The Singularity Problem in Gravitational Theory. The Spherically Symmetric Case

Stefano Chinaglia

Candidate: S. Chinaglia
Supervisor: prof. S. Zerbini
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Abstract

In this work we discuss some specific features related to the concept of singularity in the gravitational theory. We give a brief review of some various definitions for singularity, then we explore some "negative" results, in the sense they are not able to reproduce, in general, a regular solution. We present some of these approaches, namely the non-commutative geometry; the Non-Linear Electrodynamics; and the conformal approach. We later generalize these results into a no-go theorem, which is actually a fully original result. In the second part of this work, we discuss some working examples of regular solutions: we present three of them already present in literature (non-minimal Yang-Mills coupling, mimetic field approach and non-polynomial gravity), then we use such results to build up a model of a regular cosmological solution. Its generating mechanism and its main features are described, replacing the Big Bang with a bounce; the inflationary behavior at large time is also recovered. In the following two chapters, we present some different schemes to build regular solutions from the coupling between gravity and a scalar field. In particular, in chapter 7, we use a minimal coupling, while in chapter 8 we find some sufficient (though not necessary) conditions to build a regular solution, within the framework of the Horndeski theory. In both cases we are not able to find explicit results. In the ninth chapter we discuss a model of a regular black hole, coupling gravity with some fluid: in this case, an exact solution is found. We prove it is regular and we show some of its general features; we also discuss the time-dependent case, although we are only able to discuss its asymptotic behavior. We also discuss some of its problems, mainly due to instability. In the appendices we try to extend the no-go theorem to $F(R)$ theories and try to solve the instabilities of the fluid approach respectively.

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There are many people I need to thank in this work. Some of them gave me a direct support, some other indirect – but not less important. Let me open this brief parenthesis, before starting the discussion: these lines (only *these* lines!) would not discuss on black holes, but just name people, who, in one way or another, made it possible. Don't mind, the rest of the work will be written in a standard scientific formulation; but here I need to take a rest.

First of all, I would like to thank the other people in my research group. The senior staff: my supervisor, prof. Sergio Zerbini, and the other component, prof. Luciano Vanzo, prof. Guido Cognola and prof. Massimiliano "Max" Rinaldi; our post-doc, Lorenzo Sebastiani; my colleagues PhD students, (present) Aimeric Colléaux and Alessandro Casalino, (and former) Yevgeniya Rabochaya; our master students. I need to cite them all, because, some more, some less, in a way or another, they gave me some help: supporting my work, reading (and sometimes also correcting) my written pages, discussing my results (typically, in a very fruitful way) and facing with different opinions and positions. In particular, I feel to thank my supervisor and Aimeric for the long sessions of discussions we had together and the continuous improvement they provided to my work. A special thank is also to one of our master students, Marco Calzà, who gave me the right suggestion in choosing the latex format of this thesis.

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My father, my mother, my sister and my brother. Maybe they did not help me writing a paper or solving an integral, but I don't think that, without them, I would have been able to be where now I am. Especially, I should say, my mother. At the end of my master thesis, writing more or less these same words in the acknowledgments I addressed her a couple of verses; four years later, I strongly repeat them here: *When I look at your eye and see a tear / I know it is a tear which falls for joy.*

At the end of this section, I would like to cite the great Italian poet Dante. In his famous imaginary journey, he travels across Hell, Purgatory and Heaven. During the first two parts of that journey, he is guided by the shadow of the Latin poet Virgil. Finally, after nine circles of Hell and seven terraces of Purgatory, he is going to leave his guide; and there is a feeling of something completed, something finally finished. There, at the top of the mountain of the Purgatory, Virgil addresses his final words to Dante and the meaning of these words is more or less "Now you are ready to walk with your own legs":

Non aspettar mio dir più, né mio cenno;
libero, dritto e sano è tuo arbitrio,
e fallo fòra non fare a suo senno:

per ch'io te sovra te corono e mitrio.

Purgatorio XXVII, 139-142

If you're wondering what does it say: Don't wait for any word or sign from me; / your will is free, is right and it is sound / and not to follow it is a mistake: / over yourself I crown and miter you.

Coming at this point, I feel more or less like him – *si parva licet*: as someone much wiser than me already said, I'm just a dwarf standing on the shoulders of giants. I feel something is going to be completed, something is going to be finally finished and I should walk with my own legs and fly with my own wings. There is also a drop of melancholy, since this is a farewell: I do not pretend to say a farewell, but certainly this is a goodbye.

Contents

1 Introduction

1.1 Some preliminary considerations

The presence of singularities into a theory typically indicates a limitation or even a breakdown of the theory itself in that particular regime. It seems that Nature has some kind of *horror infiniti*, so that no divergence (unless only apparent ones) is allowed to be.

A well known example in this sense is the ultra-violet catastrophe in the black body radiation [1]: performing a purely classical calculation, one finds a divergent spectrum in the ultra-violet regime, which is actually not realized in nature. The observed spectrum has indeed a quite different behavior, which could be explained introducing some constant $\hbar$. It deforms the classical curve in order to close the emission spectrum and transforms the classical quadratic behavior into a Planck curve. The introduction of that constant, however, was not just a mathematical trick, but the door, which opened a new, wide regime of physics.

A similar thing happens in the central singularity for the Coulomb potential: its behavior as $1/r$ indicates the presence of a singularity for $r = 0$. While working within standard electromagnetism or Newtonian gravitation, one can circumnavigate the problem assuming that a realistic mass/charge distribution will never have a truly vanishing size, so that the Coulomb potential holds outside the distribution, in vacuo, and the potential has some different (but non singular) behavior inside the distribution. An example is given by the energy level perturbation, in the hydrogen atom, due to the dimensionality of the nucleus; the effect can be easily calculated even in the basic atomic physics.

In General Relativity one may think something similar to happen, so that there is, already inside the theory, some kind of regulator, which will prevent any need to modify the theory itself. Although at the present time there is no proof for this position to be wrong, there are some elements, which encourage to go beyond General Relativity and try to solve the central singularity.

• At a sufficiently high energy scale, gravitational effects become dominant on any other interaction. If General Relativity hold at any scale, a very compact object would become more and more compact and, unless invoking some bouncing effect, its density will increase with no upper bound.

• If we deal with a very compact object, its density may be on the order of the Planck density, or in any case a relevant fraction of it. It is believed that at such scale some quantum effect will start to be relevant not only on the matter content of the theory, but also on the spacetime itself. In that case, some modifications of the theory is expected, since General Relativity is a classical theory.

• In particular, we expect some modification of the theory, able to include a Grand Unification. Although there is no proof that gravitation will melt with the other interactions at high energy, at the present this is widely believed; some modification would then be in order.

• Finally, since a definitive theory of quantum gravity is not yet available, solving the singularities may be an affordable way to explore, in a semi-classical way, quantum gravitational effects. Since there are many candidates for the role of quantum gravity (such as string theory, loop quantum gravity etc.), a semi-classical study may be useful, in the future, to discriminate between different theories.

Regarding this last point, we have to notice that there are also a number of proposal for regular solutions directly emerging from quantum gravitational approaches, e.g. [2–5].

Many other reasons can be found, showing that a study on how to regularize divergences in General Relativity is convenient. In some sense, it takes the same approach of the Bohr atom in quantum physics [6]: a classical study is taken, but with some ad hoc assumptions, mimicking the still unknown quantum effects. It is worth to notice that such kind of approximation is investigated still today [7, 8].

Before leaving for the next section, we should notice that we only discussed the case of a black hole space, but we already know that it is not the only kind of singularity actually present in General Relativity: one, indeed, has also to deal with the cosmological singularity, appearing in the Big Bang theory when the scale factor goes to 0. In this work we are mainly interested in the black hole case, rather than the cosmological one; so, at least within this introduction, we will discuss it only incidentally.

1.2 An historical overview

The main scope of this work is to study how the problem of singularity can be solved in the gravitational theory (not necessarily beyond General Relativity). However, it is in any case important to briefly discuss the history of the problem, showing how physicists became aware of the importance of the singularities and finally tried to solve the them during the years. In particular, it is interesting to note that the problem of the cosmological singularity emerged quite soon, although at the beginning was misinterpreted, while the black hole singularity needed to wait a much longer time to be finally considered.

We also should warn that this is not a history book and that we wrote this section only for completeness and for giving the reader a general overlook on how physicists' thoughts can vary during the time.

Soon after the publication of Einstein original paper in late 1915 [9], K. Schwarzschild published his renowned work on solving Einstein Equations [10]. As known, Schwarzschild found that the metric

$$ds^2 = -c^2 \left(1 - \frac{2Gm}{c^2 r}\right) dt^2 + \frac{1}{1 - \frac{2Gm}{c^2 r}} + r^2 d\Omega^2 \ , \qquad (1.1)$$

in which m is the mass of the object, G is the Newton gravitational constant, c is the velocity of light and $d\Omega^2$ the surface element of a sphere of unitary radius. Schwarzschild found this solution in vacuo, while he also studied the case of a star of finite radius R and constant density ρ_0. In that case, he was able to find the inner solution, which has a quadratic behavior, while the external one is still the vacuum solution (1.1). The inner case of a general density distribution has been studied since the works of R.C. Tolman [11] and J.R. Oppenheimer and G.M. Volkoff [12], resulting in the celebrated Tolman-Oppenheimer-Volkoff equation.

Immediately after Einstein and Schwarzschild publications, the main interest on General Relativity was devoted to the applications to Cosmology. During the '20s and the '30s many worked appeared, exploring the new relativistic universe, giving the birth of modern cosmology: just to say a few, we remind the works of de Sitter [13,14], Friedmann [15], Lemaitre [16] and Hubble [17]. Lemaitre, in particular, was the first to introduce the concept of Big Bang in his work [18], although he misunderstood its meaning. During these years, there was minor interest to the local solutions of GR, mainly because people believed that Schwarzschild results were already enough.

Of course this should not mean that there was *no* interest in local solutions. Among the other works, it is worth to notice the famous paper of Eddington et al. [19], in which an empirical proof of the new theory was given for the first time: in that work was found that the deflection of light by the Sun was in accordance to Einstein previsions, rather than Newton ones (although many doubts were subsequently formulated, regarding the precision of 1919 measurements). This is just a famous example to say that people had already in mind the importance of local consequences of General Relativity; it only did not catch the main interest of physicists.

Einstein and his contemporaries, indeed, did not believe Schwarzschild solution being problematic. Everybody, of course, noted the two singularities of solution (1.1), i.e. those for $r = 0$ and $r = r_S \equiv 2Gm/c^2$ (the so called Schwarzschild radius); but nobody thought that an object could be so small and so dense to make such limits relevant. For example, if the whole mass of the Earth would be compressed to be smaller than its Schwarzschild radius, one should concentrate it in a sphere of less than 2 cm of diameter. This appeared to be so ridiculous, that nobody troubled on the singularities of Schwarzschild metric.

Following a first work by Eddington [20], in 1933 Lemaitre proved that the singularity at the Schwarzschild radius $r = r_S$ was just apparent and could be removed via a change of coordinates [21]. This was finally confirmed by D. Finkelstein in [22] only after World War II and later by Penrose [23] in the '60s. In these new coordinates, metric (1.1) reads

$$ds^2 = -\left(1 - \frac{2Gm}{c^2 r}\right) du^2 \pm 2dudr + r^2 d\Omega^2 \ , \tag{1.2}$$

where the sign $\pm$ is chosen, depending if u is interpreted as an advanced time coordinate $(+)$ or a retarded one $(-)$. With this choice, it is openly clear that the horizon singularity is just an apparent one and that one can extend the coordinate system to the whole spacetime. Unfortunately, one also sees that the central singularity at $r = 0$ is still present; and indeed there is no way to remove it, just changing coordinates.

It is more or less from that time in the '60s that physicists slowly became aware that gravitating objects could also reach very small sizes: the name "black hole" was coined indeed in those years. At the same time, people started trying to face and solve the problem. One of the first attempts at all was proposed by Y.S. Duan in 1954 [24], in which the author proposed some modification to the Einstein Equations, in order to cancel the gravitational divergence of the Schwarzschild term: actually, it was the birth of Non-Linear Electrodynamics, but unfortunately Duan paper remained ignored for a long time: only recently translated in English, that approach had been already rediscovered.

A widely more popular proposal for a regular metric was finally performed by J.M. Bardeen in 1968 [25]. While not providing the mechanism that generates the modification, the author was able to fix the core of the problem, smoothing the sharpness of Schwarzschild divergence near the origin:

$$ds^2 = -c^2\left(1 - \frac{2Gmr^2}{c^2(r^2+l^2)^{3/2}}\right) dt^2 + \frac{1}{1 - \frac{2Gmr^2}{c^2(r^2+l^2)^{3/2}}} + r^2 d\Omega^2 \ , \tag{1.3}$$

where l is some parameter of the theory, with dimension of a length. One immediately sees that this solution, at infinity, resembles the Schwarzschild metric, while it has a regular de Sitter core near the origin; and this, following the almost contemporary ideas by Sakharov [26], is enough to have a regular solution with only non divergent scalar invariants (we will discuss Sakharov criterion more in detail in the second chapter). This and similar tricks have been widely applied along the years and are still among the most popular ways to regularize a black hole. In particular, a very similar metric has been studied by Poisson and Israel [27,28], later slightly modified by Hayward [29]. All these metrics appear in the general form

$$ds^2 = -c^2\left(1 - \frac{2Gmr^k}{c^2(r^p + l^p)^q}\right)dt^2 + \frac{1}{1 - \frac{2Gmr^k}{c^2(r^p+l^p)^q}} + r^2 d\Omega^2 \ , \tag{1.4}$$

where p, q and k are three dimensionless parameters and l has the dimension of a length. They should be chosen so that Schwarzschild is recovered at infinity, i.e. $k = pq - 1$; after the work by Sakharov, another major requirement is that they reproduce at least a de Sitter behavior near the origin, i.e. $k \geq 2$.

It is impossible to discuss here all the attempts and trials people did, in order to regularize a black hole. As typically happens, things started slowly, with a few pioneering works, and finally, when people realized the problem, many and many works appeared, following various and different approaches. If we give a look to the landscape of regular solutions before 1990, we could find only a few of them, more or less the ones we already mentioned. If we go after 1990, so after the works by Bardeen, after Poisson and Israel, we find a very huge jungle of works. Just to mention some of them, with no pretention of completeness, one has [30–63]. For some reviews on the topic, see [64–66].

Each of these, as metric (1.3), has an asymptotic Schwarzschild behavior, either when the observer is far from the source, or when the deformation parameters are set to 0, while at the centre they present a de Sitter core (some of them have also a de Sitter term at infinity, due to the presence of a cosmological constant; its contribution, however, can be easily restored, by simply adding a constant term to the stress-energy tensor). Most of all, they also try to give the generating mechanism of the regular metric: indeed, while the first attempts of regularization were satisfied in finding a non-divergent solution, during the years became more important finding why the solution is regular, i.e. how it is generated.

Among these various approaches, we notice that there are two main, although not exhaustive, classes:

- the one, which involves a modification of the stress-energy tensor, including some kind of exotic matter; in this case, the Schwarzschild divergence is cancelled by the matter behavior; examples of this class are [33–37];

- the one, which involves a modification of the Einstein tensor; while in the first case General Relativity is preserved, here the singularity is cancelled due to a modification of gravitation itself; examples are provided by [41–45].

It is clear that this classification does not take into account every possible approach: as example, we mention [67–69], which consider a non minimal coupling among matter and gravity, so that both matter and gravity are modified. In any case, it is worth to notice that, in the absence of matter, both approaches (and any other similar to them) just reduce to the standard General Relativity in vacuo.

We should also mention that, instead of modelling the matter density so that the singularity is cancelled, some people studed directly the gravitational collapse, with quantum-like corrections either in the matter sector, or in the gravitational one [70–77]. In these cases, the resulting non singular spacetime might be different from either static wormholes, or static regular black holes with (A)dS cores (for example, it might describe a dynamical bounce). In this view, static black holes are only a special regime of these more physically complete (and complicated) spacetimes.

A side-effect of the singularity removal is expected to be also a dissolution of the information loss paradox (see e.g. [78–80] and references therein). Indeed, if matter is not destroyed by the central singularity, either it flows out (even after a very long time, for external observers) via a bounce [73–77, 81], or via evaporation [29, 54], or is eternally locked inside the black hole as a stable extremal remnant [80, 82–86].

Another specific approach deserves to be mentioned, because it turns to be different from the previous ones. It was first introduced in [81, 87] a few years ago and it is based on the so called Planck star. In that works, the authors argue that a gravitating object inside a black hole decreases its size and increases its density, until it reaches the Planck density ($\sim 10^{96}$ kg/m^3). In the standard gravitational theory, the star continues to decrease in size and increase in density, while in this approach some quantum mechanism provides the star to enter into a repulsive phase, thus decreasing its density. The result is a final explosion, which turns to happen (at least for small objects) after a time comparable with the age of the universe, so with some hope of observation.

Within this approach, one notices that the central singularity does not play any role at any time, since the point-like size is never reached. It is worth to notice that Hayward in his paper [29] qualitatively discussed a similar mechanism, although he never got into details.

Finally, we close this brief overview over more than one century of physics with the recent observations by the LIGO-VIRGO collaboration [88–94]: their result is not only the proof that gravitational waves exist, thus confirming the last great prediction of General Relativity, but also consist in a very important proof that black holes exist and can be observed: this is very important, since the combination of extreme gravitational conditions and very high matter densities make a black hole the perfect laboratory to study (and possibly test) matches between gravitational and quantum physics.

In the next years, research on black holes maybe will be not only theoretical, but will be enriched by observations; and observations will produce constraints on the various models actually proposed, giving a way to discard unphysical proposals. For now it is just a guess, but it is easy to predict that (regular) black holes will be one of the hottest topics in physics for this century.

1.3 Scope, motivations and organization

This work is mainly based on the papers [95–98], plus some original research material (in particular in chapters 4, 7 and 8). In these papers, we discussed a number of approaches, able (or unable) to produce regular solutions. In particular, [96] is able to produce an exact regular solution, coupling the gravitational sector with some exotic fluid matter.

The aim of this work is to spread some light about the singularity issue in the gravitational theory, mainly in the case of black hole spacetime and secondly also in the cosmological case. In particular, we are interested in the conditions necessary to have (or to avoid) a singular solution and, finally and most of all, we are interested in looking for exact solutions – if possible.

As we argued in the first section of this introduction, indeed, singularities do not appear to be realistic in nature: thus, looking for a way to solve them may be an interesting path for any investigation in the quantum regime of gravity. Moreover, as we noticed at the end of the previous section, the investigation on black holes will be probably enriched in the next years by many observational data, so that having already some functioning model for regular solutions may be helpful in understanding the observations. Compared to the "adventure" in quantum gravity, this goal appears to be much more modest, but still is important and, most of all, probably will be soon able to produce concrete results.

Finally, the work is organized as follows. In the second chapter, we present some different notions of regularity; some of them just pedagogical, some others, quite useful. In the third chapter we present some popular approaches, which try to solve the singularity issue, but apparently are not able to achieve this goal in general; as we will discuss, some regular solutions can be found, but there is no guarantee. In chapter 4, we finally present a no-go theorem, which establishes the conditions for having a singular solution, in the case of matter minimally coupled to gravity. In the fifth chapter, we start the "positive" part of the work, presenting a few models, which actually are able to produce regular solutions. Chapter 6 is devoted to present a model of a regular universe, for which the initial Big Bang is replaced with a bounce. In chapter 7 we present and discuss a model of black hole coupled with some scalar field, giving the general strategy to find regular solutions – although, probably, non analytical. In the eighth chapter we extend our discussion to the most general theory, finding some sufficient (though not necessary) conditions for finding regular solution with the Horndeski framework. In the ninth chapter, we finally present a model of black hole worked out from a matter fluid coupled to gravity; we will be able to prove the regularity of the solution and to carry out the calculations analytically. Finally, after some concluding remarks and general overviews on our results, the two appendices A and B respectively discuss an extension of the no-go theorem and a possible resolution of the singularity issue via some *quasi*-black hole.

Notation – Throughout this work, we will make use of the following notation and abbreviations, unless otherwise specified:

$c = 1$

$G = 1$
$\hbar = 1$
GR: General Relativity
EE: Einstein Equations
RBH: Regular Black Hole
SET: Stress-Energy Tensor
(W, N, D, S) EC: (Weak, Null, Dominant, Strong) Energy Condition
FLRW: Friedmann-Lemaitre-Robertson-Walker
LQG(C): Loop Quantum Gravity (Cosmology)
rhs (resp. lhs): right-hand side (resp. left-hand side)
w.r.t.: with respect to

Other abbreviations may be also defined along the chapters.

2 Notions of singularity

2.1 Introduction

Before illustrating examples and results of RBH, we use this chapter to introduce the notion of regularity. Indeed, there are many possibilities for what we mean for "regular", and not every definition is satisfactory, in order to avoid divergences and problems of any kind. At least at the moment, it doesn't seem to be available a sort of universal criterion, whose satisfaction guarantees regularity, stability etc. of the solution; however, good criteria may still be found.

Here we illustrate different attempts to define a notion of regularity. Some of them are too much approximate and can be easily discarded as non suitable for a large number of cases (or even for all cases), but other are much more suitable, at least for some applications.

The concept of singularity we will use throughout the rest of the work is based on the Sakharov criterion, just because it is enough for our scope. However we should at least mention here that performing different choices (and in particular resting on the geodesic (in)completeness), one may also have other potentially interesting features, such as wormholes [99–102].

Finally, it is worth to remark that regularity faces only "true" singularities, i.e. we don't care of those apparent singularities, which can be removed via a change of coordinates (as is the case of the horizon singularity).

2.2 Regularity of the metric function

The first and maybe the most trivial concept of regularity is the regularity of the metric. Assuming a static and spherically symmetric background, i.e.

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Sigma^2 \ , \tag{2.1}$$

an example of such regularity is provided by the metric function

$$f(r) \equiv 1 - \frac{2mr}{r^2 + l^2} \ , \tag{2.2}$$

where m and l are two suitable parameters: m plays the role of a mass (and it is actually the mass in the Schwarzschild limit $l \to 0$ or $r \to \infty$) and l

is some deformation parameter. Actually this object is "regular", in the sense that the function never diverges.

Apparently, this seems to be a sufficient and satisfactory notion of regularity, but actually it does not fit: indeed, if we compute the scalar invariants for metrics such as (2.2), we find them with a singular behavior. Just as example, its Ricci tensor is

$$R = -\frac{4ml^2}{r}\frac{(r^2 - 3l^2)}{(r^2 + l^2)^3} \ . \tag{2.3}$$

As one can see, this object is still singular, despite the fact that the function f has no singularities. Indeed, this notion of regularity is insufficient, within the framework of GR, since it does not fulfil the so called Sakharov criterion.

2.3 Sakharov criterion

The so called Sakharov criterion was first introduced by Sakharov in [26] (although here we formulate it in a slightly different form). It states that a black hole metric such as (2.1) is regular in the centre if and only if the metric function f has at least a de Sitter core near the origin:

$$f(r \to 0) = 1 + ar^n + ... \ , \tag{2.4}$$

where a is a constant and $n \geq 2$. It is immediately clear that metric (2.2) does not fulfil this requirement, since near the origin it behaves as $f(r \to 0) = 1 - \frac{2m}{l^2}r + ...$ with a linear term instead of a quadratic one.

Notice that requirement (2.4) holds in the case of spherical symmetry. In a more general case, with topology k, the criterion reads $f(r \to 0) = k + ar^n + ...$ with $n \geq 2$.

We can prove with no trouble that the Sakharov criterion is actually a necessary and sufficient criterion to have a RBH (at least, regular in the centre). For simplicity, here we limit ourselves to the discussion of the spherical Ricci, but our result can be rearranged also for other tensors and topologies. The extension to other topologies is indeed trivial. The extension to other scalars comes from the Riemann tensor, whose components are typically in the form $(f - 1 + ...)/r^2$, where the dots represent terms containing the first and the second derivative of f: one can intuitively see that Sakharov criterion is actually necessary and sufficient, in order to remove the divergent r^2 term at the denominator. What we prove here on the Ricci scalar can be conducted in the same way on the components of the Riemann tensor, but the whole discussion would result much heavier and messier.

In order to check that metric (2.4) is actually regular, that is the sufficiency of Sakharov criterion, focus on the origin and calculate the Ricci scalar. Its dominant part reads

$$R = -(n^2 + 3n + 2)ar^{n-2} + o(r^{n-1}) \ , \tag{2.5}$$

If Sakharov criterion holds, this is clearly regular. Once again, we stress that we fixed on the Ricci scalar for a sake of simplicity, but the same discussion holds for the components of the Riemann tensor, so for any other curvature invariant.

So, we proved that Sakharov criterion is sufficient. The only thing we are left with is to prove that it is also necessary. This will be done if we would be able to prove that, *outside* the Sakharov criterion, there is at least one divergent scalar invariant. As we said, it intuitively comes from the Riemann tensor. In order to see it formally, we consider again Ricci curvature:

$$R = -\frac{f''r^2 + 4f'r + 2f - 2}{r^2} \ . \tag{2.6}$$

Once again, the same discussion can be carried out for any other scalar, starting from the single components of the Riemann tensor. We stress that we choose the Ricci scalar for simplicity.

If we exclude Schwarzschild and Reissner-Nordström, which are singular (but for which this object vanishes identically), we have only three possibilities: $f(r \to 0)$ goes to infinity, to a constant or to 0. If $f \to \infty$, also its derivatives do and so the curvature diverges. If $f \to 0$, its derivatives can go to zero, a constant or infinity; in any case, we have a singularity, due to f, to its derivatives or to the term $-2/r^2$. We are left with a single possibility, i.e. that $f \to 1$.

To show it explicitly, assume that $f(r \to 0) \simeq h(r)$, where $1 \ll h(r)$. Since we are working only around the origin, up to a proportionality constant, we have that

$$h = \int h' \simeq h'r \qquad \Longrightarrow \qquad h' = \frac{h}{r} \ , \tag{2.7}$$

$$h = \int \int h'' \simeq \frac{1}{2}h''r^2 \qquad \Longrightarrow \qquad h'' = \frac{2h}{r^2} \ . \tag{2.8}$$

These two simplifications give the Ricci scalar as (once again, up to a proportionality term)

$$R \simeq -\frac{8h - 2}{r^2} \ , \tag{2.9}$$

which is divergent for $r \to 0$. In a similar way, we find a divergence also for $h(r) \gg 1$, while the case $h(r) = \textit{constant} \neq 1$ is trivially discarded. So $f(r \to 0) \to 1$.

Discuss now the next to leading term. Since we want to check the Sakharov criterion, we assume that $f(r \to 0) = 1 + h(r)$, where $h(r) \gg r^2$. Under this assumption and considering approximations (2.7) and (2.8), we have that (once again, up to proportionality factors)

$$R \simeq \frac{8h}{r^2} \ , \tag{2.10}$$

And this, once again, diverges. It is then necessary that $h(r) \sim r^2$ or, in alternative, $h(r) \ll r^2$; in both cases, there is no divergence. And this is enough to show that the Sakharov criterion is also necessary.

It is worth to notice that the Sakharov criterion only applies for the central singularity in a spherically (or flat, or hyperbolic, depending on the topology) symmetric space. Indeed, the Sakharov criterion is a *coordinate* criterion and not a *geometric* criterion. It does not necessarily work in different kinds of spacetimes; we may guess it does, at least in some cases, but actually there is no proof of it.

2.4 A (more) covariant version of the Sakharov criterion

In this section we propose a generalization of the Sakharov criterion for any spherically symmetric spacetime. This proposal has been already discussed in the Appendix of [66], so here we will only illustrate its highlights, without going too much in detail. For the interested reader, we remand to the original paper.

First of all, we write the metric as

$$ds^2 = \gamma_{AB}(x)dx^A dx^B + r^2(x)d\Omega_k^2 \qquad A, B \in \{0, 1\} \ , \tag{2.11}$$

where the non-angular part of the metric is sometimes called the normal metric, x^A are coordinates in the time-radial subspace and $r(x)$ is the areal radius, interpreted as a scalar field in the 2D normal space. Here $d\Omega_k^2$ is the metric element of a maximally symmetric space and reads

$$d\Omega_k^2 = \frac{d\rho}{1 - k\rho^2} + \rho^2 d\phi^2 \qquad i, j \in \{0, 1\} \ . \tag{2.12}$$

Thus, for $k = 1$ one has the 2D sphere S^2, for $k = 0$ the 2D torus T^2, and for $k = -1$, the 2D Riemann surface H^2/Γ. Topology does not play a major role in our discussion, so, in order to fix ideas, one can still think to spherically symmetric space.

Now, in order to formulate the covariant regularity criterion, we work within the metric (2.11), and we introduce the invariant scalar in the reduced spacetime (which is consequently a scalar in the whole spacetime), defined by

$$\Phi(x) = \frac{1 - Z(x)}{r^2(x)} \ , \tag{2.13}$$

where $Z \equiv \gamma^{AB}(x)\partial_A r(x)\partial_B r(x)$.

The regularity criterion then states: *a Dynamical Spherically Symmetric (DSS) spacetime is regular if and only if the invariant quantity* Φ *is* bounded *for every* x *of the associated spacetimes.*

We should notice that in GR, the invariant Φ is related to Misner-Sharp energy by

$$\Phi(x) = \frac{2E_{MS}}{r^3(x)} \ , \tag{2.14}$$

i.e. the criterion states that one has a regular DSS spacetime only if the Misner-Sharp density is finite everywhere. This is a quite strong physical result.

In order to show that this is a generalization, let us prove that criterion (2.14) reduces to the ordinary version of the criterion in the case of static spacetime. Writing the metric in the diagonal Schwarzschild gauge, namely

$$ds^2 = -b^2(r)f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 dS^2 \ , \tag{2.15}$$

the coordinate r is the areal radius, and one has $Z(r) = f(r)$. We see that the only critical point is $r = 0$ and thus, for small r, one needs to require

$$\Phi(r) = \frac{1 - f(r)}{r^2} = A + o(r) \ . \tag{2.16}$$

This states that, for a spherical horizon, one needs $f(r) = 1 - Ar^2 + o(r^3)$, i.e. the existence of a de Sitter core at the origin: which is exactly the original Sakharov proposal.

As a second example, we may also consider the non-flat FLRW spacetime

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - R_0 r^2} + r^2 d\Omega^2\right) \ , \tag{2.17}$$

where R_0 is a curvature parameter, which can be positive, negative or vanishing. Computing the invariant Φ in this case, one has

$$\Phi = \frac{R_0 + \dot{a}(t)}{a(t)} \ . \tag{2.18}$$

As a consequence, the condition "Φ bounded" gives $a(t) \neq 0$ for every t, which is known to be the regularity criterion for the FRLW spacetime.

Closing the section, we should mention that this criterion (and its original formulation) is thought for four or more dimensions (in the case of higher dimensionality, statement (2.13) is intuitively modified). On the other hand, in the 3D case, it seems to suffer of some troubles. If we apply the criterion to the well known (uncharged) BTZ solution [103]:

$$f(r) = ar^2 + m \ , \tag{2.19}$$

we obtain

$$\Phi = \frac{1 - ar^2 - m}{r} \ . \tag{2.20}$$

The case $m \neq 1$ is safe, since Φ is found to be divergent and so are the components of the Riemann tensor (or the Ricci curvature, if we fix on it). On the other hand, if $m = 1$, the Riemann tensor becomes regular, while Φ still diverges at infinity. The tricky point is that the 3D vacuum solution is just a constant (instead of the $1/r^{n-3}$ of the higher dimensional case) and constant terms are eventually related to conical singularities, rather than divergences. This warns us that in the 3D case the validity of Sakharov criterion should be taken carefully.

2.5 Geodesic completeness

This may be one of the most interesting notions of regularity, though not one of the most directly applicable. In particular, the connection between singularity and geodetic incompleteness dates the papers by Penrose and Hawking [23,104, 105] and seems to be one of the most robust argument for regularity. Moreover, as is well known, this is the origin of the energy conditions.

As in the previous section, we focus on spacetime (2.11), though the discussion may be set in a more general way. We will also use the same notation and formalism, inherited from that of [66,106–109].

Before proceeding, let us fix a few ideas on our spacetime. First we define a trapping horizon H by

$$Z(x)\Big|_H = 0 \ , \tag{2.21}$$

providing that $\partial_A Z|_H \neq 0$. $Z(x)$ has here the same meaning it has in the previous section. The vertical bar means the quantity is calculated on the horizon.

Another important scalar in the normal space is the Hayward surface gravity (associated with the dynamical horizon), which reads

$$\kappa_H = \frac{1}{2}\tilde{\Box} r\Big|_H \ . \tag{2.22}$$

This is a generalization of the Killing surface gravity. In the DSS case, it also is possible to introduce also the Kodama vector K: given the metric (2.11), its components are

$$K^A(x) = \frac{1}{\sqrt{-\gamma}}\varepsilon^{AB}\partial_B r \ , \qquad K^\theta = K^\varphi = 0 \ , \tag{2.23}$$

where ε^{AB} is the usual fully antisymmetric tensor. We may also introduce the Kodama trajectories, and the related Kodama observer, using the integral lines of Kodama vector:

$$\frac{dx^A}{d\lambda} = K^A = \frac{1}{\sqrt{-\gamma}}\varepsilon^{AB}\partial_B r \ , \tag{2.24}$$

whose result is $dr(x(\lambda))/d\lambda = 0$. Thus, in a generic spherically symmetric spacetimes, the areal radius r is conserved along Kodama trajectories. The operational interpretation goes as follows: static observers in static BH become in the dynamical case Kodama observers whose velocity

$$v_K^A = \frac{K^A}{\sqrt{Z}} \quad \text{such that} \quad \gamma_{AB}v_K^A v_K^B = -1 \ . \tag{2.25}$$

Finally, the energy measured by this Kodama observer at fixed areal radius r_0 is

$$E = -v_K^A \partial_A I = -\frac{K^A\partial_A I}{\sqrt{Z_0}} = \frac{\omega}{\sqrt{Z_0}} \ , \tag{2.26}$$

where I is the classical action of the particle, $\partial_A I$ its momentum and $\omega = -K^A\partial_A I$ its Kodama energy. We will make use of these definition in the oncoming section.

2.6 Singular and regular spacetimes

As mentioned in [110], if the spacetime is sufficiently smooth, and one assume suitable condition on the Ricci tensor, causality and boundary conditions, then there exist (some) non-spacelike inextensible geodesics: and this is usually regarded as the presence of singularities. For a complete discussion, see [110–113]. For the inflationary case see also [114] and references therein.

2.6.1 Non-spacelike geodesics in curved spacetimes

Now we recall the derivation of geodesics equation related to massless (light-like) geodesics and massive (time-like geodesics). Let us start from the most general metric $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ and let us denote by $\dot{x}^\mu = dx^\mu/d\lambda$, where λ is an affine parameter (in the massive case, it plays the role of time). The equation for non-spacelike geodesics is derived from the Lagrangian

$$\mathscr{L} = -\frac{g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}{2V} + \frac{1}{2}m^2 V \ , \tag{2.27}$$

where V is a Lagrange multiplier, implementing the reparametrization invariance and m^2 is a mass like term, positive for time-like geodesics and vanishing for light-like geodesics. In fact, the variation with respect to V gives

$$g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu = -m^2V^2 \ . \tag{2.28}$$

Light-like geodesics have $m^2 = 0$, while time-like geodesics have $V^2 m^2 = -1$. We may use this into the geodesic equation, which is obtained via a variation w.r.t. x^μ:

$$\frac{d}{d\lambda}\left(g_{\mu\alpha}(x)\dot{x}^\alpha\right) = \frac{1}{2}\partial_\mu\left(g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta\right) \ . \tag{2.29}$$

Example: the FLRW spacetime

As example, we discuss the FLRW spacetime. Indeed, some things are here quite clearer, than in the usual black hole spacetime. First of all, we recall that the FLRW metric is

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - R_0 r^2} + r^2 d\Omega^2\right) \ , \tag{2.30}$$

and the geodesic equation reads

$$-\frac{d\dot{t}}{d\lambda} = H(t)\left(\left(\dot{t}\right)^2 - \varepsilon\right) \ , \tag{2.31}$$

where time-like trajectories take $\varepsilon = 1$, and light-like ones take $\varepsilon = 0$. The solution of above differential equation then reads

$$\dot{t} = \frac{dt}{d\lambda} = \frac{\sqrt{a_0^2 + \varepsilon a(t)^2}}{a(t)} \ . \tag{2.32}$$

This result permits a quite simple discussion on the geodesic completeness in FLRW spacetimes. In fact, we may rewrite

$$\frac{d\lambda}{dt} = \frac{a(t)}{\sqrt{a_0^2 + \varepsilon a(t)^2}} \ . \tag{2.33}$$

If $a(t) > 0$ for all t, it follows that λ is a monotone function, and one has the geodesic completeness. Furthermore, a further integration gives

$$\lambda = C_0 + \int dt \frac{a(t)}{\sqrt{a_0^2 + \varepsilon a(t)^2}} \ . \tag{2.34}$$

If $a(t) > 0$, we may also write

$$\lambda = C_0 + \int_{-\infty}^{t} dt' \frac{a(t')}{\sqrt{a_0^2 + \varepsilon a(t')^2}} \ . \tag{2.35}$$

If this integral is *divergent*, all non space-like geodesics are past-complete, and no singularity affect the theory. On the other hand, if $a(t)$ is vanishing in the past, say $a(0) = 0$, we may write

$$\lambda = C_0 + \int_{0}^{t} dt' \frac{a(t')}{\sqrt{a_0^2 + \varepsilon a(t')^2}} \ . \tag{2.36}$$

If this integral is *convergent*, one has past incomplete geodesics, and thus singularities are present. This discussion, which we carried out from [66], is also present in [114].

A first elementary example of this is the flat FLRW bouncing solution

$$a(t) = 1 + A^2 t^2 > 0 \ . \tag{2.37}$$

In this case $\varepsilon = 0$, and the integral (2.35) is divergent, thus no singularities arise (of course, as we expected).

On the other hand, a second example is

$$a(t) = t^\alpha \qquad \alpha > 0 \ . \tag{2.38}$$

In this case, the integral (2.38) is convergent and there is a Big Bang singularity at $t = 0$ (also in this case, as we expected). Finally, in a FLRW spacetime, if $a(t) > 0$ for all t, there are no singularities.

3 Proposals of some approaches in building up RBHs

3.1 Introduction

With this third chapter, we start doing something less introductory. In chapter 2, we tried to give a general scheme of the problem of singularity (or regularity), while in the previous chapter we discussed some concepts of regularity, which we will use throughout the rest of this work. In this chapter, we start presenting some models of RBH. However, for various reasons, neither of these models is actually really able to produce an RBH; or, better, to be sure that the Schwarzschild singularity has been avoided.

The reason for so many models being unsuccessful are various, but in many cases they rest on the structure of the EE. Indeed, if we reduce to spherical symmetry (but the case of other topologies can be treated in the same way), the 00 component of the Einstein tensor contains only the g_{11} component of the metric and its derivative, both appearing linearly: mathematically, this means that the solution of the homogeneous equation is always part of the general solution and, physically, this means that the vacuum Schwarzschild term will always appear – unless one finds some physical constraint to avoid it. We will see this feature in this chapter, for the specific case of Non-Linear Electrodynamics and in the next one, in the form of a more general no-go theorem.

Finally, we don't pretend to present here a full list of all the approaches proposed in order to solve the singularity issue, which are not actually able to produce only regular solutions (once the parameters are suitably chosen). Here we discuss just a number of them, we found significant, due to their popularity, to their expectance and finally, because we already dealt with them. In any case, it is worth to notice that the models we are going to present here cover a huge part of the models actually present in literature. In order to check it, we remind the non exhaustive list of works [29–60] and the reviews [64–66].

3.2 The Ansoldi-Nicolini-Smailagic-Spallucci (ANSS) black hole

As first model we present in this chapter, is the one proposed in [42, 43]. In this section, we just want to give the highlights and the main results of the model, so, for any detail of calculations, we remand the interested reader to the original paper.

The key trick they use is to assume that, at least on a sufficient small spacetime scale, coordinates do not commute [115–117]:

$$[\mathbf{x}^{\mu}, \mathbf{x}^{\nu}] = i\theta^{\mu\nu} \ , \tag{3.1}$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix which determines the fundamental, discrete spacetime cell.

We won't stress here with the whole amount of consequences of eq. (3.1). We are only interested in its practical effects. Indeed, the effect of smearing is mathematically implemented as some sort of substitution: the Dirac delta density appearing in the commuting case is substituted with some distribution (say a Gaussian one). The paper then chooses the source of the gravitational field as the static, spherically symmetric, smeared, particle-like effective density

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{3/2}} \exp\left(-r^2/4\theta\right) \ , \tag{3.2}$$

where M is the mass of the particle. Of course we expect this Gaussian to appear only at a very small length scale, in order to be compatible with the phenomenological observations. We also notice that density (3.2) is only one of the possible choices, not the single one.

Putting things together, at this stage one can look for a static, spherically symmetric and asymptotically Schwarzschild solution of the EE, with (3.2) describing the energy density of the system. First of all, in order to have a well-defined SET, the covariant conservation condition $\nabla_\nu T^{\mu\nu} = 0$ is implemented; that is

$$\partial_r T^r_r = -\frac{1}{2} g^{00} \partial_r g_{00} \left(T^r_r - T^0_0\right) - g^{22} \partial_r g_{22} \left(T^r_r - T^\theta_\theta\right) \ . \tag{3.3}$$

Since the solution is required to be Schwarzschild-like, the condition $g_{00}g_{11} = -1$ is also requested. It is worth to notice that nothing, in principle, forces us to choose so and we could also look for a "dirty" black hole (i.e. $g_{00}g_{11} \neq -1$). We will discuss the fluid approach more in detail in the ninth chapter, but for the moment we limit ourselves to the simplest case.

If so, we simply need $T^r_r = T^0_0$ and thus the conservation equation fixes the value for T^θ_θ:

$$T^\theta_\theta = -\rho_\theta(r) - \frac{1}{2} r \partial_r \rho_\theta(r) \ . \tag{3.4}$$

With the SET finally known, we are also able to find the radial and the tangential pressure, which turn to be different: indeed, if we immediately find $P = -\rho_\theta$, the tangential pressure reads $P_\perp = -\left(\rho_\theta + \frac{1}{2}r\partial_r\rho_\theta\right)$. The fluid is then anisotropic.

With effective density (3.2), the EE cannot be solved explicitly, but still the metric can be written in the parametric form $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2$, where $f(r)$ is

$$f(r) = 1 - \frac{4M}{r\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \ , \tag{3.5}$$

where $\gamma(3/2, r^2/4\theta)$ is the lower incomplete Gamma function:

$$\gamma(3/2, r^2/4\theta) \equiv \int_0^{r^2/4\theta} t^{1/2}e^{-t} \ dt \ . \tag{3.6}$$

Although there is no hope to calculate this function exactly, it can be studied numerically, proving it is well behaved at any time. In particular, we are able to see analytically what happens near the origin: in that case, indeed, the exponent becomes negligible w.r.t. $t^{1/2}$, so that $\gamma(3/2, r^2/4\theta \to 0) \propto r^3$, which means the Sakharov criterion is satisfied. On the other hand, at infinity, the incomplete gamma function tends to the Gamma function, so that the non flat (dominant) part of the metric is proportional to $1/r$: Schwarzschild. Finally, fixing suitable values for the parameters, one is able to produce a star, an extremal black hole or a two-horizon black hole, as shown in [42].

However, there is a reason this approach has been listed here, where we discuss models, which actually are not able to solve the problem of the singularity.

As we already argued in [95, 96, 98], this approach is not able to avoid a singularity: indeed, the EE are still those of GR, just with an effective density at the r.h.s. If we focus on the tt component of the EE, it reads

$$\frac{f'r + f - 1}{r^2} = 8\pi\rho_\theta \ , \tag{3.7}$$

and this is a linear non-homogeneous ordinary differential equation, whose solution is given by a particular solution of the non-homogeneous equation plus the general solution of the homogeneous one; and it is clear that the homogeneous solution is just the Schwarzschild term c/r.

This means that, whatever profile we choose for the effective density, an extra term c/r will always appear in the solution. Unless we find a physical condition to eliminate it, we are forced to include it into the metric. Moreover, since c is an integration constant, it appears there is no way to cancel it with a fine tuning of the density itself, whose parameter are free, but fixed.

Finally, in the ninth chapter we will see a similar model of RBH, although we would not consider, then, any non-commutativity issue. Moreover, we already anticipate that we will be able to justify the choice of $g_{00}g_{11} = -1$ and, most of all, we will be also able to find a *physical* criterion, in order to avoid the emergence of Schwarzschild singularities.

3.3 The Non-Linear Electrodynamics black hole

As a second example, we discuss the so called Non-Linear Electrodynamics (NED) approach. The aim of this approach is to cure the Schwarzschild gravitational singularity, coupling an electromagnetic charge to the black hole, able to cancel the $1/r$ Schwarzschild divergence:

$$\mathscr{I} = \int d^4x \, \sqrt{-[g]} \left(\frac{R}{2} - 2\Lambda - \mathscr{L}(I) \right) , \tag{3.8}$$

where R is the Ricci scalar, Λ is a cosmological constant, $I = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is an electromagnetic-like tensor and $\mathscr{L}(I)$ is a suitable function of it. It is clear, and we will show it later as a check, that the case $\mathscr{L}(I) \propto I$ reduces to the standard (and singular) Reissner-Nordström solution. Before proceeding, recall that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; we will only deal with gauge invariant quantities, and we put $\Lambda = 0$, since its contribution can be easily restored.

The NED approach recovers the spirit of Born-Infeld solutions, in which a deformation is inserted, in order to cure the Coulomb central singularity [118]. It became very popular in the Nineties, after the work of E. Ayon-Beato and A. Garcia [50], who interpreted the Bardeen solution as the result of a specific NED action. This approach has been discussed in several other papers, such as [34, 119–121]. Papers working within the framework of NED have still been published also in recent years (see e.g. [31, 36, 46]).

Since the coupling in action (3.1) is minimal, the equations of motion can be easily found with standard derivation and read

$$G_\mu^\nu = -F_{\alpha\mu} \partial_I \mathscr{L} F^{\nu\alpha} + \mathscr{L} \delta_\mu^\nu , \tag{3.9}$$

$$\nabla^\mu (F_{\mu\nu} \partial_I \mathscr{L}) = 0 . \tag{3.10}$$

Eq. (3.9) is the EE, while eq. (3.10) is the modified Maxwell equation. Here we follow the lines of [98], working with tensor $F_{\mu\nu}$ and scalar I. However, it is worth to notice that there exists also another equivalent approach, the so called dual P approach; it is based on the three new gauge invariant quantities [119]

$$P_{\mu\nu} \equiv F_{\mu\nu} (\partial_I \mathscr{L}(I)) , \qquad P \equiv \frac{1}{4} P_{\mu\nu} P^{\mu\nu} , \qquad \mathscr{H} \equiv 2I \partial_I \mathscr{L} - \mathscr{L} , \tag{3.11}$$

such that $\nabla^\mu P_{\mu\nu} = 0$. The convenience of this approach is that the Maxwell equations are reduced to the ordinary form and can now be solved exactly as in the standard linear case. However, one is then forced to a non trivial reconstruction procedure, in order to recover $F_{\mu\nu}$ and I and so the Lagrangian.

Eq. (3.9)-(3.10) are written in an explicitly covariant form and therefore hold for any spacetime. However we are much more interested in what happens in the case of a simple spherical black hole, described by the metric

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega^2 \ . \tag{3.12}$$

Thus, from (3.10), one has

$$\partial_r \left(r^2 \partial_I \mathscr{L} F^{0r} \right) = 0 \ . \tag{3.13}$$

Throughout this section, we will consider $g(r) = f(r)$. This can be easily proved within the dual P approach [95] and still holds in the standard framework, because the change does not involve the gravitational sector of the theory. Then, since $I = \frac{1}{2}F_{0r}F^{0r} = -\frac{1}{2}F_{0r}^2$, one gets

$$r^2 \partial_I \mathscr{L} = \frac{Q}{\sqrt{-2I}} \ , \tag{3.14}$$

where Q is an integration constant. This allows to solve, within the NED approach, the generalized Maxwell equation. We shall use this fact and the (t,t) component of the EE, which reads

$$G_t^t = \frac{f'r + f - 1}{r^2} = 8\pi \left(-2I \partial_I \mathscr{L} + \mathscr{L} \right) = -8\pi\rho \ . \tag{3.15}$$

If we now introduce the more convenient quantity $X \equiv Q\sqrt{-2I}$, we may rewrite equation (3.14) as

$$r^2 \partial_X \mathscr{L} = 1 \ . \tag{3.16}$$

Furthermore, we have

$$\rho = X\partial_X \mathscr{L} - \mathscr{L} = \frac{X}{r^2} - \mathscr{L} \ . \tag{3.17}$$

Thus, when $\mathscr{L}(X)$ is given, then one obtains $\rho = \rho(r)$ just using eq. (3.14). This allows to discuss the problem even within the formalism of a (perfect) fluid, which turns to be much more convenient. We will discuss the perfect fluid approach in the ninth chapter.

Finally, before illustrating some applications and examples, it is worth to notice that the formal results of this section strictly rest upon the NED equations (in particular, we made use of the modified Maxwell equation); thus, our results are not (at least, trivially) extensible to other approaches.

3.3.1 Reconstruction

In the following, we present an efficient reconstruction scheme able to produce RBH solutions. In principle, one can start from the Lagrangian, deriving then the associated solution; however, it is more convenient to do the contrary, i.e. assuming the solution $f(r)$ to be known, finding then the associated Lagrangian. First of all, we not that the first of the EE can be rewritten as

$$\frac{d}{dr}\left(r(f-1)\right) = -8\pi r^2 \rho \ , \tag{3.18}$$

which immediately gives $\rho(r)$ once $f(r)$ is known. The other two equations are

$$r^2 \partial_X \mathscr{L} = 1 \qquad \text{and} \qquad \mathscr{L} = \frac{X}{r^2} - \rho \ . \tag{3.19}$$

Combining them, they state that X can be written in terms of ρ as $X = -r^3\rho'/2$, the prime indicating the radial derivative. As consequence, one obtains $r = r(X)$ and, using the second equation, can find $\mathscr{L} = \mathscr{L}(X)$, which finishes the work. The tricky point is that the step of finding $r(X)$ proves to be a highly non trivial one.

As a first example, let us discuss the Reissner-Nordström solution [122,123]: we have

$$f(r) = 1 - \frac{C}{r} + \frac{Q^2}{r^2} \ , \tag{3.20}$$

where C plays the role of a mass and Q of an electric charge. Using eq. (3.18), one has $\rho = Q^2/8\pi r^4$. In this case, it is easy to apply our algorithm and recover the standard Maxwell Lagrangian.

As a second and less trivial example, consider the general solution

$$f(r) = 1 - \frac{C}{r} - \frac{2A}{\xi} + \frac{2A}{\xi r}\arctan\left(\frac{r}{\xi}\right) - H_0^2 r^2 \ , \tag{3.21}$$

where C is an integration constant and A, ξ and H_0 are suitable parameters. This solution generalizes a black hole solution obtained from a particular Horndeski Lagrangian, namely Einstein gravity with a non minimally coupled scalar field [124]. Its effective density is

$$\rho = \frac{A}{4\pi\xi^2(\xi^2 + r^2)} + \frac{3H_0^2}{8\pi} \ , \tag{3.22}$$

thus

$$X = \frac{Br^4}{(\xi^2 + r^2)^2} \ , \qquad \text{where} \qquad B = \frac{A}{4\pi} > 0 \ . \tag{3.23}$$

Also in this case, the Lagrangian can easily be reconstructed and the result is

$$\mathscr{L}(X) = -\frac{1}{\xi^2}\left(\sqrt{B} - \sqrt{X}\right)^2 - \frac{3H_0^2}{8\pi} \ . \tag{3.24}$$

From eq. (3.23), one finds that a suitable choice for the parameters A and H_0 (e.g. $A,\ H_0 \geq 0$) is able to satisfy the WEC. Indeed, we recall that the WEC is satisfied if and only if [111]

$$\rho \geq 0 \tag{3.25}$$

$$\rho + p_k \geq 0 \qquad k = 1, 2, 3\ . \tag{3.26}$$

The issue if the WEC is satisfied or not by regular (and in this particular case by NED regular) solutions has been widely discussed, among others, by I. Dymnikova in [125] and [126]. In particular, [125] finds some conditions a Lagrangian should satisfy, in order to fulfill the WEC.

As a last example, we start from the following metric:

$$f(r) = 1 - \frac{C}{r} + \frac{4B\pi}{r^2 + \xi^2} - \frac{4B\pi}{\xi r} \arctan\left(\frac{r}{\xi}\right)\ , \tag{3.27}$$

where $B > 0$. If $C = 0$, it represents the regular black hole solution proposed by Dymnikova in [127]. The solution turns to be asymptotically flat, and its regular part has a de Sitter core and no conical singularity. However the reconstruction scheme, in this case, is not of simple application: indeed, we have

$$\rho = \frac{B}{(\xi^2 + r^2)^2}\ , \tag{3.28}$$

and since B is positive, by the way we notice that both the WEC and the DEC are satisfied. Furthermore, we have

$$X = \frac{2Br^4}{(\xi^2 + r^2)^3}\ . \tag{3.29}$$

We may rewrite it as

$$X(\xi^2 + r^2)^3 = 2B(r^2)^2\ . \tag{3.30}$$

Considering r^2 as a function of X, the only thing we are left with is solving a third degree algebraic equation. Once we have the solution, the Lagrangian reads

$$\mathscr{L}(X) = \frac{(r^2(X) - \xi^2)X}{2(r^2(X))^2}\ . \tag{3.31}$$

The point now is that the final expression turns to be very complicated, so we won't write it here. However, we are still able to write $\mathscr{L}$ with the simple parametric representation

$$\mathscr{L}(r) = \frac{(r^2 - \xi^2)}{(r^2 + \xi^2)^3}\ , \tag{3.32}$$

One may now wonder, why we enlisted the NED approach in this chapter, where we are presenting some unsuccessful trials to solve the singularity. The point is immediately said: if we go back to eq. (3.15), or (3.18), we immediately see that the r.h.s. of these equations is completely independent on the metric function. Under the assumptions of spherical symmetry and $f = g$, one can also prove it already from eq. (3.9). It is then immediate to see by direct check, say into eq. (3.15), that, if f is a solution, then also $f + \frac{c}{r}$ is a solution too, where c is an arbitrary constant; exactly as we did in the previous section.

This is not surprising: indeed, eq. (3.15) is a linear non-homogeneous ordinary differential equation, whose solution is given by a particular solution of the non-homogeneous equation and the general solution of the homogeneous one; and, once again, it is clear that the homogeneous solution is just the Schwarzschild term c/r.

This is not surprising, but is the key point: indeed, there is no hope to find a way to set $c = 0$, at least within the NED framework. c is indeed an integration constant and there is no way to fix it. In order to convince that things are so, consider the Bardeen solution [25]

$$f(r) = 1 - \frac{2Kr^2}{(r^2 + l^2)^{3/2}} \ , \tag{3.33}$$

where K is an integration constant. We choose it, because it is the function originally reconstructed in [50], later identified as a magnetic monopole solution in [129].

It sounds natural to fix K as the mass of the black hole, but this is true only if $c = 0$: indeed, we recall that the mass of a black hole is just the half of the coefficient of the $1/r$ power in the asymptotic expansion. Adding to solution (3.33) the c/r term, one has still a solution, but its asymptotic behavior is

$$f(r \to \infty) = 1 - \frac{2K - c}{r} + o(r^{-2}) \ . \tag{3.34}$$

It is then clear that the mass is now $m = K - c/2$. Since both K and c are arbitrary constants and since we have a constraint only on their sum (the only asymptotic requirement is $2K - c = 2m$), we are not free to fix any of them.

Of course, this discussion does not mean that the NED approach is not able to produce RBHs: indeed, the choice $c = 0$ is admissible, so there are actually *some* regular solutions. The point is that we cannot choose them, unless an arbitrary (but inadmissible!) fixing of the integration constants. The NED approach still deserves some interest, but with the consciousness that it is not able to solve the singularity issue.

We close this section and the presentation of the NED approach with the following remark. One can start with the $\mathscr{L}$ given in the implicit form $X = G(\mathscr{L})$, where G is a smooth known function. Taking the derivative w.r.t. X and using the fact that $r^2\partial_X\mathscr{L} = 1$, one has $r^2 = \partial_{\mathscr{L}}G(\mathscr{L})$. In principle, this gives $\mathscr{L}$ in terms of r, and the effective density may be computed $r^2\rho = G(\mathscr{L}) - r^2\mathscr{L}$.

For example, let us consider

$$X = G(\mathscr{L}) = G_0 + G_1 \mathscr{L} + \frac{G_2}{2} \mathscr{L}^2 \ , \tag{3.35}$$

with $G_{0,1,2}$ suitable constants. Then, we have

$$\mathscr{L} = \frac{r^2 - G_1}{G_2} \qquad \text{and} \qquad r^2 \rho = G_0 - \frac{G_1^2}{2G_2} + r^2 \frac{G_1}{G_2} - \frac{r^4}{2G_2} \ . \tag{3.36}$$

In order to avoid conical singularities, one has the constraint $G_0 = G_1^2/2G_2$ (see e.g. [130] and references therein). Consequently, the Lagrangian is determined by the algebraic equation

$$X = \frac{G_1^2}{2G_2} + G_1 \mathscr{L} + \frac{G_2}{2} \mathscr{L}^2 \ , \tag{3.37}$$

which can be solved and we get

$$\mathscr{L}(X) = -\frac{G_1 \mp \sqrt{2G_2 X}}{G_2} \ , \tag{3.38}$$

coupled with the associated density term

$$r^2 \rho = r^2 \frac{G_1}{G_2} - \frac{r^4}{2G_2} \ . \tag{3.39}$$

The related general solution then reads

$$f(r) = 1 - \frac{C}{r} - 8\pi \left(\frac{r^2 G_1}{3G_2} - \frac{r^3}{10 G_2} \right) \ . \tag{3.40}$$

This is a static spherically symmetric and regular solution, found in [98]. Other solutions can be found with the same technique, the above solution being one of the simplest. Apparently, up to the C/r term, this seems to be regular, but actually it is not: indeed, it contains a cubic divergent term at infinity, so that it violates the generalized Sakharov criterion we showed in the previous chapter. Moreover, one can prove that the Kretschmann scalar $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ diverges, for large r, like r^2.

A final remark is in order. Throughout this section, we considered only theories generated by actions of the kind of action (3.8). In principle, one can use also more complicated actions, as in [128]. However this strategy does not appear to be able to cancel the singularity, since it is still related to the electromagnetic part of the theory − which remains invariant.

3.4 The Modesto et al. approach

We close the chapter presenting a third way, which is not able to cure the singularity. We follow the path of [131–133] and in particular [133], with which we will share some notation.

So, we start considering the standard Schwarzschild metric, in the usual form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\Omega^2 \ , \tag{3.41}$$

where $d\Omega^2$ is the usual surface element of a sphere of unitary radius. The following step is to multiply ds^2 times some factor $S^2(r)$, still describing the same spacetime. The new metric is then

$$\tilde{ds^2} \equiv S^2(r) ds^2 = -S^2(r) \left(1 - \frac{2m}{r}\right) dt^2 + \frac{S^2(r)}{1 - \frac{2m}{r}} dr^2 + S^2(r) r^2 d\Omega^2 \ . \tag{3.42}$$

In order to preserve the spherical symmetry of the system, we should not use r as the areal radius anymore, but $R \equiv S(r)r$. With this choice, metric (3.42) becomes

$$\tilde{ds^2} = -S^2 (\partial_R r)^2 \left(1 - 2m\frac{S}{R}\right) dt^2 + \frac{S^2 (\partial_R r)^2}{1 - 2m\frac{S}{R}} dR^2 + R^2 d\Omega^2 \ . \tag{3.43}$$

What we should do now is to choose $S(r)$ in order that metric (**??**) reduces to Schwarzschild at infinity and to a de Sitter core near the origin: $\tilde{g}^{11}(R \to 0) = 1 + aR^2 + ...$ and $\tilde{g}^{11}(R \to \infty) = 1 - 2m/R + ...$ Moreover, we also keep in mind that eq. (**??**) holds only under the assumption that R is invertible: this is actually necessary, if we want to write $r(R)$ and so also $\tilde{g}^{11}$ in the new coordinates (t, R, θ, φ).

Instead of considering r and S as the true degrees of freedom of the problem, it is more convenient to consider r and R (which is just a matter of convenience, but the work is the same). This is done because the regularity of the black hole is determined by the behavior for small R (not for small S) and we expect that $r = R$ asymptotically.

Thus said, we write $g(R) \equiv \tilde{g}^{11}$ as a function of R as

$$g(R) = \frac{r^2}{R^2 (\partial_R r)^2} \left(1 - \frac{2m}{r}\right) \ , \tag{3.44}$$

where R is the independent variable and $r \equiv r(R)$ a function of it. If so, eq. (3.44) can be rewritten as

$$\frac{\partial_r R}{R\sqrt{f(R)}} = \frac{1}{\sqrt{r(r-2m)}} \ , \tag{3.45}$$

and consequently, assuming $g(R)$ being a known function, the equation can be integrated and then inverted algebraically. However, if we tried to apply this machinery to a full solution, this turns out to be too difficult to be solved: as example, taking $g(R)$ that one by Hayward, which is one of the simplest RBH metric, the integral becomes so complicated that cannot be solved analytically.

In order to find a way out, let's see what happens only for $R \to 0$, when $g(R)$ is much simpler and takes the form $g(R) = 1 + \beta^2 R^2 + ...$ Substituting into eq. (3.45), we have

$$\frac{\partial_r R}{R\sqrt{1+\beta^2 R^2}} = \frac{1}{\sqrt{r(r-2m)}} \; . \tag{3.46}$$

Maybe surprisingly, this object is integrable. After some managing, we get

$$R(r) = \frac{16\alpha}{\beta} \frac{\left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m} - 1}\right)^2}{1 - 64\alpha^2 \left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m} - 1}\right)^4} \; , \tag{3.47}$$

where α is some integration constant. Of course, we remind this holds only for $R \to 0$. By the way, $S(r)$ turns to be

$$S(r) = \frac{8\alpha}{\beta m} \frac{\left(1 + \sqrt{1 - \frac{2m}{r}}\right)^2}{1 - 64\alpha^2 \left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m} - 1}\right)^4} \; . \tag{3.48}$$

Using this result, with a little abuse of notation, we are able to write $g(R)$ as $g(r)$, i.e.

$$g(r) = \left(\frac{1 + 64\alpha^2 \left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m} - 1}\right)^4}{1 - 64\alpha^2 \left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m} - 1}\right)^4}\right)^2 \; . \tag{3.49}$$

We remind that this result does not hold for any r, because we found it using an expression of $f(R)$ which holds only for $R \to 0$.

Finally, it is time to discuss our results. Eq. (3.47) shows that $R \to 0$ only when $r \to \infty$ (the sign is not relevant, since α and β have no definite sign). Then we have the following situation:

- if $R \to \infty$, then $r \to \infty$ (indeed, we require $r(R) = R$ asymptotically);
- if $R \to 0$, then $r \to \infty$ again.

But what does it actually mean? Since the function $r(R)$ goes to ∞ both for $R \to 0$ and for $R \to \infty$, it possesses at least one minimum and, consequently, it's not invertible (unless locally). But this means that we cannot write $R(r)$ globally and so we cannot write globally $S(r)$: in other words, *there is no conformal factor $S(r)$ able to produce, starting from the Schwarzschild metric, a regular solution.*

Finally, notice that the same result holds also thinking in terms of S and r, instead of R and r. Indeed, consider eq. (3.48): since $R \to 0$, this means that $r \to \infty$ and it's not difficult to convince that

$$S(R \to 0) \simeq -\frac{m}{8\alpha\beta r^2(R)} = -\frac{\alpha\beta}{2m}R^2 \ , \tag{3.50}$$

which means that $S \to 0$ for $R \to 0$ (as expected) and consequently that $r(S) \sim 1/\sqrt{S}$ for $S \to 0$, meaning that $r \to \infty$ for $S \to 0$; but we already know that $r \to \infty$ also for $S \to 1$: i.e. the function $r(S)$ has at least one minimum and it's not invertible: we get the same conclusion that we cannot find some $S(r)$ producing a RBH.

4 A no-go theorem for RBHs

4.1 Introduction

In this chapter we discuss a new negative result concerning RBH. We show that, under suitable assumptions, once a matter Lagrangian is fixed, the Schwrzschild term c/r determines a class of solutions (differing among themselves only for a term proportional to $1/r$) and thus it is not possible to avoid a Schwarzschild singularity just with a choice "smart enough" of the matter distribution. We discuss and prove one preparatory lemma, the main theorem and a corollary.

4.2 Assumptions and statement

As first point, see which are the common assumptions for all the following propositions. As usual, we work within a static and spherically symmetric spacetime:

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega^2 \ , \tag{4.1}$$

where f and g are suitable functions and $d\Omega^2$ is the volume element of a 2-sphere of unit radius. Notice that here we chose to work within a spherical symmetry, but the extentions to other topologies, such as a torus, would be trivial. Notice also that the extension to the dynamical case doesn't make trouble, since the Schwarschild-like term is singular in the radial variable. We will briefly discuss this point at the end of the chapter.

We assume that this metric is generated by the minimally coupled action

$$I = \int d^4x\sqrt{-[g]}\left(\frac{R}{2} - \mathscr{L}(\phi_{\alpha_1\ldots\alpha_n}, \nabla\phi_{\alpha_1\ldots\alpha_n})\right) \ , \tag{4.2}$$

where $[g]$ is the determinant of the metric, $\phi_{\alpha_1\ldots\alpha_n}$ is some kind of field and $\mathscr{L}(\phi_{\alpha_1\ldots\alpha_n}, \nabla\phi_{\alpha_1\ldots\alpha_n})$ is some scalar function of it and its derivatives; its sign is just conventional. Our argument holds also in the case of more fields, but, for simplicity, here we discuss just one.

Using with action I the variational principle, we have the independent Einstein equations (we use mixed indices for a matter of convenience):

$$R^{\nu}_{\mu} - \frac{1}{2}R\delta^{\nu}_{\mu} = g^{\nu\rho}\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} - \frac{1}{2}\mathscr{L}\delta^{\nu}_{\mu} \ , \tag{4.3}$$

together with the field equations for $\phi_{\alpha_1\ldots\alpha_n}$:

$$\nabla_{\mu}\frac{\partial\mathscr{L}}{\partial(\nabla_{\mu}\phi_{\alpha_1\ldots\alpha_n})} - \frac{\partial\mathscr{L}}{\partial\phi_{\alpha_1\ldots\alpha_n}} \ . \tag{4.4}$$

We also assume that the pair of function $(\bar{f}, \bar{g})$ (not necessarily reglular) solves the system (4.2)-(4.3). Finally, we define

$$f(r) \equiv \bar{f}(r) + \frac{c}{r} \ , \tag{4.5}$$

$$g(r) \equiv \bar{g}(r) + \frac{c}{r} \ . \tag{4.6}$$

In both cases, using the same constant c. It is indeed easy to prove that, if the constants were different, the pair (f, g) would not be a solution of the system.

Finally, we write the statement of the theorem.

The following statements are equivalent:

$$\begin{aligned} &\text{(a) } (f, g) \textit{ is a solution;} \\ &\text{(b) } f = g \ ; \\ &\text{(c) } \partial_{(\bar{f},\bar{g})}\mathscr{L} = 0 \ . \end{aligned} \tag{4.7}$$

This means that the solution has a Schawrzschild-like singularity as soon as either $g_{00}g_{11} = -1$ or the Lagrangian does not depend on the solution itself. It may be quite relevant, since it imposes some constraints on any further RBH model research. As we will see later in this chapter, some examples of applications are actually provided.

Finally, before proceeding with the proof, it is worth to notice that having some $1/r$ term into the solution is enough to make it singular: indeed, it badly violates the Sakharov criterion. Moreover, while approaching to the origin, if no other singularity is present, the solution becomes asymptotically equivalent to Schwarzschild – and we already know that Schwarzschild solution is singular in the origin.

4.3 Lemma

Before proceeding with the proof of the theorem itself, we discuss a preparatory lemma:

The matter Lagrangian $\mathscr{L}(\phi_{\alpha_1\ldots\alpha_n}, \nabla\phi_{\alpha_1\ldots\alpha_n})$ does not depend on the solution (f,g) if and only if the SET T^ν_μ does not depend on (f,g) too:

$$\partial_{(f,g)}\mathscr{L} = 0 \qquad \Longleftrightarrow \qquad \partial_{(f,g)}T^\nu_\mu = 0 \ . \tag{4.8}$$

We check the implication $\Longrightarrow$ by a direct calculation. Assuming $\partial_{(f,g)}\mathscr{L} = 0$, we have that

$$\begin{aligned}\partial_g T^\nu_\mu &= \partial_g \left(g^{\nu\rho}\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} - \frac{1}{2}\mathscr{L}\delta^\nu_\mu \right) \\ &= \partial_g g^{\nu\rho}\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} + g^{\nu\rho}\partial_g\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} - \frac{1}{2}\partial_g\mathscr{L}\delta^\nu_\mu \\ &= g^{\nu\rho}\partial_g\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} + \left(\delta^\nu_1\delta^1_\mu - \frac{1}{2}\delta^\nu_\mu\right)\partial_g\mathscr{L} \\ &= 0 \ ,\end{aligned} \tag{4.9}$$

since $\partial_{(f,g)}\mathscr{L} = 0$. A smilar proof holds for $\partial_f T^\nu_\mu$:

$$\begin{aligned}\partial_f T^\nu_\mu &= \partial_f \left(g^{\nu\rho}\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} - \frac{1}{2}\mathscr{L}\delta^\nu_\mu \right) \\ &= 0 \ .\end{aligned} \tag{4.10}$$

The first part of the lemma has been proved. The proof of the implication $\Longleftarrow$ is even simpler, since, by definition of T^ν_μ we have

$$g^{\nu\rho}\partial_f\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} + \left(\delta^\nu_0\delta^0_\mu - \frac{1}{2}\delta^\nu_\mu\right)\partial_f\mathscr{L} = 0 \ , \tag{4.11}$$

$$g^{\nu\rho}\partial_g\frac{\partial\mathscr{L}}{\partial g^{\mu\rho}} + \left(\delta^\nu_1\delta^1_\mu - \frac{1}{2}\delta^\nu_\mu\right)\partial_g\mathscr{L} = 0 \ . \tag{4.12}$$

Eq. (4.11)-(4.12) hold for any choice of the indices μ,ν. If we ask, in both equations, $\mu = \nu = 3$, due to the spherical symmetry the two equation read respectively

$$-\frac{1}{2}\partial_f\mathscr{L} = 0 \ , \tag{4.13}$$

$$-\frac{1}{2}\partial_g\mathscr{L} = 0 \ . \tag{4.14}$$

However, since eq. (4.11)-(4.12) hold also for $\mu = \nu$, we immediately find $\partial_{(f,g)}\mathscr{L} = 0$. Thus also the second implication has been proved.

4.4 Proof of the theorem – part one

We start the proof proving the equivalence of statements (a) and (b): this means proving that the general solution of system (4.2)-(4.3) has a Schwarzschild-like singularity if and only if $f = g$:

$$(f, g) \text{ is a solution} \qquad \Longleftrightarrow \qquad f = g \; . \tag{4.15}$$

In order to show it, let's discuss separately the two implications.

4.4.1 (f, g) is a solution $\Rightarrow$ $f = g$

Consider the EE. Writing them explicitly both for $(\bar{f}, \bar{g})$ and (f, g), we have that

$$\frac{\bar{g}' r + \bar{g} - 1}{r^2} = T_0^0(\bar{f}, \bar{g}) \; , \tag{4.16}$$

$$\frac{\bar{f}' \bar{g} r + \bar{f}\bar{g} - \bar{f}}{\bar{f} r^2} = T_1^1(\bar{f}, \bar{g}) \; , \tag{4.17}$$

$$\frac{2\bar{f}'' \bar{f} \bar{g} r + \bar{f}' \bar{g}' \bar{f} r + 2\bar{f}^2 \bar{g}' + 2\bar{f}' \bar{f} \bar{g} - \bar{f}'^2 \bar{g} r}{4\bar{f}^2 r} = T_2^2(\bar{f}, \bar{g}) \; , \tag{4.18}$$

and also that

$$\frac{\bar{g}' r + \bar{g} - 1}{r^2} = T_0^0(f, g) \; , \tag{4.19}$$

$$\frac{\bar{f}\left(\bar{f}' \frac{\bar{g}}{\bar{f}} r + \bar{g} - 1\right) + \frac{c}{r}\left(\bar{f}' r + \bar{f} - 1\right)}{\left(\bar{f} + \frac{c}{r}\right) r^2} = T_1^1(f, g) \; , \tag{4.20}$$

$$\begin{aligned}
&\frac{2\bar{f}'' \bar{f} \bar{g} r + \bar{f}' \bar{g}' \bar{f} r + 2\bar{f}^2 \bar{g}' + 2\bar{f}' \bar{f} \bar{g} - \bar{f}'^2 \bar{g} r}{4\left(\bar{f} + \frac{c}{r}\right)^2 r} + \\
&+ \frac{c}{r} \frac{\left(2\bar{f}'' r + 4\bar{f}' + \frac{2}{r}\bar{f} + \frac{c}{r^2}\right)\bar{g} + \left(\bar{f}' r + 3\bar{f} + \frac{c}{r}\right)\bar{g}'}{4\left(\bar{f} - + \frac{c}{r}\right)^2 r} + \\
&+ \frac{c}{r^2} \frac{2\left(\bar{f}'' + \frac{2c}{r^3}\right)\left(\bar{f} + \frac{c}{r}\right) r + \left(\bar{f}' - \frac{c}{r^2}\right)\left(\bar{f} + \frac{c}{r}\right) r}{4\left(\bar{f} + \frac{c}{r}\right)^2 r} + \\
&- \frac{c}{r^2} \frac{2\left(\bar{f} + \frac{c}{r}\right)^2 + \left(\bar{f}' - \frac{c}{r^2}\right)^2 r^2}{4\left(\bar{f} + \frac{c}{r}\right)^2 r} = T_2^2(f, g) \; .
\end{aligned} \tag{4.21}$$

If we subtract side by side eq. (4.16) to (4.19), we find that

$$T_0^0(f, g) - T_0^0(\bar{f}, \bar{g}) = 0 \; . \tag{4.22}$$

In order this equation to be holding, it is necessary T_0^0 being constant with respect to c. Still this does not mean that $\partial_{(f,g)}\mathscr{L} = 0$, since we also need to prove T_j^j, with $j = 1, 2, 3$ does not depend on c too (and this turns to be a more difficult task). However we can still refer to the lemma, since eq. (4.11) and (4.12) actually hold, though only for $\mu, \nu = 0$. With this choice, they read

$$-f\partial_f^2\mathscr{L} + \frac{1}{2}\partial_f\mathscr{L} = 0 \ , \tag{4.23}$$

$$-f\partial_f\partial_g\mathscr{L} + \frac{1}{2}\partial_g\mathscr{L} = 0 \ . \tag{4.24}$$

Solving eq. (4.23) and then integrating, we find that either $\partial_f\mathscr{L} = 0$ or

$$\mathscr{L}(f, g) = -\frac{2A}{\sqrt{f}} \ , \tag{4.25}$$

where A is something not depending on f. Inserting this result into eq. (4.24), after some manipulations one finds $\partial_g A = 0$, which simply means that $\partial_g\mathscr{L} = 0$.

Now, in order to show that $\partial_f\mathscr{L} = 0$ is the only admissible choice, we only need to check if Lagrangian (4.25) violets some of our assumptions.

Inserting Lagrangian (4.25) into the tt and the rr components of the EE, we solve them with respect to $(\bar{f}, \bar{g})$:

$$\frac{\bar{g}'r + \bar{g} - 1}{r^2} = 0 \ , \tag{4.26}$$

$$\frac{\bar{f}'\bar{g}'r + \bar{f}(\bar{g} - 1)}{\bar{f}r^2} = \frac{A}{\sqrt{\bar{f}}} \ . \tag{4.27}$$

The first equation gives easily $\bar{g}(r) = 1 - a/r$, where a is a constant. Substituting it into the second and introducing the new variable $X \equiv \sqrt{\bar{f}}$, we find a linear equation:

$$2X' - \frac{a}{1 - \frac{a}{r}}X = \frac{Ar}{1 - \frac{a}{r}} \tag{4.28}$$

whose solution is finally

$$\bar{f}(r) = \left(1 - \frac{a}{r}\right)\left(1 + \frac{A}{2}\int \frac{r}{\left(1 - \frac{a}{r}\right)^{3/2}} dr\right)^2 \tag{4.29}$$

In a similar way, we solve the tt and rr components of the EE with repsect to (f, g): we have

$$f(r) = \left(1 - \frac{b}{r}\right)\left(1 + \frac{A}{2}\int \frac{r}{\left(1 - \frac{b}{r}\right)^{3/2}} dr\right)^2 \ , \tag{4.30}$$

$$g(r) = 1 - \frac{b}{r} \; , \tag{4.31}$$

where $b = a - c$. Notice that we are allowed to play this trick since eq. (4.26) and (4.27) have the same form in both cases.

If Lagrangian (4.25) fulfilled our assumptions, we should have $f(r) - \bar{f}(r) = c/r$; and we prove that this is not the case. Indeed,

$$\begin{aligned}
&f(r) - \bar{f}(r) = \\
&= \left(1 - \frac{b}{r}\right)\left(1 + \frac{A}{2}\int \frac{r}{\left(1-\frac{b}{r}\right)^{3/2}}dr\right)^2 - \left(1 - \frac{a}{r}\right)\left(1 + \frac{A}{2}\int \frac{r}{\left(1-\frac{a}{r}\right)^{3/2}}dr\right)^2 \\
&= \frac{c}{r} + \frac{Ac}{2r}\int \frac{r}{\left(1-\frac{a-c}{r}\right)^{3/2}}dr\left(2 + \frac{A}{2}\int \frac{r}{\left(1-\frac{a-c}{r}\right)^{3/2}}dr\right) + \\
&+ \frac{Ac}{r}\left(1 - \frac{a}{r}\right)\int r\frac{\left(1-\frac{a}{r}\right)^{3/2} - \left(1-\frac{a-c}{r}\right)^{3/2}}{\left(1-\frac{a}{r}\right)^{3/2}\left(1-\frac{a-c}{r}\right)^{3/2}}dr + \\
&+ \frac{A^2c}{4r}\left(1 - \frac{a}{r}\right)\left(\left(\int \frac{r}{\left(1-\frac{a-c}{r}\right)^{3/2}}\right)^2 - \left(\int \frac{r}{\left(1-\frac{a}{r}\right)^{3/2}}\right)^2\right)
\end{aligned} \tag{4.32}$$

It is not difficult to see, at this point, that $f(r) - \bar{f}(r) = c/r$ only in the trivial case of $A = 0$. In any other case, the extra term still survives, since c (which is not fixed!) appears in a highly non linear way into the integrals: it is even possible that $f(r) - \bar{f}(r) = c/r$ with $A \neq 0$, for some specific values of c, but not in general. So, eq. (4.32) states that Lagrangian (4.20) violates one of our basic assumption, namely the request that $f(r) - \bar{f}(r) = c/r$. It follows that Lagrangian (4.25) is not acceptable and finally that $\partial_{f,g}\mathscr{L} = 0$.

This result is not the end of the proof, however we are almost done. The condition for having $f = g$ within the framework of eq. (4.3), is just to have $T^0_0 = T^1_1$. But, since we proved that $\partial_{(f,g)}\mathscr{L} = 0$, this implies that

$$T^0_0 = -f\partial_f\mathscr{L} - \frac{1}{2}\mathscr{L} = \frac{1}{2}\mathscr{L} = g\partial_g\mathscr{L} - \frac{1}{2}\mathscr{L} = T^1_1 \; . \tag{4.33}$$

This means that $G^0_0 = G^1_1$ and consequently that $g'/g = f'/f$ and consequently that $f = g$. The first part of the equivalence is done.

4.4.2 (f, g) is a solution $\Leftarrow$ $f = g$

This second proof is much simpler and shorter than the previous one. First of all, since we required c to be the same for f and g, we have that $f = g$ implies $\bar{f} = \bar{g}$ and viceversa. If so, the independent EE reduce to

$$\frac{\bar{f}'r + \bar{f} - 1}{r^2} = T_0^0(\bar{f}) \ , \tag{4.34}$$

$$\frac{\bar{f}'' + 2\bar{f}'}{2r} = T_2^2(\bar{f}) \ . \tag{4.35}$$

On the other hand, if we calculate the Einstein tensor for f instead of $\bar{f}$, we have

$$\begin{aligned} G_0^0 &= \frac{\left(\bar{f}' - \frac{c}{r^2}\right) r + \left(\bar{f} + \frac{c}{r}\right) - 1}{r^2} \\ &= \frac{\bar{f}'r + \bar{f} - 1}{r^2} \end{aligned} \tag{4.36}$$

$$\begin{aligned} G_2^2 &= \frac{\left(\bar{f}'' + \frac{2c}{r^3}\right) r + 2\left(\bar{f}' + \frac{c}{r^2}\right) - 1}{2r} \\ &= \frac{\bar{f}''r + 2\bar{f}'}{2r} \ . \end{aligned} \tag{4.37}$$

Since these are the only independent components of the Einstein tensor, it is easy to conclude that $G_\mu^\nu(\bar{f}) = G_\mu^\nu(f)$. In order to complete the proof, showing so that f is also a solution, we only need to prove that the same holds for the SET. But $T_\mu^\nu(\bar{f}) = T_\mu^\nu(f)$ means that $\partial_f T_\mu^\nu = 0$ and this, according to the lemma, just requires to show that the Lagrangian does not depend on the solution: $\partial_f \mathscr{L} = 0$. Actually this happens, since $f = g$ requires, as we said, that $T_0^0 = T_1^1$ (this is why we have only two independent EE, instead of three). Explicitly, it reads

$$-f\partial_f\mathscr{L} - \frac{1}{2}\mathscr{L} = f\partial_f\mathscr{L} - \frac{1}{2}\mathscr{L} \qquad \Longrightarrow \qquad -2f\partial_f\mathscr{L} = 0 \ . \tag{4.38}$$

Since $f = 0$ is not an acceptable solution, we have $\partial_f\mathscr{L} = 0$ and so, due to the lemma, we follow the whole path in reverse, thus completing the proof.

4.5 Proof of the theorem – part two

Since we already showed the equivalence among statements (a) and (b), in order to complete the proof we only need to show that statements (a) and (c) or (b) and (c) are equivalent; in both cases, the remaining equivalence will follow immediately.

Thus said, since the discussion is simpler, we prove the equivalence among statements (b) and (c). This means we need to prove that $f = g$ if and only if the Lagrangian depends neither on $\bar{f}$ nor on $\bar{g}$, that is

$$f = g \qquad \Longleftrightarrow \qquad \partial_{(\bar{f},\bar{g})}\mathscr{L} = 0 \ . \tag{4.39}$$

Once again, discuss separately the two implications.

4.5.1 $\partial_{(\bar{f},\bar{g})}\mathscr{L} = 0 \;\Rightarrow\; \bar{f} = \bar{g}$

If $\partial_{(\bar{f},\bar{g})}\mathscr{L} = 0$, the T_0^0 and T_1^1 components of the SET become

$$T_0^0 = -\bar{f}\partial_{\bar{f}}\mathscr{L} - \frac{1}{2}\mathscr{L} = -\frac{1}{2}\mathscr{L} \ , \tag{4.40}$$

$$T_1^1 = \bar{g}\partial_{\bar{g}}\mathscr{L} - \frac{1}{2}\mathscr{L} = -\frac{1}{2}\mathscr{L} \ . \tag{4.41}$$

i.e. $T_0^0 = T_1^1$; and since $(\bar{f},\bar{g})$ is a solution of the system (4.3)-(4.4), we know that this is a sufficient condition for having $\bar{f} = \bar{g}$.

4.5.2 $\partial_{(\bar{f},\bar{g})}\mathscr{L} = 0 \;\Leftarrow\; \bar{f} = \bar{g}$

If $\bar{f} = \bar{g}$, then $T_0^0 = T_1^1$, because the condition is necessary and sufficient. But this implies that, as we said in the first part of the proof,

$$-f\partial_f\mathscr{L} - \frac{1}{2}\mathscr{L} = f\partial_f\mathscr{L} - \frac{1}{2}\mathscr{L} \qquad \Longrightarrow \qquad -2f\partial_f\mathscr{L} = 0 \ . \tag{4.42}$$

Unless the trivial (but not acceptable) case of $\bar{f} = 0$, this means that the Lagrangian does not depend on the solution, so the proof is completed. Using the equivalence among (a) and (b), we prove also the remaining equivalence.

4.5.3 Sketch of a proof for the equivalence among (a) and (c)

We close this section, sketching a proof of the equivalence among (a) and (c). This is not necessary, since equivalence among (a) and (b) and among (b) and (c) are enough to close the chain of implications, but this attempt can be viewed as a possible alternative proof.

We start with (f,g) is a solution $\Rightarrow \partial_{(\bar{f},\bar{g})}\mathscr{L} = 0$. As we showed, this implies that $\partial_f T_0^0 = 0$; and this implies that $\partial_{\bar{g}}\mathscr{L} = 0$ and either $\partial_{\bar{f}}\mathscr{L} = 0$, or $\mathscr{L} = -2A/\sqrt{f}$. We are able to discard this second option, since it violates the request that $f(r) - \bar{f}(r) = c/r$, using the same procedure we adopted before; we are left with the first choice, namely $\partial_f\mathscr{L} = 0$ and this completes the proof.

(Notice that we followed the same way we traced proving the equivalence among (a) and (b), but we did it independently, without formulating the statement (4.15).)

On the other hand, showing that (f,g) is a solution $\Leftarrow \partial_{(\bar{f},\bar{g})}\mathscr{L} = 0$ is much simpler and direct. Indeed, as we did, this implies that $\bar{f} = \bar{g}$ and consequently, repeating the demonstration of the $\Longleftarrow$ of the first equivalence, we arrive at the conclusion.

4.6 Corollary

Together with the general assumptions, we also assume that $\bar{f} = \bar{g}$ (we could also consider the other two statements, but, since they are equivalent, one is enough). If so, the general solution of the system has no more than one non-Lagrangian singularity and this is Schwarzschild-like: i.e., if $\bar{f}$ is a solution and h a singular function, then $\bar{f} + h$ is a solution if and only if $h(r) = c/r$:

$$f \equiv \bar{f} + h \text{ is a solution} \qquad \Longleftrightarrow \qquad h(r) = \frac{c}{r} \tag{4.43}$$

Notice that here we deal only with singularities, which are not generated by the Lagrangian. It may happen, as is the case of the Reissner-Nordström solution, that the solution contains other kinds of singularities, but they are always related to a specific choice of the matter Lagrangian.

Just for notational convenience, we isolate an explicit c/r term into $h(r)$: so we write it $h(r) = c/r + \tilde{h}(r)$, assuming that $\tilde{h}$ does not contain any term proportional to $1/r$ (but other divergent terms are still included). The corollary now states that $\bar{f} + c/r + \tilde{h}$ is a solution if and only if $\tilde{h} = 0$.

4.6.1 $\bar{f} + \frac{c}{r} + \tilde{h}$ is a solution $\Rightarrow$ $\tilde{h}(r) = 0$

Since we are assuming the basic assumptions to hold and that $\bar{f} = \bar{g}$, the theorem guarantees that also $\bar{f} + c/r$ and $\bar{f} + \tilde{h}$ are solutions. The first one, because $\bar{f}$ is a solution; the second one, because $\bar{f} + c/r + \tilde{h}$ is a solution (and since the constant c is free, also choosing $-c$ is admissible). So, wirte the independent EE both for $\bar{f}$ and for $\bar{f} + \tilde{h}$ respectively:

$$\frac{\bar{f}'r + \bar{f} - 1}{r^2} = T^0_0(\bar{f}) \tag{4.44}$$

$$\frac{\bar{f}''r + 2\bar{f}'}{2r} = T^2_2(\bar{f}) \tag{4.45}$$

and also

$$\frac{\bar{f}'r + \bar{f} - 1}{r^2} + \frac{\tilde{h}'r + \tilde{h}}{r^2} = T^0_0(\bar{f} + \tilde{h}) \tag{4.46}$$

$$\frac{\bar{f}''r + 2\bar{f}'}{2r} + \frac{\tilde{h}''r + 2\tilde{h}'}{2r} = T^2_2(\bar{f} + \tilde{h}) \tag{4.47}$$

Subtracting side to side the respective pairs, we are left with

$$\frac{\tilde{h}'r + \tilde{h}}{r^2} = T^0_0(\bar{f} + \tilde{h}) - T^0_0(\bar{f}) \tag{4.48}$$

$$\frac{\tilde{h}''r + 2\tilde{h}'}{2r} = T^2_2(\bar{f} + \tilde{h}) - T^2_2(\bar{f}) \tag{4.49}$$

Since $\tilde{h}$ does not contain any term in c/r, if $\tilde{h}$ were not vanishing, the lhs of both eq. (4.48) and (4.49) do not vanish, so the rhs. As consequence, $\partial_{\bar{f}} T^{\nu}_{\mu} \neq 0$ and so, due to the lemma, $\partial_{\bar{f}} \mathscr{L} \neq 0$: this is a violation of statement (b), which has been proved to be equivalent to $f = g$. Consequently, we must require $\tilde{h} = 0$.

4.6.2 $\bar{f} + \frac{c}{r} + \tilde{h}$ is a solution $\Leftarrow$ $\tilde{h}(r) = 0$

Since $\bar{f}$ is a solution and the theorem holds, then also $\bar{f} + c/r$ is a solution; which is our implication, since $\tilde{h} = 0$.

4.7 The theorem in more dimensions

The theorem and the proof we presented here strongly rest on the dimensionality $d = 4$ of spacetime. Actually this is a very realistic assumption, since any extra dimension is expected to be physically manifest only at a very small scale. However it still deserves interest looking what happens in a more general framework, with a higher dimensionality: indeed extra dimensions play a very relevant role in a number of theories of modified or quantum gravity. In this section we show that the lemma, the theorem and the corollary still hold also within a higher dimensionality.

Notice that we will only discuss static and spherically symmetric spacetimes, just making use of the extedend notion of spherical symmetry. Notice also that we won't discuss the whole proofs, but just their highlights, especially when they differ from the original 4D case.

The assumptions, the statement and the lemma don't need any further discussion: indeed, we proved them in a general form, so that they do not rest on the number of dimensions. We just require to split the metric into a temporal and radial part plus an angular part, but this comes from the request of spherical symmetry:

$$ds^2 = g^{ab} dx^a dx^b + r^2 dS_n^2 \ , \tag{4.50}$$

where dS_n^2 is the surface of a sphere of unit radius in n dimensions, r is the areal radius and $a, b = 0, 1$. We only need to recall that the EE have n instead of 4 components and that the Schwarzschild solutions in a general dimensionality n is $f(r) = 1 - \frac{c}{r^{n-3}}$; this modifies the definition of (f, g) from $(\bar{f}, \bar{g})$ in an obvious way.

4.7.1 Proof of the theorem – part one

We start the proof proving the equivalence of statements (a) and (b): as in the 4D case, this means proving that the general solution of system (4.2)-(4.3) has a Schwarzschild-like singularity if and only if $f = g$:

$$(f, g) \text{ is a solution} \qquad \Longleftrightarrow \qquad f = g \ . \tag{4.51}$$

The $\Longleftarrow$ implication is done exactly as in the 4D case, since the condition for having $f = g$ is still $\partial_{f,g}\mathscr{L} = 0$ and the lemma holds.

The $\Longrightarrow$ implication follows more or less the same path. Let us write the tt component of the EE for $(\bar{f}, \bar{g})$ and (f, g) respectively:

$$\frac{n-2}{2r^2}\left(\bar{g}'r + (n-3)(\bar{g}-1)\right) = \bar{T_0^0} \ , \tag{4.52}$$

$$\frac{n-2}{2r^2}\Big(\Big(\bar{g}' - (n-3)\frac{c}{r^{n-4}}\Big) r + (n-3)(\bar{g} + \frac{c}{r^{n-3}} - 1)\Big) = T_0^0 \ . \tag{4.53}$$

These two equations mean that $\partial_{(f,g)}T_0^0 = 0$ ($\partial_f T_0^0 = 0$ is trivial, since the lhs of eq. (4.52) does not depend on f). At this stage we are in the same position after eq. (4.22). However, since the lemma holds without any change, we are able to perform the same discussion and prove that $\partial_{f,g}\mathscr{L} = 0$. This condition implies that $T_0^0 = T_1^1$ and finally that $f = g$.

4.7.2 Proof of the theorem – part two

After we proved the equivalence among statements (a) and (b), we prove the equivalence among statements (b) and (c). After that, the chain of implications will be easily closed. This means to prove that $f = g$ if and only if the Lagrangian depends neither on $\bar{f}$ nor on $\bar{g}$, i.e.

$$f = g \qquad \Longleftrightarrow \qquad \partial_{(\bar{f},\bar{g})}\mathscr{L} = 0 \ . \tag{4.54}$$

Once again, discuss separately the two implications.

$\Longrightarrow$ If so, $\bar{G_0^0} = \bar{G_1^1}$ and this implies $\bar{T_0^0} = \bar{T_1^1}$. However this implies that $\partial_{\bar{f}}\mathscr{L} = 0$, exactly as we discussed in the 4D case.

$\Longleftarrow$ If so, as we showed in the 4D case, $\bar{T_0^0} = \bar{T_1^1}$ and, as we already discussed, this is the condition to have $\bar{f} = \bar{g}$.

4.7.3 Corollary

Assuming, together with the general assumptions, that $\bar{f} = \bar{g}$ (we could also consider the other two statements). If so, the general solution of the system has no more than one non-Lagrangian singularity and this is Schwarzschild-like: i.e., if $\bar{f}$ is a solution and h a singular function, then $\bar{f} + h$ is a solution if and only if $h(r) = c/r^{n-3}$. As in the 4D case, here we deal only with singularities, which are not generated by the Lagrangian.

Just for notational convenience, we isolate an explicit c/r^{n-3} term into $h(r)$: so we write it $h(r) = c/r^{n-3} + \tilde{h}(r)$, assuming that $\tilde{h}$ does not contain any term proportional to $1/r^{n-3}$ (but other divergent terms are still included). The corollary now states that $\bar{f}+c/r^{n-3}+\tilde{h}$ is a solution if and only if $\tilde{h}=0$.

$\Longrightarrow$ Since we are assuming the basic assumptions to hold and that $\bar{f}=\bar{g}$, the equivalence among (a) and (b) states that also $\bar{f}+c/r^{n-3}$ and $\bar{f}+\tilde{h}$ are solutions. So, wirte the G_t^t component fo the EE both for $\bar{f}$ and for $\bar{f}+\tilde{h}$ respectively:

$$(n-2)\frac{\bar{f}'r+(n-3)(\bar{f}-1)}{2r^2}=T_0^0(\bar{f}) \tag{4.55}$$

$$(n-2)\frac{\bar{f}'r+(n-3)(\bar{f}-1)}{2r^2}+(n-2)\frac{\tilde{h}'r+(n-3)\tilde{h}}{2r^2}=T_0^0(\bar{f}+\tilde{h}) \tag{4.56}$$

Subtracting side to side, we are left with

$$(n-2)\frac{\tilde{h}'r+(n-3)\tilde{h}}{2r^2}=T_0^0(\bar{f}+\tilde{h})-T_0^0(\bar{f}) \tag{4.57}$$

Since $\tilde{h}$ does not contain any term in c/r^{n-3}, a non vanishing $\tilde{h}$ makes the lhs of eq. (4.57) non vanishing too, and so the rhs. As consequence, $\partial_{\bar{f}}T_\mu^\nu\neq 0$ and so, due to the lemma, $\partial_{\bar{f}}\mathscr{L}\neq 0$: this is a violation of the equivalence among (a) and (c). Consequently, we must require $\tilde{h}=0$.

$\Longleftarrow$ Since $\bar{f}$ is a solution and the equivalence among (a) and (b) holds, then also $\bar{f}+c/r^{n-3}$ is a solution; which is our implication, since $\tilde{h}=0$.

4.8 Special cases

We studied the theorem both in four dimensions and in the case of more dimensions. We assumed to have a general number of dimensions, so even lower, but it's easy to see, from the results and the proof, that our arguments hold only for $n\geq 4$. One may wonder on what happens for $n=1,2,3$. In this section we want to look briefly what happens in these special cases, although they may be of minor interest, since the universe has no less than four dimensions.

The first thing we notice is that the lemma holds automatically also for $n=1,2,3$. Indeed, the general proof of the lemma does not involve any explicitation of the dimensionality of the problem. When we set a special number for the indices, in eq. (4.13) and (4.14) it was only an application of the spherically symmetric background (the extension to a lower dimensional case follows quite naturally). The only thing we need to be careful with is to ignore the part involving g while discussing the case of $n=1$.

4.8.1 $n=1$

The situation is trivial. Metric indeed is $ds^2=-f(x)dx^2$, which can easily be reabsorbed into the new variable $X\equiv\frac{2}{3}f^{3/2}(x)$. Since we have a single variable, playing the role of a time, this choice is admissible and we get Minkowski.

No surprise: $n = 1$ indicates the time evolution alone of a point-like space; which evidently cannot possess a proper geometry.

4.8.2 $n = 2$

Here, things are different. First of all, the metric looks like

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 \ . \tag{4.58}$$

We assume that this equation is generated by the same minimally coupled action of the general theorem, just integrating on two variables. It is not difficult to see that the EE are

$$0 = -\bar{f}\frac{\partial \mathscr{L}}{\partial \bar{f}} - \frac{1}{2}\mathscr{L} \ , \tag{4.59}$$

$$0 = \bar{g}\frac{\partial \mathscr{L}}{\partial \bar{g}} - \frac{1}{2}\mathscr{L} \ , \tag{4.60}$$

where the upper bar has the same meaning as in the rest of the chapter. These equations hold for *every* choice of the matter Lagrangian. It is clear that $G^\nu_\mu = \bar{G}^\nu_\mu$. If (f, g) are also solutions, this means that $T^\nu_\mu = \bar{T}^\nu_\mu$, i.e. $\partial_{(f,g)}T^\nu_\mu = 0$. Consequently, due to the lemma, we can state that $\partial_{(f,g)}\mathscr{L} = 0$: but this means that $T^0_0 = T^1_1$, i.e. the necessary condition to have $f = g$. So the $\Rightarrow$ of the equivalence among (a) and (b) has been proved.

On the other hand, if $\bar{f} = \bar{g}$, we still have $G^\nu_\mu = 0$, since this holds independently on the specific choice of the pair (f, g). This states that $G^\nu_\mu = \bar{G}^\nu_\mu = \bar{T}^\nu_\mu$; moreover, subtracting side to side eq. (4.59) and (4.60), we have that

$$2\bar{f}\frac{\partial \mathscr{L}}{\partial \bar{f}} = 0 \ , \tag{4.61}$$

so that $\partial_{\bar{f}}\mathscr{L} = 0$; due to the lemma, this means that $\bar{T}^\nu_\mu = T^\nu_\mu$ and this completes also the proof for the $\Leftarrow$ implication.

Also the equivalence among (b) and (c) holds, since $\bar{f} = \bar{g}$ requires $\partial_{\bar{f}}\mathscr{L} = 0$, as we saw. On the other hand, $\partial_{\bar{f}}\mathscr{L} = 0$ means that $\bar{T}^0_0 = \bar{T}^1_1$ and this is the condition to have $\bar{f} = \bar{g}$. The proof follows using the same argument we already discussed in the $n \geq 4$ case.

However, there is a difference: contrary to the $n \geq 4$ case, indeed, the corollary does not hold. Indeed, the only vacuum solution for $n = 2$ is Minkowski, so that there is no singularity: the statements of the corollary is not well posed in this framework.

4.8.3 $n = 3$

In this case, the metric is

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\theta^2 \tag{4.62}$$

Once again, we assume that this metric is generated by the minimally coupled action of the main theorem, just integrating on three variables. It is not difficult to see that the resulting EE are

$$\frac{g'}{2r} = -f\frac{\partial \mathscr{L}}{\partial f} - \frac{1}{2}\mathscr{L} \ , \tag{4.63}$$

$$\frac{f'g}{2fr} = g\frac{\partial \mathscr{L}}{\partial g} - \frac{1}{2}\mathscr{L} \ , \tag{4.64}$$

$$\frac{g}{2f}\left(f'' + \frac{f'g'}{2g} - \frac{f'^2}{2f}\right) = -f\frac{\partial \mathscr{L}}{\partial f} - \frac{1}{2}\mathscr{L} \ . \tag{4.65}$$

As seen from eq. (4.63) and (4.64), the vacuum solution is once again $f, g = 1$. We prove the theorem in the same way as we did for $n = 2$.

In order to prove the $\Rightarrow$ direction of the equivalence among (a) and (b),we see, from eq. (4.63), that $T^0_0 = \bar{T}^0_0$. From now on, the proof proceeds as in the case of $n \geq 4$: we set $\mu = \nu = 2$ in the final lemma equations (we have now two spatial dimensions), having so $-f\partial^2_f\mathscr{L} + \frac{1}{2}\partial_f\mathscr{L} = 0$ and $-f\partial_f\partial_g\mathscr{L} + \frac{1}{2}\partial_g\mathscr{L} = 0$. These equation impose the Lagrangian to give either $\partial_f\mathscr{L} = 0$ or $\partial_f\mathscr{L} \propto f^{-3/2}$; this second object prodeuces a solution, which violates the request of f and $\bar{f}$ to differ only for a vacuum term and so should be discarded (as in the $n \geq 4$ case). We have then $\partial_{(f,g)}\mathscr{L} = 0$, which implies $T^0_0 = T^1_1$, which implies $f = g$.

The implication $\Leftarrow$ is proved by recalling that, when $\bar{f} = \bar{g}$, then $\bar{G}^0_0 = \bar{G}^1_1 = \bar{f}'2r$; and since $\bar{f}$ is a solution and the free solution is $f_{\text{free}} = 1$, we immediately have that $\bar{G}^0_0 = G^0_0$ and $\bar{G}^1_1 = G^1_1$, so that eq. (4.63) holds also for f and not only for $\bar{f}$. This completes the proof.

The proof of the equivalence among (b) and (c) proceeds as in the case of $n = 2$. In the same way and for the same reason, we see that the corollary is not well posed, so we won't discuss it.

4.9 The NED example

We already showed, in the previous chapter, that NED models are not suitable in producing RBHs, since a Schwarzschild term always comes out. Here we re-present those arguments, but we will interpret them in the light of our theorem. First, we recall that the NED action is written

$$\mathscr{I} = \int d^4x\ \sqrt{-[g]}(R + \mathscr{L}(I)) \ , \tag{4.66}$$

where $I \equiv \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the electromagnetic scalar and $F_{\mu\nu}$ the electromagnetic tensor; moreover, the spacetime is in the form (4.1). The assumptions of the theorem are then satisfied, then we only need to prove that this holds also for the three statements. Just to give an idea, we check the equivalence among (b) and (c). In order to do that, it is better to make use of the dual notation we introduced. So, assuming that the Lagrangian does not depend on the solution, the tt and rr components of the EE read

$$\frac{g'}{r} + \frac{g-1}{r^2} = 2\left((\partial_P \mathscr{H}) P_{0\rho} P^{0\rho} - 2P\partial_P \mathscr{H}\right) \ , \tag{4.67}$$

$$\frac{f'g}{fr} + \frac{g-1}{r^2} = 2\left((\partial_P \mathscr{H}) P_{1\rho} P^{1\rho} + \mathscr{H} - 2P\partial_P \mathscr{H}\right) \ . \tag{4.68}$$

Subtracting them side by side, we get

$$\frac{f'g}{fr} - \frac{g'}{r} = 2(\partial_P \mathscr{H})\left(P_{0\rho} P^{0\rho} - P_{1\rho} P^{1\rho}\right) \ . \tag{4.69}$$

After a slight manipulation, eq. (4.69) reads

$$f'g - fg' = -\frac{2}{r}(\partial_P \mathscr{H})\left(P_{02}P_{02} + fgP_{12}P_{12} + \frac{1}{\sin^2\theta}(P_{03}P_{03} + fgP_{13}P_{13})\right) \ . \tag{4.70}$$

Since the lhs of the equation only depends on r, the term proportional to $1/\sin^2\theta$ should be zero. Moreover, since the electromagnetic tensor $F_{\mu\nu}$ and the matter Lagrangian do not depend on the solution, P_{03} and P_{13} vanish separately. We show that the other term is vanishing making first use of the spherical symmetry: since everything is expected to be static and spherically symmetric, we expect the non-radial components of the electric field to vanish, i.e. $F_{0\mu} \neq 0$ only for $\mu = 1$; this is transmitted also to $P_{\mu\nu}$, simply by definition: so $P_{02} = 0$. In order to show that also P_{12} vanishes, we make use of the Maxwell equations; in the dual notation, they read $\nabla_\mu P^{\mu\nu} = 0$. Fixing $\nu = 1$, it reduces to $\cot\theta P^{21} = 0$, so that the whole rhs of eq. (4.70) vanishes. The conclusion $f = g$ follows trivially.

On the other hand, assuming that $f = g$, the proof is even simpler: we only need to write explicitly the electromagnetic scalar:

$$\begin{aligned} I &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}F_{01}F^{01} \\ &= -\frac{1}{2}E_r^2 \ . \end{aligned} \tag{4.71}$$

Actually, this happens because we are dealing with a purely static situation, for which there is only a radial electric field E_r and since we already proved that $f = g$.

Before closing the section, we want to emphasize a remark: the theorem states that, under assumptions (4.1)-(4.6), statements (a), (b) and (c) are equivalent. In general, however, one is free to look for solutions, violating one of them (for instance, looking for $f \neq g$). In this case, since the theorem still holds, neither of the statements (a), (b) and (c) will be actually verified.

In the specific case of NED models, however, there is no hope to find a way out. Indeed, we are able to prove that $f = g$ only requiring the static and spherical symmetry, so that this configuration is always realized; and this provides the general solution always with an unavoidable Schwarzschild term.

4.10 Final remarks

We close this chapter with a few observations. First of all, during the whole discussion, we always dealt with static frameworks; however, at the beginning of the chapter, we noticed that the dynamical case does not make trouble. Indeed, if we write the dynamical tt and rr components of the Einstein tensor, we see that they are the same of the static case: thus the vacuum solution is still Schwarzschild, provided that the term c is constant also in time. The rest of the proof follows as in the static case, so we won't re-discuss it here.

A second remark concerns the presence of the Schwarzschild term in those black holes having $f = g$. The theorem states it holds for the general solution, but nothing forbids to a physical argument to be able to set $c = 0$. We will discuss an example later on in this work, within the final chapters.

A final remark is related to the NPG approach we will discuss in the following chapters. We will show that this approach is actually able to produce regular solutions, also satisfying the condition $f = g$. Apparently this represents a violation of our theorem, but actually this is not the case. Indeed, that Lagrangian does not satisfy action 4.2, since it cannot be written as a minimal coupling among gravity and matter: actually, NPG approach is a vacuum theory, although corrections are written as $\mathscr{L} = R + \mathscr{L}_{corr}$, any correction is genuinely geometric. Thus the assumptions of the theorem are violated and consequently it is possible for it to build RBHs with $f = g$.

5 Some examples of working RBHs

5.1 Introduction

In the previous chapters, after giving a brief look on the various notions of regularity, we discussed most of all negative results, i.e. we showed under which conditions a RBH cannot be produced. In particular, in the third chapter we analyzed some models, which actually are not able to solve the singularity issue or that, at least, still contain a singular term. Conversely, in chapter 4 we presented and proved a general no-go theorem, applicable for any minimally coupled model.

In this chapter, we finally start doing something successful, i.e. we present some approaches, which are actually able to produce regular solutions. Of course we will violate, in way or another, the assumptions of the no-go theorem, thus dealing with a non-minimal coupling among gravity and matter, or a more complicated gravitational Lagrangian (introducing for example a Gauss-Bonnet term or other, more complicated, tensors).

It is also worth to notice, before proceeding, that violating the no-go theorem assumptions is not sufficient, in order to produce regular solutions. An example is provided by [69] and [134]: in these works, the Lagrangian is written in the form $\mathscr{L} = R - Y(R)I$, where $Y(R)$ is a given function of R, thus introducing a non-minimal coupling in the theory. This approach is still not able to produce a RBH, since the authors assume to fix the function $R(r)$, thus reducing their approach to GR coupled with some effective fluid density, once again opening the door to the Schwarzschild c/r term (in a similar way to the non-commutative effective fluid approach, we discussed in the second chapter).

5.2 The Balakin-Lemos-Zayats black hole

The first model we take into consideration has been recently proposed by A.B. Balakin, J.P.S. Lemos and A.E.Zayats in [67]. They built it from a non-minimal coupling among gravity and some Yang-Mills matter field. Although one may suppose no regular solution to be available in such a framework, the authors have been able not only to find one, but also an analytical one, which turned to be truly regular.

We only briefly recall that non-minimal field theories are based on five classes, divided accordingly to the types of fields that couple with gravity. The first class deals with the coupling of scalar field(s) with the spacetime curvature. They arose at first within the Scherrer–Jordan–Thiry–Brans-Dicke theory, in which a scalar field couples in a non minimal way to the Ricci scalar (see [135, 136]) for an historical perspective. The second class works on the modelling of non-minimal interactions of the electromagnetic field with curvature, usually called non-minimal Einstein-Maxwell models (see e.g. [137–140]). The third class contains Einstein–Yang-Mills models with an $SU(n)$ symmetry [141, 142]. The fourth class discusses Einstein–Yang-Mills–Higgs models [143, 144]. The fifth class covers models containing an axion pseudoscalar field, e.g. the coupling to the electromagnetic and gravitational fields in what may be called nonminimal Einstein-Maxwell-axion models [145]. In particular, the present model, living in the third class, investigates non-minimally coupled magnetic RBHs in the presence of a cosmological constant.

5.2.1 General formalism and main equations

The framework of the paper is a nonminimal Einstein–Yang-Mills theory with $SU(2)$ symmetry and a Wu-Yang ansatz; it turns to be a generalization of the non-minimal Einstein-Maxwell theory with $U(1)$ symmetry studied in [140]. The model is based on the action

$$S_{\text{NMEYM}} = \int d^4x\sqrt{-g}\left(\frac{R+2\Lambda}{8\pi} + \frac{1}{2}F^{(a)}_{ik}F^{ik(a)} + \frac{1}{2}\mathscr{R}^{ikmn}F^{(a)}_{ik}F^{(a)}_{mn}\right) \ , \quad (5.1)$$

where $g = \det(g_{ik})$ is the determinant of the metric g_{ik}, R is the Ricci scalar and Λ is the cosmological constant. Throughout this section we uniform our notation to that of [67], so that Latin indices without parentheses indicate a sum from 0 to 3, while Latin indices with parentheses are group indices, running from 1 to 3; when repeated, they should be summed with a Kronecker delta metric. Finally, we define the non-minimal susceptibility tensor $\mathscr{R}^{ikmn}$ as

$$\begin{aligned}\mathscr{R}^{ikmn} \equiv\ & \frac{q_1}{2}R(g^{im}g^{kn} - g^{in}g^{km}) \\ & + \frac{q_2}{2}(R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}) + q_3R^{ikmn} \ ,\end{aligned} \quad (5.2)$$

where R^{ik} and R^{ikmn} are the Ricci and the Riemann tensors respectively, while q_1, q_2, q_3 are three phenomenological parameters describing the non-minimal coupling of the Yang-Mills field with the gravitational one. Finally, the $SU(2)$ Yang-Mills field is described by a triplet of vector potentials $A^{(a)}_m$, generating the Yang-Mills tensor by the usual form

$$F^{(a)}_{mn} = \nabla_m A^{(a)}_n - \nabla_n A^{(a)}_m + f^{(a)}_{\cdot(b)(c)}A^{(b)}_m A^{(c)}_n \ , \quad (5.3)$$

where $f^{(a)}_{\cdot(b)(c)}$ are the structure constants of the group.

Now one can perform the variation of the action w.r.t. the gauge field and w.r.t. the metric function. The paper considers both variations, but here we are interested only in the EE, which read

$$R_{ik} - \frac{1}{2} R g_{ik} = \Lambda g_{ik} + 8\pi T^{(\text{eff})}_{ik} \ , \tag{5.4}$$

where $T^{(\text{eff})}_{ik}$ is an effective SET, which can be divided, just for convenience of notation, in four parts, namely

$$T^{(\text{eff})}_{ik} = T^{(YM)}_{ik} + q_1 T^{(I)}_{ik} + q_2 T^{(II)}_{ik} + q_3 T^{(III)}_{ik} \ . \tag{5.5}$$

Managing with action (5.1), one is able to find explicitly the four components of the tensor, in terms of the metric function: they turn to be

$$T^{(YM)}_{ik} \equiv \frac{1}{4} g_{ik} F^{(a)}_{mn} F^{mn(a)} - F^{(a)}_{in} F_k^{\ n(a)} \ , \tag{5.6}$$

$$T^{(I)}_{ik} = R T^{(YM)}_{ik} - \frac{1}{2} R_{ik} + F^{(a)}_{mn} F^{mn(a)} + \frac{1}{2} \left[D_i D_k - g_{ik} D^l D_l \right] \left[F^{(a)}_{mn} F^{mn(a)} \right] \ , \tag{5.7}$$

$$\begin{aligned} T^{(II)}_{ik} = {} & \frac{1}{2} D_l \left[D_i \left(F^{(a)}_{kn} F^{ln(a)} \right) + D_k \left(F^{(a)}_{in} F^{ln(a)} \right) \right] \\ & - \frac{1}{2} g_{ik} \left[D_m D_l \left(F^{mn(a)} F^{l(a)}_{\ \ n} \right) - R_{lm} F^{mn(a)} F^{l(a)}_{\ \ n} \right] \\ & - F^{ln(a)} \left(R_{il} F^{(a)}_{kn} + R_{kl} F^{(a)}_{in} \right) - R^{mn} F^{(a)}_{im} F^{(a)}_{kn} - \frac{1}{2} D^m D_m \left(F^{(a)}_{in} F_k^{\ n(a)} \right) \ , \end{aligned} \tag{5.8}$$

$$\begin{aligned} T^{(III)}_{ik} = {} & \frac{1}{4} g_{ik} R^{mnls} F^{(a)}_{mn} F^{(a)}_{ls} - \frac{3}{4} F^{ls(a)} \left(F_i^{\ n(a)} R_{knls} + F_k^{\ n(a)} R_{inls} \right) \\ & - \frac{1}{2} D_m D_n \left[F_i^{\ n(a)} F_k^{\ m(a)} + F_k^{\ n(a)} F_i^{\ m(a)} \right] \ , \end{aligned} \tag{5.9}$$

where D_m is the covariant derivative in the sense of the gauge field; that is, for any arbitrary tensor $Q^{(a)\dots}_{\dots(d)}$,

$$D_m Q^{(a)\dots}_{\dots(d)} \equiv \nabla_m Q^{(a)\dots}_{\dots(d)} + f^{(a)}_{\cdot(b)(c)} A^{(b)}_m Q^{(c)\dots}_{\dots(d)} + \dots - f^{(c)}_{\ (b)(d)} A^{(b)}_m Q^{(a)\dots}_{\dots(c)} - \dots \ . \tag{5.10}$$

It is easy to see that the first of these, i.e. tensor (5.6) is just the standard SET of the pure Yang-Mills field. The other three tensors are related to the corresponding coupling constants q_1, q_2, q_3.

Apparently, there is no way to solve eq. (5.4), at least exactly. But a way is possible. First of all, one restricts to spherical metrics of the form

$$ds^2 = Ndt^2 - \frac{dr^2}{N} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{5.11}$$

where (t, r, θ, φ) are spacetime spherical coordinates and N is a function depending on the radial variable only. In order to further simplify things, we assume the gauge field to be characterized by the so called Wu-Yang ansatz [146]; that is one doesn't use the standard base of generators of $SU(2)$, i.e. $\mathbf{t}_{(1)}$, $\mathbf{t}_{(2)}$, $\mathbf{t}_{(3)}$, but instead the new basis

$$\begin{aligned}
\mathbf{t}_{(r)} &= \cos\nu\varphi\sin\theta\mathbf{t}_{(1)} + \sin\nu\varphi\sin\theta\mathbf{t}_{(2)} + \cos\theta\mathbf{t}_{(3)} \ , \\
\mathbf{t}_{(\theta)} &= \cos\nu\varphi\cos\theta\mathbf{t}_{(1)} + \sin\nu\varphi\cos\theta\mathbf{t}_{(2)} - \sin\theta\mathbf{t}_{(3)} \ , \\
\mathbf{t}_{(\varphi)} &= -\sin\nu\varphi\mathbf{t}_{(1)} + \cos\nu\varphi\mathbf{t}_{(2)} \ ,
\end{aligned} \tag{5.12}$$

satisfying the commutation relations

$$[\mathbf{t}_r, \mathbf{t}_\theta] = \mathbf{t}_\varphi \qquad [\mathbf{t}_\theta, \mathbf{t}_\varphi] = \mathbf{t}_r \qquad [\mathbf{t}_\varphi, \mathbf{t}_r] = \mathbf{t}_\theta \ . \tag{5.13}$$

Thus said, the Wu-Yang ansatz states that

$$A_0^{(a)} = 0 \qquad A_r^{(a)} = 0 \qquad A_\theta^{(a)} = \delta^{(a)}_{(\varphi)} \qquad A_\varphi^{(a)} = -\nu\sin\theta\ \delta^{(a)}_{(\theta)} \ , \tag{5.14}$$

resulting that the magnetic parameter ν is a nonvanishing integer. The field strength tensor has only one nonvanishing component:

$$F^{(r)}_{\theta\varphi} = -A^{(\varphi)}_\theta A^{(\theta)}_\varphi = \nu\sin\theta \ . \tag{5.15}$$

This is a magnetic-type solution, independent on the parameters Λ, q_1, q_2 and q_3. Notice also that all the assumptions we made here are perfectly admissible, since they are only related to the form of the Lagrangian, i.e. to the setting of a number of arbitrary, but fixable parameters.

The surprise is that, with the choice of spherical symmetry combined with the Wu-Yang ansatz, the resulting EE are just very simple ones:

$$\frac{1-N}{r^2} - \frac{N'}{r} - \Lambda = \frac{8\pi\nu^2}{r^4}\left[\frac{1}{2} - q_1\frac{N'}{r} + (13q_1 + 4q_2 + q_3)\frac{N}{r^2} - \frac{q_1+q_2+q_3}{r^2}\right] \ , \tag{5.16}$$

$$\frac{1-N}{r^2} - \frac{N'}{r} - \Lambda = \frac{8\pi\nu^2}{r^4}\left[\frac{1}{2} - q_1\frac{N'}{r} - (7q_1 + 4q_2 + q_3)\frac{N}{r^2} - \frac{q_1+q_2+q_3}{r^2}\right] \ , \tag{5.17}$$

$$\begin{aligned}
&\frac{1}{r}N' + \frac{1}{2}N'' + \Lambda = \\
&= \frac{8\pi\nu^2}{r^4}\left[\frac{1}{2} - \frac{q_1 N''}{2} - (7q_1 + 4q_2 + q_3)\left(\frac{N'}{r} - \frac{2N}{r^2}\right) + \frac{2(q_1+q_2+q_3)}{r^2}\right] \ ,
\end{aligned} \tag{5.18}$$

where a prime denotes the radial derivative. Notice that eq. (5.18) can be deduced from eq. (5.17) upon differentiation.

5.2.2 A set of parameters generating a regular solution

Looking at the field equations, one can further simplify things, assuming that $13q_1 + 4q_2 + q_3 = -(7q_1 + 4q_2 + q_3)$, i.e. that

$$10q_1 + 4q_2 + q_3 = 0 \ . \tag{5.19}$$

In this case, eq. (5.16) and (5.17) coincide, so that there is only a single independent equation.

The second choice one performs on the parameters qs is the requirement of regularity. As the authors show, it is enough to impose that $N(0) = 1$ and $N'(0) = 0$. Eq. (5.16) satisfies these conditions, when

$$4q_1 + q_2 = 0 \ . \tag{5.20}$$

Since we have three coupling constant and two constraints on them, we remain with a single independent coupling constant, say q. One may put

$$q_1 \equiv -q \ , \tag{5.21}$$

$$q_2 = 4q \ , \tag{5.22}$$

$$q_3 = -6q \ . \tag{5.23}$$

In the following, we assume that $q > 0$. Of course, only one of these three equations is really independent, while the other two are just mere consequences. Within these choices, the resulting independent EE is just

$$\frac{d}{dr}\left[r(N-1)\left(1 + \frac{2Q_m^2 q}{r^4}\right)\right] = -\frac{Q_m^2}{r^2} - \Lambda r^2 \ , \tag{5.24}$$

where

$$Q_m^2 \equiv 4\pi\nu^2 \ , \tag{5.25}$$

with Q_m being some magnetic charge. It is worth to notice, as we should expect, that the contribution of the cosmological constant is just a quadratic term into the metric.

Eq. (5.24) is finally solvable, in an even very simple way. Integrating and slightly managing it, we have

$$N(r) = 1 + \left(\frac{r^4}{r^4 + 2Q_m^2 q}\right)\left(-\frac{2M}{r} + \frac{Q_m^2}{r^2} - \frac{\Lambda}{3}r^2\right) \ . \tag{5.26}$$

It is immediate to check that this solution is actually regular and that it resembles the Reissner-Nordström–de Sitter solution at large distances (or for $q \to 0$; this corresponds to the case of a minimal coupling):

$$N(r \to \infty) = 1 - \frac{2M}{r} + \frac{Q_m^2}{r^2} - \frac{\Lambda}{3} r^2 + o(r^{-4}) \ , \tag{5.27}$$

$$N(r \to 0) = 1 + \frac{1}{2q} r^2 - \frac{M}{Q_m^2 q} r^3 + o(r^6) \ , \tag{5.28}$$

thus satisfying the Sakharov criterion for small r.

Eq. (5.26) describes a four-parameter family of exact solutions: the first one is the non-minimal parameter of the theory q, the second is the cosmological constant Λ, the third is the magnetic charge of the Wu-Yang gauge field, Q_m, and the fourth is the asymptotic mass of the object M. It is worth to notice that they do not appear as integration constants (such as the terms c/r of the previous chapters), but as Lagrangian parameters, which can be set at will *before* the calculation. It is immediate to note that, playing with their numerical values, one is able to produce a black hole, an extremal black hole or an horizonless star. The interested reader can find a detailed discussion in the original paper, while we are here just interested in the regularity property of the theory.

5.2.3 Regularity check

We close this brief presentation, showing that the non-minimal coupling of action (5.1) is actually able to bypass the presence of a singular term. We already know that the main risk, when one finds a regular solution $N(r)$, is that also $N(r) + \frac{c}{r}$ may be a solution too. But in this framework, it is easy to prove that it is not the case. Indeed, the master equation of the problem is eq. (5.24), i.e. the only component of the EE, still surviving after performing the parameters choices (5.21)-(5.23). Assuming that N is a solution and inserting $N + c/r$ in it, one has immediately that

$$c\frac{d}{dr}\left(1 + \frac{2Q_m^2 q}{r^4}\right) + \frac{d}{dr}\left[r(N-1)\left(1 + \frac{2Q_m^2 q}{r^4}\right)\right] = -\frac{Q_m^2}{r^2} - \Lambda r^2 \ , \tag{5.29}$$

which however gives, since N is assumed to be a solution,

$$c\frac{d}{dr}\left(1 + \frac{2Q_m^2 q}{r^4}\right) = 0 \ . \tag{5.30}$$

This is clearly false, unless $c = 0$, $Q_m = 0$ or $q = 0$. In other words, as long as $Q_m \neq 0$ and $q \neq 0$ (which, we recall, are parameters, not integration constants!), the solution found in [67] is a true regular solution, with no risk to present a divergent term.

5.3 The Chamseddine-Mukhanov black hole

In this second section, we present a new model of RBH, recently introduced by A.H. Chamseddine and V. Mukhanov in [147]. In that paper, they applied the so called mimetic approach [148–160], introducing into the action a scalar field term, which is cancelled due to the presence of a Lagrange multiplier, but which is able, while working at the level of the field equations, to cancel the singularity.

It is worth to notice that the paper does not proceed "alone by itself" and implements in its approach the idea of a limiting curvature [148,150,161], in which is assumed that the EE are modified well below the Planck scale. Nothing forbids this idea, because the EE have been checked experimentally only for curvatures well below the Planckian size. If the limiting curvature is below the Planck value, quantum effects such as particle production and vacuum polarization can be ignored and the theory will be under control up to the highest possible curvatures.

Finally, this approach deserves interest, because it differs from the standard RBH procedure. Usually, indeed, one looks for solution satisfying the Sakharov criterion, having then a minimum and an inner horizon. Apparently, [147] is able to produce regular solutions with no need of an internal positive region.

The theory is then described by the action

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + f(\chi)\right) , \tag{5.31}$$

where $\chi = \Box\phi$, λ is a Lagrange multiplier and we have set $8\pi G = 1$. The presence of standard matter is not crucial in the argument, so its part can be omitted into the Lagrangian. As one can immediately see, the variation of the Lagrange multiplier λ states that the scalar ϕ always satisfies the constraint

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 . \tag{5.32}$$

It is easy to see that, if the metric has the form

$$ds^2 = dt^2 - \gamma_{ik}(t, x^l)dx^i dx^k , \tag{5.33}$$

with Latin indices running from 1 to 3, then the field ϕ plays the role of a time, having indeed $\phi = \pm t + A$, with A an integration constant.

In action (5.31), the function $f(\chi)$ can be chosen so that the the derivative of the metric determinant is bounded in the synchronous coordinate system (5.33): and this opens the possibility to have a nonsingular solution. By trials-and-errors, the authors found a good choice for f to be

$$f(\chi) = 1 - \sqrt{1-\chi^2} + g(\chi) , \tag{5.34}$$

with $\chi^2 \leq 1$. The function $g(\chi)$ is less restrictive but it has at least to satisfy two necessary conditions: it must be chosen so that the χ^2 term in the expansion of f is removed; and must also remove the singularity in $df/d\chi$ at $\chi = 1$. A good (but not mandatory) choice in this sense turns to be

$$g(\chi) = \frac{1}{2}\chi^2 - \chi \arcsin \chi \ . \tag{5.35}$$

Now that we have an explicit form for the Lagrangian, it is time to find the field equation: applying the variational principle and varying w.r.t. the metric, one gets

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \tilde{T}_{\mu\nu} \ , \tag{5.36}$$

where

$$\tilde{T}_{\mu\nu} = 2\lambda\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}(\chi f' - f + g^{\rho\sigma}\partial_\rho f'\partial_\sigma\phi) - (\partial_\mu f'\partial_\nu\phi + \partial_\nu f'\partial_\mu\phi) \ , \tag{5.37}$$

where we denoted $f' = df/d\chi$. For a sake of completeness, we also enlist the field equation, obtained varying the field ϕ:

$$\frac{1}{\sqrt{\gamma}}\partial_0(2\sqrt{\gamma}\lambda) = \Box f' = \frac{1}{\sqrt{\gamma}}\partial_0(\sqrt{\gamma}f''\dot{\chi}) - \Delta f' \ , \tag{5.38}$$

where $\Delta f'$ is the covariant Laplacian of f' for the metric γ_{ik}. We also denote with γ the determinant of the space metric: $\gamma \equiv \det \gamma_{ik}$.

Starting from this equation, the paper proceeds with some manipulations, whose aim is to produce a sort of master equation for black holes with a curvature limitation. We would not reproduce here the whole discussion, mainly due to reason of space: although straightforward, indeed, the discussion appears to be quite long. So we immediately skip to the master equation, remanding to the original paper for the whole argument. One has

$$\frac{1}{12}\left(\frac{\dot{\gamma}}{\gamma}\right)^2 = \varepsilon\left(1 - \frac{\varepsilon}{\varepsilon_m}\right) \ , \tag{5.39}$$

where

$$\varepsilon \equiv \frac{1}{8\gamma}\lambda^i_k\lambda^k_i - \frac{1}{2}P \ , \tag{5.40}$$

and $\varepsilon_m \equiv 2\chi_m^2$. Here P_{ik} is the three dimensional Ricci tensor associated to the reduced metric γ_{ik}; P is its Ricci scalar; and

$$\lambda^i_k \equiv -2\int\left(P^i_k - \frac{1}{3}P\delta^i_k\right)\sqrt{\gamma}dt \ , \tag{5.41}$$

with χ_m^2 a scaling parameter, denoting the characterization of the limiting curvature.

The relevant physical point of master equation (5.39) is that it produces black hole with a limited curvature (although the theory depends on the scale parameter χ_m^2). Thus, although exact solutions seem hard to be found, any resulting black hole built from it turns to be regular: since invariants turn to be limited, any eventual singularity is just due to the coordinate choice.

5.3.1 How the scheme works

We restrict to the example case of a metric in the form

$$ds^2 = dt^2 - a^2(t)dR^2 - b^2(t)d\Omega^2 \ , \tag{5.42}$$

which is just a special case of metric (5.34). One may note that quantities depend on time: indeed, inside the black hole, the time and the radial coordinate invert their role; here this interchange has been already taken into account. With this choice, one has that

$$P_1^1 = 0 \ , \qquad P_2^2 = P_3^3 = \frac{1}{b^2} \ , \qquad P = \frac{2}{b^2} \ , \tag{5.43}$$

so that

$$\frac{\lambda_k^i}{\sqrt{\gamma}} = -\frac{2\tilde{\lambda}_{(i)}\delta_k^i}{ab^2}F(t) \ , \qquad F(t) = \int a dt \ , \tag{5.44}$$

with

$$\tilde{\lambda}_{(i)} = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \ . \tag{5.45}$$

The constant of integration of $F(t)$ is determined from the condition $|t| \gg \varepsilon_m^{-1/2}$, for which the Schwarzschild solution is valid in the leading approximation.

Redefining the time t, one can write $F(t\tau) = r_g\tau^2 + C$, where r_g is the horizon radius in the Schwarzschild metric, τ is the new time variable and C is an integration constant. In principle, it is arbitrary; but in practice it is not, due to the Schwarzschild limit: one indeed finds that $C = -\frac{3}{2}r_g$.

Changing variables, changing the metric and changing the derivative finally leads to the new equation

$$\frac{\lambda_k^i}{\sqrt{\gamma}} = \frac{2\tilde{\lambda}_{(i)}\delta_k^i r_g}{ab^2}\left(\frac{3}{2} - \tau^2\right) \ . \tag{5.46}$$

This expression, which has been derived in the region where Einstein theory is fully applicable, can also be used deeply inside the black hole. If we lay in the condition for which $\tau^2 \ll 1$, eq. (5.46) is even simplified:

$$\frac{\lambda_k^i}{\sqrt{\gamma}} = \frac{3\tilde{\lambda}_{(i)}\delta_k^i r_g}{ab^2} \ . \tag{5.47}$$

Substituting this expression into the definition of ε, one finally has

$$\varepsilon = \frac{3r_g^2}{4a^2b^4} - \frac{1}{b^2} \ . \tag{5.48}$$

This can be used into the master equation, so that one has

$$\left(\frac{\dot{\gamma}_t}{\gamma_t}\right)^2 = \frac{9r_g^2}{\gamma_t}\left(1 - \frac{3r_g^2}{4\varepsilon_m \gamma_t}\right) \ , \tag{5.49}$$

where $\gamma_t \equiv \gamma/\sin^2\theta$, i.e. is only the time-radial part of the determinant (the only relevant one, due to the spherical symmetry). This can be easily integrated to give the solution

$$\gamma_t = \frac{3r_g^2}{4\varepsilon_m}\left(1 + 3\varepsilon_m t^2\right) \ . \tag{5.50}$$

The single components of the metric can also be obtained and we get

$$a(t) = \left(\frac{3r_g^2}{4\varepsilon_m}\left(1 + 3\varepsilon_m t^2\right)\right)^{1/6} \exp\left(\frac{2}{3}\left(\sinh^{-1}\left(\sqrt{3\varepsilon_m}t\right) + \ln\left(\frac{4}{3}\sqrt{3\varepsilon_m}\right)\right)\right) \ , \tag{5.51}$$

and

$$b(t) = \left(\frac{3r_g^2}{4\varepsilon_m}\left(1 + 3\varepsilon_m t^2\right)\right)^{1/6} \exp\left(-\frac{1}{3}\left(\sinh^{-1}\left(\sqrt{3\varepsilon_m}t\right) + \ln\left(\frac{4}{3}\sqrt{3\varepsilon_m}\right)\right)\right) \ , \tag{5.52}$$

Thus, the singularity is avoided and instead of it we have a bounce of duration $\Delta t \simeq \varepsilon_m^{-1/2}$. During this time the curvature is not much different from the limiting curvature but drastically drops after that. And this is what we mainly wanted to get.

5.4 The Non-Polynomial Gravity approach

As a last example, of ways to produce RBH, we consider the so called Non-Polynomial Gravity (NPG). We present here a summary of the work of [162], in the way it is arranged in [66]. Once again, mainly due to reasons of space, we won't present here the whole discussion, but just its highlights.

The key idea of NPG is to extend the results of [97, 163–165]. There non-polynomial curvature invariants were considered, both in the context of black holes and cosmology. We will see that many different RBH can be found in this way and that it provides a unified higher dimensional (4D only, in our case) description of 2D dilaton gravity reduction models, studied for example in [166–170]. We also notice that this approach can be seen as a generalization of Lovelock-Lanczos [171–174] and Quasi-Topological [175–182]: indeed, the polynomiality and second order equations of motion are required only for specific spacetimes (namely, for our purposes, spherically symmetric or cosmological ones).

5.4.1 Cotton tensor decomposition

Here we introduce the Cotton tensor in four dimensions, defined as

$$C_{\alpha\beta\gamma} = \nabla_\alpha R_{\beta\gamma} - \nabla_\beta R_{\alpha\gamma} + \frac{1}{6}\left(g_{\alpha\gamma}\nabla_\beta R - g_{\beta\gamma}\nabla_\alpha R\right) \ , \tag{5.53}$$

while we restrict to work in the spherically symmetric spacetimes defined by the metric:

$$ds^2 = \gamma_{AB}dx^A dx^B + r^2 d\Omega_r^2 \ , \tag{5.54}$$

where $A = \{0, 1\}$, γ_{AB} represents the non-angular part of the metric and $r^2 d\Omega_r^2$ its angular part (r is the areal radius, in general a function of x^0 and x^1). Within this framework, the Cotton obeys the algebraic equation

$$\left(-3C^{\mu\alpha\beta} + C^{\mu\beta\alpha}\right) C^{\nu}{}_{\alpha\beta} = \frac{1}{2}\left(-2g^{\mu\nu} + \frac{3}{2}\sigma^{\mu\nu}\right) C^{\rho\sigma\gamma}C_{\rho\sigma\gamma} \ , \tag{5.55}$$

in which $\sigma_{\mu\nu}$ is the 4D degenerate metric of the 2-sphere $r^2\Omega_r$. Conversely, defining the non-polynomial tensor

$$u_{\alpha\beta} \equiv \frac{(-3C_{\alpha\mu\nu} + C_{\alpha\nu\mu})C_\beta{}^{\mu\nu}}{C_{\sigma\rho\delta}C^{\sigma\rho\delta}} \ , \tag{5.56}$$

allows to write the degenerate metric of Ω_r as $\sigma_{\alpha\beta} = \frac{4}{3}(u_{\alpha\beta} + g_{\alpha\beta})$. Note that this construction holds in any dimensionality $D > 3$ (see [162]).

In spacetimes (5.54), $u_{\alpha\beta}$ turns not to depend on the derivatives of the metric, and thus it is possible to build up curvature scalars from it that are polynomial and second order in the metric itself, thus constructing an effective-like action (see again [66, 162] and references therein).

5.4.2 Action

In order to produce the action functional, recall that in the class of spacetimes (5.54), $r(x)$ is a scalar field on the non-angular part of the spacetime described by γ (in the following, we will call Σ this manifold). Denoting with tildes the covariant derivatives restricted on this manifold (while the ∇ without tilde defines the standard covariant derivative on the whole spacetime), we can write the following well-known 2D decomposition of the Ricci scalar of the 4D manifold in (5.54) (see for example [183]):

$$R\rfloor = R(\gamma) + R(\Omega) + 2\left(\frac{\tilde\nabla r \cdot \tilde\nabla r - \tilde\Box r^2}{r^2}\right) \ , \tag{5.57}$$

where $R(\gamma)$ and $R(\Omega)$ are the Ricci scalars of the 2D manifolds Σ and Ω_r respectively. For notational convenience, for any scalar X, we write its restriction to (5.54) by $X\rfloor$; in this, we follow the notation of [66, 162].

The main idea of this approach is to construct a series of higher order corrections to Einstein-Hilbert action from $u_{\alpha\beta}$, but keeping at any order the structure of the Ricci scalar R and of the Einstein tensor $G_{\mu\nu}$. To do so, we consider the following action :

$$\mathscr{I} = \frac{1}{16\pi G}\int_{\mathscr{M}} d^4x\,\sqrt{-g}\left(R - 2\Lambda + \sum_{i=1}^{m} l^i\,\mathscr{R}^{i/2}\left(\alpha_i\mathscr{R} + \beta_i\mathscr{S}_i\right)\right)\,. \tag{5.58}$$

Here l represents a length scale, introduced for dimensional reasons, and the scalars $\mathscr{S}_i$ and $\mathscr{R}$ are defined respectively as

$$\mathscr{S}_i = \frac{1}{3(i-2)}\left(-\frac{4i(i+1)}{3}\nabla_\gamma u_{\alpha\beta}\nabla^\gamma u^{\alpha\beta} + R + 4\left(R^{\alpha\beta} - \nabla^\alpha\nabla^\beta\right)u_{\alpha\beta}\right)\,, \tag{5.59}$$

and

$$\mathscr{R} = \frac{2}{3}\left(R + \left(R^{\alpha\beta} - \nabla^\alpha\nabla^\beta\right)u_{\alpha\beta}\right)\,. \tag{5.60}$$

Matching things together, it is not difficult to see that on spacetime (5.54) they read respectively

$$\mathscr{S}_i\rfloor = \frac{1}{2-i}\left(R(\gamma) + \frac{(i^2+i+2)\tilde{\nabla}r\cdot\tilde{\nabla}r - 2\tilde{\Box}r^2}{r^2}\right)\,, \tag{5.61}$$

$$\mathscr{R}\rfloor = \frac{R(\Omega)}{2}\,. \tag{5.62}$$

The result of this discussion is that higher order corrections to the Einstein-Hilbert action, in the framework of NPG, can be interpreted as a sum of powers of the Ricci scalar of the horizon manifold, given by $\mathscr{R}$, multiplied by a suitable scalar $\mathscr{S}_i$, which provides the same structure as GR in (5.54) for any order of corrections i. The precise expression of the coefficients of the scalars $\mathscr{S}_i$ is explained in details in [162], so we won't discuss it here.

Finally, putting things together, the action can be also rewritten factorizing the angular part (this is certainly not surprising, since we deal with spherical symmetry) and we have

$$\mathscr{I} = \frac{\mathscr{A}_{2,1}}{16\pi G}\mathscr{I}_{2D}\,, \tag{5.63}$$

where

$$\mathscr{I}_{2D} = \sum_{i=-2}^{m} l^i \int_\Sigma d^2x \sqrt{-\gamma} \left(\alpha_i + \beta_i \left[(1-i)\tilde{\nabla} r \cdot \tilde{\nabla} r + \frac{r^2}{2-i} R(\gamma) \right] \right) r^{-i} \ , \tag{5.64}$$

and $\mathscr{A}_{2,1} = \int d^2x\sqrt{\sigma} = 4\pi$ is just the volume of the 2D sphere of unitary radius.

5.4.3 Covariant 2D equations of motion

In order to find the field equations in the reduced spacetime, we perform the variation of $\mathscr{I}_{2D}$ w.r.t. the reduced metric γ_{AB} and to the scalar field $r(x)$, both on Σ gives: we have

$$\begin{aligned}\mathscr{E}_{AB} \equiv \sum_{i=-2}^{m} l^i \left(-\frac{1}{2}\alpha_i \gamma_{AB} - \beta_i r \left(\tilde{\nabla}_A \tilde{\nabla}_B - \gamma_{AB} \tilde{\Box} \right) r + \frac{\beta_i}{2}(1-i)\gamma_{AB} \tilde{\nabla} r \cdot \tilde{\nabla} r \right) r^{-i} \\ = 8\pi T_{AB} \ ,\end{aligned} \tag{5.65}$$

and

$$\begin{aligned}\mathscr{E}^\rho_\rho = \mathcal{E}^\phi_\phi \equiv \frac{1}{4}\sum_{i=-2}^{m} l^i \left(i\alpha_i - \beta_i \left(i(1-i)\tilde{\nabla} r \cdot \tilde{\nabla} r + r^2 R(\gamma) - 2(1-i) r \tilde{\Box} r \right) \right) r^{-i-2} \\ = 8\pi T^\rho_\rho\end{aligned} \tag{5.66}$$

Any other component of $T_{\mu\nu}$ and $\mathscr{E}_{\mu\nu}$ is vanishing.

In order to find the vacuum solutions of these equations (the only ones we will consider here), the simplest way is to use the Weyl method, which is possible due to the principle of symmetric criticality [184, 185], applied to spherically symmetric spacetimes.

Restricting to the non-angular part Σ of the spacetime, we write the metric as

$$d\Sigma^2 = -a(t,r)b^2(t,r)dt^2 + \frac{dr^2}{a(t,r)} + 2b(t,r)f(t,r)dtdr \ , \tag{5.67}$$

for which the $\mathscr{S}$-term in action (5.64) gives (up to some boundary term)

$$\sqrt{-\gamma}\, r^2 \mathscr{R}^{i/2} \mathscr{S}_i \Big\rfloor = \frac{r^{1-i}}{\sqrt{1+f^2}} \left(ab' - f\frac{\dot{a}}{a} + \frac{abff'}{1+f^2} \right) \ , \tag{5.68}$$

where dots and primes have the standard meaning of derivatives w.r.t. time and radius respectively. It makes the action proportional to :

$$\mathscr{I} \propto \int_{\Sigma} dr dt \sum_{i=-2}^{m} l^i r^{-i} \left(\alpha_i b\sqrt{1+f^2} + \frac{\beta_i r}{\sqrt{1+f^2}} \left(ab' - f\frac{\dot{a}}{a} + \frac{abff'}{1+f^2} \right) \right) . \tag{5.69}$$

Since the cross term in the metric can be reabsorbed by a change of coordinates, we can derive the field equations w.r.t. a, b and f evaluating then the result for $f = 0$. Therefore, defining $\Delta_\beta = \sum_{i=-2}^{m} l^i r^{-i} \beta_i$ and $\Delta_\alpha = \sum_{i=-2}^{m} l^i r^{-i} \alpha_i$, the equations of motion of respectively a, b and f read

$$r\Delta_\beta b' = 0 \qquad \Delta_\alpha - (r\Delta_\beta\, a)' = 0 \qquad - r\Delta_\beta \frac{\dot{a}}{a} = 0 . \tag{5.70}$$

5.4.4 A regular solution: the Poisson-Isreael case

The first equation of motion gives $b(t, r) = b(t)$ and this can be reabsorbed by transformation of time coordinate. The third one gives $a(t, r) = a(r)$. Finally, the second one reads

$$a(r) = \frac{-4M + \int \Delta_\alpha}{r\Delta_\beta} = 1 - \frac{4M + \left(r\Delta_\beta - \int \Delta_\alpha\right)}{r\Delta_\beta} , \tag{5.71}$$

where M is an integration constant and

$$\int \Delta_\alpha = \sum_{i=2}^{m} r^{1-i}\, l^i \left(\frac{\alpha_i}{1-i} \right) + l\alpha_1 \log\left(\frac{r}{l}\right) + 2r - \frac{2}{3} r^3 \Lambda . \tag{5.72}$$

This is enough to prove that this model satisfies the Birkhoff theorem (see [186]). Now,if we define $\gamma_i = \beta_i - \frac{\alpha_i}{1-i}$, the general solution is given, just depending on the maximal order of correction m, on the length scale l, the integration constant M and the dimensionless coupling constants α_i and β_i:

$$a = 1 - \frac{\frac{2}{3}\Lambda\, r^{m+2} + \Big(\left(\beta_1 - \alpha_1 \log\frac{r}{l}\right) l + 4M \Big) r^{m-1} + \sum_{i=2}^{m} l^i\, r^{m-i}\, \gamma_i}{2\, r^m + \sum_{i=1}^{m} l^i\, \beta_i\, r^{m-i}} . \tag{5.73}$$

Up to the logarithmic term, these solutions all represent rational black holes (with no mass terms at the denominator). They can be made regular, just considering corrections greater than their dimensionality, as shown in [162]. In 4D, we therefore need to consider at least an order 5 correction, corresponding to $m = 3$, so that the Sakharov criterion is satisfied. Doing so, one has

$$a(r) = 1 - \frac{\frac{2}{3}\Lambda\, r^5 + r^2 \Big(4M + l\left(\beta_1 - \alpha_1 \log\frac{r}{l}\right) \Big) + \gamma_2\, l^2 r + \gamma_3\, l^3}{2r^3 + \beta_1\, l r^2 + \beta_2\, l^2 r + \beta_3\, l^3} . \tag{5.74}$$

Imposing $\gamma_2 = \gamma_3 = \alpha_1 = 0$ and $\beta_3 \neq 0$ (which is free, since these are parameters and not integration constants), one immediately finds near the origin, the behavior

$$a(r \to 0) = 1 - \left(\frac{4M + l\,\beta_1}{l^3\,\beta_3}\right) r^2 + O(r^3)\ . \tag{5.75}$$

Moreover, if we don't care of the cosmological term, i.e. we set $\Lambda = 0$; if we note that the remaining dimensionless coupling constant β_1 can be reabsorbed into l; and setting $\beta_3 = 2$; then we find the solution

$$a(r) = 1 - \frac{2\,M\,r^2}{r^3 + l^3}\ , \tag{5.76}$$

which is the well-known Poisson-Israel RBH that has been found by a semi-classical argument in [28]. This is enough to show that the spacetime is truly non-singular and that its generating action is

$$\mathscr{I}_{PI} = \frac{1}{16\pi G}\int_{\mathscr{M}} d^4x\sqrt{-g}\left(R + 2l^3\mathscr{R}^{3/2}\left(-2\mathscr{R} + \mathscr{S}_3\right)\right)\ . \tag{5.77}$$

Other examples of application can be found in [66, 162]. In particular, another example of application will be discussed, more in detail, in the next chapter. There, we will take the NPG approach into account, in order to solve the cosmological singularity, instead of the Schwarzschild one. We considered it of special importance, because, although our main efforts have been devoted to regularize black hole metrics, also the cosmological framework deserves a major interest; this is why we put the model in a chapter on its own.

5.4.5 Some final remarks

Although it is not the only possible formulation, a direct geometrical interpretation to any correction to the Einstein-Hilbert action is in some sense expected: since the full solution should also appear in vacuo, no matter is presumably allowed to be present.

Focusing on the effective action, one would expect new degrees of freedom: they are indeed necessarily present, since the Lovelock theorem [172–174] prevents to find corrections, involving only the metric field, associated with second order differential equations for any metric field (what would exclude NPG). The effective equations of motion would therefore be higher order ones for general metric field, meaning the theory would involve additional fields. If such an effective action formulation exists, it must therefore involve new degrees of freedom at least in some specific backgrounds (which is not a trouble, however, for what we do here).

We conclude wandering if, within a numerical NPG approach, a regularization can be obtained at lowest order of corrections, without involving the whole set of corrections: that is, finding only a finite number of the coefficients

α_i instead of the whole set. If so, it might be an interesting way to work out a class of possible corrections without the need to reconstruct the whole sequence of constants, with the consequent major computational interest and convenience.

6 A model of a regular Universe

6.1 Introduction

In the first chapters of this work, we saw some negative results, meaning that no regular solution can be found (or, at least, we have no physical argument do distinguish among regular and singular solutions) if certain conditions are fulfilled. On the other hand, in the previous chapter, we presented some models in literature, apparently able to avoid the singularity and produce truly and satisfactory regular solutions. We did not discuss them in detail, since it was not our aim.

In this chapter we discuss in a more detailed way a model based on the NPG approach, which is also able to reproduce the cosmological version of the mimetic approach. The model has been presented in [97], so, for any detail of the calculation we may omit here or just rapidly name, we remand to it the interested reader.

The first ones to propose the model were Helling [187] and, independently, Date and Sengupta [188]. They suggested a modification of GR Lagrangian, able to produce the same correction to the Friedmann equation than LQC, with consequently the same bounce: their approach was intended to be an effective formulation of the loop quantization procedure. In there, Helling showed that a formulation in terms of an infinite sum of curvature invariants is possible, but it was not possible to write it explicitly. Their results are in line with [189–191].

More recently, Chamseddine and Mukhanov, working within the framework of mimetic gravity, followed the same idea and in a series of papers [147, 149, 151, 152] used a non-polynomial function of the mimetic field in order to reproduce the LQC result.

What is new in [97] is that an explicit Lagrangian has been found, including a cosmological term, built only from the metric field and still leading to the LQC corrections [192–196]. See also [197–216] for other approaches able to cure the Big Bang singularity.

Finally, [97] also studies some properties of the solution, in particular finding that the parameters appearing in it are not fully independent.

6.2 Action and field equations

In this section we briefly revise the NPG model we already presented, writing it explicitly for the FLRW framework (which is the only interesting one, for this discussion). We start with the metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \ . \tag{6.1}$$

It has been shown that, for such a metric, there exist geometric tensors of order 0 [165], i.e. independent on the derivatives of the metric (corresponding, for metric (6.1), to the scale factor):

$$V_\alpha \equiv \frac{\partial_\alpha R}{\sqrt{-\partial_\sigma R \partial^\sigma R}} \qquad \text{and} \qquad V_{\alpha\beta} \equiv V_\alpha V_\beta \ . \tag{6.2}$$

Starting from these, we are able to build two second order invariants, i.e.

$$K \equiv \frac{1}{9}\left(\nabla^\alpha \nabla^\beta V_{\alpha\beta} - V_\alpha \nabla^\alpha \nabla^\beta V_\beta\right) \qquad \text{and} \qquad \Omega \equiv \frac{R}{6} - 2K \ . \tag{6.3}$$

In the following, we will deal these tensor only with their restriction to the FLRW spacetime, the only one which interests here. Of course this means a little abuse of notation, but it is quite suitable for our present scope. So, in FLRW spacetime, we have that

$$K = H^2 \qquad \text{and} \qquad \Omega = \dot{H} \ , \tag{6.4}$$

where H is the Hubble parameter and the dot denotes the standard time derivative. Notice that, in the case of flat FLRW spacetimes, there exists other invariants, which have similar properties (see e.g. [164, 165]), but the ones we have chosen are also relevant in spherically symmetric spacetimes [162].

We stress the point that K and Ω, although we use them in a coordinate form, are curvature invariants and so one is allowed to use them to build invariant quantities. This is an important result, since makes us able to write down the Lagrangian: indeed we have now a connection among geometry and the field equation. Choosing a suitable combination of the above invariants, one can write the following action:

$$\mathscr{I} = \int d^4x \sqrt{-[g]} \left(\frac{R - 2\Lambda + \mathscr{L}^{\infty}_{NPG}}{16\pi} + \mathscr{L}_{matter} \right) \ , \tag{6.5}$$

where Λ is the cosmological constant, $\mathscr{L}_{matter}$ is the matter Lagrangian and

$$\mathscr{L}^{\infty}_{NPG} = -2\Omega + \frac{4\Omega}{S}\left(1 - \sqrt{1 - S}\right) \ , \tag{6.6}$$

where the scalar S is $S = \frac{3}{2\pi\rho_c}K$, with ρ_c playing the role of some critical density, which we leave as a free parameter.

It is worth to notice that we can explicitly rewrite the geometrical part of the Lagrangian density as

$$
\begin{aligned}
&R - 2\Omega + \frac{4\Omega}{S}\left(1 - \sqrt{1-S}\right) = \\
&= 8\pi\rho_c\left(1 - \sqrt{1-S} - \sqrt{S}\arcsin(\sqrt{S})\right) + \frac{4}{\sqrt{-[g]}}\sqrt{\frac{2\pi\rho_c}{3}}\dot{B} \ ,
\end{aligned}
\tag{6.7}
$$

where

$$
B \equiv \sqrt{-[g]}\left(\csc^{-1}\left(\frac{1}{\sqrt{S}}\right) - \frac{1 - S - \sqrt{1-S}}{\sqrt{S}}\right) \ . \tag{6.8}
$$

Making now use of the so called Weyl method, one can derive from action (6.5) the Lagrange equations (in principle, one should keep also the g_{00} term different from -1, but there is no need here to do so). Assuming that matter is a perfect fluid, with equation of state $P = \omega\rho$, we have

$$
4\pi\rho_c\left(1 - \sqrt{1 - \frac{3H^2}{2\pi\rho_c}}\right) = 8\pi\rho + \Lambda \ . \tag{6.9}
$$

This correction is consistent with the ordinary theory: indeed, in the case of $H^2/\rho_c \ll 1$, one recovers the Friedmann equation of GR. On the other hand, defining $\bar{\rho} \equiv \Lambda/8\pi + \rho$, one gets the standard form of the Friedmann equation coupled with LQC corrections:

$$
H^2 = \frac{8\pi\bar{\rho}}{3}\left(1 - \frac{\bar{\rho}}{\rho_c}\right) \ . \tag{6.10}
$$

This is the only equation we actually need, since the second Friedmann equation can be derived from eq. (6.9) and from the energy conservation condition $\dot{\rho} + 3H(\rho + P) = 0$.

6.3 Exact solutions

Making explicit use of the equation of state and redefining, for a sake of convenience, the density as $\tilde{\rho} = 8\pi\rho$ and $\mu = 1/8\pi\rho_c$, one has

$$
3H^2 = (\tilde{\rho} + \Lambda) - \mu(\tilde{\rho} + \Lambda)^2 \ , \tag{6.11}
$$

$$
\frac{d\tilde{\rho}}{\tilde{\rho}} = -3(1+\omega)Hdt \ . \tag{6.12}
$$

First of all, we note that a necessary condition is in order, if we want to hope to have a bounce: in the case of bounce, indeed, H vanishes at some point; so that, since eq. (6.12) holds, we need to have $\dot{\rho} = 0$ at that point (which is expected) and, most of all, $1-\mu\Lambda = \mu\tilde{\rho}$, meaning that $\mu\Lambda < 1$. Since μ and Λ are constant, this result holds at any time. Thus said, we work out the solution: plugging the first equation into the second, one has

$$\frac{dX}{(X-\Lambda)\sqrt{X-\mu X^2}} = \pm\sqrt{3}(1+\omega)dt \ , \tag{6.13}$$

where $X = \rho + \Lambda$. Thus,

$$\frac{2\tanh^{-1}\left(\frac{\sqrt{\Lambda}\sqrt{1-X\mu}}{\sqrt{X}\sqrt{1-\Lambda\mu}}\right)}{\sqrt{\Lambda}\sqrt{1-\Lambda\mu}} = \pm\sqrt{3}(1+\omega)t + c \ , \tag{6.14}$$

where c is some integration constant. In the following, we may set $c = 0$ without any problem: indeed, contrary to what happens in the black hole case, it is not associated to a singular term, but just to the position of the bounce (which we are free to fix where we prefer). Solving it in X and thus in $\tilde{\rho}$, we find

$$\tilde{\rho}(t) = -\frac{2\Lambda(-1+\Lambda\mu)}{-1+2\Lambda\mu+\cosh\left(\left(\pm\sqrt{3}t(1+\omega)\right)\sqrt{\Lambda}\sqrt{1-\Lambda\mu}\right)}. \tag{6.15}$$

Eq. (6.11)-(6.12) admit the usual well known solution $a = a_0\tilde{\rho}^{\frac{-1}{3(1+\omega)}}$. Consequently, one has

$$a(t) = a_0\left(\frac{-1+2\Lambda\mu+\cosh\left(\left(\sqrt{3}(1+\omega)t\right)\sqrt{\Lambda}\sqrt{1-\Lambda\mu}\right)}{2\Lambda(1-\Lambda\mu)}\right)^{1/3(1+\omega)} . \tag{6.16}$$

Given this solution, one may check that scalar $\partial_\sigma R\partial^\sigma R$ is not identically vanishing and the scalars (6.3) are indeed well defined.

As a further check, we study the two limits $\mu \to 0$ and $\Lambda \to 0$. The first corresponds to the GR limit, while the second recovers the standard LQC corrections (without a cosmological constant). Respectively, we have

$$\begin{aligned} \lim_{\mu\to 0}\tilde{\rho}(t) &= \Lambda\text{csch}^2\left(\frac{1}{2}\sqrt{\Lambda}(\sqrt{3}\ t(1+\omega))\right) \ , \\ \lim_{\mu\to 0}a(t) &= a_0\left(\frac{\cosh\left(\sqrt{\Lambda}(\sqrt{3}\ t(1+\omega))\right)-1}{2\Lambda}\right)^{\frac{1}{3(1+\omega)}} \ , \end{aligned} \tag{6.17}$$

and

$$
\begin{aligned}
\lim_{\Lambda\to 0}\tilde{\rho}(t) &= \frac{4}{\left(\sqrt{3}\ t(1+\omega)\right)^2 + 4\mu}\ , \\
\lim_{\Lambda\to 0} a(t) &= a_0\left(\mu + \frac{1}{4}\sqrt{3}\ t(1+\omega)\right)^{1/3(1+\omega)}\ .
\end{aligned}
\tag{6.18}
$$

In the first case, one recovers the standard GR solution, with a Big Bang; while in the second one, one gets the LQC bounce (without cosmological constant). This is the first important consistency check.

It is now time to study our solution w.r.t. the coordinate t. For small t, we have

$$
a(t\to 0) = a_0\left(\frac{\mu}{1-\mu\Lambda}\right)^{1/3(1+\omega)}\left(1 + \frac{(1-\mu\Lambda)(1+\omega)}{4\mu}t^2 + \dots\right)\ . \tag{6.19}
$$

We see that there is a minimal value, $a(0) = a_0\left(\frac{\mu}{1-\mu\Lambda}\right)^{1/3(1+\omega)}$, corresponding to the bounce. Moreover, already eq. (6.16) shows that $a(t)$ is never vanishing: indeed, we see the hyperbolic cosine is always greater than 1, so $\cosh x - 1 \geq 0$; and since μ and Λ are both positive, the scale factor is always positive and never vanishing.

The other interesting limit is the one for t very large. We remind that $\cosh x \to e^{|x|}$, for $x \to \pm\infty$, so one has

$$
\begin{aligned}
a(t\to\infty) = &\frac{a_0}{(2\Lambda(1-\mu\Lambda))^{1/3(1+\omega)}}\times \\
&\times\left(2\mu\Lambda - 1 + \exp\left(\sqrt{3\Lambda(1-\mu\Lambda)}(1+\omega)t\right)\right)^{1/3(1+\omega)}\ .
\end{aligned}
\tag{6.20}
$$

As one should expect, the exponential becomes dominant and this corresponds to an accelerating universe. Thus, our solution (and especially our energy density) may represent dark energy (DE), with a chosen suitable scale, so to fit with observational data [217, 218].

We conclude this section briefly discussing the limits μ and Λ large. We have already seen that the product $\mu\Lambda$ must be $\mu\Lambda < 1$. This is not a problem for DE issue because $\mu = 1/8\pi\rho_c$ mimics a quantum correction and thus it can be taken "safely" small. The situation is different with Λ not small, as during inflation, and this solution, in that case, may not be interesting (*may*, but it is not *forced to*; indeed, everything depends on how much μ is small and Λ large).

Finally, regarding the scalars we used in the argument, one may wonder if they are regular at the bounce, like polynomial scalars. One can check that, given the solution (6.16), their behaviors are :

$$\lim_{t\to 0}\partial_\sigma R\partial^\sigma R = \lim_{t\to 0}K = 0 \ , \tag{6.21}$$

$$\lim_{t\to 0}\nabla^\alpha\nabla^\beta V_{\alpha\beta} = \lim_{t\to 0}V_\alpha\nabla^\alpha\nabla^\beta V_\beta = 3\lim_{t\to 0}\Omega = \frac{3(1+\omega)(1-\Lambda\mu)}{2\mu} \tag{6.22}$$

i.e. no problem arises when $\mu \neq 0$.

6.4 A few final remarks

As we mentioned at the beginning of the section, the model we presented here is not fully original in concept, since it implements the argument of [187,188]. The key point is the formulation of the Lagrangian, so that it produces only second order equations (at least in the FLRW sector, the only we were interested in), so that no additional degree of freedom is involved, without a direct geometrical interpretation.

In some sense, this does not surprise, since LQC is a quantum *geometry* theory and it should be expected any correction inspired by it or trying to mimic or to reproduce it to be of geometrical nature (of course, this discussion does not apply the ordinary matter part of the Lagrangian).

The great emphasis we put on the results of [97] should not obscure the fact that the NPG approach is an effective approach and other effective approaches may be interesting too. In particular, the mimetic approach seems to be among the most interesting and popular ones: indeed, as we noted in the previous chapter, it has been used in a number of papers e.g. the already mentioned [148–150, 152, 153, 156]. It is important to note that the auxiliary field ϕ of such approach has a direct geometrical meaning, at least in some frameworks, playing the role of time. We also mention its generalization to $F(R)$ and for $F(R)$ ghost-free models [159, 160]. Moreover, polynomial and second order corrections to Friedmann equation have been found from a Galileon inspired action in [219] and from non-polynomial invariants in [164].

Finally, it is worth to notice that it might be interesting to perform numerical analyses to see how many corrections are needed to the Lagrangian to get the bounce. Indeed we must recall that action (6.5) and Lagrangian (6.6) contain an infinite number of corrections (due to their non-polynomial nature). This forced us to reconstruct all the coefficients of the series. If a bounce can already be obtained for a truncation at some order of the non-polynomial expansion (6.6), it could be an interesting way to work out a class of possible corrections without the need to reconstruct the whole sequence of constants (similarly to what we argued in the previous chapter). Such analysis cannot be carried out analytically, due to an excessive algebraic complexity.

7 A model of a RBH coupled to a scalar field

7.1 Introduction

In this new chapter we present a specific class of models of black holes (and even RBH), generated by the coupling of GR with a scalar matter field. In principle, in this framework, one can build up a huge number of models and during the years there have been many proposals of such couplings. In some special cases, authors have been even able to write explicitly both the metric and the action, which is one of the most relevant goals one can achieve in the search of RBH.

In this chapter we would not be able to find an explicit solution, due to excessive computational difficulties. However, even if analytical procedures fail in finding exact results, we are able to produce a scheme, relatively simple to follow with numerical tools, able to reproduce regular solutions.

We close this brief introduction with a few remark. Here we decide to deal with a minimal coupling, among gravity and matter, but the most general coupling is different (as we already saw, while discussing the Balakin-LemosZayats black hole). We will deal with this possibility in the next chapter, when we will work within the most general Horndeski theory.

A second remark concerns our no-go theorem. Since we decided to work with a minimally coupled matter-gravity action, the assumptions of the theorem hold, so that we should look for solutions with $f \neq g$. Actually, as we will see in a while, this can be done quite naturally, due to the kinetic term of the scalar field Lagrangian.

7.2 General framework and equations of motion

As first thing, we write the metric in the form

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \ . \tag{7.1}$$

For clearness and for simplicity, here we use only spherical symmetry, but nothing forbids us to discuss the topological case; the extension is indeed quite trivial.

As second point, we write down the action as

$$S(R, \phi_{matter}) = \int d^4x \sqrt{-[g]} \left(R + \mathscr{L}(g_{\mu\nu}, \phi_{matter})\right) \ , \tag{7.2}$$

where $\mathscr{L}(g_{\mu\nu}, \phi_{matter})$ is the matter Lagrangian and $[g]$ is the determinant of the metric. In the following, in order to avoid an excessive notation, we will drop the subscript *matter* in the field.

Action (7.2) produces the following independent EE:

$$\frac{g'r + g - 1}{r^2} = \frac{1}{2}\mathscr{L} + f\frac{\partial \mathscr{L}}{\partial f} \tag{7.3}$$

$$\frac{f'gr + fg - f}{fr^2} = \frac{1}{2}\mathscr{L} - g\frac{\partial \mathscr{L}}{\partial g} \tag{7.4}$$

$$\frac{1}{2fr}\left(f''gr + f'g + fg' + \frac{f'g'r}{2} - \frac{f'^2gr}{2f}\right) = \frac{1}{2}\mathscr{L} - g^{22}\frac{\partial \mathscr{L}}{\partial g^{22}} \tag{7.5}$$

Along with these, we should add the Euler-Lagrange equation for the field:

$$\nabla_\mu \frac{\partial \mathscr{L}}{\partial \left(\nabla_\mu \phi\right)} - \frac{\partial \mathscr{L}}{\partial \phi} = 0 \ . \tag{7.6}$$

Since we are dealing with a simple matter Lagrangian, containing a kinetic and a potential term (and since the field is a scalar one), this reads

$$\begin{aligned} \mathscr{L}\left(\phi, \nabla_\mu \phi\right) &= \frac{1}{2}\nabla_\rho \phi \nabla^\rho \phi - V(\phi) \\ &= \frac{1}{2} g \phi'^2 - V(\phi) \ . \end{aligned} \tag{7.7}$$

Notice that the function g of the metric appears into this object: this means that the Lagrangian depends on the solution, i.e. that $\partial_g \mathscr{L} \neq 0$ and this is enough, due to our no-go theorem, to prevent any emergence of Schwarzschild singularities; a singular term in the solution may be present, but, as we argued, it would be related to the choice of the Lagrangian.

Written the Lagrangian, the field equation (7.6) reads

$$\nabla_\mu \nabla^\mu \phi = -\partial_\phi V(\phi) \qquad \text{i.e.} \qquad g\phi'' + \frac{g}{2r}\left(4 + \frac{f'}{f}r + \frac{g'}{g}r\right)\phi' = -\partial_\phi V(\phi) \ , \tag{7.8}$$

i.e. the GR version of the famous Klein-Gordon Equation (KGE).

Before proceeding, let us spend a word about the no-hair theorem and the fact that we are building RBH solutions from a scalar field (see e.g. [220] and references therein). One would expect no black hole to arise, once the field potential is fixed, although counterexamples are known (see once again [220]). Our key trick is that the potential is left free and will be fixed only at the end of the discussion.

Regarding eq. (7.3)-(7.5) and (7.8), it is worth to notice that only three of them are independent, since the remaining one is immediately found as a linear combination of the others; thus we have the right to fix one of the unknowns, such as the field or the potential. In the following, we will make use of this freedom and will indicate two different approaches, both able to produce RBH (but both unable to provide exact results).

As first step for the solution, we notice that combining eq. (7.3) and (7.4), we can rule out one component of the metric: indeed, subtracting side to side eq. (7.3) to eq. (7.4) and recalling that the Lagrangian depends on g but not on f, it is easy to find that

$$\begin{aligned} f(r) &= g(r) \exp\left(-\int r\left(\frac{\partial \mathscr{L}}{\partial g}+\frac{f}{g}\frac{\partial \mathscr{L}}{\partial f}\right)\right) \\ &= g(r)\exp\left(-\frac{1}{2}\int \phi'^2 r dr\right) \ . \end{aligned} \tag{7.9}$$

The resulting (and remaining) equations are so

$$\frac{g'r+g-1}{r^2}-\frac{1}{4}g\phi'^2=-\frac{1}{2}V(\phi) \ , \tag{7.10}$$

$$g''+\left(\frac{2}{r}-\frac{3}{4}\phi'^2 r\right)g'-\left(\frac{3}{2}\phi'^2-\frac{1}{8}\phi'^4 r^2+\phi'\phi'' r\right)g=-V(\phi) \ , \tag{7.11}$$

$$g\phi''+\frac{g}{2r}\left(4-\frac{2g'}{g}r-\frac{1}{2}\phi'^2 r^2\right)\phi'=-\partial_\phi V(\phi) \ . \tag{7.12}$$

As next step, it not is convenient to solve directly eq. (7.10), since it would give $g(r)$ in terms of ϕ' and $V(\phi)$, which are two unknowns instead of one: and we have the freedom to fix only a single function. It is more convenient to follow another way, deriving it w.r.t r. Isolating the potential, we have

$$\partial_r V(\phi)=-\frac{2g''}{r}+\frac{1}{2}g'\phi'^2+\left(\phi'\phi''+\frac{4}{r^3}\right)g-\frac{4}{r^3} \tag{7.13}$$

The aim of this machinery is to eliminate one variable into the field equations. Indeed, we can now insert eq. (7.13) into the KGE. If we recall that ϕ depends only on r and so that $\partial_r=\phi'\partial_\phi$, we have

$$\frac{g''r^2-2(g-1)}{r^2}+\frac{3}{4}\phi'^2 g' r+\left(\phi'^2+\frac{1}{8}\phi'^4 r^2+\phi'\phi'' r\right)g=0 \ . \tag{7.14}$$

At this point, there are two main possible strategies, both with some *pro* and with some *contra*. There is no way to declare which one is the simplest or the most convenient, so we will follow, separately, both them.

7.3 First approach

Eq. (7.14) is linear in the metric solution, but it is also of second order. If we tried to solve it fixing a value for the field ϕ (we have the right to do it), it would result a very complicated equation; and of course there is no guarantee that the solution would be regular.

On the other hand, we could assume the metric function g to be fixed and the field ϕ to be undetermined. In this case, we are safe about the regularity and the equation reduces to a first order one, but it is not linear.

We start considering this second option, which lays within the reconstructive approaches. Since g is assumed to be known, indeed, the only thing to do is to find $\phi(r)$ and invert it: eq. (7.10) will give us the potential $V(\phi)$. In order to do it, we start introducing the new variable $X \equiv \frac{1}{2}\phi'^2 r$. Eq. (7.14) then becomes

$$X' + \frac{1}{2}X^2 + \left(\frac{1}{r} + \frac{3g'}{2g}\right) X + \frac{g''r^2 - 2(g-1)}{gr^2} = 0 \ . \tag{7.15}$$

With the variable X, we introduce a second varibale Y, this time defined as

$$X \equiv -Y e^{-\int A} \int \frac{B}{Y} e^{\int A} \ , \tag{7.16}$$

where A and B are, for simplicity,

$$A \equiv \frac{1}{r} + \frac{3g'}{2g} \ , \tag{7.17}$$

$$B \equiv \frac{g''r^2 - 2(g-1)}{gr^2} \ . \tag{7.18}$$

Using the new variable and the new notation, eq. (7.15) reads

$$\frac{Y'}{Y^2} = \frac{1}{2} e^{-\int A} \int \frac{B}{Y} e^{\int A} \ . \tag{7.19}$$

Still this equation is not linear. However, we can finally linearize it defining a third variable $Z \equiv 1/Y$. With this new substitution, we have

$$Z'' + AZ' + \frac{1}{2}BZ = 0 \ . \tag{7.20}$$

One goal has been reached, however still eq. (7.20) is of difficult resolution, due to the fact that it is of second order. However, there is no hope to avoid this fact, because reducing its order would imply a relation among A and B we are not allowed to impose (and which, in any case, would not produce an accessible scheme, since it would produce a hugely complicated equation for g, without any guarantee to get a regular solution). However, if we define the final variable $W \equiv Ze^{\frac{1}{2}\int A}$, we can cancel the first derivative in eq. (7.19), having so a simpler object than that of eq. (7.14):

$$W'' = -\frac{1}{2}\left(B - A' - \frac{1}{2}A^2\right) W \ , \tag{7.21}$$

that is, writing explicitly A and B in terms of g,

$$W'' = \frac{1}{2g}\left(\frac{1}{2}g'' - \frac{2}{r^2} + \frac{3g}{r^2} - \frac{3g'^2}{8g} + \frac{3g'}{2r}\right) W \ . \tag{7.22}$$

Eq. (7.22) cannot be solved by analytical means, but still is simpler than the original version and a numerical analysis can be carried out. As anticipated, however, we won't embark in this whole adventure, but numerical solutions actually can be found. A simple calculation, performed with free online integration programs, shows that $W(r)$ has a dumped oscillating behavior, which seem to prevent any risk of divergence. In any case, this is not much relevant to state, since what really matters is that ϕ does not diverge.

We come then to ϕ. It is quite easy to invert all changes of variables from X to W, so that we have

$$\phi(r) = \phi_0 + \int \sqrt{-\frac{2}{W\sqrt{r}} \int W \frac{g''r^2 - 2(g-1)}{r\sqrt{r}}} \ , \tag{7.23}$$

Where ϕ_0 is an integration constant. Also for this integral, once W and g are known, it is possible to produce a scheme of numerical resolution (it is unlikely to solve this integral analytically). Once this is done and we possess $\phi(r)$, since we already possess $V(r)$, the inversion $r(\phi)$ will give the field potential.

However, even if a full solution cannot be found, approximate results are actually available. In particular, we are able to discuss the relevant limits for a black hole spacetime, i.e. those for $r \to 0$ and $r \to \infty$. Since the metric is at our will, let's expand it (in the asymptotic regime) so that reduces to Schwarzschild at infinity and to de Sitter at the centre:

$$g(r \to \infty) = 1 - \frac{2M}{r} + \frac{k}{r^n} + ... \ , \tag{7.24}$$

$$g(r \to 0) = 1 + ar^2 + hr^m + ... \ , \tag{7.25}$$

where M is the mass of the black hole, $m > 2$ and $n > 1$ are numbers and a, k and h are suitable constants (typically model-dependent).

We now use eq. (7.22)-(7.25) to solve the problem. We insert the approximated functions (7.24) and (7.25) respectively into eq. (7.22), so that we see the behavior of W at infinity and at the origin respectively:

$$W'' = -\frac{1}{2r^2}\left(1+\frac{2M}{r}-\frac{k}{r^n}\right)\times$$
$$\times\left(\frac{1}{2}+\frac{2M}{r}+\frac{3M}{2r^2}-\frac{3M^3}{r^3}-\frac{(n^2-2n+3)k}{2r^n}-\frac{2Mnk}{2r^{n+1}}\right.$$
$$\left.+\frac{3n^2k^2}{8r^{2n}}+\frac{3M^2k(2n+1)}{2r^{n+2}}-\frac{3Mnk^2(n+2)}{4r^{2n+1}}+\frac{3n^2k^3}{8r^{3n}}\right)W\ , \qquad (7.26)$$

$$W'' = \frac{1}{2}(1-ar^2-hr^m)\times$$
$$\times\left(-\frac{1}{2r^2}+\frac{11}{2}a+\frac{1}{2}mh(m+2)r^{m-2}+\frac{3}{2}hr^{m-2}\right.$$
$$\left.-\frac{3}{8}(4a^2r^2+4amhr^m+m^2h^2r^{2m-2})(1-ar^2-hr^m)\right)W\ . \qquad (7.27)$$

If we consider only the dominant terms of these equations, they are greatly simplified and we get

$$W'' = -\frac{1}{4r^2}\left(1+\frac{6M}{r}\right)W\ , \qquad (7.28)$$

$$W'' = -\frac{1}{4r^2}(1-12ar^2)W\ . \qquad (7.29)$$

Notice that the dominant term is the same for both limits: this is not a surprise, since in both cases the solution reads $g(r)\to 1+....$

It sounds incredible, but these two equations are solvable: we get, in the two cases respectively,

$$W(r\to\infty) = c_0\sqrt{r}J_0\left(\sqrt{\frac{6m}{r}}\right)+c_1\sqrt{r}Y_0\left(\sqrt{\frac{6m}{r}}\right) \qquad (7.30)$$

$$W(r\to 0) = d_0\sqrt{r}J_0\left(i\sqrt{3a}\ r\right)+d_1\sqrt{r}Y_0\left(-i\sqrt{3a}\ r\right)\ , \qquad (7.31)$$

Where $J_n(x)$ is the first kind Bessel function and $Y_n(x)$ is the second kind Bessel function. If we expand the two functions, focusing only on the dominant order, we get

$$W(r) = k_{0,\infty}\sqrt{r}+k'_{0,\infty}\ln\left(\frac{r}{r_0}\right)+...\ , \qquad (7.32)$$

where k_0, k'_0, k_∞ and k'_∞ are four suitable integration constants.

Curiously, but not too much, eq. (7.32) holds at both $r\to 0$ and $r\to\infty$. However this is somewhat expected, since the dominant term is the same (i.e. 1) in both approximate regimes.

With this result, we are also able to find the behavior of the field in the asymptotic regimes: we find, in the two limits respectively,

$$\phi(r \to \infty) = \phi_0 + \frac{2}{1-n}\sqrt{\frac{2k(n^2+n-2)}{n}}\; r^{\frac{1-n}{2}} + \dots\ , \tag{7.33}$$

$$\phi(r \to 0) = \tilde{\phi}_0 + \frac{2}{1+m}\frac{\sqrt{-2h(m^2+m-2)}}{\sqrt{m-1/2}} r^{\frac{1+m}{2}} + \dots\ . \tag{7.34}$$

Since $n > 1$ and $m > 2$, these object correctly reduce to a constant when we are exactly at $r = \infty$ and $r = 0$. Notice that the two constants ϕ_0 and $\tilde{\phi}_0$ in general are different, since eq. (7.33) and (7.34) are not a general integration, over the whole space, but only a local one.

The final step cannot be done, because we don't have the full solution for $\phi(r)$, so the inversion cannot be performed (unless locally) and the potential cannot be written (unless we follow a numerical scheme).

7.3.1 An example

It is quite clear that this first approach is not able to produce analytical solutions. In order to convince (though not to prove) that things are so, consider one of the simplest case: the Hayward-like solution:

$$g(r) = 1 - \frac{2mr^2}{r^3 + l^3}\ , \tag{7.35}$$

where l is a quantum-like parameter. This is not the true original Hayward proposal, but it has the same behavior. With this choice, expansions (7.24) and (7.25) become respectively

$$g(r \to \infty) = 1 - \frac{2m}{r} + \frac{2ml^3}{r^4} + \dots\ , \tag{7.36}$$

$$g(r \to 0) = 1 - \frac{2m}{l^3}r^2 + \frac{2m}{l^6}r^5 + \dots\ , \tag{7.37}$$

so that, using notation of eq. (7.24)-(7.25), $k = 2ml^3$, $n = 4$, $a = -2m/l^3$, $h = 2m/l^6$, $m = 5$. We don't even try to discuss eq. (7.22) and integral (7.23), because it is quite evident that no general and analytical solution can be found (as can be easily verified with some integration software).

With these results, we can write explicitly the approximate expressions (7.33) and (7.34) for the scalar field:

$$\phi(r \to \infty) = \phi_0 - \frac{2}{3}\sqrt{18ml^3}\; r^{-3/2} + \dots\ , \tag{7.38}$$

$$\phi(r \to 0) = \tilde{\phi}_0 + \frac{1}{3}\frac{\sqrt{224m/l^6}}{3} r^3 + \dots\ . \tag{7.39}$$

These two equation tell us the behavior of the scalar field only in the asymptotic regime, but already from them we are able to discuss some qualitative property of the field itself. In particular, one may notice that the field is increasing both at the origin and at infinity. However, in order to perform the inversion, first we need to know the whole behavior of the field and second we should also pay attention on the asymptotic constants ϕ_0 and $\tilde{\phi}_0$.

Finally, we stress again that this is just an example, not a proof that analytical solutions cannot be found. Analytical solutions may be available (though it sounds unlikely), but they require a more refined treatment.

7.4 Second approach

In this section, we follow an alternative path, still able to produce a scheme for further numerical calculations (which continue to be beyond the scope of this work), but different from that we already presented. Of course, though different, the two approaches are widely equivalent.

We go back to eq. (7.14), which, slightly manipulated, reads

$$g'' + \frac{3}{4}\phi'^2 g' r + \left(\phi'^2 + \frac{1}{8}\phi'^4 r^2 + \phi'\phi'' r - \frac{2}{r^2}\right) g + \frac{2}{r^2} = 0 \ . \tag{7.40}$$

Contrary to the previous case, we don't fix the solution at will, but we fix $\phi(r)$. This forces us to solve a second order (though linear) equation, but gives the advantage of a simple inversion for the field, thus easily calculating $r(\phi)$. We will discuss a brief example, at the end of the section.

As first point, we set for notational convenience:

$$X \equiv \frac{1}{2}\phi'^2 r \ , \tag{7.41}$$

$$A \equiv \frac{3}{2}X \ , \tag{7.42}$$

$$B \equiv X' + \frac{1}{2}X^2 + \frac{1}{r}X - \frac{2}{r^2} \ , \tag{7.43}$$

so that eq. (7.40) appears in the simpler form

$$g'' + Ag' + Bg + \frac{2}{r^2} = 0 \ . \tag{7.44}$$

We now look for the regularity conditions on the solution. We know that a necessary and sufficient condition, at least in a static 4D framework, is the Sakharov criterion, i.e. that

$$g(r) = 1 + \sum_{n\geq 2} \frac{1}{n!} g_n r^n \ . \tag{7.45}$$

Moreover, since the field is at our will, we choose it so that X is expansible:

$$X = x_0 + \sum_{n\geq 1} \frac{1}{n!} x_n r^n \ , \tag{7.46}$$

Notice that this procedure does not represent an excess of freedom: indeed we are not imposing an arbitrary form both for the field and the solution; we are only looking for those properties the field must fulfill, in order the solution to be regular. We have the right to do so, because what we exceed in the constraint (7.45) on the function, we get back as a constraint on the field.

The following step is to insert the two expansions into eq. (7.44). We have

$$\begin{aligned}
&\sum_{n\geq 2} \frac{1}{(n-2)!} g_n r^{n-2} + \frac{3}{2}\left(x_0 + \sum_{n\geq 1} \frac{1}{n!} x_n r^n\right) \sum_{n\geq 2} \frac{1}{(n-1)!} g_n r^{n-1} \\
&+ \left(\sum_{n\geq 1} \frac{1}{(n-1)!} x_n r^{n-1} + \frac{1}{2} x_0^2 + x_0 \sum_{n\geq 1} \frac{1}{n!} x_n r^n \right. \\
&\left. + \frac{1}{2} \sum_{n,m\geq 1} \frac{1}{n! m!} x_n x_m r^{n+m} + \frac{x_0}{r} + \sum_{n\geq 1} \frac{1}{n!} x_n r^{n-1} - \frac{2}{r^2}\right) \times \\
&\times \left(1 + \sum_{n\geq 2} \frac{1}{n!} g_n r^n\right) + \frac{2}{r^2} = 0 \ .
\end{aligned} \tag{7.47}$$

We should remark that this is an exact result, since we did not perform (up to now) any approximation).

It is clear that, in order to avoid any divergent or non-vanishing term in this equation, we must require $x_0 = 0$ and $x_1 = 0$. Thus said, the only remaining thing is to fix an expression for $\phi(r)$ satisfying these two requirements and then solving the resulting eq. (7.44). As we show in the next subsection, this will probably always result in a numerical procedure.

7.4.1 An example

In order to show that there is no reasonable hope to find analytical solutions, even following this second approach, consider a very simple shape of the scalar field:

$$\phi(r) = \frac{\phi_0}{1 + ar^2} \ , \tag{7.48}$$

where ϕ_0 and a are suitable parameters. Of course this is not the only possible choice, but it is one of the simplest, able to fulfill the two conditions $x_0 = 0$, $x_1 = 0$ and able to reduce to a constant at infinity (which is another obvious requirement, if we want to reduce asymptotically to Schwarzschild). Consequently, X reads

$$X(r) = \frac{2\phi_0^2 a^2 r^3}{(1+ar^2)^4} \ , \tag{7.49}$$

and eq. (7.44) becomes, explicitly,

$$g'' + \frac{3\phi_0^2 a^2 r^3}{(1+ar^2)^4} g' + \left(\frac{2\phi_0^2 a^2 r^2}{(1+ar^2)^5} \left(4(1-ar^2) + \frac{\phi_0^2 a^2 r^4}{(1+ar^2)^3} \right) - \frac{2}{r^2} \right) g + \frac{2}{r^2} = 0 \tag{7.50}$$

As one can easily see, there is no hope to solve this object in general: the equation, though linear, is indeed too complicated. It is however still interesting to study it in the asymptotic regimes.

At infinity things are quite easy: indeed the approximate equation reads

$$g'' + \frac{3\phi_0^2}{a^2 r^5} g' - \frac{2}{r^2} \left(1 - \frac{4\phi_0^2}{a^2 r^4} \right) g + \frac{2}{r^2} = 0 \tag{7.51}$$

The dominant term in the solution is clearly the Schwarzschild solution $1 - 2m/r$. If we now define $h \equiv g - \left(1 - \frac{2m}{r}\right)$, so that $h \ll 1/r$, the dominant part of eq. (7.51) reads

$$h'' - \frac{2}{r^2} h - \frac{8\phi_0^2}{a^2 r^6} = 0 \ . \tag{7.52}$$

Solving this equation and putting things together, one has

$$g(r) = 1 - \frac{2m}{r} + \frac{4\phi_0^2}{9a^2 r^4} + \dots \ . \tag{7.53}$$

It is worth to mention that $g(r)$ may also contain a cosmological term $+\Lambda r^2$, since we may assume the potential $V(\phi)$ also containing a (cosmological) constant. However, since an eventual restoration of it is trivial, we don't consider it here.

The study near the origin is slightly more complicated. First of all, we define the new variable h as $h \equiv g e^{\frac{3}{4}\int X}$, so that eq. (7.50) loses the term in the first derivative and reads

$$h'' + \left(\frac{7\phi_0^2 a^2 r^2}{2(1+ar^2)^5} \left(1 - \frac{1}{7} a r^2 - \frac{\phi_0^2 a^2 r^4}{14(1+ar^2)^3} \right) - \frac{2}{r^2} \right) h + \frac{2}{r^2} e^{-\phi_0^2 \frac{1+3ar^2}{8(1+ar^2)^3}} = 0 \ . \tag{7.54}$$

Also for this equation it is very difficult to find a solution, but things become much simpler if we consider only the dominant terms. In this case, we have

$$h'' + \left(\frac{7\phi_0^2 a^2}{2} r^2 - \frac{2}{r^2} \right) h + \frac{2}{r^2} e^{-\phi_0^2 \frac{1+3ar^2}{8(1+ar^2)^3}} = 0 \ . \tag{7.55}$$

It may sound incredible, but we are able to solve analytically this equation, although the resulting expression is not at all a simple one:

$$
\begin{aligned}
h(r\to 0) = & \frac{4}{3}\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{4}\right)\sqrt{r}\times \\
& \times\left(-J_{3/4}\left(\frac{1}{2}\sqrt{k}\, r^2\right)\int\sqrt{r}A(r)J{-}3/4\left(\frac{1}{2}\sqrt{k}r^2\right)\right. \\
& \left.+J_{-3/4}\left(\frac{1}{2}\sqrt{k}\, r^2\right)\int\sqrt{r}A(r)J_{3/4}\left(\frac{1}{2}\sqrt{k}\, r^2\right)\right) \\
& + c_1\Gamma\left(\frac{5}{4}\right)k^{1/8}\sqrt{2r}\; J_{-3/4}\left(\frac{1}{2}\sqrt{k}\, r^2\right) \\
& + \frac{1}{\sqrt{2}}c_2 k^{1/8}\sqrt{r}\;\Gamma\left(\frac{7}{4}\right)J_{3/4}\left(\frac{1}{2}\sqrt{k}\, r^2\right)\ ,
\end{aligned}
\tag{7.56}
$$

where $\Gamma(x)$ is the Euler Gamma Function, $J_n(x)$ are the first kind Bessel functions, and k, $c_{1,2}$ are suitable constants. Approximating and taking only the dominant terms, we get

$$
h(r\to 0) = e^{3\phi_0^2/16} + 2^{3/4}k^{1/4}c_1\frac{1}{r} + \frac{c_2\sqrt{k}}{4\times 2^{3/4}}r^2 + \frac{3}{160}e^{3\phi_0^2/16}\phi_0^2 a^2 r^4 + \dots\ , \tag{7.57}
$$

so that we have

$$
g(r\to 0) = 1 + 2^{3/4}c_1 k^{1/4}e^{-3\phi_0^2/16}\frac{1}{r} + e^{-3\phi_0^2/16}\frac{c_2\sqrt{k}}{4\times 2^{3/4}}r^2 + \frac{3}{160}\phi_0^2 a^2 r^4 + \dots\ . \tag{7.58}
$$

Apparently there is a Schwarzschild term into this solution even near the origin. However this is just an apparent term and we should set $c_1 = 0$. Indeed, if we recall our no-go theorem, we see that $g_0 + \frac{c}{r}$ is a solution if and only if $f = g$: we proved that $f \neq g$ and this implies $c_1 = 0$ (otherwise, due to its arbitrariness, the theorem would be violated). The point is that we found solution (7.58) only approximately and the approximation makes the Lagrangian (almost) independent from the metric itself: indeed, the potential $V(\phi)$ tends to dominate on the kinetic term $\frac{1}{2}\phi'^2$; and the potential does not depend on the metric, because the field ϕ is a scalar one.

In order to prove it, consider $g(r)$ an *exact* solution of the metric. Then, $\tilde{g}(r) \equiv g(r) + c/r$ is *not* a solution. This can be immediately proved by direct check. However, near the origin, $\tilde{g}$ reads

$$
g(r\to 0) = 1 + 2^{3/4}\tilde{c}k^{1/4}e^{-3\phi_0^2/16}\frac{1}{r} + e^{-3\phi_0^2/16}\frac{c_2\sqrt{k}}{4\times 2^{3/4}}r^2 + \frac{3}{160}\phi_0^2 a^2 r^4 + \dots\ , \tag{7.59}
$$

where $\tilde{c} \equiv c_1 - ce^{3\phi_0^2/16}/(2^{3/4}k^{1/4})$. One can see that eq. (7.58) and (7.59) have the same form, the latter only having $\tilde{c}$ instead of c_1. However, since both $\tilde{c}$ and c_1 are arbitrary, it turns that the two equations are equivalent and, consequently, the only admissible choice for c_1 is $c_1 = 0$. This is an example of physical argument, we invoked in chapter 4, in order to bypass the no-go theorem.

This example shows that we are quite safe about the regularity, but also shows that apparently there is no hope to find fully analytical results; though, actually, an example is not a proof. In any case, we traced a scheme that can be followed via numerical calculations.

In this sense, though both approaches appear to have more or less the same computational difficulty, the second one is probably the most viable: indeed it fixes the field $\phi(r)$ and this simplifies the inversion process; on the other hand, within the first approach, a simple invertibility is not guaranteed.

8 Sufficient conditions for a RBH within the Horndeski theory

8.1 Introduction

In the first chapters of this work, we presented and discussed most of all negative results; i.e. we showed that, under suitable conditions and for certain classes of frameworks, there is *no* RBH in the theory (or, at least, no RBH in general). However we did not prove that RBH do not exist at all, even within a classical or semi-classical framework, so there may be still room to circumnavigate any obstacle: we provided some examples in two previous chapters. In particular, since one of the equivalent statements of the no-go theorem was $f = g$, one may avoid this limitation working within a "dirty" black hole, i.e. with $f \neq g$.

In the previous chapter, we followed this path. Also, we haven't been able to find an explicit solution, but this happened only due to computational problems. What is most relevant, however, is that, under suitable condition, we proved that a theory does exists containing RBHs.

(This is not the case of NED, as we proved in chapter 3. In that case, the general solution will always remain singular, but this is related to the specific structure of the matter field equations, namely the Maxwell equations, that provide $f = g$ autonomously with respect of the gravitational (Einstein) field equations.)

In this chapter we want to extend the discussion we already afforded in the previous one, investigating the possibility of having RBH within a coupling among gravity and a scalar matter field. In the previous chapter we discussed a specific example, making use of a minimal coupling; here we allow also the possibility for the coupling to be non-minimal, and indeed we will provide a sufficient (though not necessary) condition to have a RBHs producing approach.

In order to do so, in this chapter we work within the framework of Horndeski theory. Also within this huger framework we won't be able to provide explicit solutions, once again due to excessive computational difficulties.

8.2 Horndeski theory

We start our discussion presenting the approach first introduced by G.W. Horndeski in his famous paper [221], more than fourty years ago. The original aim of Horndeski was mainly on the mathematical side: he intended to write the most general second order field equations built from gravity coupled with a scalar field ϕ.

Field equations are indeed expected to be of second order, due to the famous Ostrogradsky instability: as proved already in 1850 by M. Ostrogradsky [222], if an action produces Euler-Lagrange equations containing higher derivatives, this is associated to an instability in the solution; in particular, it may produces states with negative energy − which is unphysical. On the other hand, if field equations are (at most) of second order, nothing of this happens.

Horndeski paper did not reach much attention, at the time of its publication in 1974 (as can be easily verified by checking its citation log at INSPIRE [223]). It was completely ignored until very recently, when it was finally resurrected within researches on Galileon theories: after 2012, it became one of the most cited papers in the last years. However, we remind that Horndeski results are much more general and do not rest on the choice of a single, specific approach.

Thus said, Horndeski action is typically written in the form

$$S = \int d^4x\sqrt{-[g]}(R + \mathscr{L}_2 + \mathscr{L}_3 + \mathscr{L}_4 + \mathscr{L}_5) \ , \tag{8.1}$$

where $\mathscr{L}_{2,3,4,5}$ are respectively

$$\mathscr{L}_2 = K(\phi, X) \ , \tag{8.2}$$

$$\mathscr{L}_3 = -G_3(\phi, X)\Box\phi \ , \tag{8.3}$$

$$\mathscr{L}_4 = RG_4(\phi, X) + \partial_X G_4(\phi, X)\left((\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right) \ , \tag{8.4}$$

$$\begin{aligned}\mathscr{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}\partial_X G_5(\phi, X)\times \\ \times\left((\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right) \ ,\end{aligned} \tag{8.5}$$

where $K(\phi, X)$ and $G_{3,4,5}(\phi, X)$ are four arbitrary functions of ϕ and $X \equiv -\frac{1}{2}(\nabla_\mu\phi)(\nabla^\mu\phi)$; we also remind that the box indicates the d'Alambert operator: $\Box \equiv \nabla_\mu\nabla^\nu$. It is immediate to see that GR is included in this set of theories, just setting $K = G_3 = G_4 = G_5 = 0$.

As one can see, at least in the most general case, the coupling among gravity and matter (represented by the scalar field) is highly non minimal. In the general case it becomes then very difficult to find some explicit and analytical results, but this happens only for computational difficulties. Indeed we have not been able, although some trials and some simplifications, to produce a regular solution (we have not been able even in the simple case of minimal coupling, discussed in the previous chapter). However we are still able to find some regularity criteria, so that suitable choices of the Lagrangian, coupled with enough computational power, may provide black hole solution and even regular black hole solutions.

8.3 Conditions for a regular solution

After having written down the Lagrangian, we start trying to simplify it. Since we are looking for sufficient and not necessary conditions, we start assuming $g_{00}g_{11} = -1$. This is not the most general statement, but it will much simplify any further consideration. Of course, this assumption makes our no-go theorem hold and so, in order to find a regular solution, we expect further constraints on the matter Lagrangian, in order to violate its hypotheses (we recall that our theorem does not states that $g_{00}g_{11} = -1 \Leftrightarrow$ singular solutions). We also assume that $G_4 = 0$; that K and $G_{3,5}$ only depend on ϕ and not on X; and that they have a Taylor expansion near the origin.

We stress the point that all these assumptions are not mandatory and are done in order to simplify much the discussion. However, we stress also that we are allowed in doing them, since we are looking for a sufficient (and not necessary) criterion.

If so, the independent components of the EE reduce to the two equations

$$f'r+f-1-\frac{f'r}{2}\phi'^2(2+f)G_{5,\phi}-\frac{1}{2}r^2(K+\phi'^2G_{3,\phi}f)-\frac{1}{2}\phi'^2(1+f)G_{5,\phi}f = 0\ , \quad (8.6)$$

$$f'' + \frac{2f'}{r} - \frac{1}{2}\phi'^2G_{5,\phi}f\left(f'' + \frac{f'}{fr}\right) + \frac{f'^2}{2f}\phi'^2G_{5,\phi}f) - \phi'^2G_{3,\phi}f - K = 0\ . \quad (8.7)$$

As is typically done, the subscript "ϕ" in these equations indicates the derivative with respect to ϕ; a similar notation holds for the other fields.

At this point, we first investigate the divergence properties of the solution. If f were divergent, the dominant terms of eq. (8.6), around the origin, would read

$$(f'r + f)G_{5,\phi} + G_{3,\phi}r^2 = 0 \quad (8.8)$$

Since $G_{3,\phi}r^2$ is still negligible w.r.t. $(f'r + f)G_{5,\phi}$ (because we assumed the Gs having a Taylor expansion near the origin), it is clear that eq. (8.8) is satisfied only if the singular part of f is Schwarzschild-like. Actually, this is nothing surprising. Apparently there is a single way to circumnavigate the problem, that is working on the only "untouched" assumption of the no-go theorem: i.e. we must require the coupling not to be minimal. This immediately implies that $G_{5,\phi} \neq constant$, although (at the moment) we have no other constraint on it.

Notice that in principle this still does not guarantee that the solution will be regular; however, within the general assumption we made in this chapter, it is a necessary condition. We now prove that it is also sufficient.

Assume that $\bar{f}$ is a solution. If $\bar{f}$ contained a Schwarzschild term, then $f \equiv \bar{f} + c/r$ would be a solution too. In this case, eq. (8.6) and (8.7) read

$$\frac{c}{2r}\phi'^2\left(G_{3,\phi}r^2 + (\bar{f}'r + \bar{f} - 1)G_{5,\phi}\right) = 0 \tag{8.9}$$

$$\frac{c}{2r}\phi'^2\left(2G_{3,\phi}r^2 + G_{5,\phi}\left(\bar{f}'' + \frac{2\bar{f}-1}{r^2} + \frac{\bar{f}'^2}{\bar{f}r\left(\bar{f}+\frac{c}{r}\right)} + \frac{2\bar{f}'r^2 + \frac{2c^2}{r} + 2c\bar{f} - c}{r^3\left(\bar{f}+\frac{c}{r}\right)}\right)\right) = 0 \tag{8.10}$$

From these, we immediately see that $G_{5,\phi} \neq 0$. Indeed, if we exclude the trivial case $\phi' = 0$ and since eq. (8.9) and (8.10) hold together, if $G_{5,\phi} = 0$ a simple combination of them results in an identity: and this means that any c is allowed, which is bad. On the other hand, if $G_{5,\phi} \neq 0$, no linear combination of (8.9) and (8.10) will make the two equations valid if $c \neq 0$.

If then we assume $G_{5,\phi} \neq 0$, we are sure that the function does not diverge. However we are not done in finding regular solutions, since we already proved (Sakharov criterion) that a static and spherically symmetric black hole is regular (at the origin) if and only if the metric has a de Sitter core: $f(r \to 0) = 1 + a_2r^2 + a_3r^3 + ...$ But, in general, we only provided a condition for having non-divergent solutions, so that we may also have $f(r \to 0) = a_0 + a_1r + a_2r^2 + ...$ What we should then do is using the full expansion (with general a_0 and a_1 coefficients) into eq. (8.6) and (8.7), at least until second order and impose suitable conditions on K and $G_{3,5}$ able to set $a_0 = 1$ and $a_1 = 0$.

Studying eq. (8.6) and (8.7) until the second order in r results in the six independent equations (if you are not interested in the details of the whole argument, you may skip the following lines and go directly to eq. (8.17), where we list the final results):

$$a_0 - 1 - \frac{1}{2}a_0(1+a_0)\phi_0'^2 G_{5,\phi}^{(0)} = 0\ , \tag{8.11}$$

$$\begin{aligned} &a_1\left(2 - \phi_0'^2 - \frac{1}{2}a_0\phi_0'^2G_{5,\phi}^{(0)} - \phi_0'\phi_0''G_{5,\phi}^{(0)} - \frac{1}{2}G_{5,\phi}^{(0)}\phi_0'^2(1+2a_0)\right) \\ &- a_0\phi_0'\left(\phi_0''G_{5,\phi}^{(0)} + \frac{1}{2}\phi_0'^2G_{5,\phi\phi}^{(0)}\right) = 0\ , \end{aligned} \tag{8.12}$$

$$
\begin{aligned}
&3a_2 - \frac{1}{2}\left(K^{(0)} + \phi_0'^2 G^{(0)}_{3,\phi} a_0\right) - 2a_2\phi_0'^2 G^{(0)}_{5,\phi} - a_0 a_2 \phi_0'^2 G^{(0)}_{5,\phi} \\
&- \frac{1}{2} a_1 \phi_0' \left(2\phi_0'^3 G^{(0)}_{5,\phi\phi} + a_0\phi_0'^2 G^{(0)}_{5,\phi\phi} + a_1\phi_0' G^{(0)}_{5,\phi} + 2\phi_0'' G^{(0)}_{5,\phi}(2+a_0)\right) \\
&- \frac{1}{2} a_0 G^{(0)}_{5,\phi}(\phi_0''^2 + \phi_0'\phi_0''')(1+a_0) - 2\phi_0'\phi_0'' a_0 a_1 G^{(0)}_{5,\phi} \\
&- \phi_0'\phi_0'' \left(a_0 G^{(0)}_{5,\phi\phi}\phi_0' + a_1 G^{(0)}_{5,\phi} + a_0^2\phi_0' G^{(0)}_{5,\phi\phi}\right) - \frac{1}{2} G^{(0)}_{5,\phi}\phi_0'^2(a_2 + 2a_0a_2 + a_1^2) \\
&- \frac{1}{2}\phi_0'^3(a_1 + 2a_0a_1) G^{(0)}_{5,\phi\phi} - \frac{1}{4}\phi_0'^2 a_0(1+a_0)\left(G^{(0)}_{5,\phi\phi\phi}\phi_0'^2 + G^{(0)}_{5,\phi\phi}\phi_0''\right) \\
&- 2a_1\phi_0'\phi_0'' G^{(0)}_{3,\phi} = 0 \ ,
\end{aligned}
\tag{8.13}
$$

$$
6a_2 - K^{(0)} - 2\phi_0'^2 G^{(0)}_{5,\phi} a_2(1+a_0) + \frac{a_1^2}{2a_0}\phi_0'^0 G^{(0)}_{5,\phi} = 0 \ , \tag{8.14}
$$

$$
\begin{aligned}
&12a_3 - K,\phi^{(0)}\phi_0' - a_0\phi_0'^2 G^{(0)}_{3,\phi} - a_0\phi_0'^3 G^{(0)}_{3,\phi\phi} - 2a_0\phi_0'\phi_0'' G^{(0)}_{3,\phi} \\
&- \frac{1}{2}\phi_0'^2 G^{(0)}_{5,\phi}(2a_1a_2 + 3a_3 + 6a_0a_3) - 2\phi_0'\phi_0'' G^{(0)}_{5,\phi} a_2(1+a_0) - \phi_0'^3 G^{(0)}_{5,\phi\phi} a_2(1+a_0) \\
&- \frac{1}{2}\phi_0'^2 G^{(0)}_{5,\phi}(2a_1a_2 + 3a_3 + 6a_0a_3) - 2\phi_0'\phi_0'' a_2 G^{(0)}_{5,\phi}(1+a_0) - a_2\phi_0'^3 G^{(0)}_{5,\phi\phi}(1+a_0) \\
&+ \frac{2a_1a_2}{a_0}\phi_0'^2 G^{(0)}_{5,\phi} - \frac{a_1^3}{2a_0^2} G^{(0)}_{5,\phi}\phi_0'^2 \\
&+ \frac{a_1^2}{2a_0}\left(G^{(0)}_{5,\phi\phi}\phi_0'^3 + 2\phi_0'\phi_0'' G^{(0)}_{5,\phi}\right) = 0 \ ,
\end{aligned}
\tag{8.15}
$$

$$
\begin{aligned}
&20a_4-\frac{1}{2}\left(K^{(0)}_{,\phi}+K^{(0)}_{,\phi\phi}\phi_0'^2\right)-\frac{1}{2}\phi_0'^2G^{(0)}_{5,\phi}(2a_2^2+6a_1a_3+12a_0a_4+4a_4)\\
&-a_2\phi_0'^2G^{(0)}_{3,\phi}-a_1\phi_0'^3G^{(0)}_{3,\phi\phi}-\frac{1}{2}a_0\phi_0'^2\left(\phi_0'^2G^{(0)}_{3,\phi\phi\phi}+\phi_0''G^{(0)}_{3,\phi\phi}\right)\\
&-a_0G^{(0)}_{3,\phi}(\phi_0''^2+\phi_0'\phi_0''')-2a_0\phi_0'^2\phi_0''G^{(0)}_{3,\phi\phi}-\phi_0'\phi_0''G^{(0)}_{5,\phi}(2a_1a_2+3a_3+6a_0a_3)\\
&-a_2(1+a_0)G^{(0)}_{5,\phi}(\phi_0''^2+\phi_0'\phi_0''')-\frac{1}{2}a_2(1+a_0)\phi_0'^2\left(G^{(0)}_{5,\phi\phi\phi}\phi_0'^2+5G^{(0)}_{5,\phi\phi}\phi_0''\right)\\
&-\phi_0'^2G^{(0)}_{5,\phi}(a_2^2+3a_1a_3+6a_0a_4+2a_4)-\frac{1}{2}\phi_0'^3G^{(0)}_{5,\phi\phi}(2a_1a_2+3a_3+6a_0a_3)\\
&-\phi_0'\phi_0''G^{(0)}_{5,\phi}(2a_1a_2+3a_3+6a_0a_3)-a_2G^{(0)}_{5,\phi}(\phi_0''^2+\phi_0'\phi_0''')(1+a_0)-2\phi_0'^2\phi_0''G^{(0)}_{5,\phi\phi}a_2(1+a_0)\\
&-\frac{1}{2}\phi_0'^2\left(G^{(0)}_{5,\phi\phi\phi}\phi_0'^2+G^{(0)}_{5,\phi\phi}\phi_0''\right)(1+a_0)+\frac{3a_1a_3+2a_2^2}{a_0}\phi_0'^2G^{(0)}_{5,\phi}\\
&+\frac{2a_1a_2}{a_0}G^{(0)}_{5,\phi\phi}\phi_0'^3-\frac{2a_1^2a_2}{a_0^2}\phi_0'^2G^{(0)}_{5,\phi}+\frac{a_1^2}{4a_0^2}\left(\frac{2a_1^2}{a_0}-2a_2\right)\phi_0'^2G^{(0)}_{5,\phi}\\
&-\frac{a_1^3}{2a_0^2}\left(G^{(0)}_{5,\phi\phi}\phi_0'^3+2\phi_0'\phi_0''G^{(0)}_{5,\phi}\right)\\
&+\frac{a_1^2}{2a_0}\left(2\phi_0'^2\phi_0''G^{(0)}_{5,\phi\phi}+G^{(0)}_{5,\phi}(\phi_0''^2+\phi_0'\phi_0''')+\frac{1}{2}\left(G^{(0)}_{5,\phi\phi\phi}\phi_0'^4+G^{(0)}_{5,\phi\phi}\phi_0'^2\phi_0''\right)\right)=0\ .
\end{aligned}
\tag{8.16}
$$

In all these equations, the subscript indicates the derivative w.r.t the subwritten variable, the prime, as usual, indicates the radial derivative and the superscript (0) indicates that the quantity is evaluated at the origin (which can be done, since we assumed any quantity in the Lagrangian to be Taylor-expansible). We intend $K^{(0)}\equiv K\left(\phi(r=0)\right)$ and similarly the other functions, depending on ϕ.

Eq. (8.11)-(8.16) are only eq. (8.6) and (8.7) restricted near the origin for the orders in r^0, r^1 and r^2 respectively. We only need to look for those conditions on K and $G_{3,5}$ such that the resulting metric is regular. It is quite easy that this actually happens, if (but not *only if*) $G^{(0)}_{5,\phi}$ and $G^{(0)}_{5,\phi\phi}$ are vanishing. If so, the hugely complicated eq. (8.11)-(8.16) read in the very simple form

$$a_0=1\ , \tag{8.17}$$

$$a_1=0\ , \tag{8.18}$$

$$3a_2-\frac{1}{2}\left(K^{(0)}+\phi_0'^2G^{(0)}_{3,\phi}\right)-\frac{1}{2}\phi_0'^4G^{(0)}_{5,\phi\phi\phi}=0\ , \tag{8.19}$$

$$a_2=\frac{1}{6}K^{(0)}\ , \tag{8.20}$$

$$12a_3 - K,\phi^{(0)}\phi_0' - \phi_0'^2 G_{3,\phi}^{(0)} - \phi_0'^3 G_{3,\phi\phi}^{(0)} - 2\phi_0'\phi_0'' G_{3,\phi}^{(0)} = 0 \ , \tag{8.21}$$

$$\begin{aligned} &20a_4 - \frac{1}{2}\left(K_{,\phi}^{(0)} + K_{,\phi\phi}^{(0)}\phi_0'^2\right) - a_2\phi_0'^2 G_{3,\phi}^{(0)} - \frac{1}{2}\phi_0'^2\left(\phi_0'^2 G_{3,\phi\phi\phi}^{(0)} + \phi_0'' G_{3,\phi\phi}^{(0)}\right) \\ &- G_{3,\phi}^{(0)}(\phi_0''^2 + \phi_0'\phi_0''') - 2\phi_0'^2\phi_0'' G_{3,\phi\phi}^{(0)} - a_2\phi_0'^4 G_{5,\phi\phi\phi}^{(0)} - \phi_0'^4 G_{5,\phi\phi\phi}^{(0)} = 0 \ . \end{aligned} \tag{8.22}$$

Among these, the most interesting ones are the first two: indeed, they say exactly what we wanted to read, i.e. that $a_0 = 1$ and $a_1 = 0$. Eq. (8.20) tells us the de Sitter term, so that the metric function goes as (near the origin)

$$f(r \to 0) = 1 - \frac{1}{6}K^{(0)} r^2 + ... \tag{8.23}$$

The remaining ones are slightly less interesting, since they just specify some constraints on the behavior of K and $G_{3,5}$ and their derivatives near the origin. Notice also that, in principle, we could continue the expansion, discussing the behavior of the coefficients of the term r^3, r^4 etc. in eq. (8.6) and (8.7), but these would only determine, in terms of K and $G_{3,5}$, the values of a_3, a_4 etc. and they are not interesting for us.

Finally, summarizing the results of this section, if

- $g_{00}g_{11} = -1$;
- $K = K(\phi)$; $G_3 = G_3(\phi)$; $G_4 = 0$; $G_5 = G_5(\phi)$;
- $G_{5,\phi}^{(0)} = 0$ and $G_{5,\phi\phi}^{(0)} = 0$, but $G_5(\phi) \neq 0$;

then the solution of the field equations (8.6)-(8.7) is regular; moreover, the same Lagrangian does not produce also a singular solution.

8.4 An example: Hayward solution

In the previous section, we showed that, under suitable assumptions on the Lagrangian, the Horndeski approach is able to produce RBHs and we gave a set of sufficient conditions for it. However, these conditions cannot be implemented in practice, at least if we look for analytical results.

In order to give an idea of how calculations are complicated, let us briefly discuss the reconstructive approach for a very simple regular metric, i.e. the Hayward metric. If, even with such a simple object, finding the generating action would result in a highly non trivial computational effort, this would indicate the scarcity of hopes in finding explicit solutions.

We start recalling that an Hayward-like solution is

$$f(r) = 1 - \frac{2mr^2}{r^3 + \lambda^3} \tag{8.24}$$

where λ is a parameter, playing the role of minimal length and m is an integration constant, playing the role of a mass. This is not exactly the Hayward solution, but the discussion turns to be the same.

Working within the reconstructive approach, we have the right to choose also a second variable; it is convenient, as we suggested in the previous chapter, to fix the field ϕ, in order to clearly know what is $\phi(r=0)$. One of the simplest possible choices (though not unique) is

$$\phi(r) = \frac{\lambda^3 \phi_0}{r^3 + \lambda^3} . \tag{8.25}$$

Moreover, the totale number of degrees of freedom gives us the right to fix one more function among K and $G_{3,5}$. A good choice would be to fix at zero either K or G_3, but a more convenient one is to fix K as

$$K(\phi) = -\frac{2f'r + f'fr + f + f^2}{r^2} \phi'^2 G_{5,\phi} . \tag{8.26}$$

It is clear that, since f is known and ϕ can be easily inverted, once G_5 is known, also K is. With these choices, eq. (8.6) and (8.7) can be easily integrated and give the remaining functions $G_{3,5}$ in a parametric way:

$$G_{3,\phi} = \alpha - \frac{4m}{\phi_0} \int \frac{r^3 + \lambda^3}{r^2(r^3 - 2mr^2 - \lambda^3)} dr , \tag{8.27}$$

$$G_5 = \beta - \int \frac{f''r^2 - 2(f-1)}{\phi' r^2 \left(\frac{3}{2} f'r + f'fr + f + f^2 - \frac{1}{2} f''r^3 + \frac{f'^2 r^2}{2f} \right)} dr , \tag{8.28}$$

where α and β are two integration constants; we kept f and ϕ within an implicit form in G_5 just to avoid byzantine integrals. Once the integrals are solved, we are done, although still remains an integration in ϕ for having G_3. Eq. (8.27) and (8.28) seem to be too much complicated to hope they have some chance of integrability. Indeed, any attempt to find an analytical result came up with a failure.

Of course, this is just an attempt, not a proof that analytical solutions do not exist in this framework. Moreover, we provided only sufficient, not necessary conditions: different assumptions can be performed too, still having regular solutions. On the other hand, this example is good propaganda against the search of analytical solutions in the Horndeski framework (at least within our conditions): indeed, Hayawrd metric is one of the simplest regular metrics and even with it we are not able to reconstruct explicitly the generating action.

Finally, it is worth to mention that a numerical scheme will be able to solve the problem, only providing enough computational power. However, since this effort would not match the aim of the present work, we would not discuss it here.

9 A RBH from a matter fluid

9.1 Introduction

Throughout this work, we mainly focused on Lagrangian models, and this because it would be a very important goal having both the (regular) solution and its generating action. However, throughout the whole work, we also showed that it is very difficult to provide a complete picture in such framework, typically due to excessive computational difficulties.

In this chapter, we take a different approach, specifying only the SET. Of course, we would not give an arbitrary SET, but we will focus on an interesting and viable physical model, i.e. the perfect fluid. We will discuss the standard case, in which all the pressures (the radial and the transversal ones) coincide, but we will be more interested in the anisotropic case, in which the radial and the transversal pressures are different.

This approach is typically known as "fluid approach" and, together with the (minimally coupled) Lagrangian approach, is one of the most popular in the study of GR. However, it is worth to notice that there are many different way to approach to the singularity issue, without introducing the Lagrangian [224–230].

This chapter is mainly based on the recent paper [96] and some research work which subsequently followed. Thus, throughout the chapter, we won't present all the intermediate calculations, if they turned to be quite annoying, and we will limit ourselves in fixing the main points and illustrating the results. In any case, we may also present some calculations, if they were crucial part of the argument or of particular interest (and of course we will discuss and prove any new result, beyond the original paper).

A couple of remarks are in order, before proceeding. First, as we already mentioned in the paper, this work has nothing to do with [231]: indeed, it considers a gravitational source (which may also be a black hole) made of two different contributions and works out to separate their effects. On the other hand, we always deal with a single source.

Second, we should mention that Babichev *et al.* published some papers on black holes in presence of exotic fluids [232–234], but their objectives and results are different from ours.

Finally, contrary to what happened in the previous chapters, this time we will be able to find (some) analytical results. As we will discuss, they are not entirely free of pathologies, but still they represent some progress w.r.t. any previous (and incomplete) discussion.

9.2 Basic assumptions

As usually done, we consider spherical and static symmetry. We will discuss the dynamical properties of the framework only later in the chapter:

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega^2 \ , \tag{9.1}$$

where, as usual f and g are suitable functions and $d\Omega^2$, the surface element of a 2-sphere of unit radius. This time, however, we don't assume the metric to be generated by some matter Lagrangian, but by some perfect fluid, whose SET is written in the form

$$T^\mu_\nu = \rho u^\mu u_\nu + (P - P_\perp)\delta^1_\nu \delta^\mu_1 + P_\perp \left(\delta^\mu_\nu + u^\mu \, u_\nu\right) \ , \tag{9.2}$$

where u_μ is the four-velocity of the fluid, ρ its density, P the radial pressure and $P_\perp$ the transversal one. If $P = P_\perp$, we have an isotropic fluid; otherwise, the fluid is anisotropic. Finally, concerning the density, we ask for the following properties:

- ρ is always positive and finite, i.e. $0 \leq \rho < \infty$;
- ρ is monotonically increasing, i.e. $\rho' \leq 0$.

In the rest frame of the fluid, $u_\mu = (\sqrt{f}, 0, 0, 0)$ and the SET reads simply $T^\mu_\nu = \mathrm{diag}(-\rho, P, P_\perp, P_\perp)$. This simplifies the EE, whose independent components then are

$$\frac{d}{dr}\left(r(1-g)\right) = 8\pi r^2 \rho \ , \tag{9.3}$$

$$\frac{g}{fr}f' + \frac{g-1}{r^2} = 8\pi P \ , \tag{9.4}$$

$$\frac{f'' g}{2f} - \frac{f'^2 g}{4f^2} + \frac{f' g'}{4f} + \frac{g'}{2r} + \frac{f' g}{2fr} = 8\pi P_\perp \ . \tag{9.5}$$

Finally, we write also the conservation equation for the fluid, $\nabla_\mu T^\mu_r = 0$; i.e.

$$P' + \frac{P+\rho}{2}\frac{f'}{f} + \frac{2\left(P - P_\perp\right)}{r} = 0 \ . \tag{9.6}$$

It is easy to show that these four equations are not all independent: indeed, one can use a combination of three of them to find the fourth. As example, one can derive eq. (9.4) w.r.t r and then use the result, together with eq. (9.3) and eq. (9.4) itself (not derived) into eq. (9.6): this done, one has exactly the third EE, eq. (9.5). This leaves us with two degrees of freedom, because the unknowns are f, g, ρ, P and $P_\perp$. Imposing another constrain, such as the isotropy of the fluid (i.e. $P = P_\perp$) or requiring an equation of state (i.e. $P = P(\rho)$) still leaves a single degree of freedom. We will conveniently use this fact.

Now, in order to further simplify the discussion, we write $f = e^\alpha g$, where α is some arbitrary function. If so, eq. (9.3), (9.4) and (9.6) read respectively

$$g(r) = 1 - \frac{c}{r} - \frac{8\pi}{r}\int r^2\rho \ , \tag{9.7}$$

$$8\pi P = \frac{g\alpha'}{r} - 8\pi\rho \ , \tag{9.8}$$

$$P' + \frac{P+\rho}{2}\left(\alpha' + \frac{g'}{g}\right) + \frac{2\,(P - P_\perp)}{r} = 0 \ . \tag{9.9}$$

where c is an integration constant. Its presence is a priori unavoidable, since we see that, in the rest frame, the SET does not depend on the metric itself; and thus the no-go theorem holds and a Schwarzschild term arises. We will show, later on, the argument one may use to avoid it.

9.3 Study of the solution

In the paper [96] we discussed only the two cases $P = P_\perp$ and $\alpha = 0$. Although they are the main choices, at this stage we are actually able to discuss a wider set of cases, keeping P and $P_\perp$ different and keeping a linear equation of state. Here we present the three cases (two main cases, $P = P_\perp$ and $P \neq P_\perp$ and the special case of $P = -\rho$) separately, studying them both in the classical regime, where we can distinguish an inner and an outer part of the star, and in some deformed regime, where such identification appears to be much more difficult.

9.3.1 Case 1: $P = P_\perp$

This first choice, originally performed by Schwarzschild [10], is a very popular one, performed more and more times during the decades. Under this assumption, eq. (9.7)-(9.9) become

$$g(r) = 1 - \frac{c}{r} - \frac{8\pi}{r}\int r^2\rho \ , \tag{9.10}$$

$$\alpha' = 8\pi\frac{r(P+\rho)}{1 - \frac{c}{r} - \frac{8\pi}{r}\int r^2\rho} \ , \tag{9.11}$$

$$
\begin{aligned}
&P' + \frac{4\pi}{1 - \frac{c}{r} - \frac{8\pi}{r}\int r^2\rho}\times \\
&\times\left(P^2 r + \left(\frac{c}{8\pi r^2} + \rho r + \frac{1}{r^2}\int r^2\rho\right)P + \frac{c}{8\pi r^2}\rho + \frac{\rho}{r^2}\int r^2\rho\right) = 0\ .
\end{aligned}
\tag{9.12}
$$

Once the density is given, eq. (9.10) is solved autonomously, while eq. (9.11) and (9.12) form a closed system. This, however cannot be solved analytically, unless trivial cases.

In order to fix ideas, we discuss the classical limit of a finite star of radius R, whose density is $\rho = \rho_0\,\theta(R-r)$. If so, the three equations gain the simpler form

$$
g(r) = 1 + \frac{c}{r} - \frac{8\pi}{3}\rho_0 r^2\ , \tag{9.13}
$$

$$
\alpha' = \frac{r(P+\rho_0)}{1 + \frac{c}{3} - \frac{8\pi}{3}\rho_0 r^2}\ , \tag{9.14}
$$

$$
\begin{aligned}
&P' + \frac{4\pi}{1 - \frac{c}{r} - \frac{8\pi}{3}\rho_0 r^2}\times \\
&\times\left(P^2 r + \left(\frac{c}{8\pi r^2} + \rho_0 r + \frac{1}{3}\rho_0 r\right)P + \frac{c}{8\pi r^2}\rho_0 + \frac{1}{3}\rho_0^2 r\right) = 0\ ,
\end{aligned}
\tag{9.15}
$$

inside the star. Typically the term c does not appear in these equations, however, for the moment, we prefer to keep it. In the next section, we will show it vanishes.

On the other hand, outside, we have simply the Schwarzschild vacuum solution:

$$
g(r) = 1 - \frac{2M}{r}\ , \tag{9.16}
$$

where $M = 4\pi\int r^2\rho$ is the whole mass of the star and the integral is performed on the whole mass distribution. All the other quantities, i.e. ρ, α, P, are vanishing.

This density shape, at least for $c = 0$, is one of the special trivial cases, in which the pressure can be found explicitly: indeed eq. (9.15) becomes solvable via separation of variables and we get

$$
P(r) = \rho_0 \frac{\sqrt{1 - \frac{8\pi}{3}\rho_0 r^2} - \sqrt{1 - \frac{8\pi}{3}\rho_0 R^2}}{3\sqrt{1 - \frac{8\pi}{3}\rho_0 R^2} - \sqrt{1 - \frac{8\pi}{3}\rho_0 r^2}}\ . \tag{9.17}
$$

Coupling this with eq. (9.14) we find also $\alpha(r)$:

$$\alpha(r) = 2\ln\left(\frac{3\sqrt{1-\frac{8\pi}{3}\rho_0 R^2} - \sqrt{1-\frac{8\pi}{3}\rho_0 r^2}}{2\sqrt{1-\frac{8\pi}{3}\rho_0 r^2}}\right) , \tag{9.18}$$

so that the g_{00} component of the internal solution is

$$f(r) = \frac{1}{4}\left(3\sqrt{1-\frac{8\pi}{3}\rho_0 R^2} - \sqrt{1-\frac{8\pi}{3}\rho_0 r^2}\right)^2 , \tag{9.19}$$

while the external part is just the Schwarzschild term. It appears that $\rho = \rho_0$ is the only possible analytical case, since otherwise eq. (9.15) has no more separable variables and, apparently, there is no other way to solve eq. (9.15) analytically. However, as we showed in the paper and as we will discuss in the next sections, this approach is not able to produce RBH from some kind of gravitating matter.

Notice also that eq. (9.17)-(9.19) hold only for $c = 0$. However this choice is the only admissible one, since we found that $g_{00}g_{11} \neq -1$: thus, due to our no-go theorem, the only possible value for c is 0 (using the same argument, with obvious modifications, we already used in chapter 7).

9.3.2 Case 2: $P \neq P_\perp$

In this case, it is convenient to perform an ansatz on the equation of state. Many choices, of course, are possible, but the most interesting one is maybe the usual $P = \omega\rho$, where $-1 \leq \omega \leq 1$. This includes the cases of standard dust ($\omega = 0$), radiation ($\omega = 1$) and dark energy ($\omega = -1$). Under this choice, eq. (9.7)-(9.9) reduce to

$$g(r) = 1 - \frac{c}{r} - \frac{8\pi}{r}\int r^2\rho , \tag{9.20}$$

$$8\pi(1+\omega)\rho = \frac{g\alpha'}{r} , \tag{9.21}$$

$$\omega\rho' + \frac{1+\omega}{2}\rho\left(\alpha' + \frac{g'}{g}\right) + \frac{2\left(\omega\rho - P_\perp\right)}{r} = 0 . \tag{9.22}$$

One immediately sees that eq. (9.20) has the same form of eq. (9.7): indeed, since P does not appear in it, nothing changes.

Now, the interesting point is that, once the density profile is given (and we can give it freely, since we recall that there is still one degree of freedom untouched), $g(r)$ is known and this is enough to find also $\alpha(r)$ and so $f = ge^\alpha$. The only remaining unknown is $P_\perp$, but, putting things together in eq. (9.22), it is immediately found. The only (non trivial) difficulty is to solve all the arising integrals, whose analyticity depends on the specific choice of the profile. In practice, unless trivial choices, for a general value of ω there seems to be no analytical result.

As we did for Case 1, also here we study the classical toy model, with a definite radius R of the star and a constant density: $\rho(r) = \rho_0\ \theta(R - r)$. In this case, outside the star there is only the Schwarzschild vacuum solution $f(r) = g(r) = 1 - 2M/r$, while inside eq. (9.20)-(9.22) read (in the case of $c = 0$)

$$g(r) = 1 - \frac{8\pi}{3}\rho_0 r^2 \ , \tag{9.23}$$

$$\alpha(r) = \begin{cases} \alpha_0 - \frac{3}{2}(1+\omega)\ln\left(1 - \frac{8\pi}{3}\rho_0 r^2\right) & \text{if } r < \sqrt{\frac{3}{8\pi\rho_0}} \\ \alpha_0 - \frac{3}{2}(1+\omega)\ln\left(\frac{8\pi}{3}\rho_0 r^2 - 1\right) & \text{if } r > \sqrt{\frac{3}{8\pi\rho_0}} \end{cases} \ , \tag{9.24}$$

$$P_\perp = \omega\rho_0 + 2\pi\rho_0^2 \frac{(1+\omega)(1+3\omega)}{3\left(1 - \frac{8\pi}{3}\rho_0 r^2\right)} r^2 \ . \tag{9.25}$$

Eq. (9.25) is of particular interest, since it states that the transversal pressure is divergent on the horizon, where $g(r_H) = 0$. One may argue that this is just an effect of the toy model $\rho = constant$, but this is not so. Indeed, if we go back to eq. (9.22), in which the density profile has been kept general, and we put $P_\perp$ in evidence, we have

$$P_\perp = \omega\rho + \frac{1}{2}\omega\rho' r + \frac{1+\omega}{4g}\rho r\left(g' + 8\pi(1+\omega)\rho r\right) \ , \tag{9.26}$$

and one sees that there is only $g(r)$ at the denominator. Thus the transversal pressure always explodes at the horizon, unless also the numerator $(1 + \omega)\rho r\left(g' + 8\pi(1+\omega)\rho r\right)$ vanishes at the horizon. That is, either $\omega = -1$, or $\rho(r_H) = 0$, or

$$\omega\rho(r_H)r_H + \frac{1}{r_H^2}\int r^2\rho\Big|_{r=r_H} = 0 \ . \tag{9.27}$$

This last condition states, using eq. (9.20) and $g(r_H) = 0$, that $\rho(r_H) = -1/8\pi\omega r_H^2$; unless $\omega < 0$, this violates any reasonable request on the density. On the other hand, also $\rho(r_H) = 0$ doesn't seem to be a good choice, because either no black hole exist (so that g never vanishes), or it forces the star to be confined within the horizon itself: that is, since black holes form from previous brighter objects, every object in the universe is a black hole!

We are left with the last choice, a negative ω. However, $\omega \neq -1$ is still not able to solve the divergence, as can be checked by direct computation. We are left with the last choice, $\omega = -1$, and this is the special case we are going to study now.

9.3.3 Special case: $P \neq P_\perp$ and $\omega = -1$

With this choice, eq. (9.20)-(9.22) gain the simpler form

$$g(r) = 1 - \frac{c}{r} - \frac{8\pi}{r} \int r^2 \rho \ , \tag{9.28}$$

$$\alpha' = 0 \ , \tag{9.29}$$

$$P_\perp = -\left(\rho + \frac{1}{2}\rho' r\right) \ . \tag{9.30}$$

These equations are much simpler than the ones in the case of general ω and, as we discussed at the end of the last subsection, they also provide the tangential pressure being finite on the horizon: indeed, one immediately sees it from eq. (9.30), as long as the density profile is suitably chosen. However, contrary to the previous cases, here $g_{00}g_{11} = -1$, so our theorem holds and so the general solution has the c/r term in it. Fortunately, this is just an apparent problem, since in the next section we will show it must be set at 0.

As we did in the previous subsections, we discuss here the case of $\rho = constant$. In this framework we have a definite radius R and eq. (9.28)-(9.30) read

$$g(r) = 1 - \frac{c}{r} - \frac{8\pi}{3}\rho_0 r^2 \ , \tag{9.31}$$

$$\alpha' = 0 \ , \tag{9.32}$$

$$P_\perp = -\rho_0 \ . \tag{9.33}$$

Of course this holds only inside the star, because outside we have only the standard Schwarzschild solution. As one can see, this time we have no divergent term (as we expected) and the only thing we still have to do is to show that $c = 0$.

Notice that, in the case $\rho = constant$, this subcase coincides with a subcase of Case 1. However, as one can easily check from eq. (9.13)-(9.15), no divergence arises in this special case and all the divergent terms of Case 1 actually vanish.

9.4 $c = 0$

In order to see that the Schwarzschild term arising from the EE vanishes, we only need to discuss the first EE: if we were able to prove it vanishes there, then it will not appear in any other equation. As we briefly argued, throughout the previous section, we only need to show $c = 0$ only in the case $P \neq P_\perp$ and $\omega = -1$, since in any other case $g_{00}g_{11} \neq -1$ and thus, due to our no-go theorem, no Schwarzschild term is expected to arise. However, since the proof turns to be the same for any configuration of the fluid, we won't reduce to a single specific case.

We discuss the classical case, where there is a definite radius R for the star. In this case, outside the star $g(r) = 1 - \frac{2M}{r}$, where $M = 4\pi \int r^2 \rho$ is the total mass of the object; the identification of M is safe. On the other hand, inside the star the metric reads

$$g(r) = 1 - \frac{c}{r} - \frac{8\pi}{r} \int \tilde{r}^2 \rho(\tilde{r}) d\tilde{r} \ , \tag{9.34}$$

where we found convenient specifying that the variable r inside the integral is a mute variable and the result of the integral, after being performed, should be calculated at $\tilde{r} = r$. If now we calculate the outer and the inner solution on the boundary of the star, we have respectively

$$g_{\text{ext}}(R) = 1 - \frac{2M}{R} \tag{9.35}$$

$$g_{\text{int}}(R) = 1 - \frac{c}{R} - \frac{8\pi}{R} \int \tilde{r}^2 \rho(\tilde{r}) d\tilde{r} \bigg|_{r=R} \ . \tag{9.36}$$

But $4\pi \int \tilde{r}^2 \rho(\tilde{r}) d\tilde{r}\big|_{r=R}$ is exactly the total mass of the black hole. If so, it is immediate to conclude that $c = 0$, just invoking the continuity of the metric. Notice that this result holds for any density profile, only providing the mass distribution to be confined.

In the non-confined case, however, things turn to be the same. In that case, we don't have to deal with the continuity of the metric on the radius of the star, since there is no definite radius and the metric is automatically continuous. However, the density profile must fulfill the classical limit, i.e. it must reduce to $\rho \to \rho_{cla}\ \theta(R-r)$ when we set to zero (or infinity) the deformation parameters; and consequently, also the solution should reduce to the classical limit, in which $c = 0$.

The key point is that c does not reduce to anything, since c is an arbitrary integration constant: in other words, if $c \neq 0$ in the deformed regime, in which there is no continuity issue (the metric is already continuous), then it would be preserved in the classical limit, in which *there is* a continuity issue. Thus, $c = 0$ in any case.

Closing the section, notice that our argument works due to the reduction of the density to the something times the step function in the classical limit: $\rho \to \rho_{cla}\ \theta(R - r)$. If, for example, the classical density reduces to zero, the argument is no more available.

In order to clear the point, consider the Hayward solution [29]:

$$g(r) = 1 - \frac{2mr^2}{r^3 + 2ml^2} \ , \tag{9.37}$$

where m and l are two parameters. This is coupled together with its generating energy density:

$$\rho(r) = \frac{1}{2\pi} \frac{3m^2 l^2}{(r^3 + 2ml^2)^2} \ . \tag{9.38}$$

The classical limit is reached when $l \to 0$ and it clearly vanishes everywhere: indeed Hayward metric is just a deformation of the Schwarzschild vacuum solution, thus there is no inner part in it and so there is no continuity to be preserved; the c/r term can be summed at will and its only effect is to modify the physical meaning of parameter m (which nobody guarantees to be the mass of the star).

Finally, it is worth to notice that this discussion makes our model radically different to that proposed in [42]. Indeed, as we already anticipated in the second chapter, in that case the classical limit was just a pure Schwarzschild, with no inner part, so the presence of an extra c/r term was perfectly allowed. On the other hand, in this case, the junction condition among the inner and the outer part of the classical limit solution forbids any extra Schwarzschild term.

9.5 Buchdahl limit

Up to now, we argued that, if $P \neq P_\perp$, $\omega = -1$ is the only reasonable choice, if we want to produce black holes. However we did not rule out Case 1, we only anticipated that $P = P_\perp$ is not a good choice. Here we show that it is not reasonable to expect black hole production, under this assumption. Indeed, we show that it is not able to solve the so called Buchdahl limit [235], i.e. the divergence of the central pressure; moreover, when we set R lower than the Schwarzschild radius, the central pressure also acquires an imaginary part.

9.5.1 Buchdahl limit "from above"

First of all, we write the central pressure. In general, we are not able to know its exact behavior, since, as we already argued, eq. (9.15) cannot be analytically solved, unless trivial cases. However we are not interested in the whole solution, but only in what happens around the origin. Here we expand the density in powers, keeping only the lowest order term: $\rho(r \to 0) = \rho_0 - \kappa r^n + ...$ With it, we are able to rewrite eq. (9.15) and approximately to solve it. The calculation is quite straightforward, however it is also quite long and boring, so we only give the final result for the central pressure:

$$P(r = 0) = \frac{3R_S}{R^3} \frac{1 - \sqrt{1 - \frac{R_S}{R}}}{3\sqrt{1 - \frac{R_S}{R}} - 1} \ , \tag{9.39}$$

where $R_S \equiv 2M$ is the Schwarzschild radius. The interested reader may find the whole calculation in the original paper [96].

As one can easily check, eq. (9.39) is the same result one obtains with the toy model $\rho = constant$. It may sound surprising, but it is just a natural consequence of the request $\rho(r \to 0) = \rho_0 - \kappa r^n + ...$, where the dominant term is the constant central density. However this also means that the isotropy assumption $P = P_\perp$ is not at all able to avoid the Buchdahl limit: the central pressure becomes infinite at $R = \frac{9}{8}R_S$, independently on the profile of the star. This imposes that every object has a classical radius (strictly) larger than its Schwarzschild radius. We recall that the classical radius R is the radius the star would have if $\lambda \to \infty$. If λ is large enough, this can be still defined as the radius of the object: in this case, we don't have a black hole. On the other hand, if λ is not large enough, we still don't have a black hole, because the density spreads and the total mass contained in the sphere of radius R_S is less than the classical one (leaving untouched all the parameters but λ).

Things look quite and crucially different, working in the anisotropic subcase $\omega = -1$. In that case, indeed, we have $P_\perp = -\left(\rho + \frac{1}{2}\rho' r\right)$ and thus we only need a suitable choice of the density profile, in order to prevent any divergence and fully avoid the Buchdahl limit.

9.5.2 Buchdahl limit "from below"

Before moving to the next section, we see what happens when $R < \frac{9}{8}R_S$ in the isotropic scenario (in the anisotropic, $\omega = -1$ case, as we saw, there are no risks). Looking at eq. (9.39), we see that the inner "Buchdahl zone" is divided in two regions: the first, for $R_S < R < \frac{9}{8}R_S$, where the central pressure becomes negative; and the second, for $R < R_S$, where it acquires an imaginary part.

In the first region, the central pressure varies from $-\frac{3R_S}{R^3}$ at R_S to $-\infty$ at the Buchdahl point ($R = \frac{9}{8}R_S$). In the second, the central pressure reads

$$P(r = 0; R < R_S) = -\frac{\rho_0}{2\left(\frac{9R_S}{8R} - 1\right)} \left(1 - \frac{3R_S}{4R} \pm \frac{i}{2} \left|\sqrt{1 - \frac{R_S}{R}}\right|\right) , \qquad (9.40)$$

thus the imaginary part cannot be set to 0. But since the pressure must always be real, this means that the isotropic fluid cannot build a star, whose radius is lower than the Schwarzschild radius – i.e. a black hole. Moreover, looking at the dynamical picture, a star typically starts collapsing for $R \gg R_S$, so we expect that also the Buchdahl radius is never reached by the collapsing matter.

9.6 A specific model of RBH

In the previous sections we argued that the only hope to build a RBH from the SET (9.2) is the anisotropy assumption, i.e. $P \neq P_\perp$, coupled with a dark energy equation of state, i.e. $P = -\rho$. In this and in the following sections, we present and discuss a specific density profile, able to produce a RBH and also able to reduce to the classical toy model $\rho = constant$ as long as we set a parameter to infinity. What deserves interest in this model is that we are able to find all our results analytically and that it satisfies some energy conditions (although violating others).

We choose the density in the form

$$\rho(r) = \rho_0 \frac{1 + e^{-\lambda R^3}}{1 + e^{\lambda(r^3 - R^3)}} \ , \tag{9.41}$$

where λ plays the role of quantum-like deformation parameter; in the limit $\lambda \to \infty$, density (9.41) reduces to the classical toy density $\rho(r) = \rho_0\ \theta(R - r)$; the presence of cubic powers in the exponentials is chosen just for calculative convenience.

As an important remark, R represents the classical radius: i.e. it is the radius of the star in the limit $\lambda \to \infty$. While working with a finite λ, the matter distribution is diffused everywhere, so no radius can be rigorously defined. However, in many cases (such as astrophysical black holes) we may expect λ to be much larger than any other comparable quantity, so that the Heaviside function is just slightly deformed. In such cases, a radius can be defined quite naturally (as example, setting it so that $\rho(R) = \frac{1}{2}\rho_0$).

As another important remark, notice that λ has a true quantum or quantum-like nature: the star described by density (9.41) has no definite radius and is extended to infinity, so it cannot be simply viewed as a classical (e.g. hydrodynamical) deformation.

With this choice, we are able to calculate the resulting metric. Eq. (9.7) gives

$$\begin{aligned} g(r) &= 1 - \frac{8\pi}{r}\rho_0 \left(1 + e^{-\lambda R^3}\right) \int \frac{r^2}{1 + e^{\lambda(r^3 - R^3)}} dr \\ &= 1 - \frac{8\pi\rho_0}{3r}\left(1 + e^{-\lambda R^3}\right)\left(r^3 - \frac{1}{\lambda}\ln\left(\frac{1 + e^{\lambda\left(r^3 - R^3\right)}}{1 + e^{-\lambda R^3}}\right)\right) \ . \end{aligned} \tag{9.42}$$

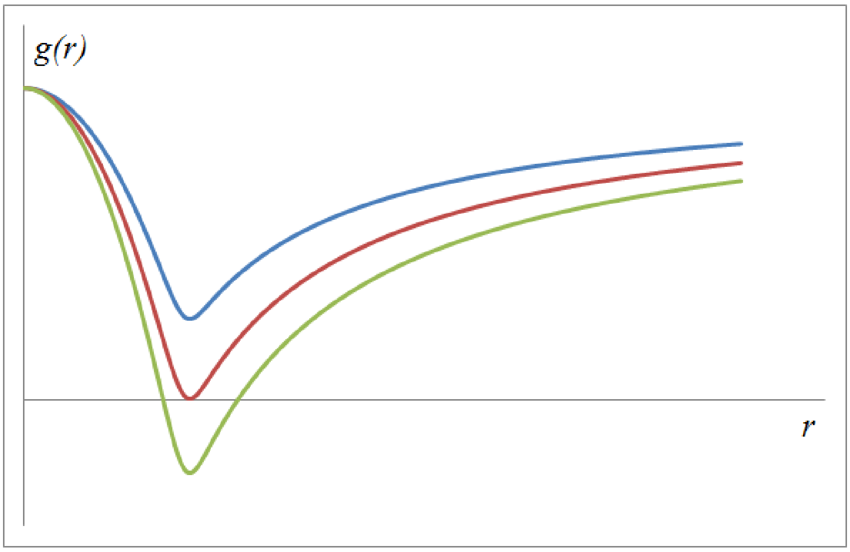
It is easy to show that it reduces to Schwarzschild outside + de Sitter inside when $\lambda \to \infty$:

$$g(r > R; \lambda \to \infty) = 1 - \frac{2M}{r} + o(\lambda^{-1}) \ , \tag{9.43}$$

$$g(r < R; \lambda \to \infty) = 1 - \frac{8\pi}{3}\rho_0 r^2 + o(\lambda^{-1}) \ . \tag{9.44}$$

Similarly, one can show that, at a fixed λ, eq. (9.42) has the same limits for $r \to \infty$ and $r \to 0$ respectively.

A final remark is in order to close the section: we said that metric (9.42) represents a black hole, so we should check it has an horizon. However the metric is too complicated to have analytical solutions for $g(r_H) = 0$, but we are able to prove that such solutions exist: indeed, manipulating parameters, one may have the picture represented in Fig. (9.1). Depending on the values of the different parameters, and most of all on ρ_0, one has different behaviors of the object.



Three different behaviors of metric function (9.42), depending on the values of the central density. The green (lower) line describes a black hole, with an outer and an inner horizon; the red (middle) line depicts an extremal black hole, with a single horizon; the blue (upper) line, with no horizons, does not represent a black hole. The figure is from [96].

Moreover, we are also able to prove that the horizon has a minimal value:

$$r_H \geq \left(R^3 + \frac{1}{\lambda} \ln 2 \right)^{1/3} . \tag{9.45}$$

The reason of eq. (9.45) lays in the minimum of the metric: if we calculate $g'(r) = 0$, we have a sum of an always strictly positive term and a term, which can change sign, depending on the parameters. Since their sum must be vanishing (because $g(r)$ has the same limit, both at infinity and 0, so a stationary point exists), this imposes a constraint among the parameters. The interested reader can recover, once again, the whole discussion in the original paper [96].

9.7 Energy conditions

In this section we show what we already anticipated, i.e. that metric (9.42) fulfills the WEC. In order to have the full discussion, we also see what happens to the NEC, the DEC and the SEC.

For the **NEC** we only need to check the positivity of the scalar $T^{\nu}_{\mu} k_\nu k^\mu$, where k_μ is a general null vector. Using our SET within the anisotropic case with $\omega = -1$, we have

$$\begin{aligned} T^{\nu}_{\mu} k_\nu k^\mu &= -(\rho + P_\perp)(k_0 k^0 + k_1 k^1) \\ &= \frac{1}{r^2}(\rho + P_\perp)\left(k_2^2 + \frac{k_3^2}{\sin^2\theta}\right) \\ &\geq 0 \ , \end{aligned} \tag{9.46}$$

meaning that $\rho + P_\perp \geq 0$, i.e. $\rho - \left(\rho + \frac{1}{2}\rho' r\right) = -\frac{1}{2}\rho' r \geq 0$. This is easily verified, since $\rho' \leq 0$ is just one of our original requests on the density.

Discuss now the **WEC**. Considering the general timelike vector X_μ, it reads

$$\begin{aligned} T^{\nu}_{\mu} X_\nu X^\mu &= -(\rho + P_\perp)(X_0 X^0 + X_1 X^1) - P_\perp \left|X_\mu X^\mu\right| \\ &= \frac{1}{r^2}(\rho + P_\perp)\left(X_2^2 + \frac{X_3^2}{\sin^2\theta}\right) + \rho \left|X_\mu X^\mu\right| \\ &\geq 0 \ , \end{aligned} \tag{9.47}$$

consisting in the two sufficient conditions

$$\rho \geq 0 \ , \tag{9.48}$$

$$\rho + \left(-\rho - \frac{1}{2}\rho' r\right) \geq 0 \ , \tag{9.49}$$

and both them are easily satisfied.

Checking the **DEC** means to prove, together with the WEC, that the vector $-T^{\mu}_{\beta} Y^\beta$ is causal and future-directed, where Y^β is a general causal and future-directed vector. We already showed that the WEC holds, so we only need to see if

$$-T^{0}_{\beta} Y^\beta \geq 0 \ , \tag{9.50}$$

$$\begin{aligned} g_{\mu\nu}(-T^{\mu}_{\beta} Y^\beta)(-T^{\nu}_{\alpha} Y^\alpha) &= \rho^2 Y_0 Y^0 + P^2 Y_1 Y^1 + P_\perp (Y_2 Y^2 + Y_3 Y^3) \\ &= -\left(\frac{1}{r^2}(\rho^2 - P_\perp^2)\left((Y_2)^2 + \frac{(Y_3)^2}{\sin^2\theta}\right) + \rho^2 \left|Y_\beta Y^\beta\right|\right) \\ &\leq 0 \ . \end{aligned} \tag{9.51}$$

The first equation reads $\rho Y^0 \geq 0$ and since both ρ and Y^0 are positive, it is immediately satisfied. Regarding the second, it is possible to see that it implies the condition $\rho \geq -\frac{1}{4}\rho' r$. In general it is not possible to state if it holds or not, since it depends on how $\rho' r$ behaves w.r.t. ρ. We can however discuss it in the case of density (9.41), having then

$$1 + \left(1 - \frac{3}{4}\lambda r^3\right) e^{\lambda(r^3 - R^3)} \geq 0 \ , \tag{9.52}$$

and it is easy to see that, for $r \to \infty$, this is badly violated.

The last thing we are left with is checking whether the **SEC** is satisfied or not. In other words, we need to check if $\left(T^\nu_\mu - \frac{1}{2}T\delta^\nu_\mu\right) X_\nu X^\mu \geq 0$, where T is the trace of the SET and X_μ a general causal vector. We have

$$\begin{aligned}\left(T^\nu_\mu - \frac{1}{2}T\delta^\nu_\mu\right) X_\nu X^\mu &= -(\rho + P_\perp)(X_0 X^0 + X_1 X^1) - \frac{1}{2}(\rho - P)\left|X_\mu X^\mu\right| \\ &= (\rho + P_\perp)\left((X_2)^2 + \frac{(X_3)^2}{\sin^2\theta}\right) + P_\perp \left|X_\mu X^\mu\right| \\ &\geq 0 \ .\end{aligned} \tag{9.53}$$

This imposes the two constraints $\rho + P_\perp \geq 0$ and $P_\perp \geq 0$, which respectively read

$$-\frac{1}{2}\rho' r \geq 0 \tag{9.54}$$

$$\rho + \frac{1}{2}\rho' r \leq 0 \tag{9.55}$$

The first equation is satisfied, due to the NEC. On the other hand, the second cannot be discussed immediately, since ρ and ρ' have opposite signs and may have very different behaviors. However, if we want to check that the SEC is violated, it is enough to show it is for some region of space. In order to do this, focus around the origin: in this limit, the density reads $\rho(r \to 0) = \rho_0 - \kappa r^n + ...$ with $\kappa > 0$, so that cond. (9.55) reads

$$\kappa \geq \frac{2\rho_0}{(n+2)r^n} \ . \tag{9.56}$$

If the SEC were fulfilled, this would hold for any r around the origin and it's clear it cannot be: thus, the SEC is violated.

We close the section observing that this discussion does not violate the Penrose-Hawking theorem [23, 104, 105]. Indeed the theorem only requires the violation of *some* condition on energy, without specifying which one. We already saw that the DEC and the SEC are actually not fulfilled, so our results still agree with the singularity theorem.

9.8 Limit $R \to 0$

It is interesting to study what happens to metric (9.42) in the limit of $R \to 0$. Classically, it corresponds to a point-like source. The interesting thing is that the properties of the solutions are preserved also in this regime.

First of all, the limit $R \to 0$ can be performed exactly and we get

$$g(r; R \to 0) = 1 - \frac{16\pi\rho_0}{3r}\left(r^3 - \frac{1}{\lambda}\ln\left(\frac{1+e^{\lambda r^3}}{2}\right)\right) , \tag{9.57}$$

which still is a RBH (setting suitable values for the parameters). Indeed, in the asymptotic regimes, g reads

$$g(r \to \infty; R \to 0) = 1 - \frac{2M_0}{r} + o(r^{-2}) , \tag{9.58}$$

$$g(r \to 0; R \to 0) = 1 - \frac{8\pi\rho_0}{3}r^2 - \frac{2\pi\rho_0}{3}\lambda r^5 + o(r^8) , \tag{9.59}$$

where $M_0 \equiv \frac{4}{3}\pi\left((\ln 2/\lambda)^{1/3}\right)^3(2\rho_0)$. This first limit is of particular interest: indeed, it states that, even if the matter distribution has a vanishing radius, the black hole still has some residual mass. This confirms that λ has a quantum meaning: indeed the only classical solution for a matter distribution of vanishing radius is *exactly* the Schwarzschild solution; here, we recover it only asymptotically. By the way, if we send λ to infinity, we see that metric (9.57) correctly reduces to Minkowski: this is not surprising, because the density does not reduces to a Dirac delta, but to an ordinary functions, which is non-vanishing only in the origin.

9.9 Limitations and problems of the static picture

Not everything is solved by this framework and also our model has some major problems. Here we discuss two of them, probably the trickiest ones. They emerged quite recently and only after the completion of paper [96], so they do not appear in it. It is worth to notice that both them produce qualitatively the same result, i.e. an instability of the metric.

9.9.1 The Cauchy Horizon

Working within the spherical framework, Sakharov criterion imposes the metric to become positive again: thus every spherical RBH has (at least) two horizons: one is the standard event horizon; the second is the inner horizon, also known as Cauchy horizon, produced by the deformation of the Schwarzschild metric.

After it, space and time, which changed sign in the negative region, gain their standard meaning again.

As proved by Maeda, Torii and Harada [236], the Cauchy horizon is related to an instability of the metric. Here we don't discuss again the whole calculation, only present the final result.

Their argument is related to a scalar field coupled to GR, but its effect can be easily adapted to our fluid framework: indeed, we only assumed the SET to be diagonal with independent components ρ, P and $P_\perp$ and this assumption fits well with a number of generating theories. Other works dealing with inner horizon instability are [237–239] and, in some sense *ante litteram*, [240, 241] (for a more modern approach, see [242, 243]).

The result of [236] is then the following: applying a perturbation on the scalar field, it turns that

$$\delta\phi'' \propto \exp\left(-f'_{(b)} e^{A_{(b)}}\, v\right) \ , \tag{9.60}$$

where the subscript (b) indicates the background (i.e. unperturbed) solution. Here we used the notation of [236], which is different from ours: indeed their metric has the form

$$ds^2 = -fe^{2A}dv^2 + 2e^A dvdr + r^2 d\Omega^2_{n-2} \ , \tag{9.61}$$

in n dimensions, with r having the standard meaning of radial coordinate. Actually this is not a major qualitative difference, since it turns that the sign of f is the same of our g_{11} (i.e. g, in our notation).

Now, if the global sign of the exponential in eq. (9.60) is negative, the perturbation has a dumping term and it's safe; this happens for $f' > 0$, which is the case of the standard event horizon. On the other hand, when $f' < 0$, the global sign becomes positive and the dominant part of the perturbation acquires an exponential increasing term.

9.9.2 Sound velocity

As we already pointed out in our recent paper [66], setting $\omega = -1$ has consequences also on the sound velocity. Indeed, the radial speed of sound c_s into a fluid reads

$$c_s^2 = \frac{dP}{d\rho} \ , \tag{9.62}$$

where P is still the radial pressure. In the case of $\omega = -1$, $c_s^2 = -1$ and this is once again associated with an instability of the solution [244]. Qualitatively this can be seen calculating the propagation equation for the perturbation: indeed, an imaginary sound velocity changes an elliptic equation into an hyperbolic one, thus causing a divergent behavior in the propagation. As one can see and as we anticipated, this result is qualitatively similar to that of the previous subsection.

It appears there is no way to circumnavigate the instability problems, within the fluid approach: indeed, we argued that $P \neq P_\perp$ and $\omega = -1$ seems to be the only way to produce a RBH in this way, thus resulting in an exponentially increasing propagation of the perturbation. Moreover, since the Sakharov criterion forces us to have a de Sitter core near the origin, the presence of a Cauchy horizon is, at the same, inevitable.

There may be three attempts to solve the problem and, so to say, save the day. The first attempt, as already pointed out by Hayward in [29], is to consider the static solution only as an approximate result, not realized in nature: thus the true picture is dynamical and, if perturbations propagate slowly enough, there is time for a RBH to live sufficiently long, before collapsing, exploding or whatever (depending on the model). In the next and final section, we will briefly discuss this possibility, though only qualitatively.

The second attempt is to avoid abruptly the Cauchy horizon: indeed the Sakharov criterion forces the metric to be $g(r) = 1 + \kappa r^2 + ...$ only in the case of spherical symmetry, which is not necessarily realized. If we move to an hyperbolic space, where curvature changes sign, the Sakharov criterion forces the metric to be $g(r) = -1 + \kappa r^2 + ...$ and this allows to completely avoid the inner horizon. The problem of this attempt is that spacetime appears to be at least locally spherical and not hyperbolic; thus it would imply that RBH as astrophysical objects were not expected to exist − which we know is (presumably) wrong.

The last attempt is the most interesting, because it proposes to completely avoid horizons: in this case, we have objects like *quasi*-black holes, i.e. very compact objects, which however may not be unstable (they also may be, but the instability of a non-black hole is much safer than the instability of a black hole). At the moment, although some literature is present (see e.g. [245]), this is just an interesting and fascinating speculation; however, due to the observation of a black hole-black hole merger event by the LIGO-VIRGO collaboration, observational results may seem to be expected in the next years.

9.10 The dynamical picture

We close the chapter proposing an analysis of what happens if we turn on also the dynamical frame of our model. We immediately declare that (probably) there is no hope to find full analytical solutions; however, we are interested only in the qualitative and asymptotic behavior.

First of all, we consider the dynamical metric

$$ds^2 = -f(r,t)dt^2 + \frac{1}{g(r,t)}dr^2 + r^2 d\Omega^2 \ . \tag{9.63}$$

Along with it, we also assume the SET to be the same (in form) of the static case

$$T^\mu_\nu = \rho u^\mu u_\nu + (P - P_\perp)\delta^1_\nu \delta^\mu_1 + P_\perp(\delta^\mu_\nu + u^\mu u_\nu) \ , \tag{9.64}$$

where u_μ is the four-velocity of the fluid, ρ its density, P its radial pressure and $P_\perp$ its tangential pressure. Of course, we also require that, at any time, the density fulfills the same properties of the static case:

- ρ is always positive and finite, i.e. $0 \le \rho < \infty$;
- ρ is monotonically decreasing, i.e. $\rho' \le 0$.

The only difference of these quantities with their static counterparts is that they have also a time dependence; in particular, this means that $u^0 u_0 \neq -1$ and $u_\mu = (u_0, u_1, 0, 0)$; of course, we still have $u_\mu u^\mu = -1$.

Due to this choice, the new EE are:

$$\frac{g'r + g - 1}{r^2} = 8\pi(\rho + P_\perp)u^0 u_0 + 8\pi P_\perp \ , \tag{9.65}$$

$$-\frac{\dot{g}}{r} = 8\pi(\rho + P_\perp)u^1 u_0 \ , \tag{9.66}$$

$$\frac{g'r + \alpha' g r + g - 1}{r^2} = 8\pi P + 8\pi(\rho + P_\perp)u^1 u_1 \ , \tag{9.67}$$

$$\frac{\alpha' g}{2r} + \frac{g'}{r} + \frac{g'' + \alpha'' g}{2} - \frac{3\alpha' g'}{4} - \frac{\dot{\alpha}\dot{g}}{4g^2 e^\alpha} + \frac{\alpha'^2 g}{4} - \frac{\dot{g}^2}{g^3 e^\alpha} + \frac{\ddot{g}}{2g^2 e^\alpha} = 8\pi P_\perp \ . \tag{9.68}$$

Along these, we also need to add the conservation equation $\nabla_\mu T^\mu_r = 0$ for the fluid itself; that is,

$$\begin{aligned} &\partial_t\left((\rho + P_\perp)u^0 u_1\right) + \partial_r\left(P + (\rho + P_\perp)u^1 u_1\right) \\ &- \frac{g' + \alpha' g}{2g}\left(P_\perp + (\rho + P_\perp)u^0 u_0\right) \\ &- \frac{2}{r}P_\perp + \frac{1}{2g}\left(\dot{g} + \dot{\alpha} g + \frac{\dot{g}}{g^2}e^{-\alpha}\right)(\rho + P_\perp)u^0 u_1 \\ &+ \left(\frac{g' + \alpha' g}{2g} + \frac{2}{r}\right)\left(P + (\rho + P_\perp)u^1 u_1\right) = 0 \ . \end{aligned} \tag{9.69}$$

For sake of simplicity and notational convenience, we already wrote $f = e^\alpha g$, with α an arbitrary function, inside eq. (9.65)-(9.69).

These equations do not appear to be easily solvable, at least trough analytical approaches. However, as we already anticipated, we are not really interested in their general solution, but only in their solution in a very specific range of time: we only look for their asymptotic behavior for $t \to \infty$.

The first thing we should do is studying the density. We recall that in the static case it was

$$\rho(r) = \rho_0 \frac{1 + e^{-\lambda R^3}}{1 + e^{\lambda(r^3 - R^3)}} , \tag{9.70}$$

with constant λ, ρ_0 and R. We chose this object because it has the property, for $\lambda \to \infty$, to reduce to the Heaviside function: $\rho(r; \lambda \to \infty) \to \rho_0 \; \theta(R - r)$; i.e. the classical toy model. For the dynamical case, it is reasonable to expect something similar: i.e. that, in the classical limit,

$$\rho(r, t; \lambda \to \infty) \to \rho_0(t) \; \theta(R(t) - r) . \tag{9.71}$$

Of course this is just another toy model, where we are assuming a sufficiently slow compression of the star, so that, at any time, the fluid is almost static and the static solution holds in an approximate way. If so, it is reasonable to write the dynamical density using expression (9.70), only introducing a time variation for its parameters. It follows that a good model for the density is

$$\rho(r, t) = \rho_0(t) \frac{1 + e^{-\lambda R^3(t)}}{1 + e^{\lambda(r^3 - R^3(t))}} . \tag{9.72}$$

If we focus on a time interval small enough, R and ρ_0 can be taken as constant, so that eq. (9.72) reduces to its static expression.

It is worth to notice that our results hold for any way of propagation of the perturbations. The only difference between a slow and a fast perturbation is the time scale of validity the static approximation and, thus, how fast are the parameters into (9.72) to change.

Now, eq. (9.65)-(9.69), coupled with density (9.72) appear to be too much complicated to have exact solutions. However we are able to find some approximations. First of all, from eq. (9.65) we immediately have

$$g(r, t) = 1 + \frac{\xi(t)}{r} + \frac{8\pi}{r} + \int r^2 (P_\perp + (\rho + P_\perp) u^0 u_0) dr , \tag{9.73}$$

where $\xi(t)$ is an arbitrary function of time. However, fortunately, it should be set at 0: indeed, studying eq. (9.73) for a suitably small time interval around some fixed time t_0, the local static approximation would present a Schwarzschild term $\xi(t_0)/r$ and we already proved that the static case cannot present such a term.

This appears to be the only exact result. However, we can greatly simplify the whole discussion, if we assume $u^0 u_0 \simeq -1$ and consequently the other combinations. In particular, we expect this approximation to hold at $t \to \infty$, since we expect the black hole to be evaporated and so to reduce to some final (and stable) condition. In principle, this could also be a full evaporation, which result in a final Minkowski state.

Within this approximation, integral (9.73) can be solved. Thus, if we assume $t = t_0 + \tau$, where t_0 is some fixed time, large enough, and τ small enough, we have that

$$g(r, t_0 + \tau) = 1 - \frac{8\pi\rho_0(t)}{3r}(1 + e^{-\lambda R^3(t)})\left(r^3 - \frac{1}{\lambda}\ln\left(\frac{1 + e^{\lambda(r^3 - R^3(t))}}{1 + e^{-\lambda R^3(t)}}\right)\right) + ... \tag{9.74}$$

We can now explicitly implement the limit. Since we did not assume any quantum mechanism to avoid the full evaporation of the black hole, it is reasonable to assume $R(\infty) = 0$ and $\dot{R}(\infty) = 0$, although there is no need of it (in any case, the qualitative picture would not change too much). Thus, the leading term reads

$$\begin{aligned} g(r, t_0 + \tau; t_0 \to \infty) =& 1 - \frac{8\pi\rho_0(\infty)}{3r}\left(r^3 - \frac{1}{\lambda}\ln(1 + e^{\lambda r^3})\right) \\ & - \frac{8\pi\dot{\rho}_0(\infty)}{3r}\left(r^3 - \frac{1}{\lambda}\ln(1 + e^{\lambda r^3})\right)\tau + ... \end{aligned} \tag{9.75}$$

This result is quite nice, because it states that the two limits $R \to 0$ and $t \to \infty$ coincide. And since we proved that the first limit still produces a RBH, so also the second does.

We have then the following picture:

• we start from (non necessarily classical) star, with some initial classical radius $R(t = 0)$ and some density distribution $\rho(r; t = 0)$; the solution is initially static;

• time passes and the star collapses to its centre; ρ_0 increases and R decreases respectively; we don't need to assume it to be "slow", since the only modification is the time scale of validity of the static approximations.

• after a sufficiently long time (we guess, when $R \lesssim \lambda$), the dynamical component becomes negligible and we are left with the static residual g_∞ alone;

• notice that this is reached only asymptotically and the true solution is still $g_\infty + ...$; perturbations continue to hold at any finite time.

The interesting point of this scheme is that, independently on how perturbations propagate in the solution, due to the gravitational collapse, the asymptotic result seems to be stable. This may be viewed as an argument to cancel (at least asymptotically) the divergent part of the perturbations. Since we are not able to study the solution at any time, so that we are not able to study the behavior of perturbations, this is much more an argument, rather than a proof.

As a final remark, it is worth to notice that such black holes never completely evaporate, but still, at any time, they present a remnant with the residual mass $M_{res} = (\rho_\infty \ln 2)/3\lambda$. This mass is 0 in the limit of $\lambda \to \infty$, thus we have another argument to enlighten the quantum nature of λ: if it were just a classical parameter, there would be no reason for it to be present also when the whole fluid is evaporated.

10 Conclusions

In this final section, we want to summarize, chapter by chapter, our results and trying to speculate on some possible future directions for any further research.

10.1 The "negative" part

We start with the negative results, with a special interest for the NED approach and its generalization in the no-go theorem. In their framework, we proved that a singular term c/r cannot be avoided in the general solution and thus that framework is not the most suitable, to produce RBHs. We reviewed some solutions already present in literature, but in any case we had to include the Schwarzschild-like term. It is worth to notice that we showed it by direct calculation and, in the case of the theorem, we gave a complete proof.

In particular, the theorem is important, because it generalizes an explicit calculation, we carried out in the NED case. In that case, as we already showed in chapter 3, we made use of the specific properties of the NED approach, while in the following chapter the discussion is carried out in general. This allows to apply the theorem even for the ANSS black hole or for the fluid approach: there is no hope to avoid the Schwarzschild term in the general solution.

It is important to stress that we did not exclude *at all* the possibility of having regular solutions. Indeed, what we need to do in such cases is to find some physical condition, able to produce a suitable behavior for the constant of integration (typically, setting it to 0). This is indeed what we did in the ninth chapter. We also do not exclude the importance of such approaches for any further study. In particular, referring to NED, we should say that its reconstruction scheme is a very powerful tool, viable for many applications. Just to make an example, in [247] a rotating NED solution has been built, which shows the interest this approach diserves (we recall that physical black holes are expected to be rotating, so this is a crucial point for future phenomenology).

We also take here the occasion to notice that the theorem was proved at Lagrangian level, but it was not necessary; with an immediate rearrangement of its statement, the proof could be carried out also working only on the SET, due to the lemma. This is why we could apply it to the fluid approach, although the formulation was not exactly the same.

Finally, for the conformal approach we also studied in the third chapter, we proved it is not able to produce RBHs, and also in this case we showed it by direct calculation. Regarding this particular approach, however, we should mention that our proof could be not definitive, because there have been some indication that (maybe, not for certain) our argument may be circumnavigated. We did not put these considerations inside the main discussion, because they are the result of some conversation and at the present there is no proof they are correct. We should also mention that we are skeptical on them, so we trust our discussion in chapter 3, but we decided to mention this possibility, at least in the conclusion, for a sake of completeness. If fruitful, things will be discussed in future papers.

10.2 The "positive" part

The second part of this work, illustrating some examples of *working* ways to produce RBHs is maybe even more interesting. In the fifth chapter, the first of this second section, we presented some model of actually true RBHs. In any case, we should mention that they are "actually true" and not just "true always and everywhere": actually, there is no proof they present some trouble or contradiction (unless the instability due to the Cauchy horizon, which is however a feature of any spherical RBH). Of course, however, we cannot exclude some problem would be found in future research, such as an extreme instability. In that case, the static form of all the three solutions we studied should be intended only as an approximate case and the discussion should be carried out in the dynamical framework – which turns to be a much more difficult task.

Among the approaches we briefly presented, the most interesting one is certainly the NPG approach; this because it is actually somewhat intuitive and simple in concept (although not so simple in realization) and also because it is able to reproduce the other two, as shown in [66] and [162]. The Yang-Mills and the Chamseddine-Mukhanov black holes appear then to be just special cases of the NPG formulation.

The same happens for the application to the cosmological sector, in which the NPG results implement quite simply the original argument of Helling and Date [187,188] and is able to build a complete discussion: it allows to interpret the LQC corrections as purely geometrical corrections to GR. Indeed, we saw the corrections leading to second order equations of motion in their FLRW sector, therefore without involving additional degrees of freedom with no direct geometrical interpretation.

In some sense, this might be expected, since the singularity is a problem also present in the vacuum (Schwarzschild) solution. It is then reasonable that any effective action will give the Einstein-Hilbert one as leading term, plus higher energy corrections; and since these correction should also appear in vacuo, they should have a geometrical interpretation.

An advantage of modifying the geometrical content of the theory is that regular solutions can be found already when a point-like mass is considered (which is not the case, e.g., in the fluid approach). Therefore, given a suitable distribution of matter, these models might provide RBH solutions. On the contrary, if one modifies the matter content in a given way, it remains to be explained why every kind of matter should behave in a specific way at small distances.

This became much relevant in the following chapter, when we discussed in more detail a cosmological model, based on NPG: that's the sixth chapter. In that chapter (and, before, in the original paper), we found an explicit covariant Lagrangian formulation for the LQC tree level correction [195] to Friedmann equation, expressed as an infinite sum of NPG corrections to Einstein-Hilbert action. We also showed that they provide a suitable effective action in FLRW because, despite their non-polynomiality for general metrics, their contributions evaluated on FLRW spacetime are just polynomial. Finally, we also found the exact solution of the model in presence of a (positive) cosmological constant and a perfect fluid with state parameter ω; we showed it represents a bounce, replacing the Big Bang singularity of GR.

Finally, despite being "just" a special case, also the mimetic approach of [147, 149, 151–158, 246] deserves some interest, in order to understand a semi-classical formulation of LQG. The mimetic approach has been indeed used in a number of works and there might be a large class of theories, having the same property to convert additional fields without a clear geometrical meaning into ones related to geometry via Lagrange multipliers.

The last chapters are once again devoted to the black hole space. In the seventh and the eighth we deal with a scalar matter field (no mimetic meaning, this time) and we are able to reproduce some conditions to have RBHs. In the two chapters we work within different frameworks, but the idea behind the scene is the same.

As we already said, we have been not able to find explicit solutions, but we have been able to find a scheme to produce them. We also showed that there is no risk to find the c/r term, because, when it appears, it is only apparent, so that the regularity can be set safely, just modelling the parameters of the scalar potential (or even the potential itself).

Regarding our discussion in chapter 7, however, we have to notice that [248], working within the same framework of us, found a regular, but unstable solution. At the present, we are not able to say, if that instability is strictly related to their specific solution (so that other ones might be stable) or it is a general feature of black holes coupled with scalar fields. In the latter case, our picture would be only a local approximation of a more complicated dynamical framework.

Finally, the ninth chapter discusses how to build a RBH, only using some exotic fluid instead of standard matter or vacuum. The most relevant results of this chapter are that the standard maximal symmetry assumption $P = P_\perp$

is not able to produce any black hole, neither singular, nor regular; and that, assumed $P \neq P_\perp$ and $P = \omega\rho$, regular solutions are available only for $\omega = -1$.

Another important feature of this ninth chapter is a physical criterion to avoid the Schwarzschild term: indeed, a c/r is present in the general function, but we must set $c = 0$ due to the continuity of the metric. This is something really important: other similar models do not possess it, since their limit is a point-like mass, instead of some matter distribution. As we argued in [96], this is enough to answer to the objections of [249].

A final remark about this chapter is its dynamical behavior. We only studied it approximately, but the final result (i.e. that the dynamical black hole has an asymptotic and quasi-static residual) deserves some attention due to an interesting possibility. Indeed, $P = -\rho$ is the equation of state for dark energy and it is credible that such objects formed already at the early stage of the universe: if so, their residuals are still within us and, since they have a non-vanishing mass, they may be a candidate for present day dark matter. This possibility has already been explored, although not in our framework [86, 250, 251]. Actually we don't have the whole dynamical analysis, so this is just a (though fascinating) guess.

11 Appendix A: Extension of the no-go theorem to the $F(R)$ case

11.1 General framework

In this appendix, we discuss the case of the no-go theorem of chapter 4, but extended to an $F(R)$ theory. $F(R)$ gravity is a very important framework (see e.g. [252–257]), so proving the theorem also in this case would be of high interest. A whole proof cannot be carried out and, in general, we are not able to say if statement (4.7) and the consequent corollary still hold. This is why we discuss it in an Appendix.

In order to explore the new framework, we take the same assumptions (4.1)-(4.6), with the obvious modifications for (4.2) and (4.3); indeed, they now become respectively

$$I = \int d^4x \sqrt{-[g]} \left(F(R) - \mathscr{L}(\phi_{\alpha_1 \dots \alpha_n}, \nabla \phi_{\alpha_1 \dots \alpha_n}) \right) , \qquad (11.1)$$

and

$$\partial_R F R^\nu_\mu - \frac{1}{2} F(R) \delta^\nu_\mu = g^{\nu\rho} \frac{\partial \mathscr{L}}{\partial g^{\mu\rho}} - \frac{1}{2} \mathscr{L} \delta^\nu_\mu , \qquad (11.2)$$

where F is some scalar function of R and all other meanings remain the same of the standard case (we refer to chapter 4 for notation).

11.2 Lemma

As we did in the standard case, first of all we prove the preparatory lemma, whose statement now reads

> The matter Lagrangian $\mathscr{L}(\phi_{\alpha_1 \dots \alpha_n}, \nabla \phi_{\alpha_1 \dots \alpha_n})$ does not depend on the solution (f, g) if and only if the SET T^ν_μ does not depend on (f, g) too:
>
> $$\partial_{(f,g)} \mathscr{L} = 0 \qquad \Longleftrightarrow \qquad \partial_{(f,g)} T^\nu_\mu = 0 \ . \qquad (11.3)$$

One may immediately notice that this is exactly the same statement (4.8). Indeed, the lemma involves only the rhs of the EE, which is not modified by the new action (11.1). It is then easy to see that the lemma itself is also proved, since also the proof of the standard case only involves the metric, the Lagrangian and the SET – which haven't been modified at all by our new assumptions. One can check it by hand, just following the same steps we already performed in chapter 4.

11.3 Proof of the theorem – part one

For a mere matter of convenience, we start proving the equivalence among statements (b) and (c). We study separately the two implications.

11.3.1 $\partial_{\bar{f},\bar{g}}\mathscr{L} = 0 \ \Rightarrow\ \bar{f} = \bar{g}$

We start writing explicitly the three independent components of eq. (11.2): we have, writing them for $\bar{f}$ and $\bar{g}$,

$$\partial_R F\left(\frac{\bar{f}''\bar{g}}{2\bar{f}} + \frac{\bar{f}'\bar{g}'}{4\bar{f}} + \frac{\bar{f}'\bar{g}}{\bar{f}r} - \frac{\bar{f}'^2\bar{g}}{4\bar{f}^2}\right) + \frac{1}{2}F = T_0^0(\bar{f},\bar{g}) \ , \tag{11.4}$$

$$\partial_R F\left(-\frac{\bar{f}''\bar{g}}{2\bar{f}} - \frac{\bar{f}'\bar{g}'}{4\bar{f}} - \frac{\bar{g}'}{r} + \frac{\bar{f}'^2\bar{g}}{4\bar{f}^2}\right) - \frac{1}{2}F = T_1^1(\bar{f},\bar{g}) \ , \tag{11.5}$$

$$\partial_R F\left(\frac{\bar{g}'r + \bar{g} - 1}{r^2} + \frac{\bar{f}'\bar{g}}{2\bar{f}r} - \frac{\bar{g}'}{2r}\right) + \frac{1}{2}F = T_2^2(\bar{f},\bar{g}) \ . \tag{11.6}$$

Combining eq. (11.4) and (11.5), we immediately find that

$$\partial_R F\left(\frac{\bar{f}'}{\bar{f}} - \frac{\bar{g}'}{\bar{g}}\right) = r\left(\bar{f}\partial_{\bar{f}}\mathscr{L} + \bar{g}\partial_{\bar{g}}\mathscr{L}\right) \ . \tag{11.7}$$

Assuming that $\partial_R F \neq 0$, since the rhs of this equation vanishes, also the lhs must, meaning immediately that $\bar{f} = \bar{g}$.

11.3.2 $\partial_{\bar{f},\bar{g}}\mathscr{L} = 0 \ \Leftarrow\ \bar{f} = \bar{g}$

In this case, the proof is quite simplified and eq. (11.4)-(11.6) read

$$\partial_R F\left(\frac{\bar{f}''}{2} + \frac{\bar{f}'}{r}\right) + \frac{1}{2}F = T_0^0(\bar{f}) \ , \tag{11.8}$$

$$-\partial_R F\left(\frac{\bar{f}''}{2} + \frac{\bar{g}'}{r}\right) - \frac{1}{2}F = T_1^1(\bar{f}) \ , \tag{11.9}$$

$$\partial_R F\frac{\bar{f}'r + \bar{f} - 1}{r^2} + \frac{1}{2}F = T_2^2(\bar{f}) \ . \tag{11.10}$$

Combining the first two, one immediately sees (unless the trivial but unacceptable case of $\bar{f} = 0$) that $\partial_{\bar{f}} \mathscr{L} = 0$.

One still has to deal with the case of $\partial_R F = 0$. However this case is of little interest, because it means that $F(R) = \textit{constant}$ (w.r.t. R) and it is clearly not able to reconstruct the standard case of $F(R) = R$. So, we won't bother it any further.

11.4 Proof of the theorem – part two

In order to prove the theorem also in this framework, it is enough to show that statements (a) and (b) are equivalent; the properties of equivalence will make the rest. However, in this case, we start facing problems.

As done in the standard proof, we prove the two equivalence separately.

11.4.1 (f, g) is a solution $\Leftarrow$ $f = g$

We start with this proof, because it is much easier to be carried out. First of all, we notice that the curvature, written in terms of f, is the same to that written in terms of $\bar{f}$. Indeed,

$$\begin{aligned} R &= -\bar{f}'' - \frac{2c}{r^3} - \frac{4\bar{f}'}{r} + \frac{4c}{r^3} - \frac{2\bar{f}}{r^2} - \frac{2c}{3} + \frac{2}{r^2} \\ &= -\bar{f}'' - \frac{4\bar{f}'}{r} - \frac{2\bar{f}}{r^2} + \frac{2}{r^2} \\ &= \bar{R} \ . \end{aligned} \tag{11.11}$$

Thus said, eq. (11.4) and (11.6), the only independent components of the EE, now read

$$\partial_R F \left(\frac{\bar{f}''}{2} + \frac{\bar{f}'}{r} \right) + \frac{1}{2} F = T_0^0(\bar{f}) \ , \tag{11.12}$$

$$\partial_R F \frac{\bar{f}' r + \bar{f} - 1}{r^2} + \frac{1}{2} F = T_2^2(\bar{f}) \ . \tag{11.13}$$

Once we know that $\partial_R F$ is the same both for the barred and the unbarred case, it is immediate to see that the lhs of these two equations are the same, if calculated for f instead that for $\bar{f}$. At this point, the argument is carried out in the same way we did for the standard case and the proof is complete.

11.4.2 (f, g) is a solution $\Rightarrow$ $f = g$

This time, things are different. If we combine the tt components of the EE, written for $(\bar{f}, \bar{g})$ and (f, g) respectively, we have

$$
\begin{aligned}
&\partial_R F\left(\frac{f''g}{2f}+\frac{f'g'}{4f}+\frac{f'g}{fr}-\frac{f'^2g}{4f^2}\right)+\frac{1}{2}F\\
&-\left(\partial_{\bar{R}} F\left(\frac{\bar{f}''\bar{g}}{2\bar{f}}+\frac{\bar{f}'\bar{g}'}{4\bar{f}}+\frac{\bar{f}'\bar{g}}{\bar{f}r}-\frac{\bar{f}'^2\bar{g}}{4\bar{f}^2}\right)+\frac{1}{2}F\right)=\\
&=T_0^0(f,g)-T_0^0(\bar{f},\bar{g})\ .
\end{aligned}
\tag{11.14}
$$

We cannot proceed with any major cancellation in this equation, because we cannot be sure that $F(R)$ and $F(\bar{R})$, written in terms of $\bar{f}$, $\bar{g}$ and c, are the same. If yes, eq. (11.14) becomes simply

$$
T_0^0(f,g)-T_0^0(\bar{f},\bar{g})=0\ , \tag{11.15}
$$

and the proof proceeds as in the standard case. The point is that this seems to be just a matter of luck: indeed the expression of the Ricci scalar, if $f \neq g$ (which in general is), is highly complicated and the c/r terms do not vanish trivially (as happens, if $f = g$). Eq. (11.14) do not simplify and we have no proof the equivalence among (a) and (b) holds; although we have been able to prove that (b) $\Rightarrow$ (a).

12 Appendix B: The *quasi*-black hole

During the course of the work, and in particular in the ninth chapter, we discussed some problems which appear to be connected with RBHs. In particular, we found that a fluid black hole solution always suffers of at least two instabilities, due to the presence of a Cauchy horizon and a negative sound velocity. We also showed that there is no way to avoid them: if we want to have a black hole, it must have an event (or apparent) horizon; and if we want it to be regular, it must have also an inner horizon. This holds for any spherical RBH. Moreover, at least within the fluid approach, we found that the squared sound velocity turns to be negative, with the consequent (in)stability problem.

In principle one could invoke the dynamical framework, thus considering the static solution only as a local approximation and working only with small time scales, compared to that of the perturbation: in that case, quasi static black holes can be found, with no much trouble caused by the perturbation. This is what we did in the final section of the ninth chapter. In this appendix, we want to give a look also to the approach of quasi black holes (QBHs). For any detail, however, we remand to the literature (such as [227, 245] and references therein).

The idea dates more than seventy years, since the first QBHs are even older than the concept of black hole itself [258, 259] and consists in having a gravitating object, whose size is near to its horizon, but a little larger: thus, working on a spherically symmetric metric (for simplicity, we will take $f \equiv -g_{00} = g^{11}$), one has

$$0 < f(r_H) \ll 1 \ , \tag{12.1}$$

where we indicate with r_H the (quasi) horizon of the QBH. If we write the metric function in the usual way, i.e.

$$f(r) = 1 - \frac{2M(r)}{r} \ , \tag{12.2}$$

this turns on a condition on the mass function M, i.e.

$$M(r_H) < \frac{1}{2} r_H \ . \tag{12.3}$$

It is clear that such a metric can be used with no trouble of a Cauchy horizon, because actually a Cauchy horizon does not exist. Inside the horizon, the metric could have any behavior, including also the regular quadratic core.

Condition (12.3) imposes an upper bound on the mass, but there is no reason to think it should be unrealistic. Indeed, take for example an Hayward-like metric, namely

$$f(r) = 1 - \frac{2mr^2}{r^3 + l^3} \ , \tag{12.4}$$

with m and l suitable parameters, so that the mass function is $M(r) = \frac{mr^3}{r^3+l^3}$. This establishes a relation between the two parameters m and l:

$$\frac{mr_H^2}{r_H^3 + l^3} < \frac{1}{2} \ . \tag{12.5}$$

Contrary to what happens in the standard black hole case, however, this time we are able to find explicitly the location of the (quasi) horizon (notice that this is not a general feature, but still depends on the shape of the mass function): indeed, it is located at the minimum of the metric and, in the Hayward-like case, this results in $r_H = 2^{1/3} l$. Thus, eq. (12.5) reads

$$m < \frac{3}{2^{5/3}} l \simeq 0.945 \ l \ . \tag{12.6}$$

If we want to build a macroscopic object with a macroscopic mass, l has to be also macroscopic. In principle, this is not a major problem, since l is a free parameter, but one might expect it to represent some quantum correction, i.e. to be small. Indeed, assuming l as the Planck length, one would get $m < 0.945 \ m_P \simeq 2 \times 10^{-6}$ Kg, where m_P is the Planck mass.

The point, however, is misleading, since the QBH has no Schwarzschild limit: indeed, the Schwarzschild solution cannot be viewed as a QBH even in approximation, because it has a true horizon (and diverges near the origin).

If we work on our solution (9.42) things appear to be more realistic: indeed, its classical limit is just simply a star, with its radius R and its central density ρ_0 (in the classical toy model, the density is constant). In that case, the metric was, we recall,

$$g(r) = 1 - \frac{8\pi\rho_0}{3r}\left(1 + e^{-\lambda R^3}\right)\left(r^3 - \frac{1}{\lambda}\ln\left(\frac{1 + e^{\lambda\left(r^3 - R^3\right)}}{1 + e^{-\lambda R^3}}\right)\right) \ . \tag{12.7}$$

We are using the same notation of chapter 9. Since in that chapter we also found the horizon was located around $r_H \simeq \left(R^3 + \frac{1}{\lambda}\ln 2\right)^{1/3}$, condition (12.3) reads

$$\frac{8}{3}\pi\rho_0(1 + e^{-\lambda R^3}) < \left(R^3 + \frac{1}{\lambda}\ln 2\right)^{-2/3} \ , \tag{12.8}$$

which becomes

$$\lambda > \frac{\ln 2}{\left(\frac{8}{3}\pi\rho_0\right)^{-3/2} - R^3} \ . \tag{12.9}$$

Here we used the fact that $1/\lambda \ll R^3$ and that the metric has a minimum at the horizon, so that

$$\frac{e^{\lambda(r^3-R^3)} - 2}{1 + e^{\lambda(r^3-R^3)}} = \frac{1}{\lambda r^3} \ln \left(\frac{1 + e^{\lambda(r^3-R^3)}}{1 + e^{-\lambda R^3}} \right) \ . \tag{12.10}$$

This can be easily proved, calculating the minimum condition $g'(r_H) = 0$.

With condition (12.9) seems that we have no gain, since we still have an upper bound on a parameter (we recall that we interpret λ as the inverse of some Planck-like volume), which is expected to go to infinity in the classical limit. However the difference with the Hayward-like case is much relevant: indeed, in metrics such as (12.4) there are only two parameters, one of which is identified with the asymptotic mass (in the limit of $r \to \infty$). Thus condition (12.3) fixes also the second parameter. On the other hand, metric (12.7) has *three* parameters: the central density ρ_0; the classical radius R; and the deformation parameter λ. The asymptotic condition fixes only one of these parameters, leaving the other untouched (what actually happens is even more general, since the only fixed thing is a relation among the parameters). Thus, a more convenient shape of eq. (12.9) is

$$\rho_0 < \frac{3}{8\pi \left(R^3 + \frac{1}{\lambda}\ln 2\right)^{2/3}} \ . \tag{12.11}$$

If we now assume $\lambda \ll R^3$ (which is compatible with a macroscopic object), λ disappears and this condition only turns to be $R > 2M$, i.e. the basic requirement for metric (12.7) not to be a black hole. It follows that no extra requirement seems to hold on λ, so that it can easily take a Planck-related value.

Finally this means that metric (12.7) is a good tool to inspect also the quantum regime of QBH, because one of its parameters (λ) can still be viewed as quantum in nature, which was the aim of this appendix. Future research will tell if metric (12.7) has or not some troubles in this framework and will illustrate its properties.

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