Renormalization of Wick polynomials for Boson fields in locally covariant AQFT

Ph.D. Thesis
submitted to the Doctoral School of Physics – XXX cycle
for the degree of Doctor of Philosophy

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Abstract

The aim of this thesis is to study renormalization of Wick polynomials of quantum Boson fields in locally covariant algebraic quantum field theory in curved spacetime. Vector fields are described as sections of natural vector bundles over globally hyperbolic spacetimes and quantized in a locally covariant framework through the known functorial machinery in terms of local $*$-algebras. These quantized fields may be defined on spacetimes with given classical background fields, also sections of natural vector bundles: The most obvious one is the metric of the spacetime itself, but we encompass also the case of generic spacetime tensors as background fields. In our framework also physical quantities like the mass of the field or the coupling to the curvature are viewed as background fields. Wick powers of the quantized vector field are then axiomatically defined imposing in particular local covariance, scaling properties and smooth dependence on smooth perturbation of the background fields. A general classification theorem is established for finite renormalization terms (or counterterms) arising when comparing different solutions satisfying the defining axioms of Wick powers. The result is then specialized to the case of spacetime tensor fields. In particular, the case of a vector Klein-Gordon field and the case of a scalar field renormalized together with its derivatives are discussed as examples. In each case, a more precise statement about the structure of the counterterms is proved. The finite renormalization terms turn out to be finite-order polynomials tensorially and locally constructed with the backgrounds fields and their covariant derivatives whose coefficients are locally smooth functions of polynomial scalar invariants constructed from the so-called marginal subset of the background fields. Our main technical tools are based on the Peetre-Slovák theorem characterizing differential operators and on the classification of smooth invariants on representations of reductive Lie groups.

This thesis is mostly based on the paper [KMM17], which was submitted to *Annales Henri Poincaré* for publication.

“Because in the end, you won’t remember the time you spent working in the office or mowing your lawn. Climb that goddamn mountain”

—Jack Kerouac

Contents

Overview

This thesis presents some results about renormalization of Wick polynomials of Boson fields in locally covariant algebraic quantum field theory in curved spacetime. Wick polynomials and time-ordered products of Wick polynomial are the building blocks for perturbative renormalization of quantum fields, both in Minkowski spacetime and in curved spacetime, where the metric is considered as a given external classical field. Taking the Klein-Gordon scalar field operator $\varphi(x)$ as an example, the Wick powers $\varphi^k(x)$ have the physical interpretation of products of k factors φ evaluated at the same point x. But these products are not well-defined in general because they correspond to ***products of distributions at a given point*** and this notion is not well-defined. To overcome this problem, in Minkowskian QFT, Wick polynomials are defined exploiting the standard ***normal ordering prescription***: Using Fourier transform, all quantum fields are written in term of creation and annihilation operators, a^* and a, and then normal ordered product of operators is defined by

$$:aa^*: = a^*a, \quad :a^*a: = a^*a, \quad :aa: = aa, \quad :a^*a^*: = a^*a^*.$$

obviously extended to products of many operators (e.g. $:aa^*a: = a^*aa$). In Minkowski spacetime the normal order prescription written in terms of creation and annihilation operators is equivalent to the standard vacuum subtraction procedure, ***i.e.***

$$:\varphi^2(x): = \lim_{y\to x} \varphi(x)\varphi(y) - \langle 0|\varphi(x)\varphi(y)|0\rangle \mathbb{I}$$

extended to higher powers with the well-known Wick rule, where $|0\rangle$ is Minkowski vacuum. But in a general curved spacetime we also do not have a proper vacuum state at our disposal. The best we can do is to replace the Poincaré vacuum $|0\rangle$ with a generic Hadamard state ω, *i.e.*

$$:\varphi^2(x): = \lim_{y\to x} \varphi(x)\varphi(y) - \omega(\varphi(x)\varphi(y))\mathbb{I}.$$

This is a good definition because, using the language of Algebraic QFT (AQFT), for any Hadamard state ω, the singularity structure of (more precisely the wave-front set of) $\omega(\varphi(x)\varphi(y))$ is under sufficient control so that we can find a distribution $H(x, y)$ that is independent of ω (as long as it remains Hadamard) such that the difference $\omega(\varphi(x)\varphi(y)) - H(x, y)$ is regular enough to properly define the Wick square by $\varphi^2(x) := \lim_{y\to x} \varphi(x)\varphi(y) - H(x, y)\mathbb{I}$, and so on for higher Wick powers (this is known as the Hadamard parametrix regularization method [HW01, HW02, BF00]). In any case, even such a procedures do not lead to a unique definition. The constructed Wick powers (or also more generally time ordered products of Wick powers) may be still affected by (finite) ambiguities, popularly called *finite-renormalization terms* (or *counterterms*). Within the divergence subtraction paradigm their nature is obvious: depending on how the regularization is carried out, $\infty - \infty$ could be any number. A given Wick product $\varphi^k(x)$, interpreted as a distributional kernel evaluated at x, can always be be redefined by adding similar counter terms of lower order multiplied with coefficients depending on x: $\varphi^k(x) = \sum_{l<k} C_l(x)\varphi^l(x)$. The structure of these coefficients C_l can be fixed by imposing some further physical constraints. The renormalization of Wick powers of scalar fields has been studied in detail in [HW01, KM16].

This work deals with the classification of Wick polynomials (study of the renormalization counterterms) of a rather general locally covariant bosonic vector-valued quantum field, in the presence of rather general classical background fields. The constant parameters that usually defining a quantum field, like the mass and coupling constants, are included in among the classical background fields (and may be restricted to be constants).

Our general settings is the following: We consider a quantum field over a time-oriented globally-hyperbolic spacetime $(M, \mathbf{g})$ of dimension n. In general, in addition to the quantum field, there are some classical assigned *background fields* on M. They influence the evolution of the quantum field, for example because they may be present in the equation of motion of the quantum field. The first necessary background field is the metric itself $\mathbf{g}$, however further tensor fields may enter the theory. Background fields are described as sections $\mathbf{b}$ of a suitable bundle $BM \to M$ (actually a ***natural*** bundle [KMS93] since we are working with the locally covariant framework [BFV03] of AQFT and so we have to deal with all bundles of background fields simultaneously and coherently for every globally-hyperbolic spacetime). With $(M, \mathbf{b})$ we denote a *background geometry*, *i.e.* a spacetime M equipped with background fields $\mathbf{b}$.

On the other hand, *quantum fields*, always in the framework of locally covariant AQFT [BFV03], are viewed as as distribution valued in a suitable algebra of observables $\mathcal{W}(M, \mathbf{b})$, *i.e.* if $\mathscr{D}(V^*M)$ is the space of compactly supported smooth section of the dual natural bundle V^*M, A *quantum V-field* is an assignment $A_{(M,\mathbf{b})}$ of an algebra-valued distribution

$$A_{(M,\mathbf{b})} : \mathscr{D}(V^*M) \to \mathcal{W}(M, \mathbf{b})$$

which respects the inclusions and isomorphisms induced by isometries. We also have two further requirement regarding quantum fields: We assume a suitable commutation relation between fields which characterize them as proper *Boson* fields and we assume that quantum fields scale homogeneously under physical scaling, *i.e.* under the dilatation of all background fields (possibly with different weights).

In this general settings we state a set of axioms that Wick powers have to satisfy, namely behaviour of *Low Powers*, *Scaling*, *Kinematic Completeness*, *Commutator Expansion*, and *Smoothness*. Starting form this set of axioms we proved our first result: If we consider two families of Wick powers $\{\tilde{A}^k\}_{k=1,2,\ldots}$ and $\{A^k\}_{k=1,2,\ldots}$ of the same quantum V-field A the the following formula holds:

$$\widetilde{A_{b_1} \cdots A_{b_k}}(x) = A_{b_1} \cdots A_{b_k}(x) + \sum_{l=0}^{k-1} \binom{k}{l} C_{k-l}[M, \mathbf{b}]_{(b_1 \cdots b_{k-l}}(x) A_{b_1} \cdots A_{b_l)}(x)$$

where $\{C_k\}_{k=1,2,\ldots}$ is family of smooth locally-covariant *symmetric* V-tensor fields of order k. Moreover C_k, for $k = 1, 2, \ldots$, are *differential operators of locally bounded order*. The last result is proved exploiting an important result of differential geometry, the *Peetre-Slovák* theorem [Slo88]. This result was firstly used in [KM16] in the study of renormalization of scalar fields and it permit to avoid analyticity requirements previously used in [HW01, HW02].

After this very general result we specialize a bit our model considering the case when both the dynamical and background fields are spacetime tensors. In this framework we proved the main result of this work. It precisely characterizes the form of the differential operator coefficients C_k in the general finite renormalization formula. These coefficients must be linear combinations of tensor valued polynomials, covariantly constructed out of the curvature tensors, the background field tensors and all of their covariant derivatives. The number of independent terms and the degrees of these polynomials are *a priori* bounded, with the bound determined by the scaling dimension of the Wick power and the ranks of the tensors involved. The coefficients of these polynomials are *locally* (in a precise sense) smooth functions (no longer just polynomial) of finitely many polynomial scalars covariantly constructed out of the subset of the background fields. Crucially these finiteness results hold only when all background fields are *admissible*. Here a background tensor field is *admissible* if its physical scaling weight and its tensor rank satisfy an inequality. Those background fields that saturate the inequality are called *marginal* and only they are allowed to appear non-polynomially in the finite renormalization terms. We stress that, despite the admissibility is a technical mathematical condition necessary in our proofs, all physical models of major interest satisfies it (in particular some examples are discussed at the end of Chapter 4). To achieve this result we used some important mathematical results: the *Thomas replacement theorem* and results from the classical *invariant theory* of the general linear and (Lorentzian) orthogonal groups (the notion of *local* smooth dependence on a set of polynomial invariants was actually born out of the necessity of dealing with the complicated orbit structure for the action of the orthogonal group on background tensors).

It is remarkable that all these results are written to encompass the case in which the bundle of dynamical fields is actually a direct sum of vector bundles: This means that we can renormalize quantum fields that are actually n-tuples of fields.

Finally, we use our main result in the study of some physically relevant models: the *vector Klein-Gordon field*, also with a tensorial coupling to the scalar curvature, and the *scalar Klein-Gordon field* with its first derivative.

Now that we have presented the main topics we are going to deal with, we briefly summarized how the thesis is organized.

In Chapter 1 we introduce all geometric structure we need in the subsequent parts of the thesis: We recall the notion of spacetime and in particular of *globally hyperbolic* spcaetime. Then, just to fix notation, we introduce the concept of vector bundle and of *jet bundle*, and important object that we use extensively in the whole thesis. After that we outline the key properties of Green hyperbolic operators, introducing also the notion of propagator. Finally, after a brief discussion on the basic concept of category theory, we introduce the most important geometric object of this thesis: The *background geometries*.

In Chapter 2 we discuss the problem of normal ordering in Algebraic QFT. We firstly present the problem of normal ordering al Minkowskian QFT and, after a brief introduction of Algebraic QFT, we discuss the problem of ambiguities in the definition of Wick polynomials in a general curved spacetime, motivating then the whole work.

Chapter 3 is devoted to present all technical results that we use to study the renormalization of quantum Boson fields: the most important one is surely the *Peetre-Slovák* theorem: after having discussed it we prove some results about physical scaling, we present some results from the classical *invariant theory* of the general linear and (Lorentzian) orthogonal groups and finally we introduce the *Thomas replacement theorem*.

The Chapter 4 is the core of the thesis. We introduce Wick powers from an axiomatic viewpoint and we prove the general renormalization formula. After that we consider a tensor Boson field and we prove that the coefficients of the formula are linear combinations of tensor valued polynomials, covariantly constructed out of the curvature tensors, the background field tensors and all of their covariant derivatives. Finally we consider three physically motivated applications of the previous result.

Geometric structures | 1

This chapter is devoted to introducing all geometric structures that we use in this work. In the first sections we briefly recall the notion of globally hyperbolic spacetime and we introduce the notions of vector and jet bundles. Then, after having recalled some basic ideas from category theory, we introduce the concept of background geometries, one of our main building blocks.

1.1 Spacetimes

In this section we present our general geometric setting by recalling some well-known notions about spacetimes.

In this thesis, a ***spacetime*** $(M, \mathbf{g})$ is a Hausdorff, second countable, connected, smooth n-dimensional manifold M endowed with a smooth Lorentzian metric $\mathbf{g}$ with signature $(- + \cdots +)$. With standard notation, TM and T^*M denote the tangent and cotangent bundle of M respectively (T_pM and T_p^*M denote the fibre at $p \in M$ of TM and T^*M respectively). For a spacetime $(M, \mathbf{g})$ we also require a key property, the ***global hyperbolicity***, which is essential in the formulation of physical theories (see for e.g. [BD15], [DMP17]). To properly define this concept we recall some standard notions of Lorentzian geometry.

Definition 1.1. Given a spacetime $(M, \mathbf{g})$, we say that a vector $v \in T_pM$ is

- ***timelike*** if and only if $\mathbf{g}(v, v) < 0$,
- ***lightlike*** if and only if $\mathbf{g}(v, v) = 0$,
- ***spacelike*** if and only if $\mathbf{g}(v, v) > 0$.

A vector is called ***causal*** if it is either timelike or lightlike.

For a fixed $p \in M$ it is possible to construct a two-folded ***light cone*** $V_p \subseteq T_pM \setminus \{0\}$ made of all causal vectors and we have the freedom to call future-directed the non-zero vectors lying in one of the two-folds. If such choice can be made smoothly varying $p \in M$, we say that $(M, \mathbf{g})$ is ***time orientable***.

Definition 1.2. Let I be an interval. A piecewise smooth curve $\gamma : I \to M$ is timelike (lightlike, spacelike) if, for every $t \in I$, the tangent vector to the curve at $\gamma(t)$ is timelike (respectively lightlike, spacelike). The curve γ is causal if, for every $t \in I$, the tangent vector to the curve at $\gamma(t)$ is causal and every tangent vector has the same time orientation.

A curve, usually denoted by γ, will be henceforth supposed to be piecewise smooth. We can now define the causal structure of a spacetime $(M, \mathbf{g})$. By $I^+(p)$ we denote the ***chronological future*** of a point $p \in M$, ***i.e.***

$$I^+(p) = \{q \in M \mid \exists\, \gamma \text{ future directed timelike curve s.t. } \gamma(0) = p,\, \gamma(1) = q\}\,,$$

while $J^+(p)$ denotes the ***causal future*** of a point $p \in M$, ***i.e.***

$$J^+(p) = \{p\} \cup \{q \in M \mid \exists\, \gamma \text{ future directed causal curve s.t. } \gamma(0) = p,\, \gamma(1) = q\}\,.$$

Similarly, we define the *chronological past* $I^-(p)$ and the *causal past* $J^-(p)$ of a point $p \in M$ by employing past directed timelike and causal curves, respectively. These definitions can be extended to a general subset $\Omega \subset M$ by setting

$$I^\pm(\Omega) = \bigcup_{p\in\Omega} I^\pm(p), \qquad J^\pm(\Omega) = \bigcup_{p\in\Omega} J^\pm(p);$$

additionally, we define $I(\Omega) = I^+(\Omega) \cup I^-(\Omega)$ and $J(\Omega) = J^+(\Omega) \cup J^-(\Omega)$. We say that a subset Ω is *achronal* if $I^+(\Omega) \cap \Omega = \emptyset$, *i.e.* a set such that every timelike curve meets it at most once. If Ω is a closed achronal subset, we define its future *domain of dependence* as

$$D^+(\Omega) = \{p \in M \mid \text{every past inextensible causal curve through } p \text{ intersects } \Omega\}\,.$$

Finally, we can define $D^-(\Omega)$ analogously and set $D(\Omega) = D^+(\Omega) \cup D^-(\Omega)$. Above a past inextensible causal curve is a causal curve which cannot be extended into a larger causal curve.

Definition 1.3. A *Cauchy surface* is a closed achronal set $\Sigma \subset M$ with $D(\Sigma) = M$. A spacetime M is called **globally hyperbolic** if it contains a Cauchy surface.

It is possible to obtain a more practical characterization of globally hyperbolic spacetimes establishing also the existence of smooth Cauchy surfaces [BD15, Thm. 3.1.4], [BS05]:

Theorem 1.1. *Let (M, g) be any time-oriented spacetime. The following two statements are equivalent:*

1. *(M, g) is globally hyperbolic;*

2. *(M, g) is isometric to $\mathbb{R} \times \Sigma$ with metric $g = -\beta \mathrm{d}t \otimes \mathrm{d}t + h_t$, where $t \in \mathbb{R}$, $\beta \in C^\infty(M)$ is strictly positive, whereas h_t is a smooth Riemannian metric on Σ depending smoothly on t. Furthermore each $\{t\} \times \Sigma$ is a smooth spacelike Cauchy surface on M.*

To conclude, in the following, with *spacetime* we always refer to a *time-oriented globally hyperbolic spacetime.*

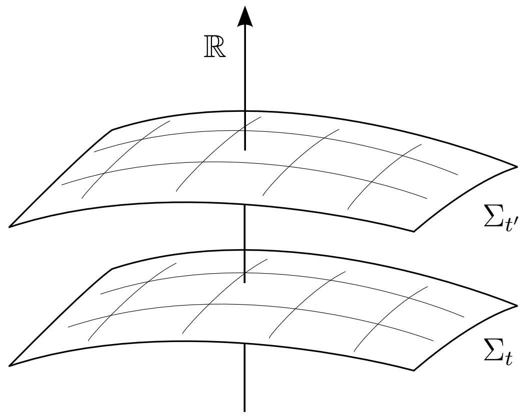


Figure 1.1: Two smooth Cauchy surfaces.

1.2 Vector bundles and jets

In this sections we briefly recall some well-know notions of differential geometry, concerning vector and jet bundles, with the main purpose to fix notations. We also introduce the definition and useful identities for the symmetrized contraction of sections of a vector bundle: This symmetrized contraction will be useful for giving index-free versions of our results.

With $VM \to M$ (or simply with VM) we denote a smooth real vector bundle over a manifold M whose fibres V_p are isomorphic to a given $\mathbb{R}^a$. We shall make use of the auxiliary tensor bundles $V^{\otimes k}M \to M$ and $V^{*\otimes l}M \to M$, which are bundles of tensor products of k copies of the bundle VM and l copies of the dual bundle V^*M respectively. In the following, we also consider two special sub-bundles, namely those of the fully symmetrized contravariant and covariant tensor products, defined by

$$V^{\otimes k}M \supset S^kVM = V^{\odot k}M \,, \quad V^{*\otimes l}M \supset S^lV^*M = V^{*\odot l}M$$

where we denoted with $\odot$ the symmetrized tensor product. We recall [Pro07, Lem.9.1.1] that fully symmetric tensors of order k are spanned by decomposable tensors of the form $v^{\odot k}$. We will take advantage of the following spaces of smooth sections:

- $\mathscr{E}(X) := \Gamma(X)$ the space of smooth sections of the bundle X;

- $\mathscr{D}(X)$ the space of smooth and compactly supported sections of the bundle X;

where X is a generic vector bundle. Obviously $\mathscr{E}(X) \supset \mathscr{D}(X)$.

Remark 1.1. In the following, we will consider bundles which are constructed as direct sum, *i.e.*,

$$VM = \bigoplus_{i=1}^{N} W_i M$$

for some vector bundles $W_i M \to M$. We stress that, in this case, using the distributivity of the tensor product with respect to the direct sum, we have

$$V^{\otimes k} M = \bigoplus_{|P|=k} \bigotimes_{i=1}^{N} W_i^{\otimes p_i} M, \quad S^k VM = \bigoplus_{|P|=k} \bigotimes_{i=1}^{N} S^{p_i} W_i M$$

where $P = (p_1, \ldots, p_N)$ is a multi-index and $|P| = p_1 + \cdots + p_N$. It is straightforward to write the analogous decomposition for $V^{*\otimes l} M$ and $S^l V^* M$. Moreover, if the bundle VM has the introduced direct sum structure, we will often take advantage of the identification: If $\mathscr{E}(VM) \ni f = \bigoplus_i f_i$ we identify $f_i \simeq \bigoplus_k \delta_i^k f_k$. With this identification we can substitute the direct sum with a standard sum:

$$f = \sum_i f_i.$$

In the whole thesis, we always use remarks to specialize our results to the case of a bundle constructed as a direct sum.

Remark 1.2. In the sequel, we sometimes write tensors in components. We use two type of indices: We use the notation with Greek indices for sections of a generic tensor bundle (for example, $v^{\mu_1 \cdots \mu_k}$ denotes a section of $V^{\otimes k} M$) and Latin indices for spacetime tensors, *i.e.*, section of tensor products of TM and T^*M bundles (for example $t^{a_1 \cdots a_k}$ denotes a section of $T^{\otimes k} M$).

We can now introduce the contraction product between symmetric tensor fields. As we said, the following product is very useful to greatly simplify the notation in our results.

Definition 1.4. The l-**contraction product** of symmetric sections

$$\cdot_l \colon \mathscr{E}(S^l V^* M) \times \mathscr{E}(S^k VM) \longrightarrow \mathscr{E}(S^{k-l} VM) \quad \text{with } k \geq l,$$

is defined pointwise on decomposable tensors $g^{\odot l} \in \mathscr{E}(S^l V^* M)$, $f^{\odot k} \in \mathscr{E}(S^k VM)$ by

$$\left(g^{\odot l} \cdot_l f^{\odot k}\right) := \binom{k}{l} \langle g, f\rangle^l f^{\odot k-l}.$$

and extended by linearity.

We can now prove some technical results, about the just introduced contraction product, that will be very useful in the subsequent part.

Proposition 1.2. *Let $k, l, s > 0$ be such that $l \leq k$ and $s \leq k - l$. For $h \in \mathscr{E}(S^s V^* M), g \in \mathscr{E}(S^l V^* M)$ and $f \in \mathscr{E}(S^k VM)$ it holds*

$$h \cdot_s (g \cdot_l f) = g \cdot_l (h \cdot_s f) \tag{1.1}$$

Proof. It is immediate using the definition. It is sufficient to prove the result for decomposable tensors and then use linearity to extend the proof to general tensors. We consider $h^{\odot s} \in \mathscr{E}(S^s V^* M)$, $g^{\odot l} \in \mathscr{E}(S^l V^* M)$, $f^{\odot k} \in \mathscr{E}(S^k VM)$. Thus

$$\begin{aligned} h^{\odot s} \cdot_s \left(g^{\odot l} \cdot_l f^{\odot k}\right) &= h^{\odot s} \cdot_s \left(\binom{k}{l} \langle g, f\rangle^l f^{\odot k-l}\right) \\ &= \binom{k-l}{s}\binom{k}{l} \langle h, f\rangle^s \langle g, f\rangle^l f^{\odot k-l-s} \\ &= \binom{k}{s}\binom{k-s}{l} \langle h, f\rangle^s \langle g, f\rangle^l f^{\odot k-l-s} \\ &= g^{\odot l} \cdot_l \left(h^{\odot s} \cdot_s f^{\odot k}\right). \end{aligned}$$

□

In the following, we often use the shorthand notation $f^k := f^{\odot k}$.

Proposition 1.3. *Let $g \in \mathscr{E}(V^* M)$, $f_i \in \mathscr{E}(VM)$ and $p_i \geq 1$ for $i = 1, \dots N$. The following relations hold*

(a) $g \cdot_1 \left(g^{l-1} \cdot_{l-1} f_i^k\right) = l g^l \cdot_l f^k$,

(b) $g \cdot_1 \left(f_1^{p_1} \odot \cdots \odot f_N^{p_N}\right) = \displaystyle\sum_{l=1}^{N} f_1^{p_1} \odot \cdots \odot \left(g \cdot_1 f_l^{p_l}\right) \odot \cdots \odot f_N^{p_N}$,

(c) $g^l \cdot_l \left(f_1^{p_1} \odot \cdots \odot f_N^{p_N}\right) =$

$$= \sum_{\substack{|Q|=l \\ q_i \leq p_i}} \left(\prod_{i=1}^{N} \binom{p_i}{q_i}\right) \left(g^l \cdot_l \left(f_1^{q_1} \odot \cdots \odot f_N^{q_N}\right)\right) f_1^{p_1-q_1} \odot \cdots \odot f_N^{p_N-q_N}.$$

Moreover, if $Q = (q_1, \ldots, q_N)$ ***and*** $P = (p_1, \ldots, p_N)$ ***are multi-indices such that*** $|Q| = |P| = l$, $h^Q \in \mathscr{E}\left(\odot_{i=1}^N S^{q_i} W_i M\right)$ ***and*** $f_i \in \mathscr{E}(W_i M)$***, then***

(d) $h^Q \cdot_l \left(f_1^{p_1} \odot \cdots \odot f_N^{p_N}\right) = 0$ ***if*** $P \neq Q$.

Proof. Relation *(a)* follows immediately from the definition of contraction product. We prove relation *(b)*. It is sufficient to prove the relation for the product $f_1 \odot f_2$ and then the relation *(b)* follows immediately using the obtained result recursively. Defining $f = f_1 + f_2$ we obtain

$$\begin{aligned} g \cdot_1 f^2 &= 2\langle g, f\rangle \odot f = 2\left(\langle g, f_1\rangle + \langle g, f_2\rangle\right) \odot (f_1 + f_2) \\ &= g \cdot_1 f_1^2 + 2\langle g, f_1\rangle \odot f_2 + 2\langle g, f_2\rangle \odot f_1 + g \cdot_1 f_2^2 \\ &= g \cdot_1 f_1^2 + 2(g \cdot_1 f_1) \odot f_2 + 2(g \cdot_1 f_2) \odot f_1 + g \cdot_1 f_2^2 \end{aligned}$$

but we also have

$$g \cdot_1 f^2 = g \cdot_1 \left(f_1^2 + 2 f_1 \odot f_2 + f_2^2\right) = g \cdot_1 f_1^2 + 2 g \cdot_1 (f_1 \odot f_2) + g \cdot_1 f_2^2$$

and then

$$g \cdot_1 (f_1 \odot f_2) = (g \cdot_1 f_1) \odot f_2 + f_1 \odot (g \cdot_1 f_2).$$

We now prove relation *(c)*. Applying recursively relation *(a)* and recalling that $\cdot_1$ acts as a derivation (relation *(b)*), we have, for $q_i \leq p_i$,

$$\begin{aligned} g^l \cdot_l &(f_1^{p_1} \odot \cdots \odot f_N^{p_N}) = \\ &= \frac{1}{l!} \underbrace{g \cdot_1 (\cdots g \cdot_1}_{l-\text{times}} \left(f_1^{p_1} \odot \cdots \odot f_N^{p_N}\right)) \\ &= \frac{1}{l!} \sum_{|Q|=l} \binom{l}{Q} \bigodot_{i=1}^N \underbrace{g \cdot_1 (\cdots g \cdot_1}_{q_i-\text{times}} f_i^{p_1}) \\ &= \sum_{|Q|=l} \bigodot_{i=1}^N g^{q_i} \cdot_{q_i} f_i^{p_i} \\ &= \sum_{|Q|=l} \left(\prod_{i=1}^N \binom{p_i}{q_i} g^{q_i} \cdot_{q_i} f_i^{q_i}\right) f_1^{p_1 - q_1} \odot \ldots \odot f_N^{p_N - q_N} \\ &= \sum_{|Q|=l} \left(\prod_{i=1}^N \binom{p_i}{q_i}\right) \left(g^l \cdot_l \left(f_1^{q_1} \odot \ldots \odot f_N^{q_N}\right)\right) f_1^{p_1 - q_1} \odot \ldots \odot f_N^{p_N - q_N} \end{aligned}$$

where the last equality holds because, if $f = \sum_i f_i$,

$$g^l \cdot_l f^l = \left(\sum_i \langle g, f_i\rangle\right)^l = \sum_{|P|=l} \binom{l}{P} \prod_i \langle g, f_i\rangle^{p_i} = \sum_{|P|=l} \binom{l}{P} \prod_i g^{p_i} \cdot_{p_i} f_i^{p_i}$$

but also

$$g^l \cdot_l f^l = \sum_{|P|=l} \binom{l}{P} g^l \cdot_l \left(f_1^{p_1} \odot \ldots \odot f_N^{p_N}\right).$$

Finally, we have to prove relation (d). Define $h = \sum_i h_i$ and $f = \sum_i f_i$ where $h_i, f_i \in \mathscr{E}(W_i M)$. Then

$$\begin{aligned} h^l \cdot_l f^l &= \left(\sum_i \langle h_i, f_i\rangle\right)^l = \sum_{|P|=l} \binom{l}{P} \prod_i \langle h_i, f_i\rangle^{p_i} \\ &= \sum_{|P|=l} \sum_{|Q|=l} \delta_{PQ} \binom{l}{P} \prod_i \langle h_i, f_i\rangle^{p_i}, \end{aligned}$$

but also

$$h^l \cdot_l f^l = \sum_{|P|=l} \sum_{|Q|=l} \binom{l}{P}\binom{l}{Q} \left(h_1^{q_1} \odot \cdots \odot h_N^{q_N}\right) \cdot_l \left(f_1^{p_1} \odot \ldots \odot f_N^{p_N}\right).$$

Thus

$$\left(h_1^{q_1} \odot \cdots \odot h_N^{q_N}\right) \cdot_l \left(f_1^{p_1} \odot \ldots \odot f_N^{p_N}\right) = 0, \quad \text{if } P \neq Q$$

and, since h^Q is a linear combination of $h_1^{q_1} \odot \cdots \odot h_N^{q_N}$, we have concluded the proof. □

Finally, we can introduce a geometric structure that we use extensively in this work: **jet bundles**. We only recall some basic notions: A complete reference for this subject is [KMS93]. Naively, given a bundle $E \to M$ and a section $f \colon M \to E$, the jet of f at a point $p \in M$ collects the information about the coordinate derivatives of f at p up to some order. The collection of all jets then forms the jet bundle associated to E.

Definition 1.5. Consider a pair of smooth manifolds M, E and the class of smooth functions $f \colon M \to E$, in particular E may be a bundle with base M and in this case the relevant set of functions f is that of smooth sections.
The **germ** of f at $p \in M$ is the equivalence class $[f]_p$ of smooth functions (sections)

$M \to E$ that are equal to f on some neighbourhood of p. The r-**jet** of f at $p \in M$, denoted by $j^r_p f$, is the equivalence class $[f]^r_p$ of smooth functions (sections) $M \to E$ that have the same Taylor expansion at p as f to order r with respect to fixed local coordinate systems in M and E (this property being independent from the choice of the coordinate patch). When $E \to M$ is a smooth bundle, $J^r E \to M$ denotes the set of r-jets varying the point in the base, itself a smooth bundle. Finally, if $\psi \in \Gamma(E)$ is a smooth section, the r**-jet extension** of ψ, denoted with $j^r\psi \in \Gamma(J^r E)$, is the section of $J^r E$ which collects the $r-$jets of ψ over each point $p \in M$.

A fiber $(J^r E)_p$ at $p \in M$ is diffeomorphic to $E_p \times \mathbb{R}^{s_r}$ where E_p is the fiber of E at p and s_r is the number of all (symmetrized) partial derivatives up to order r with respect to any local chart on the base around p. The notion of jet extension gives rise to the definition of *local adapted coordinates on jet bundles.*

Definition 1.6. Let (x^a, v^i) be a local adapted coordinate chart on a bundle $E \to M$, where x^a are local coordinates on an open domain $U \subseteq M$ and (x^a, v^i) are trivializing coordinates on the fibers over the open domain $Z \subseteq E$ projecting onto U. This charts extends to an **adapted coordinates chart** (x^a, v^i_A) on the jet bundle $J^r E$ defined as follows. Its domain is $Z^r \subseteq J^r E$ is diffeomorphic to $Z \times \mathbb{R}^{s_r}$. Moreover

$$v^i_A\left(j^r\psi(p)\right) = \partial_A v^i\left(j^r\psi(p)\right) = \frac{\partial}{\partial x^{a_1}} \cdots \frac{\partial}{\partial x^{a_l}} v^i\left(j^r\psi(p)\right)$$

for any section ψ of the bundle E and where $A = a_1 \cdots a_l$ is a multi-index of size $|A| = l$ with $l = 0, 1, \dots, r$.

1.3 Green hyperbolic operators

The dynamics of most important and physically relevant fields is ruled by wave-like equations: Generally we have a partial differential equation and we want to solve an initial value problem. We have already introduced globally hyperbolic spacetimes, which have a geometric structure suitable to assign initial data, now we outline the key properties of a class of partial differential equations which encompass all physical models that we treat in the following. In this section we mainly follow [BD15, Section 3.2].

Definition 1.7. Let F, F' be two vector bundles of rank k and k', with typical fiber V and V' respectively, over the same manifold M. A linear map $L : \Gamma(F) \to \Gamma(F')$

is a **linear partial differential operator** of order at most $s \in \mathbb{N}_0$ if for all $p \in M$ there exist

- A coordinate neighbourhood (U, ϕ) centred in p;
- Local trivializations (U, ψ) and (U, ψ') of F and F' respectively;
- A collection of smooth maps $A_I : U \to \mathrm{Hom}(V, V')$ labelled by multi-indices;

for which, given any $f \in \Gamma(F)$, on U one have

$$Lf = \sum_{|I| \leq s} A_I \partial^I f.$$

Notice that here we are implicitly using both the coordinate chart ϕ and the trivialization ψ, ψ'. Moreover, if $I = (I_0, \dots, I_{n-1})$, we used the notation $\partial^I = \prod_{\mu=0}^{n-1} \partial_\mu^{I_\mu}$, where $\partial_0, \dots, \partial_{n-1}$ are the partial derivatives with respect to the coordinates coming from the chart (U, ϕ). Furthermore, L is of order $s \in \mathbb{N}_0$ if it is of order at most s, but not of order at most $s - 1$.

Since this definition is very general, we need to introduce a class of linear partial differential operators more suitable to study initial value problems in quantum field theory.

Definition 1.8. Let F, F' be two vector bundles of rank k and k' over the same manifold M and let $L : \Gamma(F) \to \Gamma(F')$ be any linear partial differential operator of order s. We call **principal symbol** of L the map $\sigma_L : T^*M \to \mathrm{Hom}(F, F')$ locally defined as follows: For $p \in M$, mimicking Definition 1.7, consider a coordinate chart around p and local trivializations of F and F' and, for all $\zeta \in T_p^*M$, set

$$\sigma_L(\zeta) = \sum_{|I|=s} A_I(p) \zeta^I,$$

where $\zeta^I = \prod_{\mu=0}^{n-1} \zeta_\mu^{I_\mu}$ and ζ_μ are the components of ζ with respect to the chosen chart. Furthermore, given a Lorentzian manifold (M, g), we call a second order linear partial differential operator $P : \Gamma(F) \to \Gamma(F')$ **normally hyperbolic** if $\sigma_P(\zeta) = g(\zeta, \zeta)\, \mathrm{id}_{F_p}$ for all $p \in M$ and all $\zeta \in T_p^*M$.

We can write a normally hyperbolic operator P in local coordinates to better understand its structure. Let $p \in M$ and (U, ϕ) be a chart centered in p where the vector bundle F is trivial. There exist both A and A_μ, $\mu = 0, \ldots, n-1$, smooth maps from U to $\mathrm{End}(V)$ (where V is the typical fiber of F) such that, for any $f \in \Gamma(F)$, on U one has

$$Pf = g^{ab}\,\mathrm{id}_V\,\partial_a\partial_b f + A^a\partial_a f + Af. \tag{1.2}$$

From this expression we can notice that normally hyperbolic operators provide a generalization the usual d'Alembert operator constructed out with covariant derivatives $\Box_{\mathrm{g}} = g^{\mu\nu}\nabla_\mu\nabla_\nu$, since (1.2) locally agrees, up to lower order terms in the derivatives, with the local expression of d'Alembert operator. It is important to recall that, over a globally hyperbolic spacetime, initial values problems for a normally hyperbolic operator P admit unique solution [BD15, Prop. 3.2.9].

We are now in position to introduce Green operators, or *propagators*, some of the most important objects in field theory, which permit to characterize the smooth solutions of the equation $Pf = 0$ [BD15, Lemma 3.2.14]. Before that, we need an auxiliary definition.

Definition 1.9. Let (M, g) be a globally hyperbolic spacetime. and F be a vector bundle over M. We call

1. $\Gamma_{fc}(F) \subset \Gamma(F)$ the subset of smooth and *future compact* sections of F, that is $f \in \Gamma_{fc}(F)$ if $\mathrm{supp}\, f \cap J^+(p)$ is compact for all $p \in M$;

2. $\Gamma_{pc}(F) \subset \Gamma(F)$ the subset of smooth and *past compact* sections of F, that is $f \in \Gamma_{pc}(F)$ if $\mathrm{supp}\, f \cap J^-(p)$ is compact for all $p \in M$;

3. $\Gamma_{tc}(F) = \Gamma_{fc}(F) \cap \Gamma_{pc}(F)$ the space of smooth and *timelike compact* sections of F.

Definition 1.10. Let (M, g) be a globally hyperbolic spacetime and consider a vector bundle F over M. Furthermore, let $L : \Gamma(F) \to \Gamma(F)$ be a linear partial differential operator. We call **retarded** $(+)$ and **advanced** $(-)$ **Green operators** two linear maps

$$E^+ : \Gamma_{pc}(F) \to \Gamma(F), \qquad E^- : \Gamma_{fc}(F) \to \Gamma(F)$$

Fulfilling the properties listed below

1. For any $f \in \Gamma_{pc}(F)$, it holds $LE^+f = f = E^+Lf$ and $\mathrm{supp}\,(E^+f) \subset J^+(\mathrm{supp}\ f)$;

2. For any $f \in \Gamma_{fc}(F)$, it holds $LE^- f = f = E^- Lf$ and $\mathrm{supp}\,(E^- f) \subset J^-(\mathrm{supp}\ f)$.

The operator $E = E^- - E^+ : \Gamma_{tc}(F) \to \Gamma(F)$ will be referred to as advanced-minus-retarded operator or **causal propagator**. A linear partial differential operator admitting both E^+ and E^- will be called **Green hyperbolic**.

1.4 Category theory

The main purpose of this thesis is to investigate the renormalization of a quantum Boson field in a generic curved spacetime. Defining a quantum field in a General Relativity settings we have to take into account the diffeomorphism-invariance of this theory and then we have to construct *locally covariant* quantum fields. A modern and complete definition of locally covariant quantum field theory was firstly given by the authors of [BFV03]. To give a precise mathematical meaning to this definition of locally covariant quantum field theory, we need the language of *category theory*, of which we recall the fundamental concepts. We recall only the basic definitions that we use in the following, for further details the reader can refers to the standard monograph [ML98].

Definition 1.11. A **category** $\mathfrak{C}$ consists of

1. A class of $\mathrm{obj}(\mathfrak{C})$ of *objects*;

2. A class of *morphism* $\hom(\mathfrak{C})$ between objects. Each morphism connects two objects (a source object and a target object) together. If f is a morphism with source object A and target object B, we write $f : A \to B$;

3. A notion of composition of these morphisms. If $g : A \to B$ and $f : B \to C$ are two morphisms, they can be composed, resulting in a morphism $f \circ g : A \to C$.

Composition of morphism is required to be *associative*, *i.e.* if $g : A \to B$, $f : B \to C$ and $h : C \to D$ are three morphisms then

$$h \circ (f \circ g) = (h \circ f) \circ g.$$

Moreover, it is required that for every object A, there exists a morphism id_A called the *identity morphism* for A, such that for every morphism $f : A \to B$ and every morphism $g : C \to A$, we have

$$\mathrm{id}_A \circ g = g, \qquad \text{and} \qquad f \circ \mathrm{id}_A = f$$

which means that the identity morphism id_A leaves all morphisms starting from A invariant upon composition.

One of the most basic example of category is $\mathfrak{Set}$: Objects of $\mathfrak{Set}$ are sets while the morphism are maps between sets. In the subsequent part of this chapter we will introduce other categories related to the geometry of the spacetime.

Definition 1.12. Given two categories $\mathfrak{C}$ and $\mathfrak{D}$, a covariant **functor** $F : \mathfrak{C} \to \mathfrak{D}$ is a map which maps objects to objects, $A \mapsto F(A) \in \mathrm{obj}(\mathfrak{D})$ for every $A \in \mathrm{obj}(\mathfrak{C})$, and morphisms to morphisms, $(f : A \to B) \mapsto (F(f) : F(A) \to F(B)) \in \mathrm{hom}(\mathfrak{D})$ for every morphism $f : A \to B$ in $\mathrm{hom}(\mathfrak{C})$, such that:

1. For every object A in $\mathfrak{C}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$;
2. For all morphisms $f : A \to B$ and $g : B \to C$, $F(g \circ f) = F(g) \circ F(f)$.

In other words, functors are structure-preserving maps between categories.

1.5 Background geometries

In the whole thesis, our general settings is the following: We consider a quantum field over a time-oriented globally-hyperbolic spacetime $(M, \mathbf{g})$ of dimension n. In addition to the quantum field, there are some classical assigned ***background fields*** on M. They influence the evolution of the quantum field, for example because they may be present in the equation of motion of the quantum field. The first necessary background field is the metric itself $\mathbf{g}$, however further tensor fields may enter the theory. Background fields are described as sections $\mathbf{b}$ of a suitable bundle $BM \to M$. In this section we introduce all mathematical structures we need to appropriately describe background fields in a locally covariant framework [BFV03], ***i.e.*** we have to deal with all bundles of background fields simultaneously and coherently for every globally-hyperbolic spacetime. We will also take advantage of the elementary notions of category theory introduced in the previous section.

Definition 1.13. A bundle $E \to M$ is said to be **dimensionful** if it is equipped with a smooth action of the multiplicative group $\mathbb{R}^+ := (0, +\infty)$

$$\mathbb{R}^+ \times E \ni (\lambda, e) \mapsto e_\lambda \in E,$$

called **scaling**. It is assumed that every bundle diffeomorphism $E \ni e \mapsto e_\lambda \in E$ leaves fixed each fiber of E (so that the λ-parametrized family of these restrictions

to a given fiber defines a group representation of $\mathbb{R}^+$ in terms of fiber diffeomorphisms).
A dimensionful bundle is said to be **dimensionless** if the action is chosen to be everywhere trivial.

Remark 1.3. Every vector bundle or a cone sub-bundle of a vector bundle (a cone is a subset of a vector space that is invariant under multiplication by positive real numbers, *e.g.*, the cone of metrics of Lorentzian signature in the vector space of symmetric 2-tensors) can be viewed as dimensionful, since it can be equipped with a well-defined multiplication by scalars with some fixed power $p \in \mathbb{R}$ on its fibers: $t \mapsto \lambda^p t$.

Remark 1.4. The definition applies also when a scaling action is not defined. In this case the standard scaling action is assumed to be the trivial one, *i.e.*, the bundles are supposed to be dimensionless.

We can now introduce some categories that we will use in the following:

- $\mathfrak{Man}$ is a category of smooth manifolds. Here objects are connected smooth manifolds M of fixed dimension n and morphisms are smooth embeddings $\chi\colon M \to M'$.

- $\mathfrak{Bndl}$ is a category of dimensionful smooth bundles. Here objects $\pi_E\colon E \to M$ are smooth bundles over a smooth base of fixed dimension n. Since a smooth bundle is locally trivializable, its typical fiber is diffeomorphic to a fixed manifold F with possibly some additional structures (*e.g.*, a vector space structure) compatible with the smooth structure. Morphisms are smooth maps $\xi\colon E \to E'$ that are both

  (i) **fiber preserving**: $\pi_{E'} \circ \xi = \chi_\xi \circ \pi_E$ for uniquely associated smooth maps $\chi_\xi\colon M \to M'$ and preserving the additional structure of the fiber if any,

  (ii) **equivariant with respect to scaling**: $\xi(e)_\lambda = \xi(e_\lambda)$ for $\lambda \in \mathbb{R}^+$ and $e \in E$.

Definition 1.14. A natural (dimensionful) bundle is a functor $H\colon \mathfrak{Man} \to \mathfrak{Bndl}$ such that, using the notation $HM := H(M)$ for every $M \in \mathfrak{Man}$, a morphism $\chi\colon M \to M'$ has an associated morphism $H_\chi\colon HM \to HM'$ with $\pi_{HM'} \circ H_\chi = \chi \circ \pi_{HM}$ and H_χ is a local diffeomorphism (a diffeomorphism onto its image).

Given a morphism $\chi\colon M \to M'$ and exploiting the fact that H_χ is a local diffeomorphism, it is possible to construct a **pullback action on sections** of the associated bundles

$$\chi^* \colon \mathscr{E}(HM') \to \mathscr{E}(HM)$$

which is completely defined by requiring that

$$\mathbf{h}' \circ \chi = H_\chi \circ (\chi^* \mathbf{h}') \quad \text{for } \mathbf{h}' \in \mathscr{E}(HM'). \tag{1.3}$$

Since the morphism H_χ is equivariant, the scaling commutes with the pull-back, *i.e.*,

$$\chi^*(\mathbf{h}'_\lambda) = (\chi^* \mathbf{h}')_\lambda \quad \text{with } \lambda \in \mathbb{R}^+.$$

Furthermore, exploiting the compactness of the support of the elements of $\mathscr{D}(HM)$, also a natural push-forward map $\chi_* \colon \mathscr{D}(HM) \to \mathscr{D}(HM')$ arises immediately. It is defined as follows

$$(\chi_* f)(p') = H_\chi|_{\chi^{-1}(p')} f(\chi^{-1}(p')) \,, \tag{1.4}$$

for $f \in \mathscr{D}(HM)$ and $p' \in \chi(M)$ and the right-hand side is extended to the zero function for $p' \notin \chi(M)$. Finally, if $H, H' \colon \mathfrak{Man} \to \mathfrak{Bndl}$ are natural bundles, the duals H^*, H'^*, the direct sum $H \oplus H'$ and tensor product $H \otimes H'$ also define natural bundles.

Dealing with a general framework of relativistic quantum field theory, a relevant natural bundle, denoted by $B \colon \mathfrak{Man} \to \mathfrak{Bndl}$, is[1]

$$BM = \mathring{S}^2 T^* M \bigoplus_{j=1}^{K} \left(T^{*\otimes l_j} M \right) \tag{1.5}$$

where $\mathring{S}^2 T^* M \subset S^2 T^* M$ is the bundle of Lorentzian metrics over M and some choice of tensor powers l_j. We will later use BM as the bundle of background fields for a model of quantum fields. Scalar fields in particular are admitted when $k_i = l_i = 0$.

As previously observed, the bundles (1.5) are naturally dimensionful. The sections of these type of bundles represent the ***non-quantized fields*** of definite type assigned in every spacetime simultaneously and coherently. Let us state a pair of precise definitions adding some further relevant details concerning the effective action of $\mathbb{R}^+$.

[1] It is of course possible to add also tensor powers of tangent bundle without any changes in the subsequent part.

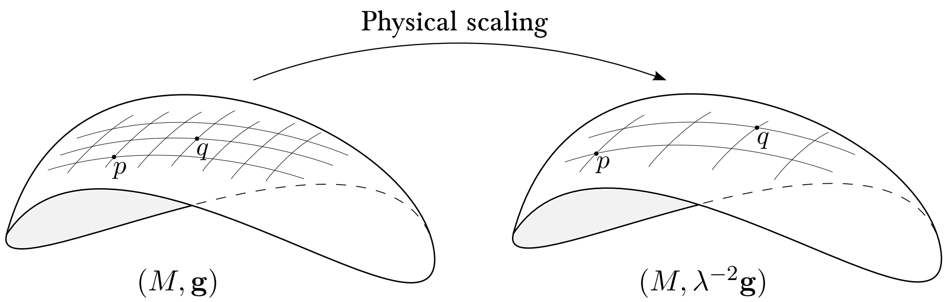


Figure 1.2: Pictorial view of the action of physical scaling on metric tensor.

Definition 1.15. Let $B\colon \mathfrak{Man} \to \mathfrak{Bndl}$ be the natural bundle of the form (1.5), with fixed l_j. A **background field** is a section $\mathbf{b}\colon M \to BM$. A pair $(M, \mathbf{b})$ is a **background geometry**, provided the section $\mathbf{b} = (\mathbf{g}, \mathbf{t}_1, \ldots, \mathbf{t}_K)$ is such that $(M, \mathbf{g})$ is a time-oriented globally hyperbolic spacetime. The action of $\mathbb{R}^+$ on the bundles of the form (1.5) is such that, for every background field,

$$\begin{gathered}(\mathbf{g}, \mathbf{t}_1, \ldots, \mathbf{t}_K) \longmapsto \left(\lambda^{-2}\mathbf{g}, \lambda^{s_1}\mathbf{t}_1, \ldots, \lambda^{s_K}\mathbf{t}_K\right) , \\ (\mathbf{g}, \mathbf{t}_1, \ldots, \mathbf{t}_K) \in \mathscr{E}(BM) \quad \lambda \in \mathbb{R}^+ ,\end{gathered} \tag{1.6}$$

for given reals s_i independent from the section and M. Each such transformation is called **physical scaling** transformation.

Definition 1.16. Referring to the natural bundle $B\colon \mathfrak{Man} \to \mathfrak{Bndl}$ of the form (1.5), we define the following associated categories:

(a) $\mathfrak{BkgG}$ is the **category of background geometries**, having time-oriented background geometries as objects and morphisms given by smooth embeddings $\chi\colon M \to M'$ that preserve the background fields, $\chi^*\mathbf{b}' = \mathbf{b}$ on M, the time orientation, and causality (every causal curve between $\chi(p)$ and $\chi(q)$ in M' is the χ-image of a causal curve between p and q in M).

(b) $\mathfrak{BkgG}^+$ is the **category of oriented background geometries** having oriented and time-oriented background geometries as objects and morphisms as in $\mathfrak{BkgG}$, but also required to preserve the spacetime orientation.

Remark 1.5. The group of physical scaling transformations acts on the above categories mapping $(M, \mathbf{b}) \mapsto (M, \mathbf{b}_\lambda)$, for any $\lambda \in \mathbb{R}^+$. By equivariance of the pullback of background fields, physical scalings actually act as functors, $\mathfrak{BkgG} \to \mathfrak{BkgG}$ and $\mathfrak{BkgG}^+ \to \mathfrak{BkgG}^+$ respectively.

Finally, for future convenience, we introduce various local coordinates systems on BM and J^rBM together with the description of a particular class of diffeomorphisms called *coordinate scalings*. Let (x^a) be local coordinates on the open set $U \subseteq M$ and let $(x^a, g_{ab}, \ldots, (t_j)_{a_1\ldots a_{l_j}}, \ldots)$ be corresponding adapted local coordinates on $Z \subseteq BM$ where by definition Z projects onto U and, more strongly, each fiber BM_x is completely included in Z if $x \in U$.

- **Covariant coordinates**. According to Definition 1.6, the chart (x^a) on U induces corresponding adapted local coordinates on J^rBM called *covariant coordinates*

  $$\left(x^a, g_{ab,A}, (t_j)_{a_1\ldots a_{l_j},A}\right) \quad \text{on the afore-mentioned domain } Z^r \subseteq J^rBM,$$

  where only $n(n+1)/2$ metric components are considered because g_{ab} is a symmetric tensor.

- **Contravariant coordinates**. Since Lorentzian metrics are non-degenerate, they admit an inverse denoted, using a standard notation, with g^{ab}. We correspondingly obtain induced coordinates g^{ab}_A on jets of the inverse-metric bundle. Using the notation $g^{AB} = g^{a_1b_1}\cdots g^{a_lb_l}$, for $|A| = |B| = l$, we define the following functions:

  $$g = |\det g_{ab}|, \qquad g^{ab,A} = g^{AB} g^{ab}_B, \qquad (t_j)^{a_1\ldots a_{l_j},A} = g^{AB}(t_j)_B^{a_1\ldots a_{l_j}},$$

  where we have chosen fully contravariant coordinates for tensor bundles. We have then obtained the set of local *contravariant coordinates*

  $$\left(x^a, g^{ab,A}, (t_j)^{a_1\ldots a_{l_j},A}\right) \quad \text{on } Z^r \subseteq J^rBM.$$

- **Rescaled contravariant coordinates.** We can obtain another coordinate set by a suitable rescaling of the previous one: We introduce a factor of the form g^α, with $\alpha \in \mathbb{R}$, to rescale the coordinates (n is the dimension of M):

  $$\left(x^a, g, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|}(t_j)^{a_1\ldots a_{l_j},A}\right) \quad \text{on } Z^r \subseteq J^rBM.$$

  It should be noticed that the coordinates g and $g^{-\frac{1}{n}} g_{ab}$ are functionally independent only up to the identity $g^{-1} = \left|\det g^{ab}\right|$. So one of the coordinates $g^{-\frac{1}{n}} g_{ab}$ is (implicitly) omitted and replaced by g. The relevance of the rescaled contravariant coordinates consists of the fact since s_j is the (physical) scaling degree of $\mathbf{t}_j$ these coordinates without the coordinate g,

$$\left(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|} (t_j)^{a_1 \ldots a_{l_j},A}\right)$$

are *invariant* under physical scaling.

- **Curvature coordinates.** Since we have a Lorentzian metric $\mathbf{g}$, we can always define the Levi-Civita connection ∇ and the Riemann tensor $\mathbf{R}$. By well-known formulas, we can also regroup the second order jet coordinates of the metric into the components of the Christoffel symbols Γ^a_{bc} and the components of the fully covariant Riemann tensor $\bar{R}_{abcd}$. An alternative way to regroup the components of the Riemann tensor is into the following fully contravariant tensor $\mathbf{S}$, with components

$$\bar{S}^{abcd} := g^{aa'} g^{bb'} \bar{R}_{a'}{}^{(c}{}_{b'}{}^{d)}.$$

We denote by $\Gamma^a_{bc,A}$ the components of the ∂_A coordinate derivatives of Γ^a_{bc}, by $\bar{S}^{abcd,A}$ the components of the symmetrized contravariant $\nabla^A = \nabla^{(a_1} \cdots \nabla^{a_l)}$ derivatives of $\mathbf{S}$, with $(\bar{t}_j)^{a_1 \ldots a_{l_j},A}$ the components of the symmetrized contravariant derivatives of $(t_j)^{a_1 \ldots a_{l_j}}$. It is possible to prove that [AT94, AT96] that

$$\left(x^a, g_{ab}, \Gamma^a_{(bc,A)}, \bar{S}^{ab(cd,A)}, (\bar{t}_j)^{a_1 \ldots a_{l_j},A}\right)$$

defines a complete coordinate system on $Z^r \subseteq J^r BM$, which we call *curvature coordinates.*

- **Rescaled curvature coordinates.** Analogously to rescaled contravariant coordinates, we can rescale the curvature coordinates obtaining a new coordinate system

$$\left(x^a, g, g^{-\frac{1}{n}} g_{ab}, \Gamma^a_{(bc,A)}, g^{\frac{3}{n}+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|} (\bar{t}_j)^{a_1 \ldots a_{l_j},A}\right).$$

As before, removing g form the set of rescaled curvature coordinates we find a set of coordinates which is fixed under physical scaling (since s_j is the scaling degrees of $\mathbf{t}_j$).

Definition 1.17. We call a diffeomorphism $M \to M$ a **coordinate scaling** around of $p \in M$ if, in a neighborhood of p whose closure is included in the domain $U \subseteq M$ of local coordinates (x^a) centered at p itself, it acts as

$$x^a \longmapsto \mu^{-1} x^a \quad (a = 1, \dots, n)$$

for some $\mu > 0$, and smoothly extends to the identity before reaching the boundary of U. More precisely, defining $t := -\ln \mu$, the **class of coordinate scaling** around p is represented by the one-parameter group of diffeomorphisms $\{\phi_t\}_{t \in \mathbb{R}}$ of the whole M leaving p fixed generated by the globally defined vector field $X^a = -h x^a \frac{\partial}{\partial x^a}$, where $h \in \mathscr{D}(M)$ vanishes before reaching the boundary of U and attains the constant value 1 in a neighborhood of p.

We stress that, unlike physical scaling, these transformation are induced by diffeomorphisms of M.

Normal ordering and AQFT | 2

In this chapter we present the problem of normal ordering in AQFT. Firstly we show why the normal ordering procedure is necessary, analysing a brief example from Minkowskian QFT. Then we show why there does not exists a unique normal ordering prescription in curved spacetime, motivating then the whole thesis. In the last section we introduce some further ideas of locally covariant AQFT.

2.1 Normal ordering and Wick polynomials

In general, in a quantum field theory, several physically relevant observables (like the stress-energy tensor) are constructed with products of fields at the same spacetime point. For this reason, field powers and polynomials have to be defined properly. We start analysing this problem in the standard (Minkowskian) theory of quantum fields (see for e.g. [PS95]). We take a massive scalar Klein-Gordon field $\varphi(x)$ in a 4-dimensional (Minkowski) spacetime as an example to explain the problem, but also observables constructed with other kind of fields present the same issue. First of all we recall that in standard Minkowskian QFT, quantum fields are operators (more precisely operator valued distributions) on a suitable Hilbert space. Exploiting spatial Fourier transform, the quantum field $\varphi(x)$ is then written in terms of creation and annihilation operators, a and a^*, which act on a Fock space, ***i.e.***

$$\varphi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(\sqrt{2\pi})^3} \frac{1}{\sqrt{2k_0}} \left(a_{\mathbf{k}} e^{-ik\cdot x} + a^*_{\mathbf{k}} e^{ik\cdot x} \right) \tag{2.1}$$

where $\mathbf{k}$ is the 3-momentum of the field and where $k_0 = \sqrt{\mathbf{k}^2 + m^2}$ with m the mass of the field. Then, we can naively try to define the most basic field polynomial, namely φ^2, in the following way

$$\varphi^2(x) := \lim_{y\to x} \varphi(x)\varphi(y). \tag{2.2}$$

But, with this definition, the squared field ***is not a well-defined object*** because its vacuum expectation value "blow-up". Indeed, using (2.1), we obtain

$$\begin{aligned}
&\varphi(x)\varphi(y) = \\
&\quad = \int \frac{\mathrm{d}^3\mathbf{k}}{(\sqrt{2\pi})^3} \frac{\mathrm{d}^3\mathbf{k}'}{(\sqrt{2\pi})^3} \frac{1}{2\sqrt{k_0 k_0'}} \left(a_{\mathbf{k}} e^{-ik\cdot x} + a^*_{\mathbf{k}} e^{ik\cdot x} \right) \left(a_{\mathbf{k}'} e^{-ik'\cdot y} + a^*_{\mathbf{k}'} e^{ik'\cdot y} \right) \\
&\quad = \int \frac{\mathrm{d}^3\mathbf{k}}{(\sqrt{2\pi})^3} \frac{\mathrm{d}^3\mathbf{k}'}{(\sqrt{2\pi})^3} \frac{1}{2\sqrt{k_0 k_0'}} \Big(a_{\mathbf{k}} a_{\mathbf{k}'} e^{-ik\cdot x} e^{-ik'\cdot y} + a_{\mathbf{k}} a^*_{\mathbf{k}'} e^{-ik\cdot x} e^{ik'\cdot y} \\
&\qquad\qquad + a^*_{\mathbf{k}} a_{\mathbf{k}'} e^{ik\cdot x} e^{-ik'\cdot y} + a^*_{\mathbf{k}} a^*_{\mathbf{k}'} e^{ik\cdot x} e^{ik'\cdot y} \Big)
\end{aligned} \tag{2.3}$$

and if we compute the expectation value of $\varphi^2(x)$ on the Minkowski vacuum $|0\rangle$

$$\langle 0|\varphi^2(x)|0\rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{k}^2 + m^2}} \longrightarrow \infty$$

and the $\varphi^2(x)$, defined with (2.2), is not a well-defined object. In Minkowskian QFT this problem is solved exploiting the so-called ***normal ordering prescription***: all observables are written in term of creation and annihilation operators and then all creation operators have to be written to the left of all annihilation operators. In other words, the normal ordered product, denoted by $:\cdot:$, of a and a^* is defined by

$$:aa^*: = a^*a, \quad :a^*a: = a^*a, \quad :aa: = aa, \quad :a^*a^*: = a^*a^*. \tag{2.4}$$

Using this prescription, there are no more divergences in the vacuum expectation value of the squared field and the expectation value of such kind of objects is now meaningful:

$$:\varphi(x)\varphi(y): = \int \frac{\mathrm{d}^3\mathbf{k}}{(\sqrt{2\pi})^3}\frac{\mathrm{d}^3\mathbf{k}'}{(\sqrt{2\pi})^3}\frac{1}{2\sqrt{k_0k_0'}}\Big(a_{\mathbf{k}}a_{\mathbf{k}'}e^{-ik\cdot x}e^{-ik'\cdot y} + a^*_{\mathbf{k}'}a_{\mathbf{k}}e^{-ik\cdot x}e^{ik'\cdot y} + a^*_{\mathbf{k}}a_{\mathbf{k}'}e^{ik\cdot x}e^{-ik'\cdot y} + a^*_{\mathbf{k}}a^*_{\mathbf{k}'}e^{ik\cdot x}e^{ik'\cdot y}\Big) \tag{2.5}$$

and if we now define the normal ordered squared field as

$$:\varphi^2(x): = \lim_{y\to x} :\varphi(x)\varphi(y):$$

we obtain

$$\langle 0|:\varphi^2(x):|0\rangle = 0.$$

This procedure can be obviously extended to define higher powers of $\varphi(x)$. From this example we can therefore see how a prescription like normal ordering is essential to properly define important observables of quantum field theories.

Remark 2.1. Since, in general, we are interested in theories defined in the general framework of a curved spacetime and since, for this reason, we do not have the global Fourier transform at our disposal, we have to re-write the normal ordering prescription (2.4) in position space. Comparing equations (2.3) and (2.5) it is easy to show that

$$:\varphi(x)\varphi(y): = \varphi(x)\varphi(y) - \int \frac{\mathrm{d}^3\mathbf{k}}{(\sqrt{2\pi})^3}\frac{\mathrm{d}^3\mathbf{k}'}{(\sqrt{2\pi})^3}\frac{1}{2\sqrt{k_0k_0'}}\left[a_{\mathbf{k}}, a^*_{\mathbf{k}'}\right]e^{-ik\cdot x}e^{ik'\cdot y}$$

where $[\cdot,\cdot]$ is the standard commutator of operators. Since

$$[a_{\mathbf{k}}, a^*_{\mathbf{k}'}] = \delta(\mathbf{k}-\mathbf{k}')\mathbb{I}$$

we have

$$
\begin{aligned}
:\varphi(x)\varphi(y): &= \varphi(x)\varphi(y) - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2k_0}e^{-ik\cdot(x-y)}\,\mathbb{I} \\
&= \varphi(x)\varphi(y) - \langle 0|\varphi(x)\varphi(y)|0\rangle \mathbb{I}
\end{aligned}
$$

and we finally obtain the expression of the normal ordered product of fields in position space

$$
:\varphi^2(x): = \lim_{y\to x} \varphi(x)\varphi(y) - \langle 0|\varphi(x)\varphi(y)|0\rangle \mathbb{I}
$$

which is a meaningful object. As we have shown in this brief example, the standard normal ordering prescription, in Minkowski spacetime, is equivalent to the well-known divergence subtraction method [KM15, Sec. 5.3.1]. In the next sections, we show how this divergence substraction method can be used in a generic curved spacetime.

An observable which is a polynomial in the field, with factors involving products of fields evaluated at the same spacetime point, is called **Wick polynomial**. The most basic Wick polynomials are **Wick powers**, namely k-times product of the field at the same point.

Now, we have to translate these ideas in the general framework of locally covariant algebraic quantum field theory (AQFT) on curved spacetimes. To do that, we start introducing some general concepts of AQFT.

2.2 Algebraic Quantum Field Theory

In this section we introduce the general ideas of algebraic quantum field theory. A complete and recent reference on this topic is [BDFY15]. In particular, in this section we mainly refer to [KM15]. We start recalling a basic definition.

Definition 2.1. A collection $\mathcal{A}$ of elements is called a unital (associative) **$*$-algebra** if $\mathcal{A}$ is endowed with a (associative) product operation endowing it with an algebra structure and if there exists $\mathbb{I} \in \mathcal{A}$ such that $a\mathbb{I} = \mathbb{I}a = a$ for all $a \in \mathcal{A}$. Furthermore it is also assigned $^* : \mathcal{A} \to \mathcal{A}$ such that

1. $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$ for all $a, b \in \mathcal{A}$ and for all $\lambda \in \mathbb{C}$,
2. $(ab)^* = b^* a^*$ for all $a, b \in \mathcal{A}$,
3. $a^{**} = a$ for all $a \in \mathcal{A}$.

A set $G \subset \mathcal{A}$ is said to *generate* the algebra $\mathcal{A}$, and the elements of G are said *generators* of $\mathcal{A}$, if each element of $\mathcal{A}$ is a finite complex linear combination of products of elements of G.

The algebraic formulation of quantum theories was first introduced by Haag in [Haa96]. In this framework there is no preferred Hilbert space but observables are viewed as abstract self-adjoint objects. These observables generate a $*$-algebra (or a C^*-algebra in some formulations). In other words, the $*$-algebra of observables is not defined simply as some concrete set of operators on a Hilbert space, but it is defined abstractly using more basic objects. In this very general framework, also the notion of quantum state is generalized introducing the notion of quantum algebraic state.

Definition 2.2. Given an unital $*$-algebra $\mathcal{A}$, an algebraic **state** ω over $\mathcal{A}$ is a $\mathbb{C}$-linear map $\omega : \mathcal{A} \to \mathbb{C}$ which is positive and normalized, *i.e.*

$$\omega(a^* a) \geq 0 \quad \text{for all } a \in \mathcal{A}, \qquad\qquad \omega(\mathbb{I}) = 1.$$

Given a $*$-algebra and an algebraic state on it, it is possible to recover the standard interpretation of quantum theories via the Gelfand-Naimark-Segal (GNS) construction. In the following, $\mathscr{L}(V)$ will denote the linear space of linear operators on the vector space V.

Definition 2.3. Let $\mathcal{A}$ be a complex algebra and let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$.

1. An algebra homomorphism $\pi : \mathcal{A} \to \mathscr{L}(\mathcal{D})$ (*i.e.* a linear and product preserving map) is called **representation** of $\mathcal{A}$ on $\mathcal{H}$ with domain $\mathcal{D}$. If $\mathcal{A}$ is furthermore unital, a representation is also required to satisfy $\pi(\mathbb{I}) = I$.

2. If finally $\mathcal{A}$ is a $*$-algebra, a $*$-**representation** of $\mathcal{A}$ on $\mathcal{H}$ with domain $\mathcal{D}$ is a representation which satisfies

   $$\pi(a)^\dagger \restriction_{\mathcal{D}} = \pi(a^*) \quad \forall a \in \mathcal{A}$$

   where † denotes the Hermitian adjoint operation in $\mathcal{H}$.

Having a $*$-algebra and an algebraic state on it, the standard interpretation of quantum theories can be recovered via the well-know GNS construction.

Theorem 2.1 (GNS construction). *If $\mathcal{A}$ is a complex unital $*$-algebra and $\omega : \mathcal{A} \to \mathbb{C}$ is a state, the following facts holds.*

1. *There is a quadrupole* $(\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega, \Psi_\omega)$ *where,:*

   (i) $\mathcal{H}_\omega$ *is a complex Hilbert space;*

   (ii) $\mathcal{D}_\omega \subset \mathcal{H}_\omega$ *is a dense subspace;*

   (iii) $\pi_\omega : \mathcal{A} \to \mathscr{L}(\mathcal{D}_\omega)$ *a* $*$*-representation of* $\mathcal{A}$ *on* $\mathcal{H}_\omega$ *with domain* $\mathcal{D}_\omega$*;*

   (iv) $\pi_\omega(\mathcal{A})\Psi_\omega\omega = \mathcal{H}_\omega$*;*

   (v) $\omega(a) = \langle \Psi_\omega\omega | \pi_\omega(a)\Psi_\omega\omega\rangle$ *for every* $a \in \mathcal{A}$*.*

2. *If* $(\mathcal{H}'_\omega, \mathcal{D}'_\omega, \pi'_\omega, \Psi'_\omega)$ *satisfies (i)-(v), then there is* $U : \mathcal{H}_\omega \to \mathcal{H}'_\omega$ *surjective and isometric such that:*

$$U\Psi_\omega = \Psi'_\omega, \qquad U\mathcal{D}_\omega = \mathcal{D}'_\omega, \qquad U\pi_\omega(a)U^{-1} = \pi'_\omega(a) \quad \text{if } a \in \mathcal{A}.$$

Proof. See [KM15, Thm. 5.1.13]. □

After this brief abstract introduction about the basic concepts of AQFT, we can define, as an example, a $*$-algebra $\mathcal{A}(M)$ which encodes the properties of a **massive vector Klein-Gordon field** A on a globally hyperbolic spacetime M. We recall that, classically, a (covariant) vector KG field is a vector field, *i.e.* a section of $\mathscr{D}(T^*M)$, which fulfils the equation of motion

$$PA_a(x) = (-\Box_\mathbf{g} + m^2 + \xi R)A_a(x) = 0$$

where $\Box_\mathbf{g} := g^{ab}\nabla_a\nabla_b$ and where $\xi \in \mathbb{R}$ is the coupling to the scalar curvature R. We recall that the KG operator P is Green hyperbolic (Definition 1.10).

Definition 2.4 (CCR algebra). The CCR algebra of the quantum field A over M is the unital $*$-algebra presented by the following generators and relations. The generators consist of (smeared abstract) field operators, $A(f)$, labelled by functions $f \in \mathscr{D}(T^*M)$ (the identity $\mathbb{I}$ is of course included in the construction of the corresponding freely generated algebra). These generators satisfy the following relations:

- $\mathbb{R}$**-linearity**: $A(af + bg) = aA(f) + bA(g)$ if $f, g \in \mathscr{D}(T^*M)$ and $a, b \in \mathbb{R}$
- **Hermiticity**: $A(f)^* = A(f)$ for $f \in \mathscr{D}(T^*M)$
- **Klein-Gordon**: $A\left((-\Box_\mathbf{g} + m^2 + \xi R)f\right) = 0$ for $f \in \mathscr{D}(T^*M)$
- **Commutation relations**: $[A(f), A(g)] = iE(f, g)\mathbb{I}$ for $f, g \in \mathscr{D}(T^*M)$

where $E(\cdot,\cdot)$ is the causal propagator (*i.e.* the advanced-minus-retarded fundamental solution defined in Def. 1.10).

The Hermitian elements of $\mathcal{A}(M)$ are the elementary observables of the free theory of vector KG field A. However, this algebra is not sufficient to fully describe the physics of a quantum vector KG field: for example it does not include observables like the stress-energy tensor of A or basic interaction terms, because these observables involves products of fields evaluated at the same spacetime point (*i.e.* Wick polynomials), like $A_a A_b(x)$ (interpreted as a symmetric tensor product of fields). This observable, smeared with a test function $f \in \mathscr{D}(S^2T^*M)$, could be formally defined as

$$A^2(f) = \int_M A^a(x)A^b(y)f_{ab}(x)\delta(x,y)\,\mathrm{d}g \tag{2.6}$$

but this object is not an element of the algebra $\mathcal{A}(M)$. So, since this algebra is too small to describe important observables, we have to enlarge it, and, at the same time, we aim to give a precise mathematical meaning to (2.6). To do this, we have to properly define Wick polynomials generalizing the divergence subtraction method outlined in Remark 2.1. The problem, if we want to develop a completely covariant theory valid on generic curved spacetimes, is that, in general, it is not possible to assign a physically distinguishable state to each spacetime (like the Poincaré vacuum in Minkowski spacetime). The key observation to overcome this problem is that it is possible to select a type of divergence in common with all physically relevant states in every spacetime. These states which mimic the type of divergence of Minkowski vacuum are called **Hadamard states**. A good reference for Hadamard states of scalar fields is [KM15], while the vector fields case was firstly studied in [SV01]. We do not present a complete analysis on Hadamard states, because it would be outside the aims of this work, but in the next section we show the basic ideas behind extending the algebra using those kind of states (or, more precisely, their singularity).

2.3 Normal ordering in AQFT

In this section we show how the Hadamard singularity can be used to properly define Wick polynomials in AQFT on curved spacetime. To explain the procedure we take as example a scalar Klein-Gordon field φ in a four dimensional spacetime, but the same ideas are also applicable for a vector field. Since this is a well-known topic in AQFT, we summarize only the main ideas. For a detailed discussion on this topic we refer to [KM15, Sec. 5.3.2] and the references therein. What we are going to present is called Hadamard parametrix regularization method, firstly studied in [HW01, HW02, BF00].

We start recalling some basic notions of Lorentzian geometry, necessary to properly introduce Hadamard states from a geometric viewpoint.

Definition 2.5. If $(M, \mathbf{g})$ is a spacetime, an open set $C \subset M$ is said a ***normal convex*** neighbourhood if there is an open set $W \subset TM$ with the form $W = \{(q, v) \mid q \in C, v \in S_q\}$ where $S_q \in T_q M$ is a star-shaped open neighborhoodbof the origin such that

$$\exp|_W : (q, v) \mapsto \exp_q v$$

is a diffeomorphism onto $C \times C$. Moreover, with $\sigma(x, y)$ we indicate the *squared geodesic distance* of x from y. It is defined as

$$\sigma(x, y) := g_x \left(\exp_x^{-1} y, \exp_x^{-1} y\right).$$

The squared geodesic distance $\sigma(x, y)$ turns out to be smoothly defined on $C \times C$ if C is a convex normal neighbourhood where we also have $\sigma(x, y) = \sigma(y, x)$.

In a convex neighbourhood C of a spacetime $(M, \mathbf{g})$ it is possible to define a *local solution* (up to smooth terms) of the Klein-Gordon equation, also known as a *parametrix*. This function (or more precisely, class of functions) has essentially the same short distance singularity of the two point function of Minkowski vacuum. In a convex neighbourhood C of a four dimensional spacetime the **Hadamard parametrix** has the form

$$H_\varepsilon(x, y) = \frac{1}{4\pi^2} \frac{u(x, y)}{\sigma_\varepsilon(x, y)} + v(x, y) \log\left(\frac{\sigma_\varepsilon(x, y)}{\lambda^2}\right) \tag{2.7}$$

where $x, y \in C$, $\lambda > 0$ is a length scale and T is a local time coordinate increasing towards the future such that

$$\sigma_\varepsilon(x, y) := \sigma(x, y) + 2i\varepsilon\left(T(x) - T(y)\right) + \varepsilon^2.$$

The cut in the complex domain of the $\log$ function is assumed along the negative axis in (2.7). The coefficients u, v are determined by recursive differential equations (se for e.g. [Mor03, App. A]). This parametrix distributionally satisfies Klein-Gordon equation in both argument up to jointly smooth functions of x and y.

Remark 2.2. The Hadamard parametrix 2.7 is constructed using only local geometric data and the parameters defining the equation of motion but does not refers to particular states, which are global objects.

Definition 2.6. If $\omega : \mathcal{A}(M) \to \mathbb{C}$ is a state, its n**-point function** is defined as

$$\omega_n(f_1, \ldots, f_n) := \omega\left(\varphi(f_1) \cdots \varphi(f_n)\right)$$

where $f_1, \ldots, f_n \in \mathscr{D}(M)$.

Definition 2.7. If (M, g) is a four dimensional spacetime, we say that a state ω on $\mathcal{A}(M)$ and its two-point function ω_2 are **Hadamard** if $\omega_2 \in \mathscr{D}'(M \times M)$ and every point of M admits an open normal neighbourhood C where

$$\omega_2(x, y) - H_{0^+}(x, y) = w(x, y) \quad \text{for some} \quad w \in \mathscr{E}(C \times C).$$

Here 0^+ indicates the standard weak distributional limit as $\varepsilon \to 0^+$.

We are now in position to define the expectation values of Wick powers $:\varphi^k:(f)$ with respect to Hadamard state ω. As it should be clear from the previous analysis, all of that can be done simultaneously for all Hadamard state (*i.e.* without considering a reference state) because the only important aspect is the singularity, which is the same for all Hadamard states. First, define for smooth functions $f_1, \ldots, f_k \in \mathscr{D}(M)$ supported in a convex normal neighbourhood C

$$:\varphi(f_1) \cdots \varphi(f_k):_H = \int_{M^n} :\varphi(x_1) \cdots \varphi(x_k):_H f_1(x_1) \cdots f_k(x_k)\, \mathrm{d}g(x_1, \ldots, x_k)$$

where the completely symmetrized formal kernels $:\varphi(x_1) \cdots \varphi(x_k):_H$ are defined recursively by

$$\begin{aligned} :\varphi(x_1):_H &= \varphi(x_1) \\ :\varphi(x_1)\varphi(x_2):_H &= \varphi(x_1)\varphi(x_2) - H_{0^+}(x_1, x_2) \\ &\vdots \\ :\varphi(x_1) \cdots \varphi(x_{k+1}):_H &= :\varphi(x_1) \cdots \varphi(x_k):_H \varphi(x_{k+1}) \\ &\quad - \sum_{l=1}^{k} :\varphi(x_1) \cdots \widetilde{\varphi(x_l)} \cdots \varphi(x_k):_H H_{0^+}(x_l, x_k + 1) \end{aligned}$$

where the element with $\widetilde{\cdot}$ is removed. It is possible to show [KM15, Sec. 5.3.2] that the expectation value $\omega(:\varphi(x_1) \cdots \varphi(x_k):_H)$ is smooth and thus we are in position to define the expectation values of Wick powers for $f \in \mathscr{D}(M)$ such that its support is included in C

$$\omega\left(:\varphi^k:_H(f)\right) = \int_{M^n} \omega\left(:\varphi(x_1)\cdots\varphi(x_k):_H\right) f(x_1)\delta(x_1,\ldots,x_k)\,\mathrm{d}g(x_1,\ldots,x_k).$$

Now this is a good definition, but it is affected by several ambiguities due to the construction of the parametrix H_ε, which means that there is not a unique definition of Wick powers. The goal of the next chapters is to study and classify these ambiguities (*i.e.* study the renormalization) for a generic boson quantum field using a locally covariant framework.

Remark 2.3. Using the ideas discussed in this section and some tools from ***Microlocal Analysis*** (see, for e.g. [KM15, Sec 5.3.3]) it is possible to extend the original algebra $\mathcal{A}(M)$ to an extended algebra $\mathcal{W}(M)$ which includes also Wick polynomials. In the rest of this thesis, we consider an extended algebra $\mathcal{W}$ without regarding how it has been constructed.

Remark 2.4. It is possible to perform the same analysis discussed in this section also for vector fields. A good reference for Hadamard state for these kind of fields is [SV01]. A more explicit construction of Hadamard parametrix for vector fields can be found, for e.g., in [BF16].

2.4 Locally covariant AQFT

In the previous sections we worked in a fixed spacetime, but now we can remove this constraint and show how the previous ideas can be set up in a locally covariant framework [BFV03]. The introduced formalism permits us to describe the *net of algebras of local quantum observables* on our background geometries (cf. Section 1.5). We explicitly only deal with $\mathfrak{BkgG}$, but everything we say can be trivially re-adapted to $\mathfrak{BkgG}^+$.

Definition 2.8. A **net of algebras (of local quantum observables)** $\mathcal{W}$ is an assignment of a complex unital $*$-algebra $\mathcal{W}(M,\mathbf{b})$ to every background geometry $(M,\mathbf{b})$ in $\mathfrak{BkgG}$ together with an assignment of an injective unital $*$-algebra homomorphism $\iota_\chi\colon \mathcal{W}(M,\mathbf{b}) \to \mathcal{W}(M',\mathbf{b}')$ to every morphism in $\mathfrak{BkgG}$, respecting compositions and associating identities to identities. In other words $\mathcal{W}\colon \mathfrak{BkgG} \to \mathfrak{Alg}$ is a functor from the category of background geometries into the category of (complex) unital $*$-algebras whose morphisms are injective unital

$*$-algebra homomorphisms. Further, we require that $\mathcal{W}$ respects (i) **scaling** and (ii) the **time slice axiom**, as described below.

(i) Physical scaling transformations $(M, \mathbf{b}) \mapsto (M, \mathbf{b}_\lambda)$ are represented in terms of $*$-algebra isomorphisms $\sigma_\lambda \colon \mathcal{W}(M, \mathbf{b}) \to \mathcal{W}(M, \mathbf{b}_\lambda)$ such that $\sigma_1 = id$ and $\sigma_\lambda \circ \sigma_{\lambda'} = \sigma_{\lambda\lambda'}$. Varying $(M, \mathbf{b})$, scaling transformations must commute with embeddings, *i.e.*, they act as natural isomorphisms $\sigma_\lambda \colon \mathcal{W} \to \mathcal{W}_\lambda$ between the $*$-algebra valued functors $\mathcal{W}$ and $\mathcal{W}_\lambda$, the latter defined by $\mathcal{W}_\lambda(M, \mathbf{b}) = \mathcal{W}(M, \mathbf{b}_\lambda)$.

(ii) Given a morphism $\chi \colon M \to M'$ between the background geometries $(M, \mathbf{b})$ and $(M', \mathbf{b}')$, if the image $\chi(M) \subseteq M'$ contains a Cauchy surface for $(M', \mathbf{g}')$, then the induced $*$-homomorphism $\iota_\chi \colon \mathcal{W}(M, \mathbf{b}) \to \mathcal{W}(M', \mathbf{b}')$ is a $*$-isomorphism.

We refer to a similar functor $\mathcal{W} \colon \mathfrak{BkgG}^+ \to \mathfrak{Alg}$ with analogous properties as a net of algebras as well.

Remark 2.5. The unit of every algebra $\mathcal{W}(M, \mathbf{b})$ will be simply denoted by 1 in place of a cumbersome notation $1_{(M,\mathbf{b})}$.

Remark 2.6. The scaling axiom is necessary because we will be required to compare local algebras defined on a given manifold which are identified by scaling transformations. These algebras must be viewed as distinct since their background fields are different. Therefore to compare them we need to assume that there is an isomorphism σ_λ identifying them. In more physically minded presentations this structure is not discussed and the said identification is hidden in the formalism.

The time slice axiom has a consequence which will play a fundamental role in the sequel, in particular for the application of the Peetre-Slovák.

Proposition 2.2. *Referring to Definition 2.8 consider* $(M, \mathbf{b}), (M, \mathbf{b}') \in \mathfrak{BkgG}$ *(resp.* $\mathfrak{BkgG}^+$*) such that* $\mathbf{b} = \mathbf{b}'$ *with identical temporal orientation outside a compact region* $K \subseteq M$*. There exists a unital* $*$*-algebra isomorphism*

$$\tau \colon \mathcal{W}(M, \mathbf{b}) \to \mathcal{W}(M, \mathbf{b}')$$

such that $\tau|_{\mathcal{W}(N,\mathbf{b}|_N)} \colon \mathcal{W}(N, \mathbf{b}|_N) \to \mathcal{W}(N, \mathbf{b}'|_N)$ *is the identity for every* $(N, \mathbf{b}|_N) \in \mathfrak{BkgG}$ *(resp.* $\mathfrak{BkgG}^+$*) satisfying* $N \cap J^+_{(M,\mathbf{b})}(K) = \emptyset$.

Proof. $J^+_{(M,\mathbf{g})}(K) = J^+_{(M,\mathbf{g}')}(K)$ since these sets are made of the union of the future-directed causal curves, defined with respect to the corresponding metrics, emanating from K itself. These sets are therefore made of the union of K and of the part external to K of all future-directed causal curves emanating from the frontier ∂K. However $\mathbf{g}|_{\partial K} = \mathbf{g}'|_{\partial K}$ because the metrics are smooth and coincide outside K, thus these curves do not depend on the chosen metric. In globally hyperbolic spacetimes $J^+_{(M,\mathbf{g})}(K)$ is always closed for K compact, so $\Delta := M \setminus J^+_{(M,\mathbf{g})}(K) = M \setminus J^+_{(M,\mathbf{g}')}(K)$ equipped with $\mathbf{g}|_\Delta = \mathbf{g}'|_\Delta$ is a smooth spacetime. $(\Delta, \mathbf{g}|_\Delta)$ is a (time-oriented with respect to the assigned temporal orientation of $(M, \mathbf{g})$) globally-hyperbolic spacetime so that $(\Delta, \mathbf{b}|_\Delta) \in \mathfrak{BkgG}$ (resp. $\mathfrak{BkgG}^+$) : it is sufficient to observe that a Cauchy surface Σ of $(M, \mathbf{g})$ in the past of K with respect to a factorization $M = \mathbb{R} \times \Sigma$ is a Cauchy surface for $(\Delta, \mathbf{g}|_\Delta)$. Let us denote by χ_Δ and χ'_Δ respectively the embeddings of Δ into $(M, \mathbf{g})$ and $(M, \mathbf{g}')$ Def.2.8 entails that they correspond to associated embeddings of unital $*$-algebras $\imath_{\chi_\Delta} : \mathcal{W}(\Delta, \mathbf{b}|_\Delta) \to \mathcal{W}(M, \mathbf{b})$ and $\imath_{\chi'_\Delta} : \mathcal{W}(\Delta, \mathbf{b}|_\Delta) \to \mathcal{W}(M, \mathbf{b}')$. Since Δ includes Cauchy surfaces in the past of K which are also simultaneously a Cauchy surfaces for $(M, \mathbf{g})$ and $(M, \mathbf{g}')$ respectively, in view of (ii) in Def.2.8, $\imath_{\chi_\Delta}$ and $\imath_{\chi'_\Delta}$ must be unital $*$-algebra isomorphisms and hence $\tau := \imath_{\chi'_\Delta} \circ \imath^{-1}_{\chi_\Delta} : \mathcal{W}(M, \mathbf{b}) \to \mathcal{W}(M, \mathbf{b}')$ is a well defined unital $*$-algebra isomorphism. Finally consider $(N, \mathbf{b}|_N) \in \mathfrak{BkgG}$ satisfying $N \cap J_(M, \mathbf{g})^+(K) = \emptyset$. Since $\mathbf{b}(x) = \mathbf{b}'(x)$ if $x \in N$, we have $(N, \mathbf{b}|_N) = (N, \mathbf{b}'|_N)$ (so that in particular $\mathcal{W}(N, \mathbf{b}|_N) = \mathcal{W}(N, \mathbf{b}'|_N)$) and thus the (co-domain restrictions of) spacetime embeddings $\chi_N : N \to N \subseteq M$ and $\chi'_N : N \to N \subseteq M$ are nothing but the identity maps and consequently $\imath_{\chi_N} = \imath_{\chi'_N} = \mathrm{id}_{\mathcal{W}(N,\mathbf{b}|_N)}$ due to (i) in Def.2.8. On the other hand we know that $\chi_N = \chi_\Delta|_N$ and $\chi'_N = \chi'_\Delta|_N$. Since the overall assignment of injective unital $*$-algebra homomorphisms to spacetime embeddings respects compositions and associates identities to identities, collecting the obtained identities we also have $\imath_{\chi_\Delta}|_{\mathcal{W}(N,\mathbf{b}|_N)} = \imath_{\chi_M} = \mathrm{id}_{\mathcal{W}(N,\mathbf{b}|_N)}$ and $\imath_{\chi'_\Delta}|_{\mathcal{W}(N,\mathbf{b}'|_N)} = \mathrm{id}_{\chi'_M} = \mathrm{id}_{\mathcal{W}(N,\mathbf{b}'|_N)} = \mathrm{id}_{\mathcal{W}(N,\mathbf{b}|_N)}$. Inserting into the definition of $\tau|_N : \mathcal{W}(N, \mathbf{b}|_N) \to \mathcal{W}(N, \mathbf{b}'|_N) = \mathcal{W}(N, \mathbf{b}|_N)$ we have

$$\tau|_{\mathcal{W}(N,\mathbf{b}|_N)} = \imath'_{\chi_N} \circ \imath^{-1}_{\chi_N} = \mathrm{id}_{\mathcal{W}(N,\mathbf{b}|_N)} \circ \mathrm{id}_{\mathcal{W}(N,\mathbf{b}|_N)} = \mathrm{id}_{\mathcal{W}(N,\mathbf{b}|_N)}$$

concluding the proof. □

Remark 2.7. There is an analogous but different unital $*$-algebra isomorphism constructed relying upon $J^-_{(M,\mathbf{g})}(K)$ instead of $J^+_{(M,\mathbf{g})}(K)$. The results established in this paper do not depend on the choice of one of these isomorphisms.

Quantum fields

We have so far discussed all the mathematical structures we need to describe background fields and the abstract notion of a net of quantum observables. At this abstract level we may introduce the definition of quantum fields as special elements of the algebras of observables $\mathcal{W}(M, \mathbf{b})$.

Definition 2.9. Fix a net of local quantum observables $\mathcal{W}$ as in Definition 2.8 and a natural vector bundle V. A **quantum V-field** is an assignment $\Phi_{(M,\mathbf{b})}$ of an algebra-valued distribution

$$\Phi_{(M,\mathbf{b})} \colon \mathscr{D}(V^*M) \to \mathcal{W}(M, \mathbf{b})$$

to each background geometry $(M, \mathbf{b}) \in \mathfrak{BkgG}$, where it is also supposed that for all background geometry $(M, \mathbf{b})$ and $f \in \mathscr{D}(V^*M)$ the quantum field is self-adjoint,

$$\Phi_{(M,\mathbf{b})}(f) = \Phi_{(M,\mathbf{b})}(f)^* \,.$$

The given definition does not yet assume any particular relation between $\Phi_{(M,\mathbf{b})}$ and $\Phi_{(M',\mathbf{b}')}$ when $(M, \mathbf{b})$ and $(M', \mathbf{b}')$ are connected by a morphism χ of $\mathfrak{BkgG}$. A quantum field Φ is said to be *locally covariant* when each pair of $\Phi_{(M,\mathbf{b})}$ and $\Phi_{(M',\mathbf{b}')}$ is in fact connected according to a natural rule arising from the definition of a natural bundle, translating into the mathematical language the ideas of locality and general covariance.

Definition 2.10. A quantum V-field Φ (Definition 2.9) with respect to the net of local quantum observables $\mathcal{W}$ as in Definition 2.8 is said to be **locally covariant** if satisfies the following identity for each morphism $\chi\colon (M, \mathbf{b}) \to (M', \mathbf{b}')$ where $\mathbf{b} = \chi^*\mathbf{b}'$,

$$\iota_\chi \left(\Phi_{(M,\mathbf{b})}(f)\right) = \Phi_{(M',\mathbf{b}')} \left(\chi_* f\right), \qquad \forall f \in \mathscr{D}(V^*M). \tag{2.8}$$

where χ_* is the push-forward with respect to a natural bundle, as in (1.4).

In the following, we require that the field Φ has a definite **scaling degree** $d_\Phi \in \mathbb{R}$ with respect to the action of physical scaling. However, when Φ has multiple components, different components of a V-field can be grouped together by their scaling degree, giving rise to the decomposition of the field bundle as in Remark 1.1. Then the role of the scaling degree is played by a globally diagonalizable endomorphism $\mathbf{d}_\phi \colon VM \to VM$, whose eigen-subspaces constitute the bundle decomposition $VM = \bigoplus_{i=1}^{N} W_i M$ and whose eigenvalues correspond to the weights of these field sub-bundles. Alternatively, once this field bundle decomposition is known, the endomorphism $\mathbf{d}_\Phi$ can be identified with its eigenvalues $(d_{\Phi_1}, \dots, d_{\Phi_N})$. Informally written, the relation between the scaling of background fields and Φ means that, for every $(\mathbf{g}, \mathbf{t}) \in \Gamma(BM)$,

$$(\mathbf{g}, \mathbf{t}_1, \dots, \mathbf{t}_N) \longmapsto \left(\lambda^{-2}\mathbf{g}, \lambda^{s_1}\mathbf{t}_1, \dots, \lambda^{s_N}\mathbf{t}_N\right) \quad \Longrightarrow \quad \Phi \longmapsto \lambda^{\mathbf{d}_\Phi}\Phi\,. \tag{2.9}$$

To formulate a precise statement valid also for V-fields exploiting our formalism we need a precise scaling procedure based on the isomorphism σ_λ introduced in Definition 2.8. If Φ is a quantum V-field, we can define the rescaled quantum V-field $S_\lambda\Phi$ as

$$(S_\lambda\Phi)_{(M,\mathbf{b})}(f) = \sigma_\lambda^{-1}(\Phi_{(M,\mathbf{b}_\lambda)}(\lambda^n f))\,, \quad \lambda \in \mathbb{R}^+\,, \tag{2.10}$$

where n is the dimension of the spacetime M and σ_λ is the algebra isomorphism introduced in Definition 2.8. It should be clear here that both $\Phi_{(M,\mathbf{b})}$ and $(S_\lambda\Phi)_{(M,\mathbf{b})}$ are element of the same algebra $\mathcal{W}(M,\mathbf{b})$ due to the presence of σ_λ^{-1} in the second case. The factor λ^n just compensates the scaling of the volume form $dg \mapsto \lambda^{-n}dg$ when $\mathbf{g} \mapsto \lambda^{-2}\mathbf{g}$. A mathematically rigorous version of (2.9) is now

$$(S_\lambda\Phi)_{(M,\mathbf{b})}(f) = \lambda^{d_\Phi}\Phi_{(M,\mathbf{b})}(f)\,, \quad (M,\mathbf{b}) \in \mathfrak{Bkg}\mathfrak{G}\,, \quad f \in \mathscr{D}(VM)\,, \quad \lambda \in \mathbb{R}^+\,.$$

Remark 2.8. As usual, it is convenient to think of the algebra-valued distribution Φ as a formal point-like field $\Phi^\mu(x)$ smeared with a test section $f \in \mathscr{D}(V^*M)$. In the sequel, we will make extensive use of the case when V is replaced by S^kV^*M. Then we may write the formal point-like field $\Phi^{\mu_1\cdots\mu_k}(x)$ smeared with a test section $f \in \mathscr{D}(S^kV^*M)$ as

$$\Phi_{(M,\mathbf{b})}(f) = \int_M \Phi^{\mu_1\cdots\mu_k}(x) f_{\mu_1\cdots\mu_k}(x)\,\mathrm{d}g(x)\,,$$

where $dg(x)$ is the volume form induced by the metric $\mathbf{g}$ on M.
Similarly, if $\mathbf{C}$ is any function that maps a background geometry $\mathbf{b}$ to a distribution on a certain space of test functions $\mathbf{C}_{(M,\mathbf{b})} \colon \mathscr{D}(S^kVM) \to \mathbb{R}$, it is heuristically convenient to use the distributional notation

$$\mathbf{C}_{(M,\mathbf{b})}(f) := \int_M C[M,\mathbf{b}]_{\mu_1\cdots\mu_k}(x) f^{\mu_1\cdots\mu_k}(x)\,\mathrm{d}g(x)\,, \quad f \in \mathscr{D}(S^kV^*M), \tag{2.11}$$

where the formal *c-number field* $C[M,\mathbf{b}]_{a_1\cdots a_k}(x)$ may not be smooth. Such a function $\mathbf{C}$ will be named a **c-number S^kV-field**. If the sections $C[M,\mathbf{b}]$ are smooth for every $(M, \mathbf{b})$ (*i.e.*, $C[M, \mathbf{b}] \in \mathscr{E}(S^k V^* M)$), $\mathbf{C}$ is said to be a **smooth** c**-number** $S^k V$**-field**. From now on we systematically identify $\mathbf{C}_{(M,\mathbf{b})}(f)$ with the corresponding trivial element, a so called *c-number*, $\mathbf{C}_{(M,\mathbf{b})}(f)1$ of $\mathcal{W}(M, \mathbf{b})$. In this sense a c-number $S^k V$-field is a sub-case of a quantum $S^k V$-field and, for instance, the scaling action (2.10) applies to these particular quantum fields as well. If in addition $\mathbf{C}$ satisfies the identity

$$\mathbf{C}_{(M,\chi^*\mathbf{b}')}(f) = \mathbf{C}_{(M',\mathbf{b}')}(\chi_* f)\,, \tag{2.12}$$

for every background morphism $\chi\colon (M, \mathbf{b}) \to (M', \mathbf{b}')$ (so $\mathbf{b} = \chi^*\mathbf{b}'$) and every test section $f \in \mathscr{D}(S^k V^* M)$, then $\mathbf{C}$ defines a **locally covariant** c-number $S^k V$-field. Using the definitions of pull-back and push forward, *i.e.*

$$\chi^*\colon \mathscr{E}(S^k V^* M') \to \mathscr{E}(S^k V^* M) \qquad \chi_*\colon \mathscr{D}(S^k V M) \to \mathscr{D}(S^k V M'),$$

it is easy to prove that if $\mathbf{C}$ is described by $C[M, \mathbf{b}] \in \mathscr{E}(S^k V^* M)$ for every choice of $(M, \mathbf{b})$ by means of (2.11), then (2.12) is equivalent to

$$C[M, \chi^*\mathbf{b}'](x) = \left(\chi^* C[M', \mathbf{b}']\right)(x)\,, \tag{2.13}$$

for every background morphism $\chi\colon (M, \mathbf{b}) \to (M', \mathbf{b}')$ (with $\mathbf{b} = \chi^*\mathbf{b}'$) and $x \in M$.

3 Technical results

In this chapter we introduce all technical tools that we need in our analysis. We aim to present all the results in the clearest possible way, without reporting some cumbersome proofs, for which we refer to the literature. The chapter is divided in four section: Firstly we introduce the most important result, the Peetre-Slovák theorem. Then we prove some results about physical scaling and we present some results of invariant theory and the Thomas replacement theorem.

3.1 Peetre-Slovák theorem

In this section we introduce the Peetre-Slovák theorem, an important result of differential geometry which is also a key result for our analysis of renormalization of Wick powers. We introduce only some basic concepts, namely differential operator of (globally and locally) bounded order and compactly supported variation of sections, and then we state the Peetre-Slovák theorem. For further details the reader can refer to the standard monographs [KMS93, Slo88].

Let $E \to M$ be a smooth bundle. We recall that the afore-mentioned r-jet extension of sections acts as a map $j^r : \Gamma(E) \ni \psi \mapsto j^r\psi \in \Gamma(J^r E)$.

Definition 3.1. Let $E \to M$ and $F \to M$ be smooth bundles over the same base M. Consider a map $D\colon \Gamma(E) \to \Gamma(F)$.

1. D is a **differential operator of globally bounded order** if there exists an integer $r \geq 0$, the order, and a smooth map

$$d\colon J^r E \to F \, ,$$

which leaves fixed the base of the transformed point ($\pi_F \circ d = \pi_{J^r E}$) such that for any section $\psi \in \Gamma(E)$ we have an associated section of the form

$$D[\psi] = d \circ j^r \psi \, .$$

2. D is a **differential operator of locally bounded order** if it satisfies a similar condition locally. Namely, if for every $y \in M$ and every $\psi_0 \in \Gamma(E)$, there exists

   - a neighborhood $U \subseteq M$ of y with compact closure;
   - an integer $r \geq 0$;
   - an open neighborhood $Z^r \subseteq J^r(E)$ of $j^r\psi_0(U)$ projecting onto U;
   - a smooth function $d\colon Z^r \to F$ which leaves fixed the base of the transformed point

   such that

$$D[\psi](x) = d \circ j^r\psi(x)$$

   for all $x \in U$ and all $\psi \in \Gamma(E)$ with $j^r\psi(U) \subseteq Z^r$.

A differential operator D transforms sections ψ to sections $D[\psi]$ with the constraint that the value $D[\psi](x)$ of the transformed section attained at a point $x \in M$ depends only on the value of the initial section ψ at the same point x together with the values of its M-derivatives at x up to a certain order r, the jet $j^r_x\psi$ evaluated at the said x. A natural question is how to characterize these type of local transformations of sections among the whole class of maps $\Gamma(E) \to \Gamma(F)$. An answer is provided by some results known as Peetre-Slovák's theorem, of which we state two versions (there is a third more complete version we do not consider here [KMS93, Slo88]).

Theorem 3.1 (Linear Peetre's Theorem). *Let $E \to M$ and $F \to M$ be vector bundles over the same base M and $\Psi \colon \Gamma(E) \to \Gamma(F)$ a map such that $\Psi[\psi](x) \in F$ depends only on the germ of ψ at x for every $\psi \in \Gamma(E)$ and $x \in M$. If Ψ is linear with respect to the natural vector space structures of $\Gamma(E)$ and $\Gamma(F)$, then Ψ is a differential operator of locally bounded order.*

In other words, if Ψ is *linear*, even if the values $\Psi[\psi](x)$ potentially depends on the germ of ψ around every considered $x \in M$, actually they only depend on the *jet* of ψ at x as it is proper of differential operators. This noticeable result for a function $\Psi \colon E \to F$, can be made stronger keeping the requirement of dependence on the germ but relaxing the *linearity* hypothesis (thus also dropping the vector space structure of the fibers of E and F) and replacing linearity for a suitable *regularity* condition. This alternate condition demands regularity of Ψ when it acts on certain smooth families of sections ψ_s parametrized by $s \in \mathbb{R}^n$ we go to introduce with the help of an auxiliary bundle used to specify the joint-smoothness of these families. Given a smooth bundle $E \to M$ and the standard projection $\pi \colon \mathbb{R}^n \times M \ni (s, x) \mapsto x \in M$, we define an associated smooth bundle, called the **pullback bundle**, $p \colon \pi^*E \to \mathbb{R}^n \times M$ whose canonical fiber is isomorphic to that of E and the base is $\mathbb{R}^n \times M$. As a set, $\pi^*E = \mathbb{R}^n \times E$ with canonical projection onto its base given by $p \colon \pi^*E \ni (s, e) \to (s, \pi_E(e)) \in \mathbb{R}^n \times M$. The smooth differentiable structure of π^*E is defined accordingly. The smooth projection $q \colon \pi^*E \ni (s, e) \mapsto e \in E$ restricts to fiber diffeomorphisms $q|_{p^{-1}(s,x)} \colon p^{-1}(s, x) \to \pi_E^{-1}(x)$. This way the following diagram is commutative,

$$
\begin{array}{ccc}
\pi^*E & \xrightarrow{\;q\;} & E \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi_E} \\
\mathbb{R}^n \times M & \xrightarrow{\;\pi\;} & M
\end{array}
$$

and can be used to abstractly define π^*E taking advantage of a certain universal property of the triple (π^*E, p, q). It is now clear that a smooth section $\sigma \in \Gamma(\pi^*E)$ uniquely defines a $\mathbb{R}^n$-parametrized jointly-smooth family of sections $\{\psi_s\}_{s\in\mathbb{R}^n} \subseteq \Gamma(E)$, where $\psi_s(x) := q \circ \sigma(s, x)$ for $(s, x) \in \mathbb{R}^n \times M$. This observation justifies the following definition.

Definition 3.2. Given smooth bundle $E \to M$ and the associated pullback bundle $\pi^*E \to \mathbb{R}^n \times M$, $\sigma \in \Gamma(\pi^*E)$ is called **smooth n-dimensional family** of sections of E. If furthermore there exists a compact subset $K \subseteq M$ such that $\sigma(s, x) = \sigma(s', x)$ if $x \notin K$ and $s, s' \in \mathbb{R}^n$, then σ is said to be a smooth **compactly supported n-dimensional variation**.

We are in a position to state the relevant definition about the necessary regularity required in Peetre-Slovák's Theorem [KMS93, Slo88].

Definition 3.3. Given smooth bundles $E \to M$ and $F \to M$, a map $\Psi\colon \Gamma(E) \to \Gamma(F)$ is **regular** if it maps smooth n-dimensional families of sections to smooth n-dimensional families of sections for every natural n. Ψ is **weakly regular** (cf. [KM16, Apx.A]) if it maps smooth compactly supported n-dimensional variations to smooth compactly supported n-dimensional variations for every natural n.

Finally, having introduced all definitions that we need, we are now able to state the main result of this section.

Theorem 3.2 (Peetre-Slovák's Theorem). *Let $E \to M$ and $F \to M$ be smooth bundles over the same base M and $\Psi\colon \Gamma(E) \to \Gamma(F)$ a map such that, $\Psi[\psi](x) \in F$ depends only on the germ of ψ at x for every $\psi \in \Gamma(E)$ and $x \in M$. If Ψ is weakly regular, then it is a differential operator of locally bounded order.*

3.2 Some technical results about physical scaling

In this section we generalize some results from [KM16, Sec.2.4] in order to consider the case of a tensor valued function. Physical scaling will be used together with the notion of *homogeneous* and *almost homogeneous* scaling degree. Since these notions are quite abstract we can present them into a general inductive definition [KM16, Def.2.3].

Definition 3.4. Consider a linear representation $\rho\colon \mathbb{R}^+ \to GL(W)$ of the multiplicative group $\mathbb{R}^+$ on a vector space W whose action is indicated by $W \ni F \mapsto F_\lambda := \rho(\lambda)F \in W$, for every $\lambda \in \mathbb{R}^+$.

(a) An element $F \in W$ is said to have **homogeneous degree** $k \in \mathbb{R}$ if

$$F_\lambda = \lambda^k F, \quad \text{for all } \lambda \in \mathbb{R}^+.$$

(b) An element $F \in W$ is said to have **almost homogeneous degree** $k \in \mathbb{R}$ **and order** $l \in \mathbb{N}$ if $l \geq 0$ is an integer such that (with the sum over j is omitted when $l = 0$)

$$F_\lambda = \lambda^k F + \lambda^k \sum_{j=1}^{l} \left(\log^j \lambda\right) G_j, \quad \text{for all } \lambda \in \mathbb{R}^+,$$

and for some $G_j \in W$ depending on F, which have respectively almost homogeneous degree k and order $l - j$. An element that is almost homogeneous of order $l = 0$ is homogeneous by definition.

We are now in position to present several technical results about physical scaling. We prove all results except the next general lemma proved in [KM16, Lem. 2.5].

Lemma 3.3. *Referring to Definition 3.4, consider a pair of vector spaces W, W' endowed with corresponding representations of $\mathbb{R}^+$. Concerning (b) below, assume also that there exists a product $W \times W' \to V$ such that (i) V admits a representation of $\mathbb{R}^+$ and (ii) the map $W \times W' \to V$ is equivariant: $F_\lambda F'_\lambda = (FF')_\lambda$ for $F \in W$, $F' \in W'$ and $\lambda \in \mathbb{R}^+$. The following facts hold.*

(a) *A linear combination of two elements $F, F' \in W$ of almost homogeneous degree k and order l is of almost homogeneous degree k and order l.*

(b) *A product of an element $F \in W$, of almost homogeneous degree k and order l, and an element $F' \in W'$, of almost homogeneous degree k' and order l', has almost homogeneous degree $k + k'$ and order $l + l'$.*

We recall that in Section 1.5 we have defined the bundle of background fields as

$$BM = \mathring{S}^2 T^* M \oplus \left(\bigoplus_{j=1}^{K} T^{*\otimes l_j} M \right) \tag{3.1}$$

and that the physical scaling transformation on the sections of $\Gamma(BM)$ is given by

$$BM \ni (p, \mathbf{g}(p), \mathbf{t}_j(p)) \longmapsto \left(p, \lambda^{-2}\mathbf{g}(p), \lambda^{s_j}\mathbf{t}_j(p)\right) \in BM,$$

where $\lambda \in \mathbb{R}^+$ defines the scaling transformation. This (globally defined) representation of the multiplicative group $\mathbb{R}^+$ can be written in local coordinates

$$x^a \mapsto x^a, \quad g_{ab} \mapsto \lambda^{-2} g_{ab}, \quad (t_j)_{a_1 \dots a_{l_j}} \mapsto \lambda^{s_j} (t_j)_{a_1 \dots a_{l_j}}.$$

This transformation lifts to a transformation of the jet bundle $J^r BM$. In local coordinates

$$x^a \mapsto x^a, \quad g_{ab,A} \mapsto \lambda^{-2} g_{ab,A}, \quad (t_j)_{a_1 \dots a_{l_j},A} \mapsto \lambda^{s_j} (t_j)_{a_1 \dots a_{l_j},A}.$$

We stress that, with respect to Definition 3.4, we are interested in the case $W = C^\infty\left(J^r BM, S^k V^* M\right)$, as we will see in the next chapter. Moreover, since we have to consider also smaller domains $Z^r \subseteq J^r BM$ (with Z^r not invariant under physical scaling), it is more convenient to consider the infinitesimal version of these transformations, which are effected by the following vector field[1]

$$e = -2 g_{ab,A} \partial^{ab,A} + s_j (t_j)_{a_1 \dots a_{l_j},A} \partial^{a_1 \dots a_{l_j},A},$$

in the sense that the induced action on tensor functions on $J^r BM$ satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=1} F_\lambda = \mathcal{L}_e F, \tag{3.2}$$

where $\mathcal{L}_e$ is the Lie derivative, $F \in W$ and $F_\lambda \in W$ is the transformation under physical scaling of F. We stress that, since the physical scaling transformation is globally defined, the vector field e is globally defined on $J^r BM$.

Lemma 3.4. *A smooth function $F \colon J^r BM \to S^k V^* M$ that has almost homogeneous degree k and order l when the action $F \to F_\lambda$ is the one induced by physical scaling transformations, satisfies*

$$\left(\mathcal{L}_e - k\right)^{l+1} F = 0.$$

[1]We use the following notation

$$\partial^{ab,A} := \frac{\partial}{\partial g_{ab,A}} \quad \partial^{a_1 \dots a_{l_j},A} := \frac{\partial}{\partial (t_j)_{a_1 \dots a_{l_j},A}}$$

and an analogous one for contravariant coordinates.

Proof. If F is an almost homogeneous function of degree k and order l, using equation (3.2), we obtain

$$\left(\mathcal{L}_e - k\right) F = G^{(l-1)},$$

where $G^{(l-1)}$ is an almost homogeneous function of degree k and order $l-1$. If we repeat this operation l we obtain an homogeneous function $G^{(0)}$ of degree k:

$$\left(\mathcal{L}_e - k\right)^l F = G^{(0)}.$$

Since, for all homogeneous function B, we have $\left(\mathcal{L}_e - k\right) B = 0$ the proof is concluded. □

Thanks to this result, we can give an infinitesimal definition of homogeneous and almost homogeneous function. This definition is very useful since we have to consider function defined on a subset $Z^r \subseteq J^r BM$ which is not invariant under physical scaling.

Definition 3.5. A smooth function $F\colon Z^r \subseteq J^r BM \to S^k V^* M$, where Z^r is an open subset which may coincide with all of $J^r BM$, is said to have **almost homogeneous degree** $k \in \mathbb{R}$ **and order** $l \in \mathbb{N}$ (with $l \geq 0$) **under physical scalings** if it satisfies the identity

$$\left(\mathcal{L}_e - k\right)^{l+1} F = 0.$$

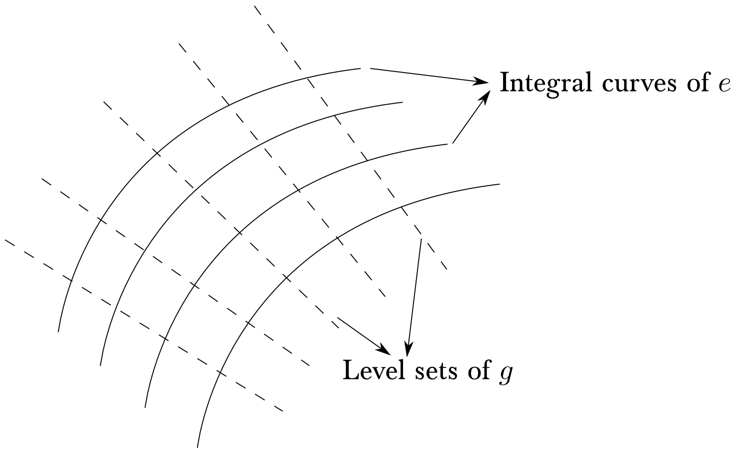
If $l = 0$, F is said to have **homogeneous degree** $k \in \mathbb{R}$.

In the contravariant coordinates $(x^a, g^{ab,A}, (t_j)^{a_1\ldots a_{l_j},A})$, defined in Section 1.5, finite and infinitesimal physical scalings take the form

$$x^a \mapsto x^a, \quad g \mapsto \lambda^{-2n} g, \quad g^{ab,A} \mapsto \lambda^{2+2|A|} g^{ab,A},$$
$$(t_j)^{a_1\ldots a_{l_j},A} \mapsto \lambda^{s_j+2|A|} (t_j)^{a_1\ldots a_{l_j},A}$$

$$e = \left(2+2|A|\right) g^{ab,A} \partial_{ab,A} + \left(s_j + 2|A|\right) (t_j)^{a_1\ldots a_{l_j},A} \partial_{a_1\ldots a_{l_j},A} \tag{3.3}$$

where, as remarked previously, we use g as coordinate in place of one of the g^{ab}. Since e is everywhere non zero its integral curves form a foliation of $J^r BM$ and hence of Z^r. Moreover, since $\mathcal{L}_e g^{-\frac{1}{2n}} = g^{-\frac{1}{2n}}$, g restricts to a global coordinate on each orbit of e and then the level sets of g form another foliation of $J^r BM$, transverse to the integral curves of e. For this reason it is convenient to study the structure of almost homogeneous function in *rescaled* coordinates:



$$\left(x^a, g, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|} (t_j)^{a_1\dots a_{l_j},A}\right).$$

Note that each of these functions but g is invariant under physical scaling. In our notation, we mean that the coordinates g and $g^{-\frac{1}{n}} g_{ab}$ are functionally independent only up to the identity $g = |\det g_{ab}|$.

Lemma 3.5. *Suppose that $Z^r \subseteq J^r BM$ is an open set equipped with any of the introduced coordinate system in Section 1.5, and $F\colon Z^r \to S^k V^* M$ is a smooth function that has almost homogeneous degree k and order l with respect to physical scaling. Then there exist uniquely defined* homogeneous of degree 0 *functions $B_j\colon Z^r \to S^k V^* M$, for $j = 0, 1, \dots, l$, such that*

$$F = g^{-\frac{k}{2n}} \sum_{j=0}^{l} \log^j \left(g^{-\frac{1}{2n}}\right) B_j.$$

In particular, using rescaled contravariant coordinates, each B_j can be taken independent of g and written in the form

$$B_j = B_j\left(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n}+\frac{1}{n}|A|} g^{ab,A}, g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|} (t_j)^{a_1\dots a_{l_j},A}\right).$$

Before proving this Lemma we need some technical results.

Lemma 3.6. *In the hypotheses of Lemma 3.5, if we define, for $l \geq 0$,*

$$G := g^{\frac{k}{2n}} F, \qquad B_l := \frac{1}{l!} \mathcal{L}_e^l G, \qquad G_{l-1} = G - \left(\log^l g^{-\frac{1}{2n}}\right) B_l$$

we have

$$(a)\ \mathcal{L}_e^{l+1} G = 0, \qquad (b)\ \mathcal{L}_e B_l = 0, \qquad (c)\ \mathcal{L}_e^l G_{l-1} = 0$$

Proof. Proof of (a). Since F is almost homogeneous of degree k and order l we have

$$\begin{aligned}0 &= (\mathcal{L}_e - k)^{l+1} F = (\mathcal{L}_e - k)^{l+1} g^{-\frac{k}{2n}} G = (\mathcal{L}_e - k)^l \left(g^{-\frac{k}{2n}} \mathcal{L}_e G\right) \\ &= (\mathcal{L}_e - k)^{l-1} (\mathcal{L}_e - k) \left(g^{-\frac{k}{2n}} \mathcal{L}_e G\right) = \cdots = g^{-\frac{k}{2n}} \mathcal{L}_e^{l+1} G\end{aligned}$$

and we have concluded the proof of point (a).

Proof of (b). It follows immediately from point (a).

Proof of (c). First of all notice that $\mathcal{L}_e^l \log^l g^{-\frac{1}{2n}} = l!$. So, by direct computation,

$$\begin{aligned}\mathcal{L}_e^l G_{l-1} &= \mathcal{L}_e^l G - \mathcal{L}_e^l \left[\log^l \left(g^{-\frac{1}{2n}}\right) B_l\right] \\ &= \mathcal{L}_e^l G - \mathcal{L}_e^{l-1} \left[\left(\mathcal{L}_e \log^l g^{-\frac{1}{2n}}\right) B_l + \log^l \left(g^{-\frac{1}{2n}}\right) \mathcal{L}_e B_l\right] \\ &= \mathcal{L}_e^l G - \mathcal{L}_e^{l-1} \left[\left(\mathcal{L}_e \log^l g^{-\frac{1}{2n}}\right) B_l\right] \\ &\ \vdots \\ &= \mathcal{L}_e^l G - \left(\mathcal{L}_e^l \log^l g^{-\frac{1}{2n}}\right) B_l \\ &= \mathcal{L}_e^l G - l! B_l \\ &= 0\end{aligned}$$

and the proof is concluded. □

Proof of Lemma 3.5. We can define $G_l = g^{\frac{k}{2n}} F$ and proceed recursively using the relation

$$G_{l-1} = G_l - \left(\log^l g^{-\frac{1}{2n}}\right) B_l, \qquad B_l = \frac{1}{l!} \mathcal{L}_e^l G_l$$

where B_l is invariant under physical scaling as proved in Lemma 3.6. Proceeding to the next step we find

$$\begin{aligned}G_{l-2} &= G_{l-1} - \left(\log^{l-1} g^{-\frac{1}{2n}}\right) B_{l-1} \\ &= G_l - \left(\log^l g^{-\frac{1}{2n}}\right) B_l - \left(\log^{l-1} g^{-\frac{1}{2n}}\right) B_{l-1}\end{aligned}$$

where B_{l-1} is again invariant under physical scaling since, using again the Lemma 3.6, we have $\mathcal{L}_e B_{l-1} = 0$. We can go on until $l = 0$ obtaining

$$G_{-1} = G_l - \sum_{j=0}^{l} \log^j \left(g^{-\frac{1}{2n}} \right) B_j .$$

where B_j is invariant under physical scaling for all j. But by definition $G_{-1} = G_0 - B_0 = 0$: Thus $G_{l<0} = B_{l<0} = 0$ and the procedure stops. Using the definition of G_l we have

$$F = g^{-\frac{k}{2n}} \sum_{j=0}^{l} \log^j \left(g^{-\frac{1}{2n}} \right) B_j$$

where all B_j are invariant under physical scaling. In particular, if we use rescaled coordinates (all invariant under physical scaling except g) we can take B_j independent form g and so we have concluded the proof. □

3.3 Invariant theory

The goal of this section is to state and prove the Equivariance Lemma 3.14, which generalizes some results proven in [KM16, Sec.2.6]. This Lemma is used in the proof of our main Theorem 4.3 to characterize all smooth $GL(n)$-equivariant (resp. $GL^+(n)$-equivariant, if we restrict ourselves to transformations that preserve spacetime orientation) tensor-valued maps that depend on a Lorentzian metric and any number of tensorial arguments.

The main difference with the previous weaker [KM16, Lem.2.8] is the allowed dependence on other tensors besides the metric. As a result of this generalization, the final characterization is a bit more complicated. In particular, while any such equivariant map is still polynomial in the metric g, its inverse g^{-1} and possibly the Levi-Civita tensor $\varepsilon(g)$, it may depend on the additional tensor arguments z in two different ways. First, being tensor-valued, any such equivariant may be ***polynomially*** and ***covariantly*** constructed from g, g^{-1}, $\varepsilon(g)$ and the tensor components of z, but the coefficients in these polynomial will be allowed to depend in an essentially arbitrary ***smooth*** way on invariant scalar polynomials built out of g, g^{-1}, $\varepsilon(g)$ and of the tensor components of z.

The precise statements and proofs of these results depend on some fundamental notions and facts from ***classical invariant theory*** of the $GL(n)$ and $O(1, n-1)$ (resp. $GL^+(n)$ and $SO(1, n-1)$ in the oriented case) groups. Invariant theory, which studies invariants of linear representations of groups and other related topics) is a highly developed subject (we will only mention [Pro07] and [GW09] as an introduction to the literature), but the majority of the literature, especially at the introductory level, focuses on polynomial invariants on representations of complex algebraic groups. Thus, it is not always easy to locate some (even classical) results in the context of real Lie groups and smooth (rather than polynomial) invariants. For the convenience of the reader, we summarize the relevant notions and results below and, when possible, try to provide reasonably concise and elementary proofs that are not easy to extract from the literature. For some of the more technical proofs, which are not strictly relevant to develop this thesis, we refer the reader directly to [KMM17].

In the following, we will use the one point space $* \cong \mathbb{R}^0$ with the trivial action of $GL(n)$ of any of its subgroup thereon.

Definition 3.6. Let X and Y be spaces carrying actions of the group G, resp. $\rho_u^{(X)}: X \to X$ and $\rho_u^{(Y)}: Y \to Y$ for $u \in G$. A map $f: X \to Y$ is said to be **equivariant** if it commutes with the action of G:

$$f \circ \rho_u^{(X)} = \rho_u^{(Y)} \circ f, \quad \text{for every } u \in G.$$

In the special case $Y = \mathbb{R}$ carrying the trivial representation, an equivariant map $f: X \to \mathbb{R}$ is called a **(scalar) invariant**. We denote the space of all scalar invariants by $\mathcal{S}_X$. When X is a vector space, we denote the subspace of **(scalar) polynomial invariants** by $\mathcal{P}_X \subseteq \mathcal{S}_X$. The subspace $\mathcal{P}_X^k \subset \mathcal{P}_X$ consists of all homogeneous polynomials of degree k.

With the above definitions, it is easy to establish a relation between scalar invariants and equivariant maps for linear group representations. We recall that if Y is a finite dimensional vector space with a linear representation ρ of the group G, the contragradient (dual) representation ρ^* of G on the dual space Y^* is defined by $\rho_u^* = \rho_{u^{-1}}^t$ for every $u \in G$, where $\cdot^t$ indicates the transposition operation.

Proposition 3.7. *Let X and Y be finite dimensional vector spaces with linear representations of the group G, and denote by Y^* the linear dual of Y equipped with the contragradient representation of G. If $f: X \to Y$ is an equivariant map, then $f^*(x, y^*) := y^* \cdot f(x)$ is a scalar invariant $f^*: X \times Y^* \to \mathbb{R}$. If $h: X \times Y^* \to \mathbb{R}$ is a scalar invariant, then $\frac{\partial h}{\partial y^*}|_{y^*=0}: X \to Y$ is an equivariant map. Moreover, for any equivariant map $f: X \to Y$, $\frac{\partial f^*}{\partial y^*}|_{y^*=0} = f$.*

Proof. We denote with, for every $u \in G$, ρ_u^X, ρ_u^Y, $\rho_u^{Y^*}$ the representation of G on X, Y, Y^* respectively (where $\rho_u^{Y^*}$ is the dual representation). Then, if $f\colon X \to Y$ is an equivariant map, we have

$$f^*\left(\rho_u^X x, \rho_u^{Y^*} y^*\right) = (\rho_u^{Y^*} y^*)\cdot f\left(\rho_u^X x\right) = (\rho_u^{Y^*} y^*)\cdot \rho_u^Y f\left(x\right) = y^* \cdot f(x) = f^*(x, y^*)$$

and f^* is a scalar invariant. Moreover, if $h\colon X \times Y^* \to \mathbb{R}$ is a scalar invariant, *i.e.*

$$h\left(\rho_u^X x, \rho_u^{Y^*} y^*\right) = h\left(x, y^*\right)$$

then we have

$$\rho_u^{Y^*} \left.\frac{\partial h}{\partial y^*}\right|_{y^*=0} \left(\rho_u^X x\right) = \left.\frac{\partial h}{\partial y^*}\right|_{y^*=0} (x)$$

and using the definition of contragradient representation

$$\left.\frac{\partial h}{\partial y^*}\right|_{y^*=0} \left(\rho_u^X x\right) = \rho_u^Y \left.\frac{\partial h}{\partial y^*}\right|_{y^*=0} (x)$$

which means that the function $\frac{\partial h}{\partial y^*}|_{y^*=0}\colon X \to Y$ is an equivariant map. □

Definition 3.7. Let M_n^p be the space of p-multilinear forms on $\mathbb{R}^n$ and consider the natural linear action of $GL(n)$ thereon. Denote by M_n^{p*} the dual of M_n^p, with the contragradient $GL(n)$ representation on it. Let T be a finite-dimensional real vector space carrying a representation of $GL(n)$.

1. If T, with respect to some linear embedding $T \hookrightarrow M_n^p \otimes M_n^{q*}$, is invariant under the action of $GL(n)$, and if (the representation carried by) T is the restriction of the action of $GL(n)$ on $M_n^p \otimes M_n^{q*}$, then T is called **tensor representation** of $GL(n)$. We call (p, q) the **(covariant, contravariant) tensor rank** of T and $p + q$ the **total tensor rank** of T.

2. If T is as in 1., but the action of $GL(n) \ni u \mapsto \rho(u)$ on T is given by a tensor representation up to a multiplication by $|\det u|^s$, then T is called **tensor density representation** of $GL(n)$. We call s the tensor **weight** of T.

3. Denote by $\eta \in M_n^2$ is the standard Minkowski metric (signature $(-+\cdots+)$), and by $\epsilon \in M_n^n$ the standard antisymmetric Levi-Civita tensor. The orthogonal subgroup $O(1, n-1) \subset GL(n)$ (resp. $SO(1, n-1) \subset GL^+(n)$) is the stabilizer subgroup of η under the action on M_n^2. A ***tensor (density) representation*** of the orthogonal group is a restriction of a tensor density representation of the general linear group.

Remark 3.1. Clearly, since for any $u \in O(1, n-1)$, $|\det u| = 1$ and $(u^{-1})^T = \eta u \eta^{-1}$ in the fundamental representation, the restriction of any two tensor density representations of $GL(n)$ to $O(1, n-1)$ or $SO(1, n-1)$ are linearly equivalent as long as their total tensor rank is the same. So it is sufficient to talk only about ***tensor*** (rather than ***tensor density***) representations of these subgroups.

Remark 3.2. Below, some results about a group G and its representations require as a hypothesis that G be ***reductive***. There are several different flavors of reductive groups (cf. [Pro07, Sec.7.3]), not all of them being equivalent, with different ones serving as natural hypotheses for different results. The general property that they share is that each representation from a certain class is completely reducible (*i.e.*, no reducible but indecomposable representations may occur). For the sake of uniformity, we specialize all results stated below to ***linearly reductive*** groups, even if the original result could be stated under looser hypotheses. First, note that a ***real (complex) algebraic*** group is a subgroup of $GL(n;\mathbb{R})$ ($GL(n;\mathbb{C})$), for some n, that is also a real (complex) algebraic subvariety (it is defined by polynomial equations). A ***real (complex) linearly reductive*** group G is a real (complex) algebraic group such that each real (complex) finite dimensional rational representation of G is completely reducible. Here ***polynomial*** and ***rational*** mean with respect to the matrix elements of the embedding of G into $GL(n;\mathbb{R})$ ($GL(n;\mathbb{C})$). Obviously, any real algebraic group gives rise to a complex algebraic group, its ***complexification***, simply by extending the defining polynomial equations from $GL(n;\mathbb{R})$ to $GL(n;\mathbb{C})$. ***A priori***, the property of being reductive is different for a real algebraic group and its complexification. Fortunately, we only need to appeal to such hypotheses for the real orthogonal groups $O(1, n-1) := O(1, n-1; \mathbb{R})$ and $SO(1, n-1) := SO(1, n-1; \mathbb{R})$, both of which are known to be linearly reductive, and so are their complexifications $O(1, n-1; \mathbb{C})$ and $SO(1, n-1; \mathbb{C})$ (see [Pro07, Sec.7.3.2], [RS90, Sec.5.2]). Unless explicitly mentioned, below we always refer to real groups and their representations on real vector spaces.

Definition 3.8. Let $L_n \subset M_n^2$ denote the space of Lorentzian bilinear forms (non-degenerate, with signature $(-+\cdots+)$), and let it inherit the natural action of $GL(n)$ (resp. $GL^+(n)$). Let $Z = \bigoplus_i Z_i$ and $T = \bigoplus_j T_j$ be finite sums of tensor density representations of $GL(n)$ (resp. $GL^+(n)$). We will refer to a smooth equivariant map

$$\tau \colon L_n \times Z \to T \tag{3.4}$$

as a $GL(n)$-**equivariant tensor density** (resp. $GL^+(n)$-**equivariant tensor densities**). The space of $GL(n)$-equivariant tensor densities will be denoted by $\mathcal{E}_{Z,T}$.

The space of $GL^+(n)$-equivariant tensor densities will be denoted by $\tilde{\mathcal{E}}_{Z,T}$. In the special case when $T = \mathbb{R}$ carries the trivial representation, we call $\mathcal{S}_Z := \mathcal{E}_{Z,\mathbb{R}}$ (resp. $\tilde{\mathcal{S}}_Z := \tilde{\mathcal{E}}_{Z,\mathbb{R}}$) the space of **scalar invariants**.

Definition 3.9. Let $Z = \bigoplus_i Z_i$ and $T = \bigoplus_j T_j$ be finite sums of tensor representations of $O(1, n-1)$ (resp. $SO(1, n-1)$). We will refer to a smooth equivariant map

$$\tau \colon Z \to T \tag{3.5}$$

as the space of $O(1, n-1)$**-isotropic tensors** (resp. $SO(1, n-1)$**-isotropic tensors**). The space of $O(1, n-1)$-isotropic tensors will be denoted by $\mathcal{I}_{Z,T}$. The space of $SO(1, n-1)$-isotropic tensors will be denoted by $\tilde{\mathcal{I}}_{Z,T}$.

Proposition 3.8. *With the notation of Definitions 3.8 and 3.9, the space of equivariant tensor densities $\mathcal{E}_{Z,T}$ (resp. $\tilde{\mathcal{E}}_{Z,T}$) is isomorphic to the space of isotropic tensors $\mathcal{I}_{Z,T}$ (resp. $\tilde{\mathcal{I}}_{Z,T}$).*

Proof. Let $\eta \in L_n \cong GL(n)/O(1, n-1) \cong GL^+(n)/SO(1, n-1)$, where the orthogonal group is interpreted as the stabilizer subgroup of η. The equivariance of $\tau \colon L_n \times Z \to T$ implies that $\tau(\eta, uz) = u\tau(\eta, z)$, whenever $u \in GL(n)$ and $u \cdot \eta = \eta$, meaning that $\tau(\eta, -) \colon Z \to T$ is $O(1, n-1)$ (resp. $SO(1, n-1)$) equivariant. On the other hand, since any $L_n \ni g = u_g \cdot \eta$ for some $u_g \in GL(n)$, the knowledge of $\tau(\eta, -)$ uniquely determines the equivariant extension $\tau(g, z) := u_g \tau(\eta, u_g^{-1} z)$. Clearly, this correspondence is bijective. □

For the fundamental representations of $O(1, n-1)$ and $SO(1, n-1)$, homogeneous polynomials, invariant linear functionals and isotropic tensors all have a very explicit description. We give this description below in several different versions, related as follows. Any ***polynomial*** on a vector space that is invariant under the action of a linear representation can be written as a sum of invariant ***homogeneous*** polynomials. Any invariant homogeneous polynomial of degree p is also naturally a linear functional on a p-fold symmetric tensor product of the original representation and vice-versa. By duality, the adjoint of a linear functional on a p-fold tensor product representation defines an equivariant map from $*$ to the dual of the p-fold tensor product representation.

Proposition 3.9. *Let M_n^p and η be as in Definition 3.9, let $V^p = (\mathbb{R}^n)^p$ be the space of $(v_1, \ldots, v_p)$ of p-copies of vectors in the fundamental representation of $O(1, n-1)$ (or $SO(1, n-1)$), and let $T = \bigoplus_j T_j$ be a finite sum of tensor representations of ranks p_j of $O(1, n-1)$ (or $SO(1, n-1)$).*

1. *Polynomials* $p(v_1, \cdots, v_p) \in \mathcal{P}_{V^p}$ *invariant under the simultaneous action of* $O(1, n-1)$ *on its arguments are generated by the contractions* $\eta_{ab} v_i^a v_j^b$*, with* $i, j = 1, \ldots, p$.
   Polynomials $p(v_1, \cdots, v_p) \in \mathcal{P}_{V^p}$ *invariant under the simultaneous action of* $SO(1, n-1)$ *on its arguments are generated by the contractions* $\eta_{ab} v_i^a v_j^b$ *and* $\epsilon_{a_1 \cdots a_p} v_{i_1}^{a_1} \cdots v_{i_p}^{a_p}$*, with* $i, j, i_k = 1, \ldots, p$.

2. *The isotropic tensors* $\mathcal{I}_n^p$ *are linear combinations of tensor products of copies of* η_{ab} *with arbitrarily permuted indices.*
   The isotropic tensors $\tilde{\mathcal{I}}_n^p$ *are spanned by tensor products of* η_{ab} *and* $\epsilon_{a_1 \cdots a_n}$ *with arbitrarily permuted indices.*

3. *All* $O(1, n-1)$*-invariant linear functionals on* M_n^p *are spanned by arbitrary complete contractions of a tensor* $t_{a_1 \cdots a_p} \in M_n^p$ *with copies of* η^{ab}*, in an arbitrary order of indices.*
   All $SO(1, n-1)$*-invariant linear functionals on* M_n^p *are spanned by arbitrary complete contractions of a tensor* $t_{a_1 \cdots a_p} \in M_n^p$ *with copies of* η^{ab} *and* $\epsilon^{a_1 \cdots a_n}$*, in an arbitrary order of indices.*

4. *All degree* k *homogeneous polynomial scalar* $O(1, n-1)$*-invariants* $p(t) \in \mathcal{P}_T^k$ *on* T *are spanned by complete contractions of tensor products*

$$(t_{j_1})_{a_1^1 \cdots a_{p_{j_1}}^1} \cdots (t_{j_k})_{a_1^k \cdots a_{p_{j_k}}^k} \tag{3.6}$$

   with copies of η^{ab}*, when* $t = \bigoplus_j t_j$.
   All degree k *homogeneous polynomial scalar* $SO(1, n-1)$*-invariants* $p(t) \in \mathcal{P}_T^k$ *on* T *are spanned by complete contractions of tensor products* (3.6) *with copies of* η^{ab} *and* $\epsilon^{a_1 \cdots a_n}$.

Proof. See [KMM17, App. C, Prop. C.3] □

Before stating and proving our Equivariance Lemma 3.14, we need the following fundamental results from invariant theory.

Proposition 3.10 (Hilbert [Pro07, Sec.14.1], [Mic08, §7.2]). *Let* G *be a linearly reductive group with a rational representation on a finite dimensional vector space* Z*. Then the algebra of polynomial scalar* G*-invariants on* Z *is finitely generated.*

Definition 3.10. Let G be a linearly reductive group with a rational representation on a finite dimensional vector space Z and let $p_i \in C^\infty(Z)$, $i = 1, \ldots, N_Z$, be a generating set for the algebra of polynomial scalar G-invariants (Proposition 3.10). A smooth function $\sigma \in C^\infty(Z)$ is said to be **stably G-invariant** if it is constant along each joint level set of the invariant polynomials p_i, $i = 1, \dots N_Z$.

Clearly, any function that is stably G-invariant is also G-invariant, but the converse is not always true. Also, it is easy to see that the definition is independent of the choice of the generating polynomials p_i. The *stability* in this definition is meant with respect to complexification, since upon replacing G with its complexification the orbits become larger, while the invariant polynomials remain the same, in a way that invariant polynomials do completely separate all closed orbits, which erases the difference between G-invariant and stably G-invariant functions. In Section 4.2.2, we discuss the action of $O(1, n-1)$ on the subspace of symmetric forms in M^2_n (in this case, the action coincides with that of $SO(1, n-1)$). There, we give an explicit list of a generating set of scalar invariant polynomials and also discuss the structure of the orbits. That case also gives an explicit example of the difference between G-invariant and stably G-invariant functions, because invariant polynomials do not separate closed orbits on symmetric bilinear forms.

The following results seem to be close to the state of the art in characterizing the *smooth* scalar invariants that apply to our cases of interest. Unfortunately, we actually require a somewhat strengthened version of these results (though see also [Sto08] for more recent work), which we state below in Proposition 3.13, but whose proof we do not discuss (Remark 3.3).

Proposition 3.11 (Luna [Lun76], [Mic08, §7.14] [KMS93, §26.3]). *Let G be a linearly reductive group with a rational representation on a finite dimensional vector space Z and let $p_i \in \mathcal{P}_Z$, $i = 1, \dots, N_Z$, be a generating set for the algebra of polynomial scalar G-invariants (Proposition 3.10). Then a smooth stably G-invariant function $\sigma \in C^\infty(Z)$ can always be written as $\sigma = \Sigma(p_1, \dots, p_{N_Z})$ where Σ is a smooth function of its arguments.*

Though, as indicated above, the statement of Luna's theorem can be found in several references, as far as we know, a proof is available only in the original reference [Lun76], written in French. However, the more recent result on the structure of invariants of finite C^k differentiability [Rum98] does use a proof that is logically similar to Luna's.

Proposition 3.12 (Richardson [Ric73, Thms.2.3,4.1]). *Let G be a linearly reductive group with a linearly reductive complexification and a rational representation on a finite dimensional vector space Z. Let $p_i \in C^\infty(Z)$, $i = 1, \dots, N_Z$, be homogeneous* *polynomials generating the algebra of polynomial scalar G-invariants on Z (Proposition 3.10). Then, there exists a $p_0 = P(p_1, \dots, p_{N_Z})$ polynomial in its arguments and, with $Z^0 = p_0^{-1}(0)$, a partition $Z \setminus Z^0 = \bigcup_j Z_j$ into finite union of disjoint connected open subsets (Z_j) where each Z_j is stable under the action of G and, for each j and for any two points $z_1, z_2 \in Z_j$ the stabilizer subgroups $G_{z_1}, G_{z_2} \subseteq G$ are conjugate in G.*

The following definition is rather technical, but is necessary to precisely capture the difference between the behavior of *smooth* invariants and *polynomial* invariants (or *analytic*, or even *stable smooth* invariants).

Definition 3.11. Let Z be a finite dimensional vector space, $p_i \in C^\infty(Z)$, $i = 1, \dots, N_Z$, be a set of homogeneous polynomials on Z, and $p_0 = P(p_1, \dots, p_{N_Z})$ a polynomial in its arguments. With $Z^0 = p_0^{-1}(0)$, consider a partition $Z \setminus Z^0 = \bigcup_{j=1}^{r_Z} Z_j$ into pairwise disjoint open sets Z_j, for some $r_Z < \infty$. We say that a function $\sigma \in C^\infty(Z)$ is **locally a smooth function of the polynomials p_i with respect to the partition** (Z_j) if there exist $\Sigma_j \in C^\infty(\mathbb{R}^{N_Z})$, $j = 1, \dots, r_Z$, such that $\sigma = \Sigma_j(p_1, \dots, p_{N_Z})$ on Z_j. We say that σ is a **function of the p_i (globally)** if we can choose $\Sigma_j = \Sigma_i$, for $i, j = 1, \dots, r_Z$. We write $\sigma = [\Sigma]_Z(p_1, \dots, p_{N_Z})$.

Proposition 3.13 (extended Luna-Richardson). *Let G be a linearly reductive group with a linearly reductive complexification and a rational representation on a finite dimensional vector space Z. Also, let $p_i \in \mathcal{P}_Z$, $i = 1, \dots, N_Z$, be homogeneous polynomials generating the algebra of polynomial scalar G-invariants on Z (Proposition 3.10). Then, there exists a $p_0 = P(p_1, \dots, p_{N_Z})$ polynomial in its arguments and, with $Z^0 = p_0^{-1}(0)$, a partition $Z \setminus Z^0 = \bigcup_{j=1}^{r_Z} Z_j$ into pairwise disjoint open G-invariant sets, such that any G-invariant function $\sigma \in C^\infty(Z)$ is locally a smooth function $\sigma = [\Sigma]_Z(p_1, \dots, p_{N_Z})$ of the polynomials p_i with respect to the partition (Z_j) (Definition 3.11).*

Remark 3.3. The proof of Proposition 3.13 follows from combining the details of the proofs of Propositions 3.11 and 3.12, which can be found in the original references [Lun76] and [Ric73] respectively. Discussing a complete proof goes beyond the scope of the current work and will be discussed elsewhere.

Combining the results presented so far allows us to finally formulate the main Equivariance Lemma that is needed in the proof of our main Theorem 4.3.

Lemma 3.14 (Equivariance). *Consider finite sums of tensor density representations $Z = \bigoplus_j Z_j$ and $T = \bigoplus_j T_j$ of $GL(n)$ (resp. $GL^+(n)$), and its natural action on L_n. Recall also (Definitions 3.8, 3.9) the notion of invariant scalars ($\mathcal{S}_Z, \tilde{\mathcal{S}}_Z \subset$*

$C^\infty(L_n \times Z)$), equivariant tensors ($\mathcal{E}_{Z,T}, \tilde{\mathcal{E}}_{Z,T} \subset C^\infty(L_n \times Z; T)$) and isotropic tensors ($\mathcal{I}_{Z,T}, \tilde{\mathcal{I}}_{Z,T} \subset C^\infty(Z;T)$), as well as their characterizations (Propositions 3.7, 3.8 and 3.9)

1. *There exist diagonalizable intertwiners $s_Z\colon Z \to Z$ and $s_T\colon T \to T$ such that $(u,z) \mapsto |\det u|^{-s_Z}(u \cdot z)$ and $(u,t) \mapsto |\det u|^{-s_T}(u \cdot t)$, for $u \in \mathrm{GL}(n)$, $z \in Z$ and $t \in T$, define* tensor *representations (*i.e.*, with density weight zero) on Z and T. Denoting these tensor representations by Z' and T', we have $\mathcal{E}_{Z,T} \cong \mathcal{E}_{Z',T'}$ (resp. $\mathcal{E}_{Z,T} \cong \mathcal{E}_{Z',T'}$).*

2. *When Z carries a tensor representation and $p \in \mathcal{S}_Z$ (resp. $\tilde{\mathcal{S}}_Z$) such that $p(g,z)$ is polynomial in z, then p is a covariantly constructed scalar that is polynomial in the tensor components of g, g^{-1} and z (resp. of g, g^{-1}, $\varepsilon(g)$ and z).*

3. *There is a finite number of invariants $p_i \in \mathcal{S}_Z$ (resp. $\tilde{\mathcal{S}}_Z$), $i = 1, \ldots, N_Z$, such that each $p_i(g,z)$ is a homogeneous polynomial in z and each $\sigma \in \mathcal{S}_Z$ (resp. $\tilde{\mathcal{S}}_Z$) is locally a smooth function $\sigma = [\Sigma]_Z(p_1, \ldots, p_{N_Z})$ of the invariant polynomials p_i, $i = 1, \ldots, N_Z$, as in Proposition 3.13.*

4. *There is a finite number of equivariant tensors $q_j \in \mathcal{E}_{Z,T}$ (resp. $\tilde{\mathcal{I}}_{Z,T}$), $j = 1, \ldots, N_{Z,T}$, whose components are homogeneous polynomials on Z, such that each $\tau \in \mathcal{I}_{Z,T}$ (resp. $\tilde{\mathcal{E}}_{Z,T}$) is of the form $\tau = \sum_{j=1}^{N_{Z,T}} \sigma^j q_j$ with $\sigma^j \in \mathcal{S}_Z$ (resp. $\tilde{\mathcal{S}}_Z$).*

Proof. After we establish point 1, we can without loss of generality assume that Z and T consist of direct sums of only *tensor* representations.

1. By hypotheses, both Z and T reduce to a sum of tensor density representations. This means that there exist diagonalizable intertwiners $s_Z\colon Z \to Z$ and $s_T\colon T \to T$ such that $(u,z) \mapsto |\det u|^{-s_Z}(u \cdot z)$ and $(u,t) \mapsto |\det u|^{-s_T}(u \cdot t)$, for $u \in \mathrm{GL}(n)$, $z \in Z$ and $t \in T$, define *tensor* representations on Z and T. Let us refer to the corresponding representations as Z' and T'. If $\tau\colon L_n \times Z \to T$ is an equivariant map with respect to the tensor density representations on Z and T, then

$$\tau'(g,z') = |\det g|^{-s_T}\, \tau(g, |\det g|^{s_Z} z') \tag{3.7}$$

defines an equivariant map $\tau'\colon L_n \times Z' \to T'$ with respect to the corresponding tensor representations. Clearly, this operation can be reversed.

2. Recall that, in our notation, $\mathcal{S}_Z \cong \mathcal{E}_{Z,\mathbb{R}}$ (resp. $\tilde{\mathcal{S}}_Z \cong \tilde{\mathcal{E}}_{Z,\mathbb{R}}$) where $\mathbb{R}$ carries the trivial representation. Then, by Proposition 3.8, we have the isomorphism

$\mathcal{S}_Z = \mathcal{I}_{Z,\mathbb{R}}$ (resp. $\tilde{\mathcal{S}}_Z = \tilde{\mathcal{I}}_{Z,\mathbb{R}}$). Under this isomorphism, an invariant $p(g,z)$ is polynomial in z iff the corresponding $p_\eta(z) = p(g=\eta, z)$ is polynomial. Moreover, by the classification Proposition 3.9, any such polynomial $p_\eta(z)$ consists of a complete contraction of products of the tensor components of z with copies of η (and also ϵ in the oriented case). Recalling the details of the restriction of tensor representations to the orthogonal subgroup (Remark 3.1), the invariant extension $p(g,z)$ of $p_\eta(z)$ clearly constitutes the same complete contraction of products of the tensor components of z, but with every occurrence of η replaced by either g_{ab} (when contracting two contravariant indices), g^{ab} (when contracting two covariant indices) or ∂ta_a^b (when contracting a covariant and a contravariant index). Respectively, a contraction with ϵ is replaced by a contraction with $\varepsilon(g)$ with its indices appropriately raised or lowered by g. Thus, we arrive at the desired conclusion about the polynomiality of $p(g,z)$ in g, g^{-1} (and resp. $\varepsilon(g)$).

3. Recall the isomorphism $\mathcal{S}_Z \cong \mathcal{I}_{Z,\mathbb{R}}$ (resp. $\tilde{\mathcal{S}}_Z \cong \tilde{\mathcal{I}}_{Z,\mathbb{R}}$) from point 2. Then, the desired conclusion follows from Proposition 3.13, noting that $O(1,n-1)$ (resp. $SO(1,n-1)$) is a ***linearly reductive*** Lie group (and so is its complexification, cf. Remark 3.2) and any tensor representation (Definition 3.7) is obviously rational. The finiteness of the number of generating invariant polynomials p_i ultimately follows from Hilbert's theorem (Proposition 3.10), which can obviously be chosen to be homogeneous.

4. It follows from Proposition 3.7 that any equivariant $\tau \in \mathcal{E}_{Z,T}$ (resp. $\tilde{\mathcal{E}}_{Z,T}$), can be written as a gradient $\tau(g,z) = \frac{\partial}{\partial t^*}\sigma(g,z,t^*)\big|_{t^*=0}$, for some invariant $\sigma \in \mathcal{S}_{Z\times T^*}$ (resp. $\tilde{\mathcal{S}}_{Z\times T^*}$) that is linear in the t^* arguments. On the other hand, point 3 implies that

$$\sigma = [\Sigma]_{Z\times T^*}(p_1,\dots,p_{N_Z},Q_1,\dots,Q_{N_{Z,T}})$$

is locally a smooth function of the invariants polynomial on $Z\times T^*$, split into the p_i that do not depend on the T^*, and the Q_i that depend on the T^* only linearly. By combining the chain rule with the notion of local dependence on polynomials (Definition 3.11), we get

$$\begin{aligned} \tau(g,z) &= \frac{\partial}{\partial t^*}[\Sigma]_Z(p_1,\dots,p_{N_Z},Q_1,\dots Q_{N_Z})\Big|_{t^*=0} \\ &= \sum_{j=1}^{N_{Z,T}} \left[\frac{\partial}{\partial Q_j}\Sigma\right](p_1,\dots,p_{N_Z},Q_1,\dots,Q_{N_Z})\Big|_{Q_j=0}\frac{\partial Q_j}{\partial t^*}\Big|_{t^*=0} \end{aligned}$$

and thus

$$\tau(g,z) = \sum_{j=1}^{N_{Z,T}} \sigma^j(g,z) q_j(g,z),$$

with the obvious definitions for σ^j and q_j. This concludes the proof. □

3.4 Thomas replacement theorem

In this section we present a version of the Thomas Replacement Theorem 3.15, which basically states that any non-linear differential operator that depends on a Lorentzian (or pseudo-Riemannian) metric and a finite number of any kind of other tensor fields while itself transforming as a tensor field under diffeomorphisms must be expressible as a function of the covariant derivatives of the Riemann curvature and the other tensor arguments. This is a rather old result, with versions of it going back to the work of Thomas [Tho34] and in some form even to earlier works of Christoffel [Chr69]. However, it has since then taken on a folk nature, making it difficult to find precise references that state the result in a form most convenient for our applications, give a complete proof, with modern notation and terminology, that is concise and without an overabundance of formalism. If one omits at least some of the above conditions, the result of Theorem 3.15 can be found in [Sch54, §III.7], [KMS93, §§28.14,33.10], and [Slo92, Thm.3]. Thus, in this section we present a modern version of Thomas Replacement Theorem, using our notations. For the details about this technical result, and also for the proof, we refer to [KMM17] (since the proof is not strictly relevant in the development of this thesis).

Let $BM \to M$ be a natural bundle of the form

$$BM = S^2T^*M \oplus T^{\otimes k_1}M \oplus \cdots \oplus T^{\otimes k_N}M, \tag{3.8}$$

where $T^{\otimes k}M$ is the bundle of $(k,0)$-tensors[2]. Consider the *curvature coordinates* introduced in Section 1.5 on J^rBM,

$$\left(x^a, g_{ab}, \Gamma^a_{(bc,A)}, \bar{S}^{ab(cd,A)}, t^{a_1\cdots a_{k_1},A}, \ldots t^{a_1\cdots a_{k_N},A}\right), \tag{3.9}$$

where the multi-indices $A = a_1 \cdots a_{|A|}$ range through the sizes $|A| = 1, \ldots, r$. All coordinates, other than $(x^a, \Gamma^a_{(bc,A)})$, correspond to components of tensor densities (Definition 3.7) transforming under $GL(n)$, where $GL(n)$ is interpreted as the

[2] It is possible to consider the bundle of (k,l)-tensors without changes in the result.

quotient of $\mathrm{Diff}_x(M)$, the subgroup of diffeomorphisms fixing the point $x \in M$, by the subgroup of diffeomorphisms with vanishing Jacobian at x.

Theorem 3.15. *Let $F\colon Z'^r_x \subseteq J^r BM \to T^{*\otimes l}M$ be a smooth bundle map that is defined on a $\mathrm{Diff}(M)$-invariant domain Z'^r_x and is $\mathrm{Diff}(M)$-equivariant, given by*

$$F_{a_1 \cdots a_l} = F_{a_1 \cdots a_l}(x, g, \Gamma, S, t) \tag{3.10}$$

in adapted coordinates (3.9) *on a chart $Z^r_x \subseteq Z'^r_x$. That is, given a diffeomorphism $\chi\colon M \to M$, we have $\chi^* \circ F = F \circ p^r\chi^*$, where on the left χ^* is the pullback along χ acting on the tensor bundle $T^{*\otimes l}M$, while on the right $p^r\chi^*$ is the r-jet prolongation of the pullback along χ acting on the bundle BM of background fields. Then, when restricted to a chart $Z^r_x \subseteq Z'^r_x$ covered by adapted coordinates* (3.9)*, F must be expressible as*

$$F_{a_1 \cdots a_l} = G_{a_1 \cdots a_l}(g, S, t), \tag{3.11}$$

where the function G is equivariant with respect to the action of $GL(n)$ on its arguments and the action of $GL(n)$ on the fibers of $T^{*\otimes l}M$.

Proof. See [KMM17, App. B]. □

To conclude this section, we present an easy example to explain the idea exploited in the proof of the previous theorem. In the hypothesis of Theorem 3.15, consider a function $F_{a_1 \cdots a_l}(x^a, T)$ which depends only on coordinates (x^a) and on some other tensors T (we do not write the indices for simplicity). If we consider the translations subset of $\mathrm{Diff}(M)$, *i.e.* if we consider the class of diffeomorphism

$$x^a \mapsto x^a + t^a$$

for some vectors t^a (such that $x^a + t^a$ is still in the domain of the local chart). Since the Jacobian matrix of these transformation is the identity matrix, the $\mathrm{Diff}(M)$-equivariance of F becomes

$$F_{a_1 \cdots a_l}(x^a + t^a, T) = F_{a_1 \cdots a_l}(x^a, T)$$

for every vector t^a, which mean that the function F cannot depend on the coordinates (x^a). The same procedure, with a different choice of the relevant class of diffeomorphisms, can be re-adapted to prove that coordinates Γ disappear from the functional form of F.

4 Wick powers of Boson fields and their renormalization

This chapter is the real core of this thesis: We study the renormalization of Wick powers of boson fields. Firstly we develop a very general model: We prove a renormalization formula for a generic Boson field and then we show that the coefficients in this formula are polynomials when we consider generic space-time tensors as background fields. In the last section we present three physically motivated models.

4.1 Wick powers of quantum Boson fields

We are now in position to discuss the core of the present thesis: In this section we study the renormalization Wick powers of a generic quantum Boson field. To begin, we outline the general setting in which we study the problem.

General settings. Our general setting is the following:

1. We start with a bundle VM which is constructed as a direct sum of vector bundles
$$VM = \bigoplus_{i=1}^{N} W_i M. \tag{4.1}$$

2. We consider a locally covariant quantum V-field
$$A_{(M,\mathbf{b})} \colon \mathscr{D}(VM) \to \mathcal{W}(M, \mathbf{b})$$
and we characterize it as a quantum ***Boson field*** in the following way. We assume that the commutator of two V-fields $[A_{(M,\mathbf{b})}(f), A_{(M,\mathbf{b})}(g)]$ is a c-number, ***i.e.***,
$$[A_{(M,\mathbf{b})}(f), A_{(M,\mathbf{b})}(g)] = C_{(M,\mathbf{b})}(f \otimes g)1, \tag{4.2}$$
where $C_{(M,\mathbf{b})} \in \mathscr{D}'(VM \times VM)$ is a distribution with some suitable properties (***e.g.***, for ***Boson fields*** it vanishes for spacelike separated arguments). Thus, Schwartz' kernel theorem implies that a unique continuous linear map $\Delta_{(M,\mathbf{b})} \colon \mathscr{D}(VM) \to \mathscr{D}'(VM)$ exists such that $[\Delta_{(M,\mathbf{b})}(g)](f) = C_{(M,\mathbf{b})}(f \otimes g)$. We require moreover that $\Delta_{(M,\mathbf{b})}(g)$ is ***regular*** in the sense that
$$\Delta_{(M,\mathbf{b})} \colon \mathscr{D}(VM) \to \mathscr{E}(V^*M), \tag{4.3}$$
where we used the fact that $\mathscr{E}(V^*M) \subset \mathscr{D}'(VM)$. There are many ways to implement this requirement in practical cases, for example our assumption holds when the dynamics of the field A is ruled by any hyperbolic field equation in view of the theorem of propagation of singularities. More generally it holds when some microlocal spectrum (cf. [KM15], [BF09, Ch.4]) hypothesis on the wavefront set of n-point functions is assumed with respect to relevant classes of states even in the absence of a field equation[1].

[1] Using the language of Microlocal Analysis, (4.3) is valid when $WF(C_{(M,\mathbf{b})}) \not\ni (x, y, p_x, p_y)$ with either $p_x = 0$ or $p_y = 0$ and this is guaranteed as soon as some standard microlocal spectrum condition on $C_{(M,\mathbf{b})}$ is valid, in particular if $C_{(M,\mathbf{b})}$ is a bisolution of a hyperbolic field equation.

If we use explicitly the decomposition (4.1) the map $\Delta_{(M,\mathbf{b})}$ can be seen as a direct sum of maps

$$\Delta_{(M,\mathbf{b})} = \bigoplus_{l=1}^{N} \sum_{j=1}^{N} \Delta^{lj}_{(M,\mathbf{b})},$$

where $\Delta^{lj}_{(M,\mathbf{b})} : \mathscr{D}(W_j M) \to \mathscr{E}(W^*_l M)$.

3. Since we assumed a bundle constructed as in (4.1), the V-field $A_{(M,\mathbf{b})}$ can be written as a N-tuple of W_i-fields

$$A_{(M,\mathbf{b})} = \left((A_1)_{(M,\mathbf{b})}, \ldots, (A_N)_{(M,\mathbf{b})}\right).$$

We assume that each W_i-field $(A_i)_{(M,\mathbf{b})}$ scales homogeneously under physical scaling with degree $d_{A_i} \in \mathbb{R}$, ***i.e.***, for every $\lambda \in \mathbb{R}^+$

$$(S_\lambda A_i)_{(M,\mathbf{b})}(f) = \lambda^{d_{A_i}} (A_i)_{(M,\mathbf{b})}(f)\,, \quad (M,\mathbf{b}) \in \mathfrak{BkgG}\,, \quad f \in \mathscr{D}(W_i M)\,.$$

We will say that the V-field $A_{(M,\mathbf{b})}$ scales homogeneously with degree

$$\mathbb{R}^N \ni \mathbf{d}_A = (d_{A_1}, \ldots d_{A_N})$$

under physical scaling.

4. We then consider the Wick powers A^k of A. These quantum fields A^k have the physical interpretation of products of k factors A evaluated at the same point x. Formally, assuming a geometric background $(M,\mathbf{b})$ has been fixed,

$$A^k_{\mu_1\ldots\mu_k}(x) = (A_{\mu_1} \cdots A_{\mu_k})(x)\,.$$

It is worth stressing that these quantum fields are not elements of the sub unital $*$-algebra generated by 1 and by elements $A(f)$ since these elements are associated with kernels formally evaluated at ***different*** points of spacetime, ***i.e.***, they are linear combinations of objects $A_{\mu_1}(x_1)\cdots A_{\mu_k}(x_k)$. Thus Wick powers need a specific definition which, as is well-known, involves some *renormalization* procedure.
Finally, we stress that, using the decomposition introduced in Remark 1.1, the Wick powers A^k can be written as a sum

$$\begin{aligned} A^k(f) &= \sum_{|P|=k} \binom{k}{P} A^k\left(f_1^{p_1} \odot \cdots \odot f_N^{p_N}\right) \\ &=: \sum_{|P|=k} \binom{k}{P} A^P\left(f_1^{p_1}, \cdots, f_N^{p_N}\right), \end{aligned} \tag{4.4}$$

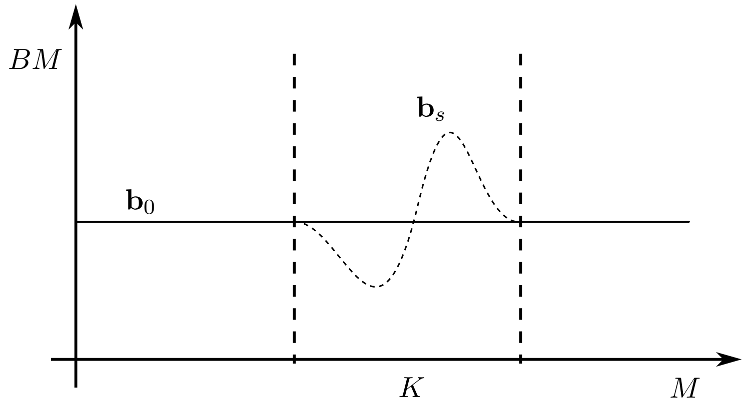


Figure 4.1: We consider compactly supported variations of background fields.

where $P = (p_1, \dots, p_N)$ is a multi-index and $\binom{k}{P} = \frac{k!}{\prod_{i=1}^N p_i!}$. The last equality is intended as a definition.

We assume an axiomatic viewpoint stating five axioms regarding Wick powers. These do not determine them, but determine the degrees of freedom due to the different possible choice of renormalization procedures and classify the finite renormalization counterterms. Regarding the 5th requirement in the definition below, for clarity we recall the notion of a compactly supported variation from Definition 3.2. If $(M, \mathbf{b})$ is a background geometry, the jointly smooth function $\mathbf{B} = \mathbf{b}_s(x)$ with $s \in \mathbb{R}^m$ and $x \in M$ defines a smooth m-dimensional ($m \geq 0$ integer) family of smooth compactly supported variations of $\mathbf{b}$ if $\mathbf{b}_s(x) = \mathbf{b}(x)$ for $x \in M$ and $\mathbf{b}_s(x) = \mathbf{b}_{s'}(x)$ for $s, s' \in \mathbb{R}^n$ and $x \notin K$ for a fixed compact $K \subseteq M$ depending on the family. According to Proposition 2.2, we can identify each algebra $\mathcal{W}(M, \mathbf{b}_s)$ with $\mathcal{W}(M, \mathbf{b})$ by means of the unital $*$-algebra isomorphism

$$\tau_s \colon \mathcal{W}(M, \mathbf{b}) \to \mathcal{W}(M, \mathbf{b}_s)\,, \tag{4.5}$$

which reduces to the identity on every spacetime $(N, \mathbf{g}_s|_N)$ if $N \cap J^+_{(M,\mathbf{g})}(K) = \emptyset$.

Definition 4.1 (Wick powers for general Boson fields). Consider a net of algebras $\mathcal{W}$ on the category of background geometries $\mathfrak{BkgG}$ (resp. $\mathfrak{BkgG}^+$) and a locally-covariant quantum V-field A (Definition 2.10) with $A_{(M,\mathbf{b})} \colon \mathscr{D}(VM) \to \mathcal{W}(M, \mathbf{b})$ for every $(M, \mathbf{b}) \in \mathfrak{BkgG}$ (resp. $\mathfrak{BkgG}^+$).
A class of **Wick powers** $\{A^k\}$ of A, for $k = 0, 1, 2, \dots$ is a family of *symmetric*

locally-covariant quantum S^kV-fields, so that each k defines an assignment of algebra-valued distribution

$$A^k_{(M,\mathbf{b})}\colon \mathscr{D}(S^kVM) \to \mathcal{W}(M,\mathbf{b})$$

to every $(M,\mathbf{b}) \in \mathfrak{BkgG}$ (resp. $\mathfrak{BkgG}^+$) respecting (2.8), satisfying the following requirements.

1. **Low powers**. $A^0 = \mathbf{1}$, the unit c-number field, $A^1 = A$, the V-field.

2. **Scaling**. With respect to the decomposition (4.4), each component A^P of the Wick power A^k is almost homogeneous of degree $\langle P, \mathbf{d}_A\rangle = p_1 d_{A_1} + \cdots p_N d_{A_N}$, with respect to the action of physical scalings S_λ in (2.10); that is, there exists an integer $l \geq 0$ and quantum k-tensor fields B_j such that

$$S_\lambda A^P = \lambda^{\langle P,\mathbf{d}_A\rangle} A^P + \lambda^{\langle P,\mathbf{d}_A\rangle} \sum_{j=1}^{l} \left(\log^j \lambda\right) B_j,$$

where each B_j is itself almost homogeneous of degree $\langle P, \mathbf{d}_A\rangle$ and order $l - j$. (Every degree is supposed to be independent from the choice of the background geometry).

3. **Kinematic completeness**. For any $(M,\mathbf{b})$, an element $a \in \mathcal{W}(M,\mathbf{b})$ satisfies

$$[a, A_{(M,\mathbf{b})}(f)] = 0 \quad \text{for every } f \in \mathscr{D}(VM)$$

iff $a = \alpha 1$, with $\alpha \in \mathbb{C}$ and 1 the unit element of the algebra.

4. **Commutator expansion**. Each Wick power A^k also satisfies the following properties[2]: For every $f \in \mathscr{D}(S^kVM), g \in \mathscr{D}(VM)$

$$\left[A^k_{(M,\mathbf{b})}(f), A_{(M,\mathbf{b})}(g)\right] = iA^{k-1}_{(M,\mathbf{b})}(\Delta_{(M,\mathbf{b})}(g) \cdot_1 f), \tag{4.6}$$

where $\Delta_{(M,\mathbf{b})}\colon \mathscr{D}(VM) \to \mathscr{E}(V^*M)$ is a given map.

5. **Smoothness**. If $(M,\mathbf{b}) \in \mathfrak{BkgG}$ (resp. $\mathfrak{BkgG}^+$), we require that there exist a class of states $\mathcal{S}_{(M,\mathbf{b})}$ on $\mathcal{W}(M,\mathbf{b})$ such that if $\omega \in \mathcal{S}_{(M,\mathbf{b})}$, the expectation values $\omega \circ \tau_s^{-1}\left(A^k_{(M,\mathbf{b}_s)}(f)\right)$ (with $f \in \mathscr{D}(S^kVM)$) can be written as

$$\omega \circ \tau_s^{-1}\left(A^k_{(M,\mathbf{b}_s)}(f)\right) = \int_M \omega_{\mu_1\cdots\mu_k}(s,x) f^{\mu_1\cdots\mu_k}(x)\, \mathrm{d}g(x),$$

[2] We recall that there is a factor k hidden in the contraction product.

for some jointly smooth kernels

$$\mathbb{R}^m \times M \ni (s,x) \mapsto \omega_{\mu_1\cdots\mu_k}(s,x) \in \mathbb{R},$$

for every smooth m-parameter family of compactly supported variations $\mathbf{b}_s$ of $\mathbf{b}$ on M and $\tau_s\colon \mathcal{W}(M,\mathbf{b}) \to \mathcal{W}(M,\mathbf{b}_s)$ defined as in (4.5) and every integer $m \geq 0$.

Remark 4.1. While the first four axioms are standard requirements, we would like to comment briefly the last axiom. We require that any Wick powers has *smooth* expectation value both with respect to x, the coordinate on spacetime manifold, and s, the parameter that labels the variations of $\mathbf{b}$. The smoothness with respect to x reflect the physical idea that a renormalized observable is smooth since we have removed all singularities in the renormalization procedure. The joint smoothness in (x,s) is a version of the *parametrized microlocal spectrum condition* that was introduced in [KM16, Def.3.5(iv)], as a substitute for the old analyticity condition of Hollands and Wald [HW01].

Remark 4.2. In components, *i.e.*, with respect to equation (4.4), the commutator expansion axiom, for $j \in 1,\dots,N$ and a multi-index P, becomes

$$\begin{gathered}\left[A_1^{p_1}\cdots A_N^{p_N}\left(f_1^{p_1},\cdots,f_N^{p_N}\right),A_j(g_j)\right] = \\ i\sum_{l=1}^{N} A_1^{p_1}\cdots A_l^{p_l-1}\cdots A_N^{p_N}\left(f_1^{p_1},\cdots,(\Delta^{lj}(g_j)\cdot_1 f_l^{p_l}),\cdots,f_N^{p_N}\right).\end{gathered} \tag{4.7}$$

To show this, fixing a background geometry $(M,\mathbf{b})$, consider $g_j \in (W^*M)_j$ and $f = \sum_j f_l$ with $f_l \in W_l M$, where we have used the identification introduced in Remark 1.1, *i.e.*, $g_j = (0,\dots,g_j,\dots,0)$ and $f_j = (0,\dots,f_j,\dots,0)$. We recall that

$$A_1^{p_1}\cdots A_N^{p_N}\left(f_1^{p_1},\cdots,f_N^{p_N}\right) := A^k\left(f_1^{p_1}\odot\cdots\odot f_N^{p_N}\right),$$

then, using Proposition 1.3,

$$\begin{aligned}\left[A_1^{p_1}\cdots A_N^{p_N}\left(f_1^{p_1},\cdots,f_N^{p_N}\right),A_j(g_j)\right] &= \\ &= \left[A^k\left(f_1^{p_1}\odot\cdots\odot f_N^{p_N}\right),A(g_j)\right] \\ &= iA^{k-1}\left(\sum_{l=1}^{N}\Delta^{lj}(g_j)\cdot_1\left(f_1^{p_1}\odot\cdots\odot f_N^{p_N}\right)\right) \\ &= i\sum_{l=1}^{N}A^{k-1}\left(f_1^{p_1}\odot\cdots\odot\left(\Delta^{lj}(g_j)\cdot_1 f_l^{p_l}\right)\odot\cdots\odot f_N^{p_N}\right),\end{aligned}$$

which by definition is equal to (4.7).

General renormalization formula for Wick products of Boson fields

We are now ready to prove a general renormalization formula for Wick powers of Boson fields. If $\{\tilde{A}^k\}_{k=1,2,\dots}$ and $\{A^k\}_{k=1,2,\dots}$ are two families of Wick powers of the same quantum V-field A, our task is now to find a formula relating these two pairs of Wick powers relying on the fact that both classes satisfy the above set of general axioms. The following theorem is a generalization of [KM16, Lem.3.3]

Theorem 4.1. *Let $\{\tilde{A}^k\}_{k=1,2,\dots}$ and $\{A^k\}_{k=1,2,\dots}$ be two families of Wick powers (Definition 4.1) of the same locally-covariant quantum V-field A (Definition 2.10) of homogeneous scaling degree $\mathbf{d}_A \in \mathbb{R}^N$. Then there exists a family of smooth locally-covariant c-number S^kV-fields $\{C_k\}_{k=1,2,\dots}$, where $C_1 = 0$, such that, for every $k = 1, 2, \dots$,*

$$\widetilde{A}^k_{(M,\mathbf{b})}(f) = A^k_{(M,\mathbf{b})}(f) + \sum_{l=0}^{k-1} A^l_{(M,\mathbf{b})}\left(C_{k-l}[M,\mathbf{b}] \cdot_{k-l} f\right), \tag{4.8}$$

where $(M,\mathbf{b}) \in \mathfrak{BkgG}$ (resp. $\mathfrak{BkgG}^+$) and $f \in \mathscr{D}(S^kV^*M)$. In components equation (4.8) *turns out to be*

$$\begin{gathered}\widetilde{A}^P_{(M,\mathbf{b})}\left(f_1^{p_1},\dots f_N^{p_N}\right) \\ = \\ A^P_{(M,\mathbf{b})}\left(f_1^{p_1},\dots f_N^{p_N}\right) \\ + \\ \sum_{l=0}^{k-1}\sum_{\substack{|Q|=l \\ q_i\le p_i}}\left(\prod_{i=1}^N \binom{p_i}{q_i}\right) A^Q_{(M,\mathbf{b})}\left(\left(C^{P-Q}_{k-l}[M,\mathbf{b}] \cdot_{k-l} f^{P-Q}\right) f_1^{q_1},\dots,f_N^{q_N}\right)\end{gathered} \tag{4.9}$$

where $Q = (q_1,\dots,q_N)$, $P = (p_1,\dots,p_N)$ are multi-indices and $C_k = \sum_{|Q|=k} C^Q_k$. Finally, for every fixed $M \in \mathfrak{Man}$,

(i) the map

$$\Gamma(BM) \ni \mathbf{b} \mapsto C_k[M,\mathbf{b}] \in \mathscr{E}(S^kV^*M)$$

is a differential operator of locally bounded order. Regarding components of the coefficients $C^Q_k[M,\mathbf{b}] \in \mathscr{E}(\bigodot_{i=1}^N S^{q_i}W_i^*M)$;

(ii) each $C^Q_k[M,\mathbf{b}]$ scales almost homogeneously of degree $\langle Q, \mathbf{d}_A\rangle$ under the physical scaling transformation on $\mathbf{b}$.

Proof. In the first part of the proof we write $A(f)$ in place of $A_{(M,\mathbf{b})}(f)$ and we adopt similar notations for the other involved fields, for the sake of notational simplicity. For all k, the difference

$$\widetilde{A}^k(f) - A^k(f) = D_k(f), \qquad f \in \mathscr{D}(S^k VM)$$

defines, by construction, a symmetric locally-covariant quantum $V^{\otimes k}$-field of order k, in particular is self-adjoint. Using the commutator expansion (4.6) in Axiom 4, it is easy to show that

$$[D_k(f), A(g)] = iD_{k-1}\left(\Delta(g) \cdot_1 f\right). \tag{4.10}$$

$D_k(f)$ is an element of the algebra $\mathcal{W}(M,\mathbf{b})$ and we go to prove that it can be expanded as a linear combination of element of the form A^l.
We proceed by induction in k. The thesis holds for $k = 1$ and $C_1 = 0$ since, using again the first axiom, $D_1(f) = 0$ for all $f \in \mathscr{D}(V^*M)$. Suppose now that (4.8) holds for $k-1$ with respect to some functions $C_i \colon \Gamma(BM) \to \mathscr{E}(S^i V^* M)$, $i = 1, 2, \ldots, k-1$, that satisfy all the desired properties. We intend to establish the validity of the thesis also for $i = k$. Consider the Wick polynomial, for $f \in \mathscr{D}(S^k VM)$,

$$W_k(f) := \sum_{l=1}^{k-1} A^l(C_{k-l}[M,\mathbf{b}] \cdot_{k-l} f).$$

We stress that the sections $C_{k-1}[M,\mathbf{b}], C_{k-2}[M,\mathbf{b}], \ldots, C_1[M,\mathbf{b}]$ appearing in the sum, by hypotheses are smooth and have all the desired properties stated in the theorem. Writing C_{k-1} in place of $C_{k-1}[M,\mathbf{b}]$, we have:

$$\begin{aligned}
&[D_k(f) - W_k(f), A(g)] = \\
&\quad = iD_{k-1}(\Delta(g) \cdot_1 f) - \sum_{l=1}^{k-1} \left[A^l(C_{k-l} \cdot_{k-l} f), A(g)\right] \\
&\quad = i\sum_{l=0}^{k-2} A^l\left(C_{k-1-l} \cdot_{k-1-l} \Delta(g) \cdot_1 f\right) - \sum_{l=1}^{k-1} iA^{l-1}(\Delta(g) \cdot_1 C_{k-l} \cdot_{k-l} f) \\
&\quad = i\sum_{l=1}^{k-1} A^{l-1}\left(C_{k-l} \cdot_{k-l} \Delta(g) \cdot_1 f\right) - A^{l-1}\left(\Delta(g) \cdot_1 C_{k-l} \cdot_{k-l} f\right) = 0,
\end{aligned}$$

where we have used Proposition 1.2. Thus, we can conclude that $[D_k(f) - W_k(f), A(g)] = 0$ for any test function g. Due to Axiom 3 we must therefore have

$$D_k(f) - W_k(f) = C_k(f)1, \tag{4.11}$$

where $C_k(f)$ is real and must define a locally-covariant c-number S^kV-field since it is difference of that type of fields. Using the smoothness axiom we have

$$\omega\left(D_k(f) - W_k(f)\right) = C_k(f)\,,$$

where the left-hand side is a distribution with smooth kernel. Therefore there is a corresponding smooth section $C_k[M,\mathbf{b}] \in \mathscr{E}(S^kV^*M)$ generating the right-hand side. The covariance requirement immediately implies that the value of the smooth section $C_k[M,\mathbf{b}]$ at any point $x \in M$ depends only on the germ around x for every fixed manifold M, using smaller and smaller neighborhoods $M' \ni x$ with $M' \subseteq M$ viewed as background geometries on their own right when equipped with the restriction of $\mathbf{b}$ to M' and using the inclusion map $\chi\colon M' \ni x' \mapsto x' \in M$ as embeddings. More strongly, if we consider any m-parameter family of compactly supported smooth deformation $\mathbf{B} = \mathbf{b}_s(x)$ of $\mathbf{b} = \mathbf{b}_0$, then the same argument tells us that $C_k[M,\mathbf{b}_s](x)$ is also jointly smooth in (s,x). Thus the map $C_k[M,\cdot]$ is weakly regular and Peetre-Slovák theorem implies that $\Gamma(BM) \ni \mathbf{b} \mapsto C_k[M,\mathbf{b}] \in \mathscr{E}(S^kV^*M)$ is a differential operator of locally bounded order. Summing up, we have proved that

$$\widetilde{A}^k(f) - A^k(f) = D_k(f) = W_k(f) + C_k(f)1 = \sum_{l=0}^{k-1} A^l\left(C_{k-l} \cdot_{k-l} f\right),$$

where *all* coefficients $C_l[M,\mathbf{b}]$ from $l = 0$ to $l = k$ have all properties stated in the thesis, but the scaling property which must be still established for C_k only. Choosing as test function $f = f_1^{p_1} \odot \cdots \odot f_N^{p_N}$ and using relation *(c)* and *(d)* from Proposition 1.3, we obtain immediately the formula (4.9).

Thanks to the *scaling* property of A^P and $\widetilde{A}^P$, C_k^Q is a linear combination of products of terms with almost homogeneous degree that add up to $\langle Q, \mathbf{d}_A\rangle$. Thus, by Lemma 3.3, C_k^Q itself has almost homogeneous degree $\langle Q, \mathbf{d}_A\rangle$, and thus

$$S_\lambda C_k^Q 1 = \lambda^{\langle Q,\mathbf{d}_A\rangle} C_k^Q 1 + \lambda^{\langle Q,\mathbf{d}_A\rangle} \sum_{j=1}^{l} \left(\log^j \lambda\right) B_j^Q\,,$$

where S_λ is the action of physical scalings on fields here applied to a c-number field, with B_j^Q some other quantum fields of almost homogeneous degree $\langle Q, \mathbf{d}_A\rangle$. Using again the kinematic completeness of A, we find that $B_j^Q = F_j^Q 1$ are also c-number fields. Now, exploiting the definition of S_λ as in (2.10), we find that

$S_\lambda C_k^Q 1 = C_k^Q 1$, and similarly for the F_j^Q. Therefore, we find that for every $x \in M$,

$$C_k^Q[M, \mathbf{b}_\lambda](x) = \lambda^{\langle Q, \mathbf{d}_A \rangle} C_k[M, \mathbf{b}](x) + \lambda^{\langle Q, \mathbf{d}_A \rangle} \sum_{j=1}^{l} \left(\log^j \lambda\right) F_j^Q[M, \mathbf{b}](x),$$

is an almost homogeneous element of degree $\langle Q, \mathbf{d}_A \rangle$ of the vector space of maps $\Gamma(BM) \to \mathscr{E}(\odot_{i=1}^N S^{q_i} W_i^* M)$ under the action $F[M, \mathbf{b}] \mapsto F[M, \mathbf{b}_\lambda]$. The proof is concluded. □

We have finally obtained a general formula, (4.8), that classifies all finite renormalizations counter-terms of Wick powers of a generic locally-covariant Boson vector field A, where the coefficients $C_k[M, \mathbf{b}]$ depend on the type of vector bundle VM and the nature of background fields $\mathbf{b}$ of the field A. For this reason, in order to study in detail these coefficients, we have to consider physically relevant models.

4.2 Tensor fields and renormalization of Wick powers

In this section we consider a class of physically relevant models and we study in detail the renormalization counter-terms C_k introduced in the last section. We choose as bundles

$$VM = \bigoplus_{i=1}^{N} T^{*\otimes k_i} M, \qquad BM = \mathring{S}^2 T^* M \oplus \left(\bigoplus_{j=1}^{K} T^{*\otimes l_j} M \right)$$

which means that we are considering as fields an N-tuple of tensor fields with different tensor ranks

$$A = (A_1, \dots, A_N) \qquad A_i \colon \mathscr{D}(T^{\otimes k_i} M) \longrightarrow \mathcal{W}(M, \mathbf{b}) \tag{4.12}$$

and we will say that A has tensor rank $\mathbf{k} = (k_1, \dots, k_N)$. As background fields we consider the metric $\mathbf{g}$ together with other (covariant) tensor fields $\mathbf{t}_j$ of rank l_j

$$(\mathbf{g}, \mathbf{t}_1, \dots, \mathbf{t}_K), \qquad \mathbf{g} \in \mathscr{E}(\mathring{S}^2 T^* M), \ \mathbf{t}_j \in \mathscr{E}(T^{*\otimes l_j} M).$$

Regarding ***physical scaling***, we assume the most general situation, ***i.e.***,

$$A_i \longmapsto \lambda^{d_{A_i}} A_i \qquad (\mathbf{g}, \mathbf{t}_1, \dots, \mathbf{t}_K) \longmapsto \left(\lambda^{-2} \mathbf{g}, \lambda^{s_1} \mathbf{t}_1, \dots, \lambda^{s_K} \mathbf{t}_K\right) \qquad \lambda \in \mathbb{R}^+$$

under physical scaling transformation, where $s_j \in \mathbb{R}$ for $j = 1, \dots, K$. We require also another property of the background fields, encoded in the following

Definition 4.2. A background field $\mathbf{t}_j$ is called **admissible** if its rank l_j and its degree under physical scaling s_j fulfill the following condition

$$l_j + s_j \geq 0.$$

If the above relation is an equality, then we also call $\mathbf{t}_j$ **marginal**. By convention, let us order the background fields such that each $\mathbf{t}_j$ for $j = 1, \ldots, K_0 \leq K$ is marginal and collectively denote them by $\mathbf{z} = (\mathbf{t}_1, \ldots, \mathbf{t}_{K_0})$. To emphasize their distinction from other background fields, we will also use the notation $\mathbf{z}_j = \mathbf{t}_j$.

Remark 4.3. We have chosen to all dynamical and background tensor fields to be purely covariant, *i.e.*, to be sections of powers of the cotangent bundle T^*M. This choice is motivated purely by convenience and the desire not to complicated our notation even further. Our main results, Theorems 4.1 and 4.3, hold in easily adapted forms also for contravariant or mixed tensors, as well as for tensors of symmetric, antisymmetric, or any other symmetry type. One does have to note that, in the definition of *admissible* and *marginal* background fields (Definition 4.2), the tensor rank l_j must be taken to be the number of covariant tensor indices minus the number of contravariant indices of $\mathbf{t}_j$.

Lemma 4.2. *Consider an admissible background field* $\mathbf{t}_j$*. Then all its rescaled coordinates have positive or null scaling weight under coordinate scaling. In particular the rescaled coordinates scale as*

$$g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|}(\bar{t}_j)^{a_1\ldots a_{l_j},A} \longmapsto \mu^{l_j+s_j+|A|}g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|}(\bar{t}_j)^{a_1\ldots a_{l_j},A}$$

Proof. Under coordinate scaling we have the following rescaling

$$(\bar{t}_j)^{a_1\ldots a_{l_j},A} \mapsto \mu^{-l_j-|A|}(\bar{t}_j)^{a_1\ldots a_{l_j},A} \qquad g \mapsto \mu^{2n}g.$$

Then the result follows immediately. □

We are finally ready to state and prove our main result, which generalizes Theorem 3.1 of [KM16].

Theorem 4.3. *Let* $\{\tilde{A}^k\}_{k=1,2,\ldots}$ *and* $\{\tilde{A}^k\}_{k=1,2,\ldots}$ *be two families of Wick powers (Definition 4.1) of the same locally-covariant quantum* V*-field* A *of homogeneous scaling degree* $\mathbf{d}_A \in \mathbb{R}^N$ *and tensor rank* $\mathbf{k}$ *(same as in Theorem 4.1), where the natural vector bundle* $V = \bigoplus_{i=1}^N W_i$ *is the* N*-tuple introduced in* (4.12)*. Assume also that all background fields* $\mathbf{b}$*, sections of the bundle* BM *(Definition 1.15), are admissible*

(Definition 4.2). Recall also from Theorem 4.1 the renormalization coefficients C_k, $k = 1, 2, \dots$ (with $C_1 = 0$) appearing in (4.8) *when comparing two families of Wick powers of A. Finally, recall the notation R_{abcd} for the Riemann tensor, ∇_a for the Levi-Civita connection of g_{ab}, and $\epsilon^{a_1 \cdots a_n}$ for the associated Levi-Civita tensor.*

Then following facts hold:

(a) *If $Q = (q_1, \dots, q_N)$ is a multi-index with $|Q| = k$ such that $\langle Q, \mathbf{d}_A + \mathbf{k} \rangle = \sum_{i=1}^N q_i(d_{A_i} + k_i) < 0$, then the corresponding component C_k^Q of the renormalization coefficient C_k vanishes.*

(b) *If A is locally covariant with respect to the category $\mathfrak{B}\mathfrak{k}\mathfrak{g}\mathfrak{G}$ (Definition 1.16), then for every background geometry $(M, \mathbf{b})$, every $x \in M$ and each $k = 1, 2, \dots$, the renormalization coefficients C_k are given by differential operators of globally bounded order*

$$C_k[M, \mathbf{b}](x) = C_k\Big(g^{ab}(x), R_{abcd}(x), \dots, \nabla_{e_1} \cdots \nabla_{e_h} R_{abcd}(x), \dots \\ \dots (t_j)_{a_1 \cdots a_{l_j}}(x), \dots, \nabla_{e_1} \cdots \nabla_{e_r} (t_j)_{a_1 \cdots a_{l_j}}(x), \dots\Big),$$

where $C_k(\cdots)$ is a tensor field covariantly constructed from its arguments, whose structure is described in more detail below.

(c) *If A is locally covariant with respect to the category $\mathfrak{B}\mathfrak{k}\mathfrak{g}\mathfrak{G}^+$ (Definition 1.16), then for every background geometry $(M, \mathbf{b})$, every $x \in M$ and each $k = 1, 2, \dots$, the renormalization coefficients C_k are given by differential operators of globally bounded order*

$$C_k[M, \mathbf{b}](x) = C_k\Big(g^{ab}(x), \varepsilon^{a_1 \cdots a_n}, R_{abcd}(x), \dots, \nabla_{e_1} \cdots \nabla_{e_h} R_{abcd}(x), \dots \\ \dots (t_j)_{a_1 \cdots a_{l_j}}(x), \dots, \nabla_{e_1} \cdots \nabla_{e_r} (t_j)_{a_1 \cdots a_{l_j}}(x), \dots\Big),$$

where $C_k(\cdots)$ is a tensor field covariantly constructed from its arguments, whose structure is described in more detail below.

In both (b) and (c), by covariantly constructed *we mean that the C_k are* equivariant *functions of their tensorial arguments, at each $x \in M$, in the sense of Lemma 3.14. That is, each $C_k(\dots)$ is a linear combination of finitely many covariantly constructed tensors that are polynomial in g_{ab}, g^{ab} and the rest of the tensorial arguments, with scalar coefficients that are smooth functions depending locally (Definition 3.11) on finitely many polynomial scalars covariantly constructed from the tensor fields $\mathbf{z}$, which consist* *of those background tensors* $\mathbf{t}_j$ *that are* marginal *according to Definition 4.2. Moreover, the functional form of the* $C_k(\cdots)$ *does not depend on* $(M, \mathbf{b})$.

Finally, each C_k *can be written as* $C_k = \sum_{|Q|=k} C_k^Q$ *with respect to the multiplet decomposition* $V = \bigoplus_{i=1}^N W_i$*, where* Q *is a multi-index and where each* C_k^Q *is homogeneous degree under physical scaling. More precisely, it scales as*

$$C_k^Q \mapsto \lambda^{\langle Q, \mathbf{d}_A \rangle} C_k^Q ,$$

when its arguments are rescaled according to

$$(t_j)_{a_1 \cdots a_{l_j}} \mapsto \lambda^{s_j} (t_j)_{a_1 \cdots a_{l_j}} , \quad g^{ab} \mapsto \lambda^2 g^{ab} ,$$
$$\varepsilon^{a_1 \cdots a_n} \mapsto \lambda^n \varepsilon^{a_1 \cdots a_n} , \quad R_{abcd} \mapsto \lambda^{-2} R_{abcd}$$

(the covariant derivatives are fixed under rescaling). These scaling properties fix the upper bound on the differential and polynomial order of C_k^Q.

Remark 4.4. Before going on to the proof, how it resembles and differs from the proof of Theorem 3.1 of [KM16], which proved a similar result but only for scalar dynamical and background fields. Generally speaking, the structure of the two proofs are similar, which are broken down into roughly the same number of steps, roughly in the same sequence. In both cases, we start out by knowing that the renormalization coefficients $C_k[M, \mathbf{b}]$ are differential operators of locally bounded order. Hence, each C_k is given by a smooth function defined on the jet bundle $J^r BM$ of the background fields, at least when applied to sections $\mathbf{b}$ whose jets fall into some open neighborhood in the jet bundle. The remaining steps gradually fix the structure of the C_k more and more rigidly, while also expanding its domain of definition on $J^r BM$, ultimately extending it to the entire jet bundle and thus showing that it is of *globally bounded* differential order. The structure of C_k is first restricted by appealing to its properties under physical scaling, using results from Section 3.2. One immediate difference in the new proof is the need to keep track of different (both physical and coordinate) scaling weights for the different components of the C_k. Next, the structure of the C_k is further restricted by its local covariance, meaning that it commutes with diffeomorphisms. The results from Section 3.4, provide the necessary tools for that, which essentially consist of a strengthened version of the Thomas Replacement Theorem reported in [KM16, Prop.2.6]. Finally, local covariance is once again used to fix the final form of the C_k, by using the results of Section 3.3, which essentially strengthen the classification of equivariant and isotropic tensors reported in [KM16, Prop.2.7, Lem.2.8].

These supporting results needed to be strengthened, compared to the ones used in [KM16], because of the transition from only scalar dynamical and background fields to tensorial ones.

Proof. We already know that, from the Peetre-Slovák theorem (see Section 3.1), the coefficients C_k^Q define a differential operator

$$\Gamma(BM) \ni \mathbf{b} \mapsto C_k^Q[M, \mathbf{b}] \in \mathscr{E}\left(\bigodot_{i=1}^{N} S^{q_i} T^{*\otimes k_i} M\right) \subseteq \Gamma\left(\bigodot_{i=1}^{N} S^{q_i} T^{*\otimes k_i} M\right)$$

of locally bounded order as established in Theorem 4.1. The rest of the proof is broken down into five steps, which are described in more detail below.

1. Physical scaling. We now take advantage from almost homogeneity under physical scaling of the components of the coefficients C_k^Q to find their functional form. Consider a Lorentzian manifold M endowed with a metric $\mathbf{g}_0$, as well as a point $y \in M$ and an open neighborhood of U of y with compact closure. We also assume that $\mathbf{g}_0$ restricted to U is flat. Consider also a coordinate system (x^d) on U centered at y. These coordinates induce adapted local coordinates on $Z^r \subseteq J^r BM$, which we write as

$$\left(x^a, g, g_{ab}, g^{ab,A}, (t_1)^{a_1 \ldots a_{l_1}, A}, \ldots, (t_K)^{a_1 \ldots a_{l_K}, A}\right).$$

Recall that the coordinates (g, g^{ab}) are functionally independent up to the identity $|\det g_{ab}| = g$. We already know that $\mathbf{b} \mapsto C_k^Q[M, \mathbf{b}](x)$ is a differential operator of locally bounded order, thus for $\mathbf{g}_0$, y and U (defined as above), there exists and integer $r \geq 0$ such that C_k^Q is a differential operator on U of local order r when acting on sections of BM close to $\mathbf{b}_0 := (\mathbf{g}_0, \mathbf{t}_j = 0)^3$. In other words, there exists a neighbourhood $Z_1^r \subseteq Z^r \subseteq J^r BM$ of $j_y^r \mathbf{b}_0$, projecting onto U, and a function

$$F_k^Q \colon Z_1^r \longrightarrow \left(\odot_{i=1}^{N} S^{q_i} T^{*\otimes k_i} M\right)$$
$$\left(x^a, g, g^{ab,A}, (t_j)^{a_1 \ldots a_{l_j}, A}\right) \longmapsto F_k^Q\left(x^a, g, g^{ab,A}, (t_j)^{a_1 \ldots a_{l_j}, A}\right),$$

[3]We stress that the flatness assumption on $\mathbf{g}_0$ is not a strong requirement because the flat metric is only the section with respect to which we consider variations. At the moment we can consider only metrics in a neighbourhood of $\mathbf{g}_0$ but we will gradually enlarge it to the whole set of Lorentzian metrics. A similar argument is also valid for all $\mathbf{t}_j$, which at the moment have to be close to the sections $\mathbf{t}_j = 0$.

such that

$$C_k^Q[M, \mathbf{b}](x) = F_k^Q\left(j^r \mathbf{b}(x)\right), \tag{4.13}$$

for any section $\mathbf{b} \in \Gamma(BM|_U)$ such that $j^r\mathbf{b}(U) \subseteq Z_1^r$. Without loss of generality, but possibly shrinking the domain of F_k, we chose it such that

$$\begin{array}{ccc} Z_1^r \simeq & U \times & W_1^r \\ & \downarrow & \searrow \\ & (x^a) & \left(g, g^{ab,A}, (t_j)^{a_1 \dots a_{l_j},A}\right). \end{array}$$

At the moment, we are very far from arguing that $Z_1^r = J^r BM$ especially because, using the Peetre-Slovák theorem we only know that the order of the differential operator C_k is locally bounded and a finite global bound may not exist. During the proof we will gradually enlarge the domain Z_1^r to eventually cover all of $J^r BM$ while maintaining the identity (4.13). The differential order r of C_k^Q may increase in the process, but will remain finite. These extensions will be labeled by an increasing index j in Z_j^r. Presently $j = 1$.

Theorem 4.1 implies that C_k^Q and hence the function F_k^Q scales almost homogeneously with degree $\langle Q, \mathbf{d}_A\rangle$ under physical scaling of the background fields. Thus, thanks to Lemma 3.4 and Lemma 3.5, there exists an integer $l > 0$ and function B_j on Z_1^r, for $h = 0, \dots, l$, such that

$$F_k^Q = g^{-\frac{\langle Q, \mathbf{d}_A\rangle}{2n}} \sum_{h=0}^{l} \log^h\left(g^{-\frac{1}{2n}}\right) B_h, \tag{4.14}$$

where each B_h is invariant under the action of physical scaling. Therefore, adopting rescaled coordinates (which are invariant under physical scaling), B_h cannot depend on g and can be written as

$$B_h = B_h\left(x^d, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n} + \frac{1}{n}|A|} g^{ab,A}, \dots, g^{\frac{l_j}{n} + \frac{s_j}{2n} + \frac{1}{n}|A|}(t_j)^{a_1 \dots a_{l_j},A}, \dots\right).$$

We now extend the domain Z_1^r to a larger domain $Z_2^r \subseteq J^r BM$. We define Z_2^r to be the smallest domain invariant under physical scaling and containing Z_1^r. That is, we can write it as

$$\begin{array}{ccccc} Z_2^r \simeq & \mathbb{R}^+ \times & U & \times W_2^r & \\ & \downarrow & \downarrow & \searrow & \\ & (g) & (x^d) & \left(g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n} + \frac{1}{n}|A|} g^{ab,A}, g^{\frac{l_j}{n} + \frac{s_j}{2n} + \frac{1}{n}|A|}(t_j)^{a_1 \dots a_{l_j},A}\right). & \end{array}$$

Up to now, we know that the identity (4.13) holds only when the germ of $\mathbf{b}$ at $x \in M$ projects onto one of the jets in the domain $Z_1^r \in J^r BM$, but the function F_k^Q, via formula (4.14), has a ***unique*** extension to Z_2^r that scales almost homogeneously and agrees with F_k^Q on Z_1^r. The identity (4.13) must remain valid also for germs at x that projects onto Z_2^r since any element of $\mathbf{b}' \in Z_2^r$, using the action of physical scaling, can be brought back to $\mathbf{b} \in Z_1^r$, *i.e.*, $\mathbf{b}' = \mathbf{b}_\lambda$ for some $\lambda > 0$. Since $C_k^Q[M, \mathbf{b}]$ scales almost homogeneously and is already defined on Z_2^r, we conclude that it must coincide there with the unique extension of F_k^Q.

2. Diffeomorphism covariance. We consider now the covariance properties of the coefficient C_k^Q under diffeomorphisms. In the previous paragraph, we fixed a point $y \in M$ and a fixed background geometry $\mathbf{b}_0$. But, since that choice was arbitrary, all the same results are also valid for any other choice of $y' \in M$, open neighborhood $U' \subseteq M$ of y' and background geometry $\mathbf{b}_0'$, so long as $\mathbf{b}_0' = \chi^* \mathbf{b}_0$ on U', where

$$\chi \colon M \to M \text{ is some diffeomorphism such that } \chi(y') = y.$$

The Peetre-Slovák theorem gives us a differential operator of order r' on a domain $Z_2'^{r'} \subseteq J^{r'} BM$. The diffeomorphism covariance of C_k^Q then implies that the order may be chosen the same, $r = r'$.
We now extend the domain Z_2^r to a larger domain $Z_3^r \subseteq J^r BM$. We define Z_3^r to be the smallest domain invariant under $\mathrm{Diff}(M)$ and containing Z_2^r. Since the coefficient C_k^Q is $\mathrm{Diff}(M)$-covariant, the function F_k^Q is itself $\mathrm{Diff}(M)$-covariant on Z_3^r. The case of orientation preserving diffeomorphisms $\mathrm{Diff}^+(M)$ is strictly analogous.
Now we can use Thomas replacement theorem (Theorem 3.15) in order to remove the dependence of F_k^Q on some of the coordinates on Z_3^r. We apply Theorem 3.15 separately to the various functions B_h appearing in (4.14), obtaining

$$\begin{aligned} &g^{-\frac{\langle Q, \mathbf{d}_A \rangle}{2n}} B_h \left(x^a, g^{-\frac{1}{n}} g_{ab}, g^{\frac{1}{n} + \frac{1}{n}|A|} g^{ab,A}, g^{\frac{l_j}{n} + \frac{s_j}{2n} + \frac{1}{n}|A|} (t_j)^{a_1 \dots a_{l_j}, A} \right) \\ &\quad = g^{-\frac{\langle Q, \mathbf{d}_A \rangle}{2n}} G_h \left(g^{-\frac{1}{n}} g_{ab}, g^{\frac{3}{n} + \frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{l_j}{n} + \frac{s_j}{2n} + \frac{1}{n}|A|} (\bar{t}_j)^{a_1 \dots a_{l_j}, A} \right), \end{aligned}$$

where each $g^{-\frac{\langle Q, \mathbf{d}_A \rangle}{2n}} G_j$ is equivariant under the action of $GL(n)$ (respectively $GL^+(n)$). In particular, G_j does not depend on the coordinates (x^a) and $(\Gamma^a_{(bc,A)})$.

Since Z_3^r is $\mathrm{Diff}(M)_U$-invariant it has the structure:

$$\begin{array}{ccccccccc} Z_3^r \simeq & U & \times & L_n & \times & \mathbb{R}^\gamma & \times & W_3^r & \\ & \downarrow & & \downarrow & & \downarrow & & \searrow & \\ & (x^a) & & (g_{ab}) & & \left(\Gamma^a_{(bc,A)}\right) & & \left(g^{\frac{3}{n}+\frac{1}{n}|A|}\bar{S}^{ab(cd,A)}, g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|}(\bar{t}_j)^{a_1\ldots a_{l_j},A}\right). \end{array}$$

The second factor describes the metrics at a fixed point $p \in U$ and coincides with the full set L_n of non-degenerate bilinear forms on $\mathbb{R}^n$ with Lorentzian signature. This is because the fiber action of the subgroup of $\mathrm{Diff}(M)$ which leaves p fixed is the action of the whole $GL(n)$ which, in turn, acts transitively on L_n. W_3^r contains at least the point with all components $g^{\frac{3}{n}+\frac{1}{n}|A|}\bar{S}^{ab(cd,A)}$ and $g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|}(\bar{t}_j)^{a_1\ldots a_{l_j},A}$ vanishing (in particular because $\mathbf{g}_0$ is flat on U). W_3^r is invariant under the said natural action of the whole $GL(n)$. The same argument applies to $\mathrm{Diff}^+(M)$ and $GL(n)^+$.

3. Covariance under coordinate scaling. Now we use the equivariance of the function F_k^Q under the action of a subgroup of $GL(n)$ (respectively $GL^+(n)$), the subgroup of coordinate scaling.[4] We can rewrite the set of coordinates over $L_n \times W_3^r$ (the remaining coordinates x^a and $\Gamma^a_{(bc,A)}$ of Z_3^r do not appear in the explicit form of F_k^Q as already established) as

$$\left(g, g^{-\frac{1}{n}}g_{ab}, z^j, q^i\right)$$

Here the coordinates are grouped together along with the following idea: The coordinates $(g^{-\frac{1}{n}}g_{ab}, z^j)$ have weight 0 under coordinate scaling (*i.e.*, z^j, $j = 1, \ldots, m_z$, are precisely the rescaled components of those undifferentiated coordinates of the background fields $\mathbf{t}_i$ satisfying $l_i + s_i = 0$, or precisely the components of the marginal background fields $\mathbf{z}$, Definition 4.2), g transforms as $g \to \mu^{2n} g$ and all remaining coordinates, here denoted by q^i, $i = 1, \ldots, m_q$, have positive weight ($d_i > 0$) under coordinate scalings. There are no coordinates with negative weight (Lemma 4.2).

Let us recall that F_k^Q is a $(\bigodot_{i=1}^N S^{q_i}T^{*\otimes k_i}M)$-valued function and that the vector $\mathbf{k} = (k_1, \ldots, k_N)$ is constructed with the tensor ranks k_i. Then the general diffeomorphism equivariance of the function F_k^Q specialized to coordinate scalings

[4] This part of the proof is analogous to part *4.* of the proof of Theorem 3.1 in [KM16]. Unfortunately, that earlier argument contained an error when eliminating logarithmic terms from F_k. This error has been corrected in the current argument, which should also be considered retroactively inserted into the proof given in [KM16].

(centered at some base point (x^a), which could be arbitrary within the domain of definition of F_k^Q), implies the identity

$$\begin{aligned} F_k^Q\left(g, g^{-\frac{1}{n}}\ g_{ab}, z^j, q^i\right) = \\ &= \mu^{-\langle Q,\mathbf{k}\rangle} F_k^Q\left(\mu^{2n} g, g^{-\frac{1}{n}} g_{ab}, z^j, \mu^{d_i} q^i\right) \\ &= g^{-\frac{\langle Q,\mathbf{d}_A\rangle}{2n}} \sum_{h=0}^{l-1} \mu^{-\langle Q,\mathbf{d}_A+\mathbf{k}\rangle} \log^h\left(\mu^{-1} g^{-\frac{1}{2n}}\right) G_h\left(g^{-\frac{1}{n}} g_{ab}, z^j, \mu^{d_i} q^i\right) \end{aligned} \tag{4.15}$$

for any point in Z_3^r and any $\mu > 0$. As we mentioned in the previous part, the limit $(g^{-\frac{1}{n}} g_{ab}, z^j, 0)$ of the argument of the functions G_h as $\mu \to 0$, belongs to the domain of the function G_h, which is smooth there. Therefore we have the Taylor expansions

$$G_h\left(g^{-\frac{1}{n}} g_{ab}, z^j, q^i\right) = \sum_{|I|<N_q} G_{h,I}\left(g^{-\frac{1}{n}} g_{ab}, z^j\right) q^I + O\left(q^{N_q}\right),$$

around $(g, g^{-\frac{1}{n}} g_{ab}, z^j, 0)$, where $I = i_1 \cdots i_{m_q}$ is a multi-index with respect to the coordinates (q^i), the coefficients $G_{h,I}$ are smooth, and $N_q > 0$ is an integer such that

$$\langle d, I\rangle = \sum_{j=1}^{m_q} d_j i_j > \langle Q, \mathbf{d}_A + \mathbf{k}\rangle \quad \text{for all } I \text{ such that } |I| \geq N_q.$$

This choice guarantees that each error term $O(q^{N_q})$ is mapped to $O(\mu^{\langle Q,\mathbf{d}_A+\mathbf{k}\rangle+1})$ under the substitution $q^i \mapsto \mu^{d_i} q^i$ as $\mu \to 0$. Thus we obtain

$$\begin{aligned} F_k^Q\left(g, g^{-\frac{1}{n}} g_{ab}, z^j, q^i\right) = \\ &= g^{-\frac{\langle Q,\mathbf{d}_A\rangle}{2n}} \sum_{h=0}^{l-1} \mu^{-\langle Q,\mathbf{d}_A+\mathbf{k}\rangle} \log^h\left(\mu^{-1} g^{-\frac{1}{2n}}\right) G_h\left(g^{-\frac{1}{n}} g_{ab}, z^j, \mu^{d_i} q^i\right) \\ &= g^{-\frac{\langle Q,\mathbf{d}_A\rangle}{2n}} \sum_{h=0}^{l-1} \sum_{|I|<N_q} \mu^{\langle d,I\rangle-\langle Q,\mathbf{d}_A+\mathbf{k}\rangle} \log^h\left(\mu^{-1} g^{-\frac{1}{2n}}\right) G_{h,I}\left(g^{-\frac{1}{n}} g_{ab}, z^j\right) q^I \\ &\qquad + \mu^{-\langle Q,\mathbf{d}_A+\mathbf{k}\rangle} O\left(\mu^{\langle Q,\mathbf{d}_A+\mathbf{k}\rangle+1}\right). \end{aligned} \tag{4.16}$$

Now, if we take the limit $\mu \to 0$, the left-hand side of (4.16) does not change, being independent of μ, and in particular remains bounded. Hence, for equality to hold, any term on the right-hand side of (4.16) that independently goes to ∞ as $\mu \to 0$ must vanish. That is, the coefficient of each $\mu^p \log^h \mu$ term with $p < 0$ or $p = 0, h > 0$ must be zero. Actually taking the $\mu \to 0$ limit on the right-hand side of (4.16) we obtain the identity

$$F_k^Q\left(g, g^{-\frac{1}{n}} g_{ab}, z^j, q^i\right) = \sum_{\langle d,I\rangle = \langle Q, \mathbf{d}_A + \mathbf{k}\rangle} g^{-\frac{\langle Q, \mathbf{d}_A\rangle}{2n}} G_{0,I}\left(g^{-\frac{1}{n}} g_{ab}, z^j\right) q^I. \qquad (4.17)$$

All terms $\langle d, I\rangle > \langle Q, \mathbf{d}_A + \mathbf{k}\rangle$ were set to zero by the limit, which by consistency means that they had zero coefficients to being with. Notice that this identity implies that the function F_k^Q scales ***homogeneously*** with degree $\langle Q, \mathbf{d}_A\rangle$ (that is, it has almost homogeneous order zero). This sum could conceivably be empty, if it happens that $\langle Q, \mathbf{d}_A + \mathbf{k}\rangle < 0$ (recall that $d_i > 0$), which can only happen if some of the combinations $d_{A_i} + k_i < 0$. In that case, $F_k^Q = 0$ and the corresponding component C_k^Q of the renormalization coefficient C_k vanishes, which proves part (a) of the theorem.

We can now enlarge again the domain of the function F_k^Q along the fibers, where the identity (4.13) holds, from Z_3^r to $Z_4^r \subseteq J^r BM$. The new domain is isomorphic to

$$\begin{array}{cccccccccc} Z_4^r \simeq & U & \times & L_n & \times & W_4 & \times & \mathbb{R}^\gamma & \times & \mathbb{R}^{m_q} \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \searrow \\ & (x^a) & & \left(g^{-\frac{1}{n}} g_{ab}\right) & & (z^j) & & \left(\Gamma^a_{(bc,A)}\right) & & \underbrace{\left(g^{\frac{3}{n}+\frac{1}{n}|A|} \bar{S}^{ab(cd,A)}, g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|} (\bar{t}_j)^{a_1\dots a_{l_j},A}\right)}_{l_j+s_j+2|A|>0} \end{array}$$

The function F_k^Q extends uniquely to Z_4^r as a covariant function under coordinate scaling. Essentially we have enlarged the factor W_3^r to $W_4 \times \mathbb{R}^{m_q}$. We can do that because all the (q^i) coordinates have positive weight under coordinate scaling, so that their domain can be extended to all of $\mathbb{R}^{m_q}$. The range of the (z^j) coordinates is limited to $W_4 \subset \mathbb{R}^{m_z}$ because these coordinates are invariant under coordinate scaling. Note that the dependence of F_k^Q on the $\mathbb{R}^\delta$ factor in Z_4^r is polynomial and remember that F_k^Q does not depend on the factor $U \times \mathbb{R}^\gamma$ (see previous part).

4. Global definition. It is now the moment to expand the domain Z_4^r to all $J^r BM$, for an appropriate choice of r. In (4.17), a generic q^I is of the form

$$\prod_{|A|,|B|} \left(\bar{S}^{ab(cd,A)}\right)^{p_{S,|A|}} \left((\bar{t}_j)^{a_1\cdots a_{l_j},B}\right)^{p_{j,|B|}},$$

where all the p-exponents are non-negative integer numbers and $p_{j,0} = 0$ if $l_j + s_j = 0$. The constraint $\langle d, I\rangle = \langle Q, \mathbf{d}_A + \mathbf{k}\rangle$ in (4.17) can be written explicitly as

$$\langle Q, \mathbf{d}_A + \mathbf{k}\rangle = \sum_{|A|,|B|} (2 + |A|)p_{S,|A|} + (s_j + l_j + |B|)p_{j,|B|}.$$

By the admissibility of the background fields (Definition 4.2), we have $s_j + l_j \geq 0$. Hence, the coefficients of the p-exponents are non-negative and grow linearly with $|A|$ and $|B$. Thus, there exists a bound on the maximum values of $|A|, |B|$ with non-zero p-exponents. Let r_k be the maximum number of derivatives of the curvature or background tensors for which the p-exponents are non-zero. Note that r_k depends only on the structure of the bundle BM and k, and not on the chosen domain Z_4^r. Then we can set $r = r_k$ in all the previous parts of the proof, *i.e.*, we end up with a domain

$$Z_4^{r_k} \subseteq J^{r_k}BM, \qquad Z_4^{r_k} = U \times L_n \times W_4 \times \mathbb{R}^\delta.$$

We can now extend one last time the domain $Z_4^{r_k}$ keeping the order of F_k globally bounded. The factor L_n is already maximal since it contains all Lorentzian metrics. At the beginning of the proof we chose as the initial domain Z_1^r a neighbourhood of the point $j_y^r(\mathbf{g}_0, \mathbf{t}_j = 0) \in J^rBM$. Recall that we later split the coordinates on J^rBM into two groups, the q-coordinates, identified by positive scaling weights ($s_j + l_j > 0$), and the z-coordinates, identified by zero scaling weights ($s_j + l_j = 0$), the components of the marginal tensor fields $\mathbf{z}$ (Definition 4.2). What was essential for the subsequent arguments was that, for each allowed value of the z-coordinates, $(z, q = 0)$ was also contained in Z_1^r, because $q \circ j_y^r(\mathbf{g}_0, \mathbf{t}_j = 0) = 0$. However, the condition $\mathbf{z}(y) = z \circ j_y^r(\mathbf{g}_0, \mathbf{t}_j = 0) = 0$ did not play a significant role. Thus, the entire proof would work without any changes had we chosen different background fields $\mathbf{t}_j$ such that still $q \circ j_y^r(\mathbf{g}_0, \mathbf{t}_j) = 0$, but $\mathbf{z}(y) = z \circ j_y^r(\mathbf{g}_0, \mathbf{t}_j)$ assuming an arbitrary value. Then, having a priori fixed $r = r_k$, the functions F_k on different Z_1^r domains would necessarily agree on overlaps (since they are merely local expressions of the globally defined differential operator C_k) and the union of all the Z_1^r domains would cover arbitrary values of the z-coordinates. Thus, having already performed the extension of the domain into the q-coordinates, we can set $W^4 = \mathbb{R}^{m_q}$ in Z_4^r. In other words we can set

$$Z_4^r = Z_4^{r_k} = \pi^{-1}(U) \tag{4.18}$$

for some open neighborhood $U \subset M$ of $y \in M$, where $\pi\colon J^{r_k}BM \to M$.

The union of all those open sets U, when y varies in M, completely covers M. Thus, the corresponding domains Z_4^r completely cover $J^{r_k}BM$. Thus, the globally defined differential operator C_k is of globally bounded order at most r_k and its components $C_k^Q\colon J^{r_k}BM \to \bigotimes_{i=1}^N S^{q_i}T^{*\otimes k_i}M$ have the form (4.13) when restricted to a domain of the form $Z_4^{r_k}$ with the functions F_k^Q satisfying (4.17).

5. $GL(n)$**-equivariance.** In this last point, we intend to give a precise form of the function F_k^Q exploiting their $GL(n)$-equivariance. From the previous discussion we know that the function F_k^Q satisfying (4.13), is defined on the domain $Z_4^{r_k} \simeq U \times \mathbb{R}^\gamma \times Z_4$, but it *depends* only on the coordinates corresponding to the factor $Z_4 = L_n \times \mathbb{R}^{m_z} \times \mathbb{R}^{m_q}$. We also know the following:

1. the dependence is ***polynomial*** with respect to the standard coordinates on the $\mathbb{R}^{m_q}$ factor;
2. the coefficients $g^{-\frac{\langle Q, \mathbf{d}_A\rangle}{2n}} G_{0,I}(g^{-\frac{1}{n}} g_{ab}, z^j)$ of these polynomials depend only on $L_n \times \mathbb{R}^{m_z}$.

Each factor in Z_4 carries a tensor density representation of $GL(n)$ (resp. $GL^+(n)$) arising from the action of the subgroup of $\mathrm{Diff}(M)$ (resp. $\mathrm{Diff}^+(M)$) which leaves fixed a given point of U. More precisely, if $u \in GL(n)$:

1. on L_n the action is given by $(u, \mathbf{g}) \mapsto |\det u|^{-\frac{2}{n}}\, u^{\otimes 2}\mathbf{g}$;
2. on $\mathbb{R}^{m_z}$, which corresponds to the rescaled components $g^{\frac{l_j}{n}+\frac{s_j}{2n}}(\bar{t}_j)^{a_1\cdots a_{l_j}}$, $j = 1, \dots, K_0$, of the marginal background tensor fields $\mathbf{z} = (\mathbf{t}_1, \dots \mathbf{t}_{K_0})$ (Definition 4.2), the action is given by $(u, \mathbf{t}_j) \mapsto |\det u|^{\frac{l_j}{n}+\frac{s_j}{2n}}\, u^{\otimes l_j}\mathbf{t}_j$;
3. on $\mathbb{R}^{m_q}$, which corresponds to the components $g^{\frac{l_j}{n}+\frac{s_j}{2n}+\frac{1}{n}|A|}(\bar{t}_j)^{a_1\cdots a_{l_j},A}$, for $l_j + s_j + 2|A| > 0$, and $g^{\frac{3}{n}+\frac{1}{n}|A|}\bar{S}^{ab(cd,A)}$, for $|A| \geq 0$, also decomposes into a direct sum of corresponding tensor density representations
$$\mathbb{R}^{m_q} = \bigoplus_\alpha \mathcal{R}_\alpha,$$
where $\mathcal{R}_\alpha$ carries a tensor density representation of rank n_α;
4. the fibers of the bundle where the functions $F_k^Q\colon Z_4 \to \bigotimes_{i=1}^N S^{q_i}T^{*\otimes k_i}M$ take their values also carry a representation of $GL(n)$ (resp. $GL^+(n)$), which

obviously decomposes into a direct sum of tensor density representations, which we will denote by

$$\mathcal{T} = \bigoplus_{\beta} \mathcal{T}_\beta,$$

where $\mathcal{T}_\beta$ has rank n_β.

Note also that, the homogeneous polynomials $\mathcal{P}^\delta$ of degree δ on $\mathbb{R}^{m_q}$ carry the representation

$$(uP)(\rho) := P\left(u^{-1}\rho\right), \quad \text{for any } u \in GL(n),\ P \in \mathcal{P}^\delta,\ \rho \in \mathbb{R}^{m_q}.$$

This representation on polynomials is made up of direct sums of symmetric tensor powers of $\mathbb{R}^{m_q}$ and hence itself also decomposes into a direct sum of tensor density representations

$$\mathcal{P}^\delta = \bigoplus_{\gamma} \mathcal{S}^\delta_\gamma,$$

where $\mathcal{S}^\delta_\gamma$ has rank n^δ_γ.

From the above remarks, it is easy to see that the equivariance of the functions F^Q_k (see Proposition 3.7 for the relation between invariant and equivariant functions) and the linear independence of the monomials q^I on $\mathbb{R}^{m_q}$ implies that the polynomial coefficients in (4.17) are themselves smooth equivariant maps

$$G_0 \colon L_n \times \mathbb{R}^{m_z} \to \mathcal{T} \otimes \mathcal{P}^{\langle Q,\mathbf{k}\rangle} = \bigoplus_{\beta,\gamma} \mathcal{T}_\beta \otimes \mathcal{S}^{\langle Q,\mathbf{k}\rangle}_\gamma. \tag{4.19}$$

Indeed, in the third part of the proof we have reduced the function F^Q_k in the form (4.17): The coefficients G_0, evaluated at $(g^{-\frac{1}{n}} g_{ab}, z^j) \in L_n \times \mathbb{R}^{m_z}$, give a polynomial which is then evaluated at $q \in \mathbb{R}^{m_q}$. But, in view of the above remarks, we have

$$g^{-\frac{kd_A}{2n}} G_0\left(g^{-\frac{1}{n}} g_{ab}, z^j\right) = \sum_{\beta,\gamma} \widetilde{G}_{\beta,\gamma}(g_{ab}, z^j),$$

$$\widetilde{G}_{\beta,\gamma} \colon L_n \times \mathbb{R}^{m_z} \to \mathcal{T}_\beta \otimes \mathcal{S}^{\langle Q,\mathbf{k}\rangle}_\gamma$$

and thus we can write

$$F^Q_k\left(g^{-\frac{1}{n}} g_{ab}, z^j, q^i\right) = \sum_{\beta,\gamma} \widetilde{G}_{\beta,\gamma}(g_{ab}, z^j)(q^i).$$

Since the function F^Q_k is equivariant under the action of $GL(n)$ the maps $\widetilde{G}_{\beta,\gamma}$ must be equivariant. Indeed, for any $(g_{ab}, z^j, q^i) \in Z_4$ we have

$$\begin{aligned}
F^Q_k(ug_{ab}, uz^j, uq^i) &= \sum_{\beta,\gamma} \widetilde{G}_{\beta,\gamma}(ug_{ab}, uz^j)(uq^i) \\
&= \sum_{\beta,\gamma} \left(u^{\otimes n_\beta} \otimes u^{\otimes n_\gamma^{\langle Q,\mathbf{k}\rangle}}\right) \widetilde{G}_{\beta,\gamma}(g_{ab}, z^j)(uq^i) \\
&= \sum_{\beta,\gamma} u^{\otimes n_\beta} \widetilde{G}_{\beta,\gamma}(g_{ab}, z^j)(u^{-1}uq^i) \\
&= \left(\bigoplus_\beta u^{\otimes n_\beta}\right) F^Q_k(g_{ab}, z^j, q^i) \\
&= u^{\otimes\langle Q,\mathbf{k}\rangle} F^Q_k(g_{ab}, z^j, q^i)
\end{aligned}$$

where we have denoted symbolically with u the action of $GL(n)$ on the arguments of F^Q_k.

Now, since the components $\widetilde{G}_{\beta,\gamma}$ are ***equivariant tensor densities*** (Definition 3.8), we can invoke the classification Lemma 3.14 to conclude that each $\widetilde{G}_{\beta,\gamma}$ is, up to an overall power of $g = |\det g_{ab}|$, a tensor of appropriate rank built covariantly out of g_{ab}, g^{ab}, $\varepsilon^{a_1\cdots a_n}(g)$ and the tensor components of **z**, $t_j^{a_1\cdots a_{l_j}}$, for $j = 1, \ldots, K_0$. To be more precise, each $\widetilde{G}_{\beta,\gamma}$ is a finite linear combination of $\mathcal{T}_\beta \otimes \mathcal{S}_\gamma^{\langle Q,\mathbf{k}\rangle}$ terms, each built from a tensor product of finitely many aforementioned ingredients (possibly repeating) followed by any number of index contractions or permutations, with coefficients being *smooth* functions of all possible polynomial scalar invariants covariantly constructed from the same ingredients,

$$\begin{aligned}
\widetilde{G}_{\beta,\gamma} = g^{\alpha_{\beta,\gamma}} \sum_m c^m_{\beta,\gamma}(g_{ab}, \varepsilon^{a_1\cdots a_n}(g), \ldots, t_j^{a_1\cdots a_{l_j}}, \ldots) \\
P^m_{\beta,\gamma}(g_{ab}, \varepsilon^{a_1\cdots a_n}(g), \ldots, t_j^{a_1\cdots a_{l_j}}, \ldots)\,.
\end{aligned}$$

Lemma 3.14 also tells us that, in each case, there are only finitely many algebraically independent polynomial scalar invariants that the coefficients $c^m_{\beta,\gamma}$ can depend on and there are only finitely many tensor valued polynomials $P^m_{\beta,\gamma}$ that are linearly independent up to a redefinition of the $c^m_{\beta,\gamma}$ coefficients. The dependence on $\varepsilon(g)$ is allowed only in the $GL(n)^+$ case. Also, note that for the contractions $G_{0,I}q^I$ to remain equivariant, all the explicit appearances of powers of $g = |\det g_{ab}|$ must cancel.

Finally, combining the above conclusions with (4.17), we can say that

$$F_k^Q = F_k^Q \Big(g^{ab}, g_{ab}, \epsilon^{a_1\cdots a_n}(g), \ldots, (z_j)_{a_1\cdots a_{l_j}}, \ldots ; g^{ab}, \\ \epsilon^{a_1\cdots a_n}(g), \ldots, S_{ab(cd,A)}, \ldots, (t_j)_{a_1\cdots a_{l_j},A}, \ldots \Big) \ , \quad (4.20)$$

where the dependence on the second group of arguments is purely polynomial, while the dependence on first group of arguments is smooth with respect to finite set of algebraically independent scalar polynomial invariants that can be formed from them by tensor products and contractions. Recall that we have used the notation $\mathbf{z}_j = \mathbf{t}_j$, for $j = 1, \ldots, K_0$, that is for those background tensor fields such that are marginal, satisfying $s_j + l_j = 0$. This completes the proof. □

After the proof of this very general model, we can move on to some more physically relevant models.

4.2.1 Vector Klein-Gordon field

We now consider a specific quantum vector field in order to investigate in detail the form of the coefficients C_k in (4.8): We focus on the **vector Klein-Gordon field**. The classical configurations of the vector KG field over an oriented globally hyperbolic spacetime $(M, \mathbf{g})$ are smooth 1-forms, *i.e.*, sections of the cotangent bundle T^*M, namely $A \in \mathscr{E}(T^*M)$. The vector KG equation, where we include also a coupling term with the curvature R, reads

$$-\Box_{\mathbf{g}} A + m^2 A + \xi R A = 0 \quad (4.21)$$

where m^2 and ξ are here smooth real-valued functions on M (they can be constant functions, but in general we admit that m^2 and ξ can vary on the spacetime). When passing to the quantum formulation, the **locally-covariant quantum vector KG field**, indicated by the same symbol A, is defined as in Definition 2.10 with $k = 1$ and $VM = T^*M$. Moreover we have the following requirements.

(a) The net of local quantum observables $\mathcal{W}$ including the vector KG field is as in Definition 2.8 is fixed according to equation (4.21), which suggests that the natural bundle of background fields is the one completely defined by

$$BM = \mathring{S}^2 T^*M \oplus \mathbb{R} \oplus \mathbb{R}, \quad (4.22)$$

so that the sections $M \to BM$ are triples $\mathbf{b} = (\mathbf{g}, m^2, \xi)$. (The metric $\mathbf{g}$ affects the theory because it enters $\Box_{\mathbf{g}}$, R (also derived) and even the Levi-Civita tensor ϵ in case one deals with the category of background geometries $\mathfrak{BkgG}^+$ instead of $\mathfrak{BkgG}$).

(b) The natural vector bundle is completely fixed by requiring

$$VM = T^*M$$

and the morphism V_χ, whose associated pushforward on test sections is exploited to define the notion of local covariance as in Definition 2.10, is nothing but the natural lift of the embeddings $\chi\colon M \to M'$ to the corresponding tangent bundles.

(c) To fix the physical scaling of background fields, *i.e.* to fix d_A, s_{m^2}, s_ξ such that

$$A_a \mapsto \lambda^{d_A} A_a, \quad m^2 \mapsto \lambda^{s_{m^2}} m^2, \quad \xi \mapsto \lambda^{s_\xi} \xi$$

under physical scaling, we consider the lagrangian density of vector KG field, *i.e.*

$$\mathscr{L}_A = \frac{1}{2} \left(\nabla_a A_b - \nabla_b A_a\right) \left(\nabla^a A^b - \nabla^b A^a\right) + \left(m^2 + \xi R\right) A_b A^b.$$

where we supposed $\hbar = c = 1$ and we require that the action of vector KG field is a physically dimensionless quantity, which mean that it must be invariant under physical scaling. Recalling that, under physical scaling, $\mathrm{d}g \mapsto \lambda^{-n} \mathrm{d}g$ where n is the spacetime dimension, we obtain the following system of equation

$$\begin{cases} 2d_A + 4 - n = 0 \\ 2d_A + s_{m^2} + 2 - n = 0 \\ 2d_A + s_\xi + 4 - n = 0 \end{cases} \implies \begin{cases} d_A = \frac{n-4}{2} \\ s_{m^2} = 2 \\ s_\xi = 0 \end{cases}$$

which means that

$$A_a \mapsto \lambda^{\frac{n-4}{2}} A_a$$

when $\mathbf{g} \mapsto \lambda^{-2}\mathbf{g}$, $m^2 \mapsto \lambda^2 m^2$ and $\xi \mapsto \xi$ according to (1.6). We recall that the presence of covariant derivatives do not change this rescaling behaviour as the coordinates are dimensionless.

(d) We stress that all background fields of this model are scalars of non-negative physical scaling weight and hence are admissible according to Definition 4.2.

Remark 4.5.

(1) The quantum vector KG field, in addition to the requirements in Definition 2.10, it is also supposed to verify (4.21) in a *distributional sense* for every background geometry

$$A_{(M,\mathbf{b})}\left((-\Box_{\mathbf{g}} + m^2 + \xi R)f\right) = 0 \quad \forall f \in \mathscr{D}(T^*M)\,. \tag{4.23}$$

Though this fact does not play any role in our work, it implies several relevant facts which are mentioned in the some of subsequent remarks. Moreover, exactly as does the Klein-Gordon equation for the scalar field, this equation of motion plays a crucial role in the construction of an explicit algebra of Wick polynomials [HW01].

(2) It is well-known [BD15, Sec.3.3.1] that the KG operator $P = -\Box + m^2 + \xi R$ is *Green hyperbolic* (Definition 1.10) for every choices of the involved given smooth functions (m^2 may attain non-positive values in particular) and thus the retarded and advanced Green operators of P exist. In particular the function $\Delta_{(M,\mathbf{b})}$ discussed in (4) Remark 4.2 in this case is the *causal propagator* of the KG equation [BD15, Sec.3.3.1]. As a consequence of the standard properties of the causal propagator, we also have that $[A_{(M,\mathbf{b})}(f), A_{(M,\mathbf{b})}(g)] = 0$ when the supports of f and g are causally disjoint.

(3) As is well-known, exactly as for the scalar field (*e.g.*, see [BD15, Sec.3.3.1]), the statement of the time-slice axiom for the locally covariant vector field A can be sharpened, based on the properties of the causal propagator of equation of motion (4.23). Namely, if O is an open neighborhood of any Cauchy surface of $(M, \mathbf{g})$ and $f \in \mathscr{D}(T^*M)$, then $A_{(M,\mathbf{b})}(f) = A_{(M,\mathbf{b})}(h)$ for a suitable $h \in \mathscr{D}(T^*M)$, depending on f, whose support is contained in O.

(4) When defining the Wick products $A^k_{(M,\mathbf{b})}(f)$, the class of states $\mathcal{S}_{(M,\mathbf{b})}$ appearing in the smoothness requirement in Definition 4.1 should be naturally interpreted as consisting of the extensions of *Hadamard states* [SV01] from the unital $*$-subalgebra $\mathcal{W}_{0,(M,\mathbf{b})} \subset \mathcal{W}_{(M,\mathbf{b})}$ to the whole ambient algebra, where $\mathcal{W}_{0(M,\mathbf{b})}$ is generated by 1 and products of elements $A_{(M,\mathbf{b})}(f)$.

Remark 4.6. We would like to stress why we do not analyse in detail the most obvious example of vector field, namely the **Proca field**. It is possible to show that the propagator of Proca equation is *not* continuous in the limit $m^2 \to 0$ [SS17, Cor. 3.3]. The propagator is used to construct the map τ_s of Axiom 5 (Definition 4.1), where we require smooth dependence on background fields for *arbitrary* compactly supported variations (*i.e.* the mass can vanish): The construction can be done with the same techniques used in [Wal94, Chapter 3] for KG field. The zero-mass limit problem of the propagator is an evidence that the Proca field case is delicate and it deserves particular attention: For this reason we preferred not to include it in this thesis and we decided to study it in a dedicated future work.

Remark 4.7. It is worth stressing that the case of **vanishing mass** $m = 0$, even if the spacetime is Minkowski one, does *not* correspond to the quantization of the electromagnetic field (within Lorenz-gauge choice). Indeed, we are dealing here with the algebraic approach and, in a given spacetime, the (Weyl) $*$-algebra of vector KG field is well defined for every choice of the function m^2 which may also attain negative values, because its definition only relies on the fact that the spacetime is globally hyperbolic and on the nature of the operator $P = -\Box + m^2 + \xi R$ which is Green hyperbolic. The existence of Hadamard states playing a role in *requirement 5* can be proved with a standard deformation argument even in Minkowski spacetime for $m^2 \leq 0$ constantly: It is enough smoothly change the function m^2 in the past of a Cauchy surface Σ until it becomes a constant function with value $m_0^2 > 0$ in the past a second Cauchy surface Σ' in the past of Σ in Minkowski spacetime. Next, in the past of Σ' one may construct the standard Poincaré-invariant vacuum for (constant) squared mass $m_0^2 > 0$ and spin-1 particles. This state can be viewed as a state over the algebra in the future of Σ when taking advantage of time slice axiom and it remains Hadamard in view of the known singularity propagation property of Hadamard states. Obviously, for the algebra of fields in the future of Σ, the constructed state is *not* the Poincaré-invariant vacuum which cannot be defined if $m^2 < 0$ (constantly) and the problem with negative-norm states would immediately arise for $m^2 = 0$ (usually removed by means of the Gupta-Bleuler treatment which also lower to 2 the physical degrees of freedom of particles associated to the field from the 3 degrees of freedom of massive spin-1 particles). This way also the $m^2 \leq 0$ theory in Minkowski spacetime admits Hadamard states, but none of them is a Poincaré-invariant vacuum. In other words, for $m^2 = 0$, our vector KG field field does *not* describe photons. In the algebraic approach, photons are described by including gauge invariance into the algebra of fields from scratch which is a more complicated procedure than the one we are discussing [Hol08, FR12]. Using some delicate adiabatic changes of mass procedures similar to the ones pointed out above it is however possible, at least for the scalar field, transform vacua states into vacua states with different masses [DHP17, DD16, DG17].

Remark 4.8. It is worth commenting on the **existence** of prescriptions of Wick polynomials that are smooth in m^2, including at $m^2 = 0$. For that, it is important to recall the precise form of the smoothness axiom (Definition 4.1, Axiom 5) and that the main candidate for such a construction is point splitting regularized with a Hadamard parametrix. That is, in the simplest $k = 2$ case, what we must check is the joint smoothness of the integral kernel $\omega_{ab}(s,x)$ in the expression

$$\lim_{y\to x} \omega \circ \tau_s^{-1} \left(A_{a\,(M,\mathbf{b}_s)}(x) A_{b\,(M,\mathbf{b}_s)}(y) - H_{ab\,(M,\mathbf{b}_s)}(x,y) \right) = \omega_{ab}(s,x), \quad (4.24)$$

where $H_{ab\,(M,\mathbf{b}_s)}(x,y)$ is the Hadamard parametrix and ω is any Hadamard state on the algebra $\mathcal{W}(M,\mathbf{b}_0)$, with $\mathbf{b}_0 = (\mathbf{g}_0, m^2 = m_0^2, \xi = \xi_0)$ and $\mathbf{b}_s$ a compactly supported variation thereof. It is well-known that, already on (even dimensional) Minkowski space with $m^2 = m_0^2$ constant, the Hadamard parametrix contains terms proportional to $\log(\mu^2\sigma(x,y))$, where $\sigma(x,y)$ is the squared geodesic distance and μ^2 is an arbitrary dimensionful constant. On the other hand, the Wightman 2-point function $\omega_{m^2}(A_a(x)A_b(y))$, where ω_{m^2} is the Fock vacuum, also contains terms proportional to $\log(m^2\sigma(x,y))$. Thus, we expect the point split regularization

$$\lim_{y\to x} \omega_{m_0^2}(A_a(x)A_b(y) - H_{ab}(x,y)) \quad (4.25)$$

to exist, because of the cancellation of singular $\sigma(x,y)$-dependent terms, and to contain terms proportional to $\log m_0^2/\mu^2$. Thus, at first glance, it might seem that the desired smoothness property in (4.24) would not hold because of a logarithmic singularity encountered as m^2 varies from m_0^2 to 0 as a function of s. However, a careful comparison of (4.24) and (4.25) reveals that they are not analogous expressions. In particular, one can never represent the family ω_{m^2} of Fock vacua as $\omega_{m_0^2} \circ \tau_s^{-1}$ for some fixed constant m_0^2 and an s-dependent compactly supported variation thereof, because the difference $m^2 - m_0^2$ would not be compactly supported. In reality, with ω fixed in (4.24), the difference between $A_{a\,(M,\mathbf{b}_0)}(x)A_{b\,(M,\mathbf{b}_0)}(y)$ and $\tau_s^{-1}(A_{a\,(M,\mathbf{b}_s)}(x)A_{b\,(M,\mathbf{b}_s)}(y))$ can be expressed using advanced and retarded propagators for the vector KG operators on $(M,\mathbf{b}_0)$ and $(M,\mathbf{b}_s)$, with the latter depending smoothly on the difference $m^2(s,x) - m_0^2$, as we argue briefly in the following paragraph. The conclusion of the above discussion is casually implied in the discussion of Section 5.2 of [HW02].

To argue that the retarded propagator with mass $m^2(s,x)$ has smooth dependence on the difference $m^2(s,x) - m_0^2$, when it has compact support, we will refer to some results from [DHP17]. More precisely, we can express the retarded propagator $\Delta^R_{m^2}$ in terms of the retarded propagator $\Delta^R_{m_0^2}$ and an operator $R_{m_0^2}$ (Lemma 3.10 in [DHP17]), where $R_{m_0^2} = [1 + \Delta^R_{m_0^2}(m^2 - m_0^2)]^{-1}$ (Proposition 3.8 in [DHP17]). This comes down to the perturbative expression, cf. Equation (43) in [DHP17],

$$\Delta^R_{m^2} = \sum_{n\geq 0} \left[-\Delta^R_{m_0^2} \left(m^2 - m_0^2 \right) \right]^n \Delta^R_{m_0^2} \, .$$

Lemma B.1 of [DHP17] uses the support properties of $\Delta^R_{m_0^2}$ to show that the above series, together with all of its functional derivatives with respect to the difference $m^2 - m_0^2$, converges when $m_0^2 = 0$ and the background spacetime is Minkowski. It is plausible that the same proof generalizes to more general globally hyperbolic spacetimes, and even when $m^2 - m_0^2$ is not compactly supported.

Explicit form of counterterms. With the concrete case of the vector KG field, Theorem 4.1 can be sharpened to give a more explicit expression for the renormalization coefficients C_k. In terms of algebra valued distributions, equation (4.8) can be rewritten as

$$\widetilde{A_{b_1} \cdots A_{b_k}}(x) = A_{b_1} \cdots A_{b_k}(x) + \sum_{l=0}^{k-1} \binom{k}{l} C_{k-l}[M, \mathbf{b}]_{(b_1 \cdots b_{k-l}}(x) A_{b_1} \cdots A_{b_l)}(x) \tag{4.26}$$

with $C_k[M, \mathbf{b}]_{b_1 \cdots b_k}(x) \in (T^*_x)^{\otimes k} M$ fully symmetric. Using Theorem 4.3 we can immediately obtain a precise form of the symmetric covariant k-tensor fields $C_k[M, \mathbf{b}]$. For example, if we choose $n = 4$ and $k = 2$ we obtain for all $f \in \mathscr{D}(S^2 T^* M)$

$$\widetilde{A}^2_{(M,\mathbf{b})}(f)$$
$$=$$
$$A^2_{(M,\mathbf{b})}(f)$$
$$+$$
$$1 \left(\left(y_1 m^2 \mathbf{g} + y_2 R \mathbf{g} + y_3 \mathbf{Ric} + y_4 \Box \xi \mathbf{g} + y_5 \nabla^2 \xi + y_6 \mathbf{g} (\nabla \xi)^2 + y_7 (\nabla \xi)^{\odot 2} \right) \cdot_2 f \right)$$

which can be written in terms of distributional fields, omitting for simplicity explicit x-dependence, as

$$\begin{aligned} \widetilde{A_a A_b} = A_a A_b + \big(& y_1 g_{ab} m^2 + y_2 g_{ab} R + y_3 R_{ab} \\ & + y_4 g_{ab} \Box \xi + y_5 \nabla_{(a} \nabla_{b)} \xi + y_6 g_{ab} \nabla^c \xi \nabla_c \xi + y_7 \nabla_{(a} \xi \nabla_{b)} \xi \big) \, , \end{aligned}$$

where $y_j(x) := Y_j(\xi(x))$ and Y_j, for $j = 1, \ldots, 7$, are dimensionless smooth functions which do not depend on the chosen spacetime. Obviously, in concrete physical theories the final values of some background fields like m^2 and ξ are taken to be everywhere constant. In this case all derivatives of these fields disappear. In particular

$$\widetilde{A_a A_b} = A_a A_b + \left(y_1 g_{ab} m^2 + y_2 g_{ab} R + y_3 R_{ab}\right) \ ,$$

where the $y_j := Y_j(\xi)$ turn out to be true *renormalization constants* independent form the chosen spacetime.

4.2.2 Vector Klein-Gordon field with tensor curvature coupling

It is possible to complicate a bit the previous example by adding a non-trivial background field. We consider a tensorial coupling to the scalar curvature in the vector KG equation, *i.e.*,

$$-\Box_{\mathbf{g}} A_a + m^2 A_a + R\xi_a^b A_b = 0. \tag{4.27}$$

Lowering the upper index of the coupling tensor, $\xi_{ab} = g_{ac}\xi_b^c$, we have a fully covariant background 2-tensor field. We will take ξ_{ab} to be symmetric, both for simplicity and because only symmetric tensorial coefficients are compatible with the existence of a Lagrangian density for (4.27). Then, the bundle of background field is now completely defined by

$$BM = \mathring{S}^2 T^* M \oplus \mathbb{R} \oplus S^2 T^* M \tag{4.28}$$

and the sections $M \to BM$ are triples $\mathbf{b} = (\mathbf{g}, m^2, \boldsymbol{\xi})$. The background field $\boldsymbol{\xi}$ is marginal[5] since the tensor index $l_M = 2$ and the physical scaling weight $s_M = -2$, hence satisfying $l_M + s_M = 0$. Clearly, ξ_{ab} is the only marginal background field. All other hypotheses remain invariant with respect to the previous example.

To apply our main Theorem 4.3, we first need to analyze the structure of the scalar polynomial invariants on the fibers of S^2T^*M under the action of $O(1, n-1)$ (or $SO(1, n-1)$) and the separability of closed orbits by these invariants. As is well known [Pro07, Sec.11.8], a generating set of the polynomial invariants is given by the contractions

$$\left(\operatorname{tr}\boldsymbol{\xi} = \xi_a^a, \operatorname{tr}\boldsymbol{\xi}^2 = \xi_a^b \xi_b^a, \ldots, \operatorname{tr}\boldsymbol{\xi}^n = \xi_{a_1}^{a_2}\xi_{a_2}^{a_3}\cdots\xi_{a_n}^{a_1}\right) \ , \tag{4.29}$$

[5] In the Lagrangian density, the curvature coupling term becomes $Rg^{ad}g^{bc}\xi_{ac}A_b A_d\sqrt{-g}$.

which, as indicated, can be interpreted as traces of successive powers of ξ_a^b, interpreted as n-dimensional endomorphisms (or $n \times n$ matrices). All higher order contractions are algebraically dependent due to the Cayley-Hamilton identity. The result obtained in Theorem 4.3 applied to this case when, for example, we choose $n = 4$ and $k = 2$ gives, omitting the x-dependence for simplicity,

$$\widetilde{A_a A_b} = A_a A_b + \left(y_1 g_{ab} m^2 + y_2 g_{ab} R + y_3 R_{ab} + y_4 m^2 \xi_{ab} + y_5 \xi_{ab} R + B_\xi\right) ,$$

with all terms that vanish when the background fields are constant collected in

$$\begin{aligned} B_\xi = {} & y_6 g_{ab} \Box \xi_c^c + y_7 \nabla_{(a} \nabla_{b)} \xi_c^c + y_8 g_{ab} \nabla^c \xi_d^d \nabla_c \xi_d^d + y_9 g_{cd} \nabla_{(a} \xi^{cd} \nabla_{b)} \xi_c^c \\ & + y_{10} \left(\nabla_{(a} \nabla_{b)} \xi_{cd}\right) \xi^{cd} + y_{11} \nabla_{(a} \xi^{cd} \nabla_{b)} \xi_{cd} + y_{12} g_{ab} \left(\Box \xi_{cd}\right) \xi^{cd} \\ & + y_{13} g_{ab} \nabla^c \xi_{de} \nabla_c \xi^{de} + y_{14} \xi_{ab} \Box \xi_c^c + y_{15} \xi_{ab} \nabla^c \xi_d^d \nabla_c \xi_d^d + y_{16} \Box \xi_{ab} \\ & + y_{17} \xi_{ab} \left(\Box \xi_{cd}\right) \xi^{cd} + y_{18} \xi_{ab} \nabla^c \xi_{de} \nabla_c \xi^{de} + y_{19} \xi_{cd} \nabla_{(a} \xi^{cd} \nabla_{b)} \xi_c^c \\ & + y_{20} \xi_{cd} \xi_{ef} \nabla_{(a} \xi^{ef} \nabla_{b)} \xi^{cd} , \end{aligned}$$

where the y_i are locally smooth functions of our invariant scalars (4.29) in the sense of Definition 3.11 and Proposition 3.13.

Now, we analyse in detail the structure of the coefficients y_i. In general, illustrating the phenomenon discussed in Section 3.3, our invariant polynomials do not separate the closed orbits of $O(1, n-1)$ (or $SO(1, n-1)$) acting on the fibers of $S^2 T^* M$. For instance, given an orthonormal basis $v_a^0, \dots, v_a^3$ with $\mathbf{v}^0$ timelike and the rest spacelike, the following symmetric tensors with distinct $\lambda_0, \dots, \lambda_4$ cannot be distinguished by invariant polynomials

$$\begin{aligned} \boldsymbol{\xi} &= -\lambda_0 v_a^0 v_b^0 + \lambda_1 v_a^1 v_b^1 + \lambda_2 v_a^2 v_b^2 + \lambda_3 v_a^3 v_b^3 \\ \boldsymbol{\xi}' &= -\lambda_1 v_a^0 v_b^0 + \lambda_0 v_a^1 v_b^1 + \lambda_2 v_a^2 v_b^2 + \lambda_3 v_a^3 v_b^3 \end{aligned}$$

even though they belong to different orbits. The orbits are distinct because any linear transformation mapping $\boldsymbol{\xi}$ to $\boldsymbol{\xi}'$ must exchange the λ_0- and λ_1-eigenvectors, hence exchanging a spacelike vector with a timelike vector, which cannot be done by any element of $O(1, n-1)$. Other examples of this kind can be constructed by looking at the complete classification of the orbit types of symmetric 2-tensors [SKM+03, Sec.5.1]. On the other hand, invariant polynomials do distinguish the orbit of $\boldsymbol{\xi}$ from the orbit of any other point in a sufficiently small neighborhood, because the case of distinct eigenvalues allows us to choose the eigenvectors smoothly under small variations, and small variations of timelike (spacelike) vectors remain timelike (spacelike). Thus, the subsets where invariant polynomials can locally distinguish orbits must be separated by a "barrier" (the Z^0 subset of Proposition 3.13). Since any continuous path from $\boldsymbol{\xi}$ to $\boldsymbol{\xi}'$ must pass through some tensor with ***degenerate*** eigenvalues, we can take Z^0 to consist of all tensors with at least two equal eigenvalues. The open sets Z_j of Proposition 3.13 can then be identified with the connected components of $Z \setminus Z_0$, where Z is a generic fiber of S^2T^*M.

The reason why the set $Z^0 \subset Z$ and the partition $Z \setminus Z^0 = \bigcup_j Z_j$ is consistent with Proposition 3.13 is that Z^0 is actually the zero-set of an invariant polynomial $p_0(\boldsymbol{\xi}) = \mathrm{disc}(\boldsymbol{\xi})$, known as the **matrix discriminant**. It is defined by requiring that, for diagonalizable tensors with eigenvalues λ_i, it takes the value

$$\mathrm{disc}\,(\boldsymbol{\xi}) = \prod_{i<j} (\lambda_i - \lambda_j)^2 \ ,$$

which can be shown to coincide with the polynomial $\mathrm{disc}\,(\boldsymbol{\xi}) = \det\left(\mathrm{tr}\,\boldsymbol{\xi}^{i+j-2}\right)_{i,j=1}^{n}$ [Par02, Lem.1]. In the $n = 4$ case, it has the explicit form

$$\mathrm{disc}\,(\boldsymbol{\xi}) = \det \begin{pmatrix} \mathrm{tr}\, I & \mathrm{tr}\,\boldsymbol{\xi} & \mathrm{tr}\,\boldsymbol{\xi}^2 & \mathrm{tr}\,\boldsymbol{\xi}^3 \\ \mathrm{tr}\,\boldsymbol{\xi} & \mathrm{tr}\,\boldsymbol{\xi}^2 & \mathrm{tr}\,\boldsymbol{\xi}^3 & \mathrm{tr}\,\boldsymbol{\xi}^4 \\ \mathrm{tr}\,\boldsymbol{\xi}^2 & \mathrm{tr}\,\boldsymbol{\xi}^3 & \mathrm{tr}\,\boldsymbol{\xi}^4 & \mathrm{tr}\,\boldsymbol{\xi}^5 \\ \mathrm{tr}\,\boldsymbol{\xi}^3 & \mathrm{tr}\,\boldsymbol{\xi}^4 & \mathrm{tr}\,\boldsymbol{\xi}^5 & \mathrm{tr}\,\boldsymbol{\xi}^6 \end{pmatrix} ,$$

where we recall that $\mathrm{tr}\,\boldsymbol{\xi}^5, \mathrm{tr}\,\boldsymbol{\xi}^6$ are algebraically dependent on lower order contractions due to the Cayley-Hamilton identity. Thus, the coefficients y_i are locally smooth functions (Definition 3.11) of the scalar polynomials invariants (4.29), ***i.e.***,

$$y_i(x) = [Y_i]_{S^2T^*M}(\mathrm{tr}\,\boldsymbol{\xi}(x), \mathrm{tr}\,\boldsymbol{\xi}^2(x), \mathrm{tr}\,\boldsymbol{\xi}^3(x), \mathrm{tr}\,\boldsymbol{\xi}^4(x)) \ ,$$

for $i = 1, \ldots, 20$, with respect to the partition $Z \setminus Z^0 = \bigcup_j Z_j$ indicated above, with Z a generic fiber of S^2T^*M.

4.2.3 Scalar field with derivative

We now consider the renormalization of Wick powers of a **scalar field with its first derivative**. The classical configurations of the scalar KG field over an oriented globally hyperbolic spacetime $(M, \mathbf{g})$ are smooth real-valued functions, ***i.e.***, sections of the bundle $T^{*\otimes 0}M = M \times \mathbb{R}$, namely $\varphi \in \mathscr{E}(T^{*\otimes 0}M) = C^\infty(M)$. Similarly to the previous case we have the following equation of motion

$$-\Box_{\mathbf{g}}\varphi + m^2\varphi + \xi R\varphi = 0, \tag{4.30}$$

where m^2 and ξ are smooth real-valued functions on M (they can be constant functions, but in general we admit that m^2 and ξ can vary on the spacetime). Since we want to consider renormalization of a scalar field with its first derivative, we construct the field Φ as the pair of fields

$$\Phi = (\varphi, \nabla_a \varphi) .$$

When passing to the quantum formulation, the ***locally-covariant quantum field*** Φ, denoted by the same symbol Φ, is defined as in Definition 2.10, with the following details.

(a) As in the previous case, the net of local quantum observables $\mathcal{W}$ including the scalar field, as in Definition 2.8, is fixed according to equation (4.30), which suggests that the natural bundle of background fields is the one completely defined by

$$BM = \mathring{S}^2 T^* M \oplus (M \times \mathbb{R}) \oplus (M \times \mathbb{R}), \tag{4.31}$$

so that the sections $M \to BM$ are triples $\mathbf{b} = (\mathbf{g}, m^2, \xi)$.

(b) The natural vector bundle is completely fixed by requiring

$$VM = (M \times \mathbb{R}) \oplus T^* M$$

and the morphism V_χ, whose associated pushforward on test sections is exploited to define the notion of local covariance as in Definition 2.10, is nothing but the natural lift of the embeddings $\chi \colon M \to M'$ to the corresponding tangent bundles.

(c) To fix physical scaling of background fields it is possible to proceed with the same strategy of Section 4.2.1. Therefore, according to its mass dimension[6], the physical scaling degree of the field Φ is

$$\mathbf{d}_\Phi = \left(\frac{n-2}{2}, \frac{n-2}{2} \right),$$

when $\mathbf{g} \mapsto \lambda^{-2}\mathbf{g}$, $m^2 \mapsto \lambda^2 m^2$ and $\xi \mapsto \xi$ according to (1.6). We recall that the presence of covariant derivatives do not change this rescaling behaviour as the coordinates are dimensionless.

[6] E.g., assuming that both the terms summed in the Lagrangian density of the scalar field $m^2 \varphi^2 \sqrt{g}$ and $\nabla^a \varphi \nabla_a \varphi \sqrt{g}$ are dimensionless in natural $\hbar = c = 1$ units.

(d) We stress that all background fields of this model are scalars of non-negative physical scaling weight and hence are admissible according to Definition 4.2.

For this specific model, using Theorem 4.1 and Theorem 4.3, we can immediately obtain a renormalization formula and a precise form of the renormalization counter-terms. For example, if we choose $n = 4$ and $k = 2$ we obtain, in terms algebra valued distributions and for brevity omitting the all dependence on the spacetime point x,

$$\begin{bmatrix} \widetilde{\varphi}^2 \\ \widetilde{\varphi \nabla_a \varphi} \\ \nabla_{(a}\widetilde{\varphi \nabla_{b)} \varphi} \end{bmatrix} = \begin{bmatrix} \varphi^2 \\ \varphi \nabla_a \varphi \\ \nabla_{(a}\varphi \nabla_{b)} \varphi \end{bmatrix} + \begin{bmatrix} \alpha_1 m^2 + \alpha_2 R + A_{\xi,m^2} \\ \beta_1 \nabla_a R + B_{\xi,m^2} \\ g_{ab}\left(\gamma_1 (m^2)^2 + \gamma_2 m^2 R + \gamma_3 R^2\right) \\ + \left(\gamma_4 m^2 + \gamma_5 \Box\right) R_{ab} + C_{\xi,m^2} \end{bmatrix}$$

where all α-, β-, and γ-coefficients are smooth functions of ξ and

$$\begin{aligned} A_{\xi,m^2} &= \alpha_3 \nabla^a \xi \nabla_a \xi + \alpha_4 \Box \xi \,, \\ B_{\xi,m^2} &= \beta_2 \nabla_a m^2 + \beta_3 m^2 \nabla_a \xi + \beta_4 R \nabla_a \xi + \beta_5 R_{ab} \nabla^b \xi \\ &\quad + \beta_6 (\nabla^b \xi \nabla_b \xi) \nabla_a \xi + \beta_7 \Box \xi \nabla_a \xi + \beta_8 \nabla^b \xi \nabla_{(b} \nabla_{a)} \xi + \beta_9 \nabla_a \Box \xi \,, \\ C_{\xi,m^2} &= \gamma_6 \nabla_{(a} \xi \nabla_{b)} m^2 + \gamma_7 m^2 \nabla_{(a} \xi \nabla_{b)} \xi + \gamma_8 R \nabla_a \xi \nabla_b \xi + \gamma_9 R_{ab} \nabla^c \xi \nabla_c \xi \\ &\quad + \gamma_{10} R_{c(a} \nabla_{b)} \xi \nabla^c \xi + \gamma_{11} g_{ab} \nabla^c \xi \nabla_c m^2 + \gamma_{12} g_{ab} m^2 \nabla^c \xi \nabla_c \xi \\ &\quad + \gamma_{13} g_{ab} R \nabla^c \xi \nabla_c \xi + \gamma_{14} g_{ab} R^{bc} \nabla_b \xi \nabla_c \xi + \gamma_{15} \nabla_{(a} \nabla_{b)} m^2 \\ &\quad + \gamma_{16} m^2 \nabla_{(a} \nabla_{b)} \xi + \gamma_{17} \Box \xi \nabla_{(a} \nabla_{b)} \xi + \gamma_{18} R \nabla_{(a} \nabla_{b)} \xi + \gamma_{19} R_{ab} \Box \xi \\ &\quad + \gamma_{20} g_{ab} \Box m^2 + \gamma_{21} g_{ab} m^2 \Box \xi + \gamma_{22} g_{ab} (\Box \xi)^2 + \gamma_{23} g_{ab} R \Box \xi \\ &\quad + \gamma_{24} \nabla_{(a} \xi \nabla_{b)} \Box \xi + \gamma_{25} \nabla_{(a} \nabla_{b)} \Box \xi + \gamma_{26} g_{ab} \nabla^c \xi \nabla_c \Box \xi + \gamma_{27} g_{ab} \Box^2 \xi \end{aligned}$$

are terms which depend on covariant derivatives of ξ and m^2. Thus, if we choose constant values for m^2 and ξ:

$$\begin{bmatrix} \widetilde{\varphi}^2 \\ \widetilde{\varphi \nabla_a \varphi} \\ \nabla_{(a}\widetilde{\varphi \nabla_{b)} \varphi} \end{bmatrix} = \begin{bmatrix} \varphi^2 \\ \varphi \nabla_a \varphi \\ \nabla_{(a}\varphi \nabla_{b)} \varphi \end{bmatrix}$$

$$+ \begin{bmatrix} \alpha_1 m^2 + \alpha_2 R \\ \beta_1 \nabla_a R \\ g_{ab}\left(\gamma_1 (m^2)^2 + \gamma_2 m^2 R + \gamma_3 R^2\right) + \left(\gamma_4 m^2 + \gamma_5 \Box\right) R_{ab} \end{bmatrix}$$

Also, if we wanted to maintain the Leibniz rule $\nabla_a\varphi^2 = 2\varphi\nabla_a\varphi$ (cf. [HW05]), we would have to require $2\beta_1 = \alpha_2$, with the further requirements $2\beta_2 = \alpha_1$, $2\beta_3 = \alpha_1'$, $2\beta_4 = \alpha_2'$, $2\beta_6 = \alpha_3'$, $2\beta_7 = \alpha_4'$, $2\beta_8 = 2\alpha_3$ and $2\beta_9 = \alpha_4$ (where $'$ denotes $\frac{d}{d\xi}$) for other coefficients in the case of non-constant m^2 and ξ.

Remark 4.9. Following the same ideas of this section, it is possible to renormalize a scalar field with derivatives of arbitrary order. If we construct the $(m+1)$-tuple

$$(\varphi, \nabla_{a_1}\varphi, \nabla_{a_1}\nabla_{a_2}\varphi, \dots, \nabla_{a_1}\cdots\nabla_{a_m}\varphi)\,,$$

i.e., if we choose as bundle of dynamical fields

$$VM = \bigoplus_{i=0}^{m} T^{*\otimes i}M,$$

we can use Theorem 4.1 and Theorem 4.3 as we did in this section to obtain a renormalization formula with all renormalization counter-terms. With the same idea it is possible to renormalize any tensor fields with an arbitrary number of derivatives.

Conclusions

In this thesis, we studied in detail Wick powers renormalization of Boson fields in the framework of locally covariant algebraic quantum field theory in globally hyperbolic curved spacetimes. The obtained result is very general and satisfactory, since it simultaneously encompasses a relevant number of physical cases.

We defined a general Boson field as a section of an arbitrary natural vector bundle of a spacetime (where *naturality* means the existence of a well defined transformation law of the fibers induced from diffeomorphisms of the spacetime). Besides the metric, the spacetime is also allowed to carry some arbitrary classical background fields (pictured as sections of natural vector bundles as well). In particular, the mass and other parameters have been viewed as such background fields.

We defined Wick powers axiomatically (Definition 4.1). Our list of axioms simply generalizes the axioms exploited for the scalar field in [KM16] which, in turn, descend from those introduced in [HW01] (with the crucial difference that the "analytic dependence" axiom in [HW01] was here replaced by a weaker "smooth dependence" axiom).

Our main result consists of a classification of all possible finite renormalization counterterms of Wick powers, which refer to the ambiguities in their axiomatic definition. As a matter of fact, the result provides a first rigorous and complete classification for non-scalar Boson fields. Part of our main result (Theorem 4.1), by an application of the Peetre-Slovák theorem, reduces finite renormalization terms of a k-th Wick power to a linear combination of Wick powers of lower order with coefficients C_k that result to be differential operators locally depending on the background fields of fixed physical scaling weight and transform covariantly under diffeomorphisms. The rest of our main result (Theorem 4.3) is specialized to the case where both the dynamical and background fields are spacetime tensors by an application of a generalized version of Thomas Replacement theorem (Section 3.4) and some fundamental results from smooth classical invariant theory of the orthogonal group $O(1;n-1)$ or $SO(1;n-1)$ (Section 3.3). This way, the coefficients C_k turn out to be linear combinations of finitely many tensor polynomials covariantly constructed out of the curvature, the background tensor fields, and all of their covariant derivatives. This finiteness result crucially depends on an *admissibility* criterion for all the background fields (Definition 4.2), which relates the physical scaling weight of a background field with its tensor rank by means of a certain inequality. The structure of these tensor polynomials is controlled by their physical scaling weights. It is possible that, for a given tensor type and scaling weight, the list of such polynomials is empty, meaning that the corresponding component of C_k vanishes. This is the result (a) of Theorem 4.3, which, in the simplest cases, tell us that a vector field A of tensor rank 1 with scaling degree $d_A < -1$ (or a scalar field with scaling degree less then 0) does not admit renormalization counterterms (all of them are identically zero). This result can be viewed as the counterpart for dynamical fields of the admissibility condition of background fields: Wick powers have non-vanishing renormalization counterterms only if the sum of the scaling degree of the quantum field and of its tensor rank is greater or equal to zero, *i.e.* $d_A + k \geq 0$. The case of a n-tuple of fields is a bit more complicated but it is explained by the precise statement (a) of Theorem 4.3, as we already said.

The strongest departure from the results of [KM16] is in the structure of the coefficients in front of these polynomial terms. These coefficients are actually allowed to depend smoothly (and not just polynomially as in [KM16]) on the background fields, but into a quite restricted way. In practice, they are permitted to locally be smooth functions only of a finite number of scalar polynomial invariants constructed covariantly from the subset of marginal background fields (those that saturate the admissibility inequality). The notion of local smooth dependence on these scalar invariants (cf. Definition 3.11 and Proposition 3.13) can be made precise only by looking at the structure of the orbits of the action of $O(1;n-1)$ or $SO(1;n-1)$ on the marginal background tensor fields. In the scalar Klein-Gordon case considered [KM16], the only marginal background field was the scalar curvature coupling ξ. We next illustrated our results in detail with two physically relevant examples, checking in particular that they satisfy all the admissibility hypotheses: the vector

Klein-Gordon field A_a (Section 4.2.1), possibly coupled to the curvature through a tensor background field ξ_{ab} (Section 4.2.2), and the case of Klein Gordon scalar field φ accompanied by its spacetime derivative $\nabla_a\varphi$ (Section 4.2.3).

A number of open issues remains. A theorem of existence for Wick polynomials should be established: We did not prove that there exist a prescription to define them that satisfies our axioms. This task should be accomplished using existing tools and results, since the standard Hadamard parametrix regularization method [HW02] is suitable for vector fields too, as discussed in Section 4.2.1. The main problem is to check that our "smooth dependence" axiom is actually satisfied by this method. Moreover, the constructed formalism should be so enlarged to cover the more delicate case of the Proca field. Here, the main problem is that the zero mass limit $m^2 \to 0$ is known not to be smooth (see [SS17] for a careful recent discussion), whereas one of our axioms for Wick powers requires regularity exactly at the zero value of the mass. Some related remarks about subtleties with regular mass dependence appear in Remark 4.8.

The last, and probably more important, open issue concerns the generalization of our results to more general kinds of fields: other kinds of Boson fields (for instance non-tensorial fields like connections) but also to fermionic fields (for instance Dirac spinor fields). The latter is probably the most important and urgent case: a strategy to study this case seems to be very similar to the one of this thesis. First, one needs to introduce an additional geometric structure, the spin structure, to properly describe spinors on a general curved spacetime in the framework of locally covariant AQFT (see for e.g. [San08]). Then, using an antisymmetric version of the contraction product defined in Def. 1.4, should be possible to establish a spinorial version of Theorem 4.3. After that, to generalize Theorem 4.3 to, for example, Dirac spinor fields, an intermediate step is an extension of Thomas replacement theorem to spinors. With these remarks we just intend to suggest to the reader that, on the one hand, extensions of our results to other quantum fields are possible by building on the ground work that we have already laid, on the other hand they would deserve an entire new work. A different and much more difficult extension would regard the renormalization of time ordered products (of Wick powers), relying on existing well-established results [HW01, HW02, HW05], which are however currently available only for the scalar case. Another difficult but interesting problem is the extension of our formalism to *gauge fields*: This is an almost entire new problem (for example gauge fields are described using a graded algebra) which for sure deserves attention.

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