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Thesis

On the Necessity of Complex Numbers in Quantum Mechanics

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Contents

Chapter 1

Introduction

Quantum theories attempt to give a thorough description of microscopical physical systems, like atoms or fundamental particles. Decades of experimental and theoretical research resulted in a pretty refined mathematical framework which encompasses several fields of Mathematics, from lattice theory to functional analysis.

Starting from the very beginning a quantum system can be characterised by its set of elementary propositions (also said elementary observables), i.e. the questions we can ask about the system and test through experiments. A very simple example of such a proposition is the following. Consider a particle and a set E of the Euclidean space $\mathbb{R}^3$, then we may wonder if the particle would be detected within it or not: the sentence *the particles is inside* E provides such a testable elementary proposition. These objects are matter of investigation of quantum logic: a careful analysis reveals the set $\mathfrak{L}$ to be a complete orthomodular lattice (see Chapter *Quantum Axiomatic* of [10]). This is not a privilege of quantum systems, it being achievable also for classical objects, but quantum mechanics is subtle, often not pleasing the common sense and makes things much more complicated. The Heisenberg Uncertainty Principle shows how the measurements of some particular couples of observables cannot be performed together, for the two experiments perturbe each other. In particular this holds true for some couples of elementary propositions, which are said to be incompatible, and a simultaneous truth value turns out to be impossible to measure. This departure from the classical behavior results in the non-distributivity of the lattice $\mathfrak{L}$. This is totally a quantum feature, in that the lattice of any classical system is always distributive (more precisely a Boolean algebra).

At this point something magical happens. Assuming other (but less relevant) technical hypotheses on the lattice $\mathfrak{L}$, which can be somehow justified from an operational point of view, all this heavily abstract framework realises itself on the more concrete structure of a Hilbert space. More precisely it turns out that $\mathfrak{L}$ is isomorphic to the lattice $\mathfrak{L}(\mathsf{H})$ of orthogonal projectors on a Hilbert space H defined over the division algebra of reals, complexes or quaternions. This is the content of the celebrated Solèr Theorem [37], which relies actually on partial results by Piron [30], Maeda-Maeda [23] and other authors.

The possibility of having a real or even quaternionic quantum mechanics is quite surprising, for, up to now, all of the known physical phenomena can be fully described within the complex Hilbert space structure. As a matter of fact, independently from the results by Piron and Solèr, the theoretical possibility of formulating quantum theories in Hilbert spaces over either $\mathbb{R}$ or $\mathbb{H}$ (or other division rings of scalars) was a matter of investigation since the early mathematical formulations of quantum mechanics, see for example Chapter 22 of [6].

However, differently from quaternionic quantum mechanics which still deserves some theoretical interest, real quantum mechanics was not considered as physically interesting almost immediately especially in view of a well-known analysis performed by Stuckelberg in the early seventies, [38], [39]. Stuckelberg provided some physical reasons for getting rid of the real Hilbert space formulation relying on the demand that every conceivable formulation of quantum mechanics should include the statement of Heisenberg's Principle. He argued that such a principle requires the existence of a natural complex structure J commuting with all the operators representing physical observables. Although in the paper this is not properly argued from a mathematical point of view, by means of a complexification procedure (see Chapter 3) the existence of such an operator is capable of lifting everything to a suitable complex Hilbert space. His analysis is definitely physically interesting, but very poor from a mathematical view-point as it assumes that all observables have pure point spectrum and some of them are bounded, in contradiction with the nature of position of momentum observables which are necessary to state Heisenberg's Principle. No discussion about domains appears. Many inferences are just heuristically justified (including the universality of J) even if they all are physically plausible. Moreover, in Stuckelberg analysis, the existence of J seems to be more a sufficient condition to guarantee the validity of the Heisenberg's inequalities rather than a necessary requirement, since everything is based on an a priori and arbitrary (though physically very plausible) model of any version of the uncertainty principle as described in Section 2 of [38]. Finally, the validity of Heisenberg's Principle cannot be viewed as a fundamental a priori condition nowadays: it needs the existence of the position observable which is a very delicate issue, both theoretically and mathematically (it is based on Mackey's imprimitivity machinery) in case of elementary relativistic systems. For massless particles like photons, the position observable simply does not exist (for these issues see Section 9.7 of [44]).

Differentely from the real case, the quaternionic setting has always been taken under serious consideration and rich literature can be found on this. The first remarkable discussion on this topic was provided by Finkelstein, Jauch, Schiminovich and Speiser in [11]. Here the authors recognise how the existence of an imaginary operator J commuting with all the observables would reduce the theory to a standard description in terms of complex Hilbert spaces (this happens again by means of a peculiar complexification procedure on quaternionic Hilbert spaces, see Chapter 3, even though this is not explicitly done in the mentioned paper). They find a pleasing candidate in the operator $J_{\mathcal{H}}$ appearing in the polar decomposition of the anti-self-adjoint generator of time-displacements $\mathcal{H} = J_{\mathcal{H}}|\mathcal{H}|$. More precisely such a $J_{\mathcal{H}}$ is not a complex structure in the general case, merely an anti-self-adjoint isometry, but can always be completed to a complex structure, even though generally not in a unique way. Anyway, as they base upon a very general situation, they do not assume (and most likely this cannot be proved in full generality) that this is actually the case and focus the analysis to another point of view: the principle of covariance under automorphisms of the division algebra.

Still in the sixties and in line with this perspective, a more interesting hint was provided by Jauch in Section 4 of [21] for the specific case of strongly-continuous unitary representations of the Poincaré group. Basing upon some solid physical arguments, the author discussed the necessity for a complex structure J commuting with the entire representation. More precisely he pointed out one of the main issues of quaternionic (and also real) quantum mechanics: the construction of observables out of anti-self-adjoint generators (more on this in Chapter 9, paragraph on Noether's Principle). Symmetries are most often realised as one-parameter groups of unitary operators $s \mapsto U_s$ on the Hilbert space. Stone's Theorem allows for a realisation as $U_s = e^{s\mathcal{A}}$ where $\mathcal{A}$ is a uniquely associated anti-self-adjoint operator. In standard complex theories, by merely taking $A := -i\mathcal{A}$ we are able to define a self-adjoint operator, i.e. an observable of the system, canonically associated with the symmetry. On quaternionic spaces this is no longer possible, as operators like qA are ill-defined, due to the non commutativity of the quaternions (see Chapter 2). As a possible way out, the author proposes the association $A := -J\mathcal{A}$, where J is some complex structure commuting with $\mathcal{A}$. As explained in the paper, a natural class of complex structures J arises naturally from the restriction of the unitary representation to the subgroup of spacetime-displacement group $\mathbb{R}^4$ and they commute with its anti-self-adjoint generators $\mathcal{P}_\mu$. At this point imposing the postulate that $-J\mathcal{P}_\mu$ must transform as a 4-vector and that $-J\mathcal{P}_0$ is positive the author affirms that J is uniquely determined within the given class. However none of this is carried out explicity and, as happens for Stuckelberg, it looks like more a sufficient rather than a necessary condition for everything to work out. Besides all this, the idea underlying this work is pretty sound and definitely solid from a physical perspective and put the bases of this work.

A similar but mathematically more solid analysis was pointed out by Cassinelli and Truini [7] in the eighties, where also some mistakes in the paper by Jauch are pointed out. In this paper the authors restrict their attention to the Euclidean space $\mathbb{R}^3$ and suppose the existence of a one-parameter group of unitary operators describing the time-displacement and of a unitary representation of the Euclidean group describing the space symmetries. Similarly to [11], the authors consider the operators $J_\mathcal{H}$ coming from the polar decompositon of the anti-self-adjoint generator of the time-displacement $\mathcal{H} = J_\mathcal{H}|\mathcal{H}|$. By assuming the operator $|\mathcal{H}|$ to represent the energy of a free particle system, it can be supposed to be strictly positive, in particular injective. This guarantees that $J_\mathcal{H}$ is a complex structure. Finally, this operator is proved to commute with the entire representation, in this way making the theory complex and giving sense to the observables $-J_\mathcal{H}\mathcal{A}$, where $\mathcal{A}$ are the generators of the Euclidean group symmetry. Unfortunately, despite its undebiable physical and mathematical value, the analysis is restricted to a proper subclass of spacetime symmetries, for the boosts (not even the non-relativistic ones) are not taken into considerations.

To conclude, notice that a similar argument is carried out also in the remarkable work by Adler [1], where a sound analysis of quantum theories on quaternionic Hilbert spaces is carried out, even though it lacks of mathematical rigour.

In this work, based on the papers [25] and [26], attempting a more general and solid approach, we provide an answer to this fundamental problem: what is the meaning of a real or quaternionic quantum theory? At least in the case of elementary systems? In the first part we analyse the properties of irreducible strongly-continuous unitary representations of the Poincaré group on real and quaternionic Hilbert spaces. If $\mathcal{P}_0$ denotes the anti-self-adjoint generator of the time-displacement ($c = 1$), we manage to prove that under suitable physical hypothesis the operator J_0 appearing in the polar decomposition $\mathcal{P}_0 = J_0|\mathcal{P}_0|$ is a complex structure and commutes with the entire representation, in this way making it possible to restate the representation over a suitable complex Hilbert space. This analysis includes and concludes the work of [21] and [7]. A particular effort was required in proving that J_0 commutes with the generators of the boosts. The symmetries associated with space-displacements or space-rotations commute by definition of the Poincaré group with the time-displacement one, thus the commutativity of their generators with J_0 is almost automatic. This is basically what happens in the analysis of [7], where only time-displacement and Euclidean group are considered.

In the second part, we try a pretty general theoretical framework. More precisely our general setting consists in an irreducible von Neumann algebra of observables over a real, complex or quaternionic Hilbert space whose set of orthogonal projectors provides the lattice of elementary propositions. Furthermore we assume the system to admit the Poincaré group as *maximal symmetry group* (in particular we are excluding *a priori* the possible existence of non-trivial gauge symmetries). Driven by this idea, we suppose the existence of a representation of the Poincaré group in terms of automorphisms of the algebra. Imposing suitable physical hypotheses and exploiting the results of the first part, we succeed in proving that any attempt to describe an elementary system over a real or quaternionic Hilbert space eventually reduces to a standard description over a complex one. More precisely we determine the existence of an imaginary operator J commuting with the entire algebra. At this point a complexification procedure makes the theory complex. It is remarkable that the existence of such a structure comes as a *necessary* condition on the system. This is the content of the Chapter 9.

In Chapter 2 we provide a summary of the main concepts of Hilbert space theory for real, complex and quaternionic Hilbert spaces. In Chapter 3 we discuss two main methods to reduce or extend the division algebra of real or quaternionic Hilbert spaces respectively. These techniques are of absolute importance for almost all of the next chapters. In particular we exploit these methods in Chapter 4, aiming to extend some pillar results of complex functional analysis such as Stone's Theorem, Polar Decomposition Theorem and Schur's Lemma to Hilbert spaces with different division algebras. All about the complex versions can be found in [24]. Furthermore we give a general definition of trace-class operator which makes sense on arbitary Hilbert spaces and makes the proof of Gleason's Theorem achievable in all cases (based on [44]). In Chapter 5, basing upon [10] and [24] we discuss briefly the ideas of quantum logic, in particular stating the basic definitions and results of lattice theory. Next, in Chapter 6 we introduce the concept of a von Neumann algebra. The real and complex cases are well-known and a good introduction to them is provided by [20] and [24], respectively. Definition and properties for the quaternionic case are introduced and discussed here and are based upon the results of Chapter 3. In particular the structure of irreducible von Neumann algebras is discussed, together with important consequences such as an extension of Gleason's and Wigner's theorems. Chapter 7 is based upon [35] and deals with the theory of Lie group unitary representations. In particular the concept of an induced Lie algebra representation is discussed, which turns out to be extremely important in the development of Chapter 9. To conclude, Chapter 8 provides a short summary on the Lorentz and Poincaré groups.

Chapter 2

Basics on Hilbert Space Theory

In this chapter we give a basic summary of the most important definitions and results of real, complex and quaternionic Hilbert space theory. For an exhaustive discussion complete with proofs of the complex case see [24]. A detailed and wide treatment of the quaternionic case can be found in [15] and [16], in particular the spectral theorem for unbounded normal operators is proved and discussed. The literature for the real case is not as wide as the complex one, for the real Hilbert structure is much poorer. Basic concepts on real Hilbert spaces can be found in [5], while for the spectral theory see for example [29].

2.1 Basic Properties of Hilbert spaces

We start with some basic definitions

***Definition* 2.1.1.** *Let $\mathbb{F}$ denote the real division algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. An additive group V is said to be*

(a) a right linear space if it is equipped with a right scalar multiplication

$$\mathsf{V} \times \mathbb{F} \in (u, q) \mapsto uq \in \mathsf{V}$$

satisfying the distributivity property with respect to the two notions of sum

$$u(p+q) = up + uq \quad \textit{and} \quad (u+v)p = up + vp \quad \textit{for all} \;\; p, q \in \mathbb{F},\;\; u, v \in \mathsf{V},$$

the associative property with respect to the product of $\mathbb{F}$

$$u(pq) = (up)q \quad \textit{for all} \;\; p, q \in \mathbb{F},\;\; u \in \mathsf{V}$$

and the identity $u1 = u$ for all $u \in \mathsf{H}$.

(b) a left linear space if it is equipped with a left scalar multiplication

$$\mathbb{F} \times \mathsf{V} \in (q, u) \mapsto qu \in \mathsf{V}$$

satisfying the distributivity property with respect to the two notions of sum

$$(p+q)u = pu + qu \quad \text{and} \quad p(u+v) = pu + pv \quad \text{for all} \;\; p,q \in \mathbb{F},\; u,v \in \mathsf{V},$$

the associative property with respect to the product of $\mathbb{F}$

$$(pq)u = p(qu) \quad \text{for all} \;\; p,q \in \mathbb{F},\; u \in \mathsf{V}$$

and the identity $1u = u$ *for all* $u \in \mathsf{H}$.

(c) a two-sided linear space if it is a both left and right linear space such that

(a) $q(up) = (qu)p$ *for all* $q,p \in \mathbb{F}$ *and* $u \in \mathsf{V}$

(b) $rq = qr$ *for any* $r \in \mathbb{R}$

Now, consider a *right* linear space H over a division algebra $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. This is called a *pre-Hilbert space* if it admits an *Hermitean scalar product*, i.e. a function

$$\mathsf{H} \times \mathsf{H} \ni (x,y) \mapsto (x|y) \in \mathbb{F} \tag{2.1}$$

satisfying the following properties:

(i) *(right linearity)* $(x|ya + y') = (x|y)a + (x|y')$ for all $x,y,y' \in \mathsf{H}$ and $a \in \mathbb{F}$

(ii) *(Hermiticity)* $(x|y) = \overline{(y|x)}$ for all $x,y \in \mathsf{H}$

(iii) *(positive-definiteness)* $(x|x) \geq 0$ and if $(x|x) = 0$ then $x = 0$

Notice that $(xa|y) = \overline{a}(x|y)$ for any $x,y \in \mathsf{H}$ and $a \in \mathbb{F}$.

***Remark* 2.1.2.** In the standard treatment of Hilbert spaces over $\mathbb{R}$ or $\mathbb{C}$ the space H is taken as a *left* linear space equipped with an Hermitean scalar product which differs from 2.1 in point (i) which now reads

$$(x|py + y') = p(x|y) + (x|y') \quad \text{for all } p \in \mathbb{F} \text{ and } x,y,y' \in \mathsf{H}$$

In this special case of real or complex division algebra the two constructions are totally equivalent, see Remark 2.2.3 for more details. We stick to the multiplication on the right in order to exploit the results of [15] and [16] for the quaternionic setting. Moreover the right multiplication is compulsory in order to define Hermitean scalar products which are linear in the right entry, in line with what usually done in the complex settings. For example notice that the division algebra $\mathbb{H}$ can be interpreted both as a left and a right linear space over $\mathbb{H}$ itself. However the canonical inner product

$$\mathbb{H} \times \mathbb{H} \ni (p,q) \mapsto \overline{p}q \in \mathbb{H}$$

is linear in the right-hand entry *only* when referring to the right linear structure of $\mathbb{H}$, due to the non-commutativity of the quaternions.

Two vectors $u, v \in \mathsf{H}$ are said to be *orthogonal* to each other if $(u|v) = 0$. Sometimes we will refer to this property as $u \perp v$.

***Proposition* 2.1.3.** *Let* H *be a pre-Hilbert space, then the Hermitean scalar product (2.1) satisfies the Cauchy-Schwarz inequality*

$$|(u|v)|^2 \leq (u|u)(v|v)$$

for any choice of vectors $u, v \in \mathsf{H}$.

The Hermitean scalar product allows us to introduce another important structure on H which is the *norm*.

***Definition* 2.1.4.** *Suppose that* $\mathbb{F} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$. *A (left, right, two-sided) linear space* V *is said to be a (left, right, two-sided) locally convex space, if it is equipped with a family of seminorms* $(p_\alpha)_{\alpha \in \mathcal{A}}$, *i.e. functions* $p_\alpha : \mathsf{V} \to [0, \infty)$ *such that*

(1) (homogeneity) $p(ua) = p(u)|a|$ $(p(au) = |a|p(u))$ *for any* $u \in \mathsf{H}$ *and* $a \in \mathbb{F}$,

(2) (triangle inequality) $p(u + v) \leq p(u) + p(v)$ *for any* $u, v \in \mathsf{V}$.

If any of the p_α *satisfies the further condition*

(3) (definiteness) if $p_\alpha(u) = 0$ *then* $u = 0$

then it is said to be a norm. If V *is equipped with only one norm it is said to be a normed space.*

Consider any $\alpha \in \mathcal{A}$, $\epsilon > 0$ and $u \in \mathsf{H}$, then we can define the *ball of radius* ϵ *and centre* u for the given seminorm as the subset

$$B_\alpha(u, \epsilon) := \{v \in \mathsf{H} \,|\, p_\alpha(u - v) < \epsilon\} \tag{2.2}$$

Let us say that a subset $U \subset \mathsf{V}$ is *open* if for every $x \in U$ there exists some $\alpha_1, \ldots, \alpha_n$ and $\epsilon_1, \ldots, \epsilon_n$ such that

$$\bigcap_{i=1}^{n} B_{\alpha_i}(x, \epsilon_i) \subset U$$

It is easy to see that the set $\mathcal{T}(\mathsf{V})$ of these open sets is a topology on V which makes the natural operations of V and the seminorms themselves continuous.

***Definition* 2.1.5.** *If a (left, right, two-sided) locally convex space is complete with respect to its natural topology it is said to be a Fréchet space. A Fréchet normed space is said to be a Banach space.*

So, let us go back to our pre-Hilbert space H and define the following function

$$\|\cdot\| : \mathsf{H} \ni u \mapsto \|u\| := \sqrt{(u|u)} \in [0, \infty) \tag{2.3}$$

Exploiting the properties of the Hermitean scalar product (2.1), most of all the Cauchy-Schwarz inequality, the function (2.3) turns out to be a well-defined norm over H, called the *canonical norm* of H. Once this norm is introduced, the original scalar product can be recovered by it as the following *polarisation identity* shows. Fix any basis $\{i_\alpha\}_{\alpha=0,\dots,d_{\mathbb{F}}}$ of standard units of $\mathbb{F}$, understood as a real linear space, with $i_0 = 1$ and $d_{\mathbb{F}} := \dim_{\mathbb{R}} \mathbb{F} - 1$, then for any $x, y \in \mathsf{H}$ we have

$$(x|y) = \frac{1}{4} \sum_{\alpha=0}^{d_{\mathbb{F}}} \left(\|x i_\alpha + y\|^2 - \|x i_\alpha - y\|^2 \right) i_\alpha. \tag{2.4}$$

The canonical norm makes H a normed space. Exploiting the properties mentioned above, it is immediate to see that the scalar product is jointly continuous, thanks to the Cauchy-Schwarz inequality.

***Remark* 2.1.6.** The identity (2.4) is just an example of polarisation formula on linear spaces. In general consider a right (left, two-sided) linear space V and a function $f : \mathsf{V} \times \mathsf{V} \to \mathbb{F}$ such that

(a) $f(x, ya + y') = f(x, y)a + f(x, y')$ for all $x, y, y' \in \mathsf{V}$ and $a \in \mathbb{F}$,

(b) $f(y, x) = \overline{f(x, y)}$ for all $x, y \in \mathsf{V}$,

and define $Q : \mathsf{V} \ni u \mapsto Q(u) := f(u, u) \in \mathbb{R}$, then

$$f(x, y) = \frac{1}{4} \sum_{\alpha=0}^{d_{\mathbb{F}}} (Q(u i_\alpha + v) - Q(u i_\alpha - v)) i_\alpha. \tag{2.5}$$

We are in a position to give the following definition

***Definition* 2.1.7.** *A pre-Hilbert space is said to be a Hilbert space if it is complete with respect to the topology induced by its canonical norm, i.e. if it is a Banach space.*

***Example* 2.1.8.** The simplest example of Hilbert space on the division algebra $\mathbb{F}$ is given by $\mathbb{F}$ itself, with Hermitean scalar product defined by

$$\mathbb{F} \times \mathbb{F} \ni (s, t) \mapsto \overline{s} t \in \mathbb{F}$$

and corresponding norm given as

$$|s| = \overline{s} s = \sqrt{\sum_{\alpha=0}^{d_{\mathbb{F}}} (s_\alpha)^2}$$

where $s = \sum_{\alpha=0}^{d_{\mathbb{K}}} s_\alpha i_\alpha$ is any element of $\mathbb{F}$.

Suppose that V is a (left, right, two-sided) linear space over the division algebra $\mathbb{F}$ and consider a finite family of linear subspaces $(\mathsf{S}_1, \dots, \mathsf{S}_n)$. We say that V is the *direct sum* of the subspaces S_i, and write

$$\mathsf{H} = \bigoplus_{i=1}^{n} \mathsf{S}_i \tag{2.6}$$

if for every $v \in \mathsf{V}$ there exists unique vectors $x_i \in \mathsf{S}_i$ such that $x = \sum_{i=1}^{n} x_i$. In the case of two linear subspaces, this turns out to be equivalent to the two conditions $\mathsf{S}_1 + \mathsf{S}_2 = \mathsf{V}$ and $\mathsf{S}_1 \cap \mathsf{S}_2 = \{0\}$.

In particular this applies to Hilbert spaces and a remarkable example is given by the *orthogonal direct sum*. Let K be any subset of H. We define the *orthogonal* of K as the set

$$\mathsf{K}^{\perp} := \{x \in \mathsf{H} \mid (x|y) = 0 \quad \forall y \in \mathsf{K}\}. \tag{2.7}$$

It is a simple matter to verify that $\mathsf{K}^{\perp}$ is a closed linear subspace of H. More precisely the following result can be proved.

***Notation* 2.1.9.** Let V be a linear space over the division algebra $\mathbb{F}$ and $\mathsf{S} \subset \mathsf{V}$ a subset. We denote by $[\mathsf{S}]$ the linear subspace of V containing all finite linear combinations of elements of S.

***Proposition* 2.1.10.** *Let H be a Hilbert space and $\mathsf{K} \subset \mathsf{H}$, then the following properties hold:*

(a) $\mathsf{K}^{\perp} = [\mathsf{K}]^{\perp} = \overline{[\mathsf{K}]}^{\perp} = \overline{[\mathsf{K}]^{\perp}}$,

(b) $\overline{[\mathsf{K}]} = \mathsf{K}^{\perp\perp}$,

(c) $\overline{[\mathsf{K}]} \oplus \mathsf{K}^{\perp} = \mathsf{H}$,

(d) if $u \in \overline{[\mathsf{K}]}$ *and* $v \in \mathsf{K}^{\perp}$, *then* $\|u+v\|^2 = \|u\|^2 + \|v\|^2$,

where the bar denotes the topological closure.

Another important result is the well-known representation theorem of Riesz.

By *functional* on H we mean any linear function $f : \mathsf{H} \to \mathbb{F}$. As an example, if we fix a vector $u \in \mathsf{H}$ then the function $f_u : \mathsf{H} \ni v \mapsto (u|v) \in \mathbb{F}$ trivially provides a linear and continuous function, thanks to the properties of the scalar product and Cauchy-Schwarz inequality. It turns out that *any* continuous functional is of this form. This is the content of the Riesz's Theorem.

***Theorem* 2.1.11** (Riesz's Representation Theorem)**.** *Let H be a Hilbert space. If $f : \mathsf{H} \to \mathbb{F}$ is a continuous functional, then there exists a unique vector $x_f \in \mathsf{H}$ such that $f = (x_f|\cdot)$. Moreover $\|f\| := \sup_{\|x\|=1} |f(x)| = \|x_f\|$.*

***Corollary* 2.1.12.** *Suppose $x \in \mathsf{H}$, then $\|x\| = \sup_{\|y\|=1} |(x|y)|$.*

Next step consists in the introduction of the concept of *Hilbert basis*.

In general if $(u_n)_{n\in\mathbb{N}} \subset \mathsf{H}$ is any countable set of vectors within the Hilbert space we say that the series

$$\sum_{n=0}^{\infty} u_n \tag{2.8}$$

is *summable* if there exists some $u \in \mathsf{H}$ such that

$$\lim_{N\to\infty} \left\| \sum_{n=0}^{N} u_n - u \right\| = 0. \tag{2.9}$$

If this is the case the vector u is said to be the *sum* of the series (2.8).

***Notation* 2.1.13.** In the following, given any countable index set $\mathcal{I}$ and set $(\lambda_i)_{i\in\mathcal{I}} \subset \mathbb{R}_+$ we say that the sum $\sum_{i\in\mathcal{I}} \lambda_i$ is convergent and write

$$\sum_{i\in\mathcal{I}} \lambda_i < \infty$$

if there exists some bijection $\mathbb{N} \ni n \mapsto i_n \in \mathcal{I}$ such that

$$\sum_{n=0}^{\infty} \lambda_{i_n} < \infty$$

in the usual sense of limit of partial sums, as in (2.8). If this is true, then it can be proved that any other bijection would give the same result, making our notation meaningful. If $(\alpha_i)_{i\in\mathcal{I}} \subset \mathbb{F}$ we say that the sum $\sum_{i\in\mathcal{I}} \alpha_i$ is *absolutely convergent* if $\sum_{i\in\mathcal{I}} |\alpha_i| < \infty$. If this is true, it turns out that for every bijection $\mathbb{N} \ni n \mapsto i_n \in \mathcal{I}$ the sum

$$\sum_{n=0}^{\infty} \alpha_{i_n}$$

is convergent within $\mathbb{F}$ in the usual sense of limit of partial sums, as in (2.8) and the limit does not depend on the chosen index representation. We denote this limit by

$$\sum_{i\in\mathcal{I}} \alpha_i .$$

We are ready to give the following result.

***Proposition* 2.1.14.** *Let $\mathsf{N} \subset \mathsf{H}$ be a set of unit-norm orthogonal vectors of H, then the following statements are equivalent:*

(a) for every $u, v \in \mathsf{H}$, at most countably many products $(u|z)(z|v)$ are non-zero for all $z \in \mathsf{N}$ and the sum

$$(u|v) = \sum_{z\in\mathsf{N}} (u|z)(z|v)$$

is absolutely convergent if restricted to the non-null elements;

(b) *for every* $u \in \mathsf{H}$*, at most countably many coefficients* $(u|z)$ *are non-zero for all* $z \in \mathsf{N}$ *and*

$$\|u\|^2 = \sum_{z \in \mathsf{N}} |(u|z)|^2$$

where the sum is convergent if restricted to the non-null elements;

(c) *for every* $u \in \mathsf{H}$*, at most countably many coefficients* $(u|z)$ *are non-zero for all* $z \in \mathsf{N}$ *and*

$$u = \sum_{z \in \mathsf{N}} z(z|u)$$

where the sum is defined as in (2.8) and (2.9) when restricted to the non-null elements and is insensitive to their indexing order;

(d) $\mathsf{N}^\perp = \{0\}$*;*

(e) $[\mathsf{N}]$ *is dense in* H*.*

If the set N *satisfies any one of the previous properties, it is said to be a Hilbert basis.*

If the cardinality of any Hilbert basis is $\mathbb{N}$, then the Hilbert space is said to be *separable*.

2.2 Linear Operators on Hilbert Spaces

In this section we discuss linear operators over Hilbert spaces. After an introduction to the basic definitions and results, a final subsection is devoted to spectral theory.

2.2.1 General Definitions and Properties

Let us start with the following definition.

***Definition* 2.2.1.** *Let* H *be a Hilbert space. A (anti-)linear function* $A : D(A) \to \mathsf{H}$ *defined over a linear subspace* $D(A)$ *of* H *is said to be a (anti-)linear operator. In both cases* $D(A)$ *is called the domain of* A*. We define the range of* A *by setting*

$$R(A) := \{Au \mid u \in D(A)\}$$

and the kernel of A *by*

$$N(A) := \{u \in D(A) \mid Au = 0\}.$$

***Remark* 2.2.2.** Of course the distinction between linear and anti-linear operator makes sense only on complex or quaternionic Hilbert space, for they coincide on real ones.

Let $A : D(A) \to \mathsf{H}$, $B : D(B) \to \mathsf{H}$ be linear operators. We will use the notation

$$A \subset B \quad \text{if} \;\; D(A) \subset D(B) \text{ and } B|_{D(A)} = A.$$

In trying to perform sum and product between these operators, we need to be careful, for the domains are not maximal. We can define their *product* as

$$ABx := A(Bx) \quad \text{for } x \in D(AB) := \{u \in D(B) \mid Bu \in D(A)\}$$

and their *sum* as

$$(A + B)x := Ax + Bx \quad \text{for } \; x \in D(A + B) := D(A) \cap D(B).$$

Of course if both A and B are maximally defined operators, then so will be their product and sum.

Now, focus on the *real* and *complex* Hilbert space cases. and consider any scalar $a \in \mathbb{F}$. We can define

$$Aq : D(A) \ni u \mapsto (Aq)(u) := (Au)q \in \mathsf{H}. \tag{2.10}$$

It is clear that such a function is a well-defined linear operator, thanks to the commutativity of the division algebra. Unfortunately this no longer holds on a quaternionic Hilbert space, since $\mathbb{H}$ is not commutative. In this case the best we can do is considering only multiplication by *real* scalars, as they are the only quaternions commuting with all the others. Let us define the following (commutative) division algebra associated with H:

$$\mathbb{F}_c := \begin{cases} \mathbb{R} & \text{if} \quad \mathbb{F} = \mathbb{R}, \mathbb{H} \\ \mathbb{C} & \text{if} \quad \mathbb{F} = \mathbb{C} \end{cases}$$

If we restrict our attention to this division algebra, the operator (2.10) makes sense on any Hilbert space. In particular Aq will obviously be maximally defined and bounded if A is so.

***Remark* 2.2.3.** Let H be a Hilbert space over the division algebra $\mathbb{F}$, then we may define

$$qu := uq \quad \text{for all } u \in \mathsf{H}, \;\text{ and }\; q \in \mathbb{F}_c.$$

Thanks to the commutativity of $\mathbb{F}_c$, this definition introduces a structure of *left* scalar multiplication on H by elements of $\mathbb{F}_c$ (see Remark (2.1.2)) which is compatible with the already defined right multiplication, in the sense that

$$q(up) = (qu)p \quad \text{for all } p \in \mathbb{F}, \;\; q \in \mathbb{F}_c, \;\; u \in \mathsf{H}.$$

Thus H becomes canonically a two-sided linear space over $\mathbb{F}_c$. Moreover it satisfies

(a) $\|qu\| = |q|\|u\|$,

(b) $(qu|v) = \overline{q}(u|v)$ and $(u|qv) = q(u|v)$

for all $u, v \in \mathsf{H}$ and $q, p \in \mathbb{F}_c$.

Concerning operators, if $A : D(A) \to \mathsf{H}$ is a linear operator on H, then sticking to this definition we define for $q \in \mathbb{F}_c$ the linear operator

$$qA : D(A) \ni u \mapsto q(Au) \in \mathsf{H}. \tag{2.11}$$

It is clear that $qA = Aq$. In the following we will make use of this definition, in particular we will use the notation qA instead of Aq, for it is more practical.

Finally, notice that in the special cases $\mathbb{F} = \mathbb{R}, \mathbb{C} = \mathbb{F}_c$, we the recover the structure of (left) Hilbert space usually exploited in literature. The two construction are totally equivalent to each other.

With these standard definitions, the sum and the product turn out to be *associative*, that is, referring to three operators A, B, C with arbitrary domains in the same Hilbert space, it holds that

$$(AB)C = A(BC) \quad \text{and} \quad (A + B) + C = A + (B + C).$$

However in general the *distributive* fails to be satisfies. Indeed we have

$$(A + B)C = AC + BC \quad \text{but} \quad A(B + C) \supset AB + AC.$$

To conclude this discussion, notice that $A \subset B$ implies both $AC \subset BC$ and $CA \subset CB$. Finally $A \subset B$ together with $B \subset A$ implies $A = B$.

Let H be a Hilbert space. We can consider the *algebraic* direct sum, i.e. the set

$$\mathsf{H} \oplus \mathsf{H} := \{(x, y) \mid x, y \in \mathsf{H}\}$$

endowed with the natural linear space structure

$$(x, y) + (u, v) = (x + u, y + v) \quad \text{and} \quad (x, y)a = (xa, ya) \quad \text{for all } x, y \in \mathsf{H} \text{ and } a \in \mathbb{F}.$$

This inherits a natural structure of Hilbert space over the division algebra of H, by defining

$$((x, y)|(u, v)) := (x|y) + (u|v) \quad \forall x, y, u, v \in \mathsf{H}.$$

Now, consider a linear operator $A : D(A) \to \mathsf{H}$, its *graph* is the linear subspace

$$\mathsf{H} \oplus \mathsf{H} \supset \mathcal{G}(A) := \{(x, Ax) \in \mathsf{H} \oplus \mathsf{H} \mid x \in D(A)\}. \tag{2.12}$$

***Definition* 2.2.4.** *Let $A : D(A) \to \mathsf{H}$ be an operator over a real, complex or quaternionic Hilbert space H. We say that A is closable if the closure of $\mathcal{G}(A)$ is the graph of an operator, denoted by $\overline{A}$. It is said to be closed if it is closable and $A = \overline{A}$. A subspace $S \subset D(A)$ is a core for a closable operator A if $\overline{A|_S} = \overline{A}$.*

It is clear that for a closable operator A, it holds that $A \subset \overline{A}$. Moreover this closed extension $\overline{A}$ is unique.

***Proposition* 2.2.5.** *Let $A : D(A) \to \mathsf{H}$ be a linear operator over H then the following statements hold:*

(a) if A is closable, then $D(\overline{A})$ is made of the elements $x \in \mathsf{H}$ such that there exist $(x_n)_{n\in\mathbb{N}} \subset D(A)$ and $z_x \in \mathsf{H}$ which satisfy $x_n \to x$ and $Ax_n \to z_x$: in this case $\overline{A}x = z_x$;

(b) A is closable if and only if there are no sequences $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \to 0$ and $Ax_n \to y \neq 0$;

(c) A is closable if and only if $aA + bI$ is closable for every $a, b \in \mathbb{F}_c$. In this case $\overline{aA + bI} = a\overline{A} + bI$.

A well-known property of operators is their connection between boundedness and continuity.

***Theorem* 2.2.6.** *Let H be a Hilbert space, then an (anti-)linear operator $A : D(A) \to \mathsf{H}$ is continuous if and only if it is bounded, i.e. there exists some $K \geq 0$ such that*

$$\|Au\| \leq K\|u\| \quad \text{for any } u \in D(A).$$

If A is bounded, then it is closable and $D(\overline{A}) = \mathsf{H}$ whenever $\overline{D(A)} = \mathsf{H}$.

Let $A : D(A) \to \mathsf{H}$ be any (anti-)linear operator on H, we can define its *operator norm* by

$$\|A\| := \sup_{x\in D(A)} \frac{\|Ax\|}{\|x\|} = \inf\{K \in \mathbb{R} \mid \|Au\| \leq K\|u\| \text{ for all } u \in D(A)\} \in [0, \infty]. \tag{2.13}$$

It is clear that A is bounded if and only if $\|A\| < \infty$. We denote by $\mathfrak{B}(\mathsf{H})$ the set of all maximally defined bounded linear operators on H

$$\mathfrak{B}(\mathsf{H}) := \{A : \mathsf{H} \to \mathsf{H} \text{ linear operator } \mid \|A\| < \infty\}.$$

It is a simple matter to show that for any couple of $A, B \in \mathfrak{B}(\mathsf{H})$ and $a, b \in \mathbb{F}_c$ it holds that

$$aA = Aa \in \mathfrak{B}(\mathsf{H}),\ \ A + B \in \mathfrak{B}(\mathsf{H}) \ \text{ and } \ AB \in \mathfrak{B}(\mathsf{H}).$$

More precisely the map $\|\cdot\|$ defined in (2.13) fulfills the following properties:

(a) $\|aA\| = |a|\|A\| = \|A\||a| = \|Aa\|$ for any $A \in \mathfrak{B}(\mathsf{H})$ and $a \in \mathbb{F}_c$,

(b) $\|A + B\| \leq \|A\| + \|B\|$ and $\|AB\| \leq \|A\|\|B\|$ for any $A, B \in \mathfrak{B}(\mathsf{H})$,

(c) if $\|A\| = 0$ for some $A \in \mathfrak{B}(\mathsf{H})$, then $A = 0$.

This makes $\mathfrak{B}(\mathsf{H})$ a two-sided linear space over the division algebra $\mathbb{F}_c$. Moreover the function $\|\cdot\|$ is a well-defined *norm* over $\mathfrak{B}(\mathsf{H})$, making it a normed space with its induced topology, called the *uniform topology*. In particular the sum and composition of operators, as well as the multiplication by elements of $\mathbb{F}_c$, are continuous functions if $\mathfrak{B}(\mathsf{H})$ is endowed with it. More precisely we have the following result.

***Proposition* 2.2.7.** *The set* $\mathfrak{B}(\mathsf{H})$ *is a (two-sided) Banach space over* $\mathbb{F}_c$*.*

The structure of $\mathfrak{B}(\mathsf{H})$ is richer than the linear space one, for we can also perform products of operators. Let us give the following definition.

***Definition* 2.2.8.** *A (two-sided) algebra* $\mathcal{A}$ *over* $\mathbb{F} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *is a two-sided linear space over* $\mathbb{F}$ *equipped with a function* $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$ *which is associative, i.e.*

$$(a \circ b) \circ c = a \circ (b \circ c) \quad \forall a, b, c \in \mathcal{A},$$

and such that, referring to the linear space structure of $\mathcal{A}$*, it satifies:*

(a) $a \circ (b + c) = a \circ b + a \circ c$ *for all* $a, b, c \in \mathcal{A}$*,*

(b) $(b + c) \circ a = b \circ a + c \circ a$ *for all* $a, b, c \in \mathcal{A}$*,*

(c) $\alpha(a \circ b) = (\alpha a) \circ b$ *and* $(a \circ b)\alpha = a \circ (b\alpha)$ *for all* $a, b \in \mathcal{A}$ *and* $\alpha \in \mathbb{F}$*.*

The algebra is said to be unital if there exists an element $1 \in \mathcal{A}$ *such that* $1 \circ a = a \circ 1 = a$ *for all* $a \in \mathcal{A}$*. We say that* $\mathcal{A}$ *is normed if it is equipped with a norm* $\|\cdot\|$ *such that* $\|a \circ b\| \leq \|a\|\|b\|$ *for all* $a, b \in \mathcal{A}$*. If a normed algebra is unital we require also* $\|1\| = 1$*. Finally a normed algebra is said to be Banach if it is complete with respect to the topology induced by the norm.*

If $\mathcal{A}$ *is an (unital, normed, Banach) algebra, a subset* $\mathcal{B} \subset \mathcal{A}$ *is said to be a (unital, normed, Banach) subalgebra of* $\mathcal{A}$ *if it is an (unital, normed, Banach) algebra over the same division algebra with respect to the operation of* $\mathcal{A}$ *(with the same unit and norm of* $\mathcal{A}$ *if present).*

If $\mathcal{A}_1, \mathcal{B}_2$ *are two (unital) algebras, a function* $f : \mathcal{A}_1 \to \mathcal{A}_2$ *is said to be an (anti-)homomorphism if it is (anti-)linear and preserves the product of the algebras, i.e.*

$$f(a \circ_1 b) = f(a) \circ_2 f(b) \quad \text{for all } a, b \in \mathcal{A}_1.$$

If the algebras are unital it also holds that $f(1_1) = 1_2$*. If the function* f *is bijective, it is said to be an automorphism.*

***Proposition* 2.2.9.** *The set* $\mathfrak{B}(\mathsf{H})$ *is a unital Banach algebra over* $\mathbb{F}_c$*.*

There are two additional ways to define convergence within the family $\mathfrak{B}(\mathsf{H})$ which follow by introducing particular seminorms on $\mathfrak{B}(\mathsf{H})$.

***Definition* 2.2.10.** *Let* H *be a Hilbert space, then:*

(a) the topology induced by the seminorms $p_x(T) := \|Tx\|$ *where* $x \in \mathsf{H}$ *is called the strong topology,*

(b) the topology induced by the seminorms $p_{u,v}(T) := |(u|Tv)|$ *where* $u, v \in \mathsf{H}$ *is called the weak topology.*

It is just a simple matter to show that the functions p_x and $p_{u,v}$ are in fact semi-norms. The following result follows.

***Proposition* 2.2.11.** *Let $A \in \mathfrak{B}(\mathsf{H})$ and a net $(A_\nu)_{\nu\in\mathcal{A}} \subset \mathfrak{B}(\mathsf{H})$ be given, then:*

(a) A_ν converges strongly to A if and only if $A_\nu x \to Ax$ for all $x \in \mathsf{H}$,

(b) A_ν converges weakly to A if and only if $(u|A_\nu v) \to (u|Av)$ for all $u, v \in \mathsf{H}$.

The following chain of implication trivially holds:

$$\text{uniform convergence} \Rightarrow \text{strong convergence} \Rightarrow \text{weak convergence.}$$

This is equivalent to the following condition: consider any subset $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$, then

$$\overline{\mathfrak{S}} \subset \overline{\mathfrak{S}}^s \subset \overline{\mathfrak{S}}^w \subset \mathfrak{B}(\mathsf{H}), \tag{2.14}$$

where the bar denotes the closure referred to the corresponding topologies.

Another extremely important definition which, besides all, will enrich the structure of $\mathfrak{B}(\mathsf{H})$, is the *Hermitean adjoint operator*. Consider a densely defined operator $A : D(A) \to \mathsf{H}$ and define

$$D(A^*) := \{z \in \mathsf{H} \mid \exists w_z \in \mathsf{H} \mid (w_z|x) = (z|Ax)\ \forall x \in D(A)\}. \tag{2.15}$$

The density of the domain of $D(A)$ assures that for every $z \in D(A^*)$ the vector w_z is *unique*. So we can consider the function $A^* : D(A^*) \ni z \mapsto w_z \in \mathsf{H}$. This map turns out to be a well-defined linear operator.

***Definition* 2.2.12.** *Let H be a real, complex or quaternionic Hilbert space and let $A : D(A) \to \mathsf{H}$ be a densely defined linear operator. The operator A^* is called the Hermitean adjoint of A.*

If the operator A belongs to $\mathfrak{B}(\mathsf{H})$ an application of Ries's Representation Theorem guarantees that A^* is maximally defined. In general we have the following properties.

***Proposition* 2.2.13.** *The following statements hold:*

(a) if A is densely defined and $A \subset B$, then $B^* \subset A^*$;

(b) if $A \in \mathfrak{B}(\mathsf{H})$ then $A^* \in \mathfrak{B}(\mathsf{H})$: in this case $(A^*)^* = A$;

(c) if A, B are densely defined and $a \in \mathbb{F}_c$, then

$$(aA)^* = \overline{a}A^* = A^*\overline{a}, \quad A^* + B^* \subset (A + B)^*, \quad A^*B^* \subset (BA)^*;$$

(d) if A is densely defined and $B \in \mathfrak{B}(\mathsf{H})$, then

$$A^* + B^* = (A + B)^*, \quad A^*B^* = (BA)^*;$$

(e) if $A \in \mathfrak{B}(\mathsf{H})$, then $\|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2$.

Thanks to points (c) and (d) of the previous proposition we easily see that the function

$$^* : \mathfrak{B}(\mathsf{H}) \ni A \mapsto A^* \in \mathfrak{B}(\mathsf{H})$$

defines an *involution* over $\mathfrak{B}(\mathsf{H})$.

***Definition* 2.2.14.** *Let $\mathcal{A}$ be an algebra over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, a function $^* : \mathcal{A} \to \mathcal{A}$ which satisfies*

(a) $(a+b)^* = a^* + b^*$ for all $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{F}$,

(b) $(\alpha a)^* = a^*\overline{\alpha}$ and $(a\alpha)^* = \overline{\alpha}a^*$ for all $a \in \mathcal{A}$ and $\alpha \in \mathbb{F}$,

(c) $(a^*)^* = a$ for all $a \in \mathcal{A}$,

(d) $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$

is said to be an involution and $(\mathcal{A}, ^*)$ is said to be a *-algebra. A Banach *-algebra is said to be a C^*-algebra if it satisfies $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

If $\mathcal{A}$ is a *-algebra, a subalgebra $\mathcal{B}$ is said to be a *-subalgebra if it is closed under the action of the involution.

If $\mathcal{A}$ is a C^*-algebra a topologically closed Banach *-subalgebra $\mathcal{B}$ is said to be a C^*-subalgebra.

Given two *-algebras $\mathcal{A}_1, \mathcal{A}_2$ an (anti-)homomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$ is said to be a *-(anti-)homomorphism if $f(a^*) = f(a)^*$ for any $a \in \mathcal{A}$.

The discussion carried out so far can be summed up with the following result

***Proposition* 2.2.15.** *The set $\mathfrak{B}(\mathsf{H})$ is a unital C^*-algebra over $\mathbb{F}_c$.*

***Remark* 2.2.16.** In the next chapter we deal with subalgebras of $\mathfrak{B}(\mathsf{H})$, hence it is important to fix the definition. When we make use of (unital, normed, Banach, *-, C^*-) subalgebras of $\mathfrak{B}(\mathsf{H})$, if not explicity written, we will always refer to the (unital, normed, Banach, *-, C^*-) algebra structure of $\mathfrak{B}(\mathsf{H})$ over $\mathbb{F}_c$ and Definitions 2.2.8 and 2.2.14. We omit the term *normed* for any subalgebra of $\mathfrak{B}(\mathsf{H})$ it being tacitly understood.

An important example of subalgebra of $\mathfrak{B}(\mathsf{H})$ is provided by the *commutant* of any given subset of linear operators. Consider any $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$ and define

$$\mathfrak{S}' := \{A \in \mathfrak{B}(\mathsf{H}) \mid AS = SA \text{ for all } S \in \mathfrak{S}\}. \tag{2.16}$$

It is clear that $\mathfrak{S}'$ is a unital subalgebra of $\mathfrak{B}(\mathsf{H})$. These objects will be a matter of analysis for the Chapter 6 on von Neumann algebras.

The following result is a pillar of functional analysis and holds true on real, complex and even quaternionic Hilbert space.

***Theorem* 2.2.17** (Closed graph theorem)**.** *Let* H *be a real, complex or quaternionic Hilbert space. A linear operator* $A : \mathsf{H} \to \mathsf{H}$ *is closed if and only if* $A \in \mathfrak{B}(\mathsf{H})$*.*

There exists a remarkable relationship between closure of an operator and the definition of Hermitean adjoint.

***Proposition* 2.2.18.** *Let* $A : D(A) \to \mathsf{H}$ *be a densely defined linear operator, then*

(a) A^* *is closed,*

(b) A *is closable if and only if* $D(A^*)$ *is dense: in this case* $\overline{A} = A^{**}$*.*

***Proposition* 2.2.19.** *Let* A *be a densely defined linear operator over a Hilbert space* H*, then*

$$N(A^*) = R(A)^{\perp} \quad \textit{and} \quad N(A) \subset R(A^*),$$

where $\subset$ *can be substituted with* $=$ *if* A *is closed.*

The following definition consists in a (almost) complete classification of the possible types of operator that arise once the Hermitean adjoint conjugation is introduced on $\mathfrak{B}(\mathsf{H})$.

***Definition* 2.2.20.** *Let* H *be a real, complex or quaternionic Hilbert space. A linear operator* $A : D(A) \to \mathsf{H}$ *is said to be*

(a) symmetric if $\overline{D(A)} = \mathsf{H}$ *and* $A \subset A^*$*,*

(b) anti-symmetric if $\overline{D(A)} = \mathsf{H}$ *and* $-A \subset A^*$*,*

(c) self-adjoint if $\overline{D(A)} = \mathsf{H}$ *and* $A = A^*$*,*

(d) anti-self-adjoint if $\overline{D(A)} = \mathsf{H}$ *and* $-A = A^*$*,*

(e) essentially self-adjoint if $\overline{D(A)} = \mathsf{H}$ *and* $(A^*)^* = A^*$*,*

(f) essentially anti-self-adjoint if $\overline{D(A)} = \mathsf{H}$ *and* $(A^*)^* = -A^*$*,*

(g) normal if $\overline{D(A)} = \mathsf{H}$ *and* $A^*A = AA^*$*,*

(h) unitary if $A^*A = AA^* = I$*,*

(i) positive if $(x|Ax) \geq 0$ *for all* $x \in D(A)$*,*

(j) an isometry if $D(A) = \mathsf{H}$ *and* $\|Ax\| = \|x\|$ *for all* $x \in D(A)$*,*

(k) a partial isometry if $D(A) = \mathsf{H}$ *and* $\|Ax\| = \|x\|$ *for all* $x \in N(A)^{\perp}$*,*

(l) an orthogonal projector if $P \in \mathfrak{B}(\mathsf{H})$*,* $P^* = P$ *and* $PP = P$*.*

Another important definition follows,

***Definition* 2.2.21.** *A anti-unitary operator U on a real, complex or quaternionic is a surjective anti-linear operator such that $\|Ux\| = \|x\|$ for all $x \in \mathsf{H}$.*

Let us see some basic properties.

***Lemma* 2.2.22.** *Let H be a real, complex or quaternionic Hilbert space and $A : D(A) \to \mathsf{H}$ a symmetric operator such that $(x|Ax) = 0$ for all $x \in D(A)$, then $Ax = 0$ for all $x \in D(A)$.*

Proof. Define $f : D(A) \times D(A) \to \mathbb{F}$ as $f(u,v) := (u|Av)$ for all $u, v \in D(A)$. This map is clearly linear in the right entry. Moreover $\overline{f(y,x)} = \overline{(y|Ax)} = (Ax|y) = (x|Ay) = f(x,y)$. Applying Remark 2.1.6 we see that

$$(u|Av) = f(u,v) = \frac{1}{4}\sum_{\alpha=0}^{d_{\mathbb{F}}}(Q(ui_\alpha + v) - Q(ui_\alpha - v))i_\alpha = 0,$$

where $Q(x) = (x|Ax) = 0$ for all $x \in D(A)$ by definition. At this point, since $u \in D(A)$ is arbitrary and $D(A)$ is dense we get $Av = 0$ for any $v \in D(A)$. $\square$

***Lemma* 2.2.23.** *Let H be complex or quaternionic and $A : D(A) \to \mathsf{H}$ an anti-symmetric operator such that $(x|Ax) = 0$ for all $x \in D(A)$, then $Ax = 0$ for all $x \in D(A)$.*

Proof. Suppose that A is anti-symmetric and take $x, y \in D(A)$, then

$$\begin{aligned}0 &= (x+y|A(x+y)) = (x|Ax) + (y|Ay) + (x|Ay) + (y|Ax) = (x|Ay) - (Ay|x) = \\ &= (x|Ay) - \overline{(x|Ay)},\end{aligned}$$

which gives $(x|Ay) = \overline{(x|Ay)}$, i.e. $(x|Ay) \in \mathbb{R}$. Similarly, if i is any imaginary unit of $\mathbb{F}$ we have

$$\begin{aligned}0 &= (x+yi|A(x+yi)) = (x|Ax) + (y|Ay) + \bar{i}(y|Ax) + (x|Ay)i = \\ &= -\bar{i}(Ay|x) + (x|Ay)i = i\overline{(x|Ay)} + (x|Ay)i,\end{aligned}$$

which gives $(x|Ay)i = -i\overline{(x|Ay)}$. Since $(x|Ay) \in \mathbb{R}$ this identity immediately implies $(x|Ay) = 0$ and concludes the proof. $\square$

***Remark* 2.2.24.** Notice that the previous lemma does not hold in the real Hilbert space case. Indeed any anti-symmetric operator satisfies $(x|Bx) = -(Bx|x) = -(x|Bx)$ and thus $(x|Bx) = 0$ without being necessary null.

***Proposition* 2.2.25.** *Let H be a complex or quaternionic Hilbert space and $A \in \mathfrak{B}(\mathsf{H})$ a bounded operator. If A is positive then it is also self-adjoint.*

Proof. Let $x \in \mathsf{H}$ be given, then $(x|Ax) = \overline{(x|Ax)} = (Ax|x) = (x|A^*x)$. Since $A - A^*$ is antisymmetric, Lemma 2.2.23 concludes the proof. $\square$

***Remark* 2.2.26.** The above proof does not apply to real Hilbert spaces because Lemma 2.2.23 does not. As already pointed out in Remark 2.2.24 any anti-self-adjoint operator on a real Hilbert space satisfies $(x|Bx) \geq 0$.

More generally, on complex Hilbert spaces we can prove the following result. Notice that the proof works only on complex Hilbert spaces since we exploit the identity $i(x|Ay) = (x|Ay)i$ which holds for any x, y and imaginary unit i and this is not fulfilled in quaternionic Hilbert spaces.

***Proposition* 2.2.27.** *Let H be a complex Hilbert space and $A : D(A) \to \mathsf{H}$ a densely defined operator. If A is positive, then it is symmetric.*

Proof. Let $x, y \in \mathsf{H}$ be given, then the positivity implies in particular that

$$\begin{aligned} \mathbb{R} \ni &(x + yi|A(x + yi)) = (x|Ax) + (y|Ay) + [(x|Ay) - (y|Ax)]i, \\ \mathbb{R} \ni &(x + y|A(x + y) = (x|Ax) + (y|Ay) + (x|Ay) + (y|Ax). \end{aligned} \tag{2.17}$$

From this we get $[(x|Ay) - (y|Ax)]i \in \mathbb{R}$, or equivalently $(x|Ay) - (y|Ax) \in \mathbb{R}i$, and $(x|Ay) + (y|Ax) \in \mathbb{R}$. The former can be restated as $\Re(x|Ay) - \Re(y|Ax) = 0$ and the latter as $\Im(x|Ay) + \Im(y|Ax)) = 0$. Putting all together we see that $(x|Ay) = \overline{(y|Ax)} = (Ax|y)$. The vectors x, y being arbitrary we have the thesis. $\square$

***Proposition* 2.2.28.** *Let $A : D(A) \to \mathsf{H}$ and $B : D(B) \to \mathsf{H}$ be symmetric operators, then the following statements hold:*

(1) A is closable and $\overline{A} \subset A^*$,

(2) if $D(A) = \mathsf{H}$ then $A \in \mathfrak{B}(\mathsf{H})$,

(3) the following assertions are equivalent

$$(A^*)^* = A^*, \quad \overline{A} = A^*, \quad \overline{A} = (\overline{A})^*,$$

(4) if $A \subset B$ and A is essentially self-adjoint, then also B is and $\overline{A} = \overline{B}$.

***Theorem* 2.2.29.** *Let A be a densely defined closed operator over H, then A^*A is a positive self-adjoint operator.*

The orthogonal projectors over a Hilbert space play a fundamental role in the logic theory of quantum mechanics.

***Definition* 2.2.30.** *Let H be a Hilbert space, the set of orthogonal projectors over H is said to be the logic of H (or $\mathfrak{B}(\mathsf{H})$) and denoted by $\mathfrak{L}(\mathsf{H})$. Two projectors P, Q are said to be compatible if they commute, i.e. $PQ = QP$. They are said to be orthogonal if $PQ = 0$. They are said to be comparable and denoted by $P \leq Q$, if $PQ = P$.*

Some basic properties follow.

***Proposition* 2.2.31.** *Let H be a real, complex or quaternionic Hilbert space and $P, Q \in \mathfrak{L}(\mathsf{H})$, then the following statements hold:*

(a) $P(\mathsf{H})$ is a closed subspace of H. In particular $P(\mathsf{H}) \oplus P(\mathsf{H})^{\perp} = \mathsf{H}$,

(b) for every closed subspace $\mathsf{K} \subset \mathsf{H}$ there exists a unique $P_{\mathsf{K}} \in \mathfrak{L}(\mathsf{H})$ such that $P_{\mathsf{K}}(\mathsf{H}) = \mathsf{K}$,

(c) if P, Q are orthogonal to each other they are compatible,

(d) $P^{\perp} := I - P \in \mathfrak{L}(\mathsf{H})$ and $(I-P)(\mathsf{H}) = P(\mathsf{H})^{\perp}$: of course $PP^{\perp} = 0$,

(e) $P \leq Q$ if and only if $P(\mathsf{H}) \subset Q(\mathsf{H})$.

Consider two projectors $P, Q \in \mathfrak{L}(\mathsf{H})$ and let $P(\mathsf{H})$ and $Q(\mathsf{H})$ be their ranges. We can construct the closed subspaces

$$P(\mathsf{H}) \cap Q(\mathsf{H}) \quad \text{and} \quad \overline{[P(\mathsf{H}) \cup Q(\mathsf{H})]}. \tag{2.18}$$

Referring to Proposition 2.2.31, point (b), we can define the orthogonal projectors

$$P \wedge Q := P_{P(\mathsf{H}) \cap Q(\mathsf{H})} \quad \text{and} \quad P \vee Q := P_{\overline{[P(\mathsf{H}) \cup Q(\mathsf{H})]}}. \tag{2.19}$$

***Proposition* 2.2.32.** *Let P, Q be two compatible orthogonal projectors on a real, complex or quaternionic Hilbert space, then $P \wedge Q = PQ$ and $P \vee Q = P + Q - PQ$.*

The definitions above can be extended to *any* family of orthogonal projectors

$$\bigvee_{i \in \mathcal{I}} P_i = P_{\bigcap_{i \in \mathcal{I}} P_i(\mathsf{H})} \qquad \bigvee_{i \in \mathcal{I}} P_i = P_{\overline{[\bigcup_{i \in \mathcal{I}} P_i]}}. \tag{2.20}$$

In the case of a countable family of orthogonal projectors, the second statement in Proposition 2.2.32 can be restated in the following way:

***Proposition* 2.2.33.** *Let $(P_n)_{n \in \mathbb{N}} \subset \mathfrak{L}(\mathsf{H})$ be a sequence of projectors such that $P_n P_m = 0$ if $n \neq m$, then $\sum_{n=0}^{N} P_n$ strongly converges within $\mathfrak{L}(\mathsf{H})$ and*

$$\bigvee_{n \in \mathbb{N}} P_n = s\text{-}\lim_{N \to \infty} \sum_{n=0}^{N} P_n. \tag{2.21}$$

***Remark* 2.2.34.** The left-hand side in (2.21) is insensitive to the indexing order of the orthogonal projectors and so the same holds for the right-hand side.

To conclude, some basic properties on the connectives $\wedge, \vee$.

***Proposition* 2.2.35.** *Let $P, Q, Q' \in \mathfrak{L}(\mathsf{H})$ be given, then the following statements hold:*

(a) De Morgan laws: $(P \wedge Q)^\perp = P^\perp \vee Q^\perp$ *and* $(P \vee Q)^\perp = P^\perp \wedge Q^\perp$,

(b) if $P \leq Q$, *then* $Q = P \vee (P^\perp \wedge Q)$,

(c) the operations $\wedge, \vee$ *are distributive, i.e.*

$$P \wedge (Q \vee Q') = (P \wedge Q) \vee (P \wedge Q')$$

if and only P, Q, Q' *are compatible with each other.*

2.2.2 Basics on Spectral Theory

The spectral theory for real and complex operators is essentially identical, while in the quaternionic case things are quite different, for major difficulties arise from the lack of commutativity of the division algebra. We will try to condense the three cases as much as possible, aware of the remarkable differences between them.

The usual definitions of spectrum and resolvent sets involve the definition of the operator $A - I\lambda$ for $\lambda \in \mathbb{F}$. Unfortunately this makes no sense on quaternionic Hilbert spaces, for the multiplication of operators by scalars is meaningless outside $\mathbb{F}_c$. The closest definition we can take into consideration is the following:

***Definition* 2.2.36.** *Let* H *be a real or complex or quaternionic Hilbert space and let* $A : D(A) \to \mathsf{H}$ *be a linear operator. For any* $\lambda \in \mathbb{F}$ *define*

$$\Delta_\lambda(A) := \begin{cases} A - \lambda I & \text{if } \mathbb{F} = \mathbb{R}, \mathbb{C} \\ A^2 - (\lambda + \overline{\lambda})A + |\lambda|^2 I & \text{if } \mathbb{F} = \mathbb{H} \end{cases}.$$

The resolvent set $\rho(A)$ *of* A *is the set of elements* $\lambda \in \mathbb{F}$ *such that*

(i) $N(\Delta_\lambda(A)) = \{0\}$,

(ii) $R(\Delta_\lambda(A))$ *is dense in* H,

(iii) $\Delta_\lambda(A)^{-1} : R(\Delta_\lambda(A)) \to \mathsf{H}$ *is bounded.*

The spectrum of A *is the complement* $\sigma(A) := \mathbb{F} \setminus \rho(A)$. *This is the union of three pairwise disjoint subsets:*

(a) the point spectrum of A*:*

$$\sigma_p(A) := \{\lambda \in \mathbb{F} \mid N(\Delta_\lambda(A)) \neq \{0\}\},$$

(b) the continuous spectrum of A*:*

$$\sigma_r(A) := \{\lambda \in \mathbb{F} \mid N(\Delta_\lambda(A)) = \{0\},\ \overline{R(\Delta_\lambda(A))} = \mathsf{H},\ \Delta_\lambda(A)^{-1} \text{ is not bounded}\},$$

(c) *the residual spectrum of* A*:*

$$\sigma_c(A) := \{\lambda \in \mathbb{F} \mid N(\Delta_\lambda(A)) = \{0\},\ \overline{R(\Delta_\lambda(A))} \neq \mathsf{H}\}.$$

As we will see, in the quaternionic case this definition is reduntant for the purpose of spectral theorem, for only a particular *slice* of $\sigma(A)$ suffices for the spectral decomposition of any normal operator.

Here follow some properties which characterise the spectra in some particular conditions.

***Proposition* 2.2.37.** *Let* H *be a Hilbert space and* A *a linear operator over* H*, then*

(a) *the set* $\sigma_p(A)$ *coincides with the set of eigenvalues of* A*, i.e. the elements* $q \in \mathbb{F}$ *such that* $Au = uq$ *for some* $0 \neq u \in D(A)$*;*

(b) *if* A *is self-adjoint, then* $\sigma(A) \subset \Re\mathbb{F}$ *and* $\sigma_r(A) = \emptyset$*;*

(c) *if* H *is complex or quaternionic and* A *is anti-self-adjoint, then* $\sigma(A) \subset \Im\mathbb{F}$ *and* $\sigma_r(A) = \emptyset$*;*

(d) *if* $P \in \mathfrak{L}(\mathsf{H})$*, then* $\sigma(P) = \{0, I\}$*.*

Now, let X be any second-countable topological space and let $\mathcal{B}(X)$ be its σ-algebra of Borelian subsets. We are interested in the set of *measurable functions* $M(X, \mathbb{F})$. The subset of *bounded measurable functions* is denoted by $M_b(X, \mathbb{F})$ and for each $f \in M_b(X, \mathbb{F})$ we denote by

$$\|f\|_\infty := \sup_{x \in X} |f(x)| < \infty$$

the *supremum norm* of f.

***Definition* 2.2.38.** *Let* H *be a real, complex or quaternionic Hilbert space and* $\Sigma(X)$ *the Borel* σ*-algebra of a second-countable topological space* X*. A function* $\Sigma(X) \ni E \mapsto P_E \in \mathfrak{L}(\mathsf{H})$ *which satisfies*

(a) $P(X) = I$*,*

(b) $P(E)P(F) = P(E \cap F)$*,*

(c) *if* $(E_n)_{n \in \mathbb{N}} \subset \mathfrak{L}(\mathsf{H})$ *satisfies* $E_n \cap E_m = \emptyset$ *for* $n \neq m$*, then*

$$s\text{-}\lim_{N \to \infty} \sum_{n=0}^{N} P(E_n) = P\left(\bigcup_{n=0}^{\infty} E_n\right).$$

is said to be a projection valued measure or PVM. The support of P *is the closed subset of* X *defined as*

$$suppP := X \setminus \bigcup \{A \text{ open subset of } X \mid P(A) = 0\}.$$

The following properties are fulfilled by any PVM.

***Proposition* 2.2.39.** *Let* H *be a Hilbert space and let* P *be a PVM defined over it. Take* $E, F \in \mathcal{B}(X)$ *and* $(E_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$*, then the following statements hold:*

(a) (monotonicity) if $E \subset F$ *then* $P(E) \leq P(F)$*,*

(b) (subadditivity) $(u|P(\cup_{n\in\mathbb{N}} E_n)\, u) \leq \sum_{n\in\mathbb{N}} (u|P(E_n)u)$ *for all* $u \in \mathsf{H}$*,*

(c) (inner continuity) $\lim_{n\to\infty}(u|P(E_n)u) = (u|P(\cup_{n\in\mathbb{N}} E_n)u)$ *if* $E_n \subset E_{n+1}$*,*

(d) (outer continuity) $\lim_{n\to\infty}(u|P(E_n)u) = (u|P(\cap_{n\in\mathbb{N}} E_n)u)$ *if* $E_n \supset E_{n+1}$*,*

(e) $P(E) = P(E \cap suppP)$*.*

An important related definition is the following.

***Definition* 2.2.40.** *Let* P *be a PVM over the Hilbert space* H*. A function* $f : X \to \mathbb{F}$ *is said to be essentially bounded with respect to* P *if*

$$P(\{x \in X \mid |f(x)| \geq K\}) = 0 \quad \text{for some } K < \infty. \tag{2.22}$$

If f *is essentially bounded, the infimum* $\|f\|_\infty^{(P)}$ *of the set of* $K \geq 0$ *such that (2.22) holds is called the essential supremum (semi)norm of* f *with respect to* P*.*

Notice that any bounded function is of course essentially bounded, more precisely

$$\|f\|_\infty^{(P)} \leq \|f\|_\infty. \tag{2.23}$$

There is another important object that is worth mentioning. Let H be a Hilbert space and P a PVM defined over it. Take any two vectors $u, v \in \mathsf{H}$ and define the map

$$\mu_{u,v} : \mathcal{B}(X) \ni E \mapsto (u|P(E)v) \in \mathbb{F}. \tag{2.24}$$

An important role is played by these functions in the particular case of $u = v$:

***Proposition* 2.2.41.** *If* $u \in \mathsf{H}$ *and* P *is a projection valued measure on* H*, then the function*

$$\mathcal{B}(X) \ni E \mapsto \mu_u(E) := (u|P(E)u) \in [0, \infty)$$

is a finite positive Borel measure.

Given this, for any fixed $u \in \mathsf{H}$, we can consider the linear space

$$\mathcal{L}^2(X, \mathbb{F}, \mu_u).$$

of square integrable functions, whose (semi)norm will be denoted by

$$\|f\|_{2,u} := \int_X |f|^2 \, d\mu_u$$

It must be remarked that in the real and complex cases the function $\mu_{u,v}$ is, respectively, a *signed and complex* finite measure, so it makes sense to integrate functions also with respect to it.

Even though in the real and complex case there is no need to do what follows, we try to compact the notation as much as possible.

In order to introduce the spectral integral we need to turn our (right linear) Hilbert space into a two-sided linear space. In the real and complex case this can be achieved by merely introducing the canonical left scalar multiplication as

$$qu := uq \quad \text{for all} \quad u \in \mathsf{H},\ q \in \mathbb{F}$$

as already discussed in Remark 2.2.3. In the quaternionic case we introduce a left multiplication in a more general way, which bases upon the underlying right-linear Hilbert space structure of H. Even though we are interested in the $\mathbb{K} = \mathbb{F} = \mathbb{H}$ case, we state the following definition in a more general way, which will become useful in the next chapter.

***Definition* 2.2.42.** *Let $\mathbb{K}$ denote either $\mathbb{C}$ or $\mathbb{H}$, then a $\mathbb{K}$-left scalar multiplication on a quaternionic Hilbert space H is a map $L : \mathbb{K} \to \mathfrak{B}(\mathsf{H})$ such that*

(a) $L_{q+p} = L_q + L_p,\ L_{qp} = L_q L_p$ *and* $L_r u = ur$,

(b) $(L_q)^* = L_{\overline{q}}$

for every $p, q \in \mathbb{K}$, $r \in \mathbb{R}$ and $u \in \mathsf{H}$.

It is easy to see that $\|L_q u\| = |q|\|u\|$ for every $u \in \mathsf{H}$.

***Remark* 2.2.43.** Once this has been introduced we can use the following notation

$$qu := L_q u \quad \text{for all } q \in \mathbb{K}$$

and get a multiplication on the left by scalars, which is also compatible with the already existing right multiplication, that is

$$q(up) = L_q(up) = (L_q u)p = (qu)p \quad \text{for all } p \in \mathbb{F},\ q \in \mathbb{K},\ u \in \mathsf{H}$$

which follows from the linearity of L_q.

A concrete way to define such a function is the following. Fix a Hilbert basis N of the quaternionic Hilbert space H, thanks to Proposition 2.1.14 the sum $\sum_{z\in\mathsf{N}} zq(z|u)$ is well-defined in H for any choice of (fixed) scalar $q \in \mathbb{F}$. Thus we can give the following:

***Proposition* 2.2.44.** *Let H be a quaternionic Hilbert space and N a Hilbert basis, then the function $L^{\mathsf{N}} : \mathbb{H} \to \mathfrak{B}(\mathsf{H})$ defined by*

$$L^{\mathsf{N}}_q u := \sum_{z\in\mathsf{N}} zq(z|u) \qquad \textit{for all } q \in \mathbb{F}$$

is a quaternionic left scalar multiplication. On the contrary for any quaternionic left scalar multiplication L there exists a Hilbert basis N such that $L = L^{\mathsf{N}}$.

***Remark* 2.2.45.** Some remarks follow:

(a) if N is a Hilbert basis associated with a quaternionic left scalar multiplication L, then $L_q z = zq$ for any $q \in \mathbb{H}$ and $z \in \mathsf{N}$;

(b) the introduction of a quaternionic left scalar multiplication makes H a two-sided linear space.

We are in a position to give the following definition.

***Definition* 2.2.46.** *Let H be a Hilbert space. A rigged projection valued measure, or rPVM is a couple (P, L) where P is a PVM and L is*

(a) the canonical left scalar multiplication discussed in Remark 2.2.3 if $\mathbb{F} = \mathbb{R}, \mathbb{C}$,

(b) a left scalar multiplication as in Proposition 2.2.44 if $\mathbb{F} = \mathbb{H}$

and satisfies $L_q P(E) = P(E) L_q$ for any $q \in \mathbb{F}$ and $E \in \mathcal{B}(\mathsf{X})$.

***Remark* 2.2.47.** Again, this definition is meaningful only on quaternionic Hilbert space, for on real and complex ones every PVM is automatically a rPVM.

If the division algebra is $\mathbb{H}$, fix any imaginary unit $i \in \mathbb{H}$ and consider the subset

$$\mathbb{C}_i := \{a + bi \in \mathbb{H} \mid a, b \in \mathbb{R}\} \subset \mathbb{H}$$

which is clearly isomorphic to the field of complex numbers. In the following a major role will be played by its upper half

$$\mathbb{C}_i^+ := \{a + bi \in \mathbb{H} \mid a, b \in \mathbb{R} \text{ with } b \geq 0\}.$$

***Notation* 2.2.48.** In the following X will denote the set $\mathbb{R}, \mathbb{C}$ or $\mathbb{C}_\imath^+$, depending on the fact that the Hilbert space is, respectively, real, complex or quaternionic.

Consider a Borel set $E \in \mathcal{B}(X)$, then the function $\chi_E : X \to \mathbb{F}$ defined as $\chi_E(x) = 1$ or 0, depending on whether $x \in E$ or not, is clearly measurable and is called the *characteristic function* of E. A *simple function* over X is a measurable function $s : X \to \mathbb{F}$ which can be expressed as

$$s = \sum_{n=0}^{N} q_n \chi_{E_n} \in M_b(X, \mathbb{F}) \tag{2.25}$$

where $q_n \in \mathbb{F}$ and $E_n \in \mathcal{B}(X)$ are pairwise disjoint. Of course the expression of s is not unique.

***Definition* 2.2.49.** *Let (P, L) be a rPVM over H and s a simple function as in (2.25), then the integral of s with respect to (P, L) is defined as*

$$\int_X s\, dP := \sum_{n=0}^{N} L_{q_n} P(E_n) \in \mathfrak{B}(\mathsf{H}).$$

It can be proved that this definition is independent from the choice of the representative expression for s, moreover

$$\|s\|_\infty^{(P)} = \left\| \int_X s\, dP \right\| \quad \text{for any simple function } s. \tag{2.26}$$

The next step consists in extending this definition to (essentially) bounded functions. The following technical lemma holds.

***Lemma* 2.2.50.** *For any $f \in M_b(X, \mathbb{F})$ there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions such that $\|s_n - f\|_\infty \to 0$.*

Using this result, together with (2.23) and (2.26), it is immediate to see that the sequence of operators $\int_X s_n\, dP$ is of Cauchy type within $\mathfrak{B}(\mathsf{H})$. Hence, thanks to the completeness of $\mathfrak{B}(\mathsf{H})$, the following definition makes sense:

***Definition* 2.2.51.** *For any $f \in M_b(X, \mathbb{F})$ the integral of f with respect to (P, L) is defined as*

$$\int_X f\, dP := \lim_{n\to\infty} \int_X s_n\, dP \in \mathfrak{B}(\mathsf{H}) \tag{2.27}$$

where the limit is to be considered in the uniform topology and $(s_n)_{n\in\mathbb{N}}$ is a sequence of step functions as in Proposition 2.2.50.

Again, it can be proved that this definition is independent from the particular choice of the sequence of simple functions converging to f. Moreover for any bounded function f and for any $u \in \mathsf{H}$, it holds that

$$\left\| \int_X f\, dPu \right\|^2 = \int_X |f|^2\, d\mu_u =: \|f\|_{2,u}. \tag{2.28}$$

In particular notice that, if $f \in M_b(X, \mathbb{F})$ and $u \in \mathsf{H}$, then it always holds that $f \in \mathcal{L}^2(X, \mathbb{F}, \mu_u)$. The found result allows us to extend the theory to generic measurable functions. Take $f \in M(X, \mathbb{F})$ and define the following set

$$\Delta_f := \left\{ u \in \mathsf{H} \mid f \in \mathcal{L}^2(X, \mathbb{F}, \mu_u) \right\} \subset \mathsf{H}. \tag{2.29}$$

***Proposition* 2.2.52.** *Let $f \in M(X, \mathbb{F})$, then the set Δ_f is a dense linear subspace of H. Moreover for any $u \in \Delta_f$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded measurable functions such that $\|f_n - f\|_{2,u} \to 0$.*

Exploiting this proposition and (2.28) we see that $\int_X f_n \, dPu$ is a Cauchy sequence within H, hence converging to some vector of H. This suggests the following definition.

***Definition* 2.2.53.** *For any $f \in M(X, \mathbb{F})$ the integral of f with respect to (P, L) is defined as*

$$\int_X f \, dPu := s\text{-}\lim_{n\to\infty} \int_X f_n \, dPu \quad \textit{for all } u \in \mathsf{H} \tag{2.30}$$

where $(f_n)_{n\in\mathbb{N}}$ is a sequence of bounded measurable functions as in Proposition 2.2.52.

Again, it can be proved that this definition is independent from the particular choice of the sequence of bounded measurable functions converging to f.

Let us list some important and useful properties about these function operators.

***Proposition* 2.2.54.** *Let $f \in M(X, \mathbb{F})$ be any measurable function, then the following statements hold:*

(a) for every $q \in \mathbb{F} \setminus \{0\}$ it holds that $\Delta_{qf} = \Delta_{fq} = \Delta_f$, $L_q(\Delta_f) = \Delta_f$ and

$$L_q \left(\int_X f \, dP \right) = \int_X qf \, dP \quad \textit{and} \quad \left(\int_X f \, dP \right) L_q = \int_X fq \, dP;$$

(b) for every measurable function f and for every $u \in \Delta_f$ it holds that

$$\left\| \int_X f \, dPu \right\|^2 = \int_X |f|^2 \, d\mu_u \tag{2.31}$$

and, if f is real valued, also

$$\left(u \middle| \int_X f \, dPu \right) = \int_X f \, d\mu_u; \tag{2.32}$$

(c) if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then for every measurable function f and for every $u \in \mathsf{H}$ and $v \in \Delta_f$

$$\left(u \middle| \int_X f \, dPv \right) = \int_X f \, d\mu_{u,v}. \tag{2.33}$$

***Proposition* 2.2.55.** *Consider two measurable functions f, g, then the following statements hold:*

(a) $D\left(\int_X f \, dP \int_X g \, dP\right) = \Delta_f \cap \Delta_{fg}$ and

$$\int_X f \, dP \int_X g \, dP \subset \int_X fg \, dP \tag{2.34}$$

where can replace the latter inclusion with an equality if and only if $\Delta_{fg} \subset \Delta_g$;

(b) $D\left(\int_X f\,dP + \int_X g\,dP\right) = \Delta_f \cap \Delta_g$ *and*

$$\int_X f\,dP + \int_X g\,dP \subset \int_X (f+g)\,dP \tag{2.35}$$

where we can replace the latter inclusion with an equality if and only if $\Delta_{f+g} = \Delta_f \cap \Delta_g$*;*

(c) $\Delta_{\overline{f}} = \Delta_f$ *and*

$$\left(\int_X f\,dP\right)^* = \int_X \overline{f}\,dP, \tag{2.36}$$

in particular we have that $\int_X f\,dP$ *is always a closed operator;*

(d) the operator $\int_X f\,dP$ *is normal and*

$$\left(\int_X f\,dP\right)^* \left(\int_X f\,dP\right) = \int_X |f|^2\,dP = \left(\int_X f\,dP\right)\left(\int_X f\,dP\right)^*; \tag{2.37}$$

(e) $\Delta_f = \mathsf{H}$ *if and only if* f *is essentially bounded with respect to* P*, in particular this holds if and only if* $\int_X f\,dP \in \mathfrak{B}(\mathsf{H})$*;*

(f) let $t : X \to X$ *be a measurable map and define* P' *as* $P'(E) := P(t^{-1}(E))$ *for every* $E \in \mathcal{B}(X)$*, then* (P', L) *is a rPVM on* H*. Moreover* $\Delta'_f = \Delta_{f\circ t}$ *and*

$$\int_X f \circ t\,dP = \int_X f\,dP' \quad \text{and}$$

for every $f \in M(X, \mathbb{F})$*.*

At this point we are ready to state the spectral theorem.

***Theorem* 2.2.56.** *Let* H *be real, complex or quaternionic Hilbert space and* $A : D(A) \to \mathsf{H}$ *be a linear operator which is self-adjoint if* H *is real or, more generally, normal if* H *is complex or quaternionic, then there exists a rPVM* $(L, P^{(A)})$ *such that*

$$A = \int_X z\,dP^{(A)}(z). \tag{2.38}$$

The PVM is uniquely determined by A *on the whole* $\mathcal{B}(X)$*. The following additional statements hold:*

(a) the support of $P^{(A)}$ *is given by* $\sigma(A) \cap X$*;*

(b) concerning the spectrum of A *we have:*

(a) $\lambda \in \sigma_p(A) \cap X$ *if and only if* $P(\{\lambda\}) \neq 0$*; moreover every isolated point of* $\sigma(A) \cap X$ *belongs to* $\sigma_p(A) \cap X$*;*

(b) $\lambda \in \sigma_c(A) \cap X$ *if and only if* $P(\{\lambda\}) = 0$ *and* $P(E) \neq 0$ *for every open subset* $E \subset X$ *containing* λ*; moreover if* $\lambda \in \sigma_c(A) \cap X$ *then for any* $\epsilon > 0$ *there exists a vector* $u_\epsilon \in \mathsf{H}$ *such that* $\|u_\epsilon\| = 1$ *and* $\|Au_\epsilon - u_\epsilon\lambda\| < \epsilon$*;*

(c) $\sigma_r(A) \cap X = \emptyset$.

Remember that, in the important case of selfadjoint operator, the left scalar multiplication L plays no role in the spectral decomposition, for the only multiplications of scalar and operators involve real numbers and thus L can be neglected from the theory. Indeed in the restricted case of a selfadjoint operator the spectral theorem can be restated in the following simpler form.

***Theorem* 2.2.57.** *Let* H *be a real, complex or quaternionic Hilbert space and* A *a self-adjoint operator over* H*, then there exists a unique PVM* $P^{(A)}$ *such that*

(a) $id \in \mathcal{L}^2\left(\mathbb{R}, \mathbb{R}, \mu_u^{(P^{(A)})}\right)$,

(b) $(u|Au) = \int_{\mathbb{R}} s\, d\mu_u^{(P^{(A)})}(s)$

for any choice of $u \in D(A)$.

Proof. The PVM $P^{(A)}$ associated to A by the spectral theorem satisfies the thesis. Suppose Q is another PVM satisfying the above requirements and take $u \in D(A)$. In particular this means that $u \in D(A')$ where A' is the self-adjoint operator defined as $\int_{\mathbb{R}} t\, dQ(t)$ and $(u|Au) = (u|A'u)$, i.e. $(u|(A - A')u) = 0$. Since $D(A) \subset D(A')$, the difference $A - A'$ is densely defined. This, together with the self-adjointness of both A and A', makes their difference a symmetric operator. Applying Lemma 2.2.22 we get $Au = A'u$ for all $u \in D(A)$, i.e. $A \subset A'$. Since A and A' are both self-adjoint it must be $A = A'$. This means that the spectral decomposition (2.38) of A can be carried out with both $P^{(A)}$ and Q. The uniqueness of such a PVM stated by the spectral theorem assures that $P^{(A)} = Q$, concluding the proof. □

Thanks to the spectral theorem we see that any normal (self-adjoint in the real Hilbert space case) operator is completely characterised by its PVM. Another important confirmation of this is the following useful result.

***Proposition* 2.2.58.** *Let* H *be a real, complex or quaternionic Hilbert space and* A *be a densely defined operator, then the following statements hold:*

(a) *if* $U \in \mathfrak{B}(\mathsf{H})$ *is unitary, then* U^*AU *is densely defined and* $(U^*AU)^* = U^*A^*U$*: in particular, if* A *is self-adjoint, then the PVM of* U^*AU *is given by*

$$P^{(U^*AU)}(E) = U^*P^{(A)}(E)U$$

for every Borelian $E \in \mathcal{B}(X)$*;*

(b) if U is unitary and A closable, then UA and AU are closable and

$$\overline{UA} = U\overline{A}, \quad \overline{AU} = \overline{A}U;$$

(c) if $B \in \mathfrak{B}(\mathsf{H})$ and A is closable, then $BA \subset AB$ implies $B\overline{A} \subset \overline{A}B$;

(d) if $A^* = A$ with PVM $P^{(A)}$, then $P^{(A)}(E) = \chi_E(A)$ and

$$P^{(A)}(E)A \subset AP^{(A)}(E)$$

for any Borelian $E \in \mathcal{B}(X)$.

Proof. The first statement in (a) can be proved through a direct application of the definitions. The second part follows by noticing that $E \mapsto U^*P^{(A)}(E)U$ defines a PVM over H and by exploiting Theorem 2.2.57. Points (b) and (c) can be proved by direct inspection while point (d) follows from the definition of spectral integral and its properties stated in Proposition 2.2.55. □

The definition of function of operators become particularly important when dealing with square root or absolute value of operators.

***Proposition* 2.2.59.** *The following statements hold:*

(a) if A is a positive self-adjoint operator with PVM $P^{(A)}$, then $\sigma(A) \cap X \subset [0, \infty)$ and the operator

$$\sqrt{A} := \int_{[0,\infty)} \sqrt{z}\, dP(z)$$

is a positive self-adjoint operator satisfying $\sqrt{A}\sqrt{A} = A$;

(b) if A is a densely defined closed operator define $|A| := \sqrt{A^*A}$, then if A is self-adjoint or anti-self-adjoint (only self-adjoint in the real case) it holds that

$$|A| = \int_X |z|\, dP(z).$$

If A is a positive self-adjoint operator then $|A| = A$.

To conclude this chapter we discuss another important concept, the joint PVM for finite families of strongly commuting self-adjoint operators.

***Definition* 2.2.60.** *A finite family of self-adjoint operators $A_1, \ldots, A_n$ is said to be strongly commuting if the respective PVMs commute with each other, i.e. if*

$$P_i(\Delta)P_j(\Sigma) = P_j(\Sigma)P_i(\Delta)$$

for all $i, j = 1, \ldots, n$ and $\Delta, \Sigma \in \mathcal{B}(X)$.

***Theorem* 2.2.61.** *Let $A_1, \dots, A_n$ be a family of strongly-commuting self-adjoint operators over a separable real, complex or quaternionic Hilbert space H, then there exists a unique PVM $P : \mathcal{B}(\mathbb{R}^n) \to \mathfrak{L}(\mathsf{H})$ such that*

$$P(\Delta_1 \times \cdots \times \Delta_n) = P_1(\Delta_1) \cdots P_n(\Delta_n) \tag{2.39}$$

for all $\Delta_i \in \mathcal{B}(\mathbb{R})$. Moreover this PVM satisfies

$$f(A_i) = \int_{\mathbb{R}^n} f(z_i)\, dP(z) \tag{2.40}$$

for all $i = 1, \dots, n$ and measurable function $f : \mathbb{R} \to \mathbb{R}$.

***Proposition* 2.2.62.** *Referring to Theorem 2.2.61, it holds that $P(E)A_i \subset A_i P(E)$ for any Borelian $E \in \mathcal{B}(\mathbb{R}^n)$ and $i = 1, \dots, n$.*

Definition 2.2.60 and Theorem 2.2.61 play an important role in quantum mechanics, making the concepts of *compatible observables* and *simultaneous measurement* mathematically meaningful.

2.2.3 Excursus on Quaternionic Algebras of Operators

As discussed at the beginning of the previous section if H is a quaternionic Hilbert space on the division algebra $\mathbb{F}$ only $\mathbb{F}_c$-linear combinations of operators are meaningful. More precisely $\mathfrak{B}(\mathsf{H})$ turns out to be a unital C^*-algebra over $\mathbb{F}_c$.

Things change when a *quaternionic left scalar multiplication*

$$\mathbb{F} \ni q \mapsto L_q \in \mathfrak{B}(\mathsf{H}) \tag{2.41}$$

as in Definition 2.2.42 is introduced on H. In line with the notation $qu := L_q u$ introduced in Remark 2.2.43, given any linear operator $A : D(A) \to \mathsf{H}$ we can construct the following linear operators

$$qA := L_q A \quad \text{and} \quad Aq := AL_q \text{ for all } q \in \mathbb{F}.$$

Notice that for $q \in \mathbb{F}_c$ the above operators reduces to the already defined (2.10) and (2.11). Moreover

$$qA = Aq \quad \text{if } q \in \mathbb{F}_c.$$

Exploiting the properties of L it is clear that the function defined by

$$\mathbb{F} \times \mathfrak{B}(\mathsf{H}) \ni (q, A) \mapsto qA \in \mathfrak{B}(\mathsf{H})$$

and by

$$\mathfrak{B}(\mathsf{H}) \times \mathbb{F} \ni (A, q) \mapsto Aq \in \mathfrak{B}(\mathsf{H})$$

define on $\mathfrak{B}(\mathsf{H})$ a *left* and a *right* multiplication by elements of $\mathbb{F}$, respectively. Moreover the following result holds.

***Proposition* 2.2.63.** *For any $q, p \in \mathbb{F}$ and $A, B \in \mathfrak{B}(\mathsf{H})$ it holds that*

(a) $(qA)p = q(Ap)$,

(b) $q(AB) = (qA)B$ *and* $(AB)q = A(Bq)$,

(c) $(qA)^* = A\overline{q}$ *and* $(Aq)^* = \overline{q}A^*$.

Proof. Points (a) and (b) follow immediately from the associativity of operators within $\mathfrak{B}(\mathsf{H})$. To prove point (b) notice that $(qA)^* = (L_qA)^* = A^*(L_q)^* = A^*L_{\overline{q}} = A^*\overline{q}$. The second identity is similar. $\square$

These results make $\mathfrak{B}(\mathsf{H})$ a (two-sided) algebra over $\mathbb{F}$. More precisely we have the following result.

***Proposition* 2.2.64.** *Let H be a quaternionic Hilbert space equipped with a quaternionic left scalar multiplication, then $\mathfrak{B}(\mathsf{H})$ is a unital C^*-algebra over $\mathbb{F}$.*

We do not make use of these structures in this work, except for another brief excursus at the end of Chapter 6 where we show how any attempt to define (two-sided) von Neumann algebras over $\mathbb{H}$ is meaningless. Thus in the following we always stick to the division algebra $\mathbb{F}_c$ when talking about the algebra structure of $\mathfrak{B}(\mathsf{H})$. In particular this applies to the concept of subalgebra of $\mathfrak{B}(\mathsf{H})$, in line with Remark 2.2.16.

Chapter 3

Mathematical Tools

In this chapter we analyse some techniques for extending or reducing the division algebra of, respectively, real and quaternionic Hilbert spaces which turns out to be useful in the next chapters. First of all they allow us to restate several standard results which hold on complex Hilbert spaces on, respectively, real and quaternionic ones. Secondly, these tools become crucial in the last chapter, where we show how real and quaternionic Hilbert space representations of physical systems eventually reduce to standard formulations over complex Hilbert spaces.

3.1 Extending the Scalars of Real Hilbert Spaces

This can be carried out in two ways: by means of an *external* and an *internal* procedure. The former consists in enlarging the set of vectors the same way of the division algebra, in order to make it possible to define a multiplication by complex or quaternion scalars. Concering the latter, instead, we work on the already available set of vectors and extend the scalar multiplication by means of the action of particular operators which act as imaginary numbers.

As usual we will denote by $\{i_\alpha\}_{\alpha=0,\ldots,d_{\mathbb{K}}}$ a basis of standard units of $\mathbb{K}$ (understood as a real-linear space), where $i_0 = 1$ and $d_{\mathbb{K}} = \dim_{\mathbb{R}} \mathbb{K} - 1$.

3.1.1 Method I: Canonical Procedure

Let H be a real Hilbert space with real scalar product $(\cdot|\cdot)$. There exists a natural way to define a complex and a quaternionic Hilbert space out of H. If we interpret $\mathbb{K} = \mathbb{C}, \mathbb{H}$ as real linear spaces we can perform the (real) tensor product

$$\mathsf{H}_{\mathbb{K}} := \mathsf{H} \otimes \mathbb{K}.$$

This is by construction a real linear space. If $(e_i)_{i\in N}$ is an algebraic basis for H, then $(e_i \otimes i_\alpha)_{i\in N,\alpha\in\{0,..,d_{\mathbb{K}}\}}$ is an algebraic basis for the *real* linear space $\mathsf{H}_{\mathbb{K}}$. In particular any element can be written in the following way

$$x = \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha \otimes i_\alpha, \tag{3.1}$$

where $x_\alpha \in \mathsf{H}$ depend on x and are uniquely determined.

Next step consists in enlarging the division algebra, i.e. in defining a (right) multiplication by elements of $\mathbb{K}$. Fix any $k_0 \in \mathbb{K}$ and consider the *real* bilinear map $\mathsf{H} \times \mathbb{K} \ni (u, k) \mapsto u \otimes (kk_0) \in \mathsf{H}_{\mathbb{K}}$. Thanks to the universal property of the tensor product there must exist a (unique) real linear map $R_{k_0} : \mathsf{H}_{\mathbb{K}} \to \mathsf{H}_{\mathbb{K}}$ such that $R_{k_0}(u \otimes k) = u \otimes (kk_0)$. Notice that, as should be, R_k reduces to the original scalar multiplication if restricted to the reals. So, consider the function

$$\mathsf{H}_{\mathbb{K}} \times \mathbb{K} \ni (x, k) \mapsto R_k x \in \mathsf{H}_{\mathbb{K}}, \tag{3.2}$$

then the following result follows easily.

***Proposition* 3.1.1.** *Let H be a real Hilbert space, then the (right) $\mathbb{R}$-linear space $\mathsf{H}_{\mathbb{K}}$ equipped with the multiplication (3.2) is a (right) $\mathbb{K}$-linear space.*

Once this $\mathbb{K}$-linear structure is defined it is just a simple matter to see that $(e_i \otimes 1)_{i\in N}$ is an algebraic basis for the $\mathbb{K}$-linear space $\mathsf{H}_{\mathbb{K}}$, where $(e_i)_{i\in N}$ is any algebraic basis for the *real* linear space H.

Of course we would like to have some sort of identification of H as a *real* subspace of $\mathsf{H}_{\mathbb{K}}$. This is clearly true and realised by means of the map

$$\mathsf{H} \ni u \mapsto u \otimes 1 \in \mathsf{H}_{\mathbb{K}} \tag{3.3}$$

which is injective and real linear. This makes it possible to identify H as a real subspace of $\mathsf{H}_{\mathbb{K}}$.

We are in a position to introduce an Hermitean scalar product on $\mathsf{H}_{\mathbb{K}}$ and make it a Hilbert space on $\mathbb{K}$.

***Proposition* 3.1.2.** *There exists a unique Hermitean scalar product $(\cdot|\cdot)_{\mathbb{K}}$ over $\mathsf{H}_{\mathbb{K}}$ such that*

$$(u \otimes s | v \otimes t)_{\mathbb{K}} = (u|v)\overline{s}t$$

for any choice of $u, v \in \mathsf{H}$ and $s, t \in \mathbb{K}$.

Proof. Fix any algebraic basis $(e_i)_{i\in N}$ of H and define $f : \mathsf{H}_{\mathbb{K}} \times \mathsf{H}_{\mathbb{K}} \to \mathbb{K}$ as the unique map which is anti-linear in the left-hand entry and linear in the right-hand one and satisfies $f((e_i \otimes 1, e_j \otimes 1)) = (e_i|e_j)$. Take $u, v \in \mathsf{H}$ and $s, t \in \mathbb{K}$, then a simple calculation shows that $f(u \otimes s, v \otimes t) = (u|v)\overline{s}t$. It is obvious that this is the only left-hand-anti-linear and right-hand-linear map satisfying the thesis. The fact that f satisfies all the assumptions in the definition of a Hermitean scalar product can be easily checked exploiting the same properties which are fulfilled by hypotheses by the Hermitean scalar products on H and $\mathbb{K}$. $\square$

The norm associated with this extended scalar product reads

$$\|u \otimes k\|_{\mathbb{K}}^2 = (u \otimes k | u \otimes k)_{\mathbb{K}} = \|u\|^2 |k|^2 .$$

Referring to expression (3.1) for the generic element $x \in \mathsf{H}_{\mathbb{K}}$, a simple calculation shows that

$$(x|y)_{\mathbb{K}} = \sum_{\alpha=0}^{d_{\mathbb{K}}} \sum_{\beta=0}^{d_{\mathbb{K}}} (x_\alpha | y_\beta) \overline{i_\alpha} i_\beta \quad \text{and} \quad \|x\|_{\mathbb{K}}^2 = \sum_{\alpha=0}^{d_{\mathbb{K}}} \|x_\alpha\|^2 . \tag{3.4}$$

Now, consider any subspace $\mathsf{M} \subset \mathsf{H}$ and define the following subset of $\mathsf{H}_{\mathbb{K}}$

$$\mathsf{M}_{\mathbb{K}} := \left\{ \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha \;\middle|\; u_\alpha \in \mathsf{K} \right\} . \tag{3.5}$$

It can be proved by direct inspection that this set is closed with respect to the sum and the $\mathbb{K}$-scalar multiplication (3.2) introduced on $\mathsf{H}_{\mathbb{K}}$, making it a $\mathbb{K}$-linear subspace of $\mathsf{H}_{\mathbb{K}}$.

We could have taken the real tensor product $\mathsf{M} \otimes \mathbb{K}$ and equipped it with a $\mathbb{K}$-scalar multiplication as we did for H in (3.2). It is just a simple matter to prove that the two structures are isomorphic to each other.

***Theorem* 3.1.3.** *Let H be a real Hilbert space, then the following statements hold:*

(a) the space $\mathsf{H}_{\mathbb{K}}$ equipped with the scalar product $(\cdot|\cdot)_{\mathbb{K}}$ is a $\mathbb{K}$-Hilbert space,

(b) a subset $\mathsf{N} \subset \mathsf{H}$ is a Hilbert basis for H if and only if $\mathsf{N} \otimes 1$ is a Hilbert basis for $\mathsf{H}_{\mathbb{K}}$,

(c) let $\mathsf{K} \subset \mathsf{H}$ be a linear subspace, then $\overline{\mathsf{K}}_{\mathbb{K}} = \overline{\mathsf{K}_{\mathbb{K}}}$: in particular K is closed or dense in H if and only if $\mathsf{K}_{\mathbb{K}}$ is, respectively, closed or dense in $\mathsf{H}_{\mathbb{K}}$.

Proof. Exploiting the expressions (3.1) and (3.4) we see that any Cauchy sequence in $\mathsf{H}_{\mathbb{K}}$ defines $d_{\mathbb{K}} + 1$ Cauchy sequences in H. The completeness of H completes the proof of (a). A simple check proves that N is maximally orthonormal in H if and only if $\mathsf{N} \otimes 1$ is maximally orthonormal in $\mathsf{H}_{\mathbb{K}}$, giving in this way point (b). Point (c) can be derived easily using (3.5), (3.1) and (3.4). $\square$

Define the following function, called the *canonical conjugation* of $\mathsf{H}_{\mathbb{K}}$

$$C : \mathsf{H}_{\mathbb{K}} \ni \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha \otimes i_\alpha \mapsto \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha \otimes \overline{i_\alpha} \in \mathsf{H}_{\mathbb{K}} . \tag{3.6}$$

It is immediate to see that $CC = I$ and that $\|Cx\| = \|x\|$ for all $x \in \mathsf{H}_{\mathbb{K}}$. Moreover this is an *anti-linear* operator in the $\mathbb{K} = \mathbb{C}$ case.

Now, take a real linear operator $A : D(A) \to \mathsf{H}$ defined over H, we can consider the (real) tensor product

$$A_{\mathbb{K}} := A \otimes I : D(A) \otimes \mathbb{K} \to \mathsf{H} \otimes \mathbb{K}$$

which is by definition (thanks to the universal property of the tensor product) the only real-linear operator on $D(A) \otimes \mathbb{K} = D(A)_{\mathbb{K}}$ satisfying $A \otimes I(u \otimes k) = A(u) \otimes k$ for all $u \in \mathsf{H}$ and $k \in \mathbb{K}$. It is a simple matter to check that $A_{\mathbb{K}}$ is also linear with respect to the $\mathbb{K}$-linear structure of $\mathsf{H}_{\mathbb{K}}$.

***Proposition* 3.1.4.** *The following statements hold:*

(a) a $\mathbb{K}$-linear operator $B : D(B) \to \mathsf{H}_{\mathbb{K}}$ satisfies $B = A_{\mathbb{K}}$ for some real linear operator $A : D(A) \to \mathsf{H}$ if and only if $CB \subset BC$ where C is defined in (3.6): if this is the case, then $CB = BC$ and A is uniquely determined by

$$D(A) = \{x \in \mathsf{H} \,|\, x \otimes 1 \in D(B)\} \quad \textit{and} \quad Ax \otimes 1 = B(x \otimes 1);$$

(b) let $A : D(A) \to \mathsf{H}$ be a real linear operator over H, then

(i) if $A' : D(A') \to \mathsf{H}$ is another real linear operator, then

$$A \subset A' \ \textit{ iff } \ A_{\mathbb{K}} \subset A'_{\mathbb{K}} \quad \textit{and} \quad (AA')_{\mathbb{K}} = A_{\mathbb{K}} A'_{\mathbb{K}};$$

(ii) $A \in \mathfrak{B}(\mathsf{H})$ if and only if $A_{\mathbb{K}} \in \mathfrak{B}(\mathsf{H}_{\mathbb{K}})$, more precisely $\|A\| = \|A_{\mathbb{K}}\|_{\mathbb{K}}$;

(iii) if $D(A)$ is dense, then $(A_{\mathbb{K}})^* = (A^*)_{\mathbb{K}}$, in particular $D((A_{\mathbb{K}})^*) = D(A^*)_{\mathbb{K}}$;

(iv) A is either closed or closable if and only if $A_{\mathbb{K}}$ is, respectively, closed or closable: in the second case $\overline{A_{\mathbb{K}}} = (\overline{A})_{\mathbb{K}}$;

(v) a subspace $\mathsf{S} \subset D(A)$ is a core for A if and only if $\mathsf{S}_{\mathbb{K}}$ is a core for $A_{\mathbb{K}}$;

(vi) $A_{\mathbb{K}}$ is symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent if and only if A is, respectively, symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent;

(vii) A is symmetric and positive if and only if $A_{\mathbb{K}}$ is positive.

Proof. Let us prove (a). Suppose first that $B = A_{\mathbb{K}}$ for some operator A defined on H and take $u \in D(B)$. Exploiting the definition of $A_{\mathbb{K}}$, in particular of $D(A)_{\mathbb{K}}$, it is clear that $Cu \in D(B)$ if $u \in D(B)$ and that $CBu = BCu$ for any $u \in D(B)$. This proves $CB \subset BC$. Using $CC = I$ we immediately get $CB = BC$. So, suppose on the contrary that $CB \subset BC$, or equivalently $CB = BC$, and define $D(A) := \{x \in \mathsf{H} | x \otimes 1 \in D(B)\}$. This is clearly a real subspace of H. We want to prove that $D(B) = D(A)_{\mathbb{K}}$. Of course we have $D(A)_{\mathbb{K}} \subset D(B)$. So, take $u = \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha \in D(B)$. Since $CB \subset BC$ we have in primis that $C(D(B)) \subset D(B)$, in particular $\frac{1}{2}(u + Cu) = x_0 \otimes 1 \in D(B)$ and $\frac{1}{2}(u - Cu) = \sum_{\alpha=1}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha \in D(B)$. Multiplying the latter by $\overline{i_1}$ we get $v := u_1 \otimes 1 \in D(B)$ if $\mathbb{K} = \mathbb{C}$ and $v := u_1 \otimes 1 + u_2 \otimes i_3 - u_3 \otimes i_2 \in D(B)$ if $\mathbb{K} = \mathbb{H}$. Now, sticking to the quaternionic case, if we apply the same argument carried on u to the vector v we get $u_1 \otimes 1 \in D(B)$ and $u_2 \otimes i_3 - u_3 \otimes i_2 \in D(B)$. Proceeding similarly, we get $u_2 \otimes 1, u_3 \otimes 1 \in D(B)$. In all cases we have proved that $u_\alpha \otimes 1 \in D(B)$ for all $\alpha = 0, \ldots, d_{\mathbb{K}}$. By definition we see that $u_\alpha \in D(A)$ for all $\alpha = 0, \ldots, d_{\mathbb{K}}$, hence $u \in D(A)_{\mathbb{K}}$.

Now, take $x \in D(A)$, then $x \otimes 1 \in D(B)$ and $B(x \otimes 1) = \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha$ for some $u_\alpha \in \mathsf{H}$. Clearly, by definition of C we have $C(x \otimes 1) = x \otimes 1$ and so

$$\sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha = B(x \otimes 1) = BC(x \otimes 1) = CB(x \otimes 1) = \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes \overline{i_\alpha},$$

which clearly implies $B(x \otimes 1) = u_0 \otimes 1$. Notice that the vector u_0 is the only vector of H satisfying this equality. This allows us to define $Ax := u_0$. A simple check shows that the map $D(A) \ni x \mapsto Ax$ is a well defined real linear operator. The operator A satisfies the requirements in the thesis.

Now, let us pass to the proof of (b). Point (i) is trivial, let us prove (ii). Take any $A \in \mathfrak{B}(\mathsf{H})$, then $D(A_{\mathbb{K}}) = D(A)_{\mathbb{K}} = \mathsf{H}_{\mathbb{K}}$ and

$$\|A_{\mathbb{K}}(u \otimes k)\|_{\mathbb{K}} = \|(Au) \otimes k\|_{\mathbb{K}} = \|Au\||k| \leq \|A\|\|u\||k| = \|A\|\|u \otimes k\|.$$

Since $H_{\mathbb{K}}$ is generated by the vectors $u \otimes k$, this gives $\|A_{\mathbb{K}}\|_{\mathbb{K}} \leq \|A\|$. On the other hand if $u \in D(A)$, then

$$\|Au\| = \|Au\||1| = \|(Au) \otimes 1\|_{\mathbb{K}} = \|A_{\mathbb{K}}(u \otimes 1)\|_{\mathbb{K}} \leq \|A_{\mathbb{K}}\|_{\mathbb{K}}\|u \otimes 1\| = \|A_{\mathbb{K}}\|_{\mathbb{K}}\|u\|,$$

which gives $\|A\|_{\mathbb{K}} \leq \|A_{\mathbb{K}}\|$ and concludes the proof. Let us prove (iii) now. Suppose that $D(A)$ is dense, which implies the density of $D(A_{\mathbb{K}})$ too thanks to Theorem 3.1.3. Thus both A^* and $(A_{\mathbb{K}})^*$ are well-defined. Applying the definition of the domain of the adjoint operator we have

$$D((A_{\mathbb{K}})^*) = \{u \in \mathsf{H}_{\mathbb{K}} \mid \exists v_u \in \mathsf{H}_{\mathbb{K}} \mid (v_u|\omega)_{\mathbb{K}} = (u|A_{\mathbb{K}}\omega)_{\mathbb{K}} \ \forall \omega \in D(A_{\mathbb{K}})\}.$$

In this case $(A_{\mathbb{K}})^*u = v_u$. So, take $u \in D((A_{\mathbb{K}})^*)$. If we choose $\omega = x \otimes 1$ for some $x \in D(A)$, then for $u = \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha$ and $v_u = \sum_{\alpha=0}^{d_{\mathbb{K}}} v_\alpha \otimes i_\alpha$ we get

$$\sum_{\alpha=0}^{d_{\mathbb{K}}} (v_\alpha|x)\overline{i_\alpha} = (v_u|\omega)_{\mathbb{K}} = (u|A_{\mathbb{K}}\omega)_{\mathbb{K}} = \sum_{\alpha=0}^{d_{\mathbb{K}}} (u_\alpha|Ax)\overline{i_\alpha},$$

which immediately implies $(v_\alpha|x) = (u_\alpha|Ax)$ for all $x \in D(A)$ and for any $\alpha = 0, \ldots, d_{\mathbb{K}}$. This is equivalent to $u_\alpha \in D(A^*)$ and $A^*u_\alpha = v_\alpha$ for all $\alpha = 0, \ldots, d_{\mathbb{K}}$. Thus $u \in D((A^*)_{\mathbb{K}})$ and $(A^*)_{\mathbb{K}}u = v_u = (A_{\mathbb{K}})^*u$. This proves $(A_{\mathbb{K}})^* \subset (A^*)_{\mathbb{K}}$. The opposite inclusion is similar, concluding the proof. Let us prove (iv). Suppose that A is closable and $A_{\mathbb{K}}$ is not. So, there must exist a sequence $(z_n)_{n\in\mathbb{N}} \subset D(A_{\mathbb{K}})$ such that $z_n \to 0$ and $A_{\mathbb{K}}z_n \to w \neq 0$. Exploiting the properties of the norm of $\mathsf{H}_{\mathbb{K}}$ we get $(z_n)_\alpha \to 0$ and

$A(z_n)_\alpha \to w_\alpha$ for every $\alpha = 0, \ldots, \mathbb{K}$. Since $w \neq 0$ by definition, there must exists some index α_0 such that $w_{\alpha_0} \neq 0$. The existence of the sequence $((z_n)_{\alpha_0})_{n\in\mathbb{N}}$ contradicts the fact that A is closable. The opposite inclusion as well as the part about the closedness can be proved similarly. Point (v) follows immediately by noticing that $(\overline{A|_S})_\mathbb{K} = \overline{(A|_S)_\mathbb{K}} = \overline{A_\mathbb{K}|_{S_\mathbb{K}}}$. The proof of (vi) is a direct consequence of (i) and (iv) and the involved definitions. To conclude let us prove (vii). Consider any $u = \sum_{\alpha=0}^{d_\mathbb{K}} u_\alpha \otimes i_\alpha \in D(A_\mathbb{K})$ and remember that $(u|A_\mathbb{K} u)_\mathbb{K} = \sum_{\alpha=0}^{d_\mathbb{K}} \sum_{\beta=0}^{d_\mathbb{K}} (u_\alpha|Au_\beta)\overline{i_\alpha} i_\beta$. Suppose first that A is symmetric and positive. The symmetry guarantees that the terms $(u_\alpha|Au_\beta)\overline{i_\alpha}i_\beta + (u_\beta|Au_\alpha)\overline{i_\beta}i_\alpha$ vanish if $\alpha \neq \beta$. It remains $(u|A_\mathbb{K}u)_\mathbb{K} = \sum_{\alpha=0}^{d_\mathbb{K}}(u_\alpha|Au_\alpha)$ which is clearly positive, A being positive. This proves the positivity of $A_\mathbb{K}$. On the contrary, suppose that $A_\mathbb{K}$ is positive and take $u = x \otimes 1$, then $0 \leq (u|A_\mathbb{K}u)_\mathbb{K} = (u|Au)$, which means $A \geq 0$. Take $u = x\otimes 1 + y \otimes i_1$, then $(u|A_\mathbb{K}u)_\mathbb{K} = (x|Ax) + (x|Ay)i_1 + (y|Ax)\overline{i_1} + (y|Ay)$. Since $A \geq 0$, in particular $(u|A_\mathbb{K}u)_\mathbb{K} \in \mathbb{R}$ and so $(x|Ay)i_1 + (y|Ax)\overline{i_1} = [(x|Ay) - (y|Ax)]i = 0$ which implies $(x|Ay) = (y|Ax) = (Ax|y)$, i.e. A is symmetric. □

Some results on the spectral properties of self-adjoint operators follow.

***Proposition* 3.1.5.** *Let A be a self-adjoint operator over H and let $P^{(A)}$ be the associated PVM, then the following statements hold:*

(a) the PVM of $A_\mathbb{K}$ is given by $P^{(A_\mathbb{K})}(E) = (P^{(A)}(E))_\mathbb{K}$ for all $E \in \mathfrak{B}(\mathbb{R})$;

(b) the spectrum of $A_\mathbb{K}$ satisfies $\sigma(A_\mathbb{K}) = \sigma(A)$, more precisely $\sigma_p(A_\mathbb{K}) = \sigma_p(A)$ and $\sigma_c(A_\mathbb{K}) = \sigma_c(A)$;

(c) if $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $f(A_\mathbb{K}) = f(A)_\mathbb{K}$.

Proof. First, notice that $P : E \to P(E)$ is a PVM over H if and only if $P_\mathbb{K} : E \to (P(E))_\mathbb{K}$ is a PVM over $\mathsf{H}_\mathbb{K}$. This can be checked by direct inspection. Now, take any $u = \sum_{\alpha=0}^{d_\mathbb{K}} u_\alpha \otimes i_\alpha \in \mathsf{H}_\mathbb{K}$, then an argument similar to the one carried out in proving point (vii) of Proposition (3.1.4) (notice that every projector $P(E)$ is symmetric) proves that

$$\mu_u^{(P_\mathbb{K})}(E) = (u|(P(E))_\mathbb{K} u)_\mathbb{K} = \sum_{\alpha=0}^{d_\mathbb{K}} (u_\alpha|P(E)u_\alpha) = \sum_{\alpha=0}^{d_\mathbb{K}} \mu_{u_\alpha}^{(P)}(E).$$

This proves that, given any measurable function $f : \mathbb{R} \to \mathbb{R}$, it holds that $u \in \Delta_f^{(P_\mathbb{K})}$ if and only if $u_\alpha \in \Delta_f^{(P)}$ for any $\alpha = 0, \ldots, d_\mathbb{K}$, i.e. $\Delta_f^{(P_\mathbb{K})} = (\Delta_f^{(P)})_\mathbb{K}$. In particular this applies to the identity function id. Take $u = \sum_{\alpha=0}^{d_\mathbb{K}} u_\alpha \otimes i_\alpha \in D(A_\mathbb{K})$ for some $u_\alpha \in D(A)$. The statement $u_\alpha \in D(A) = \Delta_{id}^{(P)}$ is equivalent to $id \in \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu_{u_\alpha}^{(P)})$. The discussion above implies $id \in \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu_u^{(P_\mathbb{K})})$. Finally, thanks again to the symmetry of A we have

$$(u|A_\mathbb{K} u)_\mathbb{K} = \sum_{\alpha=0}^{d_\mathbb{K}} (u_\alpha|Au_\alpha) = \sum_{\alpha=0}^{d_\mathbb{K}} \int_\mathbb{R} s\, d\mu_{u_\alpha}^{(P)}(s) = \int_\mathbb{R} s\, d\mu_u^{(P_\mathbb{K})}(s).$$

Applying Theorem 2.2.57 we conclude the proof of point (a). Now, let f be any measurable function as in (c), then we already know that $D(f(A_{\mathbb{K}})) = \Delta_f^{(P_{\mathbb{K}})} = (\Delta_f^{(P)})_{\mathbb{K}} = D(f(A)_{\mathbb{K}})$. So, if we consider any $u = \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha \in D(f(A_{\mathbb{K}}))$, it holds that

$$(u|f(A_{\mathbb{K}})u)_{\mathbb{K}} = \int_{\mathbb{R}} f(s)\, d\mu_u^{(P_{\mathbb{K}})} = \sum_{\alpha=0}^{d_{\mathbb{K}}} \int_{\mathbb{R}} f(s)\, d\mu_{u_\alpha}^{(P)} = \sum_{\alpha=0}^{d_{\mathbb{K}}} (u_\alpha|f(A)u_\alpha) = (u|f(A)_{\mathbb{K}}u)_{\mathbb{K}}.$$

Concluding, Lemma 2.2.22 together with the self-adjointness of both $f(A_{\mathbb{K}})$ and $f(A)_{\mathbb{K}}$ and the equality of their domains assures that $f(A_{\mathbb{K}}) = f(A)_{\mathbb{K}}$. The proof of (c) is complete. Now, recalling that for any self-adjoint operator B its spectrum is contained in the real line and satisfies $\sigma(B) = suppP^{(B)}$ (see Theorem 2.2.56) we easily see that $\sigma(A) = \sigma(A_{\mathbb{K}})$. Indeed this follows from the fact that $P(E) = 0$ if and only if $P(E)_{\mathbb{K}} = 0$ for any Borel set E. Finally, exploiting again Theorem 2.2.56 it is easy to see that the equalities hold true also for the special cases of point and continuous spectra. $\square$

3.1.2 Method II: Imaginary Operators

There exists another way to get a complex or quaternionic Hilbert space out of a real one which is based upon the existence of special operators, called *imaginary*.

***Definition* 3.1.6.** *Let H be a real, complex or quaternionic Hilbert space. An imaginary operator is an element $J \in \mathfrak{B}(\mathsf{H})$ such that $JJ = -I$ and $J^* = -J$.*

***Remark* 3.1.7.** The term *imaginary* is appropriate, for such operators act like imaginary units in $\mathbb{C}$ or $\mathbb{H}$:

$$ii = -1 \quad \text{and} \quad \bar{i} = -i.$$

In fact, this very feature will be the key to the construction of a complex or quaternionic linear space structure. It must be remarked that in the literature these objects are most often referred to as *complex structures* because they are seen as the generators of *complex* linear structures on given real linear spaces. However, since in this work we are going to exploit the properties of these operators to construct quaternionic and not merely complex linear structures, we will stick to the more general term and call them imaginary. We will then make a distinction, as follows.

Consider a real Hilbert space H, we then distinguish between two important cases:

(i) we say that H is equipped with a *complex structure* if it is coupled with one imaginary operator J_1;

(ii) we say that H is equipped with a *quaternionic structure* if it is coupled with two anti-commuting imaginary operators J_1, J_2. In this case $J_3 := -J_1J_2$ consists in another imaginary operator anti-commuting with the others.

Both cases can be collected in a single definition as follows.

***Definition* 3.1.8.** *Let $\mathbb{K}$ denote either $\mathbb{C}$ or $\mathbb{H}$. A real Hilbert space H is said to be equipped with a $\mathbb{K}$-structure if it is coupled with $d_{\mathbb{K}}$ imaginary operators $\{J_\alpha\}_{\alpha=1,\dots,d_{\mathbb{K}}}$ satisfying the above properties.*

***Notation* 3.1.9.** Conventionally, we define $J_0 = I$. In this way the $\mathbb{K}$-structure can be taken as $\{J_\alpha\}_{\alpha=0,\dots,d_{\mathbb{K}}}$, a notation that turns out to be useful in the following. A set of operators like this is denoted by $\mathfrak{J}$.

By means of such operators we can endow H with a particular structure of complex or quaternionic Hilbert space without the need of extending the set of vectors. We can define a (right) multiplication by elements of $\mathbb{K}$ in the following way.

$$R_q := \sum_{\alpha=0}^{d_{\mathbb{K}}} q_\alpha J_\alpha \in \mathfrak{B}(\mathsf{H}) \quad \text{if} \quad q = \sum_{\alpha=0}^{d_{\mathbb{K}}} q_\alpha i_\alpha \in \mathbb{K}.$$

It is immediate to see that $R_1 = I$, $R_{q+p} = R_q + R_p$, $R_{qp} = R_p R_q$ for any $p, q \in \mathbb{K}$ and $R_{ck} = cR_k$ for any $k \in \mathbb{K}$ and $c \in \mathbb{R}$. This proves that the function

$$\mathsf{H} \times \mathbb{K} \ni (x, k) \mapsto R_k x \in \mathsf{H} \tag{3.7}$$

define a right multiplication by elements of $\mathbb{K}$ on H and makes it a $\mathbb{K}$-linear space, denoted by $\mathsf{H}_{\mathfrak{J}}$.

***Remark* 3.1.10.** The choice $J_3 := -J_1 J_2$ is necessary in order to make R a *right* scalar multiplication: more precisely it guarantees that $R_q R_p = R_{pq}$. With the choice $J_3 := +J_1 J_2$, the resulting function R would satisfy $R_q R_p = R_{qp}$, thus it would give rise to a left - not right - scalar multiplication on the vectors.

At this point, in order to produce a Hilbert space, we introduce the following function

$$(x|y)_{\mathfrak{J}} := \sum_{\alpha=0}^{d_{\mathbb{K}}} (x|J_\alpha y)\overline{i_\alpha}. \tag{3.8}$$

***Lemma* 3.1.11.** *The function (3.8) is a Hermitean scalar product on $\mathsf{H}_{\mathfrak{J}}$.*

Proof. This function is clearly real linear and additive in both the entries. Let us show that it is $\mathbb{K}$-linear in the right-hand entry. We show the case $\mathbb{K} = \mathbb{H}$, the complex one being even easier. So, take for example $\alpha = 1$, then

$$\begin{aligned}(x|yi_1)_{\mathfrak{J}} = (x|J_1 y)_{\mathfrak{J}} &= (x|J_1 y) + \overline{i_1}(x| - y) + \overline{i_2}(x|J_3 y) + \overline{i_3}(x| - J_2 y) = \\ &= (x|J_1 y) + i_1(x|y) - \overline{i_3}\,\overline{i_1}(x|J_3 y) - \overline{i_2}\,\overline{i_1}(x|J_2 y) = \\ &= [(x|y) + \overline{i_1}(x|J_1 y) + \overline{i_2}(x|J_2 y) + \overline{i_3}(x|J_3 y)]i_1 = (x|y)_{\mathfrak{J}} i_1.\end{aligned} \tag{3.9}$$

The others are equivalent. The Hermiticity can be proved as follows

$$(y|x)_{\mathfrak{J}} = (y|x) + \sum_{\alpha=1}^{d_{\mathbb{K}}} \overline{i_\alpha}(y|J_\alpha x) = (x|y) + \sum_{\alpha=1}^{d_{\mathbb{K}}} \overline{i_\alpha}(-J_\alpha y|x) = (x|y) + \sum_{\alpha=1}^{d_{\mathbb{K}}} i_\alpha(x|J_\alpha y) = \\ = \overline{(x|y)_{\mathfrak{J}}}.$$

It remains to prove the strict positivity of $(\cdot|\cdot)_{\mathfrak{J}}$. So, take $x \in \mathsf{H}$ and notice that for any $\alpha = 1, \ldots, d_{\mathbb{K}}$ it holds that $(x|J_\alpha x) = (-J_\alpha x|x) = -(x|J_\alpha x)$, which means $(x|J_\alpha x) = 0$. Thus $(x|x)_{\mathfrak{J}} = (x|x)$, which concludes the proof. □

***Remark* 3.1.12.** As a byproduct of the proof of the previous lemma we see that $(x|J_\alpha x) = 0$ for any $1 \leq \alpha \leq d_{\mathbb{K}}$ and $x \in \mathsf{H}$.

It is straightforward to see that

$$x \perp_{\mathfrak{J}} y \quad \text{if and only if} \quad x \perp J_\alpha y \quad \forall \alpha = 0, .., d_{\mathbb{K}} \tag{3.10}$$

and

$$(x|y) = \Re[(x|y)_{\mathfrak{J}}] \quad \forall x, y \in \mathsf{H}, \tag{3.11}$$

which immediately implies that the norm generated by this Hermitean scalar product satisfies

$$\|x\|_{\mathfrak{J}} = \|x\| \quad \forall x \in \mathsf{H}. \tag{3.12}$$

The space $\mathsf{H}_{\mathfrak{J}}$ equipped with this structure is then a pre-Hilbert space over the division algebra $\mathbb{K}$. In fact it is also complete with respect to its topology, which, by means of (3.12) coincides with the one already exististing.

***Proposition* 3.1.13.** *Let H be a real Hilbert space with a $\mathbb{K}$-structure $\mathfrak{J}$, then the following statements hold:*

(a) the space $\mathsf{H}_{\mathfrak{J}}$ equipped with (3.8) is a $\mathbb{K}$-Hilbert space

(b) $\mathsf{N} \subset \mathsf{H}_{\mathfrak{J}}$ is a Hilbert basis; for $\mathsf{H}_{\mathfrak{J}}$ if and only if $\{J_\alpha z \mid 0 \leq \alpha \leq d_{\mathbb{K}}, z \in \mathsf{N}\}$ is a Hilbert basis for H, in particular $\mathsf{H}_{\mathfrak{J}}$ is separable if and only if H is so;

(c) a subspace $\mathsf{K} \subset \mathsf{H}$ is a subspace of $\mathsf{H}_{\mathfrak{J}}$ if and only if $J_\alpha(\mathsf{K}) \subset \mathsf{K}$ for all $1 \leq \alpha \leq d_{\mathbb{K}}$ and $\overline{\mathsf{K}}^{\mathsf{H}} = \overline{\mathsf{K}}^{\mathsf{H}_{\mathfrak{J}}}$; in particular K is closed or dense in H if and only if it is, respectively, closed or dense in $\mathsf{H}_{\mathfrak{J}}$.

Proof. Thanks to (3.12) Cauchy sequences in $\mathsf{H}_{\mathfrak{J}}$ define Cauchy sequences in H. For this reason $\mathsf{H}_{\mathfrak{J}}$ is complete in view of the completeness of H, giving point (a). Point (b) is true because a set N is maximally orthonormal in H if and only if $\{J_\alpha z\}$ is maximally orthonormal in $\mathsf{H}_{\mathfrak{J}}$ as follows from Remark (3.1.12) and Remark (3.10). Let us conclude the proof by proving (c). Of course for a subspace of H to be a subspace of $\mathsf{H}_{\mathfrak{J}}$, it is necessary and sufficient that it is closed under the multiplication by elements of $\mathbb{K}$. In other words that $ui_\alpha := J_\alpha u \in \mathsf{K}$ if $u \in \mathsf{K}$ for all $1 \leq \alpha \leq d_{\mathbb{K}}$. This is clearly equivalent to $J_\alpha(\mathsf{K}) \subset \mathsf{K}$ for all $1 \leq \alpha \leq d_{\mathbb{K}}$. Finally, the closures of such a subspace with respect to H or $\mathsf{H}_{\mathfrak{J}}$, respectively, are equal to each other, the topologies (norms) being the same. □

Notice that the identity function

$$I : \mathsf{H} \ni x \mapsto x \in \mathsf{H}_{\mathfrak{J}}$$

is evidently an isometry of metric spaces. In particular H and $\mathsf{H}_{\mathfrak{J}}$ are homeomorphic.

***Proposition* 3.1.14.** *Let H_0 be a complex or quaternionic Hilbert space, then it can always be written as $\mathsf{H}_{\mathfrak{J}}$ for some structure $\mathfrak{J}$.*

Proof. Let $(\cdot|\cdot)_0$ be the Hermitean scalar product of H_0. Consider the set $\mathsf{H} = \mathsf{H}_0$ and equip it with the $\mathbb{R}$-linear structure induced by the original one (just restrict the attention on $\mathbb{R} \subset \mathbb{F}$) and define the real bilinear map $(\cdot|\cdot) := \Re[(\cdot|\cdot)_0]$. As shown in the next section $(\mathsf{H}, (\cdot|\cdot))$ is a real Hilbert space. To conclude define the real linear operators $J_\alpha : \mathsf{H} \ni x \mapsto x i_\alpha \in \mathsf{H}$ for all $\alpha = 1, \ldots, d_{\mathbb{F}}$. These turn out to define a $\mathbb{F}$-structure $\mathfrak{J}$ on H and more precisely $\mathsf{H}_0 = \mathsf{H}_{\mathfrak{J}}$. □

***Remark* 3.1.15.** If a real Hilbert space is finite-dimensional and its dimension is odd, there are no imaginary operators in $\mathfrak{B}(\mathsf{H})$, otherwise we would obtain a contradiction from (b) in Proposition 3.1.13, and no internal division algebra extension procedure is possible. The reader may easily prove that this is the only obstruction: if the dimension of H is infinite or finite and even, an imaginary operator always exists associated with every given Hilbert basis of H.

Now, consider a real Hilbert space H with $\mathbb{K}$-structure $\mathfrak{J}$ and let $B : D(B) \to \mathsf{H}_{\mathfrak{J}}$ be a linear operator defined over $\mathsf{H}_{\mathfrak{J}}$. It is evident that B is also a real linear operator if understood as a function over H. The converse is generally false. The following proposition concerns this issue.

***Proposition* 3.1.16.** *The following statements hold:*

(a) a real linear operator $A : D(A) \to \mathsf{H}$ is linear on $\mathsf{H}_{\mathfrak{J}}$ if and only if $J_\alpha A \subset A J_\alpha$ for any $1 \leq \alpha \leq d_{\mathbb{K}}$: in this case $D(A)$ is a linear subspace of $\mathsf{H}_{\mathfrak{J}}$ as well;

(b) let $A : D(A) \to \mathsf{H}$ be a real linear operator on H such that $J_\alpha A \subset A J_\alpha$ for any $1 \leq \alpha \leq d_{\mathbb{K}}$, then

(i) $A \in \mathfrak{B}(\mathsf{H})$ if and only if $A \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$, more precisely $\|A\| = \|A\|_{\mathfrak{J}}$;

(ii) if $D(A)$ is dense, then the adjoint A^* of A defined with respect to H and the one defined with respect to $\mathsf{H}_{\mathfrak{J}}$ coincide;

(iii) A is either closed or closable with respect to H if and only if it is, respectively, closed or closable with respect to $\mathsf{H}_{\mathfrak{J}}$: in this case the closures coincide;

(iv) *let* $\mathsf{S} \subset D(A)$ *be a subspace such that* $J_\alpha(\mathsf{S}) \subset \mathsf{S}$ *for any* $1 \leq \alpha \leq d_{\mathbb{K}}$*, then* S *is a core for* A *referring to* H *if and only if it is a core for* A *referring to* $\mathsf{H}_{\mathfrak{J}}$*;*

(v) A *is symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent on* H *if and only if it is, respectively, symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent on* $\mathsf{H}_{\mathfrak{J}}$*;*

(vi) A *is symmetric and positive on* H *if and only if it is positive on* $\mathsf{H}_{\mathfrak{J}}$*.*

Proof. Let us start with (a). Suppose that A is linear on H. A sufficient and necessary condition for A to be linear also on $\mathsf{H}_{\mathfrak{J}}$ is that $D(A)$ is closed under the multiplication by the i_αs, i.e. $ui_\alpha = J_\alpha u \in D(A)$, and that $AJ_\alpha u = A(ui_\alpha) = (Au)i_\alpha = J_\alpha Au$ for every $\alpha = 1, \ldots, d_{\mathbb{K}}$ and $u \in D(A)$. All this can be condensed into $J_\alpha A \subset AJ_\alpha$ for every $\alpha = 1, \ldots, d_{\mathbb{K}}$, concluding point (a). Let us pass to point (b) and let A be as in the hypothesis, then it gives rise to a linear operator over $\mathsf{H}_{\mathfrak{J}}$ thanks to point (a). Point (i) immediately follows from the equality of H and $\mathsf{H}_{\mathfrak{J}}$ as sets and (3.12). Suppose now that $D(A)$ is dense on H, then Proposition 3.1.13 assures that $D(A)$ is dense also within $\mathsf{H}_{\mathfrak{J}}$ and so we can take the adjoint of A both on H and $\mathsf{H}_{\mathfrak{J}}$, which will be distinguished using two different notations: A^* and A^{*_J}. Exploiting the definition of the adjoint operator we have that $x \in D(A^*)$ on H if and only if there exists $u_x \in \mathsf{H}$ such that $(x|A\omega) = (u_x|\omega)$ for all $\omega \in D(A)$: in this case $A^*x = u_x$. Now, bearing in mind that $J_\alpha A \subset AJ_\alpha$ for every $\alpha = 1, \ldots, d_{\mathbb{K}}$ we get

$$(x|A\omega)_{\mathfrak{J}} = \sum_{\alpha=0}^{d_{\mathbb{K}}} \overline{i_\alpha}(x|J_\alpha A\omega) = \sum_{\alpha=0}^{d_{\mathbb{K}}} \overline{i_\alpha}(x|AJ_\alpha\omega) = \sum_{\alpha=0}^{d_{\mathbb{K}}} \overline{i_\alpha}(A^*x|J_\alpha\omega) = (A^*x|\omega)_{\mathfrak{J}}. \quad (3.13)$$

Since this holds for any $\omega \in D(A)$ we see that $x \in D(A^{*_J})$ and $A^{*_J}x = A^*x$. Now, take $x \in D(A^{*_J})$, then $(x|A\omega)_{\mathfrak{J}} = (A^{*_J}x|\omega)_{\mathfrak{J}}$ for all $\omega \in D(A)$. Taking the real part on both the sides of the last identity and using (3.11) we get $(x|A\omega) = (A^{*_J}x|\omega)$ for all $\omega \in D(A)$. This assures that $x \in D(A^*)$, concluding the proof of (ii). Points (iii) and (iv) are easy consequences of (3.12). Point (v) immediately follows from the involved definitions and points (ii) and (iii). To conclude let us prove (vi). So, suppose that A is symmetric and positive on H, then the symmetry of A implies $(x|J_\alpha Ax) = (x|AJ_\alpha x) = (-J_\alpha Ax|x) = -(x|J_\alpha Ax)$ for any $1 \leq \alpha \leq d_{\mathbb{K}}$ which yields $(x|J_\alpha Ax) = 0$, i.e. $(x|Ax)_{\mathfrak{J}} = (x|Ax)$. The positivity of A on H concludes this direction of the equivalence, showing that A is also positive on $\mathsf{H}_{\mathfrak{J}}$. To prove the opposite take again $x \in D(A)$ and suppose that $(x|Ax)_{\mathfrak{J}} \geq 0$. In particular $(x|Ax) \in \mathbb{R}$ and so, exploiting the definition of $(\cdot|\cdot)_{\mathfrak{J}}$ it must be $(x|Ax)_{\mathfrak{J}} = (x|Ax)$ which has two consequences. First of all that $(x|Ax) \geq 0$, giving the positivity of A on H. Secondly, it gives $(x|J_\alpha Ax) = 0$ for any $1 \leq \alpha \leq d_{\mathbb{K}}$, which implies the symmetry of A on H. Indeed for any $x, y \in D(A)$

we have $x + J_\alpha y \in D(A)$ and

$$0 = (x + J_\alpha y | J_\alpha A(x + J_\alpha y)) = (x + J_\alpha y | J_\alpha Ax - Ay) = (x|J_\alpha Ax) - (x|Ay) + \\ + (J_\alpha y|J_\alpha Ax) - (J_\alpha y|Ay) = -(x|Ay) + (y|Ax) = -(x|Ay) + (Ax|y), \tag{3.14}$$

which gives the symmetry of A on H. □

As a byproduct of the previous proof we have the following

***Lemma* 3.1.17.** *Let A be a symmetric operator over H such that $AJ_\alpha \subset J_\alpha A$ for any $1 \le \alpha \le d_{\mathbb{K}}$, then $(u|Au)_{\mathfrak{J}} = (u|Au)$ for any $u \in D(A)$. More precisely $(u|J_\alpha Au) = 0$ for any $u \in D(A)$ and $1 \le \alpha \le d_{\mathbb{K}}$.*

Some results about the spectral properties of self-adjoint operators follow.

***Proposition* 3.1.18.** *Let A be self-adjoint operator over H such that $J_\alpha A \subset AJ_\alpha$ for any $1 \le \alpha \le d_{\mathbb{K}}$ and let $P^{(A)}$ be the associated PVM. The following statements hold:*

(a) the PVM of A referring to $\mathsf{H}_{\mathfrak{J}}$ is still given by $P^{(A)}$;

(b) the (point, continuous) spectrum of A referring to H coincides with the (point, continuous) spectrum of A referring to H;

(c) if $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $J_\alpha f(A) \subset f(A)J_\alpha$ for all $1 \le \alpha \le d_{\mathbb{K}}$ and the definition of $f(A)$ referred to H coincides with the one referring to $\mathsf{H}_{\mathfrak{J}}$.

Proof. Let A be as in the hypothesis, then it can be interpreted as a linear operator also on $\mathsf{H}_{\mathfrak{J}}$. Let $P : E \mapsto P(E)$ and $P_{\mathfrak{J}} : E \mapsto P_{\mathfrak{J}}(E)$ be the PVMs of A when understood as a linear operator over H or $\mathsf{H}_{\mathfrak{J}}$, respectively. We want to prove that $P = P_{\mathfrak{J}}$. The proof is based on Theorem 2.2.57. Notice that $P_{\mathfrak{J}}$ is still a PVM if understood on H. We will denote by $\mu^{(P_{\mathfrak{J}})}$ and $\nu^{(P_{\mathfrak{J}})}$ the usual finite measure associated with $P_{\mathfrak{J}}$ when referred as a PVM on H or $\mathsf{H}_{\mathfrak{J}}$, respectively. So, take any $u \in \mathsf{H}$, then exploiting the self-adjointness of $P_{\mathfrak{J}}(E)$ and Lemma 3.1.17 we have $\nu_u^{(P_{\mathfrak{J}})}(E) = (u|P_{\mathfrak{J}}(E)u)_{\mathfrak{J}} = (u|P_{\mathfrak{J}}(E)u) = \mu_u^{(P_{\mathfrak{J}})}(E)$ for every Borelian E of the real line. Now, focus on $u \in D(A)$. Interpreting A as an operator on $\mathsf{H}_{\mathfrak{J}}$ this is equivalent to $id \in \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \nu_u^{(P_{\mathfrak{J}})})$. Using the equality $\nu_u^{(P_{\mathfrak{J}})} = \mu_u^{(P_{\mathfrak{J}})}$ just proved we get $\mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu_u^{(P_{\mathfrak{J}})}) = \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \nu_u^{(P_{\mathfrak{J}})}) \ni id$. Finally, we have

$$\int_{\mathbb{R}} s\, d\mu_u^{(P_{\mathfrak{J}})} = \int_{\mathbb{R}} s\, d\nu_u^{(P_{\mathfrak{J}})} = (u|Au)_{\mathfrak{J}} = (u|Au), \tag{3.15}$$

thanks again to Lemma 3.1.17. At this point, since the same properties are satisfied by P - the PVM of A on H - Theorem 2.2.57 assures that $P = P_{\mathfrak{J}}$. This gives point (a). Point (b) is a direct consequence of Point (a) and Theorem 2.2.56. Let us prove Point (c). Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function, then exploiting again $\mu_u^{(P)} = \nu_u^{(P)}$, we immediately see that the definition of $\Delta_f^{(P)}$ does not depend on the Hilbert space on which it is defined. Let us denote by $f(A)_{\mathfrak{J}}$ the (self-adjoint) integration of f

with respect to P on $\mathsf{H}_{\mathfrak{J}}$ and by $f(A)$ the one on H. Of course $f(A)_{\mathfrak{J}}$ is also a well defined linear (even self-adjoint thanks to Proposition 3.1.16) operator over H. Take any $u \in D(f(A)_{\mathfrak{J}}) = \Delta_f^{(P)} = D(f(A))$, then, since $f(A)_{\mathfrak{J}}$ is self-adjoint, it holds that

$$(u|f(A)_{\mathfrak{J}}u) = (u|f(A)_{\mathfrak{J}}u)_{\mathfrak{J}} = \int_{\mathbb{R}} f(s)\, d\nu_u^{(P)} = \int_{\mathbb{R}} f(s)\, d\mu_u^{(P)} = (u|f(A)u).$$

This gives $(u|[f(A)_{\mathfrak{J}} - f(A)]u) = 0$. Since $f(A)_{\mathfrak{J}} - f(A)$ is symmetric on $D(f(A)) = D(f(A)_{\mathfrak{J}})$, Lemma 3.1.12 assures that $f(A)_{\mathfrak{J}} = f(A)$, concluding the proof. □

***Notation* 3.1.19.** Let $A : D(A) \to \mathsf{H}$ be a linear operator such that $J_\alpha A \subset AJ_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}}$, then we will denote the operator A by $A_{\mathfrak{J}}$ when understood as a linear operator over $\mathsf{H}_{\mathfrak{J}}$. In this way we make the notation coherent with the one developed in the next section.

3.2 Reducing the Scalars of Quaternionic Hilbert Spaces

In the previous section we extended the division algebra of a real Hilbert space in order to obtain a complex or quaternionic one out of it. Now, we discuss an opposite procedure, in that we show how to obtain a real or complex Hilbert space out of a quaternionic one. Again, there exist two different procedures, analogous to the two discussed in the previous sections. Despite the analogies the procedures are pretty different. Again we will refer to $\{i_\alpha\}_{\alpha=0,\dots,d_{\mathbb{K}}}$ as a standard basis of $\mathbb{H}$.

3.2.1 Method I: Canonical Procedure

Let H be a quaternionic Hilbert space with quaternionic Hermitean scalar product $(\cdot|\cdot)$. It is even simpler in this case to get a real or complex Hilbert space out of H.

Focus on the imaginary unit i_1 (this choice is arbitrary, any one of the imaginary units i_α would be fine), then, as already seen in Chapter 2, the subset

$$\mathbb{C}_{i_1} := \{a + bi_1 \mid a, b \in \mathbb{R}\} \subset \mathbb{H}$$

is isomorphic to the division algebra $\mathbb{C}$ of complex numbers. So, by fixing the unit i_1 we can always read the complexes as a sub-algebra of the quaternions. In an even more obvious way we can read the real numbers as a sub-algebra of $\mathbb{H}$. In the following the symbol $\mathbb{K}$ will denote $\mathbb{R}$ or $\mathbb{C}_{i_1}$, indifferently. We will neglect the subscript i_1, if there no risk of confusion.

Consider any quaternion $z \in \mathbb{H}$, an immediate check shows that

$$z = \sum_{\alpha=0}^{3} \Re(zi_\alpha)\overline{i_\alpha}.$$

In particular, this can be applied to the general value attained by the scalar product.

$$(x|y) = \sum_{\alpha=0}^{3} [\Re[(x|y)i_\alpha]\overline{i_\alpha} = \sum_{\alpha=0}^{3} [\Re[(x|yi_\alpha)]]\overline{i_\alpha}. \tag{3.16}$$

Now, notice that if we restrict our attention to the subalgebra $\mathbb{K}$, then H can be easily understood as a $\mathbb{K}$-linear space, which will be denoted by $\mathsf{H}_\mathbb{K}$. The same holds for any subspace $\mathsf{K} \subset \mathsf{H}$. In order to make it a Hilbert space, we need to introduce on $\mathsf{H}_\mathbb{K}$ a Hermitian scalar product and the most natural choice consists in cutting off the sum (3.16) at degree $d_\mathbb{K}$:

$$(x|y)_\mathbb{K} := \sum_{\alpha=0}^{d_\mathbb{K}} [\Re(x|yi_\alpha)]\overline{i_\alpha} \in \mathbb{K}. \tag{3.17}$$

***Lemma* 3.2.1.** *The function (3.17) is a Hermitean scalar product on* $\mathsf{H}_\mathbb{K}$.

Proof. Suppose first that $\mathbb{K} = \mathbb{R}$, then $(x|y)_\mathbb{R} = \Re[(x|y)]$. This is clearly real bilinear and symmetric. The strict positivity is a direct consequence of $(u|u)_\mathbb{R} = \Re[(u|u)] = (u|u)$. Now, suppose $\mathbb{K} = \mathbb{C}_{i_1}$. Then $(x|yi_1)_\mathbb{C} = \Re(x|yi_1) - \Re(x|yi_1i_1)i_1 = [\Re(x|yi_1)\overline{i_1} + \Re(x|y)]i_1 = (x|y)_\mathbb{C} i_1$, which gives the right-linearity of $(\cdot|\cdot)_\mathbb{C}$. Concerning Hermiticity, we have

$$\begin{aligned}(y|x)_\mathbb{C} &= \Re(y|x) - [\Re(y|xi_1)]i_1 = \Re\overline{(y|x)} - [\Re\overline{(y|xi_1)}]i_1 = \Re(x|y) - [\Re(-i_1(x|y))]i_1 = \\ &= \Re(x|y) + \Re((x|y)i_1)i_1 = \Re(x|y) - \Re((x|y)i_1)\overline{i_1} = \overline{(x|y)_\mathbb{C}}.\end{aligned} \tag{3.18}$$

Finally, notice that $\Re((u|u)i_1) = 0$ as a consequence of $(u|u) \in \mathbb{R}$ and so $(u|u)_\mathbb{C} = \Re(u|u) = (u|u)$, which concludes the proof. □

***Notation* 3.2.2.** In the following we make use of the following sets: $\mathcal{A}_\mathbb{R} := \{0, 1, 2, 3\}$ and $\mathcal{A}_\mathbb{C} := \{0, 3\}$.

A direct consequence of the definition of the scalar product (3.16) and (3.16) is the following:

$$(x|y) = \sum_{\alpha \in \mathcal{A}_\mathbb{K}} (x|yi_\alpha)_\mathbb{K} \overline{i_\alpha} \tag{3.19}$$

which implies that

$$x \perp y \quad \text{if and only if} \quad x \perp_\mathbb{K} yi_\alpha \quad \text{for all } \alpha \in \mathcal{A}_\mathbb{K}. \tag{3.20}$$

Finally, notice that the norms of H and $\mathsf{H}_\mathbb{K}$ coincide:

$$\|x\|_\mathbb{K} = \|x\| \qquad \text{for all } x \in \mathsf{H}, \tag{3.21}$$

which immediately follows from $\Re((x|xi_\alpha)) = \Re((x|x)i_\alpha) = 0$ for $1 \leq \alpha \leq d_{\mathbb{K}}$. Thanks to this, the map

$$\mathsf{H} \ni x \mapsto x \in \mathsf{H}_{\mathbb{K}}$$

defines an isometry of metric spaces.

At this point it is useful to introduce an important set of functions: take $\alpha \in \{0,1,2,3\}$ and define

$$\mathcal{J}_\alpha : \mathsf{H} \ni x \mapsto xi_\alpha \in \mathsf{H}. \tag{3.22}$$

Exploiting these definitions, expression (3.19) can be restated as

$$(x|y) = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} (x|\mathcal{J}_\alpha y)_{\mathbb{K}} \overline{i_\alpha}, \tag{3.23}$$

while expression (3.20) becomes

$$x \perp y \quad \text{if and only if} \quad x \perp_{\mathbb{K}} \mathcal{J}_\alpha y \quad \text{for all } \alpha \in \mathcal{A}_{\mathbb{K}}. \tag{3.24}$$

We are ready to state some basic properties of the space $\mathsf{H}_{\mathbb{K}}$, which we already know to be a pre-Hilbert space over $\mathbb{K}$ with respect to the Hermitean scalar product (3.17).

***Proposition* 3.2.3.** *Let H be a quaternionic Hilbert space, then the following statements hold:*

(a) the linear space $\mathsf{H}_{\mathbb{K}}$ equipped with (3.17) is a $\mathbb{K}$-Hilbert space;

(b) $\mathsf{N} \subset \mathsf{H}$ is a Hilbert basis for H if and only if $\{\mathcal{J}_\alpha z \mid z \in \mathsf{N}, \alpha \in \mathcal{A}_{\mathbb{K}}\}$ is a Hilbert basis for $\mathsf{H}_{\mathbb{K}}$;

(c) a subspace $\mathsf{K} \subset \mathsf{H}_{\mathbb{K}}$ is a subspace of H if and only if $\mathcal{J}_\alpha(\mathsf{K}) \subset \mathsf{K}$ for any non-null $\alpha \in \mathcal{A}_{\mathbb{K}}$ and $\overline{\mathsf{K}}^{\mathsf{H}_{\mathbb{K}}} = \overline{\mathsf{K}}^{\mathsf{H}}$: in particular K is closed or dense in $\mathsf{H}_{\mathbb{K}}$ if and only if it is, respectively, closed or dense in H.

Proof. The completeness of $\mathsf{H}_{\mathbb{K}}$ with respect to its natural topology follows immediately from the completeness of H and (3.21). This proves point (a). Let us pass to point (b). Exploiting point (g), Proposition 3.2.4 and (3.24) it is clear that $\mathsf{N} \subset \mathsf{H}$ is orthonormal and maximal in H if and only if $\{\mathcal{J}_\alpha z\}$ is orthonormal and maximal in $\mathsf{H}_{\mathbb{K}}$. Finally, let K be a subspace of $\mathsf{H}_{\mathbb{K}}$. In order for K to be a subspace of H it is sufficient and necessary that $ui_\alpha = \mathcal{J}_\alpha u \in \mathsf{K}$ for any non-null $\alpha \in \mathcal{A}_{\mathbb{K}}$. This is equivalent to requiring $\mathcal{J}_\alpha(\mathsf{H}) \subset \mathsf{K}$ for any non-null $\alpha \in \mathcal{A}_{\mathbb{K}}$. The part concerning the closure follows trivially by (3.21). $\square$

Referring to the Hilbert space structure of $\mathsf{H}_{\mathbb{K}}$ we can see some basic features of the operators $\mathcal{J}_\alpha$. First notice that they give rise to linear functions on $\mathsf{H}_{\mathbb{R}}$.

***Lemma* 3.2.4.** *The following statements are true:*

(a) $\mathcal{J}_\alpha \in \mathfrak{B}(\mathsf{H}_{\mathbb{R}})$ for any $0 \leq \alpha \leq 3$,

(b) $\mathcal{J}_\alpha\mathcal{J}_\beta = -\mathcal{J}_\beta\mathcal{J}_\alpha$ *for any* $1 \leq \alpha, \beta \leq 3$ *such that* $\alpha \neq \beta$*,*

(c) $(\mathcal{J}_\alpha)^2 = -\mathcal{J}_0 = -I$ *for all* $1 \leq \alpha \leq 3$*,*

(d) $\mathcal{J}_1\mathcal{J}_2 = -\mathcal{J}_3$*,*

(e) $(\mathcal{J}_\alpha)^* = -\mathcal{J}_\alpha$ *on* $\mathsf{H}_\mathbb{R}$ *for any* $1 \leq \alpha \leq 3$*,*

(f) $(u|\mathcal{J}_\alpha v)_\mathbb{K} = -\overline{(\mathcal{J}_\alpha u|v)_\mathbb{K}}$ *for any* $u, v \in \mathsf{H}_\mathbb{K}$ *and non-null* $\alpha \in \mathcal{A}_\mathbb{K}$*,*

(g) $(u|\mathcal{J}_\alpha u)_\mathbb{K} = 0$ *for all* $u \in \mathsf{H}_\mathbb{K}$ *and non-null* $\alpha \in \mathcal{A}_\mathbb{K}$*,*

(h) $\mathcal{J}_3$ *is an anti-unitary operator on* $\mathsf{H}_\mathbb{C}$*.*

Proof. The operators $\mathcal{J}_\alpha$ are clearly well-defined linear operators over $\mathsf{H}_\mathbb{R}$. Properties (a)-(d) can be verified with little effort. Let us prove (f). Let $\alpha \neq 0$ belong to $\mathcal{A}_\mathbb{K}$ and $u, v \in \mathsf{H}$, then it holds that

$$\Re[(u|vi_\alpha)] = \Re[(u|v)i_\alpha] = \Re[i_\alpha(u|v)] = -\Re[\overline{i_\alpha}(u|v)] = -\Re[(ui_\alpha|v)].$$

If $\mathbb{K} = \mathbb{R}$ then point (f) follows immediately from this identity. So, suppose that $\mathbb{K} = \mathbb{C}$, then

$$\begin{aligned}(u|\mathcal{J}_3 v)_\mathbb{C} = (u|vi_3)_\mathbb{C} = \Re[(u|vi_3)] + \Re[(u|(vi_3)i_1)]\overline{i_1} = -\Re[(ui_3|v)] + \Re[(u|-vi_1i_3)]\overline{i_1} = \\ = -\Re[(ui_3|v)] + \Re[(ui_3|vi_1)]\overline{i_1} = -\overline{(ui_3|v)_\mathbb{K}} = -\overline{(\mathcal{J}_3 u|v)_\mathbb{K}}.\end{aligned} \tag{3.25}$$

Point (f), together with the linearity of $\mathcal{J}_\alpha$ on $\mathsf{H}_\mathbb{R}$ gives point (e). Point (g) follows immediately as a special case of point (f). Finally, point (h) follows by noticing that $\mathcal{J}_3(ui_1) = ui_1i_3 = -ui_3i_1 = ui_3\overline{i_1} = (\mathcal{J}_3 u)\overline{i_1}$ which holds for every $u \in \mathsf{H}_\mathbb{C}$. □

Now, consider a quaternionic linear operator $B : D(B) \to \mathsf{H}$. This is clearly also linear with respect to the $\mathbb{K}$-linear structure defined above. The following result discusses the opposite relation.

Proposition 3.2.5. *The following statements hold:*

(a) a $\mathbb{K}$*-linear operator* $A : D(A) \to \mathsf{H}_\mathbb{K}$ *is quaternionic linear over* H *if and only if* $\mathcal{J}_\alpha A \subset A\mathcal{J}_\alpha$ *for any non-null* $\alpha \in \mathcal{A}_\mathbb{K}$*: in this case* $D(A)$ *is a linear subspace of* H *as well;*

(b) let $A : D(A) \to \mathsf{H}$ *be a* $\mathbb{K}$*-linear operator such that* $\mathcal{J}_\alpha A \subset A\mathcal{J}_\alpha$ *for any non-null* $\alpha \in \mathcal{A}_\mathbb{K}$*, then*

  (i) $A \in \mathfrak{B}(\mathsf{H})$ *if and only if* $A \in \mathfrak{B}(\mathsf{H}_\mathbb{K})$*, more precisely* $\|A\| = \|A\|_\mathbb{K}$*;*

  (ii) if $D(A)$ *is dense, then the adjoint* A^* *of* A *defined with respect to* H *and the one defined with respect to* $\mathsf{H}_\mathbb{K}$ *coincide;*

(iii) *A is either closed or closable with respect to $\mathsf{H}_{\mathbb{K}}$ if and only if it is, respectively, closed or closable with respect to H: in this case the closures coincide;*

(iv) *let $\mathsf{S} \subset D(A)$ be a subspace such that $\mathcal{J}_\alpha(\mathsf{S}) \subset \mathsf{S}$ for any non-null $\alpha \in \mathcal{A}_{\mathbb{K}}$, then S is a core for A referring to $\mathsf{H}_{\mathbb{K}}$ if and only if it is a core for A referring to H;*

(v) *A is symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent on $\mathsf{H}_{\mathbb{K}}$ if and only if it is, respectively, symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent on H;*

(vi) *A is positive on H if and only if it is symmetric and positive on $\mathsf{H}_{\mathbb{K}}$.*

Proof. Let us start with point (a). Suppose that A is a linear operator over $\mathsf{H}_{\mathbb{K}}$ then, in order for it to be linear also on H, it is necessary and sufficient that $D(A)$ is a linear subspace of H and that $A(ui_\alpha) = (Au)i_\alpha$ for any non-null $\alpha \in \mathcal{A}_{\mathbb{K}}$ (for it is already linear with respect to i_α with $0 \leq \alpha \leq d_{\mathbb{K}}$). Exploiting the definition of $\mathcal{J}_\alpha$ and Proposition 3.2.3 it is easy to see that this is equivalent to $\mathcal{J}_\alpha A \subset A\mathcal{J}_\alpha$ for any non-null $\alpha \in \mathcal{A}_{\mathbb{K}}$. Let us pass to point (b) and take A as in the hypotheses, which thanks to point (a) can be understood as a linear operator over H as well. Point (i) follows immediately from $\mathsf{H} = \mathsf{H}_{\mathbb{K}}$ and (3.21). Let us prove (ii). Thanks to Proposition 3.2.3 $D(A)$ is dense on $\mathsf{H}_{\mathbb{K}}$ if and only if it is dense on H, hence we can consider the adjoints of A on both spaces. In the following proof we will distinguish between the two definitions by denoting them as $A^{*_{\mathbb{K}}}$ and A^*, respectively. So, take $u \in D(A^*)$: this means that there exists a vector $v_u \in \mathsf{H}$ such that $(v_u|\omega) = (u|A\omega)$ for all $\omega \in D(A)$ and $v_u = A^*u$. Exploiting (3.17) we have for all $\omega \in D(A)$ that

$$(u|A\omega)_{\mathbb{K}} = \sum_{\alpha=0}^{d_{\mathbb{K}}} \Re[(u|A\omega)i_\alpha]\overline{i_\alpha} = \sum_{\alpha=0}^{d_{\mathbb{K}}} \Re[(A^*u|\omega)i_\alpha]\overline{i_\alpha} = (A^*u|\omega)_{\mathbb{K}},$$

which implies that $u \in D(A^{*_{\mathbb{K}}})$ and $A^{*_{\mathbb{K}}}u = A^*u$. If we manage to prove that $D(A^{*_{\mathbb{K}}}) \subset D(A^*)$, the proof is complete. So, suppose that $u \in D(A^{*_{\mathbb{K}}})$, thus there exists $x_u \in \mathsf{H}_{\mathbb{K}}$ such that $(u|A\omega)_{\mathbb{K}} = (x_u|\omega)_{\mathbb{K}}$ for all $\omega \in D(A)$. Exploiting identity (3.23) and $\mathcal{J}_\alpha A \subset A\mathcal{J}_\alpha$ we have

$$(u|A\omega) = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} (u|\mathcal{J}_\alpha A\omega)_{\mathbb{K}}\overline{i_\alpha} = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} (u|A\mathcal{J}_\alpha\omega)_{\mathbb{K}}\overline{i_\alpha} = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} (x_u|\mathcal{J}_\alpha\omega)_{\mathbb{K}}\overline{i_\alpha} = (x_u|\omega),$$

which holds for any $\omega \in D(A)$. This is equivalent to $u \in D(A^*)$, concluding the proof. The proof of (iii) and (iv) is a simple application of the definition and (3.21). Points (v) follows immediately from point (ii) and the involved definitions. To conclude let us prove (vi). Suppose that A is positive on H and take any $u \in D(A)$. We have $(u|Au) \geq 0$, in particular $(u|Au) \in \mathbb{R}$. Identity (3.19) shows immediately that $(u|Au) = (u|Au)_{\mathbb{K}}$ and $(u|\mathcal{J}_\alpha Au)_{\mathbb{K}} = 0$ for any non-null $\alpha \in \mathcal{A}_{\mathbb{K}}$. The first identity proves the positivity of A on $\mathsf{H}_\mathbb{K}$ while the second has the symmetry as direct consequence. Indeed, take any $u, v \in D(A)$, then $u + \mathcal{J}_\alpha v \in D(A)$ for all non-null $\alpha \in \mathcal{A}_\mathbb{K}$ and so, exploiting also Proposition 3.2.4, we get

$$0 = (u + \mathcal{J}_\alpha v | \mathcal{J}_\alpha A(u + \mathcal{J}_\alpha v))_\mathbb{K} = (u|\mathcal{J}_\alpha A u)_\mathbb{K} + (u| - Av)_\mathbb{K} + (\mathcal{J}_\alpha v | \mathcal{J}_\alpha A u)_\mathbb{K} + \\ + (\mathcal{J}_\alpha v | - Av)_\mathbb{K} = -(u|Av)_\mathbb{K} + \overline{(v|Au)_\mathbb{K}} = -(u|Av)_\mathbb{K} + (Au|v)_\mathbb{K},$$

which concludes the proof of the symmetry. Now, suppose that A is symmetric and positive on $\mathsf{H}_\mathbb{K}$ and take any $u \in D(A)$, then for any non-null $\alpha \in \mathcal{A}_\mathbb{K}$ it holds that $(u|\mathcal{J}_\alpha A u)_\mathbb{K} = (u|A\mathcal{J}_\alpha u)_\mathbb{K} = (Au|\mathcal{J}_\alpha u)_\mathbb{K} = -\overline{(\mathcal{J}_\alpha A u|u)_\mathbb{K}} = -(u|\mathcal{J}_\alpha A u)_\mathbb{K}$ which yields $(u|\mathcal{J}_\alpha A u)_\mathbb{K} = 0$. Identity (3.23) and the positivity on $\mathsf{H}_\mathbb{K}$ conclude the proof. □

As a byproduct of this proof we have the following result.

***Lemma* 3.2.6.** *Let A be a symmetric operator over H, then $(u|Au) = (u|Au)_\mathbb{K}$ for any $u \in D(A)$. More precisely $(u|\mathcal{J}_\alpha A u)_\mathbb{K} = 0$ for any $u \in D(A)$ and any non-null $\alpha \in \mathcal{A}_\mathbb{K}$.*

Again, we have some results about the spectral properties of these operators.

***Proposition* 3.2.7.** *Let A be self-adjoint operator over H and let $P^{(A)}$ be the associated PVM, then the following statements hold:*

(a) the PVM of A referred to $\mathsf{H}_\mathbb{K}$ is still given by $P^{(A)}$;

(b) the (point, continuous) spectrum of A referred to $\mathsf{H}_\mathbb{K}$ coincides with the (point, continuous) spectrum of A referred to H;

(c) if $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then the definition of $f(A)$ referred to H coincides with the one referred to $\mathsf{H}_\mathbb{K}$.

Proof. Let A be as in the hypothesis, then it can be understood as an operator over $\mathsf{H}_\mathbb{K}$ which is still self-adjoint thanks to Proposition 3.2.5. Hence we can consider the PVM $P : E \mapsto P(E)$ of A on H and the PVM $P_\mathbb{K} : E \mapsto P_\mathbb{K}(E)$ of $A_\mathbb{K}$ on $\mathsf{H}_\mathbb{K}$. We want to prove that they are equal to each other. First, notice that P is still a PVM if understood as a family of operators over $\mathsf{H}_\mathbb{K}$. We will denote by $\mu^{(P)}$ and $\nu^{(P)}$ the standard finite measures associated with P referred to H and $\mathsf{H}_\mathbb{K}$, respectively. Fix some $u \in \mathsf{H}$, then thanks to the self-adjointness of $P(E)$ and Lemma 3.2.6 we have $\mu_u^{(P)}(E) = (u|P(E)u) = (u|P(E)u)_\mathbb{K} = \nu_u^{(P)}$ for every Borelian E of the real line. At this point, suppose that $u \in D(A)$ which is equivalent to $id \in \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu_u^{(P)})$. Exploiting the equality between the measures just proved we get $\mathcal{L}^2(\mathbb{R}, \mathbb{R}, \nu_u^{(P)}) = \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu_u^{(P)}) \ni id$. Finally, we have

$$\int_\mathbb{R} s \, d\nu_u^{(P)} = \int_\mathbb{R} s \, d\mu_u^{(P)} = (u|Au) = (u|Au)_\mathbb{K}. \tag{3.26}$$

The two properties just proved are clearly already satisfied by $P_\mathbb{K}$ for A on $\mathsf{H}_\mathbb{K}$ and so Theorem 2.2.57 guarantees that $P_\mathbb{K}(E) = P(E)$ for every Borelian E. This concludes the proof of point (a). Point (b) is a direct consequence of point (a) and Theorem 2.2.56. So, let us prove point (c). Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function then, exploiting again $\mu_u^{(P)} = \nu_u^{(P)}$, we see that the definition of $\Delta_f^{(P)}$ does not depend on the Hilbert space of definition. Now, let us denote by $f(A)_{\mathbb{K}}$ the (self-adjoint) integration of f with respect to P on $\mathsf{H}_{\mathbb{K}}$ and by $f(A)$ the same on H. Clearly $f(A)$ is also a well-defined self-adjoint operator on $\mathsf{H}_{\mathbb{K}}$. At this point, if we take $u \in D(f(A)) = \Delta_f^{(P)} = D(f(A)_{\mathbb{K}})$ then, exploiting the symmetry of $f(A)$ we have

$$(u|f(A)u)_{\mathbb{K}} = (u|f(A)u) = \int_{\mathbb{R}} f(s)\, d\mu_u^{(P)} = \int_{\mathbb{R}} f(s)\, d\nu_u^{(P)} = (u|f(A)_{\mathbb{K}}u)_{\mathbb{K}}.$$

Putting all together we get $(u|(f(A) - f(A)_{\mathbb{K}})u)_{\mathbb{K}}$. Since the operator $f(A) - f(A)_{\mathbb{K}}$ is symmetric on $D(f(A)) = D(f(A)_{\mathbb{K}})$, Lemma 2.2.22 assures that $f(A) = f(A)_{\mathbb{K}}$. □

***Notation* 3.2.8.** *Let $A : D(A) \to \mathsf{H}$ be a linear operator. We denote by $A_{\mathbb{K}}$ the operator A when understood as a linear operator on $\mathsf{H}_{\mathbb{K}}$. Again, this makes notation coherent with the one developed in the previous section.*

3.2.2 Method II: Imaginary Operators

In this subsection we discuss another way to get a real or complex Hilbert space out of a quaternionic one. Similarly to what done in the subsection about real Hilbert spaces this last method is based upon the existence of special imaginary operators. All about the complex case can be found in [15] and [16].

Let H be a quaternionic Hilbert space and consider the following two situations:

(i) we say that H is equipped with a *complex structure* if it is coupled with an imaginary operator J_1;

(ii) we say that H is equipped with a *real structure* if it is coupled with two anti-commuting imaginary operators J_1, J_2. In this case $J_3 := J_1J_2$ consists in another imaginary operator anti-commuting with the others.

***Remark* 3.2.9.** Notice the difference between this and the real case of section 3.1.2: here we choose $J_3 = J_1J_2$, not $J_3 = -J_1J_2$. The reason is that in the aforementioned section the operators J were used to define a *right* multiplication by scalars, while in the actual situation they are required to give rise to a *left* multiplication.

As done in the real case we can give a comprehensive definition. To this aim define the *commutant* of any subset $\mathbb{S} \subset \mathbb{H}$ by

$$\mathbb{S}' := \{q \in \mathbb{H} \,|\, qs = sq \quad \text{for all} \quad s \in \mathbb{S}\}.$$

In particular it holds that

$$\mathbb{R}' = \mathbb{H} \qquad \mathbb{H}' = \mathbb{R} \qquad \mathbb{C}_{i_1}' = \mathbb{C}_{i_1} \quad \text{and} \quad \mathbb{S}'' = \mathbb{S}.$$

***Definition* 3.2.10.** *Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A quaternionic Hilbert space H is said to be equipped with a $\mathbb{K}$-structure if it is coupled with $d_{\mathbb{K}'}$ imaginary operators $\{J_\alpha\}_{\alpha=1,\dots,d_{\mathbb{K}'}}$ satisfying the above properties.*

***Notation* 3.2.11.** Again, we define conventionally $J_0 = I$ in order to define a $\mathbb{K}$-structure as $\{J_\alpha\}_{\alpha=0,\dots,d_{\mathbb{K}'}}$ and denote this family by $\mathfrak{J}$.

Now, fix a $\mathbb{K}$-structure $\mathfrak{J}$ and consider the map $L : \mathbb{K}' \ni q \mapsto L_q \in \mathfrak{B}(\mathsf{H})$ defined by

$$L_q := \sum_{\alpha=0}^{d_{\mathbb{K}'}} q_\alpha J_\alpha \quad \text{if} \quad q = \sum_{\alpha=0}^{d_{\mathbb{K}'}} q_\alpha i_\alpha \in \mathbb{K}'. \tag{3.27}$$

It is immediate to see that

$$L_1 = I,\ L_{\overline{q}} = (L_q)^*,\ L_{p+q} = L_p + L_q,\ L_{qp} = L_q L_p \quad \text{and} \quad L_{cq} = cL_q$$

for any $p, q \in \mathbb{K}'$ and $c \in \mathbb{R}$ and so L gives rise to a left scalar multiplication by elements of $\mathbb{K}'$ as in Definition 2.2.42.

Now, consider the subset of H defined as

$$\mathsf{H}_{\mathfrak{J}} := \{u \in \mathsf{H} \mid L_q u = uq \ \text{ for all } q \in \mathbb{K}'\}. \tag{3.28}$$

***Remark* 3.2.12.** The vectors belonging to $\mathsf{H}_{\mathfrak{J}}$ can be understood as the *real* or *complex* vectors of H depending whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, in that they "commute" with the elements of $\mathbb{K}'$.

***Proposition* 3.2.13.** *Let H be equipped with a $\mathbb{K}$-structure, then $\mathsf{H}_{\mathfrak{J}} \neq \{0\}$ if $\mathsf{H} \neq \{0\}$.*

Proof. For the complex case we refer to Proposition 3.8 of [15]. The real case derives from Proposition 2.2.44 and Remark 2.2.45. □

It is clear that the set $\mathsf{H}_{\mathfrak{J}}$ is closed under the right multiplication by i_α with $0 \leq \alpha \leq d_{\mathbb{K}}$, in this way inheriting a $\mathbb{K}$-linear space structure. In order to introduce a Hermitean scalar product on this space we are tempted to just restrict the action of $(\cdot|\cdot)$ to $\mathsf{H}_{\mathfrak{J}}$, i.e. define

$$(u|v)_{\mathfrak{J}} := (u|v) \quad \text{for all } u, v \in \mathsf{H}_{\mathfrak{J}}. \tag{3.29}$$

The following lemma justifies this intuition.

***Lemma* 3.2.14.** *Let H be a quaternionic Hilbert space with a $\mathbb{K}$-structure $\mathfrak{J}$, then $(x|y) \in \mathbb{K}$ for any $x, y \in \mathsf{H}_{\mathfrak{J}}$.*

Proof. Take $x, y \in \mathsf{H}_{\mathfrak{J}}$, then $L_q x = xq$ and $L_q y = yq$ for every $q \in \mathbb{K}'$. Fix any $q \in \mathbb{K}'$, then

$$(x|y)q = (x|yq) = (x|L_q y) = ((L_q)^* x|y) = (L_{\overline{q}} x|y) = (x\overline{q}|y) = q(x|y).$$

Since $(x|y)$ commutes with every element of $\mathbb{K}'$ it must be an element of $\mathbb{K}$. □

This together with the fact that $(\cdot|\cdot)$ is a Hermitean scalar product on H guarantees that $(\cdot|\cdot)_\mathfrak{J}$ is a Hermitean scalar product on $\mathsf{H}_\mathfrak{J}$. Moreover notice that

$$u \perp_\mathfrak{J} v \quad \text{if and only if} \quad u \perp v \quad \text{for all} \quad u, v \in \mathsf{H}_\mathfrak{J} \tag{3.30}$$

and

$$\|x\|_\mathfrak{J} = \|x\| \quad \text{for all} \quad x \in \mathsf{H}_\mathfrak{J}. \tag{3.31}$$

A first property of $\mathsf{H}_\mathfrak{J}$ is its closedness as a subset of H which follows immediately from (3.31) and the continuity of L_q for every $q \in \mathbb{K}'$.

Now, remember that, by restricting the right scalar multiplication of H to the elements of $\mathbb{K} \subset \mathbb{H}$, the set H can be understood as a linear space over $\mathbb{K}$. The subset $\mathsf{H}_\mathfrak{J}$ can then be seen as a $\mathbb{K}$-linear subspace of H. Actually, this subspace contains all the information about H itself, let us see how. First, we need the following result.

***Lemma* 3.2.15.** *Take $\alpha \in \mathcal{A}_\mathbb{K}$, then the following statements hold:*

(a) the set $\mathsf{H}_\mathfrak{J} i_\alpha$ is a $\mathbb{K}$-linear subspace of H;

(b) if $\alpha \neq 0$, the map $\mathsf{H}_\mathfrak{J} \ni u \mapsto u i_\alpha \in \mathsf{H}_\mathfrak{J} i_\alpha$ is a $\mathbb{K}$-anti-linear isomorphism.

Proof. Let us start with point (a). If $\mathbb{K} = \mathbb{R}$ the thesis is tirival, so suppose that $\mathbb{K} = \mathbb{C}_{i_1}$. We already know that $\mathsf{H}_\mathfrak{J}$ (i.e. $\alpha = 0$) is a $\mathbb{C}$-linear space, so suppose that $\alpha \neq 0$. Consider any $v \in \mathsf{H}_\mathfrak{J} i_3$, thus $v = u i_3$ for some $u \in \mathsf{H}_\mathfrak{J}$ and so $v i_1 = u i_3 i_1 = (-u i_1) i_3$. Since $\mathsf{H}_\mathfrak{J}$ is a $\mathbb{C}$-linear space, $-u i_1 \in \mathsf{H}_\mathfrak{J}$ if $u \in \mathsf{H}_\mathfrak{J}$ and the proof of point (a) is complete. Let us pass to point (b). The map $u \mapsto u i_\alpha$ is trivially a bijection, so let us prove its anti-linearity. Again the case $\mathbb{K} = \mathbb{R}$ is trivial, so let us prove the $\mathbb{K} = \mathbb{C}$ case. Take $u \in \mathsf{H}_\mathfrak{J}$, then $(u i_1) i_3 = -u i_3 i_1 = -(u i_3) i_1 = (u i_3)\overline{i_1}$. This concludes the proof. ☐

We are ready to prove the following important key result.

***Proposition* 3.2.16.** *As a $\mathbb{K}$-linear space the space H decomposes as (see (2.6))*

$$\mathsf{H} = \bigoplus_{\alpha \in \mathcal{A}_\mathbb{K}} \mathsf{H}_\mathfrak{J} i_\alpha. \tag{3.32}$$

In particular if $v = \sum_{\alpha \in \mathcal{A}_\mathbb{K}} v_\alpha i_\alpha$ then for any $u, v \in \mathsf{H}_\mathfrak{J}$ it holds that

$$(u|v) = \sum_{\alpha \in \mathcal{A}_\mathbb{K}} \sum_{\beta \in \mathcal{A}_\mathbb{K}} \overline{i_\alpha} (u_\alpha | v_\beta)_\mathfrak{J} i_\beta \quad \textit{and} \quad \|u\|^2 = \sum_{\alpha \in \mathcal{A}_\mathbb{K}} \|u_\alpha\|^2_\mathfrak{J}. \tag{3.33}$$

Proof. Take $u \in \mathsf{H}$ and define the vector

$$\phi(u) := \frac{1}{d_{\mathbb{K}'} + 1} (2\delta_{\beta 0} - 1) \sum_{\beta=0}^{d_{\mathbb{K}'}} (J_\beta u) i_\beta. \tag{3.34}$$

A direct check shows that $\phi(u) \in \mathsf{H}_\mathfrak{J}$. Moreover it holds that $\sum_{\alpha \in \mathcal{A}_\mathbb{K}} \phi(u\overline{i_\alpha}) i_\alpha = u$. Putting all together we see that every element of H can be decomposed as $\sum_{\alpha \in \mathcal{A}_\mathbb{K}} v_\alpha i_\alpha$

for some $v_\alpha \in \mathsf{H}_\mathfrak{J}$. If we manage to prove that this decomposition is unique, then (bear in mind Lemma 3.2.15) H can be decomposed as a direct sum (with respect to $\mathbb{K}$) of the $\mathsf{H}_\mathfrak{J} i_\alpha$. Take any $u, v \in \mathsf{H}$, then referring to the just found decompositions we have

$$\begin{aligned}(u|u) &= \sum_{\alpha\in\mathcal{A}_\mathbb{K}} \sum_{\beta\in\mathcal{A}_\mathbb{K}} (u_\alpha i_\alpha | u_\beta i_\beta) = \sum_{\alpha\in\mathcal{A}_\mathbb{K}} \sum_{\beta\in\mathcal{A}_\mathbb{K}} \overline{i_\alpha}(u_\alpha|u_\beta) i_\beta \overset{(*)}{=} \sum_{\alpha\in\mathcal{A}_\mathbb{K}} (u_\alpha|u_\alpha) = \\ &= \sum_{\alpha\in\mathcal{A}_\mathbb{K}} (u_\alpha|u_\alpha)_\mathfrak{J}.\end{aligned} \tag{3.35}$$

Were the vanishing of the mixed terms in $(*)$ is a consequence of Lemma 3.2.14. In particular we have $\|v\|^2 = \sum_{\alpha\in\mathcal{A}_\mathbb{K}} \|v_\alpha\|^2_\mathfrak{J}$. So, suppose we have two different decompositions $\sum_{\alpha\in\mathcal{A}_\mathbb{K}} v_\alpha i_\alpha = v = \sum_{\alpha\in\mathcal{A}_\mathbb{K}} v'_\alpha i_\alpha$, then $0 = \sum_{\alpha\in\mathcal{A}_\mathbb{K}} (v_\alpha - v'_\alpha) i_\alpha$. Exploiting the expression for the norm above we get $\|v_\alpha - v'_\alpha\| = 0$, i.e. $v_\alpha = v'_\alpha$ for any $\alpha \in \mathcal{A}_\mathbb{K}$. This concludes the proof. $\square$

***Remark* 3.2.17.** As a byproduct of the proof of this proposition we see that the *components* of the decompositions of any vectors $u \in \mathsf{H}$ are given by $u_\alpha = \phi(u\overline{i_\alpha})$ for any $\alpha \in \mathcal{A}_\mathbb{K}$ where ϕ is defined in (3.34).

Consider a linear subspace $\mathsf{K} \subset \mathsf{H}$, this is clearly also a $\mathbb{K}$-linear subspace of H. We can define the subset

$$\mathsf{K}_\mathfrak{J} := \{u \in \mathsf{K} \mid L_q u = uq \ \text{ for all } q \in \mathbb{K}'\} = \mathsf{K} \cap \mathsf{H}_\mathfrak{J} \subset \mathsf{H}_\mathfrak{J}$$

which is clearly a linear subspace of $\mathsf{H}_\mathfrak{J}$, for K is closed under right multiplication by elements of $\mathbb{K}$.

Now, suppose also that $J_\alpha(\mathsf{K}) \subset \mathsf{K}$ for any $1 \le \alpha \le d_{\mathbb{K}'}$, i.e. $L_q(\mathsf{K}) \subset \mathsf{K}$ for any $q \in \mathbb{K}'$. Take $u \in \mathsf{K}$, so from $u\overline{i_\alpha} \in \mathsf{K}$ we get $\phi(u\overline{i_\alpha}) \in \mathsf{K}$ for any $u \in \mathsf{K}$ and $\alpha \in \mathcal{A}_\mathbb{K}$. This means that, if $u = \sum_{\alpha\in\mathcal{A}_\mathbb{K}} u_\alpha i_\alpha$ is the decomposition of u referred to (3.32), then $u_\alpha \in \mathsf{K}_\mathfrak{J}$. We have just proved the first part of the following result

***Lemma* 3.2.18.** *If K is a linear subspace of H such that $J_\alpha(\mathsf{K}) \subset \mathsf{K}$ for all $1 \le \alpha \le d_{\mathbb{K}'}$, then*

$$\mathsf{K} = \left\{ \sum_{\alpha\in\mathcal{A}_\mathbb{K}} u_\alpha i_\alpha \,\middle|\, u_\alpha \in \mathsf{K}_\mathfrak{J} \text{ for all } \alpha \in \mathcal{A}_\mathbb{K} \right\}. \tag{3.36}$$

In other words, as a $\mathbb{K}$-linear space K decomposes as

$$\mathsf{K} = \bigoplus_{\alpha\in\mathcal{A}_\mathbb{K}} \mathsf{K}_\mathfrak{J} i_\alpha.$$

On the contrary if $\mathsf{V} \subset \mathsf{H}_\mathfrak{J}$ is a linear subspace then there exists a unique linear subspace $\widehat{\mathsf{V}} \subset \mathsf{H}$ such that $J_\alpha(\widehat{\mathsf{V}}) \subset \widehat{\mathsf{V}}$ for all $1 \le \alpha \le d_{\mathbb{K}'}$ and $\mathsf{V} = \widehat{\mathsf{V}}_\mathfrak{J}$.

Proof. It remains to prove only the second part of the thesis. So, let $\mathsf{V} \subset \mathsf{H}_\mathfrak{J}$ be a linear subspace and define $\widehat{V} := \{\sum_{\alpha\in\mathcal{A}_\mathbb{K}} v_\alpha i_\alpha \mid v_\alpha \in \mathsf{V}\} \subset \mathsf{H}$. A direct inspection shows that $\widehat{V}$ is a linear subspace of H and that $J_\alpha(\widehat{V}) \subset \widehat{\mathsf{V}}$ for any $1 \leq \alpha \leq d_{\mathbb{K}'}$. Of course $\mathsf{V} \subset \widehat{\mathsf{V}} \cap \mathsf{H}_\mathfrak{J}$ so let us prove the opposite inclusion. Take $u = \sum_{\alpha \in \mathcal{A}_\mathbb{K}} u_\alpha i_\alpha \in \widehat{\mathsf{V}}$ and suppose that $u \in \mathsf{H}_\mathfrak{J}$, then since $\mathsf{H} = \oplus_{\alpha \in \mathcal{A}_\mathbb{K}} \mathsf{H}_\mathfrak{J} i_\alpha$ it must be $u_0 = u$ and $u_\alpha = 0$ for $\alpha \neq 0$ and so $u = u_0 \in \mathsf{V}$, concluding the proof. It remains to prove the uniqueness of such a subspace of H. Suppose that $\widehat{\mathsf{U}} \subset \mathsf{H}$ is a linear subspace such that $J_\alpha(\widehat{\mathsf{U}}) \subset \widehat{\mathsf{U}}$ and $\widehat{\mathsf{U}} \cap \mathsf{H}_\mathfrak{J} = V$. Of course, as $\widehat{\mathsf{U}}$ is a linear subspace of H and $\mathsf{V} \subset \widehat{\mathsf{U}}$ it must be $\widehat{\mathsf{V}} \subset \widehat{\mathsf{U}}$. On the contrary, take $u \in \widehat{\mathsf{U}}$, then, referring to (3.32) it holds that $u = \sum_{\beta \in \mathcal{A}_\mathbb{K}} u_\beta i_\beta$ for some $u_\beta \in \mathsf{H}_\mathfrak{J}$. If we manage to prove that $u_\beta \in \mathsf{V}$, the proof is complete. Let us prove this in the case $\mathbb{K} = \mathbb{R}$ first. The invariance of $\widehat{\mathsf{U}}$ under the action of the operators J_α assures for $u = u_0 + u_1 i_1 + u_2 i_2 + u_3 i_3 \in \widehat{\mathsf{U}}$ that

$$\begin{aligned}\widehat{\mathsf{U}} \ni J_2(u) i_2 &= J_2(u_0 + u_1 i_1 + u_2 i_2 + u_3 i_3) i_2 = (u_0 i_2 + u_1 i_2 i_1 + u_2 (i_2)^2 + u_3 i_2 i_3) i_2 = \\ &= -u_0 + u_1 i_1 - u_2 i_2 + u_3 i_3.\end{aligned}$$

Hence we get

$$\frac{1}{2}(u - J_2(u) i_2) = u_0 + u_2 i_2 \in \widehat{\mathsf{U}} \tag{3.37}$$

and

$$-\frac{1}{2}(u + J_2(u) i_2) i_3 = -(u_1 i_1 + u_3 i_3) i_3 = u_3 + u_1 i_2 \in \widehat{\mathsf{U}}. \tag{3.38}$$

Similarly,

$$\widehat{\mathsf{U}} \ni -J_3(u_0 + u_2 i_2) i_3 = u_0 - u_2 i_2 \quad \text{and} \quad \widehat{\mathsf{U}} \ni -J_3(u_3 + u_1 i_2) i_3 = u_3 - u_1 i_2 \tag{3.39}$$

Combining together equations (3.37), (3.38) and (3.39) we easily get $u_\alpha \in \widehat{\mathsf{U}} \cap \mathsf{H}_\mathfrak{J} = \mathsf{V}$ for any $0 \leq \alpha \leq 3$. Now, suppose that $\mathbb{K} = \mathbb{C}$ and take $u = u_0 + u_3 i_3 \in \widehat{\mathsf{U}}$, then

$$\widehat{\mathsf{U}} \ni J_1(u_0 + u_3 i_3) i_1 = (u_0 i_1 + u_3 i_1 i_3) i_1 = -u_0 + u_3 i_3.$$

As in the real case this implies easily $u_\beta \in \widehat{\mathsf{U}} \cap \mathsf{H}_\mathfrak{J} = \mathsf{V}$, concluding the proof. □

We are ready to state the first set of properties about $\mathsf{H}_\mathfrak{J}$.

***Proposition* 3.2.19.** *Let H be a quaternionic Hilbert space with a $\mathbb{K}$-structure $\mathfrak{J}$, then the following statements hold:*

(a) the space $\mathsf{H}_\mathfrak{J}$ equipped with (3.29) is a $\mathbb{K}$-Hilbert space;

(b) $\mathsf{N} \subset \mathsf{H}_\mathfrak{J}$ is a Hilbert basis for $\mathsf{H}_\mathfrak{J}$ if and only if is a Hilbert basis for H;

(c) let $\mathsf{K} \subset \mathsf{H}$ be a linear subspace such that $J_\alpha(\mathsf{K}) \subset \mathsf{K}$ for every $1 \leq \alpha \leq d_{\mathbb{K}'}$, then $\overline{\mathsf{K}_\mathfrak{J}} = \overline{\mathsf{K}}_\mathfrak{J}$; in particular $\mathsf{K}_\mathfrak{J}$ is either closed or dense in $\mathsf{H}_\mathfrak{J}$ if and only if K is, respectively, either closed or dense in H.

Proof. Let us start with point (a). We already know that $\mathsf{H}_\mathfrak{J}$ is a pre-Hilbert space, while the completeness follows immediately from (3.31), the completeness of H and the closedness of $\mathsf{H}_{\mathfrak{J}}$ within H. Let us pass to point (b). Thanks to (3.30) and (3.31), then it is clear that $\mathsf{N} \subset \mathsf{H}_{\mathfrak{J}}$ is made of orthonormal vectors in $\mathsf{H}_{\mathfrak{J}}$ if and only if it is so on H. If it is maximally orthogonal in H it is clearly maximally orthogonal in $\mathsf{H}_{\mathfrak{J}}$ too. Let us prove the opposite, thus suppose that N is maximally orthogonal in $\mathsf{H}_{\mathfrak{J}}$ and suppose that there exists some $v \in \mathsf{H}$ such that $(z|v) = 0$ for all $z \in \mathsf{N}$. Exploiting (3.33) we see that $(z|v) = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}}(z|v_\alpha)i_\alpha$ which vanishes if and only if $(z|v_\alpha) = 0$ for any $\alpha \in \mathcal{A}_{\mathbb{K}}$ thanks to Lemma 3.2.14. As N is maximally orthogonal on $\mathsf{H}_{\mathfrak{J}}$ it must be $v_\alpha = 0$ for any $\alpha \in \mathcal{A}_{\mathbb{K}}$, i.e. $v = 0$. Let us prove point (c). In what follows remember that $\|v\|_{\mathfrak{J}} = \|v\|$ for any $v \in \mathsf{H}_{\mathfrak{J}}$. So, suppose that $v \in \overline{\mathsf{K}_{\mathfrak{J}}} = \overline{\mathsf{K}} \cap \mathsf{H}_{\mathfrak{J}}$ then since $v \in \overline{\mathsf{K}}$ there must exist some sequence $(v_n)_{n\in\mathbb{N}} \in \mathsf{K}$ such that $\|v_n - v\| \to 0$. Exploiting the decompositions in Proposition 3.2.16 and Lemma 3.2.18 we can write $v_n = \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}}(v_n)_\alpha i_\alpha$ with $(v_n)_\alpha \in \mathsf{K}_{\mathfrak{J}}$ for $\alpha \in \mathcal{A}_{\mathbb{K}}$ and $v = v_0 \in \mathsf{H}_{\mathfrak{J}}$. Thus it is clear that $\|(v_n)_\alpha\|_{\mathfrak{J}} \to 0$ if $\alpha \neq 0$ and $\|(v_n)_0 - v\|_{\mathfrak{J}} \to 0$. In particular the latter limit implies $v = \lim_{n\to\infty}(v_n)_0 \in \overline{\mathsf{K}_{\mathfrak{J}}}$. This proves that $\overline{\mathsf{K}}_{\mathfrak{J}} \subset \overline{\mathsf{K}_{\mathfrak{J}}}$. On the contrary, take $v \in \overline{\mathsf{K}_{\mathfrak{J}}} \subset \mathsf{H}_{\mathfrak{J}}$, then there exists some sequence $(v_n)_n \subset \mathsf{K}_{\mathfrak{J}}$ such that $\|v_n - v\| = \|v_n - v\|_{\mathfrak{J}} \to 0$. Since $\mathsf{K}_{\mathfrak{J}} \subset \mathsf{K}$ this implies also $v \in \overline{\mathsf{K}}$. To conclude notice that $L_q v = \lim_{n\to\infty} L_q v_n = \lim_{n\to\infty} v_n q = vq$ for any $q \in \mathbb{K}'$, which gives $u \in \mathsf{H}_{\mathfrak{J}}$, concluding the proof. $\square$

The relationship between operators on H and $\mathsf{H}_{\mathfrak{J}}$ is discussed in the following result

***Proposition* 3.2.20.** *The following statements hold:*

(a) let $A : D(A) \to \mathsf{H}$ be a quaternionic linear operator such that $J_\alpha A \subset AJ_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$, then $A_{\mathfrak{J}} := A|_{D(A)_{\mathfrak{J}}}$ is a well-defined $\mathbb{K}$-linear operator on $\mathsf{H}_{\mathfrak{J}}$;

(b) if $B : D(B) \to \mathsf{H}_{\mathfrak{J}}$ is a $\mathbb{K}$-linear operator, then there exists a unique quaternionic linear operator $\widehat{B} : D(\widehat{B}) \to \mathsf{H}$ such that $J_\alpha\widehat{B} \subset \widehat{B}J_\alpha$ for every $1 \leq \alpha \leq d_{\mathbb{K}'}$ and $\widehat{B}_{\mathfrak{J}} = B$, in particular $D(B) = D(\widehat{B}) \cap \mathsf{H}_{\mathfrak{J}}$;

(c) let $B : D(B) \to \mathsf{H}_{\mathfrak{J}}$ and $S : D(S) \to \mathsf{H}_{\mathfrak{J}}$ be $\mathbb{K}$-linear operators, then

  (i) $\widehat{aB} = a\widehat{B}$ for all $a \in \mathbb{R}$;

  (ii) $\widehat{B+S} = \widehat{B} + \widehat{S}$;

  (iii) $\widehat{BS} = \widehat{B}\widehat{S}$;

  (iv) $\widehat{B} \in \mathfrak{B}(\mathsf{H})$ if and only if $B \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$, more precisely $\|\widehat{B}\| = \|B\|_{\mathfrak{J}}$;

  (v) if $D(B)$ is dense, then $(\widehat{B})^* = \widehat{B^*}$;

  (vi) B is either closable or closed if and only if $\widehat{B}$ is, respectively, either closable or closed: in this case $\overline{\widehat{B}} = \widehat{\overline{B}}$;

  (vii) let $\mathsf{S} \subset D(\widehat{B})$ be a subspace such that $J_\alpha(\mathsf{S}) \subset \mathsf{S}$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$, then S is a core for $\widehat{B}$ if and only if $\mathsf{S}_{\mathfrak{J}}$ is a core for B;

(viii) *B is symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent on $\mathsf{H}_{\mathfrak{J}}$ if and only if $\widehat{B}$ is, respectively, symmetric, anti-symmetric, (essentially) self-adjoint, (essentially) anti-self-adjoint, unitary, normal, idempotent on H;*

(ix) *$\widehat{B}$ is positive if and only if B is positive and symmetric.*

Proof. Let us start with point (a) and take A as in the hypothesis, then the inclusion $J_\alpha A \subset AJ_\alpha$ for any $1 \leq \alpha \leq d_{\mathbb{K}'}$ guarantees that $A(D(A) \cap \mathsf{H}_{\mathfrak{J}}) \subset \mathsf{H}_{\mathfrak{J}}$. Indeed if $u \in D(A) \cap \mathsf{H}_{\mathfrak{J}}$, then we have $J_\alpha(Au) = AJ_\alpha u = A(ui_\alpha) = (Au)i_\alpha$ for any $1 \leq \alpha \leq d_{\mathbb{K}'}$, i.e. $u \in \mathsf{H}_{\mathfrak{J}}$. Of course since A is quaternionic linear it is also linear with respect to the elements of $\mathbb{K}$. Let us pass to point (b) and take B a linear operator on $\mathsf{H}_{\mathfrak{J}}$. Exploiting Lemma 3.2.18 we know that there exists a unique linear subspace of H, which we denote by $D(\widehat{B})$, such that $J_\alpha(D(\widehat{B})) \subset D(\widehat{B})$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$ and $D(\widehat{B})_{\mathfrak{J}} = D(B)$. Thanks again to the same lemma we can now define the function $\widehat{B} : D(\widehat{B}) \to \mathsf{H}$ as

$$\widehat{B}(u) := \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} (Bu_\alpha)i_\alpha \quad \text{for all} \quad u = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} u_\alpha i_\alpha \in D(\widehat{B}). \tag{3.40}$$

By definition, it is clear that $\widehat{B}(u) = Bu$ if $u \in D(B)$ while the linearity follows easily by direct inspection. Take for example $\mathbb{K} = \mathbb{C}$, then for every $u \in D(\widehat{B})$ it holds that

$$\begin{aligned}\widehat{B}(ui_1) &= \widehat{B}((u_0 + u_3 i_3)i_1) = \widehat{B}(u_0 i_1 + (-u_3 i_1)i_3) = B(u_0 i_1) + B(-u_3 i_1)i_3 = \\ &= (Bu_0)i_1 + (Bu_3)i_3 i_1 = (Bu_0 + (Bu_3)i_3)i_1 = (\widehat{B}u)i_1.\end{aligned}$$

Now, take $1 \leq \alpha \leq d_{\mathbb{K}'}$ and $u \in D(\widehat{B})$, then

$$\begin{aligned}J_\alpha \widehat{B}u &= J_\alpha\left(\sum_{\beta \in \mathcal{A}_{\mathbb{K}}} (Bu_\beta)i_\beta\right) = \sum_{\beta \in \mathcal{A}_{\mathbb{K}}} (J_\alpha(Bu_\beta))i_\beta = \sum_{\beta \in \mathcal{A}_{\mathbb{K}}} ((Bu_\beta)i_\alpha)i_\beta = \\ &= \sum_{\beta \in \mathcal{A}_{\mathbb{K}}} ((\widehat{B}u_\beta)i_\alpha)i_\beta \overset{(*)}{=} \widehat{B}\left(\sum_{\beta \in \mathcal{A}_{\mathbb{K}}} (u_\beta i_\alpha)i_\beta\right) = \widehat{B}\left(\sum_{\beta \in \mathcal{A}_{\mathbb{K}}} (J_\alpha u_\beta)i_\beta\right) = \\ &= \widehat{B}J_\alpha\left(\sum_{\beta \in \mathcal{A}_{\mathbb{K}}} u_\beta i_\beta\right) = \widehat{B}J_\alpha u,\end{aligned} \tag{3.41}$$

where the passage $(*)$ is meaningful thanks to the identity $Bu = \widehat{B}u$ if $u \in D(B) \subset D(\widehat{B})$ and the linearity of $\widehat{B}$. This proves $J_\alpha \widehat{B} \subset \widehat{B}J_\alpha$. It remains to prove the uniqueness of such operator. So, suppose there exists another operator $T : D(T) \to \mathsf{H}$ such that $J_\alpha T \subset TJ_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$ and $T_{\mathfrak{J}} = B$, so in particular we have $J_\alpha(D(T)) \subset D(T)$ for any $1 \leq \alpha \leq d_{\mathbb{K}'}$ and $D(T) \cap \mathsf{H}_{\mathfrak{J}} = D(T_{\mathfrak{J}}) = D(B)$. By definition of $D(\widehat{B})$, Lemma 3.2.16 assures that $D(T) = D(\widehat{B})$. The linearity of $\widehat{B}$ and T and the identity $T_{\mathfrak{J}} = B = \widehat{B}_{\mathfrak{J}}$ together with the definition of $D(\widehat{B})$ concludes the proof. So, let us pass to the proof of the properties of point (c) and let B, S be as in the hypothesis.

Point (i) is obvious, so let us prove point (ii). It holds that $u = \sum_{\beta\in\mathcal{A}_\mathbb{K}} u_\beta i_\beta \in D(\widehat{BS})$ if and only if $u_\beta \in D(BS)$ for any $\beta \in \mathcal{A}_\mathbb{K}$ which is equivalent to $u_\beta \in D(S)$ and $Su_\beta \in D(B)$ for all $\beta \in \mathcal{A}_\mathbb{K}$. This, in turn, is equivalent to $u \in D(\widehat{S})$ and $\widehat{S}u \in D(\widehat{B})$ which is equivalent to $u \in D(\widehat{B}\widehat{S})$. Since both $\widehat{BS}$ and $\widehat{B}\widehat{S}$ are linear and coincide on $D(BS)$, they coincide on the entire domain. Point (iii) is analogous. To prove (iv) notice first that $D(B) = \mathsf{H}_\mathfrak{J}$ if and only if $D(\widehat{B}) = \mathsf{H}$ which follows immediately from the definition and properties of $D(\widehat{B})$ and decomposition (3.32). So, take $u \in D(\widehat{B})$, then exploting (3.33) we have

$$\|\widehat{B}u\|^2 = \left\|\sum_{\beta\in\mathcal{A}_\mathbb{K}} (Bu_\beta)i_\beta\right\|^2 = \sum_{\beta\in\mathcal{A}_\mathbb{K}} \|Bu_\beta\|_\mathfrak{J}^2 \leq \|B\|_\mathfrak{J}^2 \sum_{\beta\in\mathcal{A}_\mathbb{K}} \|u_\beta\|_\mathfrak{J}^2 = \|B\|_\mathfrak{J}^2\|u\|^2,$$

which yields $\|\widehat{B}\| \leq \|B\|_\mathfrak{J}$. The opposite inequality follows immediately by noticing that if we take $u = u_0 \in D(B)$ then $\|Bu_0\|_\mathfrak{J} = \|\widehat{B}u\| \leq \|\widehat{B}\|\|u\| = \|\widehat{B}\|\|u_0\|_\mathfrak{J}$. This concludes the proof of (iv). Let us pass to (v). If $D(B)$ is dense then $D(\widehat{B})$ is dense too, thanks to Proposition 3.2.19, thus we can consider both B^* and $(\widehat{B})^*$. First, notice that $\widehat{B^*} \subset (\widehat{B})^*$. Indeed if $v \in D(\widehat{B^*})$ then for every $u \in D(\widehat{B})$ expression (3.33) assures that

$$(v|\widehat{B}u) = \sum_{\alpha\in\mathcal{A}_\mathbb{K}}\sum_{\beta\in\mathcal{A}_\mathbb{K}} \overline{i_\alpha}(v_\alpha|Bu_\beta)_\mathfrak{J} i_\beta = \sum_{\alpha\in\mathcal{A}_\mathbb{K}}\sum_{\beta\in\mathcal{A}_\mathbb{K}} \overline{i_\alpha}(B^*v_\alpha|u_\beta)_\mathfrak{J} i_\beta = (\widehat{B^*}v|u), \tag{3.42}$$

which proves $\widehat{B^*} \subset (\widehat{B})^*$. Now, take $u \in D((\widehat{B})^*)$, then by definition there must exist some $v_u \in \mathsf{H}$ such that $(v_u|\omega) = (u|\widehat{B}\omega)$ for every $\omega \in D(\widehat{B})$. So, if we take $\omega = \omega_0 \in D(B) \subset D(\widehat{B})$, then

$$\sum_{\beta\in\mathcal{A}_\mathbb{K}} \overline{i_\alpha}(u_\alpha|B\omega_0)_\mathfrak{J} = (u|\widehat{B}\omega) = (v_u|\omega) = \sum_{\beta\in\mathcal{A}_\mathbb{K}} \overline{i_\alpha}((v_u)_\alpha|\omega_0)_\mathfrak{J}.$$

Since $(\cdot|\cdot)_\mathfrak{J}$ takes value within $\mathbb{K}$, the above equality implies $(u_\alpha|B\omega_0)_\mathfrak{J} = ((v_u)_\alpha|\omega_0)_\mathfrak{J}$ for every $\omega_0 \in D(B)$ and for every $\alpha \in \mathcal{A}_\mathbb{K}$. In particular this implies that $u_\alpha \in D(B^*)$ for every $\alpha \in \mathcal{A}_\mathbb{K}$, i.e. $u \in D(\widehat{B^*})$, concluding the proof. Exploting the points just proved and the involved definitions it is easy to derive point (viii). Let us prove (vii). Suppose that B is closable but $\widehat{B}$ is not, thus there must exists some sequence $(u_n)_{n\in\mathbb{N}} \in D(\widehat{B})$ such that $u_n \to 0$ but $\widehat{B}u_n \to v \neq 0$. Exploiting the results of Proposition 3.2.16 we easily get $(u_n)_\alpha \to 0$ and $B(u_n)_\alpha \to v_\alpha$ for all $\alpha \in \mathcal{A}_\mathbb{K}$. Since $v \neq 0$, there must exist at least a α' such that $v_{\alpha'} \neq 0$. The existence of the sequence $((u_n)_{\alpha'})_{n\in\mathbb{N}}$ is then in contradiction with the closability of B. On the contrary suppose that $\widehat{B}$ is closable but B is not. Again, there must exist a sequence $(x_n)_{n\in\mathbb{N}} \in D(B)$ such that $x_n \to 0$ and $Bx_n \to y \neq 0$. Since $B = \widehat{B}|_{D(B)}$ and $\|\cdot\|_\mathfrak{J} = \|\cdot\|$ on $\mathsf{H}_\mathfrak{J}$ we have $x_n \in D(\widehat{B})$ and $\widehat{B}x_n \to y \neq 0$ which is impossible, $\widehat{B}$ being closable. The fact that $\overline{\widehat{B}} = \widehat{\overline{B}}$ can be proved by direct inspection, exploiting the definition of closure, the definition of $\widehat{B}$ and

(3.33). It remains to prove point (ix). So, suppose $\widehat{B}$ is positive and take $u, v \in D(B)$, then if $0 \neq \alpha \in \mathcal{A}_{\mathbb{K}}$ and exploiting $(v|\widehat{B}v) \geq 0$ we have

$$\begin{aligned}
0 \leq &(u + vi_\alpha|\widehat{B}(u + vi_\alpha)) = (u + vi_\alpha|(\widehat{B}u) + (\widehat{B}v)i_\alpha) = (u|\widehat{B}u) + \overline{i_\alpha}(v|\widehat{B}v)i_\alpha + \\
&+ \overline{i_\alpha}(v|\widehat{B}u) + (u|\widehat{B}v)i_\alpha = (u|Bu)_{\mathfrak{J}} + (v|Bv)_{\mathfrak{J}} + \overline{i_\alpha}(v|Bu)_{\mathfrak{J}} + (u|Bv)_{\mathfrak{J}}i_\alpha = \\
&= (u|Bu)_{\mathfrak{J}} + (v|Bv)_{\mathfrak{J}} + \overline{(v|Bu)_{\mathfrak{J}}}\,\overline{i_\alpha} + (u|Bv)_{\mathfrak{J}}i_\alpha = \\
&= (u|Bu)_{\mathfrak{J}} + (v|Bv)_{\mathfrak{J}} + (Bu|v)_{\mathfrak{J}}\,\overline{i_\alpha} + (u|Bv)_{\mathfrak{J}}i_\alpha = \\
&= (u|Bu)_{\mathfrak{J}} + (v|Bv)_{\mathfrak{J}} + [-(Bu|v)_{\mathfrak{J}} + (u|Bv)_{\mathfrak{J}}]i_\alpha.
\end{aligned} \tag{3.43}$$

Taking $v = 0$ we have $(u|Bu)_{\mathfrak{J}} \geq 0$, i.e. $B \geq 0$. In general, since $-(Bu|v)_{\mathfrak{J}} + (u|Bv)_{\mathfrak{J}} \in \mathbb{K}$ and $0 \neq \alpha \in \mathcal{A}_{\mathbb{K}}$ the term

$$[-(Bu|v)_{\mathfrak{J}} + (u|Bv)_{\mathfrak{J}}]i_\alpha \tag{3.44}$$

is purely imaginary and as so, since both $(u|Bu)_{\mathfrak{J}}$ and $(v|Bv)_{\mathfrak{J}}$ belong to $\mathbb{R}$, it must must vanish: this implies $(Bu|v)_{\mathfrak{J}} = (u|Bv)_{\mathfrak{J}}$, i.e. B is symmetric. Suppose on the contrary that B is symmetric and positive and take any $u = \sum_{\beta \in \mathcal{A}_{\mathbb{K}}} u_\alpha i_\alpha \in D(\widehat{B})$, then

$$(u|\widehat{B}u) = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} \sum_{\beta \in \mathcal{A}_{\mathbb{K}}} \overline{i_\alpha}(u_\alpha|Bu_\beta)_{\mathfrak{J}} i_\beta = \sum_{\alpha \in \mathcal{A}_{\mathbb{K}}} (u_\alpha|Bu_\alpha)_{\mathfrak{J}} \geq 0, \tag{3.45}$$

where the mixed terms vanish thanks to the symmetry of B and the terms $(u_\alpha|Bu_\alpha)_{\mathfrak{J}}$ commutes with $\overline{i_\alpha}$, they being real thanks to the positivity of B. □

About the spectral properties of these operators we have the following result.

***Proposition* 3.2.21.** *Let A be a linear operator over H such that $J_\alpha A \subset AJ_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$ and let $P^{(A)}$ be its PVM, then the following statements hold:*

(a) $J_\alpha P^{(A)}(E) = P^{(A)}(E)J_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$ and Borelian $E \in \mathcal{B}(\mathbb{R})$ and the PVM of $A_{\mathfrak{J}}$ is given by $P^{(A_{\mathfrak{J}})}(E) = P^{(A)}(E)_{\mathfrak{J}}$ for any Borelian $E \in \mathcal{B}(\mathbb{R})$;

(b) the (point, continuous) spectrum of $A_{\mathfrak{J}}$ coincides with the (point, continuous) spectrum of A;

(c) if $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then $J_\alpha f(A) \subset f(A)J_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$ and $f(A_{\mathfrak{J}}) = f(A)_{\mathfrak{J}}$.

Proof. Let us start with point (a). If A is as in the hypothesis, then we can consider the operator $A_{\mathfrak{J}}$ which is self-adjoint thanks to Theorem 3.2.20. First of all we want to prove that $P(E)J_\alpha = J_\alpha P(E)$ for every real Borelian E and $1 \leq \alpha \leq d_{\mathbb{K}'}$. In order to do this, neglecting the subscript α for simplicity, notice that the function $E \mapsto Q(E) := J^*P(E)J$ is still a PVM on H which follows easily from the fact that J is unitary. Now, notice that for any $u \in \mathsf{H}$ it holds that $\mu_u^{(Q)}(E) = (u|Q(E)u) =$

$(Ju|P(E)Ju) = \mu^{(P)}_{Ju}(E)$. So, take any $u \in D(A)$, then since $Ju \in D(A)$ we have $id \in \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu^{(P)}_{Ju}) = \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu^{(Q)}_u)$. Moreover it holds that

$$(u|Au) = (u|J^*AJu) = (Ju|AJu) = \int_{\mathbb{R}} s\, d\mu^{(P)}_{Ju} = \int_{\mathbb{R}} s\, d\mu^{(Q)}_u.$$

Since the two conditions just proved hold simultenously for the PVM P and Q, Lemma 2.2.57 assures that $P(E) = Q(E) = J^*P(E)J$ for every Borelian E. In conclusion we have $J_\alpha P(E) = P(E)J_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$ which allows us to consider the family $P_{\mathfrak{J}} : E \mapsto P(E)_{\mathfrak{J}}$ on $\mathsf{H}_{\mathfrak{J}}$. An easy application of Theorem 3.2.20 together with Proposition 3.2.16 shows that this map is a PVM on $\mathsf{H}_{\mathfrak{J}}$. Next goal is showing that $P_{\mathfrak{J}}$ is in fact the PVM of $A_{\mathfrak{J}}$ on $\mathsf{H}_{\mathfrak{J}}$. We will exploit again Lemma 2.2.57. First, consider any vector $u \in D(A_{\mathfrak{J}}) \subset D(A)$, then $\mu^{(P_{\mathfrak{J}})}_u(E) = (u|P(E)_{\mathfrak{J}}u)_{\mathfrak{J}} = (u|P(E)u) = \mu^{(P)}_u(E)$ Hence, $id \in \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu^{(P)}_u) = \mathcal{L}^2(\mathbb{R}, \mathbb{R}, \mu^{(P_{\mathfrak{J}})}_u)$ and

$$(u|A_{\mathfrak{J}}u)_{\mathfrak{J}} = (u|Au) = \int_{\mathbb{R}} s\, d\mu^{(P)}_u = \int_{\mathbb{R}} s\, d\mu^{(P_{\mathfrak{J}})}_u.$$

Since $A_{\mathfrak{J}}$ is self-adjoint on $\mathsf{H}_{\mathfrak{J}}$ it admits a unique PVM on $\mathsf{H}_{\mathfrak{J}}$ satisfying the above two properties as stated by Lemma 2.2.57. Thus $P_{\mathfrak{J}}$ is the PVM of $A_{\mathfrak{J}}$ on $\mathsf{H}_{\mathfrak{J}}$. Point (b) follows from point (a) and Theorem 2.2.56 and by noticing that $P(E) = 0$ if and only if $P(E)_{\mathfrak{J}} = 0$ - which follows from Theorem 3.2.20. Let us conclude the proof by proving point (c). First notice that for every $u \in \mathsf{H}$ and $1 \leq \alpha \leq d_{\mathbb{K}'}$ it holds that $\mu^{(P)}_{J_\alpha u}(E) = (J_\alpha u|P(E)J_\alpha u) = (u|J^*_\alpha P(E)J_\alpha u) = (u|P(E)u) = \mu^{(P)}_u(E)$, thanks to the fact that $P(E)$ commutes with every J_α, as proved above. Now, take any measurable function $f : \mathbb{R} \to \mathbb{R}$ then if $u \in D(f(A))$ we have $\int_{\mathbb{R}} |f(s)|^2\, d\mu^{(P)}_{J_\alpha u} = \int_{\mathbb{R}} |f(s)|^2\, d\mu^{(P)}_u < \infty$ which means that $J_\alpha u \in D(f(A))$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$, i.e. $J_\alpha(D(f(A))) \subset D(f(A))$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$. Thus, for all $u \in D(f(A))$ we have

$$(u|J^*_\alpha f(A)J_\alpha u) = (J_\alpha u|f(A)J_\alpha u) = \int_{\mathbb{R}} f(s)\, d\mu^{(P)}_{J_\alpha u} = \int_{\mathbb{R}} f(s)\, d\mu^{(P)}_u = (u|f(A)u).$$

As a consequence we have $(u|(f(A) - J^*_\alpha f(A)J_\alpha)u) = 0$. Since both $f(A)$ and $J^*_\alpha f(A)J_\alpha$ are self-adjoint, their difference is symmetric on its (dense) natural domain $D(f(A)) \cap D(f(A)J_\alpha) = D(f(A))$. Thus Lemma 2.2.22 assures that $f(A) - J^*_\alpha f(A)J_\alpha = 0$ on $D(f(A))$, i.e. $J_\alpha f(A) = f(A)J_\alpha$ on $D(f(A))$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$. This is equivalent to $J_\alpha f(A) \subset f(A)J_\alpha$ for all $1 \leq \alpha \leq d_{\mathbb{K}'}$. At this point Theorem 3.2.20 allows us to consider the self-adjoint operator $f(A)_{\mathfrak{J}} : D(f(A)) \cap \mathsf{H}_{\mathfrak{J}} \to \mathsf{H}_{\mathfrak{J}}$ and we want to prove that $f(A)_{\mathfrak{J}} = f(A_{\mathfrak{J}})$. Consider any vector $x \in D(f(A_{\mathfrak{J}})) \subset \mathsf{H}_{\mathfrak{J}}$, then, similarly to what has been done above, since $\mu^{(P_{\mathfrak{J}})}_u = \mu^{(P)}_u$ for every $u \in \mathsf{H}_{\mathfrak{J}}$ we have

$$\int_{\mathbb{R}} |f(s)|^2\, d\mu^{(P)}_u = \int_{\mathbb{R}} |f(s)|^2\, d\mu^{(P_{\mathfrak{J}})}_u < \infty,$$

which implies $x \in D(f(A))$, i.e. $D(f(A_{\mathfrak{J}})) \subset D(f(A)) \cap \mathsf{H}_{\mathfrak{J}} = D(f(A)_{\mathfrak{J}})$. Moreover for every $u \in D(f(A_{\mathfrak{J}}))$ we have

$$(u|f(A_{\mathfrak{J}})u)_{\mathfrak{J}} = \int_{\mathbb{R}} f(s)\, d\mu_u^{(P_{\mathfrak{J}})} = \int_{\mathbb{R}} f(s)\, d\mu_u^{(P)} = (u|f(A)u) = (u|f(A)_{\mathfrak{J}}u)_{\mathfrak{J}},$$

which yields $(u|(f(A_{\mathfrak{J}}) - f(A)_{\mathfrak{J}})u)_{\mathfrak{J}} = 0$. Again, since this holds for every $u \in D(f(A_{\mathfrak{J}})) \subset D(f(A)_{\mathfrak{J}})$ the symmetry of $f(A_{\mathfrak{J}}) - f(A)_{\mathfrak{J}}$ on its natural (dense) domain $D(f(A_{\mathfrak{J}})) \cap D(f(A)_{\mathfrak{J}}) = D(f(A_{\mathfrak{J}}))$ implies $f(A_{\mathfrak{J}}) \subset f(A)_{\mathfrak{J}}$. Since both $f(A_{\mathfrak{J}})$ and $f(A)_{\mathfrak{J}}$ are self-adjoint the inclusion is in fact an equality. The proof is complete. $\square$

Chapter 4

Some Useful Results on Hilbert spaces

The aim of this chapter is to derive some well-known results, as Stone's Theorem, Polar Decomposition Theorem and Schur's Lemma in the general case of a Hilbert space over any division algebra $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. A general discussion on trace-class operators is also given. The general strategy consists in taking for granted the results in the complex case, whose proof can be found in any textbook on complex Hilbert space theory (see for instance [24]), and extend them to the peculiar cases of real and quaternionic Hilbert space exploiting the techniques discussed in the previous chapter.

4.1 The Stone Theorem

Let us start with this extremely important result of functional analysis. First, recall the following definition.

***Definition* 4.1.1.** *A one-parameter group of unitary operators is a group homomorphism $U : \mathbb{R} \to \mathfrak{B}(\mathsf{H})$, i.e. a function such that*

$$U_0 = I \quad \textit{and} \quad U_t U_s = U_{t+s} \quad \textit{for every } t, s \in \mathbb{R},$$

and U_t is unitary for every $t \in \mathbb{R}$. The group is said to be strongly continuous if the function is continuous with respect to the strong topology of $\mathfrak{B}(\mathsf{H})$.

***Remark* 4.1.2.** Some remarks about the definition follow:

(a) the strong-continuity can be restated as $\lim_{t\to 0} U_t u = u$ for every $u \in \mathsf{H}$ while the continuity at every other point $t_0 \in \mathbb{R}$ follows easily from the group properties of U;

(b) from the unitariness of the operators U_t and the group properties of U it follows that $(U_t)^* = U_{-t}$ for every $t \in \mathbb{R}$;

(c) if H is a real or quaternionic Hilbert space and $U : t \mapsto U_t \in \mathfrak{B}(\mathsf{H})$ a strongly-continuous one-parameter group then the map

$$U_{\mathbb{K}} : \mathbb{R} \ni t \mapsto (U_t)_{\mathbb{K}} \in \mathfrak{B}(\mathsf{H}_{\mathbb{K}}) \tag{4.1}$$

is a strongly-continuous one-parameter group of unitary operators;

(d) if H is a real or quaternionic Hilbert space equipped with a $\mathbb{K}$-structure $\mathfrak{J}$ and $U : t \mapsto U_t \in \mathfrak{B}(\mathsf{H})$ a strongly-continuous one-parameter group then the map

$$U_{\mathfrak{J}} : \mathbb{R} \ni t \mapsto (U_t)_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}) \tag{4.2}$$

is a strongly-continuous one-parameter group of unitary operators, provided that $JU_g = U_g J$ for every $J \in \mathfrak{J}$ and $g \in G$.

Now, focus on the complex and quaternionic cases and consider any anti-self-adjoint operator $A : D(A) \to \mathsf{H}$. Exploiting the spectral theory of Chapter 2 - in particular the spectral theorem 2.2.56 - if (P, L) is the rPVM associated to A, then we can consider the following family of operators:

$$\mathbb{R} \ni t \mapsto e^{tA} := \int_X e^{tz}\, dP(z). \tag{4.3}$$

Since A is anti-self-adjoint we know that $\sigma(A)$ is purely imaginary within $\mathbb{F}$, see Proposition 2.2.37. This implies that $\overline{e^{tz}} = e^{-tz}$ on the essential domain of integration given by $suppP = \sigma(A) \cap X$, in particular the function $z \mapsto e^{tz}$ is essentially bounded. This has two important consequences: first, the operator e^{tA} is bounded for any $t \in \mathbb{R}$; second, exploiting the properties of the function operators (see Proposition 2.2.55) it turns out that the operator e^{tA} is unitary for every $t \in \mathbb{R}$. The following important and well-known result holds.

***Theorem* 4.1.3** (Stone's Theorem - complex and quaternionic case)**.** *Let H be a complex or quaternionic Hilbert space, then the following statements hold:*

(a) the operators in (4.3) define a strongly-continuous one-parameter group of unitary operators, moreover

$$D(A) = \left\{ u \in \mathsf{H} \,\middle|\, \exists \lim_{t \to 0} \frac{U_t u - u}{t} \in \mathsf{H} \right\}, \quad Au = \lim_{t \to 0} \frac{U_t u - u}{t} \tag{4.4}$$

and $e^{tA}A = Ae^{tA}$ for any $t \in \mathbb{R}$.

(b) if U is a strongly-continuous one-parameter group of unitary operators, then there exists a unique anti-self-adjoint operator over H such that $U_t = e^{tA}$ for all $t \in \mathbb{R}$.

Proof. For the complex case we refer to Theorem 9.33 and Proposition 9.35 of [24]. So, suppose that H is quaternionic. The first part of the proof of (a) can be carried out following essentially the same proof of the complex case. Concerning the last statement, notice that $e^{tA}A \subset Ae^{tA}$ follows for any $t \in \mathbb{R}$ from the definition of e^{-tA} in (4.3) and Proposition 2.2.55. Multiplying both terms by e^{-tA} on both left- and right-hand sides we get $Ae^{-tA} \subset e^{-tA}A$. The arbitrariness of $t \in \mathbb{R}$ concludes the proof. Let us prove point (b). Consider the complexification $\mathsf{H}_{\mathbb{C}}$ as described in Section 3.2.1, then the map $U_{\mathbb{C}}$ is a strongly-continuous one-parameter group of unitary operators on $\mathsf{H}_{\mathbb{C}}$ as explained in Remark 4.1.2. At this point the complex version of Stone's Theorem assures that there exists some anti-self-adjoint operator $A_{\mathbb{C}}$ on $\mathsf{H}_{\mathbb{C}}$ satisfying $(U_t)_{\mathbb{C}} = e^{tA_{\mathbb{C}}}$ for all $t \in \mathbb{R}$. Moreover $A_{\mathbb{C}}$ fulfills (4.4) on $\mathsf{H}_{\mathbb{C}}$ and $(U_t)_{\mathbb{C}} A_{\mathbb{C}} = A_{\mathbb{C}} (U_t)_{\mathbb{C}}$ for all $t \in \mathbb{R}$. Given this, we first want to prove that this operator is actually quaternionic linear, i.e. that it commutes with $\mathcal{J}_3$ (see (3.22) and Proposition 3.2.5). So, take $u \in D(A_{\mathbb{C}})$, then point (a) in the complex case assures that $A_{\mathbb{C}} u = \lim_{t\to 0} t^{-1}(U_t u - u)$ on $\mathsf{H}_{\mathbb{C}}$, but the limit holds also on H, thanks to (3.21). This implies

$$\mathcal{J}_3 A_{\mathbb{C}} u = \mathcal{J}_3 \left(\lim_{t\to 0} \frac{U_t u - u}{t} \right) = \lim_{t\to 0} \frac{\mathcal{J}_3 U_t u - \mathcal{J}_3 u}{t} = \lim_{t\to 0} \frac{U_t \mathcal{J}_3 u - \mathcal{J}_3 u}{t}.$$

Again, (3.21) and point (a) in the complex case yields $\mathcal{J}_3 u \in D(A_{\mathbb{C}})$ and $\mathcal{J}_3 A_{\mathbb{C}} u = A_{\mathbb{C}} \mathcal{J}_3 u$ which can be condensed in $\mathcal{J}_3 A_{\mathbb{C}} \subset A_{\mathbb{C}} \mathcal{J}_3$. The operator $A_{\mathbb{C}}$ is then quaternionic linear (and anti-self-adjoint) on H. From now on we can neglect the subscript $\mathbb{C}$ of $A_{\mathbb{C}}$. Notice that A fulfills (4.4) on H and $U_t A \subset A U_t$ for any $t \in \mathbb{R}$, which follow from the corresponding properties on $\mathsf{H}_{\mathbb{C}}$ and (3.21). So, consider the operator $V_t := e^{tA}$ defined as the integration of $z \mapsto e^{tz}$ with respect to the rPVM of A on H. In particular the map $t \mapsto V_t$ is a strongly-continuous one-parameter of unitary operators on H and satisfies (4.4) and $V_t A = A V_t$ for all $t \in \mathbb{R}$ thanks to point (a). Take any $u, v \in D(A)$, then

$$\frac{d}{dt} U_t v = \lim_{h\to 0} \frac{U_{t+h} v - U_t v}{h} = \lim_{h\to 0} U_t \frac{U_h v - v}{h} = U_t A v \tag{4.5}$$

and similarly

$$\frac{d}{dt} V_t v = \lim_{h\to 0} \frac{V_{t+h} v - V_t v}{h} = \lim_{h\to 0} V_t \frac{V_h v - v}{h} = V_t A v. \tag{4.6}$$

At this point, exploiting $V_t A = A V_t$ and $U_t A = A U_t$ for all $t \in \mathbb{R}$ we have in particular that $U_t(D(A)) \subset D(A)$ for all $t \in \mathbb{R}$ and

$$\begin{aligned} \frac{d}{dt}(u|V_{-t} U_t v) = \frac{d}{dt}(V_t u | U_t v) &= \left(\frac{d}{dt} V_t u \middle| U_t v \right) + \left(V_t u \middle| \frac{d}{dt} U_t v \right) = \\ &= (V_t A u | U_t v) + (V_t u | U_t A v) = (A V_t u | U_t v) + (V_t u | U_t A v) = \\ &= (V_t u | - A U_t v) + (V_t u | U_t A v) = \\ &= -(V_t u | U_t A v) + (V_t u | U_t A v) = 0. \end{aligned} \tag{4.7}$$

This proves that $t \mapsto (u|V_{-t} U_t v)$ is constant and so $(u|V_{-t} U_t v) = (u|V_0 U_0 v) = (u|v)$. Since $u, v \in D(A)$ are arbitrary and $D(A)$ is dense, we see that $V_{-t} U_t = I$, i.e. $U_t = V_t = e^{tA}$. It remains to prove the uniqueness of the operator A. So, suppose that $U_t = e^{tB}$ for some anti-self-adjoint operator B, then since we already know that $U_t = e^{tA}$ the equality $A = B$ is an immediate consequence of point (a). $\square$

The real case is different, for there is no spectral theorem over H for anti-self-adjoint operators. However something can be said. First, consider an anti-self-adjoint operator A defined over a real Hilbert space and take its external complexification $A_{\mathbb{C}}$ defined over $\mathsf{H}_{\mathbb{C}}$. We know from Proposition 3.1.4 that $A_{\mathbb{C}}$ is anti-self-adjoint and so we can consider the strongly-continuous one-parameter group $t \mapsto e^{tA_{\mathbb{C}}}$. Consider the natural conjugation C defined in (3.6) and the following result, whose proof can be carried out easily basing upon the spectral theorem and Stone's Theorem.

***Lemma* 4.1.4.** *Let H be a complex Hilbert space, A a densely defined operator and $C : \mathsf{H} \to \mathsf{H}$ a conjugation, i.e. an anti-linear operator such that $CC = I$, $\|Cx\| = \|x\|$ for all $x \in \mathsf{H}$, then the following statements hold:*

(a) CAC is a densely defined linear operator,

(b) $(CAC)^* = CA^*C$,

(c) if A is anti-self-adjoint, then $Ce^{tA}C = e^{tCAC}$.

Thanks to this lemma we have $Ce^{tA_{\mathbb{C}}}C = e^{tCA_{\mathbb{C}}C} = e^{tA_{\mathbb{C}}}$ and so Proposition 3.1.4 assures that there exists a unique operator B_t on H such that $(B_t)_{\mathbb{C}} = e^{tA_{\mathbb{C}}}$. Exploiting (3.4), Proposition 3.1.4 and the fact that $t \mapsto e^{tA_{\mathbb{C}}}$ is a strongly-continuous one-parameter group of unitary operators on $\mathsf{H}_{\mathbb{C}}$, it follows immediately that $t \mapsto B_t$ is a strongly-continuous one-parameter group of unitary operators on H and as such it is a good candidate for defining e^{tA} on the real Hilbert space H.

***Definition* 4.1.5.** *Let H be a real Hilbert space and A an anti-self-adjoint operator and $t \in \mathbb{R}$. The operator e^{tA} is defined as the unique unitary operator on H which external complexification gives $e^{tA_{\mathbb{C}}}$.*

***Theorem* 4.1.6** (Stone's Theorem - real case)**.** *Referring to Definition 4.1.5, the thesis of Theorem 4.1.3 applies to real Hilbert spaces.*

Proof. Let us start with point (a). We already know that $t \mapsto e^{tA}$ is a strongly-continuous one-parameter of unitary operators on H, so it remains to prove (4.4). We know that this is true for the complexification $A_{\mathbb{C}}$, i.e.

$$D(A_{\mathbb{C}}) = \left\{ u \in \mathsf{H}_{\mathbb{C}} \;\middle|\; \exists \frac{d}{dt}\Big|_0 e^{tA_{\mathbb{C}}}u \in \mathsf{H}_{\mathbb{C}} \right\}, \quad A_{\mathbb{C}}u = \frac{d}{dt}\Big|_0 e^{tA_{\mathbb{C}}}u.$$

Now, notice that $u \in D(A)$ if and only if $u \otimes 1 \in D(A_{\mathbb{C}})$ if and only if

$$\exists \lim_{t\to 0} \frac{e^{tA_{\mathbb{C}}}u \otimes 1 - u \otimes 1}{t} = \lim_{t\to 0} \left[\frac{(e^{tA}u) - u}{t} \otimes 1 \right] \in \mathsf{H}_{\mathbb{C}}$$

if and only if $\exists \lim_{t\to 0} \frac{(e^{tA}u)-u}{t} \in \mathsf{H}$, where the last equivalence comes from (3.4). Moreover

$$(Au) \otimes 1 = A_{\mathbb{C}}(u \otimes 1) = \lim_{t\to 0} \left[\frac{(e^{tA}u) - u}{t} \otimes 1 \right] = \lim_{t\to 0} \left[\frac{(e^{tA}u) - u}{t} \right] \otimes 1,$$

which concludes the proof of the first point. Let us pass to Point (b). Let $t \mapsto U_t$ be a strongly-continuous one-parameter group of unitary operators on H, then the function $t \mapsto (U_t)_{\mathbb{C}}$ defines a strongly-continuous one-parameter group of unitary operators on $\mathsf{H}_{\mathbb{C}}$. The complex version of Stone's Theorem assures that $(U_t)_{\mathbb{C}} = e^{tS}$ for some anti-self-adjoint operator S on $\mathsf{H}_{\mathbb{C}}$. If we manage to prove that $CS = SC$, where C is the usual natural conjugation of $\mathsf{H}_{\mathbb{C}}$, then it holds that $S = A_{\mathbb{C}}$ for some anti-self-adjoint operator A on H and $(U_t)_{\mathbb{C}} = e^{tA_{\mathbb{C}}} = (e^{tA})_{\mathbb{C}}$ which means $U_t = e^{tA}$. Applying Lemma 4.1.4 and Proposition 3.1.4 we get $e^{tCSC} = Ce^{tS}C = C(U_t)_{\mathbb{C}}C = (U_t)_{\mathbb{C}} = e^{tS}$. Since this holds for any $t \in \mathbb{R}$ and both S and CSC are anti-self-adjoint, then the complex version of Stone's Theorem assures that $CSC = S$, i.e. $CS = SC$. The same theorem gives $(U_t)_{\mathbb{C}}S \subset S(U_t)_{\mathbb{C}}$ which translates into $U_tA \subset AU_t$. It remains to prove the uniqueness of A. So, suppose that B is another anti-self-adjoint operator over H such that $U_t = e^{tB}$, then we have $e^{tA_{\mathbb{C}}} = (U_t)_{\mathbb{C}} = (e^{tB})_{\mathbb{C}} = e^{tB_{\mathbb{C}}}$ which implies $A_{\mathbb{C}} = B_{\mathbb{C}}$, thanks again to the the complex version of Stone's Theorem. Thus $A = B$ and the proof is complete. □

***Definition* 4.1.7.** *Consider a strongly-continuous one-parameter group of unitary operators $U : \mathbb{R} \to \mathfrak{B}(\mathsf{H})$ with H a real, complex or quaternionic Hilbert space. The anti-self-adjoint operator $A : D(A) \to \mathsf{H}$ associated to U is called the generator of U.*

En passant we have also proved part of the following result.

***Proposition* 4.1.8.** *Let U be a strongly-continuous one-parameter group of unitary operators over a real or quaternionic Hilbert space H with generator A, then the generator of $U_{\mathbb{K}}$ (see (4.1)) is given by $A_{\mathbb{K}}$.*

Proof. Suppose that H is quaternionic. In the proof of Theorem 4.1.3 we saw that the generator of $U_{\mathbb{C}}$ equals the one on H. Similarly for H real, we saw in the proof of Theorem 4.1.6 that the generator of $U_{\mathbb{C}}$ is given by the complexification of the generator of U. The other cases can be proved similarly. □

4.2 The Polar Decomposition Theorem

Another technical tool, which will be very useful in the following, is the *Polar Decomposition Theorem*. Again, we will take the complex case for granted and derive the real and quaternionic versions from it by an extension or reduction of the division algebra as done in the case of Stone's Theorem.

***Theorem* 4.2.1.** *Let H be a real, complex or quaternionic Hilbert space and $A : D(A) \to \mathsf{H}$ a densely defined closed operator in H, then there exists a unique pair of operators U, P on H such that*

(1) $A = UP$, where in particular $D(P) = D(A)$;

(2) P is self-adjoint and $P \geq 0$;

(3) $U \in \mathfrak{B}(\mathsf{H})$ *is isometric on* $R(P)$ *(and thus on* $\overline{R(P)}$ *by continuity);*

(4) $N(U) \supset N(P)$.

The right-hand side of point (1) is called the polar decomposition of A. *It turns out that the following additional properties are also valid:*

(1) $P = |A|$,

(2) $N(U) = N(A) = N(P)$,

(3) $R(U) = \overline{R(U)}$.

Proof. For the complex case we refer to Theorem 10.39 of [24]. So, suppose that H is real and consider the complexification $A_{\mathbb{C}}$ on $\mathsf{H}_{\mathbb{C}}$, which is densely defined and closed, thanks to Proposition 3.1.4. Applying the complex version of this theorem we get the existence of some operators U', P' on $\mathsf{H}_{\mathbb{C}}$ satisfying all the properties of the thesis. First thing we want to prove that both U' and P' commutes with the natural conjugation C - see (3.6). Notice that $P' = \sqrt{(A_{\mathbb{C}})^* A_{\mathbb{C}}} = \sqrt{(A^*)_{\mathbb{C}} A_{\mathbb{C}}} = \sqrt{(A^*A)_{\mathbb{C}}} = (\sqrt{A^*A})_{\mathbb{C}} = |A|_{\mathbb{C}}$ where we used the self-adjointness of A^*A (see Proposition 2.2.29) and point (c) of Proposition 3.1.5. This proves that $P' = P_{\mathbb{C}}$ with $P = |A|$, in particular $P'C = CP'$. Now, let us pass to U'. Take $x \in R(P')$, then $x = P'y$ for some $y \in D(P')$ and so $U'Cx = U'CP'y = U'P'Cy = A_{\mathbb{C}}Cy = CA_{\mathbb{C}}y = CU'P'y = CU'x$. By continuity of $U'C$ and CU' we get $U'C = CU'$ on the closed $\overline{R(P')}$. Now, it is easy to see that $C(N(P')) \subset N(P')$ and so $U'C = CU'$ also on $N(P') = \overline{R(P')}^{\perp}$, trivially, thanks to $N(P') \subset N(U')$. This gives $U'C = CU'$ which means that $U' = U_{\mathbb{C}}$ for some operator $U \in \mathfrak{B}(\mathsf{H})$. Properties (1)-(4) and (1)-(3) now follow from the corresponding properties of U', P'. The uniqueness of these two operators follows from the corresponding properties fulfilled by U', P'. Indeed, suppose that V, Q is a couple of operators on H satisfying (1)-(4), then the complexifications $V_{\mathbb{C}}, Q_{\mathbb{C}}$ satisfy (1)-(4) on $\mathsf{H}_{\mathbb{C}}$. The polar decomposition theorem on $\mathsf{H}_{\mathbb{C}}$ assures that $V_{\mathbb{C}} = U' = U_{\mathbb{C}}$ and $Q_{\mathbb{C}} = P' = P_{\mathbb{C}}$, i.e. $V = U$ and $Q = P$. The real case is complete, so let us prove the quaternionic Hilbert space case. The operator A can be interpreted as a linear operator over $\mathsf{H}_{\mathbb{C}}$ and is still densely defined and closed, thanks to Proposition 3.2.5. We denote this interpretation by $A_{\mathbb{C}}$. Again, we can apply the complex version of this theorem and get $(A =)A_{\mathbb{C}} = U'P'$ for some operators U', P' on $\mathsf{H}_{\mathbb{C}}$ satisfying all the properties in the thesis. Now, notice that $A_{\mathbb{C}} = (-\mathcal{J}_3)A_{\mathbb{C}}\mathcal{J}_3 = (-\mathcal{J}_3)U'P'\mathcal{J}_3 = ((-\mathcal{J}_3)U'\mathcal{J}_3)((-\mathcal{J}_3)P'\mathcal{J}_3)$ and it is easy to prove that the couple $((-\mathcal{J}_3)U'\mathcal{J}_3), ((-\mathcal{J}_3)P'\mathcal{J}_3)$ satisfies the requirements (1)-(4) on $\mathsf{H}_{\mathbb{C}}$ (see also Lemma 3.2.4) and so, by uniqueness of the polar decomposition on complex Hilbert spaces, it must be $((-\mathcal{J}_3)U'\mathcal{J}_3) = U'$ and $((-\mathcal{J}_3)P'\mathcal{J}_3 = P'$. This is equivalent to requiring that U', P' are quaternionic linear operators on H. Properties (1)-(4) and (2)-(3) easily lift to the space H. Moreover notice that, thanks to Proposition 3.2.7, the definition of A^* and $\sqrt{A^*A}$ are independent from the division algebra under consideration and so $P' = |A|$ holds also on H, giving (1). It remains to prove the uniqueness of the found decomposition. Suppose there exist two operators V, Q on H satisfying (1)-(4), then these operators can be understood as operators on $\mathsf{H}_{\mathbb{C}}$ and clearly give rise to a polar decomposition of A on $\mathsf{H}_{\mathbb{C}}$. The uniqueness of the latter concludes the proof. $\square$

Notice that U is a *partial isometry* because $U \in \mathfrak{B}(\mathsf{H})$ and it is isometric on $N(U)^{\perp}$ (it is indeed isometric on $\overline{R(P)} = N(P^*)^{\perp} = N(P)^{\perp} = N(U)^{\perp}$).

***Proposition* 4.2.2.** *Let H be a real or quaternionic Hilbert space and A a densely defined closed operator with polar decomposition $A = UP$, then the polar decomposition of $A_{\mathbb{K}}$ on H_K is given by $U_{\mathbb{K}}P_{\mathbb{K}}$.*

Proof. Suppose that H is real. In the proof of Theorem 4.2.1 we saw that the polar decomposition of $A_{\mathbb{C}}$ is given by $U_{\mathbb{C}}P_{\mathbb{C}}$. Similarly for H quaternionic, we saw in the proof of Theorem 4.2.1 that the polar decomposition of $A_{\mathbb{C}}$ is given by $U_{\mathbb{C}}P_{\mathbb{C}}$. The remaining cases can be proved arguing in a similar way. $\square$

We conclude this section with a pair of technical proposition, the second concerning the interplay between the commutativity of one-parameter unitary groups and the commutativity of the elements appearing in the polar decomposition of the generators. That result will turn out to be very useful later.

***Proposition* 4.2.3.** *Let H be a real, complex or quaternionic Hilbert space. Consider an either self-adjoint or anti-self-adjoint operator $A : D(A) \to \mathsf{H}$ with polar decomposition $A = UP$, then the following statements hold:*

(a) if $A^* = -A$ then $B \in \mathfrak{B}(\mathsf{H})$ satisfies $Be^{tA} = e^{tA}B$ if and only if $BA \subset AB$;

(b) if $A^* = A$ with PVM $P^{(A)}$ and $B \in \mathfrak{B}(\mathsf{H})$, then the following statements are equivalent:

(i) $BA \subset AB$,

(ii) $Bf(A) \subset f(A)B$ for any measurable function $f : \mathbb{R} \to \mathbb{R}$,

(iii) $BP^{(A)}(E) = P^{(A)}(E)B$ for any Borelian $E \subset \mathbb{R}$;

(c) if $B \in \mathfrak{B}(\mathsf{H})$ satisfies $BA \subset AB$, then $BU = UB$ and $BP \subset UP$;

(d) the following commutation relations are true

$$UA \subset AU \quad \textit{and} \quad U^*A \subset AU^*\,;$$

moreover, for every measurable function $f : [0, +\infty) \to \mathbb{R}$, it holds that

$$Uf(P) \subset f(P)U \quad \textit{and} \quad U^*f(P) \subset f(P)U^*\,;$$

(e) U is respectively self-adjoint or anti-self-adjoint;

(f) if A is injective (equivalently if either P or U is injective), then U and U^* are unitary. In this case all the inclusions in (c) are identities.

Proof. Let us start with point (a). The necessity is an easy consequence of Stone's Theorem. So, suppose that $BA \subset AB$ and let us prove $Be^{tA} = e^{tA}B$. If H is complex, the proof can be found in Theorem 9.41 of [24]. So, suppose that H is real. Consider the complexification $\mathsf{H}_{\mathbb{C}}$, then the anti-self-adjoint operator $A_{\mathbb{C}}$ generates the group $t \mapsto e^{tA_{\mathbb{C}}} = (e^{tA})_{\mathbb{C}}$. The inclusion $BA \subset AB$ translates into $B_{\mathbb{C}}A_{\mathbb{C}} \subset A_{\mathbb{C}}B_{\mathbb{C}}$ and so the complex version of this result implies $B_{\mathbb{C}}e^{tA_{\mathbb{C}}} = e^{tA_{\mathbb{C}}}B_{\mathbb{C}}$ which is equivalent to $Be^{tA} = e^{tA}B$. Now, suppose that H is quaternionic and consider A, e^{tA}, B as operators on $\mathsf{H}_{\mathbb{C}}$, denoted by $A_{\mathbb{C}}, (e^{tA})_{\mathbb{C}}, B_{\mathbb{C}}$. We know that $t \mapsto (e^{tA})_{\mathbb{C}}$ is still a strongly-continuous one-parameter subgroup of unitary operators on $\mathsf{H}_{\mathbb{C}}$ and that its generator on $\mathsf{H}_{\mathbb{C}}$ is still given by $A_{\mathbb{C}}$ (see Proposition 4.1.8). Again, the complex result implies the thesis. Let us pass to point (b). Again, the proof for the complex case can be found in Theorem 9.41 of [24]. So, suppose that H is real or quaternionic and move to the complexification $\mathsf{H}_{\mathbb{C}}$. The thesis follows immediately from the complex version by noticing that for any measurable function $f : \mathbb{R} \to \mathbb{R}$ it holds that $f(A_{\mathbb{C}}) = f(A)_{\mathbb{C}}$ and that $P^{(A_{\mathbb{C}})}(E) = (P^{(A)}(E))_{\mathbb{C}}$. Let us prove (c) now. First suppose that H is complex. The operator A^*A is densely defined, positive and self-adjoint as we know. Since $P = |A| = \sqrt{A^*A}$ is self-adjoint, we have (Proposition 2.2.19) $N(|A|)^{\perp} = \overline{R(|A|)}$ and $\mathsf{H} = K(|A|) \oplus \overline{R(|A|)}$. The inclusion $BA \subset AB$ implies $BAA \subset ABA$ and thus $BAA \subset AAB$. This inclusion can be rewritten as $BA^*A \subset A^*AB$ because $A^* = \pm A$. As A^*A is self-adjoint and B bounded, the found inclusion extends to all real valued measurable functions of A^*A, i.e. $Bf(A^*A) \subset f(A^*A)B$ (see point (b)). In particular, we have $B|A| = B\sqrt{A^*A} \subset \sqrt{A^*A}B = |A|B$ which is the second of the two relations we wanted to establish. Now, take $u \in D(|A|) = D(A)$, then from the proved inclusion we immediately have $UB|A|u = U|A|Bu = ABu = BAu = BU|A|u$ which implies that $UB = BU$ on $R(|A|)$ and thus on $\overline{R(|A|)}$ by continuity. If we manage to prove that this equality holds also on $N(|A|)$, then the proof is complete for the complex Hilbert space case because $\mathsf{H} = N(|A|) \oplus \overline{R(|A|)}$. So take $u \in N(|A|)$, then $|A|Bu = B|A|u = 0$, that is $Bu \in N(|A|)$. Since $N(|A|) = N(U)$ (Theorem 4.2.1) and $N(|A|)$ is invariant under the action of B, it immediately follows that $UBx = BUx$ trivially for $x \in N(|A|)$ as wanted, concluding the proof for the complex Hilbert space case. The real and quaternionic cases follow from this result, working on the complexified space $\mathsf{H}_{\mathbb{C}}$ and exploiting Proposition 4.2.2. Let us pass to point (d). Let first suppose that H is complex and $A = -A^*$. In this case Stone's Theorem implies that e^{-tA} (which belongs to $\mathfrak{B}(\mathsf{H})$) commutes with A and thus, exploiting Point (c), we have $Ue^{-tA} = e^{-tA}U$ and $U^*e^{tA} = e^{tA}U^*$ (by taking the adjoint of the former). Since $t \in \mathbb{R}$ is arbitrary, point (a) gives $UA \subset AU$ and $U^*A \subset AU^*$. Remaining in the complex case, if $A^* = A$ and replacing A for iA everywhere in our reasoning, we again reach the same final result $UA \subset AU$ and $U^*A \subset AU^*$. Now assume $A = A^*$ (otherwise everywhere replace A for iA). As U and U^* are bounded, we conclude (point (b)) that U and U^*

commute with every real valued measurable function of A, in particular $U|A| \subset |A|U$ and $U^*|A| \subset |A|U^*$. Exploiting point (b) once again, we prove that $Uf(|A|) \subset f(|A|)U$ and $U^*f(|A|) \subset f(|A|)U^*$ for every measurable function $f : [0, +\infty) \to \mathbb{R}$. We have so far established (c) for a complex Hilbert space H. The proof for the real and quaternionic cases follows easily by moving to the complexification $\mathsf{H}_{\mathbb{C}}$ and exploiting Propositions 4.1.8, 3.1.4, 3.2.5, 3.1.5 and 3.2.7. Let us prove point (e). We prove that $U^* = \pm U$ if, respectively, $A^* = \pm A$. Since U is bounded, we have $(\pm U)|A| = \pm A = A^* = |A|U^*$. Take $u \in D(|A|)$, then $U^*|A|u = |A|U^*u = (\pm U)|A|u$ and so $U^*x = \pm Ux$ for $x \in R(|A|)$ and by continuity also for $x \in \overline{R(|A|)}$. Since $\mathsf{H} = \overline{R(|A|)} \oplus N(|A|)$ we have to prove that $U^*x = \pm Ux$ holds also for $x \in N(|A|)$. Since $N(|A|) = N(U)$ is guaranteed by Theorem 4.2.1, we have $Ux = 0$ if $x \in N(|A|)$. By proving $N(|A|) \subset N(U^*)$ we would have $U^*x = 0$, establishing $U^*x = \pm Ux$ also for $x \in N(|A|)$ as required. To this end, take $x \in N(|A|)$ and $y \in \mathsf{H}$, then we have $y = u + v$, with $u \in \overline{R(|A|)}$ and $v \in N(|A|)$. Take $|A|x_n \in R(|A|)$ such that $u = \lim_{n\to\infty} |A|x_n$, then we have $(U^*x|y) = (x|Uy) = (x|Uu) = \lim_{n\to\infty}(x|U|A|x_n) = \lim_{n\to\infty}(x||A|Ux_n) = \lim_{n\to\infty}(|A|x|Ux_n) = \lim_{n\to\infty}(0|Ux_n) = 0$. Since y is arbitrary, we have $U^*x = 0$ if $x \in N(|A|)$ as required, proving our thesis $U^*x = \pm Ux$ for all $x \in \mathsf{H}$. It remains to prove (f). We exploit here Theorem 4.2.1 several times. Since $\mathsf{H} = N(|A|) \oplus \overline{R(|A|)}$ and U is isometric on $\overline{R(|A|)}$, if $N(|A|)$ (which coincides with $N(U)$) is trivial, then U is isometric on H. Therefore it is enough proving that $R(U) = \mathsf{H}$ to end the proof of the fact that U is unitary. We know from Theorem 4.2.1 that $R(U) = \overline{R(U)}$, but since $U = \pm U^*$ we also have $\overline{R(U)} = \overline{R(U^*)} = N(U)^\perp = N(|A|)^\perp = \{0\}^\perp = \mathsf{H}$.
To conclude demonstrating the last statement of (f), observe that if U is unitary and $US \subset SU$, $U^*S \subset SU^*$ simultaneously hold (in particular $U(D(S)) \subset D(S)$ and $U^*(D(S)) \subset D(S)$), we also have $U^*USU^* \subset U^*SUU^*$, that is $SU^* \subset U^*S$. The found inclusion together with $U^*S \subset SU^*$ implies $U^*S = SU^*$. Interchanging the role of U and U^*, we also achieve $US = SU$. □

***Proposition* 4.2.4.** *Let H be a real, complex or quaternionic Hilbert space and A and B anti-self-adjoint operators in H with polar decompositions $A = U|A|$ and $B = V|B|$. If the strongly-continuous one-parameter groups generated by A and B commute, i.e.,*

$$e^{tA}e^{sB} = e^{sB}e^{tA} \quad \textit{for every } s, t \in \mathbb{R},$$

then the following statements hold:

(a) $UB \subset BU$ and $U^*B \subset BU^*$;

(b) $Uf(|B|) \subset f(|B|)U$ and $U^*f(|B|) \subset f(|B|)U^*$ for every non-negative measurable function $f : [0, +\infty) \to \mathbb{R}$;

(c) $UV = VU$ and $U^*V = VU^*$.

If any of A, $|A|$, U is injective, then the inclusions in (a) and (b) can be replaced by identities.

Proof. Point (a) of Proposition 4.2.3 gives $e^{sB}A \subset Ae^{sB}$, while applying point (c) to this inclusion gives $e^{sB}U = Ue^{sB}$. Again, applying point (a) to this equality, we get $UB \subset BU$. At this point, applying point (c) to $UB \subset BU$ gives $UV = VU$ and $U|B| \subset |B|U$. Finally, applying Proposition 4.2.3 and point (b) we easily see that $Uf(|B|) \subset f(|B|)U$ for any real valued measurable function. To conclude notice that the inclusions involving U^* can be derived from the corresponding ones for U simply by noticing that $U^* = -U$. □

Exploiting these two results we can prove the following result.

***Proposition* 4.2.5.** *Let* H *be a real or quaternionic Hilbert space equipped with a* $\mathbb{K}$-*structure* $\mathfrak{J}$. *Let* A *be a self-adjoint or anti-self-adjoint operator over* H *with polar decomposition* $A = UP$. *If* $JA \subset AJ$ *for all* $J \in \mathfrak{J}$, *then* $JU = UJ$ *and* $JP \subset PJ$ *for all* $J \in \mathfrak{J}$ *and the polar decomposition of* $A_{\mathfrak{J}}$ *is given by* $A_{\mathfrak{J}} = U_{\mathfrak{J}}P_{\mathfrak{J}}$.

Proof. Thanks to Proposition 4.2.3 we see that $JU = UJ$ and $JP \subset PJ$ for all $J \in \mathfrak{J}$, thus we can consider the operators $U_{\mathfrak{J}}, P_{\mathfrak{J}}$ on $\mathsf{H}_{\mathfrak{J}}$. Applying the results of Chapter 3 it is easy to see that points (1)-(4) are satisfied by the couple $(U_{\mathfrak{J}}, P_{\mathfrak{J}})$. Theorem 4.2.1 concludes the proof. □

4.3 The Schur Lemma

Another important result in complex functional analysis is of the so-called *Schur's Lemma*, which characterises the commutant of irreducible sets of operators. Despite the simplicity and power of its formulation on complex Hilbert space, on real and quaternionic ones it requires a slightly different formulation which makes it a little weaker.

Let us start with some basic definitions.

***Definition* 4.3.1.** *Let* H *be a real, complex or quaternionic Hilbert space. A family of operators* $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ *is said to be irreducible if* $A(\mathsf{K}) \subset \mathsf{K}$ *for all* $A \in \mathfrak{A}$ *and a closed subspace* $\mathsf{K} \subset \mathsf{H}$ *implies* $\mathsf{K} = \{0\}$ *or* $\mathsf{K} = \mathsf{H}$. $\mathfrak{A}$ *is said to be reducible if it is not irreducible.*

Since the definition refers to *closed* subspaces, our notion of irreducibility is sometimes called *topological* irreducibility.

***Lemma* 4.3.2.** *Let* $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ *be any family of operators, then*

(a) if $\mathfrak{A}$ *is irreducible, then* $\mathfrak{A}' \cap \mathfrak{L}(\mathsf{H}) = \{0, I\}$;

(b) if $\mathfrak{A}$ *is* *-*closed and satisfies* $\mathfrak{A}' \cap \mathfrak{L}(\mathsf{H}) = \{0, I\}$, *then it is irreducible.*

Proof. Let us start with point (a). Consider a projector P commuting with every element of $\mathfrak{A}$ and define $\mathsf{K} := P(\mathsf{H})$, then $A(\mathsf{K}) = AP(\mathsf{H}) = PA(\mathsf{H}) \subset P(\mathsf{H}) = \mathsf{K}$ for any $A \in \mathfrak{A}$. Since the family is irreducible it must be $\mathsf{K} = \{0\}$ or $\mathsf{K} = \mathsf{H}$ which translates immediately into $P = 0$ or $P = I$. Let us pass to the proof of point (b). Consider a closed subspace $\mathsf{K} \subset \mathsf{H}$ which is invariant under the action of the entire algebra $\mathfrak{A}$ and let P be the orthogonal projector such that $P(\mathsf{H}) = \mathsf{K}$. Take any $A \in \mathfrak{A}$ and $u \in \mathsf{H}$, then $A(Pu) \in \mathsf{K}$ by hypothesis, thus $APu = PAPu$. Since this holds for any $u \in \mathsf{H}$ we have $AP = PAP$ for every $A \in \mathfrak{A}$. Taking the adjoint we get $PA^* = PA^*P = A^*P$, where the last equality derives from the fact that $A^* \in \mathfrak{A}$ if $A \in \mathfrak{A}$. Since A is arbitrary and $\mathfrak{A}$ is *-closed we end up with $AP = PA$ for all $A \in \mathfrak{A}$, i.e. $P \in \mathfrak{A}' \cap \mathfrak{L}(\mathsf{H}) = \{0, I\}$. This can be restated as $\mathsf{K} = \{0\}$ or $\mathsf{K} = \mathsf{H}$. □

We have a first result which is valid regarless the nature of the division algebra involved.

***Proposition* 4.3.3** (Schur's Lemma for essentially self-adjoint operators)**.** *Let* H *be a real, complex or quaternionic Hilbert space and let* $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ *be irreducible. If the operator* $A : D(A) \to \mathsf{H}$ *is essentially self-adjoint and satisfies*

$$UA \subset AU \quad \textit{for all} \quad U \in \mathfrak{A} \tag{4.8}$$

then $\overline{A} \in \mathfrak{B}(\mathsf{H})$ *(the bar denoting the closure of* A*) and*

$$\overline{A} = aI \qquad \textit{for some } a \in \mathbb{R}.$$

If A *is self-adjoint, we have* $A \in \mathfrak{B}(\mathsf{H})$ *with* $A = aI$ *for some* $a \in \mathbb{R}$.

Proof. The proof is independent from the involved division algebra. Since the operators $U \in \mathfrak{A}$ are bounded, from Proposition 2.2.58 one has $U\overline{A} \subset \overline{A}U$. At this point Proposition 4.2.3 implies that the spectral measure of $P^{(\overline{A})}$ of $\overline{A}$ commutes with every $U \in \mathfrak{A}$. Since $\mathfrak{A}$ is irreducible, for any fixed $E \in \mathcal{B}(\mathbb{R})$ it holds that either $P^{(\overline{A})}(E) = 0$ (i.e. $P^{(\overline{A})}$ projects onto $\{0\}$) or $P^{(\overline{A})}(E) = I$ (i.e. $P^{(\overline{A})}$ projects onto the entire H). If $P^{(\overline{A})}((a_0, b_0]) = 0$ for all $a_0 < b_0$ in $\mathbb{R}$, we would have $P^{(\overline{A})}(\mathbb{R}) = 0$, due to the σ-additivity, which is not possible, thus $P^{(\overline{A})}((a_0, b_0]) = I$ for some $a_0 < b_0$ in $\mathbb{R}$. Notice that $P^{(\overline{A})}(\mathbb{R} \setminus (a_0, b_0]) = 0$ as a trivial consequence of the properties of the projection valued measures. Now, define $\delta_0 := b_0 - a_0$ and divide $(a_0, b_0]$ into the disjoint union of two equal-length contiguous half-open intervals. Reasoning as above we see that one and only one of them has vanishing measure, while the other satisfies $P^{(\overline{A})}((a_1, b_1]) = I$. Clearly $\delta_1 := b_1 - a_1 = \frac{1}{2}\delta_0$. Iterating this procedure we find a couple of sequences $a_0 \leq a_1 \leq \cdots \leq a_n < b_n \leq \cdots \leq b_1 \leq b_0$ within $[a_0, b_0]$ such that $\delta_n := b_n - a_n = 2^{-n}\delta_0$ and $P^{(\overline{A})}((a_n, b_n]) = I$. Being $[a_0, b_0]$ compact and $(a_n)_n, (b_n)_n \subset [a_0, b_0]$ monotone sequences, it follows that there exist $\lambda_a \leq \lambda_b \in [a_0, b_0]$ such that $a_n \to \lambda_a$ and $b_n \to \lambda_b$. Since $\delta_n \to 0$ it is easy to see that actually $\lambda_a = \lambda_b = \lambda_0$. From the outer continuity of the positive measure $E \mapsto (x|P^{(\overline{A})}(E)|x)$ we have $(x|P^{\overline{A}}(\{\lambda_0\})|x) = (x|P^{(\overline{A})}(\cap_n(a_n, b_n])x) = (x|Ix)$ for every $x \in \mathsf{H}$. Since $P^{(\overline{A})}(\{\lambda_0\}) - I$ is self-adjoint, Lemma 2.2.22 implies that $P^{(\overline{A})}(\{\lambda_0\}) = I$ and therefore

$P^{(\overline{A})}(\mathbb{R} \setminus \{\lambda_0\}) = 0$. Computing the spectral integral of $\overline{A}$ and defining $a := \lambda_0$, this result immediately implies that $\overline{A} = \int_{\mathbb{R}} \lambda P^{(\overline{A})}(\lambda) = aI$. If the operator A is self-adjoint, then it is essentially self-adjoint too and the proof still applies to $\overline{A}$. However, as A^* is closed and $A = A^*$, we have $A = A^{**} = \overline{A}$ proving the last statement. $\square$

A different and more precise result can be obtained when the class $\mathfrak{A}$ consists of a *-closed subset of $\mathfrak{B}(\mathsf{H})$.

***Theorem* 4.3.4.** *Let H be a real, complex or quaternionic Hilbert space, $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ a *-closed subset and consider a densely-defined closed operator $A : D(A) \to \mathsf{H}$ such that*

$$UA = AU, \quad UA^* = A^*U \quad \text{for all } \; U \in \mathfrak{A}. \tag{4.9}$$

If $\mathfrak{A}$ is irreducible, then

(a) if H is real or quaternionic, then $A = aI + bJ$, with $a, b \in \mathbb{R}$ and J is an imaginary operator;

(b) if H is complex, then $A = cI$, where $c \in \mathbb{C}$.

In particular $D(A) = \mathsf{H}$ and $A \in \mathfrak{B}(\mathsf{H})$ in both the cases.

Proof. The identities $AU = UA$ and $A^*U = UA^*$ imply $A^*AU = A^*UA = UA^*A$ on the natural domain. Since A is closed, the operator A^*A is densely defined and self-adjoint (Proposition 2.2.29) and so Proposition 4.3.3 applied to the self-adjoint operator A^*A implies $A^*A = aI$ for some real a. In particular $D(A^*A) = D(aI) = \mathsf{H}$ so that $D(A) = \mathsf{H}$ and thus, since A is closed, the closed graph theorem (Theorem 2.2.17) gives $A \in \mathfrak{B}(\mathsf{H})$. To go on, decompose A as $A = \frac{A+A^*}{2} + \frac{A^*-A}{2}$ where the two addends denoted by A_S and A_A are, respectively, self-adjoint and anti-self-adjoint, they belong to $\mathfrak{B}(\mathsf{H})$ and commute with the elements of $\mathfrak{A}$, this family being *-closed. In particular, $UA_S = A_SU$ for any $U \in \mathfrak{A}$ gives $A_S = aI$ for some $a \in \mathbb{R}$, thanks to Proposition 4.3.3. Now, suppose that H is complex, then the operator iA_A is self-adjoint and commutes with the operators U so, thanks again to Proposition 4.3.3 we find $iA_A = cI$ for some $c \in \mathbb{R}$, i.e. $A_A = -ci$ and the proof is complete. Now, suppose that H is real or quaternionic. The operator $A_A^2 \in \mathfrak{B}(\mathsf{H})$ is self-adjoint and commutes with the operators in $\mathfrak{A}$, hence $A_A^2 = cI$ for some $c \in \mathbb{R}$, thanks again to Proposition 4.3.3. It must be $c \leq 0$ because, if $v \in \mathsf{H}$ has unit norm, then $c = \langle v|cv\rangle = \langle v|A_AA_Av\rangle = -\langle A_Av|A_Av\rangle = -\|A_Av\|^2 \leq 0$. In particular, $c = 0$ if and only if $A_A = 0$, that is if A is self-adjoint and in this case the proof ends. In the case $c \neq 0$, define $J := \frac{A_A}{\sqrt{-c}}$. With this definition we find $J \in \mathfrak{B}(\mathsf{H})$, $J^* = -J$ and $J^*J = -I$ so that J is an imaginary operator and $A = aI + bJ$ for $a, b \in \mathbb{R}$ ending the proof again. $\square$

***Remark* 4.3.5.** *Let us see some remarkable examples.*

(a) *Let $G \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ be a unitary group representation and consider the family*

$$\mathfrak{A} := \{U_g \mid g \in G\}.$$

In this case the hypothesis (4.9) of Proposition 4.3.4 can be weakened to

$$U_g A \subset AU_g\,, \quad \forall g \in G\,.$$

Indeed, multiplying both side by $U_{g^{-1}}$ on the left and on the right, we get $AU_{g^{-1}} \subset U_{g^{-1}}A$. Since G is a group and g is generic, this implies $AU_g \subset U_g A$. Putting the two inclusions together we get $U_g A = AU_g$. Taking the adjoint of this identity we also have $U_g^* A^* \subset A^* U_g^*$, because U_g is bounded. Again, since $U_g^* = U_{g^{-1}}$ and g ranges on the whole set G we actually have $U_g A^* \subset A^* U_g$ and thus $U_g A^* = A^* U_g$ by an argument similar to above, recovering the hypotheses of (4.9)

(b) *If A is bounded, the hypothesis (4.9) of Proposition 4.3.4 can be weakened to $UA = AU$ for $U \in \mathfrak{A}$. Indeed the second identity in (4.9) immediately follows from $UA = AU$ and the *-closedness of $\mathfrak{A}$.*

(c) *In general the irreducibility of a group unitary representation on H is lost when moving from the quaternionic Hilbert space structure to the underlying real Hilbert space one $\mathsf{H}_{\mathbb{R}}$ as the following example shows. Take $\mathsf{H} = \mathbb{H}$ and $G = SO(3)$ and define the representation $G \ni R \mapsto U_R$ defined by $U_R(a, \mathbf{b}) := (a, R\mathbf{b})$ for all $(a, \mathbf{b}) \in \mathbb{H}$. This is clearly unitary and irreducible (we are working on a one-dimensional Hilbert space). Of course $\mathsf{H}_{\mathbb{R}} = \mathbb{R}^4$ and $U_R(a, \mathbf{0}) = (a, \mathbf{0})$ for all $a \in \mathbb{R}$, hence the one-dimensional subspace $\{(a, \mathbf{0}) \mid a \in \mathbb{R}\}$ is invariant under the action of U. This make the representation reducible on $\mathbb{R}^4$.*

To conclude this general part let us consider the case of *-closed subsets of $\mathfrak{B}(\mathsf{H})$ where H is equipped with a $\mathbb{K}$-structure.

***Proposition* 4.3.6.** *Let $\mathfrak{A}$ be a *-closed subset of $\mathfrak{B}(\mathsf{H})$ for a real or quaternionic Hilbert space H equipped with a $\mathbb{K}$-structure $\mathfrak{J}$. Suppose that $\mathfrak{A} \subset \mathfrak{J}'$, then*

$$\mathfrak{A}_{\mathfrak{J}} := \{U_{\mathfrak{J}} \mid U \in \mathfrak{A}\} \subset \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$$

is irreducible if $\mathfrak{A}$ is irreducible.

Proof. Notice that $\mathfrak{A}_{\mathfrak{J}}$ is *-closed if $\mathfrak{A}$ is so and the fact that, dealing with *-closed sets, irreducibility is equivalent to the non-existence of non-trivial projectors commuting with the algebra as proved in Lemma 4.3.2. So, suppose that $\mathfrak{A}$ is irreducible and take $P \in \mathfrak{L}(\mathsf{H}_{\mathfrak{J}}) \cap (\mathfrak{A}_{\mathfrak{J}})'$. We have $P = \widehat{P}|_{\mathsf{H}_{\mathfrak{J}}}$ for some $\widehat{P} \in \mathfrak{B}(\mathsf{H})$, which is clearly an orthogonal projector on H because P is, thanks to Propositions 3.1.16 and 3.2.20. These propositions also imply that P commutes with every element of $\mathfrak{A}_{\mathfrak{J}}$ if and only if $\widehat{P}$ commutes with every element of $\mathfrak{A}$. Since $\mathfrak{A}$ is irreducible we have the thesis. $\square$

4.4 Trace-class Operators and the Gleason Theorem

In this section we give some basic notions about trace-class operators for real, complex or quaternionic Hilbert space. These turn out to be extremely important in quantum mechanics, in that they are connected to the concept of *state* of the system. The general theory for the complex case can be found in Sections 4.4 and 7.4 of [24], while definitions and properties for the real and quaternionic cases are introduced and discussed here, at a basic level which suffices to state and prove Gleason's Theorem.

Before going into the details we want to remark that a general definition of trace-class operators for a generic real, complex or quaternionic Hilbert space already appears in Section 4.2 of [44], where the author defines such operators as those bounded operators $A \in \mathfrak{B}(\mathsf{H})$ such that

$$\sum_{n=0}^{\infty} |(e_n|Ae_n)| < \infty$$

for any countable (he sticks to separable Hilbert spaces) Hilbert basis $\{e_n\}_{n\in\mathbb{N}}$ of H. If this is fulfilled, says he, then the sum

$$tr(A) := \sum_{n=0}^{\infty} (e_n|Ae_n) \tag{4.10}$$

exists for any Hilbert basis $\{e_n\}_{n\in\mathbb{N}}$ and its value is independent from the basis itself. Also, if B is a bounded operator, then AB is of trace-class too and $tr(AB) = tr(BA)$.

Even though on complex Hilbert space this definition turns out to be equivalent to the standard one (Definition 4.4.1, which appears in Section 4.4 of [24]) and the cited properties are really fulfilled, when dealing with real or quaternionic spaces we run into troubles. Take for example an infinite dimensional real Hilbert space H and let J be an imaginary operator over it, then we know that $(z|Jz) = -(Jz|z) = -(z|Jz)$, i.e. $(z|Jz) = 0$. In particular J is of trace-class, according to the definition given by Varadarajan. However JJ cannot be of trace-class, it being equal to $-I$. Another problem concerns the dependence of (4.10) on the Hilbert basis. Indeed, consider the quaternionic Hilbert space $\mathsf{H} = \mathbb{H}$ with $(s|t) = \overline{s}t$ and define the operator $A : q \mapsto i_1 q$ which is clearly linear and of trace-class (the dimension is finite). If we take as Hilbert bases the sets $\{1\}$ and $\{i_2\}$ we get

$$(1|A1) = (1|i_1) = i_1 \quad \text{and} \quad (i_2|Ai_2) = (i_2|i_1 i_2) = (i_2|-i_2 i_1) = -\overline{i_2} i_2 i_1 = -i_1.$$

The trace defined with respect to the two different bases are not equal to each other.

This shows that we need something different. It turns out that the definition commonly used in the complex setting provides a good definition of trace-class operator also for real and quaternionic Hilbert spaces. Concerning the definition of trace, we saw in the previous example that this is a delicate issue in the quaternionic framework. As we will see, for our purposes the real part of the trace (4.10) is indeed independent from the chosen Hilbert basis and suffices to state Gleason's Theorem.

Bearing in mind notation 2.1.13 we can give the following definition.

***Definition* 4.4.1.** *Let H be a real, complex or quaternionic Hilbert space. An operator $A \in \mathfrak{B}(\mathsf{H})$ is said to be of trace-class if there exists some Hilbert basis N of H such that at most countably many scalars $(z||A|z)$ are non-zero and*

$$\sum_{z\in\mathsf{N}}(z||A|z) < \infty.$$

The set of these operators is denoted by $\mathfrak{B}_1(\mathsf{H})$.

***Remark* 4.4.2.** *Notice that if A is of trace-class, then $|A|$ is of trace-class as well. This follows immediately from the identity $|(|A|)| = |A|$.*

The term *trace-class* is appropriate, as will become clear from the next results. First we state the following, whose proof can be found in [24] (see Proposition 4.36).

***Proposition* 4.4.3.** *Let H be a complex Hilbert space and N a Hilbert basis of H. Then for any $A \in \mathfrak{B}_1(\mathsf{H})$, at most countably many scalars $(z|Az)$ are non-zero and*

$$tr(A) := \sum_{z\in\mathsf{N}}(z|Az) \tag{4.11}$$

is absolutely convergent. Moreover its value is independent from the chosen Hilbert basis.

Sticking to a more general situation, where the Hilbert space might be real, complex or quaternionic, we can state the following weaker result.

***Proposition* 4.4.4.** *Let H be a real, complex or quaternionic Hilbert space and N a Hilbert basis of H. Then for any $A \in \mathfrak{B}_1(\mathsf{H})$ at most countably many scalars $\Re[(z|Az)]$ are non-zero and*

$$\sum_{z\in\mathsf{N}}\Re[(z|Az)]$$

is absolutely convergent. Moreover its value is independent from the chosen Hilbert basis.

Proof. The proof for the complex case follows from Proposition 4.4.3 by taking the real part of (4.11). Suppose that H is real and consider the operator $A_{\mathbb{C}}$ on $\mathsf{H}_{\mathbb{C}}$. Consider a Hilbert basis N for H as in Definition 4.4.1, then we know from Proposition 3.1.3 that $\{z\otimes 1 \mid z \in \mathsf{N}\}$ is a Hilbert basis for $\mathsf{H}_{\mathbb{C}}$ and of course $(z\otimes 1||A_{\mathbb{C}}|z\otimes 1)_{\mathbb{C}} = (z||A|z)$, where we used the the fact that $|A_{\mathbb{C}}| = |A|_{\mathbb{C}}$, as proved by Proposition 4.2.2. By the hypothesis on N we see that at most countably many of the scalars $(z\otimes 1||A_{\mathbb{C}}|z\otimes 1)$ are non-zero and

$$\sum_{z\in\mathsf{N}}(z\otimes 1||A_{\mathbb{C}}|z\otimes 1|)_{\mathbb{C}} = \sum_{z\in\mathsf{N}}(z||A|z) < \infty.$$

This show that $A_{\mathbb{C}}$ is of trace-class on $\mathsf{H}_{\mathbb{C}}$. Thus, exploiting Proposition 4.4.3 we see that at most countably many of the scalars $(z\otimes 1|A_{\mathbb{C}}z\otimes 1)_{\mathbb{C}}$ are non-zero and

$$\sum_{z\in\mathsf{N}}(z\otimes 1|A_{\mathbb{C}}z\otimes 1)_{\mathbb{C}}$$

is absolutely convergent in $\mathbb{C}$. Moreover its value does not depend on the particular Hilbert basis used. Since $(z\otimes 1|A_{\mathbb{C}}z\otimes 1)_{\mathbb{C}}=(z|Az)$ we see that at most countably many of the scalars $(z|Az)$ are non-zero and the sum

$$\sum_{z\in\mathsf{N}}(z|Az)\left(=\sum_{z\in\mathsf{N}}(z\otimes 1|A_{\mathbb{C}}z\otimes 1)_{\mathbb{C}}\right)$$

is absolutely convergent. Now, suppose that M is another Hilbert basis for H, then we can repeat the discussion carried out for N. In particular, since $\{\omega\otimes 1\,|\,\omega\in\mathsf{M}\}$ is a Hilbert basis for $\mathsf{H}_{\mathbb{C}}$ and $A_{\mathbb{C}}$ is of trace-class on $\mathsf{H}_{\mathbb{C}}$ then

$$\sum_{\omega\in\mathsf{M}}(\omega\otimes 1|A_{\mathbb{C}}\omega\otimes 1)_{\mathbb{C}}=tr(A_{\mathbb{C}})=\sum_{z\in\mathsf{N}}(z\otimes 1|A_{\mathbb{C}}z\otimes 1)_{\mathbb{C}}.$$

At this point, since $(\omega|A\omega)=(\omega\otimes 1|A_{\mathbb{C}}\omega\otimes 1)_{\mathbb{C}}$ and $(z|Az)=(z\otimes 1|A_{\mathbb{C}}z\otimes 1)_{\mathbb{C}}$ we see that

$$\sum_{\omega\in\mathsf{M}}(\omega|A\omega)=\sum_{z\in\mathsf{N}}(z|Az),$$

concluding the proof of the real case (notice that $(\cdot|\cdot)=\Re[(\cdot|\cdot)]$ in this case).

Now, suppose that H is quaternionic and take its complex counterpart $\mathsf{H}_{\mathbb{C}}$. Let N be a Hilbert basis for H as in Definition 4.4.1, then $\{\mathcal{J}_\alpha z\,|\,z\in\mathsf{N},\alpha\in\mathcal{A}_{\mathbb{C}}\}$ is a Hilbert basis for $\mathsf{H}_{\mathbb{C}}$. As usual, let $B_{\mathbb{C}}$ denote the generic operator $B\in\mathfrak{B}(\mathsf{H})$ when understood as a $\mathbb{C}$-linear operator on $\mathsf{H}_{\mathbb{C}}$. We know that $|A_{\mathbb{C}}|=|A|_{\mathbb{C}}=|A|$ (see Proposition 4.2.2). Since $|A|$ is self-adjoint (on both spaces) it holds that $(x||A_{\mathbb{C}}|x)_{\mathbb{C}}=(x||A|x)\in\mathbb{R}$ for every $x\in\mathsf{H}=\mathsf{H}_{\mathbb{C}}$. Thus we have $(z||A_{\mathbb{C}}|z)_{\mathbb{C}}=(z||A|z)$ and

$$(\mathcal{J}_3z||A_{\mathbb{C}}|\mathcal{J}_3z)_{\mathbb{C}}=\overline{(z|-\mathcal{J}_3|A_{\mathbb{C}}|\mathcal{J}_3z)_{\mathbb{C}}}=\overline{(z||A_{\mathbb{C}}|z)_{\mathbb{C}}}=(z||A|z)$$

where we used $\mathcal{J}_3|A|\subset|A|\mathcal{J}_3$ and $-\mathcal{J}_3\mathcal{J}_3=I$. By the hypothesis on N we see that at most countably many of the scalars $(\mathcal{J}_\alpha z||A_{\mathbb{C}}|\mathcal{J}_\alpha z)_{\mathbb{C}}$ are non-zero and

$$\sum_{z\in\mathsf{N}}\sum_{\alpha\in\mathcal{A}_{\mathbb{C}}}(\mathcal{J}_\alpha z||A_{\mathbb{C}}|\mathcal{J}_\alpha z)_{\mathbb{C}}=2\sum_{z\in\mathsf{N}}(z||A|z)<\infty,$$

which proves that $A_{\mathbb{C}}$ is of trace-class on $\mathsf{H}_{\mathbb{C}}$. Hence we can apply again Proposition 4.4.3 to $A_{\mathbb{C}}$ and find that at most countably many of the scalars $(\mathcal{J}_\alpha z|A_{\mathbb{C}}\mathcal{J}_\alpha z)_{\mathbb{C}}$ are

non-zero and the sum
$$\sum_{z \in \mathsf{N}} \sum_{\alpha \in \mathcal{A}_{\mathbb{C}}} (\mathcal{J}_\alpha z | A_{\mathbb{C}} \mathcal{J}_\alpha z)_{\mathbb{C}}$$
converges absolutely in $\mathbb{C}$. Moreover its value does not depend on the chosen Hilbert basis. As above it holds that
$$(\mathcal{J}_3 z | A_{\mathbb{C}} \mathcal{J}_3 z)_{\mathbb{C}} = \overline{(z| - \mathcal{J}_3 A_{\mathbb{C}} \mathcal{J}_3 z)_{\mathbb{C}}} = \overline{(z|A_{\mathbb{C}} z)_{\mathbb{C}}}$$
and so
$$\sum_{\alpha \in \mathcal{A}_{\mathbb{C}}} (\mathcal{J}_\alpha z | A_{\mathbb{C}} \mathcal{J}_\alpha z)_{\mathbb{C}} = (z|A_{\mathbb{C}} z)_{\mathbb{C}} + \overline{(z|A_{\mathbb{C}} z)_{\mathbb{C}}} = 2\Re[(z|A_{\mathbb{C}} z)_{\mathbb{C}}] = 2\Re[(z|Az)].$$
We see that at most countably many of the scalars $\Re[(z|Az)]$ are non-zero and the sum
$$\sum_{z \in \mathsf{N}} \Re[(z|Az)] = \frac{1}{2} \sum_{z \in \mathsf{N}} \sum_{\alpha \in \mathcal{A}_{\mathbb{C}}} (\mathcal{J}_\alpha z | A_{\mathbb{C}} \mathcal{J}_\alpha z)_{\mathbb{C}}$$
converges absolutely. Now, suppose that M is a different Hilbert basis for H, then we can repeat the discussion carried out for N. In particular, since $\{\mathcal{J}_\alpha \omega \,|\, \alpha \in \mathcal{A}_{\mathbb{C}},\ \omega \in \mathsf{M}\}$ is a Hilbert basis for $\mathsf{H}_{\mathbb{C}}$ and $A_{\mathbb{C}}$ is of trace-class on $\mathsf{H}_{\mathbb{C}}$, then
$$\sum_{z \in \mathsf{N}} \sum_{\alpha \in \mathcal{A}_{\mathbb{C}}} (\mathcal{J}_\alpha z | A_{\mathbb{C}} \mathcal{J}_\alpha z)_{\mathbb{C}} = tr(A_{\mathbb{C}}) = \sum_{\omega \in \mathsf{M}} \sum_{\alpha \in \mathcal{A}_{\mathbb{C}}} (\mathcal{J}_\alpha \omega | A_{\mathbb{C}} \mathcal{J}_\alpha \omega)_{\mathbb{C}}.$$
At this point, since
$$\Re[(\omega|A\omega)] = \frac{1}{2} \sum_{\alpha \in \mathcal{A}_{\mathbb{C}}} (\mathcal{J}_\alpha \omega | A_{\mathbb{C}} \mathcal{J}_\alpha \omega)_{\mathbb{C}}, \quad \Re[(z|Az)] = \frac{1}{2} \sum_{\alpha \in \mathcal{A}_{\mathbb{C}}} (\mathcal{J}_\alpha z | A_{\mathbb{C}} \mathcal{J}_\alpha z)_{\mathbb{C}},$$
we see that
$$\sum_{\omega \in \mathsf{M}} \Re[(\omega|A\omega)] = \sum_{z \in \mathsf{N}} \Re[(z|Az)],$$
concluding the proof. □

This suggests the following definition.

***Definition* 4.4.5.** *If $A \in \mathfrak{B}(\mathsf{H})$, then the real trace of A is defined as*
$$tr_{\mathbb{R}}(A) := \sum_{z \in \mathsf{N}} \Re[(z|Az)], \tag{4.12}$$
where N is any Hilbert basis of H.

***Remark* 4.4.6.** The restriction to the *real trace*, instead of considering the entire sum as in (4.11), is necessary in order to have a meaningful definition of trace in the real and quaternionic setting. For instance, going back to the counterexample involving the operator $A : \mathbb{H} \ni q \mapsto i_1 q \in \mathbb{H}$ discussed in the introduction to this section, we see that, even though the trace (4.10) does depend on the Hilbert basis its real part does not. In any case the real trace suffices for our purposes, for we are mainly interested in self-adjoint trace-class operators and so it would be $(u|Au) \in \mathbb{R}$ anyway.

Exploiting this definition and the proof of Proposition 4.4.4 we can prove the following result.

***Proposition* 4.4.7.** *Let H be real or quaternionic and $A \in \mathfrak{B}(\mathsf{H})$, then $A \in \mathfrak{B}_1(\mathsf{H})$ if and only if $A_{\mathbb{K}} \in \mathfrak{B}_1(\mathsf{H}_{\mathbb{K}})$. Moreover the following identities hold*

$$tr_{\mathbb{R}}(A) = tr_{\mathbb{R}}(A_{\mathbb{K}}) \quad \text{if } \mathsf{H} \text{ is real,} \tag{4.13}$$

$$tr_{\mathbb{R}}(A) = \frac{1}{d_{\mathbb{K}}+1} tr_{\mathbb{R}}(A_{\mathbb{K}}) \quad \text{if } \mathsf{H} \text{ is quaternionic.}$$

Proof. Suppose first that $\mathbb{K} = \mathbb{C}$. In the proof of Proposition 4.4.4 we saw that a trace-class operator A on H gives rise to a trace-class operator $A_{\mathbb{C}}$ on $\mathsf{H}_{\mathbb{C}}$ and that (4.13) holds. Let us prove the opposite relation. So, take $A \in \mathfrak{B}(\mathsf{H})$ such that $A_{\mathbb{C}}$ is of trace-class on $\mathsf{H}_{\mathbb{C}}$. Exploiting Definition 4.4.1 and Proposition 4.4.4 and by noticing that $|T|$ is trivially of trace-class if T is of trace-class we see that, fixed any basis $\mathsf{M} \subset \mathsf{H}_{\mathbb{C}}$, at most countably many of the scalars $(\omega||A_{\mathbb{C}}|\omega)_{\mathbb{C}}$ are non-zero and the sum $\sum_{\omega\in\mathsf{M}}(\omega||A_{\mathbb{C}}|\omega)_{\mathbb{C}}$ converges. So, suppose first that H is real. If we fix any basis N of H and consider the Hilbert basis $\{z \otimes 1 \,|\, z \in \mathsf{N}\}$ for $\mathsf{H}_{\mathbb{C}}$, then $(z||A|z) = (z\otimes 1||A_{\mathbb{C}}|z \otimes 1)_{\mathbb{C}}$ and so at most countably many of the $(z||A|z)$ are non-null and the sum

$$\sum_{z\in\mathsf{N}}(z||A|z) = \sum_{z\in\mathsf{N}}(z\otimes 1||A_{\mathbb{C}}|z\otimes 1)_{\mathbb{C}}$$

converges. This implies that A is of trace-class on H. If H is quaternionic and N is a Hilbert basis of H, then $\{\mathcal{J}_\alpha z \,|\, z \in \mathsf{N},\ \alpha \in \mathcal{A}_{\mathbb{C}}\}$ is a Hilbert basis for $\mathsf{H}_{\mathbb{C}}$ and it holds that

$$(\mathcal{J}_3 z||A_{\mathbb{C}}|\mathcal{J}_3 z)_{\mathbb{C}} = \overline{(z|-\mathcal{J}_3|A_{\mathbb{C}}|\mathcal{J}_3 z)_{\mathbb{C}}} = \overline{(z||A_{\mathbb{C}}|z)_{\mathbb{C}}} = (z||A_{\mathbb{C}}|z)_{\mathbb{C}} = (z||A|z),$$

where we used the self-adjointness of $|A|, |A_{\mathbb{C}}|$ (they coincide as functions). So, at most countably many of the scalars $(z||A|z)$ are non-zero and the sum

$$\sum_{z\in\mathsf{N}}(z||A|z) = \frac{1}{2}\sum_{z\in\mathsf{N}}\sum_{\alpha\in\mathcal{A}_{\mathbb{C}}}(\mathcal{J}_\alpha z||A_{\mathbb{C}}|\mathcal{J}_\alpha z)_{\mathbb{C}}$$

converges, which implies that A is of trace-class on H. If $\mathbb{K} \neq \mathbb{C}$ the proof is analogous. $\square$

We are ready to state and prove some basic properties of trace-class operators.

***Proposition* 4.4.8.** *Let H be a real, complex or quaternionic Hilbert space, then the following statements hold:*

(a) the set $\mathfrak{B}_1(\mathsf{H})$ is a linear subspace of $\mathfrak{B}(\mathsf{H})$ and it is closed under the Hermitean adjoint operation, moreover

$$tr_{\mathbb{R}}(aA+bB) = a\,tr_{\mathbb{R}}(A) + b\,tr_{\mathbb{R}}(B) \quad \textit{and} \quad tr_{\mathbb{R}}(A^*) = tr_{\mathbb{R}}(A) \tag{4.14}$$

for all $A, B \in \mathfrak{B}_1(\mathsf{H})$ and $a, b \in \mathbb{R}$.

(b) if $A, T \in \mathfrak{B}(\mathsf{H})$ and A is of trace-class, then TA and AT are of trace-class, moreover

$$tr_{\mathbb{R}}(AT) = tr_{\mathbb{R}}(TA). \tag{4.15}$$

Proof. For the complex case we refer to Theorem 4.34 and Proposition 4.38 of [24]. So, suppose that H is real or quaternionic. Let us start with point (a) and take $A, B \in \mathfrak{B}_1(\mathsf{H})$ and $c \in \mathbb{R}$. It is easy to see that $|cA| = |c||A|$ showing that cA is of trace-class if A is so. Now, let us move to the complexified space $\mathsf{H}_{\mathbb{C}}$ and consider the complexified operators $A_{\mathbb{C}}, B_{\mathbb{C}}$. We know from Proposition 4.4.7 that $A_{\mathbb{C}}, B_{\mathbb{C}}$ are of trace-class on $\mathsf{H}_{\mathbb{C}}$ and so, exploiting the complex version of this proposition, the operators $(A_{\mathbb{C}})^* = (A^*)_{\mathbb{C}}$ and $A_{\mathbb{C}} + B_{\mathbb{C}} = (A+B)_{\mathbb{C}}$ are of trace-class on $\mathsf{H}_{\mathbb{C}}$. Proposition 4.4.7 assures that A^* and $A+B$ are of trace-class on H. The identities in (4.14) follows again by Proposition 4.4.7 and the complex version of this proposition. Let us pass to point (b). The real and quaternionic cases can be easily proved by moving to $\mathsf{H}_{\mathbb{C}}$ and exploiting the complex part of this proposition together with Proposition 4.4.7. □

***Proposition* 4.4.9.** *Let H be real or quaternionic and $\mathfrak{J}$ an $\mathbb{K}$-structure on it. Suppose $A \in \mathfrak{B}(\mathsf{H})$ satisfies $A \in \mathfrak{J}'$, then $A \in \mathfrak{B}_1(\mathsf{H})$ if and only if $A_{\mathfrak{J}} \in \mathfrak{B}_1(\mathsf{H}_{\mathfrak{J}})$. Moreover the following identities hold:*

$$tr_{\mathbb{R}}(A_{\mathfrak{J}}) = \frac{1}{d_{\mathbb{K}}+1} tr_{\mathbb{R}}(A) \quad \textit{if } \mathsf{H} \textit{ is real,} \tag{4.16}$$

$$tr_{\mathbb{R}}(A_{\mathfrak{J}}) = tr_{\mathbb{R}}(A) \quad \textit{if } \mathsf{H} \textit{ is quaternionic.}$$

Proof. Suppose H is real and take $A \in \mathfrak{B}(\mathsf{H})$ such that $AJ = JA$ for any $J \in \mathfrak{J}$, then we can consider $A_{\mathfrak{J}}(= A) \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$. Similarly $J|A| = |A|J$ for any $J \in \mathfrak{J}$ and $|A_{\mathfrak{J}}| = |A|_{\mathfrak{J}}(= |A|)$. Suppose first that $A_{\mathfrak{J}}$ is of trace-class on $\mathsf{H}_{\mathfrak{J}}$, then there exists some Hilbert basis N of $\mathsf{H}_{\mathfrak{J}}$ such that at most countably many of the scalars $(z||A_{\mathfrak{J}}|z)_{\mathfrak{J}}$ are non-zero and it holds that $\sum_{z\in\mathsf{N}}(z||A_{\mathfrak{J}}|z)_{\mathfrak{J}} < \infty$. Notice that $|A_{\mathfrak{J}}|$ is self-adjoint and so

$$(x||A_{\mathfrak{J}}|x)_{\mathfrak{J}} = \Re[(x||A_{\mathfrak{J}}|x)_{\mathfrak{J}}] = (x||A_{\mathfrak{J}}|x) = (x||A|x)$$

for any $x \in \mathsf{H}_{\mathfrak{J}} = \mathsf{H}$. In particular we have $(z||A|z) = (z||A_{\mathfrak{J}}|z)_{\mathfrak{J}} \in \mathbb{R}$ for $z \in \mathsf{N}$ and

$$(Jz||A|Jz) = (z|-J|A|Jz) = (z||A|z) = (z||A_{\mathfrak{J}}|z)_{\mathfrak{J}}$$

if $J \in \mathfrak{J}$. Thus we get

$$\sum_{z \in \mathsf{N}} \sum_{\alpha=0}^{d_{\mathbb{K}}} (J_\alpha z||A|J_\alpha z) = (d_{\mathbb{K}} + 1) \sum_{z \in \mathsf{N}} (z||A_{\mathfrak{J}}|z)_{\mathfrak{J}} < \infty.$$

Since $\{J_\alpha z \mid 0 \leq \alpha \leq d_{\mathbb{K}}, z \in \mathsf{N}\}$ is a Hilbert basis for H, then this proves that A is of trace-class on H. Moreover, since $(Jz|AJz) = (z|-JAJz) = (z|Az)$ for all $J \in \mathfrak{J}$ and the real trace does not depend on the chosen basis:

$$tr_{\mathbb{R}}(A) = \sum_{z \in \mathsf{N}} \sum_{\alpha=0}^{d_{\mathbb{K}}} (J_\alpha z|AJ_\alpha z) = (d_{\mathbb{K}} + 1) \sum_{z \in \mathsf{N}} (z|Az) = (d_{\mathbb{K}} + 1) \sum_{z \in \mathsf{N}} \Re[(z|Az)_{\mathfrak{J}}] =$$
$$= (d_{\mathbb{K}} + 1) \sum_{z \in \mathsf{N}} \Re[(z|A_{\mathfrak{J}} z)_{\mathfrak{J}}] = (d_{\mathbb{K}} + 1) tr_{\mathbb{R}}(A_{\mathfrak{J}}).$$

Suppose on the contrary that A is of trace-class on H and let N be a Hilbert basis for $\mathsf{H}_{\mathfrak{J}}$, then, again $\{J_\alpha z \mid 0 \leq \alpha \leq d_{\mathbb{K}}, z \in \mathsf{N}\}$ is a Hilbert basis for H. Since $|A|$ is of trace-class as well as A, then at most countably many of the scalars $(J_\alpha z||A|J_\alpha z)$ are non-zero and the sum $\sum_{z \in \mathsf{N}} \sum_{\alpha=0}^{d_{\mathbb{K}}} (J_\alpha z||A|J_\alpha z)$ converges. Since

$$(Jz||A|Jz) = (z||A|z) = \Re[(z||A_{\mathfrak{J}}|z)_{\mathfrak{J}}] = (z||A_{\mathfrak{J}}|z)_{\mathfrak{J}}$$

for all $J \in \mathfrak{J}$, the above sum turns out to be equal to $(d_{\mathbb{K}} + 1) \sum_{z \in \mathsf{N}} (z||A_{\mathfrak{J}}|z)_{\mathfrak{J}}$ which means that $A_{\mathfrak{J}}$ is of trace-class. The quaternionic case can be proved similarly, without particular effort. $\square$

***Proposition* 4.4.10.** *Let H be a real, complex or quaternionic Hilbert and $T \in \mathfrak{B}_1(\mathsf{H})$. Suppose that $T \geq 0$, then it holds that*

$$0 \leq tr_{\mathbb{R}}(TP) \leq tr_{\mathbb{R}}(T)$$

for any orthogonal projector $P \in \mathfrak{L}(\mathsf{H})$.

Proof. If $P \in \mathfrak{L}(\mathsf{H})$ then $TP \in \mathfrak{B}_1(\mathsf{H})$ as guaranteed by Proposition 4.4.8. Moreover, (4.15) gives

$$tr_{\mathbb{R}}(TP) = tr_{\mathbb{R}}(TPP) = tr_{\mathbb{R}}(PTP) = \sum_{z \in \mathsf{N}} \Re[(z|PTPz)] = \sum_{z \in \mathsf{N}} \Re[(Pu|TPu)] \geq 0$$

because $T \geq 0$, where N is any basis of H. Now, let N_1 be a Hilbert basis of $P(\mathsf{H})$ and N_2 be a Hilbert basis of $P(\mathsf{H})^\perp$, thus $\mathsf{N}_1 \cup \mathsf{N}_2$ is a Hilbert basis for $\mathsf{H} = P(\mathsf{H}) \oplus P(\mathsf{H})^\perp$ and so, thanks again to the positivity of T we have

$$tr_{\mathbb{R}}(TP) = \sum_{z \in \mathsf{N}} \Re[(z|TPz)] = \sum_{z \in \mathsf{N}_1} \Re[(z|Tz)] \leq \sum_{z \in \mathsf{N}} \Re[(z|Tz)] = tr_{\mathbb{R}}(T)$$

since $Pz = 0$ for any $z \in \mathsf{N}_2$, concluding the proof. □

***Lemma* 4.4.11.** *Let H be a real, complex or quaternionic Hilbert space and A_n, A, B be trace-class operators such that w-$\lim_{n\to\infty} A_n = A$. Suppose there exists a Hilbert basis N such that $|\Re[(z|A_n z)]| \leq |\Re[(z|Bz)]|$ for any $n \in \mathbb{N}$ and $z \in \mathsf{N}$, then*

$$\lim_{n\to\infty} tr_{\mathbb{R}}(A_n) = tr_{\mathbb{R}}(A). \tag{4.17}$$

Proof. Let N be the Hilbert basis in the hypothesis and equip it with the counting measure ν. Since the operators A_n, A, B are of trace-class then the sums $\sum_{z\in\mathsf{N}} \Re[(z|A_n z)]$, $\sum_{z\in\mathsf{N}} \Re[(z|Az)]$ and $\sum_{z\in\mathsf{N}} \Re[(z|Bz)]$ converges absolutely, where at most countably many of their terms are different from zero. They can be interpreted as the integral of the following (measurable) functions

$$\begin{cases} f_n : \mathsf{N} \ni z \mapsto \Re[(z|A_n z)] \in \mathbb{R} \\ f : \mathsf{N} \ni z \mapsto \Re[(z|Az)] \in \mathbb{R} \\ g : \mathsf{N} \ni z \mapsto \Re[(z|Bz)] \in \mathbb{R} \end{cases} . \tag{4.18}$$

Notice that: 1) $f_n, f, g \in \mathcal{L}^1(\mathsf{N}, \mathbb{R}, \nu)$, 2) $f_n(z) \to f(z)$ for every $z \in \mathsf{N}$ and $|f_n(z)| \leq |g(z)|$ for every $z \in \mathsf{N}$. The dominated convergence theorem assures that

$$tr_{\mathbb{R}}(A_n) = \sum_{z\in\mathsf{N}} \Re[(z|A_n z)] = \int_{\mathsf{N}} f_n(z)\, d\nu \to \int_{\mathsf{N}} f(z)\, d\nu = \sum_{z\in\mathsf{N}} \Re[(z|Az)] = tr_{\mathbb{R}}(A),$$

which concludes the proof. □

Exploiting this lemma we can prove the following important result. This becomes crucial in the characterisation of the states of quantum system as σ-probability measures - see Definition 4.4.16 and Theorem 4.4.17.

***Proposition* 4.4.12.** *Let H be a real, complex or quaternionic Hilbert space and suppose $T \in \mathfrak{B}_1(\mathsf{H})$. Consider a countable family $(P_i)_{i\in\mathbb{N}} \subset \mathfrak{L}(\mathsf{H})$ such that $P_n P_m = 0$ if $n \neq m$, then*

$$tr_{\mathbb{R}}\left(T s\text{-}\sum_{i=0}^{\infty} P_i\right) = \lim_{n\to\infty} tr_{\mathbb{R}}\left(T \sum_{i=0}^{n} P_i\right). \tag{4.19}$$

Proof. Define $Q_n := \sum_{i=0}^{n} P_i \in \mathfrak{L}(\mathsf{H})$ and $Q := \bigvee_{i\in\mathbb{N}} P_i \in \mathfrak{L}(\mathsf{H})$. Of course we have $Q_n \leq Q$, $Q = \text{s-}\lim_{n\to\infty} Q_n$ (check Proposition 2.2.33) and $TQ, TQ_n \in \mathfrak{B}_1(\mathsf{H})$ for any $n \in \mathbb{N}$ as proved in Proposition 4.4.8. Moreover we have $TQ = \text{w-}\lim_{n\to\infty} TQ_n$, indeed notice that for any $u, v \in \mathsf{H}$ it holds that

$$|(u|TQ_n v) - (u|TQv)| = |(u|T(Q_n v - Qv))| \leq \|u\|\|T\|\|Q_n v - Qv\|,$$

which goes to zero by the properties of Q. If we manage to prove that

$$|\Re[(z|TQ_n z)]| \leq |\Re[(z|TQz)]|$$

for z in some particular Hilbert basis N and for every $n \in \mathbb{N}$ then the thesis follows as a direct application of Lemma 4.4.11: $tr(TQ_n) \to tr(TQ)$. Let M_i be a Hilbert basis of $P_i(\mathsf{H})$, then $\mathsf{N}_n := \bigcup_{i=0}^{n} \mathsf{M}_i$ is a Hilbert basis for $Q_n(\mathsf{H})$. Thus $\mathsf{N}' := \bigcup_{n=0}^{\infty} \mathsf{N}_n$ is a Hilbert basis for $Q(\mathsf{H})$. If N'' is any Hilbert basis for $Q(\mathsf{H})^\perp$, then $\mathsf{N}' \cup \mathsf{N}''$ is a Hilbert basis for the whole H. Notice that $Q_n z = 0$ for any $z \in \mathsf{N}' \setminus \mathsf{N}_n$. Thus we have

$$|\Re[(z|TQ_n z)]| = \begin{cases} |\Re[(z|Tz)]| = |\Re[(z|TQz)]| & z \in \mathsf{N}_n \\ 0 \leq |\Re[(z|TQz)]| & z \in \mathsf{N}' \setminus \mathsf{N}_n \\ 0 = |\Re[(z|TQz)]| & z \in \mathsf{N}'' \end{cases}, \tag{4.20}$$

which gives the general inequality $|\Re[(z|TQ_n z)]| \leq |\Re[(z|TQz)]|$, concluding the proof. $\square$

A particularly important class of trace-class operator is the one of *density operators*.

***Definition* 4.4.13.** *A trace-class operator $A \in \mathfrak{B}(\mathsf{H})$ is said to be a density operator if it is self-adjoint, positive and has unit trace. The class of these operators is denoted by $\mathfrak{D}(\mathsf{H})$.*

***Remark* 4.4.14.** Notice that the self-adjointness condition can be omitted in the complex and quaternionic case, as it is a direct consequence of the positivity (see Proposition 2.2.25). Anyway in the real case this is not generally true and must be required in order to assures the uniqueness. Let us see this in detail. If $T \in \mathfrak{B}(\mathsf{H})$ is of trace-class then the same is true for $T^*, T - T^*$ and $T + T^*$. Let $P \in \mathfrak{L}(\mathsf{H}) \setminus \{0, I\}$ be any non-trivial projector, N_1 a Hilbert basis for $P(\mathsf{H})$ and N_2 a Hilbert basis for $P(\mathsf{H})^\perp$, then $\mathsf{N} := \mathsf{N}_1 \cup \mathsf{N}_2$ is a Hilbert basis for H and

$$tr_{\mathbb{R}}((T - T^*)P) = \sum_{z \in \mathsf{N}} (z|(T - T^*)Pz) = \sum_{z \in \mathsf{N}_1} (z|(T - T^*)z) = 0,$$

where the last equality follows from the fact that $(x|Bx) = -(Bx|x) = -(x|Bx)$ and so $(x|Bx) = 0$ for any $x \in \mathsf{H}$ if B is anti-self-adjoint. This shows that

$$tr_{\mathbb{R}}(TP) = tr_{\mathbb{R}}((T + T^*)P) + tr_{\mathbb{R}}((T - T^*)P) = tr_{\mathbb{R}}((T + T^*)P).$$

En passant we have just proved an important result:

***Proposition* 4.4.15.** *On a real Hilbert space any anti-self-adjoint trace-class operator has vanishing real trace.*

Density operators are extremely important in quantum mechanics, in that they fully characterise the states of the systems. We will see this in details in the next chapter.

For the moment being, focus on the following definition (compare with Proposition 2.2.33).

***Definition* 4.4.16.** *Let H be a real, complex or quaternionic Hilbert space, a σ-probability measure is a function $\mu : \mathfrak{L}(\mathsf{H}) \to [0,1]$ satisfying the following properties:*

(a) $\mu(I) = 1$,

(b) if $(P_n)_{n\in\mathbb{N}} \subset \mathfrak{L}(\mathsf{H})$ and $P_nP_m = 0$ for $n \neq m$, then

$$\mu\left(\bigvee_{n\in\mathbb{N}} P_n\right) = \sum_{n=0}^{\infty} \mu(P_n).$$

We can ready to state the following extremely important result.

***Theorem* 4.4.17** (Gleason's Theorem)**.** *Let H be an infinite-dimensional separable real, complex or quaternionic Hilbert space, then the following statements hold:*

(a) if $T \in \mathfrak{D}(\mathsf{H})$, then the function $\mu_T : \mathfrak{L}(\mathsf{H}) \to [0,1]$ defined by

$$\mu_T(P) := tr_{\mathbb{R}}(TP) \quad \textit{for all} \quad P \in \mathfrak{L}(\mathsf{H}) \tag{4.21}$$

is a σ-probability measure on $\mathfrak{L}(\mathsf{H})$;

(b) for every σ-probability measure $\mu : \mathfrak{L}(\mathsf{H}) \to [0,1]$ there exists a unique density operator $T \in \mathfrak{D}(\mathsf{H})$ such that $\mu = \mu_T$, where μ_T is defined as in (4.21).

Proof. Point (a) follows from Propositions 4.4.10 and 4.4.12. Point (b) is the true content of Gleason's Theorem and a statement of this result for the case of a Hilbert space with generic division algebra can be found in Theorem 4.23 of [44]. However it should be remarked that the actual analysis done by Varadarajan in his book is not fully valid since, as we already pointed out in the introduction to this section, his definition of trace-class operators is not consistent. Nevertheless, despite these difficulties, sticking to our definition of real trace, the main part of the analysis carried out by Varadarajan on σ-probability measures applies and the Gleason Theorem can be proved to hold true. More precisely, notice that for *positive self-adjoint operator* the definition of trace given by Varadarajan (denoted by tr_V) is well-posed, in that it coincides with the definition of real trace defined in this section. The principal part of the proof given in [44] shows that for any σ-probability measure μ there exists a self-adjoint positive operator T on H which is of trace-class and with unit trace with respect to tr_V which satisfies $\mu(P) = tr_V(PTP)$ for any element P in the set of orthogonal projectors (notice that PTP is positive and self-adjoint). At this point the conclusion is immediate since T is also unit-trace class with respect to our definition of trace and $tr_{\mathbb{R}}(TP) = tr_{\mathbb{R}}(PTP) = tr_V(PTP) = \mu(P)$. □

***Notation* 4.4.18.** In the following chapters, if there is no risk of confusion, we will drop the subscript $\mathbb{R}$ from $tr_{\mathbb{R}}$. From now on the term tr will always denote the real trace.

Chapter 5

Elements of Lattice Theory

In this chapter, introduced by a brief summary of quantum logic, we outline the general definitions and discuss the main results of the theory of lattices. A more detailed discussion on these topics can be found in [6], [10], [23], [30] or [24].

5.1 Quantum Mechanics and Its Propositional Calculus

The birth of quantum logic traces back to the pioneering work by Birkhoff and von Neumann in the thirties [4]. Since then, this theory has grown fast and deep, developed through decades by several important physicists and mathematicians until today. Remarkable results has been obtained, like Piron's and Solèr's theorems (see next section)

In parallel with the first successful attempts to give quantum mechanics a proper mathematical guise in terms of Hilbert spaces, several apparent conceptual inconsistencies required a completely new approach of investigation.

The main issue is a direct consequence of the existence of *incompatible* physical observables, like for instance the momentum and the position of a particle, whose simultaneous measurement is physically unfeasible. This turned out not to be a mere technical limit of the actual experimental tools, but a true constraint of nature at microscopical level: the two measurements disturb drastically each other. Thus, differentely from classical mechanics the truth value of a proposition like *the particle momentum is p and the position particle is x* is difficult to interpret, since there is no way to measure them at the same time.

Quantum logic arises then as a natural attempt to reconcile these new phenomena with the construction of some suitable propositional calculus, as it is expected to be in any brand of physics.

Birkhoff and von Neumann in their paper analysed the Hilbert space structure and recognised the set of orthogonal projectors $\mathfrak{L}(\mathsf{H})$ (see Definition 2.2.30) as the true set of *quantum propositions* and the operations $\leq, \wedge, \vee, \perp$ as the natural logical connectives among propositions. The found framework differs from the Boolean algebra structure of the logic of classical physics because of the general absense of the distributivity property among the connectives:

$$p \wedge (q \vee q') \neq (p \wedge q) \vee (p \vee q').$$

Anyway, when dealing with *compatible* observables, which instead can be tested at the same time, this property is recovered as we expect.

In this first subsection we want to discuss this logical structure, basing upon a slightly more modern formalism than the one exploited in [4]. In order to do this let us introduce briefly the traditional framework of quantum mechanics (we are neither considering superselection rules nor gauge symmetries, in particular this is the standard framework for an elementary particle with Poincaré group as maximal symmetry group).

- The mathematical setting is a *complex Hilbert space* H.
- The *observables* - i.e. the measurable quantities - of the systems are in one-to-one correspondence with the *self-adjoint operators* on H.
- For any observable A the *possible outcomes* - i.e. the values that A may attain when measured - are in one-to-one correspondence with the elements of the *spectrum* $\sigma(A)$.
- The *states* of the systems are in one-to-one correspondence with the *density operators* $\rho \in \mathfrak{D}(\mathsf{H})$.

The essence of quantum mechanics lies in its non deterministic nature: only probability distributions of the possible outcomes, together with expectation values, moments and so on, are knowable. For example, given any state $\rho \in \mathfrak{D}(\mathsf{H})$ the expecation value of any observable $A \in \mathfrak{B}(\mathsf{H})$ is given by

$$\langle A \rangle_\rho = tr(\rho A) \tag{5.1}$$

So, if we repeat a measurement of A on the system in the fixed state ρ an infinite numbers of times, then we get a distribution of outcomes with mean value $\langle A \rangle_\rho$.

As already said above, the peculiar characteristic of quantum mechanics is the existence of *incompatible observables*, i.e. couple of physical quantities that cannot be measured at the same time with one tester. This happens because the two measurements disturb drastically each other: consecutives tests of the two quantities give different values each time they are performed, making it impossible to define their *simultaneous value*. From an mathematical point of view they are represented by non-commuting operators.

- Two observables are *compatible* if they strongly-commute with each other. Otherwise, they are said to be *incompatible*.

Now, focus on the orthogonal projectors $\mathfrak{L}(\mathsf{H})$. As *self-adjoint* operators, they represent observables of the system. More precisely notice that (see Proposition 2.2.37)

$$\sigma(P) = \{0, 1\}.$$

The possible values that P can attain in experiments can be understood as {yes, no} outcomes, giving the orthogonal projector the role of *questions about the system*, whose answer is to be found through an experiment. On the contrary, any question should evidently be represented by an orthogonal projector.

- The *orthogonal projectors* on H represents all of the *statements* about the system

***Remark* 5.1.1.** Notice that, in the case of orthogonal projectors, the notion of compatibility among observables given above coincides with the one given in Definition 2.2.30.

An appealing example is provided by the spectral theorem 2.2.56. Consider any self-adjoint operator A representing a given observable and let

$$\mathfrak{B}(\mathbb{R}) \ni E \mapsto P^{(A)}(E) \in \mathfrak{L}(\mathsf{H})$$

be its PVM. Sticking to the physical meaning of the projectors just discussed, we have the following natural interpretation.

- For any observable A and Borelian $E \in \mathcal{B}(\mathbb{R})$ the statement $P^{(A)}(E)$ corresponds to: *the value of A falls within E.*

Now, focus on the states $\rho \in \mathfrak{D}(\mathsf{H})$. Exploiting the results of Chapter 4, Section 4.4, we can prove the following result.

***Proposition* 5.1.2.** *If $\rho \in \mathfrak{D}(\mathsf{H})$ then the following statements hold for $P, Q \in \mathfrak{L}(\mathsf{H})$:*

(a) $tr(\rho P) \in [0, 1]$,

(b) $tr(\rho P) \leq tr(\rho Q)$ *if* $P \leq Q$,

(c) $tr(\rho 0) = 0$ *and* $tr(\rho I) = 1$,

(d) $tr(\rho P^{\perp}) = 1 - tr(\rho P)$,

(e) $tr(\rho(P \wedge Q)) = 1 \iff tr(\rho P) = 1$ *and* $tr(\rho Q) = 1$,

(f) $tr(\rho(P \vee Q)) = 1 \Longleftarrow tr(\rho P) = 1$ *or* $tr(\rho Q) = 1$,

(g) if $(P_n)_{n\in\mathbb{N}} \subset \mathfrak{L}(\mathsf{H})$ *satisfies* $P_n P_m = 0$ *for* $n \neq m$, *then*

$$tr\left(\rho \bigvee_{n\in\mathbb{N}} P_n\right) = \sum_{n=0}^{\infty} tr(\rho P_n).$$

Proof. The only non trivial points are (e)-(f), the other ones following by the properties of trace-class operators developed in Chapter 4, Section 4.4. Let us start with point (e) and let N be a Hilbert basis of $P(\mathsf{H}) \cap Q(\mathsf{H})$ and N' be a Hilbert basis of its orthogonal complement. Then

$$1 = tr(\rho(P \wedge Q)) = \sum_{\mathsf{N}\cup\mathsf{N}'} \Re[(z|\rho(P \wedge Q)z)] = \sum_{z\in\mathsf{N}} \Re[(z|\rho z)] = \sum_{z\in\mathsf{N}} \Re[(Pz|\rho Pz)] =$$
$$= \sum_{z\in\mathsf{N}} \Re[(z|P\rho Pz)] \leq \sum_{z\in\mathsf{N}\cup\mathsf{N}'} \Re[(z|P\rho Pz)] = tr(P\rho P) = tr(\rho PP) = tr(\rho P),$$

where we used $(x|P\rho Px) = (Px|\rho Px) \geq 0$ which holds for all $x \in \mathsf{H}$ as a consequence of the positivity of ρ. This result, together with point (a) gives $tr(\rho P) = 1$. The same calculation holds for Q. Let us prove the opposite implication. Suppose that $tr(\rho P) = 1$ and take any Hilbert basis N of H, then, exploiting the cyclic property of the trace, the identity $PP = P$, the positivity and self-adjointness of ρ and points (c) and (d) we have

$$\begin{aligned} 0 = tr(\rho P^{\perp}) = tr(P^{\perp}\rho P^{\perp}) = \sum_{z\in\mathsf{N}} \Re[(z|P^{\perp}\rho P^{\perp}z)] = \sum_{z\in\mathsf{N}} (P^{\perp}z|\rho P^{\perp}z) = \\ = \sum_{z\in\mathsf{N}} (\sqrt{\rho}P^{\perp}z|\sqrt{\rho}P^{\perp}z) = \sum_{z\in\mathsf{N}} \|\sqrt{\rho}P^{\perp}z\|^2. \end{aligned} \tag{5.2}$$

This immediately gives $\|\sqrt{\rho}P^{\perp}z\|^2 = 0$ for all $z \in \mathsf{N}$. Since all the operators involved are continuous and N is a Hilbert basis, then $\sqrt{\rho}P^{\perp} = 0$ and so also $\rho P^{\perp} = 0$, which implies $\rho P = \rho$. Taking the adjoint we have the equivalent condition $P\rho = \rho$ which proves that $R(\rho) \subset P(\mathsf{H})$. Doing the same for Q we see that $R(\rho) \subset P(\mathsf{H}) \cap Q(\mathsf{H}) = P \wedge Q(\mathsf{H})$ and so

$$tr(\rho(P \wedge Q)) = tr((P \wedge Q)\rho) = \sum_{z\in\mathsf{N}} \Re[(z|(P \wedge Q)\rho z)] = \sum_{z\in\mathsf{N}} (z|\rho z) = tr(\rho) = 1.$$

Point (f). Suppose without less of generality that $tr(\rho P) = 1$ and let N be a Hilbert basis of $P(\mathsf{H})$ as above, completed with a Hilbert basis N' of $P(\mathsf{H})^{\perp}$. Similarly, since $P \vee Q$ is the projector over the subspace $\overline{[P(\mathsf{H}) \cup Q(\mathsf{H})]} \supset P(\mathsf{H})$, complete N to a Hilbert basis N_1 of this closed subspace and let N_1' be a Hilbert basis of its orthogonal. Since $\Re[(u|\rho u)] \geq 0$ for any $u \in \mathsf{H}$ we have the following

$$\begin{aligned} tr(\rho(P \vee Q)) = \sum_{z\in\mathsf{N}_1\cup\mathsf{N}_1'} \Re[(z|\rho(P \vee Q)z)] = \sum_{z\in\mathsf{N}_1} \Re[(z|\rho z)] \geq \sum_{z\in\mathsf{N}} \Re[(z|\rho z)] = \\ = \sum_{z\in\mathsf{N}} \Re[(z|\rho Pz)] = \sum_{z\in\mathsf{N}\cup\mathsf{N}'} \Re[(z|\rho Pz)] = tr(\rho P) = 1, \end{aligned} \tag{5.3}$$

which concludes the proof of point (f). □

***Remark* 5.1.3.** Notice that the opposite direction in point (f) generally fails to hold. This is crucial in quantum mechanics, in that it is allows us to perform superposition of states.

Take for example two orthogonal unit vectors u, v and let P_u, P_v be the orthogonal projectors associated with the one-dimensional subspaces generated by u and v, respectively. Consider the projector P_ω on the one-dimensional subspace generated by $\omega = u\alpha + v\beta$ with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. It is clear that $P_\omega \leq P_u \vee P_v$.

Now, as projectors over one-dimensional subspaces, the operators $\rho_u = P_u, \rho_v = P_v$ and $\rho_\omega = P_\omega$ define density operators over H, thus states of the system. So, take for example $\alpha = \beta = \frac{1}{\sqrt{2}}$, then it is easy to see that

$$tr(\rho_\omega(P_u \vee P_v)) = 1$$

but

$$tr(\rho_\omega P_u) = \frac{1}{2} \quad tr(\rho_\omega P_v) = \frac{1}{2}$$

Each state ρ_ω is said to be a *superposition* of the states ρ_u and ρ_v.

Now, bearing in mind that $tr(\rho P)$ corresponds to the *expectation value* $\langle P \rangle_\rho$ of P with respect to the state ρ and that $\sigma(P) = \{0, 1\}$, it is clear that the condition $\langle P \rangle_\rho = 1$ means that a measurement of P on ρ would give 1 *for sure*. Similarly, if $tr(\rho P) = 0$ then P would surely attain 0 as outcome. We can take this as the *truth value* of any proposition $P \in \mathfrak{L}(\mathsf{H})$ once the state *has been fixed*.

- Given a state ρ, a proposition $P \in \mathfrak{L}(\mathsf{H})$ is said to be *true* if $tr(\rho P) = 1$. It is said to be *false* if $tr(\rho P) = 0$.

***Remark* 5.1.4.** Notice that, differently from classical logic, the attributes *non-true* and *false* are *not equivalent*. The statement "P is not true" is to be interpreted as $tr(\rho P) \neq 1$, which includes but do not equal $tr(\rho P) = 0$.

Accepting this viewpoint, Proposition 5.1.2 suggests that we can interpret the function

$$\mathfrak{L}(\mathsf{H}) \ni P \mapsto tr(\rho P) \in [0, 1] \tag{5.4}$$

as a *probability assignment* to the quantum propositions.

- Given a state ρ, for any proposition $P \in \mathfrak{L}(\mathsf{H})$, the number $\mu(P) \in [0, 1]$ is the *probability* that a measurement of P gives 1 as an outcome.

***Remark* 5.1.5.** As we saw in Chapter 4 Section 4.4 - in particular Definition 4.4.16 and Theorem 4.4.17 - states are in complete correspondence with σ-probability measures on $\mathfrak{L}(\mathsf{H})$.

Referring to this viewpoint, the points in Proposition 5.1.2 inherit a natural interpretation in terms of logical connectives.

- From point (b) we see that if P is true, then Q is true as well. Thus $P \leq Q$ acts as *logical implication*.

- Point (c) constrains us to interpret $0, I$ as the *contradiction* and *tautology* propositions, respectively (think of the statements *the system does not exists* and *the system exists*).

- From point (d) we see that the property $P^\perp$ is true if and only if P is false. Thus $P^\perp$ acts as *logical negation* (keep in mind Remark 5.1.4).

- from point (e), the property $P \wedge Q$ is true if and only if both P and Q are true. So the property $P \wedge Q$ acts as *logical conjunction*.

- Frompoint (f) we see that $P \vee Q$ acts a weak form of *logical disjunction*, in that the implication holds only in one direction.
- From point (g) we see that orthogonal propositions are *mutually exclusive*: they are not correlated from a statistical point of view.

***Remark* 5.1.6.** Some remarks follow.

(a) Not all the operations above can be interpreted as the standard connectives of classical logic. For example, as already discussed in Remark 5.1.4 $P^\perp$ does *not* correspond to "P is not true", but to "P is false". Similarly, the $\vee$ operation is a *weaker* form of the classical disjunction.

(b) Point (e) makes sense also when propositions P and Q are not compatible. The given interpretation bypasses the issue of measuring at the same time the two properties by saying that P and Q are both true with respect to a given state ρ when separate measurements of P and Q would give 1 with certainty. The property $P \wedge Q$ is to be interpreted as the statement: "*the measurement of a property randomly chosen within* $\{P, Q\}$ *gives* 1".

(c) As already introduced at the beginning of this section, notice that in general the action of the connectives $\wedge, \vee$ is not distributive: it holds only among compatible propositions (check Proposition 2.2.35).

Now, we want to discuss another important principle: the *collapse of the state*. Suppose we are given a quantum system in the physical state ρ and want to measure some given property $P \in \mathfrak{L}(\mathsf{H})$. If the outcome is 1, then it is reasonable to assume that a repetition of the experiment immediately after the first measurement would give 1 for sure. For this to be possible, the state must turn into some state ρ_P with respect to which the property P is true, i.e. $tr(\rho_P P) = 1$. The most natural way to define such a collapsed state ρ_P out of ρ and P is to take its *projection* over the closed subspace $P(\mathsf{H})$.

- *The Lüders-von Neumann post-measurement axiom*: if, given a state ρ, the outcome of a measurement of $P \in \mathfrak{L}(\mathsf{H})$ is positive, then the state immediately after the measurement is given by

$$\rho_P = \frac{1}{tr(\rho P)} P \rho P \in \mathfrak{D}(\mathsf{H}).$$

***Remark* 5.1.7.** Taking into account the collapse of the state that happens at every measurement on the system, it is a simple exercise to prove that for compatible propositions and for fixed any initial state ρ the following facts happen:

(a) if a measurement of $P \wedge Q$ gives 1, then a consequent measurement of P and Q (in whichever order) gives $(1, 1)$;

(b) if a measurement of P and Q (in whichever order) gives $(1,1)$, then a consequent measurement of $P \wedge Q$ gives 1;

(c) if a measurement of P and Q (in whichever order) gives $(1,0),(0,1)$ or $(1,1)$, then a consequent measurement of $P \vee Q$ gives 1.

This is an operational way to understand the logical connectives introduced above. As expected, since it involves measurements and we know about the issues concerning incompatible observables, this interpretation makes sense only among compatible propositions.

To conclude, we want to discuss another important situation. Suppose that immediately after a measurement of P on the state ρ with positive outcome we want to measure some other proposition $Q \in \mathfrak{L}(\mathsf{H})$ compatible with P. According to the Lüders-von Neumann principle, the probability for a positive outcome is now given by

$$\mu_P(Q) = \frac{1}{tr(\rho P)} tr(P\rho P\, Q) = \frac{1}{tr(\rho P)} tr(P\rho P\, PQP) = \mu_P(PQP)$$

and $PQP \leq P$. Moreover notice that if $Q \leq P$ then $PQP = Q$. This shows that immediately after the measurement the set of relevant propositions which can be tested without perturbing the logical value of P reduces to the subset

$$\mathfrak{L}(\mathsf{H})_P := \{Q \in \mathfrak{L}(\mathsf{H}) \mid Q \leq P\} \subset \mathfrak{L}(\mathsf{H}) \tag{5.5}$$

which is the set of all of the propositions that logically imply P.

- For any $P \in \mathfrak{L}(\mathsf{H})$ the set $\mathfrak{L}(\mathsf{H})_P$ represents the set of propositions testable immediately after any measurement with positive outcome without perturbing the logical value of P.

So, consider any $Q \in \mathfrak{L}(\mathsf{H})_P$, i.e. $Q \leq P$. The properties P, Q are compatible and so any measurement of Q does not perturb a consecutive measurement of P, which remains true with respect to the new collapsed state. This suggests the introduction of a *relative negation* within $\mathfrak{L}(\mathsf{H})_P$ by defining the statement Q *is false relative to* P as the proposition Q *is false and* P *is true*. We can introduce a notation for this as follows:

$$Q^{\perp_P} := Q^{\perp} \wedge P.$$

Finally notice that

$$0 \leq P,\ Q \wedge Q',\ Q \vee Q,\ Q^{\perp_P} \in \mathfrak{L}(\mathsf{H})_P \quad \text{for any } Q, Q' \in \mathfrak{L}(\mathsf{H})_P.$$

The set $\mathfrak{L}(\mathsf{H})_P$ together with the connectives $(\leq, 0, P, \wedge, \vee, \perp_P)$ inherits the role of logic of the quantum system when the proposition P is measured, found true and kept so.

Going back to $\mathfrak{L}(\mathsf{H})$ and equipping it with all the logical connectives discussed so far, we find a structure

$$(\mathfrak{L}(\mathsf{H}), \leq, \wedge, \vee, \perp, 0, 1), \tag{5.6}$$

which fulfills the definition of

$$\textit{irreducible complete atomistic orthomodular lattice satisying the covering law} \tag{5.7}$$

The meaning of these terms will become clear in the next section. Moreover the set

$$(\mathfrak{L}(\mathsf{H})_P, \leq, \wedge, \vee, \perp_P, 0, P)$$

defines as well an irreducible complete atomistic orthomodular lattice satisying the covering law. This is how it should be, in that the role of system proposition lattice of $\mathfrak{L}(\mathsf{H})$ passes to $\mathfrak{L}(\mathsf{H})_P$ once P is measured, found positive and kept so.

At this point, once the logical structure of quantum systems has been recongnised, a natural question comes to mind: what if we *start* with an abstract structure $\mathfrak{L}$ which only resemble (5.6), without referring to any Hilbert space structure? This is the core of quantum logic, as it developed through the decades. More precisely, basing upon several assumptions, the set of propositions for a quantum system is usually *taken* as a complete atomistic orthomodular lattice satisying the covering law. If the system is also assumed to be *elementary*, i.e. not decomposable into subsystems, then the lattice is supposed to be *irreducible*, otherwise it would decompose into a direct sum of sublattices (see Theorem 5.18) and each sublattice would describe a subsystem (all these terms will become clear later on in the following sections).

A long standing problem in this theory was the so called *coordinatisation problem*: the attempt to prove that an abstract structure of this type is necessarily isomorphic to the lattice $\mathfrak{L}(\mathsf{H})$ of the orthogonal projectors of some Hilbert space. A first proof on this was provided by Piron (see Theorem 5.4.7) who demonstrated that a lattice $\mathfrak{L}$ as in (5.7) under suitable conditions must be isomorphic to the set of biorthogonal subspaces $\mathfrak{L}(\mathsf{V})$ of a generalized Hilbert space. Eventually, Solèr (see Theorem 5.4.8) improved this result and succeeded in proving that, under further subtle hypotheses, the generalised Hilbert space of Piron and Maeda-Maeda is necessarily a proper Hilbert space and $\mathfrak{L}(\mathsf{V})$ coincide with the set of orthogonal projectors.

In the next sections we give a brief description of the main concepts of lattice theory, encompassing all the useful definitions and results encountered in quantum logic.

5.2 General Definitions

Everything begins with the basic concept of *partially ordered set* which, in the general situation, is an extremely poor structure. We start enriching it by assuming the existence of a supremum and an infimum for any couple of elements.

***Definition* 5.2.1.** *A lattice is a partially ordered set* $(\mathfrak{L}, \leq)$ *such that any couple of elements* $p, q \in \mathfrak{L}$ *admits* $\inf\{p, q\} \in \mathfrak{L}$ *and* $\sup\{p, q\} \in \mathfrak{L}$.

The definition is restricted to couple of elements: if we take a generic subset $\mathfrak{S} \subset \mathfrak{L}$ then we could consider $\sup \mathfrak{S}$ and $\inf \mathfrak{S}$, which, however, might not exists.

***Definition* 5.2.2.** *A lattice* $(\mathfrak{L}, \leq)$ *is said to be complete if* $\sup \mathfrak{S}$ *and* $\inf \mathfrak{S}$ *exist within* $\mathfrak{L}$ *for any subset* $\mathfrak{S} \subset \mathfrak{L}$. *It is said to be* σ*-complete if this holds for at most countable subsets* $\mathfrak{S}$.

We introduce the following symbols

$$\bigvee \mathfrak{S} := \sup \mathfrak{S} \quad \text{and} \quad \bigwedge \mathfrak{S} := \inf \mathfrak{S},$$

which are called *join* and *meet* of $\mathfrak{S}$, respectively. When dealing with couples of elements we will use the notation $p \wedge q = \wedge\{p, q\}$ and $p \vee q = \vee\{p, q\}$ which introduce two operations on $\mathfrak{L}$. A list of simple properties for these operations follows immediately from the definition:

(a) for any $p, q \in \mathfrak{L}$ it holds that $p \leq q$ if and only if $p = p \wedge q$;

(b) the meet and the join operations are *associative*, i.e. for any $p, q, r \in \mathfrak{L}$ it holds that

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r, \quad p \vee (q \vee r) = (p \vee q) \vee r;$$

(c) the meet and the join operations are *symmetric*, i.e. for any $p, q \in \mathfrak{L}$ it holds that

$$p \wedge q = q \wedge p, \quad p \vee q = q \vee p.$$

A subset $\mathfrak{S} \subset \mathfrak{L}$ which is closed under the meet and the join operations is said to be a *sublattice*. In particular, consider any subset $\mathfrak{A} \subset \mathfrak{L}$, then we can consider the set

$$\mathfrak{A}^g := \bigcap \{\mathfrak{S} \mid \mathfrak{S} \text{ is a sublattice of } \mathfrak{L} \text{ and } \mathfrak{A} \subset \mathfrak{S}\}.$$

Of course $\mathfrak{A} \subset \mathfrak{A}^g$, moreover $\mathfrak{A}^g$ is a lattice as a direct check can show. $\mathfrak{A}^g$ is evidently the *smallest* sublattice containing $\mathfrak{L}$.

As seen in the previous section, the distributivity property is not guaranteed to hold on generic lattices, it need to be required if needed.

***Definition* 5.2.3.** *A lattice* $(\mathfrak{L}, \leq)$ *is said to be distributive if for any* $p, q, r \in \mathfrak{L}$ *it holds that*

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r).$$

***Remark* 5.2.4.** As already said, the non-distributivity of the lattice is a distinctive feature of quantum systems.

Another important definition follows.

***Definition* 5.2.5.** *A lattice* $\mathfrak{L}$ *is said to be bounded if there exist two elements* $0, 1 \in \mathfrak{L}$ *such that* $0 \leq p \leq 1$ *for any* $p \in \mathfrak{L}$. *They are referred to as the lower and the upper bound of* $\mathfrak{L}$, *respectively.*

Of course a complete lattice is automatically bounded, by taking as 0 and 1 the meet and the join of all the entire set $\mathfrak{L}$, respectively.

***Definition* 5.2.6.** *Let $\mathfrak{L}$ be a bounded lattice. A map $\perp: \mathfrak{L} \to \mathfrak{L}$ is called an orthocomplementation if it satisfies the following properties:*

(a) $p^{\perp\perp} = p$,

(b) $p \vee p^{\perp} = 1$ *and* $p \wedge p^{\perp} = 0$,

(c) if $p \leq q$ *then* $q^{\perp} \leq p^{\perp}$.

The lattice $\mathfrak{L}$ equipped with $\perp$ is said to be orthocomplemented. Two elements $p, q \in \mathfrak{L}$ are said to be orthogonal to each other if $p \leq q^{\perp}$.

Notice that the definition of orthogonal elements is well-defined as it does not depend on the chosen order. Indeed if $p \leq q^{\perp}$, then (a) and (c) assure that $q \leq p^{\perp}$.

On orthocomplemented lattices, the *De Morgan laws* are fulfilled

$$(p \wedge q)^{\perp} = p^{\perp} \vee q^{\perp}, \quad (p \vee q)^{\perp} = p^{\perp} \wedge q^{\perp}.$$

***Example* 5.2.7.** Fix any set X, the *power set* $\mathcal{P}(X)$ provides a simple example of distributive complete orthocomplemented lattice. The partial order is defined as the *inclusion* $A \subset B$, the meet and the join operations corresponds to *intersection* $A \cap B$ and *union* $A \cup B$, respectively, and the orthocomplementation is given by the *complement* $X \setminus A$.

The power set is part of an important class of lattices

***Definition* 5.2.8.** *A distributive orthocomplemented lattice is called a Boolean algebra.*

***Remark* 5.2.9.** Boolean algebras give the natural framework for the logical analysis of classical (non-quantum) physical systems.

Let $\mathfrak{A}$ be a *finite* subset of a lattice $\mathfrak{L}$. It is clear that the set made of all the possible joins and meets among elements of $\mathfrak{A}$ is a sublattice of $\mathfrak{L}$, hence it must equal the generated $\mathfrak{A}^g$, this one being the smallest sublattice of $\mathfrak{L}$ containing $\mathfrak{A}$. If this is the case, $\mathfrak{L}$ is orthocomplemented and $\mathfrak{A}$ is closed under the orthocomplementation, then it is clear that $\mathfrak{A}^g$ is also an orthocomplemented lattice where the extrema and the orthocomplementation are the same of $\mathfrak{L}$.

This allows for the introduction of the concept of *compatible* elements on an arbitrary orthocomplemented lattice.

***Definition* 5.2.10.** *Let $\mathfrak{L}$ be an orthocomplemented lattice, two elements p, q are said to be compatible if the sublattice generated by $\{p, p^{\perp}, q, q^{\perp}\}$ is a Boolean algebra. If this is the case we write $p \leftrightarrow q$.*

Notice that if $\{p, q\}$ are compatible, then the same holds for the couples $\{p^\perp, q\}$, $\{p^\perp, q^\perp\}$ and $\{p, q^\perp\}$. Moreover, if p, q are compatible, then

$$q = (q \wedge p) \vee (q \wedge p^\perp), \tag{5.8}$$

which can be proved by direct inspection.

We are in a position to list a set of important definitions. As discussed above, the distributivity property may be too strong a requirement in some interesting applications. Nevertheless it is useful to look for slightly weaker conditions. For instance, we may suppose that the distributivity property holds at least for the triple $\{p, p^\perp, q\}$ when $p \leq q$. More precisely

$$p \vee (p^\perp \wedge q) = (p \vee p^\perp) \wedge (p \vee q) = 1 \wedge q = q.$$

***Definition* 5.2.11.** *An orthocomplemented lattice $\mathfrak{L}$ is said to be weakly modular (or orthomodular) if $q = p \vee (p^\perp \wedge q)$ for any $p, q \in \mathfrak{L}$ such that $p \leq q$.*

This is an important definition because of the following result.

***Proposition* 5.2.12.** *Let $\mathfrak{L}$ be an orthocomplemented lattice, then $\mathfrak{L}$ is weakly modular if and only if $p \leq q$ implies $p \leftrightarrow q$ for any couple $p, q \in \mathfrak{L}$.*

This has an important consequence. Let $\mathfrak{L}$ be an orthomodular lattice and p any one of its elements and define:

$$\mathfrak{L}_p := \{q \in \mathfrak{L} \,|\, q \leq p\} \subset \mathfrak{L}. \tag{5.9}$$

An immediate check shows that $\mathfrak{L}_p$ is closed under the join and the meet operations of $\mathfrak{L}$, thus making it a sublattice of $\mathfrak{L}$. Of course $\{0, p\}$ play the role of bounds for $\mathfrak{L}_p$. The weak modularity is essential at this point, for it allows us to define a natural orthocomplementation on it by

$$q^{\perp_p} := q^\perp \wedge p. \tag{5.10}$$

More precisely:

***Proposition* 5.2.13.** *Let $\mathfrak{L}$ be an orthocomplemented lattice then the set (5.9) equipped with (5.10) is an orthocomplemented lattice for every $p \in \mathfrak{L}$ if and only if $\mathfrak{L}$ is weakly modular.*

Let $\mathfrak{L}$ be an orthocomplemented lattice, we can consider the subset of all the elements compatible with all the others.

***Definition* 5.2.14.** *Let $\mathfrak{L}$ be an orthocomplemented lattice, then an element $p \in \mathfrak{L}$ is said to be central if $p \leftrightarrow q$ for any $q \in \mathfrak{L}$. The set $\mathfrak{C}(\mathfrak{L})$ of all central elements is called the center of $\mathfrak{L}$. The lattice is said to be irreducible if $\mathfrak{C}(\mathfrak{L}) = \{0, 1\}$.*

Of course $\mathfrak{C}(\mathfrak{L})$ contains the bounds $0, 1$ and it is closed under the join, the meet and the orthocomplementation operations. More precisely we have the following result.

***Proposition* 5.2.15.** *For every orthocomplemented lattice $\mathfrak{L}$, referring to the structure induced by $\mathfrak{L}$, the center $\mathfrak{C}(\mathfrak{L})$ is a Boolean algebra. Moreover it holds that $\mathfrak{L} = \mathfrak{C}(\mathfrak{L})$ if and only if $\mathfrak{L}$ is a Boolean algebra.*

Another important definition concerns the concept of an *atom* within a lattice. These elements consists in some sense in the *smallest* non-null elements of the lattice.

***Definition* 5.2.16.** *Let $\mathfrak{L}$ be a bounded lattice, then*

(a) an element $a \in \mathfrak{L} \setminus \{0\}$ is said to be an atom if $0 \leq p \leq a$ for some $p \in \mathfrak{L}$ implies $p = 0$ or $p = a$;

(b) the lattice is said to be atomic if for any $p \neq 0$ there exists an atom $a \leq p$;

(c) the lattice is said to be atomistic if it is atomic and every element $p \neq 0$ equals the join of all the atoms $a \leq p$.

Another important definition is the following

***Definition* 5.2.17.** *An orthocomplemented lattice is said to satisfy the covering property if for any couple $a, p \in \mathfrak{L}$ where a is an atom satisfying $a \wedge p = 0$, then $p \leq q \leq a \vee p$ implies $q = p$ or $q = a \vee p$.*

Of course, once the lattice structure is defined, a natural concept of a morphism arises, as a map between lattices which preserves their structure.

***Definition* 5.2.18.** *Let $\mathfrak{L}_1$, $\mathfrak{L}_2$ be two lattices. A lattice homomorphism is a function $h : \mathfrak{L}_1 \to \mathfrak{L}_2$ such that*

$$h(p \wedge_1 q) = h(p) \wedge_2 h(q) \quad \text{and} \quad h(p \vee_1 q) = h(p) \vee_2 h(q)$$

for any couple $p, q \in \mathfrak{L}_1$. It the lattice is bounded then we require also

$$h(0_1) = 0_2 \quad \text{and} \quad h(1_1) = 1_2.$$

Moreover, if $\mathfrak{L}$ is orthocomplemented, then a last assumption is

$$h(p^{\perp_1}) = h(p)^{\perp_2}.$$

Finally, if the lattices are (σ-)complete, the homomorphism is required to respect the join and the meet of every (at most countable) family of elements. If h is bijective, it is said to be an isomorphism. Finally, if $\mathfrak{L}_1 = \mathfrak{L}_2$ a lattice isomorphism is called an automorphism and the group made of these objects is denoted by $Aut(\mathfrak{L}_1)$.

Actually the underlying basic structure of a lattice is the partial order, hence the first requirement on a lattice homomorphism should be the preservation of the partial order. However this is an easy consequence of the other assumptions, as a direct check can show.

***Proposition* 5.2.19.** *If $h : \mathfrak{L}_1 \to \mathfrak{L}_2$ is a lattice homomorphism then $h(p) \leq_2 h(q)$ whenever $p \leq_1 q$.*

5.3 Direct Product of Lattices

When two different lattices are given, there exists a canonical procedure to obtain a new lattice out of them and of course the same property extends to homomorphisms. Let us see this in detail.

Consider a family of lattices $(\mathfrak{L}_i)_{i\in\mathcal{I}}$, define the cartesian product

$$\mathfrak{L} := \prod_{i\in\mathcal{I}} \mathfrak{L}_i \tag{5.11}$$

and equip it with the following relation

$$(p_i)_{i\in\mathcal{I}} \leq (q_i)_{i\in\mathcal{I}} \quad \text{if and only if} \quad p_i \leq q_i \quad \text{for all} \quad i \in \mathcal{I} \tag{5.12}$$

It is easy to prove that this actually defines a partial order on $\mathfrak{L}$. Moreover each couple of elements admits infimum and supremum within $\mathfrak{L}$, which are given by

$$(p_i)_{i\in\mathcal{I}} \wedge (q_i)_{i\in\mathcal{I}} = (p_i \wedge_i q_i)_{i\in\mathcal{I}} \quad \text{and} \quad (p_i)_{i\in\mathcal{I}} \vee (q_i)_{i\in\mathcal{I}} = (p_i \vee_i q_i)_{i\in\mathcal{I}} \tag{5.13}$$

This makes $\mathfrak{L}$ a lattice, which is is clearly bounded if each $\mathfrak{L}_i$ is with bounds given by the elements

$$(0_i)_{i\in\mathcal{I}} \quad \text{and} \quad (1_i)_{i\in\mathcal{I}} \tag{5.14}$$

If, furthermore, the lattices $\mathfrak{L}_i$ are (σ-)complete, then the existence of supremum and infimum can be extended to any (countable) family of elements of $\mathfrak{L}$.

Similarly, suppose that the lattices $\mathfrak{L}_i$ are orthocomplemented, then we can introduce the following involutive function

$$\mathfrak{L} \ni (p_i)_{i\in\mathcal{I}} \mapsto ((p_i)_{i\in\mathcal{I}})^{\perp} := \left(p_i^{\perp_i}\right)_{i\in\mathcal{I}} \in \mathfrak{L} \tag{5.15}$$

which defines an orthocomplementation over $\mathfrak{L}$. In particular notice that

$$(0, \dots, p_i, \dots, 0, \dots, 0, \dots) \perp (0, \dots, 0 \dots, p_j, \dots, 0, \dots) \tag{5.16}$$

for any $p_i \in \mathfrak{L}_i$, $p_j \in \mathfrak{L}_j$ if $i \neq j$.

We can condense everything within the following result

***Proposition* 5.3.1.** *Let $(\mathfrak{L}_i)_{i\in\mathcal{I}}$ be a family of lattices, then the set (5.11) equipped with (5.12) is a lattice, called the direct product of the lattices and denoted by*

$$\prod_{i\in\mathcal{I}} \mathfrak{L}_i.$$

Moreover, this lattice is bounded, (σ-)complete and orthocomplemented if each $\mathfrak{L}_i$ is.

***Notation* 5.3.2.** When the index set $\mathcal{I}$ is finite, the direct product in Proposition 5.3.1 can also be referred to as the *direct sum* of the lattices $\mathfrak{L}_i$ and denoted by

$$\bigoplus_{i\in\mathcal{I}} \mathfrak{L}_i$$

in analogy with the direct sum of linear spaces.

The definition just given provides a mechanism to define a lattice as a direct product of a family of original ones. The opposite procedure can be performed when the center of the lattice is non-trivial.

Consider a ((σ-)complete) orthomodular lattice $\mathfrak{L}$ and let $p \in \mathfrak{C}(\mathfrak{L})$ be a central element, Proposition 5.9 assures that the subsets

$$\mathfrak{L}_p := \{q \in \mathfrak{L} \,|\, q \leq q\} \quad \text{and} \quad \mathfrak{L}_{p^\perp} := \{q \in \mathfrak{L} \,|\, q \leq p^\perp\} \tag{5.17}$$

equipped with the restriction of the partial order of $\mathfrak{L}$ and with $q^{\perp_p} := q^\perp \wedge p$ and $q^{\perp_{p^\perp}} := q^\perp \wedge p^\perp$ define two orthocomplemented lattices. So, consider the lattice $\mathfrak{L}_P \oplus \mathfrak{L}_{P^\perp}$ and take the function

$$\imath : \mathfrak{L} \ni q \mapsto (q \wedge p, q \wedge p^\perp) \in \mathfrak{L}_p \oplus \mathfrak{L}_{p^\perp}. \tag{5.18}$$

Exploiting the properties of compatible elements, it is easy to see that then the function (5.18) is an isomorphism of ((σ-)complete) orthocomplemented lattices. More generally we have the following result.

***Definition* 5.3.3.** *Let $\mathfrak{L}$ be an orthocomplemented lattice. A family $(p_i)_{i\in\mathcal{I}} \subset \mathfrak{L}$ is said to be maximally orthogonal if $p_i \perp p_j$ when $i \neq j$ and if $p \perp p_i$ for any $i \in \mathcal{I}$ implies $p \in \{0, 1\}$.*

***Definition* 5.3.4.** *An orthocomplemented lattice $\mathfrak{L}$ is said to be separable if any maximal orthogonal subset is at most countable.*

Such families allow for the construction of maximal decompositions as the following result shows.

***Theorem* 5.3.5.** *Let $\mathfrak{L}$ be an ((σ-)complete) orthomodular lattice and let $(p_i)_{i\in\mathcal{I}} \subset \mathfrak{C}(\mathfrak{L})$ be maximally orthogonal. Then the function*

$$\mathfrak{L} \ni q \mapsto (q \wedge p_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathfrak{L}_{p_i} \tag{5.19}$$

is an injective homomorphism of ((σ)-complete) orthocomplemented lattices. The map is also surjective if one of the following assertion is true:

(i) $\mathcal{I}$ is finite,

(ii) $\mathcal{I}$ is countable and $\mathfrak{L}$ is σ-finite,

(iii) $\mathcal{I}$ is uncountable and $\mathfrak{L}$ is complete.

Everything seen so far concerns the decomposition into direct sum of lattices. We now discuss the relative properties for homomorphisms.

Consider two families of lattices $(\mathfrak{L}_i)_{i\in\mathcal{I}}$ and $(\mathfrak{L}'_i)_{i\in\mathcal{I}}$ on the same index set $\mathcal{I}$ and let $h_i : \mathfrak{L}_i \to \mathfrak{L}'_i$ be an homomorphism for any index $i \in \mathcal{I}$. Consider the function

$$h : \prod_{i\in\mathcal{I}} \mathfrak{L}_i \ni (p_i)_{i\in\mathcal{I}} \mapsto (h_i(p_i))_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathfrak{L}'_i. \tag{5.20}$$

It is immediate to prove that such a function is a lattice homomorphism and that it preserves the bounds and the orthocomplement when existing.

***Proposition* 5.3.6.** *Let $h_i : \mathfrak{L}_i \to \mathfrak{L}'_i$ be a family of ((σ-)complete, orthocomplemented) lattice homomorphisms. Then the function defined in (5.20) is a ((σ-)complete, orthocomplemented) lattice homomorphism, called the direct product of the homomorphisms and denoted by $\prod_{i\in\mathcal{I}} h_i$. This function is an isomorphism if each h_i is.*

Referring to the internal decomposition of Theorem 5.3.5 we have an analogous result concerning homomorphisms.

***Theorem* 5.3.7.** *Let $h : \mathfrak{L} \to \mathfrak{L}$ be a homomorphism of ((σ-)complete) orthomodular lattices and suppose there exists a maximally orthogonal family $(p_i)_{i\in\mathcal{I}} \subset \mathfrak{C}(\mathfrak{L})$ such that $h(p_i) = p_i$ for all $i \in \mathcal{I}$. Then the following statements hold:*

(a) $h(\mathfrak{L}_{p_i}) \subset \mathfrak{L}_{p_i}$;

(b) $h_{p_i} := h|_{\mathfrak{L}_{p_i}}$ is an homomorphism of ((σ-)complete) orthocomplemented lattices; moreover it is an automorphism if h is so;

(c) referring to the identification (5.20), in the case that at least one among (i)-(iii) is satisfied, then

$$h = \prod_{i\in\mathcal{I}} h_{p_i}.$$

The condition $h(p_i) = p_i$ is essential to assure point (a), in that it implies $h(q) \leq h(p_i) = p_i$ for any $q \in \mathfrak{L}_{p_i}$, which is equivalent to $h(p_i) \in \mathfrak{L}_{p_i}$.

5.4 Representation Results for Abstract Lattices

In this section we want to analyse the most important realisation of lattices. We will focus on the case of an irreducible bounded orthomodular atomistic lattice satisfying the covering law.

We need to introduce some technical definitions first which consist in sort of a generalisation of the concept of a Hilbert space.

Let $\mathbb{D}$ be a division ring with null and unit elements given, respectively, by $0, 1$. Equip it with an *involution*, i.e. a function $^* : \mathbb{D} \to \mathbb{D}$ such that

$$(k+l)^* = k^* + l^*, (k^*)^* = k \quad \text{and} \quad (kl)^* = l^* k^*$$

for any $k, l \in \mathbb{D}$. In particular such a map is bijective.

Consider a (right) linear space V over $\mathbb{D}$, i.e. an abelian group with a right scalar multiplication

$$\mathsf{V} \times \mathbb{D} \ni (u, a) \mapsto ua \in \mathsf{V},$$

which satisfies the distributive properties with respect to the two notions of sum

$$(u+v)k = uk + vk \quad \text{and} \quad u(k+k') = uk + uk' \quad \text{if } u, v \in \mathsf{V} \text{ and } k, k' \in \mathbb{D}$$

and such that $u1 = u$ for any $u \in \mathsf{V}$.

Similarly to what we did in defining a pre-Hilbert space, define an *Hermitean inner product* over V as a map

$$\mathsf{V} \times \mathsf{V} \ni (u, v) \mapsto (u|v) \in \mathbb{D},$$

which satisfies the following assumptions:

(a) (*right linearity:*) $(u|vk + v'k') = (u|v)k + (u|v')k'$ for all $u, v \in \mathsf{V}$ and $k, k' \in \mathbb{D}$,

(b) (*Hermiticity:*) $(u|v)^* = (v|u)$ for all $u, v \in \mathsf{V}$,

(c) (*definiteness:*) $(u|u) = 0$ if and only if $u = 0$.

Such a structure is called a *generalised inner product space.*

Clearly, a real, complex or quaternionic pre-Hilbert space is a particular example of such a structure. Up to now the only differences rely on the wider generality of $\mathbb{D}$ and point (c) in the definition of Hermitean inner product, which is no longer positive definite (requirement that would not make sense here, for the division ring is not required to contain an ordered sub-ring as happens for $\mathbb{R}$ within $\mathbb{C}$ and $\mathbb{H}$).

Now, let $\mathsf{M}, \mathsf{N} \subset \mathsf{V}$ be linear subspaces, then as usual we write $\mathsf{V} = \mathsf{M} \oplus \mathsf{N}$ if for any element $v \in \mathsf{V}$ there exist two unique vectors $x \in \mathsf{M}$ and $y \in \mathsf{N}$ such that $v = x + y$. In this case V is said to be the *direct sum* of M and N and denoted by $\mathsf{V} = \mathsf{M} \oplus \mathsf{N}$. An equivalent condition is the requirement that $\mathsf{N} \cap \mathsf{M} = \{0\}$ and $\mathsf{V} = \mathsf{M} + \mathsf{N}$.

Now, consider any subset $\mathsf{M} \subset \mathsf{V}$ and define the *orthogonal subspace* as the set

$$\mathsf{M}^\perp := \{v \in \mathsf{V} \mid (v|u) = 0 \text{ for all } u \in \mathsf{M}\}. \tag{5.21}$$

It is clear that this is a subspace of V. Moreover if $\mathsf{M} \subset \mathsf{N}$, then $\mathsf{N}^\perp \subset \mathsf{N}^\perp$. In general it holds that $\mathsf{M} \subset \mathsf{M}^{\perp\perp}$ which is pretty obvious. We say that a subspace M is *biorthogonally closed* if the stronger condition $\mathsf{M}^{\perp\perp} = \mathsf{M}$ holds.

Notice that the operation of taking the orthogonal cannot go beyond the double application. This immediately follows from $\mathsf{M} \subset \mathsf{M}^{\perp\perp}$, more precisely

$$\mathsf{M}^{\perp\perp\perp} = \mathsf{M}^\perp \quad \text{for all } \mathsf{M} \subset \mathsf{V}.$$

In particular $\mathsf{M}^{\perp\perp}$ is a biorthogonally closed subspace whatever M is. It is easy to see that for any given subset M, the subspace $\mathsf{M}^{\perp\perp}$ is actually the *smallest* biorthogonally closed subspace including M, i.e. if $\mathsf{N} = \mathsf{N}^{\perp\perp}$ and $\mathsf{M} \subset \mathsf{N}$, then $\mathsf{M}^{\perp\perp} \subset \mathsf{N}$.

In the case of a Hilbert space H, exploiting Proposition 2.1.10, it follows that $\mathsf{N}^{\perp\perp} = \overline{[\mathsf{N}]}$ for any subset $\mathsf{N} \subset \mathsf{H}$. Thus a subspace is biorthogonally closed if and only if it is closed with respect to the topology of H.

So, consider the set of biorthogonally closed subspaces of a generalized inner product space

***Definition* 5.4.1.** *Let* V *be a generalised inner product space, then the set*

$$\mathfrak{L}(\mathsf{V}) := \{\mathsf{M} \subset \mathsf{V} \,|\, \mathsf{M} = (\mathsf{M}^{\perp})^{\perp}\} \tag{5.22}$$

is called the logic of V.

This set of subspaces can be given a lattice structure. The definition of partial order is quite natural

$$\mathsf{M} \leq \mathsf{N} \quad \text{if and only if} \quad \mathsf{M} \subset \mathsf{N}.$$

Exploiting the properties of $\perp$ it is easy to prove the following statements:

(a) for any $(\mathsf{M}_i)_{i\in\mathcal{I}} \subset \mathfrak{L}(\mathsf{V})$ we have; $\sup_{i\in\mathcal{I}} \mathsf{M}_i = \left(\bigcup_{i\in\mathcal{I}} \mathsf{M}_i\right)^{\perp\perp} \in \mathfrak{L}(\mathsf{H})$

(b) for any $(\mathsf{M}_i)_{i\in\mathcal{I}} \subset \mathfrak{L}(\mathsf{V})$ we have $\inf_{i\in\mathcal{I}} \mathsf{M}_i = \bigcap_{i\in\mathcal{I}} \mathsf{M}_i \in \mathfrak{L}(\mathsf{H})$.

Finally a good candidate for the orthocomplementation is obviously given by the orthogonal operation $\mathsf{M} \mapsto \mathsf{M}^{\perp}$ defined in (5.21). Notice that the trivial subsets $\{0\}$ and V are clearly biorthogonally closed and act as bounds for $\mathfrak{L}(\mathsf{V})$. More precisely we can prove the following result (see Section 34 of [23]).

***Proposition* 5.4.2.** *Let* V *be a generalized inner product space, then the following statements hold:*

(a) $\mathfrak{L}(\mathsf{V})$ *is an irreducible complete atomistic orthocomplemented lattice satisfying the covering law;*

(b) $\mathfrak{L}(\mathsf{V})$ *is weakly modular if and only if* $\mathsf{M}^{\perp} + \mathsf{M}^{\perp\perp} = \mathsf{V}$ *for every* $\mathsf{M} \subset \mathsf{V}$*;*

(c) M *is an atom if and only if* $\dim \mathsf{M} = 1$*;*

(d) $\mathfrak{L}(\mathsf{V})$ *is distributive if and only if* $\dim \mathsf{V} = 1$.

***Remark* 5.4.3.** Some remarks follow.

(a) Notice that, whatever $\mathsf{M} \subset \mathsf{V}$ is, it trivially holds that $\mathsf{M}^{\perp} \cap \mathsf{M}^{\perp\perp} = \{0\}$. Thus the condition in point (b) is equivalent to $\mathsf{M}^{\perp} \oplus \mathsf{M}^{\perp\perp} = \mathsf{V}$ for every $\mathsf{M} \subset \mathsf{V}$.

(b) In the case of a Hilbert space H we already used the notation $\mathfrak{L}(\mathsf{H})$ in Definition 2.2.30 to denote the set of orthogonal projectors over H. The apparent ambiguity is solved by the identification $P \mapsto P(\mathsf{H})$ (see the discussion and results right after Definition 2.2.30. In particular notice $\overline{[U_{i\in\mathcal{I}} P_i]} = [U_{i\in\mathcal{I}} P_i]^{\perp\perp}$ as follows from Proposition 2.1.10).

Now consider the particular case of a pre-Hilbert space H over the real, complex or quaternion real division algebra. We already know that, if H is complete (i.e. is a Hilbert space), then $\mathsf{H} = \mathsf{K}^{\perp} + \mathsf{K}^{\perp\perp}$ for any subset $\mathsf{K} \subset \mathsf{H}$, see Proposition 2.1.10. Actually also the opposite implication holds (see Theorem 34.9 in [23]).

***Proposition* 5.4.4.** *Let* H *be a pre-Hilbert space, then* $\mathfrak{L}(\mathsf{H})$ *is weakly-modular if and only if* H *is complete, i.e. is a Hilbert space.*

This result suggests the following definition.

***Definition* 5.4.5.** *A generalised inner product space* V *which satisfies the equality*

$$\mathsf{M}^{\perp} + \mathsf{M}^{\perp\perp} = \mathsf{V}$$

for all $\mathsf{M} \subset \mathsf{V}$ *is said to be a generalised Hilbert space.*

This becomes sort of a generalisation of the concept of completeness in this situation where no natural topology is introduced.

From Proposition 5.4.2 and Definition 5.4.5 we get the following.

***Corollary* 5.4.6.** *Let* V *be a generalised Hilbert space, then the logic* $\mathfrak{L}(\mathsf{V})$ *is an irreducible bounded (complete) atomistic orthomodular lattice satisfying the covering law.*

A long standing problem in quantum logic was the inverse situation, i.e. the so called *coordinatisation problem*: the attempt to prove that an abstract irreducible complete atomistic orthomodular lattice $\mathfrak{L}$ fulfilling the covering law and possibly further technical requirements, is necessarily isomorphic to the lattice $\mathfrak{L}(\mathsf{V})$ of the orthogonal projectors of a Hilbert space. The first fundamental result due to Piron (see Theorem 34.5 of [23]) who demonstrated that such a lattice $\mathfrak{L}$, if it happens to contain at least four orthogonal atoms, must be isomorphic to the lattice $\mathfrak{L}(\mathsf{V})$ of a suitable generalized Hilbert space.

***Theorem* 5.4.7** (Piron)**.** *Let* $\mathfrak{L}$ *be an irreducible complete atomistic orthomodular lattice satisfying the covering law. If* $\mathfrak{L}$ *contains at least four pairwise-orthogonal atoms, then there exists a generalised Hilbert space* V *and a lattice isomorphism* $h : \mathfrak{L} \to \mathfrak{L}(\mathsf{V})$.

In 1995 Solèr achieved the perhaps conclusive result [37], slighty generlised in the same year by Holland [18]. If V is a generalised inner product space, a set $\mathsf{N} \subset \mathsf{V}$ of orthogonal elements is said to be *orthonormal* if $(z|z) = 1$ for any $z \in \mathsf{N}$.

***Theorem* 5.4.8** (Solèr-Holland)**.** *Let* $(\mathsf{V}, (\cdot|\cdot))$ *be a generalised Hilbert space over the division ring* $\mathbb{D}$. *Suppose one of the following condition holds:*

(a) (Solér) V *contains an infinite orthonormal sequence* $(e_n)_{n\in\mathbb{N}}$,

(b) (Holland) V *contains an infinite orthogonal sequence* $(v_n)_{n\in\mathbb{N}}$ *such that* $(v_n|v_n) = k$ *for some (fixed),* $k \in \mathbb{D} \setminus \{0\}$.

then the following statements hold:

(a) $\mathbb{D} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *and the involution is given by the natural conjugation;*

(b) either $(\cdot|\cdot)$ *or* $-(\cdot|\cdot)$ *is positive definite, making* V *a pre-Hilbert space;*

(c) V *is complete with respect to its natural topology.*

The Hilbert space V *is separable if and only if* $\mathfrak{L}(\mathsf{V})$ *is a separable lattice.*

Combining Piron and Solèr theorems we get a complete characterisation of this (particular) kind of lattices. In the next chapter we will relax some of the conditions required in the above theorems when dealing with more general structure such as the logics of von Neumann algebras.

To conclude this section we state another important representation result concerning the realisation of lattices. We saw above how an irreducible complete orthomodular atomistic lattice satysfing the covering law is necessarily of the form $\mathfrak{L}(\mathsf{H})$ for some Hilbert space H. An analogous result holds for automorphisms on these lattices.

Consider an infinite-dimensional separable real, complex or quaternionic Hilbert space H and take a function $V : \mathsf{H} \to \mathsf{H}$ which is unitary in the real and quaternionic case and either unitary or anti-unitary in the complex case, then it is easy to see that the function

$$\mathfrak{L}(\mathsf{H}) \ni \mathsf{M} \mapsto V(\mathsf{M}) \in \mathfrak{L}(\mathsf{H})$$

is an automorphism of the lattice $\mathfrak{L}(\mathsf{H})$. An important result, which was proved by Wigner in the complex framework and extended by Varadarajan (see Section 4.3 of [44]) to the real and quaternionic cases, tells us that the automorphisms of $\mathfrak{L}(\mathsf{H})$ are only of this form.

***Theorem* 5.4.9** (Wigner-Varadarajan)**.** *Let* H *be an infinite-dimensional separable real, complex or quaternionic Hilbert space, then for every* $h \in Aut(\mathfrak{L}(\mathsf{H}))$ *there exists a unitary (or possibily anti-unitary in the complex case) operator* $U : \mathsf{H} \to \mathsf{H}$ *such that*

$$h(\mathsf{M}) = U(\mathsf{M}) \quad \textit{for all} \;\; \mathsf{M} \in \mathfrak{L}(\mathsf{H}). \tag{5.23}$$

Moreover if V *is another (anti-)unitary operator satisfying (5.23), then* $V = \phi U$ *where* $\phi = \pm 1$ *in the real and quaternionic cases and* $\phi = e^{i\alpha}$ *in the complex case. In particular* V, U *are either both linear or both anti-linear.*

***Remark* 5.4.10.** Some remarks follow.

(a) Interpreting $\mathfrak{L}(\mathsf{H})$ as the set of orthogonal projectors instead of closed subspaces, the action of U in (5.23) translates into

$$h(P) = UPU^{-1} \tag{5.24}$$

for any projector $P \in \mathfrak{L}(\mathsf{H})$.

(b) Also this result is of extremal importance in quantum mechanics as it allows for a realisation of symmetries in terms of (anti-)unitary operators. More on this is discussed in Chapter 6 and Chapter 9.

Chapter 6

Basics in von Neumann Algebras

In this chapter we want to analyse briefly the main concepts concerning von Neumann algebras on arbitrary Hilbert spaces. Good introductions for real and complex Hilbert spaces, covering all the concepts treated here, are given by [20] and [24], respectively. Basing upon these known result, we introduce the concept of a von Neumann algebra on quaternionic Hilbert spaces and provide a proof for the well-known Double Commutant Theorem.

6.1 Elementary Facts of von Neumann Algebras

Let $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ be any subset of the family of bounded operators over a real, complex or quaternionic Hilbert space H and recall the definition of the *commutant* of $\mathfrak{A}$, i.e. the subset

$$\mathfrak{A}' := \{T \in \mathfrak{B}(\mathsf{H}) \mid TA - AT = 0 \text{ for all } A \in \mathfrak{A}\}.$$

It is immediate from the definition that $\mathfrak{A} \subset \mathfrak{A}''$ and $\mathfrak{A}' \subset \mathfrak{T}'$ for any $\mathfrak{T} \subset \mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$. Combining the two results it follows that $\mathfrak{A}' = \mathfrak{A}'''$, hence we cannot reach beyond the second commutant by iteration. It is easy to prove the following result.

***Proposition* 6.1.1.** *If $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$, then the commutant $\mathfrak{A}'$ is closed within $\mathfrak{B}(\mathsf{H})$ with respect to the uniform, strong and weak topologies.*

Now, recall that over real, complex or quaternionic Hilbert spaces, the family $\mathfrak{B}(\mathsf{H})$ is a unital C^*-algebra over $\mathbb{F}_c$ (see Proposition 2.2.15). Referring to this structure it is easy to see that, given any $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$, the commutant $\mathfrak{A}'$ is a unital subalgebra of $\mathfrak{B}(\mathsf{H})$. Furthermore, $\mathfrak{A}'$ turns out to be closed with respect to the Hermitean adjoint if $\mathfrak{A}$ is. Finally, since the commutant is closed in the uniform topology, we have the following result.

***Proposition* 6.1.2.** *Let $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ be *-closed, then $\mathfrak{A}'$ is a unital C^*-subalgebra of $\mathfrak{B}(\mathsf{H})$.*

We are ready to give the following fundamental definition.

***Definition* 6.1.3.** *Let* H *be a real, complex or quaternionic Hilbert space. A unital* **-subalgebra* $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ *is said to be a von Neumann algebra if* $\mathfrak{M}'' = \mathfrak{M}$*. A von Neumann algebra* $\mathfrak{N} \subset \mathfrak{M}$ *is said to be a von Neumann subalgebra of* $\mathfrak{M}$*. A von Neumann algebra is said to be Abelian if its made of commuting operators.*

As a first example of von Neumann algebra consider any *-closed subset $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$. As already discussed above, $\mathfrak{A}'$ is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$. Moreover we know that $(\mathfrak{A}')'' = \mathfrak{A}'$, hence it is a von Neumann. In particular this applies to $\mathfrak{A}''$, which is then a von Neumann algebra containing $\mathfrak{A}$. We can prove that this is actually the *smallest* von Neumann algebra containing $\mathfrak{A}$. Indeed, let $\mathfrak{T} \supset \mathfrak{A}$ be another von Neumann algebra containing $\mathfrak{A}$, then we have $\mathfrak{T}' \subset \mathfrak{A}'$ and so $\mathfrak{T} = \mathfrak{T}'' \supset \mathfrak{A}''$. Therefore we have the following result.

***Proposition* 6.1.4.** *Let* $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ *be* **-closed, then* $\mathfrak{A}''$ *is the smallest von Neumann algebra including* $\mathfrak{A}$*. It is called the von Neumann algebra generated by* $\mathfrak{A}$*.*

We are ready to state and prove the most important result concerning von Neumann algebras. This is a very useful and important characterisation, for it shows how for such structures the topological and the algebraic features are deeply interconnected.

***Theorem* 6.1.5** (Double Commutant Theorem)**.** *Let* H *be a real, complex or quaternionic Hilbert space and* $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ *a unital* **-subalgebra. The following statements are equivalent to each other:*

(a) $\mathfrak{M} = \mathfrak{M}''$*,*

(b) $\mathfrak{M}$ *is weakly closed,*

(c) $\mathfrak{M}$ *is strongly closed.*

The proof for the real and complex cases can be found in any textbook, see for example Theorem 4.3.8 and Corollary 4.4.2 of [20] and Theorem 3.88 of [24], respectively. Unfortunately the procedure implemented there cannot be applied to the quaternionic case because of the following obstruction. In the standard proofs for the real and complex cases, at some point of the calculation, one encounters a unital *-algebra $\mathfrak{A}$ on some Hilbert space $\check{\mathsf{K}}$ and proceeds in defining the subset

$$[\mathfrak{A}x] := \{Ax \mid A \in \mathfrak{A}\} \subset \mathsf{K}. \tag{6.1}$$

where $x \in \mathsf{K}$ is any non-null fixed vector. At this point the crucial observation is that this object is a *linear subspace* of K due to the fact that $\mathfrak{A}$ is an algebra with respect to $\mathbb{F}_c = \mathbb{F}$. Unfortunately, this does not hold true in the quaternionic case, in that $\mathbb{F}_c = \mathbb{R} \neq \mathbb{H} = \mathbb{F}$. One may give it a try by taking the linear span of (6.1) instead of (6.1) itself, but in this case the final part of the proof would not apply. We need something different.

***Remark* 6.1.6.** The concept of a von Neumann algebra over a quaternionic Hilbert space appears also in [13], where the author provides an independent discussion on the reduction of quaternionic quantum systems to complex ones, based on [25]. The double commutant theorem is stated, though it is not proved. For a proof the author sends the reader to Theorem 5.3.1 of [19] - the double commutant theorem for complex Hilbert spaces - stating that the calculation therein holds also in a quaternionic setting. Unfortunately the mentioned proof carries the same obstruction discussed before this remark.

So, focus on the quaternionic Hilbert space case. First, we state and prove a technical result. We will exploit the results of Section 3.2.1, in particular recall the definition of the following operators

$$\mathcal{J}_\alpha : \mathsf{H} \ni x \mapsto x i_\alpha \in \mathsf{H} \quad \text{with} \quad 0 \leq \alpha \leq 3.$$

Now, consider any subset $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$, then, referring to the underline structure of $\mathsf{H}_\mathbb{R}$ and Proposition 3.2.5, it is clear that $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H}_\mathbb{R})$. Hence we can also consider the commutant of $\mathfrak{A}$ within $\mathfrak{B}(\mathsf{H}_\mathbb{R})$, i.e. when $\mathfrak{A}$ is understood as a family of bounded operators on $\mathsf{H}_\mathbb{R}$:

$$\mathfrak{A}'^{\mathbb{R}} := \{A \in \mathfrak{B}(\mathsf{H}_\mathbb{R}) \mid AS = SA \quad \text{for all} \quad S \in \mathfrak{A}\}. \tag{6.2}$$

In particular notice that $\mathcal{J}_\alpha \in \mathfrak{A}'^{\mathbb{R}}$ for all $0 \leq \alpha \leq 3$, the operators in $\mathfrak{A}$ being quaternionic-linear.

***Lemma* 6.1.7.** *Let* H *be quaternionic Hilbert space and* $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$*, then*

$$\mathfrak{A}' \subset \mathfrak{A}'^{\mathbb{R}} \subset \sum_{\alpha=0}^{3} \mathcal{J}_\alpha \mathfrak{A}' \quad (\subset \mathfrak{B}(\mathsf{H}_\mathbb{R})).$$

Proof. The first inclusion is trivial, since all quaternionic linear operators commuting with the elements of $\mathfrak{A}$ are in particular real linear. Let us pass to the second inclusion. Take $A \in \mathfrak{A}'^{\mathbb{R}}$ and define $B_\alpha := \sum_{\beta=0}^{3}(1 - 2\delta_{\alpha\beta})\mathcal{J}_\beta A \mathcal{J}_\beta$ which is clearly an element of $\mathfrak{A}'^{\mathbb{R}}$. By direct inspection it can be proved that $\mathcal{J}_\alpha B_\alpha$ also commutes with every $\mathcal{J}_\alpha$, in this way proving itself to be quaternionic linear (see Proposition 3.2.5). Therefore it actually holds that $\mathcal{J}_\alpha B_\alpha \in \mathfrak{A}'$ or, equivalently, that $B_\alpha \in \mathcal{J}_\alpha \mathfrak{A}'$ for every $0 \leq \alpha \leq 3$. At this point a simple calculation shows that $4A = \sum_{\alpha=0}^{3}(1 - 2\delta_{\alpha 0})B_\alpha \in \sum_{\alpha=0}^{3} \mathcal{J}_\alpha \mathfrak{A}'.$, concluding the proof. □

We are ready to prove the Double Commutant Theorem for quaternionic Hilbert spaces, which can be restated as follows.

***Theorem* 6.1.8.** *Let* H *be a quaternionic Hilbert space and* $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ *a unital* **-subalgebra, then*

$$\mathfrak{M}'' = \overline{\mathfrak{M}}^{s} = \overline{\mathfrak{M}}^{w} = \mathfrak{M}'^{\mathbb{R}}{}'^{\mathbb{R}},$$

where the strong and weak closures above can indifferently be taken in $\mathfrak{B}(\mathsf{H})$ *or* $\mathfrak{B}(\mathsf{H}_\mathbb{R})$*. As a consequence, the following statements are equivalent to each other:*

(a) $\mathfrak{M} = \mathfrak{M}''$,

(b) $\mathfrak{M}$ *is weakly closed,*

(c) $\mathfrak{M}$ *is strongly closed.*

***Remark* 6.1.9.** Thanks to this theorem, a unital sub-*-algebra $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H}_\mathbb{R})$ is a von Neumann algebra over H if and only if it is a von Neumann algebra over $\mathsf{H}_\mathbb{R}$.

Proof of Theorem 6.1.8. The idea of the proof is to reduce to the analogous result for algebras of operators in real Hilbert spaces. As a first step we prove that the weak and strong closures of a set $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H}_\mathbb{R})$ do not depend on the choice of either the real or the quaternionic Hilbert space structure on H. Let us prove it for the strong closures first. If an operator $A \in \mathfrak{B}(\mathsf{H}_\mathbb{R})$ is a strong-limit of elements of $\mathfrak{S}$ then it commutes with every $\mathcal{J}_\alpha$ because every element of the sequence does and so it belongs also to $\mathfrak{B}(\mathsf{H})$. To conclude, since the norms of H and $\mathsf{H}_\mathbb{R}$ coincide, we easily see that A is also a strong limit of elements of $\mathfrak{S}$ within $\mathfrak{B}(\mathsf{H})$. The opposite inclusion is similar, just remember that $\mathfrak{B}(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H}_\mathbb{R})$. Let us pass to the weak closures. Take $(A_\nu)_{\nu\in\mathcal{A}} \subset \mathfrak{S}$ any net of operators within $\mathfrak{A}$ and first suppose that it weakly converges to some $A \in \mathfrak{B}(\mathsf{H})$. This means that $(x|A_\nu y) \to (x|Ay)$ for all $x, y \in \mathsf{H}$ and thus $(x|A_\nu y)_\mathbb{R} = \Re(x|A_\nu y) \to \Re(x|Ay) = (x|Ay)_\mathbb{R}$ for every $x, y \in \mathsf{H}_\mathbb{R}$. The fact that $\mathfrak{B}(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H}_\mathbb{R})$ finally implies that A_ν weakly converges to A within $\mathfrak{B}(\mathsf{H}_\mathbb{R})$, too. Conversely, suppose that A_ν converges weakly to some $A \in \mathfrak{B}(\mathsf{H}_\mathbb{R})$, that is $(x|A_\nu y)_\mathbb{R} \to (x|Ay)_\mathbb{R}$ for any $x, y \in \mathsf{H}_\mathbb{R}$. This operator A must be quaternionic linear as every A_ν is. Indeed notice first that

$$
\begin{aligned}
(x|\mathcal{J}_\alpha Ay)_\mathbb{R} = (\mathcal{J}_\alpha^* x|Ay)_\mathbb{R} = \lim_{\nu\in\mathcal{A}}(\mathcal{J}_\alpha^* x|A_\nu y)_\mathbb{R} = \lim_{\nu\in\mathcal{A}}(x|\mathcal{J}_\alpha A_\nu y) = \\
= \lim_{\nu\in\mathcal{A}}(x|A_\nu \mathcal{J}_\alpha y)_\mathbb{R} = (x|A\mathcal{J}_\alpha y)_\mathbb{R}.
\end{aligned}
$$

At this point the arbitrariness of x, y yields $A\mathcal{J}_\alpha = \mathcal{J}_\alpha A$ for all $0 \le \alpha \le 3$, and thus A is quaternionic linear. Finally we can prove that A_ν converges weakly to A also on H. Indeed exploiting (3.23) we have

$$
(x|A_\nu y) = \sum_{\alpha=0}^{3}(x|\mathcal{J}_\alpha A_\nu y)_\mathbb{R}\overline{i_\alpha} = \sum_{\alpha=0}^{3}(x|A_\nu \mathcal{J}_\alpha y)_\mathbb{R}\overline{i_\alpha} \to \sum_{\alpha=0}^{3}(x|A\mathcal{J}_\alpha y)_\mathbb{R}\overline{i_\alpha} = (x|Ay).
$$

We have in particular established that the weak and strong closures of a unital *-subalgebra $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H}_\mathbb{R})$ do not depend on the real or quaternionic Hilbert space structure of H. At this point the double commutant theorem for real Hilbert spaces guarantees that

$$
\mathfrak{M}'^{\mathbb{R}\prime_\mathbb{R}} = \overline{\mathfrak{M}}^s = \overline{\mathfrak{M}}^w, \tag{6.3}
$$

where now the closures can indifferently be interpreted in $\mathfrak{B}(\mathsf{H}_\mathbb{R})$ or in $\mathfrak{B}(\mathsf{H})$. We are close to concluding the proof. Next step consists in proving that $(\mathfrak{M}')' \subset (\mathfrak{M}'^{\mathbb{R}})'^{\mathbb{R}}$.

So, take a quaternionic linear operator $A \in (\mathfrak{M}')'$, then A is also real linear and thus $A \in (\mathfrak{M}')'^{\mathbb{R}}$ (where $\mathfrak{M}'$ is first defined within $\mathfrak{B}(\mathsf{H})$ and next is viewed as a subset of $\mathfrak{B}(\mathsf{H}_{\mathbb{R}})$). So A is a real linear operator which commutes with every $\mathcal{J}_\alpha$ (it being quaternionic linear) and the elements of $\mathfrak{M}'$. The second inclusion in the statement of Lemma (6.1.7) implies that A also commutes with the elements of $\mathfrak{M}'^{\mathbb{R}}$. Summing everything up we have $A \in (\mathfrak{M}'^{\mathbb{R}})'^{\mathbb{R}}$ which gives $(\mathfrak{M}')' \subset (\mathfrak{M}'^{\mathbb{R}})'^{\mathbb{R}}$. Thus $(\mathfrak{M}')' \subset (\mathfrak{M}'^{\mathbb{R}})'^{\mathbb{R}} = \overline{\mathfrak{M}}^s$, the identity arising from (6.3). Since $\mathfrak{M} \subset (\mathfrak{M}')'$ we also have $\mathfrak{M} \subset (\mathfrak{M}')' \subset (\mathfrak{M}'^{\mathbb{R}})'^{\mathbb{R}} = \overline{\mathfrak{M}}^s$. Taking the strong closure (which is independent from the division algebra) of every set and noticing that $\mathfrak{S}'$ and $\mathfrak{S}'^{\mathbb{R}}$ are always strongly closed (Proposition 6.1.1) we have $\overline{\mathfrak{M}}^s \subset (\mathfrak{M}')' \subset (\mathfrak{M}'^{\mathbb{R}})'^{\mathbb{R}} = \overline{\mathfrak{M}}^s$ and so $\overline{\mathfrak{M}}^s = (\mathfrak{M}')' = (\mathfrak{M}'^{\mathbb{R}})'^{\mathbb{R}}$. This equality together with (6.3) concludes the proof. Indeed points (a),(b),(c) follows immediately. □

***Corollary* 6.1.10.** *Let H be a real, complex or quaternionic Hilbert space, then any von Neumann algebra is a unital C^*-subalgebra of $\mathfrak{B}(\mathsf{H})$.*

Proof. Since $\mathfrak{M}$ is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$, it remains to prove that it is complete with respect to the uniform topology. This follows immediately from Theorem 6.1.8 and (2.14). □

***Proposition* 6.1.11.** *Let H be a real, complex or quaternionic Hilbert spaces and $\mathfrak{M}, \mathfrak{N}$ two von Neumann algebras, then $\mathfrak{M} \cap \mathfrak{N}$ is a von Neumann algebra.*

Proof. By direct inspection it can be proved that the intersection of unital *-subalgebras of $\mathfrak{B}(\mathsf{H})$ is itself a unital *-subalgebras. Moreover the intersection of two closed sets is itself a closed set. The double commutant theorem concludes the proof. □

***Definition* 6.1.12.** *The center of a von Neumann algebra $\mathfrak{M}$ is the Abelian von Neumann algebra $\mathfrak{M} \cap \mathfrak{M}'$. It is denoted by $\mathfrak{Z}_{\mathfrak{M}}$.*

***Notation* 6.1.13.** Let us fix some notation.

(a) By *homomorphism* of von Neumann algebras we mean an homomorphism of unital C^*-algebras between von Neumann algebras. Again, by *isomorphism* we mean a bijective homomorphism and by *automorphism* an isomorphism from a given von Neumann algebra to itself. The set of automorphisms on a von Neumann algebra $\mathfrak{M}$ is a group and denoted by $Aut(\mathfrak{M})$.

(b) Consider any subset $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$, then we can consider the *unital *-algebra* generated by $\mathfrak{S}$ as the family of operators containing the identity element and all the products, the linear combinations and the Hermitean adjoints of the elements of $\mathfrak{S}$. This is of course the *smallest* unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$ containing $\mathfrak{S}$ and it is denoted by $\langle \mathfrak{S} \rangle$.

At this point of the section, given a *-closed subset $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$, we have two definitions of unital *-subalgebra generated out of $\mathfrak{S}$: $\mathfrak{S}''$ and $\langle \mathfrak{S} \rangle$. Exploiting Theorems 6.1.5 and 6.1.8 it is clear that the former is nothing but the strong (or weak) closure of the latter.

***Proposition* 6.1.14.** *Let $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$ be *-closed, then the following statements hold:*

(a) $\mathfrak{S}'' = \overline{\langle \mathfrak{S} \rangle}^s = \overline{\langle \mathfrak{S} \rangle}^w$,

(b) $\mathfrak{S}''$ *is Abelian if* $\mathfrak{S}$ *is so.*

Proof. Point (a) follows immediately from the considerations that were done before the statement of the proposition. To prove point (b) consider any $A, B \in \mathfrak{S}''$, then thanks to point (a) there exist $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \langle \mathfrak{S} \rangle$ such that $A_n v \to Av$ and $B_n v \to Bv$ for all $v \in \mathsf{H}$. Now, let C denote any operator of $\langle \mathfrak{S} \rangle$, then $ACv = \lim_{n\to\infty} A_n Cv = \lim_{n\to\infty} CA_n v = CAv$. Since $v \in \mathsf{H}$ is arbitrary we get $AC = CA$. Now, going back to $B \in \mathfrak{S}''$, using the fact that $B_n \in \langle \mathfrak{S} \rangle$ for all $n \in \mathbb{N}$ and exploiting the just proved commutation, we get $ABv = \lim_{n\to\infty} AB_n v = \lim_{n\to\infty} B_n Av = BAv$. Again, the arbitrariness of $v \in \mathsf{H}$ gives $AB = BA$, concluding the proof. □

To conclude, let us examine the interplay of the von Neumann algebra structures in H and in $\mathsf{H}_{\mathfrak{J}}$ where $\mathfrak{J}$ is a $\mathbb{K}$-structure over H. Referring to the results of Chapter 3, for any subset $\mathfrak{S} \subset \mathfrak{J}'$ we can define

$$\mathfrak{S}_{\mathfrak{J}} := \{S_{\mathfrak{J}} \mid S \in \mathfrak{S}\} \subset \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}).$$

***Proposition* 6.1.15.** *Let $\mathfrak{M}$ be a von Neumann algebra over the real or quaternionic Hilbert space H and let $\mathfrak{J}$ be a $\mathbb{K}$-structure on H such that $\mathfrak{M} \subset \mathfrak{J}'$, then*

(a) if $\mathbb{K} = \mathbb{C}$ and $\mathfrak{J} \subset \mathfrak{M}$, then $\mathfrak{M}_{\mathfrak{J}}$ is a von Neumann algebra over $\mathsf{H}_{\mathfrak{J}}$;

(b) if $\mathbb{K} \neq \mathbb{C}$ then $\mathfrak{M}_{\mathfrak{J}}$ is a von Neumann algebra over $\mathsf{H}_{\mathfrak{J}}$.

Proof. First of all notice that the set $\mathfrak{M}_{\mathfrak{J}}$ defines a unital *-subalgebra of $\mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ as follows from the results of Chapter 3. The requirement $\mathfrak{J} \subset \mathfrak{M}$ in point (a) is essential to guarantee the closedness of $\mathfrak{M}_{\mathfrak{J}}$ under *complex* combinations of operators on $\mathsf{H}_{\mathfrak{J}}$. Hence in both point (a) and point (b) it remains to show that $\mathfrak{M}_{\mathfrak{J}}$ is strongly-closed within $\mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$. We will prove the quaternionic Hilbert space case, the real one being analogous. So, take $(A_n)_{n\in\mathbb{N}} \in \mathfrak{M}_{\mathfrak{J}}$ and $A \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ such that $A_n u \to Au$ for every $u \in \mathsf{H}_{\mathfrak{J}}$, we want to prove that $A \in \mathfrak{M}_{\mathfrak{J}}$. Exploiting the definition of $\mathfrak{M}_{\mathfrak{J}}$ and Proposition 3.2.20 we know that there must exist $\widehat{A_n} \in \mathfrak{M}$ and $\widehat{A} \in \mathfrak{B}(\mathsf{H})$ such that $\widehat{A_n}|_{\mathsf{H}_{\mathfrak{J}}} = A_n$ and $\widehat{A}|_{\mathsf{H}_{\mathfrak{J}}} = A$. So, take $u \in \mathsf{H}$, then we know that $u = \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} u_\alpha i_\alpha$ with $u_\alpha \in \mathsf{H}_{\mathfrak{J}}$ as explained in Proposition 3.2.16. The same proposition assures that $\|\widehat{A}u - \widehat{A_n}u\|^2 = \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} \|Au_\alpha - A_n u_\alpha\|^2_{\mathfrak{J}} \to 0$. thus $\widehat{A}$ is the strong limit of the $\widehat{A_n} \in \mathfrak{M}$ within $\mathfrak{B}(\mathsf{H})$. Since $\mathfrak{M}$ is a von Neumann algebra, it is strongly closed and so $\widehat{A} \in \mathfrak{M}$. This implies $A \in \mathfrak{M}_{\mathfrak{J}}$, concluding the proof. □

6.2 The Unitary Group of von Neumann Algebras

Another important subset of a von Neumann algebra is its group of *unitary elements*.

***Definition* 6.2.1.** *Let $\mathfrak{M}$ be a von Neumann algebra over a real, complex or quaternionic Hilbert space H. The unitary group of $\mathfrak{M}$ is the set*

$$\mathfrak{U}(\mathfrak{M}) := \{U \in \mathfrak{M} \mid U^*U = UU^* = I\}$$

of unitary operators belonging to $\mathfrak{M}$.

A remarkable feature of this subset is the fact that it is able to generate the entire algebra, as we are going to prove.

Recall that for any $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$, by the symbol $[\mathfrak{S}]$ we denote the *linear span* of $\mathfrak{S}$ within $\mathfrak{B}(\mathsf{H})$ (with respect to $\mathbb{F}_c$).

***Proposition* 6.2.2.** *Let $\mathfrak{M}$ a von Neumann algebra over a real, complex or quaternionic Hilbert space H, then the following statements hold:*

(a) $\mathfrak{U}(\mathfrak{M})$ *is a* **-closed group of operators,*

(b) $\overline{[\mathfrak{U}(\mathfrak{M})]} = \mathfrak{U}(\mathfrak{M})'' = \mathfrak{M}$.

Proof. Point (a) can be proved by direct inspection. Let us prove (b). Let us prove $\overline{[\mathfrak{U}(\mathfrak{M})]} = \mathfrak{M}$ first. Once this is proved, then the second equality follows from

$$\mathfrak{M} = \overline{[\mathfrak{U}(\mathfrak{M})]} \subset \overline{\langle \mathfrak{U}(\mathfrak{M}) \rangle} \subset \overline{\langle \mathfrak{U}(\mathfrak{M}) \rangle}^s = \mathfrak{U}(\mathfrak{M})'' \subset \mathfrak{M},$$

where we used Proposition 6.1.14 and the fact that $\mathfrak{U}(\mathfrak{M})$ is *-closed. The proof for the complex case can be found in any textbook on von Neumann algebras (in this case, the even simpler equality $[\mathfrak{U}(\mathfrak{M})] = \mathfrak{M}$ holds), while the proof for the real case can be found in [20] (see Proposition 4.3.5 therein and the remark immediately after). So, suppose that H is quaternionic. Notice that, as pointed out in Remark 6.1.9, $\mathfrak{M}$ is a von Neumann algebra on H if and only if it is a von Neumann algebra on $\mathsf{H}_\mathbb{R}$. Moreover $\mathfrak{U}(\mathfrak{M})$ does not depend on the chosen division algebra since an operator $A \in \mathfrak{M} \subset \mathfrak{B}(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H}_\mathbb{R})$ is unitary on H if and only if it is unitary on $\mathsf{H}_\mathbb{R}$. Moreover, since in H as such as in $\mathsf{H}_\mathbb{R}$ only *real* linear combinations of operators can be carried out, then the definition of $[\mathfrak{U}(\mathfrak{M})]$ is independent from the chosen division algebra. Finally, since the norms of H and $\mathsf{H}_\mathbb{R}$ are equal to each other, it easily turns out that the uniform closure $\overline{[\mathfrak{U}(\mathfrak{M})]}$ can be taken indifferently within $\mathfrak{B}(\mathsf{H})$ or $\mathfrak{B}(\mathsf{H}_\mathbb{R})$. The real version of this very proposition concludes the proof. □

6.3 The Logic of von Neumann Algebras

Another important structure associated with a von Neumann algebra $\mathfrak{M}$ in a real, complex or quaternionic Hilbert space H is the set of orthogonal projectors in $\mathfrak{M}$.

***Definition* 6.3.1.** *Let $\mathfrak{M}$ be a von Neumann algebra over a real, complex or quaternionic Hilbert space H. The logic of $\mathfrak{M}$ is the set*

$$\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) := \{P \in \mathfrak{M} \,|\, P^* = P,\ PP = P\}$$

of orthogonal projectors belonging to $\mathfrak{M}$.

For simplicity of notation we maintain the symbol $\mathfrak{L}(\mathsf{H}) = \mathfrak{L}_{\mathfrak{B}(\mathsf{H})}(\mathsf{H})$.

Differently from the complex Hilbert space case, in the real and quaternionic ones this lattice does not contain all the information about the algebra $\mathfrak{M}$. More precisely it may hold that

$$\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' \subsetneq \mathfrak{M}$$

as the following elementary example shows.

***Example* 6.3.2.** Let H be an either real or quaternionic Hilbert space and suppose there exists an imaginary operator $J \in \mathfrak{B}(\mathsf{H})$. Define $\mathfrak{M} := \{aI + bJ \,|\, a, b \in \mathbb{R}\} \subset \mathfrak{B}(\mathsf{H})$ which is clearly a unital *-subalgebra. Now, take $A \in \overline{\mathfrak{M}}^w$, then there exists some net $(a_\nu I + b_\nu J)_{\nu \in \mathcal{A}} \subset \mathfrak{M}$ weakly-converging to it. A direct calculation shows that $a_\nu I - b_\nu J = (a_\nu I + b_\nu J)^* \to A^*$ weakly. Putting all together it immediately follows that $a_\nu \to a$ and $b_\nu \to b$ for some $a, b \in \mathbb{R}$, thus giving $A = aI + bJ \in \mathfrak{M}$. As $\mathfrak{M}$ is a weakly closed unital *-subalgebra, Theorem 6.1.5 assures that $\mathfrak{M}$ is a von Neumann algebra. On the other hand it must be $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \{0, I\}$ since any self-adjoint element of $\mathfrak{M}$ must be of the form aI for some $a \in \mathbb{R}$. The smallest von Neumann algebra containing the lattice is $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' = \{aI \mid a \in \mathbb{R}\} \subsetneq \mathfrak{M}$.

Even though in the real and quaternionic cases the lattice $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ is not big enough to characterise $\mathfrak{M}$, there exists an interesting interplay between them, which is specified in the following results.

***Proposition* 6.3.3.** *Let H be a real, complex or quaternionic Hilbert space and $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ a von Neumann algebra. Define the set*

$$\mathfrak{J}_{\mathfrak{M}} := \{J \in \mathfrak{M} \,|\, J^* = -J, -J^2 \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})\}.$$

The following statements hold true:

(a) $A = A^* \in \mathfrak{M}$ if and only if the orthogonal projectors of its PVM belongs to $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$;

(b) $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ is a complete orthomodular sublattice of $\mathfrak{L}(\mathsf{H})$;

(c) $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ contains all the self-adjoint elements of $\mathfrak{M}$;

(d) $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' + \mathfrak{J}_{\mathfrak{M}}\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' = \mathfrak{M}$;

(e) $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' \subsetneq \mathfrak{M}$ if and only if there exists $J \in \mathfrak{J}_{\mathfrak{M}} \setminus \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$;

(f) if H is complex then $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' = \mathfrak{M}$;

(g) $\mathfrak{L}_{\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''}(\mathsf{H}) = \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$.

Proof. Let us start with points (a) and (c). Suppose $A^* = A \in \mathfrak{M}$ and $B \in \mathfrak{M}'$, then in particular B commutes with A and Lemma 2.2.58 guarantees that B commutes with the PVM $P^{(A)}$ of A. The arbitrariness of B implies $P^{(A)}(E) \in \mathfrak{M}'' = \mathfrak{M}$, i.e. $P^{(A)}(E) \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ for every Borel set $E \subset \mathbb{R}$. Suppose conversely that $A^* = A \in \mathfrak{B}(\mathsf{H})$ is such that $P^{(A)}_E \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ for every Borel set E, we want to prove that $A \in \mathfrak{M}$. If $B \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'$, then in particular B commutes with the PVM of A and so, thanks again to Lemma 2.2.58, it commutes also with A and so, the operator B being arbitrary, $A \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' \subset \mathfrak{M}$. This concludes the proof of (a). The used argument proves also (c): indeed, if $A^* = A \in \mathfrak{M}$, then, thanks to the first implication of (a) its PVM belongs to $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ and so the argument above applies. The properties of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ listed in (b) are inherited from the same properties of $\mathfrak{L}(\mathsf{H})$. The proof for the complex case can be found in Proposition 7.61 of [24] and can be exploited also for the real and quaternionic setting. Let us prove point (d). First suppose $A \in \mathfrak{M}$ is anti-self-adjoint, then Theorem 4.2.1 and Proposition 4.2.3 show that $A = W|A|$ where W is an anti-self-adjoint partial isometry and $|A| := \sqrt{A^*A} \in \mathfrak{B}(\mathsf{H})$. Moreover W and $|A|$ commute with each other and with every operator commuting with A, as guaranteed again by Proposition 4.2.3. This implies that $W, |A| \in \mathfrak{M}'' = \mathfrak{M}$. Because $|A|$ is self-adjoint, $|A| \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ thanks to (c). Since W is a partial isometry, $-W^2 = W^*W$ is an orthogonal projector which clearly belongs to $\mathfrak{M}$, hence $W \in \mathfrak{I}_{\mathfrak{M}}$. To conclude the proof of (d), observe that a generic operator $A \in \mathfrak{M}$ can always be decomposed as $\frac{1}{2}(A + A^*) + \frac{1}{2}W_0|A - A^*|$ where $(W_0, |A - A^*|)$ is the polar decomposition of the anti-self-adjoint operator $A - A^* \in \mathfrak{M}$. Point (c) and the previous discussion prove that $A + A^*, |A - A^*| \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ and $W_0 \in \mathfrak{I}_{\mathfrak{M}}$ concluding the proof of (d). Let us prove (e). If there exits $J \in \mathfrak{I}_{\mathfrak{M}} \setminus \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ then $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' \subsetneq \mathfrak{M}$, evidently. So, suppose $A \in \mathfrak{M} \setminus \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$, then $A - A^* \in \mathfrak{M} \setminus \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$. Indeed, suppose by contradiction that $A - A^* \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$, then, thanks to the self-adjointness of $A + A^*$ and point (c), we get $2A \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ which contradicts the hypothesis. Referring to the polar decomposition $A - A^* = W_0|A - A^*|$, the above discussion guarantees that $W_0 \in \mathfrak{I}_{\mathfrak{M}}$. Moreover, since $A - A^* \notin \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ and $|A - A^*| \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$, W_0 cannot belong to $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$. Point (f) is a well-known result and can be found in Proposition 7.61 of [24]. To conclude let us prove point (g). Of course, since $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' \subset \mathfrak{M}$, then $\mathfrak{L}_{\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''}(\mathsf{H}) \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. On the contrary take $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, then in particular it belongs to the von Neumanna algebra $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ and so it belongs to its lattice of projectors. $\square$

In the general case it is not obvious when the orthogonal projector generate the algebra. Let us see two examples.

First, we give the following well-known result.

***Proposition* 6.3.4.** *Let* H *be an infinite-dimensional real, complex or quaternionic Hilbert space, then* $\mathfrak{L}(\mathsf{H})' = \{aI \mid a \in \mathbb{F}_c\}$. *In particular* $\mathfrak{L}(\mathsf{H})'' = \mathfrak{B}(\mathsf{H})$.

Proof. Take $A \in \mathfrak{L}(\mathsf{H})'$, then in particular A commutes with all the one-dimensional orthogonal projectors of H. Take $u \in \mathsf{H}$ and let P_u be the orthogonal projector over the one-dimensional subspace generated by u. It holds that $Au = AP_u u = P_u Au$ which implies that $Au \in P_u(\mathsf{H})$, i.e. $Au = u\lambda_u$ for some $\lambda_u \in \mathbb{F}$. Next target is proving that $\lambda_u = \lambda_v$ for all $u, v \in \mathsf{H}$. Take two linearly independent vectors u, v, then the linearity of A gives $u\lambda_u + v\lambda_v = Au + Av = A(u+v) = (u+v)\lambda_{u+v} = u\lambda_{u+v} + v\lambda_{u+v}$ and so $u(\lambda_{u+v} - \lambda_u) = v(\lambda_v - \lambda_{u+v})$. Since the vectors are linearly independent it must be $\lambda_{u+v} - \lambda_u = \lambda_v - \lambda_{u+v} = 0$, i.e. $\lambda_u = \lambda_{u+v} = \lambda_v$. Now want to prove that this scalar belongs to $\mathbb{F}_c$. If $\mathbb{F} = \mathbb{R}, \mathbb{C}$ this is obviously true, so suppose $\mathbb{F} = \mathbb{H}$. If u, v are linearly independent, of course the same holds for u, vp for any $p \in \mathbb{H} \setminus \{0\}$ and so, reasoning as above, $\lambda_v = \lambda_u = \lambda_{vp}$. Thus $v(\lambda_v p) = (Av)p = A(vp) = vp\lambda_{vp} = v(p\lambda_v)$. Since $v \neq 0$ it must be $\lambda_v p = p\lambda_v$ which implies $\lambda_v \in \mathbb{R}$ thanks to the arbitrariness of p. Now, consider a Hilbert basis $\mathsf{N} \subset \mathsf{H}$ and let $\lambda \in \mathbb{F}_c$ be the scalar such that $Az = z\lambda$ for all $z \in \mathsf{N}$. Any $u \in \mathsf{H}$ can be decomposed as $u = \sum_{z \in \mathsf{N}} z(z|u)$ and so $Au = \sum_{z \in \mathsf{N}} Az(z|u) = \sum_{z \in \mathsf{N}} z\lambda(z|u) = \sum_{z \in \mathsf{N}} z(z|u)\lambda = u\lambda$. The arbitariness of $u \in \mathsf{H}$ concludes the proof. $\square$

The second example is more interesting.

***Proposition* 6.3.5.** *Let H be real or quaternionic Hilbert space and $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$ Abelian and made of self-adjoint operators, then*

(a) $\mathfrak{S}''$ is made of self-adjoint operators,

(b) $\mathfrak{L}_{\mathfrak{S}''}(\mathsf{H})'' = \mathfrak{S}''$.

Proof. Point (b) is an easy consequence of point (a) and of point (c) of Proposition 6.3.3. So, consider any element $A \in \mathfrak{S}''$. Thanks to Proposition 6.1.14 we know that A is the weak-limit of some net $(A_\nu)_{\nu \in \mathcal{A}} \in \langle \mathfrak{S} \rangle$. Of course, since $\mathfrak{S}$ is made of commuting self-adjoint element, the same holds for $\langle \mathfrak{S} \rangle$ (notice that only *real* linear combinations of operators can be performed on real or quaternionic Hilbert spaces). Thus $(u|Av) = \lim_{\nu \in \mathcal{A}} (u|A_\nu v) = \lim_{\nu \in \mathcal{A}} (A_\nu u|v) = (Au|v)$ which implies $A^* = A$. $\square$

6.4 Hilbert Direct Sum of von Neumann Algebras

As we did in the lattice framework (see Chapter 5, Section 5.3) in this Section we discuss how to define von Neumann algebras out of already given ones and how a von Neumann algebra can be decomposed into a direct sum of von Neumann subalgebras.

Consider a set of indexes $\mathcal{I}$ and for each $i \in \mathcal{I}$ take a Hilbert space $(\mathsf{H}_i, (\cdot|\cdot)_i)$ on the (same) division algebra $\mathbb{F}$. Define the set

$$\bigoplus_{i \in \mathcal{I}} \mathsf{H}_i \tag{6.4}$$

as the family of the vectors $(v_i)_{i\in\mathcal{I}} \subset \prod_{i\in\mathcal{I}} \mathsf{H}_i$ such that at most countably many of the vectors $(v_i)_{i\in\mathcal{I}}$ is non-zero and

$$\sum_{i\in\mathcal{I}} \|v_i\|_i^2 < \infty$$

This is clearly a linear space over the division algebra $\mathbb{F}$.

Now notice that the function

$$\bigoplus_{i\in\mathcal{I}} \mathsf{H}_i \times \bigoplus_{i\in\mathcal{I}} \mathsf{H}_i \ni ((v_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}}) \mapsto ((v_i)_{i\in\mathcal{I}}|(u_i)_{i\in\mathcal{I}}) := \sum_{i\in\mathcal{I}} (v_i|u_i)_i \in \mathbb{F} \tag{6.5}$$

is well defined in that at most countably many of the terms in the series are non-zero and the sum is absolutely convergent. This function defines an Hermitean scalar product as a direct check shows. Moreover H is complete with respect to the topology induced by $(\cdot|\cdot)$, thus we have the following.

***Proposition* 6.4.1.** *The set (6.4) equipped with (6.5) is a Hilbert space over $\mathbb{F}$, called the Hilbert direct sum of the family $(\mathsf{H}_i, (\cdot|\cdot)_i)_{i\in\mathcal{I}}$.*

***Remark* 6.4.2.** *Notice that each factor H_k can be identified as a closed subspace of $\oplus_{i\in\mathcal{I}}\mathsf{H}_i$. More precisely the map*

$$\mathsf{H}_k \ni v_k \mapsto (0, \dots, 0, v_k, 0, \dots)_{i\in\mathcal{I}} \in \bigoplus_{i\in\mathcal{I}} \mathsf{H}_i \tag{6.6}$$

is an isometry between Hilbert spaces.

An important example of such decomposition is provided by the orthogonal decomposition of any Hilbert space with respect to any maximally orthogonal subset of projectors.

Let H be a Hilbert space and $(P_i)_{i\in\mathcal{I}} \subset \mathfrak{L}(\mathsf{H})$ as in Definition 5.3.3, i.e. such that

$$P_i \perp P_j \quad \text{if } i \neq j \qquad \text{and} \qquad P \perp P_i \text{ for all } i \in \mathcal{I} \Longrightarrow P = 0.$$

The subset $\mathsf{H}_i = P_i(\mathsf{H})$ is a closed subspace of H, hence it defines a Hilbert space when equipped with the restriction of the scalar product of H. The subspaces H_i are mutually orthogonal. Moreover the maximality of the family $(P_i)_{i\in\mathcal{I}}$ implies that for every $v \in \mathsf{H}$ at most countably many of the $P_i v$ are non-zero and

$$v = \sum_{i\in\mathcal{I}} P_i v,$$

where the sum is defined as in (2.8) and (2.9) when restricted to the non-null elements and is insensitive to their indexing order. Similarly

$$\|v\|^2 = \sum_{i\in\mathcal{I}} \|P_i v\|^2.$$

Putting all together it is easy to prove the following result.

***Proposition* 6.4.3.** *Let H be a real, complex or quaternionic Hilbert space and let $(P_i)_{i\in\mathcal{I}} \subset \mathfrak{L}(\mathsf{H})$ be maximally orthogonal for $\mathfrak{L}(\mathsf{H})$. Then the map*

$$\mathsf{H} \ni v \mapsto (P_i v)_{i\in\mathcal{I}} \in \bigoplus_{i\in\mathcal{I}} P_i(\mathsf{H}) \tag{6.7}$$

is a unitary operator between Hilbert spaces.

Now, suppose we are given a family of operators $A_i \in \mathfrak{B}(\mathsf{H}_i)$ such that

$$\sup_{i\in\mathcal{I}}\{\|A_i\|_i\} < \infty,$$

then the function defined by

$$\bigoplus_{i\in\mathcal{I}} A_i : \bigoplus_{i\in\mathcal{I}} \mathsf{H}_i \ni (v_i)_{i\in\mathcal{I}} \mapsto (A_i v_i)_{i\in\mathcal{I}} \in \bigoplus_{i\in\mathcal{I}} \mathsf{H}_i \tag{6.8}$$

is a linear bounded operator, called the *Hilbert direct sum* of the operators A_i.

***Remark* 6.4.4.** *As done for the vectors notice that every operator $A_k \in \mathfrak{B}(\mathsf{H}_k)$ can be identified with a bounded linear operator over $\oplus_{i\in\mathcal{I}}\mathsf{H}_i$. More precisely the map*

$$\mathfrak{B}(\mathsf{H}_k) \ni A_k \mapsto (0,\dots,0,A_i,0,\dots) \in \mathfrak{B}\left(\bigoplus_{i\in\mathcal{I}} \mathsf{H}_i\right) \tag{6.9}$$

is an isometry between unital C^*-algebras.

Some trivial properties follow. Take $a \in \mathbb{F}_c$ and $(A_i)_{i\in\mathcal{I}}, (B_i)_{i\in\mathcal{I}} \subset \mathfrak{B}(\mathsf{H})$ with $\sup_i\{\|A_i\|_i\} < \infty$ and $\sup_i\{\|B_i\|_i\} < \infty$, then the following statements hold:

(i) $I = \bigoplus_{i\in\mathcal{I}} I_i$,

(ii) $\left(\bigoplus_{i\in\mathcal{I}} A_i\right)^* = \bigoplus_{i\in\mathcal{I}} A_i^*$ (remember that $\|T\| = \|T^*\|$, see Proposition 2.2.54),

(iii) $\bigoplus_{i\in\mathcal{I}} A_i + \bigoplus_{i\in\mathcal{I}} B_i = \bigoplus_{i\in\mathcal{I}} (A_i + B_i)$,

(iv) $a\left(\bigoplus_{i\in\mathcal{I}} A_i\right) = a \bigoplus_{i\in\mathcal{I}} A_i$,

(v) $\left(\bigoplus_{i\in\mathcal{I}} A_i\right)\left(\bigoplus_{i\in\mathcal{I}} B_i\right) = \bigoplus_{i\in\mathcal{I}} A_i B_i$.

These properties ensure that the family of operator as in (6.8) give rise to a unital *-algebra over $\oplus_{i\in\mathcal{I}}\mathsf{H}_i$. More generally we have the following result.

***Proposition* 6.4.5.** *Let $(\mathsf{H}_i)_{i\in\mathcal{I}}$ be a family of Hilbert spaces on the same division algebra $\mathbb{F}$ and let $\mathfrak{M}_i \subset \mathfrak{B}(\mathsf{H}_i)$ be a von Neumann algebra for every $i \in \mathcal{I}$. The family*

$$\bigoplus_{i\in\mathcal{I}} \mathfrak{M}_i := \left\{ \bigoplus_{i\in\mathcal{I}} A_i \,\middle|\, A_i \in \mathfrak{M}_i \ \forall i \in \mathcal{I},\ \sup_{i\in\mathcal{I}}\{\|A_i\|_i\} < \infty \right\} \tag{6.10}$$

is a von Neumann algebra over $\bigoplus_{i\in\mathcal{I}}\mathsf{H}_i$, called the Hilbert direct sum of the von Neumann algebras $\mathfrak{M}_i$.

Proof. It remains to prove that $\oplus_{i\in\mathcal{I}}\mathfrak{M}_i$ is closed in the strong-topology of $\oplus_{i\in\mathcal{I}}\mathsf{H}_i$. So, suppose we have $A_i^{(n)}\in\mathfrak{M}_i$ such that

$$\bigoplus_{i\in\mathcal{I}} A_i^{(n)}(v_i)_{i\in\mathcal{I}} \to A(v_i)_{i\in\mathcal{I}}$$

for some operator $A\in\mathfrak{B}(\oplus_{i\in\mathcal{I}}\mathsf{H}_i)$ and for any vector $(v_i)_{i\in\mathcal{I}}$. Fix an index $j\in\mathcal{I}$ and a vector $u_j\in\mathsf{H}_j$. By putting $u_i=0_i$ for any $i\neq j$ we get a vector $(u_i)_{i\in\mathcal{I}}\in\oplus_{i\in\mathcal{I}}\mathsf{H}_i$. Exploiting the convergence of $\oplus_{i\in\mathcal{I}}A_i^{(n)}(u_i)_{i\in\mathcal{I}}$ and the completeness of H_j it immediately follows that $A_j^{(n)}u_j\to w_j$ for some unique $w_i\in\mathsf{H}_i$. The function $A_j:u_j\mapsto w_j$ defines a linear operator over H_i which is also bounded as follows from

$$\begin{aligned}\|A_ju_j\|_j &= \lim_{n\to\infty}\|A_j^{(n)}u_j\|_j = \lim_{n\to\infty}\|\oplus_{i\in\mathcal{I}}A_i^{(n)}(u_i)_{i\in\mathcal{I}}\| = \|A(u_i)_{i\in\mathcal{I}}\| \leq \\ &\leq \|A\|\|(u_i)_{i\in\mathcal{I}}\| = \|A\|\|u_j\|_j,\end{aligned}$$

which implies $\|A_j\|_j\leq\|A\|$. This holds for any index $j\in\mathcal{I}$ and so $\sup_{i\in\mathcal{I}}\{\|A_i\|_i\}\leq\|A\|<\infty$ and so we are allowed to construct the operator $\oplus_{i\in\mathcal{I}}A_i$, which must coincide with A itself. Moreover, by definition, every A_i is the strong-limit within $\mathfrak{B}(\mathsf{H}_j)$ of $A_i^{(n)}\in\mathfrak{M}_i$. Since $\mathfrak{M}_i$ is a von Neumann algebra, it is closed under the strong-topology and so $A_i\in\mathfrak{M}_i$ which implies $A\in\oplus_{i\in\mathcal{I}}\mathfrak{M}_i$. □

A similar composition theorem hold for homomorphisms. Consider two families of von Neumann algebras $(\mathfrak{M}_i)_{i\in\mathcal{I}}$, $(\mathfrak{M}'_i)_{i\in\mathcal{I}}$ over the same Hilbert space and a family of homomorphisms $h_i:\mathfrak{M}_i\to\mathfrak{M}'_i$. Consider the function

$$\bigoplus_{i\in\mathcal{I}} h_i : \bigoplus_{i\in\mathcal{I}}\mathfrak{M}_i \ni (A_i)_{i\in\mathcal{I}} \mapsto (h_i(A_i))_{i\in\mathcal{I}} \in \bigoplus_{i\in\mathcal{I}}\mathfrak{M}'_i. \tag{6.11}$$

The following result can be proved by direct inspection.

***Proposition* 6.4.6.** *Let $h_i:\mathfrak{M}_i\to\mathfrak{M}'_i$ with $i\in\mathcal{I}$ be a family of homomorphisms of von Neumann algebras defined over the same Hilbert space. Then the function defined in (6.11) is an homomorphism of von Neumann algebras, called the Hilbert direct sum of the homomorphisms. The function is an isomorphism if each h_i is.*

At this point we may wonder if an opposite result holds, i.e. if a von Neumann algebra can be decomposed into subalgebras as in Proposition 6.4.5.

Consider a von Neumann algebra $\mathfrak{M}$ and take $P\in\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})\cap\mathfrak{M}'$ and consider the closed subspace $\mathsf{H}_P:=P(\mathsf{H})$. If $A\in\mathfrak{M}$, then

$$A(\mathsf{H}_P)\subset\mathsf{H}_P.$$

Thus the operator

$$\mathfrak{M}\ni A_P := AP \equiv A|_{\mathsf{H}_P}$$

is a well-defined operator over H_P. Some properties can be proved by direct inspection:

(i) $A_P \in \mathfrak{B}(\mathsf{H}_P)$,

(ii) $(A^*)_P = (A_P)^*$,

(iii) $(A+B)_P = A_P + B_P$,

(iv) $(aA)_P = a(A_P)$,

where $A, B \in \mathfrak{B}(\mathsf{H})$ and $a \in \mathbb{F}_c$. These properties ensure that the subset

$$\mathfrak{M}_P := \{A_P \mid A \in \mathfrak{M}\} \subset \mathfrak{B}(\mathsf{H}_P) \tag{6.12}$$

is a unital *-algebra over H_P. Notice that $\mathfrak{M}_P \subset \mathfrak{M}$, it being $P \in \mathfrak{M}$.

***Proposition* 6.4.7.** *The subset $\mathfrak{M}_P$ defined in (6.12) is a von Neumann algebra. Moreover*

$$\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})_P = \{Q_P \mid Q \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})\} = \mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P).$$

Proof. To prove that $\mathfrak{M}_P$ is a von Neumann algebra we have only to prove that it is strongly-closed. Thus, take $(A_P^{(n)})_{n\in\mathbb{N}} \subset \mathfrak{M}_P$ and suppose that $A_P^{(n)}v \to Bv$ for some $B \in \mathfrak{B}(\mathsf{H}_P)$ and for any $v \in \mathsf{H}_P$. By completing B as the null operator on the orthogonal subspace we get some $\tilde{B} \in \mathfrak{B}(\mathsf{H})$ and of course $A^{(n)}Pv \to \tilde{B}v$ for any $v \in \mathsf{H}$. Since $A^{(n)}P \in \mathfrak{M}$ and $\mathfrak{M}$ is strongly-closed it must be $\tilde{B} \in \mathfrak{M}$, thus $B = \tilde{B}P \in \mathfrak{M}_P$. The statement about the lattice is trivial. $\square$

***Theorem* 6.4.8.** *Let $\mathfrak{M}$ be a von Neumann algebra over the real, complex or quaternionic Hilbert space H and let $(P_i)_{i\in\mathcal{I}} \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$ be maximally orthogonal for $\mathfrak{L}(\mathsf{H})$ (see Definition 5.3.3), then the function*

$$\mathfrak{M} \ni A \mapsto (AP_i)_{\in\mathcal{I}} \in \bigoplus_{i\in\mathcal{I}} \mathfrak{M}_{P_i} \tag{6.13}$$

is an isomorphism of von Neumann algebras.

***Theorem* 6.4.9.** *Referring to the same hypotheses of Theorem 6.4.8 suppose there exists some homomorphism $h : \mathfrak{M} \to \mathfrak{M}$ such that $h(P_i) = P_i$ for any $i \in \mathcal{I}$, then the following statements hold:*

(a) $h(\mathfrak{M}_{P_i}) \subset \mathfrak{M}_{P_i}$;

(b) $h_{P_i} := h|_{\mathfrak{M}_{P_i}}$ is an homomorphism, moreover it is an automorphism if h is so;

(c) it holds that $h = \bigoplus_{i\in\mathcal{I}} h_{P_i}$.

6.5 The Commutant of Irreducible von Neumann Algebras

The version of Schur's Lemma given by Theorem 4.3.4 shows that the commutant of any irreducible *-closed subset $\mathfrak{A}$ of $\mathfrak{B}(\mathsf{H})$ in a complex Hilbert space is trivial, i.e. it is given by the complex multiples of the identity operator. For real and quaternionic Hilbert spaces the picture is more complicated: in principle, basing upon Theorem 4.3.4, the commutant may contain infinitely different imaginary operators. We now examine this case in the special case of von Neumann algebras.

Let us focus on the real and quaternionic Hilbert space case now.

***Theorem* 6.5.1.** *Let $\mathfrak{M}$ be a von Neumann algebra on the real or quaternionic Hilbert space H. If $\mathfrak{M}$ is irreducible, then $\mathfrak{M}'$ is of three possible mutually exclusive types:*

(a) $\mathfrak{M}' = \{aI \mid a \in \mathbb{R}\}$ and $\mathfrak{Z}_{\mathfrak{M}} = \{aI \mid a \in \mathbb{R}\}$;

(b) $\mathfrak{M}' = \{aI + bJ \mid a, b \in \mathbb{R}\}$, where J is an imaginary operator determined uniquely up to its sign and satisfies $J \in \mathfrak{M}$, and $\mathfrak{Z}_{\mathfrak{M}} = \{aI + bJ \mid a, b \in \mathbb{R}\}$;

(c) $\mathfrak{M}' = \{aI+bJ_1+cJ_2+dJ_3 | a, b, c, d \in \mathbb{R}\}$, where the operators J_i are anti-commuting imaginary operators satisfying $J_3 = \pm J_1J_2$ and $J_i \notin \mathfrak{M}$, and $\mathfrak{Z}_{\mathfrak{M}} = \{aI \mid a \in \mathbb{R}\}$.

It is easy to see that $\mathfrak{M}' \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, respectively.

Proof. If $A \in \mathfrak{M}'$, Proposition 4.3.4 implies that $A = aI + bL$ for some $a, b \in \mathbb{R}$ and some imaginary operator $L \in \mathfrak{B}(\mathsf{H})$. As a consequence, $\mathfrak{M}'$ is a real associative unital normed algebra with the further property that $||AB|| = ||A||\,||B||$. Indeed, by direct computation we see that $||(aI + bL)x||^2 = (a^2 + b^2)||x||^2$ so that $||aI + bL||^2 = a^2 + b^2$. Furthermore, iterating the procedure, where L' is another complex structure, we get $||(aI + bL)(a'I + b'L')x||^2 = (a^2 + b^2)(a'^2 + b'^2)||x||^2 = ||aI + bL||^2\,||a'I + b'L'||^2||x||^2$ and thus $||(aI+bL)(a'I+b'L')|| = ||aI+bL||\,||a'I+b'L'||$. Thus, as proved in [42] there exists a real associative unital normed algebra isomorphism h from $\mathfrak{M}'$ to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In the first case, $\mathfrak{M}' = h^{-1}(\mathbb{R}) = \{aI \mid a \in \mathbb{R}\}$. In the second case, $\mathfrak{M}' = h^{-1}(\mathbb{C}) = \{aI + bJ \mid a, b \in \mathbb{R}\}$ where $J := h^{-1}(i)$. Furthermore, as h^{-1} is an isomorphism, $JJ = h^{-1}(jj) = h^{-1}(-1) = -I$. In the third case, $\mathfrak{M}' = h^{-1}(\mathbb{H}) = \{aI + bJ_1 + cJ_2 + dJ_3 \mid a, b, c, d \in \mathbb{R}\}$ with $J_1 := h^{-1}(i_1)$ and $J_2 := h^{-1}(i_2)$ and $J_3 := h^{-1}(\pm i_3)$ where $i_1, i_2, i_3 \in \mathbb{H}$ are the three standard imaginary units. Again, as in the complex case, we get $J_iJ_i = h^{-1}(i_ii_i) = h^{-1}(-1) = -I$. Moreover $J_1J_2 = h^{-1}(i_1i_2) = h^{-1}(-i_2i_1) = -J_2J_1$. Exploiting $J_3 = h^{-1}(\pm i_3) = \pm h^{-1}(i_1i_2) = \pm J_1J_2$ we can easily show that $J_iJ_j = -J_jJ_i$ for every $i \neq j$. Let us prove that J in the complex case and J_1, J_2 in the quaternionic one are anti-self-adjoint concluding that they are imaginary operators. The proof is the same in both cases, so we deal with J only. Since $\mathfrak{M}'$ is a *-algebra, it holds that $J^* \in \mathfrak{M}'$, in particular $J^*J \in \mathfrak{M}'$ which is clearly self-adjoint and positive. Since $\mathfrak{M}$ is irreducible, Lemma 4.3.3 guarantees that $J^*J = aI$ for some $a \geq 0$. Multiplying both sides by $-J$ on the right, using

$JJ = -I$, we get $J^* = -aJ$. Taking the adjoint on both sides yields $J = -aJ^*$ which, in particular, assures that $a \neq 0$, J being unitary and thus bijective. So, $J^* = -\frac{1}{a}J$. Summing up, $0 = J^* - J^* = \left(a - \frac{1}{a}\right) J$. As $JJ = -I$, it must be $a - \frac{1}{a} = 0$, hence $a = 1$ and $J^* = -aJ = -J$ as wanted. J is an imaginary operator. J_3 turns out to be a complex structure as well, since J_1 and J_2 are anticommuting imaginary operators such that $J_3 = \pm J_1 J_2$. To conclude, let us establish the form of the centers $\mathfrak{Z}_{\mathfrak{M}}$. The real case is obvious. In the complex case, J commutes with $\{aI + bJ \mid a, b \in \mathbb{R}\} = \mathfrak{M}'$, so it belongs to $\mathfrak{M}'' = \mathfrak{M}$ and thus $\mathfrak{Z}_{\mathfrak{M}} = \{aI + bJ \mid a, b \in \mathbb{R}\}$. This result also implies that in the complex case J is unique up to its sign. Indeed, let J' be another imaginary operator in $\mathfrak{M}'$, then it commutes with J (as it belongs also to $\mathfrak{M}$). Therefore $JJ' \in \mathfrak{M}'$ is self-adjoint and thus $JJ' = aI$, namely $J' = -aJ$, because $\mathfrak{M}$ is irreducible. Since $JJ = J'J' = -1$ we must have $a = \pm 1$. The form of $\mathfrak{Z}_{\mathfrak{M}}$ for the quaternionic case is trivial. Suppose that $U = aI + bJ_1 + cJ_2 + dJ_3 \in \mathfrak{M} \cap \mathfrak{M}'$ for some $a, b, c, d \in \mathbb{R}$, then it must be $UJ_1 = J_1 U$, that is

$$aJ_1 + bJ_1^2 + cJ_2J_1 + dJ_3J_1 = aJ_1 + bJ_1^2 + cJ_1J_2 + dJ_1J_3, \tag{6.14}$$

which yields $2cJ_1J_2 + 2dJ_1J_3 = 0$, i.e. $\pm 2cJ_3 + 2dJ_1J_3 = 0$. Multiplying on the right by J_3 we get $\mp 2c - 2dJ_1 = 0$. Taking the Hermitean conjugate we get $\mp 2c + 2dJ_1 = 0$. Combining the two identities it easily follows that $c = d = 0$. Finally, since $UJ_2 = J_2U$ we get similarly $aJ_2 + bJ_1J_2 = aJ_2 + bJ_2J_1$, which gives $2bJ_1J_2 = 0$, i.e. $b = 0$, concluding the proof. □

***Remark* 6.5.2.** The choice of the sign of J_3 in the case $\mathfrak{M}' \cong \mathbb{H}$ is totally arbitrary for the mere purpose of Theorem 6.5.1. However, in view of what follows we fix it by choosing $J_3 = -J_1J_2$ if H is real and $J_3 = J_1J_2$ if H is quaternionic. In this way the set $(J_\alpha)_{\alpha=0,\ldots,3}$ defines a quaternionic, respectively real, structure on H as defined in Chapter 3 (check Remarks 3.1.10 and 3.2.9).

6.6 The Structure of Irreducible von Neumann Algebras

We are in a position to prove a remarkable characterisation of irreducible von Neumann algebras: all of them are isomorphic to $\mathfrak{B}(\mathsf{H}')$ for some suitable Hilbert space H'. This is a trivial consequence of Schur's Lemma in the complex case, while the proof for the real and quaternoinic cases is a bit more involved. This result will play a central and crucial role in the rest of this work.

***Proposition* 6.6.1.** *Let $\mathfrak{M}$ be an irreducible von Neumann algebra over a complex Hilbert space, then $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$.*

So, let us focus on the real and quaternionic Hilbert space cases. Notice that in the $\mathfrak{M}' \cong \mathbb{C}$ and $\mathfrak{M}' \cong \mathbb{H}$ cases the operators J_i in the commutant give rise to a $\mathfrak{J}$ structure on H. Hence we can apply the results of Chapter 4 and work on the space $\mathsf{H}_{\mathfrak{J}}$ (keep in mind Remark 6.5.2).

***Theorem* 6.6.2.** *Let $\mathfrak{M}$ be an irreducible von Neumann algebra on a real or quaternionic Hilbert space H. Referring to the three cases listed in Theorem 6.5.1, the following statements hold:*

(a) if $\mathfrak{M}'$ is real then it holds that

- *(i) $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$,*
- *(ii) $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$;*

(b) if $\mathfrak{M}'$ is complex or quaternionic, then, referring to the $\mathbb{K}$-structure $\mathfrak{J}$ induced by the commutant, it holds that

- *(i) $\mathfrak{M}_{\mathfrak{J}} = \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and the map $\mathfrak{M} \ni A \mapsto A_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ is a norm-preserving weakly-bicontinuous (thus strongly-bicontinuous) isomorphism of real unital *-algebras,*
- *(ii) the map $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \ni P \mapsto P_{\mathfrak{J}} \in \mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$ is an isomorphism of complete orthocomplemented lattices.*

Proof. Let us start with point (a). From $\mathfrak{M}' = \{aI \,|\, a \in \mathbb{R}\}$ it immediately follows that $\mathfrak{M} = \mathfrak{M}'' = \mathfrak{B}(\mathsf{H})$ and thus $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$. Let us pass to point (b). We know from Proposition 6.1.15 (notice that $\mathfrak{J} \subset \mathfrak{M}$ in the complex commutant case) that $\mathfrak{M}$ can be identified with the von Neumann algebra $\mathfrak{M}_{\mathfrak{J}}$ using the map $\mathfrak{M} \ni A \mapsto A_{\mathfrak{J}} \in \mathfrak{M}_{\mathfrak{J}}$, the latter von Neumann algebra being irreducible due to Proposition 4.3.6 since the former is irreducible. Let us prove that $\mathfrak{M}_{\mathfrak{J}} = \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$. If the commutant is of complex type, then the thesis follows immediately from Proposition 6.6.1. So, suppose that $\mathfrak{M}'$ is of quaternionic type. Consider $A \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$, then there exists a unique $\widehat{A} \in \mathfrak{B}(\mathsf{H})$ such that $A = \widehat{A}_{\mathfrak{J}}$ (this is trivial if H is real, since $A_{\mathfrak{J}} = A$ and so it suffices to take $\widehat{A} = A$, while in the quaternionic Hilbert space case the thesis follows from Proposition 3.2.20). Moreover $\widehat{A}$ commutes with J_1, J_2 and J_3. Since $\mathfrak{M}' = \{aI + bJ_1 + cJ_2 + dJ_3\}$, it immediately follows that $\widehat{A} \in \mathfrak{M}'' = \mathfrak{M}$, i.e., $A \in \mathfrak{M}_{\mathfrak{J}}$. This means $\mathfrak{M}_{\mathfrak{J}} \supset \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and thus $\mathfrak{M}_{\mathfrak{J}} = \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ because the converse inclusion is obvious. The map $A \mapsto A_{\mathfrak{J}}$ is also a norm-preserving weakly-bicontinuous, strongly-bicontinuous *-isomorphism of unital *-algebras in view of Proposition 3.1.16 and Proposition 3.2.20. Finally, from the same propositions it can be easily proved that $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \ni P \mapsto P_{\mathfrak{J}} \in \mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$ is an isomorphism of orthocomplemented lattices. The only pair of properties to be proved concerns completeness of the involved lattices of orthogonal projectors and are the following ones. (1) Given a family $\{P_a\}_{a\in A} \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ such that $\mathsf{M}_a := P_a(\mathsf{H})$ defining $P = \inf_{a\in A} P_a$ – in other words P is the orthogonal projector onto $\mathsf{M} := \cap_{a\in A}\mathsf{M}_a$ – it turns out that $P_{\mathfrak{J}} = \inf_{a\in A}(P_a)_{\mathfrak{J}}$. Regarding the fact that $P_{\mathfrak{J}}$ is well defined, observe that $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ because $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ is complete and thus P commutes with $\mathfrak{J}$ since $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \mathfrak{M}$. (2) Given a family $\{P_a\}_{a\in A} \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ such that $M_a := P_a(\mathsf{H})$ defining $Q = \sup_{a\in A} P_a$ – in other words $Q \in \mathfrak{M}$ is the orthogonal projector onto $\mathsf{N} := \overline{[\cup_{a\in A}\mathsf{M}_a]}$ – it turns out that $Q_{\mathfrak{J}} = \sup_{a\in A}(P_a)_{\mathfrak{J}}$. Regarding the fact that $Q_{\mathfrak{J}}$ is well defined, observe that $Q \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ because $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ is complete and thus Q commutes with $\mathfrak{J}$ since $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \mathfrak{M}$. To prove (1), observe that $P_{\mathfrak{J}}$ is the orthogonal projector onto $\mathsf{M} \cap \mathsf{H}_{\mathfrak{J}}$ because $x = P_{\mathfrak{J}}x$ if and only if both $x \in \mathsf{H}_{\mathfrak{J}}$ (because $P_{\mathfrak{J}}$ is a projector in $\mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$) and $x = Px$, which means $x \in \mathsf{M}$. We conclude that $P_{\mathfrak{J}}$ is the orthogonal projector onto $\mathsf{M} \cap \mathsf{H}_{\mathfrak{J}} = (\cap_{a\in A}\mathsf{M}_a) \cap \mathsf{H}_{\mathfrak{J}} = \cap_{a\in A}(\mathsf{M}_a \cap \mathsf{H}_{\mathfrak{J}})$. The orthogonal projector onto the last space is $\inf_{a\in A}(P_a)_{\mathfrak{J}}$ by definition. We have obtained $P_{\mathfrak{J}} = \inf_{a\in A}(P_a)_{\mathfrak{J}}$. Property (2) is an immediate consequence of (1) and De Morgan's rule, valid for any family orthogonal projectors $\{Q_b\}_{b\in B} \subset \mathfrak{B}(\mathsf{K})$ with K real, complex or quaternionic, $\sup_{b\in B} Q_b = \left(\inf_{b\in B} Q_b^{\perp}\right)^{\perp}$, where $Q^{\perp} := I - Q$ is the orthogonal projector onto $Q(\mathsf{K})^{\perp}$. □

6.7 Gleason's Theorem on Irreducible von Neumann Algebras

Theorems 6.5.1 and 6.6.2 have remarkable consequences. First of all they allow us to import the important result due to Gleason (Theorem 4.4.17) also to irreducible von Neumann algebras on real or quaternionic Hilbert spaces.

***Theorem* 6.7.1** (Gleason's Theorem for irreducible von Neumann algebras)**.** *Let $\mathfrak{M}$ be a von Neumann algebra over an infinite-dimensional separable real, complex or quaternionic Hilbert space, then the following statements hold:*

(a) if $T \in \mathfrak{D}(\mathsf{H})$ then the function $\mu_T : \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \to [0,1]$ defined by

$$\mu_T(P) := tr(TP) \quad \text{for all } P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \tag{6.15}$$

is a σ-probability measure on $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$;

(b) if $\mathfrak{M}$ is irreducible, then for every σ-probability measure μ on $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ there exists a unique density operator $T \in \mathfrak{D}(\mathsf{H}) \cap \mathfrak{M}$ such that $\mu = \mu_T$, where μ_T is defined as in (6.15).

Proof. Point (a). Notice that the function $\mathfrak{L}(\mathsf{H}) \ni P \mapsto tr(TP)$ with $T \in \mathfrak{D}(\mathsf{H}) \cap \mathfrak{M}$ is a σ-probability measure on $\mathfrak{L}(\mathsf{H})$ as proved in Theorem 4.4.17. By taking its restriction to $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ we have a σ-probability measure on $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. Point (b). Suppose on the contrary that we are given some σ-probability measure $\mu : \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \to [0,1]$. If H is complex or if H is real or quaternionic and $\mathfrak{M}' \cong \mathbb{R}$, then Theorem 4.4.17 together with the fact that $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$ concludes the proof. So, suppose that H is real or quaternionic and $\mathfrak{M}' \cong \mathbb{C}$ or $\mathbb{H}$. Thus $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$. The latter isomorphism, as better specified stated in Theorem 6.6.2, guarantees that a σ-additive probability measure over $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ corresponds to a σ-additive measure $\mu_{\mathfrak{J}}$ over $\mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$ defined by

$$\mu_{\mathfrak{J}} : \mathfrak{L}(\mathsf{H}_{\mathfrak{J}}) \ni P_{\mathfrak{J}} \mapsto \mu_{\mathfrak{J}}(P_{\mathfrak{J}}) := \mu(P) \in [0,1]$$

Since we are on the entire lattice $\mathfrak{L}(\mathsf{H}_\mathfrak{J})$, we can apply the Gleason-Varadarajan theorem (Theorem 4.4.17) proving that there is a density operator $T_0 \in \mathfrak{D}(\mathsf{H}_\mathfrak{J})$ such that $\mu_\mathfrak{J}(P_\mathfrak{J}) = tr(T_0 P_\mathfrak{J})$ with $P_\mathfrak{J} \in \mathcal{L}(\mathsf{H}_\mathfrak{J})$. Take $T \in \mathfrak{M}$ such that $T_\mathfrak{J} = T_0$, which is still self-adjoint and positive. Proposition 4.4.9 assures that $T \in \mathfrak{B}_1(\mathsf{H}) \cap \mathfrak{M}$ and that $tr(T) = (d_\mathbb{K}+1)tr(T_0) = (d_\mathbb{K}+1)$ if H is real and $tr(T) = tr(T_\mathfrak{J}) = 1$ if H is quaternionic. Thus if we define $S \in \mathfrak{M}$ as $S := T$ in the quaternionic case and $S := (d_\mathbb{K}+1)^{-1}T$ in the real case, it holds that $tr(S) = 1$, more precisely $S \in \mathfrak{D}(\mathsf{H}) \cap \mathfrak{M}$. Similarly, since $SP \in \mathfrak{B}_1(\mathsf{H})$, exploiting again Proposition 4.4.9, we have

$$\mu(P) = \mu_\mathfrak{J}(P_\mathfrak{J}) = tr(T_0 P_\mathfrak{J}) = \begin{cases} (d_\mathbb{K}+1)^{-1} tr(TP) = tr(SP) & \mathsf{H} \text{ real} \\ tr(TP) = tr(SP) & \mathsf{H} \text{ quaternionc} \end{cases} \tag{6.16}$$

If we manage to prove that this operator is uniquely defined within $\mathfrak{M}$ then the proof is complete. So, suppose that there exists $S' \in \mathfrak{M} \cap \mathfrak{D}(\mathsf{H})$ such that $tr(S'P) = tr(SP)$ for all $P \in \mathfrak{L}_\mathfrak{M}(\mathsf{H})$. Exploiting again the isomorphism $\mathfrak{L}_\mathfrak{M}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_\mathfrak{J})$ and Proposition 4.4.9 we see that $tr(S'_\mathfrak{J} P_\mathfrak{J}) = tr(S_\mathfrak{J} P_\mathfrak{J})$ for all $P_\mathfrak{J} \in \mathfrak{L}(\mathsf{H}_\mathfrak{J})$. At this point (after a suitable normalisation) Theorem 4.4.17 assures that $S_\mathfrak{J} = S'_\mathfrak{J}$. The isomoprhism $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_\mathfrak{J})$ gives $S = S'$ concluding the proof.

□

***Remark* 6.7.2.** Notice that the uniqueness in point (b) of Theorem 6.7.1 concerns the density operators which *belong* to $\mathfrak{M}$. If $T \in \mathfrak{D}(\mathsf{H}) \setminus \mathfrak{M}$ then, as stated in point (a), the map $tr(T\cdot)$ still defines a σ-probability measure on $\mathfrak{L}_\mathfrak{M}(\mathsf{H})$, but, according to point (b), the same measure can be realised by means of a density operators that belongs to $\mathfrak{M}$ and this one is unique.

6.8 Wigner's Theorem on Irreducible von Neumann Algebras

As we did in the previous section for σ-probability measures we can now discuss the consequences of irreducibility on the automorphisms of the complete orthocompleted lattice $\mathfrak{L}_\mathfrak{M}(\mathsf{H})$.

***Remark* 6.8.1.** The following theorem is an improved version of the corresponding statement in Proposition 8.2 - (b) of [26]. The central idea of this improvement comes from Theorem 3.15 of [13], though the calculation slightly differs from the original one.

***Theorem* 6.8.2.** *Let $\mathfrak{M}$ be a von Neumann algebra over an infinite-dimensional separable real, complex or quaternionic Hilbert space, then the following statements hold:*

(1) let U be a unitary, or possibly anti-unitary in the complex case, operator on H such that $UPU^{-1} \in \mathfrak{M}$ for all $P \in \mathfrak{L}_\mathfrak{M}(\mathsf{H})$, then the function

$$h : \mathfrak{L}_\mathfrak{M}(\mathsf{H}) \ni P \mapsto UPU^{-1} \in \mathfrak{L}_\mathfrak{M}(\mathsf{H}) \tag{6.17}$$

defines an automorphism of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$*;*

(2) if $\mathfrak{M}$ *is irreducible, then any automorphism of* $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ *is of the form (6.17) where* U *satisfies the following properties:*

(a) if H *is complex, then* U *is unitary (thus* $U \in \mathfrak{M}$*) o anti-unitary (and* $U \notin \mathfrak{M}$*);*

(b) if H *is real or quaternionic then* U *is unitary, more precisely:*

(i) if $\mathfrak{M}' \cong \mathbb{R}$ *or* $\mathbb{H}$ *then* $U \in \mathfrak{M}$*,*

(ii) if $\mathfrak{M}' \cong \mathbb{C}$ *then* $UJ = JU$ *(thus* $U \in \mathfrak{M}$*) or* $UJ = -JU$ *(and* $U \notin \mathfrak{M}$*).*

Moreover, suppose U, V *are two operators as in (a) or (b) satisfying (6.17), then they are of the same kind (either both belong to* $\mathfrak{M}$ *or they do not) and differ only by elements of* $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$*.*

Proof. Point (1) follows immediately by direct inspection, so let us prove point (2). Suppose first that H is complex, then the irreducibility of $\mathfrak{M}$ together with Schur's Lemma assures that $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ and so, in particular, $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$. Theorem 5.4.9 gives point (a). Let us pass to (b) and so suppose that H is real or quaternionic. Suppose $\mathfrak{M}' \cong \mathbb{R}$ first, then $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$. Again, Theorem 5.4.9 gives the first half of point (i). If $\mathfrak{M}' \cong \mathbb{H}$, then (thanks to Theorem 6.6.2), referring to the structure $\mathfrak{J}$ induced by the commutant, it holds that $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$, where $\mathsf{H}_{\mathfrak{J}}$ is a quaternionic or a real Hilbert space depending whether H is, respectively, real or quaternionic. Exploiting the latter isomorphism, any automorphism h of the (complete orthocomplemented) lattice $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ translates into an automorphism $h_{\mathfrak{J}}$ of the (complete orthocomplemented) lattice $\mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$ simply by defining $h_{\mathfrak{J}}(P_{\mathfrak{J}}) := h(P)_{\mathfrak{J}}$ for all $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. Theorem 5.4.9 gives the existence of a unitary operator V over the (respectively quaternionic or real) Hilbert space $\mathsf{H}_{\mathfrak{J}}$ such that $h_{\mathfrak{J}}(P_{\mathfrak{J}}) = VP_{\mathfrak{J}}V^*$. Thanks to the isomorphism $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$, there exists a unique unitary operator $\widetilde{V} \in \mathfrak{M}$ such that $\widetilde{V}_{\mathfrak{J}} = V$ (remember that $\widetilde{V}_{\mathfrak{J}} = \widetilde{V}$ in the real Hilbert space case and $\widetilde{V}_{\mathfrak{J}} = \widetilde{V}|_{\mathsf{H}_{\mathfrak{J}}}$ in the quaternionic one). Moreover, $(VPV^*)_{\mathfrak{J}} = V_{\mathfrak{J}}P_{\mathfrak{J}}(V_{\mathfrak{J}})^* = h_{\mathfrak{J}}(P_{\mathfrak{J}}) = h(P)_{\mathfrak{J}}$, i.e. $VPV^* = h(P)$. Point (i) is completed. Now, suppose that $\mathfrak{M}' \cong \mathbb{C}$. Again, referring to the complex structure $\mathfrak{J}$ induced by the commutant, we can perform the isomorphisms $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$ on the complex Hilbert space $\mathsf{H}_{\mathfrak{J}}$. As above any automorphism h on $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ translates into an automorphism $h_{\mathfrak{J}}$ on $\mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$. At this point Theorem 5.4.9 assures the existence of a unitary *or anti-unitary* operator V on $\mathsf{H}_{\mathfrak{J}}$ such that $h_{\mathfrak{J}}(P_{\mathfrak{J}}) = VP_{\mathfrak{J}}V^{-1}$ for all $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. If V is unitary we can proceed as in the quaternionic commutant case and find a unitary operator $\widetilde{V} \in \mathfrak{M}$ such that $\widetilde{V}_{\mathfrak{J}} = V$ and $h(P) = VPV^*$. So, suppose that V is anti-unitary, thus $V(ui_1) = -(Vu)i_1$ for all $u \in \mathsf{H}_{\mathfrak{J}}$. If H is real, then $\mathsf{H}_{\mathfrak{J}} = \mathsf{H}$ as sets and their norms coincide, thus the function V defines a real-linear norm-preserving bijective function on H, thus a unitary operator on H (thanks to the polarisation identity (2.4)) generating h. In particular $VJu = V(ui_1) = -(Vu)i_1 = -JVu$, i.e. $VJ = -JV$. Now, suppose that H is quaternionic and, referring to the decomposition (3.32), define the function $\widetilde{V} : \mathsf{H} \to \mathsf{H}$ by

$$\widetilde{V}(x + yi_3) := -Vy + (Vx)i_3 \quad \text{for all } x, y \in \mathsf{H}_{\mathfrak{J}}.$$

This is clearly real-linear and preserves the norm as follows from (3.33) and the fact that V preserves the norm on $\mathsf{H}_{\mathfrak{J}}$. As a corollary, the function $\widetilde{V}$ is injective. Let us prove the surjectivity. Consider a vector $u + vi_3 \in \mathsf{H}$ with $u, v \in \mathsf{H}_{\mathfrak{J}}$, then, the function V being a bijection, there must exit some $x, y \in \mathsf{H}_{\mathfrak{J}}$ with $u = Vx$ and $v = Vy$. Thus $\widetilde{V}(y + (-x)i_3) = Vx + (Vy)i_3 = u + vi_3$ and so the function $\widetilde{V}$ is surjective. It is easy to prove that the inverse V^{-1} on $\mathsf{H}_{\mathfrak{J}}$ is itself an anti-linear function preserving the norm. Reasoning as we did for V, the function $\widetilde{V}^{-1} : \mathsf{H} \to \mathsf{H}$ defined by

$$\widetilde{V}^{-1}(u + vi_3) := V^{-1}v - (V^{-1}u)i_3 \quad \text{for all } u, v \in \mathsf{H}_{\mathfrak{J}}$$

is a real-linear norm-preseving bijective function on H. Moreover, by direct inspection, it holds that $\widetilde{V}^{-1}\widetilde{V} = \widetilde{V}\widetilde{V}^{-1}$, thus $\widetilde{V}^{-1}$ truly defines the inverse of $\widetilde{V}$. Now, consider any vector $x + yi_3 \in \mathsf{H}$, then, exploiting the definition of $\widetilde{V}$ and the anti-linearity of V we have

$$\begin{aligned}
\widetilde{V}((x + yi_3)i_1) &= \widetilde{V}(xi_1 + (-yi_1)i_3) = -V(-yi_1) + (V(xi_1))i_3 = \\
&= -(Vy)i_1 - (Vx)i_1 i_3 = [-Vy + (Vx)i_3]i_1 = \widetilde{V}(x + yi_3)i_1 \\
\widetilde{V}((x + yi_3)i_3) &= \widetilde{V}(-y + xi_3) = -Vx + (V(-y))i_3 = [-(Vy) + (Vx)i_3]i_3 = \\
&= \widetilde{V}(x + yi_3)i_3 \\
\widetilde{V}((x + yi_3)i_2) &= \widetilde{V}((x + yi_3)i_3 i_1) = \widetilde{V}((x + yi_3))i_3 i_1 = \widetilde{V}((x + yi_3))i_2.
\end{aligned} \tag{6.18}$$

This proves that $\widetilde{V}$ is a linear (norm-preserving invertible) function H. At this point, exploiting the polarisation identity (2.4), we can easily prove that the operator $\widetilde{V}$ is actually a *unitary* operator on H. Everything that has been done so far can be repeated for the inverse $\widetilde{V}^{-1}$.

Now, let us prove that $\widetilde{V}J = -J\widetilde{V}$ where by hypothesis $\mathfrak{M}' = \{Ia + Jb \,|\, a, b \in \mathbb{R}\}$. So, take $x + yi_3 \in \mathsf{H}$, then

$$\begin{aligned}
\widetilde{V}J(x + yi_3) &= \widetilde{V}(Jx + (Jy)i_3) = \widetilde{V}(xi_1 + (yi_1)i_3) = -V(yi_1) + (V(xi_1))i_3 = \\
&= (Vy)i_1 - (Vx)i_1 i_3 = J(Vy) - J(Vx)i_3 = -J[-Vy + (Vx)i_3] = \\
&= -J\widetilde{V}(x + yi_3).
\end{aligned}$$

The vector $x+yi_3$ being arbitrary, we have the thesis. So, to conclude this part, take any $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ and any $x+yi_3 \in \mathsf{H}$, then

$$\begin{aligned}\widetilde{V}P\widetilde{V}^{-1}(x+yi_3) &= \widetilde{V}P(V^{-1}y-(V^{-1}x)i_3) = \widetilde{V}(P_{\mathfrak{Z}}V^{-1}y-(P_{\mathfrak{Z}}V^{-1}x)i_3) = \\ &= (VP_{\mathfrak{Z}}V^{-1}x)+(VP_{\mathfrak{Z}}V^{-1}y)i_3 = h_{\mathfrak{Z}}(P_{\mathfrak{Z}})x+h_{\mathfrak{Z}}(P_{\mathfrak{Z}})yi_3 = \\ &= h(P)(x+yi_3).\end{aligned} \tag{6.19}$$

The operator $\widetilde{V}$ satisfies the thesis of the theorem. Concerning the last statement, we prove only the case of quaternionic Hilbert space with complex commutant, the other ones being trivial (exploit the corresponding statement in Theorem 5.4.9 and use again the isomorphism $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H}_{\mathfrak{Z}})$ when necessary). So, suppose that we have two unitary operators U, V on H which generates the automorphism h, i.e.

$$h(P) = UPU^{-1} = VPV^{-1} \quad \text{for any} \quad P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$$

and suppose first that $UJ = JU$ and $VJ = -JV$. Exploiting the isomorphism $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H}_{\mathfrak{Z}})$, as already seen above, we can consider the automorphism $h_{\mathfrak{Z}}$ on $\mathfrak{L}(\mathsf{H}_{\mathfrak{Z}})$ defined through $h_{\mathfrak{Z}}(P_{\mathfrak{Z}}) := h(P)_{\mathfrak{Z}}$ The operator U is equivalent to its restriction $U_{\mathfrak{Z}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{Z}})$ while the restriction $V_{\mathfrak{Z}} := V|_{\mathsf{H}_{\mathfrak{Z}}}$ defines a $\mathbb{C}_{i_1}$ linear operator $V_{\mathfrak{Z}} : \mathsf{H}_{\mathfrak{Z}} \to \mathsf{H}_{\mathfrak{Z}}i_3$. ($\mathsf{H}_{\mathfrak{Z}}i_3$ is a $\mathbb{C}_{i_1}$-linear space, check Lemma 3.2.15). Indeed notice that, V being linear, for any $x \in \mathsf{H}_{\mathfrak{Z}}$

$$J((V_{\mathfrak{Z}}x)i_3) = (J(V_{\mathfrak{Z}}x))i_3 = -(V(Jx))i_3 = -(V(xi_1))i_3 = -(Vx)i_1i_3 = ((V_{\mathfrak{Z}}x)i_3)i_1$$

and so $(V_{\mathfrak{Z}}x)i_3 \in \mathsf{H}_{\mathfrak{Z}}$ which is equivalent to $V_{\mathfrak{Z}}x \in \mathsf{H}_{\mathfrak{Z}}i_3$. Exploiting the unitariness of V and (3.31) it is easy to see that such a function is a norm-preserving bijective $\mathbb{C}_{i_1}$-linear operator. At this point we can define the function

$$W_{\mathfrak{Z}} : \mathsf{H}_{\mathfrak{Z}} \ni x \mapsto (V_{\mathfrak{Z}}x)i_3 \in \mathsf{H}_{\mathfrak{Z}},$$

which turns out to be a $\mathbb{C}_{i_1}$-*anti*-linear isomorphism which preserves the norm (i.e. an anti-unitary operator) with $W_{\mathfrak{Z}}^{-1}(y) = -V_{\mathfrak{Z}}^{-1}(yi_3)$ for $y \in \mathsf{H}_{\mathfrak{Z}}$ and

$$\begin{aligned}W_{\mathfrak{Z}}P_{\mathfrak{Z}}W_{\mathfrak{Z}}^{-1}y &= -W_{\mathfrak{Z}}P_{\mathfrak{Z}}V_{\mathfrak{Z}}^{-1}(yi_3) = -(V_{\mathfrak{Z}}P_{\mathfrak{Z}}V_{\mathfrak{Z}}^{-1}(yi_3))i_3 = -(VPV^{-1}(yi_3))i_3 = \\ &= (-h(P)(yi_3))i_3 = h(P)y = h_{\mathfrak{Z}}(P_{\mathfrak{Z}})y.\end{aligned}$$

Thus, we end up with a unitary operator $U_{\mathfrak{Z}}$ and anti-unitary operator $W_{\mathfrak{Z}}$ on the complex Hilbert space $\mathsf{H}_{\mathfrak{Z}}$, both generating the automorphism $h_{\mathfrak{Z}}$. This is not possible, since, as stated by Theorem 5.4.9, two operators generating the same automorphism must be both linear o both anti-linear. So, either both U, V commutes with J or both anti-commute with it. The fact that they must differ by a unitary element in $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$ is left as an exercise. □

***Remark* 6.8.3.** Notice that, regardless of whether $U \in \mathfrak{M}$ or $U \notin \mathfrak{M}$, it holds that $U^2 \in \mathfrak{M}$ and this generates the automorphism h^2. This feature will become important when dealing with projective unitary representations of *connected* Lie groups $G \ni g \mapsto U_g$ in Chapter 9, where every U_g turns out be *necessarily* unitary, i.e. an element of $\mathfrak{M}$.

6.9 Excursus on Quaternionic von Neumann Algebras

In the previous sections, we defined a von Neumann algebra over a *quaternionic* Hilbert space H as a *real* *-subalgebra of $\mathfrak{B}(\mathsf{H})$. This was the best we could do without forcing the given setting by introducing additional axioms into the theory. Nevertheless there is an annoying dissimilarity in the definition of von Neumann algebra between the quaternionic case on one side and real and complex ones on the other side.1 Indeed, while in the latter situation the *-subalgebra is to be taken with respect to the same division algebra of the corresponding Hilbert space, in the former this is not the case because, as already said above, quaternionic linear combinations of quaternionic-linear operators are ill-defined and it is mandatory to restrict to the reals.

Things change if on the quaternionic Hilbert space we consider also the action of a left-multiplication

$$L : \mathbb{H} \ni q \mapsto L_q \in \mathfrak{B}(\mathsf{H}) \tag{6.20}$$

as in Definition 2.2.42.

In Section 2.2.3 we saw how this makes $\mathfrak{B}(\mathsf{H})$ a (two-sided) unital by defining C^*-algebra over $\mathbb{H}$ by

$$qA := L_q A \quad \text{and} \quad Aq := AL_q \quad \text{for all } q \in \mathbb{H}. \tag{6.21}$$

Thus we may consider unital *-subalgebras of $\mathfrak{B}(\mathsf{H})$ referring to this structure instead of limiting ourselves to the $q \in \mathbb{R}$ case.

Just to fix the notation and not make confusion with the concept of subalgebra encountered so far we give the following definition where the terms quaternionic is explicitly written. We keep the name unital *-subalgebra when referring to the $\mathbb{R}$-algebra structure of $\mathfrak{B}(\mathsf{H})$.

***Definition* 6.9.1.** *Let H be a quaternionic Hilbert space equipped with a left multiplication L. A unital quaternionic *-subalgebra of $\mathfrak{B}(\mathsf{H})$ is a subset $\mathfrak{U} \subset \mathfrak{B}(\mathsf{H})$ which contains the identity and is closed under addition, left and right multiplication by quaternions as in (6.21), Hermitean conjugation and composition.*

***Remark* 6.9.2.** *Some remarks on this definition.*

(a) In other words $\mathfrak{U}$ is a unital C^*-subalgebra of $\mathfrak{B}(\mathsf{H})$, the latter interpreted as a unital C^*-algebra over $\mathbb{H}$ as in Proposition 2.2.64.

(b) One may be more cautious and define a unital **-algebra just as a unital real* **-algebra which is closed under either the left or the right multiplication by quaternions. Anyway this would not make any difference since we require* **-closedness: indeed* $(qA)^* = A^*\overline{q}$ *and* $(Aq)^* = \overline{q}A^*$.

First notice the following useful trivial characterisation.

***Proposition* 6.9.3.** *A unital* **-subalgebra* $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ *is a unital quaternionic* **-subalgebra if and only if* $L_q \in \mathfrak{M}$ *for all* $q \in \mathbb{H}$.

Proof. The proof follows immediately by noticing that $\mathfrak{M}$ is closed with respect to the composition and $I \in \mathfrak{M}$. □

Now, in a complete naive try, we can try to develop the standard theory of von Neumann algebras. As are going to see, several issues arise and make the entire attempt worthless.

As usual, if $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$ is any set of operators, then $\mathfrak{S}'$ denotes the set of all the operators on H which commute with all the elements of $\mathfrak{S}$.

A first difference arises immediately. As seen at the beginning of this chapter, if the set $\mathfrak{S}$ is closed under the Hermitean conjugation, then $\mathfrak{S}'$ turns out to be a *-subalgebra with unit of $\mathfrak{B}(\mathsf{H})$. Unfortunately this is no longer the case in the actual situation. Indeed, take for example $\mathfrak{S} = \mathfrak{B}(\mathsf{H})$, then $\mathfrak{S}' = \{aI \,|\, a \in \mathbb{R}\}$ which is clearly not closed under the left and right multiplication by quaternions. More precisely the following result holds.

***Proposition* 6.9.4.** *Let* $\mathfrak{S} \subset \mathfrak{B}(\mathsf{H})$ *be closed under the Hermitean conjugation, then* $\mathfrak{S}'$ *is a unital quaternionic* **-subalgebra of* $\mathfrak{B}(\mathsf{H})$ *if and only if* $L_p \in \mathfrak{S}'$ *for every* $p \in \mathbb{H}$.

Proof. We already know that $\mathfrak{S}'$ is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$. The closedness under quaternionic linear combinations follows immediately from Proposition 6.9.3. □

Nevertheless, despite every difficulty, we can go ahead in the usual way.

First, notice that every unital quaternionic *-subalgebra $\mathfrak{U}$ is, first of all, a unital *-subalgebra. In particular Theorem 6.1.8 immediately applies and can be restated in the following form.

***Theorem* 6.9.5.** *Let* $\mathfrak{M}$ *be a unital quaternionic* **-subalgebra of* $\mathfrak{B}(\mathsf{H})$*, then it holds that* $\mathfrak{M}'' = \overline{\mathfrak{M}}^s = \overline{\mathfrak{M}}^w$. *Moreover the following statements are equivalent:*

(a) $\mathfrak{M} = \mathfrak{M}''$,

(b) $\mathfrak{M}$ *is weakly closed,*

(c) $\mathfrak{M}$ *is strongly closed.*

Again, we can give the following definition.

***Definition* 6.9.6.** *A quaternionic von Neumann algebra is a unital quaternionic* **-subalgebra* $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ *such that* $\mathfrak{M}'' = \mathfrak{M}$.

***Remark* 6.9.7.** Notice that, since every unital quaternionic *-algebra is, first of all, a unital real *-algebra, a quaternionic von Neumann algebra is also a von Neumann algebra in the meaning of Definition 6.1.3.

***Example* 6.9.8.** The full unital quaternionic *-algebra $\mathfrak{B}(\mathsf{H})$ is a quaternionic von Neumann algebra. Indeed, as already seen above, it holds that $\mathfrak{B}(\mathsf{H})' = \{aI \mid a \in \mathbb{R}\}$ and so $\mathfrak{B}(\mathsf{H})'' = \mathfrak{B}(\mathsf{H})$. Notice that $\mathfrak{B}(\mathsf{H})'$ is not a quaternionic von Neumann algebra, for it is not closed under the multiplication by quaternions.

Of course we have the following result.

***Proposition* 6.9.9.** *A von Neumann algebra $\mathfrak{M}$ is a quaternionic von Neumann algebra if and only if $L_q \in \mathfrak{M}$ for every $q \in \mathbb{H}$.*

Concluding we can see the following no-go result

***Proposition* 6.9.10.** *Let $\mathfrak{M}$ be a quaternionic von Neumann algebra, then $\mathfrak{M}'$ cannot be a unital quaternionic *-algebra, in particular it can never be a quaternionic von Neumann algebra.*

Proof. Thanks to Proposition 6.9.4 we know that in order for both $\mathfrak{M}$ and $\mathfrak{M}'$ to be unital quaternionic *-subalgebras it is necessary that L_q belongs to both of them for any $q \in \mathbb{H}$ which is impossible due to the non commutativity of the quaternions. Indeed, take $p, q \in \mathbb{H}$ such that $pq \neq qp$, then $L_q \in \mathfrak{M}$ and $L_p \in \mathfrak{M}'$ and so we would have $L_{qp} = L_q L_p = L_p L_q = L_{pq}$ which is impossible, the map L being injective. □

***Corollary* 6.9.11.** *If $\mathfrak{M}$ is a quaternionic von Neumann algebra, then there exists $A \in \mathfrak{M}$ and $q \in \mathbb{H}$ such that $L_q A \neq A L_q$.*

Proof. The proof is straightforward. Suppose that $L_q A = A L_q$ for every $q \in \mathbb{H}$ and for all $A \in \mathfrak{M}$, then we would have $L_q \in \mathfrak{M}'$ for all $q \in \mathbb{H}$. Since $\mathfrak{M}'$ is a von Neumann algebra, Proposition 6.9.9 makes it a quaternionic von Neumann algebra. Proposition 6.9.10 concludes the proof. □

The few results discussed so far, in particular Proposition 6.9.10, shows how tricky and weak a theory of quaternionic von Neumann algebras would be in the quaternionic setting. For example the ability of getting von Neumann algebras out of the commutant of given sets is one of the simplest and at same time most powerful properties of these algebraic objects, but not even this is guaranteed to hold in this case.

Nevertheless, even though it will not carried out here, a deeper analysis of these objects may be worth a try.

Complex von Neumann Algebras as Real Ones

We can try to get rid of this lack of consistency in a completely different way. As discussed in the previous chapter the difference between the real and complex cases on one side and the quaternionic one on the other side lies in the fact that, while in the former situation unital *-subalgebras can be considered with respect to the same division algebra of the corresponding Hilbert space, this cannot be done in the latter. We tried to solve this "asymmetry" by equipping H with a left-multiplication, this way being able to perform quaternionic linear combinations of operators. Unfortunately this approach did not take us very fay.

Thus, we could try to change point of view and move the asymmetry to another place. Indeed notice that, even if not coinciding with the corresponding division algebra, $\mathbb{R}$ is the only set of scalars that can be used from both the real and quaternionic cases to perform linear combinations of operators. The anomaly can then be ascribed to the complex case, where the scalars are taken out of $\mathbb{C}$. So, we can try to consider only *real* linear combinations of operators on complex Hilbert spaces, i.e. *real* unital *-subalgebras of $\mathfrak{B}(\mathsf{H})$.

We can then define a complex von Neumann algebra as a *real* unital *-subalgebra $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ such that $\mathfrak{M} = \mathfrak{M}''$. Given this definition, it is clear that $\mathfrak{M}$ turns out to be closed under complex combinations, i.e. it is also a *complex* unital *-subalgebra, hence a von Neumann algebra in the usual sense. Of course the contrary is always true: a von Neumann algebra in the usual sense is also a *real* unital *-subalgebra satisfying $\mathfrak{M}'' = \mathfrak{M}$.

Unfortunately not all that glitters is gold. Indeed the double commutant theorem fails to hold under this weaker hypothesis. For example let H be a complex Hilbert space and $\mathfrak{M} := \{aI \,|\, a \in \mathbb{R}\}$. This is clearly a real unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$ and it is closed in the strong topology as a direct check shows. Nevertheless $\mathfrak{M}' = \mathfrak{B}(\mathsf{H})$ and so $\mathfrak{M}'' = \mathfrak{B}(\mathsf{H})' = \{aI + biI \,|\, a, b \in \mathbb{R}\}$. Hence, while of course $\mathfrak{M} = \mathfrak{M}''$ implies $\mathfrak{M} = \overline{\mathfrak{M}}^s$, the opposite is no longer true for *real* unital *-subalgebras on *complex* Hilbert spaces.

Chapter 7

Lie Group Representations in Hilbert Spaces

In this chapter we give a basic introduction to the Gårding and Nelson theories on unitary representations of Lie groups, which are of absolute importance for the development of Chapter 9. This chapter is based upon Chapter 10 of [35] and Section 12.2.11 of [24] where most of the definitions and results can be found for the special case of complex Hilbert spaces. The real and quaternionic Hilbert space cases are carried out here exploiting the corresponding complex ones and the techniques discussed in Chapter 3.

7.1 Induced Lie Algebra Representations on Gårding Domain

Consider a finite dimensional real Lie group G and the real linear space

$$C_0^\infty(G) := \{f \in C^\infty(G,\mathbb{R}) \mid \mathit{supp} f \text{ is compact}\}. \tag{7.1}$$

***Definition* 7.1.1.** *Let G be a Lie group and H a real, complex or quaternionic Hilbert space. A unitary representation of G on H is a function*

$$U : G \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$$

such that every U_g is unitary and $U_{gh} = U_g U_h$ together with $U_e = I$ holds for every $g, h \in G$. The representation is said to be strongly-continuous if it is continuous with respect to the strong topology of $\mathfrak{B}(\mathsf{H})$.

***Remark* 7.1.2.** Some remarks on this definition.

(a) The strong-continuity can be restated as $\lim_{g\to e} U_g x = x$ for every $x \in \mathsf{H}$. The continuity at every other point $g_0 \in G$ follows easily from the group properties of G.

(b) From the unitariness of the operators U_g and the group properties of U it is easy to derive $(U_g)^* = U_{g^{-1}}$ for all $g \in G$.

(c) If H is a real or quaternionic Hilbert space and $U : g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ a strongly-continuous unitary representation, then the map

$$U_{\mathbb{K}} : G \ni g \mapsto (U_g)_{\mathbb{K}} \in \mathfrak{B}(\mathsf{H}_{\mathbb{K}}) \tag{7.2}$$

is a strongly-continuous unitary representation.

(d) If H is a real or quaternionic Hilbert space equipped with a $\mathbb{K}$-structure $\mathfrak{J}$ and $U : G \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ is a strongly-continuous unitary representation, then the map

$$U_{\mathfrak{J}} : G \ni g \mapsto (U_g)_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}) \tag{7.3}$$

is a strongly-continuous unitary representation of G on $\mathsf{H}_{\mathfrak{J}}$, provided that $U_g \in \mathfrak{J}'$ for every $g \in G$.

(e) The one-parameter groups of unitary operators $\mathbb{R} \ni t \mapsto U_t \in \mathfrak{B}(\mathsf{H})$ provides the simplest examples of strongly-continuous unitary representations.

Now, fix a generic continuous function $u : G \ni g \mapsto u_g \in \mathsf{H}$ such that $\|u_g\| \leq C$ uniformly for some $C \geq 0$ and consider any $f \in C_0^\infty(G)$. Since the function f is compactly supported and the function $g \mapsto (y|u_g)$ is continuous for any $y \in \mathsf{H}$ then the integral

$$\int_G f(g)(y|u_g)\, dg$$

(where dg is the left-invariant Haar measure on G) is a well-defined element of $\mathbb{F}$. Thus we can consider the functional

$$\phi(u,f) : \mathsf{H} \ni y \mapsto \int_G f(g)(y|u_g)\, dg \in \mathbb{K}.$$

This is clearly linear and also continuous, indeed

$$|\phi(u,f)y| = \left|\int_G f(g)(y|u_g)\, dg\right| \leq \int_G |f(g)||(y|u_g)|\, dg \leq \left(\int_G |f(g)|\, dg\right) C\|y\|. \tag{7.4}$$

Exploiting Riesz's Representation Theorem we are given a unique vector u_f such that $(y|u_f) = \phi(u,f)y$ for any $y \in \mathsf{H}$. The definition of this vector suggests the following notation

$$u_f =: \int_G f(g) u_g\, dg \in \mathsf{H}. \tag{7.5}$$

By direct inspection, exploiting Corollary 2.1.12, the following inequality can be easily proved

$$\left\|\int_G f(g) u_g\, dg\right\| \leq \int_G |f(g)|\|u_g\|\, dg. \tag{7.6}$$

The situation of our main interest is when $u_g = U_g x$ for some strongly-continuous unitary representation $G \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ and $x \in \mathsf{H}$. More precisely we define

$$x[f] := \int_G f(g) U_g x \, dg \in \mathsf{H} \quad \text{for} \quad f \in C_0^\infty(G) \quad \text{and} \quad x \in \mathsf{H}. \tag{7.7}$$

These vectors are referred to as the *Gårding vectors* of the representation U.

For $h \in G$ define the *displacement operator* $L_h : C_0^\infty(G) \to C_0^\infty(G)$ as the function

$$L_h(f)(g) := f(h^{-1}g) \quad \text{for all} \quad g \in G.$$

Some properties of the Gårding vectors follow.

***Proposition* 7.1.3.** *The following statements hold:*

(a) if $q \in \mathbb{F}$ then $(x[f])q = (xq)[f]$ for any $x \in \mathsf{H}$ and $f \in C_0^\infty(G)$;

(b) if $f, g \in C_0^\infty(G)$, then $x[f+g] = x[f] + x[g]$ for all $x \in \mathsf{H}$;

(c) if $B \in \mathfrak{B}(\mathsf{H})$ then $B(x[f]) = \int_G f(g) B U_g x \, dg$ for any $x \in \mathsf{H}$ and $f \in C_0^\infty(G)$;

(d) if $B \in \mathfrak{B}(\mathsf{H})$ and $BU_g = U_g B$, then $B(x[f]) = (Bx)[f]$ for any $x \in \mathsf{H}$ and $f \in C_0^\infty(G)$;

(e) $U_h(x[f]) = x[L_h(f)]$ for every $h \in G$, $x \in \mathsf{H}$ and $f \in C_0^\infty(G)$.

Proof. Let us prove point (a). Consider any $y \in \mathsf{H}$, then

$$(y|x[f]q) = (y|x[f])q = \left(\int_G f(g)(y|U_g x) \, dg \right) q = \int_G f(g)(y|U_g x) q \, dg =$$
$$= \int_G f(g)(y|U_g(xq)) \, dg = (y|(xq)[f]),$$

which immediately gives point (a). The proof of the other points is similar, in particular point (e) follows from the left-translation-invariance of the Haar measure dg. □

***Definition* 7.1.4.** *Let U be a strongly-continuous unitary representation of a Lie group G on a real, complex or quaternionic Hilbert space H. The linear subspace of H spanned by all of the Gårding vectors $x[f]$ (7.5) is called the Gårding Domain associated with U and denoted by $D_G^{(U)}$.*

***Remark* 7.1.5.** Actually $D_G^{(U)}$ coincides with the *real* span of the vectors $x[f]$ as immediately follows from Proposition (7.1.3), point (a). Thanks to the same result we are actually even allowed to consider only finite *sums* of elements $x[f]$, with no multiplications by scalars involved.

***Theorem* 7.1.6.** *Referring to Definition 7.1.4 the following statements hold:*

(a) $D_G^{(U)}$ *is dense in* H*;*

(b) $U_g(D_G^{(U)}) \subset D_G^{(U)}$ *for all* $g \in G$*.*

Proof. Let us start with (a). Consider a sequence of positive functions $f_n \in C_0^\infty(G)$ such that $\int_G f_n(g)\,dg = 1$ and $supp f_n \to \{e\}$. Take $x \in \mathsf{H}$, then it holds that

$$\|x[f_n] - x\| = \left\| \int_G f_n(g)(U_g x - x)\,dg \right\| \overset{(*)}{\leq} \int_G f_n(g)\|U_g x - x\|\,dg \leq$$
$$\leq \left(\sup_{g \in supp f_n} \|U_g x - x\| \right) \int_{supp f_n} f_n(g)\,dg = \sup_{g \in supp f_n} \|U_g x - x\|,$$

where the inequality $(*)$ follows from (7.6). Since the support of f_n shrinks to the singleton $\{e\}$, the strong-continuity of U assures that $\|x[f_n] - x\| \to 0$. Point (b) follows immediately from Proposition 7.1.3, point (e). $\square$

Now consider the Lie algebra $\mathfrak{g}$ associated with the Lie group G. Take any $\mathsf{A} \in \mathfrak{g}$ and consider the one-parameter subgroup $\mathbb{R} \ni t \mapsto \exp(t\mathsf{A}) \in G$. As U is strongly-continuous, Stone's Theorem assures the existence of a unique anti-self-adjoint operator A on H such that

$$U_{\exp(t\mathsf{A})} = e^{tA} \quad \text{for all} \quad t \in \mathbb{R}.$$

On the other hand, define the *vector field* $X_\mathsf{A} : C_0^\infty(G) \to C_0^\infty(G)$ by

$$X_\mathsf{A}(f)(g) := \lim_{t \to 0} \frac{L_{\exp(t\mathsf{A})}(f)(g) - f(g)}{t} \quad \text{for all} \quad g \in G. \tag{7.8}$$

A direct calculation shows that the map

$$\mathfrak{g} \ni \mathsf{A} \mapsto X_\mathsf{A} \in \mathfrak{X}(G)$$

is a faithful Lie-algebra representation of $\mathfrak{g}$ in terms of vector fields on $C_0^\infty(G)$.

Consider the following technical lemma - the proof can be found in Section A.4 of [41] and works on real, complex and quaternionic Hilbert spaces.

***Lemma* 7.1.7.** *Let A be an anti-self-adjoint operator and $\mathcal{D} \subset D(A)$ be a dense subspace of H such that $e^{tA}(\mathcal{D}) \subset \mathcal{D}$ for all $t \in \mathbb{R}$, then $\mathcal{D}$ is a core for A.*

***Theorem* 7.1.8.** *The following statements hold:*

(a) if $\mathsf{A} \in \mathfrak{g}$*, then* $D_G^{(U)} \subset D(A)$ *and furthermore* $A(D_G^{(U)}) \subset D_G^{(U)}$*, more precisely*

$$A(x[f]) = x[X_\mathsf{A}(f)] \quad \textit{for all} \quad f \in C_0^\infty(G);$$

(b) the linear map defined by

$$u : \mathfrak{g} \ni \mathsf{A} \mapsto A|_{D_G^{(U)}}$$

is a Lie algebra representation in terms of anti-symmetric operators on H *defined on the common dense invariant domain* $D_G^{(U)}$ *so that, in particular,*

$$[u(\mathsf{A}), u(\mathsf{B})] = u([\mathsf{A}, \mathsf{B}]_{\mathfrak{g}}) \quad \textit{for all} \quad \mathsf{A}, \mathsf{B} \in \mathfrak{g};$$

(c) $D_G^{(U)}$ *is a core for every anti-self-adjoint generator* A *with* $\mathsf{A} \in \mathfrak{g}$*, that is*

$$A = \overline{u(\mathsf{A})} \quad \textit{for all} \quad \mathsf{A} \in \mathfrak{g};$$

(d) take $B \in \mathfrak{B}(\mathsf{H})$ *and consider the following statements:*

(i) $BU_g = U_g B$ *for all* $g \in G$*,*

(ii) $Bu(\mathsf{A}) \subset u(\mathsf{A})B$ *for all* $\mathsf{A} \in \mathfrak{g}$*,*

(iii) $B\overline{u(\mathsf{A})} \subset \overline{u(\mathsf{A})}B$ *for all* $\mathsf{A} \in \mathfrak{g}$*,*

then (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii). If, furthermore,* G *is also connected then (iii)* $\Rightarrow$ *(i). If (i) is satisfied, then* $B(D_G^{(U)}) \subset D_G^{(U)}$*.*

Proof. Let us start with point (a) and consider any Gårding vector $x[f]$. Exploiting Proposition 7.1.3 we have $e^{tA}x[f] = U_{\exp(t\mathsf{A})}x[f] = x[L_{\exp(t\mathsf{A})}f]$, hence

$$x[X_{\mathsf{A}}(f)] = \lim_{t\to 0} x\left[\frac{L_{\exp(t\mathsf{A})f} - f}{t}\right] = \lim_{t\to 0} \frac{x[L_{\exp(t\mathsf{A})}f] - x[f]}{t} = \lim_{t\to 0} \frac{e^{tA}x[f] - x[f]}{t}$$

from which $x[f] \in D(A)$ follows. In particular $Ax[f] = x[X_{\mathsf{A}}(f)] \in D_G^{(U)}$. Since $D_G^{(U)}$ is generated by this kind of vectors, point (a) is proved. Point (b) immediately follows from the analogous properties of the vector field $\mathsf{A} \mapsto X_{\mathsf{A}}$, the definition of u and point (a). Point (c) is an immediate consequence of Lemma 7.1.7 once we notice that $D_G^{(U)} \subset D(A)$ (point (a)) and $e^{tA}(D_G^{(U)}) = U_{\exp(t\mathsf{A})}(D_G^{(U)}) \subset D_G^{(U)}$ (Theorem 7.1.6). To conclude let us prove (d). If (i) holds, then $B(D_G^{(U)}) \subset D_G^{(U)}$ follows from Proposition 7.1.3. Furthermore, we have in particular that $Be^{t\overline{u(\mathsf{A})}} = BU_{\exp(t\mathsf{A})} = U_{\exp(t\mathsf{A})}B = e^{t\overline{u(\mathsf{A})}}B$ for every $\mathsf{A} \in \mathfrak{g}$ and $t \in \mathbb{R}$. Exploiting Proposition 4.2.3 we get $B\overline{u(\mathsf{A})} \subset \overline{u(\mathsf{A})}B$. Since $B(D_G^{(U)}) \subset D_G^{(U)}$ and $D(u(\mathsf{A})) = D_G^{(U)}$ we have point (ii). Now suppose that (ii) holds true, then since B is bounded and $u(\mathsf{A})$ is closable, Proposition 2.2.58 gives (iii). At this point, suppose that G is connected and let (iii) be satisfied, that is $B\overline{u(\mathsf{A})} \subset \overline{u(\mathsf{A})}B$. Lemma 4.2.3 gives $Be^{t\overline{u(\mathsf{A})}} = e^{t\overline{u(\mathsf{A})}}B$ for all $t \in \mathbb{R}$. Since the group is connected, every element $g \in G$ can be written as the product of a finite number of one-parameter subgroup elements of G and so the thesis of (i) holds true. $\square$

Now, focus on the real or quaternionic Hilbert space case. As already observed in Remark 7.1.2 the map $G \ni g \mapsto (U_g)_{\mathbb{K}} \in \mathfrak{B}(\mathsf{H}_{\mathbb{K}})$ is a strongly-continuous unitary representation of G on $\mathsf{H}_{\mathbb{K}}$. Denote by $u_{\mathbb{K}}$ the Lie algebra representation associated with $U_{\mathbb{K}}$ as in Theorem 7.1.8. Similarly, if H is equipped with a $\mathbb{K}$-structure $\mathfrak{J}$ and $U_g \in \mathfrak{J}'$

for all $g \in G$, then $g \mapsto (U_g)_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ is a strongly-continuous unitary representation over $\mathsf{H}_{\mathfrak{J}}$. Again, denote by $u_{\mathfrak{J}}$ the Lie algebra representation associated with $U_{\mathfrak{J}}$ as in Theorem 7.1.8.

***Proposition* 7.1.9.** *Let H be real or quaternionic and U a strongly-continuous unitary representation of a Lie group G, then the following statements hold:*

(a) $D_G^{(U_{\mathbb{K}})} = (D_G^{(U)})_{\mathbb{K}}$,

(b) $u_{\mathbb{K}}(\mathsf{A}) = (u(\mathsf{A}))_{\mathbb{K}}$ *for all* $\mathsf{A} \in \mathfrak{g}$.

Proof. Let us start with point (a). Suppose first that H is quaternionic and take any $x \in \mathsf{H} = \mathsf{H}_{\mathbb{K}}$ and $f \in C_0^\infty(G)$. Let $x(f)$ denote the Gårding vector defined with respect to $U_{\mathbb{K}}$, while keep the notation $x[f]$ for the one referred to H. If we manage to prove that $x(f) = x[f]$, Remark 7.1.5 concludes the proof. The vector $x[f]$ is defined as the only vector of H such that $(z|x[f]) = \int_G f(g)(z|U_g x)\,dg$ for all $z \in \mathsf{H}$. Exploiting (3.17) we have

$$
\begin{aligned}
(z|x[f])_{\mathbb{K}} &= \sum_{\alpha=0}^{d_{\mathbb{K}}} \Re[(z|x[f])i_\alpha]\overline{i_\alpha} = \sum_{\alpha=0}^{d_{\mathbb{K}}} \Re\left[\left(\int_G f(g)(z|U_g x)\,dg\right) i_\alpha\right]\overline{i_\alpha} = \\
&= \sum_{\alpha=0}^{d_{\mathbb{K}}} \Re\left[\left(\int_G f(g)(z|U_g x) i_\alpha\,dg\right)\right]\overline{i_\alpha} = \sum_{\alpha=0}^{d_{\mathbb{K}}} \int_G f(g)\Re[(z|U_g x)i_\alpha]\overline{i_\alpha}\,dg = \\
&= \int_G f(g)\sum_{\alpha=0}^{d_{\mathbb{K}}} \Re[(z|U_g x)i_\alpha]\overline{i_\alpha}\,dg = \int_G f(g)(z|U_g x)_{\mathbb{K}}\,dg = (z|x(f))_{\mathbb{K}},
\end{aligned}
$$

where in the last equality we exploited the definition of $x(f)$. Since $z \in \mathsf{H}_{\mathbb{K}}$ is arbitrary it must be $x[f] = x(f)$. So, suppose now that H is real and let $x(f) \in D_G^{(U_{\mathbb{K}})}$ with $x \in \mathsf{H}_{\mathbb{K}}$ denote a Gårding vector of $U_{\mathbb{K}}$ and $u[f] \in D_G^{(U)}$ with $u \in \mathsf{H}$ a Gårding vector of U. So, take $x \in \mathsf{H}_{\mathbb{K}}$ and $f \in C_0^\infty(G)$, then thanks to (3.1) we can always develop x as $x = \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha \otimes i_\alpha$ for some $x_\alpha \in \mathsf{H}$ and $x(f)$ as $x(f) = \sum_{\alpha=0}^{d_{\mathbb{K}}} \omega_\alpha \otimes i_\alpha$ for some $\omega_\alpha \in \mathsf{H}$. Take $z \in \mathsf{H}$, then

$$
\begin{aligned}
\sum_{\alpha=0}^{d_{\mathbb{K}}} (z|\omega_\alpha) i_\alpha &= (z \otimes 1|x(f))_{\mathbb{K}} = \int_G f(g)(z\otimes 1|(U_g)_{\mathbb{K}} x)_{\mathbb{K}}\,dg = \\
&= \sum_{\alpha=0}^{d_{\mathbb{K}}} \int_G f(g)(z\otimes 1|(U_g)_{\mathbb{K}} x_\alpha \otimes i_\alpha)\,dg = \sum_{\alpha=0}^{d_{\mathbb{K}}} \int_G f(g)(z|U_g x_\alpha) i_\alpha\,dg = \\
&= \sum_{\alpha=0}^{d_{\mathbb{K}}} \left(\int_G f(g)(z|U_g x_\alpha)\,dg\right) i_\alpha.
\end{aligned}
\tag{7.9}
$$

Since the terms in brackets is real, it easily follows that

$$(z|\omega_\alpha) = \int_G f(g)(z|U_g x_\alpha)\, dg = (z|x_\alpha[f])$$

for any $0 \leq \alpha \leq d_{\mathbb{K}}$, where the last equality follows from the definition of Gårding vector on H. This implies that

$$x(f) = \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha[f] \otimes i_\alpha \in (D_G^{(U)})_{\mathbb{K}}.$$

Let us prove the opposite inclusion. Take any $x \in (D_G^{(U)})_{\mathbb{K}}$ and suppose first that $x = u[f] \otimes i_\beta$ for some $u \in \mathsf{H}, f \in C_0^\infty(G)$ and $0 \leq \beta \leq d_{\mathbb{K}}$. Suppose $z = \sum_{\alpha=0}^{d_{\mathbb{K}}} z_\alpha \otimes i_\alpha$, then

$$\begin{aligned}(z|u[f] \otimes i_\beta)_{\mathbb{K}} &= \sum_{\alpha=0}^{d_{\mathbb{K}}} (z_\alpha|u[f])\overline{i_\alpha} i_\beta = \sum_{\alpha=0}^{d_{\mathbb{K}}} \left(\int_G f(g)(z_\alpha|U_g u)\, dg \right) \overline{i_\alpha} i_\beta = \\ &= \int_G f(g) \sum_{\alpha=0}^{d_{\mathbb{K}}} (z_\alpha|U_g u)\overline{i_\alpha} i_\beta\, dg = \int_G f(g)(z|(U_g)_{\mathbb{K}}(u \otimes i_\beta))_{\mathbb{K}}\, dg = \\ &= (z|u \otimes i_\beta(f))_{\mathbb{K}},\end{aligned}$$

where $y(f)$ denotes the Gårding vectors on $\mathsf{H}_{\mathbb{K}}$. The arbitrariness of z gives

$$x = u[f] \otimes i_\beta = (u \otimes i_\beta)(f) \in D_G^{(U_{\mathbb{K}})}.$$

In the general case it holds that $x = \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha \otimes i_\alpha$ with $x_\alpha \in D_G^{(U)}$, i.e. $x_\alpha = \sum_{h_\alpha=1}^{N_\alpha} u_{h_\alpha}[f_{h_\alpha}]$ for any $0 \leq \alpha \leq d_{\mathbb{K}}$, hence

$$x = \sum_{\alpha=0}^{d_{\mathbb{K}}} \sum_{h_\alpha=1}^{N_\alpha} u_{h_\alpha}[f_{h_\alpha}] \otimes i_\alpha.$$

By the special case just proved, each vector $u_{h_\alpha}[f_{h_\alpha}] \otimes i_\alpha$ belongs to $D_G^{(U_{\mathbb{K}})}$ and so does the entire vector x. To conclude let us prove point (b). Point (a) assures that the domain of $u_{\mathbb{K}}(\mathsf{A})$ and $u(\mathsf{A})_{\mathbb{K}}$ is the same. If H is quaternionic then the thesis follows immediately by means of the definition of the representation u and (3.21). Suppose that H is real then and suppose $x = \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha \otimes i_\alpha \in (D_G^{(U)})_{\mathbb{K}} = D_G^{(U_{\mathbb{K}})}$ for some $x_\alpha \in D_G^{(U)}$.

It holds that

$$\begin{aligned}(u(\mathsf{A}))_{\mathbb{K}}x &= \sum_{\alpha=0}^{d_{\mathbb{K}}} u(\mathsf{A})x_\alpha \otimes i_\alpha = \frac{d}{dt}\Big|_0 \sum_{\alpha=0}^{d_{\mathbb{K}}} U_{\exp(t\mathsf{A})}x_\alpha \otimes i_\alpha = \\ &= \frac{d}{dt}\Big|_0 (U_{\exp(t\mathsf{A})})_{\mathbb{K}} \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha \otimes i_\alpha = \frac{d}{dt}|_0 (U_{exp(t\mathsf{A})})_{\mathbb{K}}x = u_{\mathbb{K}}(\mathsf{A})x,\end{aligned} \tag{7.10}$$

where we used (3.4). □

The following result follows immediately from Proposition 7.1.9.

***Corollary* 7.1.10.** *Let H be a real or quaternionic Hilbert space and A and anti-self-adjoint operator, then $(e^{tA})_{\mathbb{K}} = e^{tA_{\mathbb{K}}}$ for all $t \in \mathbb{R}$.*

Proof. Let $U : t \mapsto e^{tA}$ be the strongly-continuous unitary representation of $\mathbb{R}$ on H induced by A. As stated in Remark 4.1.2 the map $U_{\mathbb{K}} : t \mapsto (e^{tA})_{\mathbb{K}}$ is a strongly-continuous unitary representation of the group $\mathbb{R}$ on $\mathsf{H}_{\mathbb{K}}$. Let B be its anti-self-adjoint generator. Exploiting Proposition 7.1.9 we see that $D_G^{(U_{\mathbb{K}})} = (D_G^{(U)})_{\mathbb{K}}$ and that the restriction of B to this domain equals the restriction of $A_{\mathbb{K}}$ on the same domain. Since $D_G^{(U)}$ is a core for A and $(D_G^{(U)})_{\mathbb{K}}$ is a core for B, Propositions 3.1.4, 3.2.5 immediately give $B = A_{\mathbb{K}}$, concluding the proof. □

Similarly we have the following result.

***Proposition* 7.1.11.** *Let H be real or quaternionic, U a strongly-continuous unitary representation of a Lie group G and $\mathfrak{J}$ a $\mathbb{K}$-structure over H such that $U_g \in \mathfrak{J}'$ for all $g \in G$, then $J(D_G^{(U)}) \subset D_G^{(U)}$ and $Ju(\mathsf{A}) \subset u(\mathsf{A})J$ for any $\mathsf{A} \in \mathfrak{g}$ and $J \in \mathfrak{J}$. Moreover*

(a) $D_G^{(U_{\mathfrak{J}})} = (D_G^{(U)})_{\mathfrak{J}}$,

(b) $u_{\mathfrak{J}}(\mathsf{A}) = (u(\mathsf{A}))_{\mathfrak{J}}$ *for all* $\mathsf{A} \in \mathfrak{g}$.

Proof. First notice that the identity $U_gJ = JU_g$ implies $J(D_G^{(U)}) \subset D_G^{(U)}$ and $Ju(\mathsf{A}) \subset u(\mathsf{A})J$ as explained in Proposition 7.1.8. In particular we can consider the operator $u(\mathsf{A})_{\mathfrak{J}}$ on $\mathsf{H}_{\mathfrak{J}}$. Let us start with the proof of point (a). Suppose first that H is real and take some $u \in \mathsf{H}_{\mathfrak{J}} = \mathsf{H}$ and $f \in C_0^\infty(G)$. Denote by $x(f)$ the Gårding vector defined through $U_{\mathfrak{J}}$ and by $x[f]$ the one defined for U. If we manage to prove that $x(f) = x[f]$, then Remark 7.1.5 concludes the proof. So, by definition we have

$$\begin{aligned}(v|u(f)) = \Re[(v|u(f))_{\mathfrak{J}}] = \Re\left[\int_G f(g)(v|(U_g)_{\mathfrak{J}}u)_{\mathfrak{J}}\,dg\right] = \int_G f(g)\Re[(v|(U_g)_{\mathfrak{J}}u)_{\mathfrak{J}}]\,dg = \\ = \int_G f(g)(v|U_gu)\,dg = (v|u[f]).\end{aligned} \tag{7.11}$$

The arbitrariness of v gives $u(f) = u[f] \in D_G^{(U)}$ and the proof is complete. So, suppose that H is quaternionic. Again we keep the notation $u(f)$ for $U_{\mathfrak{J}}$ and $w[f]$ for U. So, take $x \in D_G^{(U_{\mathfrak{J}})} \subset \mathsf{H}_{\mathfrak{J}}$ and suppose first that $x = u(f)$ for some $u \in \mathsf{H}_{\mathfrak{J}}$ and $f \in C_0^\infty(G)$. Take any $z = \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} z_\alpha i_\alpha \in \mathsf{H}$ with $z_\alpha \in \mathsf{H}_{\mathfrak{J}}$, then

$$\begin{aligned}(z|u(f)) &= \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} \overline{i_\alpha}(z_\alpha|u(f))_{\mathfrak{J}} = \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} \overline{i_\alpha} \int_G f(g)(z_\alpha|(U_g)_{\mathfrak{J}}u)_{\mathfrak{J}}\, dg = \\ &= \int_G f(g) \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} \overline{i_\alpha}(z_\alpha|U_g u)_{\mathfrak{J}}\, dg = \int_G f(g)(z|U_g u)\, dg = (z|u[f]).\end{aligned} \tag{7.12}$$

The arbitrariness of z gives $u(f) = u[f]$. Since any vector of $D_G^{(U_{\mathfrak{J}})}$ is a finite sum of elements of the type $u(f)$ this concludes the proof of $D_G^{(U_{\mathfrak{J}})} \subset D_G^{(U)} \cap \mathsf{H}_{\mathfrak{J}} = (D_G^{(U)})_{\mathfrak{J}}$. Let us prove the opposite inclusion. So, consider any $u \in D_G^{(U)} \cap \mathsf{H}_{\mathfrak{J}}$ and suppose $u = \sum_{h=1}^N u_h[f_h]$ for some $u_h \in \mathsf{H}$ and $f_h \in C_0^\infty(G)$. Notice that, even though $u \in \mathsf{H}_{\mathfrak{J}}$, the same is not guaranteed to hold for the individual addends $u_h[f_h]$. However it surely holds that $u_h = \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}}(u_h)_\alpha i_\alpha$ for some $(u_h)_\alpha \in \mathsf{H}_{\mathfrak{J}}$ and so

$$\begin{aligned}u &= \sum_{h=1}^N u_h[f_h] = \sum_{h=1}^N \left(\sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} (u_h)_\alpha i_\alpha\right)[f_h] = \sum_{h=1}^N \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} ((u_h)_\alpha[f_h]) i_\alpha = \\ &= \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} \sum_{h=1}^N ((u_h)_\alpha[f_h]) i_\alpha = \sum_{\alpha\in\mathcal{A}_{\mathbb{K}}} \left(\sum_{h=1}^N (u_h)_\alpha[f_h]\right) i_\alpha,\end{aligned} \tag{7.13}$$

where we exploited Proposition 7.1.3. Since $(u_h)_\alpha \in \mathsf{H}_{\mathfrak{J}}$ for every $\alpha \in \mathcal{A}_{\mathbb{K}}$ we know from the first part of the quaternionic Hilbert space case of this proof that $(u_h)_\alpha[f_h] = (u_h)_\alpha(f_h) \in D_G^{(U_{\mathfrak{J}})}$ and so also $\sum_{h=1}^N (u_h)_\alpha[f_h] \in D_G^{(U_{\mathfrak{J}})}$. Bearing in mind that $u \in \mathsf{H}_{\mathfrak{J}}$, it must be $\sum_{h=1}^N (u_h)_\alpha[f_h] = 0$ for $\alpha \neq 0$ and so $u = \sum_{h=1}^N (u_0)_\alpha[f_h] \in D_G^{(U_{\mathfrak{J}})}$. The proof of point (b) can be carried out with little effort. $\square$

As above we can state the following result.

***Corollary* 7.1.12.** *Let H be real or quaternionic Hilbert space with $\mathbb{K}$-structure $\mathfrak{J}$ and A an anti-self-adjoint operator such that $e^{tA} \in \mathfrak{J}'$ for all $t \in \mathbb{R}$, then $JA \subset AJ$ for all $J \in \mathfrak{J}$ and $(e^{tA})_{\mathfrak{J}} = e^{tA_{\mathfrak{J}}}$ for all $t \in \mathbb{R}$.*

Proof. Let $U : t \mapsto e^{tA}$ be the strongly-continuous unitary representation of $\mathbb{R}$ on H induced by A. First notice that $e^{tA}J = Je^{tA}$ implies $JA \subset AJ$ for all $J \in \mathfrak{J}$ as follows from Proposition 4.2.3. Thus we can consider the operator $A_{\mathfrak{J}}$ on $\mathsf{H}_{\mathfrak{J}}$ which is still anti-self-adjoint. As stated in Proposition 4.1.2 the map $U_{\mathfrak{J}} : t \mapsto (e^{tA})_{\mathfrak{J}}$ is a strongly-continuous unitary representation of $\mathbb{R}$ on $\mathsf{H}_{\mathfrak{J}}$ and thus it admits a unique anti-self-adjoint generator B on $\mathsf{H}_{\mathfrak{J}}$. Exploiting Proposition 7.1.11 we see that $D_G^{(U_{\mathfrak{J}})} = (D_G^{(U)})_{\mathfrak{J}}$ and that the restriction of B to this domain equals the restriction of $A_{\mathfrak{J}}$ on the same domain. Since $D_G^{(U)}$ is a core for A and $D_G^{(U_{\mathfrak{J}})}$ is a core for B, then Propositions 3.1.16 and 3.2.20 gives $B = A_{\mathfrak{J}}$, concluding the proof. □

The following result is very important and was proved in the complex setting by Dixmier-Malliavin [8]. The elements of $D_G^{(U)}$ coincide with the smooth vectors of the representation.

***Theorem* 7.1.13** (Dixmier-Malliavin). *Let* H *be a real, complex or quaternionic Hilbert space and* U *a strongly-continuous unitary representation over it, then* $x \in D_G^{(U)}$ *if and only if the map* $G \ni g \mapsto U_g x \in \mathsf{H}$ *is infinitely differentiable at every point* $g \in G$ *with respect to the smooth atlas of* G.

Proof. If the Hilbert space is complex, the proof of this theorem is part of the content of the original Dixmier-Malliavin paper [8]. So, suppose that H is real and consider its complexification $\mathsf{H}_{\mathbb{C}}$. As discussed before the statement of this theorem, if we consider the representation $U_{\mathbb{C}}$, then $D_G^{(U_{\mathbb{C}})} = (D_G^{(U)})_{\mathbb{C}}$. Now, take any $x \in \mathsf{H}$. The notion of differentiability looks only at the norm and $\mathbb{R}$-linearity of the Hilbert space, so it is easy to see that $g \mapsto U_g x$ is smooth if and only if $g \mapsto (U_{\mathbb{C}})_g x \otimes 1 = U_g x \otimes 1$ is smooth. Thanks to the Dixmier-Malliavin theorem for complex Hilbert spaces, this is equivalent to $x \otimes 1 \in D_G^{(U_{\mathbb{C}})} = (D_G^{(U)})_{\mathbb{C}}$ which is equivalent to $x \in D_G^{(U)}$ (see (3.5)). The quaternionc case is even easier to prove. □

7.2 The Enveloping Algebra and Its Properties

A very useful notion in quantum physical applications is the concept of *universal enveloping algebra* associated with a Lie algebra $\mathfrak{g}$. To introduce this notion we observe that any real unital algebra can be turned into a Lie algebra simply by taking the natural commutator $[a, b] := ab - ba$. There exists also an inverse procedure which allows us to canonically embed a given Lie algebra $\mathfrak{g}$ into a suitable real unital algebra $\mathcal{E}(\mathfrak{g})$ with product $\circ$ such that $[\mathsf{A}, \mathsf{B}]_{\mathfrak{g}}$ identifies with $\mathsf{A} \circ \mathsf{B} - \mathsf{B} \circ \mathsf{A}$ for any $\mathsf{A}, \mathsf{B} \in \mathfrak{g}$.

Let us construct this space. In order to do this, recall the definition of *tensor algebra* of a given real linear space V, defined as

$$T(\mathsf{V}) = \bigoplus_{n=0}^{\infty} \mathsf{V}^{\otimes n}.$$

This space is a real unital algebra where the product is given by $\otimes$ among the elements of $T(\mathsf{V})$. The space V naturally embeds into $T(\mathsf{V})$ as a real linear subspace, the identification being given by the identity function $\iota : \mathsf{V} \to T(\mathsf{V})$.

This space satisfies the so-called *universal property*: for every real unital algebra $\mathcal{A}$ and linear map $\phi : \mathsf{V} \to \mathcal{A}$, there exists a unique homomorphism of real unital algebras $\tilde{\phi} : T(\mathsf{V}) \to \mathcal{A}$ such that $\phi = \tilde{\phi} \circ \iota$. The dimension of $T(\mathsf{V})$ is of course not finite, no matter what the dimension of V is. If $\{e_1, \ldots, e_n\}$ is an algebraic basis for V, then the set containing 1 and all the possible products $e_{i_1} \otimes \cdots \otimes e_{i_k}$ is a basis for $T(V)$.

Now, take as V a Lie algebra $\mathfrak{g}$. The idea is to "get rid" of the elements

$$\mathsf{A} \otimes \mathsf{B} - \mathsf{B} \otimes \mathsf{A} - [\mathsf{A}, \mathsf{B}]_{\mathfrak{g}} \in \mathfrak{g} \subset T(\mathfrak{g})$$

through a quotient procedure. Consider the two-sided ideal $\mathcal{I}(\mathfrak{g})$ of $T(\mathfrak{g})$ generated by these elements, then the following result holds.

***Definition* 7.2.1.** *Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra $\mathcal{E}(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient real unital algebra*

$$\mathcal{E}(\mathfrak{g}) := T(\mathfrak{g})/\mathcal{I}(\mathfrak{g}).$$

whose product is denoted by $\circ$.

The quotient map $\pi_g : T(\mathfrak{g}) \to \mathcal{E}(\mathfrak{g})$ is a real unital algebra homomorphism and $\pi_{\mathfrak{g}}(1)$ is the unit of $\mathcal{E}(\mathfrak{g})$. Define $\iota_{\mathfrak{g}} := \pi_{\mathfrak{g}} \circ \iota : \mathfrak{g} \to \mathcal{E}(\mathfrak{g})$. This is a Lie algebra homomorphism because

$$\begin{aligned} \iota_{\mathfrak{g}}(\mathsf{A}) \circ \iota_{\mathfrak{g}}(\mathsf{B}) - \iota_{\mathfrak{g}}(\mathsf{B}) \circ \iota_{\mathfrak{g}}(\mathsf{A}) = \pi_{\mathfrak{g}}(\mathsf{A}) \circ \pi_g(\mathsf{B}) - \pi_{\mathfrak{g}}(\mathsf{B}) \circ \pi_{\mathfrak{g}}(\mathsf{A}) = \pi_{\mathfrak{g}}(\mathsf{A} \otimes \mathsf{B} - \mathsf{B} \otimes \mathsf{A}) = \\ = \pi_{\mathfrak{g}}([\mathsf{A}, \mathsf{B}]_{\mathfrak{g}}) = \iota_{\mathfrak{g}}([\mathsf{A}, \mathsf{B}]_{\mathfrak{g}}) \quad \text{if} \quad \mathsf{A}, \mathsf{B} \in \mathfrak{g}. \end{aligned}$$

Even though the enveloping algebra was defined by means of a particular procedure it is a *canonical* object because of the following universality result which is a consequence of the universal property of the tensor algebra.

***Theorem* 7.2.2** (Universal Property)**.** *Let $\mathcal{A}$ be any real unital algebra and $\psi : \mathfrak{g} \to \mathcal{A}$ a Lie algebra homomorphism. Then there exists a unique homomorphism of real unital algebras $\tilde{\psi} : \mathcal{E}(\mathfrak{g}) \to \mathcal{A}$ such that $\psi = \tilde{\psi} \circ \iota_{\mathfrak{g}}$.*

Exploiting this result it is immediate to see that $(\mathcal{E}(\mathfrak{g}), \iota_{\mathfrak{g}})$ is the only couple of real unital associative algebra and Lie algebra homomorphism from $\mathfrak{g}$ to $\mathcal{E}(\mathfrak{g})$ satisfying this property, up to isomorphisms.

Let $\{\mathsf{E}_1, \dots, \mathsf{E}_n\}$ be an algebraic basis for $\mathfrak{g}$, then the set containing $\pi(1)$ and all the products $\pi(\mathsf{E}_{i_1} \otimes \cdots \otimes \mathsf{E}_{i_k})$ spans the quotient $\mathcal{E}(\mathfrak{g})$ but they are not linearly independent. In order to get an algebraic basis we invoke the following (see [43]).

***Theorem* 7.2.3** (Poincaré–Birkhoff–Witt Theorem)**.** *Let $\mathfrak{g}$ be a Lie algebra of dimension n and $\{\mathsf{E}_1, \dots, \mathsf{E}_n\}$ an algebraic basis of $\mathfrak{g}$. An algebraic basis of $\mathcal{E}(\mathfrak{g})$ is made of $\pi_{\mathfrak{g}}(1)$ and all products*

$$\pi_{\mathfrak{g}}(\mathsf{E}_{i_1}) \circ \cdots \circ \pi_{\mathfrak{g}}(\mathsf{E}_{i_k})$$

where $k \in \mathbb{N}$ and $i_m \in \{1, \dots, n\}$ with the constraints $i_1 \le \cdots \le i_k$.

As a corollary, the Lie algebra homomorphism $\iota_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{E}(\mathfrak{g})$ is injective since, evidently, the kernel of $\pi_{\mathfrak{g}}$ does not contain elements of $\mathfrak{g} \setminus \{0\}$. Thus $\mathfrak{g}$ turns out to be naturally isomorphic to the Lie subalgebra of $\mathcal{E}(\mathfrak{g})$ given by $\iota_{\mathfrak{g}}(\mathfrak{g})$.

Due to the afore-mentioned canonical isomorphism, we will simply denote $\pi_{\mathfrak{g}}(1)$ by 1 and $\pi_{\mathfrak{g}}(\mathsf{A})$ by A for any $\mathsf{A} \in \mathfrak{g}$. In particular we can write the generic pure product as

$$\mathsf{A}_1 \circ \cdots \circ \mathsf{A}_n = \pi_g(\mathsf{A}_1 \otimes \cdots \otimes \mathsf{A}_n) \tag{7.14}$$

for $\mathsf{A}_i \in \mathfrak{g}$ and $n \geq 1$.

Notice that, as any unital algebra, $\mathcal{E}(\mathfrak{g})$ comes with a natural commutator $[\cdot,\cdot]$, which reduces to the one originally defined on $\mathfrak{g}$ when restriced to the Lie algebra itself:

$$\mathsf{A} \circ \mathsf{B} - \mathsf{B} \circ \mathsf{A} = [\mathsf{A}, \mathsf{B}]_{\mathfrak{g}} \tag{7.15}$$

for any $\mathsf{A}, \mathsf{B} \in \mathfrak{g}$.

The last notion we want to introduct is the idea of a *symmetric element* of $\mathcal{E}(\mathfrak{g})$. Exploiting the universal property of the tensor product on any factor $\mathfrak{g}^{\otimes n} \subset T(\mathfrak{g})$ with $n \geq 1$ we see that there exists a unique linear map $p_n : \mathfrak{g}^{n\otimes} \to \mathfrak{g}^{n\otimes}$ such that

$$p_n(\mathsf{A}_1 \otimes \cdots \otimes \mathsf{A}_n) = (-1)^n \mathsf{A}_n \otimes \cdots \otimes \mathsf{A}_1$$

for any choice of $\mathsf{A}_i \in \mathfrak{g}$. So, if $p_0 = id_{\mathbb{R}}$ we can define

$$p := \bigoplus_{n=0}^{\infty} p_n : T(\mathfrak{g}) \to T(\mathfrak{g}).$$

Notice that this map is *involutive*, i.e. $pp = id_{T(\mathfrak{g})}$, so that it is a linear space automorphism, and also fulfill the crucial property

$$p(\mathsf{A} \otimes \mathsf{B} - \mathsf{B} \otimes \mathsf{A} - [\mathsf{A}, \mathsf{B}]_{\mathfrak{g}}) = \mathsf{B} \otimes \mathsf{A} - \mathsf{A} \otimes \mathsf{B} - [\mathsf{B}, \mathsf{A}]_{\mathfrak{g}}$$

for any $\mathsf{A}, \mathsf{B} \in \mathfrak{g}$ which means tha the ideal $\mathcal{I}(\mathfrak{g})$ is invariant under the action of p. At this point, thanks to the properties of the quotient, there must exist a unique linear function

$$+ : \mathcal{E}(\mathfrak{g}) \ni \mathsf{M} \mapsto \mathsf{M}^+ \in \mathcal{E}(\mathfrak{g})$$

such that $\pi \circ p = + \circ \pi$. It is easy to see that this map is an automorphism of $\mathcal{E}(\mathfrak{g})$. This map satisfies the following properties, making it a *involution* on the real algebra $\mathcal{E}(\mathfrak{g})$:

$$(a\mathsf{M} + b\mathsf{N})^+ = a\mathsf{M}^+ + b\mathsf{N}^+, \quad (\mathsf{M} \circ \mathsf{N})^+ = \mathsf{N}^+ \circ \mathsf{M}^+, \quad \mathsf{M}^{++} = \mathsf{M}$$

for all $\mathsf{N}, \mathsf{M} \in \mathcal{E}(\mathfrak{g})$ and $a, b \in \mathbb{R}$.

***Definition* 7.2.4.** *Let $\mathfrak{g}$ be a Lie algebra, the map $\mathcal{E}(\mathfrak{g}) \ni \mathsf{M} \mapsto \mathsf{M}^+ \in \mathcal{E}(\mathfrak{g})$ is called the involution of $\mathcal{E}(\mathfrak{g})$. An element $\mathsf{M} \in \mathcal{E}(\mathfrak{g})$ is said to be symmetric if $\mathsf{M}^+ = \mathsf{M}$.*

Now, let us go back to Theorem 7.1.8. No matter what the division algebra of the involved Hilbert space is, let $\mathcal{L}(D_G^{(U)})$ be the real unital algebra of *real* linear operator from $D_G^{(U)}$ to itself. The function

$$u : \mathfrak{g} \ni \mathsf{A} \mapsto u(\mathsf{A}) \in \mathcal{L}(D_G^{(U)})$$

is a Lie-algebra homomorphism. So, thanks to Theorem (7.2.2) there must exist a unique homomorphism of real unital algebras

$$\tilde{u} : \mathcal{E}(\mathfrak{g}) \ni \mathsf{M} \mapsto \tilde{u}(\mathsf{M}) \in \mathcal{L}(D_G^{(U)})$$

which reduces to u when evaluated on $\mathfrak{g} \subset \mathcal{E}(\mathfrak{g})$.

***Proposition* 7.2.5.** *The map u defined in Theorem 7.1.8 uniquely extends to a real unital algebra representation $\tilde{u}$ of the universal enveloping algebra $\mathcal{E}(\mathfrak{g})$. In particular it satisfies $\tilde{u}(1) = I|_{D_G^{(U)}}$ and*

$$\tilde{u}(\mathsf{A}_1 \circ \cdots \circ \mathsf{A}_n) = u(\mathsf{A}_1) \cdots u(\mathsf{A}_n) \tag{7.16}$$

for all $\mathsf{A}_1, \dots, \mathsf{A}_n \in \mathfrak{g}$ and $n \geq 1$. Moreover $\tilde{u}(\mathsf{M}^+) \subset \tilde{u}(\mathsf{M})^*$ for any $\mathsf{M} \in \mathcal{E}(\mathfrak{g})$. In particular $\tilde{u}(\mathsf{M})$ is symmetric if M is symmetric.

Proof. The last two statements follows immediately from the properties of the Hermitean conjugation $T^* + S^* \subset (T+S)^*$ and $S^*T^* \subset (TS)^*$ and the fact that $u(\mathsf{A})$ is anti-symmetric for any $\mathsf{A} \in \mathfrak{g}$. □

***Remark* 7.2.6.** Some useful remarks follow.

(a) For the sake of simplicity, if there is no need to specify differently, we will denote $\tilde{u}$ simply by u.

(b) From $u(\mathsf{M}^+) \subset u(\mathsf{M})^*$ it follows that $u(\mathsf{M})$ is closable, its adjoint being densely defined.

(c) Referring to Proposition 7.1.9, it is immediate to see that

$$\widetilde{u_{\mathbb{K}}}(\mathsf{M}) = \tilde{u}(\mathsf{M})_{\mathbb{K}} \quad \text{for all } \mathsf{M} \in \mathcal{E}(\mathfrak{g}).$$

(d) Referring to Proposition 7.1.11, it is immediate to see that $J\widetilde{u}(\mathsf{M}) \subset \widetilde{u}(\mathsf{M})J$ for any $J \in \mathfrak{J}$ and

$$\widetilde{u_{\mathfrak{J}}}(\mathsf{M}) = \tilde{u}(\mathsf{M})_{\mathfrak{J}} \quad \text{for all } \mathsf{M} \in \mathcal{E}(\mathfrak{g}).$$

Now, let us go back to the representation (7.8). The space of vector fields $\mathfrak{X}(G)$ is a real unital algebra of *real* linear operator from $C_0^\infty(G)$ to $C_0^\infty(G)$. The function

$$X : \mathfrak{g} \ni \mathsf{A} \mapsto X_{\mathsf{A}} \in \mathfrak{X}(G)$$

is a Lie algebra homomorphism. Thanks to Theorem (7.2.2) there must exit a unique homomorphism of real unital associative algebras

$$\tilde{X} : \mathcal{E}(\mathfrak{g}) \ni \mathsf{M} \mapsto \tilde{X}_{\mathsf{M}} \in \mathfrak{X}(G)$$

which reduces to X when evaluated on $\mathfrak{g} \subset \mathcal{E}(\mathfrak{g})$.

There exists a particularly interesting kind of objects within the enveloping algebra $\mathcal{E}(\mathfrak{g})$, the so-called *elliptic elements*. An element $\mathsf{L} \in \mathcal{E}(\mathfrak{g})$ is said to be elliptic if the associated vector field $\tilde{X}_{\mathsf{L}}$ is an elliptic partial differential operator. Remarkable examples of elliptic symmetric elements are the *Casimir elements*, i.e. the ones of the form

$$\Delta := \sum_{i=1}^{n} \mathsf{E}_i \circ \mathsf{E}_i \in \mathcal{E}(\mathfrak{g}), \tag{7.17}$$

where $\{E_1, \dots, E_n\}$ is any algebraic basis of $\mathcal{E}(\mathfrak{g})$.

These elliptic objects play an important role in determining if a given operator $u(\mathsf{M})$ is essentially self-adjoint as the following results explain.

***Theorem* 7.2.7.** *Suppose that $\mathsf{M} \in \mathcal{E}(\mathfrak{g})$ is symmetric and $[\mathsf{M}, \mathsf{L}] = 0$ for some elliptic and symmetric element $\mathsf{L} \in \mathcal{E}(\mathfrak{g})$, then $u(\mathsf{M})$ is essentially self-adjoint. In particular, for any elliptic and symmetric element L, the operator $u(\mathsf{L})$ is essentially self-adjoint.*

Proof. The complex version is a subcase of Theorem 10.2.6 of [35]. So, suppose that H is real or quaternionic and take its complexification $\mathsf{H}_{\mathbb{C}}$ together with the representation $U_{\mathbb{C}}$. If $u_{\mathbb{C}}$ is the Lie algebra representation associated with $U_{\mathbb{C}}$ and $\widetilde{u_{\mathbb{C}}}$ its extension to $\mathcal{E}(\mathfrak{g})$, the complex version of this theorem assures that $\widetilde{u_{\mathbb{C}}}(\mathsf{M})$ is essentially self-adjoint. Exploiting Remark 7.2.6, point (c) we derive the essential self-adjointness of $\widetilde{u}(\mathsf{M})$. $\square$

***Remark* 7.2.8.** In particular, Theorem 7.2.7 applies to the Casimir elements as in (7.17).

7.3 The Nelson Domain of Analytic Vectors

There is another important subspace of H made of *good vectors* associated with a strongly-continuous unitary representation U of a Lie group G, even better than $D_G^{(U)}$.

Let H be a real, complex or quaternionic Hilbert space. A function $f : \mathbb{R}^n \supset U \to \mathsf{H}$ is said to be *real analytic* at $s_0 \in U$ if there exists a neighbourhood $V \subset U$ of s_0 where the function can be expanded as

$$f(s) = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} (s - s_0)^{\alpha} v_{\alpha} \quad \text{for all } s \in V \tag{7.18}$$

with suitable $v_\alpha \in \mathsf{H}$ for all $\alpha \in \mathbb{N}^n$.

***Definition* 7.3.1.** *Let H be a real, complex or quaternionic Hilbert space and $G \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ a strongly-continuous unitary representation on H of the Lie group G. A vector $x \in \mathsf{H}$ is said to be analytic for U if the function $g \mapsto U_g x$ is real analytic at every point $g \in G$, referring to the analytic atlas of G. The linear subspace of H made of these vectors is called the Nelson Domain of the representation and denoted by $D_N^{(U)}$.*

***Proposition* 7.3.2.** *Let H be a real or quaternionic Hilbert space and $U : G \to U_g \in \mathfrak{B}(\mathsf{H})$ a strongly-continuous unitary representation. Then the following statements hold:*

(a) $D_N^{(U_{\mathbb{K}})} = (D_N^{(U)})_{\mathbb{K}}$,

(b) $D_N^{(U_{\mathfrak{J}})} = (D_N^{(U)})_{\mathfrak{J}}$ *if* $\mathfrak{J}$ *is a* $\mathbb{K}$*-structure on* H *such that* $U_g \in \mathfrak{J}'$ *for all* $g \in G$.

Proof. We prove point (a), the other one being analogous. Suppose that H is real and take $x \in (D_N^{(U)})_{\mathbb{K}}$, i.e. $u = \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha$ with $u_\alpha \in D_N^{(U)}$. Fix any $g_0 \in G$ then, referring to any chart ψ around g_0 of the analytic atlas we can write

$$s \mapsto U_{\psi(s)} u_\alpha = \sum_{m=0}^{\infty} \sum_{|\gamma|=m} (s-s_0)^k v_{\alpha,\gamma},$$

which implies

$$\begin{aligned}(U_{\psi(s)})_{\mathbb{K}} u &= \sum_{\alpha=0}^{d_{\mathbb{K}}} U_{\psi(s)} u_\alpha \otimes i_\alpha = \sum_{\alpha=0}^{d_{\mathbb{K}}} \sum_{m=0}^{\infty} \sum_{|\gamma|=m} (s-s_0)^\gamma v_{\alpha,\gamma} \otimes i_\alpha = \\ &= \sum_{\alpha=0}^{d_{\mathbb{K}}} \left[\lim_{N\to\infty} \sum_{m=0}^{N} \sum_{|\gamma|=m} (s-s_0)^\gamma v_{\alpha,\gamma} \otimes i_\alpha \right] = \lim_{N\to\infty} \sum_{\alpha=0}^{d_{\mathbb{K}}} \sum_{m=0}^{N} \sum_{|\gamma|=m} (s-s_0)^\gamma v_{\alpha,\gamma} \otimes i_\alpha = \\ &= \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{|\gamma|=m} (s-s_0)^\gamma \sum_{\alpha=0}^{d_{\mathbb{K}}} v_{\alpha,\gamma} \otimes i_\alpha = \sum_{m=0}^{\infty} \sum_{|\gamma|=m} (s-s_0)^\gamma \left(\sum_{\alpha=0}^{d_{\mathbb{K}}} v_{\alpha,\gamma} \otimes i_\alpha \right)\end{aligned}$$

which, in turn, gives $u \in D_N^{(U_{\mathbb{K}})}$. On the contrary take any $u \in D_N^{(U_{\mathbb{K}})}$ and suppose that $u = \sum_{\alpha=0}^{d_{\mathbb{K}}} u_\alpha \otimes i_\alpha$ with $u_\alpha \in \mathsf{H}$, then we have

$$s \mapsto (U_{\psi(s)})_{\mathbb{K}} u = \sum_{m=0}^{\infty} \sum_{|\gamma|=m} (s-s_0)^\gamma v_\gamma. \tag{7.19}$$

for suitable $v_\gamma \in \mathsf{H}_{\mathbb{K}}$ which can be written as $v_\gamma = \sum_{\alpha=0}^{d_{\mathbb{K}}} v_{\alpha,\gamma} \otimes i_\alpha$. So,

$$\begin{aligned}\sum_{\alpha=0}^{d_{\mathbb{K}}} U_{\psi(s)} u_\alpha \otimes i_\alpha &= (U_{\psi(s)})_{\mathbb{K}} u = \sum_{m=0}^{\infty} \sum_{|\gamma|=m} (s-s_0)^\gamma v_\gamma \\ &= \sum_{m=0}^{\infty} \sum_{|\gamma|=m} \sum_{\alpha=0}^{d_{\mathbb{K}}} (s-s_0)^\gamma v_{\alpha,\gamma} \otimes i_\alpha = \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{|\gamma|=m} \sum_{\alpha=0}^{d_{\mathbb{K}}} (s-s_0)^\gamma v_{\alpha,\gamma} \otimes i_\alpha = \\ &= \lim_{N\to\infty} \sum_{\alpha=0}^{d_{\mathbb{K}}} \left[\sum_{m=0}^{N} \sum_{|\gamma|=m} (s-s_0)^\gamma v_{\alpha,\gamma} \right] \otimes i_\alpha .\end{aligned}$$

Now, exploiting (3.4), we notice that a sequence $\sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha^m \otimes i_\alpha$ converges within $\mathsf{H}_{\mathbb{K}}$ if and only if x_α^m converges within H for all $\alpha = 0, \dots, d_{\mathbb{K}}$ and

$$\lim_{m\to\infty} \sum_{\alpha=0}^{d_{\mathbb{K}}} x_\alpha^m \otimes i_\alpha = \sum_{\alpha=0}^{d_{\mathbb{K}}} \left(\lim_{m\to\infty} x_\alpha^m\right) \otimes i_\alpha .$$

Putting all together we see that $\sum_{m=0}^{N}\sum_{|\gamma|=m}(s-s_0)^\gamma v_{\alpha,\gamma}$ converges within H for every $\alpha = 0, \dots, d_{\mathbb{K}}$ and

$$U_{\psi(s)} u_\alpha = \sum_{m=0}^{\infty}\sum_{|\gamma|=m}(s-s_0)^\gamma v_{\alpha,\gamma}.$$

for every $\alpha = 0, \dots, d_{\mathbb{K}}$, proving that $u \in (D_N^{(U)})_{\mathbb{K}}$. If H is quaternionic the proof is even simpler. □

There exists another important definition of analyticity on vectors on a Hilbert space which refers to a given linear operator.

***Definition* 7.3.3.** *Let $A : D(A) \to \mathsf{H}$ be a linear operator over a real, complex or quaternionic Hilbert space H. A vector $x \in \cap_{n\in\mathbb{N}} D(A^n)$ is said to be analytic for A if there exists some $t_x > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{t_x^n}{n!}\|A^n x\| < \infty. \tag{7.20}$$

The linear subspace of analytic vectors is denoted by $\mathcal{A}(A)$.

From the elementary theory of power series we know that if such a t_x exists, then the convergence holds also substituting t_x by any $z \in \mathbb{C}$ such that $|z| \le t_x$, obtaining in this way an absolutely convergent series.

***Proposition* 7.3.4.** *Let H be real or quaternionic and $A : D(A) \to \mathsf{H}$ a linear operator, then the following statements hold:*

(a) $\mathcal{A}(A_{\mathbb{K}}) = \mathcal{A}(A)_{\mathbb{K}}$,

(b) $\mathcal{A}(A_{\mathfrak{J}}) = \mathcal{A}(A)_{\mathfrak{J}}$ *if $\mathfrak{J}$ is a $\mathbb{K}$-structure such that $JA \subset AJ$ for all $J \in \mathfrak{J}$.*

Proof. This is left to the reader, it being analogous to the several ones carried out so far. □

A really important result due to Nelson is the following.

***Proposition* 7.3.5.** *Let $A : D(A) \to \mathsf{H}$ be a linear operator over a real, complex or quaternionic Hilbert space. Then the following statements hold:*

(a) if A is anti-self-adjoint and $x \in \mathcal{A}(A)$ then

$$e^{tA}x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x \quad \forall t \in \mathbb{R} \quad s.t. \quad |t| \leq t_x;$$

(b) if A is (anti-)symmetric and $D(A)$ contains a set of analytic vectors whose finite span is dense in H, then A is essentially (anti-)self-adjoint.

Proof. For the complex case we refer to Theorem 5.47 and Proposition 9.25 [24]. The real and quaternionic cases can be proved by moving to the complexification space $\mathsf{H}_{\mathbb{C}}$, bearing in mind that the generator of $(e^{tA})_{\mathbb{K}}$ is given by $A_{\mathbb{K}}$ (Corollary 7.1.10) and reasoning in the usual way. □

***Theorem* 7.3.6.** *The following statements hold:*

(a) $D_N^{(U)} \subset D_G^{(U)}$,

(b) $U_g(D_N^{(U)}) \subset D_N^{(U)}$ for all $g \in G$,

(c) $D_N^{(U)}$ is dense in H,

(d) $D_N^{(U)} \subset \mathcal{A}(u(\mathsf{A}))$ for any $\mathsf{A} \in \mathfrak{g}$,

(e) $u(\mathsf{A})(D_N^{(U)}) \subset D_N^{(U)}$ for any $\mathsf{A} \in \mathfrak{g}$.

Proof. Point (a) follows from Theorem 7.1.13 and the fact that any real analitic function $f : U \to \mathsf{H}$ is also infinitely differentiable. Point (b) follows from the fact that the multiplication on G is real analytic with respect to the analytic atlas of G. Point (c)-(d)-(e) can be derived in the complex case and the proof appears in Section 12.2.11 of [24] (for the original paper see [28]) In the real and quaternionic cases the proof can be carried out moving to $\mathsf{H}_{\mathbb{C}}$ and the representation $U_{\mathbb{C}}$ and exploiting the complex version of this result together with Proposition 7.1.9 and Proposition 7.3.2. □

A final remarkable consequence of the properties of Nelson's technology and our version of Schur's lemma is the following proposition.

***Proposition* 7.3.7.** *Let H be a real, complex or quaternionic Hilbert space and $G \ni g \mapsto U_g$ an irreducible strongly-continuous unitary representation of the connected Lie group G. Take any $\mathsf{M} \in \mathcal{E}(\mathfrak{g})$ and suppose that*

(a) $[\mathsf{M}, \mathsf{A}] = 0$ for all $\mathsf{A} \in \mathfrak{g}$,

(b) $u(\mathsf{M})$ is essentially self-adjoint,

then $u(\mathsf{M}) = cI|_{D_G^{(U)}}$ for some $c \in \mathbb{R}$.

Proof. Take $x \in D_N^{(U)}$ and $\mathsf{A} \in \mathfrak{g}$. Thanks to Theorem 7.3.6 it holds that $x \in D_G^{(U)}$ and x is analytic for $u(\mathsf{A})$ (hence also for $\overline{u(\mathsf{A})}$). Exploiting Proposition 7.3.5, we have that there exists $t_{\mathsf{A},x} > 0$ such that

$$U_{\exp(t\mathsf{A})}x = e^{t\overline{u(\mathsf{A})}}x = \sum_{n=0}^{\infty} \frac{t^n}{n!} u(\mathsf{A})^n x, \quad |t| \leq t_{\mathsf{A},x} .$$

Moreover $D_N^{(U)}$ is invariant under the action of u, hence $u(\mathsf{M})x \in D_N^{(U)}$. Then there exits $t_{\mathsf{A},u(\mathsf{M})x} > 0$ such that

$$U_{\exp(t\mathsf{A})}u(\mathsf{M})x = e^{t\overline{u(\mathsf{A})}}u(\mathsf{M})x = \sum_{n=0}^{\infty} \frac{t^n}{n!} u(\mathsf{A})^n u(\mathsf{M})x, \quad |t| \leq t_{\mathsf{A},u(\mathsf{M})x} .$$

Now take a positive real $t_x < \min\{t_{\mathsf{A},x}, t_{\mathsf{A},u(\mathsf{M})x}\}$. Using $[u(\mathsf{M}), u(\mathsf{A})] = 0$ we have

$$U_{\exp(t\mathsf{A})}u(\mathsf{M})x = \sum_{n=0}^{\infty} \frac{t^n}{n!} u(\mathsf{M})u(\mathsf{A})^n x, \quad |t| \leq t_x .$$

Since $u(\mathsf{M})$ is closable, it follows directly from the equations above and the invariance of $D_G^{(U)}$ under the action of U that

$$U_{\exp(t\mathsf{A})}u(\mathsf{M})x = \sum_{n=0}^{\infty} \frac{t^n}{n!} u(\mathsf{M})u(\mathsf{A})^n x = u(\mathsf{M})U_{\exp(t\mathsf{A})}x$$

for every $|t| \leq t_x$. Actually this equality holds for every $t \in \mathbb{R}$. Indeed define $\mathcal{Z} := \{z > 0 \,|\, u(\mathsf{M})U_{\exp(t\mathsf{A})}x = U_{\exp(t\mathsf{A})}u(\mathsf{M})x, \ |t| \leq z\}$ and define $t_0 := \sup \mathcal{Z}$. Suppose that $t_0 < \infty$, then it is easy to see that the closability of $u(\mathsf{M})$ ensures that $u(\mathsf{M})U_{\exp(t_0\mathsf{A})}x = U_{\exp(t_0\mathsf{A})}u(\mathsf{M})x$, hence $t_0 \in \mathcal{Z}$. We know that $y := U_{\exp(t_0\mathsf{A})}x \in D_N^{(U)}$, so we can repeat the above reasoning finding a real $t_y > 0$ such that $u(\mathsf{M})U_{\exp(t\mathsf{A})}y = U_{\exp(t\mathsf{A})}u(\mathsf{M})y$ for every $|t| \leq t_y$. Noticing that $\exp((t+t_0)\mathsf{A}) = \exp(t\mathsf{A})\exp(t_0\mathsf{A})$, it straightforwardly follows that $u(\mathsf{M})U_{exp(t+t_0)\mathsf{A}}x = U_{exp(t+t_0)\mathsf{A}}u(\mathsf{M})x$ for $|t| \leq t_y$, hence $t_0 + t_y \in \mathcal{Z}$, which is in contradiction with the definition of t_0. This proves that $t_0 = \infty$. As is well known from the elementary theory of Lie group theory, since the G is connected, every element is the product of a finite number of elements belonging to one parameter subgroups generated by the elements of $\mathfrak{g}$, so that we have actually demonstrated that $u(\mathsf{M})U_g = U_g u(\mathsf{M})$ on $D_N^{(U)}$ for every $g \in G$. This identity implies $U_g u(\mathsf{M})|_{D_N^{(U)}} = u(\mathsf{M})|_{D_N^{(U)}} U_g$ on the natural domains thanks to the invariance of the Nelson space under the action of the group representation. In our hypotheses $u(\mathsf{M})|_{D_N^{(U)}}$ is the restriction of a closable operator and thus it is closable as well and so Proposition 2.2.58 gives $U_g \overline{u(\mathsf{M})|_{D_N^{(U)}}} = \overline{u(\mathsf{M})|_{D_N^{(U)}}} U_g$ for every g. Using Proposition 4.3.4, the irreducibility of U gives $D(\overline{u(\mathsf{M})|_{D_N^{(U)}}}) = \mathsf{H}$ and $\overline{u(\mathsf{M})|_{D_N^{(U)}}} \in \mathfrak{B}(\mathsf{H})$, more precisely $\overline{u(\mathsf{M})|_{D_N^{(U)}}} = aI + bL$ for some $a, b \in \mathbb{R}$, where $L = iI$ if H is complex, and L is a generic imaginary operator if

H is real or quaternionic. Since $\overline{u(\mathsf{M})|_{D_N^{(U)}}} \subset \overline{u(\mathsf{M})}$, the maximality of the domain gives $\overline{u(\mathsf{M})|_{D_N^{(U)}}} = \overline{u(\mathsf{M})}$. As the latter is self-adjoint, it follows that $b = 0$ and $\overline{u(\mathsf{M})} = aI$ with $a \in \mathbb{R}$ ending the proof. $\square$

Chapter 8

Lorentz and Poincaré groups

In this short chapter we give a basic description of the Lorentz and Poincaré groups.

8.1 The Lorentz and Poincaré Groups

Consider the four dimensional Euclidean space $\mathbb{R}^4$ and define the *metric tensor*

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{8.1}$$

which satisfies $\eta\eta = I$. Equip $\mathbb{R}^4$ with the following pseudo-scalar inner product:

$$x \cdot y = x^t \eta y = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} x^\mu \eta_{\mu\nu} y^\nu = -x^0 y^0 + \mathbf{x} \cdot \mathbf{y}. \tag{8.2}$$

The setting of special relativity is *Minkowski spacetime*, a four-dimensional affine space $\mathbb{M}^4$. Fixing an affine frame $(O, (\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3))$, we obtain cartesian coordinate representation

$$\phi : \mathbb{M}^4 \ni p \mapsto (x^0, \mathbf{x}) \in \mathbb{R}^4.$$

Similarly, another affine frame $(O', (\mathbf{e}'_0, \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3))$ gives rise to a different cartesian coordinate representation

$$\phi' : \mathbb{M}^4 \ni p \mapsto (x'_0, \mathbf{x}') \in \mathbb{R}^4.$$

The composition $\phi' \circ \phi^{-1} : \mathbb{R}^4 \to \mathbb{R}^4$ is an invertible affine transformation, that is

$$\phi' \circ \phi^{-1} : \mathbb{R}^4 \ni x \mapsto \Lambda x + a \in \mathbb{R}^4 \tag{8.3}$$

for some $\Lambda \in GL(4, \mathbb{R})$ and $a \in \mathbb{R}^4$. The transformation can be represented by the couple (Λ, a).

Sticking to the affine frames on $\mathbb{M}^4$ which represents physical *inertial reference frames* and supposing for the moment that $O = O'$, the principles of special relativity requires the transformations of coordinates to only those that preserve the pseudo-scalar product (8.2). i.e. those matrices $\Lambda \in G(4, \mathbb{R})$ such that

$$(\Lambda x) \cdot (\Lambda y) = x \cdot y$$

for any choice of $x, y \in \mathbb{R}^4$. In order to find a vectorless equivalent definition, take any $x, y \in \mathbb{R}^4$ and Λ as above, then

$$x^t \eta y = x \cdot y = (\Lambda x) \cdot (\Lambda y) = x^t \Lambda^t \eta \Lambda y.$$

Since x, y are arbitrary we immediately get the following (equivalent) matrix condition

$$\Lambda^t \eta \Lambda = \eta. \tag{8.4}$$

Using the notation $\Lambda = (\Lambda^\alpha{}_\beta)$ we have $\eta_{\alpha\beta} = (\Lambda^t)^\alpha{}_\mu \, \eta_{\mu\nu} \, \Lambda^\nu{}_\beta = \Lambda^\mu{}_\alpha \, \eta_{\mu\nu} \, \Lambda^\nu{}_\beta$ and so

$$\eta_{\alpha\beta} = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta, \tag{8.5}$$

which is an equivalent restatement of expression (8.4).

***Proposition* 8.1.1.** *The set of matrices fulfilling (8.4) has the following properties:*

(a) defines a closed subgroup of $GL(4, \mathbb{R})$, in particular it is a matrix Lie group;

(b) it is closed under transposition;

(c) its elements satisfy $\det \Lambda = \pm 1$ *and* $(\Lambda^0{}_0)^2 \geq 1$.

It is denoted by $\mathcal{L}$ and called the Lorentz group.

The sign of $\det \Lambda$ and $\Lambda^0{}_0$ distinguish between four disjoint subsets of $\mathcal{L}$.

$$\begin{aligned}
\mathcal{L}_+^\uparrow &:= \{\Lambda \in \mathcal{L} \mid \det \Lambda = +1, \ \Lambda^0{}_0 \geq +1\} \\
\mathcal{L}_+^\downarrow &:= \{\Lambda \in \mathcal{L} \mid \det \Lambda = +1, \ \Lambda^0{}_0 \leq -1\} \\
\mathcal{L}_-^\uparrow &:= \{\Lambda \in \mathcal{L} \mid \det \Lambda = -1, \ \Lambda^0{}_0 \geq +1\} \\
\mathcal{L}_-^\downarrow &:= \{\Lambda \in \mathcal{L} \mid \det \Lambda = -1, \ \Lambda^0{}_0 \leq -1\}
\end{aligned} \tag{8.6}$$

which corresponds to the four *connected components* of $\mathcal{L}$. These subsets are not totally independent from each other, for they are connected by the so called *time* and *parity inversion* elements of $\mathfrak{L}$.

(i) *Time inversion*: $x_0' = -x_0$

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} = \eta. \tag{8.7}$$

(ii) *Parity inversion*: $\mathbf{x}' = -\mathbf{x}$

$$P = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -\eta. \tag{8.8}$$

It can be proved that

$$\mathcal{L}_+^{\downarrow} = T\mathcal{L}_+^{\uparrow}, \quad \mathcal{L}_-^{\uparrow} = P\mathcal{L}_+^{\uparrow}, \quad \mathcal{L}_-^{\downarrow} = PT\mathcal{L}_+^{\uparrow}.$$

***Remark* 8.1.2.** The elements of $\mathcal{L}_+^{\uparrow}$ satify the following properties:

(a) they preserve the direction of time: time flows in the same direction in any two reference frames
$$\frac{\partial x_0'}{\partial x_0} \geq 0,$$
which is achieved by requiring ${\Lambda^0}_0 \geq 0$;

(b) they preserve the orientation of space: this is achieved by requiring $\det \Lambda > 0$.

We are interested in the component $\mathcal{L}_+^{\uparrow}$, which, besides all, is the only one giving rise to a group, the other three of them not even containing the identity element.

***Proposition* 8.1.3.** *The subset $\mathcal{L}_+^{\uparrow}$ is closed under tranposition and is a both open and closed subgroup of $\mathcal{L}$, in particular it is a matrix Lie group. It is called the proper orthochronus Lorentz group.*

Now, dropping the condition $O = O'$, the most general affine transformations between *arbitrary* inertial reference frames is given by

$$\mathcal{L} \times \mathbb{R}^4 \ni (\Lambda, a) : \mathbb{R}^4 \ni x \mapsto \Lambda x + a \in \mathbb{R}^4$$

for some $a \in \mathbb{R}^4$. The set of these transformations gives rise to a group with respect to the composition of functions. Let us see this. Take any $x \in \mathbb{R}^4$, then

$$(\Lambda', a')(\Lambda, a)x = (\Lambda', a')(\Lambda x + a) = \Lambda'\Lambda x + (\Lambda' a + a') = (\Lambda\Lambda', \Lambda' a + a')x$$

which implies $(\Lambda', a')(\Lambda, a) = (\Lambda'\Lambda, \Lambda' a + a') \in \mathcal{L} \times \mathbb{R}^4$. Clearly

$$(I, 0)(\Lambda, a) = (\Lambda, a)(I, 0) = (\Lambda, a)$$

and so $(I, 0)$ acts as the identity element on $\mathcal{L} \times \mathbb{R}^4$. Finally, notice that

$$(\Lambda^{-1}, -\Lambda^{-1}a)(\Lambda, a) = (\Lambda^{-1}\Lambda, \Lambda^{-1}a - \Lambda^{-1}a) = (I, 0)$$

and similarly $(\Lambda, a)(\Lambda^{-1}, -\Lambda^{-1}a) = (I, 0)$. This shows that $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$.

We can reinterpret this group as a matrix group which is more convenient for calculation purposes. Embedding $\mathbb{R}^4$ within $\mathbb{R}^5$ by means of the function $a \mapsto (a, 1)$, it is easy to see that the transformation $x \mapsto \Lambda x + a$ can be restated in matrix form as

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \\ 1 \end{bmatrix} = \left[\begin{array}{cccc|c} & & & & a^0 \\ & \Lambda & & & a^1 \\ & & & & a^2 \\ & & & & a^3 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix}. \tag{8.9}$$

We maintain the notation (Λ, a) to denote the matrix (8.9).

***Proposition* 8.1.4.** *The set of matrices of the form (8.9) is a closed subgroup of $GL(5, \mathbb{R})$, in particular it is a matrix Lie group. It is denoted by $\mathcal{P}$ and called the Poincaré group.*

Exploiting the definitions (8.6) we have the following corresponding subsets

$$\mathcal{P}^{\uparrow\downarrow}_{\pm 1} := \{(\Lambda, a) \in \mathcal{P} \mid \Lambda \in \mathcal{L}^{\uparrow\downarrow}_{\pm 1}\}, \tag{8.10}$$

which, again, correspond to the four connected components of $\mathcal{P}$.

Again, we are interested in the special case of transformations (Λ, a) with Λ preserving the direction of time and the orientation of space.

***Proposition* 8.1.5.** *The subset $\mathcal{P}^{\uparrow}_{+}$ forms a both closed and open subgroup of $\mathcal{P}$, in particular it is a matrix Lie group. It is called the proper orthochronus Poincaré group.*

8.2 The Lorentz and Poincaré Lie Algebras

In this section we aim to analyse the Lie algebras of the Lorentz and Poincaré groups. Remember that the Lie algebra of any matrix Lie group $G \subset GL(n, \mathbb{R})$ is given by the linear space

$$\mathfrak{g} := \{A \in M(n, \mathbb{R}) \mid e^{tA} \in G \ \forall t \in \mathbb{R}\}.$$

Consider the Lorentz group first. First notice that the Lie algebras of $\mathcal{L}$ and $\mathcal{L}^{\uparrow}_{+}$ are equal to each other, this follows from the fact that the latter is an open subgroup of the former.

So, a matrix $X \in M(4, \mathbb{R})$ belongs to the Lie algebra of $\mathcal{L}$ if and only if $(e^{sX})^t \eta e^{sX} = \eta$ for all $s \in \mathbb{R}$. Using $\eta\eta = I$ this is equivalent to $e^{-sX} = \eta e^{sX^t}\eta = e^{s\eta X^t \eta}$ which, in turn, is equivalent to $\eta X^t \eta = -X$.

***Proposition* 8.2.1.** *The Lie algebra of $\mathcal{L}$ is given by*

$$\mathfrak{l} = \{X \in M(4, \mathbb{R}) \mid \eta X^t \eta = -X\} \tag{8.11}$$

and coincides with the Lie algebra of $\mathcal{L}^{\uparrow}_{+}$.

Given this, we can study the Lie algebra $\mathfrak{p}$ of the Poincaré group $\mathcal{P}$. First of all notice that, as in the case of the Lorentz group, it coincides with the Lie algebra of $\mathcal{P}_+^\uparrow$.

So, consider any $X \in \mathfrak{p}$, i.e. a matrix $X \in M(5,\mathbb{R})$ such that $e^{sX} \in \mathcal{P}$ for every $s \in \mathbb{R}$. Using the matrix form of (Λ, a) it is easy to see that

$$X = \left.\frac{d}{ds}\right|_0 e^{sX} = \left[\begin{array}{c|c} Y & y \\ \hline 0 & 0 \end{array}\right] \text{ hence } X^n = \left[\begin{array}{c|c} Y^n & Y^{n-1}y \\ \hline 0 & 0 \end{array}\right] \tag{8.12}$$

for some $Y \in M(4,\mathbb{R})$ and $y \in \mathbb{R}^4$. This implies that

$$e^{sX} = \sum_{n=0}^{\infty} \frac{s^n}{n!} X^n = \left[\begin{array}{c|c} e^{sY} & z \\ \hline 0 & 1 \end{array}\right]$$

for some $z \in \mathbb{R}^4$. In order for this matrix to belong to $\mathcal{P}$ it is necessary that $e^{sY} \in \mathcal{L}$ for every $s \in \mathbb{R}$, which implies $Y \in \mathfrak{l}$. On the contrary, if $Y \in \mathfrak{l}$ and $y \in \mathbb{R}^4$ then $e^{tX} \in \mathfrak{p}$ if $X \in M(5,\mathbb{R})$ is defined as in (8.12). Concluding we have the following result.

***Proposition* 8.2.2.** *The Lie algebra of $\mathcal{P}$ is given by*

$$\mathfrak{p} = \left\{ \left[\begin{array}{c|c} Y & y \\ \hline 0 & 0 \end{array}\right] \in M(5,\mathbb{R}) \,\middle|\, Y \in \mathfrak{l},\ y \in \mathbb{R}^4 \right\} \tag{8.13}$$

and coincides with the Lie algebra of $\mathcal{P}_+^\uparrow$.

Next step consists in defining the standard basis of $\mathfrak{p}$, by finding the generators of the infinitesimal principal Poincaré transformations. In what follows let $\mathbf{e}_\nu \in \mathbb{R}^4$ be the standard basis vector in the ν-th direction.

(i) The *spacetime displacement* along the ν-th axis is given by the one-parameter subgroup

$$\mathbb{R} \ni s \mapsto T_\nu(s) = \left[\begin{array}{c|c} I & s\mathbf{e}_\nu \\ \hline 0 & 1 \end{array}\right] \in \mathcal{P}_+^\uparrow. \tag{8.14}$$

The generator associated with this subgroup is

$$\mathsf{P}_\mu := \left.\frac{d}{ds}\right|_0 T_\nu(s) = \left[\begin{array}{c|c} 0 & \mathbf{e}_\nu \\ \hline 0 & 0 \end{array}\right] \in \mathfrak{p}. \tag{8.15}$$

(ii) The counterclockwise *space rotation* about the i-th axis is given by the one-parameter subgroup

$$\mathbb{R} \ni \theta \mapsto R_i(\theta) = \left[\begin{array}{c|c} R(\theta, \mathbf{e}_i) & 0 \\ \hline 0 & 1 \end{array}\right] \in \mathcal{P}_+^\uparrow, \tag{8.16}$$

where the matrices $R(\theta, \mathbf{e}_i) \in \mathcal{L}_+^\uparrow$ are given by

$$R(\theta, \mathbf{e}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R(\theta, \mathbf{e}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$
$$R(\theta, \mathbf{e}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.17}$$

The generators associated with these subgroups are

$$\mathsf{L}_3 := \frac{d}{d\theta}\Big|_0 R_3(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathsf{L}_2 := \frac{d}{d\theta}\Big|_0 R_2(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\mathsf{L}_1 := \frac{d}{d\theta}\Big|_0 R_1(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8.18}$$

(iii) The *boost* along the i-th axis is given by the one-parameter subgroup

$$\mathbb{R} \ni z \mapsto B_i(z) = \left[\begin{array}{c|c} B(z, \mathbf{e}_i) & 0 \\ \hline 0 & 1 \end{array}\right], \tag{8.19}$$

where the matrices $B(z, \mathbf{e}_i) \in \mathcal{L}_+^\uparrow$ are given by

$$B(z, \mathbf{e}_1) = \begin{pmatrix} \cosh z & -\sinh z & 0 & 0 \\ -\sinh z & \cosh z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B(z, \mathbf{e}_2) = \begin{pmatrix} \cosh z & 0 & -\sinh z & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh z & 0 & \cosh z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$B(z, \mathbf{e}_3) = \begin{pmatrix} \cosh z & 0 & 0 & -\sinh z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh z & 0 & 0 & \cosh z \end{pmatrix}. \tag{8.20}$$

The generators associated with these subgroups are

$$\mathsf{K}_1 := \frac{d}{dz}\Big|_0 B_1(z) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathsf{K}_2 := \frac{d}{dz}\Big|_0 B_2(z) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\mathsf{K}_3 := \frac{d}{dz}\Big|_0 B_3(z) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8.21}$$

***Remark* 8.2.3.** Notice that the generators L_i and K_i of $\mathfrak{g}$ for $i = 1, 2, 3$ corresponds to analogous generators of $\mathfrak{l}$. Simply, we have to forget the 4-th dimension of the corresponding matrices. With a little abuse of language we will use the same notation for them when interpreted on the different spaces.

These generators are clearly linearly independent, more precisely:

***Proposition* 8.2.4.** *The Lie algebra $\mathfrak{p}$ is ten-dimensional and the generators $\mathsf{P}_\mu, \mathsf{L}_i, \mathsf{K}_i$ form a basis. In particular the generators $\mathsf{J}_i, \mathsf{K}_i$ form a basis of the Lie algebra $\mathfrak{l}$ which is six-dimensional. In both cases we refer to them as the elements of the canonical covariant basis.*

The elements of the canonical covariant basis satisfy the following commutation relations:

$$\begin{aligned} &[\mathsf{P}_\mu, \mathsf{P}_\nu] = [\mathsf{J}_i, \mathsf{P}_0] = 0 \\ &[\mathsf{L}_i, \mathsf{L}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathsf{L}_k, \quad [\mathsf{L}_i, \mathsf{P}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathsf{P}_k, \quad [\mathsf{L}_i, \mathsf{K}_j] = \varepsilon_{ijk}\mathsf{K}_k \\ &[\mathsf{K}_i, \mathsf{K}_j] = -\sum_{k=1}^{3} \varepsilon_{ijk}\mathsf{L}_k \\ &[\mathsf{K}_i, \mathsf{P}_0] = -\mathsf{P}_i \quad [\mathsf{K}_i, \mathsf{P}_j] = -\delta_{ij}\mathsf{P}_0 \end{aligned} \tag{8.22}$$

where $\mu, \nu = 0, \ldots, 3$ and $i, j = 1, \ldots, 3$.

8.3 The (Special) Orthogonal Subgroup

The Lorentz group is defined as the set of the linear transformations over $\mathbb{R}^4$ which preserves the inner product (8.2). A special case is given by the transformations which act trivially on the time-component, i.e. ${\Lambda^0}_\alpha = {\Lambda^\alpha}_0 = \delta_{\alpha 0}$ for any $\alpha = 0, 1, 2, 3$.

Of course the subset of these matrices is closed under composition and the inverse operations and it is easy to see that they are in 1-1 correspondence with the matrices on $\mathbb{R}^3$ which preserves the canonical scalar product

$$R\mathbf{x} \cdot R\mathbf{y} = \mathbf{x} \cdot \mathbf{y}. \tag{8.23}$$

Reasoning as in the case of the whole Lorentz group, it can be proved that these are exactly the matrices on $\mathbb{R}^3$ such that

$$R^t R = I. \tag{8.24}$$

We can state the following result

***Proposition* 8.3.1.** *The set of matrices fulfilling (8.23) has the following properties:*

(a) defines a closed subgroup of $GL(3, \mathbb{R})$, in particular it is a matrix Lie group;

(b) it is closed under transposition, more precisely $R^{-1} = R^t$;

(c) its elements satisfy $\det R = \pm 1$.

It is called the orthogonal group and denoted by $O(3)$.

An important role is played by the subset of the matrices with positive determinant, usually denoted by $SO(3)$. These are the transformations on $\mathbb{R}^3$ that preserves the scalar product and the orientation of the space. Again, the following result can be proved.

***Proposition* 8.3.2.** *The subset $SO(3)$ is a both open and closed subgroup of $O(3)$, in particular it is a matrix Lie group. It is called the special orthogonal group.*

These two groups embed naturally within $\mathcal{L}$ and $\mathcal{L}_+^\uparrow$, respectively, by means of

$$O(3) \ni R \mapsto \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right] \in \mathcal{L}, \tag{8.25}$$

this map being injective and continuous. In particular the matrices defined in (8.18) fall within this case.

***Remark* 8.3.3.** The elements of $SO(3)$ correspond to the *rotations* of the Euclidean space.

Notice that the Lie algebra of $SO(3)$ and $O(3)$ are equal to each other, the former being a open subgroup of the latter. Let us determine its form. A matrix A belongs to $\mathfrak{o}(3)$ if and only if $e^{sA} \in O(3)$ for every $s \in \mathbb{R}$. Exploiting the definition of $SO(3)$, this is equivalent to

$$e^{-sA} = (e^{sA})^{-1} = (e^{sA})^t = e^{sA^t}$$

which is equivalent to $A^t = -A$.

Notice that any matrix of this kind can be expressed as a linear combinations of the matrices (8.18), when understood as matrices on $\mathbb{R}^3$ (just forget the 0-th and 4-th dimension). With a litte abuse of language we will use the same notation when referring to these matrices on the different spaces. In conclusion we have the following result.

***Proposition* 8.3.4.** *The Lie algebra of $O(3)$ is given by*

$$\mathfrak{o}(3) = \{A \in M(3,\mathbb{R}) \mid A^t = -A\} \tag{8.26}$$

and coincides with the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$. Moreover it is three-dimensional and a basis is given by the generators L_i.

8.4 The Double Covering of the Lorentz and Poincaré groups

Even though the Poincaré group is the actual group of space-time symmetries (if one does not include gravitational effects), during the development of quantum mechanics it turned out that it does not suffice to fully describe elementary physical systems. In fact, in order to encompass particles with half-integer spin a more general group is required, which essentially consists in substituting the rotation subgroup $SO(3)$ with its double covering $SU(2)$. Let us see this in detail.

***Definition* 8.4.1.** *The unitary group $U(2)$ is the group of matrices $U \in M(2,\mathbb{C})$ such that $U^*U = I$. It has the following properties:*

(a) it defines a closed subgroup of $GL(2,\mathbb{C})$, in particular it is a matrix Lie group;

(b) it is closed under the Hermitean adjoint operation, more precisely $U^* = U^{-1}$;

(c) its element satisfy $\det U \in \mathbb{S}^1$.

Again, specializing to the matrices with unit determinant we get another group, denoted by $SU(2)$.

***Proposition* 8.4.2.** *The subset $SU(2)$ is a closed subgroup of $U(2)$, in particular it is a matrix Lie group. It is called the special unitary group.*

Attention: In this case, $SU(2)$ is not an open subset of $U(2)$ in that it is not the inverse image of an open subset. Differentely from the $O(3)$ case, the image of the determinant function on $U(2)$ ranges over the whole circle $\mathbb{S}^1 \subset \mathbb{C}$. Thus every open neighbourhood of $1 \in \mathbb{C}$ contains the image throught the continuous function det of at least one element of $U(2) \setminus SU(2)$.

So, let us determine the expression of their Lie algebras.

An element $X \in M(2,\mathbb{C})$ belongs to the Lie algebra of $U(2)$ if and only if

$$e^{-sX} = (e^{sX})^{-1} = (e^{sX})^* = e^{sX^*}, \tag{8.27}$$

which is equivalent to $X^* = -X$. Consider the matrices of $M(2, \mathbb{C})$ defined by

$$\begin{aligned} &\mathsf{S}_0 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathsf{S}_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathsf{S}_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &\mathsf{S}_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{8.28}$$

Clearly, these elements belong to the Lie algebra of $U(2)$. Moreover they are (real-)linearly independent, more precisely they provide a *basis* of the Lie algebra of $U(2)$, i.e. every element can be written as a *real*-linear combination of these matrices.

The Lie algebra of $SU(2)$ can be derived similarly. In addition to condition (8.27) we impose also

$$1 = \det e^{sX} = e^{s trX},$$

which provides the further condition $trX = 0$. It follows easily that a basis for the Lie algebra of $SU(2)$ is provided by the matrices S_i with $i = 1, 2, 3$. These are the well-known *Pauli matrices*.

***Proposition* 8.4.3.** *The Lie algebra of $U(2)$ is given by*

$$\mathfrak{u}(2) = \{X \in M(2, \mathbb{C}) \mid X^* = -X\}. \tag{8.29}$$

It is four-dimensional and a basis is given by the matrices S_μ with $\mu = 0, 1, 2, 3$.

The Lie algebra of $SU(2)$ is given by

$$\mathfrak{su}(2) = \{X \in M(2, \mathbb{C}) \mid X^* = -X,\ trX = 0\}. \tag{8.30}$$

It is three-dimensional and a basis os given by the matrices S_i, with $i = 1, 2, 3$.

The three matrices S_i satisfy the well-known commutation relations

$$[\mathsf{S}_i, \mathsf{S}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathsf{S}_k \quad i, j = 1, 2, 3. \tag{8.31}$$

Comparing the Lie algebras of $SO(3)$ and $SU(2)$, together with the commutation relations of their generators L_i and S_i, it is evident that the function

$$\psi : \mathfrak{su}(2) \ni \sum_{i=1}^{3} c_i\mathsf{S}_i \mapsto \sum_{i=1}^{3} c_i\mathsf{L}_i \in \mathfrak{so}(3). \tag{8.32}$$

is an isomorphism of Lie algebras. Exponentiating this map a surjective homomorphism of groups between $SU(2)$ and $SO(3)$ can be realised, the well-known *double covering map*, let us see this.

Since $\mathfrak{su}(2)$ is a three-dimensional linear space generated by the matrices S_i, the function

$$\omega : \mathbb{R}^3 \ni \mathbf{x} \mapsto \sum_{i=1}^{3} x_i \mathsf{S}_i \in \mathfrak{su}(2) \tag{8.33}$$

is a real-linear isomorphism. Moreover, notice that for any fixed $U \in SU(2)$, the function

$$\mathfrak{su}(2) \ni X \mapsto UXU^* \in \mathfrak{su}(2) \tag{8.34}$$

is a well-defined real-linear isomorphism. At this point the following result can be proved.

***Theorem* 8.4.4.** *There exists a surjective homomorphism $\Psi : SU(2) \to SO(3)$ such that for all $U \in SU(2)$ the following diagram commutes*

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\omega"] \arrow[d, "\Psi(U)"'] & \mathfrak{su}(2) \arrow[d, "U\cdot U^*"] \\ \mathbb{R}^3 \arrow[r, "\omega"] & \mathfrak{su}(2) \end{tikzcd} \tag{8.35}$$

Moreover $N(\Psi) = \{\pm I\}$ and $\Psi(e^X) = e^{\psi(X)}$ for all $X \in \mathfrak{su}(2)$ where ψ is defined in (8.32). This is called the double covering of $SO(3)$.

***Remark* 8.4.5.** The term *double* is appropriate. Indeed the condition $N(\Psi) = \{\pm I\}$ can be interpreted, roughly speaking, as $SU(2)$ consisting in two copies of $SO(3)$.

A similar result holds for the Lorentz and Poincaré groups which follows from the fact that both $\mathcal{L}_+^\uparrow$ and $\mathcal{P}_+^\uparrow$ includes $SO(3)$ as Lie subgroup. Exploiting the above homomorphism we are able to define a double covering also for the Lorentz and Poincaré groups. We need to introduce another important matrix Lie group

***Definition* 8.4.6.** *The special linear group $SL(2,\mathbb{C})$ is the group of matrices $A \in M(2,\mathbb{C})$ such that* $\det A = 1$. *It satisfies the following properties:*

(a) it defines a closed subgroup of $GL(2,\mathbb{C})$ and so it defines a matrix Lie group;

(b) it is closed under the Hermitean adjoint operation;

(c) it contains $SU(2)$ as a subgroup .

Let us discuss the form of its Lie algebra. A matrix $X \in M(2,\mathbb{C})$ belongs to the Lie algebra of $SL(2,\mathbb{C})$ if and only if

$$1 = \det e^{sX} = e^{s\,trX} \text{ for all } s \in \mathbb{R},$$

thus if and only if $trX = 0$. Consider the matrices of $M(2,\mathbb{C})$ defined by $\mathsf{N}_i := -i\mathsf{S}_i$ for $i = 1,2,3$:

$$\mathsf{N}_1 := -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathsf{N}_2 := -\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathsf{N}_3 := -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.36}$$

These elements, together with the matrices S_i belong to the Lie algebra of $SL(2,\mathbb{C})$ and, evidently, are (real-)linearly independent. More precisely it can be proved that every element of this Lie algebra can be written as a *real* combination of these matrices, i.e. they provide a *basis* of the Lie algebra of $SL(2,\mathbb{C})$.

***Proposition* 8.4.7.** *The Lie algebra of $SL(2,\mathbb{C})$ is given by*

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in M(2,\mathbb{C}) \mid trX = 0\}. \tag{8.37}$$

It is six-dimensional and it is generated by the matrices $\mathsf{S}_i, \mathsf{N}_i$ for $i = 1,2,3$.

By direct inspection the following commutation relations can be proved to fulfill

$$[\mathsf{S}_i, \mathsf{N}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathsf{N}_k, \quad [\mathsf{N}_i, \mathsf{N}_j] = -\sum_{k=1}^{3} \varepsilon_{ijk}\mathsf{S}_k. \tag{8.38}$$

Comparing the Lie algebra structures of $\mathfrak{su}(2,\mathbb{C})$ and $\mathfrak{l}$, it is immediate to see that the function

$$\phi : \mathfrak{sl}(2,\mathbb{C}) \ni \sum_{i=1}^{3} c_i\mathsf{S}_i + \sum_{i=1}^{3} d_i\mathsf{N}_i \mapsto \sum_{i=1}^{3} c_i\mathsf{L}_i + \sum_{i=1}^{3} d_i\mathsf{K}_i \in \mathfrak{l} \tag{8.39}$$

is an isomorphism of Lie algebras. As we did in the case of $SU(2)$ and $SO(2)$ this map can be exponentiated providing a surjective homomorphism between $SL(2,\mathbb{C})$ and $\mathcal{L}_+^\uparrow$, the *double covering map* of $SL(2,\mathbb{C})$.

Consider the Lie algebra $\mathfrak{u}(2)$. It is a four-dimensional real-linear space, generated by the matrices S_μ, $\mu = 0,1,2,3$. Thus the function

$$\Omega : \mathbb{R}^4 \ni x \mapsto \sum_{\mu=0}^{3} x_\mu\mathsf{S}_\mu \in \mathfrak{u}(2) \tag{8.40}$$

is a real-linear isomorphism. Moreover, notice that for any fixed $A \in SL(2,\mathbb{C})$, the function

$$\mathfrak{u}(2) \ni X \mapsto AXA^* \in \mathfrak{u}(2) \tag{8.41}$$

is a well-defined real-linear isomorphism. At this point the following result can be proved

***Theorem* 8.4.8.** *There exists a surjective homomorphism $\Phi : SL(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ such that for all $A \in SL(2,\mathbb{C})$ the following diagram commutes*

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, "\Omega"] \arrow[d, "\Phi(A)"'] & \mathfrak{u}(2) \arrow[d, "A\cdot A^*"] \\ \mathbb{R}^4 \arrow[r, "\Omega"] & \mathfrak{u}(2) \end{tikzcd} \tag{8.42}$$

Moreover $N(\Phi) = \{\pm I\}$ and $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{sl}(2,\mathbb{C})$, where ϕ is defined in (8.39). This is called the double covering of $\mathcal{L}_+^\uparrow$.

***Remark* 8.4.9.** Consider $A \in SL(2,\mathbb{C})$ and $a \in \mathbb{R}^4$. Exploiting the isomorphism (8.40) and the properties of the maps Ω and Φ, then the Poincaré action of $(\Phi(A), a)$ on $\mathbb{R}^4$:

$$\mathbb{R}^4 \ni x \mapsto \Phi(A)x + a \in \mathbb{R}^4 \tag{8.43}$$

is equivalent to the action of $(\Phi(A), \Omega(a))$ on $\mathfrak{u}(2)$:

$$\mathfrak{u}(2) \ni X \mapsto AXA^* + \Omega(a). \tag{8.44}$$

At this point everything can be extended to the Poincaré group $\mathcal{P}_+^\uparrow$. Consider the set of couples

$$(A, a) \in SL(2,\mathbb{C}) \times \mathbb{R}^4$$

and equip it with the product

$$(A', a')(A, a) = (A'A, \Phi(A')a + a'). \tag{8.45}$$

This is a group, with inverse and identity elements given respectively by

$$(A, a)^{-1} = (A^{-1}, -\Phi(A)^{-1}a) \quad \text{and} \quad (I, 0). \tag{8.46}$$

***Theorem* 8.4.10.** *The set $SL(2,\mathbb{C}) \times \mathbb{R}^4$ equipped with the product (8.45) is a Lie group, denoted by $SL(2,\mathbb{C}) \rtimes \mathbb{R}^4$, or $\widetilde{\mathcal{P}}_+^\uparrow$.*

As happens for $\mathcal{L}_+^\uparrow$ and $\mathcal{P}_+^\uparrow$ the Lie algebra of $SL(2,\mathbb{C}) \rtimes \mathbb{R}^4$ turns out to be generated by the generators of $\mathfrak{sl}(2,\mathbb{C})$ plus the generators of $\mathbb{R}^4$. More precisely:

***Proposition* 8.4.11.** *The Lie algebra of $SL(2,\mathbb{C}) \rtimes \mathbb{R}^4$ is ten-dimensional and is isomorphic to $\mathfrak{p}$.*

Again, exponentiation this isomorphism we get a double covering of the corresponding Lie groups, which acts trivially as follows

***Theorem* 8.4.12.** *The map $\tilde{\Phi} : SL(2,\mathbb{C}) \rtimes \mathbb{R}^4 \to \mathcal{P}_+^\uparrow$ defined for all $(A, a) \in SL(2,\mathbb{C}) \rtimes \mathbb{R}^4$ by*

$$\tilde{\Phi}((A, a)) := (\Phi(A), a) \tag{8.47}$$

is a surjective homomorphism with kernel $N(\tilde{\Phi}) = \{\pm I\}$. This is called the double covering of $\mathcal{P}_+^\uparrow$.

The existence of these double-covering maps is of great importance because of the following proposition.

***Proposition* 8.4.13.** *The following statements hold:*

(a) The Lie groups $SO(3)$, $\mathcal{L}_+^\uparrow$ and $\mathcal{P}_+^\uparrow$ are not simply connected;

(b) The Lie groups $SU(2)$, $SL(2,\mathbb{C})$ and $SL(2,\mathbb{C}) \rtimes \mathbb{R}^4$ are simply connected.

This becomes useful when dealing with projective unitary representations $G \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ on Hilbert spaces (see Definition 2.2.31). The simple connection of the Lie group plays a fundamental role in the process of removing the multipliers of U. Thus, in case G is not simply connected, a chance is given by composing the projective representation with the double covering map and work on the covering group.

Chapter 9

Elementary Relativistic Systems

9.1 General Formalism

In Chapter 5 we saw how any elementary physical system is described in terms of its lattice of elementary observables. Thanks to the results of Piron and Solèr we saw that (under suitable technical hypotheses) any one of these lattices turns out to be isomorphic to the lattice of the orthogonal projectors of some real, complex or quaterinionic Hilbert space H.

Despite its power, this approach is still debated and not fully accepted by the community, for it relies on some assumptions which are quite strong and difficult to justify on a purely physical point of view (see for example the axiom of covering law or the ad hoc hypotheses of Solèr Theorem). Nevertheless it gives a deep insight on the weird and elusive nature of the quantum world.

Even though several assumptions may seem to be deliberately taken, the pillars of the theory which are the result of decades of both mathematical and physical analysis, are pretty solid. For example, for evey physical system we accept the existence of a complete orthocomplented lattice $\mathfrak{L}$ describing its set of elementary observables, together with its logic interpretation given in Chapter 5. In particular, given any property $P \in \mathfrak{L}$, we require that the subset $\mathfrak{L}_P$ describes the lattice of the system once the property P has been measured, found positive and kept true (see Section 5.1). As stated in Proposition 5.2.13, this is equivalent to requiring the orthomodularity of the lattice $\mathfrak{L}$. Finally, in full generality, states will be considered as σ-probability measures over $\mathfrak{L}$.

Moreover, we relieve this structure of the duty of carrying the entire information about the system, leaving open the possibility that other mathematical structures are needed to fully characterise the system.

So, the theses of the Piron and Solèr theorems may be seen just as a hint on the nature of the lattice of elementary propositions. Fixing a real, complex or quaternionic Hilbert space H, the requirements listed so far are fulfilled by the projector lattice of any von Neumann algebra over H, as guaranteed by Proposition 6.3.3. As shown in the same result, this lattice is not generally capable of describing the entire von Neumann which we may suppose to be associated with the system as well. This is in line with the assumption that the lattice is not necessarily able to fully characterise the system. Coherently with this and the interpretations of Chapter 5 we give the following:

***Assumption* 9.1.1** (Observables algebra and states)**.** *With every physical system a von Neumann algebra $\mathfrak{M}$ over an infinite-dimensional separable real, complex or quaternionic Hilbert space H is associated. Moreover*

(a) the observables - i.e. the measurable quantities - of the system are in one-to-one correspondence with the self-adjoint operators on H whose PVM belongs $\mathfrak{M}$;

(b) for any observable A the possible outcomes - i.e. the values that A may attain when measured - are in one-to-one correspondence with the elements of the spectrum $\sigma(A)$;

(c) the quantum propositions (or properties, statements, ...) of the system - i.e. the questions that may be asked about it - are in one-to-one correspondence with the elements of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$;

(d) the states of the system are in one-to-one correspondence with the σ-probability measures μ over $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$.

The set of the self-adjoint elements of $\mathfrak{M}$ will be denoted by $\mathfrak{M}_h$ and the set of its states by $\mathfrak{S}$.

***Assumption* 9.1.2** (Lüders-von Neumann post measurement axiom)**.** *If the measurement of the property $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ on the state $\mu \in \mathfrak{S}$ has a positive outcome, then the state immediately after the measurement is given by*

$$\mu_P(\cdot) := \frac{\mu(P \cdot P)}{\mu(P)}.$$

***Assumption* 9.1.3** (Compatible propositions)**.** *Two observables that can be measured simultaneously are said to be compatible and are represented by strongly-commuting self-adjoint operators. In particular, the properties belonging to $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'$ are called the classical properties of the system.*

***Remark* 9.1.4.** We will sometimes refer to the von Neumann algebra $\mathfrak{M}$ as the *physical system* itself.

These assumptions require some clarifications. Given a physical system, the von Neumann algebra and its relative properties are defined *regardless* the chosen reference frame. When dealing with different reference frames the only difference will concern the physical interpretation or meaning of the elements of $\mathfrak{M}, \mathfrak{M}_h, \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ and $\mathfrak{S}$. More precisely we assume that $\mathfrak{M}_h$ contains all the (bounded) observables for the system, $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ all the elementary observables of the system and $\mathfrak{S}$ all the states of the system, but the physical meaning of all of them depends on the chosen reference frame. For example, if the operator P denotes the elementary observable *"the momentum of the* *system falls within* $[a, b]$*"* with respect to a given reference frame, from the point of view of another frame the same property may be interpreted as *"the momentum of the system falls within* $[a + \delta, b + \delta]$*"*.

Regardless the various interpretations, the general structure must be the same and this will concern the entire structure, not only the lattice of elementary observables. So, suppose $\mathcal{I}$ and $\mathcal{I}'$ are two reference frames connected by a Poincaré transformation g, then there must exist some bijective function $h_g : \mathfrak{M} \to \mathfrak{M}$ which sends bijectively observables to observables, elementary observables to elementary observables and such that the correspondence $\mu \to \mu(h_g(\cdot))$ sends bijectively states to states.

The best way to do this is by requiring the map h_g to be linear (or anti-linear in the complex case), to preserves the Hermitean adjoint and the product between operators. In general we can define:

***Definition* 9.1.5.** *A weak symmetry of the system is a* C^**-algebra automorphism (or possibly anti-automorphism in the complex case)* $h : \mathfrak{M} \to \mathfrak{M}$*.*

Such a symmetry reduces to an automorphism of lattices when restricted to $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, in this way being in line with the vision of quantum logic.

***Proposition* 9.1.6.** *If* h *is a weak symmetry for the system, then its restriction to* $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ *is an automorphism of complete orthocomplemented lattices.*

Proof. For the sake of simplicity we will omit the subscript g from h_g in this proof. First notice that if $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ then $h(P) \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. Indeed $h(P)^* = h(P^*) = h(P)$ and $h(P)h(P) = h(PP) = h(P)$. Now, suppose $P \leq Q$, which is equivalent to $PQ = P$. Thus $h(P) = h(PQ) = h(P)h(Q)$, which, in turns, is equivalent to $h(P) \leq h(Q)$. Now we have $h(P^\perp) = h(I - P) = h(I) - h(P) = I - h(P) = h(P)^\perp$. All the calculation carried out so far holds also for the inverse map h^{-1}. It remains to prove that h preserves the join and the meet of any subset of elements of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. So, take $(P_i)_{i\in\mathcal{I}} \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, then of course $h(\wedge_{i\in\mathcal{I}} P_i) \leq h(P_j)$ for any $j \in \mathcal{I}$. Suppose there exists some $R \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ such that $R \leq h(P_j)$ for any $j \in \mathcal{I}$, then applying h^{-1} to both the inequalities we get $h^{-1}(R) \leq P_j$ for any $j \in \mathcal{I}$, i.e. $h^{-1}(R) \leq \wedge_{i\in\mathcal{I}} P_i$. Finally, applying h to both sides we get $R \leq h(\wedge_{i\in\mathcal{I}} P_i)$. By the definition of infimum we see that $\wedge_{i\in\mathcal{I}} h(P_i) = h(\wedge_{i\in\mathcal{A}} P_i)$. The De Morgan laws conclude the proof. $\square$

Given a physical system $\mathfrak{M}$, the objects $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ carry the same physical information, in that each one of them can be derived (uniquely) from the other as follows from Proposition 6.3.3, point (g). Thus it is natural to suppose that two given symmetries coincide on $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ if and only if they coincide on $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$. Since $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$ equals the (weak-) strong-closure of the unital *-algebra generated by the elements of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, such a condition can be achieved by assuming that the symmetry h is continuous with respect to the (weak) strong topology of $\mathfrak{M}$.

***Definition* 9.1.7.** *A strongly-continuous weak symmetry is called a symmetry. Their set is denoted by* $Sym(\mathfrak{M})$*.*

***Proposition* 9.1.8.** *Let $h, h' \in Sym(\mathfrak{M})$ be of the same type (both linear or both anti-linear) and such that $h(P) = h'(P)$ for all $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, then they coincide on the von Neumann algebra $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$.*

Proof. Consider the von Neumann algebra $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' \subset \mathfrak{M}$ and the unital *-algebra $\langle \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})\rangle$ generated by the orthogonal projectors of $\mathfrak{M}$, i.e. the algebra containing the linear combinations and products of elements of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. Proposition 6.1.14 assures that $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' = \overline{\langle \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})\rangle}^s$. Thus, if h, h' are strongly-continuous, of the same type and coincide on $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, they coincide on the entire generated algebra. $\square$

***Remark* 9.1.9.** *How discussed in Chapter 6, if the space H is real or quaternionic, then the von Neumann algebra $\mathfrak{M}$ generally does not coincide with the one generated by the orthogonal projectors:*

$$\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' \subsetneq \mathfrak{M}.$$

In order to get the equality we have to add to the generating set the family

$$\mathfrak{I}_{\mathfrak{M}} = \{J \in \mathfrak{M} \mid J^* = -J,\ -JJ \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})\}.$$

More precisely we have

$$\mathfrak{M} = \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' + \mathfrak{I}_{\mathfrak{M}}\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''.$$

This implies that, even if two symmetries are supposed to coincide on the von Neumann algebra $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$, they may be different on the remaining operators. More precisely they may assume different values when evaluated on $\mathfrak{I}_{\mathfrak{M}}$.

A particularly important example of symmetries is provided by the so-called *Wigner symmetries.*

***Definition* 9.1.10.** *A symmetry $h : \mathfrak{M} \to \mathfrak{M}$ is said to be a Wigner symmetry if*

$$h(A) = UAU^{-1} \quad \textit{for all} \quad A \in \mathfrak{M} \tag{9.1}$$

for some unitary (or possibly anti-unitary in the complex case) operator on $\mathfrak{B}(\mathsf{H})$.

***Remark* 9.1.11.** Some remarks on this definition.

(a) In the complex case, h is linear or anti-linear if U is, respectively, linear or anti-linear.

(b) In general, given a von Neumann algebra $\mathfrak{M} \subsetneq \mathfrak{B}(\mathsf{H})$ and a (anti-)unitary operator U on H, then it is not guaranteed that $UAU^{-1} \in \mathfrak{M}$ if $A \in \mathfrak{M}$. A sufficient condition is $U \in \mathfrak{M}$.

(c) It is immediate to see that, if well-defined, the function $U \cdot U^{-1}$ truly defines a weak symmetry in the sense of Definition 9.1.5 and that it is strongly-continuous.

(d) If $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ on some real, complex or quaternionic Hilbert space, then Theorem 5.4.9 and Proposition 9.1.6 assure that any two different operators U as in (9.1) differ by a phase: $e^{i\phi}$ in the complex case, ± 1 in the real and quaternionic cases.

In our situation we are given a family of symmetries $\mathcal{P}_+^\uparrow \ni g \to h_g$. It is natural to assume that the group structure of the Poincaré group is preserved in this action, i.e. that the function $g \mapsto h_g$ is a group homomorphism: $h_e = id$ and $h_{fg} = h_f \circ h_g$ for any $g, f \in \mathcal{P}_+^\uparrow$.

Another important feature we can impose on h is to preserve the continuous nature of $\mathcal{P}_+^\uparrow$ in some preferably weak and operational manner. A physically reasonable way to achieve this is to assume that what changes continuously is the probability $\mu(h_g(P))$ when some state μ and property P are fixed.

We extend these assumptions to the wider *universal covering group* $\widetilde{\mathcal{P}}_+^\uparrow$. This is always possible, thanks to Theorem 8.4.12, by composing the homomorphism over $\mathcal{P}_+^\uparrow$ with the covering map $\widetilde{\Phi}$.

***Assumption* 9.1.12** (Poincaré invariance)**.** *The system is Poincaré invariant, i.e. there exists a group homomorphism*

$$\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto h_g \in Sym(\mathfrak{M})$$

such that, for every state $\mu \in \mathfrak{S}$ and property $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ the function

$$\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto \mu(h_g(P)) \in [0,1]$$

is continuous. Finally, the one-parameter representation

$$\mathbb{R} \ni t \mapsto h_{\exp(-ct\mathsf{P}_0)} \in Sym(\mathfrak{M}) \tag{9.2}$$

represents the time-evolution when restricted to the family of (bounded) observables.

***Proposition* 9.1.13.** *Referring to the homomorphism h of Assumption 9.1.12, each representative h_g is linear.*

Proof. The thesis follows immediately from the fact that $\widetilde{\mathcal{P}}_+^\uparrow$ is connected. The connectedness assures that every element of the group can be written as a finite product of elements of the type $\exp(t\mathsf{A})$. Each one of these, in turn, can be rewritten as $\exp(t/2\mathsf{A})\exp(t/2\mathsf{A})$. Since h is a group representation we have that $h_{\exp(t\mathsf{A})} = h_{\exp(t/2\mathsf{A})} \circ h_{\exp(t/2\mathsf{A})}$. So, whether $h_{\exp(t/2\mathsf{A})}$ is linear or anti-linear, its squared $h_{\exp(t\mathsf{A})}$ must be linear. $\square$

Two useful definitions follow. The first one concerns the existence of some properties in $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ which are *classical*, in the sense that they commute with every other element of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ (see Assumption 9.1.3):

$$P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})' \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$$

and which are left invariant under the action of the Poincaré group:

$$h_g(P) = P \quad \text{for all } g \in \widetilde{\mathcal{P}}_+^{\uparrow}.$$

The objects must behave like *classical labels* of the system as they assume the same value with respect to any reference frame (think of the electric charge or the spin for instance). As we will see later on, a particularly important subcase is given by the elements of $\mathfrak{M}' \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ $(\subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})' \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}))$, which we refer to as the *superselection projectors* of the system.

***Definition* 9.1.14.** *A superselection projector of the system is an element $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$ such that $h_g(P) = P$ for all $g \in \widetilde{\mathcal{P}}_+^{\uparrow}$.*

A more detailed discussion on these operators, with a focus on the consequences of their existence, will be carried out in the section on relativistic elementary systems.

The second definition concerns the existence of physical quantities which *do not change in time*. In this sense they consist in converved quantities, or *constants of motion*:

***Definition* 9.1.15.** *An observable A is said to be a costant of motion if*

$$h_{\exp(-tc\mathsf{P}_0)}(P^{(A)}(E)) = P^{(A)}(E)$$

for all $t \in \mathbb{R}$ and $E \in \mathfrak{B}(\mathbb{R})$, where $P^{(A)} \in \mathfrak{M}$ is the PVM associated with A.

***Definition* 9.1.16.** *A relativistic system is a couple $(\mathfrak{M}, h)$ satisfying Assumptions 9.1.1, 9.1.2, 9.1.3 and 9.1.12.*

Of course the system could support other symmetries than the Poincaré one. In general we can give the following definition:

***Definition* 9.1.17.** *Let $\mathfrak{M}$ be a physical system and G a connected Lie group. A group symmetry is a group representation*

$$G \ni r \mapsto s_r \in Sym(\mathfrak{M}).$$

It is said to be continuous if, for every state $\mu \in \mathfrak{S}$ and property $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, the function

$$G \ni r \mapsto \mu(s_r(P)) \in [0,1]$$

is continuous. The symmetry is said to be dynamical if

$$s_r \circ h_{\exp(-tc\mathsf{P}_0)} = h_{\exp(-tc\mathsf{P}_0)} \circ s_r$$

for every $r \in G$ and $t \in \mathbb{R}$.

Reasoning as in Proposition 9.1.13 the following result follows directly from the connectedness of the group.

***Proposition* 9.1.18.** *The representatives of any group symmetry are linear.*

A particularly important role is played by those group symmetries which are generated in terms of Wigner symmetries. More on this will be discussed later, for the moment being we state the following result.

***Proposition* 9.1.19.** *Suppose that $\mathfrak{M}$ is irreducible and G is a connected Lie group, then every strongly-continuous unitary representation*

$$G \ni g \mapsto U_g \in \mathfrak{M}$$

gives rise to a continuous group symmetry in terms of Wigner symmetries.

The proof follows immediately from Theorem 6.7.1 and the following Lemma.

***Lemma* 9.1.20.** *Let H be a real, complex or quaternionic Hilbert space, $T \in \mathfrak{D}(\mathsf{H})$ and $G \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ a strongly-continuous unitary representation, then*

$$\lim_{g \to e} tr(TU_gPU_g^*) = tr(TP)$$

for any choice of $P \in \mathfrak{L}(\mathsf{H})$.

Proof. Take $P \in \mathfrak{L}(\mathsf{H})$ and write $T = \sqrt{T}\sqrt{T}$ which is always possible, the operator T being positive and self-adjoint. Let N be a Hilbert basis for H, then

$$\begin{aligned} tr(TU_gPU_g^*) &= tr(\sqrt{T}\sqrt{T}U_gPU_g^*) = tr(\sqrt{T}U_gPU_g^*\sqrt{T}) = \\ &= \sum_{z\in\mathsf{N}} \Re[(z|\sqrt{T}U_gPU_g^*\sqrt{T}z)] = \sum_{z\in\mathsf{N}} \Re[(U_g^*\sqrt{T}z|PU_g^*\sqrt{T}z)]. \end{aligned} \tag{9.3}$$

Now, define the positive functions $w, f_g : \mathsf{N} \to \mathbb{R}$ by $w(z) := \Re[(z|Tz)]$ and by $f_g(z) := \Re[(U_g^*\sqrt{T}z|PU_g^*\sqrt{T}z)]$. Of course $w, f_g \in \mathcal{L}^1(\mathsf{N}, \mathbb{R}, \nu)$, where ν is the counting measure. Moreover, exploiting the strong-continuity of U and $P \leq I$ we have

$$\begin{aligned} & f_g(z) = \Re[(U_g^*\sqrt{T}z|PU_g^*\sqrt{T}z)] \to \Re[(\sqrt{T}z|P\sqrt{T}z)] = f_e(z), \\ & f_g(z) = \Re[(U_g^*\sqrt{T}z|PU_g^*\sqrt{T}z)] = (U_g^*\sqrt{T}z|PU_g^*\sqrt{T}z) \leq (U_g^*\sqrt{T}z|U_g^*\sqrt{T}z) = \\ & \qquad = (\sqrt{T}z|\sqrt{T}z) = (z|Tz) = \Re[(z|Tz)] = w(z). \end{aligned} \tag{9.4}$$

Thus we can apply the Lebesgue dominated convergence theorem and get

$$\underbrace{\sum_{z\in\mathsf{N}} \Re[(U_g^*\sqrt{T}z|PU_g^*\sqrt{T}z)]}_{tr(TU_gPU_g^*)} = \int_{\mathsf{N}} f_g(z)\,d\nu \to \int_{\mathsf{N}} f_e(z)\,d\nu = \underbrace{\sum_{z\in\mathsf{N}} \Re[(\sqrt{T}z|P\sqrt{T}z)]}_{tr(TP)}, \tag{9.5}$$

concluding the proof. □

Projective Unitary Representations

A natural question concerns the possibility of realising a given continuous group symmetry in terms of some suitable strongly-continuous unitary representation as in Proposition 9.1.19. A partial but crucial results is the following.

Consider an irreducible von Neumann algebra $\mathfrak{M}$ on some infinite-dimensional separable real, complex or quaternionic Hilbert space H and take any group symmetry $s : G \mapsto Sym(\mathfrak{M})$ defined on some connected Lie group G. We already know from Proposition 9.1.18 that each s_g is a *linear automorphism* on $\mathfrak{M}$

Now, Proposition 9.1.6 and Theorem 6.8.2 assure that $s_g(P) = U_g P U_g^{-1}$ for any $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ for some operator U_g on H. In the complex Hilbert space case this operator may be unitary (thus belong to $\mathfrak{M}$) or anti-unitary (thus not belong to $\mathfrak{M}$). In the real or quaternionic Hilbert space cases it is surely unitary, but while for a real or quaternionic type commutant it always belongs to $\mathfrak{M}$, for a complex-type commutant it may either commute or anti-commute with J (thus belong or not belong to $\mathfrak{M}$, respectively) (see the conditions of Theorem 6.8.2). Moreover if V_g is another similar operator such that $s_g(P) = V_g P V_g^{-1}$ for any $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, then U_g, V_g are of the same kind and differ by an element of $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$.

So, take $p, q \in G$, then since

$$(U_p U_q) P (U_p U_q)^{-1} = U_p U_q P U_q^{-1} U_p^{-1} = s_p \circ s_q(P) = s_{pq}(P) = U_{pq} P U_{pq}^{-1},$$

thanks to what just pointed out, $U_p U_q$ and U_{pq} are of the same kind and $U_p U_q = \Omega(p,q) U_{pq}$ for some $\Omega(p,q) \in \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$. Again, the connectedness of the group implies the existence of a neighbourhood A_e of the identity which is made of elements of the type $\exp(t\mathsf{A})$ with $t \in \mathbb{R}$ and $\mathsf{A} \in \mathfrak{g}$. Also, any other element is given as a finite product of elements of this neighbourhood. In particular we have for any $g = \exp(t\mathsf{A})$ that $g = \exp(t/2\mathsf{A}) \exp(t/2\mathsf{A})$ and so $U_g = \chi (U_{\exp(t/2\mathsf{A})})^2$ for some $\chi \in \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$. Whatever the type of $U_{\exp(t/2\mathsf{A})}$ is, its square must be a unitary element of $\mathfrak{M}$. As a consequence, since any other $g \in G$ can be written as a finite product of elements of A_e, we see that any operator U_g must be a *unitary element of* $\mathfrak{M}$.

Summing up we have just found what is generally called a *projective unitary representation.*

***Definition* 9.1.21.** *Let $\mathfrak{M}$ be an irreducible von Neumann algebra over a real, complex or quaternionic Hilbert space and G a connected Lie group. A projective unitary representation is a map $G \ni g \mapsto U_g \in \mathfrak{M}$ such that each U_g is unitary and*

$$U_g U_h = \Omega(g,h) U_{gh} \quad \textit{for all } g, h \in G \tag{9.6}$$

for some function $\Omega : G \times G \mapsto \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$ which is called the multiplier of the representation.

To conclude, we want to show that $s_g(A) = U_g A U_g^*$ actually holds for every $A \in \mathfrak{M}$, not only for the elements of the lattice $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. Suppose first that H is complex or that H is real or quaternionic and $\mathfrak{M}' \cong \mathbb{R}$. In all these cases it holds that $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ and so also $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$. In particular $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'' = \mathfrak{L}(\mathsf{H})'' = \mathfrak{B}(\mathsf{H}) = \mathfrak{M}$ (see Proposition

5.3.3). At this point, since both s_g and $U_g \cdot U_g^*$ define *linear* automorphisms on $\mathfrak{B}(\mathsf{H})$ and they coincide on the lattice $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$, Proposition 9.1.8 gives the thesis.

So, suppose that H is real or quaternionic and $\mathfrak{M}' \cong \mathbb{C}$ or $\mathfrak{M}' \cong \mathbb{H}$. In both cases we know from Theorem 6.6.2 that $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$. Exploiting the first isomorphism we see that s_g translates into the following *linear* automorphism

$$(s_g)_{\mathfrak{J}} : \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}) \ni A_{\mathfrak{J}} \mapsto s_g(A)_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}). \tag{9.7}$$

Proposition 9.1.6 assures that its restriction to $\mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$ is a (complete orthocomplemented) lattice automorphism.

Similarly, since $U_g \in \mathfrak{M}$ we can consider the unitary operator $(U_g)_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ and consider the automorphism

$$\mathfrak{B}(\mathsf{H}_{\mathfrak{J}}) \ni A_{\mathfrak{J}} \mapsto (U_g)_{\mathfrak{J}} A_{\mathfrak{J}} (U_g)^*_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}). \tag{9.8}$$

Again, its restriction to $\mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$ gives rise to a (complete orthocomplemented) lattice automorphism.

Now, notice that for any $P_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$, it holds that

$$(U_g)_{\mathfrak{J}} P_{\mathfrak{J}} (U_g)^*_{\mathfrak{J}} = (U_g P U_g^*)_{\mathfrak{J}} = (s_g(P))_{\mathfrak{J}} = (s_g)_{\mathfrak{J}}(P_{\mathfrak{J}}). \tag{9.9}$$

Since the isomorphism $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ is (in both directions) strongly-continuous (see Theorem 6.6.2) the map $(s_g)_{\mathfrak{J}}$ turns out to be strongly-continuous as well as s_g. Putting all together and exploiting again Proposition 9.1.8 we see that $(s_g)_{\mathfrak{J}}(A_{\mathfrak{J}}) = (U_g)_{\mathfrak{J}} A_{\mathfrak{J}} (U_g)^*_{\mathfrak{J}}$ for any $A_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$. This equality lifts immediately to the algebra $\mathfrak{M}$, proving the thesis.

In summary we have just proved the following result.

***Proposition* 9.1.22.** *Let $\mathfrak{M}$ be an irreducible von Neumann algebra over a real, complex or quaternionic Hilbert space, G a connected Lie group and $G \ni g \mapsto s_r \in Sym(\mathfrak{M}))$ a group symmetry, then there exists a projective unitary representation $G \ni g \mapsto U_g \in \mathfrak{M}$ such that*

$$s_g(A) = U_g A U_g^* \quad \textit{for all } A \in \mathfrak{B}(\mathsf{H}).$$

The existence of these phases $\Omega(g,h)$ prevent the function $g \mapsto U_g$ from being a proper group representation. Moreover, for any function $\chi : G \mapsto \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$, the map $g \mapsto \chi(g) U_g$ generates s as well and its multipliers are given by

$$\Omega_\chi(g,h) = \chi(g)\chi(h)\chi(gh)^{-1}\Omega(p,q). \tag{9.10}$$

Now the question is: *is there a function χ such that $\Omega_\chi = I$ identically? so that we can get rid of the multipliers?* It turns out that under suitable hypotheses on G and assuming some properties on U this can be done. In particular this holds for the *universal covering of the Poincaré group*. The first proof of this was provided by Bargmann in 1954 in the framework of complex Hilbert spaces (see [3]). Basing upon this, in 1963 Emch succeeded in extending this result to the quaternionic setting (see [12]). The solution for the real case is given here, and is based upon some techical results that can be found in [24] and [31].

Let us see this in detail. So, suppose that $G = \widetilde{\mathcal{P}}_+^\uparrow$ and that s is also continuous, i.e. $g \mapsto \mu(s_g(P))$ is continuous for every state μ and property P. Again, if H is complex or if H is real or quaternionic and $\mathfrak{M}' \cong \mathbb{R}$, then $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$. The states on $\mathfrak{L}(\mathsf{H})$ reduce to the ones given by $tr(T\,\cdot)$ with $T \in \mathfrak{D}(\mathsf{H})$ as stated by Theorem 4.4.17. In particular consider the states for which $T = P_u$ (the projector on the one-dimensional subspace generated by u) and take $P = P_v \in \mathfrak{L}(\mathsf{H})$ for some unit-norm vectors u, v. It holds that

$$\mu(s_g(P)) = tr(P_u U_g P_v U_g^*) = (u|U_g P_v U_g^* u) = (u|U_g v)(v|U_g^* u) = |(u|U_g v)|^2.$$

If H is real or quaternionic and $\mathfrak{M}' \cong \mathbb{C}$ or $\mathfrak{M}' \cong \mathbb{H}$, then $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_\mathfrak{J})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_\mathfrak{J})$. By means of the latter isomorphism, any state $\mu : \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \to [0,1]$ defines a σ-probability measure on $\mathfrak{L}(\mathsf{H}_\mathfrak{J})$ by

$$\mu_\mathfrak{J} : \mathfrak{L}(\mathsf{H}_\mathfrak{J}) \ni P_\mathfrak{J} \mapsto \mu(P) \in [0,1]. \tag{9.11}$$

The fact that this actually defines a σ-probability measure can be proved easily. Exploiting (9.7), (9.8) and (9.9), the continuity condition on μ translates into the continuity of the map $g \mapsto \mu_\mathfrak{J}((s_g)_\mathfrak{J}(P_\mathfrak{J}))$. At this point Theorem 4.4.17 applies, stating that any σ-probability measure on $\mathfrak{L}(\mathsf{H}_\mathfrak{J})$ is given by $tr(A\,\cdot)$ for some $A \in \mathfrak{D}(\mathsf{H}_\mathfrak{J})$. Thus, we can repeat the argument done above and take $A = P_x$ and $P_\mathfrak{J} = P_y \in \mathfrak{L}(\mathsf{H}_\mathfrak{J})$ with $x, y \in \mathsf{H}_\mathfrak{J}$ and prove that

$$\mu_\mathfrak{J}((s_g)_\mathfrak{J}(P_\mathfrak{J})) = tr(P_x (U_g)_\mathfrak{J} P_y (U_g)_\mathfrak{J}^*) = |(x|(U_g)_\mathfrak{J} y)_\mathfrak{J}|^2. \tag{9.12}$$

Of course, notice that the map $g \mapsto (U_g)_\mathfrak{J}$ defines a projective unitary representation also on $\mathsf{H}_\mathfrak{J}$. So, in all cases we fall within the hypotheses of the following important Lemma.

***Lemma* 9.1.23** (Bargmann, Emch, Pontrjagin)**.** *Let $U : \widetilde{\mathcal{P}}_+^\uparrow \to \mathfrak{B}(\mathsf{H})$ be a projective unitary representation over an infinite-dimensional separable real, complex or quaternionic Hilbert space H such that*

$$G \ni g \mapsto |(u|U_g v)|^2 \in \mathbb{R} \tag{9.13}$$

is continuous for all $u, v \in \mathsf{H}$. Then there exists a function $\chi : g \mapsto \mathfrak{U}(\mathbb{F}_c)$ such that the map

$$G \ni g \mapsto W_g := \chi_g U_g \in \mathfrak{B}(\mathsf{H}) \tag{9.14}$$

is a strongly-continuous unitary representation of G.

Proof. The proof for the complex case can be found in [3] or [24] (see Theorem 12.72 and Remarks 12.73 therein), while the quaternionic case can be found in [12]. So, let us discuss the real case. We affirm that there always exists an equivalent representative $\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto V_g$ (necessarily made of unitary operators as assured by Theorem 5.4.9) such that $V_e = I$, it is strongly continuous over an open neighborhood of the identity A_e and its multiplier $(g,h) \mapsto \omega(g,h)$ is continuous over $A'_e \times A'_e$ with $A'_e \subset A_e$, a smaller open neighborhood of e which can always be assumed to be connected ($\mathcal{P}$ is a Lie group and as such it is locally connected). The proof of this fact can be found within the proof of Proposition 12.44 in [24] which is valid for both complex and real Hilbert spaces since there is no distinctive role played by the imaginary unit. Since in the real case $\mathfrak{U}(\mathbb{F}_c) = \{\pm 1\}$ (which is not connected if equipped with the topology induced by $\mathbb{R}$), $\omega(g,h) \in \mathfrak{U}(\mathbb{F}_c)$ and $\omega(e,e) = (V_e)^2 V_e^{-1} = V_e = I$, the continuity of ω guarantees that $\omega(g,h) = I$ for every $g,h \in A'_e$. In other words $V_g V_h = V_{gh}$ for every $g,h \in A'_e$. The group $\mathfrak{U}(\mathsf{H})$ of unitary operators over H is a topological group with respect to the strong topology, thus the continuous function $G \ni g \mapsto V_g \in \mathfrak{U}(\mathsf{H})$ is a local homomorphism of topological groups according to Definition B, Chapter 8, Par.47 of [31]. Since, as established in [32], $\mathfrak{U}(\mathsf{H})$ is connected, the space H being infinite-dimensional and the group $\widehat{\mathcal{P}}_+^\uparrow$ simply connected, we can apply Theorem 63 of [31] proving that there exists a strongly-continuous unitary representation $G \ni g \mapsto W_g \in \mathfrak{U}(\mathsf{H})$ such that $W_g = V_g$ on some open neighborhood of the identity $A''_e \subset A'_e$. To conclude, we observe that since the Lie group G is connected, every $g \in \mathcal{P}$ can be written as $g = g_1 \cdots g_n$ for some $g_1, \ldots, g_n \in A''_e$. So, $W_g = W_{g_1} \cdots W_{g_n} = V_{g_1} \cdots V_{g_n} = \chi'_g V_g = \chi_g U_g$ for some phases $\chi, \chi' \in \mathfrak{U}(\mathbb{F}_c)$. □

So, let us go back to our continuous group symmetry $s : \widetilde{\mathcal{P}}_+^\uparrow \to Sym(\mathsf{H})$ and apply the above lemma to the projective unitary representation U (when H is complex, or real and quaternionic with $\mathfrak{M}' \cong \mathbb{R}$) or $U_{\mathfrak{J}}$ (in the remaining cases). Exploiting the (strongly-continuous) isomorphism $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ where needed, the following corollary follows immediately

***Theorem* 9.1.24.** *Let $\mathfrak{M}$ be an irreducible von Neumann algebra over an infinite-dimensional separable real, complex or quaternionic Hilbert space and $s : \widetilde{\mathcal{P}}_+^\uparrow \to Sym(\mathfrak{M})$ a continuous group symmetry, then there exists a strongly-continuous unitary representation $\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto W_g \in \mathfrak{M}$ such that*

$$s_g(A) = W_g A W_g^* \quad \textit{for all } A \in \mathfrak{M}. \tag{9.15}$$

The existence of a *strongly-continuous unitary representation* generating a given continuous symmetry is of remarkable importance, in that it allows us to apply Stone's Theorem and define the anti-self-adjoint *generators* of the one-parameter subgroups $t \mapsto U_{\exp(t\mathsf{A})}$ (where A belongs to the Lie algebra of G). This turns out to be crucial in the last section.

Gauge Symmetries

In this last paragraph we discuss briefly the concept of a *gauge symmetry*.

Consider a physical system with von Neumann algebra $\mathfrak{M}$ and let $U \in \mathfrak{U}(\mathfrak{M}')$ be any unitary operator in the commutant. This gives rise to a trivial Wigner symmetry

$$UAU^* = A \quad \text{for all} \quad A \in \mathfrak{M}$$

and as such, the action of U may appear not to be interesting from a physical point of view. Nevertheless the existence of these unitary operators in the commutant is a significant consequence of the nature of the von Neumann algebra $\mathfrak{M}$. Their existence introduces some degree of redundance in the actual description. Consider for instance any unit-norm vector $v \in \mathsf{H}$, then for any $U \in \mathfrak{U}(\mathfrak{M}')$ it holds that

$$tr(P_v Q) = (v|Qv) = (Uv|UQv) = (Uv|QUv) = tr(P_{Uv} Q) \quad \text{for all } Q \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$$

(where P_x is the projector over the one-dimensional subspace generated by $x \in \mathsf{H}$). Thus there is no way to distinguish between the vectors v and Uv: the states defined by the trace-class operators P_u, P_{Uv} (which do not necessarily belong to $\mathfrak{M}$, see Remark 6.7.2) are equal to each other.

In this sense the unitary operators of $\mathfrak{M}'$ define a new type of symmetry, which is not to be confused with the one introduced in Definition 9.1.7: a symmetry as a *redundancy in the Hilbert space description of the system*. These are called *gauge symmetries*.

***Definition* 9.1.25.** *The commutant $\mathfrak{M}'$ is referred to as the gauge algebra of the system. Its unitary group is called the gauge group.*

As shown in Proposition 6.3.3, while in the complex Hilbert space the projectors generate the algebra, in the case of a real or quaternionic Hilbert space the lattice of projectors is not able to completely determine the structure of its von Neumann algebra. In particular this applies to the commutant $\mathfrak{M}'$:

$$\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})'' \subsetneq \mathfrak{M}' \quad \text{and} \quad \mathfrak{M}' = \mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})'' + \mathfrak{J}_{\mathfrak{M}'}\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})''.$$

We see that $\mathfrak{M}'$ is determined by $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})''$ *up to* elements of $\mathfrak{J}_{\mathfrak{M}'}$, which is made of the anti-self-adjoint partial isometries whose square belong to $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})$.

We give the following definition:

***Definition* 9.1.26.** *The von Neumann algebra $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})''$ is referred to as the restricted gauge algebra of the system. Its unitary group $\mathfrak{U}(\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})'')$ is called the restricted gauge group.*

***Remark* 9.1.27.** As proved in Proposition 6.1.9, differentely from the case of the logic of projectors, the unitary operators of a von Neumann algebra are able to reconstruct the entire algebra. Thus there is no loss of information in focusing our attention to the unitary groups of $\mathfrak{M}'$ and $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})''$, rather then to the algebras themselves.

In the standard treatment of quantum theories, the Poincaré group and the gauge group $\mathfrak{U}(\mathfrak{M}')$ play two distinct but equally important roles in characterising the physical system. Its total physical symmetry group involves the action of both of them. In the next section, where we focus on the specific case of Poincaré group as *maximal symmetry group*, some restrictions on the gauge group becomes necessary.

Complete Sets of Commuting Observables (CSCO)

Even if less significant, another important concept in quantum theories is the idea of *complete sets of commuting observables*.

The idea of these objects traces back to *Dirac's Principle*. Driven by operational needs, in his analysis of quantum theories, Dirac discussed the necessity for the existence of a *finite* family of compatible observables whose simultaneous measure would be able - by means of the collapse of the state - to provide the experimenter with a uniquely defined physical state.

An equivalent statement consists in assuming that there exists a *finite* family of compatible observables which is *maximal*, in the sense that any self-adjoint operator commuting with all of them is necessarily a function of them.

Let us try to make this more appealing from a mathematical point of view. So, consider a family of (generally unbounded) compatible observables $\{A_1, \ldots, A_n\}$, i.e. such that their PVM commute with each other. This is equivalent to requiring that the operators are strongly-commuting with each others.

As seen in Theorem 2.2.61 we can (uniquely) define a joint PVM $P : \mathcal{B}(\mathbb{R}^n) \to \mathfrak{L}(\mathsf{H})$ such that

$$P(\Delta_1 \times \cdots \times \Delta_n) = P_1(\Delta_1) \cdots P_n(\Delta_n)$$

for any Borel subsets $\Delta_n \in \mathcal{B}(\mathbb{R})$.

***Remark* 9.1.28.** For any fixed $\Delta \in \mathcal{B}(\mathbb{R}^n)$, the physical interpretation of $P(\Delta)$ is: *"a simultaneous measurement of $A_1, \ldots, A_n$ gives an element of Δ"*.

The possible simultaneous outcomes are naturally given by the elements of

$$\sigma_j(A_1, \ldots, A_n) := suppP. \tag{9.16}$$

Consider the set $\mathcal{A} \subset \mathfrak{M}$ containing all the projectors $P(\Delta)$, with $\Delta \in \mathcal{B}(\mathbb{R}^n)$. This is clearly Abelian and made of self-adjoint operators. We say that $\mathcal{A}$ is *maximal* if it satisfies

$$\mathcal{A}' \cap \mathfrak{L}(\mathsf{H}) \subset \mathcal{A}''$$

i.e. if any projector commuting with all the elements of $\mathcal{A}$ belongs to (at least) the von Neumann algebra generated by $\mathcal{A}$ (this resemble the idea that any self-adjoint operator commuting with the A_i is function of them). The von Neumann algebra $\mathfrak{A} := \mathcal{A}''$ is Abelian (see Proposition 6.1.14) and from $\mathcal{A}' = (\mathcal{A}'')'$ it also follows that $\mathfrak{A}' \cap \mathfrak{L}(\mathsf{H}) \subset \mathfrak{A}$.

Notice that, differentely from the complex case, on real and quaternionic Hilbert spaces the algebra $\mathfrak{A}$ is still made of self-adjoint operators (see Proposition 6.3.5). Thus in particular we have $\mathfrak{U} = \mathfrak{L}_{\mathfrak{U}}(\mathsf{H})''$ in all cases (see Proposition 6.3.3 for the complex case and Proposition 6.3.5 for the real and quaternionic ones). We can give the following definition

***Definition* 9.1.29.** *Let $\mathfrak{M}$ be a von Neumann algebra. A complete set of commuting observables (CSCO) of $\mathfrak{M}$ is a von Neumann subalgebra $\mathfrak{A} \subset \mathfrak{M}$ such that*

(a) (self-adjoint) $\mathfrak{A} = \mathfrak{L}_{\mathfrak{A}}(\mathsf{H})''$,

(b) (Abelian) $\mathfrak{A} \subset \mathfrak{A}'$,

(c) (maximal) $\mathfrak{A}' \cap \mathfrak{L}(\mathsf{H}) \subset \mathfrak{A}$.

***Remark* 9.1.30.** As a consequence of points (a) and (b), in the real and quaternionic cases such an algebra $\mathfrak{A}$ is necessarily made of self-adjoint operators. Indeed point (b) implies that $\mathfrak{L}_{\mathfrak{A}}(\mathsf{H})$ is made of *commuting* self-adjoint operators. Proposition 6.3.5 assures that its generated von Neumann algebra $\mathfrak{L}_{\mathfrak{A}}(\mathsf{H})''$ is made of self-adjoint operators as well. Point (a) concludes the proof.

We have the following characterisation.

***Theorem* 9.1.31.** *Let $\mathfrak{M}$ be a von Neumann algebra. If there exists a CSCO $\mathfrak{A}$, then the following statements hold:*

(a) $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})' \cap \mathfrak{L}(\mathsf{H}) \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$,

(b) $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) \subset \mathfrak{M}$,

(c) $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})$ is a Boolean algebra.

Proof. Suppose that $\mathfrak{M}$ admits a CSCO $\mathfrak{A}$ and take $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})' \cap \mathfrak{L}(\mathsf{H})$. Since $\mathfrak{A} \subset \mathfrak{M}$ we have $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})' \cap \mathfrak{L}(\mathsf{H}) \subset \mathfrak{L}_{\mathfrak{A}}(\mathsf{H})' \cap \mathfrak{L}(\mathsf{H}) = \mathfrak{U}' \cap \mathfrak{L}(\mathsf{H}) \subset \mathfrak{U} \subset \mathfrak{M}$. In particular $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})' \cap \mathfrak{L}(\mathsf{H}) \subset \mathfrak{M} \cap \mathfrak{L}(\mathsf{H}) = \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$, thus point (a). Point (b) immediately follows from point (a) by noticing that $\mathfrak{M}' \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})'$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \mathfrak{M}$. Now suppose that point (b) is fulfilled, then $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) \subset \mathfrak{M}' \cap \mathfrak{M}$ and so the elements of $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})$ must commute with each other, i.e. the lattice is Boolean. $\square$

***Remark* 9.1.32.** The key property in Definition 9.1.29 is point (c). Indeed it can be proved that a von Neumann subalgebra $\mathfrak{A} \subset \mathfrak{M}$ satisfying points (a) and (b) and which is maximal *with respect to* $\mathfrak{M}$, i.e. $\mathfrak{A}' \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \mathfrak{A}$ always exists, as a direct consequence of Zorn's lemma. The proof of this is a readaptation of the one for complex Hilbert spaces carried out in [40].

Proof. First we want to prove that there always exists a von Neumann subalgebra $\mathfrak{A} \subset \mathfrak{M}$ satisfying point (a) and point (b) of Definition 9.1.29 and $\mathfrak{A}' \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \mathfrak{A}$. Consider the class $\mathcal{A}$ of all the (not necessarily von Neumann) Abelian unital *-subalgebras of $\mathfrak{M}$ which are also made of self-adjoint operators if H is real or quaternionic. This class is clearly not empty, as, for instance, the trivial subset $\{aI, \mid a \in \mathbb{F}_c\}$ fulfills all the requirements. This family is partially ordered by inclusion. If $\mathcal{A}_s$ is any linearly ordered subclass of $\mathcal{A}$, then its union $\cup\mathcal{A}_s$ is still an Abelian unital *-subalgebra of $\mathfrak{M}$ which is made of self-adjoint operators if H is real or quaternionic and as such it is an upper bound for $\mathcal{A}_s$. Zorn's Lemma assures that $\mathcal{A}$ admits a maximal element $\mathfrak{A}_m$. Notice that $(\mathfrak{A}_m)''$ is still a unital *-subalgebra of $\mathfrak{M}$; moreover it is Abelian and also made of self-adjoint operators if H is real or quaternionic (see Proposition 6.1.14 and Proposition 6.3.5). Thus, since $(\mathfrak{A}_m)'' \supset \mathfrak{A}_m$, the maximality of $\mathfrak{A}_m$ implies $\mathfrak{A}_m = (\mathfrak{A}_m)''$, i.e. $\mathfrak{A}_m$ is a von Neumann subalgebra of $\mathfrak{M}$. In particular $\mathfrak{A}_m$ satisfies Point (a) and Point (b) (Proposition 6.3.3 and 6.3.5). It remains to prove that $(\mathfrak{A}_m)' \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \mathfrak{A}_m$. So, take any $P \in (\mathfrak{A}_m)' \cap \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ and consider the unital *-algebra $\langle \mathfrak{A}_m \cup \{P\}\rangle$. This is of course a unital *-*sub*algebra of $\mathfrak{M}$. Moreover, since P is self-adjoint and commutes with all the elements of $\mathfrak{A}_m$, then the algebra $\langle \mathfrak{A}_m \cup \{P\}\rangle$ is Abelian and also made of self-adjoint operators if H is real or quaternionic and so it must be $\langle \mathfrak{A}_m \cup \{P\}\rangle \in \mathcal{A}$. Since $\mathfrak{A}_m$ is a maximal element of $\mathcal{A}$ and $\mathfrak{A}_m \subset \langle \mathfrak{A}_m \cup \{P\}\rangle$, it must be $\mathfrak{A}_m = \langle \mathfrak{A}_m \cup \{P\}\rangle$, in particular $P \in \mathfrak{A}_m$, concluding the proof. □

One may wonder if such maximal algebras always exists. The answer is negative as the following proposition shows (see again [40]).

***Proposition* 9.1.33.** *If H is complex, then a CSCO exists if and only if $\mathfrak{M}' \subset \mathfrak{M}$.*

Proof. Suppose that a CSCO exists. Since H is complex, then point (c) implies $\mathfrak{A}' = \mathfrak{L}_{\mathfrak{A}'}(\mathsf{H})'' = (\mathfrak{A}' \cap \mathfrak{L}(\mathsf{H}))'' \subset \mathfrak{A}$. Thus we have $\mathfrak{M}' \subset \mathfrak{A}' \subset \mathfrak{A} \subset \mathfrak{M}$. Suppose on the contrary that $\mathfrak{M}' \subset \mathfrak{M}$ and take any $\mathfrak{A}$ maximal with respect to $\mathfrak{M}$ as in Remark 9.1.32. Exploiting $\mathfrak{M} = \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})''$, the maximality can be easily rewritten as $\mathfrak{A}' \cap \mathfrak{M} \subset \mathfrak{A}$. We want to prove that it is actually maximal with respect to $\mathfrak{B}(\mathsf{H})$, i.e. $\mathfrak{A}' \cap \mathfrak{L}(\mathsf{H}) \subset \mathfrak{A}$ or, equivalently, $\mathfrak{A}' = \mathfrak{A}' \cap \mathfrak{B}(\mathsf{H}) \subset \mathfrak{A}$. So, since $\mathfrak{M}' \subset \mathfrak{A}'$ and $\mathfrak{M}' \subset \mathfrak{M}$ we have $\mathfrak{M}' \subset \mathfrak{A}' \cap \mathfrak{M} \subset \mathfrak{A}$. This implies $\mathfrak{M}' \subset \mathfrak{A}$, and so $\mathfrak{A}' \subset \mathfrak{M}'' = \mathfrak{M}$. To conclude notice that $\mathfrak{A}' = \mathfrak{A}' \cap \mathfrak{M} \subset \mathfrak{A}$. □

***Remark* 9.1.34.** Some remarks follow.

(a) We see that not all the von Neumann algebras admit complete sets of commuting observables. In particular, in the complex case, a physical system admits a CSCO if and only if its gauge algebra is Abelian ($\mathfrak{M}' \subset \mathfrak{M} = (\mathfrak{M}')'$) or, equivalently, if and only if its gauge group is Abelian (see Propositions 6.1.14 and 6.2.2).

(b) In the case of complex Hilbert spaces, any von Neumann algebra $\mathfrak{A}$ satisfying points (a)-(c) in Definition 9.1.29 is known in the mathematical literature by the name of MASA (maximal Abelian subalgebra) of $\mathfrak{B}(\mathsf{H})$. So, the statement "$\mathfrak{M}$ *admits a CSCO*" can be restated as "$\mathfrak{M}$ *contains a MASA of* $\mathfrak{B}(\mathsf{H})$".

From now on we will focus on *elementary* systems with Poincaré group as *maximal symmetry group*. This will require further assumptions on the couple $(\mathfrak{M}, h)$ in order to mathematically characterise all this. As we will see, this will traduce into some irreducibility mathematical conditions.

9.2 Wigner Elementary Relativistic Systems

In the common theoretical framework over complex Hilbert spaces elementary relativistic particles with no non-trivial gauge symmetries are described by choosing as von Neumann algebra $\mathfrak{M}$ the whole $\mathfrak{B}(\mathsf{H})$ and as Poincaré action a locally-faithful irreducible strongly-continuous unitary representation U of the Poincaré group. This idea traces back to the fundamental classification carried out by Wigner in the late thirties [46] and other important works like the analysis of Bargmann on projective unitary representations [3]. Of course such a definition fulfills all the requirements we gave above for a relativistic system as we will see shortly.

An important remark concerns the relation between the von Neumann algebra and the Poincaré representation. Indeed notice that the irreducibility of U together with Schur's lemma assures that

$$\{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\}'' = \mathfrak{B}(\mathsf{H}) = \mathfrak{M}. \tag{9.17}$$

This can be interpreted as the fact that the *Poincaré group fully characterises the system*, in that it generates the algebra of observable and thus all the derived physical objects. This is in line with our idea of $\widetilde{\mathcal{P}}_+^\uparrow$ as the maximal symmetry group of the system.

In this first part of the discussion we try a naive approach, by merely taking the complex standard theory and exporting it as it is to the more general case of a Hilbert space with arbitrary division algebra. This is not a compelling way to face the issue of real and quaternionic relativistic systems, in that we are just applying blindly the axioms of the complex theory, without caring about their validity in these exotic frameworks. Be aware that the definition we are giving next in the complex case is the result of several physical and mathematical considerations which may apply only on complex Hilbert spaces.

In any case, despite the strength of these assumptions, this is a valuable way to define an elementary relativistic system and, most of all, it will reveal itself as an essential tool when we will discuss a more solid approach to the problem later on.

9.2.1 General Description

Bearing in mind the previous discussion we can go ahead with our naive approach and give the following definition

***Definition* 9.2.1.** *A real, complex or quaternionic Wigner elementary relativistic system (WERS) is a relativistic system* $(\mathfrak{M}, h)$ *such that*

$$\mathfrak{M} = \{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\}'' \quad \textit{and} \quad h_g = U_g \cdot U_g^* \ \ \textit{for all } g \in \widetilde{\mathcal{P}}_+^\uparrow$$

where

$$U : \widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$$

is a strongly-continuous unitary representation of the (universal covering of the) proper orthochronus Poincaré group over an infinite-dimensional separable real, complex or quaternionic Hilbert space, respectively, which is locally-faithful and irreducible.

***Remark* 9.2.2.** Notice that $\mathfrak{M}$ is irreducible because of the irreducibility of U.

This definition is coherent with the mathematical definition of a relativistic system given in the previous section. Indeed notice that, as pointed out in Remark 9.1.11, h is a group representation in terms of automorphisms of $\mathfrak{M}$ and that $g \mapsto \mu(h_g(P))$ is continuous for any state μ and proposition P as follows from the irreducibility of $\mathfrak{M}$, Theorem 6.7.1 and Proposition 9.1.20.

Let us fix an inertial reference frame in Minkowski spacetime $\mathbb{M}^4$. Basing upon this choice, the canonical covariant basis of $\mathfrak{p}$ and its commutation relations as given in Chapter 8, Propositions 8.2.4 and (8.22) refer to the axes of this reference frame.

Exploiting the strong-continuity of U and Stone's Theorem we have the following set of strongly-continuous one-parameter groups:

$$U_{\exp(t\mathsf{P}_\mu)} = e^{t\mathcal{P}_\mu},\ U_{\exp(t\mathsf{L}_i)} = e^{t\mathcal{L}_i},\ U_{\exp(t\mathsf{K}_i)} = e^{t\mathcal{K}_i}. \tag{9.18}$$

for some anti-self-adjoint generators $\mathcal{P}_\mu, \mathcal{L}_i, \mathcal{K}_i$ with $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$.

In Chapter 7 we saw how every strongly-continuous unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ gives rise to an associated Lie algebra representation u of the corresponding Lie algebra $\mathfrak{p}$ in terms of essentially anti-self-adjoint operators on the Gårding domain $D_G^{(U)}$. In particular we have the following result.

***Definition* 9.2.3.** *The basic anti-self-adjoint generators are given by*

$$\mathcal{P}_\mu := \overline{u(\mathsf{P}_\mu)}, \quad \mathcal{L}_i := \overline{u(\mathsf{L}_i)}, \quad \mathcal{K}_i := \overline{u(\mathsf{K}_i)} \tag{9.19}$$

where $\mu = 0, 1, 2, 3$ *and* $i = 1, 2, 3$*. They satisfy the following commutation rules:*

$$\begin{aligned} &[\mathcal{P}_\mu, \mathcal{P}_\nu] = [\mathcal{L}_i, \mathcal{P}_0] = 0, \\ &[\mathcal{L}_i, \mathcal{L}_j] = \varepsilon_{ijk}\mathcal{L}_k, \quad [\mathcal{L}_i, \mathcal{P}_j] = \varepsilon_{ijk}\mathcal{P}_k, \quad [\mathcal{L}_i, \mathcal{K}_j] = \varepsilon_{ijk}\mathcal{K}_k, \\ &[\mathcal{K}_i, \mathcal{K}_j] = -\varepsilon_{ijk}\mathcal{L}_k, \\ &[\mathcal{K}_i, \mathcal{P}_0] = -\mathcal{P}_i \quad [\mathcal{K}_i, \mathcal{P}_j] = -\delta_{ij}\mathcal{P}_0. \end{aligned} \tag{9.20}$$

The proof of this is a direct consequence of (8.22) and the fact that u is a Lie-algebra representation.

The Noether Principle

In the standard formulation of quantum mechanics on complex Hilbert spaces a very important role is played by the so-called *Noether Theorem.*

Consider a complex WERS $(\mathfrak{M}, h)$ and a continuous *dynamical* group symmetry generated by a strongly-continuous unitary representation $\mathbb{R} \ni s \mapsto V_s \in \mathfrak{B}(\mathsf{H})$ (check Proposition 9.1.19). Stone's Theorem guarantees that $V_s = e^{s\mathcal{A}}$ for some (uniquely defined) anti-self-adjoint operator $\mathcal{A}$ on H. It can be proved (see Theorem 13.5 of [24]) that, since the symmetry is dynamical:

$$e^{-t\mathcal{P}_0}e^{s\mathcal{A}} = e^{s\mathcal{A}}e^{-t\mathcal{P}_0} \quad \text{for all } t, s \in \mathbb{R} \tag{9.21}$$

from which, applying Stone's Theorem, we get

$$e^{-t\mathcal{P}_0}\mathcal{A}e^{t\mathcal{P}_0} = \mathcal{A} \quad \text{for all } t \in \mathbb{R}. \tag{9.22}$$

So, if we define the self-adjoint operator $A := i\mathcal{A}$ on H it follows that

$$e^{-t\mathcal{P}_0}Ae^{t\mathcal{P}_0} = A \quad \text{for all } t \in \mathbb{R}.$$

Theorem 4.2.3 assures that

$$h_{\exp(-t\mathsf{P}_0)}(P^{(A)}(\Delta)) = e^{-t\mathcal{P}_0}P^{(A)}(\Delta)e^{t\mathcal{P}_0} = P^{(A)}(\Delta), \tag{9.23}$$

which means that A is *constant of motion*. This is the content of *Noether's Theorem* which states that conserved quantities are associated with continuous dynamical group symmetries of the system.

Unfortunately this does not hold in the real and quaternionic cases. Indeed one may still manage to get to identity (9.22), but the procedure for getting a self-adjoint operator out of the anti-self-adjoint generator can no longer be carried out, for in the former case no imaginary units exist, while in the latter the operator $i\mathcal{A}$ is not well-defined (we are not given any preferred left-scalar multiplication L). A solution to this problem may be given by substituing the imaginary unit i by an imaginary operator J (see Definition 3.1.6) which *commutes* with the generators $\mathcal{A}$ and $\mathcal{P}_0$, i.e. $A := J\mathcal{A}$. Indeed in this case we would have (see Proposition 2.2.13)

$$A^* = (J\mathcal{A})^* = \mathcal{A}^*J^* = \mathcal{A}J = J\mathcal{A} = A$$

and the conservation follows from

$$e^{-t\mathcal{P}_0}Ae^{t\mathcal{P}_0} = e^{-t\mathcal{P}_0}J\mathcal{A}e^{t\mathcal{P}_0} = Je^{-t\mathcal{P}_0}\mathcal{A}e^{t\mathcal{P}_0} = J\mathcal{A} = A.$$

Of course the existence of such an operator is not guaranteed from the start. Moreover there could exist more than one of them, in which case the physical meaning of the association generator-observable would not be much clear. Finally, notice that such an imaginary operator may generally depend on the generator $\mathcal{A}$.

The ideal situation would be the existence of an imaginary operator J commuting with the entire algebra of observables $\mathfrak{M}$, even if this would necessary rule out the case $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ because of $\mathfrak{B}(\mathsf{H})' = \mathbb{R}I$ (see Proposition 6.3.4). Anyway notice that the maximality of the algebra is never assumed to be a necessary condition.

Even though there still is not a well-defined way to define an observable out of the anti-self-adjoint space-time generators $\mathcal{P}_\mu$, notice that the operator $\mathcal{P}_\mu^*\mathcal{P}_\mu = -(\mathcal{P}_\mu)^2$, which is self-adjoint, is an observable of the system, i.e. it is affiliated to the von Neumann algebra $\mathfrak{M}$. In order to prove this consider any $B \in \mathfrak{M}'$, then in particular $e^{t\mathcal{P}_\mu} B = Be^{t\mathcal{P}_\mu}$ for all $t \in \mathbb{R}$. Proposition 4.2.3 assures that $B\mathcal{P}_\mu \subset \mathcal{P}_\mu B$. In particular $B(\mathcal{P}_\mu)^2 \subset (\mathcal{P}_\mu)^2 B$, which, thanks again to Proposition 4.2.3 is equivalent to requiring that the PVM of $(\mathcal{P}_\mu)^2$ commutes with B. The arbitrariness of B implies that this PVM belongs to $\mathfrak{M}$, concluding the proof. Thus we may make the following assumption:

***Assumption* 9.2.4** (4-momentum square observable)**.** *The observable $\mathcal{P}_\mu^*\mathcal{P}_\mu$ represents the square of the μ-th component of the (covariant) 4-momentum of the system.*

Of course, once this has been taken for granted, then the following definition follows naturally.

***Definition* 9.2.5** (Squared-mass operator)**.** *Let U be a WERS on a real, complex or quaternionic Hilbert space and $\mathcal{P}_\mu$ defined as in (9.19), then the symmetric operator*

$$M_U^2 := \left(-(\mathcal{P}_0)^2 + \sum_{i=1}^{3} (\mathcal{P}_i)^2 \right) \Big|_{D_G^{(U)}} \tag{9.24}$$

is called the squared-mass operator associated with U.

***Remark* 9.2.6.** Of course in order to fully interpret M_U^2 as the squared-mass *observable* of the system we need to show that this operator is at least essentially self-adjoint on its domain. Once this is proved, then its affiliation with $\mathfrak{M}$ comes naturally from the same property shared by all the operators $\mathcal{P}_\mu^*\mathcal{P}_\mu$. This will be proved shortly.

Consider the generator of the time-displacements $\mathcal{P}_0$, thanks to the Polar Decomposition theorem this can be decomposed as $\mathcal{P}_0 = J_0|\mathcal{P}_0|$. The self-adjoint positive operator $|\mathcal{P}_0|$ is affiliated with the algebra $\mathfrak{M}$ and so represents an observable of the system. Indeed, take $B \in \mathfrak{M}'$, then in particular $e^{t\mathcal{P}_0} B = Be^{t\mathcal{P}_0}$ for all $t \in \mathbb{R}$. Proposition 4.2.3 then assures that $B|\mathcal{P}_0| \subset |\mathcal{P}_0|B$ and furthermore that B commutes with the PVM of $|\mathcal{P}_0|$, concluding the thesis. Coherently with the interpretation of $\mathcal{P}_\mu$ as related to the (covariant) 4-momentum of the particle (the correspondence still to be determined) and with Assumption 9.2.4, the observable $H := c|\mathcal{P}_0| = c\sqrt{\mathcal{P}_0^*\mathcal{P}_0}$ has the physical dimensions of energy. Moreover, since it is physically reasonable to assume the energy of an elementary relativistic free particle to be positive, we may make the following assumption.

***Assumption* 9.2.7.** *The observable $H := c|\mathcal{P}_0|$ represents the energy of the system.*

***Remark* 9.2.8.** Notice that this Assumption is indeed coherent with Assumption 9.2.4. Indeed

$$H^2 = c^2|\mathcal{P}_0|^2 = c^2\mathcal{P}_0^*\mathcal{P}_0$$

and $\mathcal{P}_0^*\mathcal{P}_0$ is the square of the 0-th component of the (covariant) 4-momentum.

9.2.2 Emergence of the Complex Structure

We are close to state and prove the main theorem of this work, but we need some techical lemmata first. In particular we prove the essential self-adjointness of M_U^2.

***Lemma* 9.2.9.** *Consider a WRES on a real, complex or quaternionic Hilbert space, then the following statements hold:*

(a) $e^{z\mathcal{K}_i}e^{a\mathcal{P}_0}e^{-z\mathcal{K}_i} = e^{a(\cosh z)\mathcal{P}_0}e^{-a(\sinh z)\mathcal{P}_i}$ *for all* $a, z \in \mathbb{R}$;

(b) $e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}x = (\cosh z)\mathcal{P}_0 x - (\sinh z)\mathcal{P}_i x$ *for all* $x \in D_G^{(U)}, z \in \mathbb{R}$.

Proof. Take $z, a \in \mathbb{R}$, then a straightforward calculation with the one-parameter subgroups $\mathbb{R} \ni s \mapsto \exp(s\mathsf{A})$ of $\mathcal{P}$ gives (see Chapter 8, Section 8.2):

$$\exp(z\mathsf{K}_i)\exp(a\mathsf{P}_0)\exp(-z\mathsf{K}_i) = \exp(a(\cosh z)\mathsf{P}_0)\exp(-a(\sinh z)\mathsf{P}_i)\ .$$

Applying the representation U to both the sides of this identity we have the first equality. Now, take $u, v \in D_G^{(U)}$. Since the Gårding domain is invariant under U, it is easy to see that

$$\begin{aligned}
\left(e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}v\middle|\, u\right) &= \frac{d}{da}\Big|_{a=0}\left(e^{z\mathcal{K}_i}e^{a\mathcal{P}_0}e^{-z\mathcal{K}_i}v\middle|\, u\right) = \\
&= \frac{d}{da}\Big|_{a=0}\left(e^{a(\cosh z)\mathcal{P}_0}e^{-a(\sinh z)\mathcal{P}_i}v\middle|\, u\right) = \\
&= \frac{d}{da}\Big|_{a=0}\left(e^{-a(\sinh z)\mathcal{P}_i}v\middle|\, e^{-a(\cosh z)\mathcal{P}_0}u\right) = \\
&= (-(\sinh z)\ \mathcal{P}_i v|\, u) + (v\,|-(\cosh z\mathcal{P}_0)u) = \\
&= (((\cosh z)\mathcal{P}_0 - (\sinh z)\mathcal{P}_i)v|\, u)\ .
\end{aligned}$$

Since $D_G^{(U)}$ is dense, we conclude the proof. □

Now, consider the element of the enveloping algebra of $\widetilde{\mathcal{P}}_+^\uparrow$ defined by

$$\mathsf{E} := -\mathsf{P}_0 \circ \mathsf{P}_0 + \sum_{i=1}^{3}\mathsf{P}_i \circ \mathsf{P}_i \in \mathcal{E}(\mathfrak{p}). \tag{9.25}$$

It is clear that this is a Casimir element of $\mathcal{E}(\mathfrak{g})$ and that $M_U^2 = u(\mathsf{E})$.

***Lemma* 9.2.10.** *The following statements hold:*

(a) $M_U^2 = \mu I|_{D_G^{(U)}}$ *for some* $\mu \in \mathbb{R}$;

(b) if $M_U^2 \geq 0$ *then* $N(\mathcal{P}_0) = \{0\}$.

In particular M_U^2 *is essentially self-adjoint.*

Proof. Let us start with (a). Take the element $\mathsf{E} \in \mathcal{E}(\mathfrak{p})$ defined in (9.25) with $u(\mathsf{E}) = M_U^2$. Exploiting the commutation relations in (8.22) it is easy to see that $[\mathsf{E}, \mathsf{A}] = 0$ for any $\mathsf{A} \in \mathfrak{p}$, hence in particular $[\mathsf{E}, \mathsf{N}] = 0$ for any Casimir element N of $\mathcal{E}(\mathfrak{p})$. Exploiting Theorem 7.2.7 and Proposition 7.3.7 we see that $M_U^2 = \mu I|_{D_G^{(U)}}$ for some real number μ. Now, let us prove (b) and take $x \in N(\mathcal{P}_0)$. Since $D_G^{(U)}$ is a core for $\mathcal{P}_0$, then there must exist a sequence $D_G^{(U)} \ni x_n \to x$ such that $\mathcal{P}_0 x_n \to \mathcal{P}_0 x = 0$. As a consequence, taking advantage of the definition of M_U^2,

$$\sum_{k=1}^{3} (\mathcal{P}_k x_n | \mathcal{P}_k x_n) = -\mu (x_n | x_n) + (\mathcal{P}_0 x_n | \mathcal{P}_0 x_n).$$

In the limit $n \to \infty$ the right-hand side converges to $-\mu \|x\|^2$, so that

$$\lim_{n \to \infty} \sum_{k=1}^{3} \|\mathcal{P}_k x_n\|^2 = -\mu \|x\|^2. \tag{9.26}$$

Since the right-hand side is non-positive whereas the left-hand side is non-negative, we conclude that $\lim_{n\to\infty} \sum_{i=1}^{3} \|\mathcal{P}_i x_n\|^2 = -\mu\|x\|^2 = 0$. If $\mu > 0$, we find $x = 0$ and thus $N(\mathcal{P}_0) = \{0\}$. So, suppose that $\mu = 0$. In this case (9.26) gives $\lim_{n\to\infty} \sum_{i=1}^{3} \|\mathcal{P}_i x_n\|^2 = 0$ and therefore $\lim_{n\to\infty} \|\mathcal{P}_i x_n\|^2 = 0$ for every $i = 1, 2, 3$. Since $\mathcal{P}_i$ is closed, $D(\mathcal{P}_i) \ni x_n \to x$ and $\mathcal{P}_i x_n \to 0$, we conclude that $N(\mathcal{P}_0) \subset D(\mathcal{P}_i)$ and, more precisely,

$$N(\mathcal{P}_0) \subset N(\mathcal{P}_i) \quad i = 1, 2, 3. \tag{9.27}$$

To go on observe that from Stone's theorem we get

$$N(\mathcal{P}_0) = \{x \in \mathsf{H} \mid e^{t\mathcal{P}_0} x = x \ \forall t \in \mathbb{R}\} = \{x \in \mathsf{H} \mid U_{\exp(t\mathsf{P}_0)} x = x \ \forall t \in \mathbb{R}\}. \tag{9.28}$$

Since P_0 commutes with the one-parameter groups generated by P_i and L_i, we get from (9.28) that $N(\mathcal{P}_0)$ is invariant under the corresponding one-parameter subgroups unitarily represented through U. However from (9.27), which immediately implies

$$e^{b\mathcal{P}_i} x = x \text{ for every } x \in N(\mathcal{P}_0) \text{ and } b \in \mathbb{R}, \tag{9.29}$$

we also conclude that $N(\mathcal{P}_0)$ is invariant under the unitary representation of the one-parameter group generated by every K_i. Indeed, from Lemma (9.2.9) and (9.29) we see that for any $x \in N(\mathcal{P}_0)$ it holds that

$$e^{a\mathcal{P}_0} e^{-z\mathcal{K}_i} x = e^{-z\mathcal{K}_i} e^{a(\cosh z)\mathcal{P}_0} e^{-a(\sinh z)\mathcal{P}_i} x = e^{-z\mathcal{K}_i} e^{a(\cosh z)\mathcal{P}_0} x = e^{-z\mathcal{K}_i} x \quad \forall z, a \in \mathbb{R},$$

hence $e^{z\mathcal{K}_i} x \in N(\mathcal{P}_0)$, in accordance with (9.28). Since $\widetilde{\mathcal{P}}_+^{\uparrow}$ is a connected Lie group, then every $g \in \widetilde{\mathcal{P}}_+^{\uparrow}$ is the product of a finite number of elements of the one-parameter subgroups generated by the vectors of any basis of $\mathfrak{p}$. Lifting this result to the Hilbert space H by means of the representation U, we conclude that the closed subspace $N(\mathcal{P}_0)$ is invariant under U. At this point, because U is irreducible, either $N(\mathcal{P}_0) = \mathsf{H}$ or $N(\mathcal{P}_0) = \{0\}$. In the first case we would have $\mathcal{P}_0 = 0$ (and more strongly $\mathcal{P}_i = 0$ for all $i = 1, 2, 3$ from (9.27)). In this case $U_{\exp(t\mathsf{P}_0)} = I$ for every $t \in \mathbb{R}$ and thus $\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto U_g$ would not be locally-faithful contrarily to the hypotheses on U. We conclude that $N(\mathcal{P}_0) = \{0\}$ also if $\mu = 0$. □

***Remark* 9.2.11.** Once the (essential) self-adjointness of M_U^2 has been proved the interpretation of the operator as the observable representing the squared mass of the system is complete. At this point the condition $M_U^2 \geq 0$ is obliged.

We are ready to state the main theorem.

***Theorem* 9.2.12.** *Let $(\mathfrak{M}, h)$ be a WERS on a real, complex or quaternionic Hilbert space, $\mathcal{P}_0 = J_0|\mathcal{P}_0|$ the polar decomposition of $\mathcal{P}_0$ and suppose that $M_U^2 \geq 0$, then the following statements hold:*

(i) *J_0 is an imaginary operator on H;*

(ii) *$J_0 \in \mathfrak{M} \cap \mathfrak{M}'$, in particular $\mathfrak{M}' = \{aI + bJ_0 \mid a, b \in \mathbb{R}\}$;*

(iii) *$J_0 U_g = U_g J_0$ for all $g \in \widetilde{\mathcal{P}}_+^\uparrow$, so in particular $J_0(D_G^{(U)}) \subset D_G^{(U)}$;*

(iv) *$J_0\overline{u(\mathsf{A})} = \overline{u(\mathsf{A})}J_0$, more precisely $J_0 u(\mathsf{A}) \subset u(\mathsf{A})J_0$ for all $\mathsf{A} \in \mathfrak{p}$;*

(v) *the operator $J_0\overline{u(\mathsf{A})}$ is self-adjoint and its PVM belongs to $\mathfrak{M}$, i.e. it is an observable of the system;*

(vi) *if J_1 is an imaginary operator such that either $J_1 U_g = U_g J_1$ for all $g \in \mathcal{P}$ or $J_1 u(\mathsf{A}) \subset u(\mathsf{A}) J_1$ for all $\mathsf{A} \in \mathfrak{p}$, then $J_1 = \pm J_0$;*

(vii) *if $\mathbb{F} = \mathbb{C}$, then $J_0 = \pm i$.*

Proof. Let us start with point (i). Thanks to Lemma 9.2.10 we know that $N(\mathcal{P}_0) = \{0\}$, hence Proposition 4.2.1 gives $N(J_0) = \{0\}$. Exploiting Proposition 4.2.3 we get that J_0 is an anti-self-adjoint unitary operator, i.e. an imaginary operator. Next step consits in proving that $J_0 \in \mathfrak{M}$. So, take $A \in \mathfrak{M}'$, then in particular $Ae^{t\mathcal{P}_0} = e^{t\mathcal{P}_0}A$ for all $t \in \mathbb{R}$. Proposition 4.2.3 implies that $A\mathcal{P}_0 \subset \mathcal{P}_0 A$ and so $AJ_0 = J_0 A$ in view of the same proposition. This means that $J_0 \in \mathfrak{M}'' = \mathfrak{M}$. At this point, if we manage to prove point (iii), then the definition of $\mathfrak{M}$ gives $J_0 \in \mathfrak{M}'$, which together with $J_0 \in \mathfrak{M}$ just proved gives $J_0 \in \mathfrak{M} \cap \mathfrak{M}'$, hence the first part of (ii). Still assuming (iii), if H is complex, then the irreducibility of U together with Schur's Lemma and the fact that J_0 is an imaginary operator immediately gives $J_0 = \pm i$ - thus point (vii) - and so also the second part of (ii) for the complex case - again from the irreducibility of $\mathfrak{M}$ and Schur's Lemma. If, instead, H is either real or quaternionic, then the second part of (ii) relative to these cases follows from Theorem 6.6.2, (i) and the first part of (ii) (again, notice that $\mathfrak{M}$ is irreducible, it including the irreducible representation U). Finally, notice that, thanks to Theorem 7.1.8 - (d), points (iii) and the two parts of (iv) are equivalent to each other. Concluding, if we manage to prove (iii) we get points (ii),(iv) and (vii) as a direct consequence.

So, let us prove point (iii). Let $\mathsf{A} \in \mathfrak{p}$ be any basis element such that $[\mathsf{A}, \mathsf{P}_0] = 0$. This can be translated in group terms as $\exp(t\mathsf{P}_0)\exp(s\mathsf{A}) = \exp(s\mathsf{A})\exp(t\mathsf{P}_0)$ for all $s, t \in \mathbb{R}$. At this point the action of U implies $e^{s\mathcal{A}}e^{t\mathcal{P}_0} = e^{t\mathcal{P}_0}e^{s\mathcal{A}}$ for all $s, t \in \mathbb{R}$, where $\mathcal{A}$ is the anti-self-adjoint generator associated with $s \mapsto U_{\exp(s\mathsf{A})}$. Let $\mathcal{A} = J_{\mathcal{A}}|\mathcal{A}|$ be the polar decomposition of $\mathcal{A}$, then Propositions 4.2.3 and 4.2.4 ensures that (1) $J_0 e^{s\mathcal{A}} = e^{s\mathcal{A}}J_0$, (2) $J_0\mathcal{A} = \mathcal{A}J_0$, (3) $J_0|\mathcal{A}| = |\mathcal{A}|J_0$, (4) $J_0\sqrt{|\mathcal{A}|} = \sqrt{|\mathcal{A}|}J_0$ and (5) $J_0 J_{\mathcal{A}} = J_{\mathcal{A}}J_0$. Notice in particular that, thanks to point (1), J_0 commutes with the one-parameter subgroups generated by $\mathsf{P}_0, \mathsf{P}_i, \mathsf{L}_i$. All these identities will be exploited shortly.

Now let us focus on the boost generators K_i, their associated one-parameter unitary subgroups and their anti-self-adjoint generators $\mathcal{K}_i$. We want to prove that, exactly as happens for the already discussed one-parameter subgroups, $J_0 e^{z\mathcal{K}_i} = e^{z\mathcal{K}_i}J_0$ for any $z \in \mathbb{R}$. If this is proved to hold, then the thesis follows easily from the connectedness of $\widetilde{\mathcal{P}}_+^\uparrow$. Indeed, since J_0 commutes with the unitary representations of the one-parameter subgroups generated by each element of the canonical basis, it therefore commutes with every U_g of the representation: $J_0 U_g = U_g J_0$ concluding the proof of (iii). So, let us go back to the proof of $J_0 e^{z\mathcal{K}_i} = e^{z\mathcal{K}_i}J_0$.

We assume first that H is complex. Observe that the polar decomposition of the closed operator $X := e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}$ is trivially constructed out of the polar decomposition of $\mathcal{P}_0$ and reads $X = [e^{z\mathcal{K}_i}J_0 e^{-z\mathcal{K}_i}][e^{z\mathcal{K}_i}|\mathcal{P}_0|e^{-z\mathcal{K}_i}]$ since the two factors satisfy the requirements listed in the polar decomposition theorem fixing the polar decomposition of X. However it also holds that $X = [J_0][-J_0 e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}]$, hence if we succeed in proving that also the couple $U := J_0$ and $B := -J_0 e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}$ satisfies the conditions of the polar decomoposition theorem and therefore defines another polar decomposition of X, then by uniqueness of the polar decomposition we get in particular that $J_0 = e^{z\mathcal{K}_i}J_0 e^{-z\mathcal{K}_i}$ which is the thesis. Item (1) is true by construction. Item (3) is trivial, since J_0 is unitary. Item (4) is equivalent to $N(B) = \{0\}$ which is immediate because $J_0, e^{\pm z\mathcal{K}_i}, \mathcal{P}_0$ are all injective. It remains to prove (2), i.e. that B is positive and self-adjoint.

Consider the generator of the space-displacements $\mathcal{P}_i = J_{\mathcal{P}_i}|\mathcal{P}_i|$, then it holds that $J_{\mathcal{P}_i}\sqrt{|\mathcal{P}_i|} \subset \sqrt{|\mathcal{P}_i|}J_{\mathcal{P}_i}$ thanks again to Proposition 4.2.3. Furthermore, since $[\mathsf{P}_i, \mathsf{P}_0] = 0$, the identities established in the first part of this proof hold for $\mathcal{A} = \mathcal{P}_i$. So, thanks to Lemma 9.2.9 we get for any $u \in D_G^{(U)}$.

$$\begin{aligned}(u|Bu) &= \big(u| - J_0 e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}u\big) = (u|(-J_0\mathcal{P}_0)u)\cosh z - (u|(-J_0\mathcal{P}_i)u)\sinh z = \\ &= \big((u||\mathcal{P}_0|u) - \big(u\big|(-J_0\mathcal{P}_i)u\big)\tanh z\big)\cosh z.\end{aligned} \tag{9.30}$$

Since $\cosh z > 0$ and $|\tanh z| < 1$ for every $z \in \mathbb{R}$, in order to prove that $(u|Bu) \geq 0$ for $u \in D_G^{(U)}$ it suffices to prove that $(u||\mathcal{P}_0|u) \geq |(u|(-J_0\mathcal{P}_i)u)|$. Define the operator $S := -J_0 J_{\mathcal{P}_i}$, which is clearly self-adjoint thanks to point (5) above and the anti-self-adjointness of both J_0 and $J_{\mathcal{P}_i}$. Since both of them are partial isometries it holds that

$$|(x|Sx)| = |(J_0x|J_{\mathcal{P}_i}x)| \leq \|J_0x\|\|J_{\mathcal{P}_i}x\| \leq \|x\|^2 = (x|x).$$

Now, thanks to point (4) above, we have $J_0J_{\mathcal{P}_i}\sqrt{|\mathcal{P}_i|} \subset J_0\sqrt{|\mathcal{P}_i|}J_{\mathcal{P}_i} = \sqrt{|\mathcal{P}_i|}J_0J_{\mathcal{P}_i}$ from which it follows that $S\sqrt{|\mathcal{P}_i|} \subset \sqrt{|\mathcal{P}_i|}S$. Now, take $u \in D_G^{(U)} \subset D(\mathcal{P}_i) = D(|\mathcal{P}_i|)$, then from $|\mathcal{P}_i| = \sqrt{|\mathcal{P}_i|}\sqrt{|\mathcal{P}_i|}$ we have in particular $u \in D(\sqrt{|\mathcal{P}_i|})$ and $\sqrt{|\mathcal{P}_i|}u \in D(\sqrt{|\mathcal{P}_i|})$. Thus

$$\begin{aligned}|(u| - J_0\mathcal{P}_iu)| &= |(u| - J_0J_{\mathcal{P}_i}|\mathcal{P}_i|u)| = |(u|S|\mathcal{P}_i|u)| = |(u|S\sqrt{|\mathcal{P}_i|}\sqrt{|\mathcal{P}_i|}u)| = \\ &= |(u|\sqrt{|\mathcal{P}_i|}S\sqrt{|\mathcal{P}_i|}u)| = |(\sqrt{|\mathcal{P}_i|}u|S\sqrt{|\mathcal{P}_i|}u)| \leq (\sqrt{|\mathcal{P}_i|}u|\sqrt{|\mathcal{P}_i|}u) = \\ &= (u||\mathcal{P}_i|u).\end{aligned} \tag{9.31}$$

Thanks to this inequality, it sufficies to prove that

$$(u||\mathcal{P}_0|u) \geq (u||\mathcal{P}_i|u) \quad \forall u \in D_G^{(U)} \tag{9.32}$$

to conclude from (9.30) that $B \geq 0$ on $D_G^{(U)}$. The proof of positivity of B on its full domain will follow by extending this result.

Thanks to Lemma 9.2.9, with $\mu \geq 0$ we have $-\mathcal{P}_0^2u = \mu u - \sum_{i=1}^3 \mathcal{P}_i^2 u$ for any $u \in D_G^{(U)}$, from which

$$(\mathcal{P}_0u|\mathcal{P}_0u) = (u| - \mathcal{P}_0^2u) = \mu - \sum_{i=1}^{3}(u|\mathcal{P}_i^2u) = \mu + \sum_{i=1}^{3}(\mathcal{P}_iu|\mathcal{P}_iu) \geq (\mathcal{P}_ku|\mathcal{P}_ku) \tag{9.33}$$

for any $k = 1, 2, 3$ where we supposed, without loss of generality, that $\|u\| = 1$. In other words,

$$\|\mathcal{P}_0u\| \geq \|\mathcal{P}_ku\| \quad \forall u \in D_G^{(U)},\ \forall k = 1, 2, 3. \tag{9.34}$$

Our next step consists in proving that (9.34) extends to the whole $D(\mathcal{P}_0) \cap D(\mathcal{P}_i)$, which is actually equal to $D(\mathcal{P}_0)$. Let us prove this last statement. We know that $\mathcal{P}_0$ and $\mathcal{P}_i$ are the closures of their restrictions to $D_G^{(U)}$. So, if $v \in D(\mathcal{P}_0)$, then there exists $(v_n)_n \subset D_G^{(U)}$ such that $v_n \to v$ and $\mathcal{P}_0v_n \to \mathcal{P}_0v$. Thanks to (9.34), we see that $(\mathcal{P}_kv_n)_n$ is a Cauchy sequence in H, thus converging to some $y \in \mathsf{H}$. Since $\mathcal{P}_k$ is closed it holds that $v \in D(\mathcal{P}_k)$ and $y = \mathcal{P}_kv$. This gives $D(\mathcal{P}_0) \subset D(\mathcal{P}_k)$. Now, we have $\|\mathcal{P}_kv\| = \lim_{n\to\infty}\|\mathcal{P}_kv_n\| \leq \lim_{n\to\infty}\|\mathcal{P}_0v_n\| = \|\mathcal{P}_0v\|$, hence (9.34) is valid also on $D(\mathcal{P}_0) \cap D(\mathcal{P}_k) = D(\mathcal{P}_0)$. This result implies (9.32), hence the positivity of B on the Gårding domain, as we are going to prove.

Notice that the spectral measures of $i\mathcal{P}_0$ and $i\mathcal{P}_k$ commute with each other. This follows by the identity $e^{t\mathcal{P}_0}e^{s\mathcal{P}_k} = e^{s\mathcal{P}_k}e^{t\mathcal{P}_0}$ together with Stone's Theorem and Lemma 4.2.3. As H is separable, this guarantees the existence of a *joint spectral measure* E on $\mathbb{R}^2$ (see Theorem 2.2.61) such that

$$f(i\mathcal{P}_0) = \int_{\mathbb{R}^2} f(\lambda_1)dE(\lambda) \quad f(i\mathcal{P}_k) = \int_{\mathbb{R}^2} f(\lambda_2)dE(\lambda)$$

for every measurable function f on $\mathbb{R}^2$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Moreover $E(\Delta)\mathcal{P}_0 \subset \mathcal{P}_0 E(\Delta)$ for every Borelian $\Delta \subset \mathbb{R}^2$, hence in particular it holds that $E(\Delta)(D(\mathcal{P}_0)) \subset D(\mathcal{P}_0)$.

Now, if $v \in D(i\mathcal{P}_0) = D(\mathcal{P}_0)$, exploiting (9.34) which is valid on the whole $D(\mathcal{P}_0) = D(\mathcal{P}_0) \cap D(\mathcal{P}_k)$ as proved above, we get $E(\Delta)v \in D(\mathcal{P}_0)$ and

$$\begin{aligned} \infty > \int_\Delta |\lambda_1|^2 d\mu_v(\lambda) &= \int_{\mathbb{R}^2} |\lambda_1|^2 d\mu_{E_\Delta v}(\lambda) = ||i\mathcal{P}_0 E(\Delta)v||^2 \geq ||i\mathcal{P}_k E(\Delta)v||^2 = \\ &= \int_{\mathbb{R}^2} |\lambda_2|^2 d\mu_{E_\Delta v}(\lambda) = \int_\Delta |\lambda_2|^2 d\mu_v(\lambda)\,, \end{aligned} \tag{9.35}$$

for every Borelian $\Delta \subset \mathbb{R}^2$. So that $\int_\Delta (|\lambda_1|^2 - |\lambda_2|^2)d\mu_v \geq 0$ for every Borelian Δ. As a consequence $|\lambda_1|^2 - |\lambda_2|^2 \geq 0$ almost everywhere $\mathbb{R}^2$ with respect to the measure μ_v, which implies $|\lambda_1| \geq |\lambda_2|$ almost everywhere on $\mathbb{R}^2$ with respect to μ_v. As an immediate consequence, if $v \in D_G^{(U)} \subset D(|\mathcal{P}_0|)$,

$$(v||\mathcal{P}_0|v) = (v|i|\mathcal{P}_0|v) = \int_{\mathbb{R}^2} |\lambda_1| d\mu_v \geq \int_{\mathbb{R}^2} |\lambda_2| d\mu_v = (v||i\mathcal{P}_k|v) = (v||\mathcal{P}_k|v)\,. \tag{9.36}$$

This proves that $B \geq 0$ on $D_G^{(U)}$. Let us finally extend this property to the entire domain of B. The operator B is the composition of $-J_0$, which is unitary, and $e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}$, which is easily seen to be the closure of its restriction to the Gårding domain ($e^{z\mathcal{K}_i}$ is a bijection of $D_G^{(U)}$ to itself and this domain is a core for $\mathcal{P}_0$). The operator B is then closed with $D_G^{(U)}$ as a core (see Proposition 2.2.58). This immediately implies that B is positive on its domain, it being positive on a core.

To conclude let us prove that B is self-adjoint. The symmetry is a direct consequence of the positivity of B and Proposition 2.2.27: $B \subset B^*$. Now, since J_0 is bounded, from $B = -J_0 e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}$ we get (see Proposition 2.2.13)

$$B^* = e^{z\mathcal{K}_i}\mathcal{P}_0^* e^{-z\mathcal{K}_i}(-J_0)^* = -e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i} J_0$$

so that $B \subset B^*$ can be rephrased as

$$-J_0 e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i} \subset -e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i} J_0.$$

Applying J_0 on the left and $-J_0$ on the right one of both the sides of the above inclusion, we find $-e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i} J_0 \subset -J_0 e^{z\mathcal{K}_i}\mathcal{P}_0 e^{-z\mathcal{K}_i}$, that is $B^* \subset B$ and thus $B = B^*$. We have so far established that U, B satisfy the requirement listed in requirements (1)-(4) of Theorem 4.2.1, so that J_0 commutes with the unitary representation of the one-parameter groups generated by K_i. This concludes the proof for the complex case.

Now, suppose that H is real or quaternionic and move to the complexified space $\mathsf{H}_{\mathbb{C}}$. The representation U gives rise to a strongly-continuous unitary representation $U_{\mathbb{C}}$ by $(U_{\mathbb{C}})_g = (U_g)_{\mathbb{C}}$. The anti-self-adjoint generators of the one-parameter subgroups are given by the complexification of the corresponding anti-self-adjoint generators of U on H, in particular $D_G^{(U_{\mathbb{C}})} = (D_G^{(U)})_{\mathbb{C}}$ (see Proposition 7.1.9 and Corollary 7.1.10). Moreover remember that $f(|\mathcal{A}_{\mathbb{C}}|) = f(|\mathcal{A}|_{\mathbb{C}}) = f(|\mathcal{A}|)_{\mathbb{C}}$ for any anti-self-adjoint generator $\mathcal{A}$ and real-valued measurable function f and that the polar decomposition of $\mathcal{A}_{\mathbb{C}}$ corresponds to the complexification of the polar decomposition of $\mathcal{A}$ (see Proposition 4.2.2). Finally, notice that

$$-(\mathcal{P}_0)^2_{\mathbb{C}} + \sum_{i=1}^{3} (\mathcal{P}_i)^2_{\mathbb{C}} = \mu I_{\mathbb{C}} \quad \text{on } D_G^{(U_{\mathbb{C}})}. \tag{9.37}$$

Of course, the complexified representation $U_{\mathbb{C}}$ is not necessarily irreducible if U is. This hypothesis was important in order to prove that $\mathcal{P}_0$ is injective, that J_0 is a unitary operator and that $-\mathcal{P}_0^2 + \sum_{i=1}^3 \mathcal{P}_i^2 = \mu I$ for some $\mu \in \mathbb{R}$. Anyway in the case of $U_{\mathbb{C}}$, referring to $(\mathcal{P}_0)_{\mathbb{C}}$, to $(J_0)_{\mathbb{C}}$, which is the partial isometry appearing in the (complex) polar decomposition of $(\mathcal{P}_0)_{\mathbb{C}}$, and to (9.37), these three hypotheses are fulfilled from the start as they follow from the relative ones of U and this holds regardless the irreducibility of $U_{\mathbb{C}}$. Thus we can apply the entire calculation carried out above for the complex case to the relevant complexified operators and prove that $(J_0)_{\mathbb{C}}(e^{t\mathcal{K}_i})_{\mathbb{C}} = (e^{t\mathcal{K}_i})_{\mathbb{C}}(J_0)_{\mathbb{C}}$ which immediately translates into $J_0 e^{t\mathcal{K}_i} = e^{t\mathcal{K}_i} J_0$ for all $t \in \mathbb{R}$, concluding the proof.

So, let us prove point (v) now. As already said above, the fact that $J_0\overline{u(\mathsf{A})} = \overline{u(\mathsf{A})}J_0$ is equivalent to the already proved statement in (iii). Since J_0 is bounded, it holds that $(J_0\overline{u(\mathsf{A})})^* = \overline{u(\mathsf{A})}^* J_0^* = \overline{u(\mathsf{A})}J_0 = J_0\overline{u(\mathsf{A})}$: this proves that $J_0\overline{u(\mathsf{A})}$ is self-adjoint. It remains to prove that its PVM is contained in $\mathfrak{M}$. Take $B \in \mathfrak{M}'$, then $Be^{t\overline{u(\mathsf{A})}} = BU_{\exp(t\mathsf{A})} = U_{\exp(t\mathsf{A})}B = e^{t\overline{u(\mathsf{A})}}B$, from which it follows that $B\overline{u(\mathsf{A})} \subset \overline{u(\mathsf{A})}B$ thanks to Proposition 4.2.3. This inclusion, together with $J_0 \in \mathfrak{M}$, gives $B[J_0\overline{u(\mathsf{A})}] \subset [J_0\overline{u(\mathsf{A})}]B$ and so, Proposition 4.2.3 guarantees that the PVM of $J_0\overline{u(\mathsf{A})}$ commutes with B. The operator $B \in \mathfrak{M}'$ being generic and $\mathfrak{M} = \mathfrak{M}''$ we have the thesis.

To conclude, let us prove (vi). First of all, notice that, in view of Proposition 7.1.8, $J_1 u(\mathsf{A}) \subset u(\mathsf{A})J_1$ implies $J_1 U_g = U_g J_1$ for every $g \in \widetilde{\mathcal{P}}_+^{\uparrow}$. So, in any case it holds that $J_1 \in \mathfrak{M}'$. Thanks to point (ii) it must be $J_1 = a + bJ_0$ for some $a, b \in \mathbb{R}$. Since J_1 is anti-self-adjoint and unitary it is a simple matter to prove that $a = 0$ and $b = \pm 1$.

$\square$

***Remark* 9.2.13.** Sticking to the interpretation of M_U^2 as the self-adjoint operator representing the squared mass of the system, the condition $M_U^2 \geq 0$ encompasses also the case of *massless particles*. Thus, in our framework this possibility is naturally taken into account, bypassing the issues about the existence of the position observable discussed in the introduction.

***Corollary* 9.2.14.** *In the same hypotheses of Theorem 9.2.12, it holds that* $J_0(D_G^{(U)}) = D_G^{(U)}$ *and* $J_0(u(\mathsf{M}) + J_0 u(\mathsf{N})) = (u(\mathsf{M}) + J_0 u(\mathsf{N}))J_0$ *for all* $\mathsf{M}, \mathsf{N} \in \mathcal{E}(\mathfrak{p})$*. More strongly*

$$J_0\overline{u(\mathsf{M}) + J_0 u(\mathsf{N})} = \overline{u(\mathsf{M}) + J_0 u(\mathsf{N})}J_0 \quad \textit{for all } \mathsf{M}, \mathsf{N} \in \mathcal{E}(\mathfrak{p}).$$

Moreover if $\overline{u(\mathsf{M}) + J_0 u(\mathsf{N})}$ *is self-adjoint, then it is an observable of the system.*

Proof. Thanks to point (iii) of Theorem 9.2.12 we already know that $J_0(D_G^{(U)}) \subset D_G^{(U)}$. Since $J_0 J_0 = -I$ we immediately see that actually $J_0(D_G^{(U)}) = D_G^{(U)}$. The property $J_0 u(\mathsf{A}) \subset u(\mathsf{A})J_0$ which is valid for any $\mathsf{A} \in \mathfrak{p}$ immediately extends to any element $\mathsf{M} \in \mathcal{E}(\mathfrak{p})$ thanks to the definition and properties of the representation u. Exploiting $J_0 J_0 = -I$ again, the inclusion in $J_0 u(\mathsf{M}) \subset u(\mathsf{M})J_0$ can be replaced by an equality. With a trivial extension of this argument we get $J_0(u(\mathsf{M}) + J_0 u(\mathsf{N})) = (u(\mathsf{M}) + J_0 u(\mathsf{N}))J_0$. Taking the closures of both sides, since J_0 is unitary, we get $J_0\overline{u(\mathsf{M}) + J_0 u(\mathsf{N})} = \overline{u(\mathsf{M}) + J_0 u(\mathsf{N})}J_0$. Regarding the last sentence suppose that $\overline{u(\mathsf{M}) + J_0 u(\mathsf{N})}$ is self-adjoint. Since, as just proved, it commutes with J_0, the same does its PVM thanks to Proposition 4.2.3 and thus the latter belongs to $\{aI + bJ_0 \mid a, b \in \mathbb{R}\}' = \mathfrak{M}'' = \mathfrak{M}$, where we used point (ii) of Theorem 9.2.12. □

Theorem 9.2.12, in particular point (v), gives a definitive answer to the question concerning the association of conserved observables to one-parameter symmetries. As desired we found an imaginary operators commuting with the *entire* algebra $\mathfrak{M}$. In particular we can give the following interpretation.

***Assumption* 9.2.15** (4-momentum, angular momentum, boost observables)**.** *The self-adjoint operators defined by*

$$P_\mu := J_0 \mathcal{P}_\mu, \quad L_i := J_0 \mathcal{L}_i, \quad K_i := -J_0 \mathcal{K}_i$$

correspond to the (covariant) 4-momentum, angular momentum and boosts observables, respectively, associated with the system.

***Remark* 9.2.16.** The definitions in Assumption 9.2.15 are coherent with Assumptions 9.2.4 and 9.2.7. Indeed, exploiting the properties of J_0 and Proposition 2.2.13 we see that

$$(P_\mu)^2 = P_\mu^* P_\mu = (J_0 \mathcal{P}_\mu)^*(J_0 \mathcal{P}_\mu) = \mathcal{P}_\mu^* J_0^* J_0 \mathcal{P}_\mu = \mathcal{P}_\mu^* \mathcal{P}_\mu$$

and clearly

$$cP_0 = cJ_0 \mathcal{P}_0 = cJ_0 J_0 |\mathcal{P}_0| = -c|\mathcal{P}_0| = -H$$

as should be (P_μ is the covariant 4-momentum).

Of course, the existence of such an imaginary operator J_0 prevents the algebra $\mathfrak{M}$ from being maximal on $\mathfrak{B}(\mathsf{H})$. Anyway something similar holds, as the following corollary of Theorem 9.2.12 shows.

***Corollary* 9.2.17.** *Referring to Theorem 9.2.12 the following statements hold:*

(a) if H *is complex, then* $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ *and* $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$*;*

(b) if H *is real or quaternionic, then referring to the identifications of Theorem 6.6.2, it holds that*

$$\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}) \quad \textit{and} \quad \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_{\mathfrak{J}}) \tag{9.38}$$

where $\mathfrak{J} = \{I, J_0\}$ *is the complex structure associated with* $\mathfrak{M}'$*.*

Proof. The proof is immediate from Theorem 9.2.12 and Theorem 6.6.2. □

We see that, even if we start from a real or quaternionic Hilbert space, we necessarily end up with an equivalent description in terms of a maximal operator algebra over a complex Hilbert space. More precisely notice that the representation $\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto U_g \in \mathfrak{M}$ is in unquely determined by

$$\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto (U_g)_{\mathfrak{J}} \in \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}), \tag{9.39}$$

which is a strongly-continuous unitary representation (see Remark 7.1.2). Moreover the representation (9.39) is clearly locally-faithful and irreducible on $\mathsf{H}_{\mathfrak{J}}$ as follows from Proposition 4.3.6. At this point Schur's Lemma for complex Hilbert spaces assures that

$$\{(U_g)_{\mathfrak{J}} \,|\, g \in \widetilde{\mathcal{P}}_+^\uparrow\}'' = \mathfrak{B}(\mathsf{H}_{\mathfrak{J}}).$$

Finally, the representation of $h : \widetilde{\mathcal{P}}_+^\uparrow \to Sym(\mathfrak{M})$ generated by U is uniquely determined by the the analogous representation

$$h_{\mathfrak{J}} : \widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto (U_g)_{\mathfrak{J}} \cdot (U_g)_{\mathfrak{J}}^* \tag{9.40}$$

on $\mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ induced by $U_{\mathfrak{J}}$.

***Corollary* 9.2.18.** *Let* H *be real or quaternionic, then, referring to Corollary 9.2.17 and (9.40), any WERS* $(\mathfrak{M}, h)$ *is equivalent to the WERS* $(\mathfrak{B}(\mathsf{H}_{\mathfrak{J}}), h_{\mathfrak{J}})$ *on the complex Hilbert space* $\mathsf{H}_{\mathfrak{J}}$*.*

9.3 A Physically More Accurate Approach

In the previous section we gave a definition of elementary relativistic system over a real or quaternionic Hilbert space just by *mimicking* the standard formulation on complex Hilbert spaces. However this approach is pretty naive under different perspectives. Consider for instance the assumed existence of a strongly-continuous unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ as in the definition of WERS. In the general situation one would like to start with an abstract definition of Poincaré symmetry, given by a group representation in terms of automorphisms of some von Neumann algebra $\mathfrak{M}$ associated with the system as in Assumption 9.1.12 and eventually recover it as a *corollary* of the basic assumptions of the theory.

In this section we want to develop a more meticulous and solid approach. In order to characterise the elementariness of the system we will put some further conditions on $(\mathfrak{M}, h)$ which, again, will eventually lead us to an equivalent description in terms of a complec WERS.

9.3.1 General Description

In this part we stick to the general framework assumed in Section 9.1 for general relativistic systems and try to better characterise it in order to encompass the concept of an *elementary system*. This can be done by looking at how a relativistic system $(\mathfrak{M}, h)$ may decompose into relativistic subsystems and prevent this from happening by putting suitable conditions on the algebra and/or the group representation. Further requirements will descend from the assumption that the Poincaré group acts as *maximal symmetry group* for the system.

Superselection Rules

The first important constraint on the system concerns the existence of superselection projectors (see Definition 9.1.14).

As a fist characterisation of elementary systems, we can suppose that for such objects, *every* element of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$ is necessarily a superselection projector, i.e. it is left invariant by the action of the Poincaré group. Indeed, in the attempt to outline the features of elementary quantum systems, it is reasonable to assume that such an object will show classical features which are as much elementary as possible. Thus the only *classical observables* we may allow to exist are the *classical labels*, in that they are invisible from the perspective of both $\mathfrak{M}$ and h.

***Assumption* 9.3.1.** *Any element of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$ is a superselection projector, i.e. it is left invariant by the action of the Poincaré group.*

So, suppose such an observable exists, say $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$, and consider the orthogonal direct sum decomposition $\mathsf{H} = \mathsf{H}_P \oplus \mathsf{H}_{P^\perp}$. As proved in Proposition 6.4.7 the sets

$$\mathfrak{M}_P := \{AP \mid A \in \mathfrak{M}\}, \quad \mathfrak{M}_{P^\perp} := \{AP^\perp \mid A \in \mathfrak{M}\}$$

define von Neumann algebras over the corresponding Hilbert spaces and the sets

$$\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})_P := \{Q \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \mid Q \leq P\}, \quad \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})_{P^\perp} := \{Q \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \mid Q \leq P^\perp\}$$

equal the corresponding logics $\mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P)$ and $\mathfrak{L}_{\mathfrak{M}_{P^\perp}}(\mathsf{H}_{P^\perp})$. Moreover the restrictions of any automorphism h_g to $\mathfrak{M}_P$ or $\mathfrak{M}_{P^\perp}$, denoted by $(h_g)_P$ and $(h_g)_{P^\perp}$, respectively, give rise to automorphisms of the corresponding restricted von Neumann algebras (see Theorem 6.4.9).

Finally, referring to the decomposition of Theorems 5.3.5, 5.3.7, 6.4.8, 6.4.9 and Proposition 6.4.7 we have the following

***Proposition* 9.3.2.** *If $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$ and Assumption 9.3.1 is satisfied, then the following statements hold:*

(a) $\mathfrak{M} = \mathfrak{M}_P \oplus \mathfrak{M}_{P^\perp}$;

(b) $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P) \oplus \mathfrak{L}_{\mathfrak{M}_{P^\perp}}(\mathsf{H}_{P^\perp})$

(c) $h_g = (h_g)_P \oplus (h_g)_{P^\perp}$

Now, focus on the sublattice $\mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P)$ and consider any σ-probability measure μ over it. Define the function

$$\mu' : \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \ni Q \mapsto \mu'(Q) := \mu(QP) \in [0,1].$$

It is easy to see that this defines a state over $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. Moreover it clearly satisfies

$$\mu'(Q) = \mu(Q) \quad \text{for any } Q \in \mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P).$$

The same can be done for $P^\perp$. In particular notice that

$$\mu((h_g)_P(Q)) = \mu'(h_g(Q)) \quad \text{for any } Q \in \mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P).$$

and thus $h_P : g \mapsto (h_g)_P$ is continuous with respect to any σ-probability measure and projector of $\mathfrak{M}_P$, this holding true for $g \mapsto h_g$. Again, the same can be done for $P^\perp$.

Thus the couples $(\mathfrak{M}_P, h_P)$ and $(\mathfrak{M}_{P^\perp}, h_{P^\perp})$ define relativistic systems themselves (check Definition 9.1.16) and in some sense can be understood as relativistic *subsystems* of $(\mathfrak{M}, h)$. Another argument supporting this point of view comes next.

Consider a state of the original system $\mu : \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \to [0,1]$ and define

$$\mu_P : \mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P) \ni Q \mapsto \mu_P(Q) := \frac{\mu(Q)}{\mu(P)}$$

and similarly

$$\mu_{P^\perp} : \mathfrak{L}_{\mathfrak{M}_{P^\perp}}(\mathsf{H}_{P^\perp}) \ni Q' \mapsto \mu_{P^\perp}(Q') := \frac{\mu(Q')}{\mu(P^\perp)}.$$

Referring to Proposition 9.3.2 every projector $Q \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ decomposes into $Q = QP + QP^\perp = QP \vee QP^\perp$, since $QP, QP^\perp$ are orthogonal to each other (see Proposition 2.2.32). Exploiting the properties of the σ-probability measures the following proposition can be easily proved.

***Proposition* 9.3.3.** *Referring to Proposition 9.3.2 the following statements hold:*

(a) μ_P, $\mu_{P^\perp}$ *define σ-probability measures on* $\mathfrak{L}_{\mathfrak{M}_P}(\mathsf{H}_P)$ *and* $\mathfrak{L}_{\mathfrak{M}_{P^\perp}}(\mathsf{H}_{P^\perp})$*, respectively;*

(b) $\mu(Q) = \mu(P) \cdot \mu_P(Q_P) + (1 - \mu(P)) \cdot \mu_{P^\perp}(Q_{P^\perp})$.

Thus we see that every state on $\mathfrak{M}$ decomposes as a *classical probabilistic mixture* of states of the lattices of the "subsystems". Thus, the systems $(\mathfrak{M}_P, h_P)$ and $(\mathfrak{M}_{P^\perp}, h_{P^\perp})$ turn out to be *more elementary* than $(\mathfrak{M}, h)$ itself. We can prevent this from happening by assuming that if such a superselecion projector exists, then it is trivial. More precisely:

***Assumption* 9.3.4** (absence of superselection rules)**.** *The system admits no superselection projectors. In particular* $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}' = \{0, I\}$.

***Remark* 9.3.5.** Some remarks follow

(a) Assumption 9.3.4 implies Assumption 9.3.1.

(b) Assumption 9.3.4 is satisfied automatically by any WERS; Indeed in that case, the algebra $\mathfrak{M}$ is irreducible by construction and so Lemma 4.3.2 gives the thesis.

(c) The assumption $P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ is stronger than $P \in \mathfrak{L}(\mathsf{H})$ but is necessary in order to develop the entire analysis above. Moreover notice that an element $P \in \mathfrak{L}(\mathsf{H})$ would not even be an observable in general, making its physical meaning difficult to interpret;.

(d) In particular there could still exist some $P \in \mathfrak{L}(\mathsf{H}) \cap \mathfrak{M}' \setminus \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$ different from 0 and I.

Gauge and CSCO

At the end of Section 9.1 we discussed the concept of a gauge group and a restricted gauge groups associated with any physical system $\mathfrak{M}$. Now, dealing with elementary relativistic systems, we can put some restrictions on them.

As already pointed out in the paragraph on gauge symmetries at the beginning of this chapter, in the usual treatment of quantum elementary particles on complex Hilbert spaces the Poincaré group and the gauge group $\mathfrak{U}(\mathfrak{M}')$ play two distinct roles, both of them significant from a physical point of view. The total physical symmetry group of the system involves the action of both of them. Sticking to the particular case of Poincaré group as *maximal symmetry group* of the system, the gauge action is assumed to be trivial. Thus, one usually requires that $\mathfrak{M}' = \mathbb{C}I$ which, in particular, implies $\mathfrak{M}' \subset \mathfrak{M}$. Recall that the latter condition is equivalent to the existence of a CSCO for $\mathfrak{M}$ as proved in Proposition 9.1.33. Actually, thanks to Assumption 9.3.4 these three conditions are equivalent to each other, as can be proved immediately.

So, in the real and quaternionic cases we may borrow this idea and try to apply it, preferably in the weakest and most cautious way possible. Thus, instead of imposing very strong assumptions like $\mathfrak{M}' = \mathbb{F}_c I$ or $\mathfrak{M}' \subset \mathfrak{M}$ which involve the entire gauge algebra $\mathfrak{M}'$, we may focus our attention on the *restricted* gauge algebra $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})''$, which, remember, equals $\mathfrak{M}'$ in the complex case. (Remember that, as already stated in Remark 9.1.27, using the restricted gauge algebra $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})''$ or the restricted gauge group $\mathfrak{U}(\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})'')$ is exactly the same)

Thus, bearing this in mind and sticking to the (still vague) idea of Poincaré group as maximal symmetry group, there may be three different consequent assumptions we can make on the system (all of them equivalent in the complex settings).

(a) (*complex analogous:* $\mathfrak{M}' = \mathbb{C}I$)

We may assume that the restricted gauge algebra is trivial: more precisely we may suppose that $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})'' = \mathbb{F}_c I$, or, equivalently, $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) = \{0, I\}$.

(b) (*complex analogous:* $\mathfrak{M}' \subset \mathfrak{M}$)

With the same idea in mind, we may be more cautious, and suppose that if some restricted gauge algebra exists then it is contained within the algebra of the system: $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H})'' \subset \mathfrak{M}$. This in particular implies $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H})$. Since, of course, $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) \subset \mathfrak{M}'$, we get $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$. Assumption 9.3.4 implies $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) = \{0, I\}$.

(c) (*complex analogous: existence of CSCO*)

Finally, we could assume that the system admits a CSCO $\mathfrak{A} \subset \mathfrak{M}$. If this is the case, then Theorem 9.1.31 assures that $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) \subset \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$. Again, exploiting Assumption 9.3.4, we get $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) = \{0, I\}$.

We see that, even if they are different from each other, the three possible assumptions above lead to the same conclusion.

***Assumption* 9.3.6** (triviality of the restricted gauge algebra)**.** *It holds that* $\mathfrak{L}_{\mathfrak{M}'}(\mathsf{H}) = \{0, I\}$, *in particular the von Neumann algebra* $\mathfrak{M}$ *is irreducible.*

***Remark* 9.3.7.** Some remarks follows.

(a) Assumption 9.3.6 *implies* Assumption 9.3.4.

(b) Again, notice that the definition of WERS automatically encompasses this property, it being irreducible from the start.

At this point we can exploit Theorems 6.5.1 and 6.6.2 and, referring to the corresponding decompositions, state the following result.

***Corollary* 9.3.8.** *Let* $(\mathfrak{M}, h)$ *be a relativistic system satisfying Assumption 9.3.6, then*

(1) If H *is complex, then* $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ *and* $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$*;*

(2) If H *is real or quaternionic then there are two possibilities:*

  (a) if $\mathfrak{M}' \cong \mathbb{R}$ *then* $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ *and* $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$,

  (b) if $\mathfrak{M}' \cong \mathbb{C}$ *or* $\mathbb{H}$ *then* $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ *and* $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_{\mathfrak{J}})$,

  where $\mathfrak{J}$ *is the* $\mathbb{K}$*-structure associated with the commutant* $\mathfrak{M}'$*.*

The found form of the algebra of observables has several remarkable consequences. Let us see them in detail.

Operatorial Nature of States

As a first consequence we discuss the classification of states. The algebra $\mathfrak{M}$ being irreducible, we can apply Proposition 6.7.1 and prove the following result.

***Corollary* 9.3.9.** *Let $(\mathfrak{M}, h)$ be a relativistic system satisfying Assumption 9.3.6, then the following statements hold:*

(a) if $T \in \mathfrak{D}(\mathsf{H})$ then the function $\mu_T : \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \to [0,1]$ defined by

$$\mu_T(P) := tr(TP) \quad \text{for all } P \in \mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \tag{9.41}$$

is a state of the system;

(b) for every state $\mu \in \mathfrak{S}$ there exists a unique density operator $T \in \mathfrak{D}(\mathsf{H}) \cap \mathfrak{M}$ such that $\mu = \mu_T$, where μ_T is defined as in (9.41).

Operatorial Nature of Poincaré Symmetry

In this subsection we want to discuss how the irreducibility of the algebra affects the form of the representation $h : \widehat{\mathcal{P}}_+^\uparrow \to Sym(\mathfrak{M})$.

First, some considerations. As already discussed in introducing WERSs., when dealing with elementary systems with $\widehat{\mathcal{P}}_+^\uparrow$ as maximal symmetry group, we assume that the physical system is totally characterised by the Poincaré symmetry itself. We think of such systems as concrete *manifestations* of the spacetime-symmetries themselves.

In line with this idea we have already put some conditions on h. For instance with Assumption 9.3.1 we assume that the elements of $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \cap \mathfrak{M}'$ can at most represent classical labels and as such they are left invariant by the action of h.

Sticking to the same principle we may now assume that, *at least locally around the identity*, the features of $\widehat{\mathcal{P}}_+^\uparrow$ are preserved when lifted through the action of the representation h.

***Assumption* 9.3.10.** *The homomorphism $h : \widehat{\mathcal{P}}_+^\uparrow \to Sym(\mathfrak{M})$ is locally-faithful.*

We can now go back to the consequences of irreducibility.

***Proposition* 9.3.11.** *Let $(\mathfrak{M}, h)$ be a relativistic system satisfying Assumptions 9.3.6 and 9.3.10, then there exists a locally-faithful strongly-continuous unitary representation*

$$\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto U_g \in \mathfrak{M} \tag{9.42}$$

such that $h_g(A) = U_g A U_g^*$ for all $A \in \mathfrak{M}$.

Proof. The existence of such a strongly-continuous representation is guaranteed by Theorem 9.1.24. The local faithfulness is an immediate consequence of Assumption 9.3.10. □

***Remark* 9.3.12.** Some remarks follow.

(a) Assumption 9.3.10 is fulfilled by any WERS by definition.

(b) Remember that the representation (9.42) is not uniquely defined, in that the operators U_g are defined up to elements of $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$.

This result gives an important realisation of the abstract representation h. Moreover it provides us with a really natural way to complete our characterisation of elementary systems. As already said above, we want the representation h to fully characterise the system. In particular it should be able to generate the algebra $\mathfrak{M}$ somehow. Since the representatives h_g are now described by operators, it seems natural to assume that the elements of $\mathfrak{M}$ are generated as the result of combining the operators U_g in all the possible ways: linear combinations, products, strong or weak limits. Of course this reconstruction should not depend on the particular family of operators $\{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\}$ representing h. Since two of them differ only by mean of phases, the elements of $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$ must be included in the generating set. Indeed notice that the physical meaning is not carried by the operators U_g, but by the automorphisms h_g, or in other words by "the operators U_g *up to phases*".

We can make the following assumption.

***Assumption* 9.3.13.** *Referring to any realisation of h as in Proposition 9.3.11 it holds that*

$$\mathfrak{M} \subset \overline{\left\langle \{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) \right\rangle}^s = \left\{ \{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) \right\}''. \tag{9.43}$$

***Remark* 9.3.14.** We could have been more cautious in imposing Assumption 9.3.13, which is no doubt a really strong requirement. An apparently weaker condition may be

$$\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \overline{\left\langle \{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) \right\rangle}^s = \left\{ \{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) \right\}''. \tag{9.44}$$

Actually the two choices are perfectly equivalent in our framework.

Proof. Of course (9.43) implies (9.44). Let us prove the opposite. If H is complex, or if it is real or quaternionic but with commutant $\mathfrak{M}' \cong \mathbb{R}$ then $\mathfrak{M} = \mathfrak{B}(\mathsf{H})$ and $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) = \mathfrak{L}(\mathsf{H})$ and so the thesis follows from Proposition 6.3.4. So, suppose that H is real or quaternionic and $\mathfrak{M}' \cong \mathbb{C}$ or $\mathbb{H}$. Let $\mathfrak{J}$ be the $\mathbb{K}$-structure characterising the commutant of $\mathfrak{M}$. Since $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) \subset \mathfrak{M}$ and $U_g \in \mathfrak{M}$ for all $g \in \widetilde{\mathcal{P}}_+^\uparrow$ then the family

$$\mathfrak{F} := \overline{\left\langle \{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) \right\rangle}^s$$

is made of operators commuting with the elements of $\mathfrak{J}$. In particular we can consider the corresponding family $\mathfrak{F}_{\mathfrak{J}} \subset \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$. The condition $\mathfrak{L}_{\mathfrak{M}}(\mathsf{H}) \subset \mathfrak{F}$ is equivalent to $\mathfrak{L}(\mathsf{H}_{\mathfrak{J}}) \subset \mathfrak{F}_{\mathfrak{J}} \subset \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$. Exploiting again Proposition 6.3.4 we immediately see that $\mathfrak{F}_{\mathfrak{J}} = \mathfrak{B}(\mathsf{H}_{\mathfrak{J}})$ or, equivalently, $\mathfrak{F} = \mathfrak{M}$. □

An immediate consequence follows.

***Proposition* 9.3.15.** *Referring to Proposition 9.3.11, if Assumption 9.3.13 holds true, then the family of operators on* H *defined by*

$$\{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})$$

is irreducible. In particular if H *is complex or if* H *is real or quaternionic with* $\mathfrak{M}' \cong \mathbb{R}$ *or* $\mathbb{H}$*, then*

$$\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto U_g \in \mathfrak{M}$$

is irreducible on H.

Proof. The first part follows from the irreducibility of $\mathfrak{M}$ and Assumption 9.3.13. The second part from the fact that $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) = \mathfrak{U}(\mathbb{F}_c I)$ in the considered cases. □

***Definition* 9.3.16.** *A real, complex or quaternionic elementary relativistic system (ERS) is a relativistic system* $(\mathfrak{M}, h)$ *satisfying assumptions 9.3.6, 9.3.10 and 9.3.13.*

Notice that the assumptions made so far give rise to a generalisation of the concept of a WRES. More precisely:

***Proposition* 9.3.17.** *The following statements hold:*

(a) every WERS is a ERS;

(b) every complex ERS is a complex WERS;

(c) every real ERS such that $\mathfrak{M}' \cong \mathbb{R}$ *or* $\mathbb{H}$ *is a real WERS;*

(d) every quaternionic ERS such that $\mathfrak{M}' \cong \mathbb{R}$ *or* $\mathbb{H}$ *is a quaternionic WERS.*

Proof. Point (a) was proved throughout the entire discussion. Points (b), (c) and (d) can be proved in the same way. Indeed in all the three cases $\mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}}) = \mathfrak{U}(\mathbb{F}_c I)$ and so, exploiting Assumption 9.3.13 we have

$$\mathfrak{M} \subset \left\{\{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \mathfrak{U}(\mathfrak{Z}_{\mathfrak{M}})\right\}'' = \{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\}'' \subset \mathfrak{M}$$

and the representation $g \mapsto U_g$ is irreducible as stated in Proposition 9.3.15. A check on the definition of WERS concludes the proof. □

9.3.2 Emergence of the Complex Structure

We are in a position to state and prove our second main result, establishing that, even relying on the more accurate definition of elementary relativistic system given in Definition 9.3.16, we eventually achieve a *complex* Wigner elementary relativistic system. Again the initial theory can be naturally rephrased into a better *complex* theory.

Consider an ERS $(\mathfrak{M}, h)$ as in Definition 9.3.16 on a real or quaternionic Hilbert space H. Consider the locally-faithful strongly-continuous unitary representation of Proposition 9.3.11:

$$U : \widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto U_g \in \mathfrak{M}. \tag{9.45}$$

We can follow the analysis of Section 9.2 (the irreducibility of U is not required for the next few points). In particular we can consider the one-parameter spacetime-displacements subgroups:

$$\mathbb{R} \ni t \mapsto U_{\exp(t\mathsf{P}_\mu)} \in \mathfrak{M} \tag{9.46}$$

for $\mu = 0, 1, 2, 3$. Stone's Theorem guarantees the existence of anti-self-adjoint operators $\mathcal{P}_\mu$ on H such that

$$U_{\exp(t\mathsf{P}_\mu)} = e^{t\mathcal{P}_\mu} \quad \text{for all } t \in \mathbb{R}. \tag{9.47}$$

In particular, the polar decomposition of the time-displacement generator

$$\mathcal{P}_0 = J_0|\mathcal{P}_0| \tag{9.48}$$

plays a crucial role. On the Gårding domain $D_G^{(U)}$ of the representation U we can define the symmetric operator

$$M_U^2 = -\mathcal{P}_0^2 + \sum_{i=1}^{3} \mathcal{P}_i^2, \tag{9.49}$$

whose physical meaning is still to be intended as the *squared mass* of the system.

Now focus on the cases $\mathfrak{M}' \cong \mathbb{R}$ and $\mathfrak{M}' \cong \mathbb{H}$. In both cases, as stated in Proposition 9.3.17 our real or quaternionic ERS $(\mathfrak{M}, h)$ is actually a real or quaternionic WERS: more precisely U is also irreducible. Thus Theorem 9.2.12 applies and imply that, if $M_U^2 \geq 0$, then J_0 of (9.48) is an imaginary operator and

$$\mathfrak{M}' = \{aI + bJ_0\} \cong \mathbb{C}.$$

This is clearly a contradiction with the hypotheses $\mathfrak{M}' \cong \mathbb{R}$ or $\mathbb{H}$. So, the commutant of $\mathfrak{M}$ cannot be of real or quaternionic type. By exclusion we have the following result.

***Lemma* 9.3.18.** *Let $(\mathfrak{M}, h)$ be a real or quaternionic ERS and suppose that, referring to the unitary representation of Proposition 9.3.11, it holds that $M_U^2 \geq 0$, then $\mathfrak{M}' \cong \mathbb{C}$.*

Now, focus on the surviving case $\mathfrak{M}' \cong \mathbb{C}$, more precisely

$$\mathfrak{M}' = \{aI + bJ \mid a, b \in \mathbb{R}\}.$$

In this case, referring to the complex structure $\mathfrak{J}$ induced by the commutant

$$\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_\mathfrak{J}) \quad \text{and} \quad \mathfrak{L}_\mathfrak{M}(\mathsf{H}) \cong \mathfrak{L}(\mathsf{H}_\mathfrak{J}). \tag{9.50}$$

Consider again the representation (9.45). As stated by Proposition 9.3.15 the family $\{U_g \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\} \cup \{J\}$ is irreducible on the Hilbert space H. Thus, exploiting Remark

7.1.2 and the isomorphism $\mathfrak{M} \cong \mathfrak{B}(\mathsf{H}_\mathfrak{J})$, it is easy to see that the function

$$\widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto (U_g)_\mathfrak{J} \in \mathfrak{B}(\mathsf{H}_\mathfrak{J}) \tag{9.51}$$

is an *irreducible* locally-faithful strongly-continuous unitary representation on $\mathsf{H}_\mathfrak{J}$ which uniquely determine U. It being irreducible, it satisfies (thanks to Schur's Lemma)

$$\{(U_g)_\mathfrak{J} \mid g \in \widetilde{\mathcal{P}}_+^\uparrow\}'' = \mathfrak{B}(\mathsf{H}_\mathfrak{J}). \tag{9.52}$$

Defining $(h_\mathfrak{J})_g(P_\mathfrak{J}) := h_g(P)_\mathfrak{J}$ for all $g \in \widetilde{\mathcal{P}}_+^\uparrow$ and $P \in \mathfrak{L}_\mathfrak{M}(\mathsf{H})$, the function

$$h_\mathfrak{J} : \widetilde{\mathcal{P}}_+^\uparrow \ni g \mapsto (h_\mathfrak{J})_g \in Sym(\mathfrak{B}(\mathsf{H}_\mathfrak{J})) \tag{9.53}$$

is a group symmetry generated by the unitary representation $U_\mathfrak{J}$ and as such it totally determines h itself. Moreover such a symmetry is necessarily continuous thanks to Proposition 9.1.19. Summing up we see that $(\mathfrak{B}(\mathsf{H}_\mathfrak{J}), h_\mathfrak{J})$ is a complex WERS and that:

***Lemma* 9.3.19.** *Any real or quaternionic ERS $(\mathfrak{M}, h)$ with complex commutant $\mathfrak{M}' = \{aI + bJ \mid a, b \in \mathbb{R}\}$ is equivalent to the complex WERS $(\mathfrak{B}(\mathsf{H}_\mathfrak{J}), h_\mathfrak{J})$ defined in (9.50), (9.51) and (9.53).*

A last comment concerns the nature of the operator J appearing in the equivalence above.

Consider the one-parameter spacetime-displacements in (9.47). From Proposition 7.1.12 we know that the generators of $t \mapsto (U_{\exp(t\mathsf{P}_\mu)})_\mathfrak{J}$ are given by $(\mathcal{P}_\mu)_\mathfrak{J}$ for any $\mu = 0, 1, 2, 3$. Moreover from Proposition 7.1.11 we see that $D_G^{(U_\mathfrak{J})} = (D_G^{(U)})_\mathfrak{J}$. Putting all together we get

$$M^2_{U_\mathfrak{J}} = (M^2_U)_\mathfrak{J}.$$

Thus, the operator M^2_U being symmetric, Propositions 3.1.16 and 3.2.20 assures that $M^2_U \geq 0$ if and only if $M^2_{U_\mathfrak{J}} \geq 0$.

As a final ingredient, referring to (9.48) and exploiting Proposition 4.2.5, the polar decomposition of $(\mathcal{P}_0)_\mathfrak{J}$ is given by

$$(\mathcal{P}_0)_\mathfrak{J} = (J_0)_\mathfrak{J} |\mathcal{P}_0|_J$$

At this point, since $g \mapsto (U_g)_\mathfrak{J}$ is irreducible on $\mathsf{H}_\mathfrak{J}$, assuming the positivity of the squared-mass operator M^2_U, we have $M^2_{U_\mathfrak{J}} \geq 0$ and thus we can apply Theorem 9.2.12 to $U_\mathfrak{J}$ and prove that $(J_0)_\mathfrak{J} = \pm i = \pm J_\mathfrak{J}$ on $\mathsf{H}_\mathfrak{J}$. Exploiting the isomorphism $\mathfrak{M} \cong \mathfrak{M}_\mathfrak{J}$ we see that

$$J = \pm J_0,$$

in particular J_0 is an imaginary structure.

***Theorem* 9.3.20.** *Let $(\mathfrak{M}, h)$ be a real or quaternionic ERS and suppose that, referring to the unitary representation of Proposition 9.3.11, it holds that $M^2_U \geq 0$, then the commutant is of complex type. More precisely the following statements hold:*

(1) referring to (9.48), the operator J_0 *is an imaginary operator;*

(2) $\mathfrak{M}' = \{aI + bJ_0 \mid a, b \in \mathbb{R}\}$.

Finally, referring to Lemma 9.3.18, the ERS is equivalent to the WERS $(\mathfrak{B}(\mathsf{H}_{\mathfrak{J}}), h_{\mathfrak{J}})$*, where* $\mathfrak{J}$ *is the complex structure associated with the commutant.*

***Remark* 9.3.21.** As happened for the WERSs (see Remark 9.2.13), also in the case of ERSs the *massless* case is naturally taken into account.

The analysis is complete. Even starting from a more solid approach for relativistic elementary systems, we eventually end up with an equivalent description in terms of complex Hilbert spaces.

Chapter 10

Conclusions

This work attempts a mathematically solid discussion on the formulation of quantum theories for elementary relativistic systems. In particular it is established that it is not physically sustainable to formulate the theory on real or quaternionic Hilbert spaces because some physical natural requirements give rise to an essentially unique complex structure which commutes with all observables of the theory. This structure permits us to reformulate the whole theory in a suitable complex Hilbert space. This formulation is less redundant than the initial real or quaternionic one, since differently from the real or the quaternionic case, all self-adjoint operators represent observables. The final result is in agreement with the final picture of Solèr theorem which however relies on different physical hypotheses. This complex structure permits also to associate conserved quantities to the anti-self-adjoint generators of the Poincaré group allowing for the formulation of a quantum version of Noether's theorem. The results are valid also for massless particles where the position observable cannot be defined and the physical analysis by Stuckelberg, leading to similar conclusions, cannot by applied. The description of an elementary relativistic system is discussed within two different frameworks. The former is closely related to the Wigner idea of elementary particle (9.2.1), the second (9.3.16) is based on a finer analysis and takes several technical subtleties into account, like the fact that representations of continuous symmetries are in general merely projective-unitary and not unitary. Both frameworks lead to the identical final result. It is however necessary to stress that the chosen notion of elementary system does not encompass relevant physical situations where the commutant of the algebra of observables is more complicated, as it happens in the description of quarks, since the commutant includes a representation of $SU(3)$. However this situation is neither considered by the Wigner notion of elementary particle in complex Hilbert spaces.

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