l : \mathbb{G}^{(1)} \to \mathbb{G}^{(3)}$ is H-linear such that*

$$\|h \circ l\| \leq \|h\| \|l\|.$$

Theorem 2.2.4 ([95], Theorem 3.2)**.** *Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be Carnot groups of step κ with homogeneous norms $\|\cdot\|_1, \|\cdot\|_2$. Let $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ be a homomorphism. Then the following are equivalent:*

1. *l is H-linear map.*

2. *l is $\mathbb{C}^\infty$ and Lipschitz.*

3. *l is $\mathbb{C}^\infty$ and $l(\mathfrak{g}_i^{(1)}) \subset \mathfrak{g}_i^{(2)}$ for every $i = 1, \dots, \kappa$ where $\mathfrak{g}^{(j)}$ is the Lie algebra of $\mathbb{G}^{(j)}$ for $j = 1, 2$.*

Remark 2.2.5. Any H-linear map can be represented by a matrix with diagonal blocks. This basically follows from $l(\mathfrak{g}_i^{(1)}) \subset \mathfrak{g}_i^{(2)}$ for every $i = 1, \dots, \kappa$, which stated in Theorem 2.2.4. Indeed if $\mathfrak{g}^{(1)} = \mathfrak{g}_1^{(1)} \oplus \cdots \oplus \mathfrak{g}_\kappa^{(1)}$ and $\mathfrak{g}^{(2)} = \mathfrak{g}_1^{(2)} \oplus \cdots \oplus \mathfrak{g}_s^{(2)}$ we have that

$$l_{|\mathfrak{g}_i^{(1)}} : \mathfrak{g}_i^{(1)} \to \mathfrak{g}_i^{(2)}$$

for $i = 1, \dots, \max\{\kappa, s\}$ (taking into account that spaces $\mathfrak{g}_i^{(1)}$ and $\mathfrak{g}_i^{(2)}$ are null spaces when i is greater than the degree of nilpotency of the group).

We point out that the general explicit computation of the coefficient of l with respect to a fixed basis can be very involved because the group operation given by the Baker-Campbell-Hausdorff formula becomes a large polynomial expression as the step of the group increases.

Let us consider a simple example taken from [82].

Example 2.2.1. Let $\mathbb{H}^1 = \mathbb{R}^3$ and consider the basis $X_1 = \partial_{p_1} - \frac{1}{2}p_2\,\partial_{p_3}, X_2 = \partial_{p_2} + \frac{1}{2}p_1\,\partial_{p_3}, Y = \partial_{p_3}$ of Lie algebra of $\mathbb{H}^1$ (whose points are denoted by $P = (p_1, p_2, p_3)$). All H-linear maps $l : \mathbb{H}^1 \to \mathbb{H}^1$ can be represented with respect to the basis (X_1, X_2, Y) with matrices of the following form

$$\begin{pmatrix} \mathcal{B} & 0 \\ 0 & \det(\mathcal{B}) \end{pmatrix}$$

where $\mathcal{B}$ is (2×2) matrix.

2.2.1 H-epimorphisms and H-monomorphisms

Concerning injective and surjective H-linear maps, we will use the classical terminology of H-epimorphism and H-monomorphisms to indicate special classes of surjective and injective H-linear maps. Indeed, we recall that a surjective linear mapping of vector spaces is characterized by the existence of a right inverse that is also linear. Analogously, injective linear mappings are characterized by the existence of a linear left inverse mapping.

The analogous characterization for either surjective or injective algebra homomorphisms is *not* true (see [70], Example 2.4). As a consequence, the existence of a right inverse homomorphism is a stronger condition than surjectivity. This motivates the following

Definition 2.2.2 ([70], Definition 2.2)**.** We say that an H-linear map is an H-epimorphism if it has a right inverse that is also an H-linear map.

We say that an H-linear map is an H-monomorphism if it has a left inverse that is also an H-linear map.

Proposition 2.2.6 (Characterization of H-epimorphisms)**.** *Let $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ be a surjective H-linear map and let $\mathbb{W}$ be its kernel. The following conditions are equivalent:*

1. *there exists a subgroup $\mathbb{M}$ complementary to $\mathbb{W}$*
2. *l is an H-epimorphism*

Moreover if 1. or 2. hold, then the restriction $l_{|\mathbb{M}} : \mathbb{M} \to \mathbb{G}^{(2)}$ is an H-isomorphism.

Proof. Let us begin with the proof of the implication 1. $\Rightarrow$ 2.

Let $h := l_{|\mathbb{M}}$ and let $Q \in \mathbb{G}^{(2)}$. We know that there exists $P \in \mathbb{G}^{(1)}$ such that $l(P) = Q$. Moreover since $\mathbb{G}^{(1)} = \mathbb{W} \cdot \mathbb{M}$, we have $P = P_{\mathbb{W}} P_{\mathbb{M}}$ (where $P_{\mathbb{W}}$ and $P_{\mathbb{M}}$ are unique) and

$$Q = l(P_{\mathbb{W}} P_{\mathbb{M}}) = l(P_{\mathbb{W}}) l(P_{\mathbb{M}}) = l(P_{\mathbb{M}}) = h(P_{\mathbb{M}}).$$

Hence h is surjective. Moreover h is injective; indeed if $h(Q) = 0$, then $Q \in \mathbb{W} \cap \mathbb{M} = \{0\}$ and so $h(Q) = 0$ iff $Q = 0$. We have shown that h is a H-isomorphism. Clearly, $h^{-1} : \mathbb{G}^{(2)} \to \mathbb{M}$ is a H-linear map and satisfies $l \circ h^{-1} = \mathrm{id}_{\mathbb{G}^{(2)}}$. Hence l is an H-epimorphism and the implication 1. $\Rightarrow$ 2. is complete.

Now we want to show the converse, i.e. 2. $\Rightarrow$ 1.

If l is an H-epimorphism, then there exists a right inverse $h : \mathbb{G}^{(2)} \to \mathbb{G}^{(1)}$ that is also a H-linear map. Let $\mathbb{M} := h(\mathbb{G}^{(2)})$. It easy to see that $\mathbb{W} \cap \mathbb{M} = \{0\}$.

Let $P \in \mathbb{G}^{(1)}$. We know that there is $Q \in \mathbb{G}^{(2)}$ such that $l(P) = Q$ and, since $Q = l(h(Q))$, we have that $l(P) = l(h(Q))$. As a consequence, $P^{-1}h(Q) \in \mathbb{W}$. Indeed,

$$l(P^{-1}h(Q)) = l(P^{-1}) l(h(Q)) = l(P^{-1}) l(P) = l(0) = 0.$$

Then

$$P = PP^{-1}h(Q) \left(P^{-1}h(Q)\right)^{-1}$$

where $PP^{-1}h(Q) \in \mathbb{M}$ and $\left(P^{-1}h(Q)\right)^{-1} \in \mathbb{W}$. Hence $\mathbb{G}^{(1)} = \mathbb{M} \cdot \mathbb{W} = \mathbb{W} \cdot \mathbb{M}$ (see Remark 2.1.2) and this concludes the proof. $\square$

Proposition 2.2.7 (Characterization of H-monomorphisms)**.** *Let $h : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ be a injective H-linear map and let $\mathbb{M}$ be its image. The following conditions are equivalent:*

1. *there exists a normal subgroup $\mathbb{W} \subset \mathbb{G}^{(2)}$ complementary to $\mathbb{M}$,*
2. *there exists an H-epimorphism $l : \mathbb{G}^{(2)} \to \mathbb{M}$ such that $l_{|\mathbb{M}} = id_{\mathbb{M}}$*
3. *h is an H-monomorphism*

We stress that the existence of H-epimorphism (or H-monomorphism) from $\mathbb{G}^{(1)}$ to $\mathbb{G}^{(2)}$ is strictly linked with the existence of complementary subgroups of $\mathbb{G}^{(1)}$ (or $\mathbb{G}^{(2)}$) and one of them is also a normal subgroup. This fact has intrinsic limitations appearing already inside Heisenberg. Indeed, low dimensional horizontal subgroups of Heisenberg groups $\mathbb{H}^n$ are not normal subgroups, hence they cannot appear as kernels of H-linear map $\mathbb{H}^n \to \mathbb{R}^d$. On the other side, surjective H-linear map $\mathbb{H}^n \to \mathbb{H}^k$ do not exist, if $n > k$ (see [6] and Proposition 2.2.10).

2.2.2 H-quotients and H-embeddings

Following [70], in this section we investigate the algebraic conditions under which either surjective or injective H-linear functions are H-epimorphisms or H-monomorphisms, respectively.

Let $\mathbb{G}$ be a real Lie group and let $\mathbb{W}$ be a Lie subgroup of $\mathbb{G}$. We recall that the quotient $\mathbb{G}/\mathbb{W}$ has a unique manifold structure that makes the projection $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{W}$ a smooth mapping. $\mathbb{G}/\mathbb{W}$ is called homogeneous manifold, (see Theorem 3.58, [107]).

If we consider a normal Lie subgroup $\mathbb{W}$, then $\mathbb{G}/\mathbb{W}$ is in addition a Lie group, according to Theorem 3.64 in [107] and in this case π is a Lie group homomorphism. Moreover by Theorem 3.14 in [107] it follows

$$d\pi : \mathfrak{g} \to \mathfrak{w} \text{ is a Lie algebra homomorphism,}$$

where $\mathfrak{g}$ and $\mathfrak{w}$ are the Lie algebras of $\mathbb{G}$ and $\mathbb{G}/\mathbb{W}$, respectively.

Definition 2.2.3 (H-quotients and H-embeddings)**.** We say that $\mathbb{G}^{(2)}$ is an H-quotient of $\mathbb{G}^{(1)}$ if there exists a normal homogeneous subgroup $\mathbb{W} \subset \mathbb{G}^{(1)}$ such that $\mathbb{G}^{(1)}/\mathbb{W}$ is H-isomorphic to $\mathbb{G}^{(2)}$.

Analogously, we say that $\mathbb{G}^{(1)}$ is an H-embedding into $\mathbb{G}^{(2)}$ if there exists a homogeneous subgroup of $\mathbb{G}^{(2)}$ which is H-isomorphic to $\mathbb{G}^{(1)}$.

Proposition 2.2.8. *Let $\mathbb{G}$ be a Carnot group and m_1 be the dimension of horizontal layer of the Lie algebra associated to $\mathbb{G}$.*

$\mathbb{R}^k$ is an H-quotient of $\mathbb{G}$ if and only if $k \le m_1$.

Now we give a characterization of H-quotients and H-embeddings:

Proposition 2.2.9. *Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be a Carnot groups. Then $\mathbb{G}^{(2)}$ is an H-quotient of $\mathbb{G}^{(1)}$ if and only if there exists a surjective H-linear map $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$.*

Moreover, $\mathbb{G}^{(1)}$ H-embeds into $\mathbb{G}^{(2)}$ if and only if there exists an injective H-linear map $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$.

The Heisenberg group $\mathbb{H}^k$ is not an H-quotient of $\mathbb{H}^n$, whenever $n > k$, indeed

Proposition 2.2.10. *Any H-linear map $l \in \mathcal{L}_H(\mathbb{H}^n, \mathbb{H}^k)$, with $n > k$, is not surjective.*

Proof. By Remark 2.2.5 we have

$$l(P) = (\mathcal{M}p^1, \lambda p^2), \quad \text{for all } P = (p^1, p^2) \in \mathbb{H}^n$$

where $\mathcal{M} : \mathbb{R}^{2n} \to \mathbb{R}^{2k}$ is a linear map with respect to the field of real numbers and $\lambda \in \mathbb{R}$. We recall that $p^1 = (p_1, \ldots, p_{2n})$ and $p^2 = p_{2n+1}$.

The homomorphism property implies that $l(PQ) = l(P)l(Q)$ for every $P = (p^1, p^2), Q = (q^1, q^2) \in \mathbb{H}^n$, i.e.

$$\left(\mathcal{M}p^1 + \mathcal{M}q^1, \lambda\left(p_{2n+1} + q_{2n+1} + \frac{1}{2}\sum_{i=1}^{n}(p_i q_{n+i} - p_{n+i}q_i)\right)\right)$$
$$= \left(\mathcal{M}p^1 + \mathcal{M}q^1, \lambda\left(p_{2n+1} + q_{2n+1} + \frac{1}{2}\sum_{i=1}^{n}\left((\mathcal{M}p^1)_i(\mathcal{M}q^1)_{n+i} - (\mathcal{M}p^1)_{n+i}(\mathcal{M}q^1)_i\right)\right)\right)$$

and consequently

$$\lambda\sum_{i=1}^{n}(p_i q_{n+i} - p_{n+i}q_i) = \lambda\sum_{i=1}^{n}\left((\mathcal{M}p^1)_i(\mathcal{M}q^1)_{n+i} - (\mathcal{M}p^1)_{n+i}(\mathcal{M}q^1)_i\right) \tag{2.3}$$

for any $p^1, q^1 \in \mathbb{R}^{2n}$. By the fact that $n > k$ we can take a non vanishing $\hat{p} = (\hat{p}_1, \dots, \hat{p}_{2n})$ in the kernel of $\mathcal{M}$. Replacing $p^1 = \hat{p}$ and $q^1 = (-\hat{p}_{n+1}, \dots, -\hat{p}_{2n}, \hat{p}_1, \dots, \hat{p}_n)$ in (2.3) we obtain that $\lambda = 0$. Then l is not surjective. □

2.3 P-differentiable functions and $\mathrm{BV}_{\mathbb{G}}$ functions

2.3.1 P-differentiability

Now we are able to define P-differentiability for functions acting between nilpotent groups (see Section 1.2.4) which was introduced by Pansu in [83]. More specifically, we recall this definition in a particular nilpotent groups, i.e. Carnot groups.

Definition 2.3.1. Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be Carnot groups with homogeneous norm $\|\cdot\|_1, \|\cdot\|_2$ and let $\mathcal{A} \subset \mathbb{G}^{(1)}$ be an open set. Then $f : \mathcal{A} \to \mathbb{G}^{(2)}$ is P-differentiable in $P \in \mathcal{A}$ if there exists a H-linear function $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ such that

$$\|(l(P^{-1}Q))^{-1}f(P)^{-1}f(Q)\|_2 = o(\|P^{-1}Q\|_1), \quad \text{as } \|P^{-1}Q\|_1 \to 0 \tag{2.4}$$

where $\lim_{t\to 0} o(t)/t = 0$. Here the H-linear map l is called P-differential of f in P.

Proposition 2.3.1. *Let $f : \mathcal{A} \subset \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ and $P \in \mathcal{A}$. If there exists a H-linear function $l : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ satisfying* (2.4)*, then l is unique.*

Definition 2.3.2. Let $\mathcal{A}$ be an open set in $\mathbb{G}^{(1)}$. We denote by $\mathbb{C}^1_H(\mathcal{A}, \mathbb{G}^{(2)})$ the set of all continuous functions $f : \mathcal{A} \to \mathbb{G}^{(2)}$ such that its P-differential $l : \mathcal{A} \to \mathcal{L}_H(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$ is continuous.

If $\mathbb{G}^{(1)} = \mathbb{G}$ and $\mathbb{G}^{(2)} = \mathbb{R}^k$ we simply denote $\mathbb{C}^1_H(\mathcal{A}, \mathbb{G}^{(2)})$ as $\mathbb{C}^1_{\mathbb{G}}(\mathcal{A}, \mathbb{R}^k)$. In particular if $k = 1$ we denote $\mathbb{C}^1_H(\mathcal{A}, \mathbb{G}^{(2)})$ as $\mathbb{C}^1_{\mathbb{G}}(\mathcal{A})$.

Notice that for a function $f : \mathcal{A} \to \mathbb{R}$ P-differentiability at $P \in \mathcal{A}$ simply means that there exists a H-linear map $l : \mathbb{G} \to \mathbb{R}$ such that

$$\lim_{Q\to P} \frac{f(Q) - f(P) - l(P^{-1}Q)}{d_{cc}(Q,P)} = 0.$$

The fundamental result where P-differentiability applies is the so-called Pansu-Rademarcher Theorem for Lipschitz functions between Carnot groups:

Theorem 2.3.2 ([83]). *Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be Carnot groups with cc-metrics denoted by, respectively, d_1, d_2. Let $f : \mathcal{A} \subset (\mathbb{G}^{(1)}, d_1) \to (\mathbb{G}^{(2)}, d_2)$ be a Lipschitz continuous function with $\mathcal{A}$ an open set.*

Then f is P-differentiable at Q for $\mathcal{L}^N$-a.e. $Q \in \mathcal{A}$.

2.3.2 $\mathbb{C}^1_{\mathbb{G}}$ functions

Once a generating family of vector fields $X_1, \dots, X_{m_1}$ is fixed, we define for any function $f : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_i f$ exist, the horizontal gradient of f as the horizontal section

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m_1} (X_i f) X_i$$

whose coordinates are $(X_1 f, \dots, X_{m_1} f)$. If $\phi = (\phi_1, \dots, \phi_{m_1}) : \mathbb{G} \to \mathbb{R}^{m_1}$ the horizontal divergence of ϕ is defined as

$$\mathrm{div}_{\mathbb{G}} \phi := \sum_{j=1}^{m_1} X_j \phi_j .$$

Remark 2.3.3. The notation we have used for the gradient in a group is partially imprecise, indeed $\nabla_{\mathbb{G}} f$ really depends on the choice of the basis $X_1, \dots, X_{m_1}$. If we choose a different base, say $Y_1, \dots, Y_{m_1}$ then in general $\sum_{i=1}^{m_1} (X_i f) X_i \neq \sum_{i=1}^{m_1} (Y_i f) Y_i$. Only if the two bases are one orthonormal with respect to the scalar product induced by the other, we have

$$\sum_{i=1}^{m_1} (X_i f) X_i = \sum_{i=1}^{m_1} (Y_i f) Y_i .$$

On the other hand, the notation $\mathrm{div}_{\mathbb{G}}$ used for the divergence is correct. Indeed, $\mathrm{div}_{\mathbb{G}}$ is an intrinsic notation and it can be computed using the previous formula for any fixed generating family.

The following proposition shows that the P-differential of a P-differentiable map f is represented by horizontal gradient $\nabla_{\mathbb{G}} f$:

Proposition 2.3.4. *If $f : \mathcal{A} \subset \mathbb{G} \to \mathbb{R}$ is P-differentiable at a point P and l is P-differential of f at P, then*

$$l(Q) = \sum_{j=1}^{m_1} X_j f(P) q_j, \quad \forall\, Q = (q_1, \dots, q_N) \in \mathcal{A}.$$

Then we can characterize $\mathbb{C}^1_{\mathbb{G}}$ maps as follows:

Proposition 2.3.5. *A continuous map $f : \mathcal{A} \to \mathbb{R}$ belongs to $\mathbb{C}^1_{\mathbb{G}}(\mathcal{A})$ if and only if its distributional derivatives $X_j f$ are continuous in $\mathcal{A}$ for $j = 1, \dots, m_1$.*

Consequently, $\mathbb{C}^1_{\mathbb{G}}(\mathcal{A}, \mathbb{R}^k)$ is the set of k-tuples $F = (F_1, \dots, F_k) : \mathcal{A} \to \mathbb{R}^k$ such that each F_i and $X_1 F_i, \dots, X_{m_1} F_i$ are continuous in $\mathcal{A}$ for $i = 1, \dots, k$. Observe that the intrinsic gradient of F is

$$\nabla_{\mathbb{G}} F(Q) := \begin{pmatrix} \nabla_{\mathbb{G}} F_1(Q) \\ \vdots \\ \nabla_{\mathbb{G}} F_k(Q) \end{pmatrix}.$$

Definition 2.3.3. Let $\mathcal{A}$ be an open set in $\mathbb{G}$, we denote by $\mathbb{C}^1_{\mathbb{G}}(\mathcal{A}, H\mathbb{G})$ the set of all sections ϕ of $H\mathbb{G}$ whose canonical coordinates $\phi_j \in \mathbb{C}^1_{\mathbb{G}}(\mathcal{A})$ for $j = 1, \dots, m_1$.

Remark 2.3.6. $\mathbb{C}^1(\mathcal{A}) \subset \mathbb{C}^1_{\mathbb{G}}(\mathcal{A})$. The following example shows that this inclusion is strict: Let $\mathbb{G} = \mathbb{H}^1$ and let $f : \mathbb{H}^1 \to \mathbb{R}$ be defined as

$$f(p_1, p_2, p_3) := p_1 - g\left(p_2, p_3 - \frac{1}{2} p_1 p_2\right), \quad \text{for } (p_1, p_2, p_3) \in \mathbb{H}^1 = \mathbb{R}^3$$

where, for any $\alpha > 0$, the map $g : \mathbb{R}^3 \to \mathbb{R}$ is

$$g(q_1, q_2) := \begin{cases} \frac{|q_1|^\alpha q_2}{q_1^4 + q_2^2} & \text{if } (q_1, q_2) \neq (0,0) \\ 0, & \text{if } (q_1, q_2) = (0,0) \end{cases}$$

Then $f \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{H}^1)$, but f is not locally Lipschitz continuous with respect to the Euclidean metric of $\mathbb{R}^3$, when $3 < \alpha < 4$. Indeed, if $\alpha > 2$, the map $g \in \mathbb{C}^0(\mathbb{R}^2) \cap \mathbb{C}^1(\mathbb{R}^2 - \{(0,0)\})$ whence

$$f \in \mathbb{C}^0(\mathbb{R}^3) \cap \mathbb{C}^1(\mathbb{R}^3 - \{(p,0,0) \,:\, p \in \mathbb{R}\}).$$

Moreover

$$\begin{aligned} X_1 f(p_1, p_2, p_3) &= 1 + p_2\, \partial_{q_2} g(p_1, 2p_1 p_2 + p_3) \\ X_2 f(p_1, p_2, p_3) &= -\partial_{q_1} g(p_1, 2p_1 p_2 + p_3) \end{aligned}$$

if $(p_2, p_3) \neq (0,0)$ and $X_1 f(p_1, 0, 0) = 1$, $X_2 f(p_1, 0, 0) = 0$. Therefore, if $\alpha > 3$, $X_1 f$ and $X_2 f$ are continuous functions on $\mathbb{R}^3$ and, by Proposition 2.3.5, $f \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{H}^1)$.

To prove that f is not locally Lipschitz continuous for $3 < \alpha < 4$, it is sufficient to notice that for any $r > 0$

$$\sup_{\{(0,p_2,p_3)\in B(0,r)\,:\,p_3\neq 0\}} \frac{|f(0,p_2,p_3) - f(0,p_2,0)|}{|p_3|} = \infty.$$

For $\mathbb{C}^1_{\mathbb{G}}$ functions we have the following Morrey type inequality (see also Lemma 3.2.2 in [105]).

Lemma 2.3.7. *Let $P \in \mathbb{G}$ and $f \in \mathbb{C}^1_{\mathbb{G}}(U(P,r_0),\mathbb{R}^k)$. Then there is $C = C(P,r_0) > 0$ such that, for each $\bar{Q} \in U(P,r_0/2)$ and $r \in (0,r_0/4)$,*

$$|f(Q) - f(\bar{Q}) - \nabla_{\mathbb{G}} f(\bar{Q})(\bar{Q}^{-1}Q)^1| \leq C d(Q,\bar{Q}) \|\nabla_{\mathbb{G}} f - \nabla_{\mathbb{G}} f(\bar{Q})\|_{\mathcal{L}^\infty(U(Q,2d(Q,\bar{Q})))}$$

for all $Q \in U(\bar{Q},r)$.

Proof. Fix $\bar{Q} \in U(P,r_0/2)$ and $r \in (0,r_0/4)$. Let $\hat{f} : U(\bar{Q},r) \to \mathbb{R}^k$ be defined as

$$Q \mapsto \hat{f}(Q) := f(Q) - \nabla_{\mathbb{G}} f(\bar{Q})(\bar{Q}^{-1}Q)^1.$$

Hence $\hat{f}(\bar{Q}) = f(\bar{Q})$ and $\nabla_{\mathbb{G}}\hat{f} = \nabla_{\mathbb{G}} f - \nabla_{\mathbb{G}} f(\bar{Q})$. Moreover, (see Theorem 1.1. in [66]) there are $p > 1$ and $\hat{C} > 0$ such that for all $Q \in U(\bar{Q},r)$

$$|\hat{f}(Q) - \hat{f}(\bar{Q})| \leq \hat{C} r \left(⨍_{U(\bar{Q},2r)} |\nabla_{\mathbb{G}}\hat{f}|^p \, d\mathcal{L}^N \right)^{1/p}$$

Then

$$\begin{aligned} |f(Q) - f(\bar{Q}) - \nabla_{\mathbb{G}} f(\bar{Q})(\bar{Q}^{-1}Q)^1| &= |\hat{f}(Q) - \hat{f}(\bar{Q})| \\ &\leq 2\hat{C}\, d(Q,\bar{Q}) \left(⨍_{U(Q,2d(\bar{Q},Q))} |\nabla_{\mathbb{G}}\hat{f}|^p \, d\mathcal{L}^N \right)^{1/p} \\ &= 2\hat{C}\, d(Q,\bar{Q}) \left(⨍_{U(Q,2d(\bar{Q},Q))} |\nabla_{\mathbb{G}} f - \nabla_{\mathbb{G}} f(\bar{Q})|^p \, d\mathcal{L}^N \right)^{1/p} \\ &\leq C d(Q,\bar{Q}) \|\nabla_{\mathbb{G}} f - \nabla_{\mathbb{G}} f(\bar{Q})\|_{\mathcal{L}^\infty(U(Q,2d(Q,\bar{Q})))}. \end{aligned}$$

□

We end this section by presenting Whitney's extension Theorem for $\mathbb{C}^1_{\mathbb{G}}$ functions which will be crucial in the Theorem 3.1.1. The proof we are going to present can be found in [47] for Carnot groups of step two only, but it is identical for general Carnot groups. For the Euclidean case the reader can see [33], Section 6.5.

Theorem 2.3.8. *Let $\mathcal{F} \subset \mathbb{G}$ be a closed set and let $f : \mathcal{F} \to \mathbb{R}^k$, $g : \mathcal{F} \to \mathbf{M}_{k\times m_1}$ be continuous functions. Let $\rho : \mathcal{F} \times \mathcal{F} \to \mathbb{R}^k$ be defined as*

$$\rho(P, Q) := \frac{f(P) - f(Q) - g(Q)(Q^{-1}P)^1}{\|Q^{-1}P\|}$$

where $g(Q)(Q^{-1}P)^1$ is the usual product between matrix and vector. For $\mathcal{K} \subset \mathcal{F}$ and $\delta > 0$ let

$$\rho_{\mathcal{K}}(\delta) := \sup\{|\rho(P, Q)| : P, Q \in \mathcal{K},\ 0 < d(P, Q) < \delta\}.$$

If, for all compact set $\mathcal{K} \subset \mathcal{F}$,

$$\lim_{\delta\to 0} \rho_{\mathcal{K}}(\delta) = 0 \tag{2.5}$$

then there exists $\hat{f} \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^k)$ such that

$$\hat{f}_{|\mathcal{F}} = f, \quad \nabla_{\mathbb{G}} \hat{f}_{|\mathcal{F}} = g.$$

Proof. **Step 1**.We start to introduce the map $\hat{\delta} : \mathbb{G} \to \mathbb{R}$ defined as

$$\hat{\delta}(p_1, \dots, p_N) := \left(\sum_{i=1}^{N} (\epsilon_i |p_i|)^{\kappa \frac{\alpha_1 \cdots \alpha_N}{\alpha_i}} \right)^{\frac{1}{\kappa \alpha_1 \cdots \alpha_N}}$$

where $\epsilon_i \in (0, 1]$ is given by (1.21) and α_i is the homogeneity of the variable p_i as in (1.9). If we consider the function $\delta : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ as

$$\delta(P, Q) := \hat{\delta}(Q^{-1}P), \quad \text{for all } P, Q \in \mathbb{G} \tag{2.6}$$

we have that δ is comparable with the metric d associated to homogeneous norm $\|\cdot\|$, because $\hat{\delta} \in \mathbb{C}^\infty(\mathbb{G} - \{0\})$ and it is homogeneous of degree 1 with respect to the dilations of $\mathbb{G}$. More precisely,

$$c_1\delta(P, Q) \le d(P, Q) \le c_2\delta(P, Q) \quad \text{for } P, Q \in \mathbb{G}$$

with $c_1 := \inf\{d(G, 0) : \hat{\delta}(G) = 1\} > 0$ and $c_2 := \sup\{d(G, 0) : \hat{\delta}(G) = 1\} < \infty$.

Let

$$r(P) := \frac{1}{20\theta} \min\{1, \mathrm{dist}(P, \mathcal{F})\}, \quad \text{for all } P \in \mathbb{G}$$

where $\theta := \frac{c_1}{c_2} > 1$ and $\mathrm{dist}(P, \mathcal{F}) := \inf\{d(P, Q) : Q \in \mathcal{F}\}$. Thanks to Vitali covering theorem (see [72]) we know that there exists a countable set $\mathcal{N} \subset \mathbb{G} - \mathcal{F}$ such that

$$\mathbb{G} - \mathcal{F} = \bigcup_{P \in \mathcal{N}} U(P, 5r(P))$$

where the balls $U(P, r(P))$ are two by two disjointed.

For each $Q \in \mathbb{G} - \mathcal{F}$ we define

$$\mathcal{N}_Q := \{P \in \mathcal{N} \,:\, U(Q, 10\theta r(Q)) \cap U(P, 10\theta r(P)) \neq \emptyset\}.$$

Here for all $Q \in \mathbb{G} - \mathcal{F}$,

$$\mathrm{Card}(\mathcal{N}_Q) \leq (120\theta + 9)^{\mathfrak{q}} \tag{2.7}$$

where $\mathfrak{q}$ is the homogeneous dimension of $\mathbb{G}$ (see (1.22)). Indeed first we note that $r(Q)$ is comparable with $r(P)$ for all $P \in \mathcal{N}_Q$. More specifically, if $\mathrm{dist}(Q, \mathcal{F}), \mathrm{dist}(P, \mathcal{F}) \leq 1$, then

$$\begin{aligned} 20\theta r(Q) &\leq \mathrm{dist}(Q, \mathcal{F}) \\ &\leq d(Q, P) + \mathrm{dist}(P, \mathcal{F}) \\ &\leq 10\theta(r(P) + r(Q)) + 20\theta r(P), \end{aligned}$$

that gives $r(Q) \leq 3r(P)$. In the similar way we obtain $r(P) \leq 3r(Q)$ and so

$$\frac{1}{3} \leq \frac{r(Q)}{r(P)} \leq 3 \quad \text{for all } P \in \mathcal{N}_Q. \tag{2.8}$$

As a consequence, if $P \in \mathcal{N}_Q$, then $P \in U(Q, 40\theta r(Q))$ and

$$U(P, r(P)) \subset U(Q, 40\theta r(Q) + r(P)) \subset U(Q, (40\theta + 3)r(Q)).$$

Now since the balls $U(P, r(P))$ are two by two disjointed and using (1.23), we conclude that

$$\begin{aligned} \mathrm{Card}(\mathcal{N}_Q)\mathcal{L}^N(U(0,1)) \left(\frac{r(Q)}{3}\right)^{\mathfrak{q}} &\leq \sum_{P \in \mathcal{N}_Q} \mathcal{L}^N(U(P, r(P))) \\ &\leq \mathcal{L}^N(U(Q, (40\theta + 3)r(Q)) \\ &= \mathcal{L}^N(U(0,1))(40\theta + 3)^{\mathfrak{q}} r(Q)^{\mathfrak{q}} \end{aligned}$$

i.e. (2.7) holds.

Step 2. Now, we construct the map $\hat{f} \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^k)$ that we are looking for.

We introduce for each $P \in \mathcal{N}$ the function $h_P : \mathbb{G} \to \mathbb{R}$ given by

$$h_P(G) := \hat{h}\left(\frac{\delta(P, G)}{5r(P)}\right)$$

where δ is defined as (2.6) and $\hat{h} : \mathbb{R} \to \mathbb{R}$ is a smooth non increasing map defined as

$$\hat{h}(t) := \begin{cases} 1 & \text{if } t \leq 1/c_1 \\ 0 & \text{if } t \geq 2/c_1 \end{cases}$$

It is clear that $h_P \in \mathbb{C}^\infty(\mathbb{G})$ and $h_P(G) \in [0,1]$. More precisely,

$$h_P(G) = \begin{cases} 1 & \text{for } G \in U(P, 5r(P)) \\ 0 & \text{for } G \in \mathbb{G} - U(P, 10\theta r(P)) \end{cases}$$

As a consequence, there exists $c_3 > 0$ such that

$$|X_j h_P(G)| \le \frac{c_3}{r(P)} \le \frac{3c_3}{2r(G)}, \quad \text{for } j = 1, \dots, m_1 \text{ and for all } G \in \mathbb{G} \tag{2.9}$$

Indeed $|X_j h_P(G)| \neq 0$ only if $G \in U(P, 10\theta r(P)) - U(P, 5r(P))$ and

$$\frac{1}{2} r(P) \le r(G) \le \frac{3}{2} r(P) \quad \text{for all } G \in U(P, 10\theta r(P)) - U(P, 5r(P)).$$

Here we used the following fact: if $\operatorname{dist}(P, \mathcal{F}), \operatorname{dist}(G, \mathcal{F}) \le 1$, then

$$20\theta r(P) \le \operatorname{dist}(P, \mathcal{F}) \le d(G, P) + \operatorname{dist}(G, \mathcal{F}) \le 10\theta r(P) + 20\theta r(G),$$

that gives $\frac{1}{2} r(P) \le r(G)$ and

$$20\theta r(G) \le \operatorname{dist}(G, \mathcal{F}) \le d(G, P) + \operatorname{dist}(P, \mathcal{F}) \le 10\theta r(P) + 20\theta r(P),$$

that gives $r(G) \le \frac{3}{2} r(P)$ for all $G \in U(P, 10\theta r(P)) - U(P, 5r(P))$.

Now we define $\sigma : \mathbb{G} \to \mathbb{R}$ as

$$\sigma(G) := \sum_{P \in \mathcal{N}} h_P(G).$$

Because $h_P = 0$ on $U(Q, 10\theta r(Q))$ if $P \notin \mathcal{N}_Q$, then the above sum is locally finite and so

$$\sigma = \sum_{P \in \mathcal{N}_Q} h_P \quad \text{in } U(Q, 10\theta r(Q)).$$

Moreover, from (2.7) and (2.9) we have that

1. $\sigma \in \mathbb{C}^\infty(\mathbb{G} - \mathcal{F})$.
2. $\sigma \ge 1$ on $\mathbb{G} - \mathcal{F}$.
3. there is $c_4 > 0$ such that $|X_j \sigma(G)| \le \frac{c_4}{r(G)}$, for all $G \in \mathbb{G} - \mathcal{F}$ and $j = 1, \dots, m_1$.

Hence we can define a partition of unity subordinated to the family $\{U(P, r(P))\}$ as

$$v_P(G) := \frac{h_P(G)}{\sigma(G)}.$$

For all $G \in \mathbb{G} - \mathcal{F}$

1. $\sum_{P\in\mathcal{N}} v_P(G) = \frac{\sigma(G)}{\sigma(G)} = 1$.
2. $\sum_{P\in\mathcal{N}} X_j v_P(G) = 0$ for $j = 1, \dots, m_1$.
3. there is $c_5 > 0$ such that $|X_j v_P(G)| \le \frac{c_5}{r(G)}$, for $P \in \mathcal{N}$ and $j = 1, \dots, m_1$.

Finally the extended function $\hat{f} = (\hat{f}_1, \dots, \hat{f}_k) : \mathbb{G} \to \mathbb{R}^k$ is

$$\hat{f}_i(G) := \begin{cases} f_i(G) & \text{if } G \in \mathcal{F} \\ \sum_{P\in\mathcal{N}} v_P(G)\left(f_i(P') + \langle g_i(P'), (P'^{-1}G)^1\rangle\right) & \text{if } G \in \mathbb{G} - \mathcal{F} \end{cases}$$

for $i = 1, \dots, k$ and where for all $Q \in \mathbb{G}$, Q' is one of the points of $\mathcal{F}$ such that $\operatorname{dist}(Q, \mathcal{F}) = d(Q, Q')$.

Step 3. We show that

$$\nabla_{\mathbb{G}} \hat{f}(Q) = g(Q), \qquad \text{for all } Q \in \mathcal{F}. \tag{2.10}$$

Fix $i = 1, \dots, k$ and $Q \in \mathcal{F}$. It is sufficient to prove that

$$|\hat{f}_i(G) - \hat{f}_i(Q) - \langle g_i(Q), (Q^{-1}G)^1\rangle| = o(d(G, Q)) \tag{2.11}$$

for $d(G, Q) \to 0$. We have two case: $G \in \mathcal{F}$ or $G \in \mathbb{G} - \mathcal{F}$. In the first case, (2.11) follows from (2.5). On the other hand, if $G \in \mathbb{G} - \mathcal{F}$ then

$$\begin{aligned}
&\left|\sum_{P\in\mathcal{N}} v_P(G)\left(f_i(P') + \langle g_i(P'), (P'^{-1}G)^1\rangle\right) - f_i(Q) - \langle g_i(Q), (Q^{-1}G)^1\rangle\right| \\
&\le \left|\sum_{P\in\mathcal{N}} v_P(G)\left(f_i(P') - f_i(Q) - \langle g_i(Q), (Q^{-1}P')^1\rangle\right)\right| \\
&\quad + \left|\sum_{P\in\mathcal{N}} v_P(G)\left(\langle g_i(P'), (P'^{-1}G)^1\rangle + \langle g_i(Q), (Q^{-1}P')^1\rangle - \langle g_i(Q), (Q^{-1}G)^1\rangle\right)\right| \\
&=: I_1 + I_2
\end{aligned}$$

Here

$$I_1 = o(d(Q, G)). \tag{2.12}$$

Indeed by (2.5) we have that $I_1 = o(d(Q, P'))$ and so (2.12) is true because if $P \in \mathcal{N}_G$, then

$$\begin{aligned}
d(Q, P') &\le d(Q, P) + d(P, P') \\
&\le 2d(Q, P) \\
&\le 2[d(Q, G) + d(G, P)] \\
&\le 2[d(Q, G) + 10\theta(r(G) + r(P))] \\
&\quad \text{(using (2.8))} \\
&\le 2[d(Q, G) + 40\theta r(G)] \\
&\le (2 + 4\theta) d(Q, G).
\end{aligned} \tag{2.13}$$

Hence it remains to estimate I_2. Notice that

$$\langle g_i(Q), (Q^{-1}P')^1\rangle - \langle g_i(Q), (Q^{-1}G)^1\rangle = -\langle g_i(Q), (P'^{-1}G)^1\rangle$$

and

$$(G^{-1}P')^1 \le d(G,P') \le (3+4\theta)d(Q,G). \tag{2.14}$$

Then recalling that $\sum_{P\in\mathcal{N}} v_P(G) = 1$ and using (2.13) and the continuity of g_i, we deduce

$$I_2 \le \left| \sum_{P\in\mathcal{N}} v_P(G)\left(\langle g_i(P') - g_i(Q), (P'^{-1}G)^1\rangle\right)\right| \le d(G,P')|g_i(P') - g_i(Q)| = o(d(Q,G)).$$

Consequently (2.11) is true and (2.10) is satisfied.

Step 4. We prove that $\hat f \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^k)$.

Fix $i = 1,\dots,k$. Because $\hat f_i \in \mathbb{C}^\infty(\mathbb{G}-\mathcal{F})$, we conclude that $\hat f_i \in \mathbb{C}^1(\mathbb{G}-\mathcal{F}) \subset \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}-\mathcal{F})$ (see Remark 2.3.6). Moreover by (2.5) $\hat f_i$ is intrinsic differentiable (see Section 2.6.2), hence continuous, in $\mathcal{F}$.

As a consequence it remains to show that $X_j\hat f_i$ are continuous in $\mathcal{F}$. Fix $j = 1,\dots,m_1$ and $Q \in \mathcal{F}$. By (2.10), if $G \in \mathcal{F}$ then

$$|X_j\hat f_i(Q) - X_j\hat f_i(G)| = |g_{ij}(Q) - g_{ij}(G)| \to 0, \quad \text{as } d(G,Q)\to 0.$$

where $g_{ij}(Q)$ is j-th component of $g_i(Q) = (g_{i1}(Q),\dots,g_{im_1}(Q))$. On the other hand, if $G \in \mathbb{G} - \mathcal{F}$ then

$$\begin{aligned}|X_j\hat f_i(G) - X_j\hat f_i(Q)| &= |X_j\hat f_i(G) - g_{ij}(Q)|\\ &\le |X_j\hat f_i(G) - g_{ij}(G')| + |g_{ij}(G') - g_{ij}(Q)|\end{aligned}$$

Since g_i is continuous and $d(G',Q) \le d(G',G) + d(G,Q) \le 2d(G,Q)$ we have that

$$|g_{ij}(G') - g_{ij}(Q)| \to 0 \quad \text{as } d(G,Q)\to 0.$$

Moreover

$$\begin{aligned}|X_j\hat f_i(G) - g_{ij}(G')| &= \left|\sum_{P\in\mathcal{N}_G}\left[X_jv_P(G)\left(f_i(P') + \langle g_i(P'), (P'^{-1}G)^1\rangle\right) + v_P(G)(g_{ij}(P') - g_{ij}(G'))\right]\right|\\ &\le R_1 + R_2 + R_3\end{aligned} \tag{2.15}$$

where

$$R_1 := \left| \sum_{P \in \mathcal{N}_G} X_j v_P(G) \left(f_i(P') - f_i(G') - \langle g_i(G'), (G'^{-1}P')^1 \rangle \right) \right|$$

$$R_2 := \left| \sum_{P \in \mathcal{N}_G} X_j v_P(G) \langle g_i(P') - g_i(G'), (P'^{-1}G)^1 \rangle \right|$$

$$R_3 := \left| \sum_{P \in \mathcal{N}_G} v_P(G)(g_{ij}(P') - g_{ij}(G')) \right|$$

We start to consider R_1. Recalling that $|X_j v_P(G)| \leq \frac{c_5}{r(G)}$ and using (2.5) and (2.7)

$$\begin{aligned} R_1 &\leq \sum_{P \in \mathcal{N}_G} |X_j v_P(G)| \left| f_i(P') - f_i(G') - \langle g_i(G'), (G'^{-1}P')^1 \rangle \right| \\ &= o(d(P', G')) \sum_{P \in \mathcal{N}_G} |X_j v_P(G)| \leq o(d(P', G'))(120\theta + 9)^{\mathfrak{q}} \frac{c_5}{r(G)} = c_6 \frac{o(d(P', G'))}{d(G, G')}. \end{aligned}$$

From (2.13) with $Q = G'$, we obtain that $d(G', P') \leq 2(1 + 2\theta) d(G', G)$ and

$$d(P', G') \leq d(P', Q) + d(Q, G') \leq (4 + 4\theta) d(Q, G). \tag{2.16}$$

Then

$$R_1 \leq c_7 \frac{o(d(P', G'))}{d(P', G')} \to 0, \quad \text{as } d(Q, G) \to 0. \tag{2.17}$$

Now we show that

$$R_2 \to 0, \quad \text{as } d(A, G) \to 0. \tag{2.18}$$

By (2.16) and (2.14)

$$R_2 \leq \sum_{P \in \mathcal{N}_G} |X_j v_P(G)| \, |g_i(P') - g_i(G')| \, |(P'^{-1}G)^1| \leq c_7 \epsilon(d(Q, G)) d(Q, G) \sum_{P \in \mathcal{N}_G} |X_j v_P(G)|$$

where $\epsilon(t) \to 0$ as $t \to 0$. Hence because $|X_j v_P(G)| \leq \frac{c_5}{r(G)}$ we obtain $R_2 \leq c_8 40\theta \epsilon(d(Q, G))$, whence (2.18) holds. Finally by (2.16) and the continuity of g_i, we conclude

$$R_3 \to 0, \quad \text{as } d(Q, G) \to 0. \tag{2.19}$$

Then putting together (2.15), (2.17), (2.18) and (2.19), one obtains

$$|X_j \hat{f}_i(G) - g_{ij}(G')| \to 0, \quad \text{as } d(G, Q) \to 0$$

and so $X_j \hat{f}_i$ is continuous in A, as requested. Hence $\hat{f} \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^k)$ and the proof of theorem is complete. □

2.3.3 BV$_{\mathbb{G}}$ functions

The following definitions relate the notion of BV function and the Caccioppoli sets in Carnot groups (see for instance [44], [47], [48]). For the Euclidean theory of BV functions and finite perimeter sets the reader can see [4] and [33].

Definition 2.3.4. We say that $f : \Omega \to \mathbb{R}$ is of bounded $\mathbb{G}$-variation in an open set $\Omega \subset \mathbb{G}$ and we write $f \in BV_{\mathbb{G}}(\Omega)$, if $f \in \mathcal{L}^1(\Omega)$ and

$$\|\nabla_{\mathbb{G}} f\|(\Omega) := \sup \left\{ \int_\Omega f \operatorname{div}_{\mathbb{G}} \phi \, d\mathcal{L}^N : \phi \in \mathbb{C}^1_c(\Omega, H\mathbb{G}), |\phi(P)| \le 1 \right\} < +\infty.$$

Analogously the space $BV_{\mathbb{G},loc}(\Omega)$ is defined in the usual way.

Theorem 2.3.9 (Compactness)**.** *$BV_{\mathbb{G},loc}(\mathbb{G})$ is compactly embedded in $\mathcal{L}^p_{loc}(\mathbb{G})$ for $1 \le p < \frac{\mathfrak{q}}{\mathfrak{q}-1}$, where $\mathfrak{q}$ is the homogeneous dimension of $\mathbb{G}$ defined as* (1.22)*.*

As in the Euclidean case, an important property of $BV_{\mathbb{G}}$ functions is the lower semicontinuity of the $\mathbb{G}$-variation with respect to the $\mathcal{L}^1_{loc}$ convergence:

Theorem 2.3.10. *Let $f, f_k \in \mathcal{L}^1(\Omega)$ be such that $f_k \to f$ in $\mathcal{L}^1_{loc}(\Omega)$. Then*

$$\|\nabla_{\mathbb{G}} f\|(\Omega) \le \liminf_{k\to\infty} \|\nabla_{\mathbb{G}} f_k\|(\Omega).$$

In the setting of Carnot groups, the structure theorem for $BV_{\mathbb{G}}$ functions reads as follows.

Theorem 2.3.11. *If $f \in BV_{\mathbb{G},loc}(\Omega)$ then $\|\nabla_{\mathbb{G}} f\|$ is a Radon measure on Ω. Moreover, there is a $\|\nabla_{\mathbb{G}} f\|$ measurable horizontal section $\sigma_f : \Omega \to H\mathbb{G}$ such that $|\sigma_f(P)| = 1$ for $\|\nabla_{\mathbb{G}} f\|$-a.e. $P \in \Omega$ and*

$$\int_\Omega f \, div_{\mathbb{G}} \xi \, d\mathcal{L}^N = \int_\Omega \langle \xi, \sigma_f \rangle \, d\|\nabla_{\mathbb{G}} f\|,$$

for every $\xi \in \mathbb{C}^1_0(\Omega, H\mathbb{G})$. Finally the notion of gradient $\nabla_{\mathbb{G}}$ can be extended from regular functions to functions $f \in BV_{\mathbb{G}}$ defining $\nabla_{\mathbb{G}} f$ as the vector valued measure

$$\nabla_{\mathbb{G}} f := -\sigma_f \llcorner \|\nabla_{\mathbb{G}} f\| = (-(\sigma_f)_1 \llcorner \|\nabla_{\mathbb{G}} f\|, \dots, -(\sigma_f)_{m_1} \llcorner \|\nabla_{\mathbb{G}} f\|),$$

where $(\sigma_f)_i$ are the components of σ_f with respect to the base X_i.

Following [30] and [31], we give a classical definition of sets of finite perimeter:

Definition 2.3.5. A set $E \subset \mathbb{G}$ is a locally finite $\mathbb{G}$-perimeter set, or a $\mathbb{G}$-Caccioppoli set, if $\chi_E \in BV_{\mathbb{G},loc}(\mathbb{G})$, where χ_E is the characteristic function of the set E. In this case the measure $\|\nabla_{\mathbb{G}} \chi_E\|$ will be called $\mathbb{G}$-perimeter measure of E and will be denoted by $|\partial E|_{\mathbb{G}}$. Moreover we call generalized intrinsic normal of ∂E in Ω the vector $\nu_E(P) := -\sigma_{\chi_E}(P)$.

The $\mathbb{G}$-perimeter is invariant under group translations, that is

$$|\partial E|_{\mathbb{G}}(\mathcal{A}) = |\partial(\tau_P E)|_{\mathbb{G}}(\tau_P \mathcal{A}),$$

for every $P \in \mathbb{G}$ and for any Borel set $\mathcal{A} \subset \mathbb{G}$. Moreover recalling that $\mathfrak{q}$ is the homogeneous dimension of $\mathbb{G}$ given by (1.22), then the $\mathbb{G}$-perimeter is homogeneous of degree $\mathfrak{q}-1$ respect to the dilations, that is

$$|\partial(\delta_\lambda E)|_{\mathbb{G}}(\mathcal{A}) = \lambda^{1-\mathfrak{q}} |\partial E|_{\mathbb{G}}(\delta_\lambda \mathcal{A}),$$

for any Borel set $\mathcal{A} \subset \mathbb{G}$.

Fundamental estimates in geometric measure theory are the so-called relative and global isoperimetric inequalities for Caccioppoli sets. The proof is established in [48], Theorem 1.18.

Theorem 2.3.12. *There exists a constant $C > 0$ such that for any $\mathbb{G}$-Caccippoli set $E \subset \mathbb{G}$, for every $P \in \mathbb{G}$ and $r > 0$*

$$\min\{\mathcal{L}^N\left(E \cap U_{cc}(P,r)\right), \mathcal{L}^N\left(U_{cc}(P,r) - E\right)\}^{(\mathfrak{q}-1)/\mathfrak{q}} \leq C|\partial E|_{\mathbb{G}}(U_{cc}(P,r))$$

and

$$\min\{\mathcal{L}^N\left(E\right), \mathcal{L}^N\left(\mathbb{G} - E\right)\}^{(\mathfrak{q}-1)/\mathfrak{q}} \leq C|\partial E|_{\mathbb{G}}(\mathbb{G})$$

where U_{cc} denotes the open ball with respect to the cc-distance d_{cc}.

The perimeter measure is concentrated in a subset of topological boundary of E, the so-called reduced boundary $\partial^*_{\mathbb{G}} E$.

Definition 2.3.6 (Reduced boundary)**.** Let $E \subset \mathbb{G}$ be a $\mathbb{G}$-Caccioppoli set. We say that $P \in \partial^*_{\mathbb{G}} E$ if

1. $|\partial E|_{\mathbb{G}}(U_{cc}(P,r)) > 0$, for all $r > 0$
2. there exists $\lim_{r\to 0} \fint_{U_{cc}(P,r)} \nu_E \, d|\partial E|_{\mathbb{G}}$
3. $\left|\lim_{r\to 0} \fint_{U_{cc}(P,r)} \nu_E \, d|\partial E|_{\mathbb{G}}\right| = 1$

The reduce boundary of a set $E \subset \mathbb{G}$ is invariant under group translations, i.e.

$$Q \in \partial^*_{\mathbb{G}} E \qquad \text{if and only if} \qquad \tau_P Q \in \partial^*_{\mathbb{G}}(\tau_P E)$$

and also

$$\nu_E(Q) = \nu_{\tau_P E}(\tau_P Q).$$

Lemma 2.3.13 (Differentiation Lemma, [3])**.** *If $E \subset \mathbb{G}$ is a $\mathbb{G}$-Caccioppoli set, then*

$$\lim_{r\to 0} \fint_{U_{cc}(P,r)} \nu_E \, d|\partial E|_{\mathbb{G}} = \nu_E(P), \quad \text{for } |\partial E|_{\mathbb{G}}\text{-a.e. } P$$

and consequently $|\partial E|_{\mathbb{G}}$-a.e. $P \in \mathbb{G}$ belongs to reduced boundary $\partial^*_{\mathbb{G}} E$.

The perimeter measure equals a constant times the spherical $(\mathfrak{q}-1)$-dimensional Hausdorff measure restricted to the reduced boundary, indeed

Theorem 2.3.14 ([40], Theorem 4.18)**.** *Let $\mathbb{G}$ be a Carnot group of step 2, endowed with the invariant distance d associated to the homogeneous norm* (1.30)*. Let $E \subset \mathbb{G}$ be a $\mathbb{G}$-Caccioppoli set. Then*

$$|\partial E|_{\mathbb{G}} = c\,\mathcal{S}^{\mathfrak{q}-1} \llcorner \partial^*_{\mathbb{G}} E$$

where $\mathcal{S}^{\mathfrak{q}-1}$ denotes the $(\mathfrak{q}-1)$-dimensional spherical Hausdorff measure.

Finally, as it is usual in the literature, we can also define the measure theoretic boundary $\partial_{*,\mathbb{G}} E$:

Definition 2.3.7. Let $E \subset \mathbb{G}$ be a measurable set. We say that P belongs to measure theoretic boundary $\partial_{*,\mathbb{G}} E$ of E if

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^N(E \cap U_{cc}(P,r))}{\mathcal{L}^N(U_{cc}(P,r))} > 0 \qquad \text{and} \qquad \limsup_{r\to 0^+} \frac{\mathcal{L}^N(E^c \cap U_{cc}(P,r))}{\mathcal{L}^N(U_{cc}(P,r))} > 0.$$

If $E \subset \mathbb{G}$ is $\mathbb{G}$-Caccioppoli set, then

$$\partial^*_{\mathbb{G}} E \subset \partial_{*,\mathbb{G}} E \subset E.$$

Moreover, $\mathcal{S}^{\mathfrak{q}-1}(\partial_{*,\mathbb{G}} E - \partial^*_{\mathbb{G}} E) = 0$.

2.4 $\mathbb{G}$-regular surfaces

2.4.1 $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surfaces

A notion of intrinsic regular surfaces with different topological dimension, is introduced and studied in Heisenberg groups by Franchi, Serapioni and Serra Cassano (see [44], [45], [46]) and then extended in Carnot groups [70] (see also [69], [68]).

Following [70], we have the following definitions

Definition 2.4.1. Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be Carnot groups such that $\mathbb{G}^{(2)}$ is an H-quotient of $\mathbb{G}^{(1)}$.

We say that a subset $S \subset \mathbb{G}^{(1)}$ is $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$*-regular surface of* $\mathbb{G}^{(1)}$ if for all $P \in S$, there exists an open neighborhood $\mathcal{U}$ of P and a continuously P-differentiable mapping $F : \mathcal{U} \to \mathbb{G}^{(2)}$ such that

$$S \cap \mathcal{U} = F^{-1}(0)$$

and the P-differential of F in P is an H-epimorphism from $\mathbb{G}^{(1)}$ to $\mathbb{G}^{(2)}$ for every $P \in \mathcal{U}$.

Definition 2.4.2. Let $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ be Carnot groups such that $\mathbb{G}^{(1)}$ H-embeds into $\mathbb{G}^{(2)}$.

We say that a subset $S \subset \mathbb{G}^{(2)}$ is $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$*-regular surface of* $\mathbb{G}^{(2)}$ if for all $P \in S$, there exists an open neighborhood $\mathcal{U} \subset \mathbb{G}^{(2)}$ of P and $\mathcal{V} \subset \mathbb{G}^{(1)}$ of $0 \in \mathbb{G}^{(1)}$ along with a continuously P-differentiable topological embedding $F : \mathcal{V} \to \mathbb{G}^{(2)}$, such that

$$S \cap \mathcal{U} = F(\mathcal{V})$$

and the P-differential of F in Q is an H-monomorphism from $\mathbb{G}^{(1)}$ to $\mathbb{G}^{(2)}$ for every $Q \in \mathcal{V}$.

Definition 2.4.3. When a subset S is either $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular in $\mathbb{G}^{(1)}$ or $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular in $\mathbb{G}^{(2)}$, we simply say that it is $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular without further specification, or we can say that it is an intrinsically regular surface.

Remark 2.4.1. We stress that, in [70], $\mathbb{G}^{(2)}$ is only *graded group*, i.e. a connected, simply connected (and finite dimensional) Lie group, whose Lie algebra $\mathfrak{g}$ admits a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_\kappa$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for $1 \leq i, j \leq \kappa$.

It is clear that a Carnot group is a graded group. However, when we consider the $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface of $\mathbb{G}^{(1)}$, thanks to Proposition 8.3 in [70], it always follows $\mathbb{G}^{(2)}$ is a Carnot group. This is *not* the case of $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface of $\mathbb{G}^{(2)}$.

Now we can define the “tangent cone” to $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface as in [34], i.e.

Definition 2.4.4. Let $\mathbb{G}$ be a Carnot group and let $S \subset \mathbb{G}$ with $P \in \mathbb{G}$. The homogeneous tangent cone of S at P is the homogeneous subset

$$\begin{aligned} \mathrm{Tan}(S,P) =\Big\{ & Q \in \mathbb{G} \ \Big| \ Q = \lim_{h\to\infty} \delta_{\lambda_h}(P^{-1}P_h), \ \text{ for some sequences } (\lambda_h)_h \subset \mathbb{R}^+, \\ & (P_h)_h \subset S, \ \text{ where } P_h \to P \Big\} \end{aligned}$$

Proposition 2.4.2 ([70], Corollary 10.3)**.** *Let S be a $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface of $\mathbb{G}^{(1)}$. Then for all $P \in S$*

$$Hdim(Tan(S,P)) = Hdim(\mathbb{G}^{(1)}) - Hdim(\mathbb{G}^{(2)})$$

We observe that if $P \in \bar{S}$, then $\mathrm{Tan}(S,P)$ is not empty.

Proposition 2.4.2 suggests that the Hausdorff dimension of a $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface of $\mathbb{G}^{(1)}$ should coincide with that of its homogeneous tangent cones, but this problem still claims to be investigated.

Moreover the fact that tangent cones to a $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface of $\mathbb{G}^{(1)}$ have a fixed Hausdorff dimension does not mean that they are all algebraically H-isomorphic (see Example 10.4 in [70]).

It is apparent that the notion of $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regularity in higher codimension allows us a certain amount of freedom in the choice of $\mathbb{G}^{(2)}$, but not all codomains are “good” to be considered. For instance, the family of $(\mathbb{H}^n, \mathbb{H}^k)$-regular surfaces is empty whenever $n > k$. This follows by the fact that there are no surjective H-linear maps between $\mathbb{H}^n$ onto $\mathbb{H}^k$ (Proposition 2.2.10).

Quite precisely in [46], the authors show

Theorem 2.4.3. *The only intrinsically regular surfaces of the Heisenberg group $\mathbb{H}^n$ are contained in the following list*

1. $(\mathbb{H}^n, \mathbb{R}^k)$*-regular surfaces, for* $k = 1, \dots, n$.
2. $(\mathbb{R}^k, \mathbb{H}^n)$*-regular surfaces, for* $k = 1, \dots, n$.

Theorem 2.4.4 ([70], Theorem 12.6)**.** *The only intrinsically regular surfaces of the complexified Heisenberg group $\mathbb{H}^1_2$ (see Example* 1.4.5*) are contained in the following list*

1. $(\mathbb{H}^1_2, \mathbb{R}^k)$*-regular surfaces, for* $k = 1, 2$.
2. $(\mathbb{R}^k, \mathbb{H}^1_2)$*-regular surfaces, for* $k = 1, 2$.

2.4.2 $\mathbb{G}$-regular surfaces

Here we study the $(\mathbb{G}, \mathbb{R}^k)$-regular surfaces with $k \le m_1$ (see Proposition 2.2.8). We simply call them k-codimensional $\mathbb{G}$-regular surfaces.

If $\mathbb{G}^{(2)} = \mathbb{R}^k$, Definition 2.4.1 takes the form

Definition 2.4.5. $S \subset \mathbb{G}$ is a k-codimensional $\mathbb{G}$-regular surface if for every $P \in S$ there exist a neighborhood $\mathcal{U}$ of P and a function $F \in \mathbb{C}^1_{\mathbb{G}}(\mathcal{U}, \mathbb{R}^k)$ such that

$$S \cap \mathcal{U} = \{Q \in \mathcal{U} : F(Q) = 0\}$$

and the $(k \times m_1)$ matrix $\nabla_{\mathbb{G}} F(Q)$ has rank k for all $Q \in \mathcal{U}$.

Remark 2.4.5. The condition that rank of $\nabla_{\mathbb{G}} F(Q)$ is maximum for every $Q \in \mathcal{U}$ is equivalent to suppose that the P-differential of F at P is surjective for every $Q \in \mathcal{U}$.

If $k = 1$ we say that $S \subset \mathbb{G}$ is a $\mathbb{G}$-regular hypersurface (i.e. topological codimension 1 surface).

We stress that the class of $\mathbb{G}$-regular surfaces is deeply different from the class of Euclidean regular surfaces. In [59], the authors give an example of $\mathbb{G}$-regular surfaces in $\mathbb{H}^1 = \mathbb{R}^3$ that are (Euclidean) fractal sets. Conversely in $\mathbb{R}^3 = \mathbb{H}^1$, the Euclidean plane $\{P = (p_1, p_2, p_3) : p_3 = 0\}$ is a simply example of smooth Euclidean submanifold, that isn't $\mathbb{G}$-regular hypersurfaces at origin.

Definition 2.4.6. $\Gamma \subset \mathbb{G}$ is $(\mathfrak{q}-1)$-dimensional $\mathbb{G}$-rectifiable if there exists a sequence of $\mathbb{G}$-regular hypersurface $(S_h)_{h\in\mathbb{N}}$ such that

$$\mathcal{H}^{\mathfrak{q}-1}\left(\Gamma - \bigcup_{h\in\mathbb{N}} S_h\right) = 0.$$

Both the notions of $\mathbb{G}$-regular surfaces and of $\mathbb{G}$-rectifiable sets are independent of the chosen invariant distance d. Indeed, from the equivalence of invariant distances (see Proposition 1.3.15) and from Definition 2.3.2 and Proposition 2.3.4, if S is a $\mathbb{G}$-regular surface with respect to an invariant distance d so it is with respect to any other invariant distance. The same holds if S is a $\mathbb{G}$-rectifiable set.

In [47], the rectifiability theorem is proved for step 2 Carnot groups and Blow-up Theorem is the main key of the proof and also the reason of the restriction to step 2. Indeed there is a counterexample regarding a particular step 3 Carnot group, called Engels groups, for which Blow-up Theorem does not hold (see Example 3.2 in [47]). Recently in [71] the author extends Blow-up Theorem (and Rectifiability Theorem) in a much larger class of step 2 Carnot groups, called Carnot groups of type $\star$.

Theorem 2.4.6 (Blow-up Theorem)**.** *Let $\mathbb{G}$ be a Carnot group of step 2 and let $E \subset \mathbb{G}$ be a set with locally finite $\mathbb{G}$-perimeter. If $P \in \partial^*_{\mathbb{G}} E$ then*

$$\lim_{r\to 0} \chi_{E_{r,P}} = \chi_{S^+_{\mathbb{G}}(\nu_E(P))} \quad \textit{in } \mathcal{L}^1_{loc}(\mathbb{G})$$

where $E_{r,P} := \delta_{1/r}(\tau_{P^{-1}}E) = \{Q \, : \, \tau_P(\delta_r(Q)) \in E\}$ and

$$S^+_{\mathbb{G}}(\nu_E(P)) := \{Q = (q^1, q^2) \in \mathbb{G} \,|\, \langle \nu_E(P), q^1 \rangle \geq 0\}.$$

Moreover for all $R > 0$

$$\lim_{r\to 0} |\partial E_{r,P}|_{\mathbb{G}}(U_{cc}(0,R)) = |\partial S^+_{\mathbb{G}}(\nu_E(P))|_{\mathbb{G}}(U_{cc}(0,R))$$

and

$$|\partial S^+_{\mathbb{G}}(\nu_E(P))|_{\mathbb{G}}(U_{cc}(0,R)) = \mathcal{H}^{N-1}(T^g_{\mathbb{G}}(\nu_E(0)) \cap U_{cc}(0,R))$$

where $T^g_{\mathbb{G}}(\nu_E(0)) := \{Q = (q^1, q^2) \in \mathbb{G} \,|\, \langle \nu_E(0), q^1 \rangle = 0\}$ is the topological boundary of $S^+_{\mathbb{G}}(\nu_E(0))$.

Theorem 2.4.7 (Rectifiability Theorem)**.** *Let $\mathbb{G}$ be a Carnot group of step 2 and let $E \subset \mathbb{G}$ be a set with locally finite $\mathbb{G}$-perimeter. Then*

$$\partial^*_{\mathbb{G}} E \textit{ is } (\mathfrak{q}-1)\textit{-dimensional } \mathbb{G}\textit{-rectifiable},$$

that is $\partial^*_{\mathbb{G}} E = N \bigcup_{h\in\mathbb{N}} K_h$, where $\mathcal{H}^{\mathfrak{q}-1}(N) = 0$ and K_h is a compact subset of a $\mathbb{G}$-regular hypersurfaces S_h. Moreover $\nu_E(P)$ is the $\mathbb{G}$ normal to S_h at P, for all $P \in K_h$.

2.5 Intrinsic graphs

The intrinsic graphs came out naturally in [44], while studying level sets of Pansu differentiable functions from $\mathbb{H}^k$ to $\mathbb{R}$. They have been introduced with different degrees of generality in [9], [13], [23], [38], [41], [46].

Definition 2.5.1. Let $\mathbb{M}$ be a homogeneous subgroup of $\mathbb{G}$. We say that a set $S \subset \mathbb{G}$ is an intrinsic (left) $\mathbb{M}$-graph if S intersects each left coset of $\mathbb{M}$ in one point at most.

If $\mathbb{W}$ and $\mathbb{M}$ are complementary homogeneous subgroups in $\mathbb{G}$, there is a one to one correspondence between left coset of $\mathbb{M}$ and a points of $\mathbb{W}$, hence S is a left $\mathbb{M}$-graph if and only if there is $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ such that

$$S = \operatorname{graph}(\phi) := \{A \cdot \phi(A) \mid A \in \mathcal{E}\}.$$

By uniqueness of the components along $\mathbb{W}$ and $\mathbb{M}$, if $S = \operatorname{graph}(\phi)$ then ϕ is uniquely determined among all functions from $\mathbb{W}$ to $\mathbb{M}$. Hence the very existence of intrinsic graphs depends on the possibility of splitting $\mathbb{G}$ as a product of complementary subgroups hence it depends on the structure of the algebra $\mathfrak{g}$.

We call graph map of ϕ, the function $\Phi : \mathcal{E} \to \mathbb{G}$ defined as

$$\Phi(A) := A \cdot \phi(A) \quad \forall A \in \mathcal{E}. \tag{2.20}$$

Hence $S = \Phi(\mathcal{E})$ is equivalent to $S = \operatorname{graph}(\phi)$.

Let us point out that an intrinsic regular graph can be very irregular from the Euclidean point of view: indeed, there are examples of intrinsic regular graphs in $\mathbb{H}^1$ which are fractal sets in the Euclidean sense [59].

The concept of intrinsic graph is preserved by translation and dilation, i.e.

Proposition 2.5.1 (see Proposition 2.2.18 in [39])**.** *If S is an intrinsic graph then, for all $\lambda > 0$ and for all $Q \in \mathbb{G}$, $\tau_Q S$ and $\delta_\lambda S$ are intrinsic graphs. In particular, if $S = \operatorname{graph}(\phi)$ with $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$, then*

1. *For all $\lambda > 0$, $\delta_\lambda S = \operatorname{graph}(\phi_\lambda)$ with $\phi_\lambda : \delta_\lambda \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ and $\phi_\lambda(A) := \delta_\lambda \phi(\delta_{1/\lambda} A)$, for $A \in \delta_\lambda \mathcal{E}$.*
2. *For any $Q \in \mathbb{G}$, $\tau_Q S = \operatorname{graph}(\phi_Q)$ where $\phi_Q : \mathcal{E}_Q \subset \mathbb{W} \to \mathbb{M}$, $\mathcal{E}_Q = \{A : \mathbf{P}_\mathbb{W}(Q^{-1}A) \in \mathcal{E}\}$ and $\phi_Q(A) := (\mathbf{P}_\mathbb{M}(Q^{-1}A))^{-1} \phi(\mathbf{P}_\mathbb{W}(Q^{-1}A))$, for all $A \in \mathcal{E}_Q$.*

Remark 2.5.2. From Proposition 2.5.1 and the continuity of the projections $\mathbf{P}_\mathbb{M}$ and $\mathbf{P}_\mathbb{W}$ it follows that the continuity of a function is preserved by translations. Precisely given $Q \in \mathbb{G}$ and $\phi : \mathbb{W} \to \mathbb{M}$, then the translated function ϕ_Q is continuous in $A \in \mathbb{W}$ if and only if the function ϕ is continuous in the corresponding point $\mathbf{P}_\mathbb{W}(Q^{-1}A)$.

Remark 2.5.3. The algebraic expression of the translated function $\phi_Q : \mathcal{E}_Q \subset \mathbb{W} \to \mathbb{M}$ is more explicit when $\mathbb{W}, \mathbb{M}$ are normal subgroups of $\mathbb{G}$. Precisely

1. If $\mathbb{W}$ is normal in $\mathbb{G}$ then $\phi_Q(A) = Q_{\mathbb{M}}\phi((Q^{-1}A)_{\mathbb{W}})$, for $A \in \mathcal{E}_Q = Q\mathcal{E}Q_{\mathbb{M}}^{-1}$
2. If $\mathbb{M}$ is normal in $\mathbb{G}$ then $\phi_Q(A) = (Q^{-1}A)_{\mathbb{M}}^{-1}\phi((Q_{\mathbb{W}}^{-1}A)$, for $A \in \mathcal{E}_Q = Q_{\mathbb{W}}\mathcal{E}$
3. If both $\mathbb{W}$ and $\mathbb{M}$ are normal in $\mathbb{G}$ then we get the well known Euclidean formula
$$\phi_Q(A) = Q_{\mathbb{M}}\phi((Q_{\mathbb{W}}^{-1}A), \quad \text{for } A \in \mathcal{E}_Q = Q_{\mathbb{W}}\mathcal{E}.$$

Following [70], we want to give a proof of Implicit Function Theorem. This theorem states that any level set of $\mathbb{C}^1_{\mathbb{G}}$ function can be locally parameterized through the intrinsic graph of a continuous map.

Theorem 2.5.4 (Implicit Function Theorem, see [70], Theorem 1.3)**.** *Let $\mathcal{U}$ be an open subset of $\mathbb{G}$. Let $f \in \mathbb{C}^1_{\mathbb{G}}(\mathcal{U}, \mathbb{R}^k)$ and assume that $\nabla_{\mathbb{G}} f(Q)$ has rank k for all $Q \in \mathcal{U}$. We assume that for a fixed $P \in \mathcal{U}$ there are complementary subgroups $\mathbb{W}$ and $\mathbb{H}$ of $\mathbb{G}$ where $\mathbb{W} = ker(\nabla_{\mathbb{G}} f(P))$. Then there are $\mathcal{I} \subset \mathbb{W}$ and $\mathcal{J} \subset \mathbb{H}$, open and such that $P_{\mathbb{W}} \in \mathcal{I}$ and $P_{\mathbb{H}} \in \mathcal{J}$ and a continuous function $\phi : \mathcal{I} \to \mathcal{J}$ such that*
$$\{Q \in \mathcal{I}\mathcal{J} : \ f(Q) = f(P)\} = \text{graph}\,(\phi)$$
where $\mathcal{I}\mathcal{J} = \{AB \,:\, A \in \mathcal{I}, B \in \mathcal{J}\}$. In particular, the mapping ϕ is $1/\kappa$ Hölder continuous.

Now we show some basic notions about the degree theory (see [65], [92]) which is crucial for the proof of Theorem 2.5.4.

Definition 2.5.2. Let $\mathcal{U}$ be an open bounded set of $\mathbb{R}^k$ and let $F \in \mathbb{C}^1(\bar{\mathcal{U}}, \mathbb{R}^k)$ and $A \notin F(Z_F) \cap F(\partial\mathcal{U})$, where Z_F is the set of critical points of F, i.e. points at which the Jacobian J_F of F vanishes. Then the degree of F at A with respect to $\mathcal{U}$ is defined by
$$\deg(F, \mathcal{U}, A) := \sum_{x \in F^{-1}(A)} \text{sgn}\,(J_F(x)),$$
where $\text{sgn}(t) = 1$ for $t > 0$ and $\text{sgn}(t) = -1$ for $t < 0$.

This definition can be extended to the continuous case as follows.

Definition 2.5.3. Let $F \in \mathbb{C}(\bar{\mathcal{U}}, \mathbb{R}^k)$ and $A \notin F(Z_F) \cap F(\partial\mathcal{U})$. We define $\deg(F, \mathcal{U}, A)$, the degree of F at A with respect to $\mathcal{U}$, to be $\deg(\tilde{F}, \mathcal{U}, A)$ for any $\tilde{F} \in \mathbb{C}^1(\bar{\mathcal{U}}, \mathbb{R}^k)$ such that $|F(x) - \tilde{F}(x)| < dist(A, \partial\mathcal{U})$ for every $x \in \bar{\mathcal{U}}$. Here $\text{dist}(A, \partial\mathcal{U}) := \inf\{d(A, A') : A' \in \partial\mathcal{U}\}$.

Note that the last definition is well defined because if $\tilde{F}_1, \tilde{F}_2 \in \mathbb{C}^1(\bar{\mathcal{U}}, \mathbb{R}^k)$ are such that $|F(x) - \tilde{F}_1(x)|, |F(x) - \tilde{F}_2(x)| < dist(A, \partial\mathcal{U})$ for every $x \in \bar{\mathcal{U}}$, then $\deg(\tilde{F}_1, \mathcal{U}, A) = \deg(\tilde{F}_2, \mathcal{U}, A)$.

Finally we need the following lemma:

Lemma 2.5.5 (Mean Value Inequality, see [70], Theorem 1.2)**.** *Let $\mathcal{U}$ be an open subset of $\mathbb{G}$ and let $f \in \mathbb{C}^1_{\mathbb{G}}(\mathcal{U}, \mathbb{R}^k)$. Let $\mathcal{U}_1, \mathcal{U}_2$ be open subsets of $\mathbb{G}$ such that $\mathcal{U}_2 \Subset \mathbb{G}$ and*

$$\{P \in \mathbb{G} \,:\, dist(P, \mathcal{U}_1) \le c(\mathbb{G}) diam(\mathcal{U}_1)\} \subset \mathcal{U}_2 \tag{2.21}$$

where $c(\mathbb{G}) > 0$ is a suitable constant given by Lemma 4.6 in [70]. Then there is a constant C only depending on $\mathbb{G}$, on $c(\mathbb{G})$ and on the modulus of continuity β of f on $\bar{\mathcal{U}}_2$, i.e.

$$\beta(t) = \max_{\substack{P,Q \in \bar{\mathcal{U}}_2 \\ \|P^{-1}Q\| \le t}} |f(P) - f(Q)|$$

such that

$$\frac{|f(P) - f(Q) - \nabla_{\mathbb{G}} f(P)\,(P^{-1}Q)\,|}{\|P^{-1}Q\|} \le C$$

for every $P, Q \in \bar{\mathcal{U}}_1$ with $P \ne Q$.

We presently have all the tools to prove Theorem 2.5.4.

Proof. Fix $P \in \mathbb{G}$ and let $P_{\mathbb{W}}$ and $P_{\mathbb{H}}$ the unique components of P along $\mathbb{W}$ and $\mathbb{H}$ respectively. We show only the existence of ϕ.

Step 1. Let $t, s > 0$ and let $\mathcal{I} := B(P_{\mathbb{W}}, t) \cap \mathbb{W}$, $\mathcal{J} := B(P_{\mathbb{H}}, s) \cap \mathbb{H}$ such that $\mathcal{I}, \mathcal{J} \subset U(P, r_0)$. Here $r_0 > 0$ is a constant which will be specified later.

We would like to prove that there is $\alpha > 0$ such that

$$\alpha^{-1} \|B'^{-1}B\| \le |f(AB') - f(AB)| \le \alpha \|B'^{-1}B\| \tag{2.22}$$

for every $A \in \mathcal{I}$ and every $B, B' \in \mathcal{J}$.

By Proposition 2.2.6, the restriction $\nabla_{\mathbb{G}} f(P)_{|\mathbb{H}} : \mathbb{H} \to \mathbb{R}^k$ is an H-isomorphism and so

$$\min_{\substack{B \in \mathbb{H} \\ \|B\| = 1}} |\nabla_{\mathbb{G}} f(P)(B)| > 0.$$

Moreover by the continuity of $\mathcal{U} \times \mathbb{H} \ni (Q, B) \mapsto |\nabla_{\mathbb{G}} f(Q)(B)|$, there is $r > 0$ such that

$$\alpha_1 := \min_{Q \in U(P,r)} \min_{\substack{B \in \mathbb{H} \\ \|B\| = 1}} |\nabla_{\mathbb{G}} f(Q)(B)| > 0 \quad \text{and} \quad \alpha_2 := \max_{Q \in U(P,r)} \max_{\substack{B \in \mathbb{H} \\ \|B\| = 1}} |\nabla_{\mathbb{G}} f(Q)(B)| > 0 \tag{2.23}$$

with $U(P,r) \subset \mathcal{U}$. Now if we consider $r_0 := r/(2+2c(\mathbb{G}))$ where $c(\mathbb{G}) > 0$ is a suitable constant given by Lemma 4.6 in [70], we have that (2.21) is satisfied. Consequently, by Lemma 2.5.5 we conclude that there is $C > 0$ such that

$$|f(G) - f(Q) - \nabla_{\mathbb{G}} f(G)\left(G^{-1}Q\right)| \leq C\|G^{-1}Q\|$$

for every $G, Q \in U(P, r_0)$. Moreover using (2.23) it follows

$$|f(G) - f(Q)| \leq (C + \alpha_1)\|G^{-1}Q\|, \quad \text{for all } Q, G \in B(P, r_0) \tag{2.24}$$

and for every $A \in \mathcal{I}$ and every $B', B \in \mathcal{J}$

$$|f(AB') - f(AB)| \geq (\alpha_2 - C)\|B'^{-1}B\|. \tag{2.25}$$

Here we used the fact $\|B'^{-1}B\| = \|(AB')^{-1}(AB)\|$. Now putting together (2.24) and (2.25) and possibly considering a smaller radius of $\mathcal{J}$, (2.22) follows.

Step 2. For each $A \in \mathcal{I}$ we define the continuous map $F_A : \mathcal{J} \to \mathbb{R}^k$ given by

$$F_A(B) := f(AB) \quad \text{for all } B \in \mathcal{J}.$$

Obviously $F_{P_{\mathbb{W}}}(P_{\mathbb{H}}) = f(P)$ and by (2.22) we have

$$F_{P_{\mathbb{W}}}(B) \neq f(P), \quad \text{for all } B \in \partial\mathcal{J}$$

i.e. $f(P) \notin F_{P_{\mathbb{W}}}(\partial\mathcal{J})$. Moreover using again (2.22) we have that $F_{P_{\mathbb{W}}}$ is injective and by Theorem 3.3.3 in [65]

$$\deg(F_{P_{\mathbb{W}}}, \mathcal{J}, f(P)) \in \{-1, 1\}. \tag{2.26}$$

Now by the continuity of $\mathcal{I} \ni A \mapsto \max_{B\in\partial\mathcal{J}} |f(AB) - f(P_{\mathbb{W}}B)|$, possibly considering a smaller radius of $\mathcal{I}$, we can assume that

$$\max_{B\in\partial\mathcal{J}} |f(AB) - f(P_{\mathbb{W}}B)| < \frac{s}{2\alpha}$$

for every $A \in \mathcal{I}$. Therefore using (2.22), we have that

$$|F_A(B) - f(P)| \geq |F_{P_{\mathbb{W}}}(B) - f(P)| - |F_{P_{\mathbb{W}}}(B) - F_A(B)| > \frac{s}{\alpha} - \frac{s}{2\alpha} = \frac{s}{2\alpha} \tag{2.27}$$

for every $B \in \partial\mathcal{J}$ and every $A \in \mathcal{I}$. Consequently $f(P) \notin F_A(\partial\mathcal{J})$.

Now for any arbitrary $A \in \mathcal{I}$ we define the continuous curve $\gamma : [0,1] \to \mathcal{I}$ given by $\gamma(\lambda) := P_{\mathbb{W}}\delta_\lambda(P_{\mathbb{W}}^{-1}A)$. Notice that since $\mathbb{W}$ is homogeneous subgroup of $\mathbb{G}$, then γ has image in $\mathcal{I}$. Then the map $\Gamma : [0,1] \times \mathcal{J} \to \mathbb{R}^k$ defined as

$$\Gamma(\lambda, B) := f(\gamma(\lambda)B)$$

is a homotopy between $F_{P_{\mathbb{W}}}$ and F_A, i.e. Γ is a continuous map such that $\Gamma(0,B) = f(\gamma(0)B) = F_{P_{\mathbb{W}}}(B)$ and $\Gamma(1,B) = f(\gamma(1)B) = F_A(B)$ for all $B \in \mathcal{J}$. Moreover, by (2.27), $\Gamma(\lambda, B) \neq f(P)$ for each $\lambda \in [0,1]$ and each $B \in \partial\mathcal{J}$. Thus, by the homotopy invariance of degree (see Theorem 3.16 in [92]) and using (2.26), we obtain that

$$\deg\left(F_A, \mathcal{J}, f(P)\right) = \deg\left(F_{P_{\mathbb{W}}}, \mathcal{J}, f(P)\right) \neq 0$$

and consequently there is at least one element $B \in \mathcal{J}$, depending on A, such that $F_A(B) = f(AB) = f(P)$. By the injectivity of F_A, B is unique and so there is a map $\phi : \mathcal{I} \to \mathcal{J}$, uniquely defined as $\phi(A) = B$ and such that

$$f(P) = F_A(B) = F_A(\phi(A)) = f(A\phi(A)).$$

Therefore the proof of the existence of a map ϕ is complete. $\square$

Now we present a general result about the $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surfaces (see Section 2.4.1) which is an immediate consequence of the last theorem when we consider the $\mathbb{G}$-regular surfaces.

Proposition 2.5.6 ([70], Corollary 1.5). *Every $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface is locally an intrinsic graph.*

A consequence of this proposition is that the topological codimension of $(\mathbb{G}^{(1)}, \mathbb{G}^{(2)})$-regular surface corresponds to the topological dimension of $\mathbb{G}^{(2)}$.

2.5.1 Intrinsic Lipschitz graphs

It is well-know that in the Euclidean setting a Lipschitz graph $S := \{(A, \phi(A)) \,:\, A \in \omega\}$, with $\phi : \omega \subset \mathbb{R}^{N-1} \to \mathbb{R}$ can be equivalently defined

1. by means of cones : there exists $C > 0$ such that

$$\mathbb{C}\left((A, \phi(A)); \frac{1}{C}\right) \cap S = \{(A, \phi(A))\}$$

   for each $A \in \omega$, where $\mathbb{C}((A, \phi(A)); \frac{1}{C}) = \{(B,t) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, |B - A| \leq \frac{1}{C}|t - \phi(A)|\}$.

2. in a metric way : there exists $C > 0$ such that $|\phi(A) - \phi(B)| \leq C|A - B|$ for all $A, B \in \omega$.

In this section, first we present the two corresponding definitions in the case of Carnot groups and then we show that these notions are equivalent (see Proposition 2.5.10).

The intrinsic Lipschitz functions in $\mathbb{G}$ are functions, acting between complementary subgroups of $\mathbb{G}$, with graphs non intersecting naturally defined cones. Precisely a $\mathbb{M}$-graph S is said to be an intrinsic Lipschitz $\mathbb{M}$-graph if S intersects intrinsic cones with axis $\mathbb{M}$, fixed opening and vertex on S only in the vertex.

We begin with two definitions of intrinsic (closed) cones. The first one, Definition 2.5.4, is more general because it does not require that $\mathbb{M}$ is a complemented subgroup.

Definition 2.5.4. Let $\mathbb{M}$ be a homogeneous subgroup of $\mathbb{G}$ and $Q \in \mathbb{G}$. The cones $X(Q, \mathbb{M}, \alpha)$ with axis $\mathbb{M}$, vertex Q, opening $\alpha \in [0,1]$ are defined as

$$X(Q, \mathbb{M}, \alpha) = Q \cdot X(0, \mathbb{M}, \alpha)$$

where $X(0, \mathbb{M}, \alpha) = \{P : \mathrm{dist}(P, \mathbb{M}) \le \alpha \|P\|\}$ and $\mathrm{dist}(P, \mathbb{M}) := \inf\{\|P^{-1}Q'\| : Q' \in \mathbb{M}\}$.

Definition 2.5.5. If $\mathbb{W}, \mathbb{M}$ are complementary subgroups in $\mathbb{G}$, $Q \in \mathbb{G}$ and $\beta \ge 0$. We can define the cones $C_{\mathbb{W},\mathbb{M}}(Q, \beta)$ with base $\mathbb{W}$ and axis $\mathbb{M}$, vertex Q, opening β are given by

$$C_{\mathbb{W},\mathbb{M}}(Q, \beta) = Q \cdot C_{\mathbb{W},\mathbb{M}}(0, \beta)$$

where $C_{\mathbb{W},\mathbb{M}}(0, \beta) = \{P : \|P_{\mathbb{W}}\| \le \beta \|P_{\mathbb{M}}\|\}$.

Remark 2.5.7. It is clear that

$$\mathbb{G} = X(0, \mathbb{M}, 1) = \overline{\cup_{\beta>0} C_{\mathbb{W},\mathbb{M}}(0, \beta)} \quad \text{and} \quad C_{\mathbb{W},\mathbb{M}}(0, 0) = X(0, \mathbb{M}, 0) = \mathbb{M}.$$

Moreover for all $\lambda > 0$ we have that $\delta_\lambda(C_{\mathbb{W},\mathbb{M}}(0, \beta_1)) = C_{\mathbb{W},\mathbb{M}}(0, \beta_1)$ and if $0 < \beta_1 < \beta_2$, then

$$C_{\mathbb{W},\mathbb{M}}(Q, \beta_1) \subset C_{\mathbb{W},\mathbb{M}}(Q, \beta_2)$$

The cones $C_{\mathbb{W},\mathbb{M}}(Q, \beta)$ are equivalent to cones $X(Q, \mathbb{M}, \alpha)$:

Proposition 2.5.8. *If $\mathbb{W}, \mathbb{M}$ are complementary subgroups in $\mathbb{G}$ then, for any $\alpha \in (0,1)$ there is $\beta \ge 1$, depending on $\alpha, \mathbb{W}$ and $\mathbb{M}$, such that*

$$C_{\mathbb{W},\mathbb{M}}(Q, 1/\beta) \subset X(Q, \mathbb{M}, \alpha) \subset C_{\mathbb{W},\mathbb{M}}(Q, \beta).$$

For a proof of Proposition 2.5.8 see Proposition 3.1.1 in [39].

Now we introduce the basic definitions of this paragraph.

Definition 2.5.6. Let $\mathbb{M}$ be an homogeneous subgroup, not necessarily complemented in $\mathbb{G}$.

1. An $\mathbb{M}$-graph S is an intrinsic Lipschitz $\mathbb{M}$-graph if there is $\alpha \in (0,1)$ such that,

$$S \cap X(P, \mathbb{M}, \alpha) = \{P\}, \quad \text{for all } P \in S.$$

2. If there is a subgroup $\mathbb{W}$ such that $\mathbb{M}, \mathbb{W}$ are complementary subgroups in $\mathbb{G}$, we say that $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ is intrinsic Lipschitz in $\mathcal{E}$ when graph (ϕ) is an intrinsic Lipschitz $\mathbb{M}$-graph.

3. We say that $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ is intrinsic C_L-Lipschitz in $\mathcal{E}$ for some $C_L \geq 0$ if for all $C_1 > C_L$

$$C_{\mathbb{W},\mathbb{M}}(P, 1/C_1) \cap \operatorname{graph}(\phi) = \{P\} \quad \text{for all } P \in \operatorname{graph}(\phi). \tag{2.28}$$

   The Lipschitz constant of ϕ in $\mathcal{E}$ is the infimum of the $C_1 > 0$ such that (2.28) holds.

We will call a set $S \subset \mathbb{G}$ an intrinsic Lipschitz graph if there exists an intrinsic Lipschitz function $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ such that $S = \operatorname{graph}(\phi)$ for suitable complementary subgroups $\mathbb{W}$ and $\mathbb{M}$.

The intrinsic Lipschitz $\mathbb{M}$-graphs and the intrinsic Lipschitz functions are well behaved with respect to left translations, indeed

Theorem 2.5.9. *Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in $\mathbb{G}$ and $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$. Then for all $Q \in \mathbb{G}$, ϕ is intrinsic C_L-Lipschitz if and only if $\phi_Q : \mathcal{E}_Q \subset \mathbb{W} \to \mathbb{M}$ is intrinsic C_L-Lipschitz, where ϕ_Q is defined in Proposition* 2.5.1.

We observe that the geometric definition of intrinsic Lipschitz graphs has equivalent analytic forms (see Proposition 3.1.3. in [39]):

Proposition 2.5.10. *Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in $\mathbb{G}$, $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ and $C_L > 0$. Then the following statements are equivalent:*

1. *ϕ is intrinsic C-Lipschitz in $\mathcal{E}$.*
2. $\|\mathbf{P}_{\mathbb{M}}(Q^{-1}Q')\| \leq C_L \|\mathbf{P}_{\mathbb{W}}(Q^{-1}Q')\|$ *for all* $Q, Q' \in \operatorname{graph}(\phi)$.
3. $\|\phi_{Q^{-1}}(A)\| \leq C_L \|A\|$ *for all* $Q \in \operatorname{graph}(\phi)$ *and* $A \in \mathcal{E}_{Q^{-1}}$.

Moreover if $\mathbb{W}$ is a normal subgroup of $\mathbb{G}$, the conditions 1.-2.-3. in Proposition 2.5.10 take a more explicit form. Indeed for all $Q = A\phi(A), Q' = B\phi(B) \in \operatorname{graph}(\phi)$ we have $\mathbf{P}_{\mathbb{M}}(Q^{-1}Q') = \phi(A)^{-1}\phi(B)$ and $\mathbf{P}_{\mathbb{W}}(Q^{-1}Q') = \phi(A)^{-1}A^{-1}B\phi(A)$. Consequently, if $\mathbb{W}$ is a normal subgroup of $\mathbb{G}$ then ϕ is intrinsic C_L-Lipschitz if and only if

$$\|\phi(A)^{-1}\phi(B)\| \leq C_L \|\phi(A)^{-1}A^{-1}B\phi(A)\| \quad \forall A, B \in \mathcal{E}. \tag{2.29}$$

Remark 2.5.11. If $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ is intrinsic C_L-Lipschitz in $\mathcal{E}$ then it is continuous. Indeed if $\phi(0) = 0$ then by the condition 3. of Proposition 2.5.10 ϕ is continuous in 0. To prove the continuity in $A \in \mathcal{E}$, observe that $\phi_{Q^{-1}}$ is continuous in 0, where $Q = A\phi(A)$.

Remark 2.5.12. A map ϕ is intrinsic C_L-Lipschitz if and only if the distance of two points $Q, Q' \in \text{graph}(\phi)$ is bounded by the norm of the projection of $Q^{-1}Q'$ on the domain $\mathcal{E}$. Precisely $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{V}$ is intrinsic C_L-Lipschitz in $\mathcal{E}$ if and only if there exists a constant $C_1 > 0$ satisfying

$$\|Q^{-1}Q'\| \leq C_1 \|\mathbf{P}_{\mathbb{W}}(Q^{-1}Q')\|,$$

for all $Q, Q' \in \text{graph}\,(\phi)$. Moreover the relations between C_1 and the Lipschitz constant C_L of ϕ follow from (2.1). In fact if ϕ is intrinsic C_L-Lipschitz in $\mathcal{E}$ then

$$\|Q^{-1}Q'\| \leq \|\mathbf{P}_{\mathbb{W}}(Q^{-1}Q')\| + \|\mathbf{P}_{\mathbb{M}}(Q^{-1}Q')\| \leq (1 + C_L)\|\mathbf{P}_{\mathbb{W}}(Q^{-1}Q')\|$$

for all $Q, Q' \in \text{graph}\,(\phi)$. Conversely if $\|Q^{-1}Q'\| \leq c_0(1 + C_L)\|\mathbf{P}_{\mathbb{W}}(Q^{-1}Q')\|$ then

$$\|\mathbf{P}_{\mathbb{M}}(Q^{-1}Q')\| \leq C_L \|\mathbf{P}_{\mathbb{W}}(Q^{-1}Q')\|$$

for all $Q, Q' \in \text{graph}\,(\phi)$, i.e. the condition 2. of Proposition 2.5.10 holds.

We observe that in Euclidean spaces intrinsic Lipschitz maps are the same as Lipschitz maps. The converse is not true (see Example 2.3.9 in [38]) and if $\phi : \mathbb{W} \to \mathbb{M}$ is intrinsic Lipschitz then this does not yield the existence of a constant C such that

$$\|\phi(A)^{-1}\phi(B)\| \leq C\|A^{-1}B\| \quad \text{for } A, B \in \mathbb{W}$$

not even locally.

In Proposition 3.1.8 in [39] the authors proved that the intrinsic Lipschitz functions, even if non metric Lipschitz, nevertheless are Hölder continuous.

Proposition 2.5.13 (see Proposition 3.1.8 in [39])**.** *Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in $\mathbb{G}$ and $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ be an intrinsic C_L-Lipschitz function. Then, for all $r > 0$,*

1. *there is $C_1 = C_1(\phi, r) > 0$ such that*

$$\|\phi(A)\| \leq C_1 \quad \text{for all } A \in \mathcal{E} \text{ with } \|A\| \leq r$$

2. *there is $C_2 = C_2(C_L, r) > 0$ such that ϕ is locally $1/\kappa$-Hölder continuous i.e.*

$$\|\phi(A)^{-1}\phi(B)\| \leq C_2 \|A^{-1}B\|^{1/\kappa} \quad \text{for all } A, B \text{ with } \|A\|, \|B\| \leq r.$$

Proof. We start to show the condition 1. Fix $A \in \mathcal{E}$ such that $\|A\| \leq R$. With this notation

$$\phi(A)^{-1}A^{-1}B\phi(B) = (\phi(A)^{-1}A^{-1}B\phi(B))_{\mathbb{W}}(\phi(A)^{-1}A^{-1}B\phi(B))_{\mathbb{M}}$$

by the uniqueness of the components, we know that

$$(\phi(A)^{-1}A^{-1}B\phi(B))_{\mathbb{W}} = (\phi(A)^{-1}A^{-1}B)_{\mathbb{W}} \quad \text{and} \quad (\phi(A)^{-1}A^{-1}B\phi(B))_{\mathbb{M}} = (\phi(A)^{-1}A^{-1}B)_{\mathbb{M}}\phi(B)$$

and consequently by Proposition 2.5.10 we obtain that

$$\|(\phi(A)^{-1}A^{-1}B)_{\mathbb{M}}\phi(B)\| \leq C_L\|(\phi(A)^{-1}A^{-1}B)_{\mathbb{W}}\| \quad \text{for all } B \in \mathcal{E}. \tag{2.30}$$

Moreover using the triangle inequality and (2.1), it follows

$$\begin{aligned}\|\phi(B)\| &= \|B^{-1}A\phi(A)\phi(A)^{-1}A^{-1}B\phi(B)\| \\ &\leq \|B^{-1}A\phi(A)\| + \|\phi(A)^{-1}A^{-1}B\phi(B)\| \\ &\leq \|B^{-1}A\phi(A)\| + \|(\phi(A)^{-1}A^{-1}B)_{\mathbb{M}}\phi(B)\| + \|(\phi(A)^{-1}A^{-1}B)_{\mathbb{W}}\|.\end{aligned}$$

Hence since (2.30)

$$\|\phi(B)\| \leq \|B^{-1}A\phi(A)\| + (1+C_L)\|(\phi(A)^{-1}A^{-1}B)_{\mathbb{W}}\|.$$

Thanks to the limitations on $\|A\|$ and on $\|B\|$ we get the condition 1.

Now we prove the condition 2. Thanks to Lemma 2.1.5 and the condition 1. we know that there exist $C_1, C_2 > 0$ such that

$$\|\phi(A)^{-1}A^{-1}B\phi(A)\| \leq C_1\|A^{-1}B\|^{\frac{1}{\kappa}}, \tag{2.31}$$

$$\|\phi(A)^{-1}B^{-1}A\phi(A)\| \leq C_2\|B^{-1}A\|^{\frac{1}{\kappa}} = C_2\|A^{-1}B\|^{\frac{1}{\kappa}}, \tag{2.32}$$

for every $A, B \in \mathcal{E}$ with $\|A\|, \|B\| \leq R$. Moreover using again (2.1) we have

$$c_0\|(\phi(A)^{-1}A^{-1}B)_{\mathbb{W}}\| \leq \|\phi(A)^{-1}A^{-1}B\phi(A)\|$$

As a consequence, from (2.31)

$$\|(\phi(A)^{-1}A^{-1}B)_{\mathbb{W}}\| \leq \frac{C_1}{c_0}\|A^{-1}B\|^{\frac{1}{\kappa}} \tag{2.33}$$

and from (2.30) and (2.33)

$$\|(\phi(A)^{-1}A^{-1}B)_{\mathbb{M}}\phi(B)\| \leq \frac{C_LC_1}{c_0}\|A^{-1}B\|^{\frac{1}{\kappa}}. \tag{2.34}$$

Finally, putting together (2.31), (2.32), (2.33) and (2.34) we conclude that

$$\begin{aligned}\|\phi(A)^{-1}\phi(B)\| &\leq \|\phi(A)^{-1}B^{-1}A\phi(A)\| + \|\phi(A)^{-1}A^{-1}B\phi(B)\| \\ &\leq C_2\|A^{-1}B\|^{\frac{1}{\kappa}} + \|(\phi(A)^{-1}A^{-1}B)_{\mathbb{W}}\| + \|(\phi(A)^{-1}A^{-1}B)_{\mathbb{M}}\phi(B)\| \\ &\leq \left(C_2 + \frac{C_1}{c_0} + \frac{C_LC_1}{c_0}\right)\|A^{-1}B\|^{\frac{1}{\kappa}} \\ &=: C_3\|A^{-1}B\|^{\frac{1}{\kappa}}\end{aligned}$$

for every $A, B \in \mathcal{E}$ with $\|A\|, \|B\| \leq R$. $\square$

2.5.2 Intrinsic difference quotients

We give another characterization of intrinsic Lipschitz functions in terms of boundedness of appropriately defined intrinsic difference quotients. We present some results proved in [94].

For simplicity first we propose the definition in the particular case of a function vanishing in the origin of the Carnot group and then we get the general definition extending the particular case in a translation invariant way.

Let Y be an element of the Lie algebra $\mathfrak{w}$ of $\mathbb{W}$ and $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ with $0 \in \mathcal{E}$ and $\phi(0) = 0$. We define the *difference quotients* of ϕ from 0 in direction Y as

$$\Delta_Y \phi(0,t) := \delta_{1/t}\phi(\delta_t \exp Y)$$

for all $t > 0$ such that $\delta_t \exp Y \in \mathcal{E}$. Then we extend this definition to any $A \in \mathcal{E}$. Let $P = A \cdot \phi(A) \in \operatorname{graph}(\phi)$, then the translated function $\phi_{P^{-1}}$ defined in Proposition 2.5.1 vanishes in $0 \in \mathcal{E}_{P^{-1}}$ and we define

$$\Delta_Y \phi(A,t) := \Delta_Y \phi_{P^{-1}}(0,t) = \delta_{1/t}\phi_{P^{-1}}(\delta_t \exp Y) \tag{2.35}$$

for all $t > 0$ such that $\delta_t \exp Y \in \mathcal{E}_{P^{-1}}$.

The more explicit form of the translated function $\phi_{P^{-1}}$ allows a more explicit form of difference quotients and we have the following

Definition 2.5.7. Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in $\mathbb{G}$, $\mathfrak{w}$ be the Lie algebra of $\mathbb{W}$ and $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$. If $A \in \mathcal{E}$ and $Y \in \mathfrak{w}$, then we define the intrinsic difference quotients of ϕ at A along Y as

$$\Delta_Y \phi(A,t) = \delta_{1/t}\left((\mathbf{P}_{\mathbb{M}}(\phi(A)\delta_t(\exp Y)))^{-1} \phi(A\mathbf{P}_{\mathbb{W}}(\phi(A)\delta_t(\exp Y)))\right)$$

for all $t > 0$ such that $A\mathbf{P}_{\mathbb{W}}(\phi(A)\delta_t(\exp Y)) \in \mathcal{E}$.

Remark 2.5.14. Definition 2.5.7 gives the same notion of difference quotient as proposed in (2.35). Indeed, if $\phi(A) = 0$ then $\mathbf{P}_{\mathbb{M}}(\phi(A)\delta_t \exp Y) = 0$ and $A\mathbf{P}_{\mathbb{W}}(\phi(A)\delta_t \exp Y) = A\delta_t \exp Y$ and consequently

$$\Delta_Y \phi(A,t) = \delta_{1/t}\left((\mathbf{P}_{\mathbb{M}}(\phi(A)\delta_t(\exp Y)))^{-1} \phi(A\mathbf{P}_{\mathbb{W}}(\phi(A)\delta_t \exp Y))\right) = \delta_{1/t}\phi(A\delta_t \exp Y).$$

Moreover if $P = A\phi(A)$ then $\phi_{P^{-1}}(0) = 0$ and so we obtain

$$\Delta_Y \phi(A,t) = \Delta_Y \phi_{P^{-1}}(0,t) = \delta_{1/t}(\phi_{P^{-1}}(\delta_t \exp Y)).$$

Remark 2.5.15. Recalling Remark 2.5.3 we get

1. If $\mathbb{W}$ is normal in $\mathbb{G}$ and $Y \in \mathfrak{w}$ then

$$\Delta_Y \phi(A,t) = \delta_{1/t}\left(\phi(A)^{-1}\phi\left(A\phi(A)\delta_t \exp Y \phi(A)^{-1}\right)\right)$$

because $\mathbf{P}_{\mathbb{M}}(\phi(A)\delta_t \exp Y) = \phi(A)$ and $A\mathbf{P}_{\mathbb{W}}(\phi(A)\delta_t \exp Y) = A\phi(A)\delta_t \exp Y \phi(A)^{-1}$.

2. If $\mathbb{M}$ is normal in $\mathbb{G}$ and $Y \in \mathfrak{w}$ then

$$\Delta_Y\phi(A,t) = \delta_{1/t}\left((\delta_t \exp Y)^{-1}\phi(A)^{-1}(\delta_t \exp Y)\phi\left(A\delta_t \exp Y\right)\right)$$

because $\mathbf{P}_{\mathbb{M}}(\phi(A)\delta_t \exp Y) = (\delta_t \exp Y)^{-1}\phi(A)\delta_t \exp Y$ and $\mathbf{P}_{\mathbb{W}}(\phi(A)\delta_t \exp Y) = \delta_t \exp Y$.

3. If both $\mathbb{W}$ and $\mathbb{M}$ are normal in $\mathbb{G}$ then we get the well known expression for the difference quotient:

$$\Delta_Y\phi(A,t) = \delta_{1/t}\left(\phi(A)^{-1}\phi\left(A\delta_t \exp Y\right)\right).$$

Next Proposition gives a characterization of intrinsic Lipschitz functions:

Proposition 2.5.16. *Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in $\mathbb{G}$, $\mathfrak{w}$ be the Lie algebra of $\mathbb{W}$ and $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$. Then the following are equivalent:*

1. *ϕ is intrinsic C_L-Lipschitz in $\mathcal{E}$*
2. *there is $C_L > 0$ such that, for all $Y \in \mathfrak{w}$ and for all $A \in E$*

$$\|\Delta_Y\phi(A,t)\| \le C_L\|\exp Y\|.$$

Proof. 1. $\Rightarrow$ 2. By (2.35) we have that if $P = A \cdot \phi(A) \in \operatorname{graph}(\phi)$, then

$$\|\Delta_Y\phi(A,t)\| = \|\Delta_Y\phi_{P^{-1}}(0,t)\| = \frac{1}{t}\|\phi_{P^{-1}}(\delta_t \exp Y)\|$$

and consequently by Proposition 2.5.10

$$\|\Delta_Y\phi(A,t)\| = \frac{1}{t}\|\phi_{P^{-1}}(\delta_t \exp Y)\| \le \frac{C_L}{t}\|\delta_t \exp Y\| = C_L\|\exp Y\|$$

for all $t > 0$ and $Y \in \mathfrak{w}$. So the implication 1. $\Rightarrow$ 2. is complete.

Now we want to show the converse, i.e. 2. $\Rightarrow$ 1. Fix $A \in \mathcal{E}$ and let $P := A \cdot \phi(A)$. For any $B \in \mathcal{E}_{P^{-1}}$, let $Y \in \mathfrak{w}$ be such that $B = \exp Y$. Then

$$\|\phi_{P^{-1}}(B)\| = \|\phi_{P^{-1}}(\exp Y)\| = \|\Delta_Y\phi(A,1)\| \le C_L\|\exp Y\| = C_L\|B\|.$$

Hence the condition 3. of Proposition 2.5.10 holds and ϕ is intrinsic C_L-Lipschitz. $\square$

The following definition shows as, in perfect analogy with Euclidean calculus, intrinsic derivatives of functions acting between complementary subgroups of $\mathbb{G}$ can be obtained as limits of intrinsic difference quotients (when these limits exist).

Definition 2.5.8. Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in $\mathbb{G}$, $\mathfrak{w}$ be the Lie algebra of $\mathbb{W}$ and $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$. If $A \in \mathcal{E}$, the *intrinsic directional derivative* of ϕ at A along $Y \in \mathfrak{w}$, is

$$D_Y\phi(A) := \lim_{t\to 0^+} \Delta_Y\phi(A,t) = \lim_{t\to 0^+} \Delta_{-Y}\phi(A,t) \tag{2.36}$$

provided the two limits on the right exist and are equal.

Remark 2.5.17. This Definition follows directly from Remark 2.5.14. Indeed

$$\phi(A) = 0 \quad \Longrightarrow \quad \Delta_Y\phi(A,t) = \delta_{1/t}\phi(\delta_t \exp Y)$$

hence, if the limits in (2.36) exist, then

$$\phi(A) = 0 \quad \Longrightarrow \quad D_Y\phi(A) = Y\phi(A).$$

2.6 Intrinsic differentiability

From now on we specialize our setting and we study an appropriate notion of differentiability for a continuous function $\phi : \mathbb{W} \to \mathbb{H}$ when $\mathbb{H}$ is an horizontal subgroup. We recall that when $\mathbb{H}$ is horizontal, $\mathbb{W}$ is always a normal subgroup since, as observed in Remark 2.1.3, it contains the whole strata $\mathbb{G}^2, \dots, \mathbb{G}^\kappa$.

Our first preliminary results are Proposition 2.6.7 and Proposition 2.6.8 where equivalent formulations of Definition 2.6.2 are given.

2.6.1 Intrinsic linear functions

First we present the notion of intrinsic linear map.

Definition 2.6.1. Let $\mathbb{W}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$. Then $\ell : \mathbb{W} \to \mathbb{H}$ is an *intrinsic linear function* if ℓ is defined on all of $\mathbb{W}$ and if $\operatorname{graph}(\ell) = \{A\ell(A) : A \in \mathbb{W}\}$ is a homogeneous subgroup of $\mathbb{G}$.

Intrinsic linear functions can be algebraically characterized as follows.

Proposition 2.6.1 (see Propositions 3.1.3 and 3.1.6 in [38])**.** *Let $\mathbb{W}$ and $\mathbb{M}$ be complementary subgroups in $\mathbb{G}$. Then $\ell : \mathbb{W} \to \mathbb{M}$ is an intrinsic linear function if and only if*

$$\ell(\delta_\lambda A) = \delta_\lambda(\ell(A)), \qquad \textit{for all } A \in \mathbb{W} \textit{ and } \lambda \geq 0$$

$$\ell(AB) = (\mathbf{P}_{\mathbb{H}}(\ell(A)^{-1}B))^{-1}\ell(\mathbf{P}_{\mathbb{W}}(\ell(A)^{-1}B)), \qquad \textit{for all } A, B \in \mathbb{W}. \tag{2.37}$$

In particular, if $\mathbb{W}$ *is normal in* $\mathbb{G}$ *then*

$$\ell(AB) = \ell(A)\ell\big(\ell(A)^{-1}B\ell(A)\big), \qquad \text{for all } A, B \in \mathbb{W}. \tag{2.38}$$

Moreover ℓ *is a polynomial function and it is intrinsic Lipschitz with Lipschitz constant* $C_L := \sup\{\|\ell(A)\| \,:\, \|A\| = 1\}$*. Note that* $C_L < +\infty$ *because* ℓ *is continuous. Moreover*

$$\|\ell(A)\| \le C_L\|A\|, \qquad \text{for all } A \in \mathbb{W}.$$

For a proof of Proposition 2.6.1 see Proposition 3.1.3 and 3.1.6 in [38].

Remark 2.6.2. H-linear functions (see Definition 2.2.1) and intrinsic linear functions are in general different. Indeed the intrinsic linear functions are not necessarily group homomorphisms between their domains and codomains, as the following example shows. Let $\mathbb{W}, \mathbb{H}$ be the complementary subgroup of Heisenberg group $\mathbb{H}^2$ defined as $\mathbb{H} := \{(p_1, 0, 0, 0, 0) \,:\, p_1 \in \mathbb{R}\}$ and $\mathbb{W} := \{(0, p_2, p_3, p_4, p_5) \,:\, p_2, p_3, p_4, p_5 \in \mathbb{R}\}$. For any fixed $\alpha \in \mathbb{R}$ the function $\ell : \mathbb{H} \to \mathbb{W}$ given by

$$\ell(p_1, 0, 0) = (0, 0, \alpha p_1, 0, -\alpha p_1^2/2)$$

is intrinsic linear because graph$\,(\ell) = \{(t, 0, \alpha t, 0, 0) \,|\, t \in \mathbb{R}\}$ is a 1-dimensional homogeneous subgroup of $\mathbb{H}^2$ but ℓ is not a group homomorphism from $\mathbb{H}$ to $\mathbb{W}$.

Moreover in [38] the authors show that if $\mathbb{W}, \mathbb{H}$ are both normal subgroups of $\mathbb{G}$, then (2.37) becomes

$$\ell(AB) = \ell(A)\ell(B), \qquad \text{for all } A, B \in \mathbb{W} \tag{2.39}$$

i.e. ℓ is homogeneous homomorphism from $\mathbb{W}$ to $\mathbb{H}$. As a consequence, by Proposition 2.6.1 and (2.39) we obtain the following statement: if $\mathbb{G}$ is direct product of $\mathbb{W}$ and $\mathbb{H}$ then

$$\ell : \mathbb{W} \to \mathbb{H} \text{ is intrinsic linear function} \quad \Longleftrightarrow \quad \ell \text{ is } H\text{-linear function} \tag{2.40}$$

We present a characterization of intrinsic linear function:

Proposition 2.6.3. *Let* $\mathbb{W}$ *and* $\mathbb{H}$ *be complementary subgroups in* $\mathbb{G}$ *with* $\mathbb{H}$ *horizontal. Let* $\ell : \mathbb{W} \to \mathbb{H}$ *be an intrinsic linear function. Then* ℓ *depends only on the variables in the first layer* $\mathbb{W}^1 := \mathbb{W} \cap \mathbb{G}^1$ *of* $\mathbb{W}$*. That is*

$$\ell(A) = \ell(A^1) \qquad \text{for all } A = (A^1, \dots, A^\kappa) \in \mathbb{W}. \tag{2.41}$$

Moreover $\ell_{|\mathbb{W}^1} : \mathbb{W}^1 \to \mathbb{H}$ *is euclidean linear and*

$$\|\ell(A)\| \le C_L\|A^1\| = C_L|A^1|. \tag{2.42}$$

Proof. In order to prove (2.41) first we prove that for all $(x^1,\dots,x^\kappa)\in\mathbb{W}$

$$\ell(x^1,\dots,x^\kappa)=\ell(x^1,\dots,x^{\kappa-1},0). \tag{2.43}$$

Let $A_\kappa:=(0,\dots,0,x^\kappa)\in\mathbb{W}$. From (1.13) we have $\ell(A_\kappa)^{-1}\cdot A_\kappa\cdot\ell(A_\kappa)=A_\kappa$ and by (2.38)

$$\ell(A_\kappa\cdot A_\kappa)=\ell(A_\kappa)\cdot\ell\Big(\ell(A_\kappa)^{-1}\cdot A_\kappa\cdot\ell(A_\kappa)\Big)=\ell(A_\kappa)\cdot\ell(A_\kappa)=2\ell(A_\kappa).$$

Because $A_\kappa\cdot A_\kappa=(0,\dots,0,2x^\kappa)=\delta_{2^{1/\kappa}}A_\kappa$, then $2\ell(A_\kappa)=\ell(A_\kappa\cdot A_\kappa)=\ell(\delta_{2^{1/\kappa}}A_\kappa)=2^{1/\kappa}\ell(A_\kappa)$; hence

$$\ell(A_\kappa)=0. \tag{2.44}$$

Because $(x^1,\dots,x^\kappa)=A_\kappa\cdot(x^1,\dots,x^{\kappa-1},0)$, from (2.38) and (2.44) we get

$$\begin{aligned}\ell(x^1,\dots,x^\kappa)&=\ell\Big(A_\kappa\cdot(x^1,\dots,x^{\kappa-1},0)\Big)=\ell(A_\kappa)\cdot\ell\Big(\ell(A_\kappa)^{-1}\cdot(x^1,\dots,x^{\kappa-1},0)\cdot\ell(A_\kappa)\Big)\\&=\ell(x^1,\dots,x^{\kappa-1},0),\end{aligned}$$

and (2.43) is proved.

In the next step we prove that

$$\ell(x^1,\dots,x^\kappa)=\ell(x^1,\dots,x^{\kappa-2},0,0) \tag{2.45}$$

Let $A_{\kappa-1}:=(0,\dots,0,x^{\kappa-1},0)\in\mathbb{W}$. From (1.13), there is $\hat{x}^\kappa$, depending on $\ell(A_{\kappa-1})$ and $A_{\kappa-1}$, such that

$$\ell(A_{\kappa-1})^{-1}\cdot A_{\kappa-1}\cdot\ell(A_{\kappa-1})=(0,\dots,0,x^{\kappa-1},\hat{x}^\kappa).$$

From (2.38), (2.44) and the fact that $(0,\dots,0,x^{\kappa-1},\hat{x}^\kappa)=(0,\dots,0,\hat{x}^\kappa)\cdot A_{\kappa-1}$ we get

$$\begin{aligned}\ell(0,\dots,0,x^{\kappa-1},\hat{x}^\kappa)&=\ell((0,\dots,0,\hat{x}^\kappa)\cdot A_{\kappa-1})\\&=\ell(0,\dots,0,\hat{x}^\kappa)\cdot\ell\Big(\ell(0,\dots,0,\hat{x}^\kappa)^{-1}\cdot A_{\kappa-1}\cdot\ell(0,\dots,0,\hat{x}^\kappa)\Big)\\&=\ell(A_{\kappa-1})\end{aligned}$$

and consequently

$$\ell(A_{\kappa-1}\cdot A_{\kappa-1})=\ell(A_{\kappa-1})\cdot\ell\Big(\ell(A_{\kappa-1})^{-1}\cdot A_{\kappa-1}\cdot\ell(A_{\kappa-1})\Big)=\ell(A_{\kappa-1})\cdot\ell(A_{\kappa-1}).$$

Because $A_{\kappa-1}\cdot A_{\kappa-1}=(0,\dots,0,2x^{\kappa-1},0)=\delta_{2^{1/\kappa-1}}A_{\kappa-1}$ we have

$$2\ell(A_{\kappa-1})=\ell(A_{\kappa-1})\cdot\ell(A_{\kappa-1})=\ell(A_{\kappa-1}\cdot A_{\kappa-1})=\ell(\delta_{2^{1/\kappa-1}}A_{\kappa-1})=2^{1/\kappa-1}\ell(A_{\kappa-1}).$$

Then $\ell(A_{\kappa-1})=0$ and also $\ell(0,\dots,0,x^{\kappa-1},\hat{x}^\kappa)=0$.

Because $(x^1,\dots,x^\kappa)=(x^1,\dots,x^{\kappa-2},0,0)\cdot(0,\dots,0,x^{\kappa-1},\bar{x}^\kappa)$ for appropriate $\bar{x}^\kappa$, we obtain (2.45) from (2.38).

This procedure can be iterated to get (2.41). □

2.6.2 Intrinsic differentiability and graph distance

For functions acting between complementary subgroups of a Carnot group $\mathbb{G}$ there is an appropriate notion of differentiability here denoted *intrinsic differentiability*. In this section we recall definition and main features of *intrinsic differentiability*.

Following [95], we use intrinsic linear functions to define it as in the usual definition of differentiability.

Definition 2.6.2. Let $\mathbb{W}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and let $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ with $\mathcal{O}$ open in $\mathbb{W}$. For $A \in \mathcal{O}$, let $P := A \cdot \phi(A)$ and $\phi_{P^{-1}} : \mathcal{O}_{P^{-1}} \subset \mathbb{W} \to \mathbb{H}$ be the translated function defined in Proposition 2.5.1. We say that ϕ is *intrinsic differentiable in* A if there is a intrinsic linear $d\phi_A : \mathbb{W} \to \mathbb{H}$ such that

$$\|d\phi_A(B)^{-1}\phi_{P^{-1}}(B)\| = o(\|B\|) \qquad \text{for } \|B\| \to 0 \tag{2.46}$$

The function $d\phi_A$ is called the *intrinsic differential of* ϕ *at* A.

Remark 2.6.4. Definition 2.6.2 is a natural one because of the following observations.

(i) Let $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ be intrinsic differentiable in $A \in \mathcal{O}$. Then the intrinsic linear functional satisfying (2.46) is unique and ϕ is continuous at A (see Theorem 3.2.8 and Proposition 3.2.3 in [38]).

(ii) The notion of intrinsic differentiability is invariant under group translations. More precisely, let $P := A\phi(A), Q := B\phi(B) \in \operatorname{graph}(\phi)$, then ϕ is intrinsic differentiable in A if and only if $\phi_{QP^{-1}} \equiv (\phi_{P^{-1}})_Q$ is intrinsic differentiable in B.

The analytic definition of intrinsic differentiability of Definition 2.6.2 has an equivalent geometric formulation. Indeed intrinsic differentiability in one point is equivalent to the existence of a tangent subgroup to the graph.

Theorem 2.6.5 (Theorem 3.2.8. in [38])**.** *Let $\mathbb{W}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$ with $\mathbb{H}$ one dimensional and let $\phi : \mathcal{O} \to \mathbb{H}$ with $\mathcal{O}$ relatively open in $\mathbb{W}$. If ϕ is intrinsic differentiable in $A \in \omega$ and $d\phi_A$ is the intrinsic differential of ϕ at A, set $\mathbb{T} := \operatorname{graph}(d\phi_A)$. Then*

1. *$\mathbb{T}$ is an homogeneous subgroup of $\mathbb{G}$;*
2. *$\mathbb{T}$ and $\mathbb{M}$ are complementary subgroups in $\mathbb{G}$;*
3. *$P \cdot \mathbb{T}$ is the tangent coset to $\operatorname{graph}(\phi)$ in $P := A\phi(A)$.*

Conversely, if $P := A\phi(A) \in \operatorname{graph}(\phi)$ and if there is $\mathbb{T}$ such that (1), (2), (3) hold, then ϕ is intrinsic differentiable in A and the differential $d\phi_A : \mathbb{W} \to \mathbb{H}$ is the unique intrinsic linear function such that $\mathbb{T} := \operatorname{graph}(d\phi_A)$.

Remark 2.6.6. P-differentiability and intrinsic differentiability are in general different notions. Indeed let $\mathbb{W}, \mathbb{H}$ be the complementary subgroups of $\mathbb{H}^2$ defined as $\mathbb{H} := \{(p_1, 0, 0) : p_1 \in \mathbb{R}\}$ and $\mathbb{W} := \{(0, p_2, p_3, p_4, p_5) : p_2, p_3, p_4, p_5 \in \mathbb{R}\}$. As we said in Remark 2.6.2 the map $\ell : \mathbb{H} \to \mathbb{W}$ defined as

$$\ell(p_1, 0, 0) = (0, 0, \alpha p_1, 0, -\alpha p_1^2/2), \quad \text{ for any fixed } \alpha \in \mathbb{R}.$$

is an example of intrinsic linear map. Obviously ℓ is intrinsic differentiable in $P = 0$ but ℓ is not P-differentiable in 0. On the other hand, the homogeneous homomorphism $h : \mathbb{H} \to \mathbb{W}$ given by

$$h(P) = (0, 0, \alpha p_1, 0, 0), \quad \text{ for any fixed } \alpha \in \mathbb{R}$$

is P-differentiable in $P = 0$ but h is not intrinsic differentiable in 0.

Moreover writing explicit $\phi_{P^{-1}}$ in Definition 2.6.2 when $\mathbb{W}, \mathbb{H}$ are both normal subgroups of $\mathbb{G}$, then $\phi : \mathbb{W} \to \mathbb{H}$ is intrinsic differentiable in $P \in \mathbb{W}$ if

$$\|\ell(Q)^{-1}\phi(P)^{-1}\phi(PQ)\| = o(\|Q\|), \quad \text{ as } \|Q\| \to 0.$$

Consequently, by also (2.40) we get the following statement: if $\mathbb{G}$ is direct product of $\mathbb{W}$ and $\mathbb{H}$, then

$$\phi : \mathbb{W} \to \mathbb{H} \text{ is P-differentiable} \quad \Longleftrightarrow \quad \phi \text{ is intrinsic differentiable}$$

If we consider an horizontal valued function ϕ, the more explicit form of its translated function $\phi_{P^{-1}}$ allows a more explicit form of intrinsic differentiability and we have the following

Proposition 2.6.7. *Let $\mathbb{H}$ and $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$ with $\mathbb{H}$ horizontal. Let $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ with $\mathcal{O}$ open in $\mathbb{W}$. Then ϕ is intrinsic differentiable in $A \in \mathcal{O}$ if and only if there is an intrinsic linear $d\phi_A : \mathbb{W} \to \mathbb{H}$ such that*

$$\|\phi(B) - \phi(A) - d\phi_A(A^{-1}B)\| = o(\|\phi(A)^{-1}A^{-1}B\phi(A)\|) \quad \text{as } B \to A. \tag{2.47}$$

Proof. First notice that $\phi(B) - \phi(A) - d\phi_A(A^{-1}B) = d\phi_A(A^{-1}B)^{-1}\phi(A)^{-1}\phi(B)$ because both $d\phi_A$ and ϕ are valued in the horizontal subgroup $\mathbb{H}$. Because $\mathbb{W}$ is normal in $\mathbb{G}$, then for $P := A\phi(A)$ and for all $B' \in \mathcal{O}_{P^{-1}}$

$$\phi_{P^{-1}}(B') = \phi(A)^{-1}\phi(A\phi(A)B'\phi(A)^{-1}).$$

Then (2.46) yields that $\phi : \mathcal{O} \to \mathbb{H}$ is intrinsic differentiable in $A \in \mathcal{O}$ if there is an intrinsic linear map $d\phi_A : \mathbb{W} \to \mathbb{H}$ such that

$$\|d\phi_A(B')^{-1}\phi(A)^{-1}\phi(A\phi(A)B'\phi(A)^{-1})\| = o(\|B'\|) \quad \text{as } \|B'\| \to 0,$$

that, setting $B := A\phi(A)B'\phi(A)^{-1}$ is equivalent to

$$\|d\phi_A\left(\phi(A)^{-1}A^{-1}B\phi(A)\right)^{-1}\phi(A)^{-1}\phi(B)\| = o(\|\phi(A)^{-1}A^{-1}B\phi(A))^{-1}\|) \quad \text{as } \|A^{-1}B\| \to 0.$$

Finally, from Proposition 2.6.3 we know that $d\phi_A$ depends only on the variables in the first layer of $\mathbb{W}$. The group operation on the first layer is commutative hence $d\phi_A\left(\phi(A)^{-1}A^{-1}B\phi(A)\right) = d\phi_A\left(A^{-1}B\right)$ and (2.47) is proved.

□

The quantity $\|\phi(A)^{-1}A^{-1}B\phi(A)\|$ in (2.47) plays the role of a "quasi distance" between A and B, depending on the function ϕ. To make things more precise we begin with the following definition.

Definition 2.6.3. Let $\mathbb{H}$ and $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$ with $\mathbb{H}$ horizontal. For $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$, $\rho_\phi : \mathcal{O} \times \mathcal{O} \to \mathbb{R}^+$ is defined as

$$\rho_\phi(A,B) := \frac{1}{2}\left(\|\phi(A)^{-1}A^{-1}B\phi(A)\| + \|\phi(B)^{-1}B^{-1}A\phi(B)\|\right)$$

for all $A, B \in \mathcal{O}$.

Then we have

Proposition 2.6.8. *Let $\mathbb{H}$ and $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$ with $\mathbb{H}$ horizontal. Let $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ with $\mathcal{O}$ open in $\mathbb{W}$. Then ϕ is intrinsic differentiable in $A \in \mathcal{E}$ if and only if there is an intrinsic linear function $d\phi_A : \mathbb{W} \to \mathbb{H}$ such that*

$$\|\phi(B) - \phi(A) - d\phi_A(A^{-1}B)\| = o(\rho_\phi(A,B)) \quad \textit{as } B \to A. \tag{2.48}$$

Proof. By Proposition 2.6.7 it is sufficient to prove that the following statements are equivalent:

1. there is an intrinsic linear function $d\phi_A : \mathbb{W} \to \mathbb{H}$ such that (2.47) is true.
2. there is an intrinsic linear function $d\phi_A : \mathbb{W} \to \mathbb{H}$ such that (2.48) is true.

The implication **1.** $\Longrightarrow$ **2.** follows from the elementary fact that when $\mathbb{W}$ is a normal subgroup then

$$(\Phi(A)^{-1}\Phi(B))_{\mathbb{W}} = \phi(A)^{-1}A^{-1}B\phi(A) \quad \text{and} \quad (\Phi(A)^{-1}\Phi(B))_{\mathbb{H}} = \phi(A)^{-1}\phi(B) \tag{2.49}$$

and so

$$\|\phi(A)^{-1}A^{-1}B\phi(A)\| \le 2\rho_\phi(A,B) \quad \text{for all } B \in \mathcal{O}. \tag{2.50}$$

Hence it remains to show **2**. $\Longrightarrow$ **1**.
In particular it is sufficient to prove that there are two positive constants $C_0 > 0$ and r such that

$$\rho_\phi(A,B) \le C_0 \|\phi(A)^{-1}A^{-1}B\phi(A)\|, \tag{2.51}$$

for all $B \in \mathcal{O}$ such that $\|A^{-1}B\| \le r$.

First we would like to show that there exist two positive constants $C_1 = C_1(\mathbb{G}), C_2 = C_2(\mathbb{G})$ such that

$$\rho_\phi(A,B) \le C_1\|Q\| + C_2\left(\|Q\|^{\frac{1}{\kappa}}\|G\|^{\frac{\kappa-1}{\kappa}} + \|Q\|^{\frac{\kappa-1}{\kappa}}\|G\|^{\frac{1}{\kappa}}\right) \tag{2.52}$$

with $Q = (\Phi(B)^{-1}\Phi(A))_{\mathbb{W}}$ and $G = (\Phi(B)^{-1}\Phi(A))_{\mathbb{H}}$ for all $B \in \mathcal{O}$ such that $\|A^{-1}B\| \le r$. Let $P \in \mathbb{G}$ and $P' := (P^{-1})_{\mathbb{W}}(P^{-1})_{\mathbb{H}}P_{\mathbb{H}}$. Then $P'_{\mathbb{W}} = (P^{-1})_{\mathbb{W}}$ and $P' = P_{\mathbb{H}}^{-1}P_{\mathbb{W}}^{-1}P_{\mathbb{H}}$. As a consequence, using (2.1) it follows that there is $c_0 > 0$ such that

$$c_0\|P'_{\mathbb{W}}\| \le \|P'\| = \|P_{\mathbb{H}}^{-1}P_{\mathbb{W}}^{-1}P_{\mathbb{H}}\|$$

and by (2.2) there exists $C_3 > 0$ such that

$$\begin{aligned}\|(P^{-1})_{\mathbb{W}}\| &\le \frac{1}{c_0}\|P_{\mathbb{H}}^{-1}P_{\mathbb{W}}^{-1}P_{\mathbb{H}}\| \\ &\le \frac{1}{c_0}\|P_{\mathbb{W}}\| + \frac{C_3}{c_0}\left(\|P_{\mathbb{W}}\|^{\frac{1}{\kappa}}\|P_{\mathbb{H}}\|^{\frac{\kappa-1}{\kappa}} + \|P_{\mathbb{W}}\|^{\frac{\kappa-1}{\kappa}}\|P_{\mathbb{H}}\|^{\frac{1}{\kappa}}\right).\end{aligned}$$

Then if we choose $P := \Phi(B)^{-1}\Phi(A)$, (2.52) follows with $C_1 = \frac{c_0+1}{2c_0}$ and $C_2 = \frac{C_3}{2c_0}$.

Now, by (2.52), applying Young's inequality in a standard way we conclude that there is $C_4 = C_4(\mathbb{G},\epsilon) > 0$

$$\rho_\phi(A,B) \le C_4\|(\Phi(A)^{-1}\Phi(B))_{\mathbb{W}}\| + \epsilon\|(\Phi(A)^{-1}\Phi(B))_{\mathbb{H}}\|$$

and so by (2.49)

$$\rho_\phi(A,B) \le C_4\|\phi(A)^{-1}A^{-1}B\phi(A)\| + \epsilon\|\phi(A)^{-1}\phi(B)\| \tag{2.53}$$

for all $B \in \mathcal{O}$ such that $\|A^{-1}B\| \le r$, $\epsilon > 0$. Hence it remains to show that

$$\|\phi(A)^{-1}\phi(B)\| \le C_5\rho_\phi(A,B) \tag{2.54}$$

for all $B \in \mathcal{O}$ such that $\|A^{-1}B\| \le r$. Indeed, by the assumptions,

$$\phi(A)^{-1}\phi(B) = d\phi_A(\phi(A)^{-1}A^{-1}B\phi(A))d\phi_A(\phi(A)^{-1}A^{-1}B\phi(A))^{-1}\phi(A)^{-1}\phi(B)$$

Thus by (2.42) and (3.2.1), (2.54) is true. Now putting together (2.53) and (2.54), there is a suitable constant $C_0 > 0$ such that (2.51) holds. $\square$

Proposition 2.6.9. *Let $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ be an intrinsic C_L-Lipschitz map in every relatively compact subset of $\mathcal{O}$. Then ρ_ϕ is a quasi-metric in every relatively compact subset of $\mathcal{O}$.*

Proof. Let $\mathcal{F} \Subset \mathcal{O}$. It is sufficient to show that if ϕ is intrinsic C_L-Lipschitz continuous in $\mathcal{F}$ then ρ_ϕ is a quasi-metric in $\mathcal{F}$ and so the thesis follows by a standard covering argument.

It is clear that ρ_ϕ is symmetric and $A = B$ yields $\rho_\phi(A, A) = 0$. Then it remains to prove the the weaker triangular inequality, i.e. there is $C_1 > 0$ such that

$$\rho_\phi(A, B) \le C_1(\rho_\phi(A, D) + \rho_\phi(D, B)), \quad \text{for all } A, B, D \in \mathcal{F}. \tag{2.55}$$

Using (2.1), (2.29) and (2.49), for all $A, B, D \in \mathcal{F}$ it follows

$$\begin{aligned} c_0\rho_\phi(A, B) &\le \|\Phi(A)^{-1}\Phi(B)\| \\ &\le \|\Phi(A)^{-1}\Phi(D)\| + \|\Phi(D)^{-1}\Phi(B)\| \\ &\le \|\phi(A)^{-1}A^{-1}D\phi(A)\| + \|\phi(D) - \phi(A)\| + \|\phi(D)^{-1}D^{-1}B\phi(D)\| \\ &\quad + \|\phi(B) - \phi(D)\| \\ &\le 2(1 + C_L)(\rho_\phi(A, D) + \rho_\phi(D, B)). \end{aligned} \tag{2.56}$$

Therefore if we put $C_1 := 2(1 + C_L)/c_0$ we obtain (2.55). □

More precisely from (2.56) we conclude that

$$c_0\rho_\phi(A, B) \le d(\Phi(A), \Phi(B)) \le 2(1 + C_L)\, \rho_\phi(A, B)$$

for every $A, B \in \mathcal{F}$. That is ρ_ϕ is equivalent to the metric d restricted to the graph map Φ. In this way, we call ρ_ϕ “graph distance”.

The quantity $\|\phi(A)^{-1}A^{-1}B\phi(A)\|$, or better a symmetrized version of it, will play the role of a quasi distance on $\mathcal{E}$, depending on ϕ. This fact will be much used in the of this thesis.

2.6.3 Uniformly Intrinsic differentiability

In order to introduce notion of intrinsic $\mathbb{C}^1$ functions for functions acting between complementary subgroups it is possible to introduce a stronger, i.e. uniform, notion of intrinsic differentiability in the general setting of Definition 2.6.2. We will not use here this more general notion and we limit ourselves to introduce the notion of uniform intrinsic differentiability for functions valued in a horizontal subgroup of $\mathbb{G}$.

Definition 2.6.4. Let $\mathbb{H}$, $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$ with $\mathbb{H}$ horizontal. Let $\mathcal{O}$ be open in $\mathbb{W}$ and $\phi : \mathcal{O} \to \mathbb{H}$. We say that ϕ is *uniformly intrinsic differentiable in* $A_0 \in \mathcal{O}$ (or ϕ *is u.i.d. in* A_0) if ϕ is intrinsic differentiable in A_0 and if

$$\lim_{r\to 0^+} = \sup\left\{ \frac{\|\phi(B) - \phi(A) - d\phi_{A_0}(A^{-1}B)\|}{\|\phi(A)^{-1}A^{-1}B\phi(A)\|} : \quad A \neq B,\ \|A_0^{-1}A\| < r,\ \|A_0^{-1}B\| < r \right\} = 0. \tag{2.57}$$

Analogously, ϕ is u.i.d. in $\mathcal{O}$ if it is u.i.d. in every point of $\mathcal{O}$.

It is clear from (2.47) that if ϕ is u.i.d. at A_0, then it is intrinsic differentiable at A_0 and the intrinsic differential $d\phi(A_0)$ of ϕ at A_0 is unique.

In the sequel we show some properties about uniformly intrinsic differentiable functions. First we introduce the following definition:

Definition 2.6.5 ([67])**.** Let $\mathcal{U} \subset \mathbb{G}$ be an open set. If $\beta \in (0,1)$ we denote as $h^\beta(\mathcal{O})$ the set of all *little Hölder continuity* functions of order β, i.e. the set of maps $f \in \mathbb{C}^0(\mathcal{O})$ satisfying

$$\lim_{r\to 0^+} \sup\left\{ \frac{\|f(B)^{-1}f(B')\|}{\|B^{-1}B'\|^\beta} : B, B' \in \mathcal{O}\,,\, 0 < \|B^{-1}B'\| < r \right\} = 0.$$

$h^\beta_{loc}(\mathcal{O})$ denotes the set of function $f \in C^0(\mathcal{O})$ such that $f \in h^\beta(\mathcal{O}')$ for all open set $\mathcal{O}' \Subset \mathcal{O}$.

Remark 2.6.10. Let $\beta_1 > \beta_2$. If $f \in h^{\beta_1}(\mathcal{O})$ then $f \in h^{\beta_2}(\mathcal{O})$.

Proposition 2.6.11. *Let $\mathbb{H}$, $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$ with $\mathbb{H}$ horizontal. Let $\mathcal{O}$ be open in $\mathbb{W}$ and $\phi : \mathcal{O} \to \mathbb{H}$ be u.i.d. in $\mathcal{O}$. Then*

1. *ϕ is intrinsic Lipschitz continuous in every relatively compact subset of $\mathcal{O}$.*
2. *the map ρ_ϕ (see Definition* 2.6.3*) is a quasi-metric in every relatively compact subset of $\mathcal{O}$.*
3. *$\phi \in h^{1/\kappa}_{loc}(\mathcal{O})$.*
4. *the function $A \mapsto d\phi_A$ is continuous in $\mathcal{O}$.*

Proof. (1) For each $A_0 \in \mathcal{O}$ there is $r = r(A_0) > 0$ s.t. for all $A, B \in \mathcal{U}(A_0, r) \cap \mathcal{O}$

$$\|\phi(B) - \phi(A) - d\phi_{A_0}(A^{-1}B)\| \le \|\phi(A)^{-1}A^{-1}B\phi(A)\|.$$

Moreover from Proposition 2.6.3, because the intrinsic linear function $d\phi_{A_0}$ depends only on the variables on the first layer of $\mathbb{W}$, we have

$$\|d\phi_{A_0}(A^{-1}B)\| \le C_L|(A^{-1}B)^1| \le C_L\|\phi(A)^{-1}A^{-1}B\phi(A)\|,$$

where C_L is the intrinsic Lipschitz constant of $d\phi_{A_0}$. Finally

$$\begin{aligned}\|\phi(B) - \phi(A)\| &\le \|\phi(B) - \phi(A) - d\phi_{A_0}(A^{-1}B)\| + \|d\phi_{A_0}(A^{-1}B)\| \\ &\le (1 + C_L)\|\phi(A)^{-1}A^{-1}B\phi(A)\|.\end{aligned}$$

Then (1) follows by a standard covering argument.

(2) The proof follows from Proposition 2.6.9 and from the condition (1).

(3) For $A_0 \in \mathcal{O}$ and $r > 0$ let

$$\rho(r) := \sup\left\{ \frac{\|\phi(B) - \phi(A) - d\phi_{A_0}(A^{-1}B)\|}{\|\phi(A)^{-1}A^{-1}B\phi(A)\|} \,:\, A \neq B \in \mathcal{U}(A_0, r) \cap \mathcal{O} \right\}$$

We have $\lim_{r\to 0} \rho(r) = 0$ because ϕ is u.i.d. at A. Moreover, by (1) of this Proposition, we know that ϕ is intrinsic Lipschitz in $\mathcal{U}(A_0, r) \cap \mathcal{O}$ and by Proposition 2.5.13 (1) we have that $\|\phi(A)\| < C_1$ for all $A \in \mathcal{U}(A_0, r) \cap \mathcal{O}$.

From (2.2) with $P = A^{-1}B$ and $Q = \phi(A)$ we deduce the existence of $C = C(C_1) > 0$ such that

$$\|\phi(A)^{-1}A^{-1}B\phi(A)\| \leq C\|A^{-1}B\|^{1/\kappa} \qquad \text{for all } A, B \in \mathcal{U}(A_0, r) \cap \mathcal{O}.$$

Therefore using (2.42)

$$\begin{aligned}
&\frac{\|\phi(B) - \phi(A)\|}{\|A^{-1}B\|^{1/\kappa}} \\
&\leq \frac{\|\phi(B') - \phi(B) - d\phi_{A_0}(A^{-1}B)\|}{\|\phi(A)^{-1}A^{-1}B\phi(A)\|}\, \frac{\|\phi(A)^{-1}A^{-1}B\phi(A)\|}{\|A^{-1}B\|^{1/\kappa}} + \frac{\|d\phi_{A_0}(A^{-1}B)\|}{\|A^{-1}B\|^{1/\kappa}} \\
&\leq C\rho(r) + C_2 r^{1-1/\kappa}
\end{aligned}$$

for all $A, B \in \mathcal{U}(A_0, r) \cap \mathcal{O}$ with $A \neq B$. Hence $\frac{\|\phi(B)-\phi(A)\|}{\|A^{-1}B\|^{1/\kappa}} \to 0$ for $r \to 0$ and the proof of (2) is complete.

(4) Fix $A_0 \in \mathcal{O}$. By Proposition 2.6.3 we denote $D^\phi\phi(A_0)$ the unique $k \times m_1$ matrix for which $d\phi_{A_0}(A) = D^\phi\phi(A_0)x^1$, for all $A = (x^1, \dots, x^\kappa) \in \mathbb{W}$.

Suppose that the thesis is not true. Then there exist a $\delta > 0$ and a sequence $(A^j) \subset \mathcal{O}$ such that $A^j \to A_0$ and

$$\|D^\phi\phi(A^j) - D^\phi\phi(A_0)\| \geq 3\delta \tag{2.58}$$

where $\|D^\phi\phi(A_0)\|$ is the usual matrix norm. Because ϕ is u.i.d. at A_0 we can consider $\mathcal{U}(A_0, r) \cap \mathcal{O}$ (for $r > 0$) such that

$$\sup\left\{ \frac{\|\phi(B) - \phi(A) - d\phi_{A_0}(A^{-1}B)\|}{\|\phi(A)^{-1}A^{-1}B\phi(A)\|} \,:\, A \neq B,\ A, B \in \mathcal{U}(A_0, r) \cap \mathcal{O} \right\} \leq \delta \tag{2.59}$$

There is no loss of generality if we suppose that $A^j = (x^{1,j}, \dots, x^{\kappa,j}) \in \mathcal{U}(A_0, r) \cap \mathcal{O}$ for each j.

Now we show that for all A^j there is $B^j = (y^{1,j}, \dots, y^{\kappa,j})$ such that

$$\|\phi(A^j)^{-1}(A^j)^{-1}B^j\phi(A^j)\| = |y^{1,j} - x^{1,j}|. \tag{2.60}$$

Fix $A^j = (x^{1,j}, \dots, x^{\kappa,j})$ and let $I := \mathcal{U}(A^j, \delta_1) \cap \mathcal{O}$ with $\delta_1 > 0$. We define

$$\begin{aligned} \mathcal{A} &:= \Big\{ B = (y^1, \dots, y^\kappa) \in I \ : \|\phi(A^j)^{-1}(A^j)^{-1}B\phi(A^j)\| = |y^1 - x^{1,j}| \Big\} \\ &= \Big\{ B = (y^1, \dots, y^\kappa) \in I \ : \ |y^s - x^{s,j} + \mathcal{P}^s(\phi(A^j), (A^j)^{-1}B)| = 0 \text{ for } \ s = 2, \dots, \kappa \Big\} \end{aligned}$$

where $\mathcal{P}$ is given by Corollary 1.3.10.

We would like to prove that there is $\delta_2 > 0$ with the property that for each $y^1 \in \mathbb{R}^{m_1}$ with $|y^1 - x^{1,j}| \le \delta_2$ there is $(y^2, \dots, y^\kappa)$ with $y^{s,j} \in [x_{m_{s-1}+1} - \delta_1, x_{m_{s-1}+1} + \delta_1] \times \cdots \times [x_{m_s} - \delta_1, x_{m_s} + \delta_1]$ for $s = 2, \dots, \kappa$ such that

$$y^s = x^{s,j} - \mathcal{P}^s(\phi(A^j), (A^j)^{-1}B) \quad \text{for} \quad s = 2, \dots, \kappa$$

i.e. $(y^1, \dots, y^\kappa) \in \mathcal{A}$.

First for all $y^1 \in \mathbb{R}^{m_1}$ with $|y^1 - x^{1,j}| \le \delta_2$ we consider the function $\gamma_{2,A} : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ given by

$$\gamma_{2,A}(y_2) := x^{2,j} - \mathcal{P}^2(\phi(A^j), (A^j)^{-1}B).$$

Using (1.17) we have

$$|\gamma_{2,A}(y_2) - x^{2,j}| = |\mathcal{P}^2(\phi(A^j), (A^j)^{-1}B)| \le C_1|y^1 - x^{1,j}| \le C_1\delta_2$$

Consequently, $\gamma_{2,A}$ is a map defined in $[x_{m_1+1} - \delta_1, x_{m_1+1} + \delta_1] \times \cdots \times [x_{m_2} - \delta_1, x_{m_2} + \delta_1]$ into itself where

$$\delta_2 \le \frac{\delta_1}{C_1}.$$

Therefore the fixed point theorem guarantees that $\gamma_{2,A}$ has a fixed point $y^2 = y_2(x^{1,j})$ if $|y^1 - x^{1,j}| \le \delta_2$, so that

$$y^2 = x^{2,j} - \mathcal{P}^2(\phi(A^j), (A^j)^{-1}B).$$

Proceeding in the same way we obtain that there exists $(y^2, \dots, y^\kappa)$ such that $B = (y^1, \dots, y^\kappa) \in \mathcal{A}$, i.e. (2.60) is true.

Now using the intrinsic differentiability of ϕ at A^j we can find a sequence of points $B^j = (y^{1,j}, \dots, y^{\kappa,j}) \in \mathcal{U}(A_0, r) \cap \mathcal{O}$ such that (2.60) holds and also

$$\frac{\|\phi(B^j) - \phi(A^j) - d\phi_{A^j}((A^j)^{-1}B^j)\|}{\|\phi(A^j)^{-1}(A^j)^{-1}B^j\phi(A^j)\|} \le \delta. \tag{2.61}$$

Since (2.61) and the uniformly intrinsic differentiability of ϕ at A_0, the vectors $(y^{1,j}-x^{1,j})$ and $(D_i^{\phi}\phi(A^j) - D_i^{\phi}\phi(A_0))$ are linearly dependent for $i = 1, \dots, m_1$ and consequently

$$\|d\phi_{A^j}((A^j)^{-1}B^j)\| = \|D^{\phi}\phi(A^j) - D^{\phi}\phi(A_0)\||y^{1,j} - x^{1,j}|.$$

Moreover by (2.58) and (2.60), we have that $\|d\phi_{A^j}((A^j)^{-1}B^j)\| \geq 3\delta\|\phi(A^j)^{-1}(A^j)^{-1}B^j\phi(A^j)\|$. Then using also (2.61), we obtain

$$\begin{aligned}
&\frac{\|\phi(B^j) - \phi(A^j) - d\phi_{A_0}((A^j)^{-1}B^j)\|}{\|\phi(A^j)^{-1}(A^j)^{-1}B^j\phi(A^j)\|} \geq \\
&\geq \frac{\|D^{\phi}\phi(A^j) - D^{\phi}\phi(A_0)\||y^{1,j} - x^{1,j}|}{\|\phi(A^j)^{-1}(A^j)^{-1}B^j\phi(A^j)\|} - \frac{\|\phi(B^j) - \phi(A^j) - d\phi_{A^j}((A^j)^{-1}B^j)\|}{\|\phi(A^j)^{-1}(A^j)^{-1}B^j\phi(A^j)\|} \\
&\geq \frac{3\delta|y^{1,j} - x^{1,j}|}{|y^{1,j} - x^{1,j}|} - \delta \geq 2\delta
\end{aligned}$$

which contradicts (2.59).

$\square$

Chapter 3

Intrinsic Regular Surfaces in Carnot groups

In this chapter we give some equivalent conditions in order that the intrinsic graph are locally regular surfaces. All the results contained here have been obtained in [32] in collaboration with R. Serapioni.

First we study the Carnot groups of step κ. More specifically, in Theorem 3.1.1 we characterize $\mathbb{G}$-regular intrinsic graphs as graphs of uniformly intrinsic differentiable functions $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ where $\mathbb{W}, \mathbb{H}$ are complementary subgroups of $\mathbb{G}$, with $\mathbb{H}$ horizontal. This result generalizes Theorem 1.2 in [7] proved in Heisenberg groups (see also [105]).

More precisely, we show that the uniform intrinsic differentiability is the additional condition to obtain locally

$$\text{graph}\,(\phi) \text{ is a } \mathbb{G}\text{-regular surface} \quad \Longleftrightarrow \quad \phi \text{ is uniformly intrinsic differentiable map}$$

A crucial role in proof of Theorem 3.1.1 is played by the matrix that represents the intrinsic differential of ϕ (Proposition 2.6.3) and which we call intrinsic gradient $D^{\phi}\phi$ of ϕ. This is the correct intrinsic replacement of Euclidean gradient for $\mathbb{C}^1$ surfaces. For instance, we remark that intrinsic regular parametrizations have continuous intrinsic gradient (see condition 3. of Proposition 2.6.11), exactly like parametrizations of regular $\mathbb{C}^1$ surfaces have continuous gradient.

Moreover we stress that the class of uniform intrinsic differentiable functions is a large class of functions. It actually includes the class of $\mathbb{C}^1$ functions (see Theorem 3.1.5).

In Section 3.2 we consider a more restrictive class of surfaces: the $\mathbb{G}$-regular hypersurfaces in groups of class $\mathcal{B}$.

Here we consider a continuous map $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$, where $\mathbb{H}$ is one dimensional subgroup of $\mathbb{G}$ (and consequently horizontal). Therefore, we can identify ϕ with a real valued map (see Remark 3.2.1).

Table 3.1: $\mathbb{C}^1$ surfaces in Euclidean spaces and corresponding definitions in Carnot groups

Euclidean spaces	**Carnot groups**
$S = \{P : F(P) = 0\} \subset \mathbb{R}^N$ $F \in \mathbb{C}^1(\mathbb{R}^N, \mathbb{R}^k)$ ∇F has rank k	$S = \{P : F(P) = 0\} \subset \mathbb{G}$ $F \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^k)$ $\nabla_{\mathbb{G}} F$ has rank k
$S = \text{graph}(\phi) = \{(A, \phi(A)) : A \in W\}$ $\phi : W \to H$ $H = \mathbb{R}^k$ and $W = \mathbb{R}^{N-k}$ H and W are complementary linear subspaces ϕ and $\nabla\phi$ are continuous	$S = \text{graph}(\phi) = \{A \cdot \phi(A) : A \in \mathbb{W}\}$ $\phi : \mathbb{W} \to \mathbb{H}$ $\mathbb{H}$ and $\mathbb{W}$ are complementary homogeneous subgroups ϕ and $D^\phi\phi$ are continuous

If we consider ϕ a real valued continuous function defined on a one codimensional homogeneous subgroup of Heisenberg groups, then it is known, after the results in [7], [15], that the intrinsic differentiability of ϕ is equivalent to the existence and continuity of suitable 'derivatives' $D^\phi_j\phi$ of ϕ. The non linear first order differential operators D^ϕ_j were introduced by Serra Cassano et al. in the context of Heisenberg groups (see [95] and the references therein). Following the notations in [95], the operators D^ϕ_j are denoted as *intrinsic derivatives* of ϕ and, as we said, from an intrinsic point of view, the operator $D^\phi\phi$ is the gradient of ϕ. Moreover, in the first Heisenberg group $\mathbb{H}^1$ the intrinsic derivative $D^\phi\phi$ is to the classical Burgers' equation.

In [7], [13], [15] (see also [12], [14]) the authors study the suitable notions of weak solution for the non-linear first order PDEs' system

$$D^\phi\phi = w \quad \text{in } \omega, \tag{3.1}$$

being w a prescribed continuous function and $\omega \subset \mathbb{R}^{N-1}$. In particular in [7], [15] the authors solve this problem in $\mathbb{H}^k$ when ϕ and w are continuous functions; while in [13] w is only a bounded measurable function (see also [1], [2]).

In particular in [7] it was introduced the concept of broad* solution of the system (3.1) (see Definition 3.2.2). In $\mathbb{H}^1$ this notion extends the classical notion of broad solution for Burger's equation through characteristic curves provided ϕ and w are locally Lipschitz continuous. In our case ϕ and w are supposed to be only continuous, so the classical theory breaks down. On the other hand broad* solution of the system (3.1) can be constructed with a continuous datum w.

In Section 3.2.2 we define the appropriate notion of intrinsic derivative in a group of class $\mathcal{B}$ and we extend Theorem 1.3 and Theorem 5.7 in [7] proved in $\mathbb{H}^k$. More precisely, in

Theorem 3.2.7 we prove that the intrinsic graph of a real valued function ϕ defined on a one codimensional homogeneous subgroup of a group of class $\mathcal{B}$ is a regular surface if and only if ϕ is broad* solution of (3.1) and it is 1/2-little Hölder continuous (see Definition 2.6.5). We also show that these assumptions are equivalent to the fact that ϕ and its intrinsic gradient can be uniformly approximated by $\mathbb{C}^1$ functions.

Finally, if the vertical layer is 1 dimensional, we can omit the little Hölder continuity of ϕ because each assumption implies it (see Theorem 3.2.8).

3.1 An characterization of $\mathbb{G}$-regular surfaces

In this section we study the relation between the following facts: the intrinsic graph of $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ is a $\mathbb{G}$-regular intrinsic graphs and ϕ is uniformly intrinsic differentiable in $\mathcal{O}$. If $\mathbb{H}$ is a horizontal subgroup we have the following theorem:

Theorem 3.1.1. *Let $\mathbb{G}$ be a Carnot group, $\mathbb{W}$ and $\mathbb{H}$ complementary subgroups, with $\mathbb{H}$ horizontal and k dimensional. Let $X_1,\dots,X_k$ be left invariant horizontal vector fields in $\mathfrak{g}$ such that $\mathbb{H}=\exp(\mathrm{span}\{X_1,\dots,X_k\})$. Finally, let $\mathcal{O}$ be open in $\mathbb{W}$, $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ and $S := \mathrm{graph}\,(\phi)$.*
Then the following are equivalent:

1. *there are $\mathcal{U}$ open in $\mathbb{G}$ and $F=(F_1,\dots,F_k)\in \mathbb{C}^1_{\mathbb{G}}(\mathcal{U};\mathbb{R}^k)$ such that*
$$\begin{aligned}&S=\{P\in\mathcal{U} : F(P)=0\}\\ &\det\left(X_iF_j\right)(Q)\neq 0, \qquad \textit{for all } Q\in\mathcal{U}.\end{aligned}$$
2. *ϕ is uniformly intrinsic differentiable in $\mathcal{O}$.*

Proof. Let us begin with the proof of the implication (1) $\Rightarrow$ (2). Denote $\nabla_{\mathbb{G}} f = \big(\mathcal{M}_1 \mid \mathcal{M}_2\big)$ where
$$\mathcal{M}_1 := \begin{pmatrix} X_1 f_1 \dots X_k f_1\\ \vdots \quad\ddots\quad \vdots\\ X_1 f_k \dots X_k f_k\end{pmatrix}, \qquad \mathcal{M}_2 := \begin{pmatrix} X_{k+1} f_1 \dots X_{m_1} f_1\\ \vdots \quad\ddots\quad \vdots\\ X_{k+1} f_k \dots X_{m_1} f_k\end{pmatrix}.$$
Fix $A\in\mathcal{O}$, let $P=A\phi(A):=\Phi(A)$ and $r>0$ be such that $I_r(A):=B(A,r)\cap\mathbb{W}\subset\mathcal{O}$.

Let $B,B'\in I_r(A)$ then
$$\begin{aligned}&|\nabla_{\mathbb{G}} f(\Phi(B))\big(\Phi(B)^{-1}\Phi(B')\big)^1|\\ &\qquad = |f(\Phi(B'))-f(\Phi(B))+\nabla_{\mathbb{G}} f(\Phi(B))\big(\Phi(B)^{-1}\Phi(B')\big)^1|\\ &\qquad \le C_1\rho(\delta_r)\,\|\Phi(B)^{-1}\Phi(B')\|\\ &\qquad \le C_1\rho(\delta_r)\left(\|\phi(B')-\phi(B)\|+\|\phi(B)^{-1}B^{-1}B'\phi(B)\|\right)\end{aligned}\tag{3.2}$$

where C_1 is given by Lemma 2.3.7 and

$$\rho(\delta) := \big\{\|\nabla_{\mathbb{G}} f(\cdot) - \nabla_{\mathbb{G}} f(P)\|_{\mathcal{L}^\infty(U(P,2d(P,P')))} : P' \in \Phi(I_\delta(A))\big\}.$$

Observe that $\big(\Phi(B)^{-1}\Phi(B')\big)^1 = \big(\phi(B') - \phi(B), (B^{-1}B')^1\big)$ and so

$$\nabla_{\mathbb{G}} f(\Phi(B))\big(\Phi(B)^{-1}\Phi(B')\big)^1 = \mathcal{M}_1(\Phi(B))\big(\phi(B') - \phi(B)\big) + \mathcal{M}_2(\Phi(B))\,(B^{-1}B')^1. \tag{3.3}$$

Now we would like to prove that there is a constant $C > 0$ such that

$$\|\phi(B') - \phi(B)\| \le C\|\phi(B)^{-1}B^{-1}B'\phi(B)\|. \tag{3.4}$$

Because $\nabla_{\mathbb{G}} f$ is uniformly continuous we deduce that

$$\lim_{r\to 0} \rho(\delta_r) = 0. \tag{3.5}$$

Using (3.2) and (3.3) we get

$$\begin{aligned}
&\Big\|\phi(B') - \phi(B) + \mathcal{M}_1^{-1}(\Phi(B))\mathcal{M}_2(\Phi(B))\,(B^{-1}B')^1\Big\| \\
&\quad = \Big\|\mathcal{M}_1^{-1}(\Phi(B))\Big(\mathcal{M}_1(\Phi(B))\big(\phi(B') - \phi(B)\big) + \mathcal{M}_2(\Phi(B))\,(B^{-1}B')^1\Big)\Big\| \\
&\quad = \big\|\mathcal{M}_1^{-1}(\Phi(B))\nabla_{\mathbb{G}} f(\Phi(B))\big(\Phi(B)^{-1}\Phi(B')\big)^1\big\| \\
&\quad \le C_1\rho(\delta_r)\,\|\mathcal{M}_1^{-1}(\Phi(B))\|\,\Big(\|\phi(B') - \phi(B)\| + \|\phi(B)^{-1}B^{-1}B'\phi(B)\|\Big)
\end{aligned} \tag{3.6}$$

for every $B, B' \in I_{\delta_r}(A)$. Now by (3.5) we can take

$$C_1\rho(\delta_{\hat r})\,\|\mathcal{M}_1^{-1}(\Phi(B))\| \le \frac{1}{2}$$

for a certain $\hat r \in (0, r_0/4)$, and so

$$\begin{aligned}
\big\|\phi(B') - \phi(B)\big\| &\le \Big\|\phi(B') - \phi(B) + \mathcal{M}_1^{-1}(\Phi(B))\mathcal{M}_2(\Phi(B))\,(B^{-1}B')^1\Big\| \\
&\qquad + \Big\|\mathcal{M}_1^{-1}(\Phi(B))\mathcal{M}_2(\Phi(B))\,(B^{-1}B')^1\Big\| \\
&\le \frac{1}{2}\big(\|\phi(B') - \phi(B)\| + \|\phi(B)^{-1}B^{-1}B'\phi(B)\|\big) + C_2|(B^{-1}B')^1| \\
&\le \frac{1}{2}\big(\|\phi(B') - \phi(B)\| + \|\phi(B)^{-1}B^{-1}B'\phi(B)\|\big) + C_2\|\phi(B)^{-1}B^{-1}B'\phi(B)\|
\end{aligned}$$

for any $B, B' \in I_{\delta_{\hat r}}(A)$. This inequality imply that (3.4) holds.

Now we are able to show the thesis, i.e. ϕ is uniformly intrinsic differentiable at A. First, using (3.6) and (3.4) we remark that there exists a constant $C_3 > 0$ for which

$$\left\| \phi(B') - \phi(B) + \mathcal{M}_1^{-1}(\Phi(B))\mathcal{M}_2(\Phi(B))\,(B^{-1}B')^1 \right\| \le C_3\rho(\delta_r)\|\phi(B)^{-1}B^{-1}B'\phi(B)\|.$$

Consequently,

(3.7)

$$\frac{1}{\|\phi(B)^{-1}B^{-1}B'\phi(B)\|}\left\| \phi(B') - \phi(B) + \mathcal{M}_1^{-1}(\Phi(A))\mathcal{M}_2(\Phi(A))\,(B^{-1}B')^1 \right\|$$
$$\le C_3\rho(\delta_r) + \sup_{I_{\delta_r}(A)} \left\| \mathcal{M}_1^{-1}(\Phi(\cdot))\mathcal{M}_2(\Phi(\cdot)) - \mathcal{M}_1^{-1}(\Phi(A))\mathcal{M}_2(\Phi(A)) \right\|$$

for any $B, B' \in I_{\delta_r}(A)$ with $r \le \hat{r}$. Finally (2.57) follows from (3.5), (3.7) and the fact that f is of class $\mathbb{C}^1_{\mathbb{G}}$. That is ϕ is uniformly intrinsic differentiable at A and moreover

$$D^\phi\phi(A) = -\mathcal{M}_1^{-1}(\Phi(A))\mathcal{M}_2(\Phi(A)). \tag{3.8}$$

So the implication (1) $\Rightarrow$ (2) is complete.

Now we want to show the converse, i.e. (2) $\Rightarrow$ (1).

Let $f : S \to \mathbb{R}^k$ and $g : S \to \mathbf{M}_{k\times m_1}$ be given by

$$f(Q) := 0, \qquad g(Q) := \left(\; \mathbb{I}_k \mid D^\phi\phi(\Phi^{-1}(Q)) \right)$$

for all $Q \in S$, where $D^\phi\phi(\Phi^{-1}(Q))$ is the unique $k \times m_1$ matrix associated to intrinsic differential $d\phi_{(\Phi^{-1}(Q))}$ of ϕ at $\Phi^{-1}(Q)$.

For any $\mathcal{K}$ compact in S let

$$\rho_{\mathcal{K}}(\delta) := \sup\left\{ \frac{|g(Q)(Q^{-1}Q')^1|}{\|Q^{-1}Q'\|} : Q, Q' \in \mathcal{K}, 0 < \|Q^{-1}Q'\| < \delta \right\}.$$

Observe that

$$\lim_{\delta\to 0} \rho_{\mathcal{K}}(\delta) = 0. \tag{3.9}$$

Indeed because $\|\phi(B)^{-1}B^{-1}B'\phi(B)\| \le \frac{1}{c_0}\|\Phi(B)^{-1}\Phi(B')\|$, we have that

$$\frac{|g(Q)(Q^{-1}Q')^1|}{\|Q^{-1}Q'\|} \le \frac{1}{c_0}\,\frac{\|\phi(B') - \phi(B) - D^\phi\phi(B)(B^{-1}B')^1\|}{\|\phi(B)^{-1}B^{-1}B'\phi(B)\|}$$

when $B = \Phi^{-1}(Q)$ and $B' = \Phi^{-1}(Q')$. Moreover by the u.i.d. of ϕ in $\mathcal{O}$, for all $B \in \Phi^{-1}(\mathcal{K})$

$$\sup\left\{ \frac{\|\phi(B') - \phi(B) - D^\phi\phi(B)(B^{-1}B')^1\|}{\|\phi(B)^{-1}B^{-1}B'\phi(B)\|} \;:\; B' \in \Phi^{-1}(\mathcal{K}),\, 0 < \|\phi(B)^{-1}B^{-1}B'\phi(B)\| < \delta \right\} \to 0$$

as $\delta \to 0$ and so by compactness of $\mathcal{K}$ we conclude that

$$\sup\left\{\frac{\|\phi(B') - \phi(B) - D^\phi\phi(B)(B^{-1}B')^1\|}{\|\phi(B)^{-1}B^{-1}B'\phi(B)\|} \,:\, B, B' \in \Phi^{-1}(\mathcal{K}),\, 0 < \|\phi(B)^{-1}B^{-1}B'\phi(B)\| < \delta\right\} \to 0$$

as $\delta \to 0$. Hence (3.9) holds.

By Whitney's extension Theorem (see Theorem 2.3.8) there is a function $\hat{f} \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^k)$ such that

$$\hat{f} = f = 0 \qquad \text{in } S$$
$$\nabla_{\mathbb{G}}\hat{f}(Q) = g(Q) = \left(\, \mathbb{I}_k \mid D^\phi\phi(\Phi^{-1}(Q))\,\right) \qquad \text{for all } Q \in S$$

and so $\operatorname{rank}\nabla_{\mathbb{G}}\hat{f}(Q) = k$ for all $Q \in S$. Consequently the proof is complete. □

Remark 3.1.2. It follows from Theorem 3.1.1 and from Proposition 2.6.11 that if $S = \operatorname{graph}(\phi)$ is $\mathbb{G}$-regular then the intrinsic gradient of ϕ is continuous.

Remark 3.1.3. Under the same assumptions of Theorem 3.1.1 with the additional condition that $\mathbb{H}$ is a 1-dimensional subgroup of $\mathbb{G}$, the intrinsic gradient of ϕ defined in (3.8) is

$$D^\phi\phi(A) = -\left(\frac{X_2 f}{X_1 f}, \dots, \frac{X_{m_1} f}{X_1 f}\right)(\Phi(A)) \tag{3.10}$$

We denote the *horizontal normal* to S at $P \in S$, the unit vector $\nu_S(P)$ defined as

$$\nu_S(P) := -\frac{\nabla_{\mathbb{G}} f(P)}{|\nabla_{\mathbb{G}} f(P)|}.$$

Consequently, the horizontal normal to S at $\Phi(A) \in S$ is

$$\nu_S(\Phi(A)) = \left(\pm\frac{1}{\sqrt{1 + |D^\phi\phi(A)|^2}}, \frac{D^\phi\phi(A)}{\sqrt{1 + |D^\phi\phi(A)|^2}}\right) \in \mathbb{R}^{m_1}.$$

From Theorem 3.1.1, Proposition 2.6.11 it follows that

Corollary 3.1.4. *Under the same assumptions of Theorem* 3.1.1*, if* $S = \operatorname{graph}(\phi)$ *is a* $\mathbb{G}$*-regular surface then*

$$\phi \in h^{1/\kappa}_{loc}(\mathcal{O}).$$

Finally we show that $\mathbb{C}^1$ functions are uniformly intrinsic differentiable too.

Theorem 3.1.5. *Under the same assumptions of Theorem* 3.1.1 *with the additional condition that* $\phi \in \mathbb{C}^1(\mathcal{O}, \mathbb{H})$*. Then* ϕ *is u.i.d. in* $\mathcal{O}$*.*

Proof. Let $f : \mathbb{G} \to \mathbb{R}^k$ defined as for all $P = P_{\mathbb{W}} \cdot \exp(\sum_{i=1}^k t_i X_i) \in \mathbb{G}$

$$f(P) = (t_1 - \phi_1(P_{\mathbb{W}}), \dots, t_k - \phi_k(P_{\mathbb{W}})).$$

It is clear that $f(P) = 0$, for all $P \in \operatorname{graph}(\phi)$, and that $f \in \mathbb{C}^1(\mathbb{G}, \mathbb{R}^k)$. Hence $f \in \mathbb{C}^1_{\mathbb{G}}(\mathbb{G}, \mathbb{R}^k)$ and rank $\nabla_{\mathbb{G}} f = k$. Indeed

$$X_l f_j(P) = \frac{d}{ds} f_j(P \exp(sX_l))_{|s=0} = \begin{cases} 1, & \text{if } j = l \\ 0, & \text{if } j \neq l \end{cases}$$

Now using Theorem 3.1.1, we obtain that ϕ is u.i.d. in $\mathcal{O}$. □

3.2 1-Codimensional Intrinsic graphs

In this section we characterize uniformly intrinsic differentiable maps $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$, when $\mathbb{V}$ is one dimensional and horizontal, in terms of existence and continuity of suitable intrinsic derivatives of ϕ. Intrinsic derivatives are first order non linear differential operators depending on the structure of the ambient space $\mathbb{G}$ and on the choice of two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$.

In order to do this we have to restrict the ambient space $\mathbb{G}$ under consideration to a subclass of Carnot groups of step two. These groups, denoted here as groups of class $\mathcal{B}$, are described in the Section 1.4 where we follow the notations of Chapter 3 of [17].

3.2.1 The intrinsic gradient

Let $\mathbb{G} = (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$ be a group of class $\mathcal{B}$ as in Definition 1.4.1 and $\mathbb{W}$, $\mathbb{V}$ be complementary subgroups in $\mathbb{G}$ with $\mathbb{V}$ horizontal and one dimensional.

We are going to characterize uniformly intrinsic differentiable functions $\phi : \mathbb{W} \to \mathbb{V}$ (see Definitions 2.6.2 and 2.6.4) in terms of existence and continuity of *intrinsic derivatives* of ϕ.

These intrinsic derivatives are non linear first order differential operators that can be explicitly written in terms only of the matrices $\mathcal{B}^{(s)}$.

Remark 3.2.1. To keep notations simpler, through all this section we assume, without loss of generality, that the complementary subgroups $\mathbb{W}$, $\mathbb{V}$ are

$$\mathbb{V} := \{(x_1, 0 \dots, 0)\}, \qquad \mathbb{W} := \{(0, x_2, \dots, x_{m+n})\}. \tag{3.11}$$

This amount simply to a linear change of variables in the first layer of the algebra $\mathfrak{g}$. If we denote $\mathcal{M}$ a non singular $m \times m$ matrix, the linear change is

$$(x, y) \mapsto (\xi, y) = (\mathcal{M}x, y)$$

The new composition law $\star$ in $\mathbb{R}^{m+n}$, obtained by writing $\cdot$ in the new coordinates, is

$$(\xi, y) \star (\xi', y') := (\xi + \xi', y + y' + \frac{1}{2}\langle \tilde{\mathcal{B}}\xi, \xi' \rangle),$$

where $\tilde{\mathcal{B}} := (\tilde{\mathcal{B}}^{(1)}, \dots, \tilde{\mathcal{B}}^{(n)})$ and $\tilde{\mathcal{B}}^{(s)} = (\mathcal{M}^{-1})^T \mathcal{B}^{(s)} \mathcal{M}^{-1}$ for $s = 1, \dots, n$. It is easy to check that the matrices $\tilde{\mathcal{B}}^{(1)}, \dots, \tilde{\mathcal{B}}^{(n)}$ are skew-symmetric and that $(\mathbb{R}^{m+n}, \star, \delta_\lambda)$ is a Carnot group of class $\mathcal{B}$ isomorphic to $\mathbb{G} = (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$.

When $\mathbb{V}$ and $\mathbb{W}$ are defined as in (3.11) we identify $(0, x_2, \dots, x_{m+n}) \in \mathbb{W}$ with $(x_2, \dots, x_{m+n}) \in \mathbb{R}^{m+n-1}$ and we use the same symbol ω for an open set in $\mathbb{R}^{m+n-1}$ and an open set in $\mathbb{W}$. Moreover we identify $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$ with the real valued $\psi : \omega \subset \mathbb{R}^{m+n-1} \to \mathbb{R}$ defined as

$$\phi(0, x_2, \dots, x_{m+n}) := (\psi(x_2, \dots, x_{m+n}), 0, \dots, 0). \tag{3.12}$$

Proposition 3.2.2. *If $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$ is continuous then, for all $A = (x, y), B = (x', y') \in \omega$, the "graph distance" $\rho_\phi : \omega \times \omega \to [0, +\infty)$, defined in Definition 2.6.3, takes the form*
(3.13)

$$\begin{aligned}\rho_\phi(A, B) := & \frac{1}{2} \max \left\{ |x' - x|, \, \epsilon \left| y' - y + \langle \mathcal{B}(x' - x), (\psi(A), 0, \dots, 0) \rangle - \frac{1}{2} \langle \mathcal{B}x, x' \rangle \right|^{\frac{1}{2}} \right\} \\ & + \frac{1}{2} \max \left\{ |x - x'|, \, \epsilon \left| y - y' + \langle \mathcal{B}(x - x'), (\psi(B), 0, \dots, 0) \rangle - \frac{1}{2} \langle \mathcal{B}x', x \rangle \right|^{\frac{1}{2}} \right\}\end{aligned}$$

where $\epsilon \in (0, 1]$ is given by (1.30).

Proof. By direct computation it can be checked

$$\begin{aligned}\phi(A)^{-1}A^{-1}B\phi(A) &= \left(x' - x, y' - y + \langle \mathcal{B}(x' - x), (\psi(A), 0) \rangle - \frac{1}{2} \langle \mathcal{B}x, x' \rangle \right) \\ \phi(B)^{-1}B^{-1}A\phi(B) &= \left(x - x', y - y' + \langle \mathcal{B}(x - x'), (\psi(B), 0) \rangle - \frac{1}{2} \langle \mathcal{B}x', x \rangle \right)\end{aligned} \tag{3.14}$$

for all $A = (x, y), B = (x', y') \in \omega$.

□

It is clear from (3.14) that $\rho_\phi(A, B)$ defined as (3.13) is the symmetrized version of $\|\phi(A)^{-1}A^{-1}B\phi(A)\|$. Then we have

Proposition 3.2.3. *Let $\mathbb{W}$ and $\mathbb{V}$ be complementary subgroups of $\mathbb{G}$ with $\mathbb{V}$ horizontal. Let $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$ with ω open in $\mathbb{W}$. Then ϕ is u.i.d. in $A_0 \in \omega$ if and only if ϕ is intrinsic differentiable in A_0 and if*

$$\lim_{r \to 0^+} \sup \left\{ \frac{\|\phi(B) - \phi(A) - d\phi_{A_0}(A^{-1}B)\|}{\rho_\phi(A, B)} : \quad A \neq B, \, \|A_0^{-1}A\| < r, \, \|A_0^{-1}B\| < r \right\} = 0.$$

Proof. The proposition follows from Proposition 2.6.8; indeed there is $C > 0$ such that

$$C\rho_\phi(A,B) \le \|\phi(A)^{-1}A^{-1}B\phi(A)\| \le 2\rho_\phi(A,B)$$

for all $A, B \in \omega$ such that $\|A^{-1}B\| \le r$ (for $r > 0$). $\square$

We show here that (3.10) can be written as a combination of left invariant vector fields defined as restrictions of the vector fields X_j, Y_s in (1.29) which are a basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. More precisely, we recall that if $\mathcal{B}^{(s)} = (b_{ij}^{(s)})_{i,j=1}^m$, then

$$X_j(x,y) = \partial_{x_j} + \frac{1}{2}\sum_{s=1}^{n}\left(\sum_{i=1}^{m} b_{ji}^{(s)} x_i\right)\partial_{y_s}, \qquad Y_s(x,y) = \partial_{y_s},$$

for $j = 1, \dots, m$, and $s = 1, \dots, n$.

Proposition 3.2.4. *With the notations of Definition* 1.4.1*, let* $\mathbb{G} := (\mathbb{R}^{m+n}, \cdot, \delta_\lambda)$ *be a Carnot group of class* $\mathcal{B}$*,* $\mathbb{V}$ *and* $\mathbb{W}$ *the complementary subgroups defined in* (3.11)*. Let* Ω *be open in* $\mathbb{G}$ *and* $f \in \mathbb{C}^1_{\mathbb{G}}(\Omega)$ *with* $X_1 f > 0$*.*

(i) Let $S := \{P \in \Omega : f(P) = 0\}$ *and assume that* $S \neq \emptyset$*. Then there are* ω *open in* $\mathbb{W}$ *and* $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$ *such that* $S = \operatorname{graph}(\phi)$*. Moreover* ϕ *is uniformly intrinsic differentiable in* ω *and, for all* $A := (x,y) \in \omega$*, the intrinsic gradient* $D^\phi\phi(A) = \Big(D_2^\phi\phi(A), \dots, D_m^\phi\phi(A)\Big)$ *takes the following form*

$$D_j^\phi\phi(A) = X_j\psi(A) + \psi(A)\sum_{s=1}^{n} b_{j1}^{(s)} Y_s\psi(A) \tag{3.15}$$

for $j = 2, \dots, m$*, where the equality* (3.15) *has to be meant in distributional sense.*

(ii) Let $E := \{P \in \Omega : f(P) < 0\}$*. Then* E *has locally finite* $\mathbb{G}$*-perimeter in* Ω *and its* $\mathbb{G}$*-perimeter measure* $|\partial E|_{\mathbb{G}}$ *has the integral representation*

$$|\partial E|_{\mathbb{G}}(\mathcal{F}) = \int_{\Phi^{-1}(\mathcal{F})} \sqrt{1 + |D^\psi\psi|^2}\, d\mathcal{L}^{m+n-1} \tag{3.16}$$

for every Borel set $\mathcal{F} \subset \Omega$*.*

Proof. The existence of $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$ such that $S = \operatorname{graph}(\phi)$ follows from Implicit Function Theorem. Moreover ϕ is uniformly intrinsic differentiable in ω by Theorem 3.1.1. Let $\psi : \omega \subset \mathbb{R}^{m+n-1} \to \mathbb{R}$ be the real valued continuous function associated to ϕ as in (3.12).

By a standard approximation argument (see Theorem 2.1 in [45]) for any $\mathcal{U} \Subset \Omega$ and $0 < \epsilon < 1$, there are functions $f_\epsilon : \mathcal{U} \to \mathbb{R}$ such that $f_\epsilon \in \mathbb{C}^1(\mathcal{U})$, $X_1 f_\epsilon > 0$ on $\mathcal{U}$,

$$f_\epsilon \to f \quad \text{and} \qquad X_j f_\epsilon \to X_j f \quad \text{for } j = 1, \dots, m \tag{3.17}$$

uniformly on $\mathcal{U}$ as $\epsilon \to 0^+$.

Because $f_\epsilon \in \mathbb{C}^1(\mathcal{U})$, by the standard implicit function theorem, there is $\omega' \subset \omega$ and functions $\phi_\epsilon : \omega' \subset \mathbb{W} \to \mathbb{V}$ such that $f_\epsilon(A \cdot \phi_\epsilon(A)) = 0$ for all $A \in \omega'$, and

$$\phi_\epsilon \to \phi \tag{3.18}$$

uniformly on ω' for $\epsilon \to 0^+$.

Finally, we denote as $\psi_\epsilon : \omega' \subset \mathbb{R}^{m+n-1} \to \mathbb{R}$ the functions associated to $\phi_\epsilon : \omega' \subset \mathbb{W} \to \mathbb{V}$ as in (3.12) and as $\Phi_\epsilon : \omega' \to \mathbb{G}$ the graph map of ϕ_ϵ given by $A \mapsto \Phi_\epsilon(A) := A \cdot \phi_\epsilon(A)$. Clearly, $\psi_\epsilon \in \mathbb{C}^1(\omega')$ and $\psi_\epsilon \to \psi$ uniformly on ω' for $\epsilon \to 0^+$.

Differentiating the equality $f_\epsilon\big(A \cdot \phi_\epsilon(A)\big) = 0$, for $A = (0, x_2, \dots, x_m, y_1, \dots, y_n) \in \omega'$, we get

$$\begin{aligned}
\partial_{x_j}\psi_\epsilon(A) &= -\frac{\partial_{x_j} f_\epsilon(\Phi_\epsilon(A)) - \frac{1}{2}\psi_\epsilon(A)\sum_{s=1}^n b_{j1}^{(s)}\partial_{y_s} f_\epsilon(\Phi_\epsilon(A))}{X_1 f_\epsilon(\Phi_\epsilon(A))}, \quad && j = 2, \dots, m\\
\partial_{y_s}\psi_\epsilon(A) &= -\frac{\partial_{y_s} f_\epsilon(\Phi_\epsilon(A))}{X_1 f_\epsilon(\Phi_\epsilon(A))}, && s = 1, \dots, n
\end{aligned}$$

and consequently, using also the skew symmetry of the matrices $\mathcal{B}^{(s)}$,

$$\begin{aligned}
&-\frac{X_j f_\epsilon(\Phi_\epsilon(A))}{X_1 f_\epsilon(\Phi_\epsilon(A))}\\
&= -\frac{\partial_{x_j} f_\epsilon(\Phi_\epsilon(A)) + \frac{1}{2}\sum_{s=1}^n \left(\psi_\epsilon(A) b_{j1}^{(s)} + \sum_{i=2}^m x_i b_{ji}^{(s)}\right)\partial_{y_s} f_\epsilon(\Phi_\epsilon(A))}{X_1 f_\epsilon(\Phi_\epsilon(A))}\\
&= -\frac{\partial_{x_j} f_\epsilon(\Phi_\epsilon(A)) - \frac{1}{2}\psi_\epsilon(A)\sum_{s=1}^n b_{j1}^{(s)}\partial_{y_s} f_\epsilon(\Phi_\epsilon(A))}{X_1 f_\epsilon(\Phi_\epsilon(A))} -\\
&\qquad\qquad - \frac{\sum_{s=1}^n \left(\psi_\epsilon(A) b_{j1}^{(s)} + \frac{1}{2}\sum_{i=2}^m x_i b_{ji}^{(s)}\right)\partial_{y_s} f_\epsilon(\Phi_\epsilon(A))}{X_1 f_\epsilon(\Phi_\epsilon(A))}\\
&= \partial_{x_j}\psi_\epsilon(A) + \sum_{s=1}^n \left(\psi_\epsilon(A) b_{j1}^{(s)} + \frac{1}{2}\sum_{i=2}^m x_i b_{ji}^{(s)}\right)\partial_{y_s}\psi_\epsilon(A).
\end{aligned}$$

Then, from (3.10)

$$D_j^{\phi_\epsilon}\phi_\epsilon(A) = \partial_{x_j}\psi_\epsilon(A) + \sum_{s=1}^n \left(\psi_\epsilon(A) b_{j1}^{(s)} + \frac{1}{2}\sum_{i=2}^m x_i b_{ji}^{(s)}\right)\partial_{y_s}\psi_\epsilon(A)$$

Letting $\epsilon \to 0^+$ and using (3.17) and (3.18), (3.15) holds as an equality of distributions. Finally using again Implicit Function Theorem in [45], we know that

$$|\partial E|_{\mathbb{G}}(\bar{\mathcal{U}}) = \int_{I_\delta} \frac{|\nabla_{\mathbb{G}} f(\Phi(A))|}{X_1 f(\Phi(A))}\, d\mathcal{L}^{m+n-1}(A)$$

and, consequently, the integral representation (3.16) is true because $D^{\psi}_j\psi = -\frac{X_j f}{X_1 f}\circ\Phi$ for all $j = 2,\dots,m$ (see Remark 3.1.3). This completes the proof. □

From this proposition and Remark 3.1.3, we have that if graph$\,(\phi)$ is a $\mathbb{G}$-regular hypersurfaces then the intrinsic gradient of ψ (where ψ is the function associated to ϕ as in (3.12)) takes the explicit form as in (3.15). This fact motivates the following definition of intrinsic gradient operator.

Definition 3.2.1. Let ω be open in $\mathbb{R}^{m+n-1}$ and $\psi : \omega \subset \mathbb{R}^{m+n-1} \to \mathbb{R}$ be continuous. The *intrinsic horizontal gradient* D^{ψ} associated with ψ is the family of $(m-1)$ first order differential operators, or continuous vector fields,

$$
\begin{aligned}
&D^{\psi} := (D^{\psi}_2,\dots,D^{\psi}_m)\\
&D^{\psi}_j := \partial_{x_j} + \sum_{s=1}^{n}\left(\psi b^{(s)}_{j1} + \frac{1}{2}\sum_{i=2}^{m} x_i b^{(s)}_{ji}\right)\partial_{y_s} = X_{j|\mathbb{W}} + \psi\sum_{s=1}^{n} b^{(s)}_{j1} Y_{s|\mathbb{W}}
\end{aligned}
\tag{3.19}
$$

for $j = 2,\dots,m$.

The following Proposition gives a further geometric characterization of the vector field $D^{\psi}\psi$. Indeed, at least for regular maps, each intrinsic derivative $D^{\psi}_j\psi$ is the derivative of ψ along the integral curves of the vector field D^{ψ}_j.

Proposition 3.2.5. *Let ω be an open subset of $\mathbb{R}^{m+n-1}$ and let $\psi : \omega \to \mathbb{R}$ be intrinsic differentiable in ω. Assume that, for $j = 2,\dots,m$ and $\delta > 0$, $\gamma^j : [-\delta,\delta] \to \omega$ be an integral curve of the vector field D^{ψ}_j such that*

$$t \mapsto \psi(\gamma^j(t))$$

is of class $\mathbb{C}^1([-\delta,\delta])$. Then we have

$$\lim_{t\to 0}\frac{\psi\big(\gamma^j(t)\big)-\psi\big(\gamma^j(0)\big)}{t} = D^{\psi}_j\psi(\gamma^j(0)). \tag{3.20}$$

Proof. Fix $j = 2,\dots,m$. Let $A = (x,y) \in \omega$ such that $\gamma^j(0) = A$. We know that $\gamma^j(t) = \big(\gamma^j_2(t),\dots,\gamma^j_{m+n}(t)\big)$ is given by

$$\gamma^j_h(t) = \begin{cases} x_h & h = 2,\dots,m\,,\ h \neq j\\ x_h + t & h = j\\ y_h + \frac{1}{2}t\sum_{i=2}^{m} x_i b^{(h)}_{ji} + b^{(h)}_{j1}\int_0^t \psi(\gamma^j(r))\,dr & h = m+1,\dots,m+n \end{cases} \tag{3.21}$$

Now we would like to show that

$$\rho_{\phi}(\gamma^j(t), A) \le C|t| \tag{3.22}$$

for a certain $C \geq 1$. First we observe that

$$\begin{aligned}\rho_\phi(\gamma^j(t), \gamma^j(0)) &= \frac{1}{2}\max\left\{|t|, \epsilon\sum_{s=1}^n \left| b_{1j}^{(s)}\int_0^t \psi(\gamma^j(r))\,dr - tb_{1j}^{(s)}\psi\big(\gamma^j(0)\big)\right|^{\frac{1}{2}}\right\}\\ &\quad + \frac{1}{2}\max\left\{|t|, \epsilon\sum_{s=1}^n \left| b_{1j}^{(s)}\int_0^t \psi(\gamma^j(r))\,dr - tb_{1j}^{(s)}\psi\big(\gamma^j(t)\big)\right|^{\frac{1}{2}}\right\}\\ &=: \frac{1}{2}\max\left\{|t|,\ ,\epsilon\sum_{s=1}^n \big|\Delta_s(t)\big|^{\frac{1}{2}}\right\} + \frac{1}{2}\max\left\{|t|,\ ,\epsilon\sum_{s=1}^n \big|\Theta_s(t)\big|^{\frac{1}{2}}\right\}\end{aligned}$$

So it is sufficient to show that $\big|\Delta_s(t)\big| \leq \hat{C}t^2$ and $\big|\Theta_s(t)\big| \leq \hat{C}t^2$ for all $s = 1, \dots, n$ and for a certain $\hat{C} > 0$; indeed for $h = m+1, \dots, m+n$, we have that the map $t \mapsto \gamma_h^j(t)$ is of class $\mathbb{C}^2$ (because of (3.21) and the hypothesis that $t \mapsto \psi(\gamma^j(t))$ is $\mathbb{C}^1$) and we also note that $\sum_{i=2}^m x_i b_{ji}^{(s)}$ is a finite sum of real numbers for each $s = 1, \dots, n$.

Then for $s = 1, \dots, n$

$$\Delta_s(t) = b_{1j}^{(s)}\left(\int_0^t \psi(\gamma^j(r))\,dr - t\psi(\gamma^j(A))\right) = O(t^2)$$

and

$$\begin{aligned}\Theta_s(t) &= b_{1j}^{(s)}\left(\int_0^t \psi(\gamma^j(r))\,dr - t\psi(\gamma^j(t))\right)\\ &= b_{1j}^{(s)}\left(\int_0^t \Big(\psi(\gamma^j(r)) - \psi(A)\Big)\,dr - t\Big(\psi(\gamma^j(t)) - \psi(A)\Big)\right)\\ &= O(t^2).\end{aligned}$$

As a consequence, (3.22) holds with $C := \max\{1, \sqrt{\hat{C}}\}$. Finally from (3.22) we get

$$\frac{\big|\psi\big(\gamma^j(t)\big) - \psi\big(\gamma^j(0)\big) - D_j^\psi\psi(A)t\big|}{t} \leq C\,\frac{\big|\psi\big(\gamma^j(t)\big) - \psi(A) - d\psi_A\big(A^{-1}\gamma^j(t)\big)\big|}{\rho_\phi(\gamma^j(t), A)}$$

where $d\psi_A$ is the intrinsic differential of ψ at A. By letting $t \to 0$ and using the intrinsic differentiable of ψ at A, we obtain the thesis (3.20). □

3.2.2 Broad* solutions and D^ψ-exponential maps

In this section we introduce a generalized notion of solution of the equation

$$D^\psi\psi = w$$

in an open set $\omega \subset \mathbb{W}$ and we study its relations with the notion of uniform intrinsic differentiability.

Definition 3.2.2. Let $\omega \subset \mathbb{R}^{m+n-1}$ be an open set and $w : \omega \to \mathbb{R}^{m-1}$ be a continuous function. For any $\psi \in \mathbb{C}(\omega)$, let D^ψ be the differential operators defined in (3.19). We say that $\psi \in \mathbb{C}(\omega)$ is a *broad* solution* in ω of the system

$$D^\psi \psi = w$$

if for every $A \in \omega$ and $j = 2, \dots, m$ there are $0 < \delta_2 < \delta_1$ and $m - 1$ maps, called the exponential maps of D^ψ,

$$\begin{aligned} \exp_A(\cdot D_j^\psi)(\cdot) : \quad & [-\delta_2, \delta_2] \times I_{\delta_2}(A) \to I_{\delta_1}(A) \\ & (t, B) \mapsto \exp_A(tD_j^\psi)(B) \end{aligned}$$

such that if $\gamma_B^j(t) := \exp_A(tD_j^\psi)(B)$,

1. $\gamma_B^j \in \mathbb{C}^1([-\delta_2, \delta_2])$
2. $\begin{cases} \dot\gamma_B^j = D_j^\psi \circ \gamma_B^j \\ \gamma_B^j(0) = B \end{cases}$
3. $\psi(\gamma_B^j(t)) - \psi(\gamma_B^j(0)) = \int_0^t w_j(\gamma_B^j(r))\, dr \qquad$ for all $B \in I_{\delta_2}(A)$.

Remark 3.2.6. If the exponential maps of D^ψ at A exist, then the map

$$[-\delta_2, \delta_2] \ni t \longmapsto \psi\Big(\exp_A(tD_j^\psi)(B)\Big)$$

is of class $\mathbb{C}^1$ for every $j = 2, \dots, m$ and for each $B \in I_{\delta_2}(A)$.

See also Theorems 1.2 and 1.3 of [7] when $\mathbb{G}$ is an Heisenberg group.

Theorem 3.2.7. *Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ be a group of class $\mathcal{B}$ with $\mathbb{V}$ one dimensional and let $\phi : \omega \to \mathbb{V}$ be a continuous function, where ω is a relatively open subset of $\mathbb{W}$ and $\psi : \omega \to \mathbb{R}$ is the map associated to ϕ as in* (3.12)*. Then the following conditions are equivalent:*

1. *$S := \mathrm{graph}\,(\phi)$ is a $\mathbb{G}$-regular hypersurface.*
2. *ϕ is u.i.d. in ω.*
3. *there exists $w \in \mathbb{C}^0(\omega, \mathbb{R}^{m-1})$ such that, in distributional sense,*

$$D^\psi \psi = w \ \text{ in } \omega$$

and there is a family $(\psi_\epsilon)_{\epsilon>0} \subset \mathbb{C}^1(\omega)$ such that, for any open $\omega' \Subset \omega$,

$$\psi_\epsilon \to \psi \quad \text{and} \quad D^{\psi_\epsilon}\psi_\epsilon \to w \quad \text{uniformly on } \omega', \ \text{ as } \epsilon \to 0^+.$$

4. $\psi \in h^{1/2}_{loc}(\omega)$ *and* ψ *is a broad* solution of* $D^{\psi}\psi = w$ *in* ω.

Proof. We split the proof in several steps.

(**1**) $\iff$ (**2**) see Theorem 3.1.1.

(**2**) $\implies$ (**3**) We begin proving that for all $A \in \omega$ there are $\delta = \delta(A) > 0$ with $I_\delta(A) \Subset \omega$ and a family $(\psi_\epsilon)_{\epsilon>0} \subset \mathbb{C}^1(I_\delta(A))$ such that

$$\psi_\epsilon \to \psi \ \text{ and } \ D^{\psi_\epsilon}\psi_\epsilon \to D^{\psi}\psi \qquad \text{uniformly on } \ I_\delta(A), \ \ \text{as } \ \epsilon \to 0. \tag{3.23}$$

Fix $A \in \omega$. Let $P := \Phi(A)$ where Φ is the graph map of ϕ defined as (2.20). Because $S := \operatorname{graph}(\phi)$ is a $\mathbb{G}$-regular hypersurface we know that there are $r > 0$, $\delta > 0$ and $f \in \mathbb{C}^1_{\mathbb{G}}(U(P,r))$ such that $f \circ \Phi = 0$ in $I_\delta(A)$.

Now we use some results proved in Implicit Function Theorem in [45] (see Theorem 2.1 in [45]). Arguing as in Step 1 of this theorem we can prove the existence of $0 < r' < r$ and of a family $(f_\epsilon)_{\epsilon>0} \subset \mathbb{C}^1(U(P,r'))$ such that

$$f_\epsilon \to f \ \text{ and } \ \nabla_{\mathbb{G}} f_\epsilon \to \nabla_{\mathbb{G}} f \qquad \text{uniformly on } U(P,r'), \ \ \text{as } \ \epsilon \to 0.$$

Moreover as in Step 3 of Implicit Function Theorem in [45] there is $(\psi_\epsilon)_{\epsilon>0} \subset \mathbb{C}^1(I_\delta(A))$ satisfying

$$\psi_\epsilon \to \psi \ \text{ and } \ -\frac{\widehat{\nabla_{\mathbb{G}}} f_\epsilon}{X_1 f_\epsilon} \circ \Phi_\epsilon \to -\frac{\widehat{\nabla_{\mathbb{G}}} f}{X_1 f} \circ \Phi = D^{\psi}\psi \qquad \text{uniformly on } I_\delta(A), \ \ \text{as } \ \epsilon \to 0$$

where $\widehat{\nabla_{\mathbb{G}}} f := (X_2 f, \dots, X_m f)$ and Φ_ϵ are the graph map of $\phi_\epsilon = (\psi_\epsilon, 0, \dots, 0)$ defined as $\Phi_\epsilon(A) = A \cdot \phi_\epsilon(A)$. Hence $f_\epsilon \circ \Phi_\epsilon \equiv 0$.

Moreover, the set $S_\epsilon := \{Q \in U(P,r') \,:\, f_\epsilon(Q) = 0\} \supset \Phi_\epsilon(I_\delta(A))$ is an Euclidean $\mathbb{C}^1$ surface. Hence, because $f_\epsilon \in \mathbb{C}^1(U(P,r')) \subset \mathbb{C}^1_{\mathbb{G}}(U(P,r'))$ (see Remark 2.3.6) and because of Theorem 3.1.1 we have that ϕ_ϵ (i.e. the parametrization of S_ϵ) is uniformly intrinsic differentiable and by Remark 3.1.3 we get

$$D^{\psi_\epsilon}\psi_\epsilon = -\frac{\widehat{\nabla_{\mathbb{G}}} f_\epsilon}{X_1 f_\epsilon} \circ \Phi_\epsilon,$$

completing the proof of (3.23).

From what proved up to now we know that for all $B \in \omega$ there is $\delta = \delta(B) > 0$ such that $I_\delta(B) \Subset \omega$ and there is family $\mathcal{A} = \mathcal{A}(B)$ of $\mathbb{C}^1$ functions defined as $\mathcal{A} := \{\psi_{\epsilon,B} : I_\delta(B) \to \mathbb{R}\}_{0<\epsilon<1}$ such that (3.23) holds.

Now let $\hat{\mathcal{F}} := \{\operatorname{Int}(I_\delta(B)) : B \in \omega\}$ be open covering of ω. Then there exists a locally finite covering $\mathcal{F}$ of $\hat{\mathcal{F}}$ and $\{\theta_h : \omega \to \mathbb{R} \,|\, h \in \mathbb{N}\}$ a partition of the unity subordinate to $\mathcal{F}$.

Let $\psi_{\epsilon,B_h} : I_h \to \mathbb{R}$, with $\psi_{\epsilon,B_h} \in \mathcal{A}$. We consider $\psi_{\epsilon,h} := \psi_{\epsilon,B_h} : \mathbb{R}^{m+n-1} \to \mathbb{R}$ where from now on, if necessary, we use the convention of extending functions by letting them vanish outside their domain. Let $\psi_\epsilon := \sum_{h=1}^{\infty} \theta_h \psi_{\epsilon,h}$; by construction $\psi_\epsilon \in \mathbb{C}^1(\omega)$ and

$$D^{\psi_\epsilon}\psi_\epsilon = \sum_{h=1}^{\infty}\Big(\psi_{\epsilon,h}D^{\psi_\epsilon}\theta_h + \theta_h D^{\psi_\epsilon}\psi_{\epsilon,h}\Big) \quad \text{on } \omega$$

Because the partition is locally finite, there are only a finite number of $h_1, \dots, h_l$ such that $\overline{\omega'} \cap \operatorname{spt}\theta_{h_\tau} \neq \emptyset$ for each $\tau = 1, \dots, l$ and $\overline{\omega'} \subset \bigcup_{\tau=1}^{l} \operatorname{spt}\theta_{h_\tau}$. Then

$$\psi_\epsilon = \sum_{\tau=1}^{l} \theta_{h_\tau}\psi_{\epsilon,h_\tau} \ \text{ and } \ \psi = \sum_{\tau=1}^{l} \theta_{h_\tau}\psi \ \text{ on } \overline{\omega'}$$

$$D^{\psi_\epsilon}\psi_\epsilon = \sum_{\tau=1}^{l}\Big(\phi_{\epsilon,h_\tau}D^{\psi_{\epsilon,h_\tau}}\theta_{h_\tau} + \theta_{h_\tau}D^{\psi_{\epsilon,h_\tau}}\psi_{\epsilon,h_\tau}\Big) \text{ on } \overline{\omega'}$$

Putting together the last equalities and (3.23) we get

$$\psi_\epsilon \to \psi \quad \text{and} \quad D^{\psi_\epsilon}\psi_\epsilon \to \sum_{\tau=1}^{l}\Big(\psi D^{\psi}\theta_{h_\tau} + \theta_{h_\tau}D^{\psi}\psi\Big) =: w \quad \text{uniformly on } \overline{\omega'}, \ \text{ as } \epsilon \to 0$$

Note that for $j = 2, \dots, m$

$$[D^{\psi}\theta_{h_\tau}]_j := \partial_{x_j}\theta_{h_\tau} + \sum_{s=m+1}^{N} P_{j,s}(\psi)\partial_{x_s}\theta_{h_\tau}$$

where $P_{j,s}(\psi)$ are polynomials depending only on ψ, $\mathcal{Q}^2, \dots, \mathcal{Q}^\kappa$. Using (3.23) we can see that $\sum_{\tau=1}^{l} \psi D^{\psi}\theta_{h_\tau} = 0$ and so finally $w = D^{\psi}\psi \in \mathbb{C}^0(\omega, \mathbb{R}^{m-1})$. This completes the proof of the implication (2) $\implies$ (3).

(3) $\implies$ **(4)**

The proof of $\psi \in h^{1/2}_{loc}(\omega)$ is the content of Theorem 3.2.8.

To prove that ψ is a broad* solution we have to show that for each $A \in \omega$ there exist $\delta_1, \delta_2 > 0$ with $\delta_1 > \delta_2$ such that for $j = 2, \dots, m$ there is the exponential maps $\exp_A(tD_j^{\psi})(B) \in I_{\delta_1}(A) \Subset \omega$ for all $(t, B) \in [-\delta_2, \delta_2] \times I_{\delta_2}(A)$; moreover,

$$w_j(B) = \frac{d}{dt}\psi\big(\exp_A(tD_j^{\psi})(B)\big)_{|t=0}$$

for all $B \in I_{\delta_2}(A)$.

Fix $j = 2, \dots, m$. For $\epsilon > 0$ we consider the Cauchy problem

$$\begin{cases} \dot\gamma^j_{B,\epsilon}(t) = D_j^{\psi_\epsilon}(\gamma^j_{B,\epsilon}(t)) \\ \\ \gamma^j_{B,\epsilon}(0) = B \end{cases}$$

which has a solution $\gamma_\epsilon : [-\delta_2(\epsilon), \delta_2(\epsilon)] \times I_{\delta_2(\epsilon)(A)} \to I_{\delta_1}(A)$. By Peano's estimate on the existence time for solutions of ordinary differential equations we get that $\delta_2(\epsilon)$ is greater than $C/\|D^{\psi_\epsilon}\psi_\epsilon\|_{L^\infty(I_{\delta_2(\epsilon)}(A))}$, with C depending only on δ_1. So it is sufficient to take $\delta_2 > 0$ such that $\delta_2 \le \delta_2(\epsilon)$ for all ϵ. Because γ_ϵ are uniformly continuous on the compact $[-\delta_2, \delta_2] \times I_{\delta_2}(A)$, by Ascoli-Arzelá Theorem, we have a sequence $(\epsilon_h)_h$ such that $\epsilon_h \to 0$ as $h \to \infty$ and $\gamma_{\epsilon_h} \to \gamma$ uniformly on $[-\delta_2, \delta_2] \times I_{\delta_2}(A)$. Obviously,

$$\gamma^j_{B,\epsilon_h}(t) = B + \int_0^t D_j^{\psi_{\epsilon_h}}(\gamma^j_{B,\epsilon_h}(r))\, dr$$

$$\psi_{\epsilon_h}(\gamma^j_{B,\epsilon_h}(t)) - \psi_{\epsilon_h}(\gamma^j_{B,\epsilon_h}(0)) = \int_0^t D_j^{\psi_{\epsilon_h}}\psi_{\epsilon_h}(\gamma^j_{B,\epsilon_h}(r))\, dr$$

and for $h \to \infty$ using that all the involved convergences are uniform we conclude

$$\gamma^j_B(t) = B + \int_0^t D_j^{\psi}(\gamma^j_B(r))\, dr$$

$$\psi(\gamma^j_B(t)) - \psi(\gamma^j_B(0)) = \int_0^t D_j^{\psi}\psi(\gamma^j_B(r))\, dr$$

i.e. the conditions of the Definition 3.2.2 are satisfied.

(4) $\Longrightarrow$ **(2)**.

Let us fix $A = (\bar{x}, \bar{y}) \in \omega$ and set

$$w(A) := \big(D_2^\psi\psi, \dots, D_m^\psi\psi\big)(A) \,\in\, \mathbb{R}^{m-1}.$$

First let $B = (x, y), B' = (x', y') \in I_\delta(A)$ for a sufficiently small $\delta > 0$.

Here we can not integrate along the vector field D_j^ψ; however this obstacle can be solved using the exponential maps, more precisely by posing

$$B_i := \exp_A(\bar{D}_i)(B_{i-1}) \quad \text{ for } i = 2, \dots, m$$

where $\bar{D} := (\bar{D}_2, \dots, \bar{D}_m)$ is the family of vector fields given by $\bar{D}_j = (x'_j - x_j)D_j^\psi$ for $j \in \{2, \dots, m\}$. A computation gives that

$$B_j = (x'_2, \dots, x'_j, x_{j+1}, \dots, x_m, y^{B_j})$$

with

$$\begin{aligned} y_s^{B_j} &= y_s + \sum_{l=2}^{j}\Big(-b_{1l}^{(s)}\int_0^{x'_l - x_l}\psi\Big(\exp_A(rD_l^\psi(B_{l-1}))\Big)\, dr + \frac{1}{2}(x'_l - x_l)\Big(\sum_{i=2}^{l} x'_i b_{li}^{(s)} + \sum_{i=l+1}^{m} x_i b_{li}^{(s)}\Big)\Big) \\ &= y_s^{B_{j-1}} - b_{1j}^{(s)}\int_0^{x'_j - x_j}\psi\Big(\exp_A(rD_j^\psi(B_{j-1}))\Big)\, dr + \frac{1}{2}(x'_j - x_j)\Big(\sum_{i=2}^{j} x'_i b_{ji}^{(s)} + \sum_{i=j+1}^{m} x_i b_{ji}^{(s)}\Big) \end{aligned}$$

for $s = 1, \dots, n$. Observe that $B_2, \dots, B_m$ are well defined for a sufficiently small δ. Because ψ is of class $\mathbb{C}^1$ (see Remark 3.2.6) we have

$$\begin{aligned}\psi(B') - \psi(B) &= [\psi(B') - \psi(B_m)] + \sum_{l=2}^{m} [\psi(B_l) - \psi(B_{l-1})] \\ &= [\psi(B') - \psi(B_m)] + \sum_{l=2}^{m} \big(\bar{D}_l\psi(B_{l-1}) + o(|x'_l - x_l|)\big)\end{aligned}$$

Notice that in the last equality we used the fact

$$\begin{aligned}\sum_{l=2}^{m} \big(\psi(B_l) - \psi(B_{l-1})\big) &= \sum_{l=2}^{m} \left(\int_0^1 (\bar{D}_l\psi)(\exp_A(r\bar{D}_l)(B_{l-1}))\, dr \right) \\ &= \sum_{l=2}^{m} \big(\bar{D}_l\psi(B_{l-1}) + o(|x'_l - x_l|)\big)\,.\end{aligned}$$

Now since

$$\lim_{\delta \to 0} \frac{\sum_{l=2}^{m} \bar{D}_l\psi(B_{l-1}) - \bar{D}\psi(A)}{|x' - x|} = 0$$

we have

$$\begin{aligned}\psi(B') - \psi(B) &= \psi(B') - \psi(B_m) + \bar{D}\psi(A) + \left(\sum_{l=2}^{m} \bar{D}_l\psi(B_{l-1}) - \bar{D}\psi(A) \right) + o(|x' - x|) \\ &= \psi(B') - \psi(B_m) + \langle w(A), x' - x \rangle + \left(\sum_{l=2}^{m} \bar{D}_l\psi(B_{l-1}) - \bar{D}\psi(A) \right) + o(\rho_\phi(B, B')).\end{aligned}$$

Consequently, it is sufficient to show that $\psi(B') - \psi(B_m) = o(\rho_\phi(B, B'))$. First we observe that

$$\frac{|\psi(B') - \psi(B_m)|}{\rho_\phi(B, B')} \le C_\psi(\delta) \frac{|y' - y^{B_m}|^{1/2}}{\rho_\phi(B, B')}$$

with

$$C_\psi(\delta) := \sup \left\{ \frac{|\psi(A') - \psi(A'')|}{|A'^{-1}A''|^{1/2}} \;:\; A' \neq A'',\; A', A'' \in I_\delta(A) \right\}. \tag{3.24}$$

We know also that $\lim_{\delta \to 0} C_\psi(\delta) = 0$ because ψ is $\mathbb{C}^1$ (see Remark 3.2.6). So it is evident that remains to prove $|y' - y^{B_m}|^{\frac{1}{2}} / \rho_\phi(B, B')$ is bounded in a proper neighborhood of A.

If we put $\mathcal{B}_M = \max\{ b_{ij}^{(s)} \,|\, i, j = 1, \dots, m\,, s = 1, \dots, n\}$ then

$$
\begin{aligned}
|y'-y^{B_m}| \le &\sum_{s=1}^{n}\bigg| y'_s - y_s + \sum_{l=2}^{m}\Big(b_{1l}^{(s)} \int_0^{x'_l-x_l} \psi\Big(\exp_A(rD_l^{\psi}(B_{l-1}))\Big)\,dr +\\
&\quad -\frac{1}{2}(x'_l-x_l)\Big(\sum_{i=2}^{l} x'_i b_{li}^{(s)} + \sum_{i=l+1}^{m} x_i b_{li}^{(s)}\Big)\Big)\bigg|\\
\le &\sum_{s=1}^{n}\bigg| y'_s - y_s + \frac{1}{2}(\psi(B)+\psi(B'))\sum_{l=2}^{m}(x'_l-x_l)b_{1l}^{(s)} - \frac{1}{2}\langle \mathcal{B}^{(s)}x, x'-x\rangle\bigg|\\
&+\sum_{s=1}^{n}\bigg| -\frac{1}{2}(x'_l-x_l)\Big(\sum_{i=2}^{l} x'_i b_{li}^{(s)} + \sum_{i=l+1}^{m} x_i b_{li}^{(s)}\Big) + \frac{1}{2}\langle \mathcal{B}^{(s)}x, x'-x\rangle\bigg|\\
&+\sum_{s=1}^{n}\bigg| -\frac{1}{2}(\psi(B)+\psi(B'))\sum_{l=2}^{m}(x'_l-x_l)b_{1l}^{(s)} + \sum_{l=2}^{m} b_{1l}^{(s)} \int_0^{x'_l-x_l} \psi\Big(\exp_A(rD_l^{\psi}(B_{l-1}))\Big)\,dr\bigg|\\
\le &\frac{1}{\epsilon^2}\rho_\phi(B,B')^2 + \frac{1}{2}n\mathcal{B}_M|x'-x|^2 +\\
&+\sum_{s=1}^{n}\bigg| -\frac{1}{2}(\psi(B)+\psi(B'))\sum_{l=2}^{m}(x'_l-x_l)b_{1l}^{(s)} + \sum_{l=2}^{m} b_{1l}^{(s)} \int_0^{x'_l-x_l} \psi\Big(\exp_A(rD_l^{\psi}(B_{l-1}))\Big)\,dr\bigg|
\end{aligned}
$$

where ϵ is given by (3.13). Note that we have used

$$
\begin{aligned}
&\frac{1}{2}\langle \mathcal{B}^{(s)}x, x'-x\rangle - \frac{1}{2}(x'_l-x_l)\Big(\sum_{i=2}^{l} x'_i b_{li}^{(s)} + \sum_{i=l+1}^{m} x_i b_{li}^{(s)}\Big)\\
&\qquad\qquad = -\frac{1}{2}(x'_l-x_l)\Big(\sum_{i=2}^{l} x'_i b_{li}^{(s)} + \sum_{i=l+1}^{m} x_i b_{li}^{(s)} - \sum_{i=2}^{m} x_i b_{li}^{(s)}\Big)\\
&\qquad\qquad \le \frac{1}{2}n\mathcal{B}_M|x'-x|^2.
\end{aligned}
$$

Finally, the last term

$$
\begin{aligned}
&\sum_{s=1}^{n}\bigg| -\frac{1}{2}(\psi(B)+\psi(B'))\sum_{l=2}^{m}(x'_l-x_l)b_{1l}^{(s)} + \sum_{l=2}^{m} b_{1l}^{(s)} \int_0^{x'_l-x_l} \psi\Big(\exp_A(rD_l^{\psi}(B_{l-1}))\Big)\,dr\bigg|\\
&\qquad \le R_1(B,B') + R_2(B,B') + R_3(B,B')
\end{aligned}
$$

where

$$R_1(B,B') := \sum_{s=1}^{n}\sum_{l=2}^{m}\left| b_{1l}^{(s)} \int_0^{x'_l - x_l} \psi\Big(\exp_A(rD_l^{\psi}(B_{l-1}))\Big)\,dr - \frac{1}{2} b_{1l}^{(s)}\Big(\psi(B_{l-1})+\psi(B_l)\Big)(x'_l - x_l)\right|$$

$$R_2(B,B') := \frac{1}{2}\sum_{s=1}^{n}\left|\sum_{l=2}^{m} b_{1l}^{(s)}(x'_l - x_l)\Big(\psi(B_{l-1})-\psi(B)\Big)\right|$$

$$R_3(B,B') := \frac{1}{2}\sum_{s=1}^{n}\left|\sum_{l=2}^{m} b_{1l}^{(s)}(x'_l - x_l)\Big(\psi(B_{l})-\psi(B')\Big)\right|$$

We would show that there exist $C_1, C_2, C_3 > 0$ such that

$$R_1(B,B') \le C_1|x'-x|^2 \tag{3.25}$$

$$R_2(B,B') \le C_2|x'-x|^2 \tag{3.26}$$

for all $B, B' \in I_\delta(A)$, and that for all r there is a $\delta_r \in (0,\delta]$ such that for all $\delta_0 \in (0,\delta_r]$

$$R_3(B,B') \le C_3|x'-x|^2 + \frac{1}{4} n\mathcal{B}_M(m-1)r|y'-y^{B_m}| \tag{3.27}$$

These estimates are sufficient to conclude; in fact, choosing $r := \frac{1}{n(m-1)\mathcal{B}_M}$ and using (3.25), (3.26) and (3.27) we deduce that

$$|y'-y^{B_m}| \le \frac{1}{\epsilon^2}\rho_\phi(B,B')^2 + \frac{1}{2}n\mathcal{B}_M|x'-x|^2 + C_1|x'-x|^2 + C_2|x'-x|^2 + C_3\rho_\phi(B,B')^2 + \frac{1}{4}|y'-y^{B_m}|$$

Hence there is $C_4 > 0$ such that

$$|y'-y^{B_m}|^{\frac{1}{2}} \le C_4\rho_\phi(B,B')$$

which is the thesis.

We start to consider $R_1(B,B')$. Fix $l = 2,\dots,m$. For $t \in [-\delta,\delta]$ we define

$$g_l(t) := \sum_{s=1}^{n} b_{1l}^{(s)}\left(\int_0^t \psi(\exp_A(rD_l^{\psi})(B_{l-1}))\,dr - \frac{1}{2}t\Big(\psi(B_{l-1}) + \psi\Big(\exp_A(tD_l^{\psi})(B_{l-1})\Big)\Big)\right)$$

working as in Proposition 3.2.5 we show the existence of $C_l > 0$ such that

$$|g_l(t)| \le C_l t^2, \quad \forall t \in [-\delta,\delta]$$

So set $t = x'_l - x_l$ we get

$$|g_l(x'_l - x_l)| \le C_l(x'_l - x_l)^2$$

and consequently (3.25) follows from

$$\sum_{l=2}^{m}|g_l(x'_l-x_l)|\leq\sum_{l=2}^{m}C_l(x'_l-x_l)^2\leq C_1|x'-x|^2.$$

Now we consider $R_2(B,B')$. Observe that

$$\begin{aligned}\frac{1}{2}\sum_{s=1}^{n}\left|\sum_{l=2}^{m}b_{1l}^{(s)}(x'_l-x_l)\Big(\psi(B_{l-1})-\psi(B)\Big)\right|&\leq\frac{1}{2}n\mathcal{B}_M\sum_{l=3}^{m}|x'_l-x_l|\Big|\psi(B_{l-1})-\psi(B)\Big|\\
&\leq\frac{1}{2}n\mathcal{B}_M\sum_{l=3}^{m}|x'_l-x_l|\Big(\sum_{i=2}^{l-1}|\psi(B_i)-\psi(B_{i-1})|\Big)\\
&\leq\frac{1}{2}n\mathcal{B}_M\sum_{l=3}^{m}|x'_l-x_l|\Big(\sum_{i=2}^{l-1}\Big|\int_0^1(\bar{D}_i\psi)(\exp_A(r\bar{D}_i(B_{i-1})))\,dr\Big|\Big)\\
&\leq\frac{1}{2}n\mathcal{B}_M\sum_{l=3}^{m}|x'_l-x_l|\Big(\sum_{i=2}^{l-1}\Big|(x'_i-x_i)(D_i^{\psi}\psi(B_{i-1})+o(1))\Big|\Big)\\
&\leq\frac{1}{2}n\mathcal{B}_MC|x'-x|^2\end{aligned}$$

Then (3.26) follows with $C_2:=\frac{1}{2}n\mathcal{B}_MC$. Finally we have

$$\begin{aligned}R_3(B,B')&\leq\frac{1}{2}n\mathcal{B}_M\sum_{l=2}^{m}|x'_l-x_l|\Big|\psi(B_l)-\psi(B')\Big|\\
&\leq\frac{1}{2}n\mathcal{B}_M\sum_{l=2}^{m}\Bigg(|x'_l-x_l|\Big(\sum_{i=l}^{m-1}|\psi(B_i)-\psi(B_{i+1})|+|\psi(B_m)-\psi(B')|\Big)\Bigg)\\
&\leq\frac{1}{2}n\mathcal{B}_M\sum_{l=2}^{m}\Bigg(|x'_l-x_l|\Big(\sum_{i=l}^{m-1}\Big|\int_1^0(\bar{D}_{i+1}\psi)(\exp_A(r\bar{D}_{i+1}(B_i)))\,dr\Big|+|\psi(B_m)-\psi(B')|\Big)\Bigg)\\
&\leq\frac{1}{2}n\mathcal{B}_MC|x'-x|^2+\frac{1}{2}n\mathcal{B}_M\sum_{l=2}^{m}|x'_l-x_l||\psi(B_m)-\psi(B')|\end{aligned}$$

Moreover, we define C_ψ as in (3.24), then we observe that $C_\psi(\delta)\to 0$ for $\delta\to 0$ since ψ is $\mathbb{C}^1$. Fix $l=2,\dots,m$ and note that

$$\begin{aligned}\Big|\psi(B_m)-\psi(B')\Big||x'_l-x_l|&\leq\frac{1}{2}\Big(2C_\psi(\delta)|y'-y^{B_m}|^{\frac{1}{2}}|x'_l-x_l|\Big)\\
&\leq\frac{1}{2}\Big(C_\psi(\delta)^2|y'-y^{B_m}|+|x'_l-x_l|^2\Big)\end{aligned}$$

Then for all $r > 0$ there is a $\delta_r > 0$ such that for all $\delta_0 \in (0, \delta_r]$ we have $C_\psi(\delta_0)^2 \le r$ and

$$\begin{aligned} R_3(B, B') &\le \frac{1}{2} n \mathcal{B}_M C |x' - x|^2 + \frac{1}{4} n \mathcal{B}_M \sum_{l=2}^{m} \Big(r|y' - y^{B_m}| + |x'_l - x_l|^2 \Big) \\ &\le C_3 |x' - x|^2 + \frac{1}{4} n \mathcal{B}_M (m-1) r |y' - y^{B_m}| \end{aligned}$$

where $C_3 := \frac{1}{4} n \mathcal{B}_M (2C + 1)$. So the inequality (3.27) is true and the (4) $\implies$ (2) is proved. □

We prove now that the solutions of the system $D^\psi \psi = w$ when $w \in \mathbb{C}^0(\omega, \mathbb{R}^{m-1})$ are Hölder continuous.

Theorem 3.2.8. *Let $\psi \in \mathbb{C}(\omega)$ where ω is open in $\mathbb{R}^{m+n-1}$. Assume that there exists $w := (w_2, \dots, w_m) \in \mathbb{C}^0(\omega, \mathbb{R}^{m-1})$ such that, in distributional sense,*

$$D^\psi \psi = w \ \ \textit{in} \ \omega$$

and there is a family $(\psi_\epsilon)_{\epsilon > 0} \subset \mathbb{C}^1(\omega)$ such that, for any open $\omega' \Subset \omega$,

$$\psi_\epsilon \to \psi \quad \textit{and} \quad D^{\psi_\epsilon} \psi_\epsilon \to w \qquad \textit{uniformly on} \ \ \omega', \quad \textit{as} \ \epsilon \to 0^+.$$

Then, for $\omega' \Subset \omega'' \Subset \omega$ there exists $\alpha : (0, +\infty) \to [0, +\infty)$ depending only on ω'', $\|\psi\|_{\mathcal{L}^\infty(\omega'')}$, $\|D^\psi \psi\|_{\mathcal{L}^\infty(\omega'')}$, on $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(n)}$ and on the modulus of continuity of w on ω'' such that

$$\lim_{r \to 0} \alpha(r) = 0 \tag{3.28}$$

and

$$\sup \left\{ \frac{|\psi(A) - \psi(A')|}{|A - A'|^{1/2}} : \ A, A' \in \omega', 0 < |A - A'| \le r \right\} \le \alpha(r). \tag{3.29}$$

Proof. It is sufficient to prove the theorem for $\psi \in \mathbb{C}^1(\omega)$. Indeed from the uniform convergence of ψ_ϵ and $D^{\psi_\epsilon} \psi_\epsilon$, we can estimate $\|\psi_\epsilon\|_{\mathcal{L}^\infty(\omega')}$, on $\|D^{\psi_\epsilon} \psi_\epsilon\|_{\mathcal{L}^\infty(\omega')}$ uniformly in ϵ for any $\omega' \Subset \omega$. Moreover the uniform convergence of $D^{\psi_\epsilon} \psi_\epsilon$ allows the choice of a modulus of continuity for $D_j^{\psi_\epsilon} \psi_\epsilon$ which is indipendent of ϵ for all j. Therefore there is $\alpha : (0, +\infty) \to [0, +\infty)$, not depending on ϵ, such that (3.28) and (3.29) follow.

We are going to prove that for each point of ω' there are sufficiently small rectangular neighborhoods $I \Subset I' \Subset \omega$ and a function $\alpha : (0, +\infty) \to [0, +\infty)$ such that $\lim_{r \to 0} \alpha(r) = 0$ and

$$\sup \left\{ \frac{|\psi(A) - \psi(A')|}{|A - A'|^{1/2}} : \ A, A' \in I', 0 < |A - A'| \le r \right\} \le \alpha(r). \tag{3.30}$$

By a standard covering argument the general statement follows.

Precisely we are going to prove (3.43) with α defined as

$$\alpha(r) := \frac{3\,(1+h)}{\mathcal{B}_M}\delta(r) + N r^{1/2} \tag{3.31}$$

where if we put

$$K := \sup_{A=(x,y)\in I'} \sum_{i=2}^{m} |x_i|, \quad M := \|\psi\|_{\mathcal{L}^\infty(I')}, \quad N := \|D^\psi \psi\|_{\mathcal{L}^\infty(I')}$$

and $\beta : (0,+\infty) \to [0,+\infty)$, increasing, such that $\lim_{r\to 0^+} \beta(r) = 0$ and

$$|w(A) - w(A')| \le \beta(|A - A'|) \qquad \text{for all } A, A' \in I'$$

then $h := (n\mathcal{B}_M(K+M))^{1/2}$, $\mathcal{B}_M = \max\{b_{il}^{(s)} : i, l = 1, \dots, m \text{ and } s = 1, \dots, n\}$ and

$$\delta(r) := \max\{r^{1/4}; (\mathcal{B}_M \beta(E r^{1/4}))^{1/2}\}.$$

Here $E > 0$ is a constant such that $|y - y'| + \mathcal{B}_M(K+2M)|y-y'|^{1/4} \le E|y-y'|^{1/4}$.

We split the proof in several steps.

Step 1. By standard considerations on ordinary differential equations, we know that for each point of ω' there are $r_0 > 0$ and rectangular neighborhoods $I \Subset I' \Subset \omega$ such that for all $A = (x,y) \in I$ there is a unique solution $\gamma_A^j \in \mathbb{C}^1([x_j - r_0, x_j + r_0], I')$ of the Cauchy problem

$$\begin{cases} \dot\gamma_A^j(t) = w_j(\gamma_A^j(t)) = X_j\psi(\gamma_A^j(t)) + \psi(\gamma_A^j(t)) \sum_{s=1}^n b_{j1}^{(s)} Y_s \psi(\gamma_A^j(t)) \\ \gamma_A^j(x_j) = A. \end{cases}$$

More precisely,

$$\begin{aligned} &\gamma_A^j(t) = \left(x_2, \dots, x_{j-1}, t, x_{j+1}, \dots, x_m, y_{1,A}^j(t), \dots, y_{n,A}^j(t)\right), \qquad \text{where} \\ &y_{s,A}^j(t) = y_s + \frac{1}{2}(t - x_j)\sum_{i=2}^m x_i b_{ji}^{(s)} + b_{j1}^{(s)} \int_{x_j}^t \psi(\gamma_A^j(r))\, dr, \qquad \text{for } s = 1, \dots, n. \end{aligned} \tag{3.32}$$

Moreover observe that

$$\frac{d^2}{dt^2} y_{s,A}^j(t) = \frac{d}{dt}\Big[\frac{1}{2}\sum_{i=2}^m x_i b_{ji}^{(s)} + b_{j1}^{(s)} \psi(\gamma_A^j(t))\Big] = b_{j1}^{(s)} w_j(\gamma_A^j(t)). \tag{3.33}$$

Step 2. Assume $A, B \in I$ with $A = (x,y)$ and $B = (x,y')$. We prove that

$$\frac{|\psi(A) - \psi(B)|}{|y - y'|^{1/2}} \le \frac{3}{\mathcal{B}_M}\delta, \tag{3.34}$$

where $\delta := \delta(|y-y'|)$. Suppose on the contrary that (3.34) is not true, i.e.

$$\frac{|\psi(A)-\psi(B)|}{|y-y'|^{1/2}} > \frac{3}{\mathcal{B}_M}\delta.$$

Let $b_{j1}^{(s)} \neq 0$ for some $s \in \{1,\dots,n\}$ and $j = 2,\dots,m$ and let γ_A^j and γ_B^j with

$$\gamma_A^j(t) = (x_2,\dots,x_{j-1},t,x_{j+1},\dots,x_m,y_1(t),\dots,y_n(t))$$

and

$$\gamma_B^j(t) = (x_2,\dots,x_{j-1},t,x_{j+1},\dots,x_m,y_1'(t),\dots,y_n'(t)).$$

Suppose that $y_s \geq y_s'$ (for the other case it is sufficient to exchange the roles of A and B). By (3.32) and (3.33), for $t \in [x_j - r_0, x_j + r_0]$ we have

$$\begin{aligned}
&y_s(t) - y_s'(t) - (y_s - y_s')\\
&= b_{j1}^{(s)} \int_{x_j}^{t} \psi(\gamma_A^j(r)) - \psi(\gamma_B^j(r))\, dr\\
&= \int_{x_j}^{t} \left[\dot{y}_s(x_j) - \dot{y}_s'(x_j) + \int_{x_j}^{r'} (\ddot{y}_s(r) - \ddot{y}_s'(r))\, dr\right] dr'\\
&= b_{j1}^{(s)}(t-x_j)(\psi(A)-\psi(B)) + b_{j1}^{(s)} \int_{x_j}^{t}\int_{x_j}^{r'} \big(w_j(\gamma_A^j(r)) - w_j(\gamma_B^j(r))\big)\, dr dr'.
\end{aligned}$$

Now using the following facts

$$\max_r |\dot{y}_s(r)| = \max_r \Big|\frac{1}{2}\sum_{i=1}^{m} x_i b_{ji}^{(s)} + b_{j1}^{(s)}\psi(\gamma_A^j(r))\Big| \leq \mathcal{B}_M\left(\frac{1}{2}K + M\right)$$

and

$$\begin{aligned}
|\gamma_A^j(r) - \gamma_B^j(r)| &\leq |\gamma_A^j(x_j) - \gamma_B^j(x_j)| + |r - x_j|(\max_r |\dot{y}_s(r)| + \max_r |\dot{y}_s'(r)|)\\
&\leq |A - B| + |t - x_j|(\max_r |\dot{y}_s(r)| + \max_{\hat{r}} |\dot{y}_s'(r)|)\\
&\leq |y-y'| + |t-x_j|\mathcal{B}_M(K+2M)
\end{aligned}$$

we obtain

$$\begin{aligned}
&y_s(t) - y_s'(t) - (y_s - y_s')\\
&\leq b_{j1}^{(s)}(t-x_j)(\psi(A)-\psi(B)) + |b_{j1}^{(s)}|(t-x_j)^2 \sup_r \beta\big(|\gamma_A^j(r) - \gamma_B^j(r)|\big)\\
&\leq b_{j1}^{(s)}(t-x_j)(\psi(A)-\psi(B)) + |b_{j1}^{(s)}|(t-x_j)^2 \beta\left(|y-y'| + |t-x_j|\mathcal{B}_M(K+2M)\right)
\end{aligned} \tag{3.35}$$

So if $b_{j1}^{(s)}(\psi(A)-\psi(B))>0$ put $t:=x_j-\frac{|y-y'|^{1/2}}{\delta}$ in (3.35) and $t:=x_j+\frac{|y-y'|^{1/2}}{\delta}$ otherwise. Observe that in both cases we conclude that

$$-|b_{j1}^{(s)}||\psi(A)-\psi(B)|<-3\delta|y-y'|^{1/2} \tag{3.36}$$

Now if $|y-y'|$ is "sufficiently small" γ_A^j and γ_B^j are well defined (it is sufficient to take $r_0\geq|y-y'|^{1/4}\geq|y-y'|^{1/2}/\delta=|t-x_j|$) and using (3.35), (3.36) and the definition of β we obtain in both cases

$$\begin{aligned} y_s(t)-y'_s(t) &\leq y_s-y'_s+|b_{j1}^{(s)}||y-y'|^{1/2}\frac{-|\psi(A)-\psi(B)|}{\delta}+\\ &\quad+\frac{1}{\delta^2}|b_{j1}^{(s)}||y-y'|\beta\left(|y-y|+\mathcal{B}_M(K+2M)\frac{|y-y'|^{1/2}}{\delta}\right)\\ &\leq y_s-y'_s-3|y-y'|^{1/2}|y-y'|^{1/2}+|b_{j1}^{(s)}||y-y'|\frac{\beta\left(E|y-y'|^{1/4}\right)}{\delta^2}\\ &\leq y_s-y'_s-3|y-y'|+\mathcal{B}_M\left(\frac{1}{\mathcal{B}_M}\right)|y-y'|=-|y-y'|<0. \end{aligned} \tag{3.37}$$

This leads to a contradiction, indeed if $y_s>y'_s$, then $y_s(\cdot)$ and $y'_s(\cdot)$ are solutions of the same Cauchy problem
(3.38)

$$\dot{y}_s(r)=\frac{1}{2}\sum_{i=2}^m x_i b_{ji}^{(s)}+b_{j1}^{(s)}\psi(x_2,\dots,x_{j-1},r,x_{j+1},\dots,x_m,y_1,\dots,y_{s-1},y_s(r),y_{s+1},\dots,y_n)$$

with initial data $y_s(x_j)=y_s$ and y'_s respectively, but two such solutions cannot meet, while $y_s(x_j)-y'_s(x_j)>0$ and $y_s(t)-y'_s(t)<0$ for a certain $t\in(x_j-r_0,x_j+r_0)$ with r_0 sufficiently large.

On the other hand if $y_s=y'_s$, by (3.37) we conclude that $y_s(t)\neq y'_s(t)$ for $t=x_j+\frac{|y-y'|^{1/2}}{\delta}$ or $t=x_j-\frac{|y-y'|^{1/2}}{\delta}$. Then we have the contradiction because $y_s(\cdot)$ and $y'_s(\cdot)$ are solutions of the same Cauchy problem (3.38) with initial data $y_s(t)$ and $y'_s(t)$ but two such solutions cannot meet, while $y_s=y'_s$.

Hence (3.34) follows.

Step 3. Now let $A,A',B\in I$ with $A=(x,y)$, $A'=(x',y')$ and $B=(x,y')$. We want to show that

$$\frac{|\psi(B)-\psi(A')|}{|x-x'|^{1/2}}\leq N|A-A'|^{1/2}+\frac{3h\delta}{\mathcal{B}_M} \tag{3.39}$$

where $\delta=\delta(|A-A'|)$. We suppose on the contrary that

$$\frac{|\psi(B)-\psi(A')|}{|x-x'|^{1/2}}>N|A-A'|^{1/2}+\frac{3h\delta}{\mathcal{B}_M}. \tag{3.40}$$

Set

$$D_j := \gamma^j_{D_{j-1}}(x_j) \quad \text{for } j = 2, \dots, m$$

with $D_1 := A'$. A computation gives that

$$D_j = (x_2, \dots, x_j, x'_{j+1}, \dots, x'_m, y^{D_j})$$

with

$$y^{D_j}_s = y'_s + \sum_{l=2}^{j} \left(b^{(s)}_{l1} \int_{x'_l}^{x_l} \psi\left(\gamma^j_{D_{j-1}}(r)\right) dr + \frac{1}{2}(x_l - x'_l)\Big(\sum_{i=2}^{l} x_i b^{(s)}_{li} + \sum_{i=l+1}^{m} x'_i b^{(s)}_{li} \Big) \right)$$

for $s = 1, \dots, n$ and consequently, recalling that $h = (n\mathcal{B}_M(K + M))^{1/2}$

$$\left| y' - y^{D_m} \right| \le n\mathcal{B}_M\,(M + K)\,|x - x'| = h^2|x - x'|. \tag{3.41}$$

Moreover we have

$$\sum_{j=2}^{m} |\psi(D_{j-1}) - \psi(D_j)| = \sum_{j=2}^{m} \left| \int_{x'_j}^{x_j} w_j(\gamma^j_{D_{j-1}}(t))\, dt \right| \le N|x - x'|$$

Then for $x' - x$ sufficiently small (and precisely when $N|x - x'|^{1/2} \le |x - x'|^{1/4} \le \delta$) by (3.40) and (3.41) we get

$$\begin{aligned}
|\psi(B) - \psi(D_m)| &\ge |\psi(B) - \psi(A')| - \sum_{l=2}^{m} |\psi(D_{l-1}) - \psi(D_l)| \\
&> \left(N|A - A'|^{1/2} + \frac{3h}{\mathcal{B}_M}\delta - N|x - x'|^{1/2} \right) |x - x'|^{1/2} \\
&\ge \frac{3h}{\mathcal{B}_M}\delta|x - x'|^{1/2} \\
&\ge \frac{3}{\mathcal{B}_M}\delta|y' - y^{D_m}|^{1/2}
\end{aligned}$$

so that we are in the first case again (see (3.34) with the couple $B = (x, y'), D_m = (x, y^{D_j})$ instead A, B respectively) which we have seen is not possible. Hence (3.39) holds.

Step 4. Using Step 2. and Step 3. we deduce that

$$\begin{aligned}
\frac{|\psi(A) - \psi(A')|}{|A - A'|^{1/2}} &\le \frac{|\psi(A) - \psi(B)|}{|y - y'|^{1/2}} + \frac{|\psi(B) - \psi(A')|}{|x - x'|^{1/2}} \\
&\le \frac{3\,(1 + h)}{\mathcal{B}_M}\delta(|A - A'|) + N|A - A'|^{1/2} \\
&= \alpha(|A - A'|)
\end{aligned}$$

for all $A = (x, y), A' = (x', y'), B = (x, y') \in I$. Then according to (3.43) and (3.31) we have that $\lim_{r\to 0} \alpha(r) = 0$ and we are able to control α with only $K, M, N, \mathcal{B}_M$ and β.

$\square$

3.2.3 The case $n = 1$

Let $\mathbb{G}$ be a group of class $\mathcal{B}$ as above but we also assume that the vertical layer is 1 dimensional, i.e. $n = 1$. We prove a Hölder regularity result for broad* solutions of the system $D^{\psi}\psi = w$ in ω.

Theorem 3.2.9. *Let $\mathbb{G} = \mathbb{R}^m \times \mathbb{R}$ be a group of class $\mathcal{B}$ with a one dimensional vertical layer. Let $\mathbb{W}$ and $\mathbb{V}$ be complementary subgroups with $\mathbb{V}$ horizontal and one dimensional. Let ω be an open subset of $\mathbb{W}$, $\psi : \omega \to \mathbb{R}$ and $w : \omega \to \mathbb{R}^{m-1}$ be continuous functions. We assume that ψ is a broad* solution of the system $D^{\psi}\psi = w$ in ω.*

Then, for $\omega' \Subset \omega'' \Subset \omega$ there exists $\alpha : (0, +\infty) \to [0, +\infty)$ depending only on ω'', $\|\psi\|_{\mathcal{L}^\infty(\omega'')}$, $\|D^{\psi}\psi\|_{\mathcal{L}^\infty(\omega'')}$, on the matrix $\mathcal{B}^{(1)}$ and on the modulus of continuity of w on ω'' such that $\lim_{r\to 0} \alpha(r) = 0$ and

$$\sup\left\{ \frac{|\psi(A) - \psi(A')|}{|A - A'|^{1/2}} \, : \, A, A' \in \omega', \, 0 < |A - A'| \le r \right\} \le \alpha(r) \tag{3.42}$$

Proof. We are going to show that for each point of ω' there are sufficiently small rectangular neighborhoods $I \Subset I' \Subset \omega$ and a function $\alpha : (0, +\infty) \to [0, +\infty)$ such that $\lim_{r\to 0} \alpha(r) = 0$ and

$$\sup\left\{ \frac{|\psi(A) - \psi(A')|}{|A - A'|^{1/2}} \, : \, A, A' \in I', 0 < |A - A'| \le r \right\} \le \alpha(r). \tag{3.43}$$

By a standard covering argument the general statement follows.

Fix $A_0 = (x_0, y_0) \in \omega'$. Then since ψ is a broad* solution of $D^{\psi}\psi = w$ in ω, there exist $0 < r_2 < r_1$ and a family of exponential maps at A_0

$$\exp_{A_0}(\cdot D^{\phi}_j)(\cdot) : [-r_2, r_2] \times I_{r_2}(A_0) \to I_{r_1}(A_0)$$

for $j = 2, \dots, m$, such that the conditions of the Definition 3.2.2 are satisfied.

Denote

$$I := I_{r_2}(A_0), \quad I' := I_{r_1}(A_0), \quad K := \sup_{A=(x,y)\in I'} |x|, \quad M := \|\psi\|_{\mathcal{L}^\infty(I')}, \quad N := \|w\|_{\mathcal{L}^\infty(I')}$$

and $\mathcal{B}_M = \max\{b^{(1)}_{il} : i, l = 1, \dots, m\}$. Let also β be the modulus of continuity of w on I'. We are going to prove (3.42) with α defined as

$$\alpha(r) := \frac{3\,(1 + h)}{\mathcal{B}_M}\delta(r) + N r^{1/2} \tag{3.44}$$

where $h := (\mathcal{B}_M(K + M))^{1/2}$ and

$$\delta(r) := \max\{r^{1/4}; (\mathcal{B}_M \beta(E r^{1/4}))^{1/2}\}.$$

Here $E > 0$ is a constant such that $|y-y'|+\mathcal{B}_M(K+2M)|y-y'|^{1/4} \le E|y-y'|^{1/4}$. Henceforth we denote $b_{il} = b_{il}^{(1)}$.

Step 1. Let $A = (x, y) \in I$ and let us denote by $\gamma_A^j(t) := \exp_{A_0}(tD_j^\psi)(A)$ if $t \in [-\delta_2, \delta_2]$. More precisely

$$\gamma_A^j(t) = (x_2, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_m, y(t)), \qquad \text{where}$$
$$y(t) = y + \frac{1}{2} t \sum_{i=2}^m x_i b_{ji} + b_{j1} \int_0^t \psi(\gamma_A^j(r))\, dr$$

and consequently $t \mapsto y(t)$ is a solution of the Cauchy problem:

$$\begin{cases} \frac{d^2}{dt^2} y(t) = \frac{d}{dt}\Big[\frac{1}{2}\sum_{i=2}^m x_i b_{ji} + b_{j1}\psi(\gamma_A^j(t))\Big] = b_{j1} w_j(\gamma_A^j(t)), \qquad t \in [-\delta_2, \delta_2] \\ y(0) = y, \\ \frac{d}{dt} y(0) = \frac{1}{2}\sum_{i=2}^m x_i b_{ji} + b_{j1}\psi(A) \end{cases}$$

Moreover observe that

$$\exp_{A_0}(\cdot D_j^\psi)(\cdot) : [-r_2, r_2] \times I \to I'$$

provided

$$r_2 < \frac{r_1}{2 + \frac{1}{2} K \mathcal{B}_M + M \mathcal{B}_M}. \tag{3.45}$$

Indeed if $(t, A) \in [-r_2, r_2] \times I$ then by definition of broad* solution

$$\gamma_A^j(t) - A_0 = (x_2 - (x_0)_2, \dots, x_j + t - (x_0)_j, \dots, x_m - (x_0)_m, y(t) - y_0) \quad \in I'$$

provided (3.45) follows.

Step 2. Assume $A, B \in I$ with $A = (x, y)$, $B = (x, y')$ and $A \neq B$. We prove that

$$\frac{|\psi(A) - \psi(B)|}{|y - y'|^{1/2}} \le \frac{3}{\mathcal{B}_M}\delta \tag{3.46}$$

where $\delta = \delta(|y - y'|)$. Suppose on the contrary that (3.46) is not true, i.e.

$$\frac{|\psi(A) - \psi(B)|}{|y - y'|^{1/2}} > \frac{3}{\mathcal{B}_M}\delta$$

Let $b_{j1} \neq 0$ for some $j = 2, \dots, m$ and let γ_A^j and γ_B^j with

$$\gamma_A^j(t) = (x_2, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_m, y(t))$$

and

$$\gamma_B^j(t) = (x_2, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_m, y'(t)).$$

As in Step 2 of Theorem 3.2.8, if $y > y'$ (for the other case it is sufficient to exchange the roles of A and B), we obtain the existence of $t \in [-\delta_2, \delta_2]$ such that

$$y(t) - y'(t) < 0.$$

More specifically, if $b_{j1}(\phi(A) - \phi(B)) > 0$ we have $t := -\frac{(y-y')^{1/2}}{\delta}$ and $t := \frac{(y-y')^{1/2}}{\delta}$ otherwise. On the other hand in both cases we conclude that

$$-|b_{j1}||\psi(A) - \psi(B)| \le -3\delta(y - y')^{1/2}. \tag{3.47}$$

Now if $t = \frac{(y-y')^{1/2}}{\delta}$ let

$$t^* := \sup\{\, t \in [0, \delta_2] \,:\, y(t) - y'(t) > 0\}$$

where the set $\{\, t \in [0, \delta_2] \,:\, y(t) - y'(t) > 0\}$ is not empty because $y(0) - y'(0) = y - y' > 0$. Moreover $0 < t^* < t < \delta_2$ and

$$y(t^*) = y'(t^*) \tag{3.48}$$

and so

$$\psi(x_2, \dots, x_{j-1}, x_j + t^*, x_{j+1}, \dots, x_m, y(t^*)) = \psi(x_2, \dots, x_{j-1}, x_j + t^*, x_{j+1}, \dots, x_m, y'(t^*)). \tag{3.49}$$

On the other hand if $t = -\frac{(y-y')^{1/2}}{\delta}$ we can assume $t^* := \inf\{\, t \in [-\delta_2, 0] \,:\, y(t) - y'(t) > 0\}$. Then t^* satisfies (3.48) and (3.49) and $-\delta_2 < t < t^* < 0$.

Let us prove now that

$$\psi(\gamma_A^j(t^*)) \ne \psi(\gamma_B^j(t^*))$$

and so by (3.49) we have a contradiction.

First we notice that

$$\begin{aligned} |\gamma_A^j(r) - \gamma_B^j(r)| &\le |\gamma_A^j(0) - \gamma_B^j(0)| + |r|(\max_r |\dot y(r)| + \max_r |\dot y'(r)|) \\ &\le |y - y'| + |t^*|\mathcal{B}_M(K + 2M) \end{aligned}$$

for all $|r| \le |t^*|$. Now if $b_{j1}(\psi(A) - \psi(B)) < 0$ using the last inequality, (3.47) and the definition of β we deduce

$$\begin{aligned}
& b_{j1}\big(\psi(\gamma_A^j(t^*)) - \psi(\gamma_B^j(t^*))\big) \\
&\qquad = b_{j1}(\psi(A) - \psi(B)) + b_{j1}\int_0^{t^*} \big(w_j(\gamma_A^j(r)) - w_j(\gamma_B^j(r))\big)\, dr \\
&\qquad \le -3(y-y')^{1/2}\delta + |b_{j1}||t^*|\,\beta\left(|\gamma_A^j(r) - \gamma_B^j(r)|\right) \\
&\qquad \le -3(y-y')^{1/2}\delta + |b_{j1}||t^*|\beta\left(|y-y'| + \mathcal{B}_M(K+2M)|t^*|\right) \\
&\qquad \le -3(y-y')^{1/2}\delta + \mathcal{B}_M|t|\,\beta\left(|y-y'| + \mathcal{B}_M(K+2M)\frac{|y-y'|^{1/2}}{\delta}\right) \\
&\qquad \le -3(y-y')^{1/2}\delta + \mathcal{B}_M(y-y')^{1/2}\delta\,\frac{\beta\left(E(y-y')^{1/4}\right)}{\delta^2} \\
&\qquad \le (-3+1)(y-y')^{1/2}\delta < 0,
\end{aligned}$$

i.e. $\psi(\gamma_A^j(t^*)) \neq \psi(\gamma_B^j(t^*))$. In a similar way, if $b_{j1}(\phi(A) - \phi(B)) < 0$ we obtain the contradiction. Therefore (3.46) follows.

Step 3. Now let $A, A', B \in I$ with $A = (x,y)$, $A' = (x',y')$ and $B = (x,y')$. We want to prove that

$$\frac{|\psi(B) - \psi(A')|}{|x-x'|^{1/2}} \le N|A-A'|^{1/2} + \frac{3h}{\mathcal{B}_M}\delta(|A-A'|). \tag{3.50}$$

To show this inequality we follow the arguments in Step 3 of Theorem 3.2.8. Observe that, differently from Step 3 of Theorem 3.2.8, here ψ is not a $\mathbb{C}^1$ function, hence we cannot integrate along the vector field D_j^ψ, i.e. we cannot define intermediate points $B_j := \gamma^j_{B_{j-1}}(x_j)$ as in Theorem 3.2.8. We use the properties of the exponential maps, defining

$$B_j := \exp_{A_0}\left((x_j - x_j')D_j^\psi\right)(B_{j-1}) \quad \text{ for } j = 2, \dots, m$$

with $B_1 := A'$.

Then using (3.46) and (3.50) we conclude that

$$\begin{aligned}
\frac{|\psi(A) - \psi(A')|}{|A-A'|^{1/2}} &\le \frac{|\psi(A) - \psi(B)|}{|y-y'|^{1/2}} + \frac{|\psi(B) - \psi(A')|}{|x-x'|^{1/2}} \\
&\le \frac{3\,(1+h)}{\mathcal{B}_M}\delta(|A-A'|) + N|A-A'|^{1/2} \\
&= \alpha(|A-A'|)
\end{aligned}$$

for all $A = (x,y), A' = (x',y'), B = (x,y') \in I'$. Hence according to (3.42) and (3.44) we have that $\lim_{r\to 0}\alpha(r) = 0$ and we are able to control α with only $K, M, N, \mathcal{B}_M$ and β.

$\square$

This last theorem shows that the assumption $\psi \in h^{1/2}_{loc}(\omega)$ in Theorem 3.2.7 can be omitted if we are inside a group of class $\mathcal{B}$ with one dimensional vertical layer.

See also Theorems 1.2 of [15] when $\mathbb{G}$ is an Heisenberg group.

Corollary 3.2.10. *Under the same assumptions of Theorem 3.2.7, if the vertical layer is 1 dimensional then the following conditions are equivalent:*

1. $S := \mathrm{graph}\,(\phi)$ *is a $\mathbb{G}$-regular hypersurface.*
2. ϕ *is u.i.d. in ω.*
3. *there exists $w \in \mathbb{C}^0(\omega, \mathbb{R}^{m-1})$ such that, in distributional sense,*

$$D^{\psi}\psi = w \ \ in \ \omega$$

and there is a family $(\psi_\epsilon)_{\epsilon>0} \subset \mathbb{C}^1(\omega)$ such that, for any open $\omega' \Subset \omega$, we get

$$\psi_\epsilon \to \psi \quad and \quad D^{\psi_\epsilon}\psi_\epsilon \to w \quad \ uniformly\ on \ \ \omega', \quad as\ \epsilon \to 0.$$

4. ψ *is a broad* solution of $D^{\psi}\psi = w$ in ω.*

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