Choice, extension, conservation

From transfinite to finite proof methods in abstract algebra

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Abstract

Maximality principles such as the ones going back to Kuratowski and Zorn ensure the existence of higher type ideal objects the use of which is commonly held indispensable for mathematical practice. The modern turn towards computational methods, which can be witnessed to have a strong influence on contemporary foundational studies, encourages a reassessment within a constructive framework of the methodological intricacies that go along with invocations of maximality principles.

The common thread that can be followed through the chapters of this thesis is explained by the attempt to put the widespread use of ideal objects under constructive scrutiny. It thus walks the tracks of a revised Hilbert's programme which has inspired a reapproach to constructive algebra by finitary means, and for which Scott's entailment relations have already shown to provide a vital and utmost versatile tool.

In this thesis several forms of the Kuratowski-Zorn Lemma are introduced and proved equivalent over constructive set theory; the notion of Jacobson radical is brought from commutative rings to a general ideal theory for single-conclusion entailment relations; a flexible conservation criterion of Scott for multi-conclusion entailment relations is put into action; elementary and constructive variants for algebraic extension theorems such as Sikorski's on the injectivity of complete atomic Boolean algebras are phrased and proved in terms of entailment relations; and a point-free version of Sikora's theorem on spaces of orderings of groups is obtained by a revisitation with syntactical means of some of the classical criteria for groups to be orderable.

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Publications included in this thesis

This thesis contains material which has already been published or has been submitted for publication:

- Peter Schuster and Daniel Wessel. "A general extension theorem for directed-complete partial orders". Preprint, submitted. 2017.
- Peter Schuster and Daniel Wessel. "Logical completeness and Jacobson radicals". Preprint. 2017.
- Davide Rinaldi, Peter Schuster, and Daniel Wessel. "Eliminating disjunctions by disjunction elimination". In: *Bulletin of Symbolic Logic* 23.2 (2017), pp. 181–200.
- Davide Rinaldi, Peter Schuster, and Daniel Wessel. "Eliminating disjunctions by disjunction elimination". In: *Indagationes Mathematicae* 29.1 (2018). Virtual Special Issue – L.E.J. Brouwer, fifty years later, pp. 226–259.
- Davide Rinaldi and Daniel Wessel. "Extension by conservation. Sikorski's theorem". Preprint, submitted. 2016. URL: `https://arxiv.org/pdf/1612.07345.pdf`.
- Davide Rinaldi and Daniel Wessel. "Some constructive extension theorems for distributive lattices". Preprint, submitted. 2017.
- Daniel Wessel. "Ordering groups syntactically". Preprint, submitted. 2017.
- Peter Schuster and Daniel Wessel. "Suzumura consistency, an alternative approach". In: *Journal of Applied Logic–IfCoLog* (2017). To appear.

Preface

Maximality principles such as the ones going back to Kuratowski and Zorn stipulate the existence of higher type ideal objects without so much as accounting for an effective procedure to bear witness. This concerns a commonplace lack of computational justification which forces a strong ontological commitment, yet prevents any means of epistemic access. Throughout mathematics, typical examples arise from the alleged need for totality where partiality abounds, e.g., if maps are to be extended from sub- to ambient structures in a coherent manner, the classical solution to which may sometimes nearly be held to ridicule the intuitive and 'procedural' concept of a function. If taking a rather practical stance, one is confronted with foundational issues that are inherently difficult to address.

Needless to say, this describes unfounded worries from a classical point of view. But even a finitist adversary might vindicate all this quickly, presumably on the grounds of deeming those ideal objects to be mere fictions. Yet the sheer amount of deep results obtained by ideal methods, generations worth of achievements, and last but not least everyday curricula can hardly be denied, disregarded, or overruled—all of which reinforces a commonly held belief that ideal objects are indispensable for mathematical practice: be it for a far-reaching development of contemporary abstract algebra, or even to address matter-of-fact questions stemming from an economic theory of preferences. But then again, dropping the subject on the former, whether ideal methods should have found their way into the latter discipline might very well be worth an argument.

I hasten to add, with all due emphasis, that this thesis does not set out to do away with misconceptions, and it does not intend to clarify conclusively whether the above rests on a misconception at all. Nor does it advocate a large-scale reapproach. But the overall turn towards computational methods, which today can be witnessed to have a strong influence on the foundations of mathematics, seems to encourage a reassessment of the methodological intricacies that go along with invocations of maximality principles in a constructive framework. It is here that some steps shall be taken. The common thread which can be followed through the chapters of this thesis is explained by the attempt to put the widespread use of ideal objects under constructive scrutiny.

Genesis and context

The studies that have led to this thesis took motivation from Bell's [33] dictum that the Kuratowski-Zorn lemma (henceforth KZL) be "constructively neutral"—as opposed to the Axiom of Choice, which has long been known to be incompatible with intuitionistic logic—along with a methodological discussion and an assessment that deems KZL to be of comparatively little use unless applied in a classical setting. It appeared reasonable that a similar analysis could be bestowed upon Raoult's principle of *Open Induction*, which in recent times has caused an attentional shift [36, 73, 223]. Eventually, such an analysis has not been carried out thoroughly, yet to a large extent the later development of this thesis can be traced back as to have originated in this context. It stems from the endeavour to rephrase several prominent applications of KZL (e.g., characterizations of injective objects, and orderability criteria for algebraic structures) in a manner which puts strong emphasis on constructions which sometimes are lurking in the background; and which allows to unveil their computational underpinning, if at least from a liberal, non-formal point of view.

With its aims declared, this thesis walks on the tracks of a revised Hilbert's programme [243, 257] which has inspired a reapproach to constructive algebra by finitary means [94], and for which Scott's entailment relations [229] have already shown to provide a vital and utmost versatile tool ([62, 82]; see Chapter 3 for a wealth of references, and Chapter 4 for a thorough introduction). The notion of ideal element, which above has already been insinuating Hilbert's terminology, will here be understood as model of an entailment relation. While we cannot circumvent transfinite methods in order to assert the existence of such an object in general, it could well be argued that entailment relations are "couched in cognitively accessible terms" [126]. They allow for a replacement of ideal semantics, which more often than not necessitates classical reasoning, in a direct manner by formal, syntactical correspondents. For instance, rather than resorting to model existence principles, we aim at asserting consistency, and at providing an elementary proof for any claim of the latter. It is then tempting to say that we obtain constructive versions of classical theorems. Surely it is in order to quote Coquand and Lombardi [82], the work of whom has had an unmistakable influence on this thesis:

> *"When we say that we have a constructive version of an abstract algebraic theorem, this means that we have a theorem the proof of which is constructive, which has a clear computational content, and from which we can recover the usual version of the abstract theorem by an immediate application of a well classified non-constructive principle."*

The development of this thesis considerably owes to joint work with Peter Schuster and Davide Rinaldi, and it naturally reflects on its content and structure, about which will be reported in the following.

Content and structure

Chapter 1 is based on joint work with Schuster [224]. Here we single out an immediate consequence of the Kuratowski-Zorn Lemma KZL, from which quite a few abstract extension theorems can be deduced more directly, save certain definitions by cases. In Section 1.3 we demonstrate that our General Extension Theorem (GET) is equivalent to a suitable form of KZL over constructive set theory **CZF**, which we take from unpublished work by Aczel [6]. In Section 1.5 we discuss a variety of induction principles that take serious the one-step extension principle [33] which is at the heart of many an application of KZL in algebraic contexts.

Chapter 2 is based on joint work with Schuster [225]. This chapter has experimental character: by analogy, we generalize a concept of extension narrower than simple containment from the theory of quasi-orders, where this is common and indeed inevitable, to an abstract ideal theory for single-conclusion entailment relations; the ideals are the subsets saturated with respect to the corresponding algebraic closure operators. A proof pattern called *Zorn Scheme* that over constructive set theory derives from Raoult's principle of Open Induction (Section 2.2.2) then prompts variants of Lindenbaum's Lemma in Section 2.3.2, and helps to describe the intersection of all complete ideals above a given ideal in a computationally meaningful way. To this end in Section 2.3.3 we carry over from commutative ring theory a point-free version of the Jacobson radical, which moreover prompts a syntactical counterpart of Lindenbaum's Lemma. In Section 2.4, our results turn out to have applications in commutative algebra, lattice theory, formal logic and order theory; the running example is in fact the deducibility relative to intuitionistic and classical logic. In Section 2.5 we finally relate all this to multi-conclusion entailment relations, which thus leads over to the following chapter.

Chapter 3 is a main pillar of this thesis. It concentrates on the interplay of single- and multi-conclusion entailment relations, and is based on joint work with Rinaldi and Schuster [209, 210]. We show how a fairly general syntactical conservation theorem that covers plenty of the semantic approaches to conservation follows from a versatile criterion due to Scott (Lemma

3.3). We work with multi-conclusion entailment relations as extending single-conclusion entailment relations. Additional axioms with disjunctions in positive position can be eliminated by reducing them to the corresponding disjunction elimination rules, which in turn prove admissible in a wealth of mathematical instances. In deduction terms this means to fold up branchings of proof trees by way of properties of the relevant mathematical structures. Applications in Section 3.4 include syntactical counterparts of the theorems or lemmas known under the names of Artin-Schreier, Krull-Lindenbaum and Szpilrajn, as well as of the spatiality of coherent locales.

Chapter 4 continues by addressing further aspects of Scott multi-conclusion entailment relations, and may as well be considered an introduction to the second part of this thesis. We discuss the Completeness Theorem (CT) for entailment relations along with a few of its consequences, and explain some limitations of the semantic method over $\mathbf{CZF}$. For instance, we show that CT necessitates classical logic (Corollary 4.15). To give an idea of the line of reasoning employed in the remaining chapters, the reader will find a first and thorough case study on *paths as ideal objects* in Section 4.5. What need be taken from this chapter is the technique of adequate and hereditary predicates introduced in Section 4.4, which can help to provide concrete, non-inductive descriptions for inductively generated entailment relations, as well as the concept of an atomic conjunction (Definition 4.23), which brings the lattice-theoretic notion of an atom to the context of entailment relations.

Chapter 5 gives an account of *inverse limits* as spectra of suitable entailment relations. This chapter takes its main motivation from an idea of Rinaldi, and grew out of our joint endeavour to find a constructive version of Sikorski's extension theorem. In Section 5.2.1 we first discuss a specific entailment relation the ideal elements of which are choice functions. Putting additional axioms on top of this entailment relations leads to a formal description of inverse limits as spectra in Section 5.2.2. After a brief introduction to the main concepts of universal algebra in Section 5.3, we are ready to prove an elementary and constructive version of the fact that profinite algebras are pure-injective objects in their corresponding categories (Theorem 5.32).

Chapter 6 is based on joint work with Rinaldi and puts together our manuscripts [212, 213]. By employing entailment relation, we are able to turn Sikorski's extension theorem, the most general form of which states that every complete Boolean algebra is an injective object in the category of distributive lattices, into a constructive and syntactical conservation result (Proposition 6.15). Moreover, a slight modification then leads to an elementary version of Monteiro's theorem [20, 182] in Section 6.5.3, regarding extension of dominated maps. With respect to the latter, we introduce the concept of *reductive lattices* in Section 6.33. This allows to give a conservation criterion for the corresponding entailment relation of bounded lattice map.

Chapter 7 is based on [254], in which attention is turned to ordered algebraic structures. The classical criterion for a group to be orderable will here be given a constructive equivalent in terms of a consistency statement for a suitable entailment relation. We hasten to add that group orderability has long been considered in the context of entailment relations, ever since their applicability for constructive algebra has been pointed out [79]. On the one hand, the contribution consists of a slight but straightforward generalization to arbitrary groups. In the abelian case this leads to a constructive version of Levi's theorem that an abelian group is linearly orderable if and only if it is torsion-free (Proposition 7.10). On the other hand, prompted by Cederquist and Coquand's fundamental theorem for entailment relations, in Section 7.4 we develop a finitary and point-free version of Sikora's theorem [235], which asserts that the space of orderings of $\mathbb{Z}^n (n > 1)$ is a Cantor space. The classical counterpart will be reobtained in Section 7.6.

Chapter 8 concludes the main part of this thesis with several perspectives, and points out some directions which future developments may take. Naturally, the content of this chapter does not intend to give a definitive account. We break the symmetry of entailment relations and allow for arbitrary sets of succedents. This requires us to explain a conditional form of model existence as well as a general form of completeness under additional hypotheses on the entailment relation. To build a bridge between the first and the final chapter, in Section 8.2 we further give an elementary—yet classically equivalent—form of Baer's criterion, along with several applications, the proofs of which to a definite extent are extracted from the classical ones. This provides an occasion to reevaluate the example discussed before in Section 5.3.1.

Chapter 9 is based on joint work with Schuster, and presents our publication [226]. We take a new approach to the notion of Suzumura consistency, which is known as a sufficient and necessary condition for a binary relation to have an order extension. We advocate the use of equivalent but negation-free forms of Suzumura consistency and of the related notion of compatible extension. From a methodological perspective, our proposals enable to work more abstractly, in the algebra of relations, and to give more direct proofs. To illustrate this we reconsider various forms and proofs of the order extension principle. As a complement in Section 9.8.4 we adopt to quasi-orders Bell's argument that Gödel–Dummett logic is necessary for order extension. Even though this chapter represents a digression in view of the preceding content, all this gave rise to the development of Chapter 2, and provided us with an example for a non-conservative extension in Chapter 3, whence it is included here.

This dissertation is cumulative and based on several self-contained manuscripts. This has the perhaps favourable effect that its chapters can be read almost independently and in any order, except that after Chapter 4 we do not give an introduction to entailment relations all over again. Therefore, before taking a look at Chapters 5, 6, 7, and 8, it is recommendable to throw a glance at Chapter 4 first, which is meant to provide useful tools, and which intends to serve as a belated but thorough introduction to this later part of the thesis. In particular, we will use adequate and hereditary predicates repeatedly in order to characterize inconsistent subsets, and decisive subsets, which will also be introduced in Chapter 4, are central for the purpose of Chapter 7.

Method and foundations

We work in constructive set theory **CZF** [3–5, 7, 8] which is based on intuitionistic logic and provides a framework for the development of Bishop-style constructive mathematics [41, 181]. Due to this methodological choice, certain assumptions sometimes have to be made explicit that otherwise, i.e., in classical Zermelo-Fraenkel set theory (**ZF**) would be trivial. For instance, a set S is *discrete* if

$$\forall x, y \in S\,(\, x = y \,\vee\, x \neq y \,);$$

and a subset T of a set S is *detachable* if

$$\forall x \in S\,(\, x \in T \,\vee\, x \notin T \,).$$

These are instances of the *restricted principle of excluded middle* (REM)

$$\varphi \vee \neg\varphi$$

where φ is a *bounded* formula, i.e., one in which all quantifiers occur only in one of the forms $\exists x \in y$ or $\forall x \in y$. We make use of class notation and terminology [7] throughout. A class is said to be *predicative* [7] if it can be defined by a bounded formula. All classes in Chapter 1 are supposed to be predicative; this assumption can be relaxed in Chapter 2. Given a class $\mathscr{E}$, a *partial order* of $\mathscr{E}$ is a subclass $\leqslant$ of $\mathscr{E} \times \mathscr{E}$ that satisfies the usual axioms of a partial order: reflexivity, transitivity, antisymmetry. For instance, if S is a set, then the class $\mathrm{Pow}(S)$ of all subsets of S is partially ordered by the subset relation. Crucially, **CZF** promotes a predicative setting and does not postulate the Powerset axiom! However, recall that **CZF** has **Exponentiation**, i.e., if S and T are sets, then so is the class of all functions $f : S \to T$. It follows that if S is a set, then the class $\mathrm{Fin}(S)$ of all *finite* subsets of S is a set as well. Here, a set S is *finite* if there is $n \geqslant 0$ and a surjective function $\{1, \dots, n\} \to S$.[1] Sometimes we write $U \subseteq_{\mathrm{f}} S$ in order to assert that U is a finite subset of a set S.

In the following, whenever we write that a certain principle holds *classically*, then we mean that REM is adopted in order to prove the implication in question; in other words, we thus work in **CZF**+ REM, which theory proves the same theorems as **ZF** does [7, Corollary 4.2.8]. Similarly, whenever we require principles beyond **CZF**, e.g., the Completeness Theorem for entailment relations (CT), then this will be indicated appropriately.

Axioms

For the reader's convenience and sake of reference, we briefly recall the axioms and axiom schemes of constructive set theory **CZF**. The language of **CZF** is the first–order language of **ZF** with the non–logical symbols $\in$ and $=$. The logical symbols are all the intuitionistic operators $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, and $\forall$. In particular, $\neg\varphi$ is defined as $\varphi \rightarrow \bot$. A formula of **CZF** is *bounded* or is a Δ_0*–formula* if all occurring quantifiers are bounded, i.e., of the form $\exists x \in y$ or $\forall x \in y$, where $\exists x \in y\, \varphi$ and $\forall x \in y\, \varphi$ stand for $\exists x\, (x \in y \wedge \varphi)$ and $\forall x\, (x \in y \rightarrow \varphi)$, respectively. In addition to the axioms for intuitionistic first–order logic with equality, the axioms of **CZF** are the following set–theoretic axioms and axiom schemes [3–5, 7, 8].

1. **Extensionality**
$$\forall a\, \forall b\, (\forall y\, (y \in a \leftrightarrow y \in b) \rightarrow a = b)$$
2. **Pair**
$$\forall a\, \forall b\, \exists x\, \forall y\, (y \in x \leftrightarrow y = a \vee y = b)$$
3. **Union**
$$\forall a\, \exists x\, \forall y\, (y \in x \leftrightarrow \exists z \in a\, (y \in z))$$
4. **Bounded Separation**
$$\forall a\, \exists x\, \forall y\, (y \in x \leftrightarrow y \in a \wedge \varphi(y))$$
for every Δ_0*–formula* $\varphi(y)$ in which x is not free.
5. **Subset Collection**
$$\begin{aligned}&\forall a\, \forall b\, \exists c\, \forall u\, (\forall x \in a\, \exists y \in b \varphi(x, y, u) \rightarrow \\ &\exists d \in c\, (\forall x \in a\, \exists y \in d\, \varphi(x, y, u) \wedge \forall y \in d \exists x \in a\, \varphi(x, y, u)))\end{aligned}$$
for every formula $\varphi(x, y)$.
6. **Strong Collection**
$$\begin{aligned}&\forall a\, (\forall x \in a\, \exists y\, \varphi(x, y) \rightarrow \\ &\exists b\, (\forall x \in a\, \exists y \in b\, \varphi(x, y) \wedge \forall y \in b\, \exists x \in a\, \varphi(x, y)))\end{aligned}$$
for every formula $\varphi(x, y)$.
7. **Strong Infinity**
$$\exists a\ (\mathrm{Ind}\,(a) \wedge \forall b\ (\mathrm{Ind}\,(b) \rightarrow \forall x \in a\, (x \in b)))$$
where the following abbreviations are used:
$\mathrm{Empty}(y)$ for $\forall z \in y\ \bot$,
$\mathrm{Succ}(x, y)$ for $\forall z\ (z \in y \leftrightarrow z \in x \vee z = x)$,
$\mathrm{Ind}(a)$ for $\exists y \in a\, \mathrm{Empty}(y) \wedge \forall x \in a\, \exists y \in a\, \mathrm{Succ}(x, y)$.

[1] For the sake of a slicker wording we thus deviate from the prevalent terminology of constructive mathematics and set theory [7, 8, 40, 41, 169, 181]: (1) to call 'subfinite' or 'finitely enumerable' a finite set in the sense above, i.e., a set T for which there is a surjection from $\{1, \dots, n\}$ to T for some $n \geq 0$; and (2) to reserve the term 'finite' to sets which are in bijection with $\{1, \dots, n\}$ for a necessarily unique $n \geq 0$.

8. **Set Induction**

$$\forall a\,(\forall x \in a\,\varphi(x) \;\rightarrow\; \varphi(a)) \;\rightarrow\; \forall a\,\varphi(a)$$

for every formula $\varphi(x)$.

The axiom schema of **Replacement**

$$\forall x \in a\,\exists! y \varphi(x,y) \;\rightarrow\; \exists b\,\forall y\,(\,y \in b \;\leftrightarrow\; \exists x \in a\,\varphi(x,y)\,)$$

for all formulae $\varphi(x,y)$, where b is not free in $\varphi(x,y)$, is a consequence of Strong Collection. Myhill's axiom of **Exponentiation** (Exp) [186] is a consequence of Subset Collection.

From **CZF** we can extend to *Intuitionistic Set Theory* **IZF** if we allow for unbounded separation and postulate the Powerset Axiom [7, 30]. We obtain classical Zermelo-Fraenkel set theory **ZF** by further admitting the (unbounded) principle of excluded middle.

Most of what follows is likely to be formalizable in a weaker fragment of **CZF**, e.g., in *Elementary Constructive Set Theory* **ECST** [7], potentially strengthened by the **Finite Powers Axiom** (FPA), which states that for every set A and for each $x \in \omega$, the class of all functions $x \rightarrow A$ is a set [7]; here ω denotes the set of natural numbers (the smallest inductive set, provided for by the axiom of Strong Infinity). Another viable option would be to strengthen **ECST** with Exp, which results in a subtheory of **CZF** [7]. However, we adopt an intuitive understanding of constructive set theory; moreover, our main focus will not be on determining the strength of new principles over **CZF**, but on providing constructive and elementary versions of classical theorems. Therefore, with the exception of Chapter 4, we favor to work over **CZF** rather than over a potential fragment, even if this may cause a slight loss of generality.

In order to apply a result of Aczel in Chapter 4, we have to go beyond **CZF** and postulate DC + uREA. DC denotes the axiom of **Dependent Choice** which states that for all sets a and set relations $R \subseteq a \times a$, if R is *total*, i.e., if

$$\forall x \in a\,\exists y \in a\,xRy$$

and $a_0 \in a$, then there exists a function $f : \omega \rightarrow a$ such that $f(0) = a_0$ and

$$\forall n \in \omega\,f(n)Rf(n+1).$$

We use infix notation, i.e., xRy stands for $(x,y) \in R$. uREA is a variant of the **Regular Extension Axiom** (REA). Recall that a set A is *transitive* if $\forall x \in A\,\forall y \in x\,y \in A$. A set A is said to be *regular* if it is inhabited, transitive, and for any $u \in A$ and every total relation $R \subseteq u \times A$ there exists $v \in A$ such that

$$\forall x \in u\,\exists y \in v\,xRy \;\wedge\; \forall y \in v\,\exists x \in u\,xRy.$$

REA is the principle that every set is a subset of a regular set. It was introduced to accomodate inductive definitions in **CZF** [3, 5, 7]. Among several variants [203] of REA there is uREA, which states that every set is a subset of a *union-closed* regular set [2], i.e., a regular set A such that, for every set a, if $a \in A$ then $\bigcup a \in A$.

Short contents

Contents

Chapter 1

A general extension theorem for directed-complete partial orders

This chapter is based on [224].

1.1 Introduction

An invocation of the Kuratowski-Zorn Lemma (KZL) [159, 259] often takes place within an indirect proof of a universal statement. Supposing towards a contradiction that there be any counterexample, the maximal counterexample provided by KZL helps—by what Bell calls "one-step extension argument" [33]—to the desired contradiction. Crucially though, this one-step argument does not depend on maximality, and in fact a more general method is hovering in the background, which a priori is not limited to hypothetical counterexamples only. An alternative approach thus seems desirable which at once is more affirmative inasmuch as it really focuses on the one-step argument, but still is in the spirit of KZL. To this end, we distill a general extension theorem (GET) for directed-complete partial orders, the intended meaning being that the poset under consideration consists of partial extensions of which one is to be proved total. The principal hypothesis of GET encodes the one-step argument which can also be found in proofs of specific extension theorems such as the ones going back to Hahn and Banach: that every partial extension can be extended by any potential element of its domain—which, of course, is impossible for any maximal extension. As compared with the typical indirect proof by KZL of such an extension theorem, GET allows for a fairly direct proof relative to a certain type of definition by cases. This is possible because GET already postulates the existence of a total extension rather than a maximal one.

In this chapter we proceed as follows. First, in Section 1.2, we explain the concept of an extension pattern on a partially ordered class, and we provide several elementary examples. Then, in Section 1.3, we phrase our general extension principle GET and prove it equivalent over constructive set theory **CZF** both to a certain variant of the Hausdorff Maximal Principle as well as to a form of KZL which is suitable for **CZF**. In Section 1.4 we focus on a specific application from module theory, obtaining a classical proof of Baer's criterion by means of GET. Finally, we obtain from GET a classically equivalent induction principle in Section 1.5.

We make use of class notation and terminology [7], notably when it comes to phrasing KZL over **CZF**. A class is said to be *predicative* [7] if it can be defined by a bounded formula. In this chapter, by a class we will invariably mean a predicative class. We denote classes by script letters. Given a (predicative) class $\mathscr{E}$, a *partial order* of $\mathscr{E}$ is a (predicative) subclass $\leqslant$ of $\mathscr{E} \times \mathscr{E}$ that satisfies the usual axioms of a partial order: reflexivity, transitivity, antisymmetry. For instance, if S is a set, then the class $\mathrm{Pow}(S)$ of all subsets of S is partially ordered by the subset relation.

1.2 Extension

Typically, when attempting to prove an extension theorem, e.g. in case of the Hahn-Banach theorem, an application of KZL takes the following form. Given a map $f : Y \to Z$ and $Y \subseteq X$, we are asked for an extension of f, i.e., another map $g : X \to Z$ for which $f(x) = g(x)$ for all $x \in Y$. When structural properties need to be preserved, this rarely is a trivial task, but a way out is provided by KZL. So one considers the collection E of intermediate extensions of f, and proceeds by showing E to be closed under unions of directed subsets. Due to KZL, the directed-complete partially ordered set E then has a maximal element, and a "one-step extension argument" [33] helps to the desired conclusion that the maximal element indeed has domain X. This one-step argument is captured by what Bell calls the *extension principle* for a family E of partial functions on a set X:

$$\forall x \in X \; \forall f \in E \; \exists g \in E \; (\; f \subseteq g \;\wedge\; x \in \mathrm{dom}(g) \;).$$

Under this extension principle, if f is a maximal element of E, then in fact $\mathrm{dom}(f) = X$, which is to say that f is total.

Example 1.1. Let R be a relation with $\mathrm{dom}(R) = X$, and let E be the set of subfunctions of R. If the domain $\mathrm{dom}(f)$ is a detachable subset of X whenever $f \in E$, then a simple definition by cases allows for an extension principle for E [33]. For let $f \in E$ and $x \in X$. If $x \in \mathrm{dom}(f)$, then an extension g of f by x is trivially given by $g = f$; and in case of $x \notin \mathrm{dom}(f)$ we may set $g = f \cup \{ (x, y) \}$ for any y such that $(x, y) \in R$. On the other hand, KZL applies to E and thus gives a subfunction of R with the same domain. This is how AC (appropriately formulated and with classical reasoning) may be deduced from KZL with the aid of an extension principle.

If EP denotes the assertion that for every relation R there is an extension principle for the set of subfunctions of R, then KZL+EP entails AC in $\mathbf{CZF}$. As is well known [7, 8, 101, 122], AC implies REM. While KZL is "constructively neutral" [33], it is EP which implies REM [33].

The situation one encounters may change, but the overall strategy with KZL remains quite the same, if only there is a one-step argument at hand.[1] We are thus led to rephrase Bell's extension principle in a somewhat more general fashion.

Definition 1.2. *Let $(\mathscr{E}, \leqslant)$ be a partially ordered class. An* extension pattern $(\mathscr{X}, \Vdash)$ *on $\mathscr{E}$ is given by a class $\mathscr{X}$ together with a class relation $\Vdash \subseteq \mathscr{X} \times \mathscr{E}$ satisfying the* extension property

$$\forall x \in \mathscr{X} \; \forall e \in \mathscr{E} \; \exists e' \in \mathscr{E} \left(\; e \leqslant e' \;\wedge\; x \Vdash e' \; \right).$$

An element e of $\mathscr{E}$ is said to be total *if*

$$\forall x \in \mathscr{X} \; (\, x \Vdash e \,).$$

We use extension data *as a name for the elements of $\mathscr{X}$.*

The intended meaning of an extension pattern is best explained in analogy with Bell's principle for a family of functions. Where the latter keeps track of the domain of a function, general extension data $x \in \mathscr{X}$ are related to elements $e \in \mathscr{E}$ in a similar manner but by means of an arbitrary relation $\Vdash$.[2]

Lemma 1.3. *Let $\mathscr{E}$ be a partially ordered class. If $e \in \mathscr{E}$ is maximal, then e is total for every extension pattern $(\mathscr{X}, \Vdash)$ on $\mathscr{E}$.*

Proof. Let $e \in \mathscr{E}$ be maximal and let $x \in \mathscr{X}$. Then there is $e' \in \mathscr{E}$ with $e \leqslant e'$ and $x \Vdash e'$. Since e is maximal, we actually have $e = e'$, hence $x \Vdash e$. Therefore $x \Vdash e$ for all $x \in \mathscr{X}$ which means that e is total. □

[1] While to our knowledge a one-step argument was made explicit first by Szpilrajn [242], more recent explicit occurrences include [192], of course on top of [33].

[2] Our choice of notation follows the one for Sambin's *Basic Pairs* [219] by which is meant a relation $\Vdash$ between sets X and S.

Example 1.4. Let E be a set of partial functions on a set X. This E is a poset, naturally ordered by inclusion. Consider every element x of X as extension data and define $\Vdash$ by

$$x \Vdash e \quad \text{iff} \quad x \in \mathrm{dom}(e).$$

Bell's extension principle says that this is an extension pattern. However, for this to go through constructively, in general we need $\mathrm{dom}(e)$ to be a detachable subset of X for every $e \in E$. In fact, this is the prime example of an extension pattern, and one encounters it under various circumstances once one sets out to capture specific one-step principles in terms of extension patterns.

Example 1.5 ("Trivial Pattern")**.** Let $\mathscr{E}$ be a partially ordered class. By stipulating $\mathscr{X} = \emptyset$ we get an extension pattern for which all elements $e \in \mathscr{E}$ are total. In particular, if $\mathscr{E}$ is a partially ordered class with extension pattern, then the collection of all total elements need not form a set. Furthermore, total elements need not be maximal.

Example 1.6 ("Maximal Pattern")**.** Let $\mathscr{E}$ be a partially ordered class with decidable partial order. This $\mathscr{E}$ works as a class of extension data for itself by way of a definition by cases:

$$x \Vdash e \quad \text{iff} \quad (x \geqslant e \to x = e) \qquad \text{and} \qquad e' = \begin{cases} x & \text{if } x \geqslant e \\ e & \text{otherwise} \end{cases}$$

for all $x, e \in \mathscr{E}$. We then indeed have the extension property

$$e \leqslant e' \quad \text{and} \quad x \Vdash e'.$$

In fact, if $x \geqslant e$, then $e' = x$ and thus $x \Vdash e'$; if $x \ngeqslant e$, then $e' = e$ and thus again $x \Vdash e'$. With respect to the relation $\Vdash$, the total elements for this pattern are precisely the maximal ones.

Example 1.7. Recall that a partially ordered class $\mathscr{E}$ is said to be *directed* if every pair of elements has an upper bound, i.e.,

$$\forall x \in \mathscr{E}\, \forall y \in \mathscr{E}\, \exists e \in \mathscr{E}\, \big(x \leqslant e \wedge y \leqslant e \big).$$

Consider this as extension property for the pattern on $\mathscr{E}$ which is defined by again taking $\mathscr{E}$ to be a class of extension data for itself and stipulating

$$x \Vdash e \quad \text{iff} \quad x \leqslant e.$$

An element e of $\mathscr{E}$ is total for this pattern if and only if e is the greatest element of $\mathscr{E}$.

1.3 Equivalence

Let $(\mathscr{E}, \leqslant)$ be a partially ordered class. From now on, by a *directed subset* in $\mathscr{E}$ we understand an inhabited subset D of $\mathscr{E}$ such that every pair of elements of D has an upper bound in D. We say that $\mathscr{E}$ is a *directed-complete* partially ordered class, for short a *dcpo*, if $\mathscr{E}$ is such that every directed subset D of $\mathscr{E}$ has a *least upper bound* $\bigvee D \in \mathscr{E}$ [2, 7]. Most dcpo's under consideration in this context are made of certain subsets of a fixed set, ordered by inclusion, for which suprema of directed families simply are unions. A dcpo $\mathscr{E}$ is said to be *set-generated* if there is a subset G of $\mathscr{E}$ such that, for every $e \in \mathscr{E}$,

$$G_e = \{\, g \in G : g \leqslant e \,\}$$

is a directed set with

$$\bigvee G_e = e.$$

Remark 1.8. A directed-complete partially ordered *set* E is set-generated, of course: take $G = E$. Conversely, if we admit the Powerset axiom, if G is a generating set for a dcpo $\mathscr{E}$, then the class

$\mathrm{Pow}(G)$ of all subsets of G is a set, and so is the class $\mathscr{D}(G)$ of all directed subsets of G. Then, still with the Powerset axiom, being the surjective image of a function

$$\mathscr{D}(G) \to \mathscr{E}, \quad D \mapsto \bigvee D,$$

we see that a set-generated dcpo $\mathscr{E}$ is a set.

One of the suitable forms of KZL over constructive set theory reads as follows [6].

KZL Every inhabited set-generated dcpo has a maximal element.

An extension pattern is hidden in many an indirect proof of an extension theorem by KZL, which provides a maximal element that in fact proves total. With our *General Extension Theorem* we extract the essence of this method.

GET Every inhabited set-generated dcpo with extension pattern has a total element.

Proposition 1.9. KZL *implies* GET.

Proof. Lemma 1.3. □

By contrast, total elements for an extension pattern need not be maximal, as was seen above with the trivial pattern in Example 1.5. The notions of totality and maximality hence do not necessarily coincide—unlike maximality, totality may depend on the pattern.

More often than not, there is an explicit method available, the application of which provides an extension e' of a given element e of $\mathscr{E}$ by arbitrary extension data $x \in \mathscr{X}$. We denote this extension e' by $f(x,e)$; indeed, we then have a class function

$$f : \mathscr{X} \times \mathscr{E} \to \mathscr{E}$$

satisfying the condition

$$\forall x \in \mathscr{X}\, \forall e \in \mathscr{E}\, \big(\, e \leqslant f(x,e) \,\wedge\, x \Vdash f(x,e)\,\big).$$

We say that an extension pattern as such is *functional*. Whether or not a pattern is functional solely depends on how extension data relate to elements $e \in \mathscr{E}$, i.e., f is not to be considered as an addendum to the definition of extension pattern, even though we could have demanded it in the first place.[3] It rather is a requirement on how X and $\Vdash$ capture the one-step argument.

Example 1.10. Let E be an ω-dcpo [11], i.e., a directed-complete partially ordered set such that every countable ascending sequence $\{\, e_n \,\}_{n \in \mathbb{N}}$ in E has a least upper bound, and let e_0 be a distinguished element of E. Let $(X, \Vdash, f)$ be a functional extension pattern on E which is *monotone*, insofar as that

$$\forall x \in X\, \forall e, e' \in E\, (\, x \Vdash e \,\wedge\, e \leqslant e' \,\to\, x \Vdash e'\,).$$

We did not stipulate this property as part of our definition of extension pattern, but it appears to be a rather natural condition: think of E as being ordered by extension, e.g. in case of a poset of partial functions, then extents should step-wise increase along with extension. Now, for every sequence $\alpha : \mathbb{N} \to X$ there is $e \in E$ such that

$$\forall n \in \mathbb{N}\, (\, \alpha(n) \Vdash e\,).$$

[3] It is interesting to note that the resulting principle fGET applies such as to convert every extension pattern into a functional one. However, to code additional information into extension data requires a certain definition by cases. Below we give an argument that shows GET and fGET constructively equivalent.

This is because we may define

$$\begin{aligned} e_1 &= f(\alpha(0), e_0) \\ &\vdots \\ e_{n+1} &= f(\alpha(n), e_n) \end{aligned}$$

and take $e = \bigvee \{ e_n \}_{n \in \mathbb{N}}$, for which we then have

$$\{ \alpha(n) : n \in \mathbb{N} \} \subseteq \{ x \in X : x \Vdash e \} .$$

In particular, if α is surjective, then this e is total for the pattern. To put it in a nutshell, if E is an (inhabited) ω-dcpo, then for every functional and monotone extension pattern on E, that has a countable set of extension data, there is a total element.

Notice that in Example 1.10, the pattern being functional helps to get around invoking choice so as to obtain an ascending sequence the supremum of which is total for the sequence α.

As for the reformulation of GET in case of a functional pattern, we have the following.

fGET Every inhabited set-generated dcpo with functional extension pattern has a total element.

Clearly, GET implies fGET. It is straightforward to show that these principles are classically equivalent. In fact, by means of fGET, every set-generated dcpo has a total element for the maximal pattern from Example 1.6, which indeed is functional, provided the partial order is decidable. The total element in question then is a maximal one.

As it turns out, with a more refined argument we are able to show that GET and fGET are equivalent even over $\mathbf{CZF}$. To this end, we make use of the following variant of the Hausdorff Maximal Principle for directed (rather than linearly ordered) subsets of a partially ordered set.

MDP Every inhabited partially ordered set has a maximal directed subset.

Proposition 1.11. fGET *implies* MDP.

Proof. Let $(E, \leqslant)$ be an inhabited partially ordered set. Consider the (predicative) partially ordered class $\mathscr{E}$ of all directed subsets of E, ordered by inclusion. We claim that $\mathscr{E}$ is an inhabited set-generated dcpo. A generating set for $\mathscr{E}$ is given by the set G of all *finite* directed subsets of E. Indeed, if $D \in \mathscr{E}$, then

$$\{ D_0 \in G : D_0 \subseteq D \}$$

is a set and it is directed: if D_0 and D_1 are finite directed subsets of D, then (being finite) they have a greatest element $x \in D_0$ and $y \in D_1$, respectively. Then, since D is directed, there is $z \in D$ such that $x, y \leqslant z$ and we have $D_0 \cup D_1 \cup \{ z \} \in G$ as well as $D_0 \cup D_1 \cup \{ z \} \subseteq D$. Moreover, $D = \bigcup \{ D_0 \in G : D_0 \subseteq D \}$, since G contains every singleton subset of E, in particular. Next, if $\mathscr{D}$ is a directed subset of $\mathscr{E}$, then $\bigcup \mathscr{D} \in \mathscr{E}$, whence $\mathscr{E}$ is directed-complete.

Now we describe an extension pattern on $\mathscr{E}$ with corresponding relation being the one from Example 1.6. For the class of extension data we take $\mathscr{E}$ itself and stipulate, for $C, D \in \mathscr{E}$,

$$C \Vdash D \quad \equiv \quad (\, C \supseteq D \rightarrow C = D \,).$$

Furthermore, there is a class function

$$f : \mathscr{E} \times \mathscr{E} \to \mathrm{Pow}(E), \quad f(C, D) = D \cup \{ z \in C : C \supseteq D \} .$$

In order to have an extension pattern, we need to verify $\mathrm{ran}(f) \subseteq \mathscr{E}$ as well as that for all $C, D \in \mathscr{E}$ we have

$$D \subseteq f(C, D) \qquad \text{and} \qquad C \Vdash f(C, D).$$

As regards the range of f, we need to show that $f(C,D)$ is directed. If $x,y \in f(C,D)$, then we can distinguish several cases. We may have both $x,y \in D$ in which case nothing needs to be checked, since D is directed. But if, say, $x \in \{ z \in C : C \supseteq D \}$, then $C \supseteq D$. This implies $x,y \in C$, and C is directed. Next, by the very definition of f we have $D \subseteq f(C,D)$. So it remains to show $C \Vdash f(C,D)$. To this end, suppose that $C \supseteq f(C,D)$. Since $f(C,D) \supseteq D$, we get $C \supseteq D$ and it follows from the definition of f that $f(C,D) = C$, as required. By way of fGET, there is an element $D \in \mathscr{E}$ which is total for the pattern just defined. This D is a maximal directed subset of E. □

Proposition 1.12. MDP *implies* KZL.

Proof. Let $\mathscr{E}$ be an inhabited set-generated dcpo and let G be a generating subset of $\mathscr{E}$. This G is inhabited since $\mathscr{E}$ is, and we restrict the partial order on $\mathscr{E}$ to G. According to MDP, there is a maximal directed subset D of G. Since $\mathscr{E}$ is a dcpo, this D has a least upper bound $\bigvee D \in \mathscr{E}$. We claim that $\bigvee D$ is a maximal element of $\mathscr{E}$. Indeed, if $e \in \mathscr{E}$ is such that $\bigvee D \leqslant e$, then we have an inclusion of sets

$$D \subseteq \{ g \in G : g \leqslant e \} .$$

This is because of $D \subseteq G$ and since for every $g \in D$ we have

$$g \leqslant \bigvee D \leqslant e.$$

Now take into account that $\{ g \in G : g \leqslant e \}$ is directed by our very assumption on G being a generating set for $\mathscr{E}$. Therefore, by maximality of D among directed subsets of G, we get

$$D = \{ g \in G : g \leqslant e \} ,$$

whence

$$\bigvee D = \bigvee \{ g \in G : g \leqslant e \} = e,$$

as required. □

Remark 1.13. Recall that all the classes considered in this chapter are supposed to be predicative. This allows for the above restriction of the order on $\mathscr{E}$ to the subset G by bounded separation, and thus to obtain a partially ordered *set*.

Remark 1.14. Under assumption of the Powerset axiom, all of the above proofs go through unchanged if the proviso "set-generated" is being dropped.

Corollary 1.15. *The following are equivalent:* KZL, GET, fGET, *and* MDP.

1.4 Application

We place ourselves in $\mathbf{CZF} + \mathsf{REM}$ in order to allow for a certain argument by cases. Let R be a ring. In the following, ideals of R and R-modules are understood to be left ideals and left R-modules, respectively. Recall that an R-module M is *injective* if every R-homomorphism $A \to M$ can be extended along injective R-homomorphisms $i : A \to B$.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\forall"'] & B \arrow[d, dashed, "\exists"] \\ & M \end{tikzcd}$$

By means of Baer's criterion, injectivity of a module M can be tested by considering R-homomorphisms $I \to M$ on ideals I of R only.

Baer's criterion Let R be a ring. An R-module M is injective already if every R-homomorphism $I \to M$, defined on an ideal I of R, extends onto R.

Proposition 1.16 (REM)**.** GET *implies Baer's criterion.*

Proof. Let M be an R-module with the property that every R-homomorphism $I \to M$, defined on an ideal I of R, extends onto R. Let $\varphi : A \to M$ and let $A \to B$ be an injective R-homomorphism; we assume that the latter is the inclusion mapping and A therefore is a submodule of B. As in the proof by Zorn's Lemma (see, e.g., [192, 247]), we consider the set E of partial extensions of φ, ordered by inclusion.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "\varphi"'] & A' \arrow[r, hook] \arrow[d, dashed, "\varphi'"'] & B \arrow[dl, dotted] \\ & M & \end{tikzcd}$$

Of course, E is readily shown to be directed-complete. The one-step extension principle, which helps to show that a maximal intermediary extension of φ is total, now encodes in an extension pattern for GET as follows. We have a set X of extension data an element of which is a triple

$$(\psi, x, \nu) \in E \times B \times \mathrm{Hom}_R(R, M),$$

subject to the condition

$$\nu\big|_{I(\psi,x)} = \nu(\psi, x), \tag{$*$}$$

where

$$I(\psi, x) = (\mathrm{dom}(\psi) : x) = \{\, r \in R : rx \in \mathrm{dom}(\psi) \,\}$$

and

$$\nu(\psi, x) : I(\psi, x) \to M, \quad r \mapsto \psi(rx).$$

The relation $\Vdash \subseteq X \times E$ is then defined by

$$(\psi, x, \nu) \Vdash \psi' \qquad \text{iff} \qquad \psi \neq \psi' \quad \text{or} \quad x \in \mathrm{dom}(\psi').$$

In case of $(\psi, x, \nu) \nVdash \psi'$, we have $\psi = \psi'$ and $x \notin \mathrm{dom}(\psi')$, and the one-step extension χ of ψ' by (ψ, x, ν) can be constructed as follows (e.g. [247]):

$$\chi : \mathrm{dom}(\psi') + Rx \to M, \quad y + rx \mapsto \psi'(y) + \nu(r).$$

Then, if ψ is total for this pattern, we can directly verify that it is defined everywhere on the R-module B. For if $x \in B$, we merely need to consider the R-homomorphism $\nu(\psi, x)$ defined on the ideal $I(\psi, x)$. Due to the assumption on M, this $\nu(\psi, x)$ is extended by some $\nu \in \mathrm{Hom}_R(R, M)$. We then have $(\psi, x, \nu) \in X$ and $(\psi, x, \nu) \Vdash \psi$ by totality, from which we infer $x \in \mathrm{dom}(\psi)$. ☐

The extension pattern with which we have deduced Baer's criterion was defined so as to be functional. Alternatively, we could simply have set

$$X = B \qquad \text{and} \qquad x \Vdash \psi \quad \text{iff} \quad x \in \mathrm{dom}(\psi)$$

for all $x \in B$ and $\psi \in E$, with E as before. If we then had proceeded as before, we would have had to choose ν in order to perform the extension step. The above use of more complex data—here, triples (ψ, x, ν) rather than only elements x—offers a way around this inasmuch as the choice of ν is anticipated.

1.5 Induction

Back to **CZF**. A subclass $\mathscr{F}$ of a dcpo $\mathscr{E}$ is *closed* if the supremum of every directed subset of $\mathscr{F}$ again belongs to $\mathscr{F}$, in which case $\mathscr{F}$ is a dcpo itself. If $\mathscr{E}$ has a functional extension pattern $(\mathscr{X}, \Vdash, f)$ and $\mathscr{F}$ moreover is such that $f(x, e) \in \mathscr{F}$ whenever $x \in \mathscr{X}$ and $e \in \mathscr{F}$, then the extension pattern restricts on $\mathscr{F}$ which thus has a total element by fGET. The *relative* version of our extension theorem then reads as follows.

rfGET Let $\mathscr{E}$ be an inhabited set-generated dcpo with functional extension pattern $(\mathscr{X}, \Vdash, f)$ and let $\mathscr{F}$ be a closed subclass of $\mathscr{E}$ such that

$$\forall x \in \mathscr{X}\, \forall e \in \mathscr{E}\, (\, e \in \mathscr{F} \,\rightarrow\, f(x,e) \in \mathscr{F}\,). \tag{†}$$

If $\mathscr{F}$ is inhabited and set-generated, then $\mathscr{F}$ has a total element.

This is an equivalent form of fGET, of course, because every dcpo may be considered a closed subclass of itself.

Conversely, a subclass $\mathscr{O}$ of $\mathscr{E}$ is said to be *open* if it cannot contain the supremum of a directed subset D of $\mathscr{E}$ unless it meets this D in at least one element, i.e., if

$$\bigvee D \in \mathscr{O} \,\rightarrow\, D \between \mathscr{O}$$

for every directed subset D of $\mathscr{E}$. Here, we write $D \between \mathscr{O}$ if the intersection $D \cap \mathscr{O}$ is inhabited.[4] Classically speaking, it is easy to see that the notions of closed and open subset of a dcpo are complementary to each other. Dualising the relative version of GET results in a principle for induction on functional extension patterns.

ifGET Let $\mathscr{E}$ be a set-generated dcpo with functional extension pattern $(\mathscr{X}, \Vdash, f)$ and let $\mathscr{O}$ be an open subclass of $\mathscr{E}$ such that

$$\forall x \in \mathscr{X}\, \forall e \in \mathscr{E}\, (\, f(x,e) \in \mathscr{O} \,\rightarrow\, e \in \mathscr{O}\,). \tag{‡}$$

If $\mathscr{O}$ contains all total elements, then $\mathscr{E} = \mathscr{O}$.

Respective forms hold also for extension patterns $(\mathscr{X}, \Vdash)$ which need not be functional. In order to state them in an analogous way, let us write $\boldsymbol{f}(x,e)$ for the subclass of $\mathscr{E}$ consisting of all *x-extensions* of a given element $e \in \mathscr{E}$, i.e.,

$$\boldsymbol{f}(x,e) = \{\, e' \in \mathscr{E} : e \leqslant e' \wedge x \Vdash e' \,\}\,.$$

In case of a functional pattern, we have $f(x,e) \in \boldsymbol{f}(x,e)$, of course. But the latter moreover includes every total element which might be above e. The relative version of GET for closed subclasses is immediate.

rGET Let $\mathscr{E}$ be an inhabited set-generated dcpo with extension pattern $(\mathscr{X}, \Vdash)$ and let $\mathscr{F}$ be a closed subclass of $\mathscr{E}$ such that

$$\forall x \in \mathscr{X}\, \forall e \in \mathscr{E}\, (\, e \in \mathscr{F} \,\rightarrow\, \boldsymbol{f}(x,e) \subseteq \mathscr{F}\,). \tag{†'}$$

If $\mathscr{F}$ is inhabited and set-generated, then $\mathscr{F}$ has a total element.

iGET Let $\mathscr{E}$ be a set-generated dcpo with extension pattern $(\mathscr{X}, \Vdash)$ and let $\mathscr{O}$ be an open subclass of $\mathscr{E}$ such that

$$\forall x \in \mathscr{X}\, \forall e \in \mathscr{E}\, (\, \boldsymbol{f}(x,e) \between \mathscr{O} \,\rightarrow\, e \in \mathscr{O}\,). \tag{‡'}$$

If $\mathscr{O}$ contains all total elements, then $\mathscr{E} = \mathscr{O}$.

We proceed by showing how these principles relate to each other.

Proposition 1.17.

1. iGET *implies* GET.
2. ifGET *implies* fGET.

[4] We have adopted this notation from Giovanni Sambin.

3. rGET *classically implies* iGET.

4. rfGET *classically implies* ifGET.

Proof. 1. Given an extension pattern $(\mathscr{X}, \Vdash)$ on $\mathscr{E}$, we consider the subclass

$$\mathscr{O} = \{ e \in \mathscr{E} : \exists e' \in \mathscr{E} (e \leqslant e' \wedge \forall x \in \mathscr{X} (x \Vdash e')) \}$$

of *totally extendable* elements. This $\mathscr{O}$ contains all total elements. Since $\mathscr{O}$ is downwards monotone, i.e.,

$$\forall e, e' \in \mathscr{E} (e \leqslant e' \wedge e' \in \mathscr{O} \to e \in \mathscr{O})$$

it is open (recall that directed subsets are to be inhabited) and it satisfies ($\ddagger'$). Thus $\mathscr{E} = \mathscr{O}$, whence every element of $\mathscr{E}$ is totally extendable.

2. Employ a similar argument as for the preceding item.

3. Given an extension pattern $(\mathscr{X}, \Vdash)$ on a set-generated dcpo $\mathscr{E}$, let $\mathscr{O}$ be an open subclass of $\mathscr{E}$ containing all total elements and such that ($\ddagger'$) holds. Working classically, suppose that there is $e \in \mathscr{E}$ such that $e \notin \mathscr{O}$. Consider $\mathscr{F} = \mathscr{E} - \mathscr{O}$. This $\mathscr{F}$ is an inhabited (set-generated, and even a set by Remark 1.8) closed subclass of $\mathscr{E}$ for which ($\dagger'$) holds. Therefore, $\mathscr{F}$ has a total element, however all of which should belong to $\mathscr{O}$.

4. Employ a similar argument as for the preceding item. □

We do not know whether there are constructive proofs for the above classical implications, nor if it can be shown that iGET implies ifGET over **CZF**. Anyway, keep in mind that GET and rGET, as well as fGET and rfGET, are mere reformulations of each other, respectively. Along with Corollary 1.15 and Proposition 1.17, we see that all principles considered in this chapter are classically equivalent.

Corollary 1.18 (REM)**.** *The following are equivalent:* KZL, MDP, GET, rGET, iGET, fGET, rfGET, ifGET.

Let us briefly compare our induction principles on extension patterns with Raoult's principle of *Open Induction* [201]. A subclass $\mathscr{P}$ of a partially ordered class $(\mathscr{E}, \leqslant)$ is said to be *progressive* if

$$\forall e \in \mathscr{E} (\forall e' \in \mathscr{E} (e' > e \to e' \in \mathscr{P}) \to e \in \mathscr{P}),$$

where $e' > e$ is understood to be the conjunction of $e \leqslant e'$ and $e \neq e'$. Here is a version of Open Induction for set-generated dcpo's.

OI Let $\mathscr{E}$ be a set-generated dcpo. If $\mathscr{P}$ is an open and progressive subclass of $\mathscr{E}$, then $\mathscr{P} = \mathscr{E}$.

For instance, let $(\mathscr{X}, \Vdash)$ be an extension pattern on $\mathscr{E}$, and let $\mathscr{O} \subseteq \mathscr{E}$ be open, contain every total element, and satisfy ($\ddagger'$). Suppose that totality is a decidable property on $\mathscr{E}$ insofar as that for every element $e \in \mathscr{E}$, either e is total or else there is certain data $x \in \mathscr{X}$ for which $x \nVdash e$. Now, if $e \in \mathscr{E}$ is such that $e' \in \mathscr{O}$ whenever $e' > e$, we also have $e \in O$. For either is e total, by which $e \in \mathscr{O}$ is immediate, or there is $x \in \mathscr{X}$ with $x \nVdash e$. In case of the latter, by extension there is $e' \in \mathscr{E}$ such that $e < e'$ and $x \Vdash e'$. It follows that $\boldsymbol{f}(x, e) \between \mathscr{O}$. As $\mathscr{O}$ is supposed to satisfy ($\ddagger'$), we get $e \in \mathscr{O}$. This shows how OI implies iGET under the above proviso that one can tell for each $e \in E$ whether e is total or has a witness to the opposite. Compare [131, 197, 223].

1.6 Conclusion

An extension pattern lies at the heart of many a proof by means of KZL, be it for obtaining a maximal element (with the maximal pattern) or showing any such maximal element to be total for a specific set of extension data. Our principle GET is not just KZL in disguise; the one-step principle on which it rests helps to circumvent the proof by contradiction typical of invocations of KZL.

The principle GET derives from a straightforward application of KZL, and subsumes KZL by virtue of an appropriate pattern. It thus is an instance with generalizing attitude. As pointed out before, being directly implied by KZL, our extension theorems do not have any nonconstructive consequences. However, defining an extension pattern typically requires a definition by cases—the very applicability of GET hence goes along with the law of excluded middle. But here we did not set out finding constructive extension theorems. From a didactical viewpoint, GET moreover is a tool which applies quite in the manner of KZL, but sheds fresh light on how we make use of maximal principles with extension arguments in everyday practice.

Chapter 2

Logical completeness and Jacobson radicals

This chapter is based on [225].

2.1 Introduction

Szpilrajn [242] has ascribed to Banach, Kuratowski and Tarski his proof of the order extension principle for strict partial orders. Following Kuratowski's [159] method of eliminating transfinite numbers, he invokes what now is known as the Kuratowski–Zorn lemma (KZL) to ensure the existence of a linear extension of a strict partial order. Szpilrajn's proof is further based on the key lemma that if a strict partial order $<$ does not yet compare two points a and b, then there is an extension $<'$ of $<$ which too is a strict partial order but for which $a <' b$. In particular, a maximal extension as given by KZL thus cannot fail to be linear.[1]

Order extension principles play a seminal role in mathematical economics, social choice, and the theory of preference relations; see, for example, [13, 48, 61]. Arrow [13] has formulated the order extension principle for quasi-orders the proof of which is due to Hansson [129]. In this context, more often than not a specific notion of extension $\preccurlyeq$ narrower than simple containment $\subseteq$ is employed [48, 61], which indeed makes good sense for preference relations [13]. As a side effect this conceptual choice rules out [129, p. 453] the trivial linearisation by $E \times E$ of any relation whatsoever on a set E, and thus makes those principles true forms [137] of the Axiom of Choice.[2] More precisely, given binary relations R and S on X with $R \subseteq S$, the additional condition for S to extend R in the aforementioned stronger sense, $R \preccurlyeq S$, is equivalent [226] to

$$S \cap R^\circ \subseteq R$$

where R° stands for the inverse relation, sometimes called *reciprocation* [112]:

$$R^\circ = \{ (b, a) : (a, b) \in R \} .$$

Now the universal relation $S = E \times E$ extends R with respect to $\preccurlyeq$ precisely when R is symmetric, which hardly is the case for order relations.

All this can be developed in quite an abstract, algebraic manner [226]. Our intention in this chapter is to take further steps by analogy, leave behind the topic of order extension, and focus on a basic concept which in fact is at the heart of the above: single-conclusion entailment relations, finitary coverings [65, 67], or algebraic closure operators. In this much wider context, a proof pattern that derives from Raoult's principle of Open Induction prompts an appropriate form of

[1] The key lemma is one of the "one-step extension" arguments Bell [33] has singled out as typical for such proofs. We have studied this [224] for a general extension theorem on dcpo's.

[2] The Axiom of Choice, however, is not required for syntactical conservation of linearity for Horn sequents [190, 209, 210] and for numerous related issues in algebra [92, 169].

Lindenbaum's Lemma which not only resembles the aforementioned extension principle for quasi-orders, but also has applications reaching far beyond, encompassing several prominent results from commutative algebra and lattice theory. Our approach thus makes possible a universal treatment similar to the one carried out in [208], from which in fact we take further motivation. While [208] concentrates on finitary coverings that satisfy the pivotal condition of *Encoding* which represents prime ideals, here instead we consider certain *complete* ideals which roughly correspond to maximal ideals. We generalize the concept of *Jacobson radical* from commutative rings—thus regaining one of the computationally meaningful characterizations [169] by abstract, yet altogether elementary means—to a general ideal theory for single-conclusion entailment relations, in which context we even obtain a syntactial counterpart of Lindenbaum's Lemma.

This chapter is structured as follows. In Section 2.2 we collect certain tools that will be needed later on. First we make some observations on preframes, thus providing abstract grounds for our extension principle. Next we discuss, and prove equivalent, several consequences of Raoult's principle of *Open Induction* including the *Zorn Scheme* used in this chapter. Finally, we give preliminaries of single-conclusion entailment relations and define the notion of *(strong) reductivity*. In Section 2.3 we introduce the concept of *tight extension*, and show under suitable hypotheses that the class of ideals for a reductive single-conclusion entailment relations is a set-generated directed-complete partially ordered class with respect to tight extension. This will lead to a variant of Lindenbaum's Lemma as well as to a generalization of several concepts from commutative ring theory to strongly reductive single-conclusion entailment relations. In particular, we obtain an explicit characterization of the set of all elements common to every complete ideal of a strongly reductive single-conclusion entailment relation. Concrete examples and applications are studied in Section 2.4. In the final Section 2.5 we discuss certain aspects of the conservation of multi-conclusion entailment relations over their single-conclusion counterparts.

2.2 Key instruments

2.2.1 Order

Let $(\mathscr{E}, \leqslant)$ be a partially ordered class. Recall that a subset D of $\mathscr{E}$ is *directed* if it is inhabited and every pair of elements $x, y \in D$ has an upper bound in D. A partially ordered class $\mathscr{E}$ is *directed-complete* (a *dcpo* for short), if every directed subset D has a least upper bound (a join) $\bigvee D$ in $\mathscr{E}$. A subclass $\mathscr{F}$ of $\mathscr{E}$ is a *sub-dcpo* if it is a dcpo with respect to the induced ordering. A class function $f : \mathscr{E} \to \mathscr{F}$ between dcpo's $\mathscr{E}$ and $\mathscr{F}$ is *Scott-continuous* if it preserves directed sets and their joins. That is, if D is a directed subset of $\mathscr{E}$, then the image $f(D)$ of D under $\mathscr{F}$ is a directed subset of $\mathscr{F}$, and $f(\bigvee D) = \bigvee f(D)$. Every Scott-continuous map $\mathscr{F}$ is monotone, while monotonicity suffices for $\mathscr{F}$ to preserve directed subsets. A *preframe* [25, 149] is a dcpo which has finite meets which distribute over joins of directed subsets:

$$\bigvee D \wedge x = \bigvee \{\, y \wedge x : y \in D \,\}\,.$$

In particular, every preframe has a top element 1, being the empty meet. We will not go into the theory of preframes later on, let alone provide any further development of this theory over **CZF**. However, we have now available appropriate tools for the following.

Definition 2.1. *Let $(\mathscr{E}, \leqslant, \wedge)$ be a preframe, and let $^\circ : \mathscr{E} \to \mathscr{E}$ be a Scott-continuous class function. We define a binary subrelation $\preccurlyeq$ of $\leqslant$ by*

$$x \preccurlyeq y \quad \equiv \quad x \leqslant y \ \text{ and } \ y \wedge x^\circ \leqslant x.$$

If $\mathscr{E}$ denotes the class of quasi-orders on a set E, ordered by $\subseteq$, and if we take $^\circ$ as reciprocation, then the concept of extension [129] as explained in the introduction is a special case of Definition 2.1. In this vein, the following abstract argument can indeed be used to prove a version of Szpilrajn's extension theorem for quasi-orders [226].

Proposition 2.2. *Let $(\mathscr{E}, \leqslant, \wedge)$ be a preframe, and let $^\circ : \mathscr{E} \to \mathscr{E}$ be a Scott-continuous class function. Every sub-dcpo $\mathscr{F}$ of $\mathscr{E}$ with respect to $\leqslant$ is a sub-dcpo also with respect to $\preccurlyeq$. In particular, $(\mathscr{E}, \preccurlyeq)$ is a dcpo.*

Proof. We first show that the subrelation $\preccurlyeq$ is a partial ordering on $\mathscr{E}$. Clearly, reflexivity and antisymmetry are inherited from $\leqslant$. As for transitivity, if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \leqslant z$ by transitivity of $\leqslant$, and

$$z \wedge x^\circ = (z \wedge x^\circ) \wedge x^\circ \leqslant (z \wedge y^\circ) \wedge x^\circ \leqslant y \wedge x^\circ \leqslant x,$$

since both $^\circ$ and $\wedge$ preserve the order. Now let $\mathscr{F}$ be a sub-dcpo of $\mathscr{E}$. This $\mathscr{F}$ is partially ordered by $\preccurlyeq$ too. If D is a directed subset of $\mathscr{F}$ with respect to $\preccurlyeq$, then D is directed with respect to $\leqslant$ as well, for which it thus has join $\bigvee D$. We show that $\bigvee D$ is the least upper bound of D also with respect to $\preccurlyeq$. Of course, if $x \in D$, then $x \leqslant \bigvee D$. Furthermore, since D is directed, for every $y \in D$ there is $z \in D$ with $x, y \preccurlyeq z$; in particular $y \leqslant z$ and $z \wedge x^\circ \leqslant x$. Together this yields $y \wedge x^\circ \leqslant z \wedge x^\circ \leqslant x$, whence

$$(\bigvee D) \wedge x^\circ = \bigvee_{y \in D} (y \wedge x^\circ) \leqslant x,$$

and therefore $x \preccurlyeq \bigvee D$. Next, suppose that there is $z \in \mathscr{F}$ with $x \preccurlyeq z$ for every $x \in D$. Then we have $\bigvee D \leqslant z$ and

$$z \wedge \big(\bigvee D\big)^\circ = z \wedge \big(\bigvee_{x \in D} x^\circ\big) = \bigvee_{x \in D} (z \wedge x^\circ) \leqslant \bigvee_{x \in D} x = \bigvee D$$

because $^\circ$ is Scott-continuous. Therefore $\bigvee D \preccurlyeq z$, as required. □

But why consider sub-dcpo's $\mathscr{F}$ of $\mathscr{E}$ in the first place? The reason is that $^\circ$ may not restrict on $\mathscr{F}$, that is to say, there can be $x \in \mathscr{F}$ such that $x^\circ \notin \mathscr{F}$. However, the induced relation $\preccurlyeq$ makes $\mathscr{F}$ a dcpo whatsoever.

Yet another notion will be important: Suppose that $\mathscr{E}$ also has binary joins providing $\mathscr{E}$ with a lattice structure. We say that an element $x \in \mathscr{E}$ is *complete* if

$$x \vee x^\circ = 1.$$

The following observation is of interest when it comes to applications of the Kuratowski-Zorn lemma and Raoult's principle of Open Induction [201], see Section 2.2.2 below.

Lemma 2.3. *Let $\mathscr{E}$ be a preframe with distributive binary joins, and let $^\circ : \mathscr{E} \to \mathscr{E}$ be a Scott-continuous class function. For every $x, y \in \mathscr{E}$, if $y \leqslant x \vee x^\circ$, then $x = y$ already if $x \preccurlyeq y$. In particular, every complete element is maximal with respect to $\preccurlyeq$.*

Proof. If $y \leqslant x \vee x^\circ$ and $x \preccurlyeq y$, then $x \leqslant y$ as well. Moreover

$$y = y \wedge y \leqslant y \wedge (x \vee x^\circ) = (y \wedge x) \vee (y \wedge x^\circ) \leqslant x,$$

whence $x = y$. If x is complete, i.e., if $x \vee x^\circ = 1$, then $y \leqslant x \vee x^\circ$ for every $y \in \mathscr{E}$. Thus, if x is complete and $x \preccurlyeq y$, then $x = y$ by the preceding observation. □

2.2.2 Some consequences of Open Induction

Let $\mathscr{E}$ be a dcpo. Recall that $\mathscr{E}$ is said to be *set-generated* [2] if there is a subset G of $\mathscr{E}$ such that, for every $x \in \mathscr{E}$,

$$G_x = \{\, g \in G : g \leqslant x \,\}$$

is directed and

$$\bigvee G_x = x.$$

In general, set-generated classes—rather than just dcpo's—play a distinctive role in constructive set theory; see, e.g., [10, 144, 248]. For instance, if S is a set, then the class $\mathrm{Pow}(S)$, partially ordered with respect to $\subseteq$, is a set-generated dcpo, with generating subset $G = \mathrm{Fin}(S)$.

Remark 2.4. If we make the Powerset axiom available, for instance, by way of REM [7], then every set-generated dcpo $\mathscr{E}$ actually is a set. Indeed, if G is a generating set for $\mathscr{E}$, and if $\mathrm{Pow}(G)$ is a set, then so is the class $\mathscr{D}(G)$ of all directed subsets of G. Therefore, since $\mathscr{E}$ is the surjective image of a function

$$\mathscr{D}(G) \to \mathscr{E}, \quad D \mapsto \bigvee D,$$

we see that $\mathscr{E}$ is a set by Replacement.

Over **CZF** one of the possible forms [6] of Kuratowski's and Zorn's famous maximality principle can be phrased as follows:

Kuratowski-Zorn Lemma (KZL) Let $\mathscr{E}$ be a set-generated dcpo. Over every element x of $\mathscr{E}$ there is one which is maximal.

Now let $\mathscr{O}$ be a subclass of $\mathscr{E}$. One says that $\mathscr{O}$ is *progressive* if

$$\forall x \in \mathscr{E} \left(\forall y \in \mathscr{E} \, (\, y > x \, \to \, y \in \mathscr{O} \,) \, \to \, x \in \mathscr{O} \, \right),$$

where $y > x$ is understood as the conjunction of $x \leqslant y$ and $x \neq y$. Furthermore, $\mathscr{O}$ is *open* if

$$\bigvee D \in \mathscr{O} \, \to \, D \between \mathscr{O}$$

for every directed subset D of $\mathscr{E}$, where $D \between \mathscr{O}$ means that D and $\mathscr{O}$ have an element in common.[3]

Raoult [201] has coined the following principle, of which we consider a reformulation for set-generated dcpo's, suitable for **CZF**.

Open Induction (OI) Let $\mathscr{E}$ be a set-generated dcpo. If $\mathscr{O}$ is an open and progressive subclass of $\mathscr{E}$, then $\mathscr{E} = \mathscr{O}$.

Raoult [201] has deduced OI from KZL over **ZF**. In fact, both principles are equivalent by complementation and thus with classical logic. See further [36, 66, 73, 208, 222].

We say that a subclass $\mathscr{O}$ of $\mathscr{E}$ is (downward) *monotone* if

$$\forall x, y \in \mathscr{E} \, (\, x \leqslant y \wedge y \in \mathscr{O} \, \to \, x \in \mathscr{O} \,).$$

Every monotone subclass is an open subclass. In fact, if $\mathscr{O}$ is monotone, and if D is a directed subset of $\mathscr{E}$ such that $\bigvee D \in \mathscr{O}$, then even $D \subseteq \mathscr{O}$; keep in mind that in this chapter every directed subset is required to have an element.

The following principle is a direct consequence of OI.

Monotone Induction (MI) Let $\mathscr{E}$ be a set-generated dcpo. If $\mathscr{O}$ is a monotone and progressive subclass of $\mathscr{E}$, then $\mathscr{E} = \mathscr{O}$.

MI turns out useful for replacing proofs by means of KZL with proofs by OI, because predicates expressing "total extendability" are monotone. Incidentally, this suggests how to bring MI close to a form resembling KZL.

Zorn Scheme (ZS) Let $\mathscr{E}$ be a set-generated dcpo. If $\mathscr{C}$ is a subclass of $\mathscr{E}$, then

$$\forall x \in \mathscr{E} \, \exists y \in \mathscr{E} \, (\, x < y \, \vee \, x \in \mathscr{C} \,) \, \to \, \forall x \in \mathscr{E} \, \exists y \in \mathscr{E} \, (\, x \leqslant y \, \wedge \, y \in \mathscr{C} \,).$$

[3] We have adopted this notation from Giovanni Sambin.

Intuitively, if every element x of a dcpo $\mathscr{E}$ either falls under the concept that constitutes a certain subclass $\mathscr{C}$ of $\mathscr{E}$ or else is strictly exceeded, then over every element of $\mathscr{E}$ there is one which does fall under the concept in question. Notice how this describes the common strategy for proofs by means of KZL. Yet proofs of the latter kind usually appeal to an argument by contradiction, for instance when it comes to verify total some maximally extended partial functional for proving the Hahn-Banach theorem. In comparison, with ZS one would have to know about the "extent" of every such object right in the first place.

Next we show that ZS is a direct consequence of MI.

Proposition 2.5. MI *implies* ZS*.*

Proof. Let $\mathscr{C}$ be a subclass of $\mathscr{E}$ such that

$$\forall x \in \mathscr{E}\, \exists y \in \mathscr{E}\, (\ x < y\ \vee\ x \in \mathscr{C}\ ). \tag{$\dagger$}$$

We define another subclass $\mathscr{O}$ of $\mathscr{E}$ by

$$x \in \mathscr{O}\ \equiv\ \exists z \in \mathscr{E}\, (\ x \leqslant z\ \wedge\ z \in \mathscr{C}\ ).$$

Reflexivity of $\leqslant$ implies $\mathscr{C} \subseteq \mathscr{O}$, and $\mathscr{O}$ is monotone: if $x \leqslant y$ and $y \in \mathscr{O}$, then

$$x \leqslant y\ \wedge\ \exists z \in \mathscr{E}\, (\ y \leqslant z\ \wedge\ z \in \mathscr{C}\ ),$$

and therefore

$$\exists z \in \mathscr{E}\, (\ x \leqslant z\ \wedge\ z \in \mathscr{C}\ )$$

by transitivity of $\leqslant$, whence $x \in \mathscr{O}$. As for $\mathscr{O}$ being progressive, consider x such that

$$\forall y \in \mathscr{E}\, (\ y > x\ \rightarrow\ y \in \mathscr{O}\ ). \tag{$\ddagger$}$$

By ($\dagger$) we either have $x \in \mathscr{C}$ and thus $x \in \mathscr{O}$ anyway, or else there is $y \in \mathscr{E}$ with $x < y$. In the latter case we have $y \in \mathscr{O}$ by assumption ($\ddagger$), whence again $x \in \mathscr{O}$, as $\mathscr{O}$ is monotone. Now, by way of MI, we get $\mathscr{E} = \mathscr{O}$, which is to say that

$$\forall x \in \mathscr{E}\, \exists z \in \mathscr{E}\, (\ x \leqslant z\ \wedge\ z \in \mathscr{C}\ ). \qquad \square$$

Classically, ZS is utterly useful—as useful as KZL, of course. In the above setting of section 2.2.1, the subclass which is of primary interest consists of all *complete* elements:

$$x \in \mathscr{C}\ \equiv\ (\ x \vee x^\circ = 1\ ).$$

But then, in order for ZS to be applicable in the first place, we need to make sure that for every element which is not complete there indeed is another one by which it is strictly exceeded. In this vein, let us say that a dcpo $\mathscr{E}$ has a *strong maximality test* if the following holds:

$$\forall x \in \mathscr{E}\, \big(\forall y \in \mathscr{E}\, (\, x \leqslant y\ \rightarrow\ x = y\,)\ \vee\ \exists z \in \mathscr{E}\, (\, x < z\,)\big)$$

Proposition 2.6. ZS *implies* KZL *for set-generated dcpo's with strong maximality test.*

Proof. Stipulate

$$x \in \mathscr{C}\ \equiv\ \forall y \in \mathscr{E}\, (\ x \leqslant y\ \rightarrow\ x = y\ ),$$

which means that x is maximal in $\mathscr{E}$. Then, if $\mathscr{E}$ has a strong maximality test, for every $x \in \mathscr{E}$ either we have $x \in \mathscr{C}$ right away, or there is $y \in \mathscr{E}$ with $x < y$. Then, by way of ZS, for every $x \in \mathscr{E}$ there is $y \in \mathscr{E}$ with $x \leqslant y$ and such that $y \in \mathscr{C}$. In other words, over every element of $\mathscr{E}$ there is one that is maximal. $\square$

Corollary 2.7. *Over* **ZF***, the principles* OI*,* MI*,* ZS*, and* KZL *are equivalent.*

Before we proceed, a caveat is in order. Whereas KZL allegedly is *constructively neutral* [33]—as opposed to the Axiom of Choice, which is known to imply REM over **CZF** [7, 8, 101, 122, 202]—OI, along with its direct consequences MI and ZS, can hardly be worse. As a rule of thumb, induction principles are constructively "less harmful" than maximality principles. In the following, however, any application of ZS can straightforwardly be replaced by one of KZL, *mutatis mutandis.*

2.3 Ideals and extensions

2.3.1 Entailment

We are interested in the semantics of a particular class of entailment relations, some preliminaries for which we now take the opportunity to briefly recall. Let S be an inhabited set. A *single-conclusion entailment relation* [83, 103, 115, 132–134, 170, 209, 210, 229, 234, 244, 255], see also [67, 188, 218, 220], is a relation

$$\rhd \subseteq \operatorname{Fin}(S) \times S$$

between *finite* subsets U and elements a of S which in the following sense is *reflexive*, *monotone*, and *transitive*, respectively:

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{U \rhd a}{U, U' \rhd a}\ (\mathrm{M}) \qquad \frac{U \rhd b \quad U', b \rhd a}{U, U' \rhd a}\ (\mathrm{T})$$

It is common to write U, V for the union $U \cup V$ and sometimes to simply write a where it actually should read a singleton set $\{ a \}$. If T is an *arbitrary* subset of S, then stipulating

$$T^{\rhd} = \{ a \in S : \exists U \in \operatorname{Fin}(T) \,(U \rhd a) \}$$

yields an *algebraic closure operator* on the class $\operatorname{Pow}(S)$ of all subsets of S. A *saturated* subset I of S is one for which $I = I^{\rhd}$. A saturated subset will also be called an *ideal*. In terms of entailment, ideals are the *deductively closed* subsets of S, i.e., I is an ideal if and only if I splits entailment:

$$\frac{I \supseteq U \quad U \rhd a}{I \ni a}$$

Ideals need not be proper subsets of S. For instance, the underlying set S itself is an ideal.

If T is an arbitrary subset of S, then the *quotient* [116, 194] of $\rhd$ with respect to T is the single-conclusion entailment relation $\rhd_T$ defined by

$$U \rhd_T a \;\equiv\; \exists V \in \operatorname{Fin}(T) \,(U, V \rhd a).$$

An element $a \in S$ is said to be *convincing* [208] for $\rhd$ if

$$\forall b \in S \,(\, a \rhd b \,).$$

This amounts to $\{ a \}^{\rhd} = S$ in terms of the saturation. If T is an arbitrary subset of S, then by its *reciprocation* T° we understand the set of all convincing elements for the quotient $\rhd_T$, thus

$$T^{\circ} = \{ a \in S : \forall b \in S(\, a \rhd_T b \,) \}\,.$$

Again, in terms of the saturation,

$$a \in T^{\circ} \qquad \text{if and only if} \qquad (T, a)^{\rhd} = S.$$

Reciprocation is (upward) monotone, i.e., if $T \subseteq T'$, then $T^{\circ} \subseteq T'^{\circ}$. Moreover, notice that

$$(T^{\rhd})^{\circ} = T^{\circ}$$

which is because of $(T^{\rhd}, a)^{\rhd} = (T, a)^{\rhd}$ for every $a \in S$. While $S^{\circ} = S$, it is *not* required that the reciprocation of a proper ideal be saturated too, and in general it isn't.

With our notation, an element a is convincing if and only if $a \in \{ a \}^{\circ}$, while the set of all convincing elements is given by $\emptyset^{\circ}$, which might as well be empty. If a is convincing, then $\{ a \}^{\circ} = S$. Moreover, if a is convincing, then an ideal I is proper if and only if $a \notin I$.

We say that $(S, \rhd)$ is *trivial* if $S \subseteq \emptyset^{\circ}$, which is to say that every element is convincing:

$$\forall a, b \in S \,(a \rhd b).$$

We say that an element $a \in S$ is *complemented* if there is $b \in S$ such that $a \in \{ b \}^{\circ}$, which is to say that $\{ a, b \}^{\rhd} = S$, and *par abus de langage* that S as a whole is complemented if every element of S is complemented.[4] For example, if S has a convincing element, then S is complemented. Furthermore, $a \in \{ b \}^{\circ}$ is equivalent to $b \in \{ a \}^{\circ}$.

The following conditions on single-conclusion entailment relations will be crucial.

Definition 2.8. *Let $\rhd$ be a single-conclusion entailment relation on S. We say that $\rhd$ is* reductive *if, for every subset T of S,*

$$a \in T^{\circ} \implies \exists U \in \mathrm{Fin}(T^{\rhd}) \left(a \in U^{\circ} \right).$$

$\rhd$ is strongly reductive *if*

$$a \in T^{\circ} \implies \exists b \in T^{\rhd} \left(a \in \{ b \}^{\circ} \right).$$

Strongly reductive single-conclusion entailment relations play a distinctive role in [195]. The definition of reductivity does not prevent U from being empty, in which case a is supposed to be convincing. The definition of strong reductivity is the special case where U is a singleton set. Both definitions bring about certain properties of $(S, \rhd)$. In the following, by a *dense* subset of S will be meant one the saturation of which equals S. So S is the only dense ideal.

Lemma 2.9.

1. *If $\rhd$ is reductive, then S has a finite dense subset.*

2. *If $\rhd$ is strongly reductive, then S is complemented.*

Proof. Both assertions follow from the fact that $S^{\circ} = S$, and since S is supposed to be inhabited. ☐

As a final preliminary remark, we point out that we do not require from the outset that S comes equipped with any algebraic structure, or is closed under certain connectives. What is required for the main part of this note can be expressed in terms of the entailment relation only. Furthermore, while it is certainly instructive to construe of $\rhd$ in terms of logical inference, see Example 2.20 and Example 2.25 below, S need not consist of formulae as specified by a certain formal language [229], by which the use of entailment relations is not at all limited to logic [62].

2.3.2 A variant of Lindenbaum's Lemma

In the following, let S be an inhabited set with a single-conclusion entailment relation $\rhd$. We introduce a partial order on the class of all ideals of $\rhd$ as follows. Let I and J be ideals of $\rhd$, the latter being an *extension* of the former, i.e., $I \subseteq J$. We say that this containment is *tight* if in addition

$$J \cap I^{\circ} \subseteq I.$$

This is an instance of Definition 2.1, and we write $I \preccurlyeq J$ in order to indicate tight extension. For instance, an inclusion $I \subseteq J$ of ideals is tight in each of the cases $I = J$, $I^{\circ} \subseteq I$, and $I = S$. Furthermore, an ideal I is tightly contained in S if and only if $I^{\circ} \subseteq I$.

Lemma 2.10. *Let I be an ideal of $\rhd$. If $I^{\circ} \between I$, then $I = S$.*

Proof. If $a \in I^{\circ} \cap I$, then $S = (I, a)^{\rhd} \subseteq I$. ☐

Corollary 2.11. *Between* proper *ideals I and J, we have $I \subseteq J$ if and only if $I \preccurlyeq J$, in fact $J \cap I^{\circ} = \emptyset$ in any such case.*

Proof. Let $I \subseteq J$. If $a \in J \cap I^{\circ}$, then $J \between J^{\circ}$, and thus $J = S$ by Lemma 2.10, yet J is supposed to be proper. Thus $J \cap I^{\circ} = \emptyset \subseteq I$, and therefore $I \preccurlyeq J$. ☐

[4] With reference to [195], this circumstance may also be addressed by saying that $\rhd$ has *denial*.

Corollary 2.12. *Suppose that $\rhd$ is strongly reductive, and let I and J be ideals. If I is inhabited and proper, then $I \preccurlyeq J$ implies that J is proper.*

Proof. Suppose that $J = S$, whence $I^\circ \subseteq I$ due to tightness. If $\rhd$ is strongly reductive, then S is complemented by Lemma 2.9. Therefore, if $a \in I$, then there is $b \in S$ such that $S = \{ a, b \}^\rhd \subseteq (I, b)^\rhd$, from which we read off $b \in I^\circ \subseteq I$, and thus $I = S$ by Lemma 2.10. $\square$

All this is well in accordance with the motivating ideas that stem from the theory of preference relations, as explained in the introduction, where a narrower concept of extension makes sure to avoid the trivial extension of a quasi-order by the universal relation on the underlying set.

At this point, the reader may rightfully claim that our concept of strong containment teeters on the brink of triviality. However, it does have certain virtues even from a classical perspective: by involving tightness we automatically keep track of proper extensions, and by allowing ideals to be improper we are able to phrase several prominent extension theorems succinctly, and in a clear and concise manner, see Section 2.4.

In the following, we write $\mathfrak{Id}(\rhd)$ for the class of all ideals of $\rhd$. We further say that an ideal is *subdetachable* if it is proper or dense. Note that if I is subdetachable, then

$$\forall a \in \emptyset^\circ \,(\, a \in I \,\vee\, a \notin I \,),$$

and the converse holds if S has a convincing element.

Theorem 2.13. *If $\rhd$ is reductive, then $\big(\mathfrak{Id}(\rhd), \preccurlyeq\,\big)$ is a dcpo. If, in addition, every ideal is subdetachable, then $\mathfrak{Id}(\rhd)$ is set-generated.*

Proof. We apply Proposition 2.2 to the sub-dcpo $\mathfrak{Id}(\rhd)$ of $\mathrm{Pow}(S)$, the latter of which is a dcpo with regard to $\subseteq$. We thus need to verify that reciprocation is Scott-continuous on $\mathrm{Pow}(S)$. If D is a directed set of subsets of S, then $\bigvee D = \bigcup D$, whence we have to show that reciprocation commutes with directed union, i.e., $(\bigcup D)^\circ = \bigcup_{T \in D} T^\circ$. Since $^\circ$ is monotone, it suffices to concentrate on the left-to-right containment. But if $a \in (\bigcup D)^\circ$, then, since $\rhd$ is reductive, there is a finite subset U of $(\bigcup D)^\rhd$ such that $a \in U^\circ$. Since D is directed, we can find $T \in D$ such that $U \subseteq T^\rhd$ and therefore $a \in T^\circ$, as required.

Now suppose that we can decide for every ideal of $\rhd$ whether it is proper or dense, in which case we intend to show that $\mathfrak{Id}(\rhd)$ is set-generated. We consider the *set* G of all *finitely generated* ideals, i.e.,

$$G = \left\{\, I \in \mathfrak{Id}(\rhd) : \exists U \in \mathrm{Fin}(S)\,(\, I = U^\rhd \,)\,\right\}.$$

If J is a proper ideal, then

$$G_J = \{\, I \in G : I \preccurlyeq J \,\}$$

is directed. For if J is proper, then $\emptyset^\rhd \in G_J$ by Corollary 2.11, whence G_J is inhabited. If we have $I_1 \in G_J$ and $I_2 \in G_J$, and, say $I_1 = U_1^\rhd$ and $I_2 = U_2^\rhd$, where U_1 and U_2 are finite subsets of S, then we claim that the finitely generated ideal $I_3 = (U_1 \cup U_2)^\rhd$ belongs to G_J as well, to which end we have to show that J is a tight extension of the latter. In any case, I_3 is contained in J, and thus (Corollary 2.11) tightly contained. The final step is to show

$$J = \bigcup G_J$$

which, however, is trivial: if $a \in J$, then $\{ a \}^\rhd$ is tightly contained in J for the same reasons as were just mentioned. It remains to consider S as an ideal. Since $\rhd$ is reductive, S has a finite dense subset U by Lemma 2.9, from which it straightforwardly follows that G_S is directed as well. $\square$

We say that an ideal C is

1. *complete* if $C \cup C^\circ = S$, and
2. *consistent* if $C \cap C^\circ = \emptyset$.

By Lemma 2.10, every proper ideal is consistent; conversely, if C is consistent and C° is inhabited, then C is proper.

Proposition 2.14. *If C is a consistent complete ideal, then C is detachable and $C^\circ = S \setminus C$.*

Corollary 2.15. *The class of all proper complete ideals of $\rhd$ is a set.*

Proof. The class of detachable subsets of S is a set, hence so is the class of all proper complete ideals of $\rhd$ by way of bounded separation. □

Of course, the underlying set S is a complete ideal itself. However, unless every complete ideal is subdetachable, or we work with the Powerset axiom, in general we cannot state that the full class of complete ideals is a set; see Remark 2.19 below.

Next, we say that an ideal I is *subcomplete* if it either is complete or else there is a witness to the opposite:

$$\exists a \in S\,(\,a \notin I \,\wedge\, a \notin I^\circ\,).$$

A subcomplete ideal which is not itself complete, and thus proper, is properly contained in another proper ideal, as follows. Of course, this is trivial from a classical point of view.

Lemma 2.16. *Let I be an ideal, and let $a \in S$ such that $a \notin I \cup I^\circ$. Then I is properly and tightly contained in the proper ideal $(I,a)^\rhd$.*

Proof. Given an element a as indicated, both containments $I \subseteq (I,a)^\rhd$ and $(I,a)^\rhd \subseteq S$ are proper. The former one is tight by way of Lemma 2.10. □

In analogy to Lemma 2.3, every complete ideal is maximal with respect to tight extension. The next observation may further motivate our interest in complete ideals.

Proposition 2.17. *Let I be a proper ideal of $\rhd$. The following are equivalent.*

1. *I is complete.*
2. *I is subcomplete and maximal for tight extension.*

Proof. As mentioned before, every complete ideal I is subcomplete, and I being maximal for tight extension is a consequence of Lemma 2.3. As for the converse, let I be a subcomplete ideal, and suppose that I is maximal for tight extension. Due to subcompleteness, we know that either I is complete, or there is an element a of S such that both $a \notin I$ and $a \notin I^\circ$. However, since the latter disjunct would allow for a proper and tight extension of I according to Lemma 2.16, it leads to a contradiction, thus leaving us with the former one. □

Here is a variant of Lindenbaum's Lemma for reductive single-conclusion entailment relations. The general assumption of subcompleteness is required in order to make ZS applicable.

Theorem 2.18 (ZS). *Suppose that every ideal is subcomplete and subdetachable. If $\rhd$ is reductive, then every ideal is tightly contained in some complete ideal.*

Proof. The partially ordered class $\mathfrak{Id}(\rhd)$ of ideals is a set-generated dcpo with respect to tight containment (Theorem 2.13). Every ideal I is either complete, or else properly and tightly contained in another ideal J (Lemma 2.16). Invoking ZS, every ideal is tightly contained in a complete ideal. □

Over **ZF**+ZS, we can drop the additional assumptions of subcompleteness and subdetachability, and simply state that if $\rhd$ is reductive, then every ideal is tightly contained in a complete ideal.

Remark 2.19. The additional hypothesis for every ideal to be subdetachable in general implies REM. It thus signifies a considerable obstruction with regard to the overall applicability of Theorem 2.18 in a constructive setting. To give an example, write $0 = \emptyset$ and $1 = \{ 0 \}$, take $S = 1$ for our domain of discourse, and consider $\rhd$ to be given by the single entailment

$$1 \rhd 0.$$

This entailment relation actually coincides with the membership relation, i.e., $U \rhd a$ if and only if $a \in U$, so every subset of 1 is an ideal for $\rhd$. Now let φ be an arbitrary bounded formula in the first-order language of set theory and consider its *intuitionistic truth value*

$$I_\varphi = \{ a \in 1 : \varphi \} ,$$

which is an ideal of course, and for which φ is equivalent to $0 \in I_\varphi$ and thus to $I_\varphi = S$. Therefore, if every ideal of $\rhd$ were either proper or dense, then plainly this would give rise to REM. Furthermore, since every element of S is convincing, it follows that every ideal of $\rhd$ is complete. Therefore, if the class of all complete ideals of $\rhd$ would be a set, then so would be $\mathrm{Pow}(1)$, which with Exponentiation would give rise to the Powerset axiom [3].

Example 2.20. Let's put all this in the context of logical theories [66, 208]. Let $\mathscr{L}$ be an arbitrary language of first-order predicate logic and take for S the corresponding set of formulae φ in $\mathscr{L}$. For a finite subset U and element φ of S put

$$U \rhd \varphi \equiv U \vdash_i \varphi$$

where $\vdash_i$ stands for deducibility with *intuitionistic logic*. We take for granted familiarity with the common notational conventions [217]. Recall that negation is considered an abbreviation:

$$\neg\varphi \equiv \varphi \to \bot.$$

If T is a set of formulae in $\mathscr{L}$, then

$$T^\circ = \left\{ \varphi : \neg\varphi \in T^\rhd \right\} .$$

In fact, if $\neg\varphi \in T^\rhd$, then $\psi \in (T, \varphi)^\rhd$ for every $\psi \in S$ in view of *ex falso quodlibet*, by means of which falsum $\bot$ is a convincing element. Conversely, if $\varphi \in T^\circ$, then $\bot \in (T, \varphi)^\rhd$ and thus $\neg\varphi \in T^\rhd$. An ideal I for $\rhd$ is nothing but a deductively closed subset, or, briefly, a *theory*. According to the above, if I is a theory, then $\varphi \in I^\circ$ if and only if $\neg\varphi \in I$. A theory I thus is complete in the sense of reciprocation precisely when, for every $\varphi \in S$,

$$\varphi \in I \vee \neg\varphi \in I,$$

which is to say that I is a complete theory in the usual sense. A theory I is proper if and only if it is consistent, that is, $\bot \notin I$.

Finally, as can be seen from the concrete description of reciprocation, this $\rhd$ is strongly reductive. By classical meta-reasoning, employing ZS and REM, Theorem 2.18 implies a variant of *Lindenbaum's Lemma*:

Every theory I is tightly contained in a complete theory C.

Since every theory contains $\bot \to \bot$, we obtain a more conventional form from Corollary 2.12, since if I consistent, and thus proper, then so is any complete theory C tightly containing I.

2.3.3 Jacobson radical

In this section we address the following question: do the elements which are common to every complete ideal of an entailment relation share a certain property, expressible solely in terms of the entailment relation at hand? For strongly reductive single-conclusion entailment relations it turns out they do, in a manner which closely resembles the situation in ring theory, as will be seen later on.

Let T be an arbitrary subset of S and consider

$$\mathfrak{J}(T) = \{ a \in S : \forall b \in S \ (\ b \in (T, a)^\circ \ \to \ b \in T^\circ \ ) \} .$$

This will turn out an abstract version of the definition of *Jacobson radical* for commutative rings [17, 169], and distributive lattices [85, 146], which will be recalled and discussed later on in Section 2.4.

Observe first that $\mathfrak{J}(T) = \mathfrak{J}(T^\rhd)$, because of $T^\circ = (T^\rhd)^\circ$. Furthermore, $\mathfrak{J}$ is expansive, i.e., if T is an arbitrary subset of S, then $T \subseteq \mathfrak{J}(T)$. Indeed, for if $a \in T$ and $b \in (T, a)^\circ$, then $S = (T, a, b)^\rhd = (T, b)^\rhd$, whence $b \in T^\circ$. With regard to the empty subset, we have

$$a \in \mathfrak{J}(\emptyset) \qquad \text{if and only if} \qquad \forall b \in S \ (\ b \in \{ a \}^\circ \ \to \ b \in \emptyset^\circ \ ).$$

This means that every element a of $\mathfrak{J}(\emptyset)$ comes with a criterion for density: if b complements $a \in \mathfrak{J}(\emptyset)$, then b is convincing.

Proposition 2.21. *Let S be complemented, and let T be a subset of S. If $\mathfrak{J}(T) = S$, then $T^\circ = S$. If I is an inhabited ideal and $\mathfrak{J}(I) = S$, then $I = S$.*

Proof. Let $b \in S$ and pick $a \in S$ such that $\{ a, b \}^\rhd = S$. By assumption, we have $a \in \mathfrak{J}(T)$, and since $b \in (T, a)^\circ$ we get $b \in T^\circ$. Therefore, if I is an inhabited ideal with $c \in I$ and such that $\mathfrak{J}(I) = S$, then $c \in I^\circ$ in particular, whence $I \between I^\circ$ and thus $I = S$ by Lemma 2.10. □

Instantiating the following proposition, we see that if $\rhd$ is *strongly* reductive, then $\mathfrak{J}(\emptyset)$ consists precisely of those elements of S that are common to every complete ideal. To this end, we further need to assume that every ideal I is subdetachable, i.e., $I \neq S$ or $I = S$. By Corollary 2.15 this additional assumption yields that all complete ideals form a set.

Theorem 2.22 (ZS)**.** *Suppose that every ideal is subcomplete and subdetachable. If $\rhd$ is strongly reductive, then*

$$\mathfrak{J}(I) = \bigcap \{ C : I \subseteq C \}$$

where C ranges over all the complete ideals. In particular, $\mathfrak{J}(I)$ is an ideal.

Proof. Let $a \in \mathfrak{J}(I)$ and let C be a complete ideal containing I. By completeness of C we know that either $a \in C$ right away, or else $a \in C^\circ$. In the latter case, since $\rhd$ is strongly reductive, there is $b \in C$ with $a \in \{ b \}^\circ$, or equivalently $b \in \{ a \}^\circ$. It follows that $b \in (I, a)^\circ$, and therefore $b \in I^\circ$ as $a \in \mathfrak{J}(I)$. By monotonicity of reciprocation we get $b \in C^\circ$ and thus $C \between C^\circ$. This implies $C = S$ by Lemma 2.10, whence $a \in C$, anyway.

As regards the converse, suppose that an element a of S belongs to every complete ideal C that contains I. We need to show $a \in \mathfrak{J}(I)$. To this end, let $b \in S$ be such that $b \in (I, a)^\circ$. We consider the ideal $J = (I, b)^\rhd$ which, by Theorem 2.18 and ZS, is tightly contained in a complete ideal C. Tightness of C over J means

$$J \supseteq C \cap J^\circ = C \cap \left((I, b)^\rhd \right)^\circ = C \cap (I, b)^\circ.$$

By $a \in (I, b)^\circ$ and $a \in C$ we obtain $a \in J$. We can now calculate

$$S = (I, a, b)^\rhd \subseteq ((I, b)^\rhd, a)^\rhd = (I, b)^\rhd$$

from which we read off $b \in I^\circ$. Since every intersection of saturated sets again is saturated, we further see that $\mathfrak{J}(I)$ is an ideal. □

We take note from the proof of Theorem 2.22 that $\rhd$ is required to be strongly reductive just in order to show that $\mathfrak{J}(I)$ is contained in the intersection of all complete ideals containing I, which does not require ZS, whereas for the reverse inclusion in view of Theorem 2.18 a reductive single-conclusion entailment relation suffices but we have to make use of ZS.

Corollary 2.23 (ZS). *Under the hypotheses of Theorem 2.22, if T is an arbitrary subset of S, then $T^\circ = \mathfrak{J}(T)^\circ$.*

Proof. Since $\mathfrak{J}$ is expansive and reciprocation is monotone, we may concentrate on showing $\mathfrak{J}(T)^\circ \subseteq T^\circ$. If $a \in \mathfrak{J}(T)^\circ$, then, since $\mathfrak{J}(T)$ is an ideal and $\rhd$ is strongly reductive, there is $b \in \mathfrak{J}(T)$ such that $a \in \{ b \}^\circ$. This implies $a \in T^\circ$. □

Corollary 2.24 (ZS). *Suppose that every ideal is subcomplete and subdetachable. If $\rhd$ is strongly reductive, then $\mathfrak{J}$ restricts to a closure operator on ideals,*

$$\mathfrak{J} : \mathfrak{Id}(\rhd) \to \mathfrak{Id}(\rhd).$$

Every fixed point I of $\mathfrak{J}$ is the intersection of all complete ideals containing I. Moreover, if I is a proper inhabited ideal, then so is $\mathfrak{J}(I)$.

Proof. Since $\mathfrak{J}$ is expansive, and monotone by way of Theorem 2.22, it remains to show that $\mathfrak{J}(\mathfrak{J}(T)) \subseteq \mathfrak{J}(T)$, so let $a \in \mathfrak{J}(\mathfrak{J}(T))$. If $b \in (T, a)^\circ$, then $b \in (\mathfrak{J}(T), a)^\circ$, thus $b \in \mathfrak{J}(T)^\circ$, and finally $b \in T^\circ$ by Corollary 2.23. As regards the add-on, by Proposition 2.21 we know that if I is an inhabited ideal, then $\mathfrak{J}(I)$ is proper whenever I is. □

Example 2.25. We continue the discussion of logical theories as carried out in Example 2.20. Consider again a language of first-order predicate logic, let S be the corresponding set of formulae, and let $\rhd$ stand for deducibility with intuitionistic logic. We claim that if I is a theory, then

$$\mathfrak{J}(I) = \{ \varphi : \neg\neg\varphi \in I \} .$$

To see this, let $\varphi \in \mathfrak{J}(I)$. We have $\neg\varphi \in (I, \varphi)^\circ$, and therefore $\neg\varphi \in I^\circ$. Because of $I^\circ = \{ \psi : \neg\psi \in I \}$ (Example 2.20) we obtain $\neg\neg\varphi \in I$. Conversely, consider φ such that $\neg\neg\varphi \in I$, and let $\psi \in (I, \varphi)^\circ$. We need to show $\psi \in I^\circ$, to which end we simply have to recall the provable rule

$$\frac{I, \varphi \vdash_i \neg\psi}{I, \neg\neg\varphi \vdash_i \neg\psi}$$

which rests on Brouwer's observation [54] that

$$\vdash_i \neg\neg\neg\varphi \to \neg\varphi$$

holds for every formula φ. In view of this, $\psi \in I^\circ$ is immediate.

Thus, if we pass with $\rhd$ to *classical* logic, then every theory is $\mathfrak{J}$-fixed, which can be considered a syntactical version of Lindenbaum's Lemma. With ZS and REM at hand, Theorem 2.22 subsequently yields that every theory is the intersection of all complete theories containing it. Note that for classical logic and a theory I we also have

$$\mathfrak{J}(I) = \left\{ \varphi : (I, \varphi)^{\rhd} \subseteq I \right\}$$

from which $\mathfrak{J}(I) = I$ follows immediately.

2.3.4 Local entailment relations

Recall that in commutative algebra a ring with exactly one maximal ideal is said to be *local* [17]. The purpose of this section is to carry over this concept to the context of single-conclusion entailment relations, and thus to put flesh on the analogy with commutative algebra.

Let S again be our default set with entailment relation $\rhd$. Recall that $\emptyset^\circ$ is the set of convincing elements. In this section we work under the *additional assumption that S has a convincing element.* In other words, we suppose that $\emptyset^\circ$ is inhabited. The set $S \setminus \emptyset^\circ$ of non-convincing elements is a proper subset of S.

Definition 2.26. *We say that $\rhd$ is* local *if there is a unique proper complete ideal for $\rhd$.*

Assuming that the convincing elements form a detachable subset of S, we will see that if $S \setminus \emptyset^\circ$ is saturated, then S is local with unique proper complete ideal $S \setminus \emptyset^\circ$. As for the converse, we seem to require transfinite methods. We are now going to make this precise, and give an explicit characterization for a strongly reductive single-conclusion entailment relation to be local, making use of our intersection principle from the preceding section.

Lemma 2.27. *Suppose that $\emptyset^\circ$ is detachable. If $S \setminus \emptyset^\circ$ is an ideal, then $\rhd$ is local with unique proper complete ideal $S \setminus \emptyset^\circ$.*

Proof. Suppose that $I = S \setminus \emptyset^\circ$ is an ideal. This I is complete because, by assumption, for every $a \in S$ either $a \in \emptyset^\circ$ which implies $a \in I^\circ$, or else $a \notin \emptyset^\circ$ and thus $a \in I$. If C is a proper complete ideal, then clearly $C \subseteq I$. On the other hand, if $a \in I$, then $a \in C$ or $a \in C^\circ$ by completeness of C. However, if $a \in C^\circ$, then $S = (C, a)^\rhd \subseteq I$, but I is proper. □

Lemma 2.28 (ZS)**.** *Suppose that every ideal is subcomplete and subdetachable, and let $\rhd$ be reductive. If $\rhd$ is local with unique proper complete ideal C, then $C = S \setminus \emptyset^\circ$.*

Proof. Clearly, if $a \in C$, then a cannot be convincing, for C would be dense. Conversely, if $a \in S \setminus \emptyset^\circ$, then $\{ a \}^\rhd$ is a proper ideal and by Theorem 2.18 contained in a proper complete ideal C' (taking into account also Corollary 2.12). This C' must coincide with C since $\rhd$ is local. □

Proposition 2.29 (ZS)**.** *Suppose that every ideal is subcomplete, subdetachable, and inhabited. If $\rhd$ is strongly reductive, then $\rhd$ is local if and only if*

$$S \neq \emptyset^\circ \quad \textit{and} \quad S = \emptyset^\circ \cup \mathfrak{J}(\emptyset). \tag{$*$}$$

Proof. Recall from Theorem 2.22 that if $\rhd$ is strongly reductive, then $\mathfrak{J}(\emptyset) = \mathfrak{J}(\emptyset^\rhd)$ is the intersection of all complete ideals. Thus, if $\rhd$ is local, then $\mathfrak{J}(\emptyset) = S \setminus \emptyset^\circ$ by Lemma 2.28. Since every ideal is subdetachable, it follows that $S = \emptyset^\circ \cup \mathfrak{J}(\emptyset)$, which is the right hand conjunct of $(*)$, and $S \neq \emptyset^\circ$ follows from the assumption that every ideal has an element. Conversely, if $(*)$ holds, then $\mathfrak{J}(\emptyset)$ need be proper, for if $\mathfrak{J}(\emptyset) = S$, then $\emptyset^\circ = S$ by Proposition 2.21. It follows that $\mathfrak{J}(\emptyset) = S \setminus \emptyset^\circ$, whence $\rhd$ is local by Lemma 2.27. □

For instance, every quotient $\rhd_C$ with respect to a complete ideal C of $\rhd$ satisfies the right hand conjunct of $(*)$. Indeed, for every $a \in S$ two cases can be distinguished, as usual: either $a \in C^\circ$, and this just means that a is convincing for $\rhd_C$, or else $a \in C$, in which case for every $b \in S$ we have $\{ a, b \}^{\rhd_C} = S$ if and only if $b \in C^\circ$. Since every convincing element of $\rhd$ is convincing also for $\rhd_C$, our additional working assumption carries over from $\rhd$ to the quotient $\rhd_C$, and we obtain the following.

Corollary 2.30 (ZS)**.** *Under the hypotheses of Proposition 2.29, if $\rhd$ is strongly reductive and C is a proper complete ideal, then the quotient $\rhd_C$ is local.*

Proof. It only remains to check $S \neq \emptyset^{\circ_C}$, where we write T°_C} for the reciprocation of a subset T of S with respect to the quotient $\rhd_C$. In particular, $\emptyset^{\circ_C} = C^\circ$. Therefore, if $S = \emptyset^{\circ_C}$, then Proposition 2.14 implies $S = S \setminus C$, but C is supposed to have an element. □

2.3.5 Operations

Before turning our attention to some concrete case studies, we consider operations on S which are compatible with the single-conclusion entailment relation in a sense to be made precise.

2.3.5.1 Binary operations

Let us first consider binary operations

$$\mu : S \times S \to S, \quad (x, y) \mapsto x \circ y$$

with which S becomes a *magma*. The following pivotal condition to be imposed on *finitary coverings* [207, 208] lies at the heart of many a syntactical conservation theorem [209, 210] of multi- over single-conclusion entailment relations.

Encoding.

$$\frac{U, a \rhd c \qquad U, b \rhd c}{U, a \circ b \rhd c}$$

An ideal I is *prime* [208] if

$$\forall a, b \in S\,(\, a \circ b \in I \;\to\; a \in I \vee b \in I\,).$$

It should come as no surprise that even in this abstract setting every complete ideal turns out prime, provided that the binary operation satisfies Encoding. This is an abstract version of the statement that *every maximal ideal is prime* in commutative ring theory. The simple reason is that in the presence of Encoding reciprocations are *multiplicative* [208] subsets of S:

Lemma 2.31. *Suppose that Encoding holds. If T is an arbitrary subset of S, then*

$$\forall a, b \in S\,(\, a \in T^\circ \wedge b \in T^\circ \;\to\; a \circ b \in T^\circ\,)$$

Proof. If $a, b \in T^\circ$, then we get

$$S = S \cap S = (T, a)^{\rhd} \cap (T, b)^{\rhd} \subseteq (T, a \circ b)^{\rhd}$$

by Encoding, and therefore $a \circ b \in T^\circ$. □

Proposition 2.32. *A complete ideal is prime for every operation which satisfies Encoding.*

Proof. Let C be a complete ideal, and let $a, b \in S$ be such that $a \circ b \in C$. By completeness of C we know that $a \in C$ or $a \in C^\circ$, as well as $b \in C$ or $b \in C^\circ$. As we need to show that $a \in C$ or $b \in C$, it suffices to consider the case when both $a \in C^\circ$ and $b \in C^\circ$. However, under this assumption we get $a \circ b \in C^\circ$ by Lemma 2.31. But then $C \between C^\circ$, hence $C = S$ and $a, b \in C$ anyway. □

Remark 2.33. Under the hypotheses of Theorem 2.22, if also Encoding holds, then every $\mathfrak{J}$-fixed ideal is the intersection of all prime ideals that contain it. But actually this holds for *every* ideal, no matter whether $\rhd$ is strongly reductive or not. This is due to the Universal Krull-Lindenbaum Theorem [208] which asserts that if Encoding holds, then

$$\bigcap \{\, P : P \supseteq I \,\} = I$$

for every ideal I, where P ranges over the (proper) prime ideals of $\rhd$. However, we believe that (strong) reductivity is as crucial for complete ideals as Encoding is for prime ideals.

2.3.5.2 Unary operations

Next we consider unary operations

$$\sim\, : S \to S, \quad a \mapsto\, \sim a.$$

Unary operations have also been considered in the context of finitary coverings [208]. If $\sim$ is such that

$$\forall a \in S\,(\, \sim a \in \{\, a \,\}^\circ\,)$$

or, directly in terms of the saturation, if $\sim$ is such that for every $a \in S$ we have

$$\{\, a, \sim a \,\}^{\triangleright} = S,$$

then we say that $\sim$ is a *dedicated negation*. In this case, if T is a subset of S and $\sim a \in T$, then $a \in T^{\circ}$. So an ideal I is complete whenever

$$\forall a \in S \; (\; a \in I \vee \sim a \in I \;).$$

For a subset T of S we write

$$\sim T = \{\, \sim a : a \in T \,\}$$

for the image of T under this operation. If $\sim$ is a dedicated negation, then

$$\sim T \subseteq T^{\circ}.$$

The following shows that if S is local, and $a \in S$, then either a is convincing or $\sim a$ is.

Lemma 2.34. *Suppose that S is local in the sense that*

$$\forall a \in S \left(\, a \in \emptyset^{\circ} \;\vee\; a \in \mathfrak{J}(\emptyset) \,\right). \tag{$*$}$$

If $\sim$ is a dedicated negation, then

$$\forall a \in S \,(\, a \in \emptyset^{\circ} \vee \, \sim a \in \emptyset^{\circ} \,).$$

Proof. This follows from $(*)$ simply because $\sim a \in \{\, a \,\}^{\circ}$. ☐

The following will be useful when it comes to applications in the next section. Recall that an operation $\sim : S \to S$ is *involutive* if

$$\forall a \in S \,(\, \sim\sim a = a \,),$$

in which case

$$\sim T = \{\, a \in S : \sim a \in T \,\}$$

Proposition 2.35. *Let $\sim : S \to S$ be an involutive dedicated negation, and T a subset of S. If $T^{\circ} \subseteq \sim T$, then T is $\mathfrak{J}$-fixed.*

Proof. Let T be such that $T^{\circ} \subseteq \sim T$. As $\mathfrak{J}$ is expansive, it suffices to show $\mathfrak{J}(T) \subseteq T$. To this end, let $a \in \mathfrak{J}(T)$. Since $\sim$ is dedicated, we have $\sim a \in \{\, a \,\}^{\circ}$, thus $\sim a \in (T, a)^{\circ}$ by monotonicity of reciprocation, and therefore $\sim a \in T^{\circ}$ since $a \in \mathfrak{J}(T)$. Now we have $\sim a \in \sim T$ by $T^{\circ} \subseteq \sim T$, and thus $a = \sim\sim a \in T$. ☐

Remark 2.36. Under the hypotheses of Theorem 2.22, if $\sim$ is an involutive dedicated negation, then $T^{\circ} \subseteq \sim T$ implies that T is an ideal by way of being $\mathfrak{J}$-fixed.

2.4 Applications

In this section we turn our attention to several concrete applications which show that requiring a single-conclusion entailment relation to be reductive is not too far-fetched an assumption. First we consider the motivating example from commutative ring theory, and see how our findings lead to elementary descriptions of the concepts of Jacobson radical and Jacobson ring. We discuss some of the examples studied thoroughly in [208], but emphasis is shifted from prime ideals to complete ideals. Moreover, Theorem 2.18 and Theorem 2.22 will turn out equivalent to the Axiom of Choice over **ZF**. For simplicity's sake, to get by without assuming ideals to be subcomplete and subdetachable, we invoke REM, i.e., work in **CZF**+REM; recall that this proves the same theorems as **ZF**.

2.4.1 Rings and ideals

2.4.1.1 Krull's Lemma

Let R be a commutative ring (with 1), and let R^* denote the set of units of R, that is,

$$R^* = \{ a \in R : \exists b \in R\, (ab = 1) \}.$$

We take $S = R$ as domain of discourse, and consider the single-conclusion entailment relation $\rhd$ of *ideal* of R. This $\rhd$ is an example of an *inductively generated* entailment relation, as it is the least one to contain all instances of the following axioms:

$$\begin{gathered} \rhd 0 \\ a, b \rhd a + b \\ a \rhd ab \end{gathered}$$

If U is a finite subset and a an element of $S = R$, then

$$U \rhd a \qquad \text{if and only if} \qquad a \in \langle U \rangle,$$

where $\langle U \rangle$ denotes the ideal of R (of course, now in the ring-theoretic sense) that is generated by U, i.e., $\langle U \rangle$ consists of all the sums of products $r_1 a_1 + \cdots + r_n a_n$ with $a_1, \ldots, a_n \in U$ and $r_1, \ldots, r_n \in R$. In particular, the ideals of R are precisely the $\rhd$-saturated subsets of R. If T is an arbitrary subset of R, then

$$a \in T^\circ \qquad \text{if and only if} \qquad 1 \in \langle T, a \rangle;$$

recall that $I = S$ if and only if $1 \in I$, for every ideal I of R. In particular,

$$a \in \{ b \}^\circ \qquad \text{iff} \qquad \exists r, s \in R\, (1 = ra + sb).$$

Hence this $\rhd$ is strongly reductive. In fact, if $a \in T^\circ$, which is to say that there are $a_1, \ldots a_n \in T$ and $r_1, \ldots, r_n, r \in R$ such that

$$1 = r_1 a_1 + \ldots r_n a_n + ra,$$

then $a \in \{ b \}^\circ$ for $b = \sum_{i=1}^n r_i a_i \in T^\rhd$. An element $a \in R$ is convincing precisely when it is a unit.

With ZS and REM at hand, Theorem 2.18 leads over to the following variant of *Krull's Lemma* for commutative rings [157]:

In a commutative ring every ideal is tightly contained in a complete ideal.

Next let us bring back to mind the interplay of completeness and maximality. Recall that a *maximal ideal* of a commutative ring is an ideal that is maximal among the proper ideals of R.

Proposition 2.37. *Let I be an ideal of a commutative ring R.*

1. *If I is such that for every ideal J,*

$$I \subseteq J \implies (I = J \vee J = R),$$

then I is complete with respect to reciprocation $^\circ$.

2. *If I is proper and complete, then I is maximal among the proper ideals, that is, for every ideal J,*

$$(I \subseteq J \wedge J \neq R) \implies I = J.$$

Proof. 1. For every $a \in R$, by assumption, either $I = \langle I, a \rangle$, which implies $a \in I$, or else $\langle I, a \rangle = R$, which is to say that $a \in I^\circ$.

2. Let J be another proper ideal and suppose that $I \subseteq J$. By completeness, for every $a \in J$ either $a \in I$ anyway, or else $a \in I^\circ$ and thus $a \in J^\circ$, which however would imply $J = R$ (Lemma 2.10). □

Corollary 2.38 (REM)**.** *The proper complete ideals of a commutative ring R are precisely the maximal (proper) ideals. In particular, R is local if and only if the entailment relation $\rhd$ of ideal of R is local.*

With Theorem 2.22 and Corollary 2.38 we thus regain the classical description of the Jacobson radical of an ideal as the description of all maximal ideals containing it.

Corollary 2.39 (ZS + REM)**.** *If I is an ideal of a commutative ring R, then*

$$\bigcap \{ C : I \subseteq C \} = \mathfrak{J}(I)$$

where C ranges over all the maximal (proper) ideals of R.

Remark 2.40. Recall from Corollary 2.12 that if I is proper, then so is every complete ideal that contains I. Taking into account Corollary 2.38 and working with ZS + REM, we thus obtain a more familiar reading of Krull's Lemma: every proper ideal is contained in a maximal (proper) ideal. Hence, as "Krull implies Zorn" [24, 136], over **ZF** the statement of Theorem 2.18 is equivalent to the Axiom of Choice. Similarly, keep in mind that if I is proper, then so is $\mathfrak{J}(I)$, due to Proposition 2.21. Therefore, still working with ZS + REM, Theorem 2.22 implies that there is a proper complete ideal C containing I. This C is maximal among proper ideals according to Corollary 2.38. It follows that over **ZF** the statement of Theorem 2.22 is equivalent to the Axiom of Choice, as well.

We take the opportunity to point out that our formal description of $\mathfrak{J}(I)$ directly leads over to the well-known and computationally meaningful characterization of the Jacobson radical [169].

Proposition 2.41. *Let I be an ideal of a commutative ring R.*

1. $\mathfrak{J}(I) = \{ a \in R : \forall b \in R \; (\; 1 \in \langle a, b \rangle \; \to \exists c \in I \, (\, 1 \in \langle b, c \rangle \,) \,) \, \}$.
2. $\mathfrak{J}(\emptyset) = \{ a \in R : \forall b \in R \; (\; 1 - ab \in R^* \;) \, \}$.

Proof. 1. Suppose that $a \in \mathfrak{J}(I)$, and let $b \in R$ such that $1 \in \langle a, b \rangle$. This is to say that $b \in \{ a \}^\circ$, hence also $b \in (I, a)^\circ$, and thus $b \in I^\circ$. Since $\rhd$ is strongly reductive, there is $c \in I$ such that $1 \in \langle b, c \rangle$. Conversely, in order to show $a \in \mathfrak{J}(I)$, consider an arbitrary $b \in (I, a)^\circ$. There are $d \in I$ and $r, s \in R$ such that $1 = d + ra + sb$. Writing $b' = d + sb$, we have $1 \in \langle a, b' \rangle$. Therefore, by assumption, there is $c \in I$ with $1 \in \langle b', c \rangle \subseteq \langle I, b \rangle$. In other words, $b \in I^\circ$, as required.

2. Recall that $\mathfrak{J}(\emptyset) = \mathfrak{J}(\emptyset^\rhd)$. We may thus resort to the preceding item. It is straightforward to verify that the description provided before is equivalent to the one that is claimed in case of $I = \emptyset^\rhd = \{ 0 \}$. □

2.4.1.2 Local rings

Recall once more that in classical mathematics a ring R is said to be local if and only if it has a unique maximal (proper) ideal. Applying Proposition 2.29 in the present context, we regain several well-known and concrete criteria for a ring to be local [150, 169].

Proposition 2.42 (ZS + REM)**.** *The following are equivalent.*

1. $0 \neq 1$ *and* $\forall a \in R \, (\, a \in R^* \vee \forall b \in R \, (\, 1 - ab \in R^* \,) \,)$.
2. $0 \neq 1$ *and* $\forall a \in R \, (\, a \in R^* \vee 1 - a \in R^*)$.

3. $0 \neq 1$ *and* $\forall a, b \in R\,(\, a + b \in R^* \,\to\, (\, a \in R^* \,\vee b \in R^* \,)\,)$

4. R is a local ring.

5. The entailment relation of ideal of R is local.

Proof. The first item directly implies the second, from which the third item follows by instantiation with $(a+b)^{-1}a$. The third item in turn yields that R is local with maximal ideal $R \setminus R^*$. Keeping in mind Corollary 2.38, we arrive back at the first item with Proposition 2.29, by noticing that $0 = 1$ is equivalent to $R = \emptyset^\circ$, and reformulating the respective condition $(*)$ with Proposition 2.41(2). $\square$

We hasten to add that in modern constructive algebra [169, 256] the defining condition for a ring R to be *local* is taken to be the right hand conjunct of the third item of Proposition 2.42. Note that the trivial ring (for which $0 = 1$) is in fact local according to this definition [169]. A ring for which the right hand conjunct of the first item of Proposition 2.42 holds is said to be a *residually discrete local ring* [169]. Constructively, this is a stronger condition than being local in the above sense of constructive algebra [169].[5] In view of Proposition 2.41(2), the right hand conjunct of condition $(*)$ in Proposition 2.29 generalizes the concept of residually discrete local rings to entailment relations.

2.4.1.3 Jacobson rings

In the context of Hilbert's Nullstellensatz, Goldman [121] and Krull [158] independently considered rings R in which every prime ideal is the intersection of all maximal ideals containing it. In this section, we see how this concept fits into the present context.

Let again R be a commutative ring. Recall that a *radical ideal* of R is an ideal I such that

$$\forall a \in R\,(\, a^2 \in I \,\to\, a \in I\,).$$

We next consider the single-conclusion entailment relation $\rhd$ of radical ideal that is generated on R by all instances of the following axioms:

$$\begin{aligned} &\rhd 0 \\ a &\rhd ab \\ a, b &\rhd a + b \\ a^2 &\rhd a \end{aligned}$$

If U is a finite subset and a an element of R, then

$$U \rhd a \qquad \text{if and only if} \qquad a \in \sqrt{\langle U \rangle},$$

where $\langle U \rangle$ denotes the ideal generated by U (see Section 2.4.1), and the *radical* of an ideal I of R is

$$\sqrt{I} = \left\{\, r \in R : \exists \ell \geqslant 1\,(r^\ell \in I)\,\right\}.$$

This single-conclusion entailment relation is strongly reductive, and a subset I of R is saturated for $\rhd$ if and only if it is a radical ideal. Furthermore, $\rhd$ satisfies Encoding for multiplication [208, 222, 223]:

$$\frac{U, a \rhd c \qquad U, b \rhd c}{U, ab \rhd c}$$

Therefore, by Proposition 2.32, every complete ideal is prime.

Proposition 2.43 (ZS + REM)**.** *Let R be a commutative ring. The following are equivalent.*

[5] We are grateful to Henri Lombardi for having pointed this out to us.

1. *Every radical ideal of R is $\mathfrak{J}$-fixed.*
2. *Every prime ideal of R is the intersection of all complete ideals containing it.*

Proof. In view of Theorem 2.22, and since every prime ideal is a radical ideal, it suffices to concentrate on showing that the second item implies the first. Thus, suppose that every prime ideal of R is the intersection of all complete ideals containing it. Now let I be a radical ideal, and consider the set (keep in mind that we are working classically) $\mathscr{P}_I$ of all prime ideals of R that contain I. Since Encoding holds, $I = \bigcap \mathscr{P}_I$ by the Universal Krull-Lindenbaum Theorem [208] (see Remark 2.33). By assumption, for every $P \in \mathscr{P}_I$,

$$P = \bigcap \{ C : C \supseteq P \}$$

where C ranges over the complete ideals of R. Combining this, we obtain

$$I = \bigcap \mathscr{P}_I = \bigcap_{P \in \mathscr{P}_I} \bigcap \{ C : C \supseteq P \} = \bigcap \{ C : C \supseteq I \} = \mathfrak{J}(I),$$

which makes use of the fact that every complete ideal is prime (Proposition 2.32). This shows that I is $\mathfrak{J}$-fixed. □

Rings in which the second item of Proposition 2.43 is satisfied, with "maximal" in place of "complete", have been called *Jacobson rings* by Krull [158], and have independently been introduced as *Hilbert rings* by Goldman [121]. Since $\mathfrak{J}(I) = \mathfrak{J}(\sqrt{I})$, the first item of Proposition 2.43 can be replaced by demanding that, for every ideal I of R,

$$\sqrt{I} = \mathfrak{J}(I),$$

which was Krull's defining property for Jacobson rings: the two notions of radical coincide. The first item of Proposition 2.43 can thus be seen as an elementary characterization of Jacobson rings. We briefly return to this in Section 2.5.

2.4.2 Filters

Let L be a bounded distributive lattice, with top and bottom elements 1 and 0, respectively. We take $S = L$ as domain of discourse, and consider the single-conclusion entailment relation $\rhd$ of *filter* of L that is inductively generated by all instances of the following axioms:

$$\begin{aligned} &\rhd 1 \\ a, b &\rhd a \wedge b \\ a &\rhd a \vee b \end{aligned}$$

If U is a finite subset and a an element of $S = L$, then

$$U \rhd a \qquad \text{if and only if} \qquad a \in \langle U \rangle,$$

where $\langle U \rangle$ denotes the filter generated by U, i.e., $\langle U \rangle$ consists of all the $a \in L$ for which there are $a_1, \dots, a_n \in U$ such that $a_1 \wedge \cdots \wedge a_n \leqslant a$. In particular, the filters of L are precisely the $\rhd$-saturated subsets of L. If T is an arbitrary subset of L, then

$$a \in T^\circ \qquad \text{if and only if} \qquad 0 \in \langle T, a \rangle;$$

recall that $F = L$ if and only if $0 \in F$, for every filter F of L. In particular,

$$a \in \{ b \}^\circ \qquad \text{iff} \qquad a \wedge b = 0.$$

As for ideals, this single-conclusion entailment relation is strongly reductive. With ZS and REM, Theorem 2.18 implies the following variant of Krull's Lemma for distributive lattices:

In a distributive lattice every filter is tightly contained in a complete filter.

Arguments similar to those put forward in the proof of Proposition 2.41 yield the following description of $\mathfrak{J}(F)$ for a filter F of L. It is taken from [85, 146].

Proposition 2.44. *Let $F \subseteq L$ be a filter.*

1. $\mathfrak{J}(F) = \{\, a \in L : \forall b \in L\,(\, a \wedge b = 0 \;\rightarrow\; \exists c \in F\,(\, b \wedge c = 0\,)\,)\,\}$.
2. $\mathfrak{J}(\emptyset) = \{\, a \in L : \forall b \in L\,(\, a \wedge b = 0 \;\rightarrow\; b = 0\,)\,\}$.

Before turning our attention to complemented distributive lattices, i.e., Boolean algebras, we briefly consider semi-normal lattices, which allow for a decomposition of completeness into primality and $\mathfrak{J}$-fixedness. This requires some preparation. From [208] we take that Encoding holds for $\vee$, i.e.,

$$\frac{U, a \rhd c \qquad U, b \rhd c}{U, a \vee b \rhd c}$$

Thus, by Lemma 2.32, every complete filter F is prime:

$$\forall a, b \in L\,(\, a \vee b \in F \;\rightarrow\; a \in F \;\vee\; b \in F\,).$$

Mind that we do not require prime filters to be proper.[6] Next, following [146], but dualizing in order to put focus on the meet operation, we say that a distributive lattice L is *semi-normal* if, whenever $a \wedge b = 0$, there are $c, d \in L$ such that $a \wedge d = b \wedge c = 0$ and $c \vee d \in \mathfrak{J}(\emptyset)$. The line of argument for the proof of the next proposition follows [146, Lemma 1.12].

Proposition 2.45 (REM)**.** *Let L be semi-normal, and let $F \subseteq L$ be a filter. The following are equivalent.*

1. *F is complete.*
2. *F is prime and $\mathfrak{J}$-fixed.*

Proof. Every complete filter turns out $\mathfrak{J}$-fixed by means of Proposition 2.21 and is prime due to Encoding. Conversely, to show that every prime and $\mathfrak{J}$-fixed filter F is complete, we reason classically. Thus, let $a \in L$ and suppose that $a \notin F$. Since F is $\mathfrak{J}$-fixed, $a \notin \mathfrak{J}(F)$, so there is $b \in L$ such that $a \wedge b = 0$, but $b \wedge c \neq 0$ for every $c \in F$. Since L is semi-normal, there are $c, d \in L$ such that $a \wedge d = b \wedge c = 0$ and $c \vee d \in \mathfrak{J}(\emptyset)$. Then, since F is $\mathfrak{J}$-fixed and prime, either $c \in F$ or $d \in F$. The former is impossible due to $b \wedge c = 0$. Therefore $d \in F$, which implies $a \in F^\circ$ because of $a \wedge d = 0$. ☐

Needless to say, one may dually consider ideals of distributive lattices, rather than filters, and obtain similar results.

Now suppose that L is a Boolean algebra, i.e., every $a \in L$ has a complement $-a \in L$ for which $a \wedge -a = 0$ and $a \vee -a = 1$. Thus we have a dedicated and involutive negation $a \mapsto -a$. In particular,

$$-F = \{\, a \in L : -a \in F\,\}\,.$$

The reciprocal of a filter is obtained by taking complements:

Lemma 2.46. *Let L be a Boolean algebra. If F is a filter of L, then $F^\circ = -F$.*

Proof. Let F be a filter of L. Since complementation gives a dedicated negation, we know that $-F \subseteq F^\circ$. Conversely, let $a \in F^\circ$, which is to say that $0 \in \langle F, a\rangle$. It follows that there is $b \in F$ such that $a \wedge b = 0$. This implies $b \leqslant -a$, whence $-a \in F$, since F is a filter. Thus $a \in -F$. ☐

[6] Filters which are understood to be prime in this sense have also been called *partially prime* [183].

Recall that a proper filter F of a Boolean algebra is said to be an *ultrafilter* if, for every $a \in L$, either $a \in F$ or $-a \in F$ (but not both). Combining Proposition 2.35 with Lemma 2.46 yields the following.

Proposition 2.47. *Let L be a Boolean algebra. If F is a filter of L, then*

1. *F is $\mathfrak{J}$-fixed, and*
2. *F is proper and complete if and only if it is an ultrafilter.*

In particular, every Boolean algebra is a *Jacobson lattice* [85], by which here we understand a distributive lattice every filter of which is $\mathfrak{J}$-fixed. Furthermore, with ZS, REM and Theorem 2.22, we obtain the following classical variant of the *Boolean ultrafilter theorem*:

Every filter of a Boolean algebra is the intersection of all complete filters containing it.

It is worth pointing out that our notion of complete filter coincides with Mulvey's concept of partial maximality [183].[7] Recall that a (not necessarily proper) filter P of a Boolean algebra is said to be *partially maximal* if, for every filter F that contains P, one has that

$$\forall a \in F \,(\, a \in P \,\vee\, 0 \in F \,).$$

Proposition 2.48. *Let L be a Boolean algebra, let $P \subseteq L$ be a filter. The following are equivalent.*

1. *P is complete.*
2. *P is partially maximal.*

Proof. If P is complete and $P \subseteq F$, then, for every $a \in F$, by completeness either $a \in P$ or $-a \in P$. In the latter case, $-a \in F$ and thus $0 = a \wedge -a \in F$ since F is a filter. Conversely, suppose that P is partially maximal, and let $a \in L$. Consider $F = \langle P, a \rangle$. Since P is partially maximal, either $a \in P$, or else $0 \in F$, which amounts to $a \in P^{\circ}$. So P is complete. $\square$

2.4.3 Order extension

2.4.3.1 Bounded quasi-orders

We return to the topic of order extension, the subject matter which has prompted the present chapter and provided motivation [226], and see how it fits into this general framework. Let E be a set with two distinguished elements 0 and 1. Our objects are *bounded quasi-orders* on E, i.e., reflexive transitive binary relations $\leqslant$ such that $0 \leqslant a$ and $a \leqslant 1$ for every $a \in E$. With the set $S = E \times E$ as our domain of discourse, we consider the single-conclusion entailment relation $\rhd$ of bounded quasi-order that is inductively generated by all instances of the following axioms:

$$\begin{aligned} &\rhd (0, a) \\ &\rhd (a, 1) \\ &\rhd (a, a) \\ (a, b), (b, c) &\rhd (a, c) \end{aligned}$$

Let T be an arbitrary subset of S and $(a, b) \in S$. We have

$$T \rhd (a, b) \qquad \text{if and only if} \qquad (a, b) \in \operatorname{rtc}\big(T \cup (\{\, 0 \,\} \times E) \cup (E \times \{\, 1 \,\})\big)$$

where rtc means the *reflexive-transitive closure*, or, in other words, the *hull* of a binary relation. The ideals clearly are the bounded quasi-orders on E, and $(1, 0)$ is a convincing element for $\rhd$.

[7] We are grateful to Thierry Coquand for having pointed this out to us.

Therefore, as regards reciprocation, unfolding the non-inductive description of $\rhd$ in terms of the hull, we obtain that $(a,b) \in T^\circ$ holds if and only if there are elements

$$1 = a_0, a_1, \ldots, a_m = a \qquad \text{and} \qquad b = b_0, b_1, \ldots, b_n = 0$$

of E such that $(a_i, a_{i+1}) \in T \cup (\{ 0 \} \times E)$ for every $i < m$, and $(b_j, b_{j+1}) \in T \cup (E \times \{ 1 \})$ for every $j < n$. If R is a bounded quasi-order, then

$$(a,b) \in R^\circ \qquad \text{if and only if} \qquad (1,a) \in R \wedge (b,0) \in R$$

and therefore, since $(T^\rhd)^\circ = T^\circ$, for arbitrary $T \subseteq S$,

$$(a,b) \in T^\circ \qquad \text{if and only if} \qquad \{ (1,a), (b,0) \} \subseteq T^\rhd.$$

This entailment relation is reductive, because $(a,b) \in \{ (1,a), (b,0) \}^\circ$. For instance, if R is a bounded partial order, i.e., an *antisymmetric* bounded quasi-order, then $R^\circ = \{ (1,0) \}$. For every bounded quasi-order R we have

$$R^\circ \subseteq \{ (a,b) : (b,a) \in R \},$$

for if $(1,a) \in R$ and $(b,0) \in R$, then by transitivity we get $(b,a) \in R$ since $(0,1) \in R$.[8] Therefore, every complete ideal for $\rhd$ is a *linear* bounded quasi-order. By means of ZS, REM, and Theorem 2.18, we obtain the following variant of Hansson's theorem [129]:

Every bounded quasi-order is tightly contained in a linear bounded quasi-order.

2.4.3.2 Szpilrajn's theorem

Next we consider Szpilrajn's extension principle in its original form for *strict partial orders*, i.e., *irreflexive* transitive relations. Once again, let E be a set. We take the product $S = E \times E$ as our domain of discourse and consider the entailment relation $\rhd$ of strict partial order of E which is inductively generated by all instances of the following axioms of *irreflexivity* and transitivity

$$(a,a) \rhd (x,y)$$
$$(a,b), (b,c) \rhd (a,c)$$

respectively. We note that the *proper* ideals for $\rhd$ are precisely the strict partial orders of E. Next, let

$$\Delta = \{ (a,b) \in S : a = b \}$$

be the *diagonal*. Now a relation $T \subseteq S$ is irreflexive if and only if $T \cap \Delta = \emptyset$. We take the following non-inductive description for the entailment relation of strict partial order from [210].

Lemma 2.49. *Let U be a finite subset of S and let $(a,b) \in S$. The following are equivalent.*

1. $U \rhd (a,b)$
2. $\mathrm{tc}(U) \between \{ (a,b) \} \cup \Delta$,

where $\mathrm{tc}(U)$ *denotes the* transitive closure *of U.*

From the generating axioms we obtain with transitivity (T) all instances of the axiom of *asymmetry* which here takes the following form:

$$(a,b), (b,a) \rhd (x,y).$$

This amounts to saying that the involutive swap mapping

$$\sim : S \to S, \quad (a,b) \mapsto (b,a)$$

is a dedicated negation. Reciprocation can be characterized as follows.

[8] In this case, notice further that $R^\circ \subseteq \sim R$, where $\sim (a,b) = (b,a)$ is the involutive swap mapping. However, this $\sim$ is not in general a dedicated negation. To see this, consider a set E with pairwise distinct elements $0, a, b, 1$. We do not have $(1,0) \in \{ (a,b), (b,a) \}^\rhd$.

Lemma 2.50. *Let T be a subset of S and let $(a,b) \in S$. The following are equivalent.*

1. $(a,b) \in T^\circ$
2. $\operatorname{tc}(T) \between \{\, (b,a) \,\} \cup \Delta$

Proof. Suppose that $(a,b) \in T^\circ$, i.e., $(T,(a,b))^{\rhd} = S$. In particular, $(b,a) \in (T,(a,b))^{\rhd}$. It follows that there is a finite subset U of T such that, according to Lemma 2.49, either

$$(b,a) \in \operatorname{tc}(U,(a,b)) \qquad \text{or} \qquad \operatorname{tc}(U,(a,b)) \between \Delta.$$

By case analysis, unfolding the definition of transitive closure, it follows that

$$(b,a) \in \operatorname{tc}(U) \qquad \text{or} \qquad \operatorname{tc}(U) \between \Delta,$$

which since $U \subseteq T$ implies $\operatorname{tc}(T) \between \{\, (b,a) \,\} \cup \Delta$. The converse follows from Lemma 2.49 with asymmetry: if U is a finite subset of T such that $U \rhd (b,a)$, then, for every $(x,y) \in S$, transitivity (T) with $(a,b),(b,a) \rhd (x,y)$ yields $U,(a,b) \rhd (x,y)$, whence $(T,(a,b))^{\rhd} = S$. ☐

Corollary 2.51. *Every strict partial order of E is $\mathfrak{J}$-fixed.*

Proof. If T is a strict partial order, then $\operatorname{tc}(T) = T$ and $T \cap \Delta = \emptyset$, in view of which the second item of Lemma 2.50 amounts to $\sim (a,b) \in T$, so that $T^\circ \subseteq \sim T$. Proposition 2.35 asserts that T is $\mathfrak{J}$-fixed. ☐

Because of Lemma 2.50, the entailment relation $\rhd$ is strongly reductive, since if $(a,b) \in T^\circ$, then $(b,a) \in T^{\rhd}$, and $\{\, (a,b),(b,a) \,\}^{\rhd} = S$ is due to asymmetry. Furthermore, the proper complete ideals C of $\rhd$ are precisely the *linear* strict partial orders of E, i.e., those such that for all $a,b \in E$, if $a \neq b$, then $(a,b) \in C$ or $(b,a) \in C$. Thus, with ZS and REM, combining Theorem 2.22 with Corollary 2.51, we reobtain Dushnik and Miller's classical result [106], for which Corollary 2.51 provides a syntactical counterpart:

Every strict partial order of E is the intersection
of all linear strict partial orders that contain it.

The examples in this Section 2.4 have been considered also in the context of conservation criteria for *multi-conclusion* extensions of single-conclusion entailment relations [209, 210].

2.5 Conservation

In this final section we shed some light on certain aspects of multi-conclusion entailment relations as extending their single-conclusion counterparts [103, 115, 140, 170, 229, 234, 255]. We start with a brief summary, referring to [209, 210] for a thorough account, which extends and builds on ideas and results from proof theory [190], and dynamical and constructive algebra [87, 92, 169]. The importance of entailment relations for constructive algebra and point-free topology has been advocated ever since Coquand and Cederquist's seminal paper [62]. Entailment relations are also related to resolution calculi [90, 258].

Let S be a set. Recall that a *multi-conclusion entailment relation* [229–231] is a relation

$$\vdash \subseteq \operatorname{Fin}(S) \times \operatorname{Fin}(S)$$

between finite subsets U and V of S which is *reflexive, monotone, and transitive*:

$$\frac{U \between V}{U \vdash V}\ (\mathrm{R}) \qquad\qquad \frac{U \vdash V}{U,U' \vdash V,V'}\ (\mathrm{M}) \qquad\qquad \frac{U \vdash V,a \quad U',a \vdash V'}{U,U' \vdash V,V'}\ (\mathrm{T})$$

making use of the usual shorthand notations. A *model* of $\vdash$ is a subset α of S that splits entailment as follows:

$$\frac{\alpha \supseteq U \quad U \vdash V}{\alpha \between V}$$

A multi-conclusion entailment relation $\vdash$ *extends* a single-conclusion entailment relation $\rhd$ on S if, for every finite subset U and element a of S, $U \rhd a$ implies $U \vdash a$. In case the converse holds as well, this extension is said to be *conservative*. It can be shown [209, 210] that an extension $\vdash$ of $\rhd$ is conservative if and only if the following holds:

$$\frac{a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell \qquad U, b_1 \rhd c \quad \ldots \quad U, b_\ell \rhd c}{U, a_1, \ldots, a_k \rhd c} \tag{\#}$$

Example 2.52. Let $\rhd$ be a single-conclusion entailment relation on S and suppose that C is a proper complete ideal of $\rhd$. Stipulate, for finite subsets U and V of S,

$$U \vdash_C V \;\equiv\; U \between C^\circ \;\vee\; V \between C$$

It may be instructive to think of this $\vdash_C$ in terms of material implication: either U contains a falsity or V contains a truth, taking S to consist of certain states of affairs of which C gathers those that are agreed to be true.[9] It is straightforward to verify that $\vdash_C$ is an entailment relation, taking into account that $C^\circ = S \setminus C$ (Proposition 2.14).

This $\vdash_C$ is an extension of the quotient $\rhd_C$. In fact, suppose that $U \rhd_C a$, i.e., there is a finite subset V of C with $U, V \rhd a$. Now either $a \in C$, and $U \vdash_C a$ is immediate, or else $a \in C^\circ$, and thus $U \between C^\circ$. To see the latter, as C is detachable (Proposition 2.14) it suffices to rule out $U \subseteq C$, in which case by $V \subseteq C$ we would get $a \in C$ since C is an ideal of $\rhd$, whence $C = S$ by Lemma 2.10, in contradiction to C being proper.

It can be shown directly that if $\rhd$ is reductive, then $\vdash_C$ is conservative over $\rhd_C$. To see this, suppose that $U \vdash_C a$, which is to say that $U \between C^\circ$ or $a \in C$. In the latter case $U \rhd_C a$ by $U, a \rhd a$. In the former case, if $b \in U \cap C^\circ$, then, since $\rhd$ is reductive, there is a finite subset V of C such that $b \in V^\circ$, that is, $(V, b)^{\rhd} = S$. In particular $V, b \rhd a$, and thus again $U \rhd_C a$.

However, all this requires a proper complete ideal of $\rhd$ to begin with, the existence of which, as we have seen, may well depend on assumptions beyond **CZF**.

For the following we make use of the conservation criterion recalled above [209, 210] as well as of the notion of Jacobson radical as developed in Subsection 2.3.3.

Proposition 2.53. *Let $\rhd$ be a single-conclusion entailment relation, and let $\vdash$ be a multi-conclusion entailment relation that extends $\rhd$.*

1. *Let $\rhd$ be reductive. If $\vdash$ is conservative over $\rhd$, then every proper complete ideal of $\rhd$ is a model of $\vdash$.*

2. (ZS + REM) *Let $\rhd$ be strongly reductive such that every ideal of $\rhd$ is $\mathfrak{J}$-fixed. If every proper complete ideal of $\rhd$ is a model of $\vdash$, then $\vdash$ is conservative over $\rhd$.*

Proof. 1. Let $\rhd$ be reductive and let $\vdash$ be conservative over $\rhd$. Let C be a proper complete ideal of $\rhd$. In order to show that C is a model of $\vdash$, consider finite subsets U and V of S, and assume that $U \vdash V$ and $U \subseteq C$. By Proposition 2.14, C is detachable with complement C°. Hence to prove $C \between V$ it suffices to rule out the alternative $V \subseteq C^\circ$. In this case, since $\rhd$ is reductive, for every $b \in V$ there is a finite subset U_b of C such that $b \in U_b^\circ$. Now let $c \in S$ be arbitrary. By the definition of reciprocation, for every $b \in V$ we have $U_b, b \rhd c$, so that by monotonicity of $\rhd$ we get $W, b \rhd c$ for every $b \in V$, where $W = \bigcup_{b \in V} U_b$. Then, by way of conservation (#), we obtain $U, W \rhd c$ and thus $c \in C$ since C is an ideal of $\rhd$ and $U, W \subseteq C$. As c was considered arbitary, it follows that $C = S$. Yet C is supposed to be proper!

[9] This is to be compared with Carnap's concept of *involution* [58], to whom, among others, the development of multi-conclusion entailment relations can be traced back [152, 234].

2. Let $\rhd$ be strongly reductive such that every ideal of $\rhd$ is $\mathfrak{J}$-fixed, and suppose that every proper complete ideal of $\rhd$ is a model of $\vdash$. We want to show that $\vdash$ is conservative over $\rhd$, to which end we consider a finite subset U and element a of S with $U \vdash a$. If C is a proper complete ideal that contains U, then, since C is a model of $\vdash$, we get $a \in C$. From Theorem 2.22 it follows that $a \in \mathfrak{J}(U^{\rhd})$. Since $U^{\rhd}$ is $\mathfrak{J}$-fixed, we get $a \in U^{\rhd}$, which is to say that $U \rhd a$, as required. $\square$

Let's support the seemingly *ad hoc* hypotheses of Proposition 2.53(2) with some concrete examples.

Example 2.54. We return to the setting of Section 2.4.1.3. Thus, let $S = R$ be a commutative ring and consider once again the entailment relation $\rhd$ of radical ideal. Let the multi-conclusion entailment relation $\vdash$ be generated by the axioms of $\rhd$ plus all instances of the axioms of *properness* and *primality*

$$\begin{aligned} 1 &\vdash \\ ab &\vdash a, b \end{aligned}$$

respectively. The models of $\vdash$ are the proper prime ideals of the ring R. Let us say that R is a *Jacobson ring* if every radical ideal is $\mathfrak{J}$-fixed with respect to $\rhd$. In view of Proposition 2.43, this classically amounts to the customary definition of a Jacobson ring [121, 158]. Every proper complete ideal C of $\rhd$ is prime, hence is a model of $\vdash$. Invoking ZS and REM, Proposition 2.53(2) now tells us that if R is a Jacobson ring, then $\vdash$ is conservative over $\rhd$. However, this can as well be shown constructively for arbitrary rings and with syntax only [209, 210].

Example 2.55. As in Section 2.4.2, let L be a distributive lattice. To extend the single-conclusion entailment relation $\rhd$ of filter of L, we consider multi-conclusion axioms

$$\begin{aligned} 0 &\vdash \\ a \vee b &\vdash a, b \end{aligned}$$

The models of $\vdash$ are the proper prime filters of L. Conversely, since Encoding holds, every (proper) complete ideal of $\rhd$ is (proper and) prime (Proposition 2.32). If L is a Jacobson lattice, i.e., such that every filter of L is $\mathfrak{J}$-fixed, then $\vdash$ is conservative over $\rhd$ according to Proposition 2.53(2). For instance, this holds if L is a Boolean algebra. However, as in the preceding Example 2.54, all this can be shown constructively for arbitrary distributive lattices and by means of syntax only [209, 210].

Example 2.56. Consider the single-conclusion entailment relation $\rhd$ of strict partial order on a set E from Section 2.4.3.2. On top of the axioms of $\rhd$ we put multi-conclusion axioms

$$\begin{aligned} (a,a) &\vdash \\ &\vdash (a,b), (b,a) \qquad (a \neq b) \end{aligned}$$

with side condition as indicated. The models of $\vdash$ are the strict linear orders of E. Since every proper complete ideal of $\rhd$ is a model of $\vdash$, and since every ideal of $\rhd$ is $\mathfrak{J}$-fixed by Corollary 2.51, it follows from Proposition 2.53(2) that $\vdash$ is conservative over $\rhd$. Again, we refer to [210] for a purely syntactical treatment.

Chapter 3

Eliminating disjunctions by disjunction elimination

This chapter is based on [209, 210].

3.1 Introduction

As is well-known, certain additional axioms in which disjunctions occur in positive position such as

$$\begin{array}{rcl@{\qquad}rcl} P(x * y) & \to & P(x) \vee P(y) & P(e) & \to & \bot \\ \top & \to & Q(z) \vee Q(\sim z) & Q(z) \wedge Q(\sim z) & \to & \bot \end{array}$$

are extremely useful in proof practice: they make possible quicker and slicker proofs in the special cases specified by the axioms. Examples include the characteristic axioms of integral domain, local ring, linear order, ordered field and valuation ring. The use of such axioms, however, is said [228] to obstruct the extraction of computational content from classical proofs.[1,2]

To reduce the general case to the special case, moreover, one needs to have at hand—in the terminology of Hilbert's Programme—the *ideal objects* characterised by the axioms, as there are prime ideals, prime filters, ultrafilters, complete theories and linear orders. Yet the existence of these ideal objects is tied together—again in Hilbert's terms—with *transfinite methods* (Axiom of Choice, Well-Ordering Theorem, Ultrafilter Theorem, Zorn's Lemma, etc.) in the appropriate mathematical forms shaped by Artin–Schreier, Hahn–Banach, Krull–Lindenbaum, Szpilrajn and others.

This method is related to *semantic* conservation[3] proofs with adequate completeness theorems at hand: by suitably embedding any given model of the base theory T into a model of the extended theory T^* [233]. In fact, if $T^* \vdash \varphi$, then $T^* \vDash \varphi$ by soundness; whence $T \vDash \varphi$ by embedding, and thus $T \vdash \varphi$ by completeness. While completeness and embedding normally need transfinite methods, in some cases Boolean-valued models can be used for constructive arguments [80].

More often than not one can also put and prove a *syntactical* conservation theorem the proof of which contains a proof-theoretic conversion algorithm that works at least for what is known as Horn sequents [148] or definite Horn clauses [217]. This approach is not new and has already proved practicable in different but related contexts: for example, in point-free topology such as locale theory and formal topology [62, 63, 72, 75, 184, 185]; in constructive algebra, especially with dynamical methods [81, 92, 164, 167–169, 256]; and in proof theory, e.g. in proof analysis [189, 190].

[1] This ample and active field of research has been displayed both in monographs [153, 227] and in survey papers [37, 155], which list of references is by no means exhaustive; early original references are, e.g., [57, 154].

[2] Falsity can be seen as nullary disjunction; and binary and nullary disjunctions cover all finite disjunctions.

[3] We prefer to write 'conservation' rather than 'conservativity', which choice of terminology follows, e.g., [109, 239].

Towards a considerable generalisation (Theorem 3.4) we now employ a method pointed out in [62]: that is, to work with Scott's entailment relations. More specifically, we invoke the extremely efficient 'sandwich criterion' Scott [229] has proved equivalent to syntactical conservation of a multi-conclusion entailment relation extending a single-conclusion entailment relation (Theorem 3.2 below). This criterion has turned out to hold in numerous cases including the ones abstracted before [208].

In a nutshell, applying Scott's criterion means to eliminate the additional axioms with disjunctions in positive position by reducing them to the corresponding disjunction elimination rules, which have proved admissible in all mathematical instances considered so far.[4] In deduction terms this means to fold up branchings of proof trees by way of properties of the relevant mathematical structures.

Perhaps it is in order to remember a saying by Scott [230, pp. 793–4]:

> *Un*fortunately, in my opinion, both because of the aim of Gentzen's own work and in the light of later applications, the Gentzen systems have been very much oriented toward proof-theoretic analyses—especially the problems of establishing the so-called *cut elimination theorem*. For me this was misleading. It took me a long time to realize that cut is *not* eliminable—except in very special circumstances. This is not to say that cut elimination is uninteresting or unimportant, but there does seem to be a simple and basic point to make with the aid of Gentzen's idea which may not be so generally appreciated.

On method

All but Section 3.5 can be expressed within Elementary Constructive Zermelo–Fraenkel Set Theory **ECST** [7, 8]. This is a fragment of Constructive Zermelo–Fraenkel Set Theory **CZF**, which is based on intuitionistic logic and does not contain the Axiom of Power Set, let alone the Axiom of Choice. To supply semantics (Section 3.5) sometimes requires to use classical logic, to speak of power sets or to invoke Zorn's Lemma; for simplicity's sake we refer to **ZFC** in any such case.

3.2 Relation

3.2.1 Consequence

Let S be a set and $\rhd \subseteq \mathrm{Pow}(S) \times S$. All but one of Tarski's axioms of consequence [244] can be put as

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{\forall b \in U(V \rhd b) \quad U \rhd a}{V \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{\exists U_0 \in \mathrm{Fin}(U)(U_0 \rhd a)}\ (\mathrm{A})$$

where $U, V \subseteq S$ and $a \in S$. Since Sambin's [218][5] these axioms have also characterised a finitary covering or Stone covering in formal topology; see further [65, 67, 187, 188, 219, 220].

The notion of consequence has allegedly been described first by Hertz [132–134]. We do not employ the one of Tarski's axioms by which he requires that S be countable. This axiom aside, Tarski has rather characterised the set of consequences of a set of propositions, which corresponds to the *algebraic closure operator* $U \mapsto U^{\rhd}$ on $\mathrm{Pow}(S)$ correlated to a relation $\rhd$ as above, viz.

$$U^{\rhd} \equiv \{a \in S : U \rhd a\}\,.$$

Rather than with Tarski's notion, we henceforth work with its restriction to finite subsets, that is, the notion of a *single-conclusion entailment relation*. This is a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ that

[4] Strictly speaking these are provable implications rather than admissible rules in the formal reading of Lorenzen's term [172]; for the sake of a slicker notation we still write them with horizontal lines.

[5] This also is from where we have taken the symbol $\rhd$, which is further used [64, 252] to denote a 'consecution' [204].

satisfies

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{V \rhd b \quad V', b \rhd a}{V, V' \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{U, U' \rhd a}\ (\mathrm{M})$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, b \equiv V \cup \{b\}$.

Our focus thus is on *finite* subsets of S, for which we reserve the letters $U, V, W, \ldots$; we also sometimes write $a_1, \ldots, a_n$ in place of $\{a_1, \ldots, a_n\}$. Redefining

$$T^{\rhd} \equiv \{a \in S : \exists U \in \mathrm{Fin}(T)(U \rhd a)\} \tag{3.1}$$

for *arbitrary* subsets T of S gives back an algebraic closure operator on $\mathrm{Pow}(S)$. Hence the single-conclusion entailment relations correspond exactly to the relations satisfying Tarski's axioms above.

3.2.2 Entailment

Let S be a set and $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$. Scott's [229] axioms of entailment can be put as

$$\frac{U \between W}{U \vdash W}\ (\mathrm{R}) \qquad \frac{V \vdash W, b \quad V', b \vdash W'}{V, V' \vdash W, W'}\ (\mathrm{T}) \qquad \frac{U \vdash W}{U, U' \vdash W, W'}\ (\mathrm{M})$$

for finite $U, V, W \subseteq S$ and $b \in S$, where $U \between W$ means that U and W have an element in common.[6] To be precise, any such $\vdash$ is a *multi-conclusion entailment relation*, where 'multi' includes 'empty'. The axioms are symmetric: that is, $\vdash$ satisfies the axioms if and only if so does the converse relation $\dashv$.

This fairly general notion of entailment has been introduced by Scott [229–231], building on Hertz's and Tarski's work (see above), and of course on Gentzen's sequent calculus [117, 118]. Shoesmith and Smiley [234] trace multi-conclusion entailment relations back to Carnap [58]. Before Scott, Lorenzen has developed analogous concepts formally [170, 171, 173, 174]; he has even listed [170, pp. 84–5] counterparts of the axioms (R), (T) and (M) for single- and multi-conclusion entailment [83, 84, 86].[7] The relevance of the notion of entailment relation to point-free topology and constructive algebra has been pointed out in [62]; this has been used e.g. in [72, 76, 82, 87, 89, 90, 190, 207]. Consequence and entailment have caught interest from various angles [19, 103, 115, 138, 139, 196, 234, 255].

It is in order to point out a major virtue Scott's treatment of entailment [229] has in comparison with the ones of his predecessors: the base set S may be any set whatsoever, and especially need not consist of formulas. The recommended reading of $U \vdash V$ still is as a sequent à la Gentzen, or rather as

$$\bigwedge_{b \in U} v(b) \to \bigvee_{c \in V} v(c)$$

where v is a distinguished predicate on S, normally the one that is of primary interest in the given context. In more logical terms one may view the elements of S as propositional variables and v as a valuation. Although this can be made precise by semantics, see Section 3.5, let us stress that—apart from heuristics—it is by no means constituent for our syntactical considerations, i.e. for all but Section 3.5.

Just as for sequents, it is common to abbreviate $\emptyset \rhd a$ by $\rhd a$, and to use $\vdash V$ and $U \vdash$ as shorthands of $\emptyset \vdash V$ and $U \vdash \emptyset$, respectively. One occasionally even writes $\vdash$ in place of $\emptyset \vdash \emptyset$.

3.2.3 Generation

Let $\mathscr{E} \subseteq \mathrm{Pow}(X \times Y)$ be a class of relations between sets X and Y. We order $\mathscr{E}$ by inclusion $\subseteq$, and call every $R \in \mathscr{E}$ an *$\mathscr{E}$-relation.* Let $R \subseteq X \times Y$, and $(x_i, y_i) \in X \times Y$ with $i \in I$. We say

[6] We have adopted this notation from Giovanni Sambin.

[7] Stefan Neuwirth has pointed this out to us.

that R *is the* $\mathscr{E}$*-relation generated by the axioms* $x_i R y_i$ *with* $i \in I$ if R is the least $\mathscr{E}$-relation to which (x_i, y_i) belongs for every $i \in I$. Note that we thus do not incur circularity inasmuch as we suppose R to be given.

We will apply this in the two cases in which $\mathscr{E}$ either consists of all single-conclusion entailment relations on a given set S, or else of all multi-conclusion entailment relations on S. In these cases the axioms rather are axiom schemes in the sense that every parameter is tacitly understood as ranging over its domain. For example, if $\circ$ is a binary operation on S, then by saying that $\rhd$ is generated by the axiom $a, b \rhd a \circ b$ we mean that $\rhd$ is generated by *all* axioms $a, b \rhd a \circ b$ with $a, b \in S$.

To actually construct an entailment relation generated by axioms, in a non-circular way, one can *inductively generate* it from the axioms by closing up with respect to (R), (T) and (M). This anyway is how we deal with our applications (Section 3.4), but is not always necessary for making proofs work. For example, for proving our main result (Theorem 3.4) it is enough to know that the entailment relations under consideration literally are the least entailment relations that satisfy the required axioms.

3.3 Conservation

Let $\rhd$ and $\vdash$ stand for a single-conclusion and a multi-conclusion entailment relation, respectively.

3.3.1 Back and forth

Given $\vdash$, its *trace* $\rhd_\vdash$ is defined by

$$U \rhd_\vdash a \equiv U \vdash a$$

and in fact is a single-conclusion entailment relation.

Given $\rhd$ and $\vdash$, it makes sense to say that

1. $\vdash$ is an *extension* of $\rhd$ if $\rhd \subseteq \rhd_\vdash$;
2. $\vdash$ is *conservative* over $\rhd$ if $\rhd \supseteq \rhd_\vdash$.

By the very definitions, every $\vdash$ is a conservative extension of its trace $\rhd_\vdash$.

Given $\rhd$, there are $\vdash_\rhd^{\min}$ and $\vdash_\rhd^{\max}$ as follows [229]:[8]

$$\begin{aligned} U \vdash_\rhd^{\min} V &\equiv \exists b \in V (U \rhd b) \\ U \vdash_\rhd^{\max} V &\equiv \forall W \in \mathrm{Fin}(S) \forall c \in S \big(\forall b \in V (W, b \rhd c) \to W, U \rhd c\big) \end{aligned}$$

Remark 3.1.

1. Both $\vdash_\rhd^{\min}$ and $\vdash_\rhd^{\max}$ are multi-conclusion entailment relations, and $\vdash_\rhd^{\min} \subseteq \vdash_\rhd^{\max}$.
2. Both $\vdash_\rhd^{\min}$ and $\vdash_\rhd^{\max}$ are conservative extensions of $\rhd$, which is the trace of either relation.

We note in passing[9] that the trace $\rhd_\vdash$ of a multi-conclusion entailment relation $\vdash$ both is

1. the least single-conclusion entailment relation over which $\vdash$ is conservative and
2. the least single-conclusion entailment relation $\rhd$ such that $\vdash_\rhd^{\max}$ contains $\vdash$.

For later use we observe that $a_1, \ldots, a_k \vdash_\rhd^{\max} b_1, \ldots, b_\ell$ is tantamount to

$$\frac{W, b_1 \rhd c \quad \ldots \quad W, b_\ell \rhd c}{W, a_1, \ldots, a_k \rhd c} \tag{3.2}$$

for all finite $W \subseteq S$ and $c \in S$.

[8] This definition of $\vdash_\rhd^{\max}$ is not identical but equivalent to the one given in [229].
[9] These observations have been made by Hajime Ishihara.

3.3.2 Unfolding sandwiches

Given $\rhd$, the $\vdash$ which are extensions of $\rhd$ (respectively, which are conservative over $\rhd$) are closed upwards (respectively, closed downwards) with respect to $\subseteq$. Hence the $\vdash$ which are conservative extensions of $\rhd$ form an interval. This interval has endpoints $\vdash_{\rhd}^{\min}$ and $\vdash_{\rhd}^{\max}$ according to the following 'sandwich criterion' for conservative extension given by Scott [229]:

Theorem 3.2. *A multi-conclusion entailment relation $\vdash$ is a conservative extension of the single-conclusion entailment relation $\rhd$ if and only if $\vdash$ lies between $\vdash_{\rhd}^{\min}$ and $\vdash_{\rhd}^{\max}$, which is to say that*

$$\rhd = \rhd_{\vdash} \iff \vdash_{\rhd}^{\min} \subseteq \vdash \subseteq \vdash_{\rhd}^{\max} .$$

Lorenzen [174, Satz 14, Satz 15] already had $\vdash_{\rhd}^{\min}$ and $\vdash_{\rhd}^{\max}$ as well as $\Rightarrow$ of Theorem 3.2 [83, 84].[10]

By proof inspection we could make Scott's criterion slightly more precise, as follows:

Lemma 3.3.

1. *$\vdash$ is an extension of $\rhd$ if and only if $\vdash_{\rhd}^{\min} \subseteq \vdash$.*

2. *If $\vdash \subseteq \vdash_{\rhd}^{\max}$, then $\vdash$ is conservative over $\rhd$.*

3. *If $\vdash$ is an extension of $\rhd$, then $\vdash$ is conservative over $\rhd$ if and only if $\vdash \subseteq \vdash_{\rhd}^{\max}$.*

Proof.

1. If $\rhd \subseteq \rhd_{\vdash}$, then $U \vdash_{\rhd}^{\min} V$ implies $U \vdash b$ for some $b \in V$; whence $U \vdash V$ by (M). Conversely, if $\vdash_{\rhd}^{\min} \subseteq \vdash$ and $U \rhd b$, then $U \vdash_{\rhd}^{\min} b$ and thus $U \vdash b$ or, equivalently, $U \rhd_{\vdash} b$.

2. If $\vdash \subseteq \vdash_{\rhd}^{\max}$ and $U \vdash a$, then $U \vdash_{\rhd}^{\max} a$, which amounts to $U \rhd a$ because $\rhd$ is the trace of $\vdash_{\rhd}^{\max}$.

3. Suppose that $\vdash$ conservatively extends $\rhd$, and let $U \vdash V$ where $V = \{ b_1, \ldots, b_\ell \}$. To prove $U \vdash_{\rhd}^{\max} V$, let $W \in \mathrm{Fin}(S)$ and $c \in S$ such that

$$W, b_1 \rhd c \qquad \ldots \qquad W, b_\ell \rhd c .$$

Since $\vdash$ extends $\rhd$, we have as well

$$W, b_1 \vdash c \qquad \ldots \qquad W, b_\ell \vdash c .$$

With $U \vdash V$ at hand we can successively cut $b_1, \ldots, b_\ell$ and thus obtain $W, U \vdash c$, which by conservation implies $W, U \rhd c$ as required. □

In view of part 2 of Lemma 3.3, to have that conservation follows from $\vdash \subseteq \vdash_{\rhd}^{\max}$ it is not necessary that $\vdash$ be an extension of $\rhd$.

Remembering the recommended disjunctive reading of the conclusion V of any sequent $U \vdash V$, in the light of Lemma 3.3 a possible interpretation of extension and conservation is as follows:

1. extension as that disjunctions introduced from $\rhd$ can be expressed as sequents of $\vdash$;

2. conservation as that disjunctions expressed as sequents of $\vdash$ can be eliminated in terms of $\rhd$.

[10] Stefan Neuwirth pointed this out to us.

3.3.3 Adding axioms

Let the single-conclusion entailment relation $\rhd$ be generated by axioms. Let the multi-conclusion entailment relation $\vdash$ be generated by the axioms of $\rhd$, of course with $\vdash$ in place of $\rhd$, and by *additional axioms* of the form

$$\varphi : \quad a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell$$

where $k, \ell \geq 0$. In any such situation we say that $\vdash$ *extends* $\rhd$, and list the additional axioms if needed. This is legitimate inasmuch as if $\vdash$ extends $\rhd$, then $\vdash$ is an extension of $\rhd$ in the sense of 3.3.1. By the *conservation criterion* of an axiom φ as above we understand

$$\frac{W, b_1 \rhd c \quad \ldots \quad W, b_\ell \rhd c}{W, a_1, \ldots, a_k \rhd c}\ (\mathrm{E}_\varphi)$$

for all finite subsets W of S and $c \in S$.[11]

Theorem 3.4. *If $\vdash$ extends $\rhd$, then $\vdash$ is conservative over $\rhd$ precisely when, for every additional axiom φ of $\vdash$, the conservation criterion E_φ holds for $\rhd$.*

Proof. As $\vdash$ is an extension of $\rhd$, the former is conservative over the latter if and only if $\vdash \subseteq \vdash_\rhd^{\max}$ (Lemma 3.3). Now recall that $\vdash$ is the least multi-conclusion entailment relation that satisfies not only the axioms of $\rhd$ but also the additional axioms of $\vdash$; and that $\vdash_\rhd^{\max}$ as an extension of $\rhd$ already satisfies the former axioms. Hence $\vdash \subseteq \vdash_\rhd^{\max}$ is tantamount to $\vdash_\rhd^{\max}$ too satisfying all additional axioms, i.e.

$$a_1, \ldots, a_k \vdash_\rhd^{\max} b_1, \ldots, b_\ell$$

for every additional axiom φ as above. In view of (3.2), this is equivalent to E_φ for $\rhd$. $\square$

Given an axiom such as φ above, let $\vdash_\varphi$ denote the multi-conclusion entailment relation that extends $\rhd$ with the single additional axiom φ. If $\vdash_\varphi$ is conservative over $\rhd$, we say—par abus de langage—that φ is *conservative* over $\rhd$. The related special case of Theorem 3.4 reads as follows:

Corollary 3.5. *An axiom φ is conservative over $\rhd$ if and only if E_φ holds for $\rhd$.*

By reduction to E_φ we can thus eliminate from proof trees occurrences of an additional axiom φ, roughly as follows; note that the result of this conversion does not contain $\vdash$ at all:

$$\dfrac{\dfrac{W \rhd a_1 \ \ldots \ W \rhd a_k \qquad \varphi : \ a_1, \ldots, a_k \vdash b_1, \ldots, b_l}{W \vdash b_1, \ldots, b_l} \qquad W, b_1 \rhd c \ \ldots \ W, b_l \rhd c}{W \vdash c}$$

$$\downarrow$$

$$\dfrac{W \rhd a_1 \ \ldots \ W \rhd a_k \qquad \dfrac{W, b_1 \rhd c \ \ldots \ W, b_l \rhd c}{W, a_1, \ldots, a_k \rhd c}\,(\mathrm{E}_\varphi)}{W \rhd c}$$

By Corollary 3.5 the following is an equivalent formulation of Theorem 3.4.

Corollary 3.6. *If $\vdash$ extends $\rhd$, then $\vdash$ is conservative over $\rhd$ precisely when every additional axiom φ of $\vdash$ is conservative over $\rhd$.*

The next lemma will prove useful for modifying the base of an instance of conservation; we hasten to add that item 1 below is the single-conclusion version of [62, Lemma 2].

[11] In the terminology of [190], this E_φ corresponds to the 'single-succedent left rule' of the 'mathematical axiom' φ.

Lemma 3.7. *Let $\rhd$ be a single-conclusion entailment relation on S that is generated by axioms. For any subset D of S, let $\rhd'$ be generated by the axioms of $\rhd$, and by the extra axioms $\rhd' d$ with $d \in D$.*

1. *We have $U \rhd' a$ if and only if $U, V \rhd a$ for a finite subset $V \subseteq D$.*
2. *If an axiom*
$$\varphi : \quad a_1, \dots, a_k \vdash b_1, \dots, b_\ell$$
is conservative over $\rhd$, then it is conservative over $\rhd'$.

Proof. 1. 'only if': Since $\rhd'$ is generated from $\rhd$ by adding axioms, we may argue inductively. We consider first the axioms generating $\rhd'$. If $U \rhd' a$ by way of an axiom $U \rhd a$ of $\rhd$, then $V = \emptyset$ will do. If we have at hand $\rhd' d$ for a certain $d \in D$, then $d \rhd d$ by reflexivity of $\rhd$. Next we address (R), (M), and (T). Our claim certainly applies to (R); by induction it also applies to (M): if $U, U' \rhd' a$ and there is a finite subset V of D such that $U, V \rhd a$, then $U, U', V \rhd a$ by (M) for $\rhd$. As for (T), consider the following instance:

$$\frac{U \rhd' b \quad U', b \rhd' a}{U, U' \rhd' a}$$

If $U, V \rhd b$ and $U', V', b \rhd a$ for finite subsets V and V' of D, then $U, U', V, V' \rhd a$ by (T) for $\rhd$.

'if': Let V be a finite subset of D such that $U, V \rhd a$. Then $U, V \rhd' a$ as well, because $\rhd'$ extends $\rhd$. We further have $\rhd' d$ for every $d \in V$. By repeated application of (T) we get $U \rhd' a$.

2. By Corollary 3.5 we have to deduce from the conservation criterion for $\rhd$ its counterpart for $\rhd'$:
$$\frac{W, b_1 \rhd' c \quad \dots \quad W, b_\ell \rhd' c}{W, a_1, \dots, a_k \rhd' c}$$
By part 1, the premisses say that for every $i \leqslant \ell$ there is a finite $V_i \subseteq D$ such that $W, V_i, b_i \rhd c$. By (M) and the conservation criterion for $\rhd$ this yields
$$W, V_1, \dots, V_\ell, a_1, \dots, a_k \rhd c\,;$$
whence the desired conclusion $W, a_1, \dots, a_k \rhd' c$ is witnessed (see part 1) by $V_1 \cup \dots \cup V_\ell$. □

Lemma 3.8. *Let $\rhd$ be a single-conclusion entailment relation on S that is generated by axioms. Let $\mathscr{C}$ be a set of finite subsets of S. Let $\rhd'$ be generated by the axioms of $\rhd$, and by the extra axioms $C \rhd' d$ with $C \in \mathscr{C}$ and $d \in S$.*

1. *We have $U \rhd' a$ if and only if*
$$U \rhd a \quad \textit{or} \quad \exists C \in \mathscr{C}\, \forall c \in C\, (\, U \rhd c\,).$$
2. *For every $C \in \mathscr{C}$ the empty-conclusion axiom*
$$\nu : \quad C \vdash$$
is conservative over $\rhd'$.

Proof. 1. 'only if': Since $\rhd'$ is generated from $\rhd$ by adding axioms, we may argue inductively. We consider first the axioms generating $\rhd'$. If $U \rhd' a$ by way of an axiom $U \rhd a$ of $\rhd$, then the conclusion is immediate. If we have at hand $C \rhd' d$ with $C \in \mathscr{C}$ and $d \in S$, then of course $C \rhd c$ for every $c \in C$ by reflexivity of $\rhd$. As (R) and (M) are easily dealt with, we may focus on (T), of which we consider the following instance:

$$\frac{U \rhd' b \quad U', b \rhd' a}{U, U' \rhd' a}$$

and suppose that we have

$$U \rhd b \quad \text{or} \qquad \exists C \in \mathscr{C}\, \forall c \in C\, (\, U \rhd c\,)$$

as well as

$$U', b \rhd a \quad \text{or} \quad \exists C' \in \mathscr{C}\, \forall c' \in C'\, (\, U', b \rhd c').$$

Now we have to check each of the possible combinations, but actually it suffices to do the case

$$U \rhd b \quad \text{and} \quad \exists C' \in \mathscr{C}\, \forall c' \in C'\, (\, U', b \rhd c').$$

Applying (T) for $\rhd$, we get $U, U' \rhd c'$ for every $c' \in C'$. This means that $C' \in \mathscr{C}$ is as required.

'if': If $U \rhd a$, then of course $U \rhd' a$, since $\rhd'$ contains the generating axioms of $\rhd$. On the other hand, suppose that there is $C \in \mathscr{C}$ such that for every $c \in C$ we have $U \rhd c$. Recall that we have an axiom $C \rhd' a$. Therefore, by repeated application of (T), we obtain $U \rhd' a$.

2. By Corollary 3.5 it suffices to verify the corresponding conservation criterion

$$\frac{}{W, C \rhd' d}\ (\mathrm{E}_\nu)$$

where $C \in \mathscr{C}$. Since we have axioms $C \rhd' d$, this E_ν follows by (M). □

Lemma 3.8.2 may perhaps be compared with *elimination of empty succedents* in Gentzen systems [217, p. 66, Lemma 3.1.8]. It has applications in the context of order relations, e.g., to bring into the scope of our method Szpilrajn's extension principle for strict partial orders (Section 3.4.3).

3.4 Application

3.4.1 Weak order relations

Our approach subsumes the existing syntactical treatment [190] of order extension, the semantics of which is that every (proper) quasi-order can be extended to a linear one [129].

3.4.1.1 Quasi-orders

As a binary relation $\leq$ on a set X is a quasi-order if $\leq$ is reflexive and transitive, the single-conclusion entailment relation $\rhd$ of *quasi-order* on $S = X \times X$ is generated by the corresponding axioms of *reflexivity* and *transitivity*:

$$\rho : \quad \rhd (a, a) \qquad\qquad \tau : \quad (a, b), (b, c) \rhd (a, c)$$

The multi-conclusion entailment relation $\vdash_\lambda$ of *linear quasi-order* extends $\rhd$ with the single additional axiom of *linearity*:

$$\lambda : \quad \vdash (a, b), (b, a)$$

The conservation criterion of λ reads as follows:

$$\frac{W, (a, b) \rhd (r, s) \qquad W, (b, a) \rhd (r, s)}{W \rhd (r, s)}\ (\mathrm{E}_\lambda)$$

The closure operator corresponding to $\rhd$ assigns to a subset T of S its *reflexive-transitive closure* T^*. With this at hand, or following the proof of [190, Theorem 5.1], one readily verifies that E_λ holds for $\rhd$; whence $\vdash_\lambda$ is conservative over $\rhd$ (Theorem 3.4). This can equally be seen by restricting to single-conclusion instances an alternative description of $\vdash_\lambda$ in terms of cycles [190, Section 7]. Reflexivity ρ is necessary for conservation, by way of the special case $a = b = r = s$ of E_λ.

3.4.1.2 Bounded quasi-orders

We say that a quasi-order $(X, \leq)$ with distinguished elements $0, 1$ is *bounded* if $0 \leq s$ and $r \leq 1$ for all $r, s \in X$; and that $\leq$ is *proper* if $1 \not\leq 0$. Accordingly, the single-conclusion entailment relation $\rhd'$ of *bounded quasi-order* on $S = X \times X$ is generated by the axioms ρ and τ as above plus the following:

$$\beta_0 : \quad \rhd(0, s) \qquad\qquad \beta_1 : \quad \rhd(r, 1)$$

The multi-conclusion entailment relation $\vdash'$ of *linear proper bounded quasi-order* extends $\rhd'$ with the additional axioms of linearity λ and *properness*:

$$\pi : \quad (1, 0) \vdash$$

The conservation criterion of π reads as follows:

$$\frac{}{W, (1, 0) \rhd (r, s)}\ (\mathrm{E}_\pi)$$

By transitivity τ it is easy to see that E_π holds for $\rhd'$. As E_λ holds for $\rhd$ (Section 3.4.1.1), E_λ holds for $\rhd'$ too (Corollary 3.5 , Lemma 3.7). In all, $\vdash'$ is conservative over $\rhd'$ (Theorem 3.4).

3.4.1.3 Discussion

A proof-theoretic analysis of order relations is carried out in [190], with sequent calculi **GPO** and **GLO** which correspond to the theories of quasi-order and linear quasi-order, respectively. It is shown that a single-conclusion sequent derivable in **GLO** is derivable already in **GPO** [190, Theorems 5.1].

This conservation result is then carried over to nondegenerate nontrivial quasi-orders [190, Theorem 5.2]. While *nondegenerate* means $1 \not\leq 0$, i.e. what we have called 'proper', a quasi-order $\leq$ with distinguished elements 0 and 1 is said *nontrivial* if $0 \leq 1$. In terms of the single-conclusion entailment relation of quasi-order, *nontriviality* means to add the axiom

$$\rhd(0, 1)$$

to reflexivity ρ and transitivity τ. With this add-on, conservation of linearity λ carries over (Lemma 3.7), whereas the conservation of properness π depends on the presence of β_0 and β_1.

The conservation of linearity for quasi-orders is an instance of the Universal Krull–Lindenbaum principle, to which we now turn our attention.

3.4.2 Universal Krull–Lindenbaum

In the sequel $\rhd$ and $\vdash$ always stand for a single-conclusion and a multi-conclusion entailment relation, respectively.

3.4.2.1 Universal Krull

Let S come with a (partial) binary operation $* : S \times S \to S$ and with a distinguished element $e \in S$. Given $\rhd$, let $\vdash$ extend $\rhd$ with additional axioms

$$\mu : \quad a * b \vdash a, b \qquad\qquad \pi : \quad e \vdash$$

In this situation Theorem 3.4 reads as follows:

Corollary 3.9. *$\vdash$ is a conservative extension of $\rhd$ precisely when*

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, a * b \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, e \rhd c}\,(\mathrm{E}_\pi)$$

for all finite $W \subset S$ and $a, b, c \in S$.

The conservation criteria E_μ and E_π have occurred [208] as '$\rhd$ satisfies *Encoding*' and as 'e is *convincing* for $\rhd$', respectively. They can be compared with *disjunction elimination*[12] and *ex falso quodlibet*, especially if S is a bounded distributive lattice such as an intuitionistic Lindenbaum algebra, with $\vee$ as $*$ and $\bot$ as e, and $\rhd$ is the single-conclusion entailment relation of filter or theory, see Section 3.4.2.3.

It is noteworthy that the axiom of *contraction*

$$a * a \rhd a$$

is necessary for conservation, by the special case $a = b = c$ of E_μ; see Section 3.4.2.9 for details.

The converse of μ is the following axiom of *magma*, which in fact is a single-conclusion axiom:

$$a, b \vdash a * b$$

Assuming that $*$ be commutative and associative, which is the case in many an application below, we can relate the axioms above to the multi-conclusion axiom of *regularity* [83] or *exchange*:

$$a * x, b * y \vdash a * y, b * x$$

Remark 3.10. Regularity and contraction together imply μ. Magma and μ together imply regularity.

3.4.2.2 Universal Lindenbaum

Let S come with a (partial) unary operation $\sim$. Given $\rhd$, let $\vdash$ extend $\rhd$ with additional axioms

$$\eta : \quad \vdash a, \sim a \qquad \nu : \quad a, \sim a \vdash$$

In this situation Theorem 3.4 reads as follows:

Corollary 3.11. *$\vdash$ is a conservative extension of $\rhd$ precisely when*

$$\frac{W, a \rhd c \quad W, \sim a \rhd c}{W \rhd c}\,(\mathrm{E}_\eta) \qquad \frac{}{W, a, \sim a \rhd c}\,(\mathrm{E}_\nu)$$

Corollary 3.11 can be compared with *excluded middle* and *non-contradiction*, especially if S is a Boolean algebra such as a Lindenbaum algebra in classical logic, with complement or negation $\neg$ as $\sim$. In these cases $\rhd$ again is the entailment relation of filter or theory (Section 3.4.2.3). Further applications of Corollary 3.11 include Artin and Schreier's theorem, see Section 3.4.2.5.

3.4.2.3 Distributive lattices

Krull's Lemma for distributive lattices says that every proper filter (respectively, proper ideal) can be extended to a proper prime filter (respectively, proper prime ideal). To view the corresponding conservation statement, let L be a bounded lattice with meet $\wedge$ and join $\vee$, and with bottom and top element 0 and 1, respectively. The entailment relation $\rhd$ of *filter* on $S = L$ is generated by the axioms

$$\rhd 1 \qquad a, b \rhd a \wedge b \qquad a \rhd a \vee b \tag{3.3}$$

[12] In the context of formal languages, E_μ can further be compared with the 'proof-by-cases property' [64, 95, 252].

The multi-conclusion entailment relation $\vdash$ of *proper prime filter* extends $\rhd$ with additional axioms

$$\mu: \quad a \vee b \vdash a, b \qquad \pi: \quad 0 \vdash \tag{3.4}$$

for which the conservation criteria read as follows:

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, a \vee b \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, 0 \rhd c}\,(\mathrm{E}_\pi)$$

The closure operator corresponding to $\rhd$ assigns to a subset T of S the filter generated by T. Thus 0 is convincing, while E_μ, i.e. Encoding for $\rhd$, follows from L being distributive [208, p. 4.2]. Therefore, with $\vee$ as $*$ and 0 as e, Universal Krull implies that the multi-conclusion entailment relation of proper prime filter on a distributive lattice is a conservative extension of the single-conclusion entailment relation of filter.

Dually, the single-conclusion entailment relation $\rhd'$ of *ideal* on $S = L$ is generated by the axioms

$$\rhd' 0 \qquad a, b \rhd' a \vee b \qquad a \rhd' a \wedge b$$

This $\rhd'$ extends to the multi-conclusion entailment relation $\vdash'$ of *proper prime ideal* by adding the following axioms:

$$\mu': \quad a \wedge b \vdash' a, b \qquad \pi': \quad 1 \vdash'$$

The closure operator corresponding to $\rhd'$ assigns to a subset T of S the ideal generated by T. Reasoning dually to the case of filters shows that the multi-conclusion entailment relation of proper prime ideal is a conservative extension of the single-conclusion entailment relation of ideal.

While this approach fits Universal Krull, if L is a Boolean algebra with complement $-$, then we may instead add the axioms

$$\eta: \quad \vdash a, -a \qquad \nu: \quad a, -a \vdash$$

to both $\rhd$ and $\rhd'$. Now Universal Lindenbaum applies, giving rise to conservation over $\rhd$ and $\rhd'$ of the multi-conclusion entailment relations of *proper complete filter* and *proper complete ideal*, respectively. This conservation corresponds to Lindenbaum's Lemma for Boolean algebras, which says that every proper filter (respectively, proper ideal) can be extended to a proper complete filter (respectively, proper complete ideal).

3.4.2.4 Commutative rings

The original form of Krull's Lemma, for commutative rings [157], says that every proper filter can be extended to a proper prime filter, which can be carried over from ideals to filters. In order to display the corresponding conservation results, let $\rhd$ be the single-conclusion entailment relation of *radical ideal* (or *reduced ring*) on a commutative ring S which is generated by the axioms of *ideal* (or *zero*)

$$\rhd 0 \qquad a, b \rhd a + b \qquad a \rhd ab \tag{3.5}$$

together with the characteristic axiom of *radical ideal*

$$a^2 \rhd a\,. \tag{3.6}$$

The corresponding closure operator assigns to every subset T of S the radical of the ideal generated by T. The following hold for all finite $W \subseteq S$ and $a, b, c \in S$ (see e.g. [208, Lemma 19] for a proof of the first one):

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, ab \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, 1 \rhd c}\,(\mathrm{E}_\pi)$$

By Universal Krull, with multiplication as $*$ and 1 as e, the following axioms of *prime ideal* (or *integral domain*) are conservative over $\rhd$:

$$\mu: \quad ab \vdash a, b \qquad \pi: \quad 1 \vdash$$

The five axioms for $\vdash$ stem from [62], and conservation of $\vdash$ over $\rhd$ is essentially known from dynamical algebra [92].

Dually, the single-conclusion entailment relation of *filter* (or *unit*) on a commutative ring S is generated by the following axioms:

$$\rhd 1 \qquad a, b \rhd ab \qquad ab \rhd a$$

The corresponding closure operator assigns to every subset T of S the filter generated by T. The following hold for all finite $W \subseteq S$ and $a, b, c \in S$ (see e.g. [208, Lemma 20] for a proof of the first one):

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, a + b \rhd c}\,(\mathrm{E}_\mu) \qquad\qquad \frac{}{W, 0 \rhd c}\,(\mathrm{E}_\pi)$$

By Universal Krull, now with addition $+$ in place of $*$ and 0 as e, the axioms of *prime filter* (or *local ring*) are conservative over $\rhd$:

$$\mu: \quad a + b \vdash a, b \qquad\qquad \pi: \quad 0 \vdash$$

This again is essentially known from dynamical algebra [167].

3.4.2.5 Ordered fields

Artin and Schreier's Theorem [15], saying that every proper quadratic preorder on a field can be extended to a total order, was used to solve Hilbert's 17th Problem in the affirmative [14]. Towards the corresponding conservation result in terms of entailment relations, let the single-conclusion entailment relation $\rhd$ of *quadratic preorder* on a field S of char $\neq 2$ be generated by the following axioms:

$$\rhd a^2 \qquad a, b \rhd a + b \qquad a, b \rhd ab$$

The corresponding closure operator assigns to every subset T of S the quadratic preorder generated by T. The following hold for all finite $W \subseteq S$ and $a, b, c \in S$ (see [208, Lemma 24] for a proof of the first one):

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, a + b \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, -1 \rhd c}\,(\mathrm{E}_\pi)$$

By Universal Krull, with addition $+$ as $*$ and -1 as e, the following axioms are conservative over $\rhd$:

$$\mu: \quad a + b \vdash a, b \qquad \pi: \quad -1 \vdash$$

Equivalently so are the axioms of *total order* on $S \setminus \{0\}$:

$$\eta: \quad \vdash a, -a \qquad \nu: \quad a, -a \vdash$$

We thus not only have an instance of Universal Krull but also one of Universal Lindenbaum, with minus $-$ in place of $\sim$. A related set of axioms [62] already contains μ and ν.

Once more, this conservation statement is essentially known from dynamical algebra [168]. There is vast literature on computational and continuous aspects of the Artin–Schreier Theorem, Hilbert's 17th Problem and related results, see e.g. [29, 42, 97, 98, 200].

3.4.2.6 Valuation rings

Let R be a subring of a field K, and let $R[U]$ denote the subring of K containing R that is generated by $U \subseteq K$. Take the single-conclusion entailment relation $\rhd$ on $S = K$ that is generated by the axioms of *subring of K containing R*

$$\rhd r \quad (r \in R) \qquad a, b \rhd a + b \qquad a, b \rhd ab \tag{3.7}$$

together with the following axiom of *integral closure*:

$$s_1, \ldots, s_n \rhd a \quad (a^n + s_1 a^{n-1} + \cdots + s_n = 0,\ n \geq 1) \tag{3.8}$$

The corresponding closure operator assigns to every subset T of S the integral closure $\overline{R[T]}$ of $R[T]$; and by [208, Lemma 23] we have

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, ab \rhd c}\,(\mathrm{E}_\mu)$$

for all finite $W \subseteq S$ and $a, b, c \in S$. By Corollary 3.5, the additional axiom

$$\mu: \quad ab \vdash a, b$$

is conservative over $\rhd$. As there is no convincing element for $\rhd$, this is a partial instance of Universal Krull, with multiplication in place of $*$.

Up to this point everything equally works for subrings of a ring K rather than a field K. For a field K, however, an alternative generation [62, 87] of $\vdash$ makes use of the axiom of *valuation*:

$$\eta: \quad \vdash a, a^{-1} \quad (a \neq 0)$$

Given the axioms of $\rhd$, this η is equivalent to μ as above whenever the field K is *discrete* in the sense that the characteristic axiom of a field holds in the form

$$a = 0 \vee \exists b\,(ab = 1)\,.$$

Hence η too is conservative over $\rhd$ for discrete fields K, and we have a partial instance of Universal Lindenbaum as well. This approach has been followed in the context of Kronecker's Theorem [166] and Dedekind's Prague Theorem [87], and more generally to study valuations in a point-free way [78].

3.4.2.7 Ordered vector spaces

Let S be a vector space over the field $\mathbb{Q}$ of rationals. Let the single-conclusion entailment relation $\rhd$ on S be generated by the axioms of *additive submonoid*

$$\rhd 0 \qquad a, b \rhd a + b \tag{3.9}$$

together with the axiom of *positive cone*:

$$n \cdot a \rhd a \quad (n \in \mathbb{N}, n \geq 1) \tag{3.10}$$

The corresponding closure operator assigns to every $T \subseteq S$ the *positive cone* generated by T, and

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, a + b \rhd c}\,(\mathrm{E}_\mu)$$

holds for $\rhd$. By Theorem 3.4, the multi-conclusion entailment relation $\vdash$ extending $\rhd$ with the additional axiom

$$\mu: \quad a + b \vdash a, b$$

is conservative over $\rhd$. The axiom $n \cdot a \rhd a$ of $\rhd$ follows from E_μ by induction and thus is necessary for conservation. A different set of axioms [62] for $\vdash$ includes μ as above and the following axiom:

$$\nu: \quad a, -a \vdash$$

From this one can go to the point-free treatments [62, 63, 72, 75] of the Hahn–Banach theorem in succession to [184, 185]. Unlike for many of the ring-theoretic applications mentioned above, it is not clear whether one can make computational use of the Hahn–Banach theorem itself [72].

3.4.2.8 Weak order relations revisited

Some syntactic variants of order extension for weak orders may also be viewed as instances of Universal Krull–Lindenbaum. To discuss this we take up Section 3.4.1, adopting also some notation.

Let X be a set. The swap operation

$$\sim(a, b) = (b, a)$$

on $S = X \times X$ fits Universal Lindenbaum in parts, giving rise to Section 3.4.1.1. In fact, the axiom η becomes linearity λ, and thus E_η holds (Section 3.4.1.1) by reflexivity ρ and transitivity τ, whereas E_ν does not hold in general.

On the same $S = X \times X$ we have the *partial* binary operation $\circ$ of composition defined by

$$(a, b) \circ (b, d) = (a, d)\,.$$

With $\circ$ in place of $*$, the axiom μ becomes *cotransitivity*

$$\kappa : \quad (a, d) \vdash (a, b), (b, d)\,,$$

the conservation criterion of which reads

$$\frac{W, (a, b) \rhd (r, s) \quad W, (b, d) \rhd (r, s)}{W, (a, d) \rhd (r, s)}\ (\mathrm{E}_\kappa)$$

and can be proved by transitivity τ only. By reflexivity ρ, (conservation of) linearity λ is a special case of (conservation of) cotransitivity κ. With $(1, 0)$ as e, Section 3.4.1.2 thus is an instance of Universal Krull as a whole.

Still on $S = X \times X$ there further is the *total* binary operation $*$ of composition defined by

$$(a, b) * (c, d) = (a, d)\,,$$

with which S is a semigroup. This operation $*$ gives rise to the additional axiom

$$\mu : \quad (a, d) \vdash (a, b), (c, d)\,.$$

The corresponding conservation criterion is

$$\frac{W, (a, b) \rhd (r, s) \quad W, (c, d) \rhd (r, s)}{W, (a, d) \rhd (r, s)}\ (\mathrm{E}_\mu)$$

holds by transitivity τ only. Before working this out in detail (Lemma 3.12 below) we note in passing that *irreflexive* relations satisfying the weakened form

$$\mu' : \quad (a, d), (c, b) \vdash (a, b), (c, d)$$

of axiom μ are known as *interval orders* in utility theory [51, 110, 238].[13]

Let $\rhd$ be the single-conclusion entailment relation of *transitive relation* on S that is generated by transitivity τ only, the closure operator of which assigns to a subset W of S its *transitive closure* W^+.

Lemma 3.12. E_μ *holds for the single-conclusion entailment relation* $\rhd$ *of transitive relation.*

Proof. Suppose that $(r, s) \in \big(W, (a, b)\big)^+ \cap \big(W, (c, d)\big)^+$. In order to show $(r, s) \in \big(W, (a, d)\big)^+$, the following diagram offers a one-glance proof, easily to be made precise:

$$\begin{tikzcd}
 & a \arrow[r] \arrow[dr, dotted] & b \arrow[dr, dashed] & \\
r \arrow[ur, dashed] \arrow[dr, dashed] & & & s \\
 & c \arrow[r] & d \arrow[ur, dashed] &
\end{tikzcd}$$

[13] Furthermore, this axiom μ' is part of the definition of *semi order* [176, 232].

The upper two dashed arrows indicate chains in $W \cup \{ (a,b) \}$, while the two lower dashed arrows indicate chains in $W \cup \{ (c,d) \}$. There are $n, m \geqslant 1$ and certain elements $x_0, \dots, x_n$ and $y_0, \dots, y_m$ of X such that

$$(r,s) = (x_0, x_n) \quad \text{and} \quad \forall i < n \ \ (x_i, x_{i+1}) \in W \cup \{ (a,b) \} \tag{1}$$

and

$$(r,s) = (y_0, y_m) \quad \text{and} \quad \forall j < m \ \ (y_j, y_{j+1}) \in W \cup \{ (c,d) \}\,. \tag{2}$$

We may assume that both pairs (a,b) and (c,d) occur at least once, respectively. Hence, let $i_0 < n$ be the index corresponding to the first occurence of (a,b) among the pairs (x_i, x_{i+1}) in (1), and let $j_0 < m$ correspond to the last occurence of (c,d) among the pairs (y_j, y_{j+1}) in (2), all such that

$$(x_{i_0}, x_{i_0+1}) = (a,b) \qquad \text{and} \qquad (y_{j_0-1}, y_{j_0}) = (c,d).$$

We get a new chain

$$r, x_1, \dots, x_{i_0-1}, a, d, y_{j_0+1}, \dots, y_{m-1}, s$$

which by construction witnesses $(r,s) \in \big(W, (a,d)\big)^+$. $\square$

By Corollary 3.5 the axiom μ is conservative over the single-conclusion entailment relation of transitive relation; and (conservation of) cotransitivity κ follows from (conservation of) μ by setting $b = c$.

Over the single-conclusion entailment relation of quasi-order obtained by adding reflexivity ρ, the axiom μ remains conservative (Lemma 3.7) and yields conservation of the axiom of *strong linearity* [180]

$$\sigma : \quad \vdash (a,b), (c,a)\,.$$

Here (conservation of) linearity λ follows as the special case $b = c$ of (conservation of) σ.

Let us say that a relation satisfying axiom μ is *strongly cotransitive*. Notice that strong linearity is to strong cotransitivity just as linearity is to cotransitivity: the respective former implies the latter with transitivity at hand, while the converse holds in the presence of reflexivity:

$$\begin{tikzcd}
\mu \arrow[r, shift left, "\text{refl}"] \arrow[d] & \sigma \arrow[l, shift left, "\text{trans}"] \arrow[d] \\
\kappa \arrow[r, shift left, "\text{refl}"] & \lambda \arrow[l, shift left, "\text{trans}"]
\end{tikzcd}$$

3.4.2.9 Axioms necessary for conservation

The single-conclusion entailment relations that we studied above as instances of Universal Krull-Lindenbaum are arguably not the most ‘natural’. In the case of commutative rings (Section 3.4.2.4), for example, the axioms of ideal (3.5) are at first glance more natural before one adds the axiom of radical ideal (3.6); similarly, in the case of valuation rings (Section 3.4.2.6) the axioms of a subring (3.7) are perhaps more natural without the extra axiom of integral closure (3.8).

But in either case the seemingly more natural entailment relation $\rhd$ lacks the conservation criterion

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, a * b \rhd c}\ (\mathrm{E}_\mu)$$

with respect to the given operation $*$ (which in both cases is multiplication). In fact, the supposedly unnatural extra axioms are among those which shortly turn out necessary for E_μ, and thus for the additional axiom $a * b \vdash a, b$ being conservative (Corollary 3.5).

In the sequel we say that a single-conclusion entailment relation on an algebraic structure S is *natural* if it is generated by axioms which only on the right-hand side of $\rhd$ have terms built by means of the given operations on S. For instance, $a \rhd a * b$ is a natural axiom, but $a * b \rhd a$ is not natural.

We now show that in most of the instances of Universal Krull-Lindenbaum studied before the chosen single-conclusion entailment relation is in fact the least one that both contains its apparent 'natural core' and satisfies E_μ with respect to the given operation $*$.

Lemma 3.13. *Let $\rhd$ be a single-conclusion entailment relation on a set S.*

1. *If $*$ is a binary operation on S satisfying E_μ, then we have for all finite $U \subseteq S$ and $a, b \in S$:*
   a) Contraction
   $$a * a \rhd a$$
   b) Cut
   $$\frac{U \rhd a * b \quad U, b \rhd a}{U \rhd a}$$
2. *If $\sim$ is a unary operation on S which satisfies E_η, then we have* consequentia mirabilis*:*

$$\frac{U, \sim a \rhd a}{U \rhd a} \qquad \frac{U, a \rhd \sim a}{U \rhd \sim a} \tag{3.11}$$

Proof. With E_μ at hand it is straightforward to verify both contraction and cut:

$$\dfrac{\dfrac{a \in \{a\}}{a \rhd a}\,(\mathrm{R}) \quad \dfrac{a \in \{a\}}{a \rhd a}\,(\mathrm{R})}{a * a \rhd a}\,(\mathrm{E}_\mu) \qquad\qquad \dfrac{U \rhd a * b \quad \dfrac{\dfrac{a \in U, a}{U, a \rhd a} \quad U, b \rhd a}{U, a * b \rhd a}\,(\mathrm{E}_\mu)}{U \rhd a}\,(\mathrm{T})$$

By the appropriate instances of (R) again, (3.11) is an immediate consequence of E_η. □

In the cases of distributive lattices, ordered fields and weak order extensions (Sections 3.4.2.3, 3.4.2.5 and 3.4.2.8) the single-conclusion entailment relations under consideration are already natural and satisfy E_μ. In the following we thus may focus on Sections 3.4.2.4, 3.4.2.6 and 3.4.2.7.

Commutative rings Let S be a commutative ring, and let $\rhd$ be a single-conclusion entailment relation satisfying both the (natural) axioms of ideal (3.5) and E_μ with multiplication $\cdot$ in place of $*$. By Lemma 3.13, $\rhd$ also satisfies contraction, which in this case is the (unnatural) axiom of radical ideal (3.6).

Valuation rings Let R be a subring of a discrete field K, and let $\rhd$ be a single-conclusion entailment relation on K that satisfies E_μ with multiplication $\cdot$ in place of $*$ on top of the (natural) axioms (3.7) of subring of K containing R. By Lemma 3.13 we have, for all $a \in K^*$,

$$\frac{U, a^{-1} \rhd a}{U \rhd a}$$

(if you prefer to use cut, note that $\rhd 1$ and thus $U \rhd aa^{-1}$). In particular, if $a \in \overline{R[U]}$, which is to say that $a \in R[U, a^{-1}]$, then $U, a^{-1} \rhd a$ because $\rhd$ satisfies (3.7), and thus $U \rhd a$ by the above. Hence $\rhd$ also satisfies the axiom of integral closure (3.8).

Ordered vector spaces Let S be a $\mathbb{Q}$–vector space, and let $\rhd$ be a single-conclusion entailment relation satisfying the (natural) axioms (3.9) of additive submonoid as well as E_μ with addition $+$ in place of $*$. By Lemma 3.13, $\rhd$ satisfies contraction, which in this case reads $2a \rhd a$. By induction one can even get $na \rhd a$, which is to say that $\rhd$ also satisfies the (unnatural) axiom (3.10) of *positive cone*, in fact, E_μ applied to $a \rhd a$ and $na \rhd a$ yields $(n+1)a \rhd a$.

3.4.3 Strict order relations

Szpilrajn's theorem in its original form [242] is about *strict* partial orders: that is, *irreflexive* transitive relations R. This form, too, can be brought within the scope of our conservation criterion, as follows.

Let X be a set. Once more we work with the single-conclusion entailment relation $\rhd$ of transitive relation on $S = X \times X$ that is generated by transitivity τ only, and the closure operator of which assigns to a subset W of S its transitive closure W^+.

The single-conclusion entailment relation $\rhd'$ of *strict partial order*, extending $\rhd$, is generated by the only axiom τ of $\rhd$, and by the extra axioms

$$(c, c) \rhd' (a, b)$$

with $a, b, c \in X$. By Lemma 3.8.1 with $\mathscr{C} = \{ \{ (c, c) \} : c \in X \}$, the closure operator corresponding to $\rhd'$ can be described as follows:

Lemma 3.14. *Let U be a finite subset of S and let $(a, b) \in S$. The following are equivalent.*

1. $U \rhd' (a, b)$
2. $(a, b) \in U^+$ *or there is* $c \in X$ *such that* $(c, c) \in U^+$.

The multi-conclusion entailment relation $\vdash'$ of *strict linear order* extends $\rhd'$ with the additional axioms of *irreflexivity* ι and *restricted linearity* λ' as follows:

$$\iota : \quad (c, c) \vdash' \qquad \lambda' : \quad \vdash' (a, b), (b, a) \qquad [a \neq b]$$

The latter axiom is restricted to pairs of distinct elements, in order to not render $\vdash'$ trivial. Notice that irreflexivity ι is the special case $a = b$ of *asymmetry*

$$\nu : \quad (a, b), (b, a) \vdash' \quad ,$$

which in turn—as is well-known—follows by (T) from irreflexivity ι and transitivity τ.

By Lemma 3.8.2 with $\mathscr{C} = \{ \{ (c, c) \} : c \in X \}$ we have the conservation criterion for irreflexivity:

$$\frac{}{W, (c, c) \rhd' (a, b)}\ (\mathrm{E}_\iota)$$

Lemma 3.15. $\mathrm{E}_{\lambda'}$ *holds for the single-conclusion entailment relation $\rhd'$ of strict partial order.*

Proof. We have to prove

$$\frac{W, (a, b) \rhd' (c, d) \quad W, (b, a) \rhd' (c, d)}{W \rhd' (c, d)}\ (\mathrm{E}_{\lambda'})$$

where $a \neq b$, to which end we make use of Lemma 3.14. Writing

$$\Delta = \{ (x, x) : x \in X \}$$

for the diagonal in $S = X \times X$, suppose that

$$\Big[(c, d) \in \big(W, (a, b)\big)^+ \vee \big(W, (a, b)\big)^+ \between \Delta \Big] \wedge \Big[(c, d) \in \big(W, (b, a)\big)^+ \vee \big(W, (b, a)\big)^+ \between \Delta \Big]$$

again with $a \neq b$. Now the reader may check all possible combinations. For each case it is a simple matter of pasting chains in order to see that either $(c, d) \in W^+$ or $W^+ \between \Delta$, as required. □

By Corollary 3.5 the multi-conclusion entailment relation $\vdash'$ of strict *linear* order thus is a conservative extension of the single-conclusion entailment relation $\rhd'$ of strict *partial* order. This is a syntactical and constructive variant of Szpilrajn's order extension principle in its original form [242].

3.5 Semantics

Now we place ourselves within **ZFC**. As before let S be a set, and write $U, V, W, \ldots$ for finite subsets.

3.5.1 Lindenbaum's Lemma and completeness

According to [229], a (multi-conclusion) entailment relation $\vDash$ on S is *complete* if for each $a \in S$ either $\vDash a$ or $a \vDash$, and *consistent* if for no $a \in S$ both $\vDash a$ and $a \vDash$. Note that [229] if $\vDash$ is inconsistent, then $\vDash$ holds in the sense that $\emptyset \vDash \emptyset$. Conversely, if $\vDash$ holds, then $\vDash$ is inconsistent unless $S = \emptyset$.

The complete consistent entailment relations $\vDash$ are just the *valuations* on S, i.e. predicates $v \in 2^S$. More precisely, if $\vDash$ corresponds to v, then

$$U \vDash V \iff (\bigwedge_{b \in U} v(b) \Rightarrow \bigvee_{c \in V} v(c))\,. \tag{3.12}$$

The following [229] surely is one of the most general versions of *Lindenbaum's Lemma*:

Theorem 3.16. *Each entailment relation $\vdash$ on a set S equals the intersection of all complete consistent entailment relations $\vDash$ on S with $\vDash \supseteq \vdash$.*

For an arbitrary subset P of S we set

$$U \Vdash_P V \iff (P \supseteq U \Rightarrow V \between P)\,.$$

Now let $\vdash$ denote an arbitrary entailment relation. If $\vdash \subseteq \Vdash_P$, then P is said to be an *ideal element* [90] of S or a *model* [76] of $\vdash$, for short $P \in \mathrm{Mod}(\vdash)$. It is appropriate to say that $\vdash$ *has enough models* if

$$U \vdash V \iff \forall P \in \mathrm{Mod}(\vdash)(U \Vdash_P V) \tag{3.13}$$

for all finite subsets U and V of S. Clearly, only $\Leftarrow$ of (3.13) is crucial, as $\Rightarrow$ is the definition of a model.

As the valuations v on S are just the subsets P of S, in view of (3.12) we have the following:

Remark 3.17. The complete consistent entailment relations $\vDash$ (which contain $\vdash$) are precisely the relations of the form $\Vdash_P$ where P is a—not necessarily finite—subset of S (respectively, a model of $\vdash$).

Hence Lindenbaum's Lemma (Theorem 3.16) is tantamount to the following *Completeness Theorem* for entailment relations [90]:

Theorem 3.18. *Every entailment relation $\vdash$ on any set S has enough models.*

The models of the converse relation $\dashv$ are exactly the complements of the models of $\vdash$. For example, the prime filters of a distributive lattice or commutative ring are exactly the complements of the prime ideals.

Interpretations An *interpretation* $(S, \vdash) \to (S', \vdash')$ of entailment relations is a function $f : S \to S'$ that preserves entailment as follows [90]:

$$U \vdash V \implies f(U) \vdash' f(V).$$

If this implication always is an equivalence, then f is said to be a *conservative* interpretation [79]. The inverse image mapping induced by an interpretation restricts to a map of models

$$f^{-1} : \mathrm{Mod}(\vdash') \to \mathrm{Mod}(\vdash), \quad P \mapsto f^{-1}(P).$$

By completeness (Theorem 3.18), an interpretation f is conservative if and only if the induced mapping f^{-1} on models is surjective [79]. This provides other means, related to yet different from the ones used in this chapter, to address extension theorems as syntactical conservation results. Examples include the Hahn-Banach theorem [62, 72], measure theory on lattices [77], and Sikorski's and Monteiro's extension theorems for finite Boolean algebras [212, 213].

Constructive semantics A constructive semantics is possible, to be expressed in **CZF**.[14] If S is a bounded distributive lattice, then a natural choice of a (multi-conclusion) entailment relation is

$$U \vdash_{\leqslant} V \equiv \bigwedge U \leq \bigvee V$$

with (R) and (M) being automatic but (T) equivalent to distributivity [229]. In this case the models of $\vdash$ are nothing but the proper prime filters of S. In fact, one readily shows the following:

Remark 3.19. $\vdash_{\leqslant}$ equals the entailment relation $\vdash$ of proper prime filter generated by the axioms (3.3)+(3.4).

In terms of interpretations, the identity map on S is a conservative interpretation between $\vdash_{\leqslant}$ and $\vdash$.

Concerning the converse, the following seminal theorem [62, Theorem 3] now is already called 'fundamental theorem of entailment relations' [169, XI, Theorem 5.3]: [15]

Theorem 3.20. *For every entailment relation $\vdash$ on a set S there is a distributive lattice D together with a conservative interpretation $i : (S, \vdash) \to (D, \vdash_{\leqslant})$ such that if L is an arbitrary distributive lattice and $j : (S, \vdash) \to (L, \vdash_{\leqslant})$ is an interpretation, then there is a unique lattice homomorphism $f : D \to L$ such that $f \circ i = j$:*

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "j"'] & D \arrow[d, dashed, "f"] \\ & L \end{tikzcd}$$

This can also be seen as the constructive essence of Theorem 3.18. In fact, (3.13) follows in **ZFC** where every distributive lattice has enough prime filters by the adequate variant of Krull's Lemma.

3.5.2 Extension and conservation semantically

Back to **ZFC**, let $\rhd$ and $\vdash$ be a single-conclusion and a multi-conclusion entailment relation, respectively. The models of $\rhd$ are exactly the subsets of S which are closed under the associated algebraic closure operator (3.1). The corresponding counterpart of Theorem 3.18 is trivial, with $\rhd$ in place of $\vdash$ and for singleton V: the closure of U equals the intersection of the closed supersets of U.

The next statements, however, largely rely on completeness (Theorem 3.18).

Lemma 3.21.

1. *$\vdash$ is an extension of $\rhd$ if and only if, for every finite $U \subseteq S$,*

$$\bigcap \{ P \in \mathrm{Mod}(\vdash) : P \supseteq U \} \supseteq U^{\rhd} \, .$$

[14] For the question whether in **CZF** the models of a given entailment relation form at least a 'set-generated class' we refer to [10, 34, 144]; see also Chapter 4. When consulting this literature we suggest to observe that—as Hajime Ishihara has kindly pointed out to us—an entailment relation R on S is a specific sort of 'set of rules' on S; that the latter can be viewed as generating sets for the former; and that the models of R are exactly the subsets of S that are 'R–closed'. Roughly speaking, nondeterministic inductive definitions are to deterministic ones as multi-conclusion entailment relations are to single-conclusion ones.

[15] The anonymous referee of [209] has kindly indicated this name to us, as well as the fact that both this theorem and the analogous one for single-conclusion entailment relations, with semilattices in place of lattices, were already known to Lorenzen [170].

2. $\vdash$ is conservative over $\rhd$ if and only if, for every finite $U \subseteq S$,

$$\bigcap\{P \in \mathrm{Mod}(\vdash) : P \supseteq U\} \subseteq U^{\rhd} \,.$$

In fact, by Theorem 3.18 for singleton V we have that, for every finite $U \subseteq S$,

$$\bigcap\{P \in \mathrm{Mod}(\vdash) : P \supseteq U\} = U^{\rhd\vdash} \,.$$

Mind that in Lemma 3.21.2 the multi-conclusion entailment relation $\vdash$ need not be an extension of $\rhd$.

Lemma 3.22. *$\vdash$ is an extension of $\rhd$ if and only if every model of $\vdash$ is a model of $\rhd$.*

Proof. Suppose that $\vdash$ extends $\rhd$ and let $P \in \mathrm{Mod}(\vdash)$. If $U \rhd a$, then we have $U \vdash a$ by extension; whence if $U \subseteq P$, then $a \in P$. So $P \in \mathrm{Mod}(\rhd)$. Conversely, suppose that $\mathrm{Mod}(\vdash) \subseteq \mathrm{Mod}(\rhd)$ and let $U \rhd a$. In order to show $U \vdash a$, we use completeness (Theorem 3.18). To this end let $P \in \mathrm{Mod}(\vdash)$ with $U \subseteq P$. Since $P \in \mathrm{Mod}(\rhd)$ by hypothesis, we get $a \in P$ as required. □

A model I of $\rhd$ is *finitely generated* if there is a finite subset U of S such that $I = U^{\rhd}$. The following gives a semantic account of conservation.

Theorem 3.23. *If $\vdash$ extends $\rhd$, then the following are equivalent.*

1. *$\vdash$ is conservative over $\rhd$.*
2. *For every subset J of S and every $a \in S$, every* maximal *member of*

   $$\{\, I \in \mathrm{Mod}(\rhd) : J \subseteq I \,\wedge\, a \notin I \,\}$$

   is a model of $\vdash$.
3. *If I is a (finitely generated) model of $\rhd$, then*

   $$\bigcap\{P \in \mathrm{Mod}(\vdash) : P \supseteq I\} = I \,. \tag{3.14}$$

In fact, it suffices to assume that (3.14) hold only for the finitely generated models I of $\rhd$.

Proof. 1. $\Longrightarrow$ 2. Let $I \in \mathrm{Mod}(\rhd)$ be maximal such that $J \subseteq I$ and $a \notin I$. To see that I is a model of $\vdash$, let $U \vdash V$ and suppose that $U \subseteq I$ but $I \cap V = \emptyset$. Because of maximality, for every $b \in V$ there is a finite subset U_b of I such that $U_b, b \rhd a$. If $\vdash$ is conservative over $\rhd$, then By item 1, $\bigcup_{b \in V} U_b, U \rhd a$. We thus obtain $a \in I$, a contradiction.

2. $\Longrightarrow$ 3. Let $I \in \mathrm{Mod}(\rhd)$. It suffices to show that if $a \notin I$, then there is a model P of $\vdash$ such that $I \subseteq P$ and $a \notin P$. But if indeed $a \notin I$, then, by means of Zorn's Lemma, there is a model P of $\rhd$ which is maximal such that $I \subseteq P$ and $a \notin P$. By item 2, this P actually is a model of $\vdash$.

3. $\Longrightarrow$ 1. Let U be a finite subset of S. Since $\vdash$ extends $\rhd$, every model of $\vdash$ is a model of $\rhd$ (Lemma 3.22). Therefore, if $P \in \mathrm{Mod}(\vdash)$, then $U \subseteq P$ if and only if $U^{\rhd} \subseteq P$. By item 3 applied to $I = U^{\rhd}$ we get $\bigcap\{\, P \in \mathrm{Mod}(\vdash) : P \supseteq U \,\} = U^{\rhd}$; whence $\vdash$ is conservative over $\rhd$ by Lemma 3.21.2. □

More often than not—so already in the foregoing proof—the characterisation of conservation from Theorem 3.23.3 occurs in its contrapositive form, viz.

> *for every $I \in \mathrm{Mod}(\rhd)$ and $a \in S$, if $I \not\ni a$, then there is $P \in \mathrm{Mod}(\vdash)$ with $P \supseteq I$ and $P \not\ni a$.*

While this perfectly fits Zorn's Lemma, to arrive at the original form of Theorem 3.23.3 (and Theorem 3.16 and Theorem 3.18) it is perhaps more natural [66, 208, 222, 223] to use Raoult's Open Induction [201]; for the use of Open Induction in diverse contexts see [36, 73].

In the situation of Section 3.4.1.1, for example, Theorem 3.23.3 reads as

> *every quasi-order R on a set X equals the intersection of all linear quasi-orders on X that contain R,*

the contrapositive of which is as follows [129, 190, 242]:

> *every proper quasi-order R on a set X can be extended to a proper linear quasi-order on X .*

More generally, if $\vdash$ is to $\rhd$ as in Section 3.4.2.1, then Theorem 3.23.3 is the conclusion of the *Universal Krull–Lindenbaum Theorem* (UKL) [208, Theorem 14, Corollary 15]. This UKL has been abstracted from Krull's [157] and Lindenbaum's [244, p. 394] results in algebra and logic, respectively.

Encoding, the crucial hypothesis of UKL in [208], is the first conservation criterion E_μ from Section 3.4.2.1. We note in passing that E_μ is indeed necessary for UKL. The second conservation criterion E_π is irrelevant inasmuch as (3.14) remains invariant [208] whether or not the intersection is limited to proper subsets of $\vdash$; this limitation would be imposed by requiring that the models of $\vdash$ respect the axiom $\pi : e \vdash$, which is to say that they miss the convincing element e.

3.5.3 Yet another case: finitary formal topologies

Given a spectral topological space X with a basis of compact opens $(B_a)_{a\in S}$ closed under finite intersections, one can define a relation of consequence on the index set S by setting

$$U \rhd a \equiv B_a \subseteq \bigcup_{b\in U} B_b\,. \tag{3.15}$$

As X is spectral, both the set X and its topology can be regained from $\rhd$; see below.

A general abstract version of this kind of consequence, introduced as 'formal topology' [218–220], has proved practicable as a constructive and predicative substitute for the notion of topological space. The finitary case—see, for instance, [65, 67, 187, 188]—corresponding to coherent locales and spectral spaces can be described in terms of single-conclusion entailment relations as follows.

While for purely syntactical considerations we first work in **ECST**, for semantical purposes we later need classical logic and/or the Axiom of Choice, and thus move to **ZF** and **ZFC**, respectively.

Let $(S,\cdot,1)$ be a commutative monoid, and let $\rhd$ be a single-conclusion entailment relation on S. For the sake of an easier reading we usually denote the monoid operation $\cdot$ simply by juxtaposition.

Suppose that the structure

$$\mathscr{S} = (S,\rhd,\cdot,1)$$

be a *finitary formal topology*, that is, *weakening* (W), *contraction* (C) and *localisation* (L) hold:

$$\frac{U\rhd a}{U\rhd ab}\ (\mathrm{W}) \qquad \frac{U\rhd a^2}{U\rhd a}\ (\mathrm{C}) \qquad \frac{U\rhd a}{Ub\rhd ab}\ (\mathrm{L})$$

Note that (W) and (C) are tantamount to axioms; and that (L) is equivalent to *Stability* (S): that is,

$$a\rhd ab\ \ (\mathrm{W}) \qquad a^2\rhd a\ \ (\mathrm{C}) \qquad \frac{U\rhd a \quad V\rhd b}{UV\rhd ab}\ (\mathrm{S})$$

Let the multi-conclusion entailment relation $\vdash$ extend $\rhd$ with the additional axioms of *formal point*:

$$\mu:\quad ab\vdash a,b \qquad \pi:\quad 1\vdash\ .$$

This is yet another instance of Universal Krull (Section 3.4.2.1), with $\cdot$ as $*$ and 1 as e:

Lemma 3.24. *The conservation criteria* E_μ *and* E_π *hold for every finitary formal topology:*

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, ab \rhd c}\ (\mathrm{E}_\mu) \qquad \frac{}{W, 1 \rhd c}\ (\mathrm{E}_\pi)$$

Proof. As for E_μ, let $U, a \rhd c$ and $U, b \rhd c$. By (L) and (C),

$$Uc, ac \rhd c \quad \text{and} \quad Ua, ab \rhd ac\,.$$

So $Uc, Ua, ab \rhd c$ by (T), and finally $U, ab \rhd c$ by appropriate applications of (W), (T) and (M).
As for E_π, by (W) we have $1 \rhd c1 = c$, and thus $W, 1 \rhd c$ by (M). □

Hence we have conservation, by Corollary 3.9:

Proposition 3.25. $\vdash$ *is a conservative extension of* $\rhd$ *for every finitary formal topology.*

Given a finitary formal topology $\mathscr{S}$ as above one can retrieve a topological space structure. A *formal point* of $\mathscr{S}$ is a—not necessarily finite—subset α of S which *splits* $\rhd$ in the sense that

$$U \rhd a \implies (a \in \alpha \Rightarrow \alpha \between U) \tag{3.16}$$

and for which the following hold:

$$\frac{a \in \alpha \quad b \in \alpha}{ab \in \alpha} \qquad \frac{}{1 \in \alpha} \tag{3.17}$$

The collection $\mathscr{P}t(\mathscr{S})$ of formal points of $\mathscr{S}$ has a spectral topology with a basis $(B_a)_{a\in S}$ defined by

$$B_a = \{\alpha \in X : a \in \alpha\}\,,$$

which are compact opens and for which—by (W), (3.16) and (3.17)—we have

$$B_{ab} = B_a \cap B_b \quad \text{and} \quad B_1 = \mathscr{P}t(\mathscr{S})\,. \tag{3.18}$$

If X is a spectral space with a basis $(B_a)_{a\in S}$ of compact opens such that $\rhd$ is as in (3.15), and $\cdot$ and 1 satisfy (3.18), then $\mathscr{P}t(\mathscr{S})$ is homeomorphic to X.
A finitary formal topology $\mathscr{S}$ as above has *enough points* or is *spatial* if

$$\forall \alpha \in \mathscr{P}t(S)(a \in \alpha \Rightarrow \alpha \between U) \implies U \rhd a\,. \tag{3.19}$$

The converse of (3.19) is automatic by the part (3.16) of the definition of formal point.

Lemma 3.26 (ZF). *By complementation, the models of* $\vdash$ *correspond exactly to the formal points of* $\mathscr{S}$*.*

In particular, (3.19) is tantamount to

$$\forall P \in \mathrm{Mod}(\vdash)(P \supseteq U \Rightarrow a \in P) \implies U \rhd a\,. \tag{3.20}$$

If $\vdash$ has enough models, then (3.20) is equivalent to conservation (Lemma 3.21.2). Hence we have:

Proposition 3.27 (ZF). *If* $\vdash$ *has enough models, then* $\vdash$ *is conservative over* $\rhd$ *precisely when* $\mathscr{S}$ *has enough points.*

In all, by Theorem 3.18 and Proposition 3.25 we obtain the following [207]:

Theorem 3.28 (ZFC). *Every finitary formal topology* $\mathscr{S}$ *has enough points.*

Conservation of $\vdash$ over $\rhd$ thus is the syntactical underpinning of "coherent locales are spatial" [147].

Spatiality of the more comprehensive class of *locally Stone formal topologies* [187] can be proved in **ZFC** as well. In fact every finitary formal topology is locally Stone; whence what we recall by Theorem 3.28 is a special case of [187, Proposition 5.1].[16] [17]

[16] We are grateful to the anonymous referee of [210] for indicating this to us.

[17] With regard to the spatiality of *countably presented* formal topologies, we further refer to [246].

3.6 Limitation

3.6.1 A non-conservative extension: Suzumura consistency

The property of Suzumura consistency (see Chapter 9), which plays a crucial role in the mathematics of preference relations and social choice theory [48, 49, 61, 241], allows us to give a counterexample to conservation in **ECST**. A binary relation R on a set X is *(Suzumura) consistent*[18] if

$$R^+ \cap R^{-1} \subseteq R$$

where as before R^+ stands for the transitive closure of R, and

$$R^{-1} = \{ (y, x) : (x, y) \in R \}$$

denotes the inverse relation of R as usual. On $S = X \times X$ the single-conclusion entailment relation $\rhd$ of *consistent relation* is generated by the axioms

$$\gamma : \quad (x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n), (x_n, x_0) \rhd (x_0, x_n) \qquad [n \geqslant 0] \, .$$

The closure operator corresponding to $\rhd$ assigns to a subset R of S its *consistent closure* $\mathrm{cc}(R)$ defined as follows [46, 48, 226]:

$$\mathrm{cc}(R) = R^+ \cap (R \cup R^{-1}) \, .$$

The models of $\rhd$ are exactly the consistent relations.

Every transitive relation is consistent, but consistency is strictly weaker than transitivity. For instance, if a set X has two distinct elements x and y, then the relation $R = \{ (x, y), (y, x) \}$ on X is consistent, but fails to be transitive; in fact, this R lacks reflexivity and thus is non-linear. Yet every *linear* consistent relation R on any set X whatsoever is transitive. Indeed, if R is linear, then $R \cup R^{-1} = X \times X$; if, in addition, R is consistent, then R is transitive, because

$$R^+ = R^+ \cap (X \times X) = R^+ \cap (R \cup R^{-1}) = \mathrm{cc}(R) = R \, .$$

To rephrase this argument within the context of entailment relations, let the multi-conclusion entailment relation $\vdash$ of *linear consistent relation* extend $\rhd$ with the additional axiom of linearity

$$\lambda : \quad \vdash (a, b), (b, a) \, .$$

Unless X has at most one element, $\vdash$ is not conservative over $\rhd$. To see this observe first that

$$\tau_\vdash : \quad (a, b), (b, c) \vdash (a, c)$$

holds by (T) applied to the instances

$$\vdash (a, c), (c, a) \quad \text{and} \quad (a, b), (b, c), (c, a) \vdash (a, c)$$

of λ and γ, respectively. In particular, every model of $\vdash$ is a transitive relation.

If $\vdash$ were conservative over $\rhd$, then

$$\tau_\rhd : \quad (a, b), (b, c) \rhd (a, c).$$

would follow. Hence every model of $\rhd$—that is, every consistent relation on X—would be transitive, too. This, as we have seen above, is not the case whenever X has at least two distinct elements.

[18] In fact, as shown in Chapter 9, this is a classically equivalent, positive way of defining Suzumura consistency [226].

3.6.2 Calibrating Universal Krull-Lindenbaum

Placing ourselves in **ZF**, we now characterise the strength of the *Universal Krull–Lindenbaum Theorem* (UKL) [208, Theorem 14, Corollary 15]. We thus address an issue that was raised but remained unattended in [208]. First we need to rephrase UKL in terms of entailment relations:

UKL *Let S be a set with a binary operation $*$, and let $\rhd$ be a single-conclusion entailment relation on S. Let $\vdash$ be the multi-conclusion entailment relation extending $\rhd$ with the additional axiom*

$$\mu : \quad a * b \vdash a, b\,.$$

If μ is conservative over $\rhd$, then, for every model I of $\rhd$,

$$\bigcap \{\, P \in \mathrm{Mod}(\vdash) : P \supseteq I \,\} = I.$$

This UKL is an instance of semantical conservation (Theorem 3.23) . More generally one can put the following *Intersection Principle*, which shortly will turn out equivalent to UKL.

IP *Let S be a set with a single-conclusion and a multi-conclusion entailment relation $\rhd$ and $\vdash$, respectively. If $\vdash$ is a conservative extension of $\rhd$, then, for every model I of $\rhd$,*

$$\bigcap \{\, P \in \mathrm{Mod}(\vdash) : P \supseteq I \,\} = I\,.$$

When proving the items of Theorem 3.23 equivalent, for simplicity's sake we made use of Zorn's Lemma. However, a slightly refined argument shows that the *Completeness Theorem* for entailment relations (Theorem 3.18, for short CT) actually suffices—hence so does the well-known *Prime Ideal Theorem* for distributive lattices (PIT).

Proposition 3.29. *Over* **ZF***, the following are equivalent:* UKL, PIT, CT *and* IP.

Proof. Notice first that PIT is among the consequences of UKL [208], and that PIT proves CT [62, 90]. Since UKL is an instance of IP, we may concentrate on showing that CT implies IP.

To this end, let S be a set with a single-conclusion and a multi-conclusion entailment relation $\rhd$ and $\vdash$, respectively; and assume that the latter be a conservative extension of the former. Let I be a model of $\rhd$, and let the single-conclusion entailment relation $\rhd_I$ be defined as follows:

$$U \rhd_I a \;\equiv\; \exists U_0 \in \mathrm{Fin}(I)\; (\, U, U_0 \rhd a \,).$$

Because I is a model of $\rhd$, we have $a \in I$ if and only if $\rhd_I a$. Similarly, a multi-conclusion entailment relation $\vdash_I$ is readily defined by

$$U \vdash_I V \;\equiv\; \exists U_0 \in \mathrm{Fin}(I)\; (\, U, U_0 \vdash V \,).$$

Since $\vdash$ is a conservative extension of $\rhd$, precisely so is $\vdash_I$ over $\rhd_I$. Therefore, $a \in I$ if and only if $\vdash_I a$, and with completeness (applied to the entailment relation $\vdash_I$) we see that $a \in I$ is tantamount to $a \in \bigcap \mathrm{Mod}(\vdash_I)$. Finally, notice that $P \in \mathrm{Mod}(\vdash_I)$ if and only if both $P \in \mathrm{Mod}(\vdash)$ and $P \supseteq I$. $\square$

Chapter 4

Some aspects of entailment relations

4.1 Introduction

This chapter is meant to serve several purposes. To begin with, a certain amount of preliminaries might be helpful for the remaining chapters, if only to avoid repetition, and for future reference. Moreover, the occasion seems right to provide space for a few general and elementary aspects (some of which may undisputably rank as "folklore") of entailment relations which came about along with the overall content of this thesis.

We start with a review and explanation of Scott's notion of entailment relation in Section 4.2, and in Section 4.3 we discuss semantics, keeping an eye on limitations over **CZF**. For instance, we address some issues concerning the size of model classes, and we give a decomposition of the completeness theorem (CT) as REM together with a certain representation principle (Rep) for entailment relations. Section 4.4 is about a generalization of Scott's conservation criterion and presents results which in parts are due to Rinaldi. Section 4.5 concentrates on a first and thorough case study: paths of finitary branching trees as ideal elements. This exemplifies the line of reasoning in the remaining chapters. The final Section 4.6 discusses a pattern that underlies many a concrete instance.

4.2 Entailment

Let S be a set. An *entailment relation* [229–231] on S is a relation

$$\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$$

between finite subsets of S, which is *reflexive*, *monotone*, and *transitive* in the following sense, written in rule notation:

$$\frac{U \between V}{U \vdash V}\ (\mathrm{R}) \qquad \frac{U \vdash V}{U, U' \vdash V, V'}\ (\mathrm{M}) \qquad \frac{U \vdash V, a \quad U, a \vdash V}{U \vdash V}\ (\mathrm{T})$$

Transitivity (T) is an abstract form of Gentzen's *cut* rule for sequent calculus. In (R) the notation $U \between V$ means that $U \cap V$ is inhabited. We write U, V rather than $U \cup V$, as well as just a where it actually should read a singleton set $\{ a \}$. Moreover, it is common to use $U \vdash$ and $\vdash V$ as shorthands of $U \vdash \emptyset$ and $\emptyset \vdash V$, respectively. We refer to the elements of S as *abstract statements*, but take into account that S need not consist of syntactic objects in the proper sense, that is to say, formulae in the sense specified by a certain formal language. An intuitive explanation can best be given as a conditional assertion, i.e., to read

$$a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell$$

just as a Gentzen sequent: conjunctively for *antecedents* a_i, and disjunctively for *succedents* b_j. We refer to [234] for an historical account of the development of entailment relations; see also [209] and Chapter 3.

As has already been indicated in the preceding chapter, we employ rule notation to provide for a shorthand. In the tradition of formal topology, the conditions, "though written in the shape of rules, must be understood as requirements of validity: if the premises hold, also the conclusion must hold" [91, p. 77].

Example 4.1. Let X and S be sets. Let $\Vdash \subseteq X \times S$ be a relation.[1] There is a canonical entailment relation on S associated with $\Vdash$ which is defined as follows. First, for every $a \in S$ write

$$\mathrm{ext}(a) = \{ x \in X : x \Vdash a \} .$$

Then, if U and V are finite subsets of S, stipulate

$$U \vdash V \;\equiv\; \bigcap_{a \in U} \mathrm{ext}(a) \subseteq \bigcup_{b \in V} \mathrm{ext}(b).$$

Simple set-theoretic reasoning shows that this $\vdash$ satisfies reflexivity, monotonicity, and transitivity, hence is an entailment relation. Of course, it is possible to define an entailment relation on X in a dual manner, as well. Suppose now that X is a subset of $\mathrm{Pow}(S)$ and define $\Vdash \subseteq X \times S$ by

$$x \Vdash a \;\equiv\; a \in x.$$

As for the entailment relation which is associated with $\Vdash$ as above, we have, for all finite subsets U and V of S,

$$U \vdash V \qquad \text{if and only if} \qquad \forall x \in X \,(U \subseteq x \to x \between V).$$

As a consequence of the *Completeness Theorem* (CT) for entailment relations, in fact *every* entailment relation on a set S arises in this way from a *set* X of subsets of S. This, however, in general requires classical reasoning.

If an entailment relation $\vdash$ is such that $U \vdash V$ for *every* pair of finite subsets U and V of S, then we say that $\vdash$ *collapses*, which notion is in the tradition of dynamical algebra [92]. In presence of monotonicity (M), an entailment relation $\vdash$ collapses if and only if $\emptyset \vdash \emptyset$. For yet another characterization, consider the following.

Lemma 4.2. *Let $\vdash$ be an entailment relation on a set S. The following are equivalent.*

1. *$\vdash$ collapses.*

2. *There is a finite subset U of S such that $A \vdash B$ whenever A and B are finite subsets of S with $A \cup B = U$.*

Proof. One implication is apparent. As for the other, write $U = \{ a_1, \dots, a_n \}$ and argue by induction on n. If U is empty, then $\vdash$ collapses by assumption on U. Next suppose that $U = \{ a_1, \dots, a_n, a_{n+1} \}$ has the propertion in question. Let A and B be finite subsets of S such that $A \cup B = \{ a_1, \dots, a_n \}$. According to the assumption, we have both $A \vdash B, a_{n+1}$ and $A, a_{n+1} \vdash B$, and we obtain $A \vdash B$ by cut. Thus we see that $U' = \{ a_1, \dots, a_n \}$ is such that $A \vdash B$ whenever $A \cup B = U'$. The inductive hypothesis now implies that $\vdash$ collapses. ☐

Remark 4.3. Nothing prevents us from considering entailment relations on an empty domain of discourse, i.e., on $S = \emptyset$. For instance, as pointed out in [234], there are two distinct entailment relations $\vdash_0 = \emptyset$ and $\vdash_1 = \{ \emptyset \vdash_1 \emptyset \}$ on $\emptyset$, the latter of which by definition collapses, while the former does not. But to say that these are all entailment relations on $\emptyset$ necessitates REM. See also Remark 4.19.

[1] This choice of notation is motivated by Sambin's *Basic Pairs* [219].

If there is at least one pair of finite subsets U and V of S for which $U \vdash V$ does *not* hold, briefly $U \nvdash V$, then we say that $\vdash$ is *consistent*. If U is a finite subset of S for which $U \vdash$, then U is *inconsistent*. We say that U is *consistent* if $U \nvdash$. Mind that if U is *not* consistent, then we cannot in general conclude that U is inconsistent, because this would necessitate REM (see Remark 4.19). To understand and characterize an *inductively generated* entailment relation in non-inductive terms, often it suffices to explain inconsistency.

Usually we start with a set $A \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$ of *initial entailments* or *axioms*, meant to generate an entailment relation $\vdash$ on S. Since $\mathrm{Pow}(S) \times \mathrm{Pow}(S)$ is an entailment relation, and so is in turn every intersection of entailment relations, we may then consider the least one to contain A. But in order to avoid issues of impredicativity, instead of taking up the aforementioned static point of view, we consider $\vdash$ instead to be an *inductive definition* [1] generated by A along with the rules for reflexivity, monotonicity, and transitivity.[2] We have then at hand a method of *proof by induction* for generated entailment relations. In order to verify that the entailment relation generated by a given set of axioms is subject to a certain property, we may concentrate on showing this property to hold for every axiom, and that it persists under applications of reflexivity, monotonicity, and transitivity. This strategy, which will prove useful throughout this text, shows the property in question to actually define an entailment relation which thus contains the one that is generated by the given set of axioms.

Needless to say, similar observations apply in an analogous manner to inductively generated *single-conclusion* entailment relations.

Example 4.4. Let R be a commutative ring with 1. Take $S = R$ for our domain of discourse and consider the entailment $\vdash$ on S inductively generated by all instances of the following axioms:

$$\begin{aligned} &\vdash 0 \\ a &\vdash ab \\ a, b &\vdash a + b \\ 1 &\vdash \\ ab &\vdash a, b \end{aligned}$$

This is the entailment relation of *proper prime ideal* of R. Let U and V be finite subsets of S. It can be shown [62] that $U \vdash V$ holds if and only if the ideal $\langle U \rangle$ that is generated by U meets the multiplicative monoid generated by V. It follows that $\vdash$ being consistent amounts to R being a non-trivial ring. Furthermore, the direct description of $\vdash$ shows that, for every finite subset U of S, we have $U \vdash$ if and only if $\langle U \rangle = R$.

Remark 4.5. Since $\mathrm{Fin}(S)$ is a set in **CZF**, so is every entailment relation which is generated by a set of axioms. This is not the case for **ECST**, upon which a further axiom needs to be imposed for $\mathrm{Fin}(S)$ to be a set. Mind that every entailment relation contains the overlap relation between finite subsets U and V of the underlying set. Over **ECST**, this relation is a proper class.

Next we consider semantics of entailment relations.

[2] It is not necessary to go beyond **CZF** in order to provide for an inductively generated $\vdash$ to be a set, carrying out, e.g., the approach with *construction sequences* [196]. A very powerful characterization for inductively generated entailment relations results from the *fundamental theorem* of entailment relations [62, 169], see Section 4.3.5, which shows the canonical entailment relation on a distributive lattice to be typical.

4.3 Spectra

4.3.1 Ideal elements

Let $\vdash$ be an entailment relation on a set S. An *ideal element* (or *model*, *point*)[3] [62, 90] of $\vdash$ is a subset $\alpha \subseteq S$ which *splits* entailments, i.e., in rule notation,

$$\frac{\alpha \supseteq U \qquad U \vdash V}{\alpha \between V}$$

If $\vdash$ is an inductively generated entailment relation, then for a subset α of S to be an ideal element of $\vdash$ it suffices that α split every axiom: consider the entailment relation $\vDash$ as described in Example 4.1 with $X = \{\alpha\}$, i.e., for every pair of finite subsets U and V of S, put

$$U \vDash V \;\equiv\; U \subseteq \alpha \;\rightarrow\; \alpha \between V.$$

If α splits every initial entailment of $\vdash$, then $\vdash$ is contained in $\vDash$ by minimality, thus α splits every entailment, hence is an ideal element. We write

$$\mathfrak{Spec}(\vdash)$$

for the collection of all ideal elements of $\vdash$, and say that $\mathfrak{Spec}(\vdash)$ is the *spectrum* of $\vdash$. Rather than forming sets, spectra may well be proper classes in **CZF**. For instance, if $\vdash$ is defined by

$$U \vdash V \;\equiv\; U \between V.$$

then $\mathfrak{Spec}(\vdash) = \mathrm{Pow}(S)$. In other words, *every* subset of S is ideal for overlap. Notwithstanding such minor ontological shortcomings over **CZF**, model classes at least are *predicative*, i.e., defined by bounded formulae (but we need the entailment relation at hand to be a set). Moreover, it is easy to see that $\mathfrak{Spec}(\vdash)$ is directed-complete, i.e., if $\mathscr{D}$ is a directed set of ideal elements of $\vdash$, then $\bigcup \mathscr{D}$ is a model as well, due to the fact that we are dealing with finite sets of antecedents only. The situation can further be remedied by providing for $\mathfrak{Spec}(\vdash)$ to be a *set-generated* dcpo (cf. Chapter 1). As a consequence, suitable forms of the Kuratowski-Zorn Lemma over **CZF** ensure that models can be maximally extended.

The following is a general lemma that will be needed later on. It is but a slight generalization of [62, Lemma 2].

Lemma 4.6. *Let S be a set with entailment relation $\vdash$. Let A and B be arbitrary subsets of S. Generate an entailment relation $\vdash'$ on S by putting further axioms*

$$\vdash' a \qquad (a \in A)$$

$$b \vdash' \qquad (b \in B)$$

on top of $\vdash$. If U and V are finite subsets of S, then

$$U \vdash' V \qquad \text{if and only if} \qquad \exists W_0 \in \mathrm{Fin}(A)\, \exists W_1 \in \mathrm{Fin}(B)\, \big(U, W_0 \vdash V, W_1\big).$$

Moreover, for every subset α of S, the following are equivalent:

1. $\alpha \in \mathfrak{Spec}(\vdash')$
2. $\alpha \in \mathfrak{Spec}(\vdash)$ *and* $A \subseteq \alpha \subseteq S \setminus B$.

Proof. The first claim is (almost) apparent. It can be verified by a simple inductive argument, and with a series of cuts. The second claim can be read off directly from the additional axioms. □

[3]The term 'ideal element' refers back to the informal discussion in the preface; 'model' of course is logical terminology; and 'point' is topologically motivated, since every entailment relation gives rise to a spectral space by means of the fundamental theorem, cf. Section 4.3.5. More dynamically, sometimes we even write that α 'is ideal for' or 'models' an axiom.

4.3.2 Set generation and maximality

With our focus on the syntactical point of view of entailment relations, we need not really be bothered by the fact that spectra tend to form proper classes in **CZF**. It should further be taken into account that whenever we draw semantical consequences from observations on mere syntax, to a large extent classical reasoning is necessary, and this causes a move to classical set theory within which spectra clearly are sets, anyway. However, even in **CZF**, as far as semantics is concerned, one may be interested in those models of an entailment relation that are maximal elements of their corresponding spectra. Of course, to be able to generally postulate the existence of maximal ideals we require a principle like KZL. In order to remain true to our forms and phrasings of maximality and extension principles as have been pointed out in Chapter 1, we need to ensure that spectra are set-generated dcpo's. The principle in question is the following.

Set generation for entailment relations (ER). Let S be a set with entailment relation $\vdash$. The class $\mathfrak{Spec}(\vdash)$ of ideal elements of $\vdash$ is a set-generated dcpo.

We do not intend to fully calibrate the strength of this principle. It is a set existence axiom and can perhaps be compared to *finitary* NID, which in turn postulates that the class of closed subsets of a finitary non-deterministic inductive definition is a set-generated class [34, 144].[4] We now make use of an abstract result of Aczel [2]. Let S and S' be sets and consider a class function

$$\Gamma : \mathrm{Pow}(S) \to \mathrm{Pow}(S').$$

This operator Γ is *monotone* if, for all sets $Y, Y' \in \mathrm{Pow}(S)$,

$$Y \subseteq Y' \quad \text{implies} \quad \Gamma(Y) \subseteq \Gamma(Y'),$$

and *finitary* if, for every $Y \in \mathrm{Pow}(S)$ and every $a \in \Gamma(Y)$ there is a *finite* subset U of Y such that $a \in \Gamma(U)$. Suppose that we further have an assignment

$$R : S' \to \mathrm{Pow}(S).$$

[4]Let S be a set. A *rule* on S is a pair (a, b) of subsets a and b of S. A rule is said to be *nullary* if a is empty, *elementary* if a is a singleton, and *finitary* if a is finite. A subset α of S is *closed under* the rule (a, b) if

$$a \subseteq \alpha \;\to\; \alpha \between b.$$

If $\mathscr{R}$ is a set of rules on S, then a subset α of S is *$\mathscr{R}$-closed* if it is closed under each rule of $\mathscr{R}$. Here is the principle of *non-deterministic inductive definitions* [248]:

NID. For any set S and any set $\mathscr{R}$ of rules on S, the class $\mathrm{Clos}_{\mathscr{R}}(S)$ is set-generated.

If the rules in $\mathscr{R}$ are required to be nullary, elementary, or finitary, then the corresponding principles are referred to as nullary NID, elementary NID, and finitary NID, respectively. Nullary NID is equivalent to Fullness [144]. Finitary NID is equivalent to SGA [248]. If $\mathscr{R}$ is set of rules, then $\mathrm{Clos}_{\mathscr{R}}(S)$ may be empty. For instance, if $x \in S$ and $\mathscr{R} = \{ (\emptyset, \{ x \}), (\{ x \}, \emptyset) \}$. In general, the question as to whether $\mathscr{R}$ allows for S to have an $\mathscr{R}$-closed subset in general can only be answered affirmatively invoking a suitable choice principle. The full Axiom of Choice (AC) can be expressed in terms of non-deterministic inductive definitions. For instance, the following items are mere reformulations of each other:

1. $\forall a \, \forall b \, [\, \forall x \in a \, \exists y \in b \, \varphi(x, y) \;\to\; \exists f : a \to b \, \forall x \in a \, \varphi(x, f(x)) \,]$, where φ is a bounded predicate.
2. Let φ be a bounded predicate, let a and b be sets such that $\forall x \in a \, \exists y \in b \, \varphi(x, y)$. Let

   $$S = \{ (x, y) \in a \times b : \varphi(x, y) \}$$

   and consider the following sets of rules

   $$\mathscr{R}_\mathrm{s} = \{ (\{ (x, y), (x, y') \}, \emptyset) : \varphi(x, y) \wedge \varphi(x, y') \wedge y \neq y' \}$$

   and

   $$\mathscr{R}_\mathrm{t} = \{ (\emptyset, \{ (x, y) \in \{ x \} \times b : \varphi(x, y) \}) : x \in a \}$$

   on S. Let $\mathscr{R} = \mathscr{R}_\mathrm{s} \cup \mathscr{R}_\mathrm{t}$. Then $\mathrm{Clos}_{\mathscr{R}}(S)$ is inhabited.

In fact, any $\mathscr{R}$-closed subset of S is a choice function. If $\{ y \in b : \varphi(x, y) \}$ is a *finite* set for every $x \in a$, then we are in the scope of entailment relations, and $\mathscr{R}$ can be considered a set of generating axioms for an entailment relation $\vdash$ the ideal elements of which precisely are the choice functions sought after. See Chapter 5.

With this information at hand, a subset α of S is said to be Γ, R*-closed* if

$$\forall x \in \Gamma(\alpha)\, \alpha \between R_x.$$

We write $\mathfrak{Clos}(\Gamma, R)$ for the class of all Γ, R-closed subsets of S.

The following is taken from [2, Theorem 37].

Theorem 4.7 ($\mathsf{CZF} + \mathsf{uREA} + \mathsf{DC}$)**.** *Given Γ and R as above, if Γ is monotone and finitary, then $\mathfrak{Clos}(\Gamma, R)$ is a set-generated dcpo.*

Corollary 4.8 ($\mathsf{CZF} + \mathsf{uREA} + \mathsf{DC}$)**.** *If S is a set with entailment relation $\vdash$, then the class $\mathfrak{Spec}(\vdash)$ of ideal elements of $\vdash$ is a set-generated dcpo.*

Proof. In order to apply Theorem 4.7, we need to define a suitable monotone and finitary operator Γ together with an assignment R as above, and in such a manner that the class of ideal elements of $\vdash$ coincides with the class of Γ, R-closed sets. To this end, take the given set S and, for every subset Y of S, define $\Gamma(Y)$ to be the set of all entailments $U \vdash V$ the antecedents of which belong to Y. To be precise, let

$$S' = \mathrm{Fin}(S) \times \mathrm{Fin}(S)$$

and

$$\Gamma : \mathrm{Pow}(S) \to \mathrm{Pow}(\vdash), \quad \Gamma(Y) = \{\, (U, V) \in S' : U \vdash V \,\wedge\, U \subseteq Y \,\}.$$

This Γ clearly is monotone, and it is finitary by way of definition: if $(U, V) \in \Gamma(Y)$, then U is a finite subset of Y for which $(U, V) \in \Gamma(U)$. In addition to that, let R be the second projection, i.e.,

$$R : S' \to \mathrm{Pow}(S), \quad R_{(U,V)} = V.$$

It is straightforward to show that a subset α of S is Γ, R-closed if and only if it is an ideal element of $\vdash$. According to Theorem 4.7, it follows that $\mathfrak{Spec}(\vdash)$ is a set-generated dcpo, $\square$

Remark 4.9. Suppose that G generates $\mathfrak{Spec}(\vdash)$, let U and V be finite subsets of S, and let α be an ideal element. In order to show that α splits (U, V), i.e., that if $U \subseteq \alpha$, then $\alpha \between V$, we could resort to generating elements. That is, if every $\gamma \in G_\alpha$ splits (U, V), then so does α. In fact, since $\alpha = \cup G_\alpha$ is a directed union of ideal elements, if $U \subseteq \alpha$, then there is $\gamma \in G_\alpha$ with $U \subseteq \gamma$. Now, if $\gamma \between V$, then we obtain $\alpha \between V$.[5]

We have the following maximality principle by way of KZL and Aczel's abstract result over $\mathsf{CZF} + \mathsf{uREA} + \mathsf{DC}$.

Maximal ideal principle (Max)**.** Let S be a set with entailment relation $\vdash$. Every ideal element of $\vdash$ is contained in a maximal ideal element.

Corollary 4.10 ($\mathsf{CZF} + \mathsf{uREA} + \mathsf{DC}$)**.** KZL *implies* Max.

Proof. If $\vdash$ has an ideal element, say α, then $\mathfrak{Spec}(\vdash_\alpha)$ is an inhabited set-generated dcpo, according to Corollary 4.8, where $\vdash_\alpha$ is defined as follows:

$$U \vdash_\alpha V \;\equiv\; \exists W \in \mathrm{Fin}(\alpha)\,(\, U, W \vdash V\,).$$

Then, by means of KZL (cf. Chapter 1), this $\vdash_\alpha$ has a maximal ideal element β which is ideal for $\vdash$ and contains α (see Lemma 4.6). $\square$

[5] We are grateful to Takako Nemoto for having pointed this out to us.

Max generalizes a variety of concrete algebraic maximality principles. For instance, and to mention only two, both the maximal ideal theorem for distributive lattices as well as the maximal ideal theorem for commutative rings follow from Max with MEL, and both these principles are well-known to be classically equivalent to AC [137]. But keep in mind that KZL is "constructively neutral" [33], hence so is Max.

Recall that a partially ordered class $\mathscr{E}$ is *flat* [2, 7] if

$$\forall x, y \in \mathscr{E}\,(\,x \leqslant y \;\rightarrow\; x = y\,).$$

In other words, a partially ordered class $\mathscr{E}$ is flat if and only if every member of $\mathscr{E}$ is maximal. Odd as it may seem, most of the entailment relations that will be considered later on do have flat spectra.

Corollary 4.11. ER *implies that every flat spectrum is a set.*

Proof. We adapt and make use of an argument put forward in [2, 7]. Since it is set-generated, the dcpo $\mathfrak{Spec}(\vdash)$ of ideal elements has a subset G of generators, all of which are ideal elements of $\vdash$. In particular, if $\alpha \in \mathfrak{Spec}(\vdash)$, then there is $g \in G$ with $g \subseteq \alpha$, whence $g = \alpha$ since $\mathfrak{Spec}(\vdash)$ is flat. To put it briefly, the set G coincides with the class of ideal elements of $\vdash$. □

With a similar argument it can be shown that if $\vdash$ is an entailment relation on a set S, then the class $\mathfrak{Min}(\vdash)$ consisting of all *minimal* points is a set, provided that $\mathfrak{Spec}(\vdash)$ is set-generated.[6] For if G generates the spectrum and α is an ideal element, then there is $g \in G$ such that $g \subseteq \alpha$, whence $g = \alpha$ in case α is minimal. Therefore, $\mathfrak{Min}(\vdash) \subseteq G$ is a set by bounded separation. For instance, under ER the class of minimal prime ideals of a commutative ring is a set; cf. [10, Corollary 7.3].

Example 4.12. In constructive algebra [82, 169, 256] a commutative ring R has *Krull dimension* $\leqslant \ell - 1$ if and only if, for any sequence $x_1, \dots, x_\ell$ of elements of R, there are $a_1, \dots, a_\ell \in R$ and $m_1, \dots, m_\ell \in \mathbb{N}$ such that

$$x_1^{m_1}(\cdots(x_\ell^{m_\ell}(1 + a_\ell x_\ell) + \cdots) + a_1 x_1) = 0.$$

This makes a dimension theory with an "explicit computational content in the form of existence of some algebraic identities" possible [82]. If R has Krull dimension $\leqslant 0$, then for all $x \in R$ there are $a \in R$ and $m \in \mathbb{N}$ such that

$$x^m(1 + ax) = 0. \tag{4.1}$$

Consider the entailment relation $\vdash$ of proper prime ideal of R. With a simple argument it follows at once that if R has Krull dimension $\leqslant 0$, then $\mathfrak{Spec}(\vdash)$ is flat.

In order for the class of proper prime ideals of a ring with Krull dimension $\leqslant 0$ to be a set, the above discussion actually need not be carried out over **CZF** + uREA + DC. Example 4.12 builds on a constructive notion of Krull dimension for distributive lattices [82]. It is possible to explain a notion of dimension for an arbitrary entailment relation by considering the dimension of the lattice that it generates (see below for the *fundamental theorem* of entailment relations). For instance, a distributive lattice L has Krull dimension $\leqslant 0$ if and only if L is a Boolean algebra [82]. Recall that for a distributive lattice L there is a correspondence between lattice maps $L \to \mathbf{2}$ and detachable prime filters. If L is Boolean, in fact every prime filter of L is detachable, in which case the class of prime filters of L is a set. This observation can be applied in order to show already that by strengthening **CZF** with the principle of completeness for entailment relations we obtain REM.

The situation is considerably less difficult for single-conclusion entailment relations than it is for their multi-conclusion counterparts. In particular, set existence axioms beyond **CZF** are

[6] We are grateful to Hajime Ishihara for having pointed this out to us.

not required, and without further ado it can be shown that if S is a set with single-conclusion entailment relation $\rhd$, then the class of all ideals of $\rhd$ is a set-generated dcpo. In fact, a generating set G is given by the collection of all *finitely generated* ideals of $\rhd$, i.e., put

$$G = \{ U^{\rhd} : U \in \mathrm{Fin}(S) \},$$

which is a set by replacement. If I is an ideal of $\rhd$ and U is a finite subset of I, then the saturation $U^{\rhd}$ is an ideal of $\rhd$ as well, and we have $U \subseteq U^{\rhd} \subseteq I$.[7]

4.3.3 Completeness and model existence

For elementary and syntactical results about entailment relations to be considered constructive versions of certain classical theorems, we require a principle that shows the equivalence of both approaches over an accordingly strengthened set theory. The following makes this possible and goes back to Scott [229], see also [62, 90].

Completeness Theorem (CT) Let S be a set with entailment relation $\vdash$. For every pair of finite subsets U and V of S, the following are equivalent.

1. $U \vdash V$
2. $\forall \alpha \in \mathfrak{Spec}(\vdash)\, (U \subseteq \alpha \rightarrow \alpha \between V)$

The implication from the first to the second item is *soundness* and nothing but a definitional consequence. The converse is inherently non-constructive, it is a form of the Axiom of Choice, classically equivalent to the prime ideal theorem for distributive lattices, and it gives rise to the restricted principle of excluded third (REM). From a constructive point of view, it may still be used for heuristics.

Lemma 4.13. *Let B be a Boolean algebra, and let $\vdash$ be the entailment relation of proper prime filter of B. The class $\mathfrak{Spec}(\vdash)$ is a set.*

Proof. There is a bijective correspondence between detachable proper prime filters of B and lattice homomorphisms $B \to \mathbf{2}$. The latter form a set by way of exponentiation and bounded separation. In a Boolean algebra, every proper prime filter is detachable. ☐

Proposition 4.14. CT *implies that every bounded formula is equivalent to a negated bounded formula.*

Proof. The main line of reasoning follows [32, p. 161 sq.]. Let φ be a bounded formula and consider

$$B_\varphi = \{ \{ a \in 1 : \varphi \}, 1 \},$$

where $1 = \{ 0 \}$ as usual. This B_φ is a set by bounded separation and pairing. We equip B_φ with a Boolean algebra structure as follows. The top element of B_φ is 1, and the bottom element is $\{ a \in 1 : \varphi \}$. Meets are intersections, joins are unions, and the complement of $x \in B_\varphi$ is

$$-x = \{ a \in 1 : a \in x \rightarrow \varphi \}.$$

Note that φ is equivalent to $\{ a \in 1 : \varphi \} = 1$, which in turn amounts to saying that B_φ is trivial.

Next we consider the entailment relation $\vdash$ of proper prime filter on B_φ. By Lemma 4.13, the class $\mathfrak{Spec}(\vdash)$ is a set. For every $x \in B_\varphi$ we have

$$\vdash x \qquad \text{if and only if} \qquad x = 1.$$

[7] This in fact shows the collection of ideals of a single-conclusion entailment relation to be a *strongly set-generated* class in the sense of [144].

By CT, this reads

$$\bigcap \mathfrak{Spec}(\vdash) = \{ 1 \} \tag{4.2}$$

and thus yields that

$$B_\varphi \text{ is trivial} \qquad \text{if and only if} \qquad \mathfrak{Spec}(\vdash) = \emptyset\,.$$

As for the latter, if B_φ is trivial, then it has no proper (prime) filter. Conversely, if $\mathfrak{Spec}(\vdash) = \emptyset$, then

$$B_\varphi = \{ 1 \}$$

by (4.2). In all, φ is equivalent to the negated bounded formula

$$\mathfrak{Spec}(\vdash) = \emptyset. \qquad \square$$

As Brouwer has observed [54, p. 253], for every formula ψ the implication

$$\neg\neg\neg\psi \to \neg\psi$$

holds in intuitionistic logic. In view of this and Proposition 4.14, CT implies

$$\neg\neg\varphi \to \varphi$$

for every bounded formula φ, which amounts to REM by Brouwer's other observation [18, 53] that

$$\neg\neg(\psi \vee \neg\psi)$$

holds for every formula ψ in intuitionistic logic.[8]

Corollary 4.15. CT *implies* REM*.*

Next we can justify our remark from Example 4.1 that every entailment relation on a set S arises from a certain *set* of subsets of S.

Representation principle (Rep). Let S be a set with entailment relation $\vdash$. There is a subset X of $\mathrm{Pow}(S)$ such that, for every pair of finite subsets U and V of S,

$$U \vdash V \qquad \text{if and only if} \qquad \forall x \in X\,(\,U \subseteq x \;\to\; x \between V\,). \tag{ρ}$$

Proposition 4.16. *The following are equivalent.*

1. CT
2. $\mathsf{Rep} + \mathsf{REM}$

Proof. CT implies REM (Corollary 4.15) which in turn implies Pow in presence of Exp [7, Proposition 10.1.1]. It follows that $\mathfrak{Spec}(\vdash)$ is a set and we can take $X = \mathfrak{Spec}(\vdash)$ in order to obtain Rep. As for the converse, notice first that if a certain set X represents $\vdash$, then $X \subseteq \mathfrak{Spec}(\vdash)$. In fact, if $U \subseteq x \in X$ and $U \vdash V$, then $x \between V$ is immediate from (ρ). Working classically with REM, we may now concentrate on the contrapositive of CT. Thus, let U and V be finite subsets of S and suppose that $U \nvdash V$. With (ρ) it follows that there is $x \in X$ such that $U \subseteq x$ and $x \cap V = \emptyset$. As we have noticed before, this x actually is an ideal element, i.e., $x \in \mathfrak{Spec}(\vdash)$. $\square$

In the proof of Proposition 4.16 we have already hinted at the following.

Model existence lemma (MEL). Let S be a set with entailment relation $\vdash$. If $\vdash$ is consistent, then $\mathfrak{Spec}(\vdash)$ is inhabited.

[8] We refer to [245] for the early history of intuitionistic logic, and to [250] for a study of the interplay between *tertium non datur*, *ex falso quodlibet* and *reductio ad absurdum*.

In comparison to CT, model existence for entailment relations does of course not fare better from a constructive point of view. For instance, it gives rise to the principle of choice for families of finite sets, see Chapter 5. However, if the underlying set of finite inconsistent subsets of S is detachable, then MEL can be proven in **CZF** at least in case the underlying set S is countable. We hasten to add that this is just a simple variant of Lindenbaum's Lemma for propositional logic, asserting that each consistent set of propositional formulae is contained in a maximally consistent set. See, e.g., [96, Lemma 1.5.7],

Proposition 4.17. *Let S be a countable set with* consistent *entailment relation $\vdash$. If the set of finite inconsistent subsets of S is detachable, then $\vdash$ has an ideal element.*

Proof. We can write $S = \{ a_0, a_1, a_2, \dots \}$ and define a chain of finite subsets of S inductively as follows. Put $S_0 = \emptyset$ and let

$$S_{i+1} = \begin{cases} S_i, & \text{if } S_i, a_i \vdash \\ S_i \cup \{ a_i \}, & \text{if } S_i, a_i \nvdash \end{cases}$$

for $i \geqslant 0$. Take into account that all of these sets S_i are defined to be consistent. We write

$$\alpha = \bigcup_{i \in \mathbb{N}} S_i.$$

Note that every finite subset of α is consistent. Moreover, for every $a \in S$ either we have $a \in \alpha$ or else there is a finite subset U of α such that $U, a \vdash$. Indeed, for if $a \in S$ then we can write $a = a_{i_0}$ with a certain $i_0 \in \mathbb{N}$. Now, since the set of finite inconsistent subsets of S is supposed to be detachable, either we have $S_{i_0}, a \vdash$ right away, or else $S_{i_0}, a \nvdash$ in which case by definition $S_{i_0+1} = S_{i_0} \cup \{ a \}$ and thus $a \in \alpha$. Our next claim is

$$\alpha \in \mathfrak{Spec}(\vdash).$$

In fact, let U and V be finite subsets of S and suppose that $U \vdash V$ and $U \subseteq \alpha$. According to the preceding observation, for every $b \in V$ either we have $b \in \alpha$, whence $\alpha \between V$ anyway, or else there is a finite subset U_b of α such that $U_b, b \vdash$. Therefore, it suffices to rule out that the latter disjunct holds for every $b \in V$. To this end, suppose that for every $b \in V$ there is a finite subset U_b of α with $U_b, b \vdash$. Considering the union $U' = \bigcup_{b \in V} U_b$, for every $b \in V$ we have $U', b \vdash$. We can successively cut $U \vdash V$ and this yields $U, U' \vdash$. However, this would be to say that α contains the finite inconsistent subset $U \cup U'$, yet any finite subset of α is consistent, as was pointed out before. ☐

Example 4.18. Let R be a commutative ring. Recall from Example 4.4 the entailment relation $\vdash$ of proper prime ideal of R. If R is non-trivial and countable and such that for finitely generated ideals I of R we can decide $1 \in I$, then R has a proper prime ideal by way of Proposition 4.17. In fact, following the proof of Proposition 4.17, this R has a maximal ideal.

As a matter of fact, working within a constructive setting and without any further assumption on the underlying set S, inconsistency with respect to an entailment relation hardly ever needs to be a decidable property for finite subsets U of S.

Remark 4.19. With regard to what has been indicated before, it is worth noting that the following are equivalent:

1. REM
2. For every set S, if $\vdash$ is an entailment relation on S, then

$$\{ U \in \mathrm{Fin}(S) : U \vdash \}$$

is a detachable subset of $\mathrm{Fin}(S)$.

In fact, since $U \vdash$ is nothing but a shorthand notation for $(U, \emptyset) \in \vdash$, the second item is an immediate consequence of the first. We thus concentrate on showing the converse. Let φ be a bounded formula and define a relation $\vdash_\varphi$ between finite subsets U and V of an arbitrary set S (we might as well take $S = \emptyset$) by stipulating

$$U \vdash_\varphi V \; \equiv \; U \between V \,\vee\, \varphi.$$

It is straightforward to show that $\vdash_\varphi$ indeed is an entailment relation (cf. Lemma 4.27). We have $U \vdash_\varphi$ if and only if φ. Therefore, if inconsistency with respect to $\vdash_\varphi$ is decidable, then $\varphi \vee \neg\varphi$. As φ was taken arbitrary for this argument, REM follows.

Needless to say, Remark 4.19 does of course not preclude entailment relations which do allow for the set of finite inconsistent subsets to be detachable. For instance, the entailment relation of positive cone of a torsion-free abelian group of finite rank has a decidable inconsistency predicate by way of the constructive *Positivstellensatz*, cf. Chapter 7. Sometimes, while we may not get by without employing classical reasoning at all, some weak instance of excluded middle, e.g., the *limited principle of omniscience* (LPO), may already suffice for inconsistency to be a decidable property, cf. Section 4.5.

4.3.4 Interpretation

Let S and S' be sets with entailment relations $\vdash$ and $\vdash'$, respectively, In order to relate $\vdash$ to $\vdash'$, we need a notion of morphism between entailment relations. A natural and practicable one is given by functions between the underlying sets which preserve entailments. The following goes back to [79].

Let $f : S \to S'$ be a function. We say that f is an *interpretation* if, for every pair of finite subsets U and V of S,

$$U \vdash V \qquad \text{implies} \qquad f(U) \vdash f(V). \tag{4.3}$$

We write

$$f : (S, \vdash) \to (S', \vdash')$$

in order to denote an interpretation.

Lemma 4.20. *If $f : (S, \vdash) \to (S', \vdash')$ is an interpretation, then the inverse image mapping $f^{-1} : \mathrm{Pow}(S) \to \mathrm{Pow}(S)$ restricts on ideal elements, i.e.,*

$$f^{-1} : \mathfrak{Spec}(\vdash') \to \mathfrak{Spec}(\vdash), \quad \beta \mapsto f^{-1}(\beta).$$

Proof. Let β be an ideal element of $\vdash'$ and write $\alpha = f^{-1}(\beta)$. In order to show that α is an ideal element of $\vdash$, let U and V be finite subsets of S and suppose that $U \subseteq \alpha$ and $U \vdash V$. We have $\beta \supseteq f(U)$, as well as $f(U) \vdash' f(V)$ by interpretation. It follows that $\beta \between f(V)$, whence $\alpha \between V$. □

If (4.3) is an equivalence, then we say that f is *conservative.* This will be an important concept for later developments in view of the following Theorem 4.21, which has been taken from [79]. To the best of my knowledge there is no detailed proof for Theorem 4.21 available in the literature. This gives an opportunity to provide one here.[9] Keep in mind that along with CT we have at hand REM and MEL.

Theorem 4.21 (CT). *If $f : (S, \vdash) \to (S', \vdash')$ is an interpretation, then the following are equivalent.*

1. *f is conservative.*

[9] Alternatively, it is possible to apply the fundamental theorem of entailment relations (see Section 4.3.5) and give a lattice theoretic argument using prime filters. This was communicated to us by Thierry Coquand and gave rise to our rather elementary proof.

2. f^{-1} *is surjective on ideal elements.*

Proof. 1 $\implies$ 2: Suppose that f is conservative and let $\alpha \in \mathfrak{Spec}(\vdash)$. We have to exhibit $\beta \in \mathfrak{Spec}(\vdash')$ with $f^{-1}(\beta) = \alpha$, and in order to do so, we first define an auxiliary relation $\vdash'_{f(\alpha)}$ on S' by stipulating $U' \vdash'_{f(\alpha)} V'$ if and only if there are $W'_0 \in \mathrm{Fin}(f(\alpha))$ and $W'_1 \in \mathrm{Fin}(f(S) \setminus f(\alpha))$ such that

$$U', W'_0 \vdash' V', W'_1.$$

Recall from Lemma 4.6 that this is an entailment relation with the property that if β is a subset of S', then $\beta \in \mathfrak{Spec}(\vdash'_{f(\alpha)})$ if and only if $\beta \in \mathfrak{Spec}(\vdash')$ and

$$f(\alpha) \subseteq \beta \subseteq (S' \setminus f(S)) \cup f(\alpha). \tag{4.4}$$

Suppose that $\vdash'_{f(\alpha)}$ collapses, i.e., let $a_1, \dots, a_k \in \alpha$ and $b_1, \dots, b_\ell \in S \setminus \alpha$ be such that

$$f(a_1), \dots, f(a_k) \vdash' f(b_1), \dots, f(b_\ell).$$

Since f is conservative, this implies

$$a_1, \dots, a_k \vdash b_1, \dots, b_\ell.$$

But since $\alpha \in \mathfrak{Spec}(\vdash)$ and $a_1, \dots, a_k \in \alpha$, we arrive at a contradiction. Thus $\vdash'_{f(\alpha)}$ is consistent. Using MEL, there is $\beta \in \mathfrak{Spec}(\vdash'_{f(\alpha)})$ and we claim that $\alpha = f^{-1}(\beta)$. On the one hand, we have

$$\alpha \subseteq f^{-1}(f(\alpha)) \subseteq f^{-1}(\beta)$$

by the left hand containment of (4.4). On the other hand, if $a \in f^{-1}(\beta)$, then we must have $f(a) \in f(\alpha)$ by the right hand containment of (4.4), and therefore $\vdash'_{f(\alpha)} f(a)$. This is to say that there are $a_1, \dots, a_k \in \alpha$ and $b_1, \dots, b_\ell \in S \setminus \alpha$ such that

$$f(a_1), \dots, f(a_k) \vdash' f(b_1), \dots, f(b_\ell), f(a).$$

Conservativity implies

$$a_1, \dots, a_k \vdash b_1, \dots, b_\ell, a.$$

Finally, since $\alpha \in \mathfrak{Spec}(\vdash)$, we get $\alpha \between \{ b_1, \dots, b_\ell, a \}$, which leaves us with $a \in \alpha$, as required.

2 $\implies$ 1: Let f^{-1} be surjective on ideal elements. In order to show that f is conservative, let U and V be finite subsets of S such that $f(U) \vdash' f(V)$. We need to show $U \vdash V$, to which end we employ completeness for entailment relations. Thus, let $\alpha \in \mathfrak{Spec}(\vdash)$ with $U \subseteq \alpha$. Since f^{-1} is surjective, there is an ideal element β of $\vdash'$ such that $f^{-1}(\beta) = \alpha$. It follows that $f(U) \subseteq \beta$, and since $\beta \in \mathfrak{Spec}(\vdash')$ we get $\beta \between f(V)$. The latter implies $\alpha \between V$. □

Example 4.22. Let A be a set, $\mathbf{2} = \{ 0, 1 \}$, and take $S = A \times \mathbf{2}$. On S we consider the entailment relation $\vdash$ which is inductively generated by all instances of the following axioms of *single-values* and *totality*.

$$(a,0), (a,1) \vdash \tag{s}$$

$$\vdash (a,0), (a,1) \tag{t}$$

To begin with, an ideal element α of $\vdash$ is a subrelation of $A \times \mathbf{2}$. While axiom (s) forces α to be single-valued, if α models (t), then α is a total relation. In a nutshell, ideal elements of $\vdash$ are nothing but functions $A \to \mathbf{2}$. This example has long been put forward [79] and underlies most of those that are yet to come. $\vdash$ can be characterized by empty-conclusion instances, and for every finite subset U of $A \times \mathbf{2}$ we have $U \vdash$ if and only if there is $a \in A$ such that $\{ (a,0), (a,1) \} \subseteq U$.

Suppose now that $f : A \to B$ is a function. On $S' = B \to \mathbf{2}$ we consider the entailment relation $\vdash'$ of function $B \to \mathbf{2}$, and have an interpretation

$$i_f : (S,\vdash) \to (S',\vdash'), \quad (a,i) \mapsto (f(a),i).$$

Taking for granted the above characterization of inconsistency, it is straightforward to show that i_f is conservative if and only f is *injective*. Semantically, by way of Theorem 4.21, we see that f is injective if and only if the induced map i_f^{-1} of ideal elements is onto, which in turn is equivalent to $\mathbf{2}$ being an *injective set*. Over **CZF** the latter assertion gives rise to RWEM [9], i.e., excluded middle for negated bounded formulae,

$$\neg\varphi \vee \neg\neg\varphi$$

so classical reasoning is necessary in order to infer extendability from the conservation argument.

It is clear that interpretations compose, whence there is a category: objects are sets equipped with a distinguished entailment relations, and morphisms are interpretations. We refer to [90] for a discussion on several categorical equivalences.[10]

4.3.5 Fundamental theorem

Let L be a bounded distributive lattice. Recall that a canonical choice for an entailment relation on L is given by stipulating, for every pair of finite subsets U and V of L,

$$U \vdash V \equiv \bigwedge U \leqslant \bigvee V.$$

A subset α of L is an ideal element of $\vdash$ if and only if α is a prime filter. Up to conservative interpretation, every entailment relation is of this form!

We take the following seminal result from [62], see also [90] and [169].

Fundamental theorem (FT). Let S be a set with entailment relation $\vdash$. There is a distributive lattice $L(S)$, together with a map $i : S \to L(S)$ such that, for every pair of finite subsets U and V of S,

$$U \vdash V \qquad \text{if and only if} \qquad \bigwedge_{a\in U} i(a) \leqslant \bigvee_{b\in V} i(b), \tag{$*$}$$

and whenever $f : S \to L'$ is an interpretation of $\vdash$ in another distributive lattice L', i.e., satisfying $(*)$ with f in place of i, then there is a lattice map $f' : L(S) \to L'$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "f"'] & L(S) \arrow[d, dashed, "f'"] \\ & L' \end{tikzcd}$$

[10] Different notions of mapping between entailment relations prove practicable, too. Let S and S' be sets, each equipped with an entailment relation $\vdash$ and $\vdash'$, respectively. An *approximable relation* [62] between $\vdash$ and $\vdash'$ is a relation $\succ$ between finite subsets of S' and elements of S such that

1. If $V' \supseteq_{\mathrm{f}} U'$ and $U' \succ a$, then $V' \succ a$, i.e. $\succ$ is monotone.
2. If $U' \vdash' b'_1, \dots, b'_\ell$ and $U', b'_j \succ a$ for $1 \leqslant j \leqslant \ell$, then $U' \succ a$.
3. If $a_1, \dots, a_k \vdash b_1, \dots, b_\ell$ and $U' \succ a_i$ for $1 \leqslant i \leqslant k$, then there are finite subsets $V'_1, \dots, V'_\ell$ of S' such that $V'_j \succ b_j$ for $1 \leqslant j \leqslant \ell$, and $U' \vdash' b'_1, \dots, b'_\ell$ for any choice of elements $b'_1 \in V'_1, \dots, b'_\ell \in V'_\ell$.

For instance, every interpretation $f : (S,\vdash) \to (S',\vdash')$ gives way to an approximable relation by stipulating, for finite subsets U' of S' and $a \in S$,

$$U' \succ a \quad \equiv \quad U' \vdash' f(a).$$

Here we shall be content with the more specific notion of interpretation in terms of entailment-preserving functions.

FT is a constructive theorem, and has an abundance of important applications and consequences [62].

There is now a correspondence between ideal elements of $\vdash$ and prime filters of the generated distributive lattice $L(S)$ [62, 90]. If P is a prime filter of $L(S)$, then its inverse image $i^{-1}(P)$ along the canonical interpretation i is an ideal element of $\vdash$. Conversely, if $\alpha \in \mathfrak{Spec}(\vdash)$, then

$$P_\alpha = \Big\{ \, x \in L(S) : \exists U \in \mathrm{Fin}(\alpha) \, (\, \bigwedge i(U) \leqslant x \,) \, \Big\}$$

is a prime filter of the generated lattice $L(S)$ such that $\alpha = i^{-1}(P_\alpha)$. Now recall that every distributive lattice L canonically defines a spectral space [147] the points of which are the prime filters P of L.

In this manner, every entailment relation gives rise to a spectral space. Syntactical properties of the former can be understood topologically by means of the latter. For instance, if the generated lattice is complemented, i.e., a Boolean algebra, then its atoms correspond with the isolated points of the space it gives rise to. This will be applied in Chapter 7.

4.3.6 Atoms in the generated lattice

For the following we need an observation that can be made prior to the construction for the proof of FT [62, 169]: if $\vdash$ is an entailment relation on a set S, then every element x of the generated distributive lattice $L(S)$ can be put in "disjunctive normal form", i.e., there is a finite subset $\mathscr{U}$ of finite subsets U of S such that

$$x = \bigvee_{U \in \mathscr{U}} \bigwedge_{a \in U} i(a).$$

Let L be a distributive lattice. Recall from [169] that an *atom* of L is a non-zero element $x \in L$ such that

$$\forall y \in L \, \big(\, (x \wedge y = 0) \, \vee \, (x \wedge y = x) \, \big).$$

If L is discrete, then a non-zero element $x \in L$ is an atom if and only if it is minimal among non-zero elements [123, 156, 169]. However, we go with the former definition which in turn directly suggests the next.

Definition 4.23. *Let $\vdash$ be an entailment relation on a set S. We say that a finite subset A of S is* decisive *if, for every $b \in S$,*

$$A, b \vdash \quad or \quad A \vdash b.$$

We say that a finite consistent decisive subset of S is an atomic conjunction.

Thus, a finite subset A of S is decisive if and only if, for every abstract statement $b \in S$, either b is a consequence of A or else $A \cup \{ \, b \, \}$ is inconsistent. Moreover, the empty subset is an atomic conjunction if and only if $\vdash$ is consistent and *complete* [229], which is to say that for every $a \in S$ either $a \vdash$ or $\vdash a$.

Proposition 4.24. *Let S be a set with entailment relation $\vdash$. The following are equivalent.*

1. *$(S, \vdash)$ has an atomic conjunction.*
2. *The generated distributive lattice $L(S)$ has an atom.*

Proof. Suppose that $A \subseteq S$ is an atomic conjunction and consider

$$x = \bigwedge_{a \in A} i(a).$$

This x is non-zero, since A is supposed to be consistent. Now let $y \in L(S)$, which can be written in disjunctive normal form [62, 169]

$$y = \bigvee_{V \in \mathscr{V}} \bigwedge_{b \in V} i(b)$$

for a certain finite set $\mathscr{V}$ of finite subsets V of S. Since A is decisive, for every $V \in \mathscr{V}$ and $b \in V$ we have either $A, b \vdash$ or else $A \vdash b$, according to which either $x \wedge i(b) = 0$ or $x \leqslant i(b)$. It now follows that

$$x \wedge y = \bigvee_{V \in \mathscr{V}} \bigwedge_{b \in V} x \wedge i(b) \in \{\, 0, x \,\}\,.$$

Conversely, if $L(S)$ has an atom x, this atom too can be written in normal form

$$0 \neq x = \bigvee_{U \in \mathscr{U}} \bigwedge_{a \in U} i(a),$$

where we may suppose that $\mathscr{U}$ is a singleton, i.e., $\mathscr{U} = \{\, U \,\}$. This U is consistent; moreover, if $b \in S$, then $x \wedge i(b) = 0$ or $x \leqslant i(b)$ since x is an atom, which by the fundamental theorem translates back to $U, b \vdash$ or $U \vdash b$, so U is decisive. In all U is an atomic conjunction. □

Remark 4.25. Let S be a set with entailment relation $\vdash$ and suppose that A is an atomic conjunction. We claim that

$$A^{\vdash} = \{\, a \in S : A \vdash a \,\} \tag{4.5}$$

is an ideal element of $\vdash$. In fact, let $U \vdash V$ and suppose that $U \subseteq A^{\vdash}$. Since A is decisive, for every $b \in V$ either we have $A \vdash b$, whence $b \in A^{\vdash}$, or else $A, b \vdash$. We thus have to rule out that the latter holds for every $b \in V$. But if this were indeed the case, with a successive application of transitivity we could obtain $A \vdash$, yet A is supposed to be consistent. It follows that in order to show that the generated lattice is atomless, it suffices to show that there can be no finitely generated ideal element as specified by (4.5). This strategy will be put into action in Chapter 7.

4.4 Counterparts for inconsistency predicates

Let S be a set equipped with an inductively generated entailment relation $\vdash$. If the corresponding generated lattice is Boolean, in order to give a *non-inductive* characterization of $\vdash$, it can be shown that being able to do so for empty-conclusion instances suffices. As a matter of fact, all but few of the entailment relations that will be studied later on do allow for a characterization in this manner. Here we put together some notions and tools that can help to explain inconsistency.[11]

In the following, by a finitary predicate Φ on S will be meant a subset of $\mathrm{Fin}(S)$, and we use $\Phi(U)$ as shorthand for $U \in \Phi$. The *canonical inconsistency predicate* $\Phi_{\vdash}$ for $\vdash$ is defined by stipulating

$$\Phi_{\vdash}(U) \;\equiv\; U \vdash$$

where U is a finite subset of S. Due to monotonicity of $\vdash$, this $\Phi_{\vdash}$ is monotone too, in the sense that if U and V are finite subsets of S with $U \subseteq V$, then $\Phi_{\vdash}(U)$ implies $\Phi_{\vdash}(V)$.

Suppose now that we have a monotone finitary predicate Φ on S. In this section we are interested in characterizing entailment relations $\vdash$ on S for which the canonical inconsistency predicate $\Phi_{\vdash}$ coincides with Φ.

Definition 4.26. *Let Φ be a finitary predicate on S. We introduce two relations $\vdash_{\Phi}^{\min}$ and $\vdash_{\Phi}^{\max}$ between finite subsets U and V of S, as follows.*

$$\begin{aligned} U \vdash_{\Phi}^{\min} V \quad &\equiv \quad U \between V \,\vee\, \Phi(U) \\ U \vdash_{\Phi}^{\max} V \quad &\equiv \quad \forall U' \supseteq_{\mathrm{f}} U \,\big(\forall b \in V \,(\, \Phi(U', b)\,) \,\to\, \Phi(U')\,\big). \end{aligned}$$

Lemma 4.27. *If Φ is a finitary monotone predicate on S, then*

1. *$\vdash_{\Phi}^{\min}$ is an entailment relation, and*

[11] For the content of this section I am indebted to Davide Rinaldi, for having shared and communicated his insights while we were trying to tackle different ends of the same problem. Rinaldi's results now form parts of [211].

2. $\vdash_\Phi^{\max}$ *is an entailment relation.*

Proof. 1. Since $\vdash_\Phi^{\min}$ contains the overlap relation, reflexivity is apparent. Monotonicity for $\vdash_\Phi^{\min}$ is a consequence of Φ being monotone. With respect to cut (T), let $a \in S$ and U and V be finite subsets of S such that both $U \vdash_\Phi^{\min} V, a$ and $U, a \vdash_\Phi^{\min} V$. This is to say that

$$U \between (V,a) \vee \Phi(U) \qquad \text{and} \qquad (U,a) \between V \vee \Phi(U,a).$$

Keep in mind that $U \between (V,a)$ amounts to $U \between V$ or $a \in U$, whence $U \vdash_\Phi^{\min} V$ follows immediately.

2. Suppose first that $a \in U \cap V$ and let $U' \supseteq_{\mathrm{f}} U$. If $\Phi(U', b)$ for every $b \in V$, then $\Phi(U', a)$ in particular and thus $\Phi(U)$. This means that $\vdash_\Phi^{\max}$ is reflexive. Next, suppose that $U \vdash_\Phi^{\max} V$ and let U' and V' be finite subsets of S. In order to show $U, U' \vdash_\Phi^{\max} V, V'$, let $U'' \supseteq_{\mathrm{f}} U \cup U'$, and suppose that for every $b \in V \cup V'$ we have $\Phi(U'', b)$. In particular, for every $b \in V$ we have $\Phi(U'', b)$, whence $\Phi(U'')$ is an immediate consequence of the assumption $U \vdash_\Phi^{\max} V$. It remains to consider cuts. Suppose that both $U \vdash_\Phi^{\max} V, a$ and $U, a \vdash_\Phi^{\max} V$. In order to show $U \vdash_\Phi^{\max} V$, let $U' \supseteq U$ and suppose that for every $b \in V$ we have $\Phi(U', b)$. By way of the latter entailment, we have $\Phi(U', a)$. Now that we know $\Phi(U', b)$ for every $b \in V \cup \{ a \}$, we get $\Phi(U')$ from the former entailment. $\square$

Given a finitary monotone predicate Φ on S, what are the *canonical* inconsistency predicates that correspond to the entailment relations $\vdash_\Phi^{\min}$ and $\vdash_\Phi^{\max}$, respectively? It is Φ in both cases:

Lemma 4.28. *Let Φ be a finitary monotone predicate on S. For every finite subset U of S, the following are equivalent.*

1. $\Phi(U)$

2. $U \vdash_\Phi^{\min}$

3. $U \vdash_\Phi^{\max}$

Proof. Clearly, $U \vdash_\Phi^{\min}$ if and only if $\Phi(U)$. Similarly, we have $U \vdash_\Phi^{\max}$ if and only if $\forall U' \supseteq_{\mathrm{f}} U\,(\,\Phi(U')\,)$ which by instantiation and monotonicity in fact is equivalent to $\Phi(U)$. $\square$

The following is due to Rinaldi and generalizes Scott's "sandwich criterion" for multi-conclusion entailment relations as extending their single-conclusion counterparts, see [209, 229, 234], Chapter 3, and [211].

Theorem 4.29. *Let $\vdash$ be an entailment relation, and let Φ be a monotone predicate on S. The following are equivalent.*

(i) $\Phi = \Phi_\vdash$

(ii) $\vdash_\Phi^{\min} \subseteq \vdash \subseteq \vdash_\Phi^{\max}$.

Proof. Suppose that $\Phi_\vdash = \Phi$, and let U and V be finite subsets of S. If $U \vdash_\Phi^{\min} V$, then $U \between V$ or $\Phi(U)$, both cases of which imply $U \vdash V$. We thus concentrate on showing $\vdash \subseteq \vdash_\Phi^{\max}$. Accordingly, let $U \vdash V$, let $U' \supseteq U$, and suppose that for every $b \in V$ we have $\Phi(U', b)$. Since $\Phi = \Phi_\vdash$, we have $U', b \vdash$ for every $b \in V$. Therefore, by a series of cuts with $U \vdash V$ we get $U' \vdash$, and $\Phi(U')$ follows, as required.

Conversely, if $\vdash$ is an entailment relation between $\vdash_\Phi^{\min}$ and $\vdash_\Phi^{\max}$, then, for every finite subset U of S, we have a chain of implications

$$\Phi(U) \implies U \vdash_\Phi^{\min} \implies U \vdash \implies U \vdash_\Phi^{\max} \implies \Phi(U)$$

by way of the preceding Lemma 4.28, and therefore $\Phi = \Phi_\vdash$. $\square$

It is worth mentioning that Theorem 4.29 can also be obtained by means of a cut-elimination principle for entailment relations [211].

In order to capture inconsistency in a succinct manner, we next we introduce the notion of an adequate and hereditary predicate.

Definition 4.30. *Let $\vdash$ be an entailment relation, and let Φ be a finitary monotone predicate on S. We say that*

1. *Φ is* hereditary *for $\vdash$ if*

$$\frac{a_1, \ldots, a_n \vdash b_1, \ldots, b_m \qquad \Phi(U, b_1) \quad \ldots \quad \Phi(U, b_m)}{\Phi(U, a_1, \ldots, a_n)}$$

where U is a finite subset of S.

2. *Φ is* adequate *for $\vdash$ if $\Phi \subseteq \Phi_{\vdash}$.*

Lemma 4.31. *Let Φ be a finitary monotone predicate on S.*

1. *Φ is adequate for $\vdash$ if and only if $\vdash_{\Phi}^{\min} \subseteq \vdash$.*
2. *Φ is hereditary for $\vdash$ if and only if $\vdash \subseteq \vdash_{\Phi}^{\max}$.*

Proof. 1. Suppose that Φ is adequate for $\vdash$ and let U and V be finite subsets of S. If $U \vdash_{\Phi}^{\min} V$, then $\Phi(U)$ or $U \between V$, both cases of which imply $U \vdash V$.

Conversely, suppose that $\vdash_{\Phi}^{\min} \subseteq \vdash$. If $\Phi(U)$, then $U \vdash_{\Phi}^{\min}$, and thus $U \vdash$. This means that Φ is adequate for $\vdash$.

2. Suppose that Φ is hereditary for $\vdash$. If U and V are finite subsets of S such that $U \vdash V$, let $U' \supseteq_{\mathrm{f}} U$ and suppose that $\Phi(U', b)$ for every $b \in V$. Since Φ is hereditary, it follows that $\Phi(U', U)$ which apparently simplifies to $\Phi(U')$, as required for $U \vdash_{\Phi}^{\max} V$.

Conversely, suppose that $\vdash \subseteq \vdash_{\Phi}^{\max}$. Thus, if $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$, then we have as well $a_1, \ldots, a_n \vdash_{\Phi}^{\max} b_1, \ldots, b_m$. If $\Phi(U, b_j)$ for $1 \leqslant j \leqslant m$, then by monotonicity we get $\Phi(U, a_1, \ldots, a_n, b_j)$ for $1 \leqslant j \leqslant m$, and therefore $\Phi(U, a_1, \ldots, a_n)$. $\square$

Corollary 4.32. *The following are equivalent.*

(i) *Φ is adequate and hereditary.*

(ii) *$\Phi = \Phi_{\vdash}$.*

Proof. Put together Theorem 4.29 and Lemma 4.31. $\square$

Corollary 4.33.

1. *Let Φ be adequate for $\vdash$. If $\Phi(\emptyset)$, then $\vdash$ collapses.*
2. *Let Φ be hereditary for $\vdash$. If $\vdash$ collapses, then $\Phi(\emptyset)$.*

Now we have at hand a strategy that helps characterizing inconsistency. Once an andequate predicate has been formulated, it remains to verify that it is hereditary. In general, this may require a bit of work. Fortunately, it suffices to concentrate only on generating axioms.

Lemma 4.34. *Let $\vdash$ be an inductively generated entailment relation, and let Φ be a finitary monotone predicate on S. The following are equivalent.*

(i) *Φ is hereditary for $\vdash$.*

(ii) *Φ is hereditary for every axiom of $\vdash$.*

Proof. Of course, if Φ is hereditary for $\vdash$, then so it is for every initial entailment. Conversely, suppose that Φ is hereditary for every initial entailment. This allows for an argument by induction, whence we need to address the rules. The case for reflexivity (R) is trivial, while the case for monotonicity (M) is handled by the assumption that Φ is monotone. We thus concentrate on transitivity (T). Suppose that we have

$$\dfrac{\dfrac{\vdots}{a_1,\ldots,a_n \vdash b_1,\ldots,b_m,c} \qquad \dfrac{\vdots}{a_1,\ldots,a_n,c \vdash b_1,\ldots,b_m}}{a_1,\ldots,a_n \vdash b_1,\ldots,b_m}$$

and let U be a finite subset of S with $\Phi(U,b_j)$ for $1 \leqslant j \leqslant m$. Employing the inductive hypothesis with regard to the right-hand entailment above the lower inference line, we get $\Phi(U,a_1,\ldots,a_n,c)$. Next we consider $U' = U \cup \{\, a_1,\ldots,a_n \,\}$, and have, since Φ is monotone, $\Phi(U',b_i)$ for $1 \leqslant i \leqslant m$, as well as $\Phi(U',c)$. We may now employ the hypothesis with regard to the left-hand entailment above, leading us to $\Phi(U',a_1,\ldots,a_n)$. This just means $\Phi(U)$, as required. $\square$

Example 4.35. Scott's criterion that we have put into action in Chapter 3 is a special case of Theorem 4.29 (see also [211]). Let $\rhd$ be a single-conclusion entailment relation, *extended* by a multi-conclusion entailment relation $\vdash$, i.e, for every finite subset U and element a of S we have that $U \rhd a$ implies $U \vdash a$. In case the converse holds as well, this extension is *conservative*, see Chapter 3. For every element $a \in S$ we define an entailment relation $\vdash_a$ as follows:

$$U \vdash_a V \;\equiv\; U \vdash V, a.$$

We further define a monotone predicate Φ_a by way of

$$\Phi_a(U) \;\equiv\; U \rhd a.$$

Since $\vdash$ extends $\rhd$, notice that Φ_a is adequate for $\vdash_a$, i.e. $\Phi_a \subseteq \Phi_{\vdash_a}$. We claim that the following are equivalent.

(i) For every $a \in S$ the predicate Φ_a is hereditary for $\vdash_a$.

(ii) $\vdash$ is a conservative extension of $\rhd$.

In fact, if $U \vdash a$, then $\Phi_{\vdash_a}(U)$. If Φ_a is hereditary (and since it is adequate) for $\vdash_a$, then we get $\Phi_a(U)$ by Corollary 4.32 and this means nothing but $U \rhd a$. Thus $\vdash$ is conservative over $\rhd$.

Conversely, suppose that $\vdash$ is a conservative extension of $\rhd$, and let $a \in S$. In order to show that Φ_a is hereditary, we consider

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m$$

and assume to have

$$\Phi_a(U,b_1) \quad \ldots \quad \Phi_a(U,b_m),$$

which by definition of Φ_a means

$$U,b_1 \rhd a \quad \ldots \quad U,b_m \rhd a.$$

Since $\vdash$ extends $\rhd$ we have

$$U,b_1 \vdash a \quad \ldots \quad U,b_m \vdash a.$$

By repeated application of transitivity we obtain $U,a_1,\ldots,a_n \vdash a$. Since $\vdash$ is conservative over $\rhd$ we get $U,a_1,\ldots,a_n \rhd a$. This translates back to $\Phi_a(U,a_1,\ldots,a_n)$.

Remark 4.36. Related concepts have been studied in the context of disjunctive logic programming [215, 258]. Let us briefly compare our approach with the concept of hyperresolution from clausal logic, following [90]. Let again S be a set with entailment relation. A finite subset X of S is called a *clause*. A *clause set* is a set of clauses. Let us say that a clause set W is closed under the *hyperresolution rule* if

$$\frac{a_1, X_1 \quad \dots \quad a_n, X_n \qquad a_1, \dots, a_n \vdash Y}{X_1, \dots, X_n, Y}$$

which is to say that whenever $a_1, \dots, a_n \vdash Y$ and $\{ a_i \} \cup X_i \in W$ for every $i \in \{ 1, \dots, n \}$, then we have $\cup_{i=1}^n X_i \cup Y \in W$. Now let Φ be a finitary monotone predicate on S. This Φ can be construed as a clause set. The following are equivalent:

1. Φ is hereditary for $\vdash$.

2. Φ is closed under the hyperresolution rule with respect to the opposite $\dashv = \vdash^\circ$.

Monotone subsets of $\mathrm{Fin}(S)$ which are closed under the hyperresolution rule with respect to the opposite of $\vdash$ are called *conjunctive states* [90]. We can thus say that Φ as above is hereditary if and only if it is a conjunctive state. Conjunctive states are the ideals of a certain coverage relation [147] with which an explicit construction of the frame generated by an entailment relation can be given [90].

4.5 Infinite paths as ideal elements

We turn our attention to a first and thorough case study that brings into action some of the methods developed so far. Moreover, the opportunity arises to examine some aspects of non-constructivity of MEL, even in case the underlying domain of discourse is a countable set, by showing this principle to give rise to a version of the Weak König Lemma (WKL).

While outlining this section's preliminaries we closely follow [88], which in fact provides overall motivation for what follows.

4.5.1 Trees

Let A be a finite discrete inhabited set, for instance $A = \{ 0, 1 \}$. Let

$$A^* = \bigcup_{n \geqslant 0} A^n$$

be the set of finite sequences of elements of A. The *length* of $u \in A^n$ is $|u| = n$. The only sequence of length 0 is the *empty sequence* $\langle\rangle$. The *n-th finite initial segment*

$$\overline{u}n = \langle u(0), \dots, u(n-1) \rangle$$

of u with $|u| \geqslant n$ has length n. Note that $\overline{u}0 = \langle\rangle, \overline{u}|u| = u$, and $\overline{\overline{u}n}m = \overline{u}m$ whenever $m \leqslant n$. Each sequence $\langle a \rangle$ of length 1 is identified with its only element a. The *concatenation* of u and v is denoted by their juxtaposition uv. We write $u \leqslant w$ if u is a *restriction* of w, i.e.,

$$u \leqslant w \;\equiv\; \exists v \, (uv = w).$$

Since A is discrete, this puts a decidable partial order on A^*. Clearly, the empty sequence $\langle\rangle$ is the least element of A^* with respect to this order. A subset T of A^* is *unbounded* if

$$\forall n \exists u \, (|u| \geqslant n \,\wedge\, u \in T).$$

A subset T of A^* is *closed under restrictions* if

$$\forall u, w \, (u \leqslant w \,\wedge\, w \in T \,\to\, u \in T).$$

If T is closed under restrictions, then T being inhabited is tantamount to $\langle\rangle \in T$. Moreover, any such T is unbounded precisely when

$$\forall n \exists u \, (\, |u| = n \, \wedge \, u \in T \,).$$

A *tree* is an inhabited subset T of A^* which is closed under restrictions. A tree is a *spread* if every element of T has an *immediate successor* in T, i.e.,

$$\forall u \, \big(\, u \in T \, \to \, \exists a \in A \, (\, ua \in T \,) \big).$$

A *linear spread* is a spread which is linearly ordered with respect to $\leqslant$. A linear spread has exactly one element of any given length.

4.5.2 Paths are ideal elements

Let again A be a finite discrete inhabited set. We make use of the following predicate for finite sequences $u, v \in A^*$:

$$N(u,v) \; \equiv \; \exists i \, \big(\, i \leqslant \min \{ \, |u|, |v| \, \} \, \wedge \, u(i) \neq v(i) \, \big).$$

Since A is discrete, $N(u,v)$ is decidable, and we make the following simple observation.

Lemma 4.37.

$$\forall u, v \, \big(\, \neg N(u,v) \, \to \, (\, u \leqslant v \, \vee \, v \leqslant u \,) \big)$$

Proof. Since $\leqslant$ is decidable, we have $u \leqslant v$ or $u \nleqslant v$. In the latter case, if $\neg N(u,v)$, it follows that $v \leqslant u$. □

Let $T \subseteq A^*$ be an unbounded detachable tree. The set of sequences of a given length n belonging to this tree will be denoted $T(n)$, viz.

$$T(n) = \{ \, u \in T : |u| = n \, \} \, .$$

Since a detachable subset of a finite set is finite, this $T(n)$ is finite. We now take the set $S = T$ for our domain of discourse and consider the entailment relation $\vdash$ on S that is inductively generated by all instances of the following axioms of *single-values*, *monotonicity*, and *totality*:

$$\begin{aligned} u, v &\vdash && \text{(s } [N(u,v)]) \\ w &\vdash u && \text{(m } [u \leqslant w]) \\ &\vdash T(n) && \text{(t } [n > 0]) \end{aligned}$$

with side conditions as indicated. Notice that

$$\vdash \langle\rangle$$

is a consequence of the above axioms and need not be postulated. In fact, for every $u \in T$ such that $|u| = 1$ we have $u \vdash \langle\rangle$ as an instance of (m). We can repeatedly cut the corresponding instance of totality $\vdash \{ \, u \in T : |u| = 1 \, \}$ and obtain $\vdash \langle\rangle$.

Due to the axioms for totality (t) and single-values (s), this entailment relation enjoys what in the following we will loosely call *back-and-forth* property.

Lemma 4.38. *Let U and V be finite subsets of T and let $w \in T$. The following are equivalent.*

1. *$U \vdash V, w$*
2. *$U, w' \vdash V$ for every $w' \in T(|w|) \setminus \{ \, w \, \}$.*

Proof. For every $w' \in T(|w|)$, if $w \neq w'$, then there is $i \leqslant |w|$ such that $w(i) \neq w'(i)$, whence $N(w, w')$ and we have an axiom $w, w' \vdash$. This entailment can be used to cut $U \vdash V, w$ and we obtain $U, w' \vdash V$.

Conversely, if $U, w' \vdash V$ for every $w' \in T(|w|) \setminus \{ w \}$, then we can cut the corresponding instance of totality $\vdash T(|w|)$ repeatedly, until we have $U \vdash V, w$. □

It follows from Lemma 4.38 that in order to provide a non-inductive description of $\vdash$, it suffices to characterize empty conclusion entailment only, and hence to find an adequate and hereditary predicate. Before doing so, let us briefly discuss semantics.

Lemma 4.39. *Let α be a subset of T. The following are equivalent.*

1. $\alpha \in \mathfrak{Spec}(\vdash)$
2. *α is a linear spread.*

Proof. Suppose that α models $\vdash$. Because of $\vdash \langle\rangle$, this α is inhabited. Furthermore, it is closed under restrictions by way of axiom (m), whence α is a tree. In order to show that α is a spread, we have to show that every element u of α has an immediate successor in α. Due to totality (t), we know that there is $w \in \alpha$ with $|w| = |u| + 1$. Now that we have $u, w \in \alpha$, we know that $N(u, w)$ is impossible, and $u \leqslant w$ follows from Lemma 4.37. The same argument applies to show that α is linear.

Conversely, if α indeed is a linear spread, then α has an element of any given length, thus is ideal for (t). Moreover, α is closed under restrictions, by which it is ideal for (m). Finally, every linear spread is ideal for the axiom of single-values (s). □

According to Lemma 4.39 we say that $\vdash$ is the entailment relation of *linear spread* of T. Recall that by an *infinite path* in a tree T one usually understands an infinite sequence $\beta : \mathbb{N} \to A$ for which all finite initial segments

$$\overline{\beta}n = \langle \beta(0), \ldots, \beta(n-1) \rangle$$

belong to T [88]. Now, apparently every such β gives way to a linear spread

$$\alpha = \left\{ \overline{\beta}n : n \geqslant 0 \right\} \in \mathfrak{Spec}(\vdash)$$

in the above sense. As for the converse, i.e., in order to extract an infinite path from a linear spread, suppose that $\alpha \in \mathfrak{Spec}(\vdash)$. Due to totality (t) and single-values (s), for every $n \in \mathbb{N}$ there is exactly one u in T such that $|u| = n$ and $u \in \alpha$, and if $u, v \in \alpha$ and $|u| \leqslant |v|$, then $u \leqslant v$. By means of replacement, there is a function f assigning to each $n \in \mathbb{N}$ the unique finite sequence $f_n \in \alpha$ such that $|f_n| = n$. With this data at hand, we define an infinite sequence

$$\beta : \mathbb{N} \to A, \quad n \mapsto f_{n+1}(n)$$

for which all finite initial segments

$$\overline{\beta}n = \langle \beta(0), \beta(1), \ldots, \beta(n-1) \rangle = \langle f_1(0), f_2(1), \ldots, f_n(n-1) \rangle = f_n$$

belong to T.

Keeping in mind this digression on the semantics of the entailment relation of linear spread, we next aim at an explicit description of $\vdash$. First we need to introduce another shorthand notation: if U is a finite subset of T, let

$$|U| = \max \{ |u| : u \in U \cup \{ \langle\rangle \} \}.$$

Lemma 4.40. *Let U be a finite subset of T. If $n \geqslant |U|$, then*

$$U \vdash \{ v \in T : |v| = n \wedge \forall u \in U \, (u \leqslant v) \}.$$

Proof. Notice first that if U is empty, then we merely reclaim the generating axiom of totality (t). Since the underlying set A is discrete, we can write

$$T(n) = \{ v \in T(n) : \forall u \in U \, (u \leqslant v) \} \cup \{ v \in T(n) : \exists u \in U \, N(u,v) \} .$$

Focussing on the right-hand set of this partition, the corresponding axiom of totality

$$\vdash T(n)$$

can thus be cut with instances of single-values

$$u, v \vdash$$

where $u \in U, v \in T(n)$ and $N(u,v)$. Doing so repeatedly, we obtain

$$U \vdash \{ v \in T(n) : \forall u \in U \, (u \leqslant v) \} . \qquad \square$$

Intuitively, Lemma 4.40 asserts that in order to extend a finite subset U of T coherently, i.e., without causing inconsistency, it is necessary to choose an element among the common successors of U. Lemma 4.40 already hints at an inconsistency predicate for $\vdash$. We stipulate, for finite subsets U of T,

$$\Phi(U) \equiv \exists n \left(n \geqslant |U| \wedge \{ v \in T(n) : \forall u \in U \, (u \leqslant v) \} = \emptyset \right).$$

It is easy to see that this predicate Φ is monotone: let $U \subseteq U'$ and let $n \geqslant |U|$ be such that $\{ v \in T(n) : \forall u \in U \, (u \leqslant v) \}$ is empty. Consider $n' = |U'|$ and suppose that there is $v \in T(n')$ such that $u \leqslant v$ whenever $u \in U'$. Since T is closed under restrictions, we have $\overline{v}n \in T(n)$. Furthermore, for every $u \in U$ it follows that $u \leqslant \overline{v}n$. This, however, contradicts our assumption and we may conclude $\Phi(U')$.

According to Lemma 4.40, if U is a finite subset of T such that $\Phi(U)$, then U is inconsistent, so Φ is an adequate predicate for $\vdash$. What is the intuitive meaning of Φ? In order for a finite subset U of T to be consistent, from a semantical point of view it should be extendable to a linear spread. The following circumstances might hinder this: for instance, U may contain an incomparable pair, whence may not even be a finite chain in the first place. Yet even if U is linearly ordered, if it happens to have an element with only finitely many successors, then it is impossible to extend U to a linear spread of T. Both these circumstances are captured in terms of Φ.

Lemma 4.41. *Φ is hereditary for $\vdash$.*

Proof. It suffices to show that Φ is hereditary for generating axioms. With regard to single-values (s), let $u, v \in T$ and suppose that $N(u,v)$, i.e., let $i \leqslant \min \{ |u|, |v| \}$ be such that $u(i) \neq v(i)$. For an arbitrary sequence $w \in T(n)$, where $n = \max \{ |u|, |v| \}$, we cannot have both $u \leqslant w$ and $v \leqslant w$. Thus, if $N(u,v)$ and U is a finite subset of T, then $\Phi(U,u,v)$. Next we consider monotonicity (m). Let U be a finite subset of T, let $u \leqslant w$, and suppose that $\Phi(U,u)$. Accordingly, there is $n \geqslant |U \cup \{ u \}|$ such that

$$\{ v \in T(n) : \forall u \in U \cup \{ u \} \; (u \leqslant v) \} = \emptyset.$$

Let $m = \max \{ n, |w| \}$, hence $m \geqslant |U \cup \{ w \}|$ and clearly

$$\{ v \in T(m) : \forall u \in U \cup \{ w \} \; (u \leqslant v) \} = \emptyset.$$

For if the latter set were inhabited, e.g., by $v \in T(m)$, then $\overline{v}n \in T(n)$ would contradict the formerly displayed one being empty, due to the fact that $u \leqslant w$. It remains to consider totality (t). Let U again be a finite subset of T, let $n > 0$, and suppose that for every $w \in T(n)$ we have $\Phi(U,w)$. This means that for every $w \in T(n)$ there is $k_w \geqslant \max \{ |U|, n \}$ such that

$$\{ v \in T(k_w) : \forall u \in U \cup \{ w \} \; (u \leqslant v) \} = \emptyset. \qquad (\dagger)$$

Consider

$$m = \max \{ k_w : w \in T(n) \} .$$

for which we claim that it witnesses $\Phi(U)$. Clearly, $m \geqslant |U|$. Now suppose that there is a particular $v \in T(m)$ such that

$$\forall u \in U \, (\, u \leqslant v \,). \tag{‡}$$

Keeping in mind $\overline{v}n \in T(n)$ and

$$|v| = m = \max \{ k_w : w \in T(n) \} \geqslant k_{\overline{v}n} \geqslant \max \{ |U|, n \} \geqslant n,$$

assertion (‡) implies

$$\forall u \in U \cup \{ \overline{v}n \} \, \big(\, u \leqslant \overline{v}k_{\overline{v}n} \, \big).$$

However, this contradicts (†). ☐

Since Φ is both adequate and hereditary for $\vdash$, it follows that $\Phi_\vdash = \Phi$. In order to recapitulate, we obtain a characterization of empty-conclusion entailment as follows.

Proposition 4.42. *Let U be a finite subset of T. The following are equivalent.*

1. *U is inconsistent.*
2. *There is $n \geqslant |U|$ such that $\{ \, v \in T(n) : \forall u \in U \, (\, u \leqslant v \,) \, \}$ is empty.*

Consistency of $\vdash$ is an immediate consequence of Proposition 4.42.

Corollary 4.43. *$\vdash$ is consistent.*

Proof. According to Proposition 4.42, $\emptyset \vdash \emptyset$ being inferrable is tantamount to $\Phi(\emptyset)$. This would be to say that there is $n \geqslant 0$ such that $T(n)$ is empty. However, T is supposed to have elements of any given length. ☐

Bringing into play the back-and-forth property of $\vdash$ as stated in Lemma 4.38, Proposition 4.42 leads over to a non-inductive description of $\vdash$, as follows.

Proposition 4.44. *The following are equivalent.*

1. *$u_1, \ldots, u_k \vdash v_1, \ldots, v_\ell$*
2. *For any choice of elements*

$$v_1' \in T(|v_1|) \setminus \{ \, v_1 \, \}, \ldots, v_\ell' \in T(|v_\ell|) \setminus \{ \, v_\ell \, \}$$

there is $n \geqslant |\{ \, u_1, \ldots, u_k, v_1, \ldots, v_\ell \, \}|$ such that

$$\{ \, w \in T(n) : \forall u \in \{ \, u_1, \ldots, u_k, v_1', \ldots, v_\ell' \, \} \, (\, u \leqslant w \,) \, \} = \emptyset.$$

4.5.3 Entailment on universal trees

Let again A be a finite discrete inhabited set. As opposed to the preceding section, here we consider the entailment relation $\vdash$ of linear spread on the *full tree*

$$A^* = \bigcup_{n \geqslant 0} A^n.$$

The axiom of totality (t) does now appear unrestricted. In other words, we are interested in the case $T = A^*$. Accordingly, let $\vdash$ on A^* be inductively generated by all instances of the following axioms:

$$u, v \vdash \qquad (\mathrm{s}\ [N(u,v)])$$
$$w \vdash u \qquad (\mathrm{m}\ [u \leqslant w])$$
$$\vdash A^n \qquad (\mathrm{t}\ [n > 0])$$

with side-conditions as indicated. This entailment relation has an explicit description in somewhat simpler terms than the one provided above for a given but arbitrary tree $T \subseteq A^*$.

Proposition 4.45. *Let U be a finite subset of A^*. The following are equivalent.*

1. *$U \vdash$*
2. *$\exists u, v \in U\, N(u, v)$.*

Proof. It suffices to show that the second item defines a hereditary predicate for $\vdash$ and we only consider the axiom of totality (t). We have to show that if U is a finite subset of S such that $\Phi(U, w)$ whenever $w \in A^n$, then $\Phi(U)$. Suppose that for every $w \in A^n$ there are $u, v \in U \cup \{ w \}$ such that $N(u, v)$. In any such case, if both $u, v \in U$, then $\Phi(U)$ is immediate. We may thus concentrate on showing that

$$\forall w \in A^n\, \exists u \in U\, N(u, w) \tag{$*$}$$

too implies $\Phi(U)$. To this end, we describe how to obtain a single element $w \in A^n$ with which $\Phi(U, w)$ directly leads to $\Phi(U)$. To begin with, we consider an arbitrary element $w_1 \in A^n$. Because of $(*)$, there are $u_1 \in U$ and $i_1 \leqslant \min\{ |u_1|, n \}$ such that $u_1(i_1) \neq w_1(i_1)$. We may suppose that i_1 is minimal in this regard, which is to say that if $i < i_1$, then $u_1(i) = w_1(i)$, or in other words $\bar{u}_1(i_1 - 1) = \bar{w}_1(i_1 - 1)$. Next we consider $w_2 \in A^n$ defined by cases as follows:

$$w_2(i) = \begin{cases} u_1(i) & 1 \leqslant i \leqslant i_1 \\ u_1(i_1) & i_1 < i \leqslant n. \end{cases}$$

Notice that $\bar{w}_2 i_1 = \bar{u}_1 i_1$. According to $(*)$, there are $u_2 \in U$ and $i_2 \leqslant \min\{ |u_2|, n \}$ with $u_2(i_2) \neq w_2(i_2)$, where i_2 again may be considered minimal, i.e., if $i < i_2$, then $u_2(i) = w_2(i)$. Now, if $i_2 \leqslant i_1$, then $w_2(i_2) = u_1(i_2)$, whence $u_2(i_2) \neq u_1(i_2)$ and u_1, u_2 witness $\Phi(U)$. Else, if $i_1 < i_2$, then we define another finite sequence $w_3 \in A^n$ by cases as follows:

$$w_3(i) = \begin{cases} u_1(i) & 1 \leqslant i \leqslant i_1 \\ u_2(i) & i_1 < i \leqslant i_2 \\ u_2(i_2) & i_2 < i \leqslant n. \end{cases}$$

Proceeding in this manner, we have at hand a method for obtaining a series of elements $u_1, \ldots, u_k \in U$ along with corresponding indices

$$1 \leqslant i_1 < i_2 < \cdots < i_{k-1} < i_k = n$$

with which we can define yet another finite sequence $w \in A^n$ by cases as follows:

$$w(i) = \begin{cases} u_1(i) & 1 \leqslant i \leqslant i_1 \\ u_2(i) & i_1 < i \leqslant i_2 \\ u_3(i) & i_2 < i \leqslant i_3 \\ \vdots & \\ u_k(i) & i_{k-1} < i \leqslant i_k \end{cases}$$

Finally, and once more because of $(*)$, there are $u \in U$ and $i \leqslant \min\{ |u|, n \}$ with $u(i) \neq w(i)$. According to the definition of $w(i)$, it now follows that $\Phi(U)$. ☐

Example 4.46. Let $u, v \in S$. If $|u| \leqslant |v|$, then the following are equivalent.

1. $u \leqslant v$
2. $v \vdash u$

In fact, if $u \leqslant v$, then $v \vdash u$ is an axiom. Conversely, since $\leqslant$ is decidable, it suffices to show that if $v \vdash u$, then $u \nleqslant v$ is contradictory. But if $u \nleqslant v$, then there is $i \leqslant |u|$ such that $u(i) \neq v(i)$, whence $N(u, v)$. According to Proposition 4.45, it follows that we have $u, v \vdash$. Cut yields $v \vdash$, yet there are no inconsistent singletons for $\vdash$.

Turning Proposition 4.45 upside down, with Lemma 4.37 we get the following.

Corollary 4.47. *Let U be a finite subset of A^*. The following are equivalent.*

1. *U is consistent.*
2. *U is a chain.*

Moreover, the set of finite inconsistent subsets of A^* is detachable.

Example 4.48. Suppose that A has at least two distinct elements a and a', and let C be an inhabited finite consistent subset of A^*. By Corollary 4.47, this C is a chain, so let $u \in C$ be its greatest element. Then both $C \cup \{ ua \}$ and $C \cup \{ ua' \}$ are chains, as well. Therefore

$$C, ua \nvdash \qquad \text{and} \qquad C, ua' \nvdash$$

hence C cannot be an atomic conjunction. For if C was decisive, then we had $C \vdash ua$ as well as $C \vdash ua'$. By cut with $ua, ua' \vdash$ we could obtain $C \vdash$, yet C is supposed to be consistent. It follows that the entailment relation of linear spread generates an atomless Boolean algebra in case the underlying set A has at least two distinct elements. It follows further that the spectrum of $\vdash$ is a Cantor space. Cf. the discussion in Section 7.6.

Last but not least, we turn our attention to the more restrictive case of entailment with regard to unbounded *binary* trees $T \subseteq \{ 0, 1 \}^*$, briefly addressing the question as to whether the corresponding inconsistency predicate is decidable.

4.5.4 Omniscience

We conclude this section with a brief glance at König's lemma. There is a vast amount of literature on this subject, and by no means is it meant to be mastered here. Let $T \subseteq \{ 0, 1 \}^*$ be an unbounded detachable binary tree and consider the entailment relation $\vdash$ of linear spread of T. In this context MEL makes an assertion about the existence of infinite paths in T.

Weak König Lemma (WKL). Every unbounded detachable binary tree has an infinite path.

Corollary 4.49. MEL *implies* WKL.

Constructive reverse mathematics has given a direct decomposition of WKL into the lesser limited principle of omniscience (LLPO) and a weak form of dependent choice [35] over **BISH**, see also [142, 143]. It follows that Φ cannot be a decidable predicate for finite subsets of T. Yet excluded middle is not required in its full generality; for instance, it suffices to make use of Bishop's limited principle of omniscience.

Limited principle of omniscience (LPO). Let $\alpha : \mathbb{N} \to \{ 0, 1 \}$ be a binary sequence. Either there exists $n \in \mathbb{N}$ such that $\alpha(n) = 1$, or else $\alpha(n) = 0$ for every $n \in \mathbb{N}$.

Lemma 4.50. *Let T be an unbounded detachable tree and let $\vdash$ be the entailment relation of linear spread of T.* LPO *implies that the set of finite inconsistent subsets of T is detachable.*

Proof. Let U be a finite subset of T. We define a binary sequence α by distinction of cases as follows:

$$\alpha(n) = \begin{cases} 0, & \text{if } \exists v \in T(n + |U|)\, \forall u \in U\, (\, u \leqslant v\,), \\ 1, & \text{else.} \end{cases}$$

By way of LPO, either there is $n \in \mathbb{N}$ such that $\alpha(n) = 1$ and therefore $\Phi(U)$. Otherwise, we have $\alpha(n) = 0$ for every $n \in \mathbb{N}$ and this implies $\neg\Phi(U)$. □

Proposition 4.51. LPO *implies* WKL.

Proof. Let T be an unbounded detachable binary tree and consider the entailment relation $\vdash$ of linear spread of T. According to the preceding Lemma 4.50, the corresponding inconsistency predicate is decidable by way of LPO. Applying Proposition 4.17, we see that this entailment relation has an ideal element. $\square$

Proposition 4.51 is not optimal in view of that the *weak limited principle of omniscience* (WLPO) suffices to derive WKL. Over intuitionistic set theory **IZF** this WLPO sits strictly in between LPO and WKL [130].

From WKL it is possible to infer a suitable form of the *compactness theorem* for entailment relations, and then to show the classical equivalence of WKL and CT for entailment relations on countable sets.[12] It might be interesting to attend to these matters in a more thorough and systematic manner, e.g., by further addressing the question as to the interplay of CT and Brouwer's fan theorem (FAN) in a constructive setting. Suitable forms of completeness, compactness, and Lindenbaum's lemma for propositional logic have been shown equivalent to FAN over a formal system of intuitionistic second-order arithmetic **WKV** ("Weak Kleene–Vesley") [163] in the vein of (classical) reverse mathematics [237].

Remark 4.52. A considerable generalization of the results of this section is possible and this will be carried out in the following Chapter 5. It will be shown that to every *inverse system* $\mathscr{S}$ of finite inhabited sets an entailment relation $\vdash$ can be associated in such a manner that the points of $\vdash$ correspond to the elements of the *inverse limit* of $\mathscr{S}$. As it turns out, the entailment relation of linear spread is just a special case.

4.6 A pattern for maximality

In this final section we quickly present a strategy that underlies most of the case studies the following chapters are concerned with, but no claim as to this being the definitive pattern is made.

Let S be a set with single-conclusion entailment relation $\rhd$. We assume to have at hand a binary symmetric relation $\bowtie$ on S which is finitary, i.e., such that for every $a \in S$ the image

$$\Diamond a = \{\, b \in S : a \bowtie b \,\}$$

is a finite subset of S. On top of $\rhd$ we put multi-conclusion axioms of *single-values* and *totality*

$$a, b \vdash \qquad (\mathrm{s}\ [a \bowtie b])$$
$$\vdash a, \Diamond a \qquad (\mathrm{t})$$

with side condition on the former as indicated. This $\vdash$ has a back-and-forth property as follows.

Lemma 4.53. *For every pair of finite subsets U and V of S, and $a \in S$, the following are equivalent.*

1. $U \vdash V, a$
2. $U, b \vdash V$ *for every* $b \in \Diamond a$.

Proof. Cut with single-values (s) and totality (t), respectively. $\square$

We aim at describing $\vdash$ in terms of $\rhd$ and $\bowtie$. This will be achieved by way of an adequate and hereditary predicate for $\vdash$. To this end, we first introduce an auxiliary predicate on finite subsets U of S by stipulating

$$\mathsf{mv}(U) \;\equiv\; \exists a, b \in S\,(\, a \bowtie b \;\wedge\; U \rhd a \;\wedge\; U \rhd b\,).$$

[12] See [82, Annex] for a sketch on how to obtain compactness for propositional logic by way of LLPO and dependent choice.

This mv is monotone by monotonicity of $\rhd$, and it is adequate, because if $U \rhd a$ and $U \rhd b$ for $a \bowtie b$, then $U \vdash$ follows by cut with (s) since $\vdash$ extends $\rhd$. On top of this we define, still for finite subsets U of S,

$$\Phi(U) \;\equiv\; \exists a_1, \ldots, a_n \in S \, \forall b_1 \in \{\, a_1 \,\} \cup \Diamond a_1 \ldots \forall b_n \in \{\, a_n \,\} \cup \Diamond a_n \, \mathsf{mv}(U, b_1, \ldots, b_n)$$

where $n \geqslant 0$. For instance, if $a \bowtie b$, then $\mathsf{mv}(a, b)$ and therefore $\Phi(a, b)$. This Φ is monotone as well. The main point is that it makes a complete characterization of $\vdash$ possible, which is a consequence of the following.

Proposition 4.54. *Φ is adequate and hereditary for $\vdash$.*

Proof. Since mv is adequate, so is Φ, which can be seen by successive application of cut with (t). As for Φ being hereditary, it suffices to go through the axioms. Every single-conclusion entailment $U \rhd a$ serves as an axiom for $\vdash$. It is straightforward to show that Φ is hereditary for entailments of this form, keeping in mind that $\rhd$ is transitive. Φ being hereditary for single-values (s) is obvious. It remains to consider totality (t). Let U be a finite subset of S, let $a \in S$ and suppose that for every $b \in \{\, a \,\} \cup \Diamond a$ we have $\Phi(U, b)$. This means that for every $b \in \{\, a \,\} \cup \Diamond a$ there are $a_1, \ldots, a_n \in S$ such that whenever $b_1 \in \{\, a_1 \,\} \cup \Diamond a_1, \ldots, b_n \in \{\, a_n \,\} \cup \Diamond a_n$ we have $\mathsf{mv}(U, b, b_1, \ldots, b_n)$. In order to witness $\Phi(U)$, we simply have to collect this data. □

Proposition 4.55. *$\mathfrak{Spec}(\vdash)$ is flat.*

Proof. We have to show that every ideal element of $\vdash$ is maximal. Thus, let α and β be ideal elements of $\vdash$ and suppose that $\alpha \subseteq \beta$. In order to show $\alpha = \beta$, let $a \in \beta$. Since α is ideal for $\vdash a, \Diamond a$ it suffices to rule out $\alpha \between \Diamond a$. To this end, suppose that $b \in \alpha \cap \Diamond a$. In particular, we have $b \in \beta$, whence both $a, b \in \beta$. However, β is ideal for $a, b \vdash$ where $a \bowtie b$, yet ideal elements do not have inconsistent subsets. □

Provided that we work with ER, Proposition 4.55 and Corollary 4.11 together ensure that $\mathfrak{Spec}(\vdash)$ is a set, rather than a proper class. Even without REM, it thus makes sense to quantify over all ideal elements of $\vdash$, in order to describe the subset of S consisting of those elements which belong to every ideal element. We obtain yet another intersection principle, similar to those discussed in the preceding chapters.

Corollary 4.56 (CT)**.**

$$\bigcap \mathfrak{Spec}(\vdash) = \{\, a \in S : \forall b \in \Diamond a \, \Phi(b) \,\} \,.$$

Proof. By CT we have $a \in \bigcap \mathfrak{Spec}(\vdash)$ if and only if $\vdash a$. According to the back-and-forth property from Lemma 4.53, the latter is equivalent to $\forall b \in \Diamond a \, (\, b \vdash \,)$. Since Φ is adequate and hereditary, by Corollary 4.32 we have $b \vdash$ if and only if $\Phi(b)$. □

Actually we do not need to invoke ER in order to ensure that $\mathfrak{Spec}(\vdash)$ is a set, the reason being that $\vdash$ generates a Boolean algebra.

Proposition 4.57. *The generated lattice L for $\vdash$ is complemented.*

Proof. Let $i : S \to L$ be the standard interpretation of $(S, \vdash)$ in the generated lattice. The axiom of totality (t) implies

$$i(a) \vee \bigvee_{b \in \Diamond a} i(b) = 1_L$$

while according to axiom (s) we have

$$i(a) \wedge i(b) = 0_L$$

whenever $a \bowtie b$, and therefore

$$i(a) \wedge \bigvee_{b \in \Diamond a} i(b) = \bigvee_{b \in \Diamond a} i(a) \wedge i(b) = 0.$$

This means that $i(a)$ has complement

$$-i(a) = \bigvee_{b \in \Diamond a} i(b).$$

Every element of L can be expressed in disjunctive normal form in terms of elements each of which is complemented, whence every element of L is complemented. In fact, if $x \in L$, then we can write

$$x = \bigvee_{U \in \mathscr{U}} \bigwedge_{a \in U} i(a)$$

for a certain finite set $\mathscr{U}$ of finite subsets U of S, and we have

$$-x = \bigwedge_{U \in \mathscr{U}} \bigvee_{a \in U} \bigvee_{b \in \Diamond a} i(b). \qquad \square$$

Since in a Boolean algebra every prime filter is an ultrafilter, we can also take from Proposition 4.57 that every ideal element of $\vdash$ is maximal.

Our final remark in this chapter concerns consistent decisive subsets.

Remark 4.58. Recall from Remark 4.25 that if A is an atomic conjunction for $\vdash$, then

$$A^{\vdash} = \{ a \in S : A \vdash a \} \in \mathfrak{Spec}(\vdash).$$

In turn, we now see that if $\vdash$ is generated by means of a pattern as above, then every finitely generated ideal element is decisive. Indeed, suppose that A is a finite consistent subset of S for which $A^{\vdash}$ is an ideal element, and let $a \in S$. Since $A^{\vdash}$ is ideal for totality (t), we know that either $a \in A^{\vdash}$ right away, or else $A^{\vdash} \between \Diamond a$. In case of the second alternative, there is $b \in S$ such that $a \bowtie b$ and $A \vdash b$. The latter entailment can then be cut with single-values (s), i.e., with $a, b \vdash$, and we obtain $A, a \vdash$.

4.7 Conclusion

It is in order to recap. The main purpose of this chapter was to give a thorough introduction to Scott's entailment relations. We have learned about the fundamental connection with distributive lattices, and we have seen some limitations of the semantic method over **CZF**. We have introduced the notion of an adequate and hereditary predicate, which in practice can help explaining inductively generated entailment relations in a non-inductive manner. A case study on paths as ideal elements not only illustrated this, but also indicated the line of reasoning in the following chapters. Finally, we have seen a pattern for generating (multi-conclusion) entailment relations on top of single-conclusion entailment relations in such a way that the corresponding model class is flat. Later on, the reader will easily recognize how this pattern is instantiated. The remaining chapters are solely devoted to further case studies of rather concrete and algebraic instances. In the following chapter, we develop a syntactical approach to inverse limits. This will help for an elementary and constructive version of Sikorski's extension theorem for complete atomic Boolean algebras.

Chapter 5

Inverse limits as spectra

5.1 Introduction

This chapter grew out of a certain discontentment with the somewhat limited applicability of our version of Sikorski's extension theorem [212] for finite Boolean algebras only, which called for a generalization to not necessarily finite codomains. The initial impetus then came from Davide Rinaldi, who first considered axioms for an entailment relation describing lattice maps into a *profinite* Boolean algebra, and asked for an explicit, non-inductive description. As this turned out more intricate than anticipated, the original question lent itself to a considerable generalization with the notion of *pure injectivity* in the context of universal algebra.

We first give an account of *choice for families of finite sets* in simple terms of entailment relations. We obtain an elementary consistency statement which with MEL leads over to its classical counterpart. Imposing further axioms results in the entailment relation of *inverse limit* and, in the setting of universal algebra, leads to the entailment relation describing homomorphisms to a *profinite algebra*. The main result in this chapter is a constructive variant of the classical theorem that every profinite algebra is *pure-injective* among algebras of the same type [178]. This will be helpful for our version of Sikorski's theorem that will be presented in Chapter 6.

5.2 Choice and entailment for inverse systems

5.2.1 Choice

We begin by describing the entailment relation of *choice function* on a family of finite sets, on top of which we will put further structure preserving axioms in several algebraic contexts later on.

Let I be an inhabited set, and let $\mathscr{S} = \{ S_\imath : \imath \in I \}$ be an I-indexed family of finite inhabited sets. We take the disjoint union

$$S = \bigcup_{\imath \in I} \{ \imath \} \times S_\imath$$

as our domain of discourse, and consider the entailment relation $\vdash$ inductively generated by all instances of the following axioms:

$$(\imath, a), (\imath, b) \vdash \qquad (\mathrm{s}\ [a \neq b])$$

$$\vdash \{ (\imath, a) : a \in S_\imath \} \qquad (\mathrm{t})$$

with side-condition as indicated. With regard to semantics, an ideal element of $\vdash$ indeed is nothing but a choice function

$$\mathfrak{c} : I \to \bigcup_{\imath \in I} S_\imath$$

which is to say that for all $\imath \in I$ we have $\mathfrak{c}(\imath) \in S_\imath$.

Remark 5.1. With an additional axiom

$$(\imath, a), (\jmath, a) \vdash$$

where $\imath \neq \jmath$, ideal elements can be forced to be injective. In this manner, entailment relations provide for a "purely syntactical proof of the Marriage Lemma [128]" [74]. Unmistakably, the treatment in [74] precedes the approach taken here.

In the presence of transitivity (T), the interplay of single-values (s) and totality (t) brings about the following "back-and-forth" property of our entailment relation.

Lemma 5.2. *Let U and V be finite subsets of S and let $(\imath, a) \in S$. The following are equivalent.*

(i) $U \vdash V, (\imath, a)$

(ii) $U, (\imath, b) \vdash V$ *for every* $b \in S_\imath \setminus \{ a \}$.

Proof. Apply transitivity (T) with (s) and (t), respectively. □

Let U be a finite subset of S. We stipulate

$$\Phi(U) \equiv \exists \imath \in I \, \exists a, b \in S_\imath \left(a \neq b \wedge (\imath, a), (\imath, b) \in U \right).$$

Apparently, this Φ is monotone. Moreover, $\Phi(U)$ is sufficient for inconsistency of U by way of the axiom of single-values (s) and monotonicity of entailment (M). In other words, Φ is *adequate* (see Definition 4.30). That is, if U is a finite subset of S with $\Phi(U)$, then $U \vdash$. As regards the converse, actually we only need to show that Φ is *hereditary* for the axiom of totality (t), which circumstance is addressed in the following lemma.

Lemma 5.3. *Let U be a finite subset of S and let $\imath \in I$. If $\Phi(U, (\imath, a))$ for every $a \in S_\imath$, then $\Phi(U)$.*

Proof. This is a simple argument. Pick $a \in S_\imath$. According to the assumption, we have $\Phi(U, (\imath, a))$, which is to say that there are $\jmath \in I$ and *distinct* elements b, b' of $S_\jmath$ such that

$$(\jmath, b), (\jmath, b') \in U \cup \{ (\imath, a) \}.$$

If both $(\jmath, b) \in U$ and $(\jmath, b') \in U$, then $\jmath$ witnesses $\Phi(U)$ right away. Otherwise it follows that $\jmath = \imath$, and since $b \neq b'$ either $(\imath, b) \in U$ or $(\imath, b') \in U$. Bringing into play the assumption once more, we have either $\Phi(U, (\imath, b))$ or $\Phi(U, (\imath, b'))$, respectively. Both cases now actually read $\Phi(U)$. □

Lemma 5.3 asserts that Φ is hereditary for the axiom of totality (t). Because Φ is trivially hereditary for single-values (s), and since it provides an adequate predicate for $\vdash$, we obtain from Corollary 4.32 a non-inductive description for inconsistent subsets of S. By way of the back-and-forth property (Lemma 5.2), from this we obtain an explicit description for arbitrary entailments.

Proposition 5.4. *Let U be a finite subset of S. The following are equivalent.*

1. $U \vdash$
2. $\Phi(U)$.

It follows that a finite set of statements is inconsistent if and only if it arises from the axiom of single-values (s) by an application of monotonicity (M). This will not be the case anymore once further axioms have been put on top of (t) and (s)! The preceding proposition with which $\vdash$ has been characterized in terms of Φ immediately yields the consistency of $\vdash$.

Corollary 5.5. *The entailment relation $\vdash$ of choice functions is consistent.*

Proof. We do not have $\Phi(\emptyset)$, hence $\vdash$ is consistent by Corollary 4.33. □

We obtain the principle of *choice for families of finite sets* [137, Form 62] as a semantical consequence of consistency by means of MEL (see Chapter 4).

5.2.2 Inverse systems

Let $(I, \leqslant)$ be a partially ordered set. Recall that I is said to be *directed* if it is inhabited and if for every pair of elements $\imath, \jmath \in I$ there is $\kappa \in I$ such that $\imath \leqslant \kappa$ and $\jmath \leqslant \kappa$. By an I-indexed *inverse system* we here understand a family $\{ S_\imath \}_{\imath \in I}$ of finite discrete inhabited sets together with a family $\{ f_{\imath\jmath} : S_\jmath \to S_\imath \}_{\imath \leqslant \jmath}$ of *transition maps* such that

$$f_{\imath\jmath} \circ f_{\jmath\kappa} = f_{\imath\kappa} \qquad \begin{tikzcd} & S_\jmath \arrow[dr, "f_{\imath\jmath}"] & \\ S_\kappa \arrow[ur, "f_{\jmath\kappa}"] \arrow[rr, "f_{\imath\kappa}"'] & & S_\imath \end{tikzcd}$$

whenever $\imath \leqslant \jmath$ and $\jmath \leqslant \kappa$. The maps $f_{\imath\jmath}$ are also said to be *compatible*. The *inverse limit* $\varprojlim_{\imath \in I} S_\imath$ of this system is given by the set of all choice functions

$$\alpha : I \to \bigcup_{\imath \in I} S_\imath$$

which are such that if $\imath \leqslant \jmath$, then $f_{\imath\jmath} \circ \alpha(\jmath) = \alpha(\imath)$. We now aim at replacing the classical question as to whether $\varprojlim_{\imath \in I} S_\imath$ is inhabited with a constructive substitute, showing instead a suitable entailment relation to be consistent.[1]

In the following, let I be a directed partially ordered set and let

$$(\{ S_\imath \}_{\imath \in I}, \{ f_{\imath\jmath} : S_\jmath \to S_\imath \}_{\imath \leqslant \jmath})$$

be a fixed but arbitrary I-indexed inverse system of finite discrete inhabited sets. As we did in the preceding section, we take the disjoint union

$$S = \bigcup_{\imath \in I} \{ \imath \} \times S_\imath$$

as our domain of discourse. On top of the axioms from the preceding Section 5.2.1, we put axioms of *transition*, and thus consider the entailment relation $\vdash$ that is inductively generated by all instances of the following axioms:

$$\begin{aligned} (\imath, a), (\imath, b) &\vdash && (\mathrm{s}\ [a \neq b]) \\ (\jmath, a) &\vdash (\imath, f_{\imath\jmath}(a)) && (f\ [\imath \leqslant \jmath]) \\ &\vdash \{ (\imath, a) : a \in S_\imath \} && (\mathrm{t}) \end{aligned}$$

with side-conditions as indicated. For instance, if $S_\imath = \{ a \}$ is a singleton set, then we do not have a corresponding axiom for single values on the index $\imath$, while totality reduces to a tautology, i.e., $\vdash (\imath, a)$.

It is straightforward to see that

$$\mathfrak{Spec}(\vdash) = \varprojlim_{\imath \in I} S_\imath .$$

By abuse of notation, in the following we sometimes write, for $a \in S_\imath$ and $\imath \leqslant \jmath$,

$$f_{\imath\jmath}^{-1}(a) = \{ (\jmath, b) \in S : f_{\imath\jmath}(b) = a \} .$$

Lemma 5.6. *Let $a \in S_\imath, b \in S_\jmath$ and suppose that $\imath \leqslant \jmath$. If $f_{\imath\jmath}(b) \neq a$, then $(\imath, a), (\jmath, b) \vdash$.*

Proof. Under the assumptions of the lemma, we have entailments

$$(\imath, a), (\imath, f_{\imath\jmath}(b)) \vdash \qquad \text{and} \qquad (\jmath, b) \vdash (\imath, f_{\imath\jmath}(b))$$

as instances of single-valuedness (s) and transition (f), respectively. By cut we get $(\imath, a), (\jmath, b) \vdash$. ☐

[1] Postulating the existence of an element of the inverse limit of an inverse system of finite discrete inhabited sets is classically equivalent to PIT [137], and hence to CT.

Proposition 5.7. *Let $\imath_1, \dots, \imath_n, \kappa \in I$ be indices such that $\imath_i \leqslant \kappa$ for every $i \in \{1, \dots, n\}$. If $a_1 \in S_{\imath_1}, \dots, a_n \in S_{\imath_n}$, then*

$$(\imath_1, a_1), \dots, (\imath_n, a_n) \vdash \bigcap_{i=1}^{n} f_{\imath_i \kappa}^{-1}(a_i).$$

Proof. Notice first that we can write (t) for S_κ as

$$\vdash \bigcap_{i=1}^{n} f_{\imath_i \kappa}^{-1}(a_i) \cup \{ (\kappa, b) : \exists i \in \{1, \dots, n\} \; f_{\imath_i \kappa}(b) \neq a_i \},$$

since every set in the system is supposed to be discrete. If $b \in S_\kappa$ is such that there is $i \in \{1, \dots, n\}$ with $f_{\imath_i \kappa}(b) \neq a_i$, then

$$(\imath_i, a_i), (\kappa, b) \vdash$$

according to the preceding Lemma 5.6. We successively cut the former instance of totality and establish the claim. □

Next we define an inconsistency predicate Φ for $\vdash$. Let U be a finite subset of S. This U can be written explicitly as $U = \{ (\imath_1, a_1), \dots, (\imath_n, a_n) \}$ and we stipulate

$$\Phi(U) \equiv \exists \kappa \in I \left(\imath_1, \dots, \imath_n \leqslant \kappa \wedge \bigcap_{i=1}^{n} f_{\imath_n \kappa}^{-1}(a_i) = \emptyset \right).$$

Thus U is inconsistent if the elements chosen by U do not have a common preimage under transition in a certain set S_κ above those already indexed in U. Yet in other words, if any choice of an element S_κ interferes along transition with those previously made by U, then U cannot be extended by choice of an element of S_κ, hence cannot be considered a finite approximation of an ideal element of our entailment relation.

Corollary 5.8. Φ *is adequate for* $\vdash$.

Proof. If $\Phi(U)$, then the entailment that holds by Proposition 5.7 has an empty set of conclusions, whence U is inconsistent. □

Proposition 5.9. Φ *is hereditary for* $\vdash$.

Proof. It suffices to show that Φ is hereditary for every initial entailment (Lemma 4.34). First we consider totality (t). Thus, let $U = \{ (\imath_1, a_1), \dots, (\imath_n, a_n) \}$ be a finite subset of S, let $\jmath \in I$, and write $S_\jmath = \{ b_1, \dots, b_m \}$. Suppose that $\Phi(U, (\jmath, b_j))$ for every $j \in \{1, \dots, m\}$, witnessed by indices $\kappa_1, \dots, \kappa_m \in I$, respectively. In particular, we keep in mind that for each $j \leqslant m$ we have $\imath_1, \dots, \imath_n, \jmath \leqslant \kappa_j$. By directedness, there is κ such that $\kappa_1, \dots, \kappa_m \leqslant \kappa$ and we claim that this κ witnesses $\Phi(U)$. Clearly, $\imath_1, \dots, \imath_n \leqslant \kappa$. Suppose that $c \in \bigcap_{i=1}^{n} f_{\imath_i \kappa}^{-1}(a_i)$. Then $f_{\jmath\kappa}(c) \in S_\jmath$, whence $f_{\jmath\kappa}(c) = b_j$ for a certain $j \leqslant m$. Now notice that

$$f_{\kappa_j \kappa}(c) \in \bigcap_{i=1}^{n} f_{\imath_i \kappa_j}^{-1}(a_i) \cap f_{\jmath \kappa_j}^{-1}(b_j),$$

because $f_{\imath_i \kappa_j}(f_{\kappa_j \kappa}(c)) = f_{\imath_i \kappa}(c) = a_i$ for every $i \leqslant n$, and since $f_{\jmath \kappa_j}(f_{\kappa_j \kappa}(c)) = b_j$. Now we have a contradiction, for the latter intersection is supposed to be empty. Thus, Φ is hereditary for (t). Next, Φ being hereditary for (s) is apparent. As regards transition (f), consider again U as above, let $a \in S_\jmath$, and suppose that $\Phi(U, (\imath, f_{\imath\jmath}(a)))$. This is to say that there is $\kappa \in I$ such that $\imath_1, \dots, \imath_n, \imath \leqslant \kappa$ and

$$\bigcap_{i=1}^{n} f_{\imath_i \kappa}^{-1}(a_i) \cap f_{\imath\kappa}^{-1}(f_{\imath\jmath}(a)) = \emptyset.$$

Since I is directed, there is $\kappa' \in I$ such that $\imath, \kappa \leqslant \kappa'$. We claim that $\bigcap_{i=1}^n f_{\imath_i\kappa'}^{-1}(a_i) \cap f_{\jmath\kappa'}^{-1}(a) = \emptyset$. For if $c \in S'_\kappa$ is such that $f_{\jmath\kappa'}(c) = a$ and $f_{\imath_i\kappa'}(c) = a_i$ for $i \leqslant m$, then we have $f_{\imath_i\kappa}(f_{\kappa\kappa'}(c)) = f_{\imath_i\kappa'}(c) = a_i$ as well as $f_{\imath\kappa}(f_{\kappa\kappa'}(c)) = f_{\imath\jmath}(f_{\jmath\kappa'}(c)) = f_{\imath\jmath}(a)$, whence

$$f_{\kappa\kappa'}(c) \in \bigcap_{i=1}^n f_{\imath_i\kappa}^{-1}(a_i) \cap f_{\imath\kappa}^{-1}(f_{\imath\jmath}(a)),$$

contrary to the assumption. □

Corollary 5.10. *Let U be a finite subset of S. The following are equivalent.*

1. $U \vdash$
2. $\Phi(U)$

Finally, here is how to describe entailment non-inductively in terms of transition maps.

Theorem 5.11. *The following are equivalent.*

(i) $(\imath_1, a_1), \dots, (\imath_n, a_n) \vdash (\jmath_1, b_1), \dots, (\jmath_m, b_m)$

(ii) For all $b'_1 \in S_{\jmath_1} \setminus \{ b_1 \}, \dots, b'_m \in S_{\jmath_m} \setminus \{ b_m \}$ *there is* κ *such that*

$$\imath_1, \dots, \imath_n, \jmath_1, \dots, \jmath_m \leqslant \kappa \qquad \textit{and} \qquad \bigcap_{i=1}^n f_{\imath_i\kappa}^{-1}(a_i) \cap \bigcap_{j=1}^m f_{\jmath_j\kappa}^{-1}(b'_j) = \emptyset.$$

Proof. Consider Corollary 5.10 and take into account the back-and-forth property. □

There are no inconsistent singletons if we happen to have at hand a *surjective* inverse system, i.e., one for which every transition map $f_{\imath\jmath} : S_\jmath \to S_\imath$ is onto.

Example 5.12. The following is basic material in algebraic number theory; we closely follow and refer to [191]. Let p be a prime number. Certain complications arising with the definition of the set $\mathbb{Z}_p$ of *p-adic numbers* as given by formal infinite series

$$f = \sum_{i=0}^{\infty} a_i p^i$$

where $0 \leqslant a_i < p$, leads to viewing them rather as sequences of residue classes

$$\overline{s}_n = s_n \bmod p^n \in \mathbb{Z}/p^n\mathbb{Z},$$

where $n \in \mathbb{N} - \{ 0 \}$. There are canonical projections

$$\dots \xrightarrow{\lambda_3} \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{\lambda_2} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\lambda_1} \mathbb{Z}/p\mathbb{Z}$$

for which one has

$$\lambda_n(\overline{s}_{n+1}) = \overline{s}_n$$

Associating to every p-adic integer $f = \sum_{i=0}^{\infty} a_i p^i$ the sequence of residue classes

$$\overline{s}_n = \sum_{i=0}^{n-1} a_i p^i \bmod p^n \in \mathbb{Z}/p^n\mathbb{Z},$$

yields a bijection

$$\mathbb{Z}_p \cong \varprojlim_{n \in \mathbb{N} - \{ 0 \}} \mathbb{Z}/p^n\mathbb{Z}.$$

This is how one obtains the *ring* $\mathbb{Z}_p$ of p-adic integers [191]. Now let us bring into play the fundamental theorem of entailment relations. We see that every p-adic integer corresponds to a prime filter of a certain distributive lattice L. The lattice in question is the one that is generated by the entailment relation which is inductively defined on the disjoint union

$$S = \bigcup_{n>0} \{ n \} \times \mathbb{Z}/p^n\mathbb{Z}$$

by all instances of

$$\begin{aligned} (n, \overline{s}), (n, \overline{s'}) &\vdash && (p^n \nmid s - s') \\ (n+1, \overline{s}) &\vdash (n, s \bmod p^n) \\ &\vdash \{ (n, \overline{0}), (n, \overline{1}), \dots, (n, \overline{p^n - 1}) \} \end{aligned}$$

with side condition as indicated. We have

$$\mathbb{Z}_p \cong \mathfrak{Spec}(\vdash).$$

The lattice generated by this entailment relation in fact is a Boolean algebra, hence every p-adic integer corresponds to an ultrafilter. From a classical topological point of view, $\mathbb{Z}_p$ is a Stone space.

5.2.3 Cofinality and conservation

Let $(I, \leqslant)$ be a directed partially ordered set. Suppose that I' is a subset of I which is directed with respect to the induced ordering. Recall that I' is said to be *cofinal in* I [205] in case

$$\forall \imath \in I \, \exists \imath' \in I' \, (\imath \leqslant \imath').$$

Let $(\{ S_\imath \}_{\imath \in I}, \{ f_{\imath\jmath} : S_\jmath \to S_\imath \}_{\imath \leqslant \jmath})$ be an I-indexed inverse system of finite discrete inhabited sets. If $I' \subseteq I$ is cofinal in I, then we have a *cofinal subsystem*

$$\big(\{ S'_\imath \}_{\imath' \in I'}, \{ f_{\imath'\jmath'} : S_{\jmath'} \to S_{\imath'} \}_{\imath' \leqslant \jmath'} \big).$$

indexed over I'. Now we have two inductively generated entailment relations $\vdash$ and $\vdash'$ as above, considered over $S = \bigcup_{\imath \in I} \{ \imath \} \times S_\imath$ and $S' = \bigcup_{\imath' \in I'} \{ \imath' \} \times S_{\imath'}$, respectively. By way of containment, there is an interpretation of entailment relations

$$\iota : (S', \vdash') \hookrightarrow (S, \vdash).$$

Proposition 5.13. *ι is conservative.*

Proof. It suffices to show conservation for inconsistent subsets of S'. Accordingly, suppose that $(\imath_1, a_1) \dots, (\imath_n, a_n) \vdash$, where $\imath_1, \dots, \imath_n \in I'$. Due to Theorem 5.11, there is $\kappa \in I$ such that $\imath_1, \dots, \imath_n \leqslant \kappa$ and $\bigcap_{i=1}^n f_{\imath_i \kappa}^{-1}(a_i) = \emptyset$. Since we have a cofinal subsystem, we can find $\kappa' \in I'$ with $\kappa \leqslant \kappa'$. Clearly $\bigcap_{i=1}^n f_{\imath_i \kappa'}^{-1}(a_i) = \emptyset$, which translates back to $(\imath_1, a_1) \dots, (\imath_n, a_n) \vdash'$, again by way of Theorem 5.11. $\square$

Remark 5.14. With CT, it follows from Theorem 4.21 that the induced mapping on ideal elements

$$\iota^{-1} : \mathfrak{Spec}(\vdash) \to \mathfrak{Spec}(\vdash'), \quad \alpha \mapsto \alpha \cap S'$$

is surjective. We can further show that cofinality causes ι^{-1} to be injective. Indeed, let α and β be ideal elements of $\vdash$ and suppose that $\alpha \cap S' \subseteq \beta \cap S'$. We need to verify $\alpha \subseteq \beta$. To this end, let $(\imath, a) \in \alpha$. Since I' is cofinal in I, there is $\imath' \in I'$ with $\imath \leqslant \imath'$. Since α is ideal for (t), we know that there is $b \in S'_\imath$ such that $(\imath', b) \in \alpha \cap S'$, whence $(\imath', b) \in \beta \cap S'$. Furthermore, we have an instance of transition (f)

$$(\imath', b) \vdash (\imath, f_{\imath\imath'}(b))$$

which implies $(\imath, f_{\imath\imath'}(b)) \in \alpha \cap \beta$. It remains to show $f_{\imath\imath'}(b) = a$. However, if these elements differ, then we have an instance of single-values (s)

$$(\imath, a), (\imath, f_{\imath\imath'}(b)) \vdash$$

yet as an ideal element α does not have inconsistent subsets. Since $S_\imath$ is discrete, this observation suffices for $f_{\imath\imath'(b)} = a$. It follows that $\alpha \subseteq \beta$, as required.

We have thus established a one-to-one correspondence

$$\varprojlim_{\imath \in I} S_\imath \cong \varprojlim_{\imath' \in I'} S_{\imath'}.$$

It is worth noting that we do not need to pick, for every $\imath \in I$, an element $\imath' \in I'$ such that $\imath \leqslant \imath'$. We refer to [205], where this move is required for defining a correspondence in the first place.

5.3 Some universal algebra

We take a step back from our considerations of inverse limits. In this section we consider algebras in the sense of *Universal Algebra* [70, 124]. First we need to provide several basic and preliminary notions, to which end we closely follow the clear presentation of [125]. By an n-ary operation on a set A, where $n \geqslant 0$ is a natural number, we mean a mapping

$$\omega : A^n \to A$$

where A^n is the n-fold cartesian product of A. Here, the number n is said to be the *arity* of ω. The choice of our tools requires us to restrict attention to operations of *finite* arity only, yet we do not restrict to finite families of operations. Briefly, a *universal algebra* is given by a set together with a set of operations. Universal algebras are classified by their *type*, specifying operations and their arity, accordingly. To be precise, the type of a universal algebra is given by a set T together with a mapping

$$T \to \mathbb{N}, \quad \omega \mapsto n_\omega$$

assigning to each element $\omega \in T$ its formal arity. Now, a *universal algebra* (or simply an *algebra*) of type T is given by a set A together with a mapping

$$T \to \bigcup_{n \in \mathbb{N}} A^{A^n}, \quad \omega \mapsto (\,\omega_A : A^{n_\omega} \to A\,)$$

assigning to each $\omega \in T$ an operation ω_A on A of arity n_ω. A *subalgebra* A' of a universal algebra A is a subset A' of A which is closed under the operations. That is, if $a_1, \ldots, a_n \in A'$ and ω has arity n, then $\omega_A(a_1, \ldots, a_n) \in A'$. If A and B are algebras of the same type T, then a *homomorphism* from A to B is a mapping

$$\varphi : A \to B$$

such that

$$\varphi(\omega_A(a_1, \ldots, a_{n_\omega})) = \omega_B(\varphi(a_1), \ldots, \varphi(a_{n_\omega}))$$

for all $\omega \in T$ and elements $a_1, \ldots, a_{n_\omega} \in A$. For the set of all homomorphisms between algebras A and B of the same type we write

$$\mathrm{Hom}(A, B).$$

If A and B are algebras of the same type, then the product $A \times B$ apparently too carries a structure of the same type with component-wise operation. In the following, all algebras will be considered to have the same type T, which remains fixed and thus will be understood implicitly. All algebras are denoted and referred to by their corresponding and underlying sets of elements.

Now let A and B be algebras, and let B be finite. We take the product

$$S = A \times B$$

for our domain of discourse and consider the entailment relation $\vdash$ which is inductively generated by all instances of the following axioms:

$$
\begin{aligned}
(a,b),(a,b') &\vdash && \text{(s } [b \neq b']) \\
(a_1,b_1),\ldots,(a_{n_\omega},b_{n_\omega}) &\vdash (\omega_A(a_1,\ldots,a_{n_\omega}),\omega_B(b_1,\ldots,b_{n_\omega})) && (\omega) \\
&\vdash \{\,(a,b) : b \in B\,\} && \text{(t)}
\end{aligned}
$$

with side conditions as indicated. As for the semantics of $\vdash$, it is clear that

$$\mathfrak{Spec}(\vdash) = \mathrm{Hom}(A,B).$$

We thus say that $\vdash$ is the entailment relation of homomorphism $A \to B$. Given the apparent level of generality, we do not expect to find an utmost simple, explicit and non-inductive description of $\vdash$. Yet it still suffices to be able to give an explanation of inconsistency, or to understand in which case U *cannot* be considered a finite approximation of a homomorphism $A \to B$. In order to do so, it turns out helpful to make use of the *single-conclusion* entailment relation $\rhd$ of *subalgebra* of $A \times B$, generated by all instances of the algebraic axiom

$$(a_1,b_1),\ldots,(a_{n_\omega},b_{n_\omega}) \rhd (\omega_A(a_1,\ldots,a_{n_\omega}),\omega_B(b_1,\ldots,b_{n_\omega})).$$

Mind that this $\rhd$ is a subrelation of $\vdash$ inasmuch as $U \rhd (a,b)$ implies $U \vdash (a,b)$.

Let U again be a finite subset of S. We introduce the following shorthand notation:

$$\mathsf{mv}(U) \;\equiv\; \exists a \in A\, \exists b,b' \in B\, \big(\, b \neq b' \;\wedge\; U \rhd (a,b) \;\wedge\; U \rhd (a,b')\,\big).$$

Intuitively, if U is a finite subset of S, then $\mathsf{mv}(U)$ holds if and only if the subalgebra $U^{\rhd}$ of $A \times B$, finitely generated by U, assigns distinct values to at least one element of A, whence there is no way for U to be be extended to a homomorphism. We say that U has *multiple values*. In this case U is inconsistent:

Lemma 5.15. *For every finite subset U of S, if $\mathsf{mv}(U)$, then $U \vdash$.*

Proof. Suppose that there are $a \in A$ and $b,b' \in B$ such that $b \neq b'$ and both $U \rhd (a,b)$ and $U \rhd (a,b')$. Likewise, we then have $U \vdash (a,b)$ and $U \vdash (a,b')$. Since $b \neq b'$, we have an axiom $(a,b),(a,b') \vdash$, with which we can cut the former entailment and obtain $U \vdash$. ☐

How about the converse, though? It is tempting and appears rather natural to conjecture that finite inconsistent subsets have to have multiple values, and it does appear so in particular with hands-on experience of concrete examples. However, as it turns out, this is just a special case, sometimes to hold, but in general the formal Nullstellensatz of $\vdash$ is not as easily obtained as a first educated guess would make one believe.

5.3.1 A digression on abelian groups

Groups rank among the prime examples of universal algebras: they come equipped with a binary operation of multiplication, along with a unary one for taking inverses, and a nullary operation that gives rise to the identity element of the group. Now let H and G be *abelian* groups; we suppose that G is finite. We consider the entailment relation $\vdash$ of group homomorphism $H \to G$, inductively generated on $H \times G$ by all instances of the following axioms:

$$
\begin{aligned}
(x,g),(x,h) &\vdash && \text{(s } [g \neq h]) \\
(x,g),(y,h) &\vdash (x+y,g+h) && (+) \\
&\vdash (0_H,0_G) && (0) \\
&\vdash \{\,(x,g) : g \in G\,\} && \text{(t)}
\end{aligned}
$$

with side condition as indicated. As to be expected, we do not need to postulate axioms forcing every ideal element of $\vdash$ to preserve the inverse operation. In Example 5.17 below, we explain how the corresponding entailment already is a consequence of the given set of axioms. We need an auxiliary lemma first.

Lemma 5.16. *Let $g \in G$. If $g \neq 0_G$, then $(0_H, g) \vdash$.*

Proof. Apply transitivity to $(0_H, 0_G), (0_H, g) \vdash$ and $\vdash (0_H, 0_G)$. $\square$

Example 5.17. Suppose that G is discrete. If $\alpha : H \to G$ is a group homomorphism, then of course

$$\alpha(-x) = -\alpha(x)$$

for every $x \in H$. In terms of entailment relations, this elementary fact reads

$$(x, a) \vdash (-x, -a)$$

which can be inferred as follows, and to which end we have assumed G to be discrete. First, for every $b \in G$ such that $b \neq -a$ we have

$$(x, a), (-x, b) \vdash (0_H, a + b)$$

Furthermore, $(0_H, a + b) \vdash$ holds by the preceding lemma because $a + b \neq 0_G$, and transitivity yields $(x, a), (-x, b) \vdash$. Therefore, since G is discrete, we can successively cut totality

$$\vdash \{ (-x, b) : b \neq -a \}, (-x, -a)$$

and obtain $(x, a) \vdash (-x, -a)$. Soundness yields the aforementioned universal argument on group homomorphisms.

Recall that an abelian group G is *divisible* if, for every non-zero integer n and every $a \in G$, there exists $b \in G$ such that $a = nb$.

Proposition 5.18 (CT)**.** *Let G be a finite abelian group. The following are equivalent.*

1. *For every abelian group H, if $(x_1, g_1), \dots, (x_k, g_k) \in H \times G$ and $x \in H$ are such that if for every $g \in G$ there are $m_1, \dots, m_k, m \in \mathbb{Z}$ such that*

$$\sum_{i=1}^{k} m_i x_i + mx = 0_H \quad \textit{and} \quad \sum_{i=1}^{k} m_i g_i + mg \neq 0_G$$

   then there are $n_1, \dots, n_k \in \mathbb{Z}$ such that

$$\sum_{i=1}^{k} n_i x_i = 0_H \quad \textit{and} \quad \sum_{i=1}^{k} n_i g_i \neq 0_G.$$

2. *For every abelian group H, if $\vdash$ is the entailment relation of group homomorphism $H \to G$, and $U = \{ (x_1, g_1), \dots, (x_k, g_k) \}$ is a finite subset of $H \times G$, then the following are equivalent:*

   (i) $U \vdash$

   (ii) There are $m_1, \dots, m_k \in \mathbb{Z}$ such that

$$\sum_{i=1}^{k} m_i x_i = 0_H \quad \textit{and} \quad \sum_{i=1}^{k} m_i g_i \neq 0_G.$$

3. *G is divisible.*

4. *G is trivial.*

Proof. 1. $\Longrightarrow$ 2. We show that the condition on U as expressed by item (ii) is both adequate and hereditary for $\vdash$. With regard to adequacy, notice that if (ii) holds as indicated, then we have $U \vdash (0_H, g)$, where $g = \sum_{i=1}^{k} m_i g_i \neq 0_G$. Because of $(0_H, g) \vdash$, we obtain $U \vdash$ by transitivity. Conversely, (ii) being hereditary for axioms (s), (0), and (+) is immediate. As for totality (t), this is precisely what the hypothesis (1.) asserts.

2. $\Longrightarrow$ 3. We consider the entailment relation $\vdash$ of abelian group homomorphism $\mathbb{Z} \to G$. Suppose that if $U = \{ (n_1, g_1), \ldots, (n_k, g_k) \}$ is a finite subset of $\mathbb{Z} \times G$, then $U \vdash$ if and only if there are $m_1, \ldots, m_k \in \mathbb{Z}$ such that

$$\sum_{i=1}^{k} m_i n_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} m_i g_i \neq 0_G.$$

In order to show G divisible, let $g \in G$ and let $n > 0$. We claim that the singleton set $\{ (n, g) \}$ is consistent. For suppose that $(n, g) \vdash$. Then there is $m \in \mathbb{Z}$ such that

$$mn = 0 \quad \text{and} \quad mg \neq 0_G.$$

But now $m = 0$ is immediate, whence $mg = 0_G$. Now that we have $(n, g) \nvdash$, applying MEL yields an ideal element, i.e., a group homomorphism $\alpha : \mathbb{Z} \to G$ such that

$$g = \alpha(n) = n\alpha(1).$$

Therefore, G is divisible.

3. $\Longrightarrow$ 4. It is well known that if a finite group G is divisible, then it is trivial due to Lagrange's theorem: let $g \in G$ be arbitrary and consider the index $n = |G|$. If G is divisible, then there is $g' \in G$ such that $g = ng'$. Since $ng' = 0$, we have $g = 0$.

4. $\Longrightarrow$ 1. *Ex falso quodlibet.* □

Let us return to our discussion of the preceding subsection, given algebras A and B of the same type, where B is finite. Proposition 5.18 shows that for the entailment relation of homomorphism $A \to B$ we cannot expect in general to have, for every finite subset U of $A \times B$, that $U \vdash$ implies $\mathsf{mv}(U)$.

Remark 5.19. Interestingly, it can be shown that if an arbitrary (not necessarily finite!) abelian group is divisible, then the first item of Proposition 5.18 holds.[2] More generally, this very argument can be phrased in such a manner that it applies to arbitrary divisible modules over principal ideal domains, and not only to abelian groups. Completeness for a generalized notion of entailment relation—allowing for infinite rather than only for finite sets of succedents—leads over to the classical theorem that over a principle ideal domain R, a module over R is injective if and only if it is divisible, invoking of course Baer's criterion for injective modules [247].

5.3.2 Inconsistency

Back to our general setting, where A and B are algebras of the same type, the latter algebra being finite. We study the entailment relation $\vdash$ of homomorphism $A \to B$, considered over the set $S = A \times B$. Next we define, for finite subsets U of S,

$$\Phi(U) \equiv \exists a_1, \ldots, a_n \in A \, \forall b_1, \ldots, b_n \in B \, \mathsf{mv}\big(U, (a_1, b_1), \ldots, (a_n, b_n)\big),$$

where $n \geqslant 0$. Thus, for $n = 0$ we understand $\Phi(U)$ simply as $\mathsf{mv}(U)$. Notice further that this Φ is monotone, since $\rhd$ is. We claim that Φ is both adequate and hereditary for $\vdash$, hence gives rise to an explicit description of $\vdash$.

Lemma 5.20. *Φ is adequate for $\vdash$.*

[2] Davide Rinaldi kindly pointed this out to me, which then led to Proposition 5.18. At the time of writing this text, Rinaldi's argument—along with the question as to the overall applicability of generalized entailment relations for constructive algebra—is subject to research. See Chapter 8 for several perspectives.

Proof. Let U be a finite subset of S and suppose that there are $a_1, \ldots, a_n \in A$ such that for whatever choice of elements $b_1, \ldots, b_n \in B$ we have

$$\mathsf{mv}\big(U, (a_1, b_1), \ldots, (a_n, b_n)\big),$$

thus

$$U, (a_1, b_1), \ldots, (a_n, b_n) \vdash$$

by Lemma 5.15, still for any choice of elements $b_1, \ldots, b_n \in B$. Fix $b_1, \ldots, b_{n-1}$ and write $V = \{ (a_1, b_1), \ldots, (a_{n-1}, b_{n-1}) \}$. Then, for every $b \in B$ we have

$$U, V, (a_n, b) \vdash$$

Now we can successively cut totality

$$\vdash \{ (a_n, b) : b \in B \} ,$$

in order to obtain

$$U, V \vdash$$

and then to argue by induction subsequently. □

Lemma 5.21. Φ *is hereditary for* $\vdash$.

Proof. As usual, we go through the generating axioms only, which suffices for showing Φ to be hereditary overall. For the axiom of single-values, nothing really needs to be checked. We go on to (ω) in which regard it is easy to see that if, say,

$$\Phi(U, (\omega_A(a_1, \ldots, a_{n_\omega}), \omega_B(b_1, \ldots, b_{n_\omega}))),$$

where U is a finite subset of S, then

$$\Phi(U, (a_1, b_1,), \ldots, (a_{n_\omega}, b_{n_\omega}))$$

because of transitivity of $\rhd$. As for totality (t), let $a \in A$ and suppose that for every $b \in B$ we have $\Phi(U, (a, b))$. This is to say that for every $b \in B$ there are $a_1, \ldots, a_n \in A$ such that for any choice of elements $b_1, \ldots, b_n \in B$ we have

$$\mathsf{mv}\big(U, (a, b), (a_1, b_1), \ldots, (a_n, b_n)\big).$$

We put $a_{n+1} = a$ and now have at hand elements $a_1, \ldots, a_{n+1} \in A$ such that for whatever choice of $b_1, \ldots, b_{n+1} \in B$ we have

$$\mathsf{mv}\big(U, (a_1, b_1), \ldots, (a_{n+1}, b_{n+1})\big),$$

just as required for $\Phi(U)$. □

Due to the back-and-forth property that $\vdash$ enjoys, we have obtained not only a characterization of inconsistent subsets of S, but at the same time a formal Nullstellensatz of $\vdash$. However, notice that Φ characterizes a finite subset U of S as inconsistent not in terms of U only, but according to its behaviour with respect to the existence of certain elements $a_1, \ldots, a_n$ of A that may or may not belong to U. This fact is likely to get in the way of a general conservation result.

5.3.3 Interpretation

Let A_1 and A_2 and B be algebras of the same type. Let $\vdash_1$ and $\vdash_2$ be the entailment relations of homomorphism $A_1 \to B$ and $A_2 \to B$, with underlying sets $S_1 = A_1 \times B$ and $S_2 = A_2 \times B$, respectively. Every homomorphism $f : A_1 \to A_2$ induces an interpretation of entailment relations, the corresponding mapping on ideal elements of which is given by composition with f. Here are the precise statements.

Lemma 5.22. *Every homomorphism $f : A_1 \to A_2$ induces an interpretation*

$$i_f : (S_1, \vdash_1) \to (S_2, \vdash_2), \quad (a, b) \mapsto (f(a), b)$$

of entailment relations.

Proof. Since $\vdash_1$ is inductively generated, it suffices to show that i_f interprets every initial entailment. This is clear both for totality (t) and single-values (s). As regards the structural axioms,

$$(a_1, b_1), \ldots, (a_{n_\omega}, b_{n_\omega}) \vdash_1 (\omega_{A_1}(a_1, \ldots, a_{n_\omega}), \omega_B(b_1, \ldots, b_{n_\omega})),$$

we have

$$\begin{aligned} i_f(\omega_{A_1}(a_1, \ldots, a_{n_\omega}), \omega_B(b_1, \ldots, b_{n_\omega})) = & \\ &= (f(\omega_{A_1}(a_1, \ldots, a_{n_\omega})), \omega_B(b_1, \ldots, b_{n_\omega})) \\ &= (\omega_{A_2}(f(a_1), \ldots, f(a_{n_\omega})), \omega_B(b_1, \ldots, b_{n_\omega})) \end{aligned}$$

from which it follows that i_f indeed interprets (ω). $\square$

Of course, i_f is nothing but a homomorphism $A_1 \times B \to A_2 \times B$ of product algebras. Now, if U is a finite inconsistent subset of S_1, i.e., if $U \vdash_1$, then $i_f(U) \vdash_2$ by interpretation. With respect to the converse, due to the back-and-forth property which both $\vdash_1$ and $\vdash_2$ enjoy, it actually suffices to check conservation on empty conclusion entailments only. Conservation has the following semantical reading.

Proposition 5.23 (CT)**.** *Let $f : A_1 \to A_2$ be a homomorphism. The following are equivalent.*

1. *The induced interpretation i_f is conservative.*
2. *For every homomorphism $\alpha : A_1 \to B$ there is a homomorphism $\beta : A_2 \to B$ such that $\alpha = \beta \circ f$.*

Proof. Recall (Theorem 4.21) that an interpretation $(S_1, \vdash_1) \to (S_2, \vdash_2)$ of entailment relations is conservative if and only if the induced mapping $\mathfrak{Spec}(\vdash_2) \to \mathfrak{Spec}(\vdash_1)$ is surjective on ideal elements. In the present context ideal elements are nothing but homomorphisms, between which the induced mapping simply is given by composition with f. $\square$

5.3.4 Conservation and extension

Recall that a formula in a first-order language $\mathscr{L}$ is said to be *positive primitive* if it is (equivalent to) an existentially quantified conjunction of atomic formulae [214]. In our present context, the language $\mathscr{L}$ at hand is determined by first-order logic with identity, the only non-logical axioms being the function symbols corresponding to the given type; there are no relation symbols. With universal algebra taken as a branch of model theory, every algebra A may be considered an $\mathscr{L}$-structure.

Positive primitive formulae φ (with n indeterminates, say) are preserved by homomorphisms, i.e., if $f : A \to B$ is a homomorphism (here, of algebras), and $a_1, \ldots, a_n \in A$, then the validity of $\varphi(a_1, \ldots, a_n)$ in A entails that of $\varphi(f(a_1), \ldots, f(a_n))$ in B. Conversely, a homomorphism $f : A \to B$ for which the validity of $\varphi(a_1, \ldots, a_n)$ in A *follows from* that of $\varphi(f(a_1), \ldots, f(a_n))$ in

B is said to be a *pure* homomorphism [214]. For instance, every pure homomorphism is injective. We refer to [214] for an excellent survey of basic model-theoretic aspects of the notion of purity and its mathematical origins.

Back to our default algebras. Let A_1, A_2 and B be of the same type, and let B be finite. We consider the entailment relations $\vdash_1$ and $\vdash_2$ of algebra homomorphism $A_1 \to B$ and $A_2 \to B$, respectively. Spelling out the definition of our inconsistency predicate Φ shows that it can indeed be given by a positive primitive formula with parameters in B, whence should be reflected by pure homomorphisms between A_1 and A_2. This merely is a brief sketch of an argument which we now unfold up to a point at which the import of purity becomes apparent.

Theorem 5.24. *If $f : A_1 \to A_2$ is a pure homomorphism, then the induced interpretation of entailment relations*

$$i_f : (S_1, \vdash_1) \to (S_2, \vdash_2), \quad (a,b) \mapsto (f(a), b)$$

is conservative.

Proof. It suffices to show conservation for finite inconsistent subsets. To this end, we consider a finite subset $U = \{ (a_1, b_1), \ldots, (a_k, b_k) \}$ of S_1 and suppose that

$$i_f(U) \vdash_2$$

As we have seen in Section 5.3.2, this means that there are $x_1, \ldots, x_n \in A_2$ such that for any choice of $\mathbf{y} = (y_1, \ldots, y_n) \in B^n$ there are $a_{\mathbf{y}} \in A_2$ and $b_{\mathbf{y},1} \in B$ and $b_{\mathbf{y},2} \in B$ with $b_{\mathbf{y},1} \neq b_{\mathbf{y},2}$ and such that

$$(f(a_1), b_1), \ldots, (f(a_k), b_k), (x_1, y_1), \ldots, (x_n, y_n) \rhd (a_{\mathbf{y}}, b_{\mathbf{y},1}) \tag{$*_1$}$$

as well as

$$(f(a_1), b_1), \ldots, (f(a_k), b_k), (x_1, y_1), \ldots, (x_n, y_n) \rhd (a_{\mathbf{y}}, b_{\mathbf{y},2}). \tag{$*_2$}$$

It follows that for every $\mathbf{y} = (y_1, \ldots, y_n) \in B^n$ we can find certain *terms* in the corresponding language (for instance, by an inductive argument)

$$t_{\mathbf{y},1} \in \mathrm{T{\small ER}}_{\mathscr{L}} \quad \text{and} \quad t_{\mathbf{y},2} \in \mathrm{T{\small ER}}_{\mathscr{L}}$$

along with elements

$$z(\mathbf{y})_1, \ldots, z(\mathbf{y})_{\ell(\mathbf{y})} \in \{ (f(a_1), b_1), \ldots, (f(a_k), b_k), (x_1, y_1), \ldots, (x_n, y_n) \}$$

such that the following equations:

$$t_{\mathbf{y},1}(z(\mathbf{y})_1, \ldots, z(\mathbf{y})_{\ell(\mathbf{y})}) = (a_{\mathbf{y}}, b_{\mathbf{y},1})$$

and

$$t_{\mathbf{y},2}(z(\mathbf{y})_1, \ldots, z(\mathbf{y})_{\ell(\mathbf{y})}) = (a_{\mathbf{y}}, b_{\mathbf{y},2})$$

hold in the product algebra $A_2 \times B$. Next we consider them component-wise. Write

$$B^n = \{ \mathbf{y}_1, \ldots, \mathbf{y}_{m^n} \}.$$

We see that in A_2 the following positive primitive formula holds:

$$\exists x_1, \ldots, x_n, u_1, \ldots, u_{m^n} \bigwedge_{i=1}^{m^n} \bigwedge_{j=1,2} t_{\mathbf{y}_i,j}(v_{i,1}, \ldots, v_{i,\ell(\mathbf{y}_i)}) = u_i$$

where

$$v_{i,1}, \ldots, v_{i,\ell(\mathbf{y}_i)} \in \{ \pi_{A_2} z(\mathbf{y}_i)_1, \ldots, \pi_{A_2} z(\mathbf{y}_i)_{\ell(\mathbf{y}_i)} \}.$$

and where $\pi_{A_2} : S_2 \to A_2$ denotes the canonical projection. Since f is pure, it follows that there are

$$x'_1, \ldots, x'_n, u'_1, \ldots, u'_{m^n} \in A_1$$

such that

$$\bigwedge_{i=1}^{m^n} \bigwedge_{j=1,2} t_{\mathbf{y}_i,j}(v'_{i,1},\dots,v'_{i,\ell(\mathbf{y}_i)}) = u'_i$$

with certain elements

$$v'_{i,1},\dots,v'_{i,\ell(\mathbf{y}_i)} \in \{\, a_1,\dots,a_k, x'_1,\dots,x'_n \,\}\,.$$

in the corresponding places. We claim that $x'_1,\dots,x'_n$ witness $U \vdash_1$. Indeed, let again $\mathbf{y} = (y_1,\dots,y_n)$ be an arbitrary element of B^n, say $\mathbf{y} = \mathbf{y}_i$. Thus, there are

$$v'_{i,1},\dots,v'_{i,\ell(\mathbf{y}_i)} \in \{\, a_1,\dots,a_k, x'_1,\dots,x'_n \,\}$$

such that

$$t_{\mathbf{y}_i,j}(v'_{i,1},\dots,v'_{i,\ell(\mathbf{y}_i)}) = u'_i$$

for $j = 1$ and $j = 2$. We can match this with the corresponding atomic formulae that hold in B, i.e., with

$$t_{\mathbf{y},1}(\pi_B z(\mathbf{y})_1,\dots,\pi_B z(\mathbf{y})_{\ell(\mathbf{y})}) = b_{\mathbf{y},1}$$

and

$$t_{\mathbf{y},2}(\pi_B z(\mathbf{y})_1,\dots,\pi_B z(\mathbf{y})_{\ell(\mathbf{y})}) = b_{\mathbf{y},2}$$

where $\pi_B : S_2 \to B$ is the projection. Then, bringing all this together and translating it back in terms of $\rhd$, we have at hand elements $x'_1,\dots,x'_n \in A_1$ such that, for any choice of $y_1,\dots,y_n \in B$, there are $u \in A_1$ and distinct elements $b, b' \in B$ such that

$$(a_1,b_1),\dots,(a_k,b_k),(x'_1,y_1),\dots,(x'_n,y_n) \rhd (u,b)$$

as well as

$$(a_1,b_1),\dots,(a_k,b_k),(x'_1,y_1),\dots,(x'_n,y_n) \rhd (u,b'),$$

just as required for $U \vdash_1$. $\square$

Let us say that an algebra C is *pure-injective* if homomorphisms $A_1 \to C$ factor through pure morphisms $A_1 \to A_2$,

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[dr, "\forall"'] & A_2 \arrow[d, dashed, "\exists"] \\ & C \end{tikzcd}$$

This is to say that if $g : A_1 \to C$ is a homomorphism of algebras and if $f : A_1 \to A_2$ is pure, then there is $h : A_2 \to C$ with $h \circ f = g$. Every injective algebra is a pure-injective algebra, in particular.

Corollary 5.25 (CT). *Every finite algebra is pure-injective.*

Proof. Let B be a finite algebra, let $g : A_1 \to B$ be a homomorphism, and suppose that $f : A_1 \to A_2$ is pure. Consider the interpretation

$$i_f : (S_1, \vdash_1) \to (S_2, \vdash_2)$$

of entailment relations describing homomorphisms $A_1 \to B$ and $A_2 \to B$, respectively. Since f is pure, it follows from Theorem 4.21 and Theorem 5.24 that the induced mapping

$$i_f^{-1} : \mathfrak{Spec}(\vdash_2) \to \mathfrak{Spec}(\vdash_1), \quad \alpha \to \alpha \circ f$$

is surjective. $\square$

Corollary 5.25 is an instance of a more general model-theoretic result: if $\mathscr{L}$ is a first-order language with equality, then every finite $\mathscr{L}$-structure is injective with respect to pure homomorphisms of $\mathscr{L}$-structures [178]. This result leads over to *profinite* $\mathscr{L}$-structures, i.e., structures that arise as inverse limit of an inverse system of finite $\mathscr{L}$-structures [178], by showing that *profinite $\mathscr{L}$-structures are retracts of ultraproducts of finite $\mathscr{L}$-structures* [179]. Rather than mimicking this argument, we again concentrate on providing an elementary and constructive substitute in terms of conservation.

5.4 Profinite algebras

5.4.1 Inverse limits of algebras

Let I be a directed partially ordered set. Let $\{ B_\imath \}_{\imath \in I}$ be an I-indexed family of finite inhabited discrete algebras $B_\imath$, all of which have the same type T, and let $\{ f_{\imath\jmath} : B_\jmath \to B_\imath \}$ be a set of compatible maps. We write

$$P = \varprojlim_{\imath \in I} B_\imath$$

for the inverse limit which carries the structure of an algebra of type T, as well (we refer to [26] for an account of (topological aspects of) profinite universal algebras). Let A be an algebra of type T. We take the disjoint union

$$S = \bigcup_{\imath \in I} A \times B_\imath \times \{ \imath \}$$

for our domain of discourse and study the entailment relation $\Vdash$ on S that is inductively generated by all instances of the following axioms:

$$\begin{aligned}
(a, b, \imath), (a, b', \imath) &\Vdash && \text{(s } [b \neq b']) \\
(a_1, b_1, \imath), \dots, (a_{n_\omega}, b_{n_\omega}, \imath) &\Vdash (\omega_A(a_1, \dots, a_{n_\omega}), \omega_{B_\imath}(b_1, \dots, b_{n_\omega}), \imath) && (\omega) \\
(a, b, \jmath) &\Vdash (a, f_{\imath\jmath}(b), \imath) && (f\ [\imath \leqslant \jmath]) \\
&\Vdash \{ (a, b, \imath) : b \in B_\imath \} && \text{(t)}
\end{aligned}$$

with side condition as indicated. Moreover, for every $\imath \in I$ we have the entailment relation $\vdash_\imath$ of algebra homomorphism $A \to B_\imath$ from the preceding section. We aim at describing $\Vdash$ "locally" in terms of $\vdash_\imath$.

As for the semantics of $\Vdash$, we take note of the following.

Lemma 5.26. *The following are equivalent.*

1. *There is $\alpha \in \mathfrak{Spec}(\Vdash)$.*
2. *For every $\imath \in I$ there is $\alpha_\imath \in \mathfrak{Spec}(\vdash_\imath)$ such that if $\imath \leqslant \jmath$, then $f_{\imath\jmath} \circ \alpha_\jmath = \alpha_\imath$.*

Proof. If $\alpha \in \mathfrak{Spec}(\Vdash)$, then, for every $\imath \in I$, let

$$\alpha_\imath = \{ (x, b) \in A \times B_\imath : (x, b, \imath) \in \alpha \} .$$

This gives a family of ideal elements as required.

Conversely, suppose that there is such a compatible family of ideal elements $\alpha_\imath \in \mathfrak{Spec}(\vdash_\imath)$. By taking the disjoint union we obtain an ideal element α as required, i.e.,

$$\alpha = \coprod_{\imath \in I} \alpha_\imath \in \mathfrak{Spec}(\Vdash). \qquad \square$$

If $U = \{ (x_1, a_1) \dots, (x_k, a_k) \}$ and $V = \{ (y_1, b_1), \dots, (y_\ell, b_\ell \}$ are finite subsets of $A \times B_\imath$ such that $U \vdash_\imath V$, then by an inductive argument it is easy to verify that a similar entailment is inferrable with respect to $\Vdash$ once every occurring formal statement has been "labelled" with index $\imath$, i.e., we have

$$(x_1, a_1, \imath), \dots, (x_k, a_k, \imath) \Vdash (y_1, b_1, \imath), \dots, (y_\ell, b_\ell, \imath)$$

We now aim at a converse. In order to provide a non-inductive description for $\Vdash$, it still suffices to be able to characterize inconsistency. We intend to give this characterization in terms of the entailment relations $\vdash_\imath$ for which an explicit description has already been made available. First we need two auxiliary lemmas.

Lemma 5.27. *Let A and B_1 and B_2 be algebras of the same type. Let $\vdash_1$ and $\vdash_2$ be the entailment relations of homomorphism $A \to B_1$ and $A \to B_2$ on $S_1 = A \times B_1$ and $S_2 = A \times B_2$, respectively. Let $f : B_1 \to B_2$ be a homomorphism. Suppose that*

$$(x_1, a_1), \ldots, (x_k, a_k) \vdash_2 (y_1, b_1), \ldots, (y_\ell, b_\ell).$$

Let

$$b'_1 \in B_2 \setminus \{ b_1 \}, \ldots, b'_\ell \in B_2 \setminus \{ b_\ell \}.$$

For any choice of elements

$$u_1 \in f^{-1}(a_1), \ldots, u_k \in f^{-1}(a_k) \quad \text{and} \quad v_1 \in f^{-1}(b'_1), \ldots, v_\ell \in f^{-1}(b'_\ell)$$

we have

$$(x_1, u_1), \ldots, (x_k, u_k), (y_1, v_1), \ldots, (y_\ell, v_\ell) \vdash_1$$

Proof. We argue by induction, going first through the axioms that generate $\vdash_2$. As concerns single-values (s), our claim is straightforwardly verified. Next we consider an instance of algebraic closure (ω)

$$(x_1, a_1), \ldots, (x_{n_\omega}, a_{n_\omega}) \vdash_2 (\omega_A(x_1, \ldots, x_{n_\omega}), \omega_{B_2}(a_1, \ldots, a_{n_\omega})).$$

Let $u_1 \in f^{-1}(a_1), \ldots, u_{n_\omega} \in f^{-1}(a_{n_\omega})$, let $b \neq \omega_{B_2}(a_1, \ldots, a_{n_\omega})$ and let $v \in f^{-1}(b)$. With regard to $\vdash_1$ we have axioms

$$(x_1, u_1), \ldots, (x_{n_\omega}, u_{n_\omega}) \vdash_1 (\omega_A(x_1, \ldots, x_{n_\omega}), \omega_{B_1}(u_1, \ldots, u_{n_\omega}))$$

and

$$(\omega_A(x_1, \ldots, x_{n_\omega}), \omega_{B_1}(u_1, \ldots, u_{n_\omega})), (\omega_A(x_1, \ldots, x_{n_\omega}), v) \vdash_1$$

the latter of which is an instance of (s). Now apply transitivity. The axiom of totality (t) is straightforwardly dealt with, too. We are left to address the rules. As usual, we concentrate on transitivity (T) only. Consider the instance

$$\dfrac{\dfrac{\vdots}{\{ (x_i, a_i) \} \vdash_2 \{ (y_j, b_j) \}, (z, c)} \qquad \dfrac{\vdots}{\{ (x_i, a_i) \}, (z, c) \vdash_2 \{ (y_j, b_j) \}}}{\{ (x_i, a_i) \} \vdash_2 \{ (y_j, b_j) \}}$$

Now let $b'_j \neq b_j$, let $v_j \in f^{-1}(b'_j)$, and let $u_i \in f^{-1}(a_i)$. Reasoning inductively, by way of the two entailments above the lower inference line, we know that for each $b \in B_2$ and $w \in f^{-1}(b)$ we have

$$\{ (x_i, u_i) \}, \{ (y_j, v_j) \}, (z, w) \vdash_1$$

Notice that

$$\vdash_1 \{ (z, w) : w \in \bigcup_{b \in B_2} f^{-1}(b) \}$$

is an instance of totality. Therefore, by repeated application of transitivity, we obtain

$$\{ (x_i, u_i) \}, \{ (y_j, v_j) \} \vdash_1$$

as required. □

Lemma 5.28. *Let $x \in A$, let $\imath, \jmath \in I$, and let $a \in B_\imath$. If $\imath \leqslant \jmath$, then*

$$(x, a, \imath) \Vdash \{ (x, b, \jmath) : f_{\imath\jmath}(b) = a \}$$

Proof. Since $B_\jmath$ is discrete, totality can be written

$$\Vdash \{ (x, b, \jmath) : f_{\imath\jmath}(b) = a \} \cup \{ (x, b, \jmath) : f_{\imath\jmath}(b) \neq a \}$$

Moreover, for every $b \in B_\jmath$ we have an axiom of transition

$$(x, b, \jmath) \Vdash (x, f_{\imath\jmath}(b), \imath)$$

as well as, in case $f_{\imath\jmath}(b) \neq a$, an axiom for single-values

$$(x, a, \imath), (x, f_{\imath\jmath}(b), \imath) \Vdash$$

Therefore, for every $b \in B_\jmath$, if $f_{\imath\jmath}(b) \neq a$, then

$$(x, a, \imath), (x, b, \jmath) \Vdash$$

Repeated application of transitivity yields

$$(x, a, \imath) \Vdash \{ (x, b, \jmath) : f_{\imath\jmath}(b) = a \}$$ □

Before we embark on proving the following theorem with which we describe $\Vdash$ in terms of its "components" $\vdash_\imath$, let us first do heuristics as to what may cause a finite subset U of S to be inconsistent for $\Vdash$. From the point of view of semantics, if U is inconsistent, then U cannot be considered a finite approximation of a homomorphism $A \to P$. In other words, any attempt towards extending U shall fail. This can happen for the following reasons: U may already assign multiple values at one component, or there is an index $\kappa \in I$ above those occurring in U at which any choice of an element of B_κ interferes along transition with those previously made with U. Actually, the former case reduces to the latter, but anyway: inconsistency can be explained "locally" at the finite algebras $B_\imath$ and their corresponding entailment relations $\vdash_\imath$.

Theorem 5.29. *Let $(x_1, a_1, \imath_1), \ldots, (x_k, a_k, \imath_k) \in S$. The following are equivalent.*

1. $(x_1, a_1, \imath_1), \ldots, (x_k, a_k, \imath_k) \Vdash$
2. *There is $\kappa \in I$ such that $\imath_1, \ldots, \imath_k \leqslant \kappa$ and for all*

$$u_1 \in f^{-1}_{\imath_1\kappa}(a_1), \ldots, u_k \in f^{-1}_{\imath_k\kappa}(a_k)$$

we have

$$(x_1, u_1), \ldots, (x_k, u_k) \vdash_\kappa$$

Proof. We follow the usual strategy and show that a suitable predicate is both adequate and hereditary. Let $U = \{ (x_1, a_1, \imath_1), \ldots, (x_k, a_k, \imath_k) \}$. Consider the predicate Φ as it is defined by way of the second item of the theorem:

$$\Phi(U) \equiv \exists \kappa \in I \left(\imath_i \leqslant \kappa \wedge \forall u_i \in f^{-1}_{\imath_i\kappa}(a_i) \ \{ (x_i, u_i) \} \vdash_\kappa \right)$$

Notice first that Φ is monotone. For suppose that $\Phi(\{ (x_i, a_i, \imath_i) \})$ is witnessed by $\kappa \in I$ and let $\{ (y_j, b_j, \jmath_j) \}$ be another finite subset of S.[3] Since I is directed, there is $\lambda \geqslant \kappa, \jmath_j$. Then let $u_i \in f^{-1}_{\imath_i\lambda}(a_i)$ and $v_j \in f^{-1}_{\jmath_j\lambda}(b_j)$. Since Φ holds for $\{ (x_i, a_i, \imath_i) \}$, we know that

$$\{ (x_i, f_{\kappa\lambda}(u_i)) \} \vdash_\kappa$$

According to Lemma 5.27, we get $\{ (x_i, u_i) \} \vdash_\lambda$. Monotonicity of $\vdash_\lambda$ yields $\{ (x_i, u_i) \}, \{ (y_j, v_j) \} \vdash_\lambda$. Thus $\Phi(\{ (x_i, a_i) \}, \{ (y_j, b_j) \})$, as required.

[3]By abuse of notation, here and in the following we write $\{ (x_i, a_i, \imath_i) \}$ where it should actually read $\{ (x_1, a_1, \imath_1), \ldots, (x_k, a_k, \imath_k) \}$, etc.

Next we show that Φ is both adequate and hereditary for $\Vdash$. As regards the former, suppose that there is $\kappa \geqslant \imath_i$ such that for all $u_i \in f^{-1}_{\imath_i\kappa}(a_i)$ we have $\{\,(x_i, u_i)\,\} \vdash_\kappa$. It is clear that in any such case we too have

$$\{\,(x_1, u_1, \kappa), \ldots, (x_k, u_k, \kappa)\,\} \Vdash$$

According to Lemma 5.28, we have

$$(x_1, a_1, \imath_1) \Vdash \left\{\,(x, u, \kappa) : u \in f^{-1}_{\imath_1\kappa}(a_1)\,\right\}$$
$$\vdots$$
$$(x_k, a_k, \imath_k) \Vdash \left\{\,(x, u, \kappa) : u \in f^{-1}_{\imath_k\kappa}(a_k)\,\right\}$$

Repeated application of transitivity yields $\{\,(x_i, a_i, \imath_i)\,\} \Vdash$. This is to say that $\Phi(U)$ implies $U \Vdash$, whence the second item of Theorem 5.29 does indeed provide an adequate predicate for $\Vdash$. It remains to show that this predicate moreover is hereditary. To this end, it suffices to address every generating axiom. In the following, let $U = \{\,(x_i, a_i, \imath_i)\,\}$, as before.

(s) For the axiom of single-values

$$(x, a, \imath), (x, a', \imath) \Vdash$$

where $a \neq a'$, one may simply take $\kappa = \imath$ in which case $f_{\imath\kappa} = \mathrm{id}_{B_\imath}$. From $\Phi((x, a, \imath), (x, a', \imath))$ we get $\Phi(U, (x, a, \imath), (x, a', \imath))$ by monotonicity of Φ.

(ω) Next we consider axioms of algebraic closure

$$(y_1, b_1, \imath), \ldots, (y_{n_\omega}, b_{n_\omega}, \imath) \Vdash (\omega_A(y_1, \ldots, y_{n_\omega}), \omega_{B_\imath}(b_1, \ldots, b_{n_\omega}), \imath)$$

Suppose that

$$\Phi(U, (\omega_A(y_1, \ldots, y_{n_\omega}), \omega_{B_\imath}(b_1, \ldots, b_{n_\omega}), \imath)) \tag{1}$$

is witnessed by $\kappa \in I$. We need to show

$$\Phi(U, (y_1, b_1, \imath), \ldots, (y_{n_\omega}, b_{n_\omega}, \imath)).$$

Thus, let $u_i \in f^{-1}_{\imath_i\kappa}(a_i)$ and let $v_j \in f^{-1}_{\jmath\kappa}(b_j)$. Take into account that

$$f_{\imath\kappa}(\omega_{B_\kappa}(v_1, \ldots, v_{n_\omega})) = \omega_{B_\imath}(f_{\imath\kappa}(v_1), \ldots, f_{\imath\kappa}(v_{n_\omega})) = \omega_{B_\imath}(b_1, \ldots, b_{n_\omega})$$

whence with (1) we get

$$\{\,(x_i, u_i)\,\}, (\omega_A(y_1, \ldots, y_{n_\omega}), \omega_{B_\kappa}(v_1, \ldots, v_{n_\omega})) \vdash_\kappa$$

Furthermore, there is an axiom

$$(y_1, v_1), \ldots, (y_{n_\omega}, v_{n_\omega}) \vdash_\kappa (\omega_A(y_1, \ldots, y_{n_\omega}), \omega_{B_\kappa}(v_1, \ldots, v_{n_\omega}))$$

Cut yields

$$\{\,(x_i, u_i)\,\}, \{\,(y_j, v_j)\,\} \vdash_\kappa$$

as required.

(f) Next we address the axioms for transition

$$(y, b, \jmath) \Vdash (y, f_{\imath\jmath}(b), \imath)$$

assuming that

$$\Phi(U, (y, f_{\imath\jmath}(b), \imath)) \tag{2}$$

is witnessed by $\kappa \in I$. Since I is directed, there is $\lambda \geqslant \jmath, \kappa$. Now let $u_i \in f^{-1}_{\imath_i\lambda}(a_i)$ and let $v \in f^{-1}_{\jmath\lambda}(b)$. Notice that

$$f_{\kappa\lambda}(u_i) \in f^{-1}_{\imath_i\kappa}(a_i) \qquad \text{and} \qquad f_{\kappa\lambda}(v) \in f^{-1}_{\imath\kappa}(f_{\imath\jmath}(b)),$$

whence

$$\{ (x_i, f_{\kappa\lambda}(u_i)) \}, (y, f_{\kappa\lambda}(v)) \vdash_\kappa$$

according to (2). We obtain

$$\{ (x_i, u_i) \}, (y, v) \vdash_\lambda$$

by way of Lemma 5.27.

(t) It remains to consider totality

$$\Vdash \{ (y, b, \jmath) : b \in B_\jmath \}$$

to which end we write $B_\jmath = \{ b_1, \dots, b_n \}$ and suppose to have

$$\Phi(U, (y, b_1, \jmath)) \quad \dots \quad \Phi(U, (y, b_n, \jmath)) \tag{3}$$

witnessed by $\kappa_1, \dots, \kappa_n \in I$, respectively. Since I is directed, there is $\lambda \geqslant \kappa_1, \dots, \kappa_n$. Let $u_i \in f^{-1}_{\imath_i\lambda}(a_i)$ and let $v_j \in f^{-1}_{\jmath\lambda}(b_j)$. We have

$$f_{\kappa_i\lambda}(u_i) \in f^{-1}_{\imath_i\kappa_i}(a_i) \qquad \text{and} \qquad f_{\kappa_i\lambda}(v_j) \in f^{-1}_{\jmath\kappa_i}(b_j).$$

Therefore, according to (3) we have

$$\begin{aligned} \{ (x_i, f_{\kappa_1\lambda}(u_i)) \}, (y, f_{\kappa_1\lambda}(v_1)) &\vdash_{\kappa_1} \\ &\vdots \\ \{ (x_i, f_{\kappa_n\lambda}(u_i)) \}, (y, f_{\kappa_n\lambda}(v_n)) &\vdash_{\kappa_n} \end{aligned}$$

With Lemma 5.27 we obtain

$$\begin{aligned} \{ (x_i, u_i) \}, (y, v_1) &\vdash_\lambda \\ &\vdots \\ \{ (x_i, u_i) \}, (y, v_n) &\vdash_\lambda \end{aligned}$$

Totality on B_λ can be written as

$$\vdash_\lambda \{ (y, v) : v \in \bigcup_{j=1}^{n} f^{-1}_{\jmath\lambda}(b_j) \}$$

Repeated application of transitivity then yields

$$\{ (x_i, u_i) \} \vdash_\lambda$$

and therefore $\Phi(U)$. The proof is now complete. □

5.4.2 Products

This subsection is a digression from our main line of reasoning, but it will find application in the next chapter. We are still in the setting of the preceding section. Dropping every instance of transition, we obtain the entailment relation that describes maps into a *product*. To be precise, for an I-indexed family of finite inhabited discrete algebras $B_\imath$, we consider the entailment relation which is generated by

$$(a, b, \imath), (a, b', \imath) \Vdash \tag{s $[b \neq b']$}$$

$$(a_1, b_1, \imath), \dots, (a_{n_\omega}, b_{n_\omega}, \imath) \Vdash (\omega_A(a_1, \dots, a_{n_\omega}), \omega_{B_\imath}(b_1, \dots, b_{n_\omega}), \imath) \tag{ω}$$

$$\Vdash \{ (a, b, \imath) : b \in B_\imath \} \tag{t}$$

with side condition as indicated. We suppose that our set I of indices is discrete. In the following, if U is a finite subset of S and $\imath \in I$, then we write

$$U_\imath = \{ (x, a) : (x, a, \imath) \in U \}.$$

Theorem 5.30. *Let U be a finite subset of S. The following are equivalent.*

1. *$U \Vdash$*
2. *There is $\imath \in I$ such that $U_\imath \vdash_\imath$*

Proof. We stipulate, for finite subsets U of S,

$$\Phi(U) \equiv \exists\imath \in I\,(\,U_\imath \vdash_\imath\,)$$

This Φ is adequate for $\Vdash$, for if $U_\imath \vdash_\imath$ can be inferred, then any derivation may just be labelled with $\imath \in I$ such as to yield $U \Vdash$. As for the converse, it is easy to verify that Φ is hereditary for the generating axioms of $\Vdash$. Here the inductive step makes use of the assumption that I is discrete and allows for an argument by cases. For instance, consider totality (t). Let U be a finite subset of S, let $x \in A$, let $\imath \in I$, and suppose that for every $a \in B_\imath$ we have $\Phi(U, (x, a, \imath))$. This is to say that for every $a \in B_\imath$ there is $\imath_a \in I$ such that

$$U_{\imath_a}, \{\,(x, a, \imath)\,\}_{\imath_a} \vdash_{\imath_a}$$

Since I is discrete, either we have $\imath_a = \imath$ for every $a \in B_\imath$, whence $U \vdash_\imath$ follows by cut with an axiom of totality for $\vdash_\imath$. Or else we have $\imath_a \neq \imath$ at least for one $a \in B$, and since $\{\,(x, a, \imath)\,\}_{\imath_a}$ is empty, the above entailment reduces to

$$U_{\imath_a} \vdash_{\imath_a}$$

at once. A similar argument handles the structural axiom (ω), while Φ being hereditary for single-values (s) is immediate. □

Lemma 5.31. *The following are equivalent.*

1. *There is $\alpha \in \mathfrak{Spec}(\Vdash)$.*
2. *For every $\imath \in I$ there is $\alpha_\imath \in \mathfrak{Spec}(\vdash_\imath)$.*

Proof. Cf. Lemma 5.26, the proof of which goes through also without transition maps and their corresponding axioms. □

5.4.3 Conservation and extension

Back to our inverse systems of algebras. Let I be a directed partially ordered set, let

$$\Big(\,\{\,B_\imath\,\}_{\imath\in I}\,, \{\,f_{\imath\jmath} : B_\jmath \to B_\imath\,\}_{\imath\leqslant\jmath}\,\Big)$$

be an I-indexed system of finite discrete inhabited algebras, all of which are of the same type T. We write

$$P = \varprojlim_{\imath\in I} B_\imath$$

for the inverse limit. Furthermore, let A_1 and A_2 algebras of type T. We have two entailment relations $\Vdash_1$ and $\Vdash_2$ of algebra homomorphism $A_1 \to P$ and $A_2 \to P$, respectively. The following is a consequence of Theorem 5.29.

Theorem 5.32. *If $f : A_1 \to A_2$ is a pure homomorphism, then the induced interpretation*

$$i_f : (S_1, \Vdash_1) \to (S_2, \Vdash_2), \quad (a, b, \imath) \mapsto (f(a), b, \imath)$$

is conservative.

Proof. It suffices to verify conservation of inconsistency. To this end, let

$$U = \{ (x_1, a_1, \imath_1), \dots, (x_k, a_k, \imath_k) \}$$

be a finite subset of S_1 and suppose that $i_f(U) \Vdash_2$. According to Theorem 5.29, this can be reduced to inconsistency with regard to an algebra B_κ, which is to say that there is $\kappa \in I$ such that $\imath_1, \dots, \imath_k \leqslant \kappa$ and whenever $u_1, \dots, u_k \in B_\kappa$ are such that $u_i \in f_{\imath_i,\kappa}^{-1}(a_i)$, then $\{ (f(x_i), u_i) \} \vdash_\kappa$. According to Theorem 5.24, since f is pure, any of the latter entailments actually is equivalent to $\{ (x_i, u_i) \} \vdash_\kappa$, respectively. By way of Theorem 5.29 again, this translates back to $U \Vdash_1$, as required. $\square$

Finally, an algebra P (of type T, say) is *profinite* if it is isomorphic to the inverse limit of an inverse system of finite algebras of type T. With completeness we obtain from Theorem 5.32 the following.

Corollary 5.33 (CT). *Every profinite algebra is pure-injective.*

Proof. Let P be a profinite algebra, say

$$P \cong \varprojlim_{\imath \in I} B_\imath$$

and suppose that $g : A_1 \to P$ is a homomorphism of algebras. Let $f : A_1 \to A_2$ pure. We take entailment relations $\Vdash_1$ and $\Vdash_2$ of homomorphisms $A_1 \to P$ and $A_2 \to P$, considered over the disjoint unions $S_1 = \bigcup_{\imath \in I} A_1 \times B_\imath \times \{ \imath \}$ and $S_2 = \bigcup_{\imath \in I} A_2 \times B_\imath \times \{ \imath \}$, and inductively generated thereupon, respectively. The homomorphism $g : A_1 \to P$ is not an ideal element of $\Vdash_1$ in the proper sense, but gives rise to one by setting

$$\alpha_g = \bigcup_{\imath \in I} \{ (x, \pi_\imath \circ g(x), \imath) : x \in A_1 \} \in \mathfrak{Spec}(\Vdash_1)$$

where $\pi_\imath : \varprojlim_{\imath \in I} B_\imath \to B_\imath$ denotes the canonical projection. Since f is pure, the interpretation

$$i_f : (S_1, \Vdash_1) \to (S_2, \Vdash_2), \quad (x, a, \imath) \mapsto (f(x), a, \imath)$$

of entailment relations is conservative, due to which the induced mapping of spectra

$$i_f^{-1} : \mathfrak{Spec}(\Vdash_2) \to \mathfrak{Spec}(\Vdash_1)$$

is onto. Therefore, there is $\beta \in \mathfrak{Spec}(\Vdash_2)$ such that

$$\alpha_g = i_f^{-1}(\beta).$$

By way of Lemma 5.26, for every $\imath \in I$ we have at hand

$$\beta_\imath \in \mathfrak{Spec}(\vdash_\imath)$$

such that if $\imath \leqslant \jmath$, then $\beta_\imath = f_{\imath\jmath} \circ \beta_\jmath$, where as before $\vdash_\imath$ is understood to be the entailment relation of homomorphism $A_2 \to B_\imath$.

$$\begin{tikzcd}
A_1 \arrow[r, "f"] \arrow[d, "g"'] & A_2 \arrow[d, "\beta_\imath"] \arrow[dl, dashed, "h"'] \\
\varprojlim_{\imath \in I} B_\imath \arrow[r, "\pi_\imath"'] & B_\imath
\end{tikzcd}$$

This compatible family of homomorphisms $\beta_\imath : A_2 \to B_\imath$ induces a homomorphism

$$h : A_2 \to \varprojlim_{\imath \in I} B_\imath$$

such that $\pi_\imath \circ h = \beta_\imath$ for every $\imath \in I$.

It remains to show that we actually have $h \circ f = g$. But it is a simple matter of writing, for every $x \in A_1$ and $a \in B_\imath$ and $\imath \in I$,

$$\begin{aligned} \pi_\imath \circ g(x) = a \quad & \equiv \quad (x, a, \imath) \in \alpha_g \\ & \equiv \quad (f(x), a, \imath) \in \beta \\ & \equiv \quad \beta_\imath \circ f(x) = a \\ & \equiv \quad \pi_\imath \circ (h \circ f)(x) = a \end{aligned}$$

in order to see that $g = h \circ f$. □

As indicated before, Corollary 5.33 is an instance of a more general model-theoretic result: if $\mathscr{L}$ is a first-order language with equality, then *every profinite $\mathscr{L}$-structure is injective with respect to pure $\mathscr{L}$-homomorphisms* [178, Theorem 8]. Needless to say, from a classical point of view we have obtained nothing more than a new (albeit perhaps conceptually simpler) proof of an instance of a known theorem. But our focus has been on the syntactical counterpart, i.e., conservation, an elementary and constructive proof of which is possible.

In the following chapter we will see an application of conservation for profinite structures: Sikorski's extension theorem for complete atomic Boolean algebras, along with Monteiro's generalization for extension under bounding semimorphisms.

Chapter 6

Some constructive extension theorems for distributive lattices

This chapter is based on [212, 213].

6.1 Introduction

Due to a time-honoured result by Sikorski (see, e.g., [236, §33] and [127]), the injective objects in the category of Boolean algebras have been identified precisely as the complete Boolean algebras. In other words, a Boolean algebra C is complete if and only if, for every morphism $f : B \to C$ of Boolean algebras, where B is a subalgebra of B', there is an extension $g : B' \to C$ of f onto B'. More generally, it has later been shown by Balbes [22], and Banaschewski and Bruns [27], that a distributive lattice is an injective object in the category of distributive lattices if and only if it is a complete Boolean algebra.

A popular proof of Sikorski's theorem proceeds as follows: by Zorn's lemma the given morphism on B has a maximal partial extension, which by a clever one-step extension principle [33, 127] can be shown to be total on B'. In turn, instantiating C with the initial Boolean algebra $\mathbf{2}$ with two elements results in a classical equivalent of the Boolean prime ideal theorem, a proper form of the Axiom of Choice [216]. Taking up a strictly constructive stance, it is even necessary to object to completeness of $\mathbf{2}$, as this implies the principle of weak excluded middle [30]. We recall further that while in classical set theory Sikorski's theorem is stronger than the Boolean prime ideal theorem [31], the latter principle is in fact equivalent to the restricted form of Sikorski's theorem for complete and atomic Boolean algebras [177].

We can give constructive meaning to Sikorski's extension theorem for finite discrete Boolean algebras by phrasing it as a syntactical conservation result. The idea is as follows. Given a distributive lattice L and a finite discrete Boolean algebra B, we generate an *entailment relation* [229–231] the models of which are precisely the lattice maps $L \to B$. This can be done in a canonical manner, and in particular so with every lattice L' containing L as a sublattice. Then we have two entailment relations $\vdash$ and $\vdash'$, the former of which can be *interpreted* in the latter. By way of a suitable form of the Axiom of Choice, with some classical logic, this interpretation being *conservative* is tantamount to the restriction of lattice maps $L' \to B$ to the sublattice L being surjective—which is extendability. The proof of conservativity, however, is elementary and constructive, and it rests upon the explicit characterization of $\vdash$ in terms of an appropriate *Formal Nullstellensatz* (see Section 6.2.3).

By introducing bounds as additional axioms, we can furthermore address Monteiro's result on extension of maps under preservation of bounds [182]. Last but not least, by employing our results on profinite structures from the preceding chapter, we can further give a generalization for complete and atomic Boolean algebras.

We hasten to add that this approach does not stem from an altogether new idea. It is quite in order to cite Mulvey and Pelletier [184, p. 41], who grasp the "*intuitive content which the*

Hahn-Banach theorem normally brings to functional analysis" in view of that

> "[...] *no more may be proved about the subspace A in terms of functionals on the seminormed space B than may already be proved by considering only functionals on the subspace A.*"

In this spirit, the Hahn-Banach theorem has been revisited by way of type theory [63], as well as in terms of entailment relations [62, 72, 75]. Further approaches to extension theorems in the guise of logical conservation include Szpilrajn's order extension principle [190]. The idea of capturing algebraic structures and in particular their ideal objects by way of entailment relations has been developed and advocated in [62, 72, 87]. The interplay of (multi-conclusion) entailment relations as extending their single-conclusion counterpart is subject of [209], of course building on [229]. Clearly, we draw inspiration from these references.

6.2 Sikorski's theorem

6.2.1 On lattices and Boolean algebras

In the following, every lattice L is considered to be distributive, and bounded, i.e., to have a top and bottom element, 1_L and 0_L, respectively. Subscripts will be written in order to emphasize to which lattice we refer, otherwise they will be omitted. We understand 1 to be the empty meet and 0 to be the empty join in L. Maps between lattices are meant to preserve structure. Mind that a lattice is discrete if and only if its order relation $\leqslant$ is decidable. An *atom* of a lattice L is an element $e \in L$ which is minimal among non-zero elements, i.e. for every $x \in L$, if $0 < x \leqslant e$, then $x = e$; of course $x < y$ is shorthand for $x \leqslant y$ and $x \neq y$. The set of all atoms of L will be denoted $\operatorname{At} L$.

We will be dealing with finite discrete Boolean algebras only, for which there is the following *Structure Theorem* [169, VII, §3, 3.3 Theorem]:

Theorem 6.1. *Every finite discrete Boolean algebra is isomorphic to the algebra of the detachable subsets of a finite discrete set.*

Crucially, every finite discrete Boolean algebra B has a finite set of atoms, and $1_B = \bigvee \operatorname{At} B$. It follows that every element of B is a finite join of atoms. We refer to [169, VII, §3].

There are several classically equivalent ways of describing atoms in a Boolean algebra [156]. They coincide, however, under assumption of discreteness. In particular, for every $e \in B$ the following are equivalent [156],[169, VII, §3].

(i) $e \in \operatorname{At} B$.

(ii) $e > 0$ and, for every $a \in B$, either $e \leqslant a$ or $e \wedge a = 0$.

(iii) $e > 0$ and, for every $a \in B$, either $e \leqslant a$ or $e \leqslant -a$.

Furthermore, if $U \subseteq \operatorname{At} B$ is finite, and $e \in \operatorname{At} B$, then $e \leqslant \bigvee U$ implies $e \in U$. Moreover, since every element of B can be written as a finite join of atoms, if $b \nleqslant b'$, then there is $e \in \operatorname{At} B$ such that $e \leqslant b \wedge -b'$.

6.2.2 Entailments for maps

Now we fix a distributive lattice L and a finite discrete Boolean algebra B, according to the conventions in the preceding Section 6.2.1. Confident that it will not lead to confusion, the order relations on L and B will both be denoted by $\leqslant$. As our domain of discourse we take $L \times B$, and study an entailment relation $\vdash$ on $L \times B$, inductively generated by the set of all instances of the following axioms.

$$
\begin{aligned}
(x,a),(x,b) &\vdash && \text{(s)}\\
(x,a),(y,b) &\vdash (x \wedge y, a \wedge b) && (\wedge)\\
(x,a),(y,b) &\vdash (x \vee y, a \vee b) && (\vee)\\
&\vdash (0_L, 0_B) && (0)\\
&\vdash (1_L, 1_B) && (1)\\
&\vdash \{\, (x,a) : a \in B \,\} && \text{(t)}
\end{aligned}
$$

where $a \neq b$ in (s).

We follow an idea outlined in [79], where suitable axioms of the kind (t) and (s) are taken to present the space of functions $\mathbb{N} \to \{\, 0, 1 \,\}$ by way of a generated entailment relation.

We read any pair (x,a) as $\varphi(x) = a$ for a generic (or yet-to-be-determined) morphism $\varphi : L \to B$ of lattices. In this regard, entailment

$$(x_1,a_1),\ldots,(x_k,a_k) \vdash (y_1,b_1),\ldots,(y_\ell,b_\ell)$$

should intuitively be read as

$$\textit{if } \varphi(x_1) = a_1 \ldots \textit{and} \ldots \varphi(x_k) = a_k, \ \textit{then } \varphi(y_1) = b_1 \ldots \textit{or} \ldots \varphi(y_\ell) = b_\ell.$$

An ideal element $\alpha \subseteq L \times B$ for $\vdash$ is a relation, in the first place. Axiom (t) forces such an α to be total, while the second axiom (s) ensures single values. The remaining axioms are to guarantee that the lattice structure is preserved. We put on record that this entailment relation really describes what we intend it to:

Lemma 6.2. *A subset $\alpha \subseteq L \times B$ is an ideal element of $\vdash$ if and only if it is a homomorphism of lattices.*

Entailments are in good accordance with our intuition about lattice maps. For instance, since every lattice map is order-preserving, we should expect that an entailment like $X \vdash (x,a)$ sets a certain bound on the set of those abstract statements (y,b), which still are consistent with X in case $x \leqslant y$.

Lemma 6.3. *For all $x,y \in L$ and $a,b \in B$, if $x \leqslant y$ and $a \nleqslant b$, then $(x,a),(y,b) \vdash$.*

Proof. We have

$$(x,a),(y,b) \vdash (x, a \wedge b)$$

by $(\wedge)$ and since $x \wedge y = x$. This entailment can be cut with

$$(x,a),(x,a \wedge b) \vdash$$

which we have as an instance of (s) because $a \neq a \wedge b$. □

It will be useful to have means for moving statements back and forth across the turnstile symbol. The idea is as follows. If a set X of statements logically forces an element $x \in L$ to take a certain value $a \in B$ under every given lattice map $L \to B$, then adjoining some statement (x,b) to X should turn out inconsistent in case $b \neq a$.

Lemma 6.4. *For every finite subset $X \subseteq L \times B$ and elements $x \in L, a \in B$, the following are equivalent.*

(i) $X \vdash (x,a), Y$

(ii) $X, (x,b) \vdash Y$ *for every $b \in B$ such that $b \neq a$.*

Proof. Suppose that $X \vdash (x,a), Y$ is inferrable. If $b \neq a$, then $(x,a),(x,b) \vdash$ is an axiom, whence we get $X,(x,b) \vdash Y$ by cut. On the other hand, if $X,(x,b) \vdash Y$ whenever $b \neq a$, then we can successively cut $\vdash \{ (x,b) : b \in B \}$, until we arrive at $X \vdash (x,a), Y$. □

Example 6.5. Lemma 6.4 has a particularly simple form for the Boolean algebra $\mathbf{2} = \{ 0,1 \}$, in which case it amounts to complementation of values: for every $x \in L$ and $i \in \mathbf{2}$ we have $(x,i) \vdash$ if and only if $\vdash (x,-i)$.

6.2.3 A formal Nullstellensatz

Following the tradition of dynamical algebra [82, 92, 165], a *formal Nullstellensatz*[1] for $\vdash$ is a theorem explicitly describing entailment in terms of certain identities within the algebraic structure at hand.

We find it useful to have an additional simple piece of notation. If X is a finite subset of $L \times B$ and $a \in B$, then let

$$X_a = \{ x \in L : (x,a) \in X \}.$$

Note that X_a is finite as well.

The following three lemmas are technical and provide for the proof of Theorem 6.10 below. First we give a condition which is sufficient for finite sets of statements to be inconsistent.

Lemma 6.6. *Let X be a finite subset of $L \times B$. If there is $e \in \mathrm{At}\, B$ such that*

$$\bigwedge_{a \geqslant e} \bigwedge X_a \leqslant \bigvee_{a \geqslant e} \bigvee X_{-a},$$

then X is inconsistent, i.e., $X \vdash$.

Proof. We can write $X = \{ (x_1,a_1),\dots,(x_k,a_k) \}$. Suppose that there is an atom e as indicated. Let $a \in \{ a_1,\dots,a_k \}$. For every $x \in X_a$ we have $X \vdash (x,a)$ by reflexivity. Since

$$\{ (x,a) : x \in X_a \} \vdash \big(\bigwedge X_a, a \big)$$

is inferrable, we get

$$X \vdash \big(\bigwedge X_a, a \big)$$

by cut. In particular, we have such an entailment whenever $a \in \{ a_1,\dots,a_k \}$ and $a \geqslant e$. Thus, writing

$$x = \bigwedge_{\substack{a \geqslant e \\ a \in \{ a_1,\dots,a_k \}}} \bigwedge X_a$$

and

$$b = \bigwedge \{ a \in \{ a_1,\dots,a_k \} : a \geqslant e \},$$

we are able to infer

$$X \vdash (x,b).$$

In a similar manner, for

$$y = \bigvee_{\substack{a \geqslant e \\ a \in \{ -a_1,\dots,-a_k \}}} \bigvee X_{-a}$$

and

$$b' = \bigvee \{ -a : a \in \{ -a_1,\dots,-a_k \} \text{ and } a \geqslant e \}$$

we are able to infer

$$X \vdash (y,b').$$

[1] Hilbert's Nullstellensatz as a proof-theoretic result appears first in [221], see also [162].

It remains to notice that on the one hand we actually have

$$x = \bigwedge_{a \geqslant e} \bigwedge X_a \qquad \text{and} \qquad y = \bigvee_{a \geqslant e} \bigvee X_{-a}.$$

Thus $x \leqslant y$, according to the assumption. On the other hand, we have

$$e \leqslant b \qquad \text{and} \qquad b' \leqslant -e$$

which implies $b \nleqslant b'$. Therefore, by way of Lemma 6.3 we get

$$(x, b), (y, b') \vdash$$

and now $X \vdash$ is immediate. □

We are going to make use of the following combinatorial principle.

Lemma 6.7. *Let A be a finite inhabited set and let L and R be subsets of A. If, for every finite subset U of A, there is an element $a \in A$ such that*

$$(\, a \in U \ \wedge\ a \in L\,) \vee (\, a \notin U \ \wedge\ a \in R\,) \tag{†}$$

then L and R have an element in common, $L \between R$.

Proof. We argue by induction on the finite number of elements of A. First we consider a singleton set $A = \{\, * \,\}$ under assumption of (†). If we instantiate with $U = \emptyset \subseteq A$, then this yields $* \in R$ immediately. If instead we instantiate with $U = \{\, * \,\}$, then we are led to the left-hand disjunct, thus $* \in L$.

Next we consider $A' = A \cup \{\, * \,\}$, where $*$ is an element not among those of A. We suppose that the principle in question is valid for A, and that (†) applies with respect to A'. In particular, for every finite subset U of A, there is $a \in A'$ such that

$$(\, a \in U \ \wedge\ a \in L\,) \vee (\, a \notin U \ \wedge\ a \in R\,).$$

Since either $a \in A$ or $a = *$ we get

$$(\, a \in U \ \wedge\ a \in L \cap A\,) \vee (\, a \notin U \ \wedge\ a \in R \cap A\,) \vee (\, * \in R\,).$$

Then the inductive hypothesis applies, whence we infer $(L \cap A) \between (R \cap A)$ or $* \in R$. Similarly, for every finite subset U of A there is $a \in A'$ such that

$$(\, a \in U \cup \{\, * \,\} \ \wedge\ a \in L\,) \vee (\, a \notin U \cup \{\, * \,\} \ \wedge\ a \in R\,).$$

Again, since either $a \in A$ or $a = *$, we get

$$(\, a \in U \ \wedge\ a \in L \cap A\,) \vee (\, a \notin U \ \wedge\ a \in R \cap A\,) \vee (\, * \in L\,)$$

which with the inductive hypothesis leads to $(L \cap A) \between (R \cap A)$ or $* \in L$. Taken together, this yields

$$(L \cap A) \between (R \cap A) \ \vee\ (\, * \in L \cap R\,),$$

whence we have $L \between R$. □

Remark 6.8. Classically, we could have given a much shorter proof of Lemma 6.7. In the classical setting, if A is finite, then the subset $R \subseteq A$ has to be finite itself. We can then instantiate (†) by $U = R$, from which the result trivially follows.

The following may be considered a cut rule for inconsistent sets of statements.[2]

[2] The following lemma in fact shows that the inconsistency predicate as being defined by Lemma 6.6 is hereditary for totality. In order to provide for a largely self-contained presentation, we refrain in this chapter from an explicit introduction of inconsistency predicates.

Lemma 6.9. *Let X be a finite subset of $L \times B$ and let $x \in L$. If for every $b \in B$ there is $e \in \operatorname{At} B$ such that*

$$\bigwedge_{a \geqslant e} \bigwedge \big(X, (x,b)\big)_a \leqslant \bigvee_{a \geqslant e} \bigvee \big(X, (x,b)\big)_{-a}$$

then there is $e' \in \operatorname{At} B$ such that

$$\bigwedge_{a \geqslant e'} \bigwedge X_a \leqslant \bigvee_{a \geqslant e'} \bigvee X_{-a}$$

Proof. If e is an atom of B, then the inequality

$$\bigwedge_{a \geqslant e} \bigwedge \big(X, (x,b)\big)_a \leqslant \bigvee_{a \geqslant e} \bigvee \big(X, (x,b)\big)_{-a} \tag{\#}$$

amounts to

$$\bigwedge_{a \geqslant e} \bigwedge X_a \wedge x \leqslant \bigvee_{a \geqslant e} \bigvee X_{-a} \tag{L_e}$$

in case $b \geqslant e$, and to

$$\bigwedge_{a \geqslant e} \bigwedge X_a \leqslant \bigvee_{a \geqslant e} \bigvee X_{-a} \vee x \tag{R_e}$$

otherwise, i.e., in case of $b \ngeqslant e$. We need to find an atom e for which both L_e and R_e hold—cut in the lattice L then allows to draw the desired conclusion. To this end, set

$$L = \{\, e \in \operatorname{At} B : L_e \,\} \qquad \text{and} \qquad R = \{\, e \in \operatorname{At} B : R_e \,\}\,.$$

In particular, for every finite subset U of $\operatorname{At} B$ our assumption applies to the finite join $\bigvee U$ for which there is $e \in \operatorname{At} B$ such that either $e \leqslant \bigvee U$ and L_e, or else $e \nleqslant \bigvee U$ and R_e. Taking into account that $e \leqslant \bigvee U$ if and only if $e \in U$, we see that the combinatorial Lemma 6.7 applies, whence we get $L \between R$. ☐

Finally, here is how to describe $\vdash$ explicitly. As it turns out, entailment $X \vdash Y$ amounts to certain inequalities in the lattice L. The formal Nullstellensatz lies at the heart of conservation.

Theorem 6.10 (Formal Nullstellensatz)**.** *For every finite subset X of $L \times B$, and every finite set of pairs $(y_1, b_1), \dots, (y_k, b_k) \in L \times B$, the following are equivalent.*

(i) $X \vdash (y_1, b_1), \dots, (y_k, b_k)$

(ii) For all $b'_1 \neq b_1, \dots, b'_k \neq b_k$ there is $e \in \operatorname{At} B$ such that

$$\bigwedge_{a \geqslant e} \bigwedge \big(X, \{\, (y_i, b'_i) \,\}_{1 \leqslant i \leqslant k}\,\big)_a \leqslant \bigvee_{a \geqslant e} \bigvee \big(X, \{\, (y_i, b'_i) \,\}_{1 \leqslant i \leqslant k}\,\big)_{-a}$$

Proof. Suppose that $X \vdash (y_1, b_1), \dots (y_k, b_k)$ can be inferred. In order to show the desired inequality to hold for every choice of $b'_i \neq b_i$, we argue by induction. This means that we have to go through the axioms, which is readily done, case by case. Next the rules need to be considered; we concentrate on transitivity only, the two remaining ones being almost trivial to check. Hence, if we have a cut

$$\dfrac{\dfrac{\vdots}{X \vdash (y_1, b_1), \dots, (y_k, b_k), (z, c)} \qquad \dfrac{\vdots}{X, (z, c) \vdash (y_1, b_1), \dots (y_k, b_k)}}{X \vdash (y_1, b_1), \dots, (y_k, b_k)}$$

then we are allowed to apply the hypothesis twice, respectively. Now let $b'_1 \neq b_1, \dots, b'_k \neq b_k$ and write $Y' = \{\, (y_1, b'_1), \dots, (y_k, b'_k) \,\}$. According to the hypothesis, there is an atom e such that

$$\bigwedge_{a \geqslant e} \bigwedge \big(X, Y', (z,c)\big)_a \leqslant \bigvee_{a \geqslant e} \bigvee \big(X, Y', (z,c)\big)_{-a}$$

Likewise (recall that we have already fixed $b'_1, \dots, b'_k$) for every $c' \neq c$ there is an atom e' such that

$$\bigwedge_{a \geqslant e'} \bigwedge \bigl(X, Y', (z, c')\bigr)_a \leqslant \bigvee_{a \geqslant e'} \bigvee \bigl(X, Y', (z, c')\bigr)_{-a}$$

But now Lemma 6.9 applies and allows us to get rid of z altogether! This yields the desired result.

As for the converse implication, assuming that assertion (ii) holds, by way of Lemma 6.6 we find that for every choice of elements $b'_1 \neq b_1, \dots, b'_k \neq b_k$ the set

$$X \cup \{\, (y_1, b'_1), \dots, (y_k, b'_k) \,\}$$

is inconsistent. In order to conclude, apply Lemma 6.4 repeatedly. □

How to formally translate this theorem into the classical extension theorem will be explained in the next section. First let us see a couple of interesting entailments which we can infer with the aid of the Nullstellensatz.

Example 6.11. The order relation on L can be expressed in terms of entailment. With Theorem 6.10 it is easy to show that for all $x, y \in L$ the following are equivalent:

(i) $x \leqslant y$

(ii) $(x, 1_B) \vdash (y, 1_B)$

(iii) $(y, 0_B) \vdash (x, 0_B)$

Example 6.12. For every $x \in L$ the following are equivalent:

(i) $x = 0_L$

(ii) $\vdash (x, 0_B)$

(iii) $(x, 1_B) \vdash$

With completeness, this leads over to the assertion that $0 \in L$ is the only element which maps to $0 \in B$ under *every* lattice map $\varphi : L \to B$; see also Section 6.2.6.

Dually, for every $x \in L$ the following are equivalent:

(i) $x = 1_L$

(ii) $\vdash (x, 1_B)$

(iii) $(x, 0_B) \vdash$

Instantiating Theorem 6.10 with empty conclusions, we have at once a simple characterization of inconsistent sets, and the converse of Lemma 6.6.

Corollary 6.13. *For every finite subset X of $L \times B$, the following are equivalent.*

(i) $X \vdash$

(ii) There is an atom $e \in \mathrm{At}\, B$ such that

$$\bigwedge_{a \geqslant e} \bigwedge X_a \leqslant \bigvee_{a \geqslant e} \bigvee X_{-a}$$

Actually, we had hinted at this result before, and one way to prove Theorem 6.10 is to take for granted the Nullstellensatz for inconsistent sets, as it is stated in Corollary 6.13, and then to generalize it by involving "multiple conclusions", all along keeping in mind Lemma 6.4. In other words, Corollary 6.13 is equivalent to the Nullstellensatz.

It is interesting to note that non-triviality of $\vdash$ is for free, given that L is non-trivial:

Corollary 6.14. *The following are equivalent.*

(i) $\emptyset \vdash \emptyset$

(ii) $1 = 0$ *in* L.

Proof. Since we have $\vdash (0_L, 0_B)$ and $\vdash (1_L, 1_B)$ as axioms, the entailment relation $\vdash$ is trivial if and only if $(0_L, 0_B), (1_L, 1_B) \vdash$ can be inferred. But if e is an arbitrary atom of B, then

$$1_L = \bigwedge_{a \geqslant e} \bigwedge \{ (0_L, 0_B), (1_L, 1_B) \}_a \leqslant \bigvee_{a \geqslant e} \bigvee \{ (0_L, 0_B), (1_L, 1_B) \}_{-a} = 0_L.$$

On the other hand, if indeed $1 = 0$ in L, then $\emptyset \vdash \emptyset$ can be inferred accordingly. □

6.2.4 Extension by conservation

Now let us see how Proposition 6.10 relates to the classical extension theorem. Suppose that L and L' are distributive lattices. Given a finite discrete Boolean algebra B, we have two entailment relations as above, which we denote by $\vdash$ and $\vdash'$, respectively, each of which describes lattice maps $L \to B$. Every lattice map $\varphi : L \to L'$ gives way to an *interpretation* (cf. Chapter 4)

$$f_\varphi : (L \times B, \vdash) \to (L' \times B, \vdash'), \quad (x, a) \mapsto (\varphi(x), a).$$

Indeed, it suffices to show that f_φ maps generating axioms for $\vdash$ to those of $\vdash'$, which is clear since φ preserves the lattice structure.

Proposition 6.15. *If* $\varphi : L \to L'$ *is an injective map of lattices and* B *a finite discrete Boolean algebra, then the induced interpretation*

$$f_\varphi : (L \times B, \vdash) \to (L' \times B, \vdash'), \quad (x, a) \mapsto (\varphi(x), a)$$

is conservative, i.e. $f_\varphi(X) \vdash' f_\varphi(Y)$ *implies* $X \vdash Y$.

Proof. It suffices to show conservation of inconsistent sets. Hence, let $X \subseteq L \times B$ and suppose that $f_\varphi(X) \vdash'$. According to Theorem 6.10, there is $e \in \operatorname{At} B$ such that

$$\bigwedge_{a \geqslant e} \bigwedge f_\varphi(X)_a \leqslant \bigvee_{a \geqslant e} \bigvee f_\varphi(X)_{-a}$$

in L'. This means

$$\varphi\Big(\bigwedge_{a \geqslant e} \bigwedge X_a \Big) \leqslant \varphi\Big(\bigvee_{a \geqslant e} \bigvee X_{-a} \Big)$$

and implies $X \vdash$ by injectivity and Theorem 6.10, once more. □

Remark 6.16. If $\varphi : L \to L'$ is a lattice map for which the induced interpretation f_φ is conservative with regard to a finite discrete Boolean algebra B, then φ is injective. In fact, recall from Example 6.11 that we have

$$\varphi(x) \leqslant \varphi(y) \qquad \text{if and only if} \qquad (\varphi(x), 1) \vdash' (\varphi(y), 1),$$

which is to say that $\varphi(x) \leqslant \varphi(y)$ if and only if $f_\varphi(x, 1) \vdash' f_\varphi(y, 1)$. Likewise, $x \leqslant y$ is equivalent to having the entailment $(x, 1) \vdash (y, 1)$. Therefore, if f_φ is conservative, then φ is injective.

Since f_φ is an interpretation, the inverse image mapping of f_φ restricts on ideal elements

$$f_\varphi^{-1} : \mathfrak{Spec}(\vdash') \to \mathfrak{Spec}(\vdash)$$

and it is easy to see that $f_\varphi^{-1}(\alpha) = \alpha \circ \varphi$. Recall from Chapter 4, that—with completeness at hand—conservation amounts to f_φ^{-1} being surjective. Thus, if $\varphi : L \to L'$ is a monomorphism of lattices, then for every $\alpha : L \to B$ there is $\beta : L' \to B$ with $\alpha = \beta \circ \varphi$. In other words, lattice maps $L \to B$ extend along embeddings:

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[dr, "\forall\alpha"'] & L' \arrow[d, dashed, "\exists\beta"] \\ & B \end{tikzcd}$$

With regard to the following, keep in mind that CT implies REM, and thus every finite Boolean algebra is discrete by application of the former.

Corollary 6.17 (CT)**.** *Every finite Boolean algebra is injective in the category of distributive lattices.*

If L is a sublattice of L' and if φ denotes inclusion of the former, then f_φ^{-1} is nothing but the restriction of lattice maps to the sublattice, and the extension is conservative if and only if restriction is surjective. But we have to emphasize again that this requires completeness!

It is well known [22, 27] that every injective distributive lattice is a Boolean algebra. In Section 6.4 we will see that complements are necessary for conservation: a finite discrete distributive lattice which lacks a complement for at least one of its elements cannot allow for a result analogous to Theorem 6.10, and cannot be injective among distributive lattices.

6.2.5 Profinite Boolean algebras

We can finally exploit the techniques from Chapter 5 and bring into action Theorem 5.29. Recall that a Boolean algebra B is said to be *profinite* if there is an inverse system

$$(\{ B_\imath \}_{\imath \in I}, \{ f_{\imath\jmath} : B_\jmath \to B_\imath \}_{\imath \leqslant \jmath})$$

of finite Boolean algebras $B_\imath$ indexed over a directed partially ordered set I such that

$$B \cong \varprojlim_{\imath \in I} B_\imath$$

Let L be a distributive lattice. Given an inverse system as above with the additional proviso that every component $B_\imath$ is discrete, for every $\imath \in I$ we have an entailment relation $\vdash_\imath$ of lattice map $L \to B_\imath$, considered over the set $S_\imath = L \times B_\imath$, inductively generated and with an explicit description as provided in this chapter. Next we take

$$S = \bigcup_{\imath \in I} L \times B_\imath \times \{ \imath \}$$

to be our domain of discourse and we consider the entailment relation $\Vdash$ that is generated by all instances of the following axioms:

$$\begin{aligned}
(x, a, \imath), (x, b, \imath) &\Vdash && \text{(s } [a \neq b]) \\
(x, a, \imath), (y, b, \imath) &\Vdash (x \wedge y, a \wedge b, \imath) && (\wedge) \\
(x, a, \imath), (y, b, \imath) &\Vdash (x \vee y, a \vee b) && (\vee) \\
(x, a, \jmath) &\Vdash (x, f_{\imath\jmath(a),\imath}) && (f[\imath \leqslant \jmath]) \\
&\Vdash (0_L, 0_{B_\imath}, \imath) && (0) \\
&\Vdash (1_L, 1_{B_\imath}, \imath) && (1) \\
&\Vdash \{ (x, a, \imath) : a \in B_\imath \} && (\text{t})
\end{aligned}$$

with side condition as indicated. The semantics of this entailment relation has been described in Chapter 5. In particular, ideal elements of $\vdash$ correspond with lattice maps

$$L \to \varprojlim_{\imath \in I} B_\imath$$

Crucially, this entailment relation can be described by means of Theorem 5.29. Together with Theorem 6.10, we obtain the following. As usual, we concentrate on characterizing inconsistency only.

Corollary 6.18. *Let $(x_1, a_1, \imath_1), \dots, (x_k, a_k, \imath_k) \in S$. The following are equivalent.*

1. $(x_1, a_1, \imath_1), \dots, (x_k, a_k, \imath_k) \Vdash$
2. *There is $\kappa \in I$ such that $\imath_1, \dots, \imath_k \leqslant \kappa$ and whenever elements*

$$b_1 \in f^{-1}_{\imath_1\kappa}(a_1), \dots, b_k \in f^{-1}_{\imath_k\kappa}(a_k)$$

are chosen, there is $e \in \mathrm{At}\, B_\kappa$ such that

$$\bigwedge_{a \geqslant e} \bigwedge \{ (x_1, b_1), \dots, (x_k, b_k) \}_a \leqslant \bigvee_{a \geqslant e} \bigvee \{ (x_1, b_1), \dots, (x_k, b_k) \}_{-a}$$

The general description of entailment derives from Corollary 6.18 by means of the back-and-forth property which $\Vdash$ enjoys, as well.

Now let L' be another distributive lattice. Likewise, there is the entailment relation $\Vdash'$ of lattice map $L' \to \varprojlim_{\imath \in I} B_\imath$ considered over the disjoint union

$$S' = \bigcup_{\imath \in I} L' \times B_\imath \times \{ \imath \}$$

and generated with axioms as above but with elements of L' in place of those of L. Recall that if $\varphi : L \to L'$ is a lattice map, then this φ induces an interpretation of entailment relations

$$i_\varphi : (S, \Vdash) \to (S', \Vdash'), \quad (x, a, \imath) \mapsto (\varphi(x), a, \imath).$$

The following is an immediate consequence of Corollary 6.18.

Corollary 6.19. *If $\varphi : L \to L'$ is an injective map of lattices, then the induced interpretation i_φ of entailment relations is conservative.*

From a classical point of view, recall that a Boolean algebra is profinite if and only if it is a *complete and atomic* Boolean algebra [147]; see also [38, 39]. Keeping this in mind, Corollary 6.19 leads over to the following version of Sikorski's theorem. We refer back to Chapter 5 as well.

Corollary 6.20 (CT)**.** *Every complete atomic Boolean algebra is injective in the category of distributive lattices.*

Let us now return to our setting that provides for finite codomains of lattice maps $L \to B$ only.

6.2.6 Further consequences of completeness

Results about entailment relations can be used to facilitate proofs of certain classical theorems. This advantage has also been pointed out in [82]. Of course, to this end we need to invoke completeness. Here are some examples, the first of which we have already mentioned above (Example 6.12).

Corollary 6.21 (CT)**.** *If L is a distributive lattice, then*

$$\bigcap_{\alpha : L \to B} \ker(\alpha) = \{ 0_L \},$$

where α runs over lattice homomorphisms.

Proof. With the Nullstellensatz it is easy to see that $\vdash (x, 0_B)$ if and only if $x = 0_L$. Now invoke completeness. $\square$

A similar result can be shown about the intersection of all dual kernels, which contains only the single element $1 \in L$.

The following is an important application, cf. [147, Prop. I.2.5].

Corollary 6.22 (CT)**.** *If L is a distributive lattice and $x, y \in L$ are such that $x \nleqslant y$, then there exists a homomorphism of lattices $\varphi : L \to \mathbf{2}$ such that $\varphi(x) = 1$ and $\varphi(y) = 0$.*

Proof. By way of Theorem 6.10 for the Boolean algebra $\mathbf{2}$, we have $x \leqslant y$ if and only if $(x, 1), (y, 0) \vdash$. Therefore, if $x \nleqslant y$, then, by completeness and classical logic, there is an ideal element witnessing $(x, 1), (y, 0) \nvdash$. This is a homomorphism $\varphi : L \to \mathbf{2}$ of lattices such that $\varphi(x) = 1$ and $\varphi(y) = 0$. $\square$

As a corollary, one obtains an easy proof of the following.

Representation theorem (CT) Let L be a distributive lattice. The map

$$\Phi : L \to \mathrm{Pow}(\mathfrak{Spec}(\vdash)), \quad x \mapsto \{\, \varphi : \varphi(x) = 1 \,\}$$

is a monomorphism of lattices. Therefore, every distributive lattice is isomorphic to a sublattice of a powerset-lattice.

6.3 Injective Heyting algebras

A (bounded) lattice L is said to be a *Heyting algebra* if, for every pair of elements $x, y \in L$, there is an element $x \to y \in L$ such that, for every $z \in L$,

$$z \leqslant x \to y \qquad \text{if and only if} \qquad z \wedge x \leqslant y.$$

It is well-known that any Heyting algebra is distributive [147]. A homomorphism of Heyting algebras is a lattice homomorphism that preserves *implication* ($\to$). Every Boolean algebra B is a Heyting algebra with $x \to y \equiv -x \vee y$. If L is a Heyting algebra, *negation* is defined by $\neg x \equiv x \to 0$. A Heyting algebra L is a Boolean algebra if and only if $\neg\neg x = x$ for every $x \in L$. An element $x \in L$ is said to be *regular* if $\neg\neg x = x$. The set $L_{\neg\neg}$ of all regular elements of L with the induced order is a Boolean algebra: it is a sub-meet-semilattice of L, with joins defined by $x \vee_{L_{\neg\neg}} y \equiv \neg\neg(x \vee y)$. We refer to [147]. It is well known that *Booleanization*

$$\neg\neg : L \to L_{\neg\neg}, \quad x \mapsto \neg\neg x$$

is a homomorphism of Heyting algebras [23].

As shown by Balbes and Horn [23], a Heyting algebra is injective (in the category of Heyting algebras) if and only if it is a complete Boolean algebra. The proof of this result employs Sikorski's theorem and argues with the Boolean algebra of regular elements of a Heyting algebra. We adopt the idea and consider the corresponding conservation result with regard to finite discrete Boolean algebras.

To this end, let L be a Heyting algebra and B a finite discrete Boolean algebra. The entailment relation of Heyting algebra morphisms $L \to B$ is inductively generated by the set of all instances of the following axioms.

$$(x, a), (x, b) \vdash \tag{s}$$
$$(x, a), (y, b) \vdash (x \wedge y, a \wedge b) \tag{$\wedge$}$$
$$(x, a), (y, b) \vdash (x \vee y, a \vee b) \tag{$\vee$}$$
$$(x, a), (y, b) \vdash (x \to y, a \to b) \tag{$\to$}$$
$$\vdash (0_L, 0_B) \tag{0}$$
$$\vdash (1_L, 1_B) \tag{1}$$
$$\vdash \{\, (x, a) : a \in B \,\} \tag{t}$$

where $a \neq b$ in (s).

Thus $\vdash$ generates from the entailment relation of lattice maps $L \to B$ by adjoining additional axioms for implication.

Even though L need not be Boolean itself, the value of an arbitrary element $x \in L$ under a Heyting algebra homomorphism $L \to B$ is determined by the value of $\neg\neg x$, and vice versa:

Lemma 6.23. *For every $(x, a) \in L \times B$ we have $(x, a) \dashv\vdash (\neg\neg x, a)$.*

Proof. Both $(x, a), (0_L, 0_B) \vdash (\neg x, \neg a)$ and $(\neg x, \neg a), (0_L, 0_B) \vdash (\neg\neg x, \neg\neg a)$ are instances of axiom $(\to)$. By cut with $\vdash (0_L, 0_B)$, and since $\neg\neg a = a$ in B, we get $(x, a) \vdash (\neg\neg x, a)$. Once we have this entailment, it follows that $(x, a), (\neg\neg x, b) \vdash$ for every $b \neq a$. Therefore, we may cut (t) for x accordingly, and infer the converse entailment, too. $\square$

Now let $\vdash'$ denote the entailment relation of lattice maps $L_{\neg\neg} \to B$, generated as before without the axiom for implication. Notice that double negation induces an interpretation

$$(L \times B, \vdash) \to (L_{\neg\neg} \times B, \vdash), \quad (x, a) \mapsto (\neg\neg x, a).$$

In the other direction we have an inclusion $L_{\neg\neg} \hookrightarrow L$ that preserves meets but not in general joins. However, an axiom of the form

$$(x, a), (y, b) \vdash' (x \vee_{L_{\neg\neg}} y, a \vee b)$$

means

$$(x, a), (y, b) \vdash' (\neg\neg(x \vee y), a \vee b)$$

which can be inferred also with regard to $\vdash$, by way of axiom $(\vee)$ and in view of Lemma 6.23. It follows that we have a conservative interpretation of entailment relations. Employing Theorem 6.10, we get the formal Nullstellensatz for this entailment relation. Here is how to describe inconsistent sets explicitly:

Corollary 6.24. *For every finite subset X of $L \times B$, the following are equivalent.*

(i) $X \vdash$

(ii) There is an atom $e \in \operatorname{At} B$ such that

$$\bigwedge_{a \geqslant e} \bigwedge \neg\neg X_a \leqslant \bigvee_{a \geqslant e} \bigvee \neg\neg X_{-a}.$$

The general Nullstellensatz for $\vdash$ derives from the description of inconsistent sets. Conservation is an immediate consequence.

6.4 From conservation to complements

At the outset, the way in which we have generated the entailment relation in Section 6.2.2 did not depend on the structure of B as a Boolean algebra, and might as well be carried out with any finite lattice D instead. At least the ideal elements would exactly be the lattice maps $L \to D$. One might thus be tempted to question whether and to what extent complements in D are necessary at all in order to allow for a corresponding conservation result. Incidentally, the entailment relation for **2**-valued maps has an important application, demonstrated in [62, Theorem 11], which may be used to resolve this question:

Proposition 6.25. *If $B, i : L \times \mathbf{2} \to B$ is the distributive lattice generated by $(L \times \mathbf{2}, \vdash)$, then B is a Boolean algebra and L embeds in B.*

Now let us say that a finite distributive lattice D is *conservative* in case the following holds: if L and L' are distributive lattices, L being a sublattice of L', then $(L \times D, \vdash) \hookrightarrow (L' \times D, \vdash')$ is a conservative extension of entailment relations, where $\vdash$ and $\vdash'$ are generated as in 6.2.2, with D in place of B, respectively.

Proposition 6.26 (CT)**.** *For every finite distributive lattice D, the following are equivalent.*

(i) D is complemented.

(ii) D is conservative.

(iii) D is injective.

Proof. We have already seen that every finite (discrete) Boolean algebra is conservative in the sense specified before, and injectivity is a classical consequence of completeness. On the other hand, let D be a finite distributive lattice and suppose that it is injective among distributive lattices. This D can be considered a sublattice of a Boolean algebra B, applying, for instance, Proposition 6.25.

$$\begin{tikzcd} D \arrow[r, hook] \arrow[dr, "\mathrm{id}_D"'] & B \arrow[d, dashed, "\exists f"] \\ & D \end{tikzcd}$$

By way of injectivity, it follows that D is the homomorphic image of a Boolean algebra, whence Boolean itself. □

What goes wrong in case D is not Boolean? Towards an answer, Proposition 6.26 might not be considered all too helpful, taking into account that its proof invokes CT. There is another, more concrete argument, which provides an explicit counterexample to conservation: let D be a finite discrete distributive lattice, and suppose that $d_0 \in D$ is not complemented. We consider the lattice $\mathbf{2}^2 = \{ (0,0), (0,1), (1,0), (1,1) \}$. Let $\vdash \subseteq \mathbf{2}^2 \times D$ be the entailment relation of D-valued lattice maps on $\mathbf{2}^2$, generated by axioms as in Section 6.2.2, with D in place of B. Let

$$X = \{ ((0,0), 0_D), ((1,1), 1_D), ((0,1), d_0) \} .$$

For every $d \in D$ we have $X, ((1,0), d) \vdash$. This is because d_0 is supposed to not have a complement, and therefore, for any $d \in D$, either we have $d_0 \wedge d \neq 0$ or $d_0 \vee d \neq 1$. Involving appropriate instances of $(\wedge)$ or $(\vee)$ as well as corresponding instances of (s), we infer that for every $d \in D$ the set $X, ((1,0), d)$ is inconsistent with respect to $\vdash$. Then we instantiate (t), which reads

$$\vdash \{ ((1,0), d) : d \in D \} ,$$

and by way of cut we get $X \vdash$. However, this set X is *not* inconsistent for the entailment relation of D-valued maps on the sublattice $\{ (0,0), (0,1), (1,1) \}$. In fact, for this very entailment relation X is an ideal element! We conclude that a finite distributive lattice, which lacks a complement for at least one of its elements, cannot be conservative either.

6.5 Monteiro's theorem

In the following, we assume every lattice L under consideration to be non-trivial, which is to say that $0_L \neq 1_L$. If L and D are lattices, by a $\vee$-*semilattice map* $d : L \to D$ we understand a function preserving joins as well as the top and bottom elements. Mind that every $\vee$-*semilattice* map is order preserving. As for extension of lattice maps for which semilattice maps provide a bound, the following generalisation of Sikorski's extension theorem is possible [182].

Monteiro's theorem (ZFC). Let C be a complete Boolean algebra. Let B be a Boolean algebra and $d : B \to C$ a $\vee$-semilattice map. If A is a Boolean subalgebra of B and $\varphi : A \to C$ a lattice map such that $\varphi \leqslant d\big|_A$, then there is an extension $\psi : B \to C$ of φ with $\psi \leqslant d$.

Even though this result does appear quite similar, Monteiro's original proof differs substantially from Sikorski's. In this section we provide a constructive version of Monteiro's result for complete and atomic Boolean algebras. To this end, we need to modify the entailment relation we used for Sikorski's theorem in order to have d-dominated lattice maps as corresponding ideal elements. The constructive version and proof will be obtained employing similar techniques as led to the proof of Sikorski's. The requirement of having a Boolean algebra emerges naturally and is applied straightforwardly.

6.5.1 Bounds as axioms

We concentrate on the initial Boolean algebra $\mathbf{2} = \{0, 1\}$ with only two elements. Later we will employ a slightly different strategy than the one leading to Sikorski's theorem by means of profiniteness. Let L and L' be distributive lattices. We are now interested in extending lattice maps $L \to \mathbf{2}$ on sublattices L of L', while maintaining bounds set by a $\vee$-semilattice map $d : L' \to \mathbf{2}$, we add

$$\vdash (x, 0) \qquad (x \in \ker d)$$

with side condition as indicated. Equivalently, we may take the axiom of totality (t) as follows:

$$\vdash (x, 0), (x, d(x)) \qquad (\mathrm{t}_d)$$

leaving the other axioms unchanged. A subset α of $L \times \mathbf{2}$ is an ideal element of $\vdash$ if and only if it is a map $\alpha : L \to \mathbf{2}$ of lattices such that $\alpha(x) \leqslant d(x)$ for every $x \in L$. In other words, an ideal element of $\vdash$ is a lattice map α which is *bounded* (or *dominated*) by d, briefly $\alpha \leqslant d$.

Here is the formal Nullstellensatz for $\vdash$. Due to the fact that we are dealing with $\mathbf{2}$ rather than with an arbitrary finite discrete Boolean algebra B, the proof actually simplifies the one of Theorem 6.10 once the additional data for the bounding morphism is being ignored.

Theorem 6.27. *Let X and Y be finite subsets of $L \times \mathbf{2}$. The following are equivalent.*

(i) $X \vdash Y$

(ii) There is $z \in \ker d$ *such that*

$$\bigwedge X_1 \wedge \bigwedge Y_0 \leqslant \bigvee X_0 \vee \bigvee Y_1 \vee z.$$

Proof. Suppose that $X \vdash Y$. In order to show the desired inequality to hold for a certain $z \in \ker d$, we proceed by induction. The axioms are quickly checked, and as far as the rules are concerned, we only need to concentrate on cut, say

$$\dfrac{\dfrac{\vdots}{X \vdash Y, (x, a)} \qquad \dfrac{\vdots}{X, (x, a) \vdash Y}}{X \vdash Y}$$

employing the inductive hypothesis accordingly. Thus, in each of both cases $a = 0$ and $a = 1$, we have

$$\bigwedge X_1 \wedge \bigwedge Y_0 \wedge x \leqslant \bigvee X_0 \vee \bigvee Y_1 \vee z$$

as well as

$$\bigwedge X_1 \wedge \bigwedge Y_0 \leqslant \bigvee X_0 \vee \bigvee Y_1 \vee x \vee z'$$

for certain elements $z, z' \in \ker d$. By way of cut in the lattice L we get

$$\bigwedge X_1 \wedge \bigwedge Y_0 \leqslant \bigvee X_0 \vee \bigvee Y_1 \vee z \vee z'$$

where $d(z \vee z') = d(z) \vee d(z') = 0$.

On the other hand, suppose that (ii) holds. Then we have

$$\big(\bigwedge X_1 \wedge \bigwedge Y_0, 1\big) \vdash \big(\bigvee X_0 \vee \bigvee Y_1 \vee z, 1\big).$$

By an inductive argument, axiom ($\wedge$) can be generalized so as to be applicable to arbitrary finite meets. It follows that we have

$$X, -Y \vdash \big(\bigwedge X_1 \wedge \bigwedge Y_0, 1\big)$$

and we get

$$X, -Y \vdash \big(\bigvee X_0 \vee \bigvee Y_1 \vee z, 1\big)$$

by cut. Similarly,

$$X, -Y, (z, 0) \vdash \big(\bigvee X_0 \vee \bigvee Y_1 \vee z, 0\big)$$

With axiom (s) we get

$$X, -Y, (z, 0) \vdash$$

Since $d(z) = 0$, we have $\vdash (z, 0)$, whence $X, -Y \vdash$ by cut. Moving $-Y$ to the right hand side by the back-and-forth principle, we obtain $X \vdash Y$. ☐

Again we have non-triviality of $\vdash$ as a direct consequence of its formal Nullstellensatz.

Corollary 6.28. $\vdash$ *is non-trivial, i.e.,* $\emptyset \nvdash \emptyset$.

Proof. According to Theorem 6.27, we have $\emptyset \vdash \emptyset$ if and only if $1_L \in \ker d$, which would amount to $0 = 1$ in $\mathbf{2}$. ☐

Here are further consequences of the Nullstellensatz, similar to the observations made in Example 6.12.

Corollary 6.29. *For every $x \in L$, the following are equivalent.*

(i) $\vdash (x, 0)$

(ii) $x \in \ker d$.

Proof. By Theorem 6.27, if $\vdash (x, 0)$, then there is $z \in \ker d$ such that $x \leqslant z$. Since d is monotone, $x \in \ker d$. On the other hand, if $d(x) = 0$, then $\vdash (x, 0)$ is an axiom. ☐

Corollary 6.30. *For every $x \in L$, the following are equivalent.*

(i) $\vdash (x, 1)$

(ii) There is $z \in \ker d$ such that $x \vee z = 1$.

Proof. Direct application of Theorem 6.27. ☐

Corollary 6.31. *For every $x \in L$, the singleton set $\{\, (x, d(x)) \,\}$ is consistent, i.e. $(x, d(x)) \nvdash$.*

Proof. Let $x \in L$ and suppose that $(x, d(x)) \vdash$. Cut with (t_d) yields $\vdash (x, 0)$, whence $d(x) = 0$ by Corollary 6.29. Therefore, we have $(x, 0) \vdash$ according to our assumption, and $\vdash (x, 0)$ is an instance of (t_d). With another cut we obtain $\emptyset \vdash \emptyset$ which is impossible by way of Corollary 6.28. ☐

Semantically, the corollary means that for every $x \in L$ there is a d-dominated lattice map $\alpha : L \to \mathbf{2}$ such that $\alpha(x) = d(x)$. This is an instance of a corollary of Monteiro's theorem [182, 8. Corollaire] for the two-element Boolean algebra.

Remark 6.32. We may consider the dual notion of $\wedge$-semilattice morphism, providing a lower (rather than an upper) bound e which is to be maintained under extension. In this case, the axiom of totality (t) would appear as

$$\vdash (x, e(x)), (x, 1) \tag{t_e}$$

We might as well consider both a lower and an upper bound at once, for which we would have to put axiom (t) as

$$\vdash \{\, (x, a) : e(x) \leqslant a \leqslant d(x) \,\} \tag{$\mathrm{t}_{e,d}$}$$

of course with the remaining axioms unchanged.

6.5.2 Conservation with bounds

Let L' be a distributive lattice and let $d : L' \to \mathbf{2}$ be a $\vee$-semilattice morphism. Let L be a sublattice of L'. We have two entailment relations, $\vdash$ and $\vdash'$ on $L \times \mathbf{2}$ and $L' \times \mathbf{2}$, respectively, the first of which generated with respect to the restricted $\vee$-semilattice map $d\big|_L : L \to \mathbf{2}$. By way of containment,

$$\iota : (L \times \mathbf{2}, \vdash) \hookrightarrow (L' \times \mathbf{2}, \vdash')$$

we have an interpretation of entailment relations. However, conservation is not immediate anymore, in view of that the formal Nullstellensatz (Theorem 6.27) involves elements z which may not belong to the sublattice L. We thus characterize lattices which do allow for conservation.

Definition 6.33. *Let L' be a distributive lattice, and let $d : L' \to \mathbf{2}$ be a $\vee$-semilattice morphism. We say that a sublattice L of L' is d-*reductive *if and only if, for all $x, y \in L$,*

$$\exists z \in \ker d \,(x \leqslant y \vee z) \;\to\; \exists z_0 \in \ker d\big|_L \,(x \leqslant y \vee z_0).$$

Example 6.34. If L' is discrete, then

$$d(x) = \begin{cases} 0 & \text{if } x = 0_{L'} \\ 1 & \text{if } x \neq 0_{L'} \end{cases}$$

defines a $\vee$-semilattice map $d : L' \to \mathbf{2}$ for which $\ker d = 0$ and thus every sublattice L of L' is d-reductive. It is clear that d is the largest $\vee$-semilattice map on L'. In particular, d dominates every lattice map $\alpha : L' \to \mathbf{2}$. In a classical setting, this d can always be defined as above by distinction of cases. It follows that Monteiro's theorem (see below) on extension of dominated maps does indeed generalize the one of Sikorski.

Lemma 6.35. *Let L be a sublattice of L' and let $d : L' \to \mathbf{2}$ be a $\vee$-semilattice map. The following are equivalent.*

(i) L is d-reductive.

(ii) The inclusion

$$(L \times \mathbf{2}, \vdash) \hookrightarrow (L' \times \mathbf{2}, \vdash')$$

is a conservative interpretation.

Proof. Suppose that L is d-reductive and let X and Y be finite subsets of $L \times \mathbf{2}$. By Theorem 6.27, if $X \vdash' Y$, then there is $z \in \ker d$ such that

$$\bigwedge X_1 \wedge \bigwedge Y_0 \leqslant \bigvee X_0 \vee \bigvee Y_1 \vee z.$$

Then, since L is d-reductive, there is $z_0 \in \ker d\big|_L$ with

$$\bigwedge X_1 \wedge \bigwedge Y_0 \leqslant \bigvee X_0 \vee \bigvee Y_1 \vee z_0$$

and this translates back to $X \vdash Y$, again by way of Theorem 6.27.

On the other hand, suppose that $\imath$ is conservative, and let $x, y \in L$ and $z \in \ker d$ be such that $x \leqslant y \vee z$. By Theorem 6.27, we see that

$$(x, 1) \vdash' (y, 1)$$

and thus $(x, 1) \vdash (y, 1)$ by conservation. Employing Theorem 6.27 once more, we get $z_0 \in \ker d\big|_L$ such that $x \leqslant y \vee z_0$. $\square$

Remark 6.36. In the general context of distributive lattices, d-reductiveness may fail bluntly, hence so may conservation and, semantically, extension of dominated lattice maps [20]. A simple counterexample is given by the sublattice $L = \{ (0,0), (0,1), (1,1) \}$ of $\mathbf{2}^2$, the latter being endowed with the map $d : \mathbf{2}^2 \to \mathbf{2}$ given by projection to the second component. For $x = (1,1) \in L$ and $y = (0,1) \in L$ we can write

$$(1,1) \leqslant (0,1) \vee (1,0),$$

in $\mathbf{2}^2$, where $d(1,0) = 0$. However, there is no inequality $x \leqslant y \vee z$ with $z \in \ker d\big|_L$.

There is one natural hypothesis which provides d-reductiveness and moves us closer to Monteiro's result:

Lemma 6.37. *If B is a Boolean algebra and $d : B \to \mathbf{2}$ a $\vee$-semilattice map, then every Boolean subalgebra of B is d-reductive.*

Proof. Let A be a Boolean subalgebra of B, let $x, y \in A$ and $z \in \ker d$, and suppose that $x \leqslant y \vee z$. We have

$$-y \wedge x \leqslant -y \wedge (y \vee z) = -y \wedge z \leqslant z,$$

whence $d(-y \wedge x) = 0$. Moreover,

$$x \leqslant y \vee x = y \vee (-y \wedge x).$$

Therefore, for the required element of the subalgebra we may take $z_0 = -y \wedge x \in A$. $\square$

Notice that in the proof of Lemma 6.37 only the fact that y has a complement was used.

6.5.3 Monteiro's theorem

It remains to adapt our strategy for profinite codomains (Chapter 5) such as to accommodate dominance and to provide for Monteiro's theorem for complete atomic Boolean algebras. Rather than doing so, we present a different route that could also have been employed towards Sikorski's theorem above. Let L be a distributive lattice and let κ be a discrete set. Suppose that $d : L \to \mathbf{2}^\kappa$ is a $\vee$-semilattice map. We now take the set

$$S = \bigcup_{\imath \in \kappa} L \times \mathbf{2} \times \{ \imath \}$$

for our domain of discourse and consider the entailment relation $\vdash$ that is inductively generated by all instances of the following axioms.

$$\begin{aligned}
(x, a, \imath), (x, b, \imath) &\Vdash && \text{(s } [a \neq b]) \\
(x, a, \imath), (y, b, \imath) &\Vdash (x \wedge y, a \wedge b, \imath) && (\wedge) \\
(x, a, \imath), (y, b, \imath) &\Vdash (x \vee y, a \vee b) && (\vee) \\
&\Vdash (0_L, 0_{B_\imath}, \imath) && (0) \\
&\Vdash (1_L, 1_{B_\imath}, \imath) && (1) \\
&\Vdash (x, 0_B, \imath) && (\text{d } [x \in \ker \pi_\imath \circ d]) \\
&\Vdash \{ (x, a, \imath) : a \in B_\imath \} && (\text{t})
\end{aligned}$$

with side condition as indicated, where $\pi_\imath : \mathbf{2}^\kappa \to \mathbf{2}$ denotes the projection corresponding to $\imath \in \kappa$. As can easily be verified, ideal elements of $\vdash$ correspond with d-dominated lattice maps $L \to \mathbf{2}^\kappa$. With an inductive argument, it is straightforward to characterize $\Vdash$, compare also 5.4.2. To this purpose, if X is a finite subset of S and $\imath \in I$, then we write

$$X_\imath = \{\, (x,a) : (x,a,\imath) \in X \,\}$$

Moreover, for every $\imath \in \kappa$ we have an entailment relation $\vdash_\imath$ that describes $\pi_\imath \circ d$-dominated lattice maps $L \to \mathbf{2}$.

Proposition 6.38. *Let X be a finite subset of S. The following are equivalent.*

1. $X \Vdash$
2. *There is $\imath \in \kappa$ such that $X_\imath \vdash_\imath$.*

The following is an immediate consequence and may be considered a constructive and generalized version of Monteiro's theorem.

Theorem 6.39. *Let L and L' be distributive lattices, L being a sublattice of L', and let κ be a discrete set. Let $d : L' \to \mathbf{2}^\kappa$ be a $\vee$-semilattice map. Let $\Vdash$ and $\Vdash'$ be the entailment relations of $d\big|_L$-dominated and d-dominated lattice maps $L \to \mathbf{2}^\kappa$ and $L' \to \mathbf{2}^\kappa$, respectively. If L is $\pi_\imath \circ d$-reductive for every $\imath \in \kappa$, then the inclusion*

$$(L \times \mathbf{2} \times \kappa, \Vdash) \hookrightarrow (L' \times \mathbf{2} \times \kappa, \Vdash')$$

is a conservative interpretation of entailment relations.

For *discrete* distributive lattices L', Theorem 6.39 directly leads over to the unbounded constructive version of Sikorski's theorem: define a $\vee$-semilattice map d as in Example 6.34 in which case both dominance and reductiveness are evident.

Instantiating Theorem 6.39 with Boolean algebras A and B in place of L and L', respectively, the former being a Boolean subalgebra of the latter, we obtain a constructive version of Monteiro's theorem by means of Lemma 6.37. From the point of view of semantics, extendability of dominated homomorphisms of Boolean algebras is a straightforward classical consequence of the Completeness Theorem for entailment relations.

Corollary 6.40 (CT). *Let C be a complete and atomic Boolean algebra. Let B be a Boolean algebra and $d : B \to C$ a $\vee$-semilattice map. If A is a Boolean subalgebra of B and $\varphi : A \to C$ a lattice map such that $\varphi \leqslant d\big|_A$, then there is an extension $\psi : B \to C$ of φ with $\psi \leqslant d$.*

6.6 Conclusion

By using Scott's notion of entailment relation, we spelled out the constructive content of Sikorski's theorem in the form that every complete and atomic Boolean algebra is injective in the category of distributive lattices. By adding further axioms, the same procedure led us to a constructive version of Monteiro's theorem. The corresponding classical version can be canonically retrieved by applying the principle of completeness for entailment relations.

Any concrete outcome aside, we stress that the ideas employed here do not strictly depend on the lattice structure, but more generally could be applied to other categories of algebraic structures and their corresponding homomorphisms. For instance, it is easy to figure out how to define entailment relations describing group homomorphisms with finite codomain. Nevertheless, a proper formal Nullstellensatz cannot always be found, as this relates to certain properties of the structure involved[3].

[3] In this chapter, being a Boolean algebra.

We keep for future research a general investigation into the approach taken here, along with further case studies on injectivity. To this end, it may be necessary to employ a notion of entailment relation which allows for infinite sets of conclusions in order to force ideal elements to be total. This could be attained in the context of dynamical algebra [92, 165, 169], an analogous logic-free approach employed in constructive algebra.

Chapter 7

Ordering groups syntactically

This chapter is based on [254].

7.1 Introduction

In this chapter, some steps are taken towards an understanding of certain topological aspects that arise in the theory of ordered groups from the point of view of constructive mathematics. A starting point of motivation is a seminal result due to Sikora [235], asserting that if $n > 1$, then the collection of orders of $\mathbb{Z}^n$ which are compatible with the abelian group structure is a Cantor space if suitably topologized. In order to prove his result, Sikora made use of the famous topological characterization of the Cantor set due to Brouwer [55]. This characterization is well-known to have an algebraic reading: any two countable atomic Boolean algebras are isomorphic [156].

Starting with an arbitrary group G, we consider the entailment relation of *positive cone* of G. Every ideal element of this entailment relation corresponds with a strict linear order of G. This, as we hasten to add, is an idea that for abelian groups has been put forward repeatedly ever since the importance of entailment relations for abstract algebra was pointed out [62, 79]. The *fundamental theorem* of entailment relations [62] then allows for a topological reading, and paves the way for a new proof of Sikora's result by means of Stone duality. However, up to this point our reasoning is constructive throughout. In particular, our approach does not require us to put forward any argument involving an abstract order conceived as a completed totality—more often than not, the existence of which is intimately linked with a suitable form of the Axiom of Choice [114, 137].

This chapter is structured as follows. First, in Section 7.2, we study the entailment relation of positive cone of a group, and we give a constructive version of the well-known orderability test for groups [114], the contrapositive of which in fact provides the *formal Nullstellensatz* for the entailment relation at hand. The case of abelian groups leads over to a constructive version of Levi's theorem in Section 7.3, to the effect that an abelian group is torsion-free if and only if the entailment relation of positive cone of G does not collapse. In Section 7.4, we consider the entailment relation of positive cone of $\mathbb{Z}^n$, and we show that the corresponding generated lattice is atomless. This carries over to discrete torsion-free finite-rank groups in Section 7.5. In Section 7.6, we briefly indicate how to obtain Sikora's theorem by a duality argument.

7.2 Entailment and ordered groups

Let G be a group, with identity element $e \in G$. We start with the single-conclusion entailment relation $\rhd$ of *normal subsemigroup* of G, inductively generated by the set of all instances of the following axioms of *multiplicative closure* and *normality*:

$$a, b \rhd ab \tag{m}$$

$$a \rhd xax^{-1} \tag{n}$$

By introducing this single-conclusion entailment relation, we provide a convenient shorthand notation which will be used later on. If U is a finite subset of G, then

$$U^{\rhd} = \{ a \in G : U \rhd a \}$$

indeed is the normal subsemigroup finitely generated by U.

Lemma 7.1. *Let U be a finite subset of G and let $a \in G$. The following are equivalent.*

1. $U \rhd a$
2. *There are $a_1, \dots, a_k \in U$ and $x_1, \dots, x_k \in G$ such that*

$$a = x_1 a_1 x_1^{-1} \cdots x_k a_k x_k^{-1}$$

where $k \geqslant 1$.

Proof. If a can be written as a product $x_1 a_1 x_1^{-1} \cdots x_k a_k x_k^{-1}$ with $a_i \in U$, then this can be disassembled along the initial entailments, such as to yield $U \rhd a$. Conversely, since $\rhd$ is generated by axioms, we can reason inductively. This is straightforward for (m) and (n), as well as for reflexivity (R) and monotonicity (M). As regards transitivity (T), suppose that

$$a = x_1 a_1 x_1^{-1} \cdots x_k a_k x_k^{-1} \qquad \text{and} \qquad b = y_1 b_1 y_1^{-1} \cdots y_\ell b_\ell y_\ell^{-1}$$

where $a_1, \dots, a_k \in U \cup \{ b \}$ and $b_1, \dots, b_\ell \in U$. Whenever we have $a_i = b$ in the former, we can substitute, in which case

$$x_i a_i x_i^{-1} = x_i y_1 b_1 (x_i y_1)^{-1} \cdots x_i y_\ell b_\ell (x_i y_\ell)^{-1},$$

and we obtain an expression as required. □

Now we restrict attention to non-identities, each of which to be considered an abstract statement. Hence we take

$$S = G \setminus \{ e \}$$

as our domain of discourse. On top of the restricted axioms of $\rhd$ we put multi-conclusion axioms describing *totality* and *single-valuedness*, that is, we consider the entailment relation $\vdash$ on S that is inductively generated by all instances of the following axioms:

$$
\begin{aligned}
a, a^{-1} &\vdash && \text{(s)}\\
a, b &\vdash ab && \text{(m } [ab \neq e])\\
a &\vdash xax^{-1} && \text{(n)}\\
&\vdash a, a^{-1} && \text{(t)}
\end{aligned}
$$

with side condition on the second axiom (m) as indicated, i.e., we postulate axiom (m) only in case $b \neq a^{-1}$. The identity element e will play the role of "falsity" later on and allows to capture inconsistency of finite subsets of S in terms of $\rhd$.

An ideal element α of $\vdash$ is nothing but the *positive cone* for a *strict linear order* $<$ of G that is compatible with the group structure—given α, it is well-known that such an order can be obtained by stipulating [114]

$$a < b \quad \equiv \quad a^{-1} b \in \alpha.$$

Accordingly, we say that $\vdash$ is the entailment relation of *positive cone* of G.

Remark 7.2. We restrict attention to *bi-orderable* groups, as opposed to *left-*, or *right-orderable* groups [69]. However, later on our focus will be on abelian groups, in which context the notions of bi-, left-, and right-orderability coincide.

Example 7.3. We take this example from [69]. Consider the fundamental group K of the Klein bottle. This group has presentation

$$\langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

The entailment relation $\vdash$ of positive cone of K has two instances of axiom (n)

$$\begin{aligned} b &\vdash aba^{-1} \\ b^{-1} &\vdash a^{-1}b^{-1}a \end{aligned}$$

which according to the generating relation reduce to

$$\begin{aligned} b &\vdash b^{-1} \\ b^{-1} &\vdash b \end{aligned}$$

By cut with totality (t) and single-valuedness (s), it follows that $\vdash$ collapses:

$$\dfrac{\dfrac{\vdash b, b^{-1} \qquad b^{-1} \vdash b}{\vdash b} \qquad \dfrac{b \vdash b^{-1} \qquad b, b^{-1} \vdash}{b \vdash}}{\vdash}$$

From the point of view of semantics, K cannot be ordered in such a way as to respect the group structure [69].

Lemma 7.4. *Let U and V be finite subsets of S and let $a \in S$. The following are equivalent.*

1. $U \vdash V, a$
2. $U, a^{-1} \vdash V$

Proof. By cut with (s) or (t), respectively. □

Due to the property described in the lemma, in order to give an explicit description of $\vdash$, it actually suffices to concentrate on characterizing empty-conclusion entailment $U \vdash$ only.

Lemma 7.5. *Let U be a finite subset of S. If $U \rhd e$, then U is inconsistent, i.e., $U \vdash$.*

Proof. In view of Lemma 7.1, suppose that $e = x_1 a_1 x_1^{-1} \cdots x_k a_k x_k^{-1}$, with $a_1, \ldots, a_k \in U$. We must have $k > 1$, thus we obtain

$$a_1, x_2 a_2 x_2^{-1} \cdots x_k a_k x_k^{-1} \vdash$$

from (s) by cut with (n). We further have

$$x_2 a_2 x_2^{-1}, \ldots, x_k a_k x_k^{-1} \vdash x_2 a_2 x_2^{-1} \cdots x_k a_k x_k^{-1}$$

and therefore

$$a_1, x_2 a_2 x_2^{-1}, \ldots, x_k a_k x_k^{-1} \vdash$$

Successive application of transitivity with (n) finally yields

$$a_1, \ldots, a_k \vdash$$

□

Example 7.6. We adapt an example taken from [114]. Let $n \geqslant 1$ be an integer and let $a \in G$. If the equation $x^n = a$ has two different solutions, then $\vdash$ collapses. To this end, notice first that for all $a, b \in G$, and $n \geqslant 1$,

$$ab^{-1} \rhd a^n b^{-n}$$

holds. For this we can argue by induction on n, starting with an instance of reflexivity (R); once we have $ab^{-1} \rhd a^{n-1}b^{-(n-1)}$, we make use of

$$a^{n-1}b^{-(n-1)} \rhd a^n b^{-(n-1)} a^{-1}$$

along with

$$a^n b^{-(n-1)} a^{-1}, ab^{-1} \rhd a^n b^{-n}$$

which both are axioms. With transitivity we obtain $ab^{-1} \rhd a^n b^{-n}$. Now suppose that x and y are such that $x^n = y^n$, yet $x \neq y$ and thus $xy^{-1} \in S$. Since $xy^{-1} \rhd x^n y^{-n}$ we have $xy^{-1} \rhd e$. Similarly we get $yx^{-1} \rhd e$. By Lemma 7.5 it follows that we have

$$xy^{-1} \vdash \qquad \text{and} \qquad yx^{-1} \vdash$$

with which we can cut the corresponding instance of totality (t)

$$\vdash xy^{-1}, yx^{-1}$$

twice. Now we see that $\vdash$ collapses.

Lemma 7.7. *Let U be a finite subset of S. If there are $a_1, \ldots, a_n \in S$ such that, for any choice of $\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$,*

$$U, a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n} \rhd e,$$

then U is inconsistent.

Proof. Suppose that

$$U, a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n} \rhd e$$

for whatever choice of $\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$. According to Lemma 7.5, we get

$$U, a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n} \vdash$$

Cut with appropriate instances of totality (t) leads to

$$U, \{ a_i : \epsilon_i = 1 \} \vdash \{ a_j : \epsilon_j = -1 \}.$$

Keep in mind that $\epsilon_1, \ldots, \epsilon_n$ were taken arbitrarily from $\{1, -1\}$. It follows that for any partition $A \cup B$ of $\{ a_1, \ldots, a_n \}$ there is a corresponding entailment

$$U, A \vdash B.$$

Now that any one of these entailments exhibits the elements of $\{ a_1, \ldots, a_n \}$ either conjunctively to the left, or disjunctively to the right of the turnstile, and since both occurs, they can be considered *eliminands* [76], whence can be cancelled (see also Lemma 4.2). In other words, what remains after successive application of transitivity is the set of *retinends*, and this is nothing but U, therefore

$$U \vdash$$

just as we were required to show. □

Proposition 7.8 (Formal Nullstellensatz)**.** *Let $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in S$. The following are equivalent.*

1. $a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell$
2. *There are $c_1, \ldots, c_n \in S$ such that, for any choice of $\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$,*

$$a_1, \ldots, a_k, b_1^{-1}, \ldots, b_\ell^{-1}, c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n} \rhd e,$$

with $n \geqslant 0$.

Proof. 1 $\implies$ 2: We are dealing with an inductively generated entailment relation, whence we may argue by induction. This means that we have to go through the axioms first, and then to address the rules. The former are straightforwardly checked. For instance, as regards (n), i.e., axioms of the form $a \vdash xax^{-1}$, notice that we have entailments

$$a \rhd xax^{-1} \qquad \text{and} \qquad xax^{-1}, xa^{-1}x^{-1} \rhd e$$

which by transitivity of $\rhd$ yield $a, xa^{-1}x^{-1} \rhd e$. Concerning the rules of reflexivity (R), monotonicity (M), and transitivity (T), we really may only concentrate on the latter. Consider an instance of transitivity

$$\dfrac{\dfrac{\vdots}{a_1, \dots, a_k, a \vdash b_1, \dots, b_\ell} \qquad \dfrac{\vdots}{a_1, \dots, a_k \vdash b_1, \dots, b_\ell, a}}{a_1, \dots, a_k \vdash b_1, \dots, b_\ell}$$

for which by induction we can suppose that there are elements $c_1, \dots, c_n$ and $d_1, \dots, d_m$ such that for any choice of $\epsilon_1, \dots, \epsilon_n, \delta_1, \dots, \delta_m \in \{1, -1\}$ we have

$$a_1, \dots, a_k, b_1^{-1}, \dots, b_\ell^{-1}, a, c_1^{\epsilon_1}, \dots, c_n^{\epsilon_n} \rhd e$$

as well as

$$a_1, \dots, a_k, b_1^{-1}, \dots, b_\ell^{-1}, a^{-1}, d_1^{\delta_1}, \dots, d_m^{\delta_m} \rhd e.$$

Collecting these elements, it is clear that

$$a, c_1, \dots, c_n, d_1, \dots, d_m$$

are such that for any choice of $\gamma_0, \gamma_1, \dots, \gamma_n, \gamma_{n+1}, \dots, \gamma_{n+m+1} \in \{1, -1\}$ we have

$$a_1, \dots, a_k, b_1^{-1}, \dots, b_\ell^{-1}, a^{\gamma_0}, c_1^{\gamma_1}, \dots, c_n^{\gamma_n}, d_1^{\gamma_{n+1}}, \dots, d_m^{\gamma_{n+m+1}} \rhd e,$$

just as required.

2 $\implies$ 1: The converse is a combination of Lemma 7.4 and Lemma 7.7. ☐

Instantiating Proposition 7.8 with the empty subset, we can characterize the collapse of $\vdash$.

Corollary 7.9. *The following are equivalent.*

1. *$\vdash$ collapses*
2. *There are $a_1, \dots, a_n \in S$ such that, for any choice of $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$,*

$$a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n} \rhd e,$$

with $n \geqslant 1$.

Proof. Suppose that $\vdash$ collapses, which by Proposition 7.8 is to say that the empty subset is inconsistent, and thus that there are $a_1, \dots, a_n \in G$ such that, for any choice of elements $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$,

$$a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n} \rhd e.$$

Yet we cannot have $n = 0$, since it is impossible to infer an element of G from the empty subset $U = \emptyset$ by means of $\rhd$. The converse is a direct consequence of the formal Nullstellensatz, too. ☐

Turning the corollary upside down and reading it contrapositively, with the completeness theorem for entailment relations and classical logic we obtain the well-known classical criterion for a group to be linearly orderable [114, 175, 193], i.e., over **ZFC**, a group G is orderable if and only if for any finite subset $\{a_1, \dots, a_n\}$ of $G \setminus \{e\}$ there are $\epsilon_i \in \{1, -1\}$ such that $e \notin \langle a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n} \rangle$, where the latter set denotes the normal subsemigroup of G which is generated by the elements $a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n}$.

In proof theory, orderability criteria for groups recently have been used to generate hypersequent calculi for varieties of lattice-ordered groups, by which new syntactic proofs of theorems which arise in the theory of ordered groups can be obtained [71].

7.3 Abelian groups and Levi's theorem

From now on, let G denote an abelian group, which in the following we will always write additively. Let $\rhd$ be the single-conclusion entailment relation of subsemigroup of G. Notice that axiom (n) now is redundant due to commutativity, hence $\rhd$ is generated by all instances of the only axiom

$$a, b \rhd a + b$$

where $a, b \in G$. If U is a finite subset of G and $a \in G$, then we have $U \rhd a$ if and only if there are $a_1, \ldots, a_k \in U$ and $n_1, \ldots, n_k \geqslant 1$, where $k \geqslant 1$, such that $\sum_{i=1}^{k} n_i a_i = a$. Next let $\vdash$ be the entailment relation of positive cone of G, with underlying set $S = G \setminus \{ 0 \}$, and generated by all instances of the following axioms:

$$a, -a \vdash \qquad \text{(s)}$$

$$a, b \vdash a + b \qquad \text{(a } [ab \neq e])$$

$$\vdash a, -a \qquad \text{(t)}$$

with side condition as indicated.

Recall that an abelian group G is *torsion-free* [181] if, for every integer $n \geqslant 1$ and every $a \in G$, if $na = 0$, then $a = 0$. In this vein, an element $a \in G$ is a *torsion element* if it is of finite order, which is to say that there is $n \geqslant 1$ such that $na = 0$.

Proposition 7.10. *The following are equivalent.*

1. *$\vdash$ collapses.*
2. *G has a non-zero torsion element.*

Proof. $1 \implies 2$: If $\vdash$ collapses, then, by way of Corollary 7.9, there are $a_1, \ldots, a_n \in S$, with $n \geqslant 1$, such that for whatever choice of $\epsilon_1, \ldots, \epsilon_n \in \{ 1, -1 \}$ we have

$$\epsilon_1 a_1, \ldots, \epsilon_n a_n \rhd 0.$$

In the context of abelian groups this means that there are $m_1, \ldots, m_n \in \mathbb{N}$, at least one of which is non-zero, such that

$$\sum_{i=1}^{n} \epsilon_i m_i a_i = 0.$$

While our claim is evident in case $n = 1$, a straightforward argument by induction on n allows for drawing the conclusion in general.

$2 \implies 1$: Conversely, suppose that $na = 0$, where $n > 1$ and $a \neq 0$. Because of $a \rhd na$ and $-a \rhd -na$, the entailment relation $\vdash$ collapses according to Corollary 7.9. □

If G is torsion-free, then G does not have a non-zero torsion element. The converse holds if G is *discrete*[1], which amounts to

$$\forall a \in G \, (\, a = 0 \ \vee \ a \neq 0 \,).$$

The contrapositive of Proposition 7.10 thus reads as follows and provides a constructive version of Levi's theorem.

Corollary 7.11. *Let G be discrete. The following are equivalent.*

1. *$\vdash$ is consistent.*
2. *G is torsion-free.*

[1] Of course we mean discrete in the sense specified above, rather than referring to the topological notion of a discrete group.

Remark 7.12. From a semantical point of view, i.e., by way of completeness for entailment relations, Corollary 7.11 gives rise to Levi's theorem that an abelian group is orderable if and only if it is torsion-free [114, 160, 161]. For sake of comparison, let us recall the well-known classical argument: the group in question can be embedded in a torsion-free *divisible* group D. With a *maximal independent* set in D, linearly ordered in an arbitrary way, it is possible in turn to obtain a linear order on G [114, 161].

For later use, we need to describe the *trace* [209] of $\vdash$, i.e., the set of all single-conclusion instances $U \vdash a$. To this end, we first take note of the following, which is straightforward to show.

Lemma 7.13. *Suppose that G is torsion-free. Let U be a finite subset of G and let $a \in G$. The following rule is provable:*

$$\frac{U, a \rhd 0 \qquad U, -a \rhd 0}{U \rhd 0}$$

Corollary 7.14. *Suppose that G is torsion-free. Let U be a finite subset of S and $a \in S$. The following are equivalent.*

1. $U \vdash a$
2. *There are $a_1, \ldots, a_k \in U$ and $n_1, \ldots, n_k \geqslant 1, m \geqslant 0$ such that*

$$\sum_{i=1}^{k} n_i a_i = ma,$$

with $k \geqslant 1$.

Proof. With Lemma 7.4, Proposition 7.8, and Lemma 7.13 we see that $U \vdash a$ is equivalent to $U, -a \rhd 0$. This in turn means that there are $b_1, \ldots, b_\ell \in U \cup \{ -a \}$ and $n_1, \ldots, n_\ell \geqslant 1$, where $\ell \geqslant 1$, such that $\sum_{j=1}^{\ell} n_j b_j = 0$. We cannot have $\ell = 1$, nor can we have $b_j = -a$ for all $1 \leqslant j \leqslant \ell$, for otherwise G would turn out to have a non-zero torsion element. $\square$

The description given in Corollary 7.14 instantiates the entailment relation of linear order for an abelian group as mentioned in [79]. As a consequence of Corollary 7.14, if G is a torsion-free abelian group, then there are no *tautologies* $\vdash a$, and neither are there inconsistent singletons $a \vdash$, due to Lemma 7.4.

As illustrated in [151], similar results can be obtained by means of certain formal systems which likewise treat group elements as abstract statements.

7.4 Entailment on integer lattices

We turn our attention to a finitary version of Sikora's theorem. In this section we consider the integer lattice $\mathbb{Z}^n$ $(n > 1)$ as an abelian group with component-wise addition. Accordingly, we take $S = \mathbb{Z}^n \setminus \{ (0, \ldots, 0) \}$ as our domain of discourse, and study the entailment relation $\vdash$ of positive cone of $\mathbb{Z}^n$, inductively generated and explicitly described as in Section 7.3.

Exhibiting an order on $\mathbb{Z}^n$ does of course not require foundationally perhaps controversial tools to begin with. For instance, the natural discrete order on $\mathbb{Z}$ gives way to the lexicographic order on $\mathbb{Z}^n$ which too respects the group structure. As indicated in [69], a plethora of orderings may be specified on $\mathbb{Z}^2$, picking some $v = (v_1, v_2) \in \mathbb{R}^2$ with irrational slope and postulating, for every $a, b \in \mathbb{Z}^2 \setminus \{ (0, 0) \}$,

$$a < b \quad \equiv \quad a \cdot v < b \cdot v$$

where $\cdot$ denotes the dot product. However, employing a classical notion of real number, for this to be linear classical reasoning is necessary [52, 56, 181].

Recall that an abelian group G is said to be *divisible* [181] if, for every non-zero integer n and every $a \in G$, there exists $b \in G$ such that $a = nb$. If G is an abelian group and $x_1, \dots, x_m$ are indeterminates, let $G\{x_1, \dots, x_m\}$ be the group of $\mathbb{Z}$-*affine forms* on G, i.e., polynomials

$$a + \sum_{i=1}^{m} z_i x_i$$

with $a \in G$ and $z_1, \dots, z_m \in \mathbb{Z}$.

The following theorem provides us with a geometric tool for locating finite consistent sets of abstract statements. We refer to [92] and [83].

Positivstellensatz. Let G be a discrete divisible linearly ordered group. Let $x_1, \dots, x_m$ be indeterminates and let $R_{=0}, R_{\geqslant 0}$, and $R_{>0}$ be finite subsets of $G\{x_1, \dots, x_m\}$. Consider the associated system $\mathscr{S}$ of sign conditions

$$z(\boldsymbol{\xi}) = 0 \text{ if } z \in R_{=0}, \qquad p(\boldsymbol{\xi}) \geqslant 0 \text{ if } p \in R_{\geqslant 0}, \qquad s(\boldsymbol{\xi}) > 0 \text{ if } s \in R_{>0}.$$

There is an algorithm giving the following answer:

1. either an algebraic certificate telling that the system $\mathscr{S}$ is impossible in G,
2. or a point $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in G^m$ realising the system $\mathscr{S}$.

An algebraic certificate is an identity

$$s + p + z = 0 \text{ in } G\{x_1, \dots, x_m\},$$

where s is a (nonempty) sum of elements of $R_{>0} \cup G_{>0}$, p is a (possibly empty) sum of elements of $R_{\geqslant 0} \cup G_{\geqslant 0}$, and z is a $\mathbb{Z}$-linear combination of elements of $R_{=0}$.

Recall that in this section $\vdash$ denotes the entailment relation of positive cone of $\mathbb{Z}^n$, where $n \geqslant 2$, considered over the set $S = \mathbb{Z}^n - \{(0, \dots, 0)\}$, every element of which is to be considered an abstract statement.

Corollary 7.15. *Let U be a finite subset of S and $b \in S$. If $U \cup \{b\}$ is consistent, then there is $a_0 \in S$ such that $a_0 \cdot b > 0$, and $a_0 \cdot a \geqslant 0$ whenever $U \vdash a$.*

Proof. We can write $U = \{a_1, \dots, a_k\}$, where for each $j \in \{1, \dots, k\}$ we have

$$a_j = (a_{j1}, \dots, a_{jn}).$$

Similarly,

$$b = (b_1, \dots, b_n).$$

To get in the setting of the Positivstellensatz, we consider $\mathbb{Q}$ as the underlying discrete divisible group, naturally ordered, and we put

$$R_{\geqslant 0} = \left\{ \sum_{i=1}^{n} a_{1i} x_i, \dots, \sum_{i=1}^{n} a_{ki} x_i \right\}$$

and

$$R_{>0} = \left\{ \sum_{i=1}^{n} b_i x_i \right\}.$$

Scaling it to an appropriate length, any realizer $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{Q}^n$ for this system of sign conditions determines an element $a_0 \in S$ as required. Thus we need to show that the first alternative postulated by the Positivstellensatz is impossible. In fact, for there would be integers $\lambda > 0$ and $\mu_1, \dots, \mu_k \geqslant 0$ such that

$$\lambda \sum_{i=1}^{n} b_i x_i + \sum_{j=1}^{k} \mu_j \sum_{i=1}^{n} a_{ji} x_i = 0.$$

Then we could collect coefficients in order to see that, for every $i \in \{1, \dots, n\}$,

$$\lambda b_i + \sum_{j=1}^{k} \mu_j a_{ji} = 0,$$

so that we have

$$\lambda b + \sum_{j=1}^{k} \mu_j a_j = 0.$$

Notice that the latter expression means $U, b \rhd 0$, whence $U, b \vdash$. However, $U \cup \{b\}$ is supposed to be consistent. $\square$

Another possible way to obtain this corollary is by a suitable version of Farkas' lemma, e.g., by an instance of the recently obtained *discrete* version of Farkas' Lemma [28].[2] Notice that Corollary 7.15 provides us with a geometric interpretation for entailment. Let U be a finite subset of S and let $b \in S$. If $U \cup \{b\}$ is consistent, then $U \nvdash -b$. If $a_0 \in S$ is such as postulated in Corollary 7.15, then the hyperplane

$$H(a_0) = \{ r \in \mathbb{Q}^n : a_0 \cdot r = 0 \}$$

separates the cone generated by U from $-b$. Similarly, we know that every inhabited consistent subset of S is located to one side of a certain hyperplane.

The next proposition is crucial for the purpose of this chapter.

Proposition 7.16. *Let U be a finite consistent subset of S. There is $a \in S$ such that*

$$U \nvdash a \qquad \textit{and} \qquad U \nvdash -a.$$

Proof. We may suppose that U is inhabited and write $b = \sum_{a \in U} a$. Notice that $b \neq 0$, since U is consistent. Since $U \vdash b$, the one-point extension $U \cup \{b\}$ is consistent, as well. Now let $a_0 \in S$ be an abstract statement according to Corollary 7.15, i.e., one such that $a_0 \cdot b > 0$, and $a_0 \cdot a \geqslant 0$ whenever $U \vdash a$. We consider the hyperplane $H(a_0)$ (as sitting in $\mathbb{Q}^n$), perpendicular to a_0, for which we can construct an integer orthogonal basis $b_1, \dots, b_{n-1}$. Next we employ the Positivstellensatz again, which for every $c \in S$ provides a way to decide

$$U \vdash c \qquad \text{or} \qquad U \nvdash c.$$[3]

If, for a certain $i \in \{1, \dots, n-1\}$, we get both

$$U \nvdash b_i \qquad \text{and} \qquad U \nvdash -b_i,$$

[2] In fact, the Positivstellensatz can be considered a variant of Farkas' lemma with regard to rational linear programming [92]. I am grateful to Stefan Neuwirth for having brought this to my attention.

[3] To this end, write $U = \{a_1, \dots, a_k\}$, where $a_j = (a_{j1}, \dots, a_{jn})$, and $c = (c_1, \dots, c_n)$. Consider over $\mathbb{Q}$ the system of sign conditions given by

$$R_{=0} = \left\{ \sum_{j=1}^{k} a_{j1} x_j - c_1, \dots, \sum_{j=1}^{k} a_{jn} x_j - c_n \right\} \qquad \text{and} \qquad R_{\geqslant 0} = \{ x_1, \dots x_n \}.$$

Any realizer for this system witnesses $U \vdash c$.

then we have already obtained an element as required. Otherwise, after relabeling if necessary, we may assume $U \vdash b_i$ for every $i \in \{1, \dots, n-1\}$. In the following, let

$$c = \sum_{i=1}^{n-1} b_i,$$

and put

$$U_{=0} = \{ a \in U : a_0 \cdot a = 0 \} \qquad \text{and} \qquad U_{>0} = \{ a \in U : a_0 \cdot a > 0 \}.$$

Notice that $U = U_{=0} \cup U_{>0}$. For every $a \in U_{>0}$, it is possible to find an integer $k_a \geqslant 0$ which is such that

$$b_1, \dots, b_{n-1}, a_0 - k_a c \vdash a. \tag{†}$$

Indeed, for we can certainly write

$$a = \sum_{i=1}^{n-1} r_i b_i + s a_0$$

for rational numbers $r_1, \dots, r_{n-1}$ among of which some may be negative, along with a strictly positive $s \in \mathbb{Q}$. Clearing the denominators, we get an equation

$$ma = \sum_{i=1}^{n-1} m_i b_i + m_0 a_0$$

with certain integers $m, m_1, \dots, m_{n-1}$, and $m_0 > 0$. Now we put

$$k_a = \max \{ |m_1|, \dots, |m_{n-1}| \}$$

with which we can write

$$ma = \sum_{i=1}^{n-1} (m_i + m_0 k_a) b_i + m_0 (a_0 - k_a c),$$

as required for (†). Next consider

$$k = \max \{ k_a : a \in U_{>0} \}.$$

Whenever $k > k_a$ for a certain $a \in U_{>0}$, notice that we have entailments

$$b_1, \dots, b_{n-1} \vdash (k - k_a)c \qquad \text{and} \qquad a_0 - kc, (k - k_a)c \vdash a_0 - k_a c$$

with which we obtain

$$b_1, \dots, b_{n-1}, a_0 - kc \vdash a$$

by successive cut with (†). It follows that we have

$$U_{=0}, b_1, \dots, b_{n-1}, a_0 - kc \vdash a \tag{‡}$$

universally for every $a \in U$. We claim that

$$U \nvdash a_0 - (k+1)c \qquad \text{and} \qquad U \nvdash -a_0 + (k+1)c.$$

In fact, on the one hand we have $a_0 \cdot (-a_0 + (k+1)c) < 0$ which by construction of a_0 forbids $U \vdash -a_0 + (k+1)c$ right away. On the other hand, let us assume that $U \vdash a_0 - (k+1)c$. Along with (‡) and finitely many cuts we obtain

$$U_{=0}, b_1, \dots, b_{n-1}, a_0 - kc \vdash a_0 - (k+1)c$$

Therefore, there are $c_1, \dots, c_\ell \in U_{=0} \cup \{ b_1, \dots, b_{n-1} \}$ and $\lambda_1, \dots, \lambda_\ell, \lambda, \mu \geqslant 0$ such that

$$\sum_{i=1}^{\ell} \lambda_i c_i + \lambda(a_0 - kc) = \mu(a_0 - (k+1)c)$$

where at least one of $\lambda_1, \dots, \lambda_\ell, \lambda$ is non-zero. Keeping in mind that $a_0 \cdot a = 0$ for every $a \in U_{=0} \cup \{ b_1, \dots, b_{n-1} \}$, it follows that $\lambda = \mu$, and thus

$$\sum_{j=1}^{\ell} \lambda_j c_j + \sum_{i=1}^{n-1} \lambda b_i = 0,$$

by which we see that $U_{=0}, b_1, \dots, b_{n-1} \vdash$. Since $U \vdash b_i$ for every $i \leqslant n-1$, it follows that $U \vdash$. But U is supposed to be consistent, whence the assumption cannot be maintained. We have thus found an element as required. $\square$

Notice that if $U \nvdash a$, then $U \nvdash ka$ for every $k \geqslant 1$. For a finite consistent subset U of S there are, in fact, infinitely many elements not in the scope of U. In other words, Proposition 7.16 tells us that a linear order on the group $\mathbb{Z}^n$ cannot be determined by finitely many elements only: if U is a finite and consistent subset of S, then the *deductive closure* of U under the *trace* [209] of $\vdash$, i.e., the set

$$U^{\vdash} = \{ a \in S : U \vdash a \}$$

of all immediate consequences of U, models the restricted entailment relation generated by the axioms for single-values (s) and additivity (a). In this manner U cannot, however, determine a total model.

The finitely many decisions we had to make with regard to a basis of the hyperplane $H(a_0)$ in the above proof of Proposition 7.16 is redundant under the assumption that U already is decisive, which would be the case for an atomic conjunction.

Corollary 7.17. *If $n \geqslant 2$, then the entailment relation $\vdash$ of positive cone of $\mathbb{Z}^n$ does not admit an atomic conjunction.*

Proof. Any finite consistent subset of S fails to be decisive for $\vdash$. $\square$

Remark 7.18. We briefly discuss a classical approach to Corollary 7.17, which is to be compared with common proofs of Sikora's theorem [69, 235]. Suppose that $A \subseteq S$ is an atomic conjunction. Since this A is supposed to be consistent, completeness exhibits a model α of $\vdash$ containing A. As explained in [69], involving an ad hoc case distinction on α, this α may be slightly perturbed in such a way as to obtain a different model α' of $\vdash$ which still contains the finite set A. Since $\alpha \neq \alpha'$, by classical reasoning there is $a \in S$ such that $a \in \alpha$ and $-a \in \alpha'$. Since A is decisive, we would have either $A \vdash a$ or else $A \vdash -a$, which in turn would yield $a, -a \in \alpha$ or $a, -a \in \alpha'$. However, models do not contain inconsistent subsets.

Here and in the following sections we write $L(G)$ for the distributive lattice generated by the entailment relation of positive cone of a group G, considered over $S = G \setminus \{ 0 \}$. The following is a finitary version of Sikora's theorem.

Corollary 7.19. *$L(\mathbb{Z}^n)$ is atomless for $n \geqslant 2$.*

How this corollary gives way to the usual version of Sikora's theorem will be explained in the final Section 7.6. Now we turn our attention to a generalization of Corollary 7.19 for torsion-free groups of finite rank.

7.5 Rank

Let G be an abelian group, and let $\vdash$ be the entailment relation of positive cone of G. Notice that the corresponding generated distributive lattice $L(G)$ is a Boolean algebra. In fact, the axioms for totality (t) and single-values (s) force $L(G)$ to be complemented: for every $a \in S = G \setminus \{ 0 \}$ we have

$$i(a) \wedge i(-a) \leqslant 0 \qquad \text{and} \qquad 1 \leqslant i(a) \vee i(-a).$$

by interpretation, whence $i(-a)$ complements $i(a)$ in $L(G)$. Since every element $x \in L(G)$ can be written in disjunctive normal form [62] in terms of elements each of which is complemented, any such x too has a complement.

Now we bring into play the notion of *rank* [21]. Recall [181] that a torsion-free abelian group is said to be of *rank* n if $G \otimes \mathbb{Q}$ is an n-dimensional discrete vector space over $\mathbb{Q}$. We first consider the case of rank 1.

Lemma 7.20. *Let G be a torsion-free abelian group of rank 1. If a and b are non-zero elements of G, then*

$$a \dashv\vdash b \qquad \text{or} \qquad a \dashv\vdash -b.$$

Here we write $a \dashv\vdash b$ for the conjunction of $a \vdash b$ and $b \vdash a$.

Proof. In a torsion-free abelian group of rank 1 any two non-zero elements have a common non-zero multiple [181]. We thus have $na = mb$ for certain integers n, m of which at least one can be taken positive. The conclusion is immediate in view of Corollary 7.14 □

Proposition 7.21. *If G is a torsion-free abelian group of rank 1, then $L(G) \cong \mathbf{2}^2$, where $\mathbf{2}^2$ is the Boolean algebra with four elements.*

Proof. By way of Lemma 7.20, for all $a, b \in S$ we have $i(a) = i(b)$ or $i(a) = -i(b)$. Now pick any non-identity a of G whatsoever, and consider an arbitrary element $x \in L(G)$. This x can be written in disjunctive normal form

$$x = \bigvee_{\mathscr{U}} \bigwedge_{b \in U} i(b)$$

for a certain finite set $\mathscr{U}$ of finite subsets U of S. It follows that $x \in \{ 0, i(a), -i(a), 1 \}$. Furthermore, keep in mind that neither $i(a) = 0$ nor $i(a) = 1$, since G is torsion-free. □

It follows at once that a torsion-free abelian group of rank 1 allows for exactly two orderings [69]—they correspond with the two prime filters of $\mathbf{2}^2$.

Corollary 7.22. $L(\mathbb{Z}) \cong \mathbf{2}^2$.

In order to address abelian groups of finite rank other than 1, we make use of the following theorem for finitely generated subgroups of finite-rank torsion-free groups [181, XI.1.5 Theorem].

Theorem 7.23. *Finitely generated subgroups of finite-rank torsion-free abelian groups are direct sums of cyclic groups.*

Proposition 7.24. *Let G be a torsion-free abelian group of finite rank $n > 1$. Let $\vdash$ be the entailment relation of positive cone of G. The generated distributive lattice $L(G)$ for $\vdash$ is atomless.*

Proof. In view of Proposition 4.24, suppose that A is a finite consistent decisive subset of S. We may assume that A is inhabited, since $\vdash$ does not have inconsistent singletons. We consider the finitely generated subgroup $\langle A \rangle$ of G. By Theorem 7.23, this is a direct sum of cyclic groups, i.e., $\langle A \rangle = \oplus_{i=1}^{k} \langle g_i \rangle$ for certain $g_1, \ldots, g_k \in G$. We claim that $k > 1$. For suppose that $\langle A \rangle$ is generated by a single element $g \in G$, and consider an arbitrary pair of non-zero elements a and b be of G. Since A is supposed to be decisive, we have $A \vdash a$ or $A \vdash -a$, as well as $A \vdash b$ or $A \vdash -b$, which by Corollary 7.14, and since A is consistent, means that there are non-zero $m_1, m_2 \in \mathbb{Z}$ such that

$$A \rhd m_1 a \qquad \text{and} \qquad A \ \rhd m_2 b.$$

Since g generates $\langle A \rangle$, there will be $n_1, n_2 \in \mathbb{Z}$ such that

$$n_1 g = m_1 a \qquad \text{and} \qquad n_2 g = m_2 b,$$

but then a and b would turn out to have a common non-zero multiple, and it would follow that G has rank 1, contrary to our assumption. Next we obtain an isomorphism $\langle A \rangle \cong \mathbb{Z}^k$, where $k > 1$, along which A gives rise to a finite consistent decisive subset of $\mathbb{Z}^k$, the existence of which, however, has already been ruled out by Corollary 7.17. $\square$

Remark 7.25. Let us briefly compare our proof with a classical approach to Proposition 7.24 as carried out elsewhere [68, 69]. Again, suppose that A is an atomic conjunction for $\vdash$ on a torsion-free abelian group of finite rank > 1. Since A is consistent, there is a model α of $\vdash$ such that $A \subseteq \alpha$. Let H denote the subgroup of G that is generated by A. We may restrict α to an order $\alpha\big|_H$, and suppose again that $H \cong \mathbb{Z}^k$ for a certain $k > 1$. Employing previous insights, this $\alpha\big|_H$ may be perturbed in such a way as to yield a different order α' of H which still contains A. One then considers the torsion-free abelian quotient group $G/I(H)$, where

$$I(H) = \{\, a \in G : \exists k \in \mathbb{Z} \,(\, ka \in H \,)\, \}$$

is the isolator subgroup of H in G. This quotient group may be linearly ordered, according to (the classical and non-constructive version) of Levi's theorem. Any order on $G/I(H)$ can be used to extend a given order on H to one on the ambient group G [69]. This observation applies to α', in particular. In this manner it is possible to obtain an order different from α, yet still containing the finite consistent set A we started with.

7.6 Concluding remarks

The following facts and results are all well-known. We switch to a classical setting, and refer to [119, 127, 156, 236]. Recall that any distributive lattice L canonically defines a spectral space the points of which are the prime filters P of L [147]. For the distributive lattice generated by an entailment relation $\vdash$, the points of this space are completely determined by the collection of ideal elements of $\vdash$ [62]. If L is a Boolean algebra, we have at hand a Stone space, i.e., a compact totally disconnected Hausdorff space. It is well-known that the atoms of a Boolean algebra correspond to the *isolated points* of its Stone space, which in case of $L(\mathbb{Z}^n)$, $n \geqslant 2$, does not have any, according to Corollary 7.19. It can be shown that the Stone space of a Boolean algebra B is metrizable if and only if B is at most countable. The latter certainly holds true in case of $L(\mathbb{Z}^n)$, which has a countable set of generators. Since any two countable Boolean algebras are isomorphic, and keeping in mind Brouwer's characterization of the Cantor set [55], Sikora's theorem [235] now is a consequence of the aforementioned facts and results:

Proposition 7.26 (Sikora)**.** *For $n \geqslant 2$, the class of orderings on $\mathbb{Z}^n$ can be equipped with a topology in such a way as to be homeomorphic to the Cantor set.*

Further and advanced lattice-theoretic tools for obtaining Sikora's and related results on spaces of orderings of groups are explained in [69].

Remark 7.27. The above discussion rests on Stone's representation theorem which is known to necessitate classical reasoning [32]. It might thus be interesting to intervene with Negri's analysis of the constructive content of Stone representation [188], applying tools from formal topology [65, 67, 218, 219]. Perhaps another interesting direction for subsequent research might stem from the relationship between entailment relations and resolution calculi [90, 258]. The universal frame construction [90], which builds on the fundamental theorem of entailment relations, could be substituted by a formal topology that arises from the hyperresolution rule [90] in such a manner that the formal points correspond with the ideal elements of the entailment relation. Presumably, the formal topology associated in this manner to the entailment relation of positive cone of $\mathbb{Z}^n$ ($n > 1$) is isomorphic to the *formal Cantor space* [111]. This would provide for another constructive version of Sikora's theorem, and is kept for future research.

Remark 7.28. The approach of this chapter carries over to algebraic structures other than groups, and can be adapted so as to take account of orderability criteria for rings and fields [114]. Notably, classical proofs can be turned "upside down" and rephrased in an affirmative manner, allowing for elementary, constructive proofs. For instance, it can be shown constructively that if K is a discrete field, then the entailment relation of positive cone of K collapses if and only if -1 is a sum of squares [211]. The reader may recognize this as a variant of Artin and Schreier's theorem [15], which can be regained by means of CT. We will keep for future research an investigation of orderability criteria for semigroups, which is intended to provide a generalization of the results obtained in this chapter [253].

Chapter 8

Perspectives

8.1 Infinitary entailment relations

An apparent obstruction towards an even wider applicability of entailment relations results from the limitation on entailment to hold only between *finite* subsets of a given domain of discourse, as well as from the classical strength of the Completeness Theorem (CT) for entailment relations, along with its many equivalences [137], beyond which we cannot go. The question arises at once: how do we adapt our methods in order to take account of principles which might classically be as strong as the full Axiom of Choice (AC)? How do our insights into conventional entailment relations carry over?

In view of our approach to Sikorski's theorem, a first and undoubtedly tempting move is to break the symmetry of entailment relations and allow for *arbitrary* rather than only finite sets of succedents, i.e., to consider relations

$$\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Pow}(S).$$

Alas, it has already been pointed out [90] that the Completeness Theorem for these *generalized* (or *infinitary*) *entailment relations* does not hold in general; entailments may therefore not be semantically determined anymore.[1] In order to address this issue, we should first specify what we understand by a generalized entailment relation. Thus, let S be a set, and let $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Pow}(S)$. We still keep $\vdash$ to be reflexive and monotone:

$$\frac{U \between V}{U \vdash V}\,(\mathrm{R}) \qquad\qquad \frac{U \vdash V}{U, U' \vdash V, V'}\,(\mathrm{M})$$

where U and U' are finite subsets of S, while this restriction need not apply to V and V', which may be arbitrary subsets of S. But once we allow for infinite sets of succedents, we feel motivated to generalize transitivity as follows:

$$\frac{U \vdash V, W \qquad \forall c \in W\,(\,U', c \vdash V'\,)}{U, U' \vdash V, V'}\,(\mathrm{T'})$$

We take this rule from [234], where it is called Cut_2. Now W need not be finite, so in order to make immediate sense of the quantifier above the inference line, it should again be emphasized that we employ rule notation just as shorthand for implication. In view of the generalized rule of transitivity (T'), for entailment relations which are generated by axioms, over **CZF** we might further have to address the inductive generation procedure carefully [91]. We do not need to adapt semantics: by an ideal element (or model) of a generalized entailment relation we still understand a subset α of S which splits every entailment.

[1] However, building on an argument of Valentini [246], Rinaldi has recently obtained a proof of completeness for infinitary entailment relations which are generated by a *countable* set of initial entailments; this result had been preceded in the context of geometric logic by Fourman and Grayson [111].

Example 8.1. We can explain the class $\mathscr{D}(S)$ of all *directed* subsets of a partially ordered set S as the class of ideal elements of a suitable generalized entailment relation. To this end, we take initial entailments

$$\begin{gathered} \vdash S \\ a, b \vdash \{\, c \in S : a, b \leqslant c \,\} \end{gathered}$$

the first of which forces ideal elements α to be inhabited, and the second provides for upper bounds for every pair of elements of α. Therefore, with a generalized form of Max (see Chapter 4), we could regain MDP (see Chapter 1). While the class of models of a generalized entailment relation still is directed-complete, we would be required to check that it is set-generated. However, Aczel's result, which has been employed in Chapter 4, would again apply without further ado.

Recall from Section 4.4 that we have defined the canonical inconsistency predicate of a conventional entailment relation by

$$\Phi_{\vdash}(U) \equiv U \vdash$$

where U is a finite subset of S, and then proceeded to characterize finitary monotone predicates which coincide with $\Phi_{\vdash}$. This makes perfect sense for infinitary entailment relations as well. $\Phi_{\vdash}$ can be extended canonically to a predicate $\Psi_{\vdash}$ on arbitrary subsets A of S by stipulating

$$\Psi_{\vdash}(A) \equiv \exists U \in \mathrm{Fin}(A)\, \Phi_{\vdash}(U).$$

In other words, we have $\Psi_{\vdash}(A)$ if and only if A contains a finite inconsistent subset. Let us now say that a subset C of S is *consistent* if $\neg\Psi_{\vdash}(C)$, which is to say that every finite subset of C is consistent in the usual sense.

Next we consider an arbitrary predicate Ψ on subsets of S, i.e., a subclass of $\mathrm{Pow}(S)$. Adopting terminology from Chapter 4, let us say (for lack of a better term) that Ψ is *strongly hereditary* (for $\vdash$) if, for all finite subsets U and every pair of arbitrary subsets V and W of S,

$$\frac{U \vdash V \qquad \forall b \in V\ \Psi(W, b)}{\Psi(U, W)} \tag{$\flat$}$$

For instance, this holds for $\Psi_{\vdash}$ and every *right-compact* entailment, i.e., provided that $U \vdash V$ implies that there is a finite subset V_0 of V such that $U \vdash V_0$. For if $V_0 = \{\, b_1, \ldots, b_n \,\}$ and if there are finite subsets $U_1, \ldots, U_n$ of W such that $U_i, b_i \vdash$ for $i \leqslant n$, then we obtain $U, U_1, \ldots, U_n \vdash$ from $U \vdash V_0$ by repeated application of transitivity, and hence $\Psi_{\vdash}(U, W)$. It is interesting to note that if $\vdash$ is generated by axioms, and if Ψ is monotone, then Ψ is strongly hereditary for $\vdash$ already if it is strongly hereditary with respect to every initial entailment. If Ψ moreover is *adequate*, i.e., if $\Psi(W)$ implies that there is $U \in \mathrm{Fin}(W)$ such that $U \vdash$, then we have $\Psi = \Psi_{\vdash}$, whence we can characterize arbitrary inconsistent subsets of S in terms of Ψ. We say in this case that Ψ is an *inconsistency predicate* for $\vdash$.

If we read ($\flat$) contrapositively and apply REM, then we obtain, for every finite subset U and every pair of arbitrary subsets V and W of S,

$$\neg\Psi(U, W) \rightarrow (\, U \vdash V \rightarrow \exists b \in V\, \neg\Psi(W, b)\,) \tag{$\sharp$}$$

This observation gives rise to an extension principle on the class of consistent subsets of S, as will become clear in the proof of the following theorem. This theorem provides a conditional version of the model existence principle (MEL) for infinitary entailment relations with strongly hereditary inconsistency predicate.

Theorem 8.2 (GET+REM)**.** *Let $\vdash$ be an infinitary entailment relation. The following are equivalent.*

1. *$\vdash$ has a strongly hereditary inconsistency predicate.*
2. *Every consistent subset of S is contained in an ideal element of $\vdash$.*

Proof. 1. $\Longrightarrow$ 2. Let C_0 be a consistent subset of S. The class

$$\mathscr{C} = \{ C \in \mathrm{Pow}(S) : C_0 \subseteq C \wedge \neg\Psi(C) \}$$

of consistent extensions of C_0 is an (inhabited) dcpo: if $\mathscr{D}$ is a directed set of consistent subsets of S, all of which contain C_0, then $\bigcup \mathscr{D}$ is consistent as well. Next we define an extension pattern on $\mathscr{C}$. Every entailment is considered extension data, and we define the relation $\Vdash$ between entailments and extensions C as follows:

$$(U \vdash V) \Vdash C \quad \equiv \quad U \subseteq C \to C \between V.$$

To say that a consistent extension C of C_0 is total for this pattern is to say that C splits all entailments, and hence that C is a model of $\vdash$. Recall that in order to actually have an extension pattern on $\mathscr{C}$, we need to verify that whenever we take extension data $U \vdash V$, and $C \in \mathscr{C}$, there is $C' \in \mathscr{C}$ such that

$$C \subseteq C' \qquad \text{and} \qquad (U \vdash V) \Vdash C'.$$

Since we reason classically, we may distinguish cases. If $(U \vdash V) \Vdash C$ then $C' = C$ will do. Otherwise, if $(U \vdash V) \nVdash C$, then this means $U \subseteq C$ and $C \cap V = \emptyset$. We thus have $\neg\Psi(U, C)$. Since Ψ is strongly hereditary, according to ($\sharp$) there is $b \in V$ such that $\neg\Psi(C, b)$. Now $C' = C \cup \{ b \} \in \mathscr{C}$ is as required. We obtain a model of $\vdash$ by means of GET.

2. $\Longrightarrow$ 1. Suppose that every consistent subset of S can be extended to a model. In order to show that the canonical inconsistency predicate Ψ is strongly hereditary, we concentrate on the contrapositive form ($\sharp$). Thus, let U be a finite and W be an arbitrary subset of S, and suppose that $\neg\Psi(U, W)$, which is to say that $U \cup W$ is consistent. According to the assumption, there is a model α of $\vdash$ such that $U \cup W \subseteq \alpha$. Since $U \vdash V$ and α is a model, there is $b \in \alpha \cap V$. Because of $W \cup \{ b \} \subseteq \alpha$, it follows $\neg\Psi(W, b)$. □

It is interesting to note that if the relation between entailments and consistent subsets as defined in the proof indeed amounts to an extension pattern, then ($\flat$) holds, by which Ψ turns out strongly hereditary with REM.

Theorem 8.2 is yet another variant of Lindenbaum's Lemma. To put it in more familiar terms: every maximal consistent subset is a model, provided that Ψ is strongly hereditary. Needless to say, we have adopted an approach by means of GET instead of KZL to build a bridge between the first chapter and the present. However, from a methodological perspective, the semantical notion of ideal element indeed seems to favour the approach by an extension pattern.[2] Notice that since Ψ is hereditary whenever $\vdash$ is a conventional, finitary entailment relation, Theorem 8.2 generalizes the principle of model existence (MEL) from Chapter 4.

Example 8.3. We revisit the leading and motivating example from Chapter 5, which can now be generalized to families $\mathscr{S} = \{ S_\imath \}$ of arbitrary inhabited sets. On the disjoint union $S = \bigcup_{\imath \in I} \{ \imath \} \times S_\imath$ we consider the infinitary entailment relation $\vdash$ which is generated by all instances of the following axioms:

$$(\imath, a), (\imath, b) \vdash \qquad \text{(s } [a \neq b])$$

$$\vdash \{ (\imath, a) : a \in S_\imath \} \qquad \text{(t)}$$

It can be directly shown that this entailment relation is consistent, and that it has a strongly hereditary inconsistency predicate. By way of Theorem 8.2, this entailment relation has an ideal element. It follows that the conditional model existence lemma is classically as strong as the Axiom of Choice (which is clear also in view of Example 8.1).

[2] On the other hand, it can be directly shown, without REM, that every maximal consistent extension of a consistent subset of S is an ideal element if ($\sharp$) holds. Moreover, the class of all consistent extensions is a set-generated dcpo: a generating set is given by all *finite* consistent subsets of S.

In the context of infinitary entailment relations, it appears that the principle of model existence does not suffice for completeness anymore, which is likely to be a consequence of the asymmetry of finite sets of antecedents as opposed to arbitrary sets of succedents. Another explanation for this semantical misbehaviour[3] arises from the fact that Lemma 4.6 does not carry over to its full extent. But if B is an arbitrary subset of S, then we can still generate an extension $\vdash_B$ of $\vdash$ by putting additional axioms

$$b \vdash_B \qquad (b \in B)$$

on top of the latter. We then have $U \vdash_B V$ if and only if $U \vdash V, B$. With regard to semantics, if α is a subset of S, then we have $\alpha \in \mathfrak{Spec}(\vdash_B)$ if and only if $\alpha \in \mathfrak{Spec}(\vdash)$ and $\alpha \cap B = \emptyset$.

Now let V be an arbitrary subset of S. We say that $\vdash$ *stabilizes on* V if, for every pair of subsets U and W of S,

$$\frac{\forall b \in W\, \exists U_b \in \mathrm{Fin}(U)\, (\, U_b, b \vdash V\,)}{\exists U_0 \in \mathrm{Fin}(U)\, \forall b \in W\, (\, U_0, b \vdash V\,)}$$

For instance, let A be a subset of S and consider the extension $\vdash^A$ which arises from $\vdash$ by putting additional axioms

$$\vdash^A a \qquad (a \in A)$$

on top of the latter, thus forcing every ideal element of $\vdash^A$ to contain the subset A. If $\vdash$ stabilizes on A, then, for every finite subset U and arbitrary subset V of S, we have

$$U \vdash^A V \qquad \text{if and only if} \qquad \exists A_0 \in \mathrm{Fin}(A)\, (\, U, A_0 \vdash V\,).$$

Next we explain a conditional form of completeness for infinitary entailment relations. Given a pair $(U, V) \in \mathrm{Fin}(S) \times \mathrm{Pow}(S)$, provided that $\vdash$ stabilizes on V, it can be semantically determined whether entailment $U \vdash V$ holds.

Corollary 8.4 (GET+REM). *Let V be a subset of S. If $\vdash$ stabilizes on V, then, for any finite subset U of S, the following are equivalent:*

1. $U \vdash V$
2. $\forall \alpha \in \mathfrak{Spec}(\vdash)\, (\, U \subseteq \alpha \to \alpha \between V\,)$

Proof. It suffices to show that the second item implies the first, to which end we concentrate on the contrapositive form. Thus, suppose that $U \nvdash V$. It is immediately clear that U is consistent with respect to $\vdash_V$. Because $\vdash$ stabilizes on V, it follows that $\vdash_V$ has a strongly hereditary inconsistency predicate. Therefore, according to the model existence principle, there is $\alpha \in \mathfrak{Spec}(\vdash_V)$ such that $U \subseteq \alpha$. To say that α is a model of $\vdash_V$ is to say that α models $\vdash$ and $\alpha \cap V = \emptyset$. We have found α as required. □

It follows that completeness holds for a generalized entailment relation $\vdash$ if and only if $\vdash$ stabilizes on every subset V of S. In particular, if $\vdash$ has a strongly hereditary inconsistency predicate, then empty conclusion entailments are semantically determined.

It should also be interesting to investigate whether and to what extent the conservation criteria of Chapter 3 carry over to this infinitary context. A variety of further extension theorems—e.g., the Hahn-Banach theorem and the full version of Sikorski's theorem, both of which can be proved with KZL followed by an application of a concrete one-step extension—then might allow to be readdressed in terms of conservation for single-conclusion entailment. In the following section, to give a hint at future developments in this direction, we revisit Baer's criterion (see Chapter 1). Here the approach with infinitary entailment relations appears to be rather promising.

[3] The deeper reason for which is to be found in the fact that not every locale is spatial. A thorough investigation will further require to put generalized entailment relations in context with geometric logic [111, 249].

8.2 Formal Baer criteria

Throughout, let R be a commutative ring with $1 \neq 0$. Further specifications will apply later. Let A and M be modules over R. We take the set $S = A \times M$ for our domain of discourse, and consider the infinitary entailment relation $\vdash$ which is generated by all instances of the following axioms:

$$\begin{aligned} (a,m),(a,n) &\vdash && \text{(s } [m \neq n]) \\ (a,m),(b,n) &\vdash (a+b, m+n) && (+) \\ (a,m) &\vdash (ra, rm) && (r) \\ &\vdash (0_A, 0_M) && (0) \\ &\vdash \{\, (a,m) : a \in M \,\} && \text{(t)} \end{aligned}$$

with side condition as indicated. An ideal element of $\vdash$ is a homomorphism of R-modules, thus

$$\mathfrak{Spec}(\vdash) = \mathrm{Hom}_R(A, M).$$

We say that $\vdash$ is the entailment relation of R-module homomorphism $A \to M$. We write $\vdash_{A,M}$ in case we need to specify which modules we refer to. For instance, with the ring R in place of A, we obtain the entailment relation $\vdash_{R,M}$ of R-homomorphism $R \to M$.

In order to provide an explicit description for inconsistent subsets of S, we first consider, for finite subsets $U = \{\, (a_1, m_1), \dots, (a_k, m_k) \,\}$ of S, the following predicate

$$\Phi(U) \;\equiv\; \exists r_1, \dots, r_k \in R \,\Big(\sum_{i=1}^{k} r_i a_i = 0_A \;\wedge\; \sum_{i=1}^{k} r_i m_i \neq 0_M \Big).$$

This predicate expresses that U has multiple values, i.e., we have $\Phi(U)$ if and only if $\mathsf{mv}(U)$ in the sense of Chapter 5: as a relation, the closure of U under (+) and (r) would force 0_A to take a non-zero value in M, whence U cannot be considered to approximate an R-homomorphism $A \to M$. In particular, $\Phi(U)$ is adequate. Next let Ψ denote the canonical extension of Φ to arbitrary subsets W of S, i.e.,

$$\Psi(W) \;\equiv\; \exists U \in \mathrm{Fin}(W)\, \Phi(U).$$

Again, in case we need to be more specific about the modules we are working with, we do so with a subscript. For instance, with R in place of A as above, we write $\Psi_{R,M}$ for the canonical extension of the inconsistency predicate $\Phi_{R,M}$ to arbitrary subsets of $R \times M$. It is straightforward to show that Ψ is strongly hereditary for the axiom of single-values (s), as well as for the structural axioms (+), (r), and (0). As regards totality (t), consider first the contrapositive form (♯) for a subset W of S,

$$\neg\Psi(W) \;\to\; \forall a \in A\, \exists m \in M\, \neg\Psi(W, (a,m)).$$

If we freely employ REM, then it suffices to verify the latter in order to show that Ψ is strongly hereditary for totality, to which end we could resort to the extension principle that is explained in the classical proof of Baer's criterion, and employ it here in an affirmative manner: first take into account that every consistent subset of $A \times M$ gives rise to a partial R-homomorphism which is defined on a certain submodule of A. Conversely, every partial homomorphism, construed as a subset of $A \times M$ is consistent with regard to $\vdash$. Suppose now that M is ideal-injective, i.e., such that every R-homomorphisms $I \to M$ which is defined on an ideal I of R extends to R. In this case, an explicit one-step extension principle is available (see Chapter 1), due to which (♯) is immediate. By way of Theorem 8.2, we know that M is ideal-injective if and only if the canonical inconsistency predicate of the entailment relation of R-homomorphism $R \to M$ is strongly hereditary. We are thus led to the following formal version of Baer's theorem, which by semantics and the model existence principle gives back the classical criterion for a module to be injective.[4]

[4] I am indebted to Rinaldi for having shared first ideas for syntactical variants of Baer's criterion already during the 2016 autumn school "Proof and Computation" in Fischbachau. His insights eventually led to the statement and proof of Theorem 8.5.

Theorem 8.5. *Let M be a module over R. If $\Psi_{R,M}$ is strongly hereditary, then, for every R-module A, $\Psi_{A,M}$ is strongly hereditary, as well.*

Proof. We concentrate only on showing that $\Psi_{A,M}$ is strongly hereditary for totality (t). To this end, let W be an arbitrary subset of $S = A \times M$, let $a \in A$, and suppose that for every $m \in M$ we have $\Psi_{A,M}(W,(a,m))$. Let W_R consist of all pairs $(r,m) \in R \times M$ such that there are $r_1,\dots,r_k \in R$ and $(a_1,m_1),\dots,(a_k,m_k) \in W$ with

$$\sum_{i=1}^{k} r_i a_i = ra \qquad \text{and} \qquad \sum_{i=1}^{k} r_i m_i = m.$$

We claim that for every $m \in M$ we have $\Psi_{R,M}(W_R,(1,m))$. For if m is an arbitrary element of M, then, according to our assumption on W, i.e., because of $\Psi_{A,M}(W,(a,m))$, there are $r_0,r_1,\dots,r_k \in R$ and $(a_1,m_1),\dots,(a_k,m_k) \in W$ such that

$$r_0 a + \sum_{i=1}^{k} r_i a_i = 0_A \qquad \text{and} \qquad r_0 m + \sum_{i=1}^{k} r_i m_i \neq 0_M.$$

By definition of W_R, the former implies $(-r_0, \sum_{i=1}^{k} r_i m_i) \in W_R$, and we see immediately that $\Psi_{R,M}((-r_0, \sum_{i=1}^{k} r_i m_i),(1,m))$, and thus $\Psi_{R,M}(W_R,(1,m))$ holds indeed. Now, since $\Psi_{R,M}$ is strongly hereditary, we obtain $\Psi_{R,M}(W_R)$. Therefore, there are $(r_1,m_1),\dots,(r_\ell,m_\ell) \in W_R$ and $s_1,\dots,s_\ell \in R$ such that

$$\sum_{j=1}^{\ell} s_j r_j = 0_R \qquad \text{and} \qquad \sum_{j=1}^{\ell} s_j m_j \neq 0_M.$$

Moreover, for every r_j there are $r_1^j,\dots,r_{k_j}^j \in R$ and $(a_1^j,m_1^j),\dots,(a_{k_j}^j,m_{k_j}^j) \in W$ such that

$$\sum_{i=1}^{k_j} r_i^j a_i^j = r_j a \qquad \text{and} \qquad \sum_{i=1}^{k_j} r_i^j m_i^j = m_j.$$

It remains to put this information together and calculate, on the one hand,

$$\sum_{j=1}^{\ell}\sum_{i=1}^{k_j} s_j r_i^j a_i^j = \sum_{j=1}^{\ell} s_j (\sum_{i=1}^{k_j} r_i^j a_i^j) = \sum_{j=1}^{\ell} s_j r_j a = 0_R$$

and, on the other hand,

$$\sum_{j=1}^{\ell}\sum_{i=1}^{k_j} s_j r_i^j m_i^j = \sum_{j=1}^{\ell} s_j (\sum_{i=1}^{k_j} r_i^j m_i^j) = \sum_{j=1}^{\ell} s_j m_j \neq 0_M.$$

This shows that we have found witnesses for $\Psi_{A,M}(W)$, as required. □

The following exposition closely follows [247].

Recall that if M is a module over R, and if $m \in M$, then m is said to be a *torsion element* if there is a non-zero element $r \in R$ such that $rm = 0_M$. We say that M is *torsion-free* if its only torsion element is 0_M.

A *zero-divisor* of R is an element r for which there is $s \in R$ such that $s \neq 0_R$ but $sr = 0_R$. Next recall that R is an *integral domain* if

$$\forall r,s \in R\,(\,rs = 0 \to r = 0 \vee s = 0\,).$$

If R is an integral domain, and r is a zero-divisor, then $r = 0_R$.

A module M over an integral domain R is said to be *divisible* if, for every $m \in M$ and for every $r \in R$ which is not a zero-divisor, there is $n \in M$ such that $m = rn$.

Now let R be a integral domain, and let $\operatorname{Frac} R$ be its *field of fractions* [17]. Let F be an R-submodule of $\operatorname{Frac} R$. If there is a non-zero element r of R such that $rF \subseteq R$, then F is called a *fractional ideal* of R. For instance, every ideal I of R, which in this context is called an *integral ideal*, is a fractional ideal. If F and F' are fractional ideals, then FF' denotes the set of all finite sums of products ff', where $f \in F$ and $f' \in F'$. This set is a fractional ideal, as well. Since R is commutative, note that $FF' = F'F$. If we have $FF' = R$, then F' is said to be *invertible*. If every non-zero integral ideal of R is invertible, then we say that R is a *Dedekind domain*.

The items of the following proposition are taken from [247, Proposition 2.7, Theorem 2.8, Proposition 2.10], respectively. In each case we follow the leading idea of the proof of the corresponding classical version.

Proposition 8.6. *Let R be a discrete integral domain, and let M be an R-module.*

1. *If M is torsion-free and divisible, then $\Psi_{R,M}$ is strongly hereditary.*
2. *If every ideal of R is a principal ideal, and if M is divisible, then $\Psi_{R,M}$ is strongly hereditary.*
3. *If R is a Dedekind domain such that for every ideal I of R either we have $I = \{ 0_R \}$ or else $I \neq \{ 0_R \}$, and if M is divisible, then $\Psi_{R,M}$ is strongly hereditary.*

Proof. For every item, we take a subset W of $R \times M$, consider an element $a \in R$, and suppose that for every $m \in M$ we have $\Psi_{R,M}(W, (a, m))$.

1. We start with $m = 0_M$ which by assumption gives $r_0, r_1, \ldots, r_k \in R$ and $(a_1, m_1), \ldots, (a_k, m_k) \in W$ such that
$$r_0 a + \sum_{i=1}^{k} r_i a_i = 0_R \qquad \text{and} \qquad \sum_{i=1}^{k} r_i m_i \neq 0_M.$$
If $r_0 = 0$, then we are done. Otherwise, if $r_0 \neq 0$, then we consider
$$m_0 = -\sum_{i=1}^{k} r_i m_i.$$
Since M is divisible, and r_0 is not a zero-divisor, we can write
$$m_0 = r_0 n_0$$
for a certain $n_0 \in M$. Then, because of $\Psi_{R,M}(W, (a, n_0))$ there are $s_0, s_1, \ldots, s_\ell \in R$ and $(b_1, n_1), \ldots, (b_\ell, n_\ell) \in W$ such that
$$s_0 a + \sum_{j=1}^{\ell} s_j b_j = 0_R \qquad \text{and} \qquad s_0 n_0 + \sum_{j=1}^{k} s_j n_j \neq 0_M.$$
Since M is torsion-free, we now multiply with r_0, which by substitution yields expressions from which we can read off $\Psi_{R,M}(W)$. To wit, if we put together all the available data, then we obtain on the one hand
$$\sum_{j=1}^{\ell} r_0 s_j b_j - \sum_{i=1}^{k} s_0 r_i a_i = r_0 (\sum_{j=1}^{\ell} s_j b_j + s_0 a) = 0_R.$$

On the other hand, since r_0 is not a zero-divisor and M is torsion-free,

$$\begin{aligned}\sum_{j=1}^{\ell} r_0 s_j n_j - \sum_{i=1}^{k} s_0 r_i m_i &= \sum_{j=1}^{\ell} r_0 s_j n_j + s_0 m_0 \\ &= \sum_{j=1}^{\ell} r_0 s_j n_j + s_0 r_0 n_0 \\ &= r_0 (\sum_{j=1}^{\ell} s_j n_j + s_0 n_0) \\ &\neq 0_M.\end{aligned}$$

2. Next we suppose that every ideal of R is a principal ideal. We consider the set

$$\left\{ r \in R : \exists r_1, \ldots, r_k \in R \, \exists (a_1, m_1), \ldots, (a_k, m_k) \in W \big(\sum_{i=1}^{k} r_i a_i = ra \big) \right\}.$$

It is straightforward to show that this set is an ideal of R; we denote it with $I_{W,a}$. Since R is a principal ideal ring, there is $r \in R$ such that $I_{W,a} = Rr$. In particular, $r \in I_{W,a}$, so there are $r_1, \ldots, r_k \in R$ and $(a_1, m_1), \ldots, (a_k, m_k) \in W$ such that

$$\sum_{i=1}^{k} r_i a_i = ra.$$

We put

$$m = \sum_{i=1}^{k} r_i m_i.$$

Now we distinguish cases. If $r \neq 0_R$, then, since M is divisible, there is $n \in M$ such that $m = rn$. We know that $\Psi_{R,M}(W, (a, n))$, hence there are $s_0, s_1, \ldots, s_\ell \in R$ and $(b_1, n_1), \ldots, (b_\ell, n_\ell) \in W$ such that

$$s_0 a + \sum_{j=1}^{\ell} s_j b_j = 0_R \qquad \text{and} \qquad s_0 n + \sum_{j=1}^{k} s_j n_j \neq 0_M.$$

From the former expression we infer that $s_0 \in I_{W,a}$, whence we can write $s_0 = s'r$ for a certain $s' \in R$. We may now substitute and obtain

$$0_R = s'ra + \sum_{j=1}^{\ell} s_j b_j = \sum_{i=1}^{k} s' r_i a_i + \sum_{j=1}^{\ell} s_j b_j$$

as well as

$$0_M \neq s'rn + \sum_{j=1}^{k} s_j n_j = s'm + \sum_{j=1}^{k} s_j n_j = \sum_{i=1}^{k} s' r_i m_i + \sum_{j=1}^{k} s_j n_j.$$

This means $\Psi_{R,M}(\{ (a_i, m_i) \}, \{ (b_j, n_j) \})$, so that we have found witnesses for the inconsistency of W, as required. It remains to consider the case $r_0 = 0_R$, i.e., $I_{W,a} = \{ 0_R \}$. We argue similarly, now employing the hypothesis $\Psi_{R,M}(W, (a, m))$. Once more we obtain certain pairs $(b_1, n_1), \ldots, (b_\ell, n_\ell) \in W$ along with $s_0, s_1, \ldots, s_\ell \in R$ such that

$$s_0 a + \sum_{j=1}^{\ell} s_j b_j = 0_R \qquad \text{and} \qquad s_0 m + \sum_{j=1}^{\ell} s_j n_j \neq 0_M.$$

As before, the former yields $s_0 \in I_{W,a}$, and therefore $s_0 = 0_R$. The conclusion can thus be drawn immediately.

3. Once again we make use of the ideal $I_{W,a}$. If we have $I_{W,a} = \{ 0_R \}$, then we start with $\Psi_{R,M}(W,(a,0_M))$, which directly leads to $\Psi_{R,M}(W)$. We thus suppose that $I_{W,a} \neq \{ 0_R \}$. Since R is a Dedekind ring, there is a fractional ideal F such that $FI_{W,a} = R$. It follows that there are $r_1, \dots, r_\ell \in I_{W,a}$ and $s_1, \dots, s_\ell \in F$ such that $\sum_{j=1}^{\ell} s_j r_j = 1$. We may suppose that every r_j is non-zero. Furthermore, for every j with $1 \leqslant j \leqslant n$ there are $(a_1^j, m_1^j), \dots, (a_\ell^j, m_\ell^j) \in W$ and $r_1^j, \dots, r_{k_j}^j \in R$ such that
$$r_j a = \sum_{i=1}^{k_j} r_i^j a_i^j.$$
Next we write
$$m_j = \sum_{i=1}^{k_j} r_i^j m_i^j.$$
Since M is divisible, there are $n_1, \dots, n_\ell \in M$ such that $m_j = r_j n_j$. Since $s_j r_j \in R$, we may put
$$m = \sum_{j=1}^{\ell} s_j r_j n_j$$
and obtain an element of M. Now, according to the assumption, we have $\Psi_{R,M}(W,(a,m))$, whence there are $(b_1, n_1), \dots, (b_k, n_k) \in W$ and $t_0, t_1, \dots, t_k \in R$ such that
$$t_0 a + \sum_{i=1}^{k} t_i b_i = 0_R \qquad \text{and} \qquad t_0 m + \sum_{i=1}^{k} t_i n_i \neq 0_M.$$
From the former expression we infer that $t_0 \in I_{W,a}$. Therefore, we have $t_0 s_j \in R$ whenever $1 \leqslant j \leqslant n$. It remains to calculate
$$t_0 a = t_0 1 a = t_0 (\sum_{j=1}^{\ell} s_j r_j) a = \sum_{j=1}^{\ell} t_0 s_j r_j a = \sum_{j=1}^{\ell} t_0 s_j (\sum_{i=1}^{k_j} r_i^j a_i^j)$$
along with
$$t_0 m = t_0 \sum_{j=1}^{\ell} s_j r_j n_j = \sum_{j=1}^{\ell} t_0 s_j m_j = \sum_{j=1}^{\ell} t_0 s_j (\sum_{i=1}^{k_j} r_i^j m_i^j)$$
in order to obtain witnesses for $\Psi_{R,M}(W)$. □

It might be interesting to address the classical converse to the third item of Proposition 8.6 by means of our methods: if R is an integral domain such that every divisible R-module is injective, then R is a Dedekind domain [247, Theorem 4.25].

Our approach to pure-injective algebras in Section 5 further begs the question as to how algebraically compact modules [199] might find a natural treatment in terms of infinitary entailment relations.

It is well-known that divisibility is necessary for a module to be injective. We regain this statement by means of model existence, as follows.

Corollary 8.7 (GET+REM)**.** *Let M be an R-module. If $\Psi_{R,M}$ is strongly hereditary, then M is divisible.*

Proof. Let $m \in M$ and let r be an element of R which is not a zero-divisor. We define $f : Rr \to M$ by $f(sr) = sm$ which gives a well-defined R-homomorphism. Construed as a subset of $R \times M$, this f is consistent with respect to the entailment relation of R-homomorphism $R \to M$. Therefore, by means of the model existence principle, it extends to an R-homomorphism $g : R \to M$. Thus
$$m = f(r) = g(r) = rg(1),$$
which shows divisibility. □

In particular, with GET+REM we see that if R is a principal ideal domain, then a module M is divisible if and only if, for every R-module A, we can prove that $\Psi_{A,M}$ is strongly hereditary. For instance, and still in the classical setting, an abelian group (construed as a $\mathbb{Z}$-module) is injective if and only if it is divisible.[5]

Example 8.8. Let K be a non-trivial field and let V be a vector space over K. Let $\vdash$ be the entailment relation of K-homomorphism $V \to K$. Let $u \in V$. We claim that the following are equivalent:

1. $\vdash (u, 0_K)$
2. $u = 0_V$.

Since $\vdash (0_V, 0_K)$ is an axiom anyway, it suffices to infer the second item from the first one. To this end, let $\mu \neq 0_K$. By cut with the axiom of single-values (s) we obtain $(u, \mu) \vdash$. According to the formal Nullstellensatz, this implies that there is $\lambda \in K$ such that $\lambda u = 0_V$ and $\lambda\mu \neq 0_K$. The latter gives $\lambda \neq 0_K$, whence we obtain $u = 0_V$ by scaling the former. From a semantical point of view, this asserts that the only element of V that maps to 0_K under *every* functional $\alpha : V \to K$ can only be 0_V, briefly

$$\bigcap_{\alpha \in V^*} \ker \alpha = \{ 0_V \},$$

where V^* denotes the dual space of V, i.e., the space of all linear maps $\alpha : V \to K$. This can be used to show that the canonical mapping

$$\iota : V \to V^{**}, \quad u \mapsto (\operatorname{ev}_u : \alpha \mapsto \alpha(u))$$

embeds V in its double dual. It is interesting to note that our method allows to go further and address bilinear forms, or, even more generally, multilinear forms over K, too. This should give rise to further applications of entailment relations in linear and multilinear algebra. We have done so already for finite fields, employing conventional finitary entailment relations. The infinitary generalization now promises to have a much wider applicability.

[5]Recall that LPO is necessary for $\mathbb{Z}$ to be a principal ideal ring [94, 181], which would be a requirement to show with Proposition 8.6 that if G is an abelian group, then $\Psi_{\mathbb{Z},G}$ is strongly hereditary.

Chapter 9

Suzumura consistency, an alternative approach

This chapter is based on [226].

9.1 Introduction

Order extension principles are originally due to Szpilrajn [242] for strict partial orders; for quasi-orders they were phrased by Arrow [13] and proved by Hansson [129]. They play a seminal role in mathematical economics, game theory, and in the theory of social choice, preferences, and utility (see, e.g., [12, 61] for an exhaustive overview, and [50] for the rising field of computational social choice). Suzumura [241] specified a notion of consistency which is sufficient and necessary for a binary relation to have an order extension. With the customary definition of consistency, however, proofs often require indirect reasoning and arguments on pairs, i.e., arguments involving specific elements of the underlying set. We now show how Suzumura consistency can be put in a logically equivalent and negation-free form, which allows for a somewhat slicker treatment. In fact, this has an interesting methodological effect: we can keep to a minimum arguments on pairs, and largely avoid proofs by contradiction—instead we argue abstractly within the algebra of relations.

Yet one cannot do with constructive means only, as J.L. Bell made clear. While Zorn's Lemma, the key tool for order extension, allegedly is "constructively neutral" [33], order extension is not: it results in Gödel–Dummett logic not only for partial orders [32] but also, as we show below (Section 9.8.4), for quasi-orders. Negri et al. [190] proved practicable a proof–theoretic study of order extension; see also [209].

This chapter is organised as follows. We first list the most necessary preliminaries in Section 9.2. In Section 9.3 we discuss the notion of consistency, and in Section 9.4 we make precise a notion of (compatible) extension intimately related with consistency. Then, in Section 9.5, we concentrate on the extendability of consistent relations to complete quasi-orders, while in Section 9.6 we rephrase a classic result of Dushnik and Miller [106] in terms of consistent relations. In Section 9.7 we present another proof of Arrow's generalization, along the lines of [13, 61, 141, 240]. In the complementary Section 9.8 we explain an alternative proof of the order extension principle by way of Open Induction [201] rather than Zorn's Lemma; carry over from partial orders to quasi-orders J.L. Bell's argument [32] that Gödel–Dummett logic is necessary for the order extension principle; and revisit Richter's theorem [206] on rationalizability of choice functions.

9.2 Preliminaries

For the purposes of this chapter, a certain amount of fairly standard terminology needs to be fixed. Until further notice, we work classically, over **ZFC**. In the following, let R and S denote binary relations on a set X, i.e., subsets of the cartesian product $X \times X$. By "relation" we shall always mean "binary relation", and henceforth we skip "binary". The *opposite* (or *reciprocation*) of R is

$$R^\circ = \{ (x,y) \in X \times X : (y,x) \in R \} .$$

Note that $(R^\circ)^\circ = R$, and if $R \subseteq S$, then $R^\circ \subseteq S^\circ$. Furthermore, $(R \cup S)^\circ = R^\circ \cup S^\circ$. The *asymmetric part* of R is $P(R) = R - R^\circ$, which is to say that

$$P(R) = \{ (x,y) : (x,y) \in R \ \wedge \ (y,x) \notin R \} .$$

The *transitive closure* of R is

$$\mathrm{tc}(R) = \bigcup_{i \geqslant 1} R^i,$$

with $R^1 = R$ and $R^{i+1} = R^i \circ R$, where

$$R \circ S = \{ (x,z) : \exists\, y \in X \ (x,y) \in R \wedge (y,z) \in S \}$$

denotes the *relational composition*.[1] A relation R is *transitive* if and only if $R \circ R \subseteq R$, which in turn holds if and only if $\mathrm{tc}(R) = R$. If R, S are relations, then $R \subseteq \mathrm{tc}(S)$ if and only if $\mathrm{tc}(R) \subseteq \mathrm{tc}(S)$; that is to say that tc is a closure operator on the powerset $\mathscr{P}(X \times X)$. As composition obeys $(R \circ S)^\circ = S^\circ \circ R^\circ$ and distributes over unions, we see that the transitive closure commutes with reciprocation, $\mathrm{tc}(R^\circ) = \mathrm{tc}(R)^\circ$.

We say that R is *complete*[2] if $R \cup R^\circ = X \times X$, which is also known as R being *linear* or *total*. Mind that a complete relation is *reflexive*, i.e., $\Delta \subseteq R$, where

$$\Delta = \{ (x,y) \in X \times X : x = y \}$$

is the *diagonal*, and that $\Delta \cap R = \Delta \cap R^\circ$. The diagonal is neutral for composition, i.e., $\Delta \circ R = R = R \circ \Delta$. The *reflexive closure* of R is $R \cup \Delta$.

A *quasi-order* (or *preorder*) is a reflexive transitive relation. The *hull* $\overline{R}$ of a relation R, viz.

$$\overline{R} = \bigcup_{i \geqslant 0} R^i$$

where $R^0 = \Delta$, is the least quasi-order which contains R. Note that $\overline{R} = \mathrm{tc}(R) \cup \Delta = \mathrm{tc}(R \cup \Delta)$.

If Y is a subset of X, then

$$R\big|_Y = R \ \cap \ (Y \times Y)$$

is the *restriction* of R on Y. The restricted diagonal is denoted by Δ_Y. Occasionally we write $R, (x,y)$ instead of $R \cup \{ (x,y) \}$. If S is a set and $X, Y \subseteq S$, then $X \between Y$ is shorthand for $X \cap Y$ being inhabited.[3]

An antisymmetric quasi-order R on X, i.e., one for which $R \cap R^\circ \subseteq \Delta$, is a *partial order*; the underlying set X in which case is called a *poset*. If R is complete, then X is said to be *linearly ordered*. By a *chain* in a poset X we understand an inhabited subset of X that is linearly ordered by the restricted relation. We say that X is *chain-complete*, if X is inhabited and every chain C has a least upper bound $\bigvee C$ in X. A *maximal element* x in X is such that

$$\forall y \in X \ (x \leqslant y \rightarrow x = y) .$$

One of the standard forms of the *Kuratowski-Zorn Lemma* reads as follows:

KZL Every chain-complete poset has a maximal element.

It is as such that the Axiom of Choice (AC) gets involved in proving the order extension principle in its full generality. In fact, a strictly weaker form of AC suffices [107, 145], but this shall not be of our concern.[4]

[1] We adhere to the traditional, Tarskian convention about composition which is customary in the context of preference relations [48], and even in certain abstract categorical settings [112].

[2] The notion of "completeness" is prevalent in the context of logical theories and Lindenbaum's Lemma.

[3] We have adopted this notation from Giovanni Sambin.

[4] The Axiom of Choice is not entirely indispensable: syntactical conservation works for Horn sequents [190]; see also [209].

9.3 Consistency

Suzumura [241] gave a sufficient and necessary condition for a relation R to have a complete quasi-order extension which preserves the asymmetric part $P(R)$. A relation R is *Suzumura consistent* if

$$\forall x, y \in X \ \big[\ (x, y) \in \mathrm{tc}(R) \ \rightarrow \ (y, x) \notin P(R)\ \big].$$

Unfolding the definition of $P(R)$, Suzumura consistency amounts to

$$\forall x, y \in X \ \big[\ (x, y) \in \mathrm{tc}(R) \ \rightarrow \ \neg\big(\ (y, x) \in R \wedge (x, y) \notin R\ \big)\ \big],$$

which (with classical logic) is equivalent to

$$\forall x, y \in X \ \big[\ (x, y) \in \mathrm{tc}(R) \wedge (y, x) \in R \ \rightarrow \ (x, y) \in R\ \big].$$

This condition on R can now be written succinctly as set containment. We replace Suzumura consistency by this equivalent, and simply call it consistency, as follows.

Definition 9.1. *A relation R is* consistent *if*

$$\mathrm{tc}(R) \cap R^{\circ} \subseteq R.$$

Remark 9.2. A relation is consistent if and only if $\overline{R} \cap R^{\circ} \subseteq R$. Moreover, a relation R is consistent if and only if every cycle in R is "reversible", by which we mean that every cycle in R forces its reciprocal to be in R as well. Really this concerns cycles of any length—here is another, equivalent way to put consistency:

$$\forall n \geqslant 0 \ \ R^n \cap R^{\circ} \subseteq R.$$

In terms of preferences, consistency "rules out ... all cycles with at least one strict preference" [48, p. 36].

Remark 9.3. Every transitive relation is consistent. In particular, Δ, $\mathrm{tc}(R)$ and $\overline{R}$ are consistent. On the other hand, it is well-known that consistency is weaker than transitivity. For instance, $R = \{\,(x, y), (y, x)\,\}$ is consistent on $X = \{\,x, y\,\}$ but not transitive unless $x = y$, in fact $\mathrm{tc}(R) = R \cup \Delta$ and $R^{\circ} = R$.

But what is missing for a consistent relation to be transitive? Compositions need to be comparable.

Proposition 9.4. *For a relation R, each of the following items implies the next.*

1. *R is transitive,*
2. *$\mathrm{tc}(R) \subseteq R \cup R^{\circ}$,*
3. *$R \circ R \subseteq R \cup R^{\circ}$.*

If R is consistent, then the above assertions are equivalent. In particular, transitivity is equivalent to consistency together with any of (2) and (3) above.

Proof. Of course, if R is transitive, then $\mathrm{tc}(R) = R$, whence (ii) follows from (i). Furthermore, from $R \circ R \subseteq \mathrm{tc}(R)$, we know that (ii) implies (iii). Next, if R is consistent and $R^2 = R \circ R \subseteq R \cup R^{\circ}$, then

$$R^2 = R^2 \cap (R \cup R^{\circ}) = (R^2 \cap R) \cup (R^2 \cap R^{\circ}) \subseteq R \cup (\mathrm{tc}(R) \cap R^{\circ}) \subseteq R$$

Therefore, transitivity is implied by (iii), given that R is consistent. □

Remember that we have defined a relation R on X to be complete if $R \cup R^{\circ} = X \times X$. The following corollary is a direct consequence of Proposition 9.4. This observation has also been made in [48].

Corollary 9.5. *A complete consistent relation is transitive.*[5] *In particular, a relation is complete and consistent if and only if it is a complete quasi-order.* □

Consider again conditions (ii) and (iii) in Proposition 9.4 above. Neither of them follows from consistency, just because a consistent relation need not be transitive. In turn, neither (ii) nor (iii) implies consistency. For example, if $R = \{ (x,y), (y,z), (z,x) \}$ on a set $\{ x,y,z \}$ with pairwise distinct elements x, y, z, then the reflexive closure $R \cup \Delta$ is not a consistent relation, yet it satisfies (ii). Therefore, consistency is independent of each of these assertions (ii) and (iii). Furthermore, as the example we have just given also shows (ii) and (iii) to be strictly weaker than transitivity, by Proposition 9.4 we have at hand a proper decomposition of transitivity.

It has been observed [46] that consistency—just as transitivity—can be expressed by means of a closure condition. Here this takes the following form.

Definition 9.6. *The* consistent closure *of R is*

$$\mathrm{cc}(R) = \mathrm{tc}(R) \cap (R \cup R^\circ).$$

Note that $R \subseteq \mathrm{cc}(R) \subseteq \mathrm{tc}(R)$. The consistent closure reverses cycles and thus "eliminates" strict preference from any such cycle. Consider for example once more a set $X = \{ x, y, z \}$ with three pairwise distinct elements x, y, z, together with the "cyclic" relation $R = \{ (x,y), (y,z), (z,x) \}$. This relation is not consistent, and the transitive closure of R is universal, i.e., $\mathrm{tc}(R) = X \times X$. The consistent closure, on the other hand, adds the opposite, but neither is reflexive nor transitive.

Lemma 9.7. *Let R and S be relations.*

1. *R is consistent if and only if $\mathrm{cc}(R) = R$.*
2. *$R \subseteq \mathrm{cc}(S)$ if and only if $\mathrm{cc}(R) \subseteq \mathrm{cc}(S)$.*

Proof.

1. We have $\mathrm{cc}(R) = (\mathrm{tc}(R) \cap R) \cup (\mathrm{tc}(R) \cap R^\circ) = R \cup (\mathrm{tc}(R) \cap R^\circ)$. Therefore, $\mathrm{cc}(R) = R$ if and only if $\mathrm{tc}(R) \cap R^\circ \subseteq R$.
2. The consistent closure is defined as intersection of transitive closure and symmetrization $R \mapsto R \cup R^\circ$. Therefore, it suffices to show that the latter satisfies the corresponding equivalence, which is immediate from the properties of reciprocation: if $R \subseteq S \cup S^\circ$, then $R^\circ \subseteq (S \cup S^\circ)^\circ = S^\circ \cup S^{\circ\circ} = S^\circ \cup S$, whence $R \cup R^\circ \subseteq S \cup S^\circ$. The converse implication is trivial. □

In other words, the assignment $R \mapsto \mathrm{cc}(R)$ defines a closure operator the fixed points of which are precisely the consistent relations. Furthermore, $\mathrm{cc}(R)$ is the least consistent relation which contains R.

9.4 Compatible Extensions

The following definition is equivalent to the one employed in the context of preference relations [104].

Definition 9.8. *Let R, S be relations. We say that S is a* compatible extension *of R if*

$$R \subseteq S \quad \textit{and} \quad S \cap R^\circ \subseteq R.$$

In fact, if $R \subseteq S$, then $S \cap R^\circ \subseteq R$ precisely when $P(R) \subseteq P(S)$ holds for the asymmetric parts.

[5] This is readily proved element-wise too. Here is another direct argument: if R is complete and consistent, then

$$\mathrm{tc}(R) = \mathrm{tc}(R) \cap (X \times X) = \mathrm{tc}(R) \cap (R \cup R^\circ) = \mathrm{cc}(R) = R,$$

whence R is transitive—see below for the *consistent closure* $\mathrm{cc}(R)$ of R.

Remark 9.9. If S is a compatible extension of $R \cup \{ (y, x) \}$, then $(x, y) \in S$ implies $(x, y) \in R$, provided that either R is reflexive or $x \neq y$. For compatibility of S over $R \cup \{ (y, x) \}$ means that

$$R \cup \{ (y,x) \} \supseteq S \cap \big(R \cup \{ (y,x) \} \big)^{\circ} = S \cap \big(R^{\circ} \cup \{ (x,y) \} \big) = \big(S \cap R^{\circ} \big) \cup \big(S \cap \{ (x,y) \} \big).$$

Compatible extension can thus be regarded as "reflecting opposite elements". On the other hand, Remark 9.9 also has the following reading: if S is a compatible extension of $R \cup \{ (y, x) \}$, and if $(x, y) \notin R$, then $(x, y) \notin S$.

The containment $R \subseteq X \times X$ is a compatible extension if and only if $R^{\circ} \subseteq R$, which is to say that R is symmetric. The reason why compatibility needs to be involved, is to avoid the universal relation to be an extension of every R [129, p. 453], and hence to be a solution of the problem of extending a relation to a complete quasi-order in the absence of any further restrictive assumption as, say, antisymmetry.[6] Mind that every relation R is a compatible extension of itself; whence a compatible extension need not necessarily be consistent. Not even a compatible extension of a consistent relation needs to be consistent.[7]

Caveat. For brevity's sake, following a certain tradition [61, 102, 241], whenever referring to an extension we will henceforth always mean a *compatible* extension.

Remark 9.10. Suppose that $R \subseteq S \subseteq T$. If T extends R, then so does S, because $S \cap R^{\circ} \subseteq T \cap R^{\circ} \subseteq R$.

Remark 9.11. Suppose that S is an extension of R. Then $S = R$ already if $S \subseteq R \cup R^{\circ}$, for in that case $S = S \cap (R \cup R^{\circ}) = (S \cap R) \cup (S \cap R^{\circ}) \subseteq R$. In particular, every complete relation R is maximal for extension, i.e., if S extends R and R is complete, then $S = R$. An extension might thus be very close; in fact, the consistent closure $\mathrm{cc}(R)$ of a relation R cannot extend R unless R itself is consistent, simply because $\mathrm{cc}(R) \subseteq R \cup R^{\circ}$.

In any case, the reflexive closure always gives an extension, which is the special case $R = S$ of the following.

Remark 9.12. Extensions carry over to reflexive closures. In fact, $R \subseteq S$ is compatible if and only if $R \subseteq S \cup \Delta$ is compatible, because

$$(S \cup \Delta) \cap R^{\circ} = (S \cap R^{\circ}) \cup (\Delta \cap R^{\circ}) = (S \cap R^{\circ}) \cup (\Delta \cap R).$$

Lemma 9.13. *Extension defines a partial order on relations.*

Proof. Extension clearly is reflexive, and inherits antisymmetry from inclusion. It remains to verify transitivity, i.e., if S extends R and T extends S, then T extends R. To this end, we calculate, using that $R^{\circ} \subseteq S^{\circ}$ whenever $R \subseteq S$,

$$T \cap R^{\circ} = (T \cap R^{\circ}) \cap R^{\circ} \subseteq (T \cap S^{\circ}) \cap R^{\circ} \subseteq S \cap R^{\circ} \subseteq R. \qquad \square$$

With the following proposition we adapt and extend an interesting result from [61]. The proof is straightforward in terms of our notion of consistency.

Proposition 9.14. *The following are equivalent for every relation R.*

1. *R is consistent.*
2. $\mathrm{cc}(R)$ *extends R.*
3. $\mathrm{tc}(R)$ *extends R.*

[6] On the other hand, if R is reflexive, then every extension of R by an antisymmetric relation S automatically is compatible, for in that case $S \cap R^{\circ} \subseteq S \cap S^{\circ} = \Delta \subseteq R$.

[7] In fact, the empty relation $\emptyset$ is consistent, and is compatibly extended by any—possibly non-consistent—relation whatsoever.

4. *$\overline{R}$ extends R.*

5. *R has a consistent extension.*

6. *R has a transitive extension.*

7. *R has a quasi-order extension.*

Proof. Notice first that whenever a quasi-order S contains R, we actually have

$$R \subseteq \mathrm{cc}(R) \subseteq \mathrm{tc}(R) \subseteq \overline{R} \subseteq S.$$

Hence, if S extends R, then (Remark 9.10) so do $\overline{R}$, $\mathrm{tc}(R)$, and $\mathrm{cc}(R)$; the latter extension is tantamount to R being consistent, by way of Remark 9.11. On the other hand, if R is consistent, then $\mathrm{cc}(R) = R$, and $\mathrm{tc}(R)$ extends R, simply by definition. Adding the diagonal does not do any harm, so $\overline{R}$ extends R, if $\mathrm{tc}(R)$ does, in which case R has a quasi-order extension. $\square$

As noticed in [61], because of $R \subseteq \mathrm{cc}(R) \subseteq \mathrm{tc}(R)$, it follows that $\mathrm{tc}(R) = \mathrm{tc}(\mathrm{cc}(R))$. In view of Lemma 9.7 and Proposition 9.14, then $\mathrm{tc}(R)$ is an extension of $\mathrm{cc}(R)$. Moreover, $\overline{R} = \overline{\mathrm{cc}(R)}$.

9.5 Extension Principles

Suzumura singled out that for a relation R to have a *complete* quasi-order extension, it suffices for R to be consistent. But as long as there are no further assumptions made on the underlying set X, some form of the Axiom of Choice has to be involved.

We still need some preparation on our way to Suzumura's variant of order extension: first we have to make sure that the consistent extensions of a relation form a chain-complete poset (Lemma 9.15 below). Then we need to verify, typical indeed for many an application of Zorn's Lemma [33], that a consistent relation can be "step-wise" extended. Once all this has been done, Zorn's Lemma may be invoked.

Lemma 9.15.

1. *Every union of a chain $(R_i)_{i\in I}$ of consistent relations is consistent.*

2. *If $(R_i)_{i\in I}$ is a chain with respect to extension, then $\bigcup_{i\in I} R_i$ is the least upper bound of $(R_i)_{i\in I}$ also with respect to extension.*

Proof. 1. This is a standard argument. Suppose that $(R_i)_{i\in I}$ is a chain of consistent relations, and let

$$(x, y) \in \mathrm{tc}(\bigcup_{i\in I} R_i) \cap (\bigcup_{i\in I} R_i)^{\circ}.$$

By the definition of tc, finitely many R_i suffice. As we have a chain, there in fact is $i_0 \in I$ with $(x, y) \in \mathrm{tc}(R_{i_0}) \cap R_{i_0}^{\circ}$. Then $(x, y) \in R_{i_0}$ since R_{i_0} is consistent, and therefore $(x, y) \in \bigcup_{i\in I} R_i$.

2. Of course $R_{i_0} \subseteq \bigcup_{i\in I} R_i$ for every $i_0 \in I$. Furthermore, as we have a chain of extensions, for every $i \in I$ either R_i extends R_{i_0} or vice versa, and in each case we have $R_i \cap R_{i_0}^{\circ} \subseteq R_{i_0}$. From this we get

$$\big(\bigcup_{i\in I} R_i\big) \cap R_{i_0}^{\circ} = \bigcup_{i\in I} (R_i \cap R_{i_0}^{\circ}) \subseteq R_{i_0}.$$

Next, if S is a relation such that S extends R_i for every $i \in I$, then of course $\bigcup_{i\in I} R_i \subseteq S$, and

$$S \cap \big(\bigcup_{i\in I} R_i\big)^{\circ} = S \cap \big(\bigcup_{i\in I} R_i^{\circ}\big) = \bigcup_{i\in I} (S \cap R_i^{\circ}) \subseteq \bigcup_{i\in I} R_i,$$

which is to say that S extends the union $\bigcup_{i\in I} R_i$. $\square$

Simply adding some pair to a consistent relation need not in general result in a consistent relation. If x, y, z are pairwise distinct, then $R = \{\, (x,y), (y,z) \,\}$ is consistent, yet $R' = R \cup \{\, (z,x) \,\}$ is not. Now one might be tempted to work with $\mathrm{cc}(R')$ instead, which, as we have seen, is the least consistent relation to contain R'. But still there is a problem, since $R \subseteq \mathrm{cc}(R')$ is not compatible, for $\mathrm{cc}(R') \cap R^\circ = R^\circ \nsubseteq R$.

The following, somewhat technical lemmas are crucial in this regard, and when it comes to proving the order extension principle. We need to provide means for extending consistent relations by suitable pairs of elements.

Lemma 9.16. *If R is a quasi-order, then for all $x, y \in X$*

1. *if $\big(\ R \circ \{\, (x,y) \,\} \circ R\ \big) \between R^\circ$, then $(y,x) \in R$,*
2. *$\mathrm{tc}\big(R, (x,y)\big) \;=\; R \;\cup\; \big(\ R \circ \{\, (x,y) \,\} \circ R\ \big)$,*
3. *if $(y,x) \in \mathrm{tc}\big(R, (x,y)\big)$, then $(y,x) \in R$, and*
4. *if $\mathrm{tc}\big(R, (x,y)\big)$ extends R, then $R, (x,y)$ is consistent.*

Proof. 1. Suppose that $(a,b) \in \big(\ R \circ \{\, (x,y) \,\} \circ R\ \big) \cap R^\circ$. This means $(a,x), (y,b) \in R$ and $(b,a) \in R$. By transitivity of R we get $(y,x) \in R$.[8]

2. One inclusion is easily verified; as for the converse inclusion we show

$$\big(R, (x,y)\big)^n \;\subseteq\; R \;\cup\; \big(\ R \circ \{\, (x,y) \,\} \circ R\ \big)$$

for every $n \geqslant 1$. We have

$$R \cup \{\, (x,y) \,\} \;=\; R \;\cup\; \Delta \circ \{\, (x,y) \,\} \circ \Delta \;\subseteq\; R \;\cup\; \big(\ R \circ \{\, (x,y) \,\} \circ R\ \big)$$

which takes care of $n = 1$ (mind that R needs to be reflexive in order for this to go through). Next we argue by induction, which gives

$$\begin{aligned}\big(R, (x,y)\big)^{n+1} &= \big(R, (x,y)\big)^n \circ \big(R, (x,y)\big) \\ &\subseteq \big[\ R \;\cup\; \big(\ R \circ \{\, (x,y) \,\} \circ R\ \big)\ \big] \circ \big(R, (x,y)\big).\end{aligned}$$

The remainder is left to the reader; take into account the transitivity of R, and

$$\{\, (x,y) \,\} \circ R \circ \{\, (x,y) \,\} \subseteq \{\, (x,y) \,\}\,.$$

3. By (2), if $(y,x) \in \mathrm{tc}\big(R, (x,y)\big)$, then $(y,x) \in R$ or $(y,x) \in R \circ \{\, (x,y) \,\} \circ R$. In case of the latter, due to the definition of relational composition, we get again $(y,x) \in R$.

4. From (3) we know that $\mathrm{tc}\big(R, (x,y)\big) \cap \{\, (y,x) \,\} \subseteq R$. Therefore, if $\mathrm{tc}\big(R, (x,y)\big)$ is an extension of R, then

$$\mathrm{tc}\big(R, (x,y)\big) \cap \big(R, (x,y)\big)^\circ = \big[\mathrm{tc}\big(R, (x,y)\big) \cap R^\circ\big] \cup\ \big[\mathrm{tc}\big(R, (x,y)\big) \cap \{\, (y,x) \,\}\big] \subseteq R.$$

□

Lemma 9.17. *If R is a quasi-order and $(y,x) \notin R$, then $R, (x,y)$ is a consistent extension of R.*

Proof. By Lemma 9.16(1), if $(y,x) \notin R$, then $\big(\ R \circ \{\, (x,y) \,\} \circ R\ \big) \cap R^\circ = \emptyset$. Now Lemma 9.16(2) implies $\mathrm{tc}\big(R, (x,y)\big) \cap R^\circ \subseteq R$, which is to say that $\mathrm{tc}\big(R, (x,y)\big)$ extends R; whence $R, (x,y)$ is consistent, according to Lemma 9.16(4). Finally, $R, (x,y)$ is an extension of R, simply because $(y,x) \notin R$. □

[8] Notice that reflexivity of R is irrelevant for this argument.

In terms of [104], Lemma 9.17 establishes that the set of quasi-orders on X is *arc-receptive.* We will now see that, with a maximal extension at hand, proving completeness boils down to just one application of Lemma 9.17, as we will end up with a quasi-order, anyway.

Lemma 9.18. *For consistent relations, ordered by extension, maximality implies completeness.*

Proof. Let R be maximal among consistent relations, ordered by extension. According to Proposition 9.14, $\overline{R}$ extends R, hence $\overline{R} = R$ by maximality, i.e., R is a quasi-order. This R cannot fail to compare any two elements from X, hence must be complete. To be precise, we need to verify $X \times X = R \cup R^\circ$, and to this end consider $x, y \in X$ such that $(x, y) \notin R^\circ$, i.e., $(y, x) \notin R$. Then $R, (x, y)$ is consistent and extends R by Lemma 9.17. Therefore, again by way of maximality, $R, (x, y) = R$, which is to say that $(x, y) \in R$. □

We are now ready to put everything together. Recall that a complete relation is consistent if and only if it is a quasi-order (Corollary 9.5). We have been working towards the following extension principle for consistent relations [241, Theorem 3]:

Consistent Extension Principle (CEP)**.** Every consistent relation can be extended to a complete quasi-order.

Proof. If R is consistent, then the set $\mathscr{E}$ of consistent extensions of R is inhabited, and it is chain-complete by Lemma 9.15. As we have seen in Lemma 9.18, every maximal element of $\mathscr{E}$ is a complete consistent extension of R, hence a complete quasi-order. The existence of at least one such maximal extension is ensured by KZL. □

It is in order to list also the following two slight variants and immediate consequences of CEP:

Transitive Extension Principle (TEP)**.** Every transitive relation can be extended to a complete quasi-order.

Quasi-Order Extension Principle (QEP)**.** Every quasi-order can be extended to a complete quasi-order.

Proof. QEP is a special case of TEP, and TEP follows from CEP as transitivity implies consistency. □

9.6 Intersection Principles

We also want to adapt a well-known observation due to Dushnik and Miller [106], which has been phrased for quasi-orders by Donaldson and Weymark [102], and Bossert [45], and put into general terms by Duggan [104]. We present a slight variation (Proposition 9.20) from which some immediate consequences can be drawn.

If R is consistent, then it has at least one complete consistent extension, according to CEP. Hence we can reasonably talk about the intersection S of all such extensions of R. Since every complete consistent extension of R is a quasi-order, S too is a quasi-order. This observation sets S apart from R whenever R happens to lack either reflexivity or transitivity. However, we will now see that S coincides with the hull $\overline{R}$, which is the intersection of all quasi-orders containing R. In fact, every pair of elements which is comparable under every complete extension of a consistent relation R must already be comparable by way of its hull.

Lemma 9.19. *For a quasi-order R, the intersection S of all complete consistent extensions of R compares the same elements as R, i.e., $R \cup R^\circ = S \cup S^\circ$.*

Proof. Of course $R \cup R^\circ \subseteq S \cup S^\circ$. In order to show the reverse inclusion, suppose $(x,y) \notin R \cup R^\circ$. i.e., $(x,y) \notin R$ and $(y,x) \notin R$. Then, according to Lemma 9.17, both $R,(x,y)$ and $R,(y,x)$ are consistent extensions of R, and both have complete consistent extensions by CEP, say $S_{(x,y)}$ and $S_{(y,x)}$, respectively, the former of which avoids (y,x), the latter (x,y) (Remark 9.9). Hence neither (x,y) nor (y,x) is common to all complete consistent extensions of R, which is to say that $(x,y) \notin S \cup S^\circ$. □

The key observation is that any pair of elements $x, y \in X$ which a quasi-order R fails to compare provides a choice: either adjoin (x,y) or go with (y,x), arbitrarily. In general there is no hope for a unique complete extension.

We can now state and prove the following *Intersection Principle* which in fact is equivalent to CEP.

Proposition 9.20. *The hull of a relation R is the intersection of all complete consistent extensions of* $\mathrm{cc}(R)$*.*

Proof. Recall that $\overline{R} = \overline{\mathrm{cc}(R)}$ extends $\mathrm{cc}(R)$, hence every extension of $\overline{R}$ is an extension of $\mathrm{cc}(R)$. Therefore

$$\bigcap \{\, T : T \supseteq \mathrm{cc}(R) \text{ is compatible} \,\} \subseteq \bigcap \{\, T : T \supseteq \overline{R} \text{ is compatible} \,\},$$

where T ranges over complete consistent relations, i.e., complete quasi-orders. As an intersection of quasi-orders all of which contain R, the left-hand side contains $\overline{R}$. On the other hand, since any intersection of extensions still is an extension, we know that the right-hand side S extends $\overline{R}$. The assertion now follows from Lemma 9.19 and Remark 9.11. □

While CEP results in Proposition 9.20, it is clear that Proposition 9.20 in turn implies CEP. In fact, if R is consistent, and $\overline{R}$ is not yet complete, then every pair (x,y) avoided by $\overline{R}$ yields a complete quasi-order which extends $\mathrm{cc}(R) = R$, and which avoids (x,y), as well.

Here is an equivalent way to put the Intersection Principle; recall that $\mathrm{cc}(R)$ is consistent even if R is not, and that $\overline{R} = \overline{\mathrm{cc}(R)}$.

Corollary 9.21. *The hull of a consistent relation R is the intersection of all complete consistent extensions of R.* □

The following is an immediate consequence. It is implicit already in Lemma 9.19.

Corollary 9.22. *Every quasi-order is the intersection of its complete consistent extensions.* □

We have been very careful in distinguishing compatible extension from simple containment: the former is a special case of the latter, so, given a quasi-order R, the intersection of all complete consistent relations containing R cannot exceed the intersection of all complete consistent *compatible* extensions of R; and R is contained in the former. Since a complete consistent relation is the same as a complete quasi-order, we thus have

Corollary 9.23. *Every quasi-order is the intersection of all complete quasi-orders containing it.* □

9.7 Relative Extensions

Arrow [13] gave a slightly more general form of the extension principle; Inada [141] provided a brief and detailed proof of this variant. A further variation was phrased by Suzumura [240] for consistent relations; recently Cato [61] suggested another generalization. We want to give a short account, focusing on Cato's result. We are not going to go into painstaking detail, and leave out a few details which can be easily verified by "chasing elements". The point we wish to make is that the algebraic method suffices at large.

Lemma 9.24. *If Q is a quasi-order on $Y \subseteq X$, and R is a quasi-order with $R\big|_Y = \Delta_Y$, then*

1. *$Q \circ R \circ Q = Q$,*
2. *$(Q \cup R)^n \subseteq Q \cup (Q \circ R) \cup (R \circ Q \circ R) \cup (R \circ Q) \cup R$ for every $n \geqslant 1$, and*
3. *$Q \cup R$ is consistent.*

Proof. We omit the symbol $\circ$ for relational composition, writing RS for $R \circ S$, etc.

1. With $R\big|_Y = \Delta$ we have $QRQ = QR\big|_Y Q = Q\Delta_Y Q = QQ = Q$.
2. This is a simple argument by induction; apply (1) and take into account transitivity of R and Q.
3. From (2) we get
$$\mathrm{tc}(Q \cup R) = Q \cup QR \cup RQR \cup RQ \cup R.$$
This can be used to show that $Q \cup R$ is consistent, which [48, 141] demonstrate in detail, but element-wise. However, an algebraic proof is possible too, the key to which is given by the *law of modularity* [112]:
$$RS \cap T \subseteq (R \cap TS^\circ)S, \tag{†}$$
which can be put equivalently as
$$RS \cap T \subseteq R(S \cap R^\circ T). \tag{‡}$$
E.g., we calculate
$$RQ \cap R^\circ \overset{(‡)}{\subseteq} R(Q \cap R^\circ R^\circ) \subseteq R(Q \cap R^\circ) \subseteq R\Delta_Y \subseteq R, \tag{$*$}$$
and then
$$RQR \cap R^\circ \overset{(†)}{\subseteq} (RQ \cap R^\circ R^\circ)R = (RQ \cap R^\circ)R \overset{(*)}{\subseteq} RR = R.$$
The remaining inclusions can be shown similarly. ☐

The first additional principle we consider reads as follows [13, 61, 141, 240]:

Relative Extension Principle (REP)**.** Let Q be a relation on $Y \subseteq X$, and let P be a relation on X such that $\overline{P}\big|_Y = \Delta_Y$. If both P and Q are consistent, then P has a complete consistent extension which restricts to a complete consistent extension on Y of Q,

Notice that the assumption is put in positive form: instead of assuming [61] that P satisfies $(x, y) \notin \mathrm{tc}(P)$ for every pair of distinct elements $x, y \in Y$, we stipulate $\mathrm{tc}(P)\big|_Y \subseteq \Delta_Y$. This makes possible a more perspicuous proof.

Proof of REP. Suppose that P and Q are consistent as in CEP. The hull $\overline{P}$ is a quasi-order on X which extends P. By means of CEP, there is a complete consistent extension Q^* of Q on Y. According to Lemma 9.24(3), the union $Q^* \cup \overline{P}$ is consistent, and it extends $\overline{P}$ on X, because of
$$Q^* \cap \overline{P}^\circ \subseteq \overline{P}^\circ\big|_Y \subseteq (\overline{P}\big|_Y)^\circ = \Delta_Y \subseteq \overline{P}.$$
Another invocation of CEP gives rise to a complete consistent extension S of $\overline{P} \cup Q^*$ on X. This S is an extension of $\overline{P}$, and the restriction of S on Y coincides with Q^*. In fact, since Q^* is complete on Y, it suffices to show that $S\big|_Y$ extends Q^* (remember Remark 9.11):
$$S\big|_Y \cap Q^{*\circ} = (S \cap Q^{*\circ})\big|_Y \subseteq \big[S \cap (\overline{P} \cup Q^*)^\circ\big]\big|_Y \subseteq (\overline{P} \cup Q^*)\big|_Y = Q^*. \qquad ☐$$

This REP is the special case $n = 1$ of the following principle:

Nested Extension Principle (NEP)**.** If $Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n$ is a chain of sets, each of which is equipped with a consistent relation P_i in such a way that $\overline{P_{i+1}}\big|_{Y_i} = \Delta_{Y_i}$ for every $i < n$, then P_n has a complete consistent extension, which restricts for every $i < n$ to a complete consistent extension on Y_i of P_i. ☐

Proof. By a straightforward inductive argument NEP follows from REP. ☐

9.8 Complements

9.8.1 Complementing Consistency

In Proposition 9.4 we have seen that if a consistent relation R ranks composed pairs, i.e., if R is consistent and such that $R \circ R \subseteq R \cup R^{\circ}$, then R is transitive. If R is even reflexive, then transitivity follows already in case R ranks (endpoints of) compositions for *some* length $n \geqslant 2$, which is to say that

$$R^n \subseteq R \cup R^{\circ}.$$

For if $\Delta \subseteq R$, then $R^2 = R^2 \circ \Delta^{n-2} \subseteq R^n$, and one can proceed with a similar argument as in the proof of Proposition 9.4.

Another condition on consistent relations, which brings about transitivity, has been given by Bossert and Suzumura [49]. Their *SC-complementarity* can be put as

$$\forall x, y, z \in X \; \left[\; xRy \wedge yRz \;\; \rightarrow \;\; \big(\; xRz \;\; \vee \;\; (\; z\mathrm{tc}(R)x \;\; \wedge \;\; \neg(zRy \wedge yRx) \,) \big) \; \right],$$

writing xRy for $(x, y) \in R$. Along with Proposition 9.4, it then follows that a consistent relation R satisfies SC-complementarity if and only if $R \circ R \subseteq R \cup R^{\circ}$.

We should point out that $R \circ R \subseteq R \cup R^{\circ}$ occurs in [49] equivalently as *TSC-complementarity*

$$\forall x, y, z \in X \left[\; xRy \wedge yRz \;\; \rightarrow \;\; \neg xNz \; \right],$$

where $N = \{ \, (x, y) : (x, y) \notin R \wedge (y, x) \notin R \, \}$ is the *non-comparable factor* of R. We have preferred to put it positively.

9.8.2 Equivalent Principles

While to prove the order-extension principles CEP, TEP, QEP, REP and NEP we have tacitly worked in customary Zermelo–Fraenkel Set Theory with the Axiom of Choice (**ZFC**), to establish their equivalence requires to drop the Axiom of Choice and move to Zermelo–Fraenkel Set Theory (**ZF**) without the Axiom of Choice. Most likely even weaker set theories would suffice, but this shall not be our concern here.

Proposition 9.25. *In* **ZF** *the following principles are equivalent:* CEP, TEP, QEP, REP *and* NEP.

Proof. We already know the following implications:

$$\mathsf{QEP} \leftarrow \mathsf{TEP} \leftarrow \mathsf{CEP} \rightarrow \mathsf{REP} \leftrightarrow \mathsf{NEP}$$

In view of this we only have to verify that each of QEP and REP implies CEP.

As for QEP implies CEP, let R be consistent. Now $\overline{R}$ is a quasi-order which, by Proposition 9.14, extends R. By QEP, this hull can be extended to a complete quasi-order S, which is an extension of R too (Lemma 9.13). As for REP implies CEP, to prove the latter apply the former with $X = Y$, $Q = R$ and $P = \Delta$. □

9.8.3 Order Extension by Open Induction

Several theorems which commonly are proved by means of Zorn's Lemma have been reproved in a more direct way via the principle of *Open Induction* [201].[9] In this vein we now present an alternative proof of CEP which rests on Open Induction. First some terminology is required.

Let $(E, \leqslant)$ be a chain-complete poset, and let O be a predicate on E.[10] One says that O is *progressive* if

$$\forall x \; \big(\; \forall y > x \; O(y) \;\; \rightarrow \;\; O(x) \; \big),$$

[9] For the use of Open Induction in diverse contexts see [36, 66, 73, 208, 222].

[10] This O may be identified with its extension $\{ \, x \in E : O(x) \, \}$ in E.

where $y > x$ is understood as the conjunction of $x \leqslant y$ and $x \neq y$. Furthermore, O is said to be *open* if

$$O(\bigvee C) \ \rightarrow \ \exists x \in C \ O(x)$$

for every chain $C \subseteq E$; recall that $\bigvee C$ stands for the least upper bound of C. For example, a predicate O is open whenever it is *downward monotone*, i.e., satisfies

$$O(x) \wedge y \leqslant x \ \rightarrow \ O(y).$$

Indeed, for if O is downward monotone, and if C is a chain such that $O(\bigvee C)$, then even $\forall x \in C \ O(x)$; note every chain is required to have an element.

Raoult [201] has coined the following principle:

Open Induction (OI)**.** If E is a chain-complete poset, and O is open and progressive, then $\forall x \ O(x)$.

Moreover, Raoult [201] has deduced OI from KZL; in fact, both principles are equivalent by complementation and thus in **ZF**—see, e.g., [208]. Here is how to prove CEP by means of OI:

We have seen (Lemma 9.13 and Lemma 9.15) that the set $\mathscr{E}$ of consistent relations on X is partially ordered and chain-complete with respect to the order of (compatible) extension. On $\mathscr{E}$ we consider the predicate O of "being completely extendable", formally, for $R \in \mathscr{E}$:

$$O(R) \quad \equiv \quad \exists S \in \mathscr{E} \ \big(\ R \subseteq S \ \wedge \ S \cap R^\circ \subseteq R \ \wedge \ S \cup S^\circ = X \times X \ \big).$$

E.g., the universal relation $X \times X$ is completely extendable, for trivial reasons. This predicate O is downward monotone, hence open. As for O being progressive, suppose that $R \in \mathscr{E}$ is such that every *strict* extension of R is completely extendable. The hull $\overline{R}$ is a consistent extension of R, which either is complete—by which R has itself as complete extension—or else fails to compare a certain pair of elements. In the latter case, say $x, y \in X$ are such that $(x, y) \notin \overline{R} \cup \overline{R}^\circ$. Then $\overline{R} \cup \{ (x, y) \}$ is a consistent extension of $\overline{R}$ by Lemma 9.17, and strictly extends $\overline{R}$ because of $(x, y) \notin \overline{R}$. Now $\overline{R} \cup \{ (x, y) \}$ is completely extendable, whence R is, as well. Then, by way of OI, we get $\forall R \in \mathscr{E} \ O(R)$, which is to say that every consistent relation R on X has a complete consistent extension.

Any concrete enough instance, i.e., one for which the underlying set X of alternatives is finite, should then allow to reduce the invocation of OI to one of what in [222] is called *Finite Induction*, which in turn can be proved by means of mathematical induction only.

9.8.4 From Order Extension to Gödel–Dummett Logic

As alluded to in the Introduction, we now briefly sketch how Gödel–Dummett logic [105, 120] is necessary for QEP, adapting to quasi-orders an argument given by Bell [32, p. 162] for partial orders. We recall that Gödel–Dummett logic,[11] which "naturally turns up in different fields in logic and computer science" [16, 108], is an intermediate logic between intuitionistic and classical logic. Roughly speaking, intuitionistic logic [96, 135] is classical logic without the law of excluded middle but with the principle *ex falso sequitur quodlibet*. Now Gödel–Dummett logic is intuitionistic logic plus

Gödel–Dummett Principle (GDP)**.** $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ for all well-formed formulas φ and ψ.

In order to adapt Bell's argument, we first make the following observation:

Lemma 9.26. *Let X be a set and $R \subseteq X \times X$ antisymmetric. Assume that X has top element 1, i.e., $xR1$ for every $x \in X$. If S is a compatible extension of R, then 1 is S-maximal.*

Proof. Since 1 is R-top we have $1R^\circ x$ for every $x \in X$. Therefore, if $1Sy$, we get $1Ry$ because of $S \cap R^\circ \subseteq R$. As R is antisymmetric, $y = 1$ follows. □

[11] As von Plato points out, Gödel–Dummett logic "was actually introduced by Skolem already in 1913" [251].

Note that if 1 is R-top and $R \subseteq S$, then of course 1 is S-top too, but this does not mean that 1 is S-maximal unless S is antisymmetric.

In the following, we work in Friedman's Intuitionistic Zermelo-Fraenkel set theory **IZF** [7, 30, 93, 113]. This **IZF** is as standard Zermelo–Fraenkel set theory (**ZF**) but with intuitionistic rather than classical logic; to make this move possible, the axiom of foundation needs to be replaced by the schema of set induction, whereas the principles of power set and full separation are part of **IZF**.

In **IZF** one thus has the so-called set of intuitionistic truth values $\Omega = \mathrm{Pow}(1)$, i.e., the set of subsets of $1 = \{ 0 \}$ partially ordered by inclusion $\subseteq$. Every formula φ in the first-order language of set theory gives rise to its truth value $V_\varphi \in \Omega$, viz.

$$V_\varphi = \{x \in 1 : \varphi\} ,$$

for which φ is equivalent to $0 \in V_\varphi$ and thus to $V_\varphi = 1$. Conversely, every $U \in \Omega$ is of the form V_φ, for φ being $U = 1$. Note that an implication $\varphi \to \psi$ between formulas φ and ψ is equivalent to $V_\varphi \subseteq V_\psi$ in Ω; and that $U \subseteq W$ in Ω amounts to $U = 1 \to W = 1$.

Bell [32, p. 162] deduced GDP from the principle that every partial order is contained in a complete one. To do so he needed that $\subseteq$ is maximal, with respect to containment, among antisymmetric relations on Ω. As quasi-orders lack antisymmetry, we have to adapt Bell's tool as follows.

Lemma 9.27. *Every compatible extension $\leqslant$ of $\subseteq$ on Ω coincides with $\subseteq$, i.e., $\subseteq$ is maximal with respect to compatible extension of relations on Ω.*

Proof. Now suppose that $\leqslant$ is a compatible extension of $\subseteq$ on Ω. By Lemma 9.26, and since 1 is $\subseteq$-top, we get

$$U \leqslant W \ \to \ (\ U = 1 \ \to \ W = 1\ )$$

or, equivalently, $U \leqslant W \to U \subseteq W$, for all $U, W \in \Omega$. □

Proposition 9.28 (**IZF**)**.** QEP *implies* GDP*.*

Proof. Applying QEP, and taking into account Lemma 9.27, we may consider $\subseteq$ on Ω to be complete, which is tantamount to $(\varphi \to \psi) \vee (\psi \to \varphi)$ for arbitrary formulas φ and ψ. □

The same assertion holds true if QEP in Proposition 9.28 is replaced by any of its equivalents from Theorem 9.25, because proving these forms equivalent is possible already in **IZF**.

To get GDP we have used the same data as Bell [32, p. 162]: the relation $\subseteq$ on the set Ω. In particular, we have invoked the consequence of QEP that every partial order $\subseteq$ can be extended to a complete quasi-order. Any such extension of $\subseteq$ on Ω, however, *a fortiori* is a partial order anyway (Lemma 9.27).

9.8.5 Further directions

By now our focus has been on methodological advantage which our choice of positive notions for consistency and compatible extension entails. Now we sketch an important application of order extension in the theory of preference relations—that is, rationalizability.

We follow [48, 206]. Let again the set X denote our domain of discourse. A *choice function* is a mapping

$$\mathfrak{c} : \mathscr{T} \to \mathrm{Pow}(X)$$

which assigns to each inhabited member $Y \in \mathscr{T}$, where $\mathscr{T} \subseteq \mathrm{Pow}(X)$, an inhabited subset $\mathfrak{c}(Y) \subseteq Y$. A relation R on X *rationalizes* $\mathfrak{c}$ if

$$\mathfrak{c}(Y) = \{\ x \in Y : \forall y \in Y\ xRy\ \}$$

for every Y in the domain of $\mathfrak{c}$; no further assumption on R is made. This version of rationalizability is known as *greatest-element rationalizability* [48] of the choice function $\mathfrak{c}$. The (indirect) *revealed preference relation* of a choice function $\mathfrak{c}$ is defined to be (the transitive closure of)

$$\{ (x,y) : \exists Y \in \mathscr{T} \, (x \in \mathfrak{c}(Y) \wedge y \in Y) \} .$$

A choice function $\mathfrak{c}$ is said to satisfy the *congruence axiom* [206] if it is rationalized by its indirect revealed preference relation. This is the setting for a fundamental application of order extension in the theory of preference relations, viz.

Richter's Theorem ([206, Theorem 1])**.** A choice function satisfies the congruence axiom if and only if it can be rationalized by a complete quasi-order.

A key step to proving this is the following observation.

Lemma 9.29. *Let $\mathfrak{c} : \mathscr{T} \to \mathrm{Pow}(X)$ be a choice function. If R is transitive and rationalizes $\mathfrak{c}$, then so does every compatible extension of R.*

Proof. We redraft an argument laid out in [48, Theorem 3.2]. Suppose that S is a compatible extension of R. Given that R rationalizes $\mathfrak{c}$, and since $R \subseteq S$, it suffices to show

$$\{ x \in Y : \forall y \in Y \, xSy \} \subseteq \{ x \in Y : \forall y \in Y \, xRy \}$$

whenever $Y \in \mathscr{T}$. To this end, let $Y \in \mathscr{T}$, and let $x \in Y$ be such that $\forall y \in Y \, xSy$. Pick any $z \in \mathfrak{c}(Y)$. As R rationalizes $\mathfrak{c}$, we have $\forall y \in Y \, zRy$, so in particular zRx. But we also know xSz. Hence xRz by compatibility of S over R. Now $\forall y \in Y \, xRy$ is immediate, since R is supposed to be transitive. ☐

Therefore, if a choice function $\mathfrak{c}$ satisfies the congruence axiom, then every compatible complete extension of its indirect revealed preference relation rationalizes $\mathfrak{c}$ as well. This is how Richter's theorem rests on order extension.[12] Conversely, it is not hard to show that if $\mathfrak{c}$ can be rationalized by means of a complete quasi-order, then it satisfies the congruence axiom. In his proof Richter applies Szpilrajn's theorem in its original reading that every *irreflexive* transitive relation, i.e., every strict partial order, is contained in one which compares every pair of distinct elements. While Cato has recently deemed this form of Szpilrajn's theorem "not useful for economic analyses because partial orders do not allow two alternatives to be indifferent" [60, p. 60], there is again a definite point to make from the methodological perspective. In order to have (strict) partial orders at hand, Richter performs a quotient construction—a move which has turned out avoidable by way of an appropriate extension principle for quasi-orders, as considered before.

Further applications of our method might be possible in the directions that research on order and extension principles has taken. For instance, the topological notion of continuity comes into play in [43, 44, 47]. In [99], transitive closure is replaced by several other closure operators, thus leading to further extension theorems. In [100] conditions for a collection of binary relations to have a common ordering extension are provided. The classic closure-complement problem has been revisited for consistent closure in [59]. Last but not least, extensions have been considered with regard to the existence of maximal elements in quasi-orders. In [198] it is shown that any maximal element of a quasi-order R is the greatest element for some complete extension of R.

[12] Incidentally, Richter mentions that "both representability and rationality have existential clauses in their definitions, so proofs of these properties are likely to involve tools like the axiom of choice and other nonconstructive techniques." [206, p. 637]

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