Essays in Bargaining and Cooperative Game Theory with an Application to Environmental Negotiations

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Abstract

Even when cooperation is clearly advantageous, attaining it is not to be taken for granted. In fact, in order to undertake a mutually beneficial joint activity, the parties must agree on the division of the gains granted by it. The self-interested nature that is supposed to characterize the same parties might then become a serious obstacle to the collectively rational choice of cooperating. Bargaining and Cooperative Game Theory are the two principal frameworks that are used by economists to investigate this puzzling but fascinating problem. In particular, if the latter proposes solutions to hypothetical bargaining problems according to normative principles such as egalitarianism and marginalism, the former examine the same problem from a positive perspective focusing on the relation between the rules of the bargaining process and its outcomes.

The present Doctoral Thesis employs both these frameworks in a complementary way. Specifically, it proposes two novel solution concepts for transferable utility games in characteristic function form and a bargaining model whose outcome is exactly one of such solutions. It further compares different solution concepts with regard to their redistributive properties and their resilience to free riding.

The Doctoral Thesis is composed by four standing alone, but interlinked, works forming the four chapters in which it is divided. Chapter 1 offers a literature review of bargaining models. Chapter 2 presents the two novel solution concepts: the Central Core and the Mid-central Core. Chapter 3 proposes the Burning Coalition Bargaining Model, a non-cooperative bargaining model whose outcome, under a specific response strategy profile, is the Mid-central Core. Finally, Chapter 4 benchmarks different solution concepts through a numerical simulation based on an environmental game.

Keywords: Bargaining Theory, Cooperative Game Theory, Coalition formation, Cooperative solution concepts, Nash program, protocols, axiomatization, outside options, partial breakdown, emissions abatement, welfare distribution, equity, free riding.

J.E.L.: C71; C78; D63; Q52.

Table of contents

List of figures

List of tables

Introduction

> "*The delicate and intricate pattern of competition and cooperation in the economic behavior... offers a challenge to the economist that is perhaps as complex as the challenges of the physicist and the chemist.*"
>
> – Stigler, 1982 –

The list of real-life examples according to which a group of decision makers could achieve a better result through some kind of cooperation among each other rather than acting isolated could be made dramatically long. Actually, the same modern system of economic production could be inserted as an element of such list, since it is undoubted that it rests on a very large degree of cooperation that has allowed to significantly expand the frontier of production possibilities. According to several authors, the capability to engage in cooperative interactions with non-relatives is even a distinguishing feature of humankind [Okasha, 2013]. Nonetheless, the antithesis of cooperation, competition, is also an intrinsic characteristic of human beings. In fact, being at the hart of the mechanism of natural selection [Darwin, 1962], it is necessarily an element shared by all living beings.

If the tension between these two antithetic forces is a basic ingredient of human interactions, it becomes naturally interesting to investigate its implications. In particular, the factors and the conditions that render cooperation feasible in a world of self-interested individuals is a theme at the core of economic thinking. When cooperation implies efficiency, as in the examples of the list previously mentioned, it could be argued that simple rationality is a sufficient condition to guarantee its implementation. However, a group of decision makers having the possibility to undertake a potentially mutually advantageous project faces two problems. The first is mainly technical and consists in finding the optimal way to implement the same project. The second, instead, relates to the distribution of gains among the participants [Curiel, 2013]. This last is the subject of interest of both Bargaining and Cooperative Game Theory (CGT).

It could be argued that this topic is, more broadly, one of the fundamental issues of Game Theory (GT). This is true, but the present work is surely rooted into the two mentioned sub-fields of Game Theory. As well explained in Ray [2007], in fact, two different approaches can be taken in analyzing the problem of cooperation among multiple agents. From one side, they can group together, forming a coalition, without any binding agreement being in place. Such coalition, therefore, would be the result of a mere consensus. This situation is analyzed through dynamic games and, particularly, through repeated games [Ray, 2007]. On the other side, instead, coalitions are supposed to be sustained by binding agreements. This is the classical domain of Cooperative Game Theory. If the assumption of enforceable agreements frees from the complexity of analysing how they are implemented, still it remains intact the problem of figuring out how to reach them.

The two mentioned sub-fields of GT, Bargaining and Cooperative Game Theory, reflect the two approaches that can be utilized to tackle such problem. The latter, also named coalitional Game Theory, can be seen as a simplification of the strategic interactions that occur in the process of building cooperation [Binmore and Eguia, 2017]. This is obtained by considering as primitives coalitions instead of players. Bargaining Theory, as intended in the present work, is the non-cooperative process, having players as primitives, through which analyzing the formation of coalitions [Osborne and Rubinstein, 1994]. In other words, they are two different representations of the same game, with coalitional GT being the normal form, whereas Bargaining Theory the extensive form representation [Serrano, 2004]. From the seminal works of Nash [1950, 1953], combining the two approaches has been a rather standard technique of investigation, that goes under the name of Nash program.

It comes with no surprise that this technique has gained wide popularity and, after more than fifty years, it still constitutes an active research strand. The reason is the mentioned importance of the topic for whose investigation it is used. Furthermore, the idea of combining Bargaining and coalitional GT, or else, the cooperative and the non-cooperative approaches, is probably the best way to serve this purpose. The simplifying setting offered by CGT allows the researcher to uncover new likely solutions, or to propose some according to normative criteria, whereas the non-cooperative approach can be used to verify their actual likelihood or, else, their implementability.

The present work follows, almost slavishly, the research method just described. For how much it might appear old-fashioned, I have personally found really fascinating to undertake this journey. Beyond the centrality of this topic, not only for the economic thinking, it is particularly astonishing to observe how, from a relatively simple setting such as a bargaining situation involving coalitions, such a great variety of models and solutions as the ones present in the dedicated literature, can arise. Likewise amazing is to note how, from such a variety, some clear patterns can be individuated. Particularly important, on this regard, it is the emerged dichotomy between egalitarianism and marginalism as the two inspiring principles of several solution concepts. Once more, this goes directly into the deep core of economic thinking.

Addressing the content of this work, it must be premised that my approach to the problem of coalitions formation was guided by a favor towards the principle of egalitarianism rather than the one of marginalism. Once the work has been completed, however, it turned out to support mostly this last principle. In order to understand the reason of this fact, it must be considered the genesis of the present work. As anticipated, it can be ascribed to that research strand that goes under the name of Nash program. In fact, at its core there is the presentation of two novel solution concepts for coalitional games and a bargaining model that is apt to sustain one of them as the result of a bargaining process among rational agents. The main idea behind these solution concepts is to relate the set of optimal points resulting from a linear minimization program with the bargaining process that rational agents undertake over the gains coming from the formation of coalitions. The seemingly lack of relation between these two elements is what, in my opinion, renders interesting the solution concepts. In fact, although they originate from observing some mathematical regularities rather than from some normative criteria that reflect the sensitivity of the researcher, nonetheless such solutions satisfy several of the axiomatic properties already present in the dedicated literature.

The thesis is divided into four main chapters. Chapter 1 is basically a literature review of coalitional bargaining models. It treats, therefore, the non-cooperative side of coalition formation. Due to the extent of the academic production in this field, the scope of the literature review has been narrowed in order to cover only that models having a relation with the rest of the present work. In particular, the review focuses on bargaining models having as underlying coalitional game a transferable utility (TU), externalities-free model. In other words, a TU coalitional game in characteristic function form. This restriction in scope provides the advantage of a deeper analysis. In particular, it allows to focus on the different facets that seemingly similar models present and on their impact on outcomes, element that is often overlooked by other reviews. A major result is the tendency of bargaining models to sustain strongly egalitarian solution concepts. A second important result is the fact that egalitarianism seems to come at the expenses of efficiency either in terms of the time required to reach an agreement and in terms of the collective level of pay-offs obtained by players. Marginalism, whose most representative solution is surely the Shapley value [Shapley, 1953], is sustained by a fewer number of models based on the assumption of an existing risk of partial breakdown of negotiations [Hart and Mas-Colell, 1992, 1996; Krishna and Serrano, 1995] or on the artifice of negotiations taking place at bilateral level through random meetings [Gul, 1989].

Chapter 2 presents the two novel solution concepts for TU games in characteristic function form: the Central Core and the Mid-central Core, both defined in the domain of balanced games. The former is a set-valued solution characterized for being a Core restriction, whereas the latter is a point-valued solution derived from the first. As anticipated, fundamental for the definition of both solutions is a linear minimization program, specifically, the centroid of the set of optimal points of such optimization problem. It must be further noted that this set of optimal solutions corresponds to the Core of the same game but with the worth of the grand coalition reduced till being the least possible value such that the game remains balanced. The Central Core is then defined as the set of efficient points, or else, the set of imputations, that grants to each player a pay-off that is greater or equal to the value of the respective element of the mentioned centroid. The Mid-central Core is simply defined as the centroid of the extreme points of the Central Core. Chapter 2 proceeds in analyzing the geometrical and axiomatic properties of both solutions. It is worth to notice that almost all the axiomatic properties pertaining to the Core are preserved by both solutions, except for the reconfirmation property. Finally, it is offered an axiomatization of the Mid-central Core that is characterized by the satisfaction of individual and group rationality, aggregate monotonicity, the strongest declination of monotonicity that a Core dependent solution can hold, and mid-point domination. This last is an adaptation of an axiom generally applied to bargaining problems to a coalitional setting and, through it, it is shown that other solution concepts can be easily characterized.

Chapter 3 switches from the cooperative to the non-cooperative side of coalitions formation. In particular, it presents the Burning Coalition Bargaining Model (BCBM), showing that, for a particular type of response strategy profile adopted by all players, the allocation obtained through the Mid-central Core is reached asymptotically as the unique equilibrium. The BCBM can be described as an alternating offer model à la Rubinstein with risk of partial breakdown. This last element, however, differs considerably from other models portraying it: Hart and Mas-Colell [1992, 1996] and Krishna and Serrano [1995]. Whereas in these models partial breakdown implies that the refusal of a proposal leads to the possibility of a player to drop out, causing all coalitions to which she belongs to become unavailable, in the BCBM players are never excluded and to dissolve (to burn) it is only the coalition object of the proposal. The chapter further presents the conditions according to which the Mid-central Core allocation is obtained asymptotically for every order of proposers selected by the initial random move.

Chapter 4 reverts to pure CGT presenting a numerical simulation based on a standard emissions reduction game. The aim is to compare the redistributive properties of different solution concepts when applied to a situation representing the formation of an International Environmental Agreement (IEA). Furthermore, the stability of the various solution concepts to deviations from potential free riders is assessed in a comparative way. Besides some of the most popular allocation rules, the revised Nash Bargaining solution and the Rawlsian Nucleolus are included in the comparison. It is easy to see that the revised Nash Bargaining solution corresponds to the Mid-central Core, whereas the Rawlsian Nucleolus is the most redistributive allocation belonging to the Central Core. The revised Nash Bargaining solution (alias Mid-central Core) is shown to be the less redistributive of all benchmarked solutions, whereas the opposite holds for the Rawlsian Nucleolus. Furthermore, this last appears to be the most resilient to free riding.

Notational conventions and basic definitions

Before approaching the mentioned chapters, that require a basic knowledge of Game Theory and of its jargon, the reader may want to refresh some basic definitions that will be widely used along the present work. Several specific definitions, such as the one of game in characteristic function form, are postponed and provided in the specific chapters where they will be employed. Occasionally, they might be repeated in more than one chapter in order to render their consultation easier for the reader. With regard to notation, the present work will follow the mathematical convention of representing sets with uppercase letter – e.g. A,B,C –, matrices with uppercase bold letters – e.g. $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ – and vectors with lowercase bold letters – e.g. $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ –. An exception will be made for point-valued solution concepts that, although being vectors, will be represented through uppercase or Greek letters. Parameters are indicated with lowercase characters, whereas functions either with lowercase or uppercase letters (generally Greek ones). For numerical sets, such as the set of reals, $\mathbb{R}$, or the set of natural numbers, $\mathbb{N}$, so as for operators, the present work will follow standard mathematical conventions.

With regard to definitions, let us start from the most basic ones.

Definition 0.0.1. (Game). A game is a description of the strategic interactions occurring among players, meant to be decision-makers endowed with some interests. A game includes the constraints on the actions that the players can take, but it excludes the specific actions that the players do take.

In the previous definition, it has been mentioned the rather vague term interests. Informally, this means that players have an objective in a game, or else, that they interact in order to achieve some goals. In particular, the various interactions that are possible in a game lead to different outcomes. A player being endowed with interests simply means that a player has well-defined preference relations over the potential outcomes. This does not exclude the possibility of a player being indifferent between two different outcomes. The existence of well-defined preference relations among players is necessary in order to represent the possible outcomes of a game through pay-offs, another fundamental ingredient of a game.

Definition 0.0.2. (Pay-off). A pay-off is a representation, in any quantifiable unit of measure, of what is obtained by a player when a certain outcome of a game is reached.

In any quantifiable measure implies that pay-offs can be expressed in monetary terms, in units of a given good or whatever other countable measure. Often, and particularly when abstract games are analyzed, units of utility, also called utils, are adopted. Two assumptions are at the base of Game Theory: rationality and common knowledge. Both of them attain to players and the first encompasses the idea of a player having well-defined preference relations. In particular, rationality states that a player is aware of the alternatives at her disposal, forms expectations about any unknowns and, having clear preferences, acts as guided by a process of optimization. This last sentence means that a player chooses her actions in a game in order to condition the outcome of the game as the one most favorable for her. Given her well-defined preference relations over outcomes, this implies that she tries to maximize her pay-off and that, for doing so, she will exploit in a consistent manner all the information she has at her disposal. Common knowledge states that all players know the rules of the game and know the other players being rational as they are, and they all know that they know and they all know that they all know that they know, and so on ad infinitum. This last aspect is fundamental for each player to form rational expectations regarding the other players' behavior and to implement rational strategies. This last is another key concept that deserves a formal definition.

Definition 0.0.3. (Strategy). A strategy is a complete plan of actions a player will undertake. The actions to be undertaken may be conditioned on the occurrences that may arise in the game.

Strategies are differentiated into pure and mixed.

Definition 0.0.4. (Pure Strategy). A pure strategy is a plan of action such that each possible action a player can do is chosen with probability 0 or 1.

Definition 0.0.5. (Mixed Strategy). A mixed strategy is a plan of action that allows a player to mix between pure strategies. More formally, if S_i is the indexed set of pure strategies of player i, with index set $\mathscr{S}$, and being s a generic element of S_i, a mixed strategy, s^m, is such that:

$$s^m = \sum_{j \in \mathscr{S}} \alpha_j s_j, \quad \sum_{j \in \mathscr{S}} \alpha_j = 1 \wedge \alpha_j \in [0,1], \forall j \in \mathscr{S}.$$

Till now, we have defined key concepts by words. Let us translate them into mathematical objects. The players of a game, G, are the elements of set $N = \{1, 2, ..., n\}$. For each player $i \in N$, there is a set, say C_i, that includes all the available actions at disposal of i. Define then the set A as the collection of all the available combinations of actions of all players: $A = \times_{i \in N} C_i$. Note that A is not the simple collection of available actions of all players, but rather it includes their combinations. There is then a function, having A as its domain, that maps onto a set of outcomes, say O. Its elements are vectors with dimensionality equal to the cardinality of N – expressed as $|N|$ – having for components a specific outcome for each player. Let us call the mentioned function μ:

$$\begin{array}{llll} \mu: & a \subset A & \rightarrow & O, \\ & a & \mapsto & \mu(a) = \boldsymbol{o}. \end{array}$$

with a being a generic element of A, or, as said, a combinations of actions represented by a tuple, whereas $\boldsymbol{o}$ is a generic element of O. The existence of well-defined preferences for players implies that there exists a function, for each player, such that the outcome – the element of vector $\boldsymbol{o}$ – for that player can be translated into a quantifiable value. Then, given o_i, there exists a function, say ϕ_i that maps onto $\mathbb{R}$. Finally, we have a last function, called pay-off function, π, that, given $\boldsymbol{o}$ and a vector collecting the preference functions for each player, $\boldsymbol{\phi}$, gives a vector of pay-offs: $\pi(\boldsymbol{o}, \boldsymbol{\phi}) = \boldsymbol{p} \in \mathbb{R}^n$ and where $p_i = \phi_i(o_i), \forall i \in N$. It has to be noted that a pay-off function is not a strictly necessary ingredient of a game. In general, the existence of preference relations among outcomes for each player is a sufficient condition for representing a game [Osborne and Rubinstein, 1994]. Preference relations are expressed through the symbols $\prec, \precsim, \sim, \succ, \succsim$. Then, outcome $\boldsymbol{o}^a$ is at least as good for player i than outcome $\boldsymbol{o}^b$, expressed as $\boldsymbol{o}^a \succsim_i \boldsymbol{o}^b$, with $\boldsymbol{o}^a$ and $\boldsymbol{o}^b$ being elements of O, if, and only if, $o_i^a \succsim_i o_i^b$. The pay-off function basically allows to quantify such relations and to use mathematical relational operators. Then, $\boldsymbol{o}^a \succsim_i \boldsymbol{o}^b \Leftrightarrow p_i^a \geq p_i^b$. Note that this result requires the function $\phi_i(\cdot)$ to be bijective for all $i \in N$. In the present work we will always assume the existence of such a pay-off function or else, we will assume that each player has a von Neumann and Morgenstern [1944] utility function $\phi_i(\cdot)$.

Once having seen the very basic elements of a game, it is opportune to introduce some more specific terminology and to distinguish different typologies of games and representations. First of all, we introduce the concept of class of games.

Definition 0.0.6. (Class). A class is a grouping of different elements into a single set or category according to some shared attributes.

From this general definition of class, it is clear that a class of games is a set that includes different games that have in common some fundamental aspects. A classical distinction that is made in the literature is between games with perfect and imperfect information[1]. The first class collects games where players are perfectly informed about the moves of the other players, whereas the second games that do not have such characteristic. The present work only deals with games belonging to the first class. Another important distinction applied to cooperative games – whose definition will be provided later on – is between cost and profit games. As the name suggests, the first class serves to group games where players are supposed to cooperate in order to share the costs of a project, generally under the assumption that cooperation has some positive impact in reducing costs. In profit games, instead, projects generate some positive profits. Again, in general cooperation entails an increase in the per capita worth of projects. Although the way of analyzing these two typologies is very similar, it must be noted that only profit games will be explicitly considered here.

Another dichotomous division of games relates to the way in which they are represented. In particular, a game can be represented in normal – also called strategic – form or in extensive form. Besides entailing a different way of graphically displaying the game, with the normal form translating into a matrix, whereas the extensive form into a tree – Figures 1 and 2 show the different representation of the same game –, there is a much more important distinction.

[1]From the definition of class just provided, it is then possible to define the class of games with perfect information and its complement: the class of games with imperfect information.

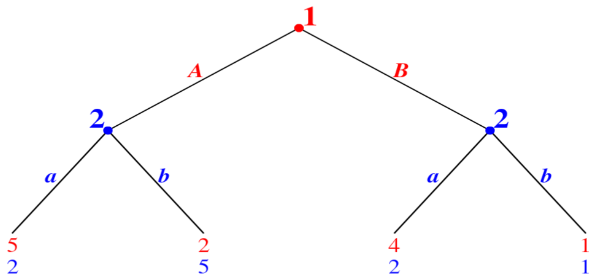

Fig. 1 Extensive form representation

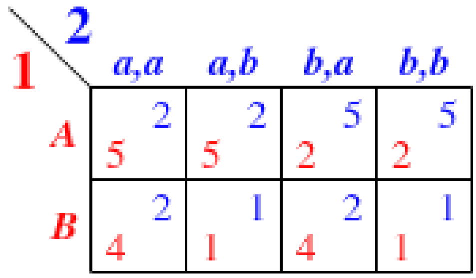

Fig. 2 Normal (strategic) form representation

A normal form representation serves to depict a situation where players have to define their plan of action once and for all. It then abstracts from the timing of the decision. This information is instead fully considered in the extensive form representation of a game. In Figure 1, in fact, it is clear that player 1 moves first and that player 2 will take her decision once having observed the move of player 1. If a dotted line was connecting the two blue nodes in Figure 1, a conventional way to express the simultaneity of the moves of the two players, then such sequentiality would be lacking. The absence of such dotted line, however, implies that the normal form representation of the shown game is a simplification insofar it discards the crucial information that player 1 moves first. The bargaining games that we will be dealing with in the present work do not feature simultaneous moves.

The distinction between the normal and the extensive form of a game is important to introduce another fundamental concept of Game Theory, namely equilibrium. It would be misleading to provide a general definition of equilibrium since different types have emerged in the literature and therefore it is necessary to have a proper definition according to the one considered. Three equilibrium concepts are relevant for the present work: Nash equilibrium (NE), sub-game perfect (Nash) equilibrium (SPE) and stationary sub-game perfect equilibrium (SSPE).

Definition 0.0.7. (Nash Equilibrium). Given a game G with player set N, a set of actions $C_i, \forall i \in N$, and a pay-off function $\pi(\cdot)$ assigning a vector of reals for each possible outcome $a \in A$ of the game, a (pure or mixed) strategy profiles combination $\boldsymbol{s} \in S$, with $S = \times_{i \in N} S_i$ constitutes a Nash equilibrium, $\boldsymbol{s}^*$, if:

$$\pi_i(s^*_{-i}, s^*_i) \geq \pi_i(s^*_{-i}, s_i), \forall s_i \in S_i, \forall i \in N.$$

The definition basically says that, given that all the other players – $(-i)$ – are adopting equilibrium strategies, then there is no a strategy profile s_i that grants to player i a higher pay-off than strategy s^*_i. When this holds for all $i \in N$, then we have a Nash equilibrium.

An analogous way of expressing this is by saying that player i cannot make any profitable deviation from strategy s_i^*. Note that $\boldsymbol{s}^* = (s_{-i}^*, s_i^*)$ corresponds to an outcome $a \in A$, as defined previously, and this, through the outcome function $\mu(\cdot)$, and the pay-off function $\pi(\cdot)$, that presupposes the existence of a function $\phi_i(\cdot), \forall i \in N$, is associated to a specific pay-off p_i for player i, that we then indicate as $\pi_i(\cdot)$. For the game represented in Figure 1 and 2, if we consider only pure strategies, we have $S_1 = \{A,B\}$ and $S_2 = \{(a,a),(a,b),(b,a),(b,b)\}$. The set of Nash equilibria, NE, has two elements: $NE = \{(A,(b,b)),(B,(b,a))\}$.

In order to understand the concept of sub-game perfect equilibrium, it is necessary to introduce the idea of history in a game in extensive form. A more formal description will be presented in chapter 1, whereas now we rely on an informal one in order to favor simplicity and understandability. A history, h, is basically a sequence of actions made by players. If an extensive form game has an infinite history, it is called an infinite horizon game, otherwise a finite horizon one. For now, let us consider only this last type. The set H collects all the possible histories of an extensive form game, including the empty set. The game represented in Figure 1 may help to clarify what h and H are: $H = \{\emptyset, \{A\}, \{B\}, \{A,a\}, \{A,b\}, \{B,a\}, \{B,b\}\}$. Each element of H is a history h. The histories leading to an outcome – a pay-off – in a game are called terminal histories. In our example, $\{\{A,a\}, \{A,b\}, \{B,a\}, \{B,b\}\}$ is the set of terminal histories. The peculiarity compared to a game in normal form is that the set of actions of a player is now defined in terms of either actions and histories: $C_i(h) = \{c_i : (c_i,h) \in H\}, \forall i \in N$. This allows to narrow the definition of strategy given before since, in an extensive game, a strategy of a player i is a plan of action for every history of the game after which player i is entitled to act (to move). We can define the set of histories after which player i is called to move as P_i with $P_i \subset H, \bigcap_{i \in N} P_i = \emptyset$ and $P = \bigcup_{i \in N} P_i \subset H^2$. Mathematically, a strategy is therefore a function that assigns an action $c \in C_i(h)$ for every element of P_i.

From what said before, an extensive game G can be fully described by the tuple (N,H,P,π), where π condenses in itself the outcome function $\mu(\cdot)$ and the preference function $\phi(\cdot)$ as previously described. Then $G = (N,H,P,\pi)$. Let us then define a sub-game.

Definition 0.0.8. (Sub-game). Given a game in extensive form $G = (N,H,P,\pi)$ a sub-game of G, named $G(h)$, is the subset of histories $H|_h \subseteq H$ that collects all the possible histories h' following h. We then have $G(h) = (N,H|_h,P|_h,\pi)$.

A sub-game is therefore the continuation of a game provided that a particular history h has been reached. We are then ready to provide a definition of sub-game Nash equilibrium.

[2] Note that terminal histories are not part of P, then it is necessarily a subset of H.

Definition 0.0.9. (Sub-game perfect Nash equilibrium). Given a game in extensive form $G = (N, H, P, \pi)$, a strategy s is a sub-game perfect Nash equilibrium of game G – s^* – if, for every history $h \in P_i$ and for every player $i \in N$, we have:

$$\pi_i|_h(s^*_{-i}|_h, s^*_i|_h) \geq \pi_i|_h(s^*_{-i}|_h, s_i), \forall s_i \in G(h).$$

Compared to NE, SPE requires that a strategy is a Nash equilibrium in every possible sub-game. The set of SPE is therefore a subset of the Nash equilibria. In particular, it allows to eliminate such Nash equilibria that originate from non-credible treats [Osborne and Rubinstein, 1994]. The example in Figure 1 may help to clarify the idea. Consider the two sub-games $G(\{A\})$ and $G(\{B\})$, having identical conditional history set $H|_h = \{\{a\}, \{b\}\}$. In $G(\{A\})$ player 2 will choose action b, granting her 5 instead of 2. The opposite hold in $G(\{B\})$ since $2 > 1$. But then player 1 will not choose action A in the first stage[3] since $\pi_1(\{A, b\}) = 2 < \pi_i(\{B, a\}) = 4$. The set of SPE in pure strategies is therefore a singleton: $\{(A, (b, b))\}$.

In order to understand the concept of stationary sub-game perfect equilibrium (SSPE), a refinement of SPE, it is opportune to introduce the idea of bargaining game, that is of central importance in chapters 1 and 3. In particular, we present the Rubinstein [1982] model that is generally considered as the archetypal bargaining model. It can be described as an infinite horizon game in extensive form of perfect information and perfect recall. Perfect information has already been defined – and we have already underlined that only such type of games is part of the present work –, whereas perfect recall simply implies that a player i, called to move at history h, knows all the previous moves that lead to that particular sub-game. Equivalently, players know the elements of the sequence h. Differently from what seen till now, the Rubinstein model is an infinite horizon game, implying $|H| = +\infty$. Three different variants of the model have been proposed: fixed discounting factor, fixed bargaining costs and fixed probability of breakdown, with a substantial equivalence of the first and of the latter variants. In the present exposition, the fixed discounting factor variant will be considered.

The Rubinstein model, also called alternating offers model for reasons that will be clear immediately, depicts two players – $N = \{1, 2\}$ – that bargain in order to split a fixed amount Ω. In the first round, one of the two is selected as proposer giving her the possibility to propose a division of Ω, $(x, \Omega - x)$, with x being the portion the proposer reserves for herself and $\Omega - x$ what offered to the opponent. This last, taking the role of responder, may accept

[3] This kind of reasoning is named backward induction.

or reject the proposal. In case of a rejection, the responder becomes the new proposer and the game repeats identically with inverted roles between players. Each Proposal\Answer phase takes one round t with $t = 0, 1, 2, ...$ If the players reach an agreement, as to say whenever the responder accepts the offer, the game ends with the players receiving what agreed, otherwise the players have to continue to bargain. Time is considered a valuable asset and this is reflected in the pay-off function: $\pi_i(\{(x, \Omega - x), \text{“Accept”}\}) = \delta^t x$, for i being the proposer and $\pi_j(\{(x, \Omega - x), \text{“Accept”}\}) = \delta^t(\Omega - x)$, for j being the responder, with $\delta \in (0, 1)$ being the common discount factor of the two players[4]. The game has infinitely many perfect Nash equilibria. In particular, for any $x^* \in [0, \Omega]$, there is a Nash equilibrium for which an agreement is reached in the first round sustained by the strategies “always propose x^* when proposer” and ”accept only an offer greater or equal than $\Omega - x^*$ when responder”. Other Nash equilibria leads to a delay in agreement [Rubinstein, 1982]. Although the game has an infinite horizon, preventing to use the backward induction device, Rubinstein [1982] has proven that there is a unique SPE equilibrium, for which the agreement is reached in the first bargaining round. Such equilibrium consists in the first proposer selecting $x^* = \frac{\Omega}{1+\delta}$ and the responder accepting the offer, thus receiving $\Omega - x^* = \frac{\delta\Omega}{1+\delta}$.

In this relatively simple setting, SSPE and SPE are coincident. However, for bargaining games where coalitions matter – a proper definition of a coalitional game and of a coalitional bargaining game will be provided later – stationarity is necessary to reduce the number of SPE equilibria. In order to understand the reason, let us introduce the formal definition of SSPE, taken from Miyakawa [2009]:

Definition 0.0.10. (Stationary sub-game perfect equilibrium). A strategy combination s^* of the game G is called a stationary sub-game perfect equilibrium point (SSPE) if it is a sub-game perfect equilibrium point with the property that, for every $t = 0, 1, 2, ...$, the t^{th} round strategy of every player depends only on the set of all active players and on the proposal at round t.

The equilibrium concepts of SPE and SSPE are actually very close, with stationarity adding a simplification, since it reduces the set of available strategies for the players. As stated in the given definition, in fact, players cannot condition their proposal and their acceptance rule on the identity of the other players or on the past actions occurred in the game. This excludes the possibility to adopt sophisticated strategies allowing only to condition the acceptance of an offer on a minimum threshold that the same offer must satisfy that is, in turn, dependent solely on t and on the subset of players still active in the game.

[4]Note that here we are adopting some simplifications either because we are considering an identical discount factor for both players and because we are considering the outside options of the players equal to zero.

Once that all the equilibrium concepts fundamental for the present work have been introduced, two last elements remain to be properly defined: Pareto dominance and the Nash bargaining solution. Although not strictly belonging to the realm of Game Theory, Pareto dominance will appear often in this work and it is therefore necessary to clarify it since the beginning. Consider to have a convex and compact set X defined in the metric space $\mathbb{R}^n$ representing a set of feasible decisions. Consider further to have a set Y in $\mathbb{R}^m$ of criterion vectors used to evaluate decisions in X. In particular, define Y as: $Y = \{\boldsymbol{y} \in \mathbb{R}^m : \boldsymbol{y} = f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n)\}$. Since Y has been defined as a set of criterion vectors through which it is possible to evaluate a decision in X, this implies that there exists a well-defined preference relation between elements of Y. Then, define the set called Pareto frontier as follows:

Definition 0.0.11. (Pareto Frontier). Given a convex and compact set of decisions $X \in \mathbb{R}^n$ and a set of criterion vectors Y in $\mathbb{R}^m$, the Pareto frontier $P(Y)$ is given by:

$$P(Y) = \left\{\boldsymbol{y} \in Y : \nexists \boldsymbol{y}' \in Y : \boldsymbol{y}' \succ \boldsymbol{y}, \boldsymbol{y}' \neq \boldsymbol{y}\right\}.$$

Once defined the Pareto frontier, we say that a point $\boldsymbol{y}$ is Pareto undominated if $\boldsymbol{y} \in P(Y)$, whereas we say that it is Pareto dominated if $\boldsymbol{y} \in Y$ but $\boldsymbol{y} \notin P(Y)$. Note that $P(Y) \subseteq Y$. If this is a very general definition of Pareto frontier and Pareto dominance, in the present work we will be mainly concerned with a more specific case. The set X of decisions is a set whose elements are called allocations, as to say a way of dividing a given sum into n components. Defining Ω as the total amount at disposal, we then have $X = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i \in N} x_i \leq \Omega\}$. Recalling the definition of $\phi(\cdot)$ previously given, this is our valuation criteria, so that $y_i = \phi_i(x_i), \forall i \in N$. Then, an allocation $\boldsymbol{x} \in X$ is Pareto undominated if $\nexists \boldsymbol{x}' \in X : \phi_i(x_i') \geq \phi_i(x_i), \forall i \in N$, with at least one inequality holding strictly.

The Nash bargaining solution [Nash, 1950] will be often mentioned given its centrality in Cooperative Game Theory and it is therefore necessary to have a clear understanding of it. Consider two persons, say i and j, bargaining in order to divide a given, divisible, good, say Ω. Each of them can interrupt the negotiation and, in such case, they both obtain a certain amount of the good, with i obtaining d_i and j, d_j. By assumption $\Omega \geq d_i + d_j$. Each of the two bargainers derives a given utility from the possession of the good described, as before, by the function $\phi_i(\cdot)$ and $\phi_j(\cdot)$. Given a bargaining problem of this type, the Nash bargaining solution is the division of the good Ω among the two bargainers such that the product of their utilities, deflated by their respective reservation utilities, is maximized. It is therefore a tuple $(x_i, x_j) : (\phi_i(x_i) - \phi_i(d_i)) \times (\phi_j(x_j) - \phi_j(d_j))$ is maximized and $x_i + x_j \leq \Omega$.

This can be easily extended to cover the case with more than two bargainers. Let us suppose to have N bargainers, each with a well defined utility function $\phi_i(\cdot)$, a given amount of good received in case of disagreement d_i and, therefore, a reservation utility $\phi_i(d_i), \forall i \in N$. Instead of representing the final allocation of the good through a tuple, we use a vector. Then, the Nash bargaining solution is that vector $\boldsymbol{x} \in \mathbb{R}^n$:

$$\boldsymbol{x} = \arg\max_{\boldsymbol{x}} \prod_{i \in N} (\phi_i(x_i) - \phi_i(d_i))$$

s.t.

$$\sum_{i \in N} x_i \leq \Omega.$$

This completes the preliminary settings necessary to understand the present work.

Chapter 1

Coalition Formation and Bargaining Protocols: A Review of the Literature

The present chapter offers a review of the vast literature regarding bargaining and coalition formation. This topic has been generally described as the attempt to provide strategic foundations to cooperative solution concepts. It can therefore be seen as the linking ring between the non-cooperative and the cooperative game theoretic approach to coalition formation. Its central role in the economic theory and its relatively long history that goes back to the Nash program have fostered a large academic production, including surveys. Nonetheless, this chapter will focus on an aspect that is often neglected in the dedicated surveys: the specificities of the bargaining protocols leading to different outcomes. Although generally downgraded to the rank of details, the differences in bargaining protocols, even when minor, can cause significant changes in fundamental aspects such as the possibility to reach full cooperation, the distribution of final pay-offs and the time taken to reach an agreement. Focused on externalities-free games, therefore on bargaining protocols sustaining solution concepts for cooperative games in characteristic function form, the chapter aims at providing a brief but exhaustive review of the topic that could result in a very useful tool for any researcher approaching the subject of coalitional bargaining.

Keywords: Bargaining, Coalition formation, Cooperative solution concepts, Nash program, Protocols.

J.E.L.: C71; C78.

1.1 Introduction

In his survey, Serrano [2004] offers an enlightening similitude between the efforts to provide micro-foundations to macroeconomic and the Nash program [Nash, 1953], whose aim is to "bridge the gap between the two counterparts of game theory (cooperative and non-cooperative)". This comparison is sufficient to shed light on the importance of the topic. If taken alone, each side of the coin, the cooperative and the non-cooperative approach, has its own weaknesses as stressed by Gul [1989]. In particular, the cooperative approach has been criticized for lacking of strategic foundations, whereas the non-cooperative one has been judged to be heavily dependent on the choice of the extensive form of the game and on the equilibrium concept adopted, choices that are far from being commonly agreed. Another problematic aspect of the non-cooperative approach is the multiplicity of equilibria that it might lead to [Gul, 1989]. Their combination, therefore, can be a useful operation to overcome the flaws of each side.

In order to understand the way in which one can strengthen the other, it is necessary to fully comprehend what the two approaches are and their differences. In particular, as Serrano [2004] underlines, the idea of cooperative game theory as a mere normative approach detached from strategic consideration that pursues cooperation and equity through desirable axioms is basically wrong. According to the same author, the proper definition of cooperative game theory is that "of a theory in which coalitions and the set of pay-offs feasible for each coalition are the primitives". A coalitional game is therefore a game in normal form where the set of pay-offs is given by the value that each coalition has attached dependently on each coalition structure (games in partition function form) or independently from it (games in characteristic function form). Cooperative game theory, therefore, operates a simplification omitting the extensive form of a game in favour of its normal, strategic form [Serrano, 2004]. What is lost in richness, however, is gained in sharpness since it restricts the attention to the fundamental aspects of a strategic situation.

On the other side, non-cooperative game theory models explicitly the game in its extensive form. Behavioural assumptions underpinning players' moves, masked in the coalitional approach, must now be stated. The bargaining process between individuals becomes the focal point. This, therefore, sheds light on the mechanism apt to lead to a certain pay-off distribution predicted by a cooperative solution concept. It could then be argued that one could start directly from the extensive form of a game. But in a multiplayer environment, where players dispose of a large sets of feasible actions, this can lead to an impossible effort of comprehension without any guideline [Winter, 2002]. Starting with the normal form of the game, either in partition or characteristic function form, offers then such a guideline. It comes with no surprise, therefore, that some of the most popular solution concepts in cooperative game theory, such as the Nash bargaining solution [Nash, 1950], the Shapley value [Shapley, 1953], the Nucleolus [Schmeidler, 1969], the Egualitarian solution [Thomson, 1983] and the Weakly- [Dutta and Ray, 1989] and Strongly-Constrained Egalitarian allocations [Dutta, 1990; Dutta and Ray, 1991] have all preceded their non-cooperative foundation.

If the cooperative approach was, and still is, a valuable simplification of strategic interactions among individuals, once several solution concepts, and the axioms upon which they are based, have been proposed, its role has lost momentum in favour of the non-cooperative side [Compte and Jehiel, 2010]. In the last decade and a half there has been a shrink of new solution concepts for games in characteristic function form[1] whereas the same cannot be said for their extensive form counterpart. A few examples may serve to confirm this: Yan [2003], Montero [2006], Compte and Jehiel [2010] and Okada [2011]. A different argument applies to games in partition function form. Although known since the work of Thrall and Lucas [1963], the higher complexity of the situation they serve to depict has considerably slowed the emergence of suitable solution concepts and this seems to be a still active research agenda. Furthermore, it is interesting to note that some of the solution concepts emerged for games in partition function form are extensions or modifications of concepts originally envisaged for games in characteristic function form. See, for example, the work of McQuillin [2009], that extends the Shapley value to partition function games, and the works of Kóczy [2007] and Bloch and Van den Nouweland [2014], whose aim is to propose a Core for such setting.

In light of what said till now, this chapter aims at reviewing the literature on bargaining and coalition formation. The focus will be on the non-cooperative side, although the link between bargaining protocols and the cooperative solutions that they support will always be pointed. Furthermore, it will adopt a narrow perspective, carefully examining how specific variations in bargaining protocols lead to different conclusions about efficiency, about the timing of coalition formation and about the distribution of pay-offs. Given the abundance of the literature on this topic, some works will necessarily be omitted and we apologize for this with their authors. Moreover, the analysis will be almost exclusively limited to bargaining games whose normal form counterpart is represented by a game in transferable utility (TU) characteristic function form. This excludes the interesting case of externalities between coalitions. The reason to apply this restriction is primarily due to the mentioned extent of the

[1]There are, however, remarkable exceptions. See, for example, González-Díaz and Sánchez-Rodríguez [2007]; Tijs et al. [2011].

topic, under the obvious consideration that scope comes at the expense of depth.

It could be argued that the excellent works of Serrano [2004], Bandyopadhyay and Chatterjee [2006] and Ray [2007] already cover this topic. This is true, but several years have passed from their publication and, in the meanwhile, the literature has done significant steps forward. Furthermore, if the work of Serrano [2004] points at the relation of cooperative solution concepts with bargaining games, the present one, as mentioned, will give prevalence to the relation between outcomes of bargaining and protocols' specificities. Moreover, two players bargaining, largely covered by Serrano [2004] will be omitted in favour of games with at least three players. Compared to Bandyopadhyay and Chatterjee [2006], this chapter focuses on "pure" coalitional bargaining, whereas the former has a large section dedicated to legislative bargaining. Initiated by the seminal paper of Baron and Ferejohn [1989], although this strand of the literature is closely connected to coalitional bargaining, it nonetheless has significant points of departure, among which the most important is its focus on games with an empty Core[2].

Section two briefly summarizes the idea of a characteristic function form game, then it sketches the fundamental elements of a game in extensive form and, starting from these lasts, it depicts an ideal coalitional bargaining model. The core of this chapter, section three, examines a sample of coalitional bargaining protocols present in the dedicated literature. Models will be divided according to the cooperative solution concepts they support, starting with the Nucleolus, proceeding with the Shapley value and concluding with a family of egalitarian solutions and with the Core. The final section is devoted to conclusions.

1.2 Structure, equilibrium concepts and other common features of coalitional bargaining models

Before describing the structure, the selection of the equilibrium concept and other common elements of coalitional bargaining games, it seems opportune to describe their simplified normal form: the characteristic function form. Introduced by the seminal work of von Neumann and Morgenstern [1944], a game in characteristic function form, G, is constituted by a 2-tuple (N,v) whose elements are the finite set of players, N, and a real valued function, v, that assigns to each non-empty element of the power set of N, here indicated as $\mathscr{P}(N)$, a real value. It could be argued that a colational game is properly defined by a 3-tuple, (N,c,v),

[2] Besides Bandyopadhyay and Chatterjee [2006], the reader interested in legislative bargaining should also consider the recent review of Binmore and Eguia [2017].

with N and v defined as previously and where c is a map form N, to a set of non-empty subsets of N itself (coalitions). Therefore:

$$\begin{aligned} c : N &\to P(\mathscr{P}(N)) \\ G(N) &\mapsto c(N) \end{aligned}$$

This clarifies that not all coalitions, or else, not all the possible subsets of N, are necessarily feasible in a game. Generally, however, the simplification (N,v) seems to prevail [Osborne and Rubinstein, 1994] and this convention will be followed here. If we indicate with S a coalition, a generic element of $\mathscr{P}(N)$[3], $v(S)$ is therefore the total pay-off that is available for division among the members of S. When no restrictions are posed on the possibility to divide $v(S)$ among coalition's members, the game is said to be a transferable utility game (TU).

For a TU game (N,v) and a coalition of players S with value $v(S)$, define $X(S)$ as the set of all feasible divisions of the worth of S among its members: $X(S) = \left\{\mathbf{x} \in \mathbb{R}^{|S|} : \sum_{i \in S} x_i \leq v(S)\right\}$. The set $X(S)$ is closed, convex and comprehensive. If we define as Γ the set of all games $G(N,v)$ in characteristic function form, a cooperative solution concept, Φ, is a mapping from an element of Γ to a set, $\Phi(N,v) \subset \mathbb{R}^{|N|}$, of feasible vectors, called pay-off profiles, such that, for each element $\mathbf{x} \in \Phi(N,v)$, there exists a coalition structure – $S_1, S_2, ..., S_k, \bigcup_{i=1}^{k} S_i = N, S_i \cap S_j = \emptyset \forall S_i, S_j \in \mathscr{P}, S_i \neq S_j$ – for which $\mathbf{x}(S_i) \in X(S_i)$ for all $i = 1, 2, ..., k$ [Serrano, 2004]. A pay-off profile is therefore a vector of values that assigns to each of the n players – since now on n will be used as $|N|$ – a pay-off under the feasibility constraint represented by the amount of pay-off available for distribution given by $v(S)$. If this is a constraint common to all solution concepts, the difference in axioms which they are based upon generates the peculiar image of each Φ. It has to be noted that solution concepts can be classified into set valued, if the cardinality of $\Phi(N,v)$ can be grater than one, or single valued, when $\Phi(N,v)$ is necessarily a singleton. Finally, note that an element of $\Phi(N,v)$, let us name it $\boldsymbol{\phi}$, is called an imputation if it satisfies the following properties: $\sum_{i=1}^{n} \phi_i = v(N)$, $\phi_i \geq v(\{i\})$, $i = 1, 2, ..., n$.

Coalitional games are generally divided into classes according to some properties hold by the characteristic function. Since these properties influence the extensive form of the game as well, it is opportune to provide their formal definition:

[3] Along the chapter, $\mathscr{P}$ will be used as a shorthand for $\mathscr{P}(N)$.

Definition 1.2.1 (Essential game)**.** A TU game (N,v) is said to be essential if

$$v(N) > \sum_{i\in N} v(\{i\}).$$

Definition 1.2.2 (Cohesivness)**.** A TU coalitional game (N,v) is said to be cohesive if

$$v(N) \geq \sum_{S\in P} v(S), \quad \forall P : \bigcup_{S\in P} S = N; S_i \cap S_j = \emptyset \text{ for } S_i \neq S_j \in P;$$

with P being a partition.

Definition 1.2.3 (Superadditivity)**.** A TU coalitional game (N,v) is said to be superadditive if

$$v(S\cup T) \geq v(S) + v(T), \quad \forall S,T \in \mathscr{P} : S\cap T = \emptyset.$$

Definition 1.2.4 (Convexity)**.** A TU coalitional game (N,v) is said to be convex if

$$v(S\cup T) \geq v(S) + v(T) - v(S\cap T), \quad \forall S,T \in \mathscr{P}.$$

Definition 1.2.5 (Normalized game)**.** Given an essential TU coalitional game (N,v), its normalized form, (N,v') is obtained by the following two steps procedure:

1) $v(S)^0 = v(S) - \sum_{i\in S} v(\{i\})$, $\forall S \in \mathscr{P}$.

2) $v(S)' = \dfrac{v(S)^0}{v(N)^0}$, $\forall S \in \mathscr{P}$.

Clearly, in a normalized game, $v(\{i\})' = 0$, $\forall i \in N$ and $v(N)' = 1$.

Definition 1.2.6 (Zero-normalized game)**.** Given an essential TU coalitional game (N,v), its zero-normalized form, (N,v'') is obtained by applying only step 1 in Definition 1.2.5.

Now that the basic features of a cooperative game have been briefly summarised, it is possible to describe the extensive form of the bargaining game.

1.2.1 The extensive form of a coalitional bargaining game

In his presentation of the concept of trembling hand perfect equilibrium, Selten [1975] offers a very clear and concise description of an extensive form game with perfect recall. Given that perfect recall, introduced by Kuhn [2016], is a standard assumption in coalitional bargaining games, – what can be considered as the most famous model of bargaining, the Rubinstein model [Rubinstein, 1982], although not coalitional, falls into this category – the description of Selten can be taken as our guideline.

An extensive game is fully described by a 6-tuple:

$$G = (H, P, F, C, \alpha, \pi),$$

where the set H represents the game tree, P is the set of players' partitions, F the information partition, C the choice partition, α is a function that assigns probability over the elements of C and π is a pay-off function that associates n real values (pay-offs), where n is the number of the players in G, to each final node of H. Selten [1975] assumes that H represents a finite tree, whereas coalitional bargaining games are generally infinite.

Selten [1975] describes H as the game tree, therefore as a collection of vertices and edges connecting them. The tree has an origin. The set of all edges except the final nodes is indicated with K, whereas Z stays for the set of endpoints. The set P, having cardinality $n+1$, collects subsets of H – $P = \{P_0, P_1, ..., P_n\}$ – each of which, in turn, collects the vertices where player i – $i = 1, 2, ..., n$ – is entitled to make a move. The set P_0 is dedicated to the random mechanism operating in the game. Each P_i, including P_0, can be further subdivided into subsets, named information sets. Let us call each of them U. Informally speaking, if we draw the game tree, the elements of U are the vertices where player i is entitled to move that lay on the same horizontal line. Therefore, for each vertex k belonging to U, player i will have the same set of moves at her disposal. Moreover, the game-play can intersect U at most once. The set Υ_i collects all the sets U belonging to player i – all the subsets of P_i – and set Υ groups together all Υ_i. Note that $\Upsilon_0 = P_0$. The last set characterizing the game G is C, that collects all sets C_U. Sets C_U, in turn, list all the possible moves c, also called actions or choices, that are available at vertex k belonging to a certain information set U. The last two elements of G are functions. The first, α, is the probability distribution of the actions belonging to the random mechanism of the game. Its argument, therefore, is c. The second, instead, π, is the pay-off function that assigns a specific value for each player of the game at each ending vertex $z \in Z : \pi(z) = (\pi_1(z), \pi_2(z), ..., \pi_n(z))$ [Selten, 1975]. In order to clarify the meaning of all the mentioned elements, let us see the extensive form representation of a very simple game with perfect and complete information.

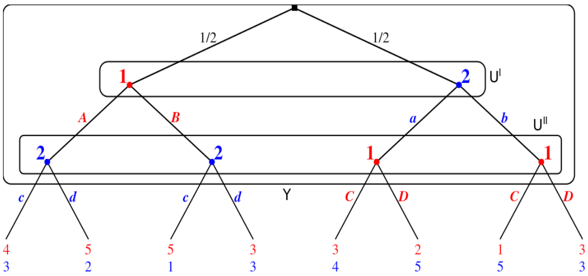

Fig. 1.1 Extensive Form Game

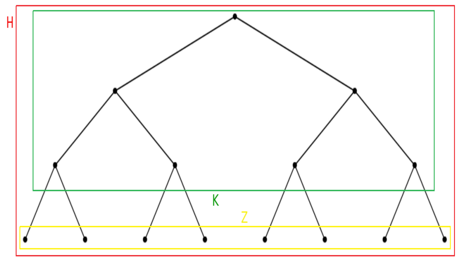

Fig. 1.2 Game Tree

Figure 1.1 and 1.2 show an hypothetical game in extensive form. From Figure 1.2, it is possible to observe the whole game tree, set H, represented by the red box. The yellow rectangle, instead, is the set K, with all nodes except the final ones, that belong to Z, represented by the yellow box. In Figure 1.1, more information are available. Set P_0 is a singleton and its sole element is the black square at the top of the tree, whereas the elements of P_1 and P_2 are, respectively, all the red and blue nodes. P, therefore, coincides with K and with Υ. Figure 1.1 further shows the information set U^I and U^{II}. Note that U^0 has not been drown but clearly coincides with P_0. The available actions for players 1 and 2 are represented by colored letters. We then have $C_1 = \{A,B,C,D\}$ and $C_2 = \{a,b,c,d\}$. The set of actions of the random mechanism, C_0, has two elements: "select player 1" and "select player 2". To each of them is attached an equal probability of 0.5: α("select player 1") $=$ α("select player 2") $= 0.5$. Finally, pay-offs are shown at the bottom of the graph. A simple example should be sufficient to understand the working of the pay-off function π: $\pi(\{$"select player 1"$,A,c\}) = (4,3)$.

If the previous paragraphs have offered a general description of a game in extensive form, it is instructive to try to characterize a coalitional bargaining game in its archetypal form relating the two representations. Given our objective of finding an archetype, the following coalitional bargaining game might have more or less elements compared to specific games present in the literature. Furthermore, the bargaining protocol depicted can be ascribed to the family of Rubinstein-type models. In general, authors use one of two types of ways for describing a coalitional bargaining game. The first way is very parsimonious and consider the game tree and the pay-off function to fully characterize the game. All the other elements, such as players' available choices, the probabilistic function of the random mechanism and the strategy space are considered to be elements of the same game tree. The game, therefore, can be described by a 2-tuple: $B = (H,\pi)$; see, for example, Kim and Jeon [2009]. A second approach, instead, considers the game tree as the result of the combination of other basic elements and therefore lists them in the tuple describing the game; see, for example, Nguyen [2015]. We will actually adopt this second, more explicit, approach.

A coalitional bargaining game is generally described by a 6-tuple:

$$B = (N, \mathscr{P}, v, \Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}).$$

The first three elements, $N, \mathscr{P}$ and v are the same as the components of a game in coalitional form: N is the set of players, $\mathscr{P}$ is the set of non-empty coalitions and v is the function that associates a value to each element of $\mathscr{P}$. Although we have defined $\mathscr{P}$ as the power set of N, it must be noted that the set of coalitions can actually be a subset of $\mathscr{P}(N)$. Nonetheless, we will continue to use this symbol in order to underline that, potentially, every subset of N can be a feasible coalition in a bargaining game. Although some works consider explicitly the way in which the value of a coalition is created – e.g. Gul [1989] –, with the characteristic function having as argument parameter values specific for each player, this is not generally the case[4]. Also the extensive form of a coalitional game, therefore, generally abstracts from the way in which the worth of a coalition is produced. The other three elements, Σ, $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, are specific of the bargaining game. Σ represents the whole strategic space. Since, usually, a coalitional bargaining game is sequential and infinite, therefore it has a temporal dimension with $t = 1, 2, ..., \infty$, we have $\Sigma = \times_{t=1}^{\infty} \sigma_t$, where $\sigma_t = \times_{i=1}^{n} \sigma_{it}$, with σ_{it} being the set of all available actions of player i at time t. If we add σ_{0t} as the set of actions, at time t, of the random mechanism, Σ corresponds to the set C described in Selten [1975].

Let us see, concretely, which are the actions at disposal of the players, starting from the random mechanism. This initiates the game by selecting the first proposer, one of the n players, according to a given probability distribution. This is why the function α has been changed into vector $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\sum_{i=1}^{n} \alpha_i = 1$. In the *rejector-proposes* protocol (e.g. Chatterjee et al. [1993]; Kim and Jeon [2009]), where the first rejector of an offer becomes the next proposer, the random mechanism operates only at the origin of the game tree by selecting an order of players that will persist for the whole game (*fixed proposers order*), whereas in the *random proposer* protocol (e.g. Okada [1996], Compte and Jehiel [2010] and Okada [2011]) every node after a rejection of an offer is dedicated to the random mechanism. It worth to note that some variants of the coalitional bargaining model have the random mechanism operating at other levels. In Gul [1989] and Nguyen [2015], for example, also the formation of a certain coalition is treated as a random event, whereas in Hart and Mas-Colell [1996] the proposer suffers from a positive probability to be excluded from the game if her proposal is refused. Our archetypal presentation, however, focuses on the simple

[4]This approach is actually followed by cooperative games as well. See, for example, Owen [1975] and Chander and Tulkens [2006b].

case in which the random mechanism operates only for the selection of a proposer.

At each time period t, there are two different information sets to which the choice of a player can belong. The available choices, therefore, vary according to which of the two states the player is in. If she has been selected as a proposer by the random mechanism, her choice can be describes by a 2-tuple: $c_p = (\boldsymbol{x}, S)$ with $\boldsymbol{x} \in \mathbb{R}^{|S|}$ and $\sum_{i \in S} x_i \leq v(S)$. A proposer, therefore, can select a coalition and propose a certain, feasible, division of its worth among the members. A proposer can always choose to pass her turn, but this is generally equivalent to propose a division that will be certainly refused. Furthermore, a proposer can be granted the possibility to choose the order of the respondents, but since this element does not generally influence the game in any way (see, for example, Okada [1996] and Compte and Jehiel [2010]), this is not a real strategic choice. If a player belongs to the coalition selected by the proposer, she will then be a respondent. In this case her action space is described by a dichotomous choice: $c_r = \{accept, reject\}$.

The last element of the 6-tuple describing B is the vector of discount factors $\boldsymbol{\delta}$ – $\boldsymbol{\delta} \in \mathbb{R}^n$ and $\delta_i \in (0,1)$,$\forall i \in N$ –, that is strongly interrelated with the role of time t in the bargaining game. In the first time period – $t = 1$ – all the three steps previously described are present in the following order: random mechanism, proposal, response. If all responders accept the offer received, the game is already at its terminal node z, provided the model allows for the formation of a single coalition[5]. The pay-off function can be described as follows:

$$\pi_i(z) = \begin{cases} \delta_i^{t-1} x_i, & \forall i \in S,\ S \in c_p \\ v(\{i\}), & \forall i \in N \setminus S,\ S \in c_p \end{cases}$$

A refusal from one of the responders gives rise to an identical sub-game tree in the random proposal protocol or to an equal sub-game tree but without the random mechanism move in the rejecter-proposes case. The game has therefore a clear recursive structure and the only change from one period to another happens through the action of discounting.

A final remark is related to players' preferences that follow in all respects the assumptions of Rubinstein [1982][6]. Being a game of complete information, every player is assumed

[5]This assumption is by no means the prevalent one in the literature. Several models – e.g. Chatterjee et al. [1993], Okada [1996] and Okada [2011] – allows for the continuation of the bargaining game with players set $N \setminus S$ after coalition S has formed. We have adopted it here merely for simplification purposes.

[6]Rubinstein [1982] preferences' assumptions are fulfilled by two models: fixed bargaining costs and fixed time discounting. We have presented a bargaining model of the second kind that is by far the most common approach in the dedicated literature.

to know her own and other players' preferences and all the other elements of the game such as the discount factors or the worth of each coalition. Furthermore, perfect recall implies that the information set is a singleton. Now that the structure of an archetypal coalitional bargaining model has been described, it should be easier to identify the relation between outcomes and protocol variants. Before, however, we will briefly discuss the selection of the equilibrium concept adopted.

1.2.2 Stationarity and sub-game perfection

Once having a set of bargaining rules (a protocol) as the archetypal type described in the previous section, and supposing to have defined a characteristic function that assigns a value to each possible coalition, the result of the strategic interaction of players in the bargaining process is sill heavily dependent on the equilibrium concept that is adopted. Nash equilibrium is clearly unsatisfactory since, even in two-players bargaining, it admits every solution that is efficient and that guarantees each player to obtain at least her disagreement pay-off [Rubinstein, 1982]. Therefore, if we consider the Rubinstein model as a balanced two-players coalitional bargaining model where the outside options are $d_i = v(\{i\})$, for $i = 1,2$, and adopting the simplifying assumption of $\delta_1 = \delta_2 = 1$, every point in the set X is a Nash equilibrium, with $X = \left\{\boldsymbol{x} \in \mathbb{R}^2 : x_i \geq v(\{i\}), \text{for } i = 1,2 \wedge \sum_{i=1}^{2} x_i = v(\{1,2\})\right\}$.

Subgame perfection, introduced by Selten [1973], has been proved to reduce the set of equilibrium points of the Rubinstein model to a singleton [Rubinstein, 1982]. However, in a 'proper' coalitional bargaining model, where the adjective proper means $|N| \geq 3$, it is not of much help to sharpen feasible equilibria if it is not coupled with stationarity. In fact, as shown in Proposition 0 of Chatterjee et al. [1993], if a coalitional bargaining game has a superadditive characteristic function, there always exists a sufficiently high discount factor $\delta^* \in (0,1)$ – assumed to be common to all players – such that each allocation $\boldsymbol{x}$ that is individually rational – $x_i \geq v(\{i\})\ \forall i \in N$ – is a subgame perfect equilibrium (SPE).

Kim and Jeon [2009] provide a concise but exhaustive definition of stationary strategy:

"A stationary strategy is a mapping from the player's position in the bargaining process to the choice set available to him for every player".

In other words, this implies that strategies are not dependent on the history of the game, but only on the position in the bargaining process that a player covers. Whereas Chatterjee et al. [1993] do not find a compelling reason to restrict the attention to a stationary subgame perfect equilibrium (SSPE) if not the fact that it is analytically tractable and that it enables to reach uniqueness, Kim and Jeon [2009] justify its adoption by the implicit stationarity of preferences and by the intrinsic recursiveness of the game.

This section has presented the main common elements of a coalitional bargaining game through the description of an archetypal sequential offer game. The next section, the core of this chapter, will present several variants of this model and their relation with some of the most well known cooperative solution concepts. We will start presenting models that departs the most from the basic protocol just sketched to come back to it at the end of the section.

1.3 Bargaining protocols' variants and results

1.3.1 The Nucleolus and the bankruptcy problem

Introduced by Schmeidler [1969], the Nucleolus does not have a straightforward definition and some steps are required in order to achieve it. First of all, it is necessary to define the notion of excess.

Definition 1.3.1 (Excess)**.** Given an allocation $\boldsymbol{x}$, with $\boldsymbol{x}$ being a vector in $\mathbb{R}^n$, the excess $e(S,\boldsymbol{x})$ of coalition S given $\boldsymbol{x}$ is defined as:

$$e(S,\boldsymbol{x}) = v(S) - \sum_{i\in S} x_i.$$

Given a coalitional game (N,v) and a feasible allocation $\boldsymbol{x}$ such that $\sum_{i\in N} x_i = v(N)$, define $\boldsymbol{e}(\boldsymbol{x})$ as the vector of the excesses, arranged in a non-increasing order, for all the non-empty coalitions of (N,v) other than N. Given a TU coalitional game (N,v), the Nucleolus (Nu) is the given by:

$$Nu = \boldsymbol{x} : Lmin_{\boldsymbol{x}} \boldsymbol{e}(\boldsymbol{x});$$

where *Lmin* is here defined as the lexicographical minimum operator. Such operator performs a component wise comparison of the excess vectors. It starts by picking all the excess vectors whose first component is minimal. Then it refines such set by discarding all the vectors whose second component is not minimal. The procedure continues till the last element. The Nucleolus is always a singleton and satisfies symmetry, covariance and consistency, where the last is the axiom that peculiarly characterizes this solution concept [Serrano, 1993].

A 3-players model with non-contingent offers

In pointing the equivalence between the Nucleolus and the solution proposed in the Talmud for the contested garment problem, Aumann and Maschler [1985] describe an implementation process leading to the Nucleolus in solving any bankruptcy case. This process, however, is not formally described as a non-cooperative bargaining model. Serrano [1993] filled the gap some years later, presenting a bargaining protocol for a three players, superadditive and normalized coalitional game.

In the first step, the random mechanism selects an order of proposers (fixed order proposer model) that will remain unvaried for the whole game. Given that, the first player in the queue will make a proposal to the other players that will then reply simultaneously. Note that, in this last step, perfect recall is violated. Another peculiar characteristic is the fact that offers are non-contingent, meaning that, if one of the two responders accepts the offer and the other does not, the accepter obtains what has been offered to her. In general, coalitional bargaining games display contingent offers where unanimity is required in order for the offer to produce an effect. Another example of non-contingent offers can be found in Chaturvedi [2016]. If the case of an offer followed by one acceptance and one refusal materializes, the game goes back to the random mechanism that selects, with equal probability, one between the former proposer and the refuser to be the new proposer. The respondent, in this case, has the option to "buy" the resources of the accepter so that her outside option will now be $\max\{v(i,j)-x_j,0\}$ where j is the index of the former accepter and i the one of the actual responder. In case the offer is rejected by both responders in the first turn, the second player in the queue (as determined by the initial random move) will become the new proposer. Obviously, if the offer is accepted by both responders, the game ends.

The model has another important peculiarity. Discounting is substituted by a fixed cost of bargaining c, that, however, applies only to the proposer whose offer has been refused by at least one player. Every new proposal, therefore, coincides with an increase of one unit of time. The game is infinite and a perpetual disagreement leads to a pay-off for each player equal to $-\infty$.

The game is solved using SSPE in pure strategies. Serrano [1993] assumes a particular relation in the worth of each coalition: $0 \leq v(2,3) \leq v(1,3) \leq v(1,2) \leq 1$, where $v(\{i,j\})$ has been substituted by $v(i,j)$ for ease of notation. This implies that, defining m_i as the sum of the marginal contributions of player i, we have $m_1 \geq m_2 \geq m_3$. Define as $v_{x\{i,k\}}(i)$ the outside option of player i if j has accepted an offer x_j. Serrano [1993] shows that the solution of the bargaining problem can be translated into a system of equations:

$$
\begin{aligned}
x_i &= 1 - x_j - x_k; && (Proposer).\\
x_j &= \frac{1}{2}[1 - x_k - v_{x\{i,j\}}(i) + v_{x\{i,j\}}(j)]; && (Responder).\\
x_k &= \frac{1}{2}[1 - x_j - v_{x\{i,k\}}(i) + v_{x\{i,k\}}(k)]; && (Responder).
\end{aligned}
$$

Proposition 1 of Serrano [1993] states the main results. The system of equation has a unique solution $\boldsymbol{x}^* = \{x_1^*, x_2^*, x_3^*\}$, the nucleolus, provided that the order of proponents mirrors the decreasing order of their sum of marginal contributions (order = $\{i, j, k\}$ if $m_i > m_j > m_k$). If the first proposer is changed, keeping the remaining order identical, the proponent will always obtain the equilibrium value x_i^*, but there is a continuum of feasible solutions, containing $\boldsymbol{x}^*$, for the pay-off of the remaining players. In each case the agreement is reached in the first round of negotiations. For other orders of proposers, without relation with m, and bargaining costs sufficiently low, there are multiple SSPE in period two only.

A generalization to n-players for bankruptcy problems

In the model just described, the Nucleolus is the unique SSPE only if a precise order of proponents is adopted. Furthermore, this result is valid only in case of three players [Serrano, 1993]. A generalization to n players is offered in Serrano [1995]. This alternative bargaining protocol relies on a particular coalitional setting: a bankruptcy problem or a surplus sharing problem. It is instructive to look at the peculiar characteristic function of these games. In a bankruptcy problem, there are n claimants, each of which claims a sum $d_i \geq 0$, with $i = 1, 2, ..., n$. The available sum to be distributed is equal to W, with $W \leq \sum_{i=1}^{n} d_i$. Defining $d(T) = \sum_{i \in T} d_i$, for T being a set in $\mathscr{P}$, the worth of a coalition $S \in \mathscr{P}$ is given by: $v(S) = \max\{0, W - d(N \setminus S)\}$. Since $W \leq d(N)$, the bankruptcy game can be described as a loss sharing problem. The other typology of game examined by Serrano [1995], the surplus sharing game, can be seen as the dual of the bankruptcy problem [Aumann and Maschler, 1985]. Here, W is the value of a joint project undertaken by the n players, each of which contributed with an amount $d_i \geq 0$ to its realization. By assumption $W \geq \sum_{i=1}^{n} d_i$. The characteristic function is defined exactly as in the bankruptcy case, and therefore it is subadditive. Consequently, the surplus sharing game is not balanced.

The bargaining game is a perfect recall, finite horizon game solved in SPE since its finite nature renders stationarity superfluous. The first random move selects the proposer and the sequence of responders. Non-contingency of offers is still valid, but what changes is the bargaining mechanism after eventual refusals. In fact, each rejecter will enter into a bilateral sub-game with the proposer consisting into an initial, fair, random draw to select who among the two players will act as a dictator. The non-selected player will get half of his claim d, plus what the dictator decides to leave her. Serrano [1995] shows that, given a pair of players $\{i, j\}$ entered into such a sub-game, the expected pay-off for player i is equal to $\max\left\{0, W - d_j\right\} + \frac{1}{2}[W - \max\{0, W - d_i\} - \max\left\{0, W - d_j\right\}]$. Defining as y the formula for the expected pay-off just shown, after a proposal $\boldsymbol{x}$ is made by player i, the final pay-offs vector $\boldsymbol{\pi}$ will be:

$$
\begin{array}{ll}
\pi_j = x_j; & \text{for each accepter j.} \\
\pi_k = y^{k-1}; & \text{for each rejecter k.} \\
\pi_i = W - \pi(N \setminus \{i\}); & \text{for proposer i.}
\end{array}
$$

where y^{k-1} is defined recursively by substituting W, in the formula for the expected pay-off, with $W - \sum_{j \in A} x_j - \sum_{k \in R_k^{k-1}} x_k$, where A is the set of accepters and R_k^{k-1} is the set of rejecters preceding rejecter k. Finally, it must be noted that the bargaining game does not have either a discount factor and bargaining costs.

As in Serrano [1993], the game has a unique SPE, coincident with the nucleolus, only if the order of players produced by the initial random move is such that it respects the magnitude of the claims/contributions of players. For example, given the order $1, 2, ..., n$, it must hold that $d_1 > d_2 > ... > d_n$. Apart from being based on peculiar characteristic form games, such as the bankruptcy game, the implementation of the Nucleolus is heavily dependent from the selected order of proposers.

1.3.2 Three roads to the Shapley value

Being one of the most popular cooperative solution concepts, the Shapley value [Shapley, 1953] has been extensively studied in the domain of non-cooperative bargaining processes. Three families of models seem to have emerged, leading, under specific conditions, to this cooperative solution. Remembering that the Shapley value is deeply rooted into marginalism, in fact it grants to each player her marginal contribution to all coalitions of a game (N, v), weighted by the probability that this contribution takes place, the formulas to compute it is:

$$\phi_i(N,v) = \sum_{S \subseteq N} \frac{(n-|S|)!(|S|-1)!}{n!}[v(S) - v(S \setminus \{i\})], \qquad \forall\, i \in N.$$

$$\phi_i(N,v) = \frac{1}{n}(v(N) - v(N \setminus \{i\})) \sum_{i \neq j} \phi_i(N \setminus \{j\}, v), \qquad \forall\, i \in N.$$

The second equation, presented in Maschler and Owen [1989], will turn out to be useful to understand the mechanism behind one of the family of non-cooperative approaches leading to the Shapley value. Finally, let us remember that this solution concept satisfies efficiency, symmetry, additivity, strong monotonicity and null-player (dummy player).

A bargaining model of bilateral random meetings

The first, following a chronological order, typology of non-cooperative game described is the one presented in Gul [1989], that could be titled "bilateral random meetings". The coalitional side presupposes that there are n players, each endowed with a valuable resource M_i for $i = 1, 2, ..., n$. Their combination forms a bundle that, in turn, generates utility according to a specific function (the characteristic function): $v(S) = v(\sum_{i \in S} M_i)$. The bargaining game, on the other side, is a perfect recall, infinite horizon model, with a peculiar aspect. At each time period t, a single random meeting between two players, say $\{i, j\}$, happens with probability $\frac{2}{n_t}(n_t - 1)$, where n_t is the number of players still in the game at time t. With equal probability, one of the two is then appointed to make a take-it-or-leave offer in terms of utility. If accepted, the responder exits the game with such offer and sells her resource to the proposer, otherwise the meeting breaks. The next turn, till there are players active in the game, opens with a new bilateral random meeting. It must be noted that pay-offs are expressed in terms of utility streams, therefore, the utility of a player, say i, associated to this game is equal to $\sum_{t=0}^{\infty}[(1-\delta)v(M_t) - r_t]\delta^t$, where δ is the common discount factor, M_t represents the bundle owned by i at period t and r_t the payment made, still at period t, to buy a player's resource[7].

The main result of Gul [1989], stated in Theorem 1, asserts that, for the common discount factor approaching unity, the SSPE unique equilibrium of the bargaining process tends to the Shapley value. Furthermore, the same process is efficient, meaning that at each random meeting the proposer will offer exactly the expected continuation pay-off to the responder

[7] Therefore, the value for player i of holding in a single period t a generic bundle composed by the resources owned by a set S of players, with $S \ni i$, is equal to $(1-\delta)v(S)$. This holds since Gul [1989] assumes that $v(S)$ expresses a discounted value of utility.

and this last will accept. Initially, Gul [1989] supposed that strict super-additivity of the characteristic function was a sufficient condition to obtain this result, namely efficiency, but an example of Hart and Levy [1999] disproved it. Although the convergence towards the Shapley value is retained under super-additivity, efficiency in the bargaining process requires the more stringent condition of strict convexity.

A model rooted into Simple Demand Commitment Games

The second type of non-cooperative bargaining game to be considered follows into the category of Simple Demand Commitment Games (SDCG). Firstly envisaged by Bennett and Van Damme [1991] and Selten [1992], Winter [1994] has shown that a particular SDCG protocol leads to the Shapley value in expected terms if the underlining coalitional game is strictly convex. Dasgupta and Chiu [1998] have subsequently simplified the protocol of Winter [1994] obtaining a more general result[8]. We will therefore describe this last protocol.

The bargaining process of Dasgupta and Chiu [1998] can be described as a finite horizon, perfect recall game with no discounting or bargaining costs. It has an underlining coalitional game (N,v) assumed to be strictly convex. In the first step, a random move selects, with equal probability, one of the possible permutations of the set N, a 'fixed' order of players. Let us call Q the list of players generated by this random move. At each next step, a single player, say i, moves. Her choice set includes two possibilities: asking an amount $d_i \in \mathbb{R}$ and passing the turn to the next player or choosing a set of players among her predecessors and forming a coalition with them. This second option entails the obligation for i to pay the chosen players their demands and it causes the game to stop, with the remaining players receiving their stand-alone pay-off[9]. Suppose the ordinal position of i in Q is equal to k and define set Q_{k^i} as the truncation of Q at position k. Therefore, a player j is in Q_{k^i} if her ordinal position in Q is lower than k. If player i decides to leave the game, she will then choose a set of players S, with $S \in \mathscr{P}(Q_{k^i})$ such that this choice maximizes her utility. This implies the following equation: $U_i = \max_{S\in\mathscr{P}(Q_{k^i})}(v(S\cup\{i\}) - \sum_{j\in S} d_j)$. The game has, at most, $|N|+1$ steps.

If we define $Q^n_{k^i}$ as the set of players following i in Q, having, therefore, $Q_{k^i} \cup \{i\} \cup Q^n_{k^i} = N$ and $Q_{k^i} \cap \{i\} \cap Q^n_{k^i} = \emptyset$, Theorem 2 of Dasgupta and Chiu [1998] shows that there is an

[8]Whereas Winter [1994] result requires three equilibrium refinement concepts, namely SPE, subgame consistency [Harsanyi et al., 1988] and strategic equilibrium [Leininger, 1986], Dasgupta and Chiu [1998] only use the first.

[9]Note that both non-selected players preceding i and the ones after i in Q will get their respective stand-alone pay-off.

SPE strategy for which, given a random permutation Q of N, player i gets $v(\{i\} \cup Q^n_{ki}) - v(Q^n_{ki}), \forall i \in Q_{n-1}$ or $v(\{i\})$ for $i \in Q_n$, where Q_{n-1} is the set of all players in Q but the last and Q_n is the singleton set of the last player in the queue. By considering all the possible permutations of N, it is clear that the expected pay-off of a player is her Shapley value of (N, v). Dasgupta and Chiu [1998] derive four results. The first, Theorem 1, states that, for a strictly convex game, there is only one SPE strategy for each player, therefore this result is the unique SPE of the game. In convex games, instead, such a strategy exists (Theorem 2), but it is not necessarily unique. Furthermore, they show (Theorem 3) that this result can be obtained for each type of game by "convexifying" it, meaning, by adding a monotonically increasing reward in the size of the formed coalition and then applying an opportune system of taxation to players' realizations to ensure budget balance. Finally, in Corollary 6, they show that, for three players (and only three) non-convex games with non-empty Core, their bargain protocol always supports an SPE equilibrium leading to an allocation inside the Core.

A model with risk of partial breakdown of negotiations

The two types of games just described departed significantly from the general model presented in the previous section. We turn now the attention towards a bargaining game whose rules are more in line with a standard Rubinstein-type model. The present game can be shortly described as a sequential bargaining game with risk of exclusion, or partial breakdown. It has been firstly proposed by Hart and Mas-Colell [1992], refined by Krishna and Serrano [1995] and finally extended to the NTU case in Hart and Mas-Colell [1996]. It is a finite horizon, perfect recall game with no discounting or fixed costs of bargaining although, as we will see, an additional parameter will partially simulate the effect of discounting.

Also the present bargaining model is based on a standard game in coalitional form (N, v) that is assumed to be 0-normalized and where v is monotonic[10]: $\forall S \in \mathscr{P} \setminus N$ and $i \notin S$, we have $v(S \cup \{i\}) \geq v(S) + v(\{i\})$. The game opens with the random mechanism selecting, with equal probability, one of the n players to be the proposer (random proposer model). This will make a feasible proposal ($\boldsymbol{x} \in \mathbb{R}^{|S|}$ and $\sum_{i \in S} x_i = v(S)$), to a coalition $S \in \mathscr{P}$. If unanimously accepted (it is, therefore, a contingent offer) the game ends with the selected players receiving their offer and the remaining getting their stand-alone pay-off $v(\{i\})$. The departure from a standard Rubinstein model with random proposer comes when a refusal takes place since the game, before reverting to the random move that selects a proposer, has another, intermediate,

[10] Since the present chapter deals with TU games, it will follow the presentation of the model given by Krishna and Serrano [1995] rather than the one in Hart and Mas-Colell [1996]. Although very similar, there are some minor differences. For example, in the latter, monotonicity is assumed, but the game is not 0-normalized.

chance move: with probability $\alpha \in [0,1)$ the actual proposer can remain in the game, but with probability $(1-\alpha)$ she is excluded, receiving her stand-alone pay-off and the game proceeds without her. This is what renders the game, potentially infinite, a finite horizon one. Further, it actually acts as a form of discounting [Hart and Mas-Colell, 1996].

On the side of results, Hart and Mas-Colell [1992] solve the model applying SSPE whereas Krishna and Serrano [1995] show that, under particular conditions, SPE is sufficient to support the uniqueness of the findings of their predecessors. Theorem 2 of Hart and Mas-Colell [1996] states that, for $0 \leq \alpha < 1$, there is a unique SSPE equilibrium in the game and the associated pay-off vector is equal to the Shapley value of the underlying cooperative game (N,v) and that, in the limit of α tending to one, this equilibrium vector will be proposed (and unanimously accepted), regardless the identity of the selected player, in the fist step. Therefore, for $\alpha \to 1$, the bargaining process is efficient. It is interesting to note that efficiency is obtained the more the model gets close to a pure Rubistein-type model with random proposer and vanishing discounting.

As anticipated, Krishna and Serrano [1995], in Theorem 3.1, extended the validity of the first part of the mentioned Theorem 2 by showing that, for $\alpha < \frac{1}{(n-1)}$, the uniqueness of the Shapley value as equilibrium pay-off holds under subgame perfection without requiring stationarity. Instead, for values of α above a minimum variable threshold value, say α^*, they prove that there is effectively a set of SPE equilibria. They point, however, that it is a strict subset of the set of individually rational vectors. This is important since it shows that, under this bargaining protocol, what asserted by the mentioned Proposition 0 of Chatterjee et al. [1993] does not hold. For symmetric games, where the adjective symmetric implies that for S and $T \in \mathscr{P}$ with $|S| = |T|$, it must hold that $v(S) = v(T)$, they managed to precisely characterise the set of SPE equilibrium points, let us call it E, and the range of α values supporting it: $E = \left\{ \boldsymbol{x} : \sum_{i=1}^{n} x_i \leq v(N) \wedge \forall i, x_i \geq \frac{v(S)}{n} \right\}$ if $\frac{1}{|S|} \leq \alpha \leq \frac{1}{(|S|-1)}$. For $\alpha < \frac{1}{(n-1)}$, E is still a singleton having the Shapley value of (N,v) as its sole vector element.

Although the present chapter does not deal with NTU games, it worth to note a result of Hart and Mas-Colell [1996], namely that when this assumption holds for the underlying coalitional game, the limits of the SSPE equilibria pay-off vectors for $\alpha \to 1$ are not the popular solution concepts for this class of games, namely the NTU Shapley and Harsanyi solutions [Harsanyi, 1958, 1963], but the consistent values of Maschler and Owen [1989].

Partial breakdown model's variants: the bidding approach

In the last section of their paper, Hart and Mas-Colell [1996] revert to the TU assumption and offer a summary description of variants of their model with related results. Basically, these variations affect two elements of the model: the identity of the player to be excluded in case of rejection and the probability of exclusion. In particular, it is relaxed the assumption that it is only the proposer the one to be potentially forced out and, furthermore, it is introduced the possibility of players having heterogeneous probabilities of dropping out. Interacting these variations generates a multiplicity of outcomes, in some of which the Shapley value looses its centrality as equilibrium result. Further variants, in the same direction, have emerged subsequently. Before analysing this literature strand, we will consider another modification of the Hart and Mas-Colell [1996] model since this has also been used as the baseline for the previously described variations. This is the model of Pérez-Castrillo and Wettstein [2001].

Excluding the initial random move, the model runs almost identically as the described version of Krishna and Serrano [1995], with the same assumptions holding for the underlying cooperative game (N,v), apart from the fact that Pérez-Castrillo and Wettstein [2001] consider only the limiting case of α being equal to zero. A rejection, therefore, excludes automatically the proposer. The real innovation results in the substitution of the initial random selection of the proposer with a bidding stage to select it. At this step, each player proposes contemporaneously (perfect recall is then dropped) a vector of offers to each of the other players: $\boldsymbol{b}_i \in \mathbb{R}^{n-1}, \forall i \in N$ where each element is $b_i^j \in \mathbb{R}, \forall j \in N \setminus \{i\}$. If we define vector $\boldsymbol{\beta}_i = \sum_{i \neq j} b_i^j - \sum_{j \neq i} b_j^i$, and the scalar $r = \arg\max_i \boldsymbol{\beta}_i$, we see that r individuates the index of the player that has made the highest bid and that, therefore, will be appointed as proposer. In case of equal maximizers, a random draw will select among them. Player $i = r$ will then have to pay immediately the bidden amount and make an offer to all other players that will answer sequentially. The game is then identical as the one described before, with $\alpha = 0$, except that each refusal is followed by a new bidding stage rather than a random move.

Pérez-Castrillo and Wettstein [2001] adopt the second formula used to define the Shapley value to prove, in Theorem 1, that the bidding mechanism just described always implement the same Shapley value in SPE for 0-normalized, monotonic TU games. In particular, all the bids will be identical and the final pay-off to each player is given by:

$$\pi_i = v(N) - v(N \setminus \{i\}) - \sum_{j \in N \setminus \{i\}} b_i^j; \qquad \text{for proposer } i.$$

$$\pi_j = \phi_j(N \setminus \{j\}) + b_i^j,\ \forall j \in N \setminus \{i\}; \qquad \text{for each accepter } j.$$

According to Pérez-Castrillo and Wettstein [2001], the bidding mechanism offers a conceptual advantage compared to the other models implementing the Shapley value since this last is not implemented in expectation but in the first round of negotiations regardless of the identity of the selected proposer. They finally show that the model can be duly modified to support the weighted Shapley value (see Kalai and Samet [1987]) in SPE.

Partial breakdown model: other variants

Before considering the various modifications of the random proposer\bidding stage models with partial breakdown, it is necessary to shortly introduce two new solution concepts. The first, called Solidarity value (SO) and introduced by Nowak and Radzik [1994], is a variant of the Shapley value that substitutes the marginal contribution of a player to a coalition with the average marginal contribution brought by all players to the same coalition, whereas the other, called in Hart and Mas-Colell [1996] the Equal Split (ES)[11], is the simple equal division of the value of the grand coalition among its members. The Solidarity value assigns to each player i the following pay-off:

$$\tau_i(N,v) = \sum_{S \ni i} \frac{(n-|S|)!(|S|-1)!}{n!} A^v(S),$$

with

$$A^v(S) = \frac{1}{|S|} \sum_{j \in S} [v(S) - v(S \setminus j)].$$

Therefore, $A^v(S)$ is the average marginal contribution of all players to coalition S. The Equal Split attributes to each player i the following pay-off:

$$\gamma_i(N,v) = \frac{v(N)}{n}.$$

Note that this solution does not take into account the values of any sub-coalitions, but just $v(N)$. Both solutions share with the Shapley value the axioms of efficiency, symmetry and additivity and both discard the null-player axiom. Whereas the first substitutes it with the axiom of A-null player (see Nowak and Radzik [1994]), the latter does it with the property of nullifying player (see van den Brink [2007])[12].

[11] van den Brink [2007] calls it the Equal Division solution.

[12] van den Brink [2007] has introduced a variant, called equal surplus sharing, obtained by dropping the nullifying player property in favour of the axiom of invariance: $\gamma_i^s(N,v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n}$. In zero-normalized games they are obviously identical.

Hart and Mas-Colell [1996] show that, holding fixed the equal chance of being a proposer, when this role is not affected by the probability of dropping out, whereas the responders, with equal probability, are, the ES solution is obtained in SSPE. Instead, when the possibility of being excluded affects all players and, particularly, the proposer drops out with probability $(1-\alpha)\theta$ whereas the responders, all equally, suffer the risk of exclusion with probability $\frac{(1-\alpha)(1-\theta)}{(s-1)}$, with s being the number of players still active and holding $\theta \in (0,1)$, the SSPE will converge towards a "compromise" between the Shapley value (SV) and the Equal Split dependent on the value of θ: $\theta\mathrm{SV} + (1-\theta)\mathrm{ES}$. Calvo [2008] considers a similar case, where a refusal gives rise to an equal probability of exclusion for each of the players still active in the game. Therefore, either the proposer and each of the responders may drop out after a rejection with probability $(1-\alpha)\frac{1}{s}$, where s is again defined as the number of players still active. For a TU game (N,v), this new setting generates a unique SSPE that is the Solidarity value. Finally, if only the proposer faces the risk of exclusion, but players have attached different probabilities both of being excluded and of being selected as proposers, the SSPE result is the weighted Shapley value, where each player weight is given by $w_i = (p_i(1-\alpha_i))_{i\in N}$, with p_i being the personal probability of being proposer and, analogously, $(1-\alpha_i)$ the one of dropping out.

There is a last general modification, and related variants, of this model to be considered: the introduction of discounting. Several authors have investigated the possibility to add, besides the risk of partial breakdown of negotiations, a discounting of players' pay-off value each time a new round of bargaining takes place. Once again, before looking at this family of models, it is opportune to introduce the cooperative solutions they support. The first, envisaged by Joosten [1996] and named egalitarian Shapley value (ESV) by van den Brink et al. [2013], has already been encountered in the previous paragraph and it is a simple convex combination of the Shapley value and the Equal Split. It will therefore grant to each player the following pay-off:

$$\xi_i(N,v) = \theta\psi_i + (1-\theta)\gamma_i, \quad \theta \in [0,1].$$

The second, still proposed by Joosten [1996]; Driessen and Radzik [2002] and named δ-discounted Shapley value (DSV) in van den Brink and Funaki [2010], awards a player with:

$$\mu_i(N,v) = \textstyle\sum_{S\subseteq N} \frac{(n-|S|)!(|S|-1)!}{n!}[\delta^{n-|S|}v(S) - \delta^{n-|S\setminus\{i\}|}v(S\setminus\{i\})],\ \delta \in [0,1].$$

When $\delta = 0$, this solutions collapses into ES, whereas when $\delta = 1$, it gives the Shapley value. Both the solutions satisfy efficiency, symmetry and additivity, while their distinguishing features are weak monotonicity for the ESV and δ-reducing player property for the DSV [van den Brink and Funaki, 2010].

van den Brink et al. [2013] show that the ESV can be implemented in SSPE with a modification of the model of Pérez-Castrillo and Wettstein [2001]. At first stage, their variant runs identical unless there is a rejection. At this point the two models depart. In van den Brink et al. [2013], $\alpha \in [0,1]$ is the probability that the proposer is eliminated and $(1-\alpha)$ is instead the possibility of a total breakdown, meaning that the bargaining game ceases with all players getting zero. In case the game reaches the second stage, instead, the possibility of total breakdown is removed and a rejection will cause the proposer to be excluded with probability equal to one, therefore the model reverts to Pérez-Castrillo and Wettstein [2001]. Although no proper discount is present here, the possibility of a total collapse can be interpreted in this way according to Dagan and Serrano [1998]. In van den Brink and Funaki [2010], instead, it can be found support for the DSV by simply adding a common discount factor δ for all players while the model remains equal in all other aspects to the one of Pérez-Castrillo and Wettstein [2001]. Calvo and Gutiérrez-López [2016] show that the simple introduction of the discount factor into the model of Hart and Mas-Colell [1996] leads to the DSV in SSPE. In particular, if the introduced discounting factor is represented by ρ, the δ in the formula of the DSV given before will take the following value: $\delta = \frac{\rho(1-\alpha)}{1-\alpha\rho}$, where α and $(1-\alpha)$ are defined as in Hart and Mas-Colell [1996]. The extension holds also for the NTU case. Finally, Kawamori [2016] shows that the same result holds even without the restriction that a proposal must be done to all active players, assumption present in the model of Hart and Mas-Colell [1996] and its variants. The proposer is now free to choose the coalition in addition to the proposal to make to its members. Note that the remaining players are free to negotiate in the next round if the coalition S called by the proposer forms, assuming $S \subset N$ and $|N \setminus S| > 1$. The result obtained by introducing a discount factor is the same as the one of Calvo and Gutiérrez-López [2016], but the greater flexibility of this model is paid by a restriction, stated in Theorem 2 of Kawamori [2016], in terms of the characteristic values of the coalitions for the result to hold.

1.3.3 Egalitarianism and the standard bargaining protocol

After having presented an archetypal model of coalitional bargaining, we have examined a series of protocols supporting different solution concepts, namely the Nucleolus, the Shapley value, the Equal Split and various combinations of these last two. Several of the models described were actually departing considerably from the given benchmark. In this last section, instead, we will revert to a family of models that is very close to our starting protocol and that resembles more faithfully the two persons bargaining model of Rubinstein. As done till here, before considering the non-cooperative side, we will briefly introduce the cooperative solution concepts related to this family of bargaining protocols.

We will start by providing a formal definition of what is probably the most influential set solution concept in cooperative game theory: the Core [Gillies, 1959].

$$C(N,v) = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i\in S} x_i \geq v(S), \forall S \in \mathscr{P} \setminus N \wedge \sum_{i\in N} x_i = v(N) \right\}.$$

Shapley and Shubik [1966] proposed a variant of the Core, the strong ε-Core, that is obtained by subtracting a constant ε to all coalition's characteristic values apart from the one of the grand coalition:

$$C_\varepsilon(N,v) = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i\in S} x_i \geq v(S) - \varepsilon, \forall S \in \mathscr{P} \setminus N \wedge \sum_{i\in N} x_i = v(N) \right\}.$$

Maschler et al. [1979] defined the intersection of all possible strong ε-Cores as the least Core: $C_l(N,v) = \cap_{\varepsilon\in\mathbb{R}} C_\varepsilon(N,v)$. Another interpretation describes the least Core as the non-empty strong ε-Core for which the value of ε is minimum (note that ε can be negative when the Core is non-empty).

In the previous section, we have encountered a solution concept strongly pervaded by the idea of egalitarianism: the Equal Split. We have seen that this solution disregards all coalitions with cardinality different from one and $|N|$. Several authors have tried to reconcile egalitarianism with the strategic dimension of a coalitional game. The Core, that satisfies personal and group rationality, together with efficiency, seems to be a perfect candidate to meet the strategic requirements; egalitarianism can then come into play by selecting the Lorenz maximal set of points inside the Core. This solution has been proposed in Hougaard et al. [2001] and, given the partial nature of the Lorenz ordering, it is a set valued solution. Before giving its mathematical description, it is necessary to define the concept of Lorenz domination that is strictly related to the set of Lorenz maximal points since these lasts are that allocations that are not Lorenz dominated by any other. Given a vector of scalars $\boldsymbol{a} \in \mathbb{R}^n$, define an equally dimensional vector $\boldsymbol{a}^l$, whose elements a^l_j are given by the mapping $f_j(\boldsymbol{a}) = \min(\sum_{i=1}^{j} a_i), \forall j = 1,2,...,n$. Now, given two vectors of scalars, $\boldsymbol{a}$ and $\boldsymbol{b} \in \mathbb{R}^n$, we say that $\boldsymbol{a} >^{L} \boldsymbol{b}$, where the symbol $>^{L}$ stays for Lorenz dominates, if $\boldsymbol{a}^{l} > \boldsymbol{b}^{l}$, meaning that $a^{l}_{j} \geq b^{l}_{j}, \forall j = 1,2,...,n$, with strict inequality holding for at least one j. Therefore, the Core-constrained Lorenz maximal imputations set of Hougaard et al. [2001] can be defined as:

$$C_{L}(N,v) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n} : \boldsymbol{x} \in C(N,v) \wedge \nexists \boldsymbol{y} \in C(N,v) : \boldsymbol{y} >^{L} \boldsymbol{x} \right\}.$$

Dutta and Ray [1989], however, were not satisfied with this formulation. Their idea was to have a solution concept that accounts for the desire of equity, as to say egalitarianism, and for selfish stimuli, where the first element is meant to be a normative principle. Using the Core conditions to satisfy the selfish side and then applying egalitarianism is inappropriate, according to them, since the same egalitarian principle should be at the base of the division of the value of every coalition. Consequently, they developed the idea of the Lorenz cores (C^{L})[13]. They are defined recursively, starting from the bottom of the cooperative pyramid, meaning that first are considered coalitions of size equal to one, then equal to two, proceeding till $|N|$. At each step, it is computed the set of Lorenz un-dominated allocations for each coalition, call it $EL(S)$, that is not necessarily a singleton. Suppose now that the Lorenz core has been defined for all coalitions with cardinality lower than $|S|$. The Lorenz core of S is then defined as:

$$C^{L}(S) = \left\{ \boldsymbol{x} \in \mathbb{R}^{|S|} : \sum_{i \in S} x_{i} = v(S) \wedge \nexists \boldsymbol{y} \in EL(T), T \subset S : \boldsymbol{y} > \boldsymbol{x}(T) \right\},$$

where $\boldsymbol{x}(T)$ is the restriction of vector $\mathbf{x}$ that includes only that allocations whose index is an element of T. Once defined the Lorenz core, the set of Lorenz maximal imputations is defined, muta mutandis, in an identical way as before:

$$C^{L}_{L}(N,v) = \left\{ \boldsymbol{x} \in \mathbb{R}^{n} : \boldsymbol{x} \in C^{L}(N,v) \wedge \nexists \boldsymbol{y} \in C^{L}(N,v) : \boldsymbol{y} >^{L} \boldsymbol{x} \right\}.$$

In Theorem 1, Dutta and Ray [1989] state the uniqueness of their egalitarian allocation. Therefore $C^{L}_{L}(N,v)$ is a singleton and it can be seen as a single valued solution[14]. It must be noted that the Lorenz core is a superset of the core, but this does not imply that the non emptiness of the core guarantees the existence of the egalitarian solution. However, for

[13] We will now provide a very short and summary description of this solution concept. The interested reader can refer to Dutta and Ray [1989] for a formal presentation and to Ray [2007] for a description of an algorithm to compute it.

[14] This does not apply to the Lorenz maximal set defined over the strong Lorenz core, a further refinement presented in Dutta and Ray [1991] that substitutes the concept of Lorenz domination with the one of strong Lorenz domination.

convex games, the authors have proved its existence and further they have shown that its location is inside the core. For this class of games, it must be noted that the Lorenz maximal point over the Lorenz core and over the "standard" Core coincide [Hougaard et al., 2001].

Fixed-order proposers protocol

As already mentioned, a bargaining game with fixed order of proposers is one in which the initial random move does not only select the current proposer among the n players, but lists them in a specific order, according to which the proposer is determined[15], that will be maintained along the whole game. Generally, these models consider a specific order as given and evaluate the outcomes of the game for different possible orders. A first model in this strand has been proposed in Selten [1988]: a zero-normalized, rejector-proposes model without discounting where only one coalition can form. Solved in SSPE, the model had to be coupled with axiomatic properties to refine the large number of equilibria due to the absence of discounting [Bandyopadhyay and Chatterjee, 2006]. Due to this partial non-cooperative nature, we will skip a further analysis of this model in favour of the one proposed in Chatterjee et al. [1993].

The protocol of Chatterjee et al. [1993] is very similar to the one of Selten [1988] apart from two major changes: it has discounting and it allows the formation of more than one coalition. This last aspect implies that, if a coalition S forms after a proposal as been unanimously (by the members of S) accepted and $|S| < |N|$, the remaining players, the members of $N \setminus S$, continue the bargaining process over the set of coalitions $\mathscr{P}(N \setminus S)$, with the order of proposers being unchanged. The addition of discounting leaded Chatterjee et al. [1993] to drop the zero normalization assumption since strategic equivalence is not preserved in this case [Bandyopadhyay and Chatterjee, 2006]. As said, the remaining assumptions underpinning the bargaining protocol[16] are identical to Selten [1988], namely, it is applied the rejecter-proposes rule. The solution concept adopted, SSPE, is the same as well.

Before looking at the results obtained by Chatterjee et al. [1993], it is instructive to consider the building blocks used to reach them, starting with the condition (lemma 1) for a stationary equilibrium response for each player i at player set $S \in \mathscr{P}$. Remembering that, after the

[15]It must be noted that in the models of Selten [1988] and Chatterjee et al. [1993] the selected order governs also the sequence of replies after a proposal. However, since a change in the order of replies is inconsequential for the model's outcome, we can restrict the attention to the sole order of proposers.

[16]It must be noted that the term protocol used in this chapter differs significantly in meaning from Chatterjee et al. [1993]. Here, it defines the whole set of bargaining rules and it is quite close to a synonym of model, whereas in Chatterjee et al. [1993] it takes a more restrictive meaning being a synonym of the order of proposers.

formation of a coalition, say P, the bargaining game continues among the remaining players $N \setminus P$, the player set S simply identifies the set of players still active in the game after a given history: $S = N \setminus P$, with P being the union of all coalitions already exited from the game at time t. Therefore, in a stationary equilibrium response vector $\boldsymbol{x}(S,\delta) \in \mathbb{R}^{|S|}$, each element must respect the following inequality:

$$x_i(S,\delta) \geq \delta \max_{i \in T \subseteq S} [v(T) - \sum_{j \in T \setminus \{i\}} x_j(S,\delta)], \quad \forall i \in S.$$

For i being a proposer, and willing to make an accepted proposal, the expression must be an equality. By solving the simultaneous set of equations for each $i \in S$, it is obtained a no-delay stationary equilibrium vector, meaning a vector for which, after every history and for any proposer, this last will make an acceptable proposal. In Proposition 1, Chatterjee et al. [1993] prove that for every $S \in \mathscr{P}$ and $\delta \in (0,1)$, such vector exists and it is unique. It must be remembered, however, that a no-delay equilibrium vector does not imply by any mean that full cooperation is obtained without delay. It is a much weaker condition, stating that at every proposal will correspond the formation of a coalition, but not necessarily the coalition with all players. Furthermore, it is dependent from the selected order of proposers.

Proposition 3 states the very demanding condition for which the grand coalition is formed with no delay for any possible order of players and for $\delta \to 1$: the game (N,v) must be dominated by its grand coalition. Domination implies that $\frac{v(S)}{|S|} \leq \frac{v(N)}{n}, \forall S \in \mathscr{P} \setminus N$. In this case the equilibrium pay-off vector will coincide with the previously seen Equal Split. Note also that, for such game, this vector is the only element of both $C_L(N,v)$ and $C_L^L(N,v)$. The stringency of this condition is made very clear in Proposition 6 that, conversely, states the requirements for having inefficiency for every order of proposers. Consider a strictly superadditive[17] game and define $\boldsymbol{x}(N,\delta)$ as the equilibrium response vector resulting from solving the simultaneous system of equations described before for $S = N$. According to Proposition 6, then, there always exists a lower bound of the discount factor, call it $\underline{\delta}$, such that for $\delta \in (\underline{\delta}, 1)$ with $\underline{\delta} < 1$, and for every proposers' order, inefficiency will arise if $\sum_{i \in N} x_i(N,\delta) > v(N)$. This last condition can materialize for superadditive games with non-empty core. Its absence, instead, implies that there will be some order of proposers leading to an efficient (without delay) bargaining outcome for δ tending to one. In Proposition 5, Chatterjee et al. [1993] show that in such case the resulting equilibrium allocation is in the Core. Strictly convex

[17] Chatterjee et al. [1993] define efficiency as the condition according to which, for a game (N,v), there is not an equilibrium allocation such that every player can be made strictly better off. Strict superadditivity implies that efficiency can be obtained only when the grand coalition forms.

games follow into this last case where an efficient, full cooperative outcome is dependent upon the selection of the order of proposers. In particular, given a strictly convex game and for a discount factor $\delta \in (\underline{\delta}, 1)$ with $\underline{\delta} < 1$ being again a lower bound, there is an order of proposers that assures a no-delay efficient equilibrium. Moreover, for such an order, the resulting allocation will converge to the Lorenz maximal allocation inside the Core[18].

Resuming briefly what seen in the previous paragraph, it is possible to state that non-emptiness of the Core is a necessary but not sufficient condition for having full cooperation. Besides it, $\sum_{i \in N} x_i(N, \delta) < v(N)$ is also a necessary – and sufficient – condition for reaching an efficient outcome, in the limit of δ tending to one, for at least some proposers' order.

Random proposer protocol

Besides the structure of the underlying coalitional game and the value of the common discounting factor, the model of Chatterjee et al. [1993] is strongly influenced by the order of proposers. Both delay and inefficiency can or cannot arise for the same game (N, v) and δ depending on the selected order. Okada [1996] tries to obviate to this problem adopting the random proposer mechanism. Therefore, either at the beginning of the bargaining process and after a rejection, a random move selects a proposer among the remaining players in the game with equal probability. The other assumptions are identical as in Chatterjee et al. [1993], apart that Okada [1996] considers the underlying coalitional game (N, v) as being essential, superadditive and zero-normalized. The equilibrium concept, SSPE in pure strategies, is also common to both papers.

By slightly changing the notation in Okada [1996] for a better comparison with Chatterjee et al. [1993], it is possible to see that the maximization program defining an equilibrium proposal is actually very similar:

$$x_i(S, \delta) \geq \delta \max_{i \in T \subseteq S} [v(T) - \sum_{j \in T \setminus \{i\}} x_j], \quad \forall i \in S;$$

s.t.

$$x_j \geq v_j^S, \quad \forall j \in T \subseteq S;$$

[18] Recall that for convex games $C_L(N, v) = C_L^L(N, v)$ and they are singletons.

where S is defined again as the set of players still active in the game. The difference, therefore, is represented by the condition $x_j \geq v_j^S$, where v_j^S is defined as the expectation pay-off for player j of the continuation of the game with players set S. In equilibrium, both inequalities $x_i(S,\delta) \geq \delta \max(\cdot)$ and $x_j \geq v_j^S$ must hold with equality. This implies that $\sum_{j \in T \setminus \{i\}} x_j$ can be substituted with $\sum_{j \in T \setminus \{i\}} v_j^S$. It remains to see what this expectation is by using the expression provided in Okada [1996]:

$$v_i^S = \frac{1}{|S|} \left\{ [v(T_i) - \delta \sum_{j \in T \setminus \{i\}} v_j^S] + \delta \sum_{i \in Q_k | k} v_i^S + \delta \sum_{i \notin P_m | m} v_i^{S \setminus P_m} \right\}.$$

The expected pay-off of the continuation of game (S,v) is composed by three additive elements multiplied by $\frac{1}{|S|}$, that reflects the equal probability of the random move. The first element, inside the square brackets, is the pay-off that player i will obtain if called to be the proposer and assuming that she will make an equilibrium proposal to coalition $T \subseteq S$. The second term, instead, displays the pay-off i gets by receiving an acceptable proposal in the next round, given that the next proposer, k, will be one that will select a coalition to which i belongs. Finally, the last term describes the continuation pay-off of i beyond the next round, provided that, therefore, the selected player m will be one proposing to a coalition that excludes i.

With regard to results, the main difference between the random proposer model compared to the fixed order one stays in the timing of agreement. We have seen that Proposition 1 of Chatterjee et al. [1993] states that a no delay equilibrium path exists and it is unique. But it should be stressed that the proposition implies that there is at least one order of proposers, possibly more, allowing for this path[19]. The random proposer model, instead, always features a no delay equilibrium path (Theorem 1 in Okada [1996]). This is mainly due to the fact that randomness eliminates the strategic opportunity that some players might enjoy given a certain order of proposers. Shifting the focus from timing to efficiency, instead, Theorem 3 proofs that, in the limit of $\delta \to 1$, the equilibrium allocation for player set S entails the formation of coalition S itself only if $x_i(S,\delta) = v_i^S = \frac{v(S)}{|S|}, \forall i \in S$. This happens only if the vector $\boldsymbol{x} \in \mathbb{R}^{|S|} = [\frac{v(S)}{|S|}, ..., \frac{v(S)}{|S|}]$ is inside the Core. This is obtained by considering that, in an SSPE equilibrium where coalition S forms, the last term in the expression for v_i^S, $\sum_{i \notin P_m | m} v_i^{S \setminus P_m}$, must necessarily be equal to zero for all $i \in S$ and by solving the resulting system of equations. At player set N, therefore, the Equal Split, provided that this is in the Core of (N,v), is the only SSPE equilibrium according to Okada's random proposer protocol.

[19] In Okada [1996] can be found an example of a four players game with a given order that necessarily entails delay.

Extensions of the random proposer protocol

The random proposer protocol just examined has found two important extensions. One, due to Okada [2011], considers the possibility that players differ both in recognition probability and in time discounting, where the first is simply defined as the probability of being called as proposer by the random move. In Okada [1996], we have seen this was equal to $\frac{1}{|S|}$, with S being the set of active players in the game. This extension, therefore, takes into account asymmetries among players in these two crucial parameters, all other elements of the bargaining protocol being equal as in Okada [1996].

No delay in agreement persists even when considering the mentioned sources of asymmetries. The crucial change brought by their introduction is the allocation that takes place when a coalition is realized. In fact, for player set S, defined as before, and assuming that the SSPE bargaining outcome entails the formation of coalition S, the expected pay-off of a player i depends on a proportion between recognition probabilities and discount factors:

$$v_i^S = \frac{\frac{\alpha_i}{1-\delta_i}}{\sum_{j\in S}\frac{\alpha_j}{1-\delta_j}},$$

where α_i is the probability of player i of becoming a proposer. Okada [2011] further shows that the condition for coalition S to be implemented is that $\sum_{i\in T} v_i^S + \sum_{j\in S\setminus T} v_j^{S\setminus T}(1-\delta_j) \geq v(S), \forall i \in S$. By substituting S with N, we then have a condition for, and the unique SSPE allocation in, the grand coalition of the underlying coalitional game (N, v). It is instructive to note that, for the grand coalition to take place, the SSPE allocation must lay on a non-empty ε-core, where ε is determined by the values of the discounting factors. In particular, when these are common, $\delta \to 1 \Rightarrow \varepsilon \to 0$, reverting to the "standard Core". The last thing that worth to be mentioned is that, by dropping the asymmetry in discounting and considering the limit of δ tending to one while keeping the asymmetry in the recognition probabilities, the SSPE allocation in the grand coalition becomes totally dependent on the same recognition probabilities: $x_i(N,v) = \alpha_i v(N), \forall i \in N$.

The second extension of the random proposer protocol that will be considered is the model of Compte and Jehiel [2010]. Their bargaining game reverts to the case of equal recognition probabilities and common discount factor, where the focus is placed on the limit of $\delta \to 1$. The peculiarities, instead, are twofold: first, only one coalition can form and players remaining outside the winning coalition get zero; second, the considered equilibrium, SSPE, is in mixed, rather than limited to pure, strategies. Both these variants go in the direction of enlarging the feasibility of the grand coalition as a bargaining outcome. The first, in fact, increases the propensity of players to cooperate since their exclusion from a coalition that forms necessarily implies a zero pay-off. Recalling the equation defining the expected continuation value in Okada [1996], this translates into nullifying the third term in the curly brackets. Mixed strategies, on the other side, increase flexibility. As seen, in Okada [1996] full cooperation materializes, asymptotically, only if the Equal Split allocation is inside the Core. In Compte and Jehiel [2010], although the SSPE allocation is still unique, this becomes dependent on the value of the characteristic functions.

As anticipated, Compte and Jehiel [2010] is mainly devoted to investigate efficiency properties for δ tending to one. They then consider games with non-empty Core. No delay holds under this new setting. Their main findings relate to the efficient SSPE allocation and to the conditions for obtaining it, that goes behind the non-emptiness of the Core. In particular, when the grand coalition forms, the unique SSPE allocation is the vector, inside the Core, that maximizes the Nash product, where the maximization is in terms of pay-off values rather than utilities:

$$\boldsymbol{x}(N,v) = \underset{x_i \in C(N,v)}{\arg\max} \prod_{i \in N} x_i.$$

Clearly this vector belongs to the set of Lorenz maximal imputations inside the Core: $\boldsymbol{x}(N,v) \in C_L(N,v)$. Further, for convex games, it will coincide with the unique element of $C_L^L(N,v)$ and, if the Equal Split solution is inside the Core, they will also be coincident. The condition for this vector to be the no delay SSPE outcome of (N,v) is expressed in Proposition 1, that reminds to Property P1 in Compte and Jehiel [2010]. For defining Property P1, consider a balanced game (N,v) and define a new game (N,v,Δ) obtained by subtracting a positive constant Δ to each characteristic value of the original game: $(N,v,\Delta) = v(S) - \Delta, \forall S \in \mathscr{P}, v \in (N,v)$. Define then scalar $\mu^C(N,v)$ as the maximum value of the product of the coordinates of a point inside the Core of a game: $\mu^C(N,v) = \max_{\boldsymbol{x} \in C(N,v)} \prod_{i \in N} x_i$. A game for which Property P1 holds is a game for which there exist a scalar Δ_0 and an open interval $(0,\Delta_0)$ such that, for $\Delta \in (0,\Delta_0)$, $\mu^C(N,v,\Delta) < \mu^C(N,v)$ and $\mu^C(N,v,\Delta)$ is a decreasing function of Δ itself. Proposition 1 of Compte and Jehiel [2010] states that Property P1 is a necessary requisite of a balanced game (N,v) for having an asymptotically efficient SSPE outcome. The authors mention some conditions that a game has to posses for P1 to hold. Among them, it is worth to note that strict convexity is a sufficient one. By considering that the value of μ^C is given by both $v(N)$ and the degree of equality attainable in the bargaining game, we see that subtracting a positive constant to $v(N)$ has clearly a negative effect on μ^C, but if this is offset by the increase in equality granted by the enlargement of the Core through the relaxation of its constraints, then full cooperation cannot take place.

The Core and the standard bargaining protocol

Although we have seen several protocols leading, under certain conditions, to multiple equilibria, the prevalence, till now, has been for models sustaining single valued solutions. Several authors, instead, focused on bargaining models explicitly supporting the Core. It is instructive to consider these works since they further clarify which are the elements of the bargaining protocols that lead to certain outcomes. Four papers appear to be particularly interesting for our purposes: Moldovanu and Winter [1995], Evans [1997], Yan [2003] and Kim and Jeon [2009][20]. The reason of their interest is twofold: besides the fact that, for balanced coalitional games, their set of equilibria coincides with the Core, they are also all based on bargaining protocols closely related with the presented archetypal model.

The first two listed papers share the absence of discounting. In particular, Moldovanu and Winter [1995] consider the same model of Selten [1988] and show that, if it is true that the set of equilibria is dependent on the order of proposers selected by the random move, the intersection of these sets for all the possible orders is exactly the Core, provided the underlying coalitional game is balanced. Therefore, the Core is sustained as an ex ante expectation set of equilibria. The model adopted by Evans [1997] is, instead, very similar to Okada [1996], but coupled with the bidding selection mechanism á la Pérez-Castrillo and Wettstein [2001]. There is, however, a significant difference between the assumptions shaping the two bidding stages, namely, in Pérez-Castrillo and Wettstein [2001] the bid was a vector of payments directed at other players, whereas here it constitutes a waste of resources. An efficient outcome, therefore, must be one in which all bids are equal to zero. In case of equal bids, there is still a random draw that selects the proposer with equal chances.

Under these settings and in absence of discounting, Evans [1997] proved (Theorem 2.1) that the set of SSPE allocations in pure strategies for a balanced game coincides with its Core. If mixed strategies are considered, however, inefficient allocations can arise as SSPE

[20] It must be also mentioned the work of Perry and Reny [1994]. This model, however, is quite peculiar since it considers continuous rather than discrete time. Given this strong departure from the models mentioned in the present chapter, we prefer to omit its description. Another interesting paper is due to Serrano and Vohra [1997] that finds an implementation mechanism for the Core in a market economy. Also this model, however, departs considerably from the scope of the present review.

equilibria. The author further examines the behaviour of the model when discounting is introduced. No efficient outcome is obtained since players will make positive bids to gain the right of proposal. However, by introducing a minor artifice, namely that offers and bids must be made in discrete units, Evans [1997] is able to restore the asymptotic support for the Core. A discrete offer is such if, given a scalar η, the elements of the proposed allocation vector are obtained by multiplying η for any arbitrary non-negative integer: $x_j \in \boldsymbol{x}(S) = k\eta, \forall j \in S, k \in \mathbb{N}, \eta \in \mathbb{R}_+$. In such case, for $\eta \to 0$ and $\delta \to 1$ and considering pure strategies, the set of SSPE allocations converges in Housdorff distance to the Core (Theorem 2.2) [Evans, 1997].

Yan [2003] basically laid the basis for the results obtained in Okada [2011]. Their bargaining protocols are almost identical, random-proposer with a variable vector $\boldsymbol{\alpha}$ of recognition probabilities, with $\boldsymbol{\alpha} \in \mathbb{R}^n$. Yan [2003], however, considers the case of a common discount factor, an essential, normalized and balanced underlying coalitional game (N, v) and adopts the same assumption of Compte and Jehiel [2010] of a single coalition being allowed to form. Remembering that normalization implies $v(N) = 1$, it shows that the ex ante unique SSPE allocation, say $\boldsymbol{x}(N, v) \in \mathbb{R}^n$, of game (N, v), once fixed δ and α is given by $\boldsymbol{x}(N, v) = \boldsymbol{\alpha}$. For $\delta \to 1$, also the ex post unique SSPE allocation will converge to this result. Recalling the result of Okada [2011] where the SSPE asymptotic allocation of a balanced game was $\boldsymbol{y}(N, v) = \boldsymbol{\alpha} v(N)$, the difference is just due to the assumption of normalization in Yan [2003]. Being the results identical, this implies that the assumption of terminating the game after a coalition is formed is not crucial for its achievement, given that Okada [2011] does not make this assumption. The last important result worth to be mentioned is that the equality between the recognition probabilities and equilibrium pay-offs vectors falls apart if the same recognition vector is not, once translated into a pay-offs vector, an element of the Core. In this case, inefficiency will arise.

The last paper to be considered is Kim and Jeon [2009]. It is based on the rejector-proposes bargaining protocol of Chatterjee et al. [1993], with a fixed order of proposers determined by the initial random move. Despite both papers adopt an identical bargaining protocol, there is a crucial distinction in results due to the fact that in Kim and Jeon [2009] mixed strategies are allowed. In the limit of $\delta \to 1$, the possibility to adopt mixed strategies nullifies the prominent role that the selected order of proposers had in the paper of Chatterjee et al. [1993]. No delay is always assured. Further, the authors show that the set of SSPE allocations, for

$\delta = 1$, is equal to the set of solutions of the following minimization program:

$$\min_{x} \sum_{i \in N} x_i;$$
s.t.
$$\sum_{i \in S} x_i \geq v(S), \quad \forall S \in \mathscr{P}.$$

For a balanced game (N, v), it is easily verified that the minimum of $\sum_{i \in N} x_i$ is equal to $v(N)$. It then follows, from the constraints of the minimization program, that the set of solutions coincides with $C(N, v)$ (Theorem 4 in Kim and Jeon [2009]). It must be further noted that, for a game with an empty Core, the same minimization program provides the set of cut-off values below which responders will refuse an offer. By using the words of Yan [2003], we can see better what these cut-off values are:

"In an SSPE players behave as if they used simple strategies, in which a player accepts a proposal if and only if she herself is offered at least a certain cut-off value, and a proposer always includes herself in the nominated coalition and offers the other coalition members their cut-off values."

As in the case of a balanced game, there can be multiple optimal vectors satisfying the given minimization program. Define E as the set collecting them. Assume that a vector $\boldsymbol{x}^* \in E$ represents the cut-off values defining the SSPE strategies of the n players of game (N, v). If the game is balanced, the final vector of pay-offs, $\boldsymbol{\pi} \in \mathbb{R}^n$, for $\delta \to 1$, will converge to $\boldsymbol{x}^*$. However, for a game with an empty Core, this is not the case. Since the equilibrium is reached without delay, this implies that the first player in the selected order, say i, will randomize the choice among the coalitions that guarantee her $\pi_i = v(S) - \sum_{j \in S \setminus \{i\}} \geq x_i^*$, holding $\delta \to 1$. Players in $N \setminus S$, are not guaranteed to obtain a pay-off equal to their cut-off value. According to Kim and Jeon [2009], once a coalition has formed, the bargaining game can continue according to the same rules and a new minimization program, with players $k \in N \setminus S$ and coalitions $T \in \mathscr{P}(N \setminus S)$, applies. However, it is not very clear if allowing mixed strategies is enough to guarantee that no strategic delay is adopted by players[21]. It could be that only by adopting the assumption à la Compte and Jehiel [2010] of a single coalition could avoid strategic delay. Further investigation on this topic seems necessary since the authors appear to have glossed over the issue.

[21] For an example of strategic delay with a fixed order of proposers, see the first example in Okada [1996].

1.4 Conclusions

The present chapter has offered a panoramic view of the vast research production during the last three decades dedicated to bargaining over coalitions. It has started from the basis by describing the main elements of a characteristic function form cooperative model, intended as the normal form of a coalitional model, and then it has proceeded by analysing its non-cooperative counterpart, or else, the extensive form of a coalitional model. The Rubinstein bargaining game, due to its popularity and its ability to describe a general bargaining framework, is by far the most adopted base model even in a coalitional setting, although significant variants are not lacking. It has therefore been presented an archetypal way to adapt this model to the case where multiple coalitions, with distinct values to be distributed among their members in form of pay-offs, are allowed. The chapter has then moved to consider the specific variants emerged in the literature. For clarity and convenience of exposition, these variants have been grouped according to the cooperative solution concepts they support.

Willing to underline some of the common features displayed by a widespread sample of the examined bargaining protocols, it is possible to start from the adopted equilibrium concept. SSPE is by far the most common given the recursive nature and the infinite temporal dimensionality of the bargaining problem they serve to depict. However, when particular characteristics render the model a finite horizon one, SPE is an option. The selection of the proposer, a crucial element in sequential bargaining, is evenly divided between the rejecter-proposes assumption, generally coupled with the fixed order of proposers drawn at the very beginning of the game, and the random proposer device, according to which the random mechanism operates at the start of each new time period. The bidding stage approach is for sure a significant variant that has gained popularity in this field. Other two major divisions seen in this strand of the literature concern the nature of the allowed strategies, pure versus mixed, and the possibility of continuing the bargaining process after a coalition has formed, with some models allowing for it and others not.

Finally, we can state some concluding remarks regarding outcomes and, particularly, regarding the relation between protocols and cooperative solution concepts. Models sustaining the nucleolus are quite peculiar, both in the coalitional side, where the characteristic function is modeled in order to represent specific instances (such as a bankruptcy problem), and in the bargaining side, with a protocol that departs significantly from the standard Rubinstein typology. Furthermore, the implementation of the nucleolus rests upon a specific order of proposers taking place.

The Shapley value is sustained by three families of models: bilateral random meetings, SDCG and partial breakdown. The first sustains the popular cooperative solution concept trough the device of random meetings, that has been accused to impose exogenously the relevance of each possible coalition and, consequently, of each player's marginal contribution. The second, instead, suffers from the drawback to lead to the Shapley value only in expected terms and from the fact that the uniqueness of this result holds solely for a restricted class of games, namely, strictly convex games. The third model type appears to be the more parsimonious in terms of assumptions. It is basically a Rubinstein bargaining model with random proposers where discounting is substituted by the risk of partial breakdown. Although it also leads to he Shapley value in expected terms, through the substitution of the random selection of the proposer with the bidding stage, this drawback can be easily avoided. However, the sustained equilibrium varies according to the assumption regarding which player suffers the risk of exclusion after the rejection of a proposal. When it is only the proposer, then we have the Shapley value, but the more this possibility becomes commonly and evenly shared among responders, the more the SSPE allocation shifts towards the equal division of the worth of the grand coalition.

Egalitarianism with regards to the distribution of final pay-offs appears to be a robust outcome of coalitional bargaining models. As seen in the last part of section three, the more a protocol resembles the original Rubinstein game, the more the previous statement holds true. This seems to come at the expense of efficiency, given that, even for superadditive and balanced games, the conditions to reach an SSPE imputation might be quite stringent, particularly when only pure strategies are allowed. The last fundamental aspect that must be underlined is therefore the role of recognition probabilities since they play a central role in determining efficient outcomes and in shaping pay-offs distribution.

A common feature of literature reviews is to suggest future directions for the research agenda. Given the apparent tension that has emerged between egalitarianism and efficiency, an interesting and promising theme of investigation seems to be the possibility to render endogenous the recognition probabilities in repeated bargaining games.

Chapter 2

The Central Core and the Mid-central Core as Novel Set-valued and Point-valued Solution Concepts for TU Coalitional Games

This chapter proposes two new solution concepts for transferable utility coalitional games that are Core restrictions, the Central Core and the Mid-central Core, with the first being set-valued and the latter point-valued. The basic idea at the root of the Central Core is to allow such Core elements that grant to each player at least the pay-off obtained from the centroid of the extreme points of the Core of the same game but with the worth of the grand coalition reduced to the minimum value such that the game remains balanced. The Mid-central Core is defined as the centroid of the extreme points of the Central Core.

Some basic geometrical properties of the Central Core are then analyzed, showing that it is a convex polytope that coincides with the Core under particular circumstances, or it is a strict subset of the Core. It is further shown that almost all fundamental axiomatic properties of the Core are preserved by these solutions. The Central Core only fails to satisfy the reconfirmation property, whereas the Mid-central Core does not also satisfy consistency and additivity.

Finally, an axiomatization of the Mid-central Core is provided through the adaptation of the mid-point domination property to a coalitional setting. The Mid-central Core is the only solution satisfying individual and group rationality together with aggregate monotonicity and a version of mid-point domination whose reference set is shaped according to the mentioned axioms.

Keywords: Cooperative Game Theory; solution concepts; axiomatization; optimization.

J.E.L.: C71; C78; D63.

2.1 Introduction

After the seminal work of von Neumann and Morgenstern [1944], transferable utility (TU) games in characteristic function form have played a central role in economics, being the main target of that research strand named Cooperative Game Theory. This sub-field of Game Theory is devoted to study games where players have the possibility to form coalitions and to divide, among the same coalition members, the worth that each coalition generates. It entails therefore a bargaining situation, since coalitions' members have to agree upon the division of the amount they have at their disposal, with the addition that they can also decide with whom to coalesce. A solution for a game in characteristic function form can be seen therefore as a tuple (or a set of tuples) (each) containing the set of coalitions that will be formed and the final pay-off, called allocation, that each player in the game will receive. Usually, however, the first element of such tuple is disregarded, or, better, taken as granted, since the coalition of all players is the implicit target. A solution, therefore, is just composed by an allocation vector (a set of allocation vectors).

For TU games in characteristic function form, the list of solution concepts present in the dedicated literature is decisively vast. These can be grouped into classical and not classical solutions[1]. The first includes, in chronological order, the stable set [von Neumann and Morgenstern, 1944], the Shapley value [Shapley, 1953], the Core [Gillies, 1959], the Kernel [Davis and Maschler, 1965] and the Nucleolus [Schmeidler, 1969], whereas, for the latter, we can mention the Egalitarian Solution [Thomson, 1983], the τ-value [Tijs, 1981], the Lorenz-maximal Core [Dutta and Ray, 1989], the Centroid of the Core [González-Díaz and Sánchez-Rodríguez, 2007] and the Alexia value [Tijs et al., 2011]. Both lists should by no means be considered exhaustive.

The Core is undoubtedly an influential solution since, when non-empty, it individuates all that allocation vectors for which no coalition can profitably deviate from full cooperation among all players. However, this set can be rather large and, therefore, scarcely predictive. It comes with no surprise that several of the non-classical solution concepts mentioned before are Core restrictions. The work of Kim and Jeon [2009] has shown that the Core can be expressed as the set of optimal solutions of a simple minimization program. More important, the same program can also be used to individuate an equilibrium demand vector in a non-cooperative bargaining model based on a coalitional game with empty Core. The

[1]The separation I am proposing can be considered rather arbitrary and it is by no means commonly accepted in the literature. Furthermore, note that the following lists consider only solutions for "pure" coalitional games, where the adjective pure implies the number of players being at least three.

basic idea is that the optimal solution vector[2] constitutes a vector of players' endogenous outside options. The rise of such endogenous outside options from strategic bargaining over coalitions has been further underlined in the literature dedicated to legislative bargaining [Binmore and Eguia, 2017].

The novel solution concepts here presented are Core restrictions based on the endogenous outside options just described. The basic idea underpinning them is that every player should get at least her potential outside option that would arise by playing the same coalitional bargaining game but excluding the grand coalition. Furthermore, it will be shown that the sum of the elements of each potential outside options vector equals the minimal worth that the grand coalition should have in order for the game to remain balanced. The two solutions, one set-valued and the other point-valued, are strictly related, being the latter the simple centroid of the extreme points of the former. Besides an analysis regarding their geometrical properties and the Core-related axioms they satisfy, a simple axiomatization of the point-valued solution is presented. It is shown, in fact, that it is uniquely characterized by the Core fundamental properties (individual and group rationality) with the addition of aggregate monotonicity and a particular declination of mid-point domination that incorporates the previous axioms.

The next section will laid down the preliminaries and it will provide a formal description of the solutions. Section three put in evidence some of their geometrical properties; section four will be dedicated to examine which axiomatic properties they satisfy whereas section five will provide an axiomatization of the Mid-central Core. The final section is devoted to conclusions.

2.2 The Central Core and the Mid-central Core

2.2.1 Preliminaries

A coalitional game in characteristic function form is defined by a 2-tuple (N, v), where N is a set of players, supposed to be finite and non-empty, – $N = \{1, 2, ..., n\}$ – and $v : 2^{|N|} \to \mathbb{R}$ is a function that maps each element of the power set of N – indicated as $\mathscr{P}(N)$ – to a real value named coalition worth: $v(S) \in \mathbb{R}, \forall S \in \mathscr{P}(N)$. By convention $v(\emptyset) = 0$. For ease of notation, we define $G = (N, v)$.

A special category of coalitional games is represented by transferable-utility (TU) games

[2] We are here assuming it is unique just for explanatory convenience, but it has not to be necessarily so.

in which $v(S)$ are sets meant to be closed lower half-spaces whose Pareto frontier is given by $\sum_{i \in S} x_i = v(S)$ [Serrano, 2004]. Given a coalition S, define the set of feasible allocations for S, $X(S,v)$, as follows: $X(S,v) = \{\boldsymbol{x} \in \mathbb{R}^s : \sum_{i \in S} x_i \leq v(S)\}$. Note that $X(S,v)$ is convex, compact and comprehensive. An allocation, $\boldsymbol{x}(S,v)$, is therefore an element of $X(S,v)$. The adjective feasible will always be implicitly assumed when talking about an allocation and therefore it will be omitted. Furthermore, $\boldsymbol{x}_S$ and X_S will be used as shorthand for, respectively, $\boldsymbol{x}(S,v)$ and $X(S,v)$. Occasionally, if no confusion arises, they will be substituted by $\boldsymbol{x}$ and X. Given a coalition S and a vector $\boldsymbol{x} \in \mathbb{R}^s, \boldsymbol{x}_T$ will be used to indicate the projection of $\boldsymbol{x}$ on T with $T \subset S$. Note further that the elements of $\boldsymbol{x}$ are always supposed to be indexed by the elements of the respective reference set: $\boldsymbol{x}_S$ will be indexed by the elements of S. Finally, $\boldsymbol{x}(S)$ will be used as a shorthand for $\sum_{i \in S} x_i$.

Some further notational conventions that will be followed along the chapter are the followings: the power set of N will be shortened from $\mathscr{P}(N)$ to $\mathscr{P}$; in representing sets such as $\{i, j, ..., k\}$, the curly brackets will be often omitted for ease of notation when no confusion arises; lower case letters in italics whose upper case equivalent represents a set will be used to indicate the cardinality of the same set: $n = |N|$ and $s = |S|$.

Being solution concepts the main object of Cooperative Game Theory, this preliminary section closes by providing a formal definition, differentiating between set-valued and point-valued.

Definition 2.2.1. (Set-valued solution concept). A set-valued solution concept is a function (occasionally a multifunction) that, from any game G in a given class Γ, maps onto a subset of $\mathbb{R}^n$:

$$\begin{array}{llcl} \phi\colon & \Omega \subset G^N & \rightarrow & P(\mathbb{R}^n) \\ & G & \mapsto & \phi(G) \end{array}$$

Definition 2.2.2. (Point-valued solution concept). A point-valued solution concept is a function that, from any game G in a given class Γ, maps onto a vector in $\mathbb{R}^n$:

$$\begin{array}{llcl} \psi\colon & \Omega \subset G^N & \rightarrow & \mathbb{R}^n, \\ & G & \mapsto & \psi(G) \end{array}$$

Each element of the set onto which a set-valued and a point-valued solution concept maps is therefore a particular type of allocation, as defined previously, where the peculiarity stays in the fact that such allocation is always defined over the whole set of players: $\boldsymbol{x}(N,v)$. Informally, we can then say that a solution concept is a map from a class of games to a set of allocations (set-valued) or to a specific allocation (point-valued). This last clarification has been made since axiomatic properties will be defined in terms of allocations. Furthermore, we define the function ξ exactly in the same way of ϕ, but we will use it in order to define a generic solution concept without specifying if it is set-valued or point-valued.

2.2.2 The novel solution concepts

As anticipated, the solution concepts we are going to present are Core restrictions. It seems opportune, therefore, to start by formally defining the Core.

Definition 2.2.3. (The Core). Given a game $G = (N, v)$, the Core of such game, $C(G)$, is the set of allocations defined as follows:

$$C(G) = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \wedge \sum_{i \in S} x_i \geq v(S), \forall S \in \mathscr{P} \right\}.$$

The Core is fully characterized by three axiomatic properties, namely, efficiency (EFF): $\boldsymbol{x}(N) = v(N)$; individual rationality (INR): $x_i \geq v(i), \forall i \in N$; group rationality (GRR)[3] $\boldsymbol{x}(S) \geq v(S), \forall S \in \mathscr{P} \setminus N, |S| \geq 2$. Clearly, every solution satisfying all these three properties must then be a subset of the Core.

Being our solutions Core restrictions, from now on we will assume that all games under consideration are balanced[4] according to the definition in Shapley [1967]. The new set-valued solution being introduced will be called Central Core and, for a game $G = (N, v)$, we will indicate the set of allocations belonging to the Central Core as $C_c(G)$.

Definition 2.2.4. (The Central Core). Given a game G, the Central Core of G, $C_c(G)$, is the portion of the Core of G included in the positive convex cone having origin in the centroid of the extreme points of the polytope[5] resulting from the following minimization program:

[3]Note that, here, we are slightly departing from the traditional definition of group rationality given by $\boldsymbol{x}(S) \geq v(S), \forall S \in \mathscr{P} \setminus N$. Clearly, this definition encompasses the case of individual rationality that becomes a mere instance of group rationality. Willing to underline the peculiarity of one-member coalitions, we prefer to keep the two axioms distinct by adding the condition $|S| \geq 2$ to group rationality.

[4]Remember that, from the Bondareva-Shapley Theorem, every coalitional game has a non-empty Core if and only if it is balanced.

[5]In this chapter, a polytope indicates a convex, compact space, defined by the intersection of a finite number of half-spaces whereas a polyhedron will indicate a convex, closed but unbounded space still defined by the intersection of a finite number of half-spaces.

$$\mathcal{P} \quad \begin{cases} \min\limits_{x_1,\ldots,x_n} & \sum\limits_{i\in N} x_i \\ s.t. & \sum\limits_{i\in S} x_i \geq v(S), \quad \forall S \in \mathscr{P} \setminus N. \end{cases}$$

Therefore, if we define Z^* as the set of optimal solutions of $\mathcal{P}$, and $E_{Z^*} \subseteq Z^*$ as the indexed set of the extreme points of Z^*, having index set $\mathscr{E}_{Z^*} = \{i \in \mathbb{N} : 1 \leq i \leq |E_{Z^*}|\}$ and elements $(\bar{\boldsymbol{z}}_i)_{i\in\mathscr{E}_{Z^*}}$, the origin of the mentioned convex cone, vector $\boldsymbol{o} \in \mathbb{R}^n$, is then given by: $\boldsymbol{o} = \frac{\sum_{i\in\mathscr{E}_{Z^*}} \bar{\boldsymbol{z}}_i}{|E_{Z^*}|}$. Define then the positive convex cone with vertex $\boldsymbol{o}$ as the set of points $Co(\boldsymbol{o}) = \{\boldsymbol{c} \in \mathbb{R}^n : c_i \geq o_i, \text{for } i = 1,2,...,n\}$. We can then define the set $C_c(G)$ as:

$$C_c(G) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in C(G) \wedge \boldsymbol{x} \in Co(\boldsymbol{o})\}.$$

Alternatively, the Central Core can be defined as follows (the proof of their equivalence will be provided later):

$$C_c(G) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}(N) = v(N) \wedge x_i \geq o_i, \forall i \in N\}.$$

From this second definition, it is clear that the Central Core is the set of all efficient vectors that weakly Pareto dominate $\boldsymbol{o}$.

Once defined the Central Core as a new set-valued solution concept for TU games in characteristic function form, we are ready to present the point-valued solution depending on it. We will call it Mid-central Core[6] and, given a game G, we will express it as $M_c(G)$. Before giving its formal definition, it must be understood that $C_c(G)$ is a polytope. Note, in fact, that its definition could have been stated as follows: $C_c(G) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in C(G) \cap Co(\boldsymbol{o})\}$. Being defined as the intersection of a convex cone and a polytope, it must necessarily be a polytope as well.

Definition 2.2.5. (The Mid-central Core). Given a balanced game G and the polytope $C_c(G)$, the Mid-central Core of G, $M_c(G)$, is defined as the centroid of the extreme points of $C_c(G)$. Therefore, defining E_{C_c} as the indexed set of extreme points of $C_c(G)$, with index set $\mathscr{E}_{C_c} = \{i \in \mathbb{N} : 1 \leq i \leq |E_{C_c}|\}$ and elements $(\bar{\boldsymbol{x}}_i)_{i\in\mathscr{E}_{C_c}}$, we have:

$$M_c(G) = \frac{\sum_{i\in\mathscr{E}_{C_c}} \bar{\boldsymbol{x}}_i}{|E_{C_c}|}.$$

[6]It must be noted that, although presented as novel, this solution has appeared under the name of Revised Nash bargaining solution in Rogna [2016]. In that paper, however, it appears merely as the result of a numerical simulation, without any proper formal definition nor axiomatization being provided.

If this is one way to define the Mid-central Core, we now provide a second one that might turn out to be useful (the proof of their equivalence will be provided later):

$$M_c(G)_i = o_i + \frac{v(N) - \sum_{i \in N} o_i}{|N|}, \forall i \in N.$$

Assume, for now, that the set of optimal solutions of the minimization program $\mathcal{P}$ is a singleton. This is, clearly, the minimum amount to satisfy individual and coalitional rationality, or else, to render cooperation stable [Friedman, 1990]. It then follows, according to this second definition, that the Mid-central Core grants to every player the necessary amount she deserves to satisfy the two mentioned properties, plus an equal share of the difference between the worth of the grand coalition and the minimum amount to render cooperation stable. The Central Core, instead, allows for every Core allocation that grants to a player at least what she receives from the optimal solution of the minimization program $\mathcal{P}$. If we relax the assumption of Z^* being a singleton ($Z^* = E_{Z^*}$), we then select a mid-point, the centroid of E_{Z^*}, as the lower bound vector defining the minimum pay-off each player should obtain. Both solution concepts, therefore, are completely characterized by two elements: the Core and the vector $\boldsymbol{o}$ that, as seen in the introduction, is the centroid of the outside options of a bargaining game played over the same coalitions but excluding $v(N)$.
In the next section we will analyse some basic geometrical properties of the Central Core and, given its dependency from it, of the Mid-central Core.

2.3 Some basic geometrical properties of the Central Core

Before starting our analysis, for ease of notation, it is opportune to rewrite the minimization program $\mathcal{P}$ in matrix form. Let $\boldsymbol{A} \in \mathbb{R}^{|\mathscr{P}| \times |N|}$ be the $((2^n - 1) \times n)$-matrix whose entries, $a_{S,i}, \forall S \in \mathscr{P}, \forall i \in N$, are equal to 1 if $i \in S$ and zero otherwise. Let vector[7] $\boldsymbol{b} \in \mathbb{R}^{|\mathscr{P}|}$ be such that $b_S = v(S), \forall S \in \mathscr{P}$. Analogously, define matrix $\boldsymbol{B}$ and vector $\boldsymbol{f}$ by simply excluding the grand coalition from $\mathscr{P}$. Clearly, $\boldsymbol{B} \in \mathbb{R}^{|\mathscr{P} \setminus N| \times |N|}$ and $\boldsymbol{f} \in \mathbb{R}^{|\mathscr{P} \setminus N|}$. Then, rewrite $\mathcal{P}$ as:

$$\mathcal{P} \quad \begin{cases} \min & z(\boldsymbol{x}) = \mathbf{1}^T \boldsymbol{x} \\ s.t. & \boldsymbol{Bx} \geq \boldsymbol{f}. \end{cases}$$

Note that $\mathbf{1}^T$ is a row vector with n components, all equals to one. Define, instead, the minimization program $\mathcal{P}2$ as follows:

[7] We will follow the convention that vectors are meant to be column vectors, whereas their transpose are row vectors.

$$\mathcal{P}2 \quad \begin{cases} \min & g(\boldsymbol{y}) = \mathbf{1}^T \boldsymbol{y} \\ s.t. & \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{b}. \end{cases}$$

We know that, for a balanced game G, the set of optimal solutions of $\mathcal{P}2$, call it $Z^*_{\mathcal{P}2}$, is actually equal to the Core of G: $Z^*_{\mathcal{P}2} = C(G)$. From this, it is possible to straightforwardly derive a simple but useful proposition.

Proposition 2.3.1. *Consider a balanced game G and define the related minimization program $\mathcal{P}$. Consider then a second game $G' = (N, v')$, with $v'(S) = v(S), \forall S \in \mathscr{P} \setminus N$ and $v'(N) = v(N) - \eta$, with η being the greatest possible scalar such that G' is balanced. Then, $Z^*_{\mathcal{P}} \equiv C(G')$.*

Proof. From the same definition of $\mathcal{P}$, it follows that, for $\boldsymbol{x}^* \in Z^*_{\mathcal{P}}$, $\boldsymbol{x}^*(S) \geq v(S) = v'(S), \forall S \in \mathscr{P} \setminus N$. Furthermore, we know that the Core is the result of a minimization program such as $\mathcal{P}2$ and that, for $\boldsymbol{x}^* \in Z^*_{\mathcal{P}}, \boldsymbol{x}^*(N) = v(N) - \eta = v'(N)$. This is clearly true from the definition of balancedness and from the way in which η has been defined. But then $Z^*_{\mathcal{P}} = \{\boldsymbol{x}^* \in \mathbb{R}^n : \boldsymbol{x}^*(N) = v'(N) \wedge \boldsymbol{x}^*(S) \geq v'(S), \forall S \in \mathscr{P} \setminus N\} = C(G')$. ■

From Proposition 2.3.1 it is possible to derive a straightforward corollary.

Corollary 2.3.1.1. *Given a game G, the minimization program $\mathcal{P}$ is always feasible and its set of optimal solutions is a non-empty polytope.*

Proof. Let us first consider the case of G being balanced. Then, the result follows immediately from Proposition 2.3.1 since G' is balanced by definition and, therefore, $C(G') \equiv Z^*_{\mathcal{P}}$ is a non-empty polytope, given the Core properties.
For G being unbalanced, it is straightforward to note that it exists a (negative) value of η such that G' is balanced. But then we are back in the previous case and we can apply the same reasoning. ■

The last step before discussing the geometrical properties of the newly introduced solution concepts is to provide an alternative and convenient definition of the Central Core.

Proposition 2.3.2. *Given a balanced game $G = (N, v)$, the following statement is true:*

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in Co(\boldsymbol{o}) \cap C(G)\} \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}(N) = v(N) \wedge x_i \geq o_i, \forall i \in N\}.$$

Proof. The condition $x_i \geq o_i$ implies that $\boldsymbol{x}$ must weakly Pareto dominate $\boldsymbol{o}$, further implying that $\boldsymbol{x} \in Co(\boldsymbol{o})$. Furthermore, from the way in which $\boldsymbol{o}$ has been defined, we know that $\boldsymbol{o}(S) \geq v(S), \forall S \in \mathscr{P} \setminus N$. Then $\boldsymbol{x}$ satisfies either individual and group rationality. Since $\boldsymbol{x}(N) = v(N)$ implies efficiency, all the Core conditions are met and, therefore, $\boldsymbol{x} \in C(G)$. Finally, $\boldsymbol{x} \in Co(\boldsymbol{o}) \wedge \boldsymbol{x} \in C(G) \equiv \boldsymbol{x} \in Co(\boldsymbol{o}) \cap C(G)$. This proofs that $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}(N) = v(N) \wedge x_i \geq o_i, \forall i \in N\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in Co(\boldsymbol{o}) \cap C(G)\}$. The converse inclusion, $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in Co(\boldsymbol{o}) \cap C(G)\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}(N) = v(N) \wedge x_i \geq o_i, \forall i \in N\}$, follows straightforwardly. Since $\boldsymbol{x} \in Co(\boldsymbol{o}) \cap C(G) \equiv \boldsymbol{x} \in Co(\boldsymbol{o}) \wedge \boldsymbol{x} \in C(G)$, this necessarily implies that $x_i \geq o_i, \forall i \in N$, from $\boldsymbol{x} \in Co(\boldsymbol{o})$, and $\boldsymbol{x} = v(N)$, from $\boldsymbol{x} \in C(G)$. This proofs the converse inclusion implying that the two sets are identical. ■

From the previous propositions, it is easy to derive several geometrical properties of the Central Core and the Mid-central Core.

Proposition 2.3.3. *Given a balanced game G, if $|C(G)| = 1$, then $\boldsymbol{o} \in C(G)$ and $C(G) \equiv C_c(G) \equiv M_c(G)$.*

Proof. First of all note that $|C(G)| = 1 \Rightarrow Z^*_{\mathcal{P}} \equiv Z^*_{\mathcal{P}2}$. For the sake of contradiction, suppose $Z^*_{\mathcal{P}} \neq Z^*_{\mathcal{P}2}$ and $|C(G)| = 1$. From Proposition 2.3.1, $Z^*_{\mathcal{P}} = C(G')$ and $\boldsymbol{o} \in C(G')$. Furthermore, $\eta \geq 0$ since G is balanced. Consider then a vector $\boldsymbol{x}$ such that $x_i = o_i + \eta$ and $x_j = o_j, \forall j \in N \setminus i$. Clearly, $\boldsymbol{x} \in C(G)$. Consider then another vector, say $\mathbf{y}$, such that $y_k = o_k + \eta; y_j = o_j, \forall j \in N \setminus k$ and $k \neq i$. It is immediate to see that also $\mathbf{y} \in C(G)$, but this contradicts that $C(G)$ is a singleton. Therefore $|C(G)| = 1 \Rightarrow Z^*_{\mathcal{P}} \equiv Z^*_{\mathcal{P}2} \Rightarrow C(G) \equiv Z^*_{\mathcal{P}}$, since $C(G) = Z^*_{\mathcal{P}2}$. The rest follows consequently from the way in which the Central Core and the Mid-central Core have been defined. ■

From this proposition, it follows immediately a simple corollary.

Corollary 2.3.3.1. *Given a balanced game G, $C(G) = Z^*_{\mathcal{P}}$ if and only if $\boldsymbol{o} \in C(G)$.*

Proof. $(\Rightarrow)$ We know that $\boldsymbol{o}$ satisfies individual and group rationality by its definition. Since $\boldsymbol{o} \in C(G)$, then $\boldsymbol{o}(N) = v(N)$, satisfying efficiency too. Being a convex combination of the extreme points of $Z^*_{\mathcal{P}}$, this implies that every other point in $Z^*_{\mathcal{P}}$ satisfies the same conditions, then they must all belong to the Core, implying $C(G) = Z^*_{\mathcal{P}}$.
$(\Leftarrow)$ This is trivially true since $\boldsymbol{o} \in Z^*_{\mathcal{P}}$ by definition and, therefore, $\boldsymbol{o} \in C(G)$ when $C(G) = Z^*_{\mathcal{P}}$. ■

It must be noted that, when $C(G) = Z^*_{\mathcal{P}}$, $C_c(G)$ is a singleton, having $\boldsymbol{o}$ as its unique element, and it coincides with the Mid-central Core: $C_c(G) \equiv M_c(G)$. Once exhausted the examination of the case when $\boldsymbol{o} \in C(G)$, or else, when $Z^*_{\mathcal{P}} \equiv Z^*_{\mathcal{P}2}$, we now turn the attention towards the situation according to which $\boldsymbol{o} \notin C(G)$, that implies $\boldsymbol{o}(N) < v(N)$.

Proposition 2.3.4. *Given a balanced game G such that $\boldsymbol{o} \notin C(G)$, then $C_c(G)$ is a polytope with n extreme points.*

Proof. Recalling the proof of Proposition 2.3.3, we know that we can individuate n points belonging to the Central Core: $\boldsymbol{x}^i : x^i_i = o_i + \eta, x^i_j = o_j, \forall j \in N \setminus i, \forall i \in N$. Clearly, there cannot be other points, say $\boldsymbol{y}$ and $\boldsymbol{z}$, belonging to the Central Core such that $\gamma\boldsymbol{y} + (1-\gamma)\boldsymbol{z} = \boldsymbol{x}^i$, for $\gamma \in (0,1)$. Therefore, we have individuated exactly n extreme points of $C_c(G)$. ∎

From Proposition 2.3.4 it is easily shown the equivalence between the two definitions of the Mid-central Core previously provided.

Corollary 2.3.4.1. *Given a balanced game G, the following statement is true:*

$$\textit{Given} \quad \boldsymbol{x} = \frac{\sum_{i \in \mathscr{E}_{C_c}} \bar{\boldsymbol{x}}_i}{|E_{C_c}|} \quad \textit{and} \boldsymbol{y} \in \mathbb{R}^n : y_i = o_i + \frac{v(N) - \sum_{i \in N} o_i}{|N|}, \forall i \in N \Rightarrow \boldsymbol{x} \equiv \boldsymbol{y}.$$

Proof. When $\boldsymbol{o} \in C(G)$, we have seen that $C_c(G) \equiv M_c(G)$ and its sole element is the same vector $\boldsymbol{o}$. Since $|E_{C_c}| = 1$ and $\boldsymbol{o}(N) = v(N)$, the identity is trivially satisfied.
When $\boldsymbol{o} \notin C(G)$, we can rewrite the LHS by using the insights of Proposition 2.3.4: $\frac{\sum_{i \in \mathscr{E}_{C_c}} \bar{\boldsymbol{x}}_i}{|E_{C_c}|} = \frac{\sum_{i \in N} \boldsymbol{x}^i}{n}$. It is easy to see that $x_i = o_i + \frac{\eta}{n}$. Considering the RHS, from Proposition 2.3.1 we know that $\boldsymbol{o}(N) = v(N) - \eta \Rightarrow x_i = o_i + \frac{\eta}{n}$; the equivalence holds again. No more cases are possible. ∎

Resuming what said till now, in the set of balanced games, the Central Core is either a singleton or a polytope with n extreme points. The last two propositions of the present section are dedicated to individuate the position of such extreme points with respect to the Core.

Proposition 2.3.5. *Given a balanced game G such that $\boldsymbol{o} \notin C(G)$, the n extreme points of $C_c(G)$ lie on the boundaries of $C(G)$.*

Proof. From the Bondareva-Shapley Theorem and the definition of η given in Proposition 2.3.1, we know that there exists a balanced collections $\mathscr{B}$ with balancing weights $\{\lambda_S\}_{S \in \mathscr{B}}$ such that $v(N) - \eta = \sum_{S \in \mathscr{B}} \lambda_S v(S)$. Recalling, again, Proposition 2.3.1, we know that $v(N) - \eta = \boldsymbol{o}(N) = \sum_{S \in \mathscr{B}} \lambda_S v(S)$, and, therefore, $\boldsymbol{o}(S) = v(S), \forall S \in \mathscr{B}$. Finally, from the definition of the n extreme points in Proposition 2.3.4, we have that $\boldsymbol{x}^i(S) = \boldsymbol{o}(S) = v(S), \forall S \in \mathscr{B}, S \not\ni i$. This necessarily implies that $\boldsymbol{x}^i$ is on a face, or else, on the boundary, of $C(G)$. ∎

Proposition 2.3.6. *Given a balanced game G, such that $\boldsymbol{o}(N) < v(N)$, the following statements are true:*

1. *The extreme points of $C_c(G)$ will all coincide with the extreme points of $C(G)$ $[E_{C_c} \equiv E_C]$ if and only if $o_i = v(i), \forall i \in N$.*

2. *If $o_i > v(i), \forall i \in N$, no one of the extreme points of the Central Core will coincide with an extreme point of the Core $[E_{C_c} \cap E_C = \emptyset]$.*

Proof. The first statement is quite trivial. $(\Rightarrow)$ Clearly, when $o_i = v(i), \forall i \in N$, Z^* is a singleton and $\boldsymbol{o}$ is its unique element. Furthermore, it is immediate to see that $C(G) \equiv C_c(G)$. Being the two sets identical, they must have the same extreme points.
The reverse causality $(\Leftarrow)$ implies the truthfulness of the following statement: $\boldsymbol{o}(N) < v(N) \wedge E_{C_c} \equiv E_C \Rightarrow o_i = v(i), \forall i \in N$. Now, since $\boldsymbol{o}(N) < v(N)$, we know that $C_c(G)$ will have exactly n extreme points and, from Proposition 2.3.4, we have an easy way to compute them: $\boldsymbol{x}^i, \forall i \in N$. Consider now a reduced game, call it G^i, such that $G^i = (N \setminus i, v^i) \to v^i(S) = v(S), \forall S \subseteq N \setminus i, v^i(T \setminus i) = v(T) - x^i_i, \forall T \subseteq N, T \ni i$. Consider the Core of G^i. Clearly, for $\boldsymbol{x} \in C(G^i), \boldsymbol{x}(N \setminus i) = \boldsymbol{o}(N \setminus i)$ since $v^i(N \setminus i) = v(N) - x^i_i = v(N) - o_i - \eta = v(N) - o_i - v(N) + \boldsymbol{o}(N) = \boldsymbol{o}(N \setminus i)$. Furthermore, it is easy to see that a vector $\boldsymbol{x} \in \mathbb{R}^n : x_i = x^i_i, x_j = y_j, \forall j \in N \setminus i$, for $\boldsymbol{y} \in C(G^i)$, would be in the Core of the original game G. But for $\boldsymbol{x}^i$ to be an extreme point of $C(G)$, it must then be true that $C(G^i)$ is a singleton and $\boldsymbol{o}(N \setminus i)$ is its unique element. Furthermore, this must be true for every reduced game G^Q, with $G^Q = (N \setminus Q, v^Q) \to v^Q(S) = v(S), \forall S \subseteq N \setminus Q, v^Q(T \setminus Q) = v(T) - \boldsymbol{x}^i(Q), \forall Q \subset N, Q \ni i, \forall i \in N$. Therefore $|C(G^Q)| = 1, \boldsymbol{o}(N \setminus Q) \in C(G^Q), \forall Q \subset N, Q \ni i, \forall i \in N \Rightarrow \boldsymbol{o}(N) < v(N) \wedge E_{C_c} \equiv E_C \Rightarrow o_i = v(i), \forall i \in N$. For the sake of contradiction, consider this to hold but with $o_i > v(i)$, for some $i \in N$ and $o_k = v(k), \forall k \in N, k \neq i$. Then, there must be a coalition $T \neq N, T \ni i, k$, such that $\boldsymbol{o}(T) = v(T)$. Furthermore, for $Q \subset T, Q \ni i, Q \not\ni k, \boldsymbol{x}(T \setminus Q) = v^Q(T \setminus Q)$ should remain binding for $\boldsymbol{o}_{N \setminus Q}$ to be the unique Core element of G^Q. But this is clearly not the case since $v^Q(T \setminus Q) < \boldsymbol{o}(T \setminus Q)$, and we have obtained the desired contradiction.
The proof of the second statement follows directly from the proof of the reverse causality just provided. Since the first statement is in the if and only if form, we have $o_i = v(i), \forall i \in N \Rightarrow |C(G^Q)| = 1, \boldsymbol{o}(N \setminus Q) \in C(G^Q), \forall Q \subset N, Q \ni i, \forall i \in N \Rightarrow \boldsymbol{o}(N) < v(N) \wedge E_{C_c} \equiv E_C$. But then, if $o_i > v(i), \forall i \in N$, it follows that $|C(G^Q)| = 1, \boldsymbol{o}(N \setminus Q) \in C(G^Q), \forall Q \subset N, Q \ni i$ must be false for all $i \in N$. This necessarily implies that no $\boldsymbol{x}^i$ is an extreme point in the Core, and, therefore, $E_{C_c} \cap E_C = \emptyset$. ■

From the two following examples, we can see why the second statement has not been written in the if and only if form and that the relation between the number of extreme points of the Central Core coinciding with the ones of the Core is not so straightforward whenever the number of elements of $\boldsymbol{o}$ equal to the worth of singleton coalitions is in between zero and n.

Example 2.3.1. Consider the game $G^1 = (N,v) : N = \{1,2,3\}, v(1,2) = 5, v(1,2,3) = 8$, all other coalitions have worth of zero. Then $\boldsymbol{o}_1 = [2.5,2.5,0], E^1_{C_c} = \{[5.5,2.5,0],[2.5,5.5,0],[2.5,2.5,3]\}, E^1_C = \{[8,0,0],[0,8,0],[5,0,3],[0,5,3]\}$. Then, the number of elements of $\boldsymbol{o}$ equal to the worth of a singleton coalition is one, but $|E^1_{C_c} \cap E^1_C| = 0$. Consider the game $G^2 = (N,w) : w(1,3) = 5, w(S) = v(S), \forall S \in \mathscr{P} \setminus \{1,3\}$. Then $\boldsymbol{o}_2 = [5,0,0], E^2_{C_c} = \{[8,0,0],[5,3,0],[5,0,3]\}, E^2_C = \{[8,0,0],[5,3,0],[5,0,3],[2,3,3]\}$. In G^2, the number of $\boldsymbol{o}_2$ elements equal to $v(i)$ is two, but $|E^2_{C_c} \cap E^2_C| = 3$.

From this section we can derive the following conclusions that relate the Central Core to the Core on the domain of balanced games. The Central Core is always a subset of the Core and the two sets will be coincident whenever the last is a singleton or when $o_i = v(i), \forall i \in N$. In every other case, the Central Core will be a strict subset of the Core and, whenever $o_i > v(i), \forall i \in N$, all the extreme points of the Core will not belong to the Central Core. This last aspect motivates the name Central Core.

2.4 The axiomatic properties satisfied by the Central Core and by the Mid-central Core

This section is dedicated to examine which axiomatic properties are satisfied by the Central Core and by the Mid-central Core. Clearly, the number of axiomatic properties present in the literature of Cooperative Game Theory is dramatically vast. In order to render such analysis manageable, we need to delimit our domain of investigation. Being both solutions subsets of the Core, it seems natural to start by investigating if the principal Core properties are preserved. Besides them, monotonicity, defined for point-valued solution concepts, is a property that has attracted considerable attention (e.g. Young [1985], Zhou [1991] and van den Brink et al. [2013]), in particular since it has a central role in characterizing a fundamental solution such as the Shapley value [Young, 1985]. For these reasons, some of the most well known properties of the Core together with monotonicity, in four of its several declinations, will be taken into consideration.

Before starting, a premise seems necessary. Being both the solutions under analysis restrictions of the Core, it is tempting to extend all the Core properties to them. This operation, however, can only be partially done. In particular, it is safe till we are considering properties that relate to all the elements of a set. In this case, obviously, the elements of a subset of this set will necessarily share the same properties. However, properties that relate to the set as an object in itself cannot be automatically extended to each of its possible subsets, since the objects will actually be different.

2.4.1 The Core properties

We will start by listing all the Core properties that will be the object of the present analysis. Subsequently, their formal definition will be provided along with the analysis of their satisfaction by the Central Core and by the Mid-central Core.

It has been already mentioned that the Core satisfies efficiency (EFF), individual rationality (INR) and group rationality (GRR). Being these the main properties defining the Core, we will group them under the name of Core fundamental properties (COP). Besides them, it satisfies invariance with regard to strategic equivalence (STI) [Peleg, 1986], together with the dummy player property (DUM). Furthermore, it satisfies anonymity (ANM), a stronger condition than symmetry [Peleg and Sudhölter, 2007] and continuity (CON) [González-Díaz and Sánchez-Rodríguez, 2007]. Peleg [1986, 1992] has characterized the Core by means of the properties of superadditivity (SUPA) and max-consistency (RGPM). Furthermore, the Core satisfies another declination of the consistency property, namely, complement-consistency (RGPC) [Thomson, 2011]. Finally, the Core satisfies the reconfirmation property (RCP) when the assumption of max-consistency (RCPM) underpins the reduced game [Peleg and Sudhölter, 2007] whereas, as it will be shown, it does not when the complement-consistency assumption (RCPC) is adopted.

Having already defined the three COP properties, we start by showing that either the Central Core and the Mid-central Core satisfy them.

Proposition 2.4.1. *The Central Core and the Mid-central Core satisfy COP properties: EFF, INR, GRR.*

Proof. This is clear by the way in which they have been defined and by the fact that they are subsets of the Core. Note, in fact, that these properties are related to Core elements. ■

We now move to examine the property of invariance with regard to strategic equivalence, starting with its definition.

Definition 2.4.1. (Invariance with regard to S-equivalence). A solution concept ξ is said to be invariant to S-equivalent transformations if, given a game $G^1 = (N, v)$, a scalar $\beta \in \mathbb{R}_{++}$, a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ and defining a game $G^2 = (N, w) : w(S) = \beta v(S) + \sum_{i \in S} \alpha_i, \forall S \in \mathscr{P}$, it holds that:

$$\forall \boldsymbol{x} \in \xi(G^1), \quad \exists \boldsymbol{y} \in \xi(G^2) : \boldsymbol{y} = \beta \boldsymbol{x} + \boldsymbol{\alpha}.$$

Proposition 2.4.2. *The Central Core and the Mid-central Core satisfy STI.*

Proof. Since the Core meets STI, and the Central Core has been defined as the intersection of $Co(\boldsymbol{o})$ with $C(G)$, we just have to show that also the former satisfies this property. $Co(\boldsymbol{o})$ is completely dependent on $\boldsymbol{o}$, defined, in turn, by Z^*. It then follows that, if Z^* satisfies STI, also $Co(\boldsymbol{o})$ does. Given a balanced game $G^1 = (N, v)$, define then $G^2 = (N, w)$ such that $w(S) = \beta v(S) + \sum_{i \in S} \alpha_i, \forall S \in \mathscr{P}, \beta \in \mathbb{R}_{++}, \boldsymbol{\alpha} \in \mathbb{R}^n$. Being $\mathcal{P}$ our reference minimization program for G^1, define analogously $\mathcal{P}_w$ as the one related to G^2. Finally, given the set Z^* of optimal solutions to $\mathcal{P}$, define $Z^*_{\beta,\boldsymbol{\alpha}}$ as follows: $Z^*_{\beta,\boldsymbol{\alpha}} = \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y} = \beta \boldsymbol{x}^* + \boldsymbol{\alpha}, \forall \boldsymbol{x}^* \in Z^*\}$. Then, the set of optimal solutions for $\mathcal{P}_w$ is given by: $Z^*_w = \left\{\boldsymbol{g} \in \mathbb{R}^n : \boldsymbol{g} = \min_{\boldsymbol{g}} z(\boldsymbol{g}) = \sum_{i \in N} g_i, \sum_{i \in S} g_i \geq w(S), \forall S \in \mathscr{P} \setminus N\right\}$. Rewrite then the set of constraints $\sum_{i \in S} g_i \geq w(S)$ as $\sum_{i \in S} g_i \geq \beta v(S) + \sum_{i \in S} \alpha_i$ and check that $\boldsymbol{g} = \boldsymbol{y}, \forall \boldsymbol{y} \in Z^*_{\beta,\boldsymbol{\alpha}}$ gives actually Z^*_w. In fact, we have $\sum_{i \in S} y_i \geq \beta v(S) + \sum_{i \in S} \alpha_i \Rightarrow \beta \sum_{i \in S} x^*_i + \sum_{i \in S} \alpha_i \geq \beta v(S) + \sum_{i \in S} \alpha_i \Rightarrow \sum_{i \in S} x^*_i \geq v(S)$. Then, we reverted to $\mathcal{P}$, for which $\boldsymbol{x}^* \in Z^*$ is optimal by definition, implying $Z^*_{\beta,\boldsymbol{\alpha}} \equiv Z^*_w$. Basically, we proved that Z^* is invariant to strategic equivalence, but then $\boldsymbol{o}$ and $Co(\boldsymbol{o})$ are as well and, consequently, $C_c(G) = Co(\boldsymbol{o}) \cap C(G)$ either. The fact that also $M_c(G)$ shares this property is another natural consequence. ■

The properties of anonymity, dummy player and superadditivity will be considered together. It must be noted that superadditivity is a weaker version of the additivity axiom that extends this last property, defined exclusively for point-valued solution concepts, to set-valued ones. When a point-valued solution is considered, they are equivalent [Peleg and Sudhölter, 2007].

Definition 2.4.2. (Anonymity). A solution concept ξ is said to be anonymous if, given a game $G = (N, v)$ and an injective function $\pi : N \to \mathscr{U}$ and defining a game $G^\pi = (\pi(N), \pi v)$, then it holds that $\xi(G^\pi) = \pi(\xi(G))$.

It should be noted that anonymity simply requires a solution concept to be independent from the names of the players [Peleg and Sudhölter, 2007].

Definition 2.4.3. (Dummy player property). A solution concept ξ is said to satisfy the dummy player property if, given a game G and defining a dummy player $i \in N$ as such player for which $v(S \cup i) - v(S) = v(i), \forall S \subseteq N \setminus \{i\}$, it then holds that, $\forall \boldsymbol{x} \in \xi(G), x_i = v(i)$.

Definition 2.4.4. (Superadditivity). A solution concept ξ is said to satisfy superadditivity if, given two games $G^1 = (N, v)$ and $G^2 = (N, w)$ and a third one, $G^3 = (N, u)$, such that

$u(S) = v(S) + w(S), \forall S \in \mathscr{P}$, it then holds that:

$$\text{For } \boldsymbol{y} \in \xi(G^1) \text{ and } \boldsymbol{z} \in \xi(G^2), \exists \boldsymbol{x} \in \xi(G^3) : \boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}.$$

Proposition 2.4.3. *The Core and the Mid-central Core satisfy ANM and DUM. SUPA is held by the Central Core but not by the Mid-central Core.*

Proof. Anonymity is clearly satisfied by the Central Core and the Mid-central Core since both solutions are totally independent from the indexing position (name) of players in N.
Dummy player is definitely a property of the elements of $C(G)$, therefore it must hold for every subset of it.
With regard to superadditivity, we will first prove that it holds for the Central Core. From Theorem 5.4 of Peleg [1986], we know that the Core satisfies superadditivity. To see why this holds also for the Central Core, consider the following three games: $G^1 = (N, v), G^2 = (N, w)$ and $G^3 = (N, u)$, with $u(S) = v(S) + w(S), \forall S \in \mathscr{P}$. We then have $C(G^3) \supseteq C(G^1) + C(G^2)$. Define further $\mathcal{P}^1, \mathcal{P}^2$ and $\mathcal{P}^3$ as the minimization programs that excludes the grand coalition over the respective games and Z_1^*, Z_2^* and Z_3^* their indexed sets of optimal solutions with index sets[8] $\mathscr{X}^1, \mathscr{X}^2$ and $\mathscr{X}^3$. We need then to show that $C_c(G^3) \supseteq C_c(G^1) + C_c(G^2)$. It is easy to see that $C_c(G^1) + C_c(G^2) = Co(\boldsymbol{o}^1) \cap C(G^1) + Co(\boldsymbol{o}^2) \cap C(G^2) = Co(\boldsymbol{o}^1 + \boldsymbol{o}^2) \cap C(G^1 + G^2) = Co(\boldsymbol{o}^1 + \boldsymbol{o}^2) \cap C(G^3)$ where the last inequality holds since, being the Core of G^3 a superset of $C(G^1 + G^2)$, the change does not affect the intersection with $Co(\boldsymbol{o}^1 + \boldsymbol{o}^2)$. For $C_c(G^3) \supseteq \left\{C_c(G^1) + C_c(G^2)\right\}$ to be true, it must then be that $Co(\boldsymbol{o}^3) \supseteq Co(\boldsymbol{o}^1 + \boldsymbol{o}^2)$, or else, that $o_i^3 \leq o_i^1 + o_i^2, \forall i \in N$. Form now vectors $\boldsymbol{e}_k^1$ and $\boldsymbol{e}_j^2$, with $\boldsymbol{e}_k^1 = \boldsymbol{B}\boldsymbol{x}_k^* - \boldsymbol{f}^1, \boldsymbol{x}_k^* \in Z_1^*, k \in \mathscr{X}^1$ and $\boldsymbol{e}_j^2$ being defined analogously. Clearly $\boldsymbol{e}_k^1, \boldsymbol{e}_j^2 \in \mathbb{R}_+^{2^n-1}$. Consider the set of constraints of $\mathcal{P}^3$: $\boldsymbol{B}\boldsymbol{g} \geq \boldsymbol{f}^3 = \boldsymbol{B}\boldsymbol{g} \geq \boldsymbol{f}^1 + \boldsymbol{f}^2$. Clearly, setting $\boldsymbol{g} = \boldsymbol{x}_k^* + \boldsymbol{y}_j^*$, for any $\boldsymbol{x}_k^* \in Z_1^*, \boldsymbol{y}_j^* \in Z_2^*$, satisfies the set of constraints. Furthermore, by translation invariance, $\mathcal{P}3$ can be rewritten as: $\min z(\boldsymbol{h}), \text{s.t. } \boldsymbol{B}\boldsymbol{h} \geq -\boldsymbol{e}_k^1 - \boldsymbol{e}_j^2$. Set Z_3^* will then be equal to: $Z_3^* = \left\{\boldsymbol{g}^* \in \mathbb{R}^n : \boldsymbol{g}^* = \boldsymbol{h}_{k,j}^* + \boldsymbol{x}_k^* + \boldsymbol{y}_j^*\right\}$, where $\boldsymbol{h}_{k,j}^*$ stays for all optima $\boldsymbol{h}$ vectors given a specific $\boldsymbol{x}_k^*$ and $\boldsymbol{y}_j^*$. Since $\boldsymbol{h}^*$ is necessarily non-positive, this implies that, or it is a vector with all components equal to zero, and then $Z_3^* = Z_1^* + Z_2^* \Rightarrow \boldsymbol{o}^3 = \boldsymbol{o}^1 + \boldsymbol{o}^2$, or, that $o_i^3 \leq o_i^1 + o_i^2, \forall i \in N$ since all the extreme points of Z^3 have components lower or equal to the ones of $Z_1^* + Z_2^*$. But then $Co(\boldsymbol{o}^3) \supseteq Co(\boldsymbol{o}^1 + \boldsymbol{o}^2)$ and therefore $C_c(G^3) \supseteq C_c(G^1) + C_c(G^2)$.
In order to prove that the Mid-central Core does not satisfy additivity – remember the equivalence between SUPA and additivity for point-valued solutions – we just need an example.

[8]We skip their formal definition for the sake of brevity.

Example 2.4.1. Consider the following two games, $G^1 = (N,v)$ and $G^2 = (N,w)$: $N = \{1,2,3\}$. If $|S| = 1$, $v(S) = w(S) = 0$. Then, for G^1, we have $v(1,2) = 5, v(1,3) = 7, v(2,3) = 8$ and $v(1,2,3) = 10$. For $G^2, w(1,2) = 10, w(1,2,3) = 12$ and all other coalitions have worth of zero. In G^1 the Core is a singleton, therefore $M_c(G^1) = \boldsymbol{o}^1 = [2,3,5]$. In G^2, $M_c(G^2) = [\frac{17}{3}, \frac{17}{3}, \frac{2}{3}]$. Then, $M_c(G^1 + M_c(G^2) = [\frac{23}{3}, \frac{26}{3}, \frac{17}{3}]$. If we now consider $G^3 = (N,u)$, with $u(S) = v(S) + w(S), \forall S \in \mathscr{P}$, we can see that its Mid-central Core is given by $M_c(G^3) = [\frac{28}{3}, \frac{31}{3}, \frac{7}{3}] \neq [\frac{23}{3}, \frac{26}{3}, \frac{17}{3}]$.

This proves that the Mid-central Core does not always satisfy additivity. ■

The next property to be taken into account is continuity. Due to its technical nature, it seems opportune to introduce an informal description before its formal definition. Continuity requires that, given two games that are close to each other, where closeness is here defined as a measure of similarity between the two games' vectors of characteristic values, the set of solutions prescribed by a solution concept for the two games are also close. A second, more direct, interpretation of this property says that a small perturbation in the characteristic values of a game should not cause large differences in outcomes [Thomson, 2009]. This second interpretation clarifies why this property should be seen as a desiderata for a solution concept: it is a measure of its "robustness".

Definition 2.4.5. (Continuity). A solution concept ξ is said to satisfy continuity if, given a game $G = (N,v)$ and a sequence of games $\{G^k = (N,v^k)\}$ such that $v^k(S) \to v(S), \forall S \subseteq N$, then, it holds that $\{\xi(G^k)\} \to \xi(G)$, where the convergence $\{\xi(G^k)\} \to \xi(G)$ is evaluated in Housdorff topology.

Proposition 2.4.4. *The Central Core and the Mid-central Core satisfy CON.*

Proof. Since we know the Core is stable [González-Díaz and Sánchez-Rodríguez, 2007] – remember that continuity and stability are equivalent [Thomson, 2009] – and that Z^* can be expressed as the Core of a game $G^\eta = (N, v^\eta)$, strictly related to $G = (N,v) : v^\eta(S) = v(S), \forall S \in \mathscr{P} \setminus N, v^\eta(N) = v(N) - \eta$, with η being defined as in Proposition 2.3.1; then Z^* is stable. The stability of a set implies the stability of its boundaries and of its extreme points. Since $\boldsymbol{o}$ depends only on these lasts and $Co(\boldsymbol{o})$ is fully determined by $\boldsymbol{o}$, they will also be stable. Finally, the intersection of two stable sets must be stable as well, and, therefore, $C_c(G)$ is continuous. Following the same line of reasoning also $M_c(G)$ is stable since, again, it is fully determined by the extreme points of a stable set. ■

As done for continuity, also for consistency, or, equivalently, for the reduced game property (RGP), a short informal description anticipates the formal definition. Consistency is the property according to which, given a game and an allocation in the set of possible allocations prescribed by a solution concept for that game, if some of the players leave the game bringing with them their allocation component and the remaining players have to repeat the game over the reduced set of players, the restriction of the original allocation vector to the set of "active" players is included in the set of feasible allocations of the reduced game solved by adopting the same solution concept. If this is the foundation of the idea of consistency, several variants have emerged in the literature – see Thomson [2011] for a review – depending on the assumptions regarding the behavior of players in the reduced game and on the possibility they have to cooperate with the left players [Bhattacharya, 2004]. We will present here two RGP variants. The first, called complement-consistency (RGPC) [Moulin, 1991], assumes that the players in the reduced game can form every coalition that includes the whole subset of left players giving them their allocation components, whereas the second, max-consistency (RGPM) [Davis and Maschler, 1965], grants to each strict subset of the "active" players the possibility to choose the most favorable coalition among all the possible subsets of left players (including the empty one).

The reconfirmation property can be still considered a property aimed at addressing the consistency of a solution. It also entails the idea of a reduced game that may follow the same assumptions seen for the consistency property. Differently from RGP, it requires that each feasible allocation vector prescribed by a solution concept for the reduced game, if combined with the elements of the allocation vector granted to the left players, constitutes a feasible allocation belonging to the solution of the original game.

Definition 2.4.6. (Complement-consistency). Given a game $G=(N,v)$ and its set of solutions $\xi(G)$ with $\boldsymbol{x}\in\mathbb{R}^n$ being an element of this last set – $\boldsymbol{x}\in\xi(G)$ – define a reduced game $G^r=(S,v^r)$ over a set $S\in\mathscr{P}\setminus\{N,\emptyset\}$ in the following way:

$$\begin{aligned}&v(\emptyset)=0;\\&v^r(T)=v(T\cup N\setminus S)-\boldsymbol{x}(N\setminus S),\quad \forall T\subseteq S,\ T\neq\emptyset.\end{aligned}$$

Consider now the set of solutions $\xi(G^r)$. Then, a solution concept ξ is said to satisfy complement-consistency if G^r remains in the same class of games as G and if, $\forall\boldsymbol{x}\in\xi(G)$, it holds that $\boldsymbol{x}_S\in\xi(G^r)$ and this is true $\forall S\in\mathscr{P}\setminus\{N,\emptyset\}$ forming reduced games (S,v^r).

Definition 2.4.7. (Max-consistency). Given a game $G=(N,v)$ and its set of solutions $\xi(G)$ with $\boldsymbol{x}\in\mathbb{R}^n$ being an element of this last set – $\boldsymbol{x}\in\xi(G)$ – define a reduced game $G^r=(S,v^r)$

over a set $S \in \mathscr{P} \setminus \{N, \emptyset\}$ in the following way:

$$v^r(\emptyset) = 0;$$
$$v^r(S) = v(N) - \boldsymbol{x}(N \setminus S);$$
$$v^r(T) = \max_{P \subseteq N \setminus S} (v(T \cup P) - \boldsymbol{x}(P)), \quad \forall T \subset S,\ T \neq \emptyset.$$

Consider now the set of solutions $\xi(G^r)$. Then, a solution concept ξ is said to satisfy max-consistency if G^r remains in the same class of games as G and if, $\forall \boldsymbol{x} \in \xi(G)$, it holds that $\boldsymbol{x}_S \in \xi(G^r)$ and this is true $\forall S \in \mathscr{P} \setminus \{N, \emptyset\}$ forming reduced games (S, v^r).

Definition 2.4.8. (Reconfirmation property). Given a game $G = (N, v)$ and its set of solutions $\xi(G)$ with $\boldsymbol{x} \in \mathbb{R}^n$ being an element of this last set – $\boldsymbol{x} \in \xi(G)$ – define a reduced game $G^r = (S, v^r)$ over a set $S \in \mathscr{P} \setminus \{N, \emptyset\}$ as in Definition 2.4.6 or as in Definition 2.4.7. Define then the set R as follows: $R = \{\boldsymbol{z} \in \mathbb{R}^n : z_i = y_i, \forall i \in S, \boldsymbol{y}_S \in \xi(G^r), z_i = x_i, \forall i \in N \setminus S\}$. Then, a solution concept ξ is said to satisfy the reconfirmation property if G^r remains in the same class of games as G and if $R \subseteq \xi(G)$, with this being true $\forall S \in \mathscr{P} \setminus \{N, \emptyset\}$ forming reduced game (S, v^r). A solution concept is said to satisfy RCPC if such conditions hold and the reduced game has been formed according to the assumptions of Definition 2.4.6, whereas it satisfies RCPM if the reduced game follows the rules of Definition 2.4.7.

Proposition 2.4.5. *Given a balanced game $G = (N, v)$, the Core satisfies RCPM, but it does not satisfy RCPC.*

Proof. From Lemma 2.3.20 in Peleg and Sudhölter [2007], we know the Core satisfies RCPM on every class of games. To see that it does not satisfy RCPC in the class of balanced games, just consider the following example:

Example 2.4.2. Let $G = (N, v)$ be: $N = \{1, 2, 3\}, v(1, 2) = 10, v(1, 2, 3) = 13$, all other coalitions have worth of zero. Then $C(G) = Conv([13, 0, 0], [0, 13, 0], [10, 0, 3], [0, 10, 3])$. Consider then player 1 leaving the game with a granted pay-off equal to 6 and consider the reduced game $G^r = (\{2, 3\}, v^r)$. From Definition 2.4.6 we then have $v(2 \cup \{1\}) = 4, v(3 \cup \{1\}) = -6, v(2, 3 \cup \{1\}) = 7$. Then, $C(G^r) = Conv([13, -6], [2, 5])$. Clearly $\boldsymbol{z} = [6, 13, -6] \in R$, with R being defined as in Definition 2.4.8. But $\boldsymbol{z} \notin C(G) \Rightarrow R \not\subseteq C(G)$.

This shows that the Core cannot always satisfy RCPC. ■

Proposition 2.4.6. *Given a balanced game $G = (N, v)$, the Central Core satisfies either max- and complement-consistency but neither RCPM nor RCPC. The Mid-central Core satisfies none of them.*

Proof. From Thomson [2011] and Peleg [1986], we know that either complement- and max-consistency are satisfied by the Core. Since both the Central Core and the Mid-central Core are subsets of the Core, this necessarily implies that, given a balanced game G, every reduced game G^r remains balanced. Therefore, in the domain of balanced games, the condition of class preservation by the reduced game is met by both solution concepts for both types of consistency property.

We now focus our attention towards the Central Core. By the fact that the Core is consistent, and that the Central Core is a subset of it, it follows that, if $\boldsymbol{x} \in C_c(G)$, then $\boldsymbol{x}_S \in C(G^r), \forall S \in \mathscr{P} \setminus \{N, \emptyset\}$. Clearly, this implies that the condition $\boldsymbol{x}_S \in C_c(G^r)$ holds if $o_i^r \leq o_i, \forall i \in S$, where $\boldsymbol{o}^r$ is the origin of our reference convex cone in the reduced game G^r. This is true for both types of consistency property. Let us start by considering max-consistency and define a reduced game $G^{r1} = (S, v^{r1})$ such that $\boldsymbol{x}_{N \setminus S} = \boldsymbol{o}_{N \setminus S}$. Clearly $\boldsymbol{x}(S) = \boldsymbol{o}(S) + \eta, \forall \boldsymbol{x}_S \in C(G^{r1})$, with η being equal to $v(N) - \boldsymbol{o}(N)$. It is immediate to see that $\boldsymbol{o}^{r1}(S) = \boldsymbol{o}(S)$ and that the centroid of the extreme points of the set of optimal solutions of the minimization program $\mathcal{P}$ related to G^{r1} must be exactly $\boldsymbol{o}_S^{r1} \equiv \boldsymbol{o}_S$. Define instead a game $G^{r2} = (S, v^{r2})$ such that $\boldsymbol{x}_{N \setminus S} \neq \boldsymbol{o}_{N \setminus S}$. It must then be that $\boldsymbol{o}(N \setminus S) + \eta \geq \boldsymbol{x}(N \setminus S) > \boldsymbol{o}(N \setminus S)$. Then, $\boldsymbol{x}(S) = \boldsymbol{o}(S) + \eta - (\boldsymbol{x}(N \setminus S) - \boldsymbol{o}(N \setminus S))$, for $\boldsymbol{x}_S \in C(G^{r2})$. Furthermore, $\boldsymbol{o}^{r2}(S) = \boldsymbol{o}(S) - (\boldsymbol{x}(N \setminus S) - \boldsymbol{o}(N \setminus S) = \boldsymbol{o}^{r1}(S) - (\boldsymbol{x}(N \setminus S) - \boldsymbol{o}(N \setminus S)$. Clearly, the minimization program $\mathcal{P}$ related to game G^{r2} is identical to the one related to game G^{r1} except that all the elements of the constraints vector $\boldsymbol{f}^{r2}$ will be lower or equal compared to the one of vector $\boldsymbol{f}^{r1}$. But then, all the extreme points of its set of optimal solutions will have coordinates that are lower or equal to the ones of the extreme points of the set of optimal solutions related to the game G^{r1}. This implies that the centroid of such extreme points – related to game G^{r2} – will also have coordinates that are lower or equal to the elements of $\boldsymbol{o}_S^{r1}$. Then $o_i^{r2} \leq o_i^{r1} = o_i, \forall i \in S$.

When the reduced game is formed according to the rules of complement-consistency, it is easy to note that the constraints vector, $\boldsymbol{f}$, in the minimization program $\mathcal{P}$ related to G^r, has elements that are lower or equal to an analogous reduced game formed through the max-consistency assumptions. Then, the same reasoning applied before can be extended to such case.

We now prove, through example, that the Mid-central Core does not satisfy either RGPM and RGPC. Since, for point-valued solutions, consistency and the reconfirmation property are equivalent [Peleg and Sudhölter, 2007], this implies that RCPM and RCPC are not satisfied either. Through the same example we show that the Central Core does not satisfy the last two properties too.

Example 2.4.3. Let $G = (N, v)$ be as in Example 2.4.2: $N = \{1, 2, 3\}, v(1, 2) = 10, v(1, 2, 3) = 13$, all other coalitions have worth of zero. Then, $\boldsymbol{o} = [5, 5, 0], C_c(G) =$

$Conv([8,5,0],[5,8,0],[5,5,3])$ and $M_c(G) = [6,6,1]$. Think then of player 1 leaving the game with her Mid-central Core allocation and consider the reduced game $G^r = (\{2,3\}, v^r)$.
Max-consistency: $v(2) = 4, v(3) = 0, v(2,3) = 7$. From this, $\boldsymbol{o}^r = [4,0], C_c(G^r) = Conv([7,0],[4,3])$ and $M_c(G^r) = [5.5,1.5] \neq [6,1]$. Furthermore, $\boldsymbol{x} = [4,3] \in C_c(G^r) \Rightarrow \boldsymbol{z} = [6,4,3] \in R$, but $\boldsymbol{z} \notin C_c(G) \Rightarrow R \nsubseteq C_c(G)$.
Complement-consistency: $v(2 \cup \{1\}) = 4, v(3 \cup \{1\}) = -6, v(2,3 \cup \{1\}) = 7$. We then have $\boldsymbol{o}^r = [4,-6], C_c(G^r) = Conv([13,-6],[4,3])$ and $M_c(G^r) = [8.5,-1.5] \neq [6,1]$. Furthermore, $\boldsymbol{x} = [13,-6] \in C_c(G^r) \Rightarrow \boldsymbol{z} = [6,13,-6] \in R$, but $\boldsymbol{z} \notin C_c(G) \Rightarrow R \nsubseteq C_c(G)$.

This completes the proof. ■

In conclusion, all the considered axiomatic properties satisfied by the Core are preserved by the Central Core, except for RCPM. The Mid-central Core, instead, fails to satisfy consistency and the reconfirmation property, in both the considered forms, and (super)additivity. It must be noted that all such properties entails stricter requirements for point-valued solutions.

2.4.2 Monotonicity in its various declinations

The present section is dedicated to an important axiomatic property for point-valued solution concepts: monotonicity. Following González-Díaz and Sánchez-Rodríguez [2007], four axioms related to monotonicity will be analyzed: strong monotonicity (STM), coalitional monotonicity (COM), aggregate monotonicity (AGM) and weak coalitional monotonicity (WCM).

Definition 2.4.9. (Strong monotonicity). A point-valued solution concept ψ is said to be strongly monotonic if, given two games, $G^1 = (N,v)$ and $G^2 = (N,w)$, such that $v(S \cup i) - v(S) \geq w(S \cup i) - w(S), \forall S \subseteq N \setminus \{i\}$, and letting $\boldsymbol{x} \in \psi(G^1)$ and $\boldsymbol{y} \in \psi(G^2)$, it holds that $x_i \geq y_i$.

Definition 2.4.10. (Coalitional monotonicity). A point-valued solution concept ψ is said to be coalitional monotonic if, given two games, $G^1 = (N,v)$ and $G^2 = (N,w)$, such that, for $T \in \mathscr{P}, v(T) \geq w(T)$ and, $\forall S \in \mathscr{P}, S \neq T, v(S) = w(S)$, and letting $\boldsymbol{x} \in \psi(G^1)$ and $\boldsymbol{y} \in \psi(G^2)$, it holds that $x_i \geq y_i, \forall i \in T$.

Definition 2.4.11. (Aggregate monotonicity). A point-valued solution concept ψ is said to satisfy aggregate monotonicity if, holding all the assumptions of Definition 2.4.10, but substituting $T \in \mathscr{P}$ with $T = N$, it holds that $x_i \geq y_i, \forall i \in T$.

Definition 2.4.12. (Weak coalitional monotonicity). A point-valued solution concept ψ is said to be weakly coalitional monotonic if, holding all the assumptions of Definition 2.4.10, it holds that $\boldsymbol{x}(T) \geq \boldsymbol{y}(T)$.

Proposition 2.4.7. *The Mid-central Core does not satisfy either STM and COM, but it satisfies AGM and WCM.*

Proof. From Theorem 2 of Young [1985], we know that the Shapley value is the only solution to satisfy both symmetry and strong monotonicity. Since the Mid-central Core satisfies symmetry, being this last a weaker axiom than anonymity, it must necessarily fail to satisfy strong monotonicity. However, since the Shapley value can be applied to games with empty Core, it remains open the possibility that, restricting the attention to balanced games, the Mid-central Core could actually satisfy this property. The next example serves to exclude such possibility.

Example 2.4.4. Consider game G^1 from Example 2.4.1 where we had: $N = \{1,2,3\}$, if $|S| = 1, v(S) = 0, v(1,2) = 5, v(1,3) = 7, v(2,3) = 8, v(1,2,3) = 10$. Now consider game $G^2 = (N,w)$ such that $w(1,2) = 7, w(1,2,3) = 13$ and all other coalitions have same worth as in G^1. Consider then player 3 and verify that $w(S \cup i) - w(S) \geq v(S \cup i) - v(S), \forall S \subseteq N \setminus \{i\}$. But $M_c(G^1) = [2,3,5]$ and $M_c(G^2) = [3+\frac{2}{3}, 4+\frac{2}{3}, 4+\frac{2}{3}]$. Then, although the sum of the marginal contributions of player 3 is higher in G^2 than in G^1, its final allocation through $M_c(G)$ is lower.

Theorem 1 in Young [1985] shows that no point-valued solution concept always belonging to the Core can be coalitional monotonic if $|N| \geq 5$. Theorem 1 in Housman and Clark [1998] extended such result for $|N| \geq 4$. This clearly excludes the possibility that the Mid-central Core satisfies COM. Examples can be found in the mentioned papers.
Aggregate monotonicity is easy to prove. Given to games $G^1 = (N,v)$ and $G^2 = (N,w)$, with $v(S) = w(S), \forall S \in \mathscr{P} \setminus N$ and $w(N) \geq v(S)$, AGM requires that, for $\boldsymbol{q} \in M_c(G^2)$ and $\boldsymbol{p} \in M_c(G^1), q_i \geq p_i, \forall i \in N$. Set then $w(N) = v(N) + \varepsilon$, with $\varepsilon \in \mathbb{R}_+$. Clearly $\boldsymbol{o}^1 \equiv \boldsymbol{o}^2$ and, therefore, $q_i - p_i = o_i + \frac{v(N) - \sum_{i \in N} o_i + \varepsilon}{n} - (o_i + \frac{v(N) - \sum_{i \in N} o_i}{n}) = \frac{\varepsilon}{n} \geq 0, \forall i \in N$.
Since WCM is a milder condition than AGM, by satisfying this last, the Mid-central Core necessarily satisfies the former. ∎

The Mid-central Core satisfies the two weaker forms of monotonicity, namely WCM and AGM, whereas it fails to satisfy either COM and STM.

2.5 An axiomatization of the Mid-central Core

In the previous section it has been presented a list of axiomatic properties in order to discuss which of them were satisfied by the Central Core and by the Mid-central Core. Except for monotonicity and for RCPC, all the mentioned properties are satisfied by the Core. What is lacking, therefore, is the identification of a set of properties that uniquely characterize the proposed solutions. This section is aimed at partially filling this gap presenting an axiomatization of the Mid-central Core.

2.5.1 The mid-point domination property

In order to accomplish the proposed task, it is necessary to introduce a further axiomatic property that has not yet been considered in the present chapter: mid-point domination (MPD). Presented in Sobel [1981] and Moulin [1983] and further considered in Chun [1990] and De Clippel [2007], this axiom has been investigated for bargaining problems where no coalitions, apart from the grand coalition and the singletons, with these lasts being considered as disagreement points, can be formed. Borrowing notation from De Clippel [2007], given a set of N players, such bargaining problem is defined by a 2-tuple $(V,\boldsymbol{d})$, with $V \subset \mathbb{R}^n$ being a convex, compact and comprehensive set whose elements are allocation vectors of utility obtainable by players through cooperation, whereas $\boldsymbol{d} \in V$ is the allocation that would prevail if no agreement is reached. By assumption, $d_i \geq 0, \forall i \in N$ and $\exists \boldsymbol{x} \in V : x_i \geq d_i, \forall i \in N$. Clearly, nothing is lost by representing this bargaining problem as a coalitional game $G = (N,v)$ that takes the following form: $v(i) = d_i, \forall i \in N; v(N) = \sum_{i \in N} x_i, \boldsymbol{x} \in V$ and $\boldsymbol{x}$ is Pareto optimal in V; $v(S) \leq \sum_{i \in S} v(i), \forall S \in \mathscr{P} \setminus N$. The assumption $\exists \boldsymbol{x} \in V : x_i \geq d_i, \forall i \in N$ necessarily implies that $v(N) \geq \sum_{i \in N} v(i)$, from which it follows that G is balanced.

Given a bargaining problem $B = (V,\boldsymbol{d})$, mid-point domination requires that a solution to such problem grants to each player at least the average of the extreme positions of the players, where extreme positions are meant to be the most advantageous and most disadvantageous ones [Chun, 1990]. In other words, if a player was holding all the bargaining power, she would naturally choose the most advantageous allocation for herself. By assuming that players hold an equal ability in bargaining, it then comes natural to suppose that the final agreement will converge towards, at least, the average of each player's dictatorial position. This description allows to provide a preliminary formal definition of this axiom.

Definition 2.5.1. (Mid-point domination). Given a bargaining problem $B = (V,\boldsymbol{d})$, a solution $\xi(G)$ satisfies the mid-point domination property if, for $\boldsymbol{x} \in \xi(B), x_i \geq \frac{\sum_{j \in N} a_{i,j}(V)}{n}, \forall i \in N$, where $\boldsymbol{a}_j(V) = \arg\max_{\boldsymbol{a} \in V} a_j, \forall j \in N$.

Although formally correct, this definition glosses over an important aspect: the identification of extreme positions. As stressed in Chun [1990], this is actually a crucial step, from which different variants of MPD may arise. In a bargaining problem B, it seems quite natural to identify the worst possible outcome for a player with her disagreement pay-off, whereas the identification of the most advantageous position is more arguable. Chun [1990] mentions two possible alternatives, the first being simply the maximal feasible pay-off obtainable by a player, whereas the second limits such feasible amount by requiring that all other players are granted their disagreement utility[9]. Practically, this translates into adding some conditions for the definition of vectors $\boldsymbol{a}_j$. In particular, if the first alternative is chosen, we have $\boldsymbol{a}_j(V) = \arg\max_{\boldsymbol{a}\in V|\boldsymbol{a}_{-j}\geq\boldsymbol{0}} a_j, \forall j \in N$, whereas, if the second one is adopted, this changes into $\boldsymbol{a}_j(V,\boldsymbol{d}) = \arg\max_{\boldsymbol{a}\in V|\boldsymbol{a}_{-j}\geq\boldsymbol{d}_{-j}} a_j, \forall j \in N$.

2.5.2 The mid-point domination property in coalitional games

From what seen till now, mid-point domination is a rather intuitive axiom, but, even in a relatively simple setting as the one described by a bargaining problem $(V,\boldsymbol{d})$, it can take different forms depending on the assumptions shaping the feasible reference set over which players act as dictators. It is interesting to note that such assumptions may be the same axioms used to characterize solution concepts. For example, the condition $\boldsymbol{a}_{-j} \geq \boldsymbol{d}_{-j}$ implies the satisfaction of individual rationality, since it restricts the feasible set to the one of individually rational allocations. It seems quite natural that, if we want a solution to satisfy an axiom such as individual rationality, this property must condition also the reference set from which MPD is derived.

In our attempt to extend MPD to a coalitional setting, this last consideration will be taken as a guide. Given the more complicated nature of a coalitional game compared to a bargaining problem of the form $(V,\boldsymbol{d})$, the identification of extreme positions must be updated. In particular, once our reference set has been defined through some axioms, players will gain bargaining power in a consecutive order. This means that, given a particular permutation of players, once the first one in the queue has chosen the best allocation for herself, the second will then get all the bargaining power over the eventual left over, then the third and so on till the last player in the queue. This should be a reasonable assumption since a player is indifferent about the way in which the remaining surplus is distributed among the other players once she has obtained her maximum pay-off. Practically, this implies that vectors $\boldsymbol{a}$s are defined over the set of all possible permutations of players, whose cardinality is equal to $n!$.

We are now ready to introduce three different versions of the mid-point domination

[9] This last is the formulation adopted in De Clippel [2007].

property for coalitional games, with the variations being originated by the underpinning axioms. Before doing that, let us introduce a new very mild axiom and some further notation.

Definition 2.5.2. (Non-negativity property). Given a coalitional game $G=(N,v)$, a solution concept $\xi(G)$ is said to satisfy the non-negativity property (NNP) if, for $\boldsymbol{x}\in\xi(G)$, it holds that $x_i\geq 0, \forall i\in N$.

Let us denote with $\mathscr{O}$ the indexed set of all the $n!$ permutations of set N, with $\mathscr{I}$ its index set and with $\mathscr{B}_i^j, i\in N, j\in\mathscr{I}$, the set of all players preceding i in permutation $\mathscr{N}_j$, with $\mathscr{N}$ being a generic element of $\mathscr{O}$. The three MPD variants differ according to the axioms used to shape the feasible reference set. The first only requires NNP, the second INR and the last either INR and GRR.

Definition 2.5.3. (MPD1). Given a coalitional game $G=(N,v)$, a solution concept $\xi(G)$ is said to satisfy MPD1 if, for $\boldsymbol{x}\in\xi(G)$, it holds that:

$$\begin{aligned}
&x_i \geq \frac{\sum_{j\in\mathscr{I}} a_{i,j}}{n!}, \forall i\in N; \quad \text{where}\\
&\boldsymbol{a}_j(v): a_{i,j} = \underset{\boldsymbol{a}\in v(N);\boldsymbol{a}_{-i}\geq\boldsymbol{0};a_i\leq v(N)-\sum_{k\in\mathscr{B}_i^j}a_k}{\arg\max} a_i, \forall i\in\mathscr{N}_j, \forall j\in\mathscr{I}.
\end{aligned}$$

Definition 2.5.4. (MPD2). Given a coalitional game $G=(N,v)$, a solution concept $\xi(G)$ is said to satisfy MPD2 if, for $\boldsymbol{x}\in\xi(G)$, it holds that:

$$\begin{aligned}
&x_i \geq \frac{\sum_{j\in\mathscr{I}} a_{i,j}}{n!}, \forall i\in N; \quad \text{where}\\
&\boldsymbol{a}_j(v): a_{i,j} = \underset{\boldsymbol{a}\in v(N);\boldsymbol{a}_{-i}\geq\boldsymbol{v(-i)};a_i\leq v(N)-\sum_{k\in\mathscr{B}_i^j}a_k}{\arg\max} a_i, \forall i\in\mathscr{N}_j, \forall j\in\mathscr{I}.
\end{aligned}$$

Definition 2.5.5. (MPD3). Given a coalitional game $G=(N,v)$, a solution concept $\xi(G)$ is said to satisfy MPD3 if, for $\boldsymbol{x}\in\xi(G)$, it holds that:

$$\begin{aligned}
&x_i \geq \frac{\sum_{j\in\mathscr{I}} a_{i,j}}{n!}, \forall i\in N; \quad \text{where}\\
&\boldsymbol{a}_j(v): a_{i,j} = \underset{\boldsymbol{a}\in v(N);\sum_{f\in S}a_f\geq v(S),\forall S\in\mathscr{P};a_i\leq v(N)-\sum_{k\in\mathscr{B}_i^j}a_k}{\arg\max} a_i, \forall i\in\mathscr{N}_j, \forall j\in\mathscr{I}.
\end{aligned}$$

2.5.3 An axiomatization of solution concepts through the mid-point domination property

Through these three extensions of the MPD property to coalitional games and the axioms that define the respective reference sets, it is possible to easily and parsimoniously characterize some well known solution concepts: the Egalitarian Solution (ES) – $\boldsymbol{x} \in \xi(G) : x_i = \frac{v(N)}{n}, \forall i \in N$ – [Thomson, 1983], the Center of the Imputation Set value (CIS) – $\boldsymbol{x} \in \xi(G) : x_i = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}, \forall i \in N$ – [Driessen and Funaki, 1991] and the Centroid of the Extreme Points of the Core (CEPC) – $\boldsymbol{x} \in \xi G : \boldsymbol{x} = \frac{\sum_{\bar{\boldsymbol{y}} \in E_C} \bar{\boldsymbol{y}}}{|E_C|}$, with E_C being the set of extreme points of $C(G)$ –.

Proposition 2.5.1. *Given a coalitional game $G = (N, v)$ such that $v(N) \geq 0$, a solution concept satisfies NNP and MPD1 if and only if it is the Egalitarian Solution.*

Proof. ($\Rightarrow$) It is immediate to see that, for i being the first player in $\mathscr{N}_j, \boldsymbol{a}_j = a_{i,j} = v(N), a_{k,j} = 0, \forall k \neq i \in N$. Since i is in the first position in exactly $(n-1)!$ of the $n!$ partitions of N, $\frac{\sum_{j \in \mathscr{I}} a_{i,j}}{n!} = \frac{(n-1)!v(N) + (n-1)!(n-1)0}{n!} = \frac{v(N)}{n}$.
($\Leftarrow$) Clearly, there are no other feasible allocations $\boldsymbol{x}$ such that $x_i \geq \frac{v(N)}{n}, \forall i \in N$. ■

Proposition 2.5.2. *Given a coalitional game $G = (N, v)$ such that $v(N) \geq \sum_{i \in N} v(i)$, a solution concept satisfies INR and MPD2 if and only if it is the Center of the Imputation Set value.*

Proof. Just consider that, for i being the first player in $\mathscr{N}_j, a_i = v(N) - \sum_{k \neq i \in N} v(k)$ and $a_k = v(k), \forall k \neq i \in \mathscr{N}_j$. The rest of the proof runs identically as the previous one. ■

Proposition 2.5.3. *Given a balanced coalitional game $G = (N, v)$, a solution concept satisfies INR, GRR and MPD3 if and only if it is the Centroid of the Extreme Points of the Core.*

Proof. ($\Rightarrow$) Clearly, every vector $\boldsymbol{a}_j$ must necessarily coincide with an extreme point of the Core of G. Therefore, if the Core is a singleton, Proposition 2.5.2 holds trivially. The same applies if the number of extreme points of the Core is maximal, as to say when the Core has $n!$ extreme points. Let us consider an intermediate case: a Core with two extreme points, $\bar{\boldsymbol{y}}_1$ and $\bar{\boldsymbol{y}}_2$. Then, there must be a player i such that $\bar{y}_1^i > \bar{y}_2^i$ and a player k such that the opposite holds, whereas $\bar{y}_1^f = \bar{y}_2^f, \forall f \neq i, j \in N$. Therefore, for all $\mathscr{N}_j$ where i precedes j, $\boldsymbol{a}_j = \bar{\boldsymbol{y}}_1$ and, conversely, for $\mathscr{N}_j$ with j preceding i, $\boldsymbol{a}_j = \bar{\boldsymbol{y}}_2$. Since there are exactly $\frac{n!}{2}$ elements in $\mathscr{O}$ with i preceding j and an equal number with the opposite holding, $\frac{\sum_{j \in \mathscr{I}} \boldsymbol{a}_j}{n!} = (\frac{n!}{2}\bar{\boldsymbol{y}}_1 + \frac{n!}{2}\bar{\boldsymbol{y}}_2) \,/\, n! = \frac{\bar{\boldsymbol{y}}_1 + \bar{\boldsymbol{y}}_2}{2} = \frac{\sum_{\bar{\boldsymbol{y}} \in E_C} \bar{\boldsymbol{y}}}{|E_C|}$. It is easy to see that the same reasoning applies for a Core with a number of extreme points between two and $n!$.

($\Leftarrow$) Since the relation $x_i \geq \frac{\sum_{j \in \mathscr{I}} a_{i,j}}{n!}$ holds with equality $\forall i \in N$, and the resulting allocation is Pareto Optimal, there cannot be other feasible allocation vectors satisfying the same set of n inequalities. ■

It is important, for a reason that will become clear later on, to make a digression regarding monotonicity. Either the Egalitarian Solution and the CIS value satisfy aggregate monotonicity [Yokote and Funaki, 2017]. González-Díaz and Sánchez-Rodríguez [2007] have proven that the Core Centroid does not satisfy this property. Through the same example they have used, it is possible to show that also the Centroid of the Extreme Points of the Core does not satisfy AGM.

Proposition 2.5.4. *In the class of balanced games, the Centroid of the Extreme Points of the Core does not always satisfy aggregate monotonicity.*

Proof. The proof will be by example.

Example 2.5.1. Consider the game $G^1 = (N, v^1)$ with $N = \{1,2,3,4\}, v^1(S) = 0$ if $|S| = 1$ or $S = \{1,2\}$ or $S = \{3,4\}$; $v^1(S) = 2$ if $S = N$ or $S = \{2,3,4\}$, otherwise $v^1(S) = 1$. Then, $C(G^1)$ is a singleton and its unique element is $[0,0,1,1] =$ CEPC(G^1). Consider game G^2, identical to G^1 except for $v^2(N) = 3$. Then $C(G^2) = Conv([1,0,1,1],[0,0,2,1],[0,0,1,2],[0,1,1,1],[1,1,1,0],[1,1,0,1],[1,2,0,0])$. It follows that CEPC($G^2$) $= [\frac{3}{7}, \frac{3}{7}, \frac{6}{7}, \frac{6}{7}]$. Either player 3 and player 4 have their allocation diminished through CEPC in G^2 compared to G^1.

Therefore, CEPC does not satisfy AGM in the class of balanced games. ■

As the reader has probably already foreseen, aggregate monotonicity is fundamental to characterize the Mid-central Core. As done in the previous definitions of mid-point domination, the introduction of a new axiom is incorporated into the same definition of MPD in order to duly modify the reference set. Let us then define MPD4 on the domain of balanced games, such that INR, GRR and AGM are all satisfied. As for the previous extensions of MPD to a coalitional setting, MPD4 preserves the principle of mediating between extreme positions. Once again, the definition of extreme positions is made dependent on other axioms, with MPD4 further adding AGM to MPD3. Therefore, it can be described as an axiom of constrained egalitarianism where egalitarianism is represented by the choice of a mid-point, whereas the constraints are the other axioms that must be respected in order to shape the reference set of extreme positions.

Definition 2.5.6. (MPD4). Given two balanced coalitional games $G^1 = (N, v^1)$ and $G^2 = (N, v^2)$ such that $v^1(S) = v^2(S), \forall S \in \mathscr{P} \setminus N$ and $v^1(N) = v^2(N) + \varepsilon$, with $\varepsilon \geq 0$, a solution concept $\xi(G)$ is said to satisfy MPD4 if, for $\boldsymbol{x} \in \xi(G^1)$ and $\boldsymbol{y} \in \xi(G^2)$, it holds that:

$$x_i \geq \frac{\sum_{j\in\mathscr{I}} a^1_{i,j}}{n!}, \quad y_i \geq \frac{\sum_{j\in\mathscr{I}} a^2_{i,j}}{n!}, \; x_i \geq y_i \forall i \in N; \quad \text{where}$$

$$\boldsymbol{a}^2_j(v^2) : a^2_{i,j} = \underset{\boldsymbol{a}^2\in v(N);\sum_{f\in S} a^2_f \geq v(S),\forall S\in\mathscr{P};a^2_i\leq v(N)-\sum_{k\in\mathscr{B}^j_i} a^2_k}{\arg\max} a^2_i, \forall i \in \mathscr{N}_j, \forall j \in \mathscr{I}.$$

$$\boldsymbol{a}^1_j(v^1) : a^1_{i,j} = \underset{\boldsymbol{a}^1\in v(N);\boldsymbol{a}^1_{-i}\geq \boldsymbol{y}_{-i}}{\arg\max} a^1_i, \forall i \in \mathscr{N}_j, \forall j \in \mathscr{I}.$$

Proposition 2.5.5. *In the set of balanced games, a solution concept satisfies INR, GRR, AGM and MPD4 if and only if it is the Mid-central Core.*

Proof. ($\Rightarrow$) Trivially, $v^2(N) = v^1(N) - \varepsilon$. Since both games are balanced by assumption, this implies that $\varepsilon \in [0,\eta]$, with η being defined as in Proposition 2.3.1. For $\varepsilon = 0, v^1(N) = v^2(N)$ and it is immediate to see that $\boldsymbol{x} \equiv \boldsymbol{y}$. For $\varepsilon \in (0,\eta)$, and i being the first player in $\mathscr{N}_j$, clearly $a^1_{i,j} = v(N) - \sum_{k\neq i\in N} y_k$, and $a^1_{k,j} = y_k, \forall k \neq i \in N$. From the proof of Proposition 2.5.1 and Proposition 2.5.2 and given that $v(N) - \sum_{k\neq i\in N} y_k = y_i + \varepsilon$, it is immediate to see that $x_i = y_i + \frac{\varepsilon}{n}, \forall i \in N$. Now, since $\varepsilon \in (0,\eta)$ by assumption, there is another game, say $G^3 = (N, v^3)$, such that $v^3(S) = v^2(S), \forall S \in \mathscr{P} \setminus N$, and $v^3(N) = v^2(N) - \varepsilon$. Therefore, by AGM, $\xi(G^2)$ must respect the same conditions towards $\xi(G^3)$ as $\xi(G^1)$ towards $\xi(G^2)$. It must then hold that, for $\boldsymbol{z} \in \xi(G^3), y_i = z_i + \frac{\varepsilon}{n}, \forall i \in N$. By applying repeatedly the same reasoning, we arrive at a game, say G^k, where $v^k(N) = v^{k-1}(N) - \varepsilon$ and $\varepsilon \to 0$, otherwise G^k would not be balanced. Then, $v^k(N) = v^1(N) - \eta \Rightarrow x_i = g_i + \frac{\eta}{n}, \forall i \in N$, for $\boldsymbol{g} \in \xi(G^k)$. By Proposition 2.5.3, $\boldsymbol{g} = \boldsymbol{o}(N,v) \Rightarrow x_i = o_i + \frac{\eta}{n} = o_i + \frac{v^1(N)-v^k(N)}{n} = o_i + \frac{v^1(N)-\sum_{i\in N} o_i}{n}, \forall i \in N$. But this is exactly the definition of the Mid-central Core.
($\Leftarrow$) The proof of the only if part is identical to the one of Proposition 2.5.3. ■

It must be noted that efficiency has never been used to characterize the mentioned solutions although they all satisfy such property. This is due to the fact that this axiom is implicitly incorporated into MPD, in line with the findings of De Clippel [2007]. Furthermore, it is immediate to see that all the axioms used to characterize the Mid-central Core are necessary. By dropping AGM and consequently replacing MPD4 with MPD3 we obtain CEPC. If only GRR is omitted, we then have the CIS value; whereas if both GRR and INR are dropped, the result is the Egalitarian Solution. If MPD is left out, there are several well known solutions satisfying INR, GRR and AGM, among which the Per-capita Prenucleolus and the Aggregate-monotonic Core [Calleja et al., 2009]. Finally, if only INR is dropped, maintaining GRR and defining an appropriate version of MPD, this might clearly result in an allocation laying outside the Core and therefore it cannot coincide with the Mid-central Core.

As a final remark, it should be noted that the Mid-central Core can be described as a mixture of different inspiring principles: a participation constraint represented by INR and GRR, a fairness consideration together with an efficiency principle guaranteed by MPD and, finally, the application of the strongest possible declination of monotonicity applicable to a Core dependent solution concept.

2.6 Conclusions

This chapter has presented two novel solution concepts for TU games in characteristic function form: the Central Core and the Mid-central Core. They are both Core restrictions, consequently fully dependent on it, with the first being a set-valued solution concept, whereas the latter a point-valued. The restriction imposed by the Central Core on the set of points belonging to the Core of a game G is given by the intersection of this last with the positive convex cone having origin in the centroid of the extreme points of an identical game but with the worth of the grand coalition diminished till being the lowest possible value such that the game remains balanced. The Mid-central Core is simply defined as the centroid of the extreme points of the Central Core.

The chapter has further analyzed some basic geometrical properties of the Central Core with particular regard to its relation with the Core. Quite intuitively, if the worth of the grand coalition of a balanced game is equal to the minimum value it could have in order for the game to be balanced, then the Central Core and the Core will be coincident if this last set is a singleton. Instead, whenever $v(N)$ is greater then this minimum value and $o_i > v(i), \forall i \in N$, the Central Core will be a strict subset of it and, furthermore, it will be a polytope with n extreme points all lying on the boundaries of the Core.

The next step has been to investigate which, among the most known axiomatic properties satisfied by the Core, are preserved by the two newly introduced solution concepts. Clearly, they satisfy all that properties pertaining to the elements of the Core, being subsets of it. Among these, we have efficiency, individual and group rationality and dummy player. Furthermore, they both satisfy invariance to strategic equivalence, anonymity and continuity. The Mid-central Core fails to be additive, whereas a weakened version of additivity for set-valued solution concepts, namely SUPA, holds for the Central Core. This last satisfy either max- and complement-consistency, whereas the reconfirmation property is not satisfied in both cases. The Mid-central Core, on the contrary, does not satisfy any of them. Finally, it fails to satisfy strong and coalitional monotonicity, but it respects both aggregate and weak coalitional monotonicity.

The last section has been dedicated to provide an axiomatization of the Mid-central Core. This has been achieved by adapting a property, usually applied to bargaining problems, to coalitional games: mid-point domination. In particular, it has been shown that individual and group rationality together with aggregate monotonicity and a version of mid-point domination whose reference set is shaped according to the mentioned axioms are sufficient and necessary conditions to characterize the Mid-central Core.

Chapter 3

The Burning Coalition Bargaining Model

Finding support for cooperative solution concepts through non-cooperative bargaining models is the basic idea of the Nash program. The present chapter pursues precisely this goal. In particular, it tries to find support for a newly introduced solution concept, the Mid-central Core, for transferable utility games in characteristic function form. It does so through a standard alternating offers model, the Burning Coalition Bargaining Model, where discounting is substituted by the risk of partial breakdown of negotiations. However, it presents a significant novelty. The risk of partial breakdown, in fact, differs from the standard characterization that it has in the dedicated literature since the rejection of a proposal triggers the possibility of dissolution of the worth of the proposed coalition, rather than the exclusion of some players. It is shown that, for balanced games, it exists a response strategy profile according to which the Mid-central Core is supported as the unique equilibrium for the value of the parameter determining the risk of partial breakdown tending to zero. The conditions for this to happen for any selected order of proposers are examined.

Keywords: Bargaining Theory, Coalition formation, Cooperative solution concepts, Nash program, partial breakdown.

J.E.L.: C71; C78.

3.1 Introduction

The seminal paper of Nash [1953], besides proposing one of the most popular solution concepts for transferable utility (TU) cooperative games, namely, the Nash bargaining solution, is of tantamount importance for initiating a new research agenda that goes under the name of Nash program [Binmore, 1986]. Such sub-field of Game Theory aims at connecting the axiomatic approach to (coalitional) bargaining problems, also called cooperative approach, to its strategic, non-cooperative, counterpart. A consistent way of differentiating the two approaches is by seeing the cooperative one as the normal form representation of a bargaining situation having coalitions as primitives, whereas the non-cooperative one entails the strategic form representation of the same game with players as primitives [Serrano, 2004]. In Gambarelli [2007] can be found an interesting way to represent games in characteristic function form as games in normal form.

Since the publication of the mentioned work of Nash, it has become quite standard in economics to pursue the goal of connecting the two approaches. In particular, although it would be misleading to try to rank them in a hierarchical way, it is well accepted to interpret the normal form representation of a coalitional bargaining game as a simplification of its strategic form [Binmore and Eguia, 2017]. Given the complexity of a multi-player bargaining process, where the worth of the amount at stake depends on the association of players that will emerge from the same bargaining process, the axiomatic approach serves as a first necessary step in order to individuate likely results. An alternative interpretation of the role of Cooperative Game Theory is that of finding solutions, as to say, results of a bargaining process, that are desirable from a social, or collective, point of view. Whatever is the interpretation given to the cooperative or axiomatic approach, the analysis of the same situation from a non-cooperative, or strategic, perspective appears to be a natural further step. If the first interpretation was the one adopted, it then becomes a mandatory phase to effectively verify the likelihood of the cooperative solution, whereas, for the second interpretation being in place, it serves to determine if the social planner has actually a viable instrument to implement her desired goal.

The aim of the present chapter is to offer a non-cooperative bargaining model that, under certain conditions, supports the point-valued solution concept introduced in the previous chapter: the Mid-central Core. This is a Core dependent solution that shares several of the axiomatic properties pertaining to the Core. The non-cooperative bargaining model here presented is a variant of the partial breakdown bargaining model introduced by Hart and Mas-Colell [1992], further studied by Krishna and Serrano [1995] and extended to cover non-transferable utility (NTU) games in Hart and Mas-Colell [1996].

Section two will review the relevant literature of non-cooperative bargaining models with a particular focus on the numerous variants of the partial breakdown bargaining model. Section three will introduce the Burning Coalition Bargaining Model (BCBM) and it will briefly describe our reference solution concept: the Mid-Central Core. Section four will describe the response strategy profile according to which the Mid-central Core is asymptotically supported by the BCBM and the conditions for this to happen for any selected order of proposers. Section five will discuss the coincidence of the Mid-Central Core with another popular solution concept: the egalitarian non-separable contribution (ENSC-) method [Driessen and Funaki, 1991]. Section six will conclude.

3.2 Review of the Relevant Literature

As anticipated in the introduction, founding cooperative solution concepts on non-cooperative bargaining games is common practice in economics. It comes with no surprise, therefore, that the academic production on such topic is dramatically vast. This is testified by the fact that several literature reviews cover this argument: Serrano [2004], Bandyopadhyay and Chatterjee [2006] and Binmore and Eguia [2017]. Such abundant production has given rise to a likewise amount of model variants. However, due to its popularity and wide acceptance, the Rubinstein's bargaining model [Rubinstein, 1982] plays a central role also in coalitional bargaining.

In two players bargaining[1], it is well known that the Rubinstein model supports the Nash bargaining solution. Furthermore, Rubinstein [1982] has shown that there are two substantially equivalent way of representing such model: fixed cost of bargaining and fixed discounting, with this last having gained more popularity in the subsequent diffusion of the model. Binmore et al. [1986] have proposed another variant. Specifically, they proved that, by adopting von Neumann-Morgenstern utility functions, the Nash bargaining outcome is obtained if a risk of breakdown of negotiations, exponentially distributed over time, is introduced[2]. In such variant, therefore, the incentive to reach an agreement is not

[1] Along the chapter it will be adopted the following terminological convention: multi-player bargaining indicates a bargaining situation entailing more than two players where only one coalition, namely, the coalition with all players can be formed; coalitional bargaining, instead, assumes both the presence of more than two players and the possibility to form coalitions that are subsets of the set of all players; two players bargaining is self-explanatory.

[2] Actually, the Nash bargaining solution is obtained asymptotically when such risk approaches zero.

due to impatience or to the perceived cost of carrying on negotiations, but rather on the possibility to miss the opportunity offered by the same joint agreement [Binmore et al., 1986].

The extension of Rubinstein's model to multi-players and to coalitional bargaining has not been straightforward, particularly in the latter case. When fixed discounting is adopted, both Sutton [1986] and Osborne and Rubinstein [1990] have recognized that there is a large multiplicity of equilibria. Actually, every efficient pay-off vector that grants to every player at least her outside option can be sustained in perfect equilibrium if players are enough patient. A first way to recover the uniqueness of the Nash result in multi-player bargaining – in the limit of the common discount factor approaching unity – has been put forth by Chae and Yang [1994] through a model where the final agreement is reached through a series of bilateral negotiations. Krishna and Serrano [1996] offer a second method to refine the mentioned abundance of equilibria, restoring the asymptotic support of the Nash bargaining solution as unique subgame perfect equilibrium (SPE). They achieve such result by adopting contingent offers, a mechanism rooted into the consistency principle of Lensberg [1988] for which unanimous acceptance of an agreement is not required since players can exit after reaching partial agreements. Miyakawa [2008] and Britz et al. [2010] prefer to adopt a model with risk of breakdown of negotiations rather then discounting. The result is similar to the previous ones with uniqueness being reached in stationary subgame perfect equilibrium (SSPE), although they find support, as the risk tends to vanish, for the asymmetric Nash bargaining solution [Kalai, 1977]. In both cases asymmetries are generated by the adoption of a random proposer model with, in the former, asymmetric recognition probabilities and, in the latter, Markow recognition probabilities.

In coalitional bargaining, the inherently more complex structure of the game generates a higher variance of outcomes. This is further increased by the fact that the adaptation of the two players Rubinstein's model to such situation is more open to variations as well. The way in which the proposer is selected, both at the beginning of the game and after a potential rejection of a proposal, the nature of the offers, contingent or not, and the possibility – or impossibility – to continue the bargaining process after one coalition has successfully formed are examples of crucial elements that can influence the results [see 1]. As in multi-player bargaining, the multiplicity of equilibria remains a serious issue. In particular, Chatterjee et al. [1993] have shown that the restriction to subgame perfection is not helpful to sharpen results for values of the common discount factor sufficiently high since, in such case, every vector of pay-offs granting to each player at least her disagreement point pay-off can be sustained as an SPE equilibrium.

Stationarity is a first necessary step to reduce the number of feasible equilibria, but, alone, it does not necessarily lead to uniqueness. In fact, assuming the underlying coalitional game being balanced, Moldovanu and Winter [1995], Evans [1997] and Kim and Jeon [2009] show, under different model specifications, that every Core allocation can be sustained as an SSPE equilibrium. Whereas the first two works do not make use of discounting, Kim and Jeon [2009] does it and assumes the rejecter-proposes protocol. A major result of this chapter is to show that the set of SSPE equilibria corresponds to the set of optimal solutions of a minimization program. By adopting discounting and the random proposer protocol, Yan [2003] obtains a unique SSPE equilibrium, according to which the proportion of the final pay-off of each player over the worth of the grand coalition will mirror her recognition probability. If the resulting allocation vector is not a Core element, delay and inefficiency will arise.

Given the result of Yan [2003], it comes with no surprise that several papers (e.g. Okada [1996], Compte and Jehiel [2010] and Okada [2011]), considering the case of equal recognition probabilities of players, obtain a strongly egalitarian equilibrium, namely, they support the Egalitarian Solution [Thomson, 1983][3]. However, when such redistributive SSPE allocation vector is not in the Core, inefficiencies arise as predicted in Yan [2003].

An important fact to note is that, whereas in multi-players – including two players – bargaining there is a perfect correspondence in results between models adopting discounting and models substituting it with a risk of negotiations' breakdown, this is basically lost in coalitional bargaining. First of all, it must be stressed that the three seminal papers introducing the risk of breakdown in coalitional bargaining, namely, Hart and Mas-Colell [1992], Krishna and Serrano [1995] and Hart and Mas-Colell [1996], adopt a partial breakdown assumption. This implies that, when a proposal is rejected, some of the players face the risk of being excluded and, consequently, all the coalitions they were belonging to, dissolve[4]. Clearly, in two or multi-players bargaining, this equates to a total breakdown of negotiations. The correspondence is interrupted since all the three models find support for the Shapley value rather than for the Egalitarian solution or other strongly redistributive Core allocations. Again, it must be noted that, in two or multi-players bargaining, assuming players are bargaining in terms of utils, the Nash bargaining solution and the Shapley value coincide.

[3]Note that the Egalitarian Solution is also know as the Equal Split [Hart and Mas-Colell, 1996] or the Equal Division [van den Brink, 2007].

[4]Note that the three mentioned papers differ in various aspects such as the pay-off obtained by the excluded players and the identity of who might be excluded.

When the player to face the risk of exclusion is the proposer, Hart and Mas-Colell [1992] shows that the Shapley value is supported, in expectation, as an SSPE for monotonic games. Krishna and Serrano [1995] refine this result proving that, for particular values of the parameter determining the probability of breakdown, the same equilibrium is obtained as an SPE. Hart and Mas-Colell [1996] extend the analysis to the case of NTU games, finding support for the consistent values of Maschler and Owen [1989], and, for the TU case, they test several variants among which the possibility that it is not the proposer to face the risk of exclusion, but the responders with equal probability. In such case, the Egalitarian solution emerges as an SSPE. In the intermediate case, when both the proposer and the responders, these lasts with equal probability, face the possibility of exclusion, the SSPE result is a convex combination of the Shapley value and the Egalitarian solution, named Egalitarian Shapley value. Further variants include the works of Calvo [2008], where all the remaining players in the game, not only the responders, are equally likely to be excluded and that supports the Solidarity value; the one of van den Brink et al. [2013] that adds the possibility of a total, besides the one of partial, breakdown and finds support for the Egalitarian Shapley value and the ones of van den Brink and Funaki [2010] and Calvo and Gutiérrez-López [2016], that, adding a common discount factor, obtain, in SSPE, the so called Discounted Shapley value. Kawamori [2016] shows that, for the underlying coalitional game respecting certain conditions, this result holds even when players are not obliged to make a proposal to every player in the game[5], being, instead, allowed to propose only to the members of a specific coalition. Finally, Pérez-Castrillo and Wettstein [2001] show that, by substituting the random selection of the proposer with a bidding stage in the model of Hart and Mas-Colell [1992], the Shapley value is obtained not in expectation but as the unique equilibrium of the game.

The aim of the present chapter is to propose a coalitional bargaining model that can be ascribed to that strand of the literature dealing with partial risk of breakdown. The novelty, however, is quite significant since, instead of assuming that some players might be excluded after a rejection, it presumes that it is only the called coalition, or, equally, its worth, to face the possibility of dissolution. This implies that a rejection can cause, with positive probability, the worth of the selected coalition to be "burned", from which the name BCBM originates.

[5]Note that such assumption is present both in Hart and Mas-Colell [1992] and Calvo and Gutiérrez-López [2016].

3.3 The Burning Coalition Bargaining Model

3.3.1 Model's description

The bargaining game we are going to consider can be fully described by a 5-tuple: $B = (N, v, \Sigma, \boldsymbol{r}, \alpha)$. The first two elements of such tuple defines the underlying coalitional game, with N being the set of players – $N = \{1, 2, ..., |N|\}$ – and $v : 2^{|N|} \to \mathbb{R}_+$ being the characteristic function that defines the worth of each possible subset of N. As usual, we have $v(\emptyset) = 0$. Note that we assumed the worth of each coalition being finite and non-negative. This last assumption, although not crucial, allows to define the set of coalitions in B as the power set of N – denoted as $\mathscr{P}$ – without having a further mapping function that defines the set of feasible coalitions. In fact, an eventual infeasible coalition can be assigned the worth of zero without altering the bargaining structure of the game.

The set Σ represents the whole strategic space of the game. Assuming, although we will see this is not the case, that the bargaining process is infinite, $\Sigma = \times_{t=1}^{\infty} \sigma_t$ with $\sigma_t = \times_{i=0}^{|N|} \sigma_{i,t}$. Note that $\sigma_{0,t}$ represents the set of moves, at each time period t, available to the random mechanism operating in the game. We will indicate with $s_{i,t}$ a specific action of player i at time t, implying that this last is an element of $\sigma_{i,t}$. The last two elements of the tuple are the parameters determining the behavior of the random move. Being this the general description of our model, let us see, concretely, how it works.

The model can be described as a standard Rubinstein-type sequential bargaining model with the rejecter-proposes rule and risk of partial breakdown, where mixed strategies are allowed. At the beginning of the game, the first move is reserved to the random mechanism that will select the order of the n players – since now on, n will be used interchangeably with $|N|$ – among the $n!$ possible ways of ordering them. The vector $\boldsymbol{r} \in \mathbb{R}^{n!}$ defines the probability that each order has to be selected. By assumption, $\boldsymbol{r} = \frac{1}{n!}$ where $\frac{1}{n!}$ indicates a vector with all components equal to $\frac{1}{n!}$[6]. Clearly, such setting implies that each player has an identical probability to be the first proposer in the game, equal to $\frac{1}{n}$. Once a player, let us say i, has been appointed as proposer, her strategic choice is given by a set of 2-tuples $(S, \boldsymbol{x}_S)$ and by the possibility to pass. In the last case the game will move to the next time period, with the following player in the queue becoming the new proposer. The first element of the tuple $(S, \boldsymbol{x}_S)$ indicates the coalition selected by i that must be an element of $\mathscr{S}_t$, the set of active coalitions at time t, and must be such that $S \ni i$. The vector $\boldsymbol{x}_S \in \mathbb{R}^s$

[6]Note that, along the chapter, it will be followed the standard convention of indicating vectors with lower case bold letters. Matrices will be indicated with upper case bold letters.

– with $s = |S|$ – has for elements the proposed allocation for each member of coalition S (including i) and must be such that $\sum_{j \in S} x_j \leq v(S)$. Note that here we are departing from the model of Hart and Mas-Colell [1996] since proposals do not have to be done to all players in N. We then have:

$$\sigma^p_{i,t} = \{(S, \boldsymbol{x}_S)\} \cup \{\text{Pass}\}, \ \forall S \in \mathscr{S}_t, S \ni i, \ \sum_{j \in S} x_j \leq v(S), \quad i \text{ is a proposer.}$$

The superscript p is used to indicate the strategy of a proposer, whereas superscript r will be used to indicate responders' strategy.

Respondents will have a simple dichotomous choice consisting in accepting or rejecting the proposal. They answer sequentially according to the order defined in the first stage of the game, starting with the player coming soon after the proposer. It must be noted that such order of answers is inconsequential and that the model would actually run identically if we were assuming that responses are given contemporaneously. Players not belonging to the coalition pointed by the proposer do not have any available action:

$$\begin{aligned} &\sigma^r_{j,t} = \{\text{accept, reject}\}, && \forall j \in S, j \neq i, \quad S \text{ being the proposed coalition.} \\ &\sigma_{k,t} = \emptyset, && \forall k \in N \setminus S. \end{aligned}$$

When unanimous acceptance occurs, the game ends and the final pay-off vector, $\boldsymbol{\pi} \in \mathbb{R}^n$, will be as follows: $\pi_i = x_i, i \in S$; $\pi_j = x_j, \forall j \in S, j \neq i$; $\pi_k = v(k), \forall k \in N \setminus S$, where $v(k)$ has been used as a shorthand for $v(\{k\})$. This implies, as in Compte and Jehiel [2010], that the formation of a coalition ends the game, preventing the possibility for excluded players to continue bargaining among themselves.

If the proposal from player i is not unanimously accepted, the move will pass to the random mechanism that, with probability α, for $\alpha \in (0, 1]$, will eliminate the proposed coalition from the set of feasible coalitions, whereas, with probability $1 - \alpha$, the game will move to the next period remaining unchanged. Here, the rejecter-proposes rule applies and, therefore, the first of the rejectors will become the new proposer. Clearly, having defined $\mathscr{S}_t$ as the set of coalitions still available at time t, we then have: $\mathscr{S}_1 \equiv \mathscr{P}$; $|\mathscr{S}_t| \leq |\mathscr{P}|, \forall t > 1$ and $|\mathscr{S}_t| = |N|$ for $t \to \infty$[7]. The last equality implies that, in a bargaining process approaching infinity, the set of available coalitions will end up being equal to the set of singleton

[7]The expression $x \to y$ indicate the limit of x tending to y.

coalitions[8]. Therefore, also in the present model, as in Hart and Mas-Colell [1996], the parameter determining the risk of breakdown, α, can be seen as a substitute for discounting.

3.3.2 The Mid-central Core

Before analyzing the model, let us refresh the previously introduced solution concept: the Mid-central Core. This is a point-valued solution concept, or else, an allocation rule, defined in the domain of balanced TU games that always lies in the Core. It is strictly related to a set-valued solution, presented in the same chapter, named the Central Core, being the centroid of the extreme points of such set. The Central Core is also defined over the domain of balanced games and it is always a subset of the Core. In order to arrive at the definition of the Mid-central Core, we have to introduce some further elements. We start by the set Z^*, assuming to have a balanced game $G = (N, v)$:

$$Z^* = \left\{ \boldsymbol{z}^* \in \mathbb{R}^n \mid \min_{z_1^*, z_2^*, \ldots, z_n^*} \sum_{i \in N} z_i^* : \sum_{i \in S} z_i^* \geq v(S), \forall S \in \mathscr{P} \setminus N \right\}.$$

Clearly, Z^* is the set of optimal solutions of a linear minimization program that individuates the minimum necessary amount to satisfy the conditions of individual and group rationality defining the Core. Such minimization program has been proposed as a Core catcher, an easy way to check the non-emptiness of the Core, in Friedman [1990]. If $\sum_{i \in N} z_i^* \geq v(N)$ was included in the set of inequalities to be satisfied, then $Z^* \equiv C(G)$, with $C(G)$ indicating the Core of the game G [Kim and Jeon, 2009]. In 2, it is shown that Z^* is always a non-empty, convex, closed and bounded set given a balanced game G[9]. Define then E_{Z^*} as the indexed set of extreme points of Z^*, having index set $\mathscr{E}_{Z^*}$ and vector $\boldsymbol{o}$ as the centroid of such points: $\boldsymbol{o} = \frac{\sum_{i \in \mathscr{E}_{Z^*}} \bar{\boldsymbol{z}}_i^*}{|E_{Z^*}|}$[10]. Furthermore, define $C(\boldsymbol{o})$ as the positive convex cone having origin in $\boldsymbol{o}$: $C(\boldsymbol{o}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \geq o_i, \forall i \in N\}$. We are now ready to state the definition of the Central Core and, consequently, of the Mid-central Core:

[8] It should be noted that, by our assumption of $v(\cdot) \in \mathbb{R}_+$, we could have actually changed the way in which the random mechanism operates after a rejection, without altering the structure of the game, in the following way: for S being the proposed coalition, at $t+1$, with probability α, $v(S) = 0$ and, with probability $1 - \alpha$, $v(S) = v(S)$. In such case, the set of available coalitions would remain unchanged along the game: $\mathscr{S}_t = \mathscr{P}, t = 1, 2, \ldots, \infty$.

[9] It is easy to see that balancedness is not actually required for these properties to hold.

[10] The bar above a vector will be used to indicate extreme points of a set.

Definition 3.3.1. (The Central Core.) Given a balanced game $G = (N, v)$, the Central Core of such game, $C_c(G)$, is the intersection between the positive convex cone having origin in $\boldsymbol{o}$ and the Core of G, $C(G)$:

$$C_c(G) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \in C(\boldsymbol{o}) \cap C(G)\}.$$

In the domain of balanced games, the Central Core is also a non-empty, convex and compact set.

Definition 3.3.2. (The Mid-central Core.) Given a balanced game $G = (N, v)$, the Mid-central Core, $M_c(G)$, is defined as the centroid of the extreme points of the Central Core of such game, $C_c(G)$.
Therefore, if we define E_{C_c} as the indexed set of extreme points of $C_c(G)$, having index set $\mathscr{E}_{C_c}$ and generic element $\bar{\boldsymbol{x}}$, the Mid-central Core can be defined as follows:

$$M_c(G) = \frac{\sum_{i \in \mathscr{E}_{C_c}} \bar{\boldsymbol{x}}_i}{|E_{C_c}|}.$$

In 2, it is proven the equivalence of this definition with the following one:

$$M_c(G)_i = o_i + \frac{v(N) - \sum_{i \in N} o_i}{n}, \forall i \in N.$$

The Mid-central Core can be described as a compromise solution concept that tries to satisfy three different, and often conflicting, classes of normative prescriptions in balanced games: stability, monotonicity and egalitarianism. Stability substantiates into the axioms of individual and group rationality that are necessarily satisfied by the Mid-central Core being this a subset of the Core. Monotonicity is satisfied in its weak and aggregate form, where this last is the strongest form that a point-valued solution concept always lying in the Core can satisfy Young [1985]; Housman and Clark [1998]. Finally, egalitarianism is represented by an adaptation of the Mid-point domination property to coalitional settings presented in previous chapter. In particular, a version of this last property whose reference set is constrained by the satisfaction of stability and aggregate monotonicity uniquely characterizes the Mid-central Core together with these last two properties.

3.4 A response strategy profile supporting asymptotically the Mid-central Core in the BCBM for balanced underling coalitional games

As a preliminary step, let us subdivide the set of all possible bargaining games $\mathscr{B}$ according to the typology of the underlying coalitional game $G = (N, v)$. For this reason, consider the set Γ as the set of all possible TU games in characteristic function form. We have already stated that we will focus only on games where all coalitions have a non-negative worth. Define then $\Gamma_+ \subset \Gamma$ as such set. Further subdivide it into Γ_+^B, the set of all balanced games in Γ_+ and into its complement: $\Gamma_+^{NB} = \Gamma_+ \setminus \Gamma_+^B$. Finally, define as $\mathscr{B}^B$ and as $\mathscr{B}^{NB}$ the sets of bargaining games, following the BCBM protocol, whose underlying coalitional game is an element of, respectively, Γ_+^B and Γ_+^{NB}. Clearly, $\mathscr{B} = \mathscr{B}^B \cup \mathscr{B}^{NB}$ is the set of all bargaining coalitional games without coalitions having negative worth.

The model runs very similar to the one of Kim and Jeon [2009], with Lemma 1, 2 and 3 presented in their paper, holding, with the due modifications, also in our case. In particular, Lemma 1 in Kim and Jeon [2009] expresses the stationarity of the response strategies, with stationarity implying a considerable simplification, namely that each player accepts a proposal if it is greater of equal to a certain threshold for all the responders:

$$s^r_{j,t} = \begin{cases} \{\text{accept}\}, & \text{if } x_j \geq \underline{x}_{j,t}, \forall j \in S, j \neq i, S \ni i, S \in \mathscr{S}_t; \\ \{\text{reject}\}, & \text{otherwise}; \end{cases}$$

given that i was the proposer and made the proposal $(S, \boldsymbol{x}_S)$ and j is a responder. Note that, in Kim and Jeon [2009], the proposer has the faculty to decide the order of responders, whereas, in the present model, such order is determined by the initial random move. However, they proved (Lemma 1) that it is always optimal for a responder to reject a proposal if this last does not satisfy the minimum requirement of any responder. This is due to the rejecter-proposes protocol, according to which being the first rejecter always grants the advantage of avoiding the risk of exclusion in the next turn. Obviously, if a rejection has already been stated by a player preceding j, the choice of j becomes simply inconsequential. This makes clear why responders' order is totally inconsequential as well, therefore no significant difference is produced by assuming that it is the proposer to choose the responders' order or that it is the random mechanism.

Let us now turn the attention towards the proposer. If she wants to make an acceptable proposal, she must give to the other players in the selected coalition at least their cut-off values: $\underline{x}_j, \forall j \in S, j \neq i$, where S is the coalition selected by i. Clearly, it is optimal for her to propose an allocation that grants the other players exactly their cut-off values, so she can reserve for herself the remaining. Furthermore, the proposer will necessarily select the coalition where such remaining is the highest:

$$s^p_{i,t} = (S, \boldsymbol{x}_S):$$
$$x_i = \max_{S \ni i, S \in \mathscr{S}_t} v(S) - \sum_{j \in S, j \neq i} x_j;$$
$$x_j = \underline{x}_{j,t}, \forall j \in S, j \neq i.$$

Note that here we are departing from the model of Kim and Jeon [2009] and, in general, from all models of coalitional bargaining that are based on the discounting of future pay-offs.

Consider the following response strategy profile:

For i being a responder, $s^r_{i,t} = accept$ if $x_i \geq o_{i,t}$ and $S \neq N$, with S being the proposed coalition that includes i. Furthermore, $o_{i,t}$ is defined as in section 3.2 for the set of coalitions $\mathscr{S}_t$.

The rest of this section will be dedicated to show that, for this response strategy profile being adopted by all players and for a bargaining game in $\mathscr{B}^B$, there is a unique efficient equilibrium resulting in the formation of the grand coalition in the first bargaining round, independently from the selected proposer provided certain conditions are met. We start with a useful Lemma, but, before, it is necessary to introduce some further notations related to the $\boldsymbol{o}$. For simplifying purposes, it will be assumed that Z^*, defined in section 3.2 as the set of optimal solutions of a simple minimization program, is a singleton, having therefore $\boldsymbol{o}$ as its unique element. Then, $\boldsymbol{o}$ must lie at the intersection of, at least, n hyperplanes. Assuming $\boldsymbol{o}$ is not a degenerate point, the binding hyperplanes will be exactly n. Recall the minimization program determining Z^* and consider its dual. In particular, define as $\boldsymbol{\lambda}$ the optimal solution vector for such dual problem. Clearly, $\boldsymbol{\lambda} \in \mathbb{R}^{2^n-2}$ since $|\mathscr{P} \setminus \{N, \emptyset\}| = 2^n - 2$.

Since $\boldsymbol{o}$ lies at the intersection of n hyperplanes, by standard duality theory, we know that exactly n elements of vector $\boldsymbol{\lambda}$ will be strictly positive, whereas all the others will be equal to zero. To every element of $\boldsymbol{\lambda}$ is associated a characteristic value and, consequently, a coalition in $\mathscr{P} \setminus \{N, \emptyset\}$. Assume this last set is indexed by the index set $\mathscr{C} = \{g \mid g \in \mathbb{N} \wedge 1 \leq g \leq 2^n - 2\}$. Then, the set of potentially forming coalitions or, equivalently, the set of coalitions constituting binding hyperplanes in our reference minimization program, call it C, is given by $C = \left\{S_g \in \mathscr{P} \setminus N \mid \lambda_g > 0, \forall g \in \mathscr{C}\right\}$. Denote then with $C_i, \forall i \in N$, the subsets of C collecting all the coalitions including player i: $C_i = \{S \mid S \ni i, \forall S \in C\}, \forall i \in N$. Once having defined the sets C and $C_i, \forall i \in N$, we are ready to state the Lemma.

Lemma 3.4.1. *Suppose to have two bargaining games in $\mathscr{B}$ that are identical in all aspects except for the worth of a single coalition. Suppose, furthermore, that $0 \leq v^2(T) < v^1(T)$ and $T \in C^1$. By assumption, $v^2(S) = v^1(S), \forall S \in \mathscr{P} \setminus T$. It is then true that $o_i^2 \leq o_i^1, \forall i \in T$.*

Proof. By the way in which $\boldsymbol{o}$ has been defined, we know it is the centroid of the extreme points defining the optimal solutions of a minimization program. Furthermore, by what assumed regarding C^1 and by standard optimization theory, $|C^1| = |N|$. This is naturally so since it collects the coalitions forming binding hyperplanes in the minimization program determining $\boldsymbol{o}^1$. Being $\boldsymbol{o}^1$ a non-degenerate point, the number of binding hyperplanes must then be equal to n. Consider the mentioned minimization program as being written in matrix form, with $\boldsymbol{A} \in \mathbb{R}^{2^n - 2 \times n}$ being a matrix whose columns represent the players in N and whose rows represent each coalition in $\mathscr{P} \setminus \{N, \emptyset\}$. Then, each cell of $\boldsymbol{A}$, call it $a_{g,i}$, is equal to one if player i is in coalition S_g and zero otherwise. Define $\boldsymbol{A}_n$ as the square matrix collecting the n rows of $\boldsymbol{A}$ corresponding to the coalitions in C^1. Further define $\boldsymbol{v}^1 \in \mathbb{R}^{2^n - 2}$ as the vector collecting the characteristic values of all coalitions in B^1 and $\boldsymbol{v}_n^1$ as the one collecting only the coalitions in C^1. Define, analogously, $\boldsymbol{v}^2$ and $\boldsymbol{v}_n^2$. Suppose, *w.l.o.g*, that the differing coalition between the two games, T, is indexed with 1. Matrix $\boldsymbol{A}$ is clearly identical for both games whereas $v_g^1 = v_g^2, \forall g > 1$. Finally, define $\mathscr{C}^1$ and $\mathscr{C}^2$ as the sets collecting the g indexes of the coalitions in, respectively, C^1 and C^2. We then have two cases.

Case 1: $\mathscr{C}^1 = \mathscr{C}^2$. This case implies that the binding hyperplanes – alias, coalitions – remain the same. Since $\boldsymbol{A}_n\boldsymbol{o}^1 = \boldsymbol{v}_n^1$ and $\boldsymbol{A}_n\boldsymbol{o}^2 = \boldsymbol{v}_n^2$, this implies $\boldsymbol{o}^1 = \boldsymbol{A}_n^{-1}\boldsymbol{v}_n^1$ and $\boldsymbol{o}^2 = \boldsymbol{A}_n^{-1}\boldsymbol{v}_n^2$. Furthermore, we have that $\boldsymbol{1}^T\boldsymbol{o}^2 = \boldsymbol{1}^T\boldsymbol{o}^1 - \lambda_1(v_{n,1}^1 - v_{n,1}^2) = \boldsymbol{1}^T\boldsymbol{o}^2 - \lambda_1(v^1(T) - v^2(T))$, with λ_1 being the dual value associated to both $v_{n,1}^1$ and $v_{n,1}^2$. Clearly, $\lambda_1 = \boldsymbol{1}^T\boldsymbol{a}_{n,1}^{-1}$, where $\boldsymbol{a}_{n,1}^{-1}$ is the first column of matrix $\boldsymbol{A}_n^{-1}$. Then, $\boldsymbol{o}_1 - \boldsymbol{o}_2 = \boldsymbol{A}_n^{-1}(\boldsymbol{v}_n^1 - \boldsymbol{v}_n^2) = \boldsymbol{a}_{n,1}^{-1}(v_{n,1}^1 - v_{n,1}^2) = \boldsymbol{a}_{n,1}^{-1}(v^1(T) - v^2(T))$. Note that $(v^1(T) - v^2(T))$ is a positive scalar and $\boldsymbol{a}_{n,1}^T\boldsymbol{a}_{n,1}^{-1} = 1$, with $\boldsymbol{a}_{n,1}^T$ being simply the first row of matrix $\boldsymbol{A}_n$[11]. Furthermore, $0 \leq a_{n,1}^{-1,i} \leq 1, \forall i \in T$, and this necessarily implies that $o_i^1 \geq o_i^2, \forall i \in T$.

Case 2: $\mathscr{C}^1 \neq \mathscr{C}^2$. This necessarily implies that $T \notin C^2 \rightarrow \boldsymbol{a}_{n,1}^T\boldsymbol{o}^1 > v(T)$. In words, coalition T and its worth do not constitute a binding hyperplane in the minimization program related to

[11]The superscript T in $\boldsymbol{a}_{n,1}^T$ stays for transpose and it serves to indicate that $\boldsymbol{a}$ is a row vector.

B^2. However, the polyhedrons defining the two minimization programs, the one related to B^1 and the one related to B^2, are identical except for the same hyperplane defined by coalition T. This is, in fact, the only hyperplane whose position has changed. This implies that the optimal solution for B^2 must lie in the negative cone having origin in $\boldsymbol{o}^1(T)$, with $\boldsymbol{o}^1(T)$ being the restriction to the elements of coalition T of the vector $\boldsymbol{o}^1$. From the definition of a negative cone, we have $\boldsymbol{a}_{n,1}^T \boldsymbol{x} \le \boldsymbol{o}^1(T)$, implying that $o_i^2 \le o_i^1, \forall i \in T$. ■

It is then possible to state the main proposition of the present chapter, relating the response strategy profile mentioned before with the equilibrium in the BCBM.

Proposition 3.4.1. *Given a bargaining game $B \in \mathscr{B}^B$ where all players adopt the response strategy profile:*

$$s_{i,t}^r = \begin{cases} \{accept\}, & \text{if } x_i \ge o_{i,t}; \\ \{reject\}, & \text{otherwise}; \end{cases} \quad \text{For } S \in \mathscr{S}_t \setminus N \text{ being the proposed coalition.}$$

the grand coalition will be chosen in the first bargaining round with the proposer, say k, offering $x_j = o_j + (1-\alpha)\frac{v(N)-\sum_{i\in N} o_i}{1+(|N|-1)(1-\alpha)}, \forall j \in N \setminus k$, and the proposal will be unanimously accepted if (Condition 1):
$v(N) > \sum_{i\in N} o_i$ and $\forall i \in N, \forall S \in C_i$ it holds that $o_i - o_i^{(1-\alpha)v(S)} \ge \frac{\sum_{j\in N} o_j - \alpha v(N) - (1-\alpha)\sum_{j\in N} o_j^{(1-\alpha)v(S)}}{1+(|N|-1)(1-\alpha)}$.

Proof. Given the response strategy profile of Proposition 3.4.1, a proposer, say i, can always choose a coalition $S \in C_i^t$ and offer $o_{j,t}$ to all other players assuring to herself a pay-off of $v(S) - \sum_{j\in S, j\neq i} o_{j,t} = o_{i,t}$. Consequently, we can exclude the possibility of i choosing a coalition $T \in \mathscr{S}_t \setminus \{C_i^t, N\}$ since, by the definition of $\boldsymbol{o}, v(T) - \sum_{j\in T, j\neq i} o_{j,t} < o_{i,t}$. Further consider that, if a coalition $T \in \mathscr{S}_t \setminus C^t$ is selected by the proposer, but its proposal gets rejected, $\boldsymbol{o}_{t+1} = \boldsymbol{o}_t$ since the worth of coalitions in C^t are not affected and, $C^{t+1} \equiv C^t$. This implies that the first responder, say j, can assure to herself $o_{j,t+1} \equiv o_{j,t}$. Furthermore, this also implies that, for a proposal to be accepted, whatever coalition S is chosen by a proposer i, $x_j \ge o_{j,t}, \forall j \in S \setminus i$.
Since $o_{j,t}$ is a lower bound for each player j in N, it follows that $\underline{x}_{j,t} \ge o_{j,t}, \forall j \in N$. Write then $\underline{x}_{j,t} = o_{j,t} + \underline{y}_{j,t}, \forall j \in N$. Remembering the optimal strategy of a proposer, $s_{i,t}^p = (S, \boldsymbol{x}_S)$: $x_i = \max_{S\ni i, S\in\mathscr{S}_t} v(S) - \sum_{j\in S, j\neq i} \underline{x}_{j,t} = \max_{S\ni i, S\in\mathscr{S}_t} v(S) - \sum_{j\in S, j\neq i} o_{j,t} - \sum_{j\in S, j\neq i} \underline{y}_{j,t}$. Since $B \in \mathscr{B}^B$ and, by assumption, $v(N) > \sum_{j\in N} o_j$, in the first bargaining round the only coalition where $v(S) - \sum_{j\in S} o_j > 0$ is actually the grand coalition and, therefore, it will be chosen with probability one. Since $o_{j,t+1} = o_{j,t}, \forall j \in N$ and, for i being the proposer, $x_i \ge \underline{x}_{i,t} = o_{i,t} + \underline{y}_{i,t}$, we then have $\underline{y}_{i,t} = (1-\alpha)(v(N) - \sum_{k\in N} o_{k,t} - \sum_{j\in N\setminus i} \underline{y}_{j,t}, \forall i \in N$. The solution of this

squared system of equations leads to $y_{i,t} = (1-\alpha)\frac{(v(N)-\sum_{k\in N} o_k)}{1+(|N|-1)(1-\alpha)}, \forall i \in N$, that is clearly identical to the solution of the Rubinstein's model – for multi-players problems – with risk of breakdown and where $v(i)$ has been substituted by $o_i, \forall i \in N$.

Condition 1, however, must be met in order to avoid strategic delay, with this last meant to be the possibility of a proposer i to select a coalition in C^i and make an unacceptable proposal ($x_j < o_j$ for some $j \in S \setminus i$). Assume player i does it. Then, her proposal will be rejected and, in the next round, the worth of $S \in C^i$, previously selected, will be equal, in expectation, to $(1-\alpha)v(S)$. Define as $\boldsymbol{o}^{(1-\alpha)v(S)}$ the vector $\boldsymbol{o}$ for the coalitions set $\mathscr{P}$ but with the worth of $v(S)$ substituted by $(1-\alpha)v(S)$. From Lemma 3.4.1, we know that $o_i \geq o_i^{(1-\alpha)v(S)}$. However, it is obviously true that $\sum_{j\in N} o_j > \sum_{j\in N} o_j^{(1-\alpha)v(S)} \Rightarrow v(N) - \sum_{j\in N} o_j < v(N) - \sum_{j\in N} o_j^{(1-\alpha)v(S)}$. Therefore, proposer i will find profitable to cause a delay if $\exists S \in C^i : o_i + \frac{v(N)-\sum_{j\in N} o_j}{1+(|N|-1)(1-\alpha)} < o_i^{(1-\alpha)v(S)} + (1-\alpha)\frac{v(N)-\sum_{j\in N} o_j^{(1-\alpha)v(S)}}{1+(|N|-1)(1-\alpha)}$. By reversing the inequality and rearranging, we obtain condition 1 that, for avoiding strategic delay given any possible order of proposers, must hold $\forall S \in C^i, \forall i \in N$. ■

In Proposition 3.4.1 we have seen a condition for always having efficiency given the mentioned response strategy profile. For $\alpha \to 0$, this condition can be reformulated. Assume $C = C^((1-\alpha)v(S)), \forall S \in C$, implying that the set of binding coalitions remains unaltered once the worth of a coalition $S \in C$ has been multiplied by $(1-\alpha)$. From Case 1 in Lemma 3.4.1, we have that $\sum_{j\in N} o_j^{(1-\alpha)v(S)} = \sum_{j\in N} o_j - \lambda_S(v(S) - (1-\alpha)v(S))$. Furthermore for $i \in S$ being the proposer, $o_i - o_i^{(1-\alpha)v(S)} = a_{n,S}^{-1,i}\alpha v(S)$. Since $\alpha \to 0$, it follows that $\alpha v(N) \to 0$ and $\frac{(1-\alpha)}{1+(n-1)(1-\alpha)} \to \frac{1}{n}$. Condition 1 will then reduce to $a_{n,S}^{-1,i}\alpha v(S) \geq \frac{\lambda_S \alpha v(S)}{n} \Rightarrow a_{n,S}^{-1,i} \geq \frac{\lambda_S}{n}, \forall S \in C^i, \forall i \in N$.

In Proposition 3.4.1, a particular class of balanced games has been omitted, namely games where $v(N) = \sum_{j\in N} o_j$. It is clear that, in such case, a proposer i will be indifferent to choose the grand coalition or a coalition $S \in C^i$.

Corollary 3.4.1.1. *Given a bargaining game $B \in \mathscr{B}^B$ with $v(N) = \sum_{j\in N} o_j$ with all players adopting the response strategy profile in Proposition* 3.4.1*, the grand coalition will form with probability $\frac{1}{|C^i|+1}$, provided Condition 1 holds true.*

Proof. It follows directly from Proposition 3.4.1 and its proof. ■

From Proposition 3.4.1 it is clear that, given a bargaining game $B \in \mathscr{B}^B$ with $v(N) > \sum_{j\in N} o_j$ and holding Condition 1, for the response strategy profile mentioned in the same proposition, the Mid-central Core is obtained as the unique equilibrium for $\alpha \to 0$. This is clear from the way in which the Mid-central Core has been defined and by the fact that, for $\alpha \to 0$, $\frac{1-\alpha}{1+(|N|-1)(1-\alpha)} \to \frac{1}{n}$. On the contrary, for $\alpha = 1$, provided the same conditions hold, the extreme points of the Central Core are obtained. In particular, for i being the first selected proposer, she will obtain the pay-off allocation corresponding to the extreme point of the Central Core most favorable to her: $x_i = o_i + v(N) - \sum_{j \in N} o_j$. Since, in the $n!$ possible permutations of the players set there are exactly $(n-1)!$ of such permutations where i is selected as first proposer whereas her allocation will be identical, and equal to o_i, whenever she will be in any position other than the first, it follows that the Mid-central Core is obtained as the ex-ante expectation pay-off: $E[x_i] = \frac{1}{n}(o_i + v(N) - \sum_{j \in N} o_i) + \frac{n-1}{n} o_i = o_i + \frac{v(N) - \sum_{j \in N} o_i}{n}$.

3.5 Coincidence of the Mid-central Core and the ENSC value

The previous sections have been dedicated to present the BCBM and a response strategy profile apt to support the Mid-central Core in such bargaining model. The present section, instead, is devoted to indicate the conditions according to which the Mid-central Core and the Equal Non-Separable Contribution value coincide. The reason of this section is fairly immediate. Having shown that the strategy profile described in Proposition 3.4.1 supports asymptotically the Mid-central Core given the BCBM, the coincidence of this solution with the ENSC value necessarily implies that also this last is supported. It is therefore interesting to understand when it occurs that the two solutions are identical.

Before tackling this problem, however, it is opportune to present the ENSC value. Introduced in Driessen and Funaki [1991], this solution concept for TU cooperative games is a variant of the Separable Contributions Remaining Benefit (SCRB)[12] method, with this last being a cost allocation rule diffused in the field of water resources management [Driessen and Tijs, 1985]. The ENSC value is obtained by the sum of two components, rooted into the idea of marginal contribution, that is at the base of several cooperative solution concepts (e.g. the Shapley value). In fact, given a balanced coalitional game $G = (N, v)$, the ENSC value is given by:

$$ENSC_i(G) = SC_i(G) + NSC(G), \qquad \forall i \in N;$$

[12]Note that this allocation is presented under the name of Equal Charge Allocation (ECA) in Straffin and Heaney [1981].

where $SC_i(G)$ is the separable contribution of player i: $SC_i(G) = v(N) - v(N \setminus i)$; whereas $NSC(G)$ is the non-separable contribution, defined as $NSC(G) = v(N) - \sum_{i \in N} SC_i(G) = (1-n)v(N) + \sum_{i \in N} v(N \setminus i)$. As done in Yokote and Funaki [2017], it is possible to make explicit the previous formula:

$$ENSC_i(G) = v(N) - v(N \setminus i) + \frac{v(N) - \sum_{j \in N}[v(N) - v(N \setminus j)]}{n}, \quad \forall i \in N;$$

that, rearranged, becomes:

$$ENSC_i(G) = \frac{1}{n}\left[v(N) + \sum_{j \in N, j \neq i} v(N \setminus j) - (n-1)v(N \setminus i)\right], \quad \forall i \in N;$$

Once having properly defined the ENSC value, it is easy to see when it coincides with the Mid-central Core.

Proposition 3.5.1. *Given a TU coalitional game $G = (N, v)$, if $G \in \Gamma_+^B$ and $|S| = n-1, \forall S \in C$ and $|C| = n$, then the Mid-central Core coincides with the ENSC value.*

Proof. From the fact that $|C| = n$, it follows that $\boldsymbol{o}$ is the unique element of the set Z^*, lying at the intersection of n hyperplanes. Furthermore, in the set $\mathscr{P}(N)$, there are exactly n coalitions of cardinality $n-1$ and, by assumption, they are all included in C, implying that they all constitute binding constraints to the minimization program defining $\boldsymbol{o}$. Recalling the definition of the square matrix $\boldsymbol{A}_n$ given in the proof of Lemma 3.4.1, it follows that, in the present case, matrix $\boldsymbol{A}_n$ will have all rows (and all columns) composed by ones except for one element that is equal to zero. For convenience, we can think to have arranged $\boldsymbol{A}_n$ so that the main diagonal has the zero elements. It is easy to see that its inverse, $\boldsymbol{A}_n^{-1}$, will then have all elements in the main diagonal equal to $-\frac{n-2}{n-1}$ and all other elements equal to $\frac{1}{n-1}$. It is immediate to check that $\boldsymbol{A}_n\boldsymbol{A}_n^{-1} = \boldsymbol{A}_n^{-1}\boldsymbol{A}_n = \boldsymbol{I}_n$. This implies that vector $\boldsymbol{\lambda}$, as defined previously, will have elements $\lambda_g = \frac{1}{n-1}, \forall g \in \mathscr{C}$ and $\lambda_g = 0, \forall g \notin \mathscr{C}$.
Clearly, $\boldsymbol{o} = \boldsymbol{A}_n^{-1}\boldsymbol{v(S)}$, with $\boldsymbol{v(S)}$ being the vector of characteristic values of the n coalitions with cardinality equal to $n-1$. Then, $\boldsymbol{v(S)} = \boldsymbol{v(N \setminus i)}, \forall i \in N$. From the given definition of the Mid-central Core, we have $M_C(G)_i = o_i + \frac{v(N) - \sum_{i \in N} o_i}{n}$. Rewrite o_i as $\boldsymbol{a}_{i,n}^{-1}\boldsymbol{v(N \setminus j)}, \forall i \in N$, with $\boldsymbol{a}_{i,n}^{-1}$ being the i^{th} column of $\boldsymbol{A}_n^{-1}$. Then, $M_C(G)_i = \boldsymbol{a}_{i,n}^{-1}\boldsymbol{v(N \setminus j)} + \frac{v(N) - \boldsymbol{1}^T\boldsymbol{A}_n^{-1}\boldsymbol{v(N \setminus j)}}{n} \to \sum_{j \in N, j \neq i} \frac{1}{n-1} v(N \setminus j) - \frac{n-2}{n-1} v(N \setminus i) + \frac{1}{n}\left[v(N) - \sum_{f \in N} \frac{1}{n-1} v(N \setminus f)\right] \to \frac{1}{n-1}\left(1 - \frac{1}{n}\right)\sum_{j \in N, j \neq i} v(N \setminus j) - \left(n - 2 + \frac{1}{n}\right) v(N \setminus i) + \frac{v(N)}{n} = \frac{1}{n}\left[v(N) + \sum_{j \in N, j \neq i} v(N \setminus j) - (n-1)v(N \setminus i)\right] = ENSC_i(G), \forall i \in N.$ ■

The coincidence between the Mid-central Core and the ENSC value has two meaningful implications. The first, already mentioned, is the common support for both solutions provided by the BCBM played according the the strategy profile of Proposition 3.4.1 when they coincide. On this regard note that, for $\alpha \to 0$, Condition 1 is always satisfied. Remember, in fact, that this condition was reducing to $a_{n,S}^{-1,i} > \frac{\lambda_S}{n}, \forall S \in C^i, \forall i \in N$. Since $a_{n,S}^{-1,i} = \frac{1}{n-1}$ for $i \in S$ and $S \in C^i$ and $\lambda_S = \frac{1}{n-1}$, the inequality necessarily holds true. From this, it follows the second implication. Although it is out of the scope of the present chapter to carefully analyze the distributive properties of the BCBM outcome and, in particular, of the Mid-Central Core, the occasional coincidence with this last and the ENSC value provides an interesting hint on such theme. In Driessen and Funaki [1991] it is shown that, for a particular class of games, named k-coalitional games, the Shapley value is a convex combination of the ENSC value and the Centre-of-Gravity of the Imputation Set (CIS) value. This last solution, for a given a TU coalitional game $G = (N, v)$, is simply defined as:

$$CIS_i(G) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}, \qquad \forall i \in N.$$

Clearly, for zero-normalized games, it coincides with the Equal Division (ED) value: $ED_i(G) = \frac{v(N)}{n}, \forall i \in N$. The ED value is, therefore, the most egalitarian solution concept, whereas the Shapley value is generally seen as the most prominent translation into cooperative game theory of marginalism. Being the Shapley value a convex combination of the CIS and the ENSC values, it follows that this last is even less redistributive than the same Shapley value. In 1, it is possible to see how several bargaining games lead to egalitarian distributions of pay-offs. In particular, variants of the Rubinstein's alternating offer model, either adopting the random-proposer or the rejecter-proposes protocol, if they assume identical recognition probabilities, obtain an efficient outcome only if the Equal Division is in the Core of the game, otherwise they only admit inefficient outcomes. Some examples are Chatterjee et al. [1993], Okada [1996] and Compte and Jehiel [2010]. On the other side, partial breakdown models such as Hart and Mas-Colell [1996] and Krishna and Serrano [1995], sustain the Shapley value, but, with slight modifications, an intermediate result between the Shapley value and the ED value is obtained. Although the class of games where the Mid-Central Core and the ENSC value coincide is surely a small subset of Γ_+ and the intersection with the set of k-coalitional games further narrows the scope of the present consideration, it is nonetheless interesting to have a bargaining model and a strategy profile leading to an efficient outcome and implementing a solution that is less redistributive than the Shapley value.

3.6 Conclusions

The present chapter offers a little contribution to the research agenda that goes under the name of Nash program. In fact, it presents a non-cooperative bargaining model, the BCBM, and a particular response strategy profile apt at supporting, asymptotically, a newly introduced solution concept for TU games in characteristic function form: the Mid-central Core. The BCBM can be described as a standard alternating offers model where the role of discounting is replaced by the risk of partial breakdown. Contrary to the standard definition adopted by similar models [Hart and Mas-Colell, 1992; Krishna and Serrano, 1995; Hart and Mas-Colell, 1996], where the risk of partial breakdown implies that a player might drop out from the game after a proposal is rejected, here the same rejection triggers the possibility that the worth of the proposed coalition vanishes. The response strategy profile according to which the Mid-central Core is supported asymptotically as the unique equilibrium of the BCBM is described, together with the conditions granting the efficiency of the bargaining process for any selected order of proposers.

The final part of the chapter is dedicated to show the potential coincidence of the Mid-central Core and the ENSC-value, another allocation rule for TU balanced games. This leads to some interesting implications regarding the redistributive properties of the BCBM. In fact, for a particular class of games, the Shapley value is a convex combination of the ENSC and the ED values, implying that the Mid-central Core – as the ENSC value – is potentially less redistributive of the Shapley value.

Chapter 4

Cooperative Game Theory Applied to IEAs: A Comparison of Solution Concepts

This chapter aims at providing a critical overview of the main solution concepts adopted by the cooperative game theoretical approach in analyzing the constitution of an international environmental agreement (IEA). The first part is mainly descriptive and focuses on the basic features of the 'global warming game' characterizing the differences of the cooperative and non-cooperative approaches to deal with this theme. It then presents the most adopted cooperative solution concepts critically analyzing their *ratio*. Furthermore, two alternative solutions, the Rawlsian Nucleolus and a 'revisited' Nash Bargaining solution are proposed, both based on the concept of Minimum Feasible Core.
The second part is dedicated to a numerical exercise based on a standard emissions game in order to compare the mentioned concepts with particular focus on their redistributive properties and on their capability to minimize the potential losses caused by free riding. The Rawlsian Nucleolus is, among the considered solutions, the one with the highest redistributive properties, outperforming, on this regard, the Chander and Tulkens solution that still tends to prioritize polluted countries. The ability of avoiding losses from free riding is shown to be strongly correlated with the redistributive properties of solution concepts till the point that their ranking perfectly coincides.

Keywords: Cooperative Game Theory, solution concepts, emissions abatement, welfare distribution, equity, free riding.

J.E.L.: C71; D63; Q52.

4.1 Introduction

Game theory (GT) is the study of mathematical modeling of rational players interacting with each other and it has been extensively applied in the environmental context, ranging from fisheries management to natural resources extraction and waste treatment [Zara et al., 2006a,b]. A subtopic that has received special attention is environmental pollution. As soon as the potential negative impact of anthropogenic emissions on the earth's ecosystem was widely recognized by the scientific and international community, different models have been envisaged in order to represent the possible interactions of countries to coordinate their efforts in abating emissions. Based on earlier models of transboundary pollution, a full class of new models has born to depict the 'global warming game' [Maler, 1989].

Although the solution concepts that have been adopted are usually already present in the game theoretical literature, it is still important to note that the 'global warming game' has its own specificities; indeed it is now possible to speak, without the risk of adding unjustified emphasis, of a new strand in that literature. Three elements, and their specific combination, can be considered the distinctive features of this particular game: the character of the players, the strategies at their disposal, and the effects produced by their cooperative behavior. Starting with the first, countries are the common unit of analysis (players, for game theory) and the game consists of constituting an international environmental treaty (IEA). Although this may appear quite natural, it is not trivial if examined more carefully. Countries, in fact, are neither the direct producers of emissions and pollution nor the direct recipients of their negative consequences. They act, therefore, as mediators between polluters (firms, consumers) and pollutees (people in general, for an anthropocentric perspective). According to the Coase Theorem [Coase, 1960], the problem can be solved by the market alone, through the law of demand and supply, but this would imply that pollution - or its complement, environmental quality - is a normal, 'priceable' good. Given that pollution is an externality[1] and environmental quality a public, rather than a private, good, polluters and pollutees cannot use the market as an exchange platform unless a superior authority has previously defined appropriate property rights.

This leads to the second specificity, that is the strategy space defining this environmental game. It is possible to differentiate at least two distinct, although interconnected,

[1]Externality is here defined as a cost or benefit affecting a party without her explicit intention to incur such cost or receiving such benefit. An externality producing costs is a negative externality whereas, when producing benefits, it is a positive externality. Here, we will be dealing exclusively with negative externalities and the adjective negative will be generally omitted for the sake of brevity.

classes of choices faced by countries. First of all, an IEA is a treaty and therefore a country has a dichotomous choice between being or not a signatory. This decision can be additionally divided into two steps, that of signing and then ratifying the treaty, as recognised, for example, in the model of Köke and Lange [2013]. In general, however, this possibility is simply disregarded and the simplifying assumption of perfect coincidence between the two steps is adopted[2]. The second strategic choice regards the optimal level of emissions or abatement to adopt. From a modeling perspective, speaking of emissions or abatement is equivalent since one is the complement of the other, so their use depends simply on author's preferences[3]. Three further clarifications are required. First, since countries bargain as representatives of their population, a model presenting a country's unique utility function implicitly relies on the assumption that the intra-state bargaining processes have already been solved. This constitutes a parallel with the role played by the representative agent in several models, where this role is here taken by the state itself. The second point regards the enforceability of state decisions. Countries are supposed to 'play' as rational actors. Therefore, besides having a single utility function, they are also supposed to possess full enforceability of their decisions. If this happens through a tax or a marketable quota mechanism (in the dedicated literature it has been largely demonstrated that the two mechanisms, under certain assumptions, are equivalent [Perman et al., 2003]) or through other instruments, is not important, but the central point to stress is that a country has a mechanism to perfectly implement the decision\strategy chosen. The third specificity concerns the outcome of cooperation which always generates positive externalities given that it reduces pollution[4].

To sum up, in this chapter IEAs are conceived to be games in which players are countries, supposed to be rational, utility maximizing agents. They have a double, interconnected, strategy space constituted by the choice of cooperate (being part of the agreement) or not cooperate (remaining outside) and a continuous choice on the level of emissions\abatement to adopt. The constitution of an IEA always generates positive externalities. Bearing these distinguishing characteristics in mind, this chapter analyses

[2]Under this assumption, therefore, a signatory is automatically a ratifier. Some examples of this approach are: Carraro and Siniscalco [1993]; Barrett [1994]; Chander and Tulkens [1995, 2006b]; Botteon and Carraro [1997].

[3]Emissions have been used by: Carraro and Siniscalco [1993]; Chander and Tulkens [1995]; Botteon and Carraro [1997]; whereas abatement has been used in Barrett [1994]. There are also examples that allow contemporaneously for both the possibilities [Diamantoudi and Sartzetakis, 2006; Sartzetakis and Strantza, 2013]. This case could still be reduced to a single choice, over net emissions, achievable through two options.

[4]An exception to this statement is represented by games based on a Stackelberg form of competition between the coalition and the countries remaining outside it, where the first move advantage is assumed to be held by the coalition itself. See, for example, Barrett [1994]; Sartzetakis and Strantza [2013].

the cooperative approach, describing the main solution concepts adopted in the literature, applied to the 'global warming game'.

4.2 Cooperative VS Non-cooperative Approach

Since this chapter deals with cooperative GT (CGT), it is opportune to give its definition and to stress the difference with the non-cooperative approach (NCGT). First of all, however, it is useful to recall a general definition of GT. This discipline can be described as the mathematical modeling of situations of conflict and cooperation that, starting from assumptions about the strategic behavioural patterns of players, provides their resulting pay-offs according to the adopted solution concepts [Zara et al., 2006a]. Bearing this definition in mind, the words of Osborne and Rubinstein [1994] clearly explain the difference between the two approaches:

"A coalitional model (a cooperative model, *ndr*) is distinguished from a non-cooperative model primarily by its focus on what groups of players can achieve rather than on what individual players can do and by the fact that it does not consider the details of how groups of players function internally".

Another way to explain this difference refers to the enforceability of agreements. NCGT models situations where this enforceability is absent, so that players are free to strategically pursue their own objectives, whereas CGT mainly considers the allocation of cooperative gains resulting from binding agreements [Zara et al., 2006a]. Adopting this last view and considering the structure of the international community, based on the principle of state sovereignty, it could seem more appropriate to adopt a non-cooperative approach to model the 'global warming game'. However, two arguments can be used to contradict this thesis.

The first stems from the words of Osborne and Rubinstein [1994]. The authors point that the two approaches differ on their focus. It is therefore a matter of perspective, a matter of what the researcher wants to investigate and which questions she wants to answer. The problem of agreement enforceability is simply disregarded. Criticising a coalitional model on the ground that it assumes enforceability in a situation where it does not exist, although usually rewarding, could actually cause a simple deviation from the focal point. In choosing this approach, in fact, the researcher already knows its limits and this criticism becomes a mere pleonasm. The second point should be limited to the case of IEAs and derives from a peculiar development of the literature in this field. The non-cooperative approach aims at investigating the constitutional process of a coalition (an IEA) under a positive perspective [Chander and Tulkens, 2006a]. For positive perspective it is meant the logical (rational) outcome af a given situation. In other words, the formation of the coalition must be in the self interest of the constituting parties that act under the typical assumption of rationality. Consequently, the most important property that the coalition must hold is stability. This concept is operationally translated into the conditions that, once a coalition has been formed, no one of the non-signatories should find rewarding to enter the agreement and no one of the signatories has an incentive to leave [d'Aspremont et al., 1983]. Mathematically, this is expressed as:

$$\Pi_{i\in S}(S) \geq \Pi_{i\notin S}(S-i) \qquad \text{and} \qquad \Pi_{i\notin S}(S) \geq \Pi_{i\in S}(S+i), \quad \forall i \in N;$$

where $\Pi_{i\in S}$ and $\Pi_{i\notin S}$ indicate, respectively, the utility a country i enjoys being, and not being, part of coalition S, whereas N is the set of all countries. When the two inequalities are satisfied, the coalition S is stable. Two things worth to be mentioned. The first is that the focus on stability clearly amounts to consider primarily the number of coalition participants or, equivalently, the number of IEA's signatories. This, in fact, seems to be the primary objective of NCGT applied to this field [Zara et al., 2006b]. The second aspect relates to the fact that the stability concept used here is reminiscent of the stability set proposed in von Neumann and Morgenstern [1944], a typical cooperative solution, although they are not coincident [Chander and Tulkens, 2006a]. In performing this analysis, however, a vast part of the non-cooperative literature, namely, the one adopting Reduced Stage Game models (RSG), considers only a particular type of free riding: external free-riding [Finus, 2008]. With this expression it is meant a country that stays outside the agreement but enjoys the positive externalities generated by it. The internal dimension of free riding, where a country joins the agreement but does not comply with it, instead, is rarely taken into consideration as recognised[5] by McEvoy and Stranlund [2009] and Finus [2008]. Disregarding this dimension implies to assume agreement enforceability. It then follows that this concept cannot be used any more as a discriminant between CGT and NCGT.

Although the border line between the two approaches is more blurred than in other fields, nonetheless they remain deeply separated in a crucial point: results. NCGT generally predicts the formation of a small stable coalition (SSC) [Carraro and Siniscalco, 1993; Barrett, 1994; Diamantoudi and Sartzetakis, 2006], whereas the cooperative one asserts that the grand coalition (GSC) can be formed [Chander and Tulkens, 1995, 2006b; Germain et al.,

[5]This dimension is explicitly addressed by Dynamic Game (DG) models such as the ones presented by McEvoy and Stranlund [2009] and Finus [2000] However, they are only a subgroup of the non-cooperative literature about IEAs.

2010]. Tulkens [1997] has dedicated a whole paper to explain in detail what causes this gap. The most important point is the fact that, in the cooperative approach, countries are given the Core solution (the comprehensive coalition) in the first step, and then are asked if they want to leave it, whereas the non-cooperative approach starts from the bottom, meaning the situation in which countries are singletons and have to agree to some form of cooperation [Tulkens, 1997]. The other main difference, whose examination is postponed, is the fact that the cooperative approach assumes a strong reaction from the side of the coalition to the potential defection of one of its members. Such defection, in fact, causes a complete breaking up of the whole coalition. Apart from this last critical aspect, it is possible to note that the main difference is a matter of perspective, as the definition of Osborne and Rubinstein [1994] underlines.

At this point, however, one might wonder which is the utility of the cooperative approach if this disregards a crucial aspect such as the one of the IEA's formation process. The answer needs a premise. Full cooperation is globally optimal. This derives from well known public goods theory [Samuelson, 1954] according to which a Pareto efficient solution in presence of a public good can be achieved only if all the interested parties are involved in the process of its allocation. From this, it follows that cooperation is desirable in itself. Cooperative game theory, in fact, is said to adopt a normative perspective [Chander and Tulkens, 2006a], rather than a positive one, meaning that it pursues cooperation as a goal, not a simple rational consequence. The aim becomes then to define the allocation that renders cooperation feasible and more acceptable. It can be said that it is an allocation, rather than a constitution, game. Its first goal, rendering cooperation feasible, is then subject to the constraint of satisfying individual rationality, whereas the second, related to acceptability, is bounded by the concepts of equity and fairness [Young, 1994]. These can be given a general, normative meaning, or a narrower, practical one. In the first case, they have to be intended as a principle of justice, a moral attitude, whereas in the second they provide indications to shape concrete behaviours. This last interpretation presupposes a synthesis and, generally, a compromise, between the abstract principle and the material conditions of the case at hand (power considerations). The satisfaction of individual rationality constitutes this last element. The solution concepts adopted by cooperative game theory can be considered as different forms of compromise between these two requirements.

4.3 Solution Concepts

This section introduces and describes the most used solution concepts of CGT applied to the IEAs' field. It therefore considers only a subset of the many solution concepts adopted by

CGT. Prevalence is given to single point rather than set solutions. The distinction between these two categories is self contained in their names. Set solutions, in fact, define the whole space in which cooperation (GSC) can be sustained, whereas a single point solution indicates a precise allocation able to perform the same task. The choice of focusing on the second category is due to this last consideration. A set solution, in fact, still leaves unsolved the problem of defining which equilibrium will emerge in the particular game at hand, providing just a space identification inside which multiple equilibria are possible (actually, every point in this space is an equilibrium). Another way to interpret the difference between the two categories recalls the distinction between feasibility and acceptability mentioned before. A set solution can be said to consider only the first element, providing an indication of all the possible allocations that are feasible, whereas a single point solution is reached by adopting a specific equity concept, according to which there will be a unique allocation.

A further consideration needs to be done regarding the nature of the games under consideration. Cooperative games are usually divided in two categories: games with (TU) and without transferable utility (NTU) [Zara et al., 2006a; Osborne and Rubinstein, 1994]. The difference is easy to figure out. TU games assume that players' utility can be transferred among them (e.g. lump-sum payments in money or goods), whereas NTU games do not allow for this. Therefore, the TU assumption implies that the worth of a coalition, calculated as the sum of the utilities of its members, can be divided among them in any possible way [Zara et al., 2006a]. Conversely, in the NTU case, coalition members are constrained to enjoy the utility that they self-generate. Obviously, in this second class of games, it will be generally more difficult to obtain cooperation since transfers cannot be used as an instrument to induce it. Given that countries are the players in the considered game and that their utility is generally proxied by GDP, there seems to be no ground to refuse the TU assumption.

4.3.1 A standard coalitional game and the characteristic function

Given a game $\Gamma(N,v)$ with a number n of players (countries, in the present case, where $n > 2$), define as N the set of all players: $N = 1,2,...,n$. Furthermore, denote a coalition S as a strict subset of $N : S \subset N$. The set N is also a coalition, namely the grand coalition, including all players. Finally, define as Σ the set that collects all the possible coalitions - among which the empty set - which will necessarily have 2^n elements. It has to be noted that the n players are also treated as single member coalitions. The following necessary step to characterize a coalitional game is to define a characteristic function v, intended as a real-valued function that assigns a value to each one of the coalitions included in the set Σ. The value $v(S)$, since we are considering a TU game, can be interpreted as the total pay-off available for distribution among the members of coalition S [Osborne and Rubinstein, 1994]. By assumption, $v() = 0$, meaning that the empty coalition has a zero value.

Definition: A TU coalitional game $\Gamma(N, v)$ in characteristic function form consists of a finite set of players N and a function v that assigns to each non empty subset S of N a real number $v(S)$, representing the utility of S available for distribution among its members.

In the vast real of games in characteristic function form, particular classes have been individuated according to the properties of the same characteristic function: convex (supermodular), and superadditive games. The first class is a subset of the second. In order for a game to be defined convex, the characteristic function must satisfy the following inequality:

$$v(S) + v(T) \leq v(S+T) + v(S \cap T), \quad \forall\, S, T \subseteq N.$$

Driessen [2013] shows the equivalence between convexity and supermodularity of v. A characteristic function is supermodular if:

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T), \quad \forall\, S \subseteq T \subseteq N \setminus \{i\} \text{ and } \forall\, i \in N.$$

From this last condition, it can be said that a cooperative convex game is one in which, starting from a given coalition, the marginal contribution a player brings to it increases monotonically by increasing the size of the same coalition. Superadditivity, instead, can be considered as a weaker version of convexity, requiring that the characterisitc function satisfies the following property[6]:

$$v(S) + v(T) \leq v(S+T), \text{ where } S \text{ and } T \text{ are disjoint coalitions: } S \cap T = \emptyset.$$

It will be seen later on why these two properties are really important in a coalitional game. For the moment it is sufficient to note that, when superadditivity holds, the characteristic value of every coalition $S \neq N$, cannot be higher than the value of the grand coalition. Obviously, this is true for convexity as well.

At this point it is necessary to define the characteristic function itself:

$$v(S) = \sum_{i \in S} \Pi_i(\phi_1, ..., \phi_n).$$

[6]Recalling the definition of convexity - $v(S) + v(T) \leq v(S+T) + v(S \cap T), \quad \forall S, T \subseteq N$ - it is clear that, since it must hold for all coalitions S and T, meaning also disjoint coalitions, it encompasses the superadditivity condition. In fact, when S and T are disjoint, $v(S \cap T) = v(\emptyset) = 0$, and therefore the inequality defining convexity collapses into the one defining superadditivity.

This is simply the sum of the utilities (indicated by $\Pi_i(.)$) of all coalition's members, given the strategies (ϕ_i) adopted by all the players in the game. It is then possible to link a coalitional form of a game with its strategic form. From standard GT, it is known that a n-players game is composed by the $2n$-tuple $(\Phi_1, ..., \Phi_n, \Pi_1, ..., \Pi_n)$ where Φ_i is the set of pure strategies of player i and $\Pi_i(\phi_1 ... \phi_n)$ is the pay-off of player i if player *1* uses the strategy $\phi_1 \in \Phi_1$ and player *2* uses $\phi_2 \in \Phi_2$,..., and player n $\phi_n \in \Phi_n$ [Ferguson, 2005]. Therefore, from a standard game in strategic form, the passage to a coalitional form game entails to give a value to each possible coalition taking into consideration the strategies adopted by its members and the ones of the non-members. In the IEA game, the set of strategies Φ_i is the level of emissions or abatement that each country will undertake. Coalition members are supposed to coordinate their strategies in order to maximize their global welfare that, as just shown, coincides with the characteristic value of the same coalition [Zara et al., 2006b]. The strategies chosen by the players outside the coalition depend on the model assumptions. This topic will be discussed in the following section together with the first solution concept: the Core. Resuming what said in this section, a n-players game in coalitional form is a game defined by the pair (N, v) that focuses on the coalitions' outcome (characteristic value) rather than on the outcome achieved by single players.

4.3.2 The Core and its various declinations: α, β and γ

In partial contradiction with what said before, the first solution concept that will be examined, the Core, is a set rather than a point solution. This deviation is due to the fact that the Core is a fundamental notion in CGT, it is useful to compute other point solutions and it is helpful in explaining the assumption regarding the non-members' behaviour in the IEA game. Assuming that $v(N)$ is higher than every other $v(S)$, it seems rational for the players of the game to form the grand coalition. The problem, in presence of transferable utility, becomes then to agree upon the amount that each player should receive (how to split the pie) [Ferguson, 2005]. An imputation $\mathbf{x} = (x_1, ..., x_n)$ is a pay-off vector that defines the sum - the amount of utility - that each player should receive, if that imputation will be accepted. The Core (N,v) is defined as the set of all imputations that satisfies the following conditions:

$$(1) \sum_{i \in N} x_i = v(N).$$

$$(2) \sum_{i \in S} x_i \geq v(S), \quad \forall\, S \subset N.$$

The first condition is a simple consequence of rationality and states that the entire value of the grand coalition should be distributed among players (efficiency condition), whereas the second one is what really defines the Core. Basically, each player should get an amount at least equal to what she could get in any of the sub-coalitions that she could form. In this way, no one has an incentive to leave the grand coalition. This second condition can be further divided into two parts. The first, quite obvious, says that every player must get more than what she could achieve playing alone: $x_i \geq v(\{i\})\ \forall i \in N$. The second part, instead, includes also the other sub-coalitions with two or more players and defines, through the condition $\sum_{i\in S} x_i \geq v(S)$, the stability of an imputation. In words, it can be said that the Core collects all the efficient imputations that satisfy stability [Ferguson, 2005]. As anticipated, neither it considers any principle of equity or fairness nor it provides a clear indication on which imputation to prefer but still it discriminates between games that can support, on the ground of stability, the grand coalition and the ones that cannot. This last case happens when the Core is an empty set.

Recalling the definition of the characteristic value of a coalition S as $v(S) = \sum_{i\in S} \Pi_i(\phi_1 ... \phi_n)$, it is clear that its definition depends from the strategies adopted by its members as well as the ones undertaken by the players 'outside'. Coalition members are supposed to act in order to maximize $v(S)$ itself. It is then required to assume which is the behavioural pattern followed by the 'outsiders'. Three such assumptions have emerged in the CGT literature. The first is the most pessimistic one, supposing that non-members will adopt the most detrimental strategy at their disposal in order to contrast S. Once this assumption is adopted, we will speak of α-characteristic function and α-Core. Therefore, $v(S)$ will be defined by a *maxmin* principle, as to say, it is the maximum pay-off that a coalition can guarantee to itself knowing that non-members will act in order to minimize it:

$$v_\alpha(S) = \max_{\phi_S \in \Phi_S} \min_{\phi_{i|i\in N\setminus S} \in \Phi_{i|i\in N\setminus S}} \Pi_S(\phi_S, \phi_{i|i\in N\setminus S}), \quad \forall i \in N \setminus S.$$

The α-Core is simply the Core under this particular assumption. The β-characteristic function (and β-Core), instead, is obtained by adopting a *minmax* principle. In words, by assuming that the coalition can achieve the minimum among the maximum pay-offs that it is able to guarantee to itself after that the strategies of the players have been fixed [Zara et al., 2006a]:

$$v_\beta(S) = \min_{\phi_{i|i\in N\setminus S} \in \Phi_{i|i\in N\setminus S}} \max_{\phi_S \in \Phi_S} \Pi_S(\phi_S, \phi_{i|i\in N\setminus S}), \quad \forall i \in N \setminus S.$$

The last Core concept, the γ-Core, has been developed by Chander and Tulkens [2006b] specifically to deal with IEAs and other environmental games. The γ-characteristic function implies two main assumptions: first, that players remaining outside coalition S do not form any other coalition, so they act as singletons, and secondly that they do not take any particular action, neither to contrast nor to favour, the formed coalition. They behave neutrally, following self-interest in a rational way:

$$v_\gamma(S) = \max_{\phi_S \in \Phi_S} \Pi_S(\phi_S, \phi_{i|i\in N\setminus S}) \quad \text{where } \phi_{i|i\in N\setminus S} \text{ results from}$$
$$\max_{\phi_{i|i\in N\setminus S} \in \Phi_{i|i\in N\setminus S}} \Pi_{i|i\in N\setminus S}(\phi_{i|i\in N\setminus S}, \phi_{N\setminus S,\{i\}}, \phi_S), \quad \forall\, i \in N \setminus S.$$

Outsiders ($is \in N \setminus S$), therefore, act in a competitive way both among each other and toward the formed coalition S, with the only aim of maximizing private utility.

Which Core is appropriate for an IEA game?

The α- and β-Core, theorized by Aumann [1959], have been discussed in the early stage of development of the environmental CGT literature, but have been almost completely abandoned after the introduction of the γ-Core. Laffont [1977] has shown that, in a game characterised by an economy with detrimental externalities (such as environmental games), the α and β assumptions coincide. Maler [1989] has been the first to discuss the problem of these assumptions when applied to the environmental field. In its “Acid Rain” game he has hypothesised that there is no upper bound to the level of pollution that countries can produce. Since, as stated in the introduction, the strategy space of a country is given by the amount of pollution it will generate, this means that, under α and β assumptions, all the non-members will produce an infinite level of pollution. Even placing some kind of ‘technical’ upper bound to the level of pollution feasibly deliverable, this does not solve the conceptual problem of why non-members should actually adopt this strategy. Generating positive externalities, a coalition favors also outsiders, therefore it should be in their interest not to contrast its formation. Secondly, such a high level of pollution is detrimental, therefore irrational, also for themselves.

Chander and Tulkens [2006b] have then envisaged a new type of Core concept, whose main assumption is the simple rationality of players. Instead of using their production economy, it is simpler to explain the idea behind it making use of the emissions’ benefit and damage functions. Under this setting, the monetary utility [Π_i] obtained by a country i is given by a benefit function [$B_i(\cdot)$] having as argument own emissions [e_i] - recall that emissions generate a benefit being a proxy for production and, consequently, consumption - minus a function [$D_i(\cdot)$] describing the environmental damage caused by pollution that, in presence of a global pollutant, will have as argument the sum of the emissions produced by all countries [$\sum_{j\in N} e_j$]:

$$\Pi_i = B_i(e_i) - D_i(\sum_{j\in N} e_j);$$

From basic optimality conditions, it is known that the pay-off of i is maximised when $B_i' = D_i'$. From what said till now, it becomes clear that the strategy space of i, under the sole assumption of rationality, becomes narrowly bounded till being a single value. It is possible to define the strategic choice of a country as a deterministic choice obtained by simply equating the first derivative of two known (by assumption) functions. Given that the damage function has the sum of all countries' emissions as its argument, this implicitly creates a strategic game that, in absence of any further assumption, takes the form of a Cournot game if the damage function is convex. The alternative would be a Stackelberg game in case a first move advantage is given to some players. In the dedicated literature, this assumption has been often used, guaranteeing the advantage to the coalition [Barrett, 1994; Sartzetakis and Strantza, 2013; Diamantoudi and Sartzetakis, 2006]. However, it has been criticised as theoretically ungrounded by Finus [2008]. It has to be noticed a strong convergence between the cooperative and the non-cooperative literature in representing the pay-off function of players. Generally, it is assumed a concave benefit and a convex damage function. The first complies with the non-satiety, but marginal declining satisfaction of consumption, that is standard in economic theory, whereas the second derives from environmental science according to which ecosystem resilience and absorption capacity suffer from saturation. The game so depicted has a single Nash equilibrium found as the solution of the maximization problem just described. Chander and Tulkens [1995] call it the disagreement point and it gives the characteristic value of the singletons' coalitions, $v(\{i\})$, also called the reservation utility of players. Till here, CGT and NCGT do not show any difference.

Once players are allowed to form (or discuss the formation of) coalitions, they will act in the interest of the same coalition, if they are members, or in their private interest if non-members. Acting in the interest of the coalition translates in maximizing the sum of the pay-off functions of all its members:

$$\max_{e_i|i\in S} \sum_{i\in S} \Pi_i \quad \rightarrow \quad \max_{e_i|i\in S} \sum_{i\in S} (B_i(e_i) - D_i(\sum_{j\in N} e_j)).$$

The coalition will therefore act as a single entity and, in this role, will play the same game just described with all the other players, that, according to the γ assumption, will keep their rational, self-interested behaviour and will act as singletons. It must be noted that the γ

assumption can be decomposed into two distinct assumptions, each related to one of the two (interconnected) strategic spaces composing the IEA game. The first relates to the amount of pollution, the choice of the level of emissions, outsiders will undertake. The second regards their possibility to associate forming one, or potentially more, competing coalitions other than the one currently existing. As said, this possibility is basically excluded. Whereas the first assumption is decisively justified, from a conceptual point of view, by rationality, the second, instead, appears more as a simplifying device[7]. In fact, several papers, among which Eyckmans and Finus [2004]; Buchner and Carraro [2005]; Eyckmans et al. [2012], have dropped it, allowing for the coexistence of more than one coalition with several players. This would imply to switch from a characteristic to a partition function form game. This chapter, however, will align with the vast majority of the literature on IEAs, both cooperative and non-cooperative, assuming that only a non-singleton coalition can be formed. A further point must be made. Recalling that a coalition always generate positive externalities, the γ assumption means that non-members, by standing as singletons, actually adopt the worst strategy at their disposal from the coalition point of view. As for the basic non-cooperative case, a unique Nash equilibrium will form (this has been called in Chander and Tulkens [1995] partial agreement Nash equilibrium - PANE). Another important property that follows from the structure of the game and from the γ assumption is that every change in the emissions level of one player will cause a partially offsetting reaction (best reply function) from the others [Finus, 2000]. A way to avoid this is to use a quadratic benefit in combination with linear damage functions, for which there is no reply to a variation in the level of emissions of other players. However, this formulation misses to capture an important feature of the pollution problem: its increasing harmful effect.

Finally, it is possible to arrive at the distinctive feature between CGT and NCGT in their application to IEAs. Given that the γ assumption basically reproduces exactly the same behavioural pattern adopted by NCGT for non-members and considering that the way a coalition acts is also identical, the difference must be searched somewhere else. Chander and Tulkens [2006b] have first tested two non-cooperative solution concepts: the strong Nash equilibrium and the coalition-proof Nash equilibrium. The first has been disregarded since it does not exist for this type of games, whereas the second, introduced by Bernheim et al. [1987], is actually the solution adopted by NCGT. Chander and Tulkens [2006b] declared to be unsatisfied with this solution since it is suboptimal (not Pareto efficient) and since it implies that a deviation of a coalition (a set of members leaving a coalition) does not

[7] Diamantoudi et al. [2002] justify this assumption on an empirical ground noticing that “IEAs are usually unique and fostered by the United Nations”.

cause any reaction from the remaining members. As stated in the introduction, the main difference of the two approaches can be described as a matter of perspective. NCGT starts from the bottom, the disagreement point, and look at which coalition can be built, whereas CGT assumes the existence of the grand coalition and examines if there are incentives for leaving it. Anyway, this is not the end of the story. The reaction of the remaining members to a deviation of a coalition (as usual, in the CGT jargon, this means also a single player, a singleton) is also central in order to theoretically justify the feasibility of the cooperative approach. In fact, when they examine the incentive to leave the grand coalition, CGT adopters consider only the pay-offs that players can achieve forming sub coalitions and the singletons' pay-offs are the ones obtained in the disagreement state. This means that the pay-offs achievable by being non members (free riders, in the NCGT jargon) are simply disregarded. The justification of this strong limitation stays on the reaction of members to deviations that implies to break the coalition and to play the disagreement strategy [Chander and Tulkens, 2006b]. Much of the controversy between the two approaches has been focused on this assumption, with NCGT supporters claiming that this threat is not credible.

4.3.3 The solution of Chander and Tulkens

The point solution proposed by Chander and Tulkens [1995] has two interesting and appealing properties. The first is that it lies in the Core, so that it preserves the individual rationality of cooperating. The second is that it uses the same elements of the countries' pay-off function (namely, the benefit and damage functions and their parameters) to define the imputation vector to be adopted. Each of its ith elements is composed by two parts: country i's pay-off obtained in the full cooperative case ($S = N$) plus a transfer: $x_i = \Pi_i(e_i^*) + T_i$, with e_i^* being the equilibrium level of emissions of country i participating to the grand coalition. The important part is constituted by the transfer T_i and the rule defining it:

$$T_i^* = [B_i(\bar{e}_i) - B_i(e_i^*)] + \frac{D_i'}{\sum_i D_i'}[\sum_{j\in N} B_j(e_j^*) - \sum_{j\in N} B_j(\bar{e}_j)];$$

$$\sum_i T_i = 0$$

where $\bar{e}$ is the equilibrium level of emissions at the disagreement point. Since the sum of T_i over the is is equal to zero, it is easy to check that $\sum_i x_i = \sum_i \Pi_i^* = v(N)$, so that the group rationality and efficiency condition is met. Explicitly writing the pay-off of a country when the grand coalition is implemented, $B_i(e_i^*) - D_i(\sum_{j\in N} e_j^*)$, helps to easily understand what will be the final imputation received and the ratio behind this transfer scheme:

$$x_i = B_i(\bar{e}_i) - D_i(\sum_{j\in N} e_j^*) + \frac{D_i'}{\sum_{j\in N} D_j'}[\sum_{j\in N} B_j(e_j^*) - \sum_{j\in N} B_j(\bar{e}_j)];$$

A country will then receive an amount equal to its benefit function valued at the disagreement point, so when its emissions and, consequently, the value of the same function, is maximum. To this, it is subtracted the value of the damage function with emissions as in the full cooperative case, so when it is the lowest. From what just said, it is clear that the term inside the square brackets is always negative. This term will then be subtracted proportionally to the magnitude of the parameter describing the importance of the environmental damage for a country compared to (divided by) the sum of the same parameter over all countries. In other words, the first, always positive, term is diminished in a way that is proportional to the vulnerability of a country to pollution. This is justified since polluttees need to pay polluters in order to induce them to cooperate by compensating them for their forgone benefits obtained by emitting. However, as explained in Chander and Tulkens [2006a], this solution is actually favourable to pollutees. In fact, they will pay polluters just up to the point that will induce them to cooperate, but the actual surplus of cooperation is retained by the same pollutees.

4.3.4 The Shapley value

The Shapley value [Shapley, 1953] is a point solution concept that has found some applications in the context of pollution problems, for example in Botteon and Carraro [1997] and Petrosjan and Zaccour [2003]. It can be considered part of a broad family of (both set and point) solution concepts that rely on the mechanism of objections and counter-objections well described in Osborne and Rubinstein [1994]. This class of solutions, differently from the Core that poses only an 'immediate' feasibility constraint, considers the chain of events that the deviation of a coalition may trigger. Also in this case feasibility is the central aspect, but it is evaluated only on the ultimate outcome produced by a deviation [Osborne and Rubinstein, 1994]. Parts of this category are: the Stable set, the Bargaining set, the Kernel, the Nucleolus and the Shapley value. The first, introduced by von Neumann and Morgenstern [1944], will not be considered since it has never been applied to environmental problems. Furthermore, it is a superset of the Core [Osborne and Rubinstein, 1994] and not only, being a set solution, allows for multiple equilibria but, for a single game, there can be more Stable sets. The following three solutions are strongly interconnected being the first a superset of the second and this of the latter [Driessen, 1988]. Only the Nucleolus, the sole point solution among the three, will be described specifically in the next section.

The mechanism of objections and counter-objections allows to define a stable state, obtained when they reciprocally nullify. At its base there is a mix of considerations about power relations and fairness. Different weights attributed to these elements give rise to the multiplicity of solution concepts mentioned. The Shapely value focuses primarily on the marginal contribution that a player brings to a coalition. The objections that a coalition member can claim to another player for a certain imputation are twofold. She can claim that, leaving the coalition, will cause a loss to that player greater, for this last, than accepting the alternative imputation she is proposing. Alternatively, she can object that there is the possibility for her and the other members to make a coalition without the accused player that will leave her better off and the remaining players at least as good as before. Basically, she can induce the others to exclude the contested player. A counter-objection is simply the same argumentation put forth by the accused player. An important consideration to be made is that the Shapley value considers at one time all the subgames present in a game. In other words, it requires that the objection\counter-objection nullification holds for all the subgames. This last sentence expresses the balanced contributions property. In order for this property to hold, it is required to assign to each player a value ψ such that:

$$\psi_i(N,v) - \psi_i(N \setminus \{j\}, v^{N\setminus\{j\}}) = \psi_j(N,v) - \psi_j(N \setminus \{i\}, v^{N\setminus\{i\}})$$

where $(N \setminus \{j\}, v^{N\setminus\{j\}})$ and $(N \setminus \{i\}, v^{N\setminus\{i\}})$ indicate the sub-games of $\Gamma(N,v)$ where players j and i are, respectively, excluded. The only value ψ that satisfies this condition is the Shapley value that, therefore, will be the imputation chosen ($\psi = x$) [Osborne and Rubinstein, 1994]. The formula for calculating it is given by:

$$\psi_i(v) = \sum_{S \in N\setminus\{i\}} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S\cup\{i\}) - v(S)]$$

where $|S|$ indicates the cardinality of the coalition S. The term in the square brackets describes the contribution player i brings to the coalition S. The sum is used to consider the (marginal) contribution that a player provides to all the possible coalitions of a game (all the sub-games). Finally, the expression preceding the square brackets is used to give a weight to each such contribution considering the probability a player has to actually 'produce' it. The denominator, in fact, is the number of all permutations of the n players, whereas the numerator expresses the number of these permutations in which the $|S|$ members of S come first than player i ($(|S|-1)!$ ways), and then the remaining $n-|S|$ players ($(n-|S|)!$ ways). ψ_i is the average contribution brought by player i to the grand coalition if the players sequentially form this coalition in a random order [Ferguson, 2005]. A possible extension of the Shapley value is to consider different probabilities in which coalitions can form. The random order just mentioned, in fact, implies to assume equal probabilities. In this case we would speak of Weighted Shapley value. A literature review describing the various weighting schemes and computation devices adopted can be found in Kalai and Samet [1987].

Regarding the properties shown by the Shapley value, it has to be pointed that it is the only solution concept contemporaneously satisfying *efficiency*, *symmetry*, *dummy axiom* and *additivity*. According to Hoàng [2012], the satisfaction of all these properties is compensated by an important drawback since the Shapley value does not always fall into the Core. However, it does in convex games [Shapley, 1971]. Applying this concept to IEA games leads to reward that countries having a high level of pollution. In fact, these are the ones whose inclusion in an agreement is most profitable from the global point of view. From another perspective, it could be said that the agreement would obtain scarce results without their presence. The solution of Chander and Tulkens takes into consideration this fact, but grants them only the sufficient benefits in order to make their participation rational. The surplus obtained from cooperation is given to polluted countries. With an imputation obtained through the Shapley value, instead, they have to give part of this surplus up to polluters.

4.3.5 The Nucleolus

The Nucleolus, introduced by Schmeidler [1969], is a point solution that, as anticipated, is contained both in the bargaining set and in the Kernel. Furthermore, when the Core is non empty, it is also an element of this last. In order to understand the mechanism of objections and counter-objections at its base, it is required to introduce the notion of the *excess* of S: $e(S,x) = v(S) - x(S)$; where $x(S) = \sum_{i \in S} x_i$ [Osborne and Rubinstein, 1994]. When $e(S,x)$ is positive, it represents the amount that the coalition will loose if that imputation will be implemented. On the contrary, when negative, it constitutes the surplus that the coalition receives from that imputation. It is then possible to define an objection having as argument an imputation $\mathbf{x}$ and a coalition S with related *excess* $e(S,x)$ to another imputation $\mathbf{y}$ if $e(S,y) > e(S,x)$ (e.g. $x(S) > y(S)$. A counter-objection is consistent if it does exist another coalition T for which $e(T,x) > e(T,y)$ and $e(T,x) \geq e(S,y)$. Compared to the Shapley value, it is possible to see that the Nucleolus uses coalitions as the main argument to make objections and counter-objections. Also in this case, the Nucleolus is defined as the equilibrium point where the two balance each other.

The other - actually the standard - way to define the Nucleolus is by saying that it individuates the imputations vector **x** for which the vector $E(x)$ is *lexicographically* minimum [Osborne and Rubinstein, 1994]. In order to understand this characterization, it is necessary to define the vector $E(x)$ and the word "lexicographically". Starting with an imputation vector **x**, it is possible to arrange the $2^n - 2$ coalitions' excesses in a non-increasing order. $E(x)$ will then be the vector collecting these excesses: $E(x) = e_l(S_l, x)$, $l = 1, ..., 2^n - 2$. Now, consider an alternative imputation, **y**, and repeat the same operation creating $E(y)$. It is then required to compare the first element (the one with the highest value, since they are ordered non-increasingly) of the two vectors. The one having a lower value will be preferred. Once it is not possible to further minimize it, switch to the second element and continue till the last. The lower bound for minimizing the first element is actually given by the second one. In fact, when the first reaches this level, further minimizing it will cause it to move on the second place given that the $E(x)$ vector must be ordered decreasingly. Therefore, as stated by Serrano [1999], the nucleolus maximises recursively the pay-off of the worst treated coalitions. The same author underlines that it can be interpreted as an application of the Rawlsian *maximin* principle [Rawls, 1971] applied to coalitions interpreted as independent subjects. The Nucleolus satisfies several properties that will be just mentioned. The first two are individual and group rationality. The third, being in the Core when this is not an empty set, is actually a proof in itself of the previous ones. The Nucleolus is unique (a point solution) and never empty. Finally, it satisfies consistency, covariance, anonimity and efficiency.

Computing the Nucleolus

The calculation of the Nucleolus requires a computational burdensome procedure even in presence of relatively simple 'games'. In their presentation of an analytic procedure to compute it, Leng and Parlar [2010] provide a review of the various algorithms present in the literature to efficiently solve the linear programming (LP) system necessary to find it. Here, it will be simply presented the standard procedure without taking into consideration the problem of computational steps.

Recall the definition of the excess, $e(S,x) = v(S) - \sum_{i \in S} x_i$, and remember that, excluding the empty and the grand coalition, there will be $l = 2^n - 2$ excesses so that it is possible to write $e_l(S_l, x)$. The Nucleolus is found by solving $\min_x \max[e_1(S_1, x), ...e_l(S_l, x)]$. A simple example, with three players, can help to further clarify the procedure. The coalitions set will be $\Sigma \setminus, N = (S_1 = 1; S_2 = 2; S_3 = 3; S_4 = 1,2; S_5 = 1,3; S_6 = 2,3)$. The minimization problem will then be:

$$\begin{aligned}
&\min_{x_1,x_2,x_3} \max[e_1, e_2, e_3, e_4, e_5, e_6] \\
&\text{s.t.} \\
&e_1 = v(S_1) - x_1 \\
&e_2 = v(S_2) - x_2 \\
&e_3 = v(S_3) - x_3 \\
&e_4 = v(S_4) - x_1 - x_2 \\
&e_5 = v(S_5) - x_1 - x_3 \\
&e_6 = v(S_6) - x_2 - x_3 \\
&x_1 + x_2 + x_3 = v(N).
\end{aligned}$$

This problem amounts to distribute the value $v(N)$ among x_1,x_2 and x_3 respecting the given conditions. Note that the displayed minimization program allows to compute the Nucleolus only if the solution is unique, otherwise it must be reiterated over the binding coalitions [Kopelowitz, 1967].

4.4 A Rawlsian solution concept bounded by individual rationality

The aim of this section is to introduce an alternative solution concept that, although lying in the Core, fosters the redistribution of utility. In order to introduce it, two additional solution concepts will be shortly described.

4.4.1 The strong ε-Core and the Least Core

The Strong ε-Core has been introduced by Shapley and Shubik [1966] as a way to find the Core even when this set is actually empty. They have shown that, for an appropriate value of $\varepsilon \in \Re$ there will always be an imputation lying in the Core. The Strong ε-Core can be defined as:

$$C_\varepsilon(N,v) = \left\{ x \in \Re^N : \sum_{i\in N} x_i = v(N);\ \sum_{i\in S} x_i \geq v(S) - \varepsilon,\ \forall S \subseteq N \right\}.$$

It can be seen that, when ε is positive and large enough, even a game with an empty Core will admit at least one element in this set. The value of ε can be interpreted as a penalty that members should pay in order to leave the grand coalition. Instead of thinking at ε as a value exogenously given, it is possible to interpret it as a variable to be minimized [Bilbao, 2000]. This amounts to solve the following system of equations:

$$
\begin{aligned}
&\min_{\varepsilon} z = \varepsilon \\
&\text{s.t.} \\
&\sum_{i \in N} x_i = v(N); \\
&\sum_{i \in S} x_i \geq v(S) - \varepsilon, \qquad \forall\, S \subset N.
\end{aligned}
$$

Its solution, that requires to find both the imputation vector and ε, being both variables, gives the Least Core. This is a point solution with redistributive properties similar to the Nucleolus. In order to understand why, let us hypothesise to have a game with non-empty Core. Furthermore, consider two coalitions with an equal number of members. One is ‘weak’, meaning that it has a low characteristic value, whereas the other is ‘strong’. In the minimization process, as said, both ε and the imputation vector will be defined. This means that, in order to have the lowest possible ε, the imputations of the members of the ‘weak’ coalition will be prioritized. Therefore, although the procedure to find (and the idea behind) the two solution concepts are quite different, the Least Core and the Nucleolus will give an imputation vector with similar characteristics.

4.4.2 The Minimum Feasible Core

The system of equations that is used to find the Least Core can be slightly changed to find another useful concept: the Minimum Feasible Core (MF Core). This is not an interesting solution in itself, since it is not efficient, but can be used to define the pure surplus generated by cooperation once the individual rationality constraint has been satisfied. Consider the following system of equations:

$$\begin{aligned}
&\max_{\eta} z = \eta \\
&\text{s.t.} \\
&\sum_{i \in N} x_i = v(N) - \eta; \\
&\sum_{i \in S} x_i \geq v(S) \qquad \forall\, S \subset N.
\end{aligned}$$

Although very similar to the program defining the Least Core, two crucial modifications have been applied: the position of the variable and its maximization rather than minimization. Basically, this solution tells which is the minimum characteristic value that the grand coalition must have in order to sustain full cooperation or, in order for the Core to be non-empty. This value is simply found as $v(N) - \eta$. The value of η can therefore be interpreted as the pure surplus (if positive) of cooperation, whereas the associated imputation vector as the minimum amount that each player should receive in order not to leave the grand coalition. A negative η, such as a positive ε in the previous case, indicates that the game has an empty Core. From what just said, it is clear that the MF Core corresponds to the set of solutions of the minimization program $\mathscr{P}2$ seen in chapter 2.

A final example can help to understand this mechanism. Let us think that each coalition $S \neq N$ is represented by an empty bottle. The bottles can have different dimensions and their volume is given by $v(S)$. The owners of a bottle are the players member of that coalition. The grand coalition, instead, is a barrel having an amount of liquid equal to $v(N)$. Now, under the assumption that one unit of liquid corresponds to a unit of volume, we need to give a certain amount of it to each player (defining an imputation) in order to fill up all the bottles. When a player is given a unit of liquid, this will contribute to fill in one unit of volume of all the bottles owned by her. Therefore, three conditions are possible. One is that there is not enough liquid to fill up all the bottles, one is that the liquid is exactly enough to do it and, finally, the last corresponds to have some spear liquid. Finding the Minimum Feasible Core tells us which would be the imputation in the middle case (how much liquid each player should receive), and, through the value of η, how much liquid we lack to reach this point (negative η) or how much there is in excess (positive η).

4.4.3 A 'revisited' Nash Bargaining solution and the Rawlsian Nucleolus

Remembering what said about the Nucleolus, this solution concept can be considered as a way to implement the Rawlsian *maximin* principle. This is actually true, but the potential flaw of this method in representing this principle is to consider coalitions as subjects. In reality, it does not really make sense to speak of the welfare of a coalition with two or more players. Welfare is an attribute of players alone. A redistributive principle should have them, and only them, as the main target. Before presenting a modified version of the Nucleolus that takes into consideration this aspect, it is opportune to further discuss the Minimum Feasible Core.

Remembering what said in the introduction, a solution concept is made of two parts. From one side it has to satisfy power relations assuring individual (and group) rationality. From the other, it has to provide a fair and equitable division, therefore, it is required to posses such a criterion of fairness and equity imbued on it. The MF Core, however, allows to completely separate the two aspects given that it provides the minimum sufficient condition to satisfy the first requirement. Basically, after that the MF Core imputation vector has been established (since now on it will be identified as $\mathbf{x}^{\eta}$), a new game can be thought regarding the way of dividing the surplus η. A first obvious solution would be to divide it in equal parts so that the final imputation would be $\mathbf{x} = \mathbf{x}^{\eta} + \frac{\eta}{n}$. This allocation sounds very appealing specially considering, as suggested, the splitting of the cooperative surplus as a cooperative bargaining problem whose starting condition is the imputation vector $\mathbf{x}^{\eta}$. This assumption would mean that the reservation utility of each country is: $r(\{i\}) = \mathbf{x}_i^{\eta}$. Therefore, it is easy to check that $\frac{\eta}{n}$ is the allocation that maximises the Nash Bargaining solution: $\arg\max_{\lambda_i} \prod_i ((x_i^{\eta} + \lambda_i \eta) - x_i^{\eta}) = \prod_i (\lambda_i \eta) = \frac{1}{n}$. Compared to the 'classical' Nash Bargaining solution, the difference stays in the alternative reservation utilized: the utility obtained in the disagreement point has been substituted by the MF Core imputation. Therefore, this solution is named 'revisited' Nash Bargaining solution[8] The fairness of this allocation, however, can be questioned. Splitting equally the surplus of cooperation, in fact, is surely equitable only if the "power game" determining the MF Core imputation vector is taken for granted. In other words, the power asymmetries at its root are considered natural and are fully justified on a moral base. Redistribution, therefore, does not find any valid reason for being implemented.

[8]Note that this solution is nothing else than the Mid-central Core. The difference in the chosen name is simply due to the fact that this chapter has been prepared, and published, before the others.

In the ‘global warming game’, however, this point of view can hardly be sustained. Indeed, it would imply to accept and to morally justify the fact that countries are affected differently from climate change on the simple base of their geographical position and that the most affected ones have to pay by themselves for this disadvantage. Furthermore, it also means to justify GDP inequalities and to wipe away the historical dimension of the pollution problem. Given the strong association between GDP level and the world share of cumulative emissions, measured in terms of CO_2 ppm, this last element is the most difficult to accept [Shukla, 1999]. A redistributive policy, in the IEA context, appears therefore an appropriate choice. However, unless introducing an altruistic attitude of countries that modifies their pay-off function, as done, for example, in Lange and Vogt [2003] and Grüning and Peters [2010], the postulate of self-interest imposes a strong lower bound on the amount that can be feasibly redistributed. This bound, as stated, is simply the MF Core imputation vector. The surplus, however, can be freely - meaning, without affecting the cooperative outcome - allocated in order to, at least partially, compensate the starting asymmetries. The problem to be solved will then be:

$$\max_{\lambda_i} \min(x_i^{\eta} + \lambda_i \eta)$$

s.t.

$$\sum_i \lambda_i = 1;$$

where $\min(x_i^{\eta} + \lambda_i \eta) = \min(x_1^{\eta} + \lambda_1 \eta, ..., x_n^{\eta} + \lambda_n \eta)$. Regarding the relation with the Nucleolus, the present solution can be considered as a modification of that concept in order to base its redistributive properties only on the singletons coalitions. Once solved for λ_i the given maximization problem, the final imputation vector will be: $\mathbf{x}^{RN} = x_i^{\eta} + \lambda_i \eta$, where the superscript “RN” stays for Rawlsian Nucleolus, the chosen name for this solution concept. A further relation with the Nucleolus can be seen by considering that the Rawlsian Nucleolus is the lexicographical minimum point of the Central Core.

4.5 A Numerical Comparison of Solution Concepts

The aim of this section is to provide a comparison of the countries' utility achieved in a standard IEA cooperative game applying the different solution concepts previously discussed: the Chander and Tulkens solution, the Shapley value, the Nucleolus, the Least Core, the revisited Nash Bargaining solution (splitting η in equal parts) and the Rawlsian Nucleolus. The model used to perform this comparison is a standard economic-environmental model with a quadratic concave emissions benefit and a quadratic convex damage functions: $B_i(e_i) = a_i(e_i - \frac{1}{2}b_i e_i^2)$; $D_i(E) = \frac{1}{2}d_i(E)^2$, with $E = \sum_i e_i$; $i = 1, ..., n$. As in Chander and Tulkens [1995], it will be adopted a γ-characteristic function, therefore, when a coalition form, the other members are supposed to pursue, as singletons, their self-interest. A country welfare function is given by:

$$\Pi_i = a_i(e_i - \frac{1}{2}b_i e_i^2) - \frac{1}{2}d_i(E)^2.$$

The model, under the assumption of symmetric countries (identical parameters), can be easily solved analytically. For convenience, three cases are treated separately: the Nash equilibrium (all countries act as singletons), the partial Nash equilibrium (PANE) and the full cooperation case (grand coalition). Solving for emissions in the three cases gives:

Nash Equilibrium

$$e_i = \frac{a}{ab + |N|d} \qquad E = \frac{|N|a}{ab + |N|d}$$

PANE

$$e_i^s = \frac{ab + (|N| - |S|)d(1 - |S|)}{b(ab + (|N| + |S^2| - |S|)d)} \qquad e_i^{ns} = \frac{ab + d(|S|^2 - |S|)}{b(ab + (|N| + |S|^2 - |S|)d)}$$

$$E = \frac{|N|a}{ab + (|N| + |S|^2 - |S|)d}$$

Grand Coalition

$$e_i = \frac{a}{ab + |N|^2 d} \qquad E = \frac{|N|a}{ab + |N|^2 d}$$

In the PANE case e_i^s and e_i^{ns} stay, respectively, for the emissions of a signatory and a non-signatory, $|S|$ indicates the number of coalition members and $|N|$ the total number of players[9]. By plugging these values in the pay-off functions, it is possible to find the welfare of each country and, consequently, the characteristic value of the coalitions.

At the beginning of the previous chapter, two properties of characteristic functions have been mentioned: convexity and superadditivity. The importance of the first stems from the Bondareva-Shapley theorem, that establishes a sufficient and necessary condition for a game to have a non-empty Core: balancedness. A game is balanced when:

$$\sum_{S\in\Sigma} \rho_S v(S) \leq v(N), \quad \text{for every balanced collection of weights } \rho_S.$$

The collection of weights $(\rho_S)_{S\in\Sigma}$ is a vector of scalars in [0,1] with dimension 2^n and it is balanced if, for every player i, the sum of it over all the coalitions that include i - 2^{n-1} coalitions - is equal to one: $\sum_{S\in\Sigma, S\ni i} \rho_S = 1$ [Osborne and Rubinstein, 1994]. Convex games are known to be always balanced, therefore they have a non-empty core [Dubey and Shapley, 1984]. Superadditivity, instead, only guarantees that the characteristic value of the grand coalition is not lower than the characteristic value of any other sub-coalition: $v(N) \geq v(S)$, $\forall\, S \subset N$. Recalling what said regarding the efficiency of an allocation referred to a public good, that is maximized when all the interested parties are included in its definition [Samuelson, 1954], it follows that a coalition should always get a benefit by expanding its membership. Superadditivity captures this, but it is not a sufficient condition to proof balancedness, therefore the Core might be empty.

[Dubey and Shapley, 1984] have proved balancedness for certain classes of not convex games, among which the production market game, the pure exchange market game and the transshipment game. Upon the introduction of certain mild restrictions - all the parameters (a, b and d) have positive values and no player generates a negative utility in any possible coalition - the present model, although not necessarily convex, always has a non-empty Core. The last claim has been proved by Helm [2001], whereas a simple numerical example, with identical players, is provided in Annexes (A1) in order to show that convexity is not a necessary property of the game. Superadditivity, instead, it is (proof is given in Annexes, A2).

A final remark is related to the restrictions on parameter values. Their strict positivity does not really need a justification since it is a necessary condition for the concavity of

[9] The subscript i on the parameters' letters has been dropped due to the assumption of symmetric countries.

the benefit function and the convexity of the damage function to hold. Setting their values in order that no player in any coalition obtains a negative utility could appear more arbitrary. However, it seems quite natural that, even in the disagreement point, countries still enjoy a positive utility. Once this condition is assured - for the case of identical players it requires that $\frac{ab}{d} > |N|(|N|-2) = \frac{B''}{D'} > |N|(|N|-2)$ - superadditivity implies that no player can be worst off in any coalition S ($|S| \geq 2$) than in the disagreement point. Furthermore, since a coalition generates positive externalities, also non members cannot experience a reduction in their level of utility. Therefore, setting appropriate values for non negative utilities in the disagreement point is sufficient to guarantee strictly positive utilities in all other cases. An analytic proof of this claim, for identical players, is given in Annexes (A3).

4.5.1 Characteristic values with asymmetric countries

The scenario with symmetric countries is not really interesting given the purpose of comparing the mentioned solution concepts. It can be checked, in fact, that, in this case, the imputation vector obtained and, therefore, countries' welfare, would be identical for all the solutions adopted. In order to give a touch of realism to the model while keeping its interpretation as simple as possible, only five countries will be considered and the parameters to vary are b and d. The first, that describes the magnitude of the marginal decrease of emissions benefits, is fundamental to determine the optimal level of the same emissions. For a given level of emissions, in fact, its magnitude is inversely correlated with the final utility achieved. It is therefore used to simulate the wealth, or the technological level, of a country. Three values will be used: High Wealth (HW) = 0.01, Medium Wealth (MW) = 0.02, and Low Wealth (LW) = 0.028. The parameter a, instead, will be kept equal for all countries and will be equal to 8. The other parameter to vary, d, represents the degree a country is affected by pollution. A higher value implies that a country is more vulnerable to the detrimental effects of climate change. Also in this case three levels will be adopted: high (HD = 0.0024), medium (MD = 0.00225) and low (LD = 0.002) vulnerability. Combining them, five types of countries are simulated. They are shown in decreasing order of '*power endowment*':

1) High Wealth - Low Damage HWLD: $a = 8$; $b = 0.01$; $d = 0.002$;

2) High Wealth - High Damage HWHD: $a = 8$; $b = 0.01$; $d = 0.0024$;

3) Medium Wealth - Medium Damage MWMD: $a = 8$; $b = 0.02$; $d = 0.00225$;

4) Low Wealth - Low Damage LWLD: $a = 8$; $b = 0.028$; $d = 0.002$;

5) Low Wealth - High Damage LWHD: $a = 8$; $b = 0.028$; $d = 0.0024$;

4.5.2 A comparison of distributive properties

In Table 4.1 it is possible to see the imputation vectors obtained with the different solution concepts. As said, each imputation corresponds to the final utility obtained by a country. The first two columns, in reality, show how pay-offs will be distributed in the Nash equilibrium (disagreement point) and in the grand coalition without any transfer scheme. The row displaying the summation of utilities testifies the benefit provided by cooperation. For a comparison between the characteristic value of the grand coalition and all the other partial coalitions see Table 4.6 in Annexes. It should also be noticed that the distribution obtained without transfer is actually the most egalitarian: the product of utilities is the highest compared with the one obtained from any other solution concept. However, this imputation does not satisfy the boundaries imposed by the MF Core, with the first two countries obtaining a lower value.
Comparing the various solution concepts, it can be first noticed that they are all efficient since the summation over all imputations is equal to the characteristic value of the grand coalition. The only one failing is the MF Core, but it has already been explained that this solution is not useful in itself. Furthermore, they all lie in the Core. The similarity between the Least Core and the Nucleolus is confirmed till the point that, with only one decimal number displayed, they appear identical. Regarding their (re)distributive properties, the Rawlsian Nucleolus is the one that advantages the most the 'weakest' countries and maximises the product of utilities. The imputation having the opposite effect is not the one obtained with the Shapley value, as could have been expected, but the one realized through the revisited Nash Bargaining solution. Dividing the cooperative surplus in equal parts advantages strong players beyond the value of their marginal contribution over all coalitions.

An interesting comparison can be done between the CT solution and the Rawlsian Nucleolus. Chander and Tulkens [2006a] affirm that their solution is the most favourable possible for pollutees. However, its redistributive properties are lower than the ones of the Rawlsian Nucleolus. One can suspect that the difference stays in the fact that this numerical example portrays differences both in the environmental damage parameter and in the emissions benefit one. This aspect, however, is not fundamental. Even allowing for symmetric benefit functions, the obtained imputation vector is different[10]. This is due to the fact that in the CT solution pollutees are required to compensate polluters for their forgone emissions benefit. This is the lower bound, whereas the Rawlsian Nucleolus uses the MF Core as lower bound. The due compensation is reduced by the increase in the utility that also

[10]This has been tested adopting a single parameter b for all countries whose value has been set equal to 0.02. Results, however, are not reported here.

Table 4.1 **Imputation Vectors From Different Solution Concepts**

Countries	**Nash Equilibrium**	**No Redistribution**	**Shapley value**
HWLD	310.7	312.6	331.5
HWHD	292.3	302.7	314.2
MWMD	100.6	125.3	118.7
LWLD	54.9	79.8	70.8
LWHD	37.1	69.9	55.1
Utility Sum.	795.5	890.4	890.4
Utility Prod.	1.86E+010	6.62E+010	4.83E+010

Countries	**Least Core**	**CT solution**	**Nucleolus**
HWLD	329.6	327.8	329.6
HWHD	311.2	312.9	311.2
MWMD	119.5	119.9	119.6
LWLD	73.9	72.1	73.9
LWHD	56.1	57.7	56.1
Utility Sum.	890.4	890.4	890.4
Utility Prod.	5.08E+010	5.12E+010	5.08E+010

Countries	**MF Core**	**Rawlsian Nucleolus**	**'R' Nash Barg.**
HWLD	326.8	326.8	334.3
HWHD	311.2	311.2	318.6
MWMD	110.1	110.1	117.5
LWLD	58.9	71.2	66.4
LWHD	46.2	71.2	53.6
Utility Sum.	853.2	890.4	890.4
Utility Prod.	3.05E+010	5.67E+010	4.46E+010

polluters enjoy thanks to a better environmental quality. However, what Chander and Tulkens claim is actually true. Furthermore, it is also appropriate their claim that this solution concept correspond to the polluters pay rule. However, another principle discussed during the Kyoto negotiation for dividing the burden of contrasting climate change refers to the concept of capacity, strictly related to the one of vulnerability [Heyward, 2007]. The damage caused by pollution differs not only according to some physical properties such as the geographical position of a country, but also given its ability to take counteractive measures (resilience).

This is likely to be positively correlated with the economic condition of a country. Unless the damage parameter already takes this into consideration, the Rawlsian Nucleolus appears to better address the vulnerability problem.

4.5.3 A comparison of incentives to, and potential losses from, free riding

This section will continue the comparison of solution concepts adopting a more non-cooperative perspective. In particular, they will be evaluated in light of their ability to prevent the damages from internal free riding. This last concept is different from the usual meaning that takes in the non-cooperative literature, where it is considered as the practice of non participating to an environmental agreement benefiting from the positive externalities generated by a coalition. In this case, instead, the participation to the grand coalition is taken as granted. However, a country can decide to cheat and to re-optimise its emissions' level taking the optimal level (from the collective point of view) adopted by the other countries as given. In choosing how much to emit, an internal free rider will face the following maximization problem:

$$\max_{e_{fr}} \Pi^{+}_{fr} = B_{fr}(e_{fr}) - D_{fr}(E^{*} - e^{*}_{fr} + e_{fr});$$

The subscript *fr* indicates the free rider and Π^{+}_{fr} is the pay-off obtained through the re-optimization. From the sum of all optimal emissions E^{*}, it is subtracted the share produced by the same free rider, e^{*}_{fr}, that will now substitute it with the result obtained from the re-optimization problem, e_{fr}. Obviously, this level will be higher and will be found through the usual optimality condition: $B'_{fr} = D'_{fr}(E^{*} - e^{*}_{fr})$. In order to free ride, instead of simply leaving the grand coalition, a free rider must still obey to the transfer scheme adopted. Although it has not been provided any formula defining a transfer for the solutions other than the one suggested by Chander and Tulkens, this is easily found: $T_i = x_i - \Pi i^{*}$. This holds for all solution concepts and it is trivial to show that the final utility obtained by a country is equal to the imputation itself: $\Pi_i = \Pi i^{*} + T_i = \Pi i^{*} + (x_i - \Pi i^{*}) = x_i$. In presence of free riding, however, this pay-off is modified in the following way:

$$\Pi_{fr} = \Pi^{+}_{fr} + (x_{fr} - \Pi^{*}_{fr}); \qquad \text{(free rider)}$$

$$\Pi_{i/fr} = \Pi^{-}_{i/fr} + (x_{i/fr} - \Pi^{*}_{i/fr}); \qquad \text{(coalition members other than free rider)}$$

The re-optimization problem faced by a free rider and its solution are independent from the imputation adopted. Moreover, the utility gain obtained by free riding is also independent from the solution concept adopted. In order to see this, just recall the definition of the final free rider pay-off given above: $\Pi^{+}_{fr} + (x_{fr} - \Pi^{*}_{fr})$. Its positive deviation from the utility that she would receive by respecting the rules, equal, as shown, to the same imputation, is given by: $\Pi^{+}_{fr} + (x_{fr} - \Pi^{*}_{fr}) - x_{fr} = \Pi^{+}_{fr} - \Pi^{*}_{fr}$. The imputations cancel out and what is left is a constant. The same holds for the loss suffered when it is another country to free ride. Table 4.7 in Annexes displays all the countries' pay-offs for the different solution concepts in presence of free riding. Table 4.2, instead, shows the gains - on the main diagonal - and the losses - on all the other cells, obtained and suffered when the country displayed on the left column free ride. As said, this table is the same for every solution concept adopted. What

Table 4.2 **Gains and Losses from Free Riding**

Free Rider	**HWLD**	**HWHD**	**MWMD**	**LWLD**	**LWHD**
HWLD	**24.75**	-13.85	-12.98	-11.54	-13.85
HWHD	-10.95	**22.50**	-12.32	-10.95	-13.14
MWMD	-5.52	-6.62	**11.83**	-5.52	-6.62
LWLD	-4.05	-4.86	-4.55	**8.98**	-4.86
LWHD	-3.86	-4.63	-4.34	-3.86	**8.19**

changes, instead, is the ratio of the gains and losses over the utility achieved in complying with the coalition rules. In order to obtain this, it is simply necessary to divide each row of the previous table by the same imputation vector (through a cell by cell, not a matrix division). Table 4.8 in Annexes displays all the coefficients so found. The values on the main diagonal can be interpreted as an index of the incentive a country has to free ride. In fact, if utility is measured in terms of GDP, this would translate in the percentage (after having been multiplied by 100) of GDP a country could obtain from a cheating behaviour. Obviously, the more favourable an imputation is to this country, the less significant the potential gain will be. Furthermore, this index, being built as a ratio of potential gains to a reference utility level, is not affected, in its representation of incentives, from the magnitude of the sole gains or from the starting conditions of a country. Proportionality should assure a balanced picture. On the other side, there are the remaining values of the described matrix. These represent the percentage loss a country would face in case another free ride. It is then a measure of risk in participating to a coalition with a given imputation vector. The more favourable is an imputation to this country, the lower will be the suffered damage (again, in terms of GDP percentage). It can be noticed that imputation vectors that favour wealthy nations reduce the risk that they will free ride. However, they also increase the damages suffered by other countries in case they will free ride. The opposite hold for imputations favouring weak countries. These last will be less tempted to free ride, but the avoided risk, at global level, will be less significant since the damage that they can inflict is lower. Finally, they will be less affected from deviations from wealthy notions that, however, will be more likely.

The problem with such a matrix is that it does not give a clear and immediate touchstone for comparing solution concepts. What is required is a single index able to measure the overall risk caused by free riding when a given imputation is implemented. This single index can be built in the following way. In order to show the necessary steps, the index coefficients table related to the Shapley value (Table 4.8 in Annexes) will be taken as an example. Let us write the transpose of it in matrix form:

$$\boldsymbol{A} = \begin{bmatrix} 0.0746 & -0.0330 & -0.0167 & -0.0122 & -0.0116 \\ -0.0441 & 0.0716 & -0.0211 & -0.0155 & -0.0147 \\ -0.1094 & -0.1038 & 0.0996 & -0.0384 & -0.0366 \\ -0.1629 & -0.1546 & -0.0779 & 0.1268 & -0.0545 \\ -0.2514 & -0.2385 & -0.1202 & -0.0882 & 0.1486 \end{bmatrix}$$

The element in the main diagonal, the incentive indexes, are extracted in order to form a vector h, keeping the same vertical order:

$$\boldsymbol{h} = \begin{bmatrix} 0.07464 \\ 0.07160 \\ 0.09961 \\ 0.12676 \\ 0.14862 \end{bmatrix}$$

Multiplying A, with the diagonal elements substituted by zeros, with h gives the vector q representing the overall risk faced by each country. In fact, each row of A displays the potential loss suffered by a country when each of the others free ride. The matrix multiplication with vector $\boldsymbol{h}$ weights the potential loss caused by a country deviation with the incentive that this country has to actually deviate. It can be contested that the vector $\boldsymbol{h}$ is used here as a measure of probability although it is actually far from being so. This critics is effectively reasonable. However, such a probability measure would be impossible to build, specially in this simple model setting. The magnitude of potential gains from free riding is therefore chosen as a second best, although with consciousness about its limitations. Once obtained the vector q, the final synthetic index is given by the summation of all its elements: $\boldsymbol{q} \times \boldsymbol{i}'$ (where $\boldsymbol{i}'$ is a vector of ones having same length as q). Basically, the overall potential loss caused by free riding in a given coalition for a given imputation vector is given by the sum of the same potential losses faced by each country. Table 4.3 reports the built indexes for each solution concept.

Table 4.3 **Overall Free Riding Potential Loss Index**

Shapley value	-0.140809
Chander and Tulkens solution	-0.136577
Least Core	-0.137407
Nucleolus	-0.137424
Rawlsian Nucleolus	-0.127944
'R' Nash Barg.	-0.146289

From the table above it can be seen that, although the more redistributive solution concepts foster the incentive to free ride of wealthy nations, whose deviation is the most detrimental, this is more than compensated by the higher imputations attributed to the other countries. The final index of free riding potential losses is the lowest for the Rawlsian Nucleolus, followed by the Chander and Tulkens solution. This last is closely followed by the Nucleolus and the Least Core, again almost identical. Finally, the Shapley value and the revisited Nash Bargaining. This classification mirrors exactly the one representing the distributive properties of solution concepts. The fact that redistribution minimizes potential free riding losses can appear counter-intuitive. On this regard it has to be noticed that it is not the absolute value of the losses to be minimized (as seen, this is constant), but the proportion each country will loose compared to its starting pay-off. If it was the absolute value of the losses to be weighted by the pseudo measure of probability of free riding, the result would have been different. However, this index appears to be justified since it can be seen as a representation of the potential losses in terms of GDP percentages. The focus is on each country and on its relative wealth, rather than in the overall value of the loss. The potential contrast with the cooperative perspective, more focused on global wealth, is settled by the fact that, when examining the risk of free riding, each country evaluates it on the base of its own potential losses.

4.6 Conclusions

This chapter has offered an overview of the most popular solution concepts derived from cooperative Game Theory that have found an application in the environmental field. In particular, the focus has been placed on a specific sub topic, namely, the constitution of an international environmental treaty to control the emissions of pollutants. After having briefly revised some fundamental concepts of CGT and having characterised the specificities of the game theoretical framework underpinning an IEA, the differences between the cooperative and the non-cooperative approach have been examined concluding that the perspective from which they look at the problem is the main point of departure. The other important element differentiating them is the way in which coalition members reply to a deviation from a cooperative behaviour of one of them.

The solution concepts taken into consideration have been the Core, together with its refinements: the Strong ε-Core and the Least Core. Moreover, solution concepts based on the idea of objections and counter-objections have been discussed: the Shapley value and the Nucleolus. An important solution in the game theoretical field of environmental economics, namely, the Chander and Tulkens solution, has also been examined. Finally, two alternative concepts have been proposed: the Rawlsian Nucleolus and a revisited Nash Bargaining solution, both based on the idea of the Minimum Feasible Core. The Rawlsian Nucleolus has been named in this way for its redistributive properties that favour the most disadvantaged, whereas the latter solution split the cooperative gain in equal shares among the cooperating parts. The difference between the solution proposed originally by Nash is that the reservation utilities applied here are the ones obtained through the MF Core.

The last part of the present chapter has been dedicated to a numerical exercise based on a standard game of emissions optimization in order to compare the properties of the mentioned solution concepts in terms of welfare distribution and ability to minimize the potential damages of internal free riding. The conclusion of the first analysis has shown that the Rawlsian Nucleolus is actually the most beneficial solution for poor countries largely affected by the detrimental consequences of climate change. The redistribution obtained through this method goes beyond the one achieved by the CT solution since the surplus is assigned on the base of final utility - prioritizing countries with the lowest - and it has the MF Core imputation as lower bound rather than emissions benefit in the disagreement point. Whereas the latter concept can be identified with the principle of polluters pay, the first further includes the criterion of vulnerability. However, it has to be underlined that such concept, if applied in a real context, could over represent the entitlements of poor countries simply given their low economic level. This would cause a detachment from the pure environmental field and could undermine the acceptability of this solution. The proposed revisited Nash Bargaining solution, instead, rewards wealthy nations, therefore high polluters, even more than the Shapley value.

The last part tries to introduce an element of non-cooperative game theory in the cooperative perspective underpinning the chapter: free riding. The link between the two approaches has been a theme largely debated, specially in the environmental context. After more than twenty years a satisfactory solution has yet to come. This chapter, therefore, does not pretend to achieve such goal. However, it offers an index, based on potential losses measured in terms of utility (GDP) proportions, that can be used as a preliminary instrument to evaluate the intrinsic free riding risk of losses present in a coalition for a given imputation vector. The ranking of the examined solution concepts under this regard mirrors exactly what emerged for the redistributive properties. The more a solution concept redistributes wealth, the lower will be the overall risk of losses due to free riding.

4.7 Annexes

A1

As a starting point, recall the supermodularity condition, remembering that proving supermodularity is equivalent to prove convexity:

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T), \quad \forall\, S \subseteq T \subseteq N \setminus \{i\} \text{ and} \,\forall\, i \in N.$$

Consider a standard environmental game where counties' pay-offs have the same form as described in section 4.5. For simplicity, consider identical countries with the following parameters' values: $a = 15$, $b = 0.02$ and $d = 0.00225$. Since countries are identical, it is irrelevant which player i is chosen to check the supermodularity condition. Furthermore, the only relevant difference between coalitions is their cardinality. Therefore, by considering the cardinality of a coalition as a variable, named s, and considering the function $f(s) = v(s) - v(s-1)$, supermodularity, in this case, requires that $\frac{\partial f}{\partial s} > 0$. In Tables 4.4 and 4.5 it is possible to observe the normalized results of the mentioned environmental game for, respectively, 6 and 12 players (the parameters are kept constant in both cases). By comparing the last columns of the two tables, it is possible to observe that the 6 players game is actually convex since the values of the column monotonically increase in the number of coalition members (first column), whereas the game with 12 players is not. In fact, the last column of Table 4.5 reaches the maximum value for the coalition with cardinality equal to eight.

A2

Consider a standard environmental game as described in section 4.5, having n players. For convenience, consider again the case where countries are identical. With $\bar{e_i^*}$ identify the optimal level of emissions of player i at the disagreement point. Clearly, when forming a coalition, players cannot improve their utility by increasing the amount of emissions compared to the current one. This stems from the fact that, by maximizing the coalition utility, each member of the same coalition must take into consideration the damage caused to the others. For each player $i \in S$, therefore, optimal emissions shift from $\bar{e_i^*} : B_i' = D_i'$ in the disagreement point to $e_i^*(S) : B_i' = \sum_{i \in S} D_i'$, when coalition S is formed. However, since being part of a coalition translates into maximizing the joint utility of its members, given by the sum of their private utilities, and given the fact that the level of emissions is a free variable, simply bounded to be non negative, nothing prevent coalition members to adopt the same amount of emissions they had when they were not coalesced. Therefore, it cannot be that

$\sum_{i \in S} \Pi_i(e_i^*(S)) < \sum_{i \in S} \Pi_i(\bar{e_i^*})$, otherwise players in S would choose the level of emission $\bar{e_i^*}$. This assures that $v(S)$ is at least equal to the sum of $v(\{i\})$ of all the members of S. By analogy, it is possible to extend the same reasoning to any union of disjoint coalitions. This proofs that $v(S+T) \geq v(S) + v(T)\ \forall\ S \cap T = \emptyset$.

A3

In section 4.5 can be found the optimal level of emissions expressed in analytic form for the disagreement point, the PANE case and the grand coalition. By plugging in these expressions into the pay-off functions of a country, it is possible to derive the conditions assuring its non-negativity. By substituting, for ease of notation, the lower case s to $|S|$ and n to $|N|$ and starting with the disagreement point, we have:

$$\Pi_i > 0 \quad \text{if} \quad a((\frac{a}{ab+nd}) - \frac{b}{2}(\frac{a}{ab+nd})^2) - \frac{d}{2}(\frac{na}{ab+nd})^2 > 0.$$

Table 4.4 **Game with 6 players**

Cardinality of S	$\Pi_i(S)$	$v(S) = \lvert S\rvert\Pi_i(S)$	$v(S) - v(S-1)$
1	0.0	0.0	0.0
2	9.5	19.1	19.1
3	29.9	89.6	70.6
4	56.0	223.8	134.2
5	83.2	416.0	192.2
6	108.4	650.2	234.2

Table 4.5 **Game with 12 players**

Cardinality of S	$\Pi_i(S)$	$v(S) = \lvert S\rvert\Pi_i(S)$	$v(S) - v(S-1)$
1	0.0	0.0	0.0
2	5.9	11.9	11.9
3	21.8	65.5	53.6
4	43.3	173.0	107.6
5	66.2	331.2	158.2
6	87.8	527.1	195.9
7	106.4	745.1	218.0
8	121.4	971.2	226.2
9	132.8	1195.1	223.9
10	140.9	1409.2	214.1
11	146.2	1607.7	198.5
12	148.7	1784.7	177.1

Solving it and eliminating the denominator (necessarily positive), it is possible to find that:

$$\Pi_i > 0 \quad \text{if} \quad \frac{ab}{d} > n(n-2).$$

For the grand coalition, instead, we have:

$$\Pi_i > 0 \quad \text{if} \quad a((\frac{a}{ab+n^2d}) - \frac{b}{2}(\frac{a}{ab+n^2d})^2) - \frac{d}{2}(\frac{na}{ab+n^2d})^2 > 0.$$

This is always true for positive values of parameters since:

$$\Pi_i > 0 \quad \text{if} \quad \frac{ab+n^2d}{2} > 0.$$

Finally, in the PANE case, for $i \in S$, we have:

$$\Pi_i > 0 \quad \text{if} \quad a((\frac{ab+(n-s)d(1-s)}{b(ab+(n+s^2-s)d)}) - \frac{b}{2}(\frac{ab+(n-s)d(1-s)}{b(ab+(n+s^2-s)d)})^2) +$$
$$-\frac{d}{2}(\frac{na}{b(ab+(n+s^2-s)d)})^2 > 0.$$

Simplifying and deleting the denominator leads to:

$$a^2b^2 + 2abd(n+s^2-s) + d^2(n+s^2-s)^2 - abdn^2 - d^2n^2s^2 > 0.$$

By separating the components of the expression it is possible to see that $d^2(n+s^2-s)^2 - d^2n^2s^2 > 0$ since $s < 1+s-\frac{s}{n}$ being $s < n$. It then remains:

$$a^2b^2 + 2abd(n+s^2-s) - abdn^2 > 0$$

that is true for $\frac{ab}{d} > n^2 - 2(n+s^2-s)$. Let us consider the case when S includes all players but one: $s = n-1$. We then have:

$$\frac{ab}{d} > n(4-n) - 4$$

that is always true for positive a, b and d and $n \geq 2$. For $s = 2$, instead, we have

$$\frac{ab}{d} > n(n-2) - 4.$$

Compared to the parameters' restriction necessary to assure non-negativity in the disagreement point ($\frac{ab}{d} > n(n-2)$), this is clearly milder and it actually becomes milder by increasing the size of S. For $s = 3$, for example: $\frac{ab}{d} > n(n-2) - 6$. This proofs that, for avoiding negative pay-offs, it is just necessary to settle appropriate parameters' values for the disagreement case: $\frac{ab}{d} > n(n-2)$.

Table 4.6 **Characteristic Values For All Coalitions**

Characteristic value		**Coalition Members**	**Characteristic value**		**Coalition Members**
v({HWLD})	310.65	1	v(16)	724.61	1,2,3
v({HWHD})	292.26	2	v(17)	479.93	1,3,4
v({MWMD})	100.56	3	v(18)	416.26	1,4,5
v({LWLD})	54.91	4	v(19)	462.86	2,3,4
v({LWHD})	37.14	5	v(20)	399.00	2,4,5
v(6)	607.12	1,2	v(21)	202.67	3,4,5
v(7)	414.14	1,3	v(22)	661.11	1,2,5
v(8)	367.78	1,4	v(23)	464.37	1,3,5
v(9)	350.76	1,5	v(24)	675.94	1,2,4
v(10)	396.14	2,3	v(25)	447.37	2,3,5
v(11)	349.66	2,4	v(26)	807.00	1,2,3,4
v(12)	332.64	2,5	v(27)	794.27	1,2,3,5
v(13)	156.97	3,4	v(28)	743.10	1,2,4,5
v(14)	139.56	3,5	v(29)	542.03	1,3,4,5
v(15)	93.42	4,5	v(30)	526.37	2,3,4,5
Grand Coalition v(N)			**890.40**		

Table 4.7 **Pay-offs under Free Riding**

Free Rider			**Shapley value**		
HWLD	**356.28**	300.38	105.73	59.29	41.24
HWHD	320.59	**336.72**	106.39	59.88	41.95
MWMD	326.02	307.60	**130.54**	65.31	48.47
LWLD	327.49	309.37	114.16	**79.81**	50.24
LWHD	327.68	309.60	114.37	66.97	**63.28**
			Chander and Tulkens solution		
HWLD	**352.57**	299.02	106.89	60.54	43.89
HWHD	316.88	**335.37**	107.56	61.14	44.60
MWMD	322.31	306.24	**131.70**	66.56	51.12
LWLD	323.78	308.01	115.32	**81.06**	52.89
LWHD	323.97	308.24	115.54	68.23	**65.93**
			Least Core		
HWLD	**354.38**	297.39	106.55	62.35	42.26
HWHD	318.68	**333.73**	107.22	62.94	42.97
MWMD	324.11	304.61	**131.36**	68.37	49.49
LWLD	325.58	306.38	114.98	**82.87**	51.25
LWHD	325.77	306.61	115.19	70.03	**64.30**
			Nucleolus		
HWLD	**354.35**	297.37	106.63	62.33	42.24
HWHD	318.66	**333.72**	107.30	62.92	42.95
MWMD	324.09	304.60	**131.44**	68.35	49.47
LWLD	325.56	306.36	115.06	**82.85**	51.23
LWHD	325.75	306.59	115.27	70.01	**64.28**
			Rawlsian Nucleolus		
HWLD	**351.57**	297.31	97.10	59.62	57.31
HWHD	315.87	**333.66**	97.77	60.21	58.02
MWMD	321.30	304.54	**121.91**	65.64	64.54
LWLD	322.77	306.31	105.53	**80.14**	66.31
LWHD	322.96	306.53	105.75	67.31	**79.35**
			‘R’ Nash Barg.		
HWLD	**359.01**	304.76	104.54	54.83	39.79
HWHD	323.32	**341.10**	105.21	55.42	40.50
MWMD	328.74	311.98	**129.35**	60.85	47.01
LWLD	330.22	313.75	112.97	**75.35**	48.78
LWHD	330.41	313.97	113.19	62.51	**61.82**
	HWLD	**HWHD**	**MWMD**	**LWLD**	**LWHD**

Table 4.8 **Index Coefficients of Free Riding Incentives and Risk**

Free Rider	**Shapley value**				
HWLD	**0.07464**	-0.04407	-0.10937	-0.16293	-0.25137
HWHD	-0.03303	**0.07160**	-0.10376	-0.15458	-0.23849
MWMD	-0.01665	-0.02108	**0.09961**	-0.07794	-0.12025
LWLD	-0.01221	-0.01546	-0.03836	**0.12676**	-0.08817
LWHD	-0.01164	-0.01474	-0.03657	-0.05448	**0.14862**
	Chander and Tulkens solution				
HWLD	**0.0755**	-0.0443	-0.1083	-0.1601	-0.2398
HWHD	-0.0334	**0.0719**	-0.1028	-0.1519	-0.2275
MWMD	-0.0168	-0.0212	**0.0986**	-0.0766	-0.1147
LWLD	-0.0123	-0.0155	-0.0380	**0.1246**	-0.0841
LWHD	-0.0118	-0.0148	-0.0362	-0.0535	**0.1418**
	Least Core				
HWLD	**0.07507**	-0.04450	-0.10862	-0.15620	-0.24681
HWHD	-0.03322	**0.07228**	-0.10305	-0.14819	-0.23416
MWMD	-0.01675	-0.02129	**0.09893**	-0.07472	-0.11807
LWLD	-0.01228	-0.01561	-0.03810	**0.12152**	-0.08657
LWHD	-0.01171	-0.01488	-0.03632	-0.05222	**0.14592**
	Nucleolus				
HWLD	**0.0751**	-0.0445	-0.1085	-0.1562	-0.2469
HWHD	-0.0332	**0.0723**	-0.1030	-0.1482	-0.2342
MWMD	-0.0167	-0.0213	**0.0989**	-0.0747	-0.1181
LWLD	-0.0123	-0.0156	-0.0381	**0.1216**	-0.0866
LWHD	-0.0117	-0.0149	-0.0363	-0.0522	**0.1460**
	Rawlsian Nucleolus				
HWLD	**0.0757**	-0.0445	-0.1179	-0.1622	-0.1946
HWHD	-0.0335	**0.0723**	-0.1119	-0.1539	-0.1846
MWMD	-0.0169	-0.0213	**0.1074**	-0.0776	-0.0931
LWLD	-0.0124	-0.0156	-0.0414	**0.1262**	-0.0683
LWHD	-0.0118	-0.0149	-0.0394	-0.0542	**0.1151**
	'R' Nash Barg.				
HWLD	**0.0740**	-0.0435	-0.1105	-0.1739	-0.2582
HWHD	-0.0328	**0.0706**	-0.1048	-0.1650	-0.2450
MWMD	-0.0165	-0.0208	**0.1006**	-0.0832	-0.1235
LWLD	-0.0121	-0.0152	-0.0387	**0.1353**	-0.0906
LWHD	-0.0115	-0.0145	-0.0369	-0.0581	**0.1527**
	HWLD	**HWHD**	**MWMD**	**LWLD**	**LWHD**

Concluding Remarks

The present Doctoral Thesis deals with the long debated theme of coalitions formation. After the seminal works of von Neumann and Morgenstern [1944] and Nash [1950], such research strand has become one of the core elements of economic investigation. Once the formation od coalitions is seen as a representation of the possibility of cooperation among self-interested individuals with the allocation of the coalitions worth potentially representing the distribution of wealth in a society, it becomes immediate to understand why so many efforts and so much interest have been dedicated to this topic.

This Doctoral Thesis investigates the mentioned theme from both a cooperative and a non-cooperative perspective. More specifically, it presents two novel solution concepts for balanced TU games in characteristic function form, one set-valued, the Central Core, and the other point-valued, the Mid-central Core. It further proposes a non-cooperative bargaining model that sustains, in SSPE, the Mid-central Core as the rational outcome of a bargaining process among self-interested players. It finally proposes a comparison between different cooperative solution concepts in relation to their redistributive properties and their resilience towards free-riding through a numerical simulation representing the attempt to reach an international agreement in order to reduce pollution.

In Chapter 1 it is offered a literature review of coalitional bargaining models from which it seems to appear a tendency towards egalitarianism even if players are assumed to be rational and self-interested. Redistribution, however, is bounded inside well-defined boundaries – generally, the Core – and this seems to cause a conflict between the egalitarian principle and efficiency. Marginalism, represented by the Shapley value, is supported from a lower number of models, often relying on particular assumptions. Chapter 2 is devoted to the presentation of the two mentioned solution concepts and to the discussion of their topological and axiomatic properties. In particular, it is shown that they are Core restrictions sharing with it almost all the axiomatic properties except for consistency. Furthermore, it is proved that the Mid-central Core satisfies aggregate and weak coalitional monotonicity, but not strong and coalitional monotonicity. Chapter 3 presents the Burning Coalition Bargaining Model, that, when the underlying coalitional game is balanced, supports the Mid-central Core in SSPE. This bargaining model can be described as a standard alternating offer model where discounting is substituted by the risk of partial breakdown, that, however, differs from what is employed in other models. In fact, the rejection of a proposal does not ignite the risk of exclusion of some players but rather the risk of unavailability of the proposed coalition in the subsequent bargaining rounds. The other novelty of the model consists in having a second, one-shot, bargaining round after a rejection, happening in the same time period. Finally, Chapter 4 is dedicated to compare different cooperative solution concepts in terms of redistributive properties and resilience to free riding. The comparison is implemented through a numerical simulation based on a game representing the constitution of an international agreement to reduce pollution. The revised Nash Bargaining solution (alias the Mid-central Core) is shown to be the less redistributive solution among the ones benchmarked, whereas the Rawlsian Nucleolus, the lexicographical minimum vertex of the Central Core, is either the more redistributive and the more resilient to free riding.

The present work has some important limitations, but, nonetheless, it offers some interesting possibilities for further investigation. First of all, it deals only with games in characteristic function form, whereas games with a coalition structure, or else, in partition function form, are excluded. Since these lasts serve to represent situations in which coalitions generate externalities, they cover a broader range of real-life scenarios. This limitation, however, can be in itself a reasonable starting point for future works. A further limitation is related to the BCBM. In particular, it has been proposed a strategy profile that, asymptotically, supports the Mid-central Core in this bargaining model. It cannot be hidden the fact that the analysis of the BCBM has been guided by the desire of finding a support for the Mid-central Core. A more neutral analysis of the BCBM could then be very interesting.

Among the potential extensions, a more thorough analysis of the Central Core and the Mid-central Core in the context of convex games could be performed. Furthermore, it can be mentioned the possibility to experimentally testing the validity of the BCBM and, particularly, of its predicted outcome. Although not formally discussed, it is easy to see that the environmental game upon which it is based the numerical simulation in Chapter 4 corresponds to a particular type of coalitional game analysed in Chapter 3. Specifically, a game where the optimal set of solutions of the minimization program defining the Central Core is a singleton and where the binding constraints are all the coalitions of cardinality equal to $n-1$. We have seen that, in such case, the Mid-central Core is equivalent to the

ENSC-value. The numerical simulation in Chapter 4 suffers from the shortage of being based on abstract values. However, it can be checked that, by using the data in Eyckmans and Tulkens [2003], where the worth of each coalition is derived from empirical data through the use of the integrated assessment model CLIMNEG, the properties of the game would be unchanged. It could then be interesting to investigate if this is a natural property of environmental games and under which conditions this property holds.

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