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The algebraic representation of OWA functions in the binomial decomposition framework and its applications in large-scale problems

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Abstract

In the context of multicriteria decision making, the ordered weighted averaging (OWA) functions play a crucial role in aggregating multiple criteria evaluations into an overall assessment to support decision makers reaching a decision. The determination of OWA weights is, therefore, an important task in this process. Solving real-life problems with a large number of OWA weights, however, can be very challenging and time consuming. In this research we recall that OWA functions correspond to the Choquet integrals associated with symmetric capacities. The problem of defining all Choquet capacities on a set of n criteria requires 2^n real coefficients. Grabisch introduced the k-additive framework to reduce the exponential computational burden. We review the binomial decomposition framework with a constraint on k-additivity whereby OWA functions can be expressed as linear combinations of the first k binomial OWA functions and the associated coefficients of the binomial decomposition framework. In particular, we investigate the role of k-additivity in two particular cases of the binomial decomposition of OWA functions, the 2-additive and 3-additive cases. We identify the relationship between OWA weights and the associated coefficients of the binomial decomposition of OWA functions. Analogously, this relationship is also studied for two well-known parametric families of OWA functions, namely the S-Gini and Lorenzen welfare functions. Finally, we propose a new approach to determine OWA weights in large-scale problems by using the binomial decomposition of OWA functions with natural constraints on k-additivity to control the complexity of the OWA weight distributions.

Keywords: *Ordered weighted averaging, OWA weights determination, Choquet integrals, symmetric capacities, binomial decomposition, k-additivity, large-scale optimization problems.*

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Publications

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Chapter 1

Introduction

The central question in this thesis is the role of ordered weighted averaging (OWA) functions in aggregation and optimization. Some research problems based on OWA functions remain a challenge for researchers. In particular, the role of k-additivity in the binomial decomposition of OWA functions requires further investigation. The relationship between OWA weights and the associated coefficients of the binomial decomposition framework must be extensively studied. In addition, we consider a research problem related to the determination of OWA weights in high dimensions, which can lead to a high computational cost. Our study aims at solving this problem by using the binomial decomposition framework with a constraint on k-additivity to equivalently express OWA functions in terms of the first k binomial OWA functions and the corresponding coefficients. This framework allows us to transform the original problem, expressed in terms of OWA weights, into a problem in which the weights are substituted by a new set of coefficients, thus leading to the dimensionality reduction in terms of the number of variables in the original problem. In the following, we present our research motivations and our proposed solutions in more detail. Finally, the introduction is closed with the outline of our thesis.

1.1 The context

In many decision making problems, decision makers have to provide their evaluations for a number of alternatives with respect to a set of criteria. The evaluation for each alternative is a process that combines multiple values according to a set of criteria and produces an overall score to support the choices of decision makers. This score can be obtained by using aggregation functions. Among different classes of aggregation functions, the ordered weighted averaging (OWA) functions introduced by Yager [115] are one of the most effective methods for aggregating data due to their flexible assignment of weights to the values of the input arguments through a reordering step. This step assigns a weight to the ordered value of an argument, instead of attaching a weight to a specific criterion or to the source where the value comes from. OWA functions are, therefore, symmetric regardless of the initial order of their input arguments. In fact, OWA weights are considered special instances of Choquet symmetric capacities [36, 50]. We recall that the problem of defining all Choquet capacities on a set of n elements requires 2^n coefficients. Grabisch [60, 63] introduced the k-additive framework in order to control the complexity level of Choquet capacities. A capacity is k-additive if it depends on coalitions of at most k cardinality ($k \leq n$). Calvo and De Baets [27], see also Bortot and Marques Pereira [21], proposed the binomial decomposition framework whereby OWA functions can be expressed as linear combinations of the first k binomial OWA functions and the associated coefficients of the binomial decomposition framework. However, studies on the role of k-additivity in the binomial decomposition of OWA functions are inadequate. We cannot fully understand how a constraint on k-additivity affects the feasible region of the coefficients of the binomial decomposition of OWA functions. Moreover, the relationship between OWA weights and these coefficients has seldom been studied. In the binomial decomposition framework, OWA weights are expressed as weighted sums of the associated coefficients of the binomial decomposition of OWA functions. However, an inverse transformation for these coefficients in terms of OWA weights is still unknown.

Another advantage of OWA functions is the fact that they allow decision makers to dynamically set certain levels of importance for the ordered values, according to the attitude of decision makers, their background and expectations on the underlying problems. This can be achieved by controlling the weight distributions. The determination of OWA weights is, therefore, an important object of study in decision making. We recall that OWA weights are characterized by two important measures called *orness* and *dispersion* [115]. The orness shows the similarity between the OWA function and the *or* (maximum) function. A higher orness means that decision makers are more optimistic while a lower orness means that they are more pessimistic. The dispersion, examples of which are entropy, variance, and disparity, is used to measure how equally all criteria are taken into account in the aggregation process.

One of the first approaches for determining OWA weights, proposed by O'Hagan [92], solved a constrained optimization problem for a set of OWA weights having maximal entropy value for a specific level of orness. In 2003, Fuller and Majlender [54] used the variance of OWA weights in their optimization problems in order to identify a specific class of OWA functions with the minimal variability OWA weights. After the pioneering work of O'Hagan [92] on the maximal entropy method and the variance-based methods of Fuller and Majlender [54], Wang and Parkan [107] proposed the minimax disparity method in which the objective is to minimize the maximum absolute difference between two adjacent weights. The minimax disparity method has received a great deal of interest in the literature [4, 44, 53, 57, 96, 106]. Most studies, however, have been limited to solving problems with small dimensions, while optimization problems in applied operational research often involve a large number of criteria. Solving these optimization problems in high dimensions requires a heavy computational load.

1.2 Research Problems

In the context of the binomial decomposition framework, OWA functions can be expressed as linear combinations of the first k binomial OWA functions and the associated coefficients of the binomial decomposition framework. OWA weights are, therefore, defined as weighted sums of the coefficients of the binomial decomposition of OWA functions. However, the main questions concerning whether these coefficients can be expressed in terms of OWA weights and how they influence OWA weights when the constraints on k-additivity are imposed remain difficult to understand. One of the objectives of our research is to study the relationship between OWA weights and the associated coefficients of the binomial decomposition framework. In particular, we focus on identifying an analytical expression for these coefficients in terms of OWA weights. We examine the feasible regions of the coefficients in two particular cases of the binomial decomposition of OWA functions, the 2-additive and 3-additive cases.

Another research direction is the determination of OWA weights. However, previous studies have given little consideration to large-scale problems. Large-scale optimization problems are common in practice when decision makers have to take into account multiple input requirements in their decision making process. The lack of methodologies for determining OWA weights in high-dimensional problems causes difficulties to decision makers in making an appropriate choice. In this work, we aim at providing a new methodology to reduce the computational complexity of the underlying problems.

1.3 Solutions

The relationship between OWA weights and the coefficients of the binomial decomposition framework can be identified if one considers two equivalent expressions of OWA functions, the formal definition and the binomial decomposition of OWA functions. The OWA functions, defined by Yager [115], are expressed as weighted sums of OWA weights and their input arguments. Alternatively, the binomial decomposition framework [21, 27] expresses OWA functions as linear combinations of binomial OWA functions and the corresponding coefficients. Solving the linear system involving OWA weights and the coefficients of the binomial decomposition framework provides an analytical expression that can be used to identify the relationship between OWA weights and the associated coefficients. In addition, we analyze the boundary and monotonicity conditions for the coefficients of the binomial decomposition of OWA functions associated with the 2-additive and 3-additive cases. We study these constraints to identify the dependency of the feasible regions on the level of k-additivity and on the increasing dimension n.

Concerning the research problem related to the determination of OWA weights in large-scale optimization problems, we focus on the minimax disparity model. We use the binomial decomposition framework, as previously described, to express OWA functions as linear combinations of the first k binomial OWA functions and their corresponding coefficients. An advantage of using the binomial decomposition framework is that it can be approximated if one considers the k-additivity framework [60, 63]. In this thesis, we use the binomial decomposition framework with a constraint on k-additivity to reduce the number of variables in the original problems. This framework allows us to transform the original problem, expressed in terms of OWA weights, into a problem in which the weights are substituted by a new set of coefficients. In the transformed representation, by exploiting the sparsity of these coefficients, we can consider only the reduced number of variables, associated with the first k-additive levels, and we can set the remaining coefficients to zero. The dimensionality of the transformed problems is significantly reduced, thus leading to a lower computational cost.

1.4 Contributions

The research is carried out to support decision makers in efficiently solving optimization problems related to OWA functions by using the binomial decomposition framework and the concept of k-additivity. We present the feasible regions of the coefficients associated with the 2-additive and 3-additive cases in n dimensions [23]. Moreover, we derive the analytical expression for the coefficients of the binomial decomposition framework in terms of OWA weights, and we apply our study to two parametric families of OWA functions, namely the S-Gini and Lorenzen welfare functions [91]. In a similar way we obtain the analytical formulations expressing the relationship between the weights of these parametric welfare functions and the corresponding coefficients of the binomial decomposition framework. We also find some interesting analogy between the binomial OWA functions and these social welfare functions in relation to the context of the binomial decomposition framework [20].

In addition, we present a novel methodology to reduce the computational cost of determining OWA weights in large-scale optimization problems [90]. The methodology, based on k-additivity of the binomial decomposition framework, significantly reduces the number of variables and therefore helps decision makers in finding optimal solutions faster. Our experiments for the minimax disparity model show that the proposed model markedly reduces the dimension up to $\left(1 - \frac{k}{n}\right)\%$, where n is the number of variables in the original problem and k is the level of k-additivity.

1.5 Outline of the thesis

Chapter 2 introduces the state of the art of OWA functions. Three main methods are then presented for obtaining a specific class of OWA functions based on: a) the characteristic measures of orness and dispersion, b) learning OWA weights from data, and c) weight-generating functions.

Chapter 3 presents some main contributions of our research. The first part describes the feasible regions of the coefficients associated with the 2-additive and 3-additive cases in n dimensions. We then prove the analytical expression for the coefficients of the binomial decomposition framework in terms of OWA weights, thus broadening our understanding of their relationship in the binomial decomposition framework. The second part proposes a new method to determine OWA weights in large-scale optimization problems by taking advantage of the sparsity of the k-additivity in the binomial decomposition framework. The experiments carried out for the minimax disparity model are numerically presented and compared to the traditional approach.

Chapter 4 discusses the representation of the binomial decomposition framework in the context of social welfare with respect to two parametric families of OWA functions, namely the S-Gini and Lorenzen welfare functions. We focus on the study of the analogy between the binomial OWA functions and the parametric welfare functions, and we derive the analytical expressions for the coefficients of this framework in terms of S-Gini weights and Lorenzen weights, respectively. We show the importance of our analytical expression with respect to the S-Gini and the Lorenzen weights to better understand the properties of their weight distributions.

Chapter 5 includes our final remarks and our suggestions for future research. Appendix A is reserved for the detailed proofs of some classical identities used in our thesis.

Chapter 2

OWA functions and weight determination

In this chapter we introduce the basic definitions of aggregation functions, with a particular focus on weighted averaging (WA) and ordered weighted averaging (OWA) functions. Recent studies on OWA functions are reviewed together with several methods for determining OWA weights related to some instances of OWA families, including the well-known minimax disparity method.

2.1 Background

In many disciplines, experts have to deal with the problem of aggregating multiple input arguments and producing a single representative output within the application context. Consider a problem of evaluating a set of proposed alternatives $A_1, \ldots, A_m$ with respect to a set of n-tuple criteria $\{1, \ldots, n\}$. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{D}^n$ denote the degree to which each alternative satisfies a set of criteria. The goal is to combine these multiple criteria evaluations into an overall score that represents the level of satisfaction of available alternatives and it can effectively guide the decision making process. The processing of combining multiple input arguments is possibly solved by using aggregation functions. Aggregation functions have been extensively studied by Beliakov et al. [13, 14], Calvo et al. [28], Fodor and Roubens [51], Grabisch et al. [68], Marichal [84], Marichal et al. [85], Mesiar et al. [88], Torra and Narukawa [105] and Yager et al. [126].

Among different classes of aggregation functions, the ordered weighted averaging functions (OWA) introduced by Yager [115] have attracted a growing interest. One of the main motivations behind the selection of the OWA functions in the aggregation process is their flexibility in providing a general class of weighted aggregation functions bounded by the *min* and the *max* functions. Moreover, the OWA functions distinguish themselves from other aggregation functions by their dynamic assignment of OWA weights to the values of the input arguments, instead of fixing a specific weight to a particular ordered position of the arguments.

Since their introduction many studies have investigated the use of OWA functions in several domains, such as multicriteria decision making (Bortot et al. [23], Yager [115, 119]), group decision making (Bortot et al. [19], Fedrizzi et al. [46, 47, 48], Dong et al. [41], Herrera-Viedma et al. [70], Palomares [93], Xu and Chen [110]), social welfare (Aristondo and Ciommi [7], Aristondo et al. [8], Bortot and Marques Pereira [20, 21]), fuzzy logic controllers (Yager [116, 123]), vision systems (López et al. [10], Marichal [84]), expert systems (Carlsson [30], O'Hagan [92]), neural networks (Yager [114, 117, 118]). A comprehensive review of OWA applications can be found in Yager and Kacprzyk [125], and Yager et al. [126].

OWA functions are extensively used in multicriteria decision making for supporting decision makers in evaluating a set of alternatives with respect to a set of criteria and aggregating their evaluations to produce a combined score. This leads decision makers to the best possible decision satisfying their requirements. Now let us consider the problem in the context of group decision making, in which two or more experts are involved in the evaluation of a set of alternatives. Experts express their individual preferences on a set of alternatives that might be heterogeneous according to their different attitudes, backgrounds and knowledge on the underlying problems. One important research direction in group decision making is the development of an appropriate measure to support experts in reaching a collective agreement.

The notion of consensus traditionally means unanimous agreement, which rarely happens in practice. Kacprzyk and Fedrizzi [71, 72, 73] proposed a "soft" consensus measure, based on numerical fuzzy preferences, to compute the degree of collective agreement when most of the experts agree on a final solution. Fedrizzi et al. [46] extended the conventional soft consensus paradigm, by defining a smooth scaling function for the linguistic quantifiers, and proposed a network dynamics for reaching soft consensus. The consensus is defined as "dynamic" due to its process of reaching collective agreement through the iterative process of preference transformation corresponding to the gradient dynamics of the full cost function of the soft consensus model.

The full cost function is a linear combination of the collective dissensus measure and the collective inertial cost function. The collective dissensus is defined as a summation of individual dissensus measures that represent the diffusion interactions between the individual preferences of an expert with respect to the preferences of the remaining experts in the group. The collective inertial cost, on the other hand, expresses the aversion to a change of the opinion of an expert from his/her original opinion towards the collective consensus trend.

A dissensus measure in the soft consensus reaching model can be represented by means of multidistance [18, 19]. Martin and Mayor [87] introduced the multidistance by extending the conventional distance between two arguments. Analogously, the multidistance is defined as a measure for computing distance for collections of more than two arguments. Among several classes of multidistances, the sum-based multidistance, with the particular case of the OWA-based multidistance, introduced by Martín and Mayor [87], shows an analogous relationship with the dissensus measure in some consensus related optimization problems on $m-$ary adjacency relations [25, 26]. The OWA-based multidistances are, therefore, used as an alternative approach to measure dissensus [18], with the OWA weighting vector is chosen in a way that more importance is given to small distance value and vice versa. Recent research by Bortot et al. [19, 22] has studied in detail the multidistance dissensus measure, which is defined on the basis of binary distances computed by means of a new subadditive scaling function. The studies have also proved that the proposed subadditive scaling function is in analogy with the conventional scaling function [46], which focuses on emphasizing small distances and at the same time attenuating large distances in preferences.

OWA functions have been used in many application domains, thereby determining OWA weights becomes an important research. The goal of the problems of determining OWA weights is to provide a methodology for decision makers to select OWA weights in a way that they can reflect different preferences of decision makers, from the optimistic to the pessimistic attitudes, with respect to specific problems. The attitudinal character of decision makers is measured by their *orness*, which is defined on the unit interval. The maximum (minimum) orness value is reached when decision makers are purely optimistic (purely pessimistic). Instead the *dispersion*, examples of which are *entropy*, *variance* and *disparity*, is used to measure the degree to which all input arguments are taken into account in the aggregation process. The determination of appropriate OWA weights is, therefore, a very important object of study when applying OWA functions in the context of decision making. Among various methods in the literature, namely [109, 111, 119], and more recently [13, 29, 81, 45], we distinguish three main methods based on: a) the characteristic measures of orness and dispersion, b) learning OWA weights from data, and c) weight-generating functions.

The remainder of this chapter is organized as follows. In Sect. 2.2 we briefly review OWA functions and their fundamental properties. Section 2.3 discusses three main methods for obtaining OWA weights and the associated OWA families. Finally, Sect. 2.4 contains some conclusive remarks.

2.2 Definition of OWA functions and main properties

Given two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, with $n \geq 2$, the increasing and decreasing reordering of the coordinates of $\boldsymbol{x}$ are denoted as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. This can be rewritten in a more general way as $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ where σ is a permutation on $\{1, \ldots, n\}$. We use the notation $\boldsymbol{x} \geq \boldsymbol{y}$ instead of stating $x_i \geq y_i$ for every $i = 1, \ldots, n$. Accordingly, we use $\boldsymbol{x} > \boldsymbol{y}$ when $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. We now introduce the basic definitions of averaging functions, weighted averaging functions, ordered weighted averaging functions and their fundamental properties.

2.2.1 Definitions

In this thesis, we assume that a point $\boldsymbol{x} = (x_1, \ldots, x_n)$ belongs to the interval domain $\mathbb{D}^n$, where $\mathbb{D} = [a, b] \subset \mathbb{R}$ is the domain that we work on.

Definition 2.1. A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is called *monotonic* if it holds that $A(\boldsymbol{x}) \leq A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ where $\boldsymbol{x} \leq \boldsymbol{y}$. On the other hand, it is called *strictly monotonic* if it holds that $A(\boldsymbol{x}) < A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ where $\boldsymbol{x} < \boldsymbol{y}$.

Definition 2.2. A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is called *idempotent* if it holds that $A(x \cdot \mathbf{1}) = x$ for all $x \in \mathbb{D}$.

Definition 2.3. A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is called *nilpotent* if it holds that $A(x \cdot \mathbf{1}) = 0$ for all $x \in \mathbb{D}$.

Definition 2.4. A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is called *symmetric* if it holds that $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$ for any permutation σ of the argument $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Definition 2.5. An *aggregation function* is a function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $\mathbb{D} = [a, b] \subset \mathbb{R}$, that aggregates $n \geq 2$ arguments and produces a single output with the following properties:

(i) boundary: $A(a, a, \cdots, a) = a$ and $A(b, b, \cdots, b) = b$;

(ii) monotonicity: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, if $\boldsymbol{x} \leq \boldsymbol{y}$ then $A(\boldsymbol{x}) \leq A(\boldsymbol{y})$.

Aggregation functions consist of four main classes: averaging, conjunctive, disjunctive and mixed functions [13]. In the context of this thesis, we focus on the averaging functions and in particular on two families: weighted averaging and ordered weighted averaging functions.

Definition 2.6. An aggregation function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is called an *averaging function* if it is idempotent.

We notice that the monotonicity and the idempotency implicitly mean the compensativeness $min(\boldsymbol{x}) \leq A(\boldsymbol{x}) \leq max(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}^n$.

Definition 2.7. A vector $\boldsymbol{w} = (w_1, \ldots, w_n)$ is called a *weighting vector* if $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$ for $i = 1, \ldots, n$.

Definition 2.8. A *Weighted Averaging (WA) function* of n arguments is an averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ with an associated weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0, 1]^n$, such that $\sum_{i=1}^{n} w_i = 1$ and

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_i. \tag{2.1}$$

Definition 2.9. An *Ordered Weighted Averaging (OWA) function* of n arguments is an averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ with an associated weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0, 1]^n$, such that $\sum_{i=1}^{n} w_i = 1$ and

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}. \tag{2.2}$$

Different OWA functions are classified according to their weighting vectors. As briefly discussed in Sect. 2.1 the OWA weights are characterized by two measures called orness and dispersion. In the following section we review these two measures and their properties.

2.2.2 Main properties

Consider an OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ with an associated weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0, 1]^n$ such that $\sum_{i=1}^{n} w_i = 1$. The OWA function A has the following fundamental properties which are monotonicity, idempotency (thus compensativeness), and symmetry. The monotonicity is inherited directly from the properties of aggregation functions while the idempotency is from the averaging function. In more detail:

(i) monotonicity: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, if $\boldsymbol{x} \geq \boldsymbol{y}$ then $A(\boldsymbol{x}) \geq A(\boldsymbol{y})$;

(ii) idempotency: for all $x \in \mathbb{D}$, it holds that $A(x \cdot \mathbf{1}) = x$;

(iii) symmetry: given $\boldsymbol{x} \in \mathbb{D}^n$ and a permutation σ on $\{1,\ldots,n\}$, A is symmetric due to $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$.

We notice that the monotonicity and the idempotency implicitly mean the compensativeness $min(\boldsymbol{x}) \leq A(\boldsymbol{x}) \leq max(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}^n$.

Concerning the weighting vector, it is characterized by the two measures of *orness* and *dispersion* introduced by Yager [115]:

$$orness(\boldsymbol{w}) = \frac{1}{n-1}\sum_{i=1}^{n}(i-1)w_i \tag{2.3}$$

$$dispersion(\boldsymbol{w}) = -\sum_{i=1}^{n} w_i ln w_i. \tag{2.4}$$

The orness measure evaluates the similarity between the OWA function and the *or (max)* operator. The dispersion measure, in contrast, indicates how equally input arguments are taken into account in the aggregation process. The highest dispersion is, therefore, associated with the uniform distribution of OWA weights.

Three special OWA weighting vectors associated with the min function, arithmetic mean and max function are $\boldsymbol{w}_* = (1, 0, \ldots, 0)$, $\boldsymbol{w}_{AM} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ and $\boldsymbol{w}^* = (0, \ldots, 0, 1)$. For these vectors the orness is equal to $0, 0.5,$ and 1; and the disparity is equal to $0, ln(n),$ and 0, respectively.

2.3 Methods for determining the OWA weights

OWA functions are characterized by their weighting vectors. The determination of appropriate OWA weights is, therefore, a very important object of study when applying OWA functions in the context of decision making. In this section we review some methods for determining the OWA weights and the associated OWA families.

2.3.1 Methods based on the measures of orness and dispersion

Several methods have been introduced to obtain optimal OWA weights based on two characterizing measures of orness and dispersion. In this regard, the pioneering work of O'Hagan [92] introduced the maximal entropy method to compute a set of the OWA weights satisfying a constraint of a specific orness value. The related optimization problem is formulated as follow:

$$
\begin{aligned}
\max_{\boldsymbol{w}} \quad & -\sum_{i=1}^{n} w_i ln w_i \\
\text{s.t.} \quad & 0 \leq w_i \leq 1, \qquad i = 1, \dots, n\,, \\
& \sum_{i=1}^{n} w_i = 1\,, \\
& \text{orness}(\boldsymbol{w}) = \eta = \frac{1}{n-1} \sum_{i=1}^{n} (i-1) w_i\,,
\end{aligned}
\tag{2.5}
$$

where η, with $0 \leq \eta \leq 1$, stands for the orness of the weighting vector.

The OWA weights that maximize the entropy function and satisfy the predefined level of orness are called MEOWA. The optimization problem proposed by O'Hagan [92] is then solved by Fuller and Majlender [53] by using a different approach. The authors applied the Lagrangian method in order to transform O'Hagan constrained optimization problem into the polynomial optimization problem which is more tractable for the derivation of analytical solutions. This is achieved by introducing two Lagrangian multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ to incorporate all constraints from the original problem into the Lagrangian dual function $L(\boldsymbol{w}, \lambda_1, \lambda_2)$ as follows

$$
\max_{\boldsymbol{w}} L(\boldsymbol{w}, \lambda_1, \lambda_2) = -\sum_{i=1}^{n} w_i ln w_i + \lambda_1 \left(\sum_{i=1}^{n} w_i - 1 \right) + \lambda_2 \left(\sum_{i=1}^{n} \frac{i-1}{n-1} w_i - \eta \right). \tag{2.6}
$$

In the new formulation, the optimal weights are analytically derived from the partial derivatives of the Lagrangian dual function as follows

$$
\begin{aligned}
\frac{\partial L}{\partial w_j} &= -ln w_j - 1 + \lambda_1 + \frac{j-1}{n-1} \lambda_2 = 0\,, \qquad \forall j = 1, \dots, n\,, \\
\frac{\partial L}{\partial \lambda_1} &= \sum_{i=1}^{n} w_i - 1 = 0\,, \\
\frac{\partial L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{i-1}{n-1} w_i - \eta = 0\,.
\end{aligned}
$$

The associated weighting vector is obtained by solving the equations of the partial derivatives. The detailed steps for solving the problem are described in [53].

In the context of determining OWA weights based on their characterizing dispersion measure, one can also use the concept of variance to propose a parameterized family of OWA functions between the min and the max functions. The variance of an OWA weighting vector in [120] is formulated as the average of the squared differences between each single weight and the arithmetic mean as follows

$$D^2(\boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{n}(w_i - E(\boldsymbol{w}))^2 = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} \tag{2.7}$$

where $E(\boldsymbol{w}) = \frac{1}{n} \cdot w_1 + \ldots + \frac{1}{n} \cdot w_n = \frac{1}{n}$ stands for the arithmetic mean of OWA weights. Fuller and Majlender [54] applied the concept of variance in their optimization problems in order to identify the OWA weights with the minimal variability. The authors then used the Karush-Kuhn-Tucker second-order conditions in order to solve the following optimization problem analytically:

$$\begin{aligned}
\min_{\boldsymbol{w}} \quad & D^2(\boldsymbol{w}) = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} \\
\text{s.t.} \quad & 0 \le w_i \le 1\,, \qquad i = 1, \ldots, n\,, \\
& \sum_{i=1}^{n} w_i = 1\,, \\
& \text{orness}(\boldsymbol{w}) = \eta = \frac{1}{n-1}\sum_{i=1}^{n}(i-1)w_i\,,
\end{aligned} \tag{2.8}$$

where η, with $0 \le \eta \le 1$, stands for the orness of the weighting vector.

After the work of O'Hagan [92] on the maximal entropy method and the variance-based methods of Yager [120], and Fuller and Majlender [54], Wang and Parkan [107] proposed the minimax disparity method, in which the objective is to minimize the maximum absolute differences between two adjacent weights, as follows:

$$\begin{aligned}
\min_{\boldsymbol{w}} \quad & \left\{ \max_{i \in \{1,\ldots,n-1\}} |w_i - w_{i+1}| \right\} \\
\text{s.t.} \quad & 0 \le w_i \le 1\,, \qquad i = 1, \ldots, n\,, \\
& \sum_{i=1}^{n} w_i = 1\,, \\
& \text{orness}(\boldsymbol{w}) = \eta = \frac{1}{n-1}\sum_{i=1}^{n}(i-1)w_i\,,
\end{aligned} \tag{2.9}$$

where η, with $0 \le \eta \le 1$, stands for the orness of the weighting vector.

One of the advantages of the minimax disparity approach is the use of a simple linear programming model in order to obtain the OWA weights. The objective function is non-linear due to the absolute difference between two adjacent weights. In order to overcome this non-linearity, the authors denote by $\delta = \max_{i \in \{1,\ldots,n-1\}} |w_i - w_{i+1}|$ the maximum absolute difference between two adjacent weights.

We then have

$$|w_i - w_{i+1}| \leq \delta, \qquad i = 1, \ldots, n-1, \tag{2.10}$$

or

$$-\delta \leq w_i - w_{i+1} \leq \delta, \qquad i = 1, \ldots, n-1. \tag{2.11}$$

This expression is equivalently rewritten by two inequations $w_i - w_{i+1} - \delta \leq 0$ and $w_i - w_{i+1} + \delta \geq 0$ where $i = 1, \ldots, n-1$. The original optimization problem is, thereby, reformulated into the linear programming problem as follows,

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \delta \\ \text{s.t.} \quad & 0 \leq w_i \leq 1\,, \qquad i = 1, \ldots, n\,, \\ & \sum_{i=1}^{n} w_i = 1\,, \\ & \text{orness}(\boldsymbol{w}) = \eta = \frac{1}{n-1} \sum_{i=1}^{n} (i-1) w_i\,, \\ & w_i - w_{i+1} - \delta \leq 0\,, \qquad i = 1, \ldots, n-1\,, \\ & w_i - w_{i+1} + \delta \geq 0\,, \qquad i = 1, \ldots, n-1\,, \end{aligned} \tag{2.12}$$

where η, with $0 \leq \eta \leq 1$, stands for the orness of the weighting vector.

Moreover, Liu [80] has proved the equivalence in the solution of the minimum variance approach suggested by Fuller and Majlender [54] and the minimax disparity method proposed by Wang and Parkan [107] under a desired orness level. Further discussion on the extensions of disparity-based models for determining OWA weights can be found in [4, 44, 57, 96, 106]. A review of a number of approaches for determining OWA weights based on the characterizing measures is briefly summarized in [52].

2.3.2 Methods based on data

Filev and Yager [49] suggested a new method to learn OWA weights from a collection of observational data, see also Yager and Filev [124]. Let us consider a collection of K samples, with each kth sample consisting of $(n+1)$-tuples $\{(x_{k1}, \ldots, x_{kn}), y_k\}$, where $(x_{k1}, \ldots, x_{kn}) \in \mathbb{D}^n$ are n-ary input arguments and $y_k \in \mathbb{D}$ is the relevant observed aggregated value. The optimal OWA weights are obtained by minimizing the instance errors between the current predicted aggregation value $A(x_{k1}, \ldots, x_{kn})$ and the actual observed value y_k, for the entire set of samples. The problem is formulated as follows

$$\min_{\boldsymbol{w}} \quad \sum_{k=1}^{K} \frac{1}{2} \left(b_{k1} w_1 + \ldots + b_{kn} w_n - y_k\right)^2 \tag{2.13}$$

$$\text{s.t.} \qquad 0 \le w_i \le 1\,, \qquad i = 1,\ldots,n\,,$$
$$\sum_{i=1}^{n} w_i = 1\,,$$

where $k = 1,\ldots,K$ denotes the k-th sample in the collection and b_{ki} denotes the $i-th$ largest element of the argument vector $(x_{k1},\ldots,x_{kn})$ for every fixed k. The authors transformed the constrained optimization problem into an unconstrained problem by expressing the OWA weights as exponential function,

$$w_i = \frac{e^{\lambda i}}{\sum_{j=1}^{n} e^{\lambda i}}. \tag{2.14}$$

The required properties of the weighting vector, that is, $\boldsymbol{w} = (w_1,\ldots,w_n) \in [0,1]^n$ and $\sum_{i=1}^{n} w_i = 1$, are implicitly satisfied by the new formulation. Other studies have focused on solving the above problem by using quadratic programming methods [11, 12, 103, 104, 114, 117].

2.3.3 Methods based on weight-generating functions

Yager [115] introduced the OWA functions and described a mechanism for obtaining OWA weights that is analogous to the procedure of combining multiple values under the guidance of linguistic quantifiers. The author considered two extreme cases of OWA functions. One extreme, associated with the *and* function, requires all the criteria to be satisfied and it corresponds to the quantifier *for all*. The other, called *or*, requires at least one of the criteria be satisfied and it corresponds to the quantifier *there exists*. In the remaining cases, the OWA functions, bounded between the *and* and the *or* functions, correspond to quantifiers, such as *few, at least half, many, most*. According to Zadeh [128], these linguistic quantifiers can be mathematically expressed by a fuzzy subset Q of the unit interval $\mathbb{I} = [0,1]$ defined as $Q : [0,1] \longrightarrow [0,1]$.

In the context of quantifiers guided aggregation, Yager [121] proposed a new approach for obtaining OWA weights by a fuzzy subset Q of the unit interval $\mathbb{I} = [0,1]$. The associated OWA functions are also called quantifier guided OWA functions. The author introduced Regular Increasing Monotone (RIM) quantifiers which are monotonic and continuous on the interval unit $Q : [0,1] \longrightarrow [0,1]$ satisfying $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \le Q(y)$ if $x \le y$.

Using the RIM quantifiers, we obtain the associated OWA weights defined as

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \qquad i = 1,\ldots,n. \tag{2.15}$$

A generating function associated with the quantifier is a function $f(x) : [0, 1] \longrightarrow [0, \infty)$ defined as

$$Q(x) = \int_0^x f(t)dt. \tag{2.16}$$

Generating functions based on quantifiers can produce OWA weights that satisfy a set of criteria in accordance to the natural linguistic expressions. This method is also applicable when the dimension of input arguments is unknown. The method, however, is not widely used in practice because of the more complex intuition of the integral behind it. Other studies that focused on this method are [79, 82, 94, 116, 122].

2.4 Discussion and conclusion

In many disciplines, aggregating multiple input arguments to produce an overall score for a set of alternatives is a crucial part in decision making process, which can effectively guide decision makers in reaching a decision. The OWA functions, introduced by Yager, are considered as one of the most effective methods for aggregating data due to their flexibility in providing a general class of weighted aggregation functions bound between two extreme cases: the *and* function, when all criteria have to be satisfied; and the *or* function, when at least one of the criteria has to be satisfied. The OWA functions can be also seen as aggregation functions guided by the linguistic quantifiers, including *most, at least one, most of.* The weights associated with the OWA functions reflect the number of criteria and their level of satisfaction that are required by the applications. Determining appropriate OWA weights has thus become an important object of study. Several methods have been introduced for obtaining OWA weights and they can be based on: a) the characteristic measures of orness and dispersion, b) learning OWA weights from observational data, and c) OWA weight-generating functions. In the next chapter we recall the binomial decomposition framework of OWA functions. We study the relationship between OWA weights and the associated coefficients of the binomial decomposition of OWA functions. We then revisit the minimax disparity model, as it has recently received great deal of attention, in the context of large-scale optimization problems, where a challenge is represented by a heavy computational load. We propose a new approach based on the binomial decomposition of OWA functions and the k-additivity framework to overcome this computational complexity.

Chapter 3

The binomial decomposition of OWA functions

OWA functions can be equivalently represented in the binomial decomposition framework. In this chapter we recall the binomial decomposition framework described in the context of the k-additivity, with a particular focus on the 2-additive and 3-additive cases. In addition, we identify the close relationship between OWA weights and the associated coefficients in the binomial decomposition framework. We derive an analytical formulation expressing these coefficients as a function of OWA weights.

We then consider one possible application of the binomial decomposition of OWA functions by revisiting the well-known minimax disparity method for determining OWA weights in the context of large-scale optimization problems, which often requires heavy computational loads. We propose a new approach based on the binomial decomposition framework, with reference to the k-additive framework. This allows us to transform the original problem, expressed in terms of OWA weights, into a problem in which the weights are substituted by a new set of coefficients. In this transformed representation, we consider only a limited number of these coefficients, associated with the first k-additive levels of the OWA function, and at the same time we set the remaining coefficients to zero, thereby reducing the computational loads in large-scale optimization problems.

3.1 Background

In many disciplines, decision makers have to deal with problems involving the aggregation and production of overall assessments from some evaluations according to a set of criteria. The ordered weighted averaging functions (OWA), introduced by Yager [115], are one of the fundamental aggregation functions in decision making theory. A review of OWA functions and their applications in decision making are discussed in [125, 126].

An alternative approach to represent the preferences of decision makers over a set of alternatives with respect to a set of criteria is the Choquet integral [36], which was introduced in the classical multiattribute utility theory model. A synthesis on the use of the Choquet integral and its applications in the context of multicriteria decision making are discussed in [5, 6, 34, 38, 58, 59, 65, 66, 67, 84, 89, 97].

One of the advantages of the Choquet integral framework comes from its *capacities* (also called *fuzzy measures*) that are able to model interactions between criteria in a flexible way. Unlike common aggregation functions, such as the weighted arithmetic mean, which assumes that the criteria are mutually independent, the Choquet integral takes into account both the importance of a criterion as a *singleton* and its relevance when interacting with other criteria in a group. The complex interactions among the criteria, which cannot be modeled by the weighted sum, have been addressed by the Choquet capacities by taking into account *weakening* and *strengthening* interactions among criteria [58]. A weakening interaction happens among criteria that share some similar features, which means that the satisfaction of one criterion is usually sufficient to the satisfaction of another. The *subadditive* capacity is, therefore, used in order to avoid the drawback of overestimated evaluations on overlapping criteria. A strengthening interaction, in which the simultaneous satisfaction of two criteria is more important than the individual satisfaction, can be modeled instead by a *superadditive* capacity. As concerns the evaluation of the complex interactions, the Choquet integral provides an aggregation method capable of measuring all possible interactions, including overlapping/supporting or no interaction among criteria.

Several studies [34, 38, 58, 59, 84, 89] have showed that the Choquet integral in the finite domain includes the weighted averaging (WA) and the ordered weighted averaging (OWA) functions as two special cases associated with the additive and symmetric capacities, respectively. Fodor et. al. [50] proved that OWA functions can be equivalently expressed as Choquet integrals with symmetric capacities.

Methods for defining the capacities of Choquet integration are reviewed in [64]. They, however, often require 2^n real coefficients to be defined. If the number of criteria is large, the definition of these coefficients can be exponentially increased. One of the possible solutions to reduce the complextity of the definition of symmetric capacities is to transform them into the Möbius representation and to apply the k-additivity framework introduced by Grabisch [60, 63], see also [61, 62]. A capacity is k-additive when its computation depends on coalitions of at most k cardinality.

Choquet integrals with symmetric capacities, expressed by means of the Möbius transform, can be reduced to OWA functions as proved in [50]. In the context of the binomial decomposition framework proposed by Calvo and De Baets [27], OWA functions can be uniquely written as linear combinations of binomial OWA functions and the associated coefficients of the binomial decomposition framework. The binomial decomposition framework was also studied in the restricted context of generalized Gini welfare functions by Bortot and Marques Pereira [21].

In this chapter we investigate the 2-additive and 3-additive cases of the binomial decomposition of OWA functions in n dimensions. We highlight the close relationship between OWA weights and the coefficients of the binomial decomposition of OWA functions. We derive an analytical expression for these coefficients in terms of OWA weights [91]. We then revisit the well-known minimax disparity method for determining OWA weights, as introduced in Sect. 2.3.1, in the context of large-scale optimization problems. We propose a new approach based on the binomial decomposition framework expressed in terms of k-additive capacities. This allows us to transform the original problem, expressed in terms of OWA weights, into a problem in which the weights are substituted by a new set of coefficients. In this transformed representation, we consider only a limited number of these coefficients, associated with the k-additivity of the OWA functions, and we set the remaining coefficients to zero [90].

The structure of the remainder of this chapter is as follows. Section 3.2 introduces the basic definitions on capacities, Choquet integration and the binomial decomposition of OWA functions, with reference to the Möbius representation framework. We then study the binomial decomposition of OWA functions in the 2-additive and 3-additive cases. Section 3.3 derives the analytical expression for the coefficients of the binomial decomposition of OWA functions in terms of OWA weights. Section 3.4 presents our proposed approach, based on the binomial decomposition and k-additivity frameworks, to solve the well-known minimax disparity method for determining OWA weights in large-scale optimization problems. Finally, Sect. 3.5 contains some conclusive remarks.

3.2 OWA functions in the binomial decomposition framework

In this section we present the binomial decomposition of OWA functions and we recall the main concepts related to the Choquet integral and the Möbius representation framework.

3.2.1 Capacities and Choquet integral

Let us consider a finite set of n elements $N = \{1, 2, \ldots, n\}$, which can represent attributes or criteria in a decision making problem. Subsets $S, T \subseteq N$ associated with cardinalities $0 \leq s, t \leq n$ are called *coalitions*, where s and t denote $|S|$ and $|T|$, respectively. In the following we recall the basic definitions of capacities and Choquet integration as in [36, 38, 58, 59, 101, 21].

Definition 3.1. A *discrete capacity* defined on $N = \{1, 2, \ldots, n\}$ is a set function $\mu : 2^N \longrightarrow [0, 1]$ satisfying the following properties:

(i) boundary: $\mu(\emptyset) = 0, \mu(N) = 1$;

(ii) monotonicity: $S \subseteq T \subseteq N$ implies $\mu(S) \leq \mu(T)$.

Since in this thesis we are interested only in discrete capacities, which are defined on finite discrete subsets, we simply use the term capacities to refer to them. The notions of capacities $\mu(S)$ and $\mu(T)$ represent the importance of the associated coalitions. The monotonicity condition implies that if a new element is added to a coalition, the capacity of the extended coalition is not less than its weight. Capacities are also called *fuzzy measures* in the multiattribute utility theory model [101]. A set of all the capacities defined on a set N of n elements requires 2^n coefficients, which are the value of $\mu(S)$ for all possible coalitions S, where $S \subseteq N$ and $s = 0, 1, \ldots, n$. This includes singletons, pairs of elements, and subsets of more elements. If the number of criteria is large, representing the importance of all possible coalitions requires a large number of real coefficients, making modeling large-scale problems more difficult.

Definition 3.2. Let us consider disjoint coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$. A capacity μ is said to be:

(i) *additive* if $\mu(S \cup T) = \mu(S) + \mu(T)$,

(ii) *subadditive* if $\mu(S \cup T) \leq \mu(S) + \mu(T)$,

(iii) *superadditive* if $\mu(S \cup T) \geq \mu(S) + \mu(T)$.

for all disjoint coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$.

Definition 3.3. The discrete *Choquet integral* $C_\mu : \mathbb{D}^n \longrightarrow \mathbb{D}$ with respect to a capacity μ is defined as follows:

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \left[\mu(A_{(i)}) - \mu(A_{(i+1)})\right] x_{(i)} \tag{3.1}$$

where $A_i = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$, with $(\cdot)$ representing a permutation on N given by $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$.

The Choquet integral consists of two particular cases: WA functions, associated with the additive capacities; and OWA functions, associated with the symmetric capacities, as discussed in [34, 38, 58, 59, 84, 89].

The Choquet integral plays an important role in the aggregation process. To give a comparison, we consider the weighted sum which assumes that the criteria are mutually independent. The Choquet integral, in contrast, takes into account both the importance of a criterion as a *singleton* and its relevance when interacting with other criteria in a group. As an example, let us consider the problem of evaluating students in a high school with respect to their scores on three subjects: mathematics, physics and literature, as described in [58, 59]. The school policy prefers students who are good at all subjects. However, it favors scientific subjects more than literature. The common weighted sum, whose weights are interpreted as the importance of different subjects, insufficiently fulfills this policy, due to its overestimated evaluation on students with respect to the scientific subjects; and its underestimated evaluation between scientific and literature subjects. Instead, as described in Def. 3.2, Choquet capacities can be subadditive and superadditive, thus providing flexible ways to properly evaluate the complex interactions among three subjects as required by the school policy.

Definition 3.4. The *Möbius transform* m_μ: $2^N \longrightarrow \mathbb{R}$ with respect to the capacity μ on the set N is a set function defined for every $T \subseteq N$ as

$$m_\mu(T) = \sum_{S \subset T} (-1)^{t-s} \mu(S) \tag{3.2}$$

where s and t are the cardinality of the coalitions S and T, respectively.

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \qquad\qquad \sum_{T \subseteq N} m_\mu(T) = 1 \tag{3.3}$$

and the monotonicity conditions take the form:

$$\sum_{S \subseteq T} m_\mu(S \cup \{i\}) \geq 0 \qquad \text{for all } T \subseteq N \setminus \{i\} \qquad \text{and } i = 1, \dots, n\,. \tag{3.4}$$

Notice that the boundary and monotonicity conditions of Möbius capacities correspond to the same conditions of the capacities defined in Definition 3.1.

Definition 3.5. The *Möbius transform* for the capacity μ: $2^N \longrightarrow \mathbb{R}$ is the inverse of the Möbius transform m_μ given as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \qquad T \subseteq N\,. \tag{3.5}$$

The inverse Möbius transform provides an alternative to express capacities in terms of m_μ. The Choquet integral, therefore, can be rewritten in terms of m_μ equivalently.

Definition 3.6. The Choquet integral with respect to the Möbius transform m_μ of capacities μ is given by [61, 84],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{T \subseteq N} m_\mu(T) \min_{i \in T}(x_i)\,. \tag{3.6}$$

We recall that the problem of defining a capacity μ on a set N of n elements requires 2^n real coefficients to represent the importance of all possible coalitions $T \subseteq N$, including singletons, pairs of elements, and subsets of more elements. When the number of criteria is large, one needs 2^n real coefficients to represent the importance of all possible coalitions. The computational load, therefore, increases exponentially. One of possible solutions is proposed by Grabisch [63]. The author suggested the use of k-additive capacities to control the level of complexity, thus reducing the exponential computational burden.

Definition 3.7. A capacity μ defined on the set N is said to be *k-additive* if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ such that $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.

We then rewrite the capacity $\mu(T)$, as expressed in (3.5), with the restriction given by k-additivity, as follows

$$\mu(T) = \sum_{S \subseteq T,\, s \leq k} m_\mu(S) \qquad T \subseteq N \tag{3.7}$$

where the Möbius transform is subject to the boundary (3.3) and monotonicity (3.4), conditions given as

$$m_\mu(\emptyset) = 0 \qquad\qquad \sum_{T\subseteq N,\, t\leq k} m_\mu(T) = 1 \tag{3.8}$$

$$\sum_{S\subseteq T,\, s\leq k-1} m_\mu(S\cup\{i\}) \geq 0 \qquad\qquad T\subseteq N\setminus\{i\} \qquad i = 1,\ldots,n\,. \tag{3.9}$$

We now examine the capacities in the k-additive case, focusing on two special cases $k = 1$ and $k = 2$.

In the 1-additive case, the capacities for coalitions $T \subseteq N$ are expressed as a sum of the capacities of the singletons

$$\mu(T) = \sum_{\{i\}\subset T} \mu(\{i\}) \tag{3.10}$$

and we, therefore, need only n real coefficients to define capacities on the set N.

In the 2-additive case, the capacities are given by

$$\mu(T) = \sum_{S\subseteq T,\, s\leq 2} m_\mu(S) \qquad\qquad T\subseteq N \tag{3.11}$$

which reduces to

$$\mu(T) = \sum_{\{i\}\subset T} m_\mu(\{i\}) + \sum_{\{i,j\}\subset T} m_\mu(\{i,j\}) \tag{3.12}$$

By replacing the Möbius transform m_μ of the singletons and the pairs of criteria with their values, as given in (3.2), the above expression can be written as

$$\begin{aligned}\mu(T) &= \sum_{\{i\}\subset T} \mu(\{i\}) + \sum_{\{i,j\}\subset T}\ \sum_{S\subset\{i,j\}} (-1)^{2-s}\mu(S) \\ &= \sum_{\{i\}\subset T} \mu(\{i\}) + \sum_{\{i,j\}\subset T} \Big(\mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\})\Big).\end{aligned} \tag{3.13}$$

In the second summation, if we expand them and group all components μ of the same singletons together, we obtain

$$\mu(T) = \sum_{\{i\}\subset T} \mu(\{i\}) + \sum_{\{i,j\}\subset T} \mu(\{i,j\}) - (t-1)\sum_{\{i\}\subset T} \mu(\{i\}) \tag{3.14}$$

which can be reduced to

$$\mu(T) = \sum_{\{i,j\}\subset T} \mu(\{i,j\}) - (t-2)\sum_{\{i\}\subset T} \mu(\{i\}). \tag{3.15}$$

In this representation we notice that the 2-additive capacities on the set N depend only on the capacities of the singletons $\mu(\{i\})$ and the pairs of criteria $\mu(\{i,j\})$, thus requiring $\frac{n(n+1)}{2}$ real coefficients to express the importance of all coalitions. In the general k-additive cases, the required number of coefficients is equal to $\sum_{j=1}^{k}\binom{n}{j}$, which is much less than the original number of 2^n.

In the following we recall the basic definition of symmetric capacities. Fodor et. al. [50] proved that the Choquet integrals associated with symmetric capacities correspond to the OWA functions.

Definition 3.8. A capacity μ is called symmetric if the value $\mu(T)$ depends only on the cardinality of the coalition $T \subseteq N$

$$\mu(T) = \mu(t) \qquad \text{where} \qquad t = |T|\,. \tag{3.16}$$

Let $S, T \subseteq N$ be two coalitions with the associated cardinalities s and t, respectively. If their cardinalities are the same, it holds that

$$\mu(S) = \mu(T) \qquad \text{if} \qquad s = t\,. \tag{3.17}$$

Analogously, the Möbius transform m_μ with respect to the symmetric capacities μ takes the form

$$m_\mu(T) = m_\mu(t) \qquad \text{where} \qquad t = |T|\,. \tag{3.18}$$

The capacity $\mu(T)$, as described in (3.5), takes the following form in the symmetric case

$$\mu(t) = \sum_{s=1}^{t}\binom{t}{s}m_\mu(s) \qquad\qquad t = 1, \dots, n \tag{3.19}$$

where the boundary and monotonicity conditions as in (3.3) and (3.4) can be written as

$$m_\mu(0) = 0 \qquad\qquad \sum_{s=1}^{n}\binom{n}{s}m_\mu(s) = 1 \tag{3.20}$$

$$\sum_{s=1}^{t}\binom{t-1}{s-1}m_\mu(s) \geq 0 \qquad\qquad t = 1, \dots, n\,. \tag{3.21}$$

The Choquet integrals with symmetric capacities correspond to the OWA functions, as proved by Fodor et. al. [50],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n}[\mu(n-i+1) - \mu(n-i)]x_{(i)} = \sum_{i=1}^{n} w_i\, x_{(i)} = A(\boldsymbol{x}) \tag{3.22}$$

where the OWA weights are given by $w_i = \mu(n-i+1) - \mu(n-i)$. The OWA weights expressed in terms of symmetric capacities are non-negative $w_i \geq 0$, for $i = 1, \dots, n$, due to the monotonicity of the capacity μ. The total sum of OWA weights is written as

$$\begin{aligned} \sum_{i=1}^{n} w_i &= \sum_{i=1}^{n} (\mu(n-i+1) - \mu(n-i)) \\ &= (\mu(n) - \mu(n-1)) + (\mu(n-1) - \mu(n-2)) + \ldots + (\mu(1) - \mu(0)) . \end{aligned} \tag{3.23}$$

This sum is expressed in terms of telescoping series and so it can be reduced to $\sum_{i=1}^{n} w_i = \mu(n) - \mu(\emptyset) = 1$.

3.2.2 The binomial decomposition framework

In the following we recall the definitions of the binomial decomposition of OWA functions introduced by Calvo and De Baets [27], see also Bortot and Marques Pereira [21]. We consider the binomial decomposition of OWA functions in the k-additive framework introduced by Grabisch [60, 63], see also [61, 62], with a particular focus on the 2-additive and 3-additive cases [23].

Proposition 3.9. *The* binomial weights w_{ji}*, with* $i, j = 1, \dots, n$*, are defined by the following weighting vector*

$$w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}} \tag{3.24}$$

where the binomial weights w_{ji}*,* $i, j = 1, \dots, n$*, are zero when* $i + j > n + 1$*, due to the convention that* $\binom{p}{q} = 0$ *when* $p < q$*, with* $p, q = 0, 1, \dots$*.*

Proof The binomial weights are evidently bounded in the unit interval $w_{ji} \in [0, 1]$.

The sum of all binomial weights takes the following form, due to Pascal's rule,

$$\sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}} = \frac{1}{\binom{n}{j}} \sum_{i=1}^{n} \left[\binom{n-i+1}{j} - \binom{n-i}{j} \right] . \tag{3.25}$$

Simplifying the summation, we obtain $\binom{n}{j}$ that leads to $\sum_{i=1}^{n} w_{ji} = 1$. □

Let us give examples of binomial weights w_{ji}, $i, j = 1, \dots, n$, in dimensions $n = 2, 3, 4, 5, 6$. In what follow we denote the vector of the binomial weights w_{ji}, $i, j = 1, \dots, n$, associated with the binomial OWA functions C_j, with $j = 1, \dots, n$, by $\boldsymbol{w}_j$.

$n = 2$

$$\begin{aligned} \boldsymbol{w}_1 &= (\tfrac{1}{2}, \tfrac{1}{2}) \\ \boldsymbol{w}_2 &= (1, 0) \end{aligned}$$

$n = 3$

$$\begin{aligned} \boldsymbol{w}_1 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) \\ \boldsymbol{w}_2 &= (\tfrac{2}{3}, \tfrac{1}{3}, 0) \\ \boldsymbol{w}_3 &= (1, 0, 0) \end{aligned}$$

$n = 4$

$$\begin{aligned} \boldsymbol{w}_1 &= (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \\ \boldsymbol{w}_2 &= (\tfrac{3}{6}, \tfrac{2}{6}, \tfrac{1}{6}, 0) \\ \boldsymbol{w}_3 &= (\tfrac{3}{4}, \tfrac{1}{4}, 0, 0) \\ \boldsymbol{w}_4 &= (1, 0, 0, 0) \end{aligned}$$

$n = 5$

$$\begin{aligned} \boldsymbol{w}_1 &= (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}) \\ \boldsymbol{w}_2 &= (\tfrac{4}{10}, \tfrac{3}{10}, \tfrac{2}{10}, \tfrac{1}{10}, 0) \\ \boldsymbol{w}_3 &= (\tfrac{6}{10}, \tfrac{3}{10}, \tfrac{1}{10}, 0, 0) \\ \boldsymbol{w}_4 &= (\tfrac{4}{5}, \tfrac{1}{5}, 0, 0, 0) \\ \boldsymbol{w}_5 &= (1, 0, 0, 0, 0) \end{aligned}$$

$n = 6$

$$\begin{aligned} \boldsymbol{w}_1 &= (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}) \\ \boldsymbol{w}_2 &= (\tfrac{5}{15}, \tfrac{4}{15}, \tfrac{3}{15}, \tfrac{2}{15}, \tfrac{1}{15}, 0) \\ \boldsymbol{w}_3 &= (\tfrac{10}{20}, \tfrac{6}{20}, \tfrac{3}{20}, \tfrac{1}{20}, 0, 0) \\ \boldsymbol{w}_4 &= (\tfrac{10}{15}, \tfrac{4}{15}, \tfrac{1}{15}, 0, 0, 0) \\ \boldsymbol{w}_5 &= (\tfrac{5}{6}, \tfrac{1}{6}, 0, 0, 0, 0) \\ \boldsymbol{w}_6 &= (1, 0, 0, 0, 0, 0) \end{aligned}$$

Proposition 3.10. *The relationship between two distinct binomial weights w_{j,i_1} and w_{j,i_2} having the same index j is*

$$\frac{w_{j,i_1}}{w_{j,i_2}} = \prod_{m=1}^{i_2-i_1} \frac{n-i_2+m}{n+1-j-i_2+m} \tag{3.26}$$

where $1 \leq i_1 < i_2 \leq n$ and $1 \leq j \leq n$.

The relationship between two distinct binomial weights $w_{j_1,i}$ and $w_{j_2,i}$ having the same index i is

$$\frac{w_{j_1,i}}{w_{j_2,i}} = \frac{j_1}{j_2} \prod_{m=1}^{j_2-j_1} \frac{n-j_2+m}{n+1-i-j_2+m} \tag{3.27}$$

where $1 \leq j_1 < j_2 \leq n$ and $1 \leq i \leq n$.

Proof. From (3.39), we have

$$\begin{aligned} \frac{w_{j,i_1}}{w_{j,i_2}} &= \frac{\binom{n-i_1}{j-1}}{\binom{n}{j}} \cdot \frac{\binom{n}{j}}{\binom{n-i_2}{j-1}} \\ &= \frac{(n-i_1)!}{(j-1)!(n+1-j-i_1)!} \cdot \frac{(j-1)!(n+1-j-i_2)!}{(n-i_2)!} \\ &= \frac{(n-i_1)!}{(n-i_2)!} \cdot \frac{(n+1-j-i_2)!}{(n+1-j-i_1)!}. \end{aligned} \tag{3.28}$$

Since it holds

$$\frac{(n-k_1)!}{(n-k_2)!} = \prod_{m=1}^{k_2-k_1} (n-k_2+m) \qquad \text{where } 1 \leq k_1 < k_2 \leq n \tag{3.29}$$

we rewrite (3.28) as

$$\frac{w_{j,i_1}}{w_{j,i_2}} = \prod_{m=1}^{i_2-i_1} \frac{n-i_2+m}{n+1-j-i_2+m}. \tag{3.30}$$

Analogously, we prove that the (3.27) is true. □

Proposition 3.11. *The binomial weights $w_{ji} \in [0,1]$, with $i,j = 1,\ldots,n$, have the following cumulative property,*

$$\sum_{i=1}^{k} w_{j-1,i} \leq \sum_{i=1}^{k} w_{ji} \qquad k = 1,\ldots,n \tag{3.31}$$

for each $j = 2,\ldots,n$.

Proposition 3.12. *Let B denote a real n by n matrix whose columns are the vectors $\boldsymbol{w}_j$ of the binomial weights $w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}}$ corresponding to each value $j = 1, \ldots, n$, such that $B = (\boldsymbol{w}_1^T \, \boldsymbol{w}_2^T \, \ldots \boldsymbol{w}_n^T)$. Any vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ can be written uniquely as the linear combination of coefficients $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and the binomial weights w_{ji} in the following linear system,*

$$w_i = \sum_{j=1}^{n} w_{ji}\alpha_j \qquad\qquad i = 1, \ldots, n \tag{3.32}$$

or equivalently as,

$$\boldsymbol{w} = B\boldsymbol{\alpha}. \tag{3.33}$$

Proof. The matrix B, whose elements are the binomial weights $w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}}$, has null weights when $i + j > n + 1$. Therefore, the matrix B is an upper triangular matrix with respect to its secondary diagonal. It is evident that the matrix is full rank and invertible. Hence there always exists a unique vector of coefficients α_j, with $j = 1, \ldots, n$, satisfying a set of equations $w_i = \sum_{j=1}^{n} w_{ji}\alpha_j$, where $i = 1, \ldots, n$. In other words, any vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in R^n$ can be written uniquely as the linear combination of coefficients $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$.

Another way to prove that the linear system has a unique solution is to use the Cramer's rule. Let D be the determinant of the matrix B and let D_j be the determinant of the matrix M_j formed by replacing the j-column values with the vector $\boldsymbol{w}$. The coefficient α_j, with $j = 1, \ldots, n$, are given by

$$\alpha_j = \frac{D_j}{D} \qquad\qquad j = 1, \ldots, n. \tag{3.34}$$

The determinant of the matrix B depends on the elements lying on the secondary diagonal.

$$D = (-1)^{\frac{(n-1)(n+4)}{2}} \cdot w_{1,n} \cdot w_{2,n-1} \cdot \ldots \cdot w_{n,1} = (-1)^{\frac{(n-1)(n+4)}{2}} \prod_{i=1,\ldots,n} w_{i,n-i+1}. \tag{3.35}$$

By substituting the binomial weights into the product and by simplifying, we obtain the following determinant D,

$$D = (-1)^{\frac{(n-1)(n+4)}{2}} \prod_{i=1,\ldots,n} \frac{1}{\binom{n}{i}}. \tag{3.36}$$

Since the determinant D is non-zero, the system of linear equations (3.32) has the unique solution of coefficients α_j, with $j = 1, \ldots, n$,

$$\alpha_j = \frac{D_j}{D} \qquad j = 1, \ldots, n \tag{3.37}$$

where D_j is the determinant of the matrix M_j formed by replacing the j-column values of the matrix B with the vector $\boldsymbol{w}$.

In particular, for $j = 1$, the matrix M_1 is triangular. Therefore,

$$D_1 = (-1)^{\frac{(n-1)(n+4)}{2}} w_n \prod_{i=2,\ldots,n} w_{i,n-i+1} = \frac{w_n}{w_{1,n}} D \tag{3.38}$$

and it is evident that $\alpha_1 = \frac{D1}{D} = \frac{w_n}{w_{1,n}}$.

On the other hand, for $j \neq 1$, the matrix M_j, where $j = 2, \ldots, n$, is not triangular, therefore we cannot obtain the analytical formulation of the determinant D_j and of the coefficients α_j, with $j = 1, \ldots, n$, by using the Cramer's rule. □

Definition 3.13. The *binomial OWA functions* $C_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, with $j = 1, \ldots, n$, are defined as

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}}\, x_{(i)} \qquad j = 1, \ldots, n \tag{3.39}$$

where w_{ji}, $i, j = 1, \ldots, n$, are the binomial weights.

Proposition 3.14. *Any OWA function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \cdots + \alpha_n C_n(\boldsymbol{x}) \tag{3.40}$$

where the coefficients α_j, $j = 1, \ldots, n$, are subject to the boundary condition (3.20) which can be written as

$$\sum_{j=1}^{n} \alpha_j = 1 \tag{3.41}$$

and the monotonicity condition (3.21) which reduces to

$$\sum_{j=1}^{i} \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \alpha_j \geq 0 \qquad i = 1, \ldots, n. \tag{3.42}$$

If we rewrite the boundary condition as $\alpha_1 = 1 - \sum_{j=2}^{n} \alpha_j$ and substitute α_1 into the monotonicity condition, we obtain the compound boundary and monotonicity (BM) conditions *for the $n-1$ coefficients $\alpha_2, \ldots, \alpha_n$ as follows*

$$\sum_{j=2}^{n} \left[1 - n\frac{\binom{i-1}{j-1}}{\binom{n}{j}}\right] \alpha_j \leq 1 \qquad i = 1, \ldots, n. \tag{3.43}$$

The detailed proof of Proposition 3.14 is given in Calvo and De Baets [27], see also Bortot and Marques Pereira [21].

In the next part, we investigate the binomial decomposition of OWA functions with a restriction on the level of k-additivity for $k = 2$ and $k = 3$. We then examine the BM conditions for the related coefficients of the binomial decomposition in the 2-additive and 3-additive cases.

3.2.3 The 2-additive OWA functions

We now consider the binomial decomposition of OWA functions in the 2-additive case.

Proposition 3.15. *Any 2-additive OWA function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ can be written uniquely as*

$$A(\boldsymbol{x}) = (1 - \alpha_2)\bar{\boldsymbol{x}} + \alpha_2 C_2(\boldsymbol{x}) \tag{3.44}$$

where $C_2(\boldsymbol{x})$ is the binomial OWA function given by

$$C_2(\boldsymbol{x}) = \sum_{i=1}^{n} w_{2i} x_{(i)} = \sum_{i=1}^{n} \frac{2(n-i)}{n(n-1)} x_{(i)} \tag{3.45}$$

and the coefficients α_2 is subject to the BM conditions,

$$\left[1 - n\frac{\binom{i-1}{1}}{\binom{n}{2}}\right]\alpha_2 \leq 1 \qquad\qquad i = 1, \ldots, n. \tag{3.46}$$

Example 3.1. *As an example to illustrate the BM conditions (3.46) in the 2-additive case, we consider the cases $n = 3, 4, 5, 6$,*

$$n = 3 \quad \begin{cases} \alpha_2 \leq 1 \\ 0 \leq 2 \\ \alpha_2 \geq -1 \end{cases} \qquad n = 4 \quad \begin{cases} \alpha_2 \leq 1 \\ \alpha_2 \leq 3 \\ \alpha_2 \geq -3 \\ \alpha_2 \geq -1 \end{cases} \tag{3.47}$$

$$n = 5 \quad \begin{cases} \alpha_2 \leq 1 \\ \alpha_2 \leq 2 \\ 0 \leq 2 \\ \alpha_2 \geq -2 \\ \alpha_2 \geq -1 \end{cases} \qquad n = 6 \quad \begin{cases} \alpha_2 \leq 1 \\ 3\alpha_2 \leq 5 \\ \alpha_2 \leq 5 \\ \alpha_2 \geq -5 \\ 3\alpha_2 \geq -5 \\ \alpha_2 \geq -1 \end{cases} \tag{3.48}$$

We notice that the BM conditions (3.46) for any dimension n reduce to

$$-1 \leq \alpha_2 \leq 1 \tag{3.49}$$

which corresponds to the cases $i = 1$ and $i = n$ due to their dominance power against the other conditions when $i = 2, \ldots, n-1$. The BM conditions are, therefore, independent of n. The feasible region of the coefficient α_2 is the line segment specified by $-1 \leq \alpha_2 \leq 1$.

3.2.4 The 3-additve OWA functions

Analogously, we now consider the binomial decomposition of OWA functions in the 3-additive case.

Proposition 3.16. *Any 3-additive OWA function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ can be written uniquely as*

$$A(\boldsymbol{x}) = (1 - \alpha_2 - \alpha_3)\bar{\boldsymbol{x}} + \alpha_2 C_2(\boldsymbol{x}) + \alpha_3 C_3(\boldsymbol{x}) \tag{3.50}$$

where $C_2(\boldsymbol{x})$ is given in (3.45) and $C_3(\boldsymbol{x})$ is the binomial OWA function defined as

$$C_3(\boldsymbol{x}) = \sum_{i=1}^{n} w_{3i} x_{(i)} = \sum_{i=1}^{n} \frac{3(n-i)(n-i-1)}{n(n-1)(n-2)} x_{(i)} \tag{3.51}$$

and the coefficients α_2 and α_3 are subject to the BM conditions,

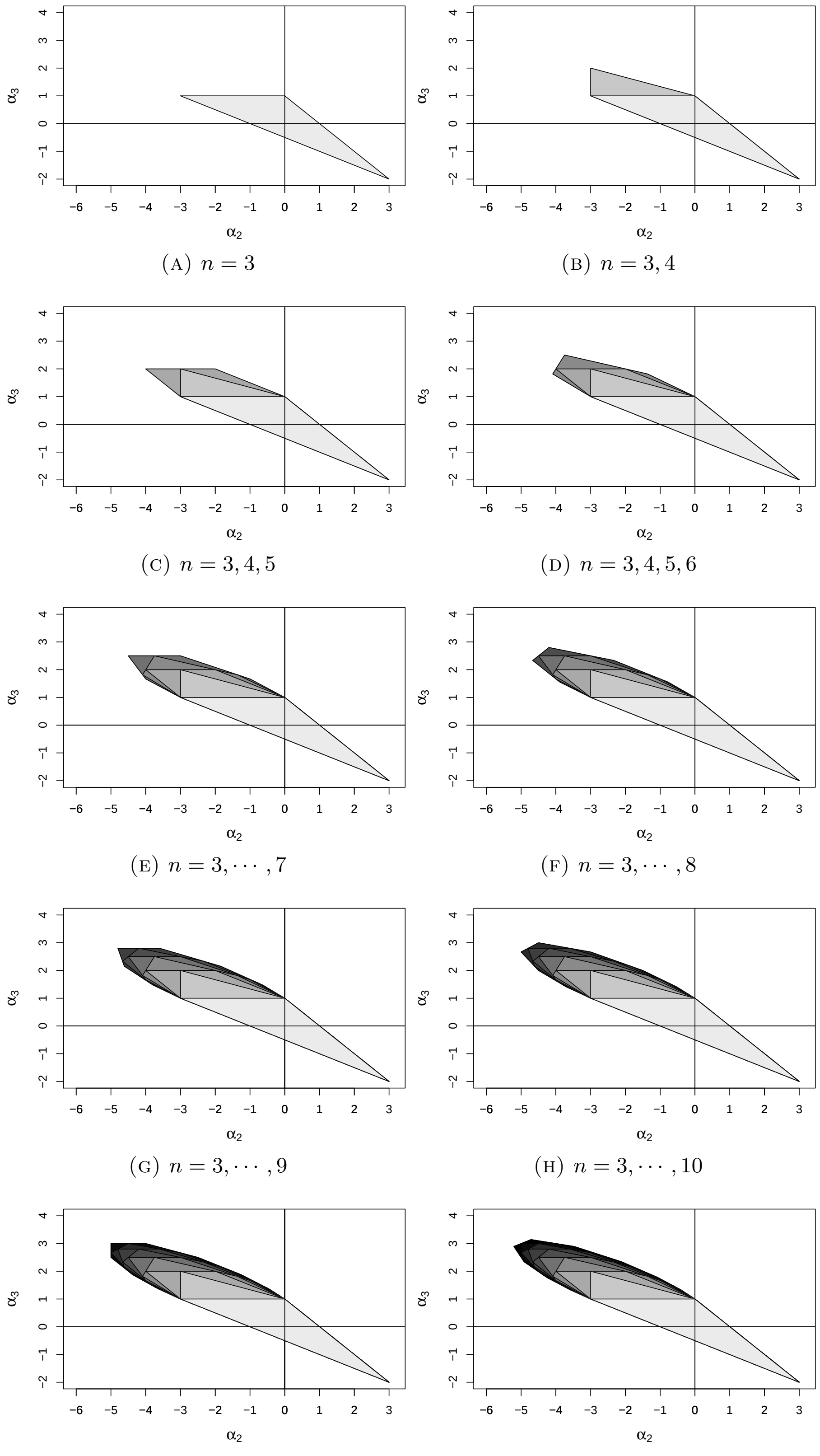
$$\left[1 - n\frac{\binom{i-1}{1}}{\binom{n}{2}}\right]\alpha_2 + \left[1 - n\frac{\binom{i-1}{2}}{\binom{n}{3}}\right]\alpha_3 \leq 1 \qquad i = 1, \ldots, n. \tag{3.52}$$

Example 3.2. *We now illustrate the BM conditions (3.52) for the coefficients α_2 and α_3 in dimensions $n = 3, 4, 5, 6$,*

$$n = 3 \quad \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \qquad n = 4 \quad \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_2 + 3\alpha_3 \leq 3 \\ \alpha_2 \geq -3 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{3.53}$$

$$n = 5 \quad \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \leq 2 \\ \alpha_3 \leq 2 \\ \alpha_2 + \alpha_3 \geq -2 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \qquad n = 6 \quad \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ 3\alpha_2 + 3\alpha_3 \leq 5 \\ 2\alpha_2 + 7\alpha_3 \leq 10 \\ 2\alpha_2 - \alpha_3 \geq -10 \\ 3\alpha_2 + 4\alpha_3 \geq -5 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{3.54}$$

We see that the BM conditions in the 3-additive case are dependent on n. The feasible region which is specified by the intersection of a set of halfspaces is a convex polygon with n vertices and n edges. Fig. 3.1 gives a useful illustration of the feasible region with respect to the coefficients α_2 and α_3. As the dimension n increases from 3 to 12, the feasible region is expanding to the upper left area and becomes asymptotic, valid only when $n \to \infty$. Further discussion about the asymptotic form of the feasible region can be found in [23].



(A) $n = 3$ (B) $n = 3, 4$ (C) $n = 3, 4, 5$ (D) $n = 3, 4, 5, 6$ (E) $n = 3, \cdots, 7$ (F) $n = 3, \cdots, 8$ (G) $n = 3, \cdots, 9$ (H) $n = 3, \cdots, 10$ (I) $n = 3, \cdots, 11$ (J) $n = 3, \cdots, 12$

FIGURE 3.1: Feasible regions associated with the BM conditions (3.52)

3.3 The analytical expression for the coefficients in the binomial decomposition

In the previous section, we reviewed the OWA functions and their alternative representation under the binomial decomposition framework. Any OWA function can be expressed as a linear combination of the coefficients α_j, with $j = 1, \ldots, n$, and the binomial OWA functions C_j, with $j = 1, \ldots, n$. The linear combination shows the relationship between OWA weights and the coefficients α_j, with $j = 1, \ldots, n$. In this section, our aim is to derive an analytical expression for these coefficients in terms of OWA weights.

The binomial decomposition, as described in (3.40), expresses the linear combination of OWA weights and the coefficients α_j, with $j = 1, \ldots, n$, and it can be written as the following linear system,

$$\begin{cases} w_1 &= w_{11}\alpha_1 + w_{21}\alpha_2 + \cdots + w_{n-1,1}\alpha_{n-1} + w_{n,1}\alpha_n \\ w_2 &= w_{12}\alpha_1 + w_{22}\alpha_2 + \cdots + w_{n-1,2}\alpha_{n-1} + w_{n,2}\alpha_n \\ \cdots & \\ w_n &= w_{1n}\alpha_1 + w_{2n}\alpha_2 + \cdots + w_{n-1,n}\alpha_{n-1} + w_{n,n}\alpha_n \end{cases} \tag{3.55}$$

where the binomial weights are given by $w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}}, i, j = 1, \ldots, n$, and the coefficients α_j, with $j = 1, \ldots, n$, are subject to the conditions (3.52).

The coefficient matrix of the linear system is composed of the binomial weights $w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}}, i, j = 1, \ldots, n$, with the first $n - j + 1$ weights being positive and non-linear decreasing, and the last $j - 1$ weights equal to zero. The linear system can be therefore simplified into the reduced *row echelon form*,

$$\begin{cases} w_1 &= w_{11}\alpha_1 + w_{21}\alpha_2 + \cdots + w_{n-1,1}\alpha_{n-1} + w_{n,1}\alpha_n \\ w_2 &= w_{12}\alpha_1 + w_{22}\alpha_2 + \cdots + w_{n-1,2}\alpha_{n-1} \\ \cdots & \\ w_n &= w_{1n}\alpha_1 \end{cases} \tag{3.56}$$

where the binomial weights $w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}}, i, j = 1, \ldots, n$, and the coefficients α_j, with $j = 1, \ldots, n$, are subject to the BM conditions (3.43).

The triangular form, where $w_i = \sum_{j=1}^{n-i+1} w_{ji}\alpha_j$ associated with the coefficient matrix, which is full rank and invertible, shows that there always exists a unique vector of coefficients α_j, with $j = 1, \ldots, n$, satisfying the linear system for any OWA function, see also Proposition 3.12. The coefficients α_j, with $j = 1, \ldots, n$, of the linear system can be obtained by the backward substitution rule. It is evident that α_1 can be determined by

w_n; α_2 can be determined by w_{n-1}, w_n; ...; and α_n can be determined by $w_1, \ldots, w_n$. In general, each $\alpha_j, j = 1, \ldots, n$, determined by the set of OWA weights $\{w_{n-j+1}, \ldots, w_n\}$, can be expressed as follows

$$\alpha_1 = \frac{1}{w_{1,n}} w_n \tag{3.57}$$

$$\alpha_2 = \frac{1}{w_{2,n-1}} \left[w_{n-1} - w_{1,n-1}\alpha_1\right] \tag{3.58}$$

$$\ldots$$

$$\alpha_{n-1} = \frac{1}{w_{n-1,2}} \left[w_2 - \sum_{k=1}^{n-2} w_{k,2}\alpha_k\right] \tag{3.59}$$

$$\alpha_n = \frac{1}{w_{n,1}} \left[w_1 - \sum_{k=1}^{n-1} w_{k,1}\alpha_k\right]. \tag{3.60}$$

The generic formulation for the coefficients α_j, with $j = 1, \ldots, n$, can be seen as a function of all the preceding α_k, with $k = 1, \ldots, j-1$, given by

$$\alpha_j = \frac{1}{w_{j,n-j+1}} \left(w_{n-j+1} - \sum_{k=1}^{j-1} w_{k,n-j+1}\alpha_k \right) \tag{3.61}$$

where $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$ and $\sum_{i=1}^n w_i = 1$.

Each α_j, with $j = 1, \ldots, n$, in the above expression is a linear combination of the preceding coefficients $\alpha_1, \ldots, \alpha_{j-1}$. If we substitute OWA weights for the preceding coefficients, we obtain the analytical expression for the coefficients α_j, with $j = 1, \ldots, n$, in terms of the set of OWA weights $\{w_{n-j+1}, \ldots, w_n\}$.

α_1 is determined by w_n:

$$\alpha_1 = \frac{1}{w_{1,n}} w_n\,. \tag{3.62}$$

α_2 is determined by w_{n-1}, w_n:

$$\begin{aligned} \alpha_2 &= \frac{1}{w_{2,n-1}} \left[w_{n-1} - w_{1,n-1}\alpha_1\right] \\ &= \frac{1}{w_{2,n-1}} \left[w_{n-1} - \frac{w_{1,n-1}}{w_{1,n}} w_n\right] \\ &= \frac{1}{w_{2,n-1}} \left[w_{n-1} - w_n\right] \end{aligned} \tag{3.63}$$

where we use the fact that $w_{1,i} = \frac{1}{n}$ for every $i = 1, \ldots, n$.

α_3 *is determined by* w_{n-2}, w_{n-1}, w_n:

$$\begin{aligned}\alpha_3 &= \frac{1}{w_{3,n-2}}\left[w_{n-2} - w_{1,n-2}\alpha_1 - w_{2,n-2}\alpha_2\right] \\ &= \frac{1}{w_{3,n-2}}\left[w_{n-2} - \frac{w_{1,n-2}}{w_{1,n}}w_n - \frac{w_{2,n-2}}{w_{2,n-1}}\left(w_{n-1} - w_n\right)\right] \\ &= \frac{1}{w_{2,n-2}}\left[w_{n-2} - 2w_{n-1} + w_n\right] \qquad (3.64)\end{aligned}$$

where we use the fact that $w_{1,i} = \frac{1}{n}$ for every $i = 1, \ldots, n$ and Proposition 3.10 to simplify the ratio $\frac{w_{2,n-2}}{w_{2,n-1}}$ to 2.

α_4 *is determined by* $w_{n-3}, w_{n-2}, w_{n-1}, w_n$:

$$\begin{aligned}\alpha_4 &= \frac{1}{w_{4,n-3}}\left[w_{n-3} - w_{1,n-3}\alpha_1 - w_{2,n-3}\alpha_2 - w_{3,n-3}\alpha_3\right] \\ &= \frac{1}{w_{4,n-3}}\left[w_{n-2} - \frac{w_{1,n-3}}{w_{1,n}}w_n - \frac{w_{2,n-3}}{w_{2,n-1}}\left(w_{n-1} - w_n\right) - \frac{w_{3,n-3}}{w_{3,n-2}}\left(w_{n-2} - 2w_{n-1} + w_n\right)\right] \\ &= \frac{1}{w_{4,n-3}}\left[w_{n-3} - 3w_{n-2} + 3w_{n-1} - w_n\right] \qquad (3.65)\end{aligned}$$

where we use the fact that $w_{1,i} = \frac{1}{n}$ for every $i = 1, \ldots, n$ and Proposition 3.10 to simplify both ratios $\frac{w_{2,n-3}}{w_{2,n-1}}$ and $\frac{w_{3,n-3}}{w_{3,n-2}}$ to 3.

The above mathematical formulations represent the analytical expressions for the coefficients $\alpha_1, \alpha_2, \alpha_3,$ and α_4. They follow the specific pattern in which each α_j, where $j = 1, 2, 3, 4$, is expressed in terms of OWA weights $\{w_{n-j+1}, \ldots, w_n\}$. We now generalize this result for all coefficients α_j, with $j = 1, \ldots, n$.

Proposition 3.17. *Consider an OWA function with the associated weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$ *such that* $\sum_{i=1}^n w_i = 1$. *The coefficients* α_j*, with* $j = 1, \ldots, n$*, of the binomial decomposition of the OWA function can be written in terms of OWA weights as follows,*

$$\alpha_j = \frac{1}{w_{j,n-j+1}} \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} w_{n-p} \qquad (3.66)$$

where $j = 1, \ldots, n$.

Proof. In order to prove that the formulation (3.66) is the analytical expression for the coefficients α_j, with $j = 1, \ldots, n$, of the binomial decomposition of OWA functions, we use the principle of strong induction [69].

1. Base case: Let us prove the base case for $j = 1$. It follows immediately, by solving the last equation of the linear system related to w_n, that is $\alpha_1 = \frac{1}{w_{1,n}} w_n$, which is equivalent to (3.66) in the case of $j = 1$.

2. Induction hypothesis: For some fixed $k \geq 1$, we assume that (3.66) is true for the coefficients α_j, with $j = 1, \ldots, k$ (according to the strong induction method [69]). Therefore the coefficients $\alpha_1, \ldots, \alpha_k$ are the solutions of the equations related to the OWA weights $w_{n-1}, \ldots, w_{n-k+1}$ of the linear system as follows

$$\begin{aligned}
\alpha_1 &= \frac{1}{w_{1,n}} w_n \\
\alpha_2 &= \frac{1}{w_{2,n-1}} \left[\binom{1}{0} w_{n-1} - \binom{1}{1} w_n \right] \\
\alpha_3 &= \frac{1}{w_{3,n-2}} \left[\binom{2}{0} w_{n-2} - \binom{2}{1} w_{n-1} + \binom{2}{2} w_n \right] \\
&\vdots \\
\alpha_k &= \frac{1}{w_{k,n-k+1}} \left[\binom{k-1}{0} w_{n-k+1} - \binom{k-1}{1} w_{n-k+2} + \cdots + (-1)^{k-1} \binom{k-1}{k-1} w_n \right].
\end{aligned}$$

3. Induction:

We need to prove that the expression (3.66) is also true for α_{k+1}

$$\begin{aligned}
\alpha_{k+1} &= \frac{1}{w_{k+1,n-k}} \sum_{p=0}^{k} (-1)^{k-p} \binom{k}{k-p} w_{n-p} \\
&= \frac{1}{w_{k+1,n-k}} \left[\binom{k}{0} w_{n-k} - \binom{k}{1} w_{n-k+1} + \cdots + (-1)^k \binom{k}{k} w_n \right]. \qquad (3.67)
\end{aligned}$$

According to the linear system (3.56), α_{k+1} is obtained from $\alpha_1, \ldots, \alpha_k$,

$$\begin{aligned}
\alpha_{k+1} &= \frac{1}{w_{k+1,n-k}} \left[w_{n-k} - w_{1,n-k}\alpha_1 - w_{2,n-k}\alpha_2 - \cdots - w_{k,n-k}\alpha_k \right] \\
&= \frac{1}{w_{k+1,n-k}} \Bigg[w_{n-k} - w_{1,n-k} \cdot \frac{1}{w_{1,n}} w_n - w_{2,n-k} \cdot \frac{1}{w_{2,n-1}} \left[\binom{1}{0} w_{n-1} - \binom{1}{1} w_n \right] - \cdots \\
&\quad - w_{k,n-k} \cdot \frac{1}{w_{k,n-k+1}} \left[\binom{k-1}{0} w_{n-k+1} - \binom{k-1}{1} w_{n-k+2} + \cdots + (-1)^{k-1} \binom{k-1}{k-1} w_n \right] \Bigg].
\end{aligned}$$

The ratio $\frac{w_{1,n-k}}{w_{1,n}} = 1 = \binom{k}{0}$ due to $w_{1,n-k} = w_{1,n} = \frac{1}{n}$.

By applying the results of Proposition 3.10, the remaining ratios, $\frac{w_{2,n-k}}{w_{2,n-1}}$, $\frac{w_{3,n-k}}{w_{3,n-2}}$, $\ldots$, $\frac{w_{k-1,n-k}}{w_{k-1,n-k+2}}$, $\frac{w_{k,n-k}}{w_{k,n-k+1}}$, are simplified to $\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k-2}, \binom{k}{k-1}$, respectively. Substituting all the simplified ratios into the expression of α_{k+1}, we get

$$\begin{aligned}
\alpha_{k+1} &= \frac{1}{w_{k+1,n-k}}\Bigg[w_{n-k} - \binom{k}{0} w_n - \binom{k}{1}\left[\binom{1}{0} w_{n-1} - \binom{1}{1} w_n\right] \\
&\quad - \binom{k}{2}\left[\binom{2}{0} w_{n-2} - \binom{2}{1} w_{n-1} + \binom{2}{2} w_n\right] - \cdots \\
&\quad \cdots - \binom{k}{k-2}\left[\binom{k-2}{0} w_{n-k+2} - \binom{k-2}{1} w_{n-k+3} + \cdots + (-1)^{k-2}\binom{k-2}{k-2} w_n\right] \\
&\quad - \binom{k}{k-1}\left[\binom{k-1}{0} w_{n-k+1} - \binom{k-1}{1} w_{n-k+2} + \cdots + (-1)^{k-1}\binom{k-1}{k-1} w_n\right]\Bigg].
\end{aligned}$$

In order to further simplify the expression, we group the coefficients, $c_n, \ldots, c_{n-k}$, associated with the OWA weights $w_n, \ldots, w_{n-k}$ in the expression of α_{k+1}, and obtain the following results.

- *Coefficients of* w_{n-k}:

  $c_{n-k} = 1$, which is immediately derived from the expression.

- *Coefficients of* w_{n-k+1}:

  $c_{n-k+1} = -\binom{k}{k-1}\binom{k-1}{0}$.

- *Coefficients of* w_{n-k+2}:

  $c_{n-k+2} = +\binom{k}{k-1}\binom{k-1}{1} - \binom{k}{k-2}\binom{k-2}{0}$.

- *Coefficients of* w_{n-k+3}:

  $c_{n-k+3} = -\binom{k}{k-1}\binom{k-1}{2} + \binom{k}{k-2}\binom{k-2}{1} - \binom{k}{k-3}\binom{k-3}{0}$.

  $\vdots$

- *Coefficients of* w_{n-2}:

$$\begin{aligned}
c_{n-2} &= -(-1)^{k-3}\binom{k}{k-1}\binom{k-1}{k-3} - (-1)^{k-4}\binom{k}{k-2}\binom{k-2}{k-4} \\
&\quad -(-1)^{k-5}\binom{k}{k-3}\binom{k-3}{k-5} - (-1)^{k-6}\binom{k}{k-4}\binom{k-4}{k-6}\cdots
\end{aligned}$$

$$\ldots - (-1)^1 \binom{k}{3}\binom{3}{1} - (-1)^0 \binom{k}{2}\binom{2}{0}.$$

- *Coefficients of w_{n-1}*:

$$\begin{aligned}
c_{n-1} &= -(-1)^{k-2}\binom{k}{k-1}\binom{k-1}{k-2} - (-1)^{k-3}\binom{k}{k-2}\binom{k-2}{k-3} \\
&\quad -(-1)^{k-4}\binom{k}{k-3}\binom{k-3}{k-4} - (-1)^{k-5}\binom{k}{k-4}\binom{k-4}{k-5}\ldots \\
&\quad \ldots - (-1)^1\binom{k}{2}\binom{2}{1} - (-1)^0\binom{k}{1}\binom{1}{0}.
\end{aligned}$$

- *Coefficients of w_n*:

$$\begin{aligned}
c_{n} &= -(-1)^{k-1}\binom{k}{k-1}\binom{k-1}{k-1} - (-1)^{k-2}\binom{k}{k-2}\binom{k-2}{k-2} \\
&\quad -(-1)^{k-3}\binom{k}{k-3}\binom{k-3}{k-3} - (-1)^{k-4}\binom{k}{k-4}\binom{k-4}{k-4}\ldots \\
&\quad \ldots - (-1)^2\binom{k}{2}\binom{2}{2} - (-1)^1\binom{k}{1}\binom{1}{1} - (-1)^0\binom{k}{0}.
\end{aligned}$$

It can be shown that the coefficients associated with particular OWA weights w_{n-p}, with $p = 0, \ldots, k$, in the expression of α_{k+1} are equal to

$$c_{n-p} = \begin{cases} 1, & \text{if } p = k, \\ \sum_{i=0}^{k-p-1} (-1)^{k-p-i}\binom{k}{i+1}\binom{k-i-1}{p}, & \text{otherwise.} \end{cases} \tag{3.68}$$

The expression of α_{k+1} can be rewritten as the linear combination of the OWA weights, $\{w_{n-k}, \ldots, w_n\}$ and their associated coefficients as follows,

$$\alpha_{k+1} = \frac{1}{w_{k+1,n-k}}\left[w_{n-k} + \sum_{p=0}^{k-1}\left(w_{n-p}\sum_{i=0}^{k-p-1}(-1)^{k-p-i}\binom{k}{i+1}\binom{k-i-1}{p}\right)\right] \tag{3.69}$$

The inner summation in (3.69) can be rewritten as

$$\sum_{i=0}^{k-p-1}(-1)^{k-p-i}\binom{k}{i+1}\binom{k-i-1}{p} = (-1)^{k-p}\sum_{i=0}^{k-p-1}(-1)^{-i}\frac{k!}{(i+1)!(k-i-1)!}\cdot\frac{(k-i-1)!}{p!(k-i-p-1)!}$$

$$
\begin{aligned}
&= (-1)^{k-p}\frac{k!}{p!}\sum_{i=0}^{k-p-1}(-1)^{-i}\frac{1}{(i+1)!(k-i-p-1)!} \\
&= (-1)^{k-p}\frac{k!}{p!(k-p)!}\sum_{i=0}^{k-p-1}(-1)^{-i}\frac{(k-p)!}{(i+1)!(k-p-i-1)!} \\
&= (-1)^{k-p}\binom{k}{k-p}\sum_{i=0}^{k-p-1}(-1)^{-i}\binom{k-p}{i+1}.
\end{aligned}
$$

If we substitute n for $k-p$, the summation $\sum_{i=0}^{k-p-1}(-1)^{-i}\binom{k-p}{i+1}$, with $k=1,\dots,n$ and $p=0,\dots,k-1$, can be equivalently written as $\sum_{i=0}^{n-1}(-1)^{-i}\binom{n}{i+1}$, with $n \geq 1$.

By applying the results of Lemma 2, we have that $\sum_{i=0}^{k-p-1}(-1)^{-i}\binom{k-p}{i+1} = 1$. Therefore, we obtain

$$
\alpha_{k+1} = \frac{1}{w_{k+1,n-k}}\left[w_{n-k} + \sum_{p=0}^{k-1} w_{n-p}(-1)^{k-p}\binom{k}{k-p}\right] \tag{3.70}
$$

which is equivalent to (3.67).

To summarize, we have proved that the formulation (3.66), given by

$$
\alpha_j = \frac{1}{w_{j,n-j+1}}\sum_{p=0}^{j-1}(-1)^{j-p-1}\binom{j-1}{j-p-1}w_{n-p} \tag{3.71}
$$

is the analytical expression for the coefficients α_j, where $j=1,\dots,n$, in terms of OWA weights.

Let us give some examples of the analytical expression for the coefficients α_j, where $j = 1,\dots,n$, in terms of OWA weights $\boldsymbol{w} = (w_1,\dots,w_n)$ for dimensions $n = 4, 6, 8$.

Example 1. In the case $n = 4$, the analytical expression for the coefficients α_j, with $j = 1,\dots,n$, is given as

$$
\alpha_j = \frac{1}{w_{j,4-j+1}}\sum_{p=0}^{j-1}(-1)^{j-p-1}\binom{j-1}{j-p-1}w_{4-p}. \tag{3.72}
$$

We obtain the analytical expression for the coefficients as follows

$$\begin{cases} \alpha_1 = 4w_4 \\ \alpha_2 = 6(w_3 - w_4) \\ \alpha_3 = 4(w_2 - 2w_3 + w_4) \\ \alpha_4 = 1(w_1 - 3w_2 + 3w_3 - w_4) \end{cases} \tag{3.73}$$

in which the coefficients α_j, with $j = 1, \ldots, n$, are given explicitly as functions of the set of the OWA weights $\{w_{n-j+1}, \ldots, w_n\}$.

Example 2. In the case $n = 6$, the analytical expression for the coefficients α_j, with $j = 1, \ldots, n$, is given as

$$\alpha_j = \frac{1}{w_{j,6-j+1}} \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} w_{6-p}. \tag{3.74}$$

We obtain the analytical expression for the coefficients as follows

$$\begin{cases} \alpha_1 = 6w_6 \\ \alpha_2 = 15(w_5 - w_6) \\ \alpha_3 = 20(w_4 - 2w_5 + w_6) \\ \alpha_4 = 15(w_3 - 3w_4 + 3w_5 - w_6) \\ \alpha_5 = 6(w_2 - 4w_3 + 6w_4 - 4w_5 + w_6) \\ \alpha_6 = 1(w_1 - 5w_2 + 10w_3 - 10w_4 + 5w_5 - w_6) \end{cases} \tag{3.75}$$

in which the coefficients α_j, with $j = 1, \ldots, n$, are given explicitly as functions of the set of the OWA weights $\{w_{n-j+1}, \ldots, w_n\}$.

Example 3. In the case $n = 8$, the analytical expression for the coefficients α_j, with $j = 1, \ldots, n$, is given as

$$\alpha_j = \frac{1}{w_{j,8-j+1}} \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} w_{8-p}. \tag{3.76}$$

We obtain the analytical expression for the coefficients as follows

$$\begin{cases} \alpha_1 = 8w_8 \\ \alpha_2 = 28(w_7 - w_8) \\ \alpha_3 = 56(w_6 - 2w_7 + w_8) \\ \alpha_4 = 70(w_5 - 3w_6 + 3w_7 - w_8) \\ \alpha_5 = 56(w_4 - 4w_5 + 6w_6 - 4w_7 + w_8) \\ \alpha_6 = 28(w_3 - 5w_4 + 10w_5 - 10w_6 + 5w_7 - w_8) \\ \alpha_7 = 8(1w_2 - 6w_3 + 15w_4 - 20w_5 + 15w_6 - 6w_7 + w_8) \\ \alpha_8 = 1(w_1 - 7w_2 + 21w_3 - 35w_4 + 35w_5 - 21w_6 + 7w_7 - w_8) \end{cases} \tag{3.77}$$

in which the coefficients α_j, with $j = 1, \dots, n$, are given explicitly as functions of the set of the OWA weights $\{w_{n-j+1}, \dots, w_n\}$.

3.4 Simplifying the minimax disparity model for determining OWA weights in large-scale problems

In Sect. 2.3.1 we reviewed methods for determining OWA weights based on the measures of orness and dispersion. Among these available methods, the disparity proposed by Wang and Parkan [107] is well-known for determining OWA weights. The usual academic instances of the minimax disparity model focus on solving problems with small dimensions ($n = 3, 4, 5, 6$). However, in applied operational research, optimization problems are often much more complex and lead to a heavy computational demand when there are hundreds or thousands of variables. In order to overcome the complexity of high-dimensional problems, we consider the binomial decomposition framework, with reference to the k-additive framework, which we described in Sect. 3.2. In the binomial decomposition framework, the original problem, expressed in terms of OWA weights, can be reformulated by a new set of coefficients of the binomial decomposition of OWA functions. We impose the level of complexity of the OWA weight distribution by means of the k-additivity, thereby leads to a significant reduction in the number of coefficients. Preliminary experiments show that the solution found for the proposed model can still be a good approximated solution for the original model, while the computational demand in high-dimensional problems can be significantly reduced.

3.4.1 The minimax disparity model for determining OWA weights

In this section, we briefly recall the specific class of OWA functions whose weights are determined by the minimax disparity methods. In 2005 Wang and Parkan [107] revisited the maximum entropy method introduced by O'Hagan [92] and proposed the minimax disparity procedure to determine the OWA weights for the convex optimization problem

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \left\{ \max_{i\in\{1,\ldots,n-1\}} |w_i - w_{i+1}| \right\} \\ \text{s.t.} \quad & 0 \le w_i \le 1\,, \qquad i = 1,\ldots,n\,, \\ & \sum_{i=1}^{n} w_i = 1\,, \\ & \text{orness}(\boldsymbol{w}) = \eta = \frac{1}{n-1}\sum_{i=1}^{n}(i-1)w_i\,, \end{aligned} \tag{3.78}$$

where η, with $0 \le \eta \le 1$, stands for the orness of the weighting vector.

The objective function is non-linear due to the absolute difference between two adjacent weights. In order to overcome this non-linearity, the authors introduced a new variable called $\delta = \max_{i\in\{1,\ldots,n-1\}} |w_i - w_{i+1}|$ and described the original problem equivalently as

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \delta \\ \text{s.t.} \quad & 0 \le w_i \le 1\,, \qquad i = 1,\ldots,n\,, \\ & \sum_{i=1}^{n} w_i = 1\,, \\ & \text{orness}(\boldsymbol{w}) = \eta = \frac{1}{n-1}\sum_{i=1}^{n}(i-1)w_i\,, \\ & w_i - w_{i+1} - \delta \le 0\,, \qquad i = 1,\ldots,n-1\,, \\ & w_i - w_{i+1} + \delta \ge 0\,, \qquad i = 1,\ldots,n-1\,, \end{aligned} \tag{3.79}$$

where η, with $0 \le \eta \le 1$, stands for the orness of the weighting vector.

The formulation (3.79) is easy to solve in practice due to its linearity. Many researchers, therefore, revisited this method and suggested numerous extensions [4, 44, 57, 96, 106]. In particular, Liu [80] proved the equivalence of the solutions of the minimax disparity model and the minimum variance method suggested by Fuller and Majlender [54].

3.4.2 The minimax disparity model under the binomial decomposition framework

In Sect. 3.4.1, we have briefly reviewed the minimax disparity method for determining OWA weights, expressed in terms of OWA weights. We now transform the minimax disparity model (3.79) into a problem in which the weights are substituted by a new set of coefficients α_j, with $j = 1, \ldots, n$,

$$\begin{aligned}
\min_{\boldsymbol{\alpha}} \quad & \delta \\
\text{s.t.} \quad & \sum_{j=1}^{i} \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \alpha_j \geq 0 \qquad\qquad i = 1, \ldots, n\,, \\
& \sum_{i=1}^{n} \alpha_j = 1\,, \\
& \text{Orness}(\boldsymbol{\alpha}) = \eta = \sum_{j=1}^{n} \frac{n-j}{(n-1)(j+1)} \cdot \alpha_j\,, \\
& \sum_{j=1}^{n-i+1} w_{ji}\alpha_j - \sum_{j=1}^{n-i} w_{j,i+1}\alpha_j - \delta \leq 0\,, \qquad\qquad i = 1, \ldots, n-1, \\
& \sum_{j=1}^{n-i+1} w_{ji}\alpha_j - \sum_{j=1}^{n-i} w_{j,i+1}\alpha_j + \delta \geq 0\,, \qquad\qquad i = 1, \ldots, n-1\,,
\end{aligned} \tag{3.80}$$

where $0 \leq \eta \leq 1$ stands for the orness of the weighting vector. Notice that the first two constraints correspond to the boundary and monotonicity conditions of the OWA weighting vector $\boldsymbol{w} = (w_1, ..., w_n) \in [0, 1]^n$, with $\sum_{i=1}^{n} w_i = 1$. Moreover, the number of constraints is equal to $3n$ either in the original model (3.79) or in the proposed model (3.80).

In Table 3.1 we report the empirical results of our proposed model with full-dimension coefficients α_j, with $j = 1, \ldots, n$, for the case $n = 10$. The coefficients α_j, with $j = 1, \ldots, n$, for the central orness values $0.3, 0.4, 0.5, 0.6$ and 0.7 have high sparsity. In particular, the sparsity is 90% for orness $\eta = 0.5$ and is 80% for orness $\eta = 0.3, 0.4, 0.6, 0.7$. This suggests that using a smaller number of coefficients α_j, with $j = 1, \ldots, n$, one can exploit the sparsity of the model and speed up the performance of solvers when the number of criteria in an optimization problems is large. In the following we develop this positive finding by introducing our proposed approach for solving the minimax disparity model in large-scale problems.

3.4.3 Simplifying the minimax disparity model in large-scale problems

In this section, we discuss about the challenges that arise when solving the minimax disparity model in the context of large-scale optimization problems. The empirical results in the literature are obtained for small dimensions ($n = 3, 4, 5, 6$). In real-life scenarios, we usually encounter high-dimensional problems. In this context, the optimization problems formulated directly in terms of OWA weights require large computational resources. Our objective is to make the minimax disparity methods for determining OWA weights more tractable in high-dimensional problems. We, therefore, propose a new approach to overcome this computational complexity. The empirical results in Table 3.1 suggest that by using a small k-additivity ($k \leq n$), one can exploit the sparsity of the model and speed up the performance of solvers. In this section, we transform the original problem, expressed in terms of OWA weights, into a problem in which the weights are substituted by a new set of coefficients. In this transformed representation, we consider the k-additivity OWA functions, so as to reduce the dimensionality of the problem.

In Fig. 3.2 the performance of the CPLEX solver is shown, with respect to the original method and to our proposed method for $k = n$ and $k = 2$. The graph shows the average running time (out of 300 runs), including the standard error, for various orness degrees $\eta = 0.50, 0.45, 0.40, 0.35, 0.30$ and dimensions $n = 10, 20, 30, 40$. The proposed method with k-additivity equal to n facilitates the solver for some orness degrees. Notably, for orness $\eta = 0.5$, this method is always faster than the conventional one due to its high sparsity (90%). For the other degrees of orness, the proposed model with k-additivity equal to n and the conventional model do not differ significantly in terms of running time. Even though both models have the same number of constraints and variables, the constraints of the coefficients α_j, with $j = 1, \ldots, n$, related to the monotonicity condition in the model (3.80) are more complex than those in the model (3.79). As a result, the proposed model with full dimensionality requires more computation time than the conventional method for some orness values.

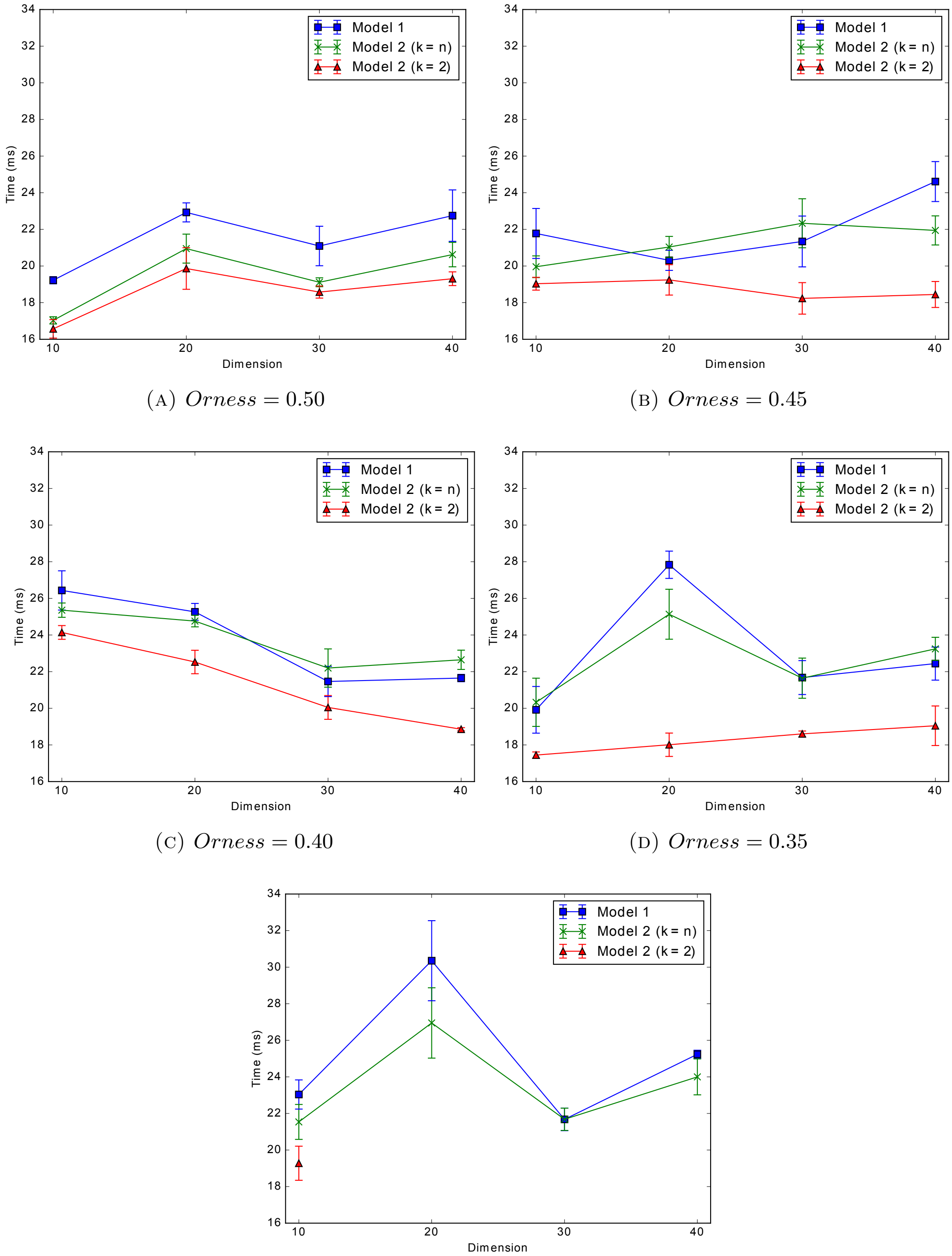
However, we note that by applying the proposed model with lower k-additivity ($k < n$), the performance improves substantially. The number of variables used in our model reduces by $(1 - \frac{k}{n})$%. As shown in Fig. 3.2 the solver spends less time to obtain the optimal weights with our proposed method with 2-additive case ($k = 2$). When the orness value differs significantly from the central orness, for instance $\eta = 0.30$, the model with the 2-additivity is adequate for identifying OWA weights for $n = 10$ while the optimization problems associated with dimensions $n = 20, 30, 40$ have no solution. In these cases, the 2-additivity or the number of coefficients α_1 and α_2 is too small to have all the constraints satisfied [64]. Therefore, a larger number of coefficients $\alpha_1, \ldots, \alpha_k$ is needed for modeling in dimensions $n = 20, 30, 40$.

Table 3.1: The coefficients α_j, with $j = 1, \ldots, n$, of our proposed method for $n = 10$

η	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
α_1	10	4.3	2.71	1.98	1.49	1	0.51	0.02	0	0	0
α_2	-45	-5.4	-1.93	-0.98	-0.49	0	0.49	0.98	0	0	0
α_3	120	0	0	0	0	0	0	0	0	0	0
α_4	-210	0	0	0	0	0	0	0	3	0	0
α_5	252	12.6	0	0	0	0	0	0	0	0	0
α_6	-210	-16.8	0	0	0	0	0	0	-9	0	0
α_7	120	2.4	0	0	0	0	0	0	13.71	8.4	0
α_8	-45	9	1.29	0	0	0	0	0	-9.64	-13.5	0
α_9	10	-6.5	-1.57	0	0	0	0	0	3.43	7.5	0
α_{10}	-1	1.4	0.5	0	0	0	0	0	-0.5	-1.4	1

Experiments show that the proposed approach with the reduced number of variables can assist decision makers in finding OWA weights faster for some degrees of orness. In the remaining cases, decision makers can exploit the flexibility of the model and choose the k-additivity that provides the best trade-off between the computational complexity of OWA weights and the accuracy of the approximated, and possibly suboptimal, OWA weights.

As an example, we consider our proposed model with the orness value equal to 0.2. If the k-additivity increases from 3 to 10, we obtain better objective values, as expected (see Fig. 3.3). However, it is evident that the k-additivity $k = 7$ leads to the best trade-off between the accuracy of the optimal value and the dimensionality reduction of the optimization problem.



(A) $Orness = 0.50$

(B) $Orness = 0.45$

(C) $Orness = 0.40$

(D) $Orness = 0.35$

(E) $Orness = 0.30$

FIGURE 3.2: Computation time of the original model (Model 1) and our proposed model (Model 2) with various degrees of orness for $n = 10, 20, 30, 40$

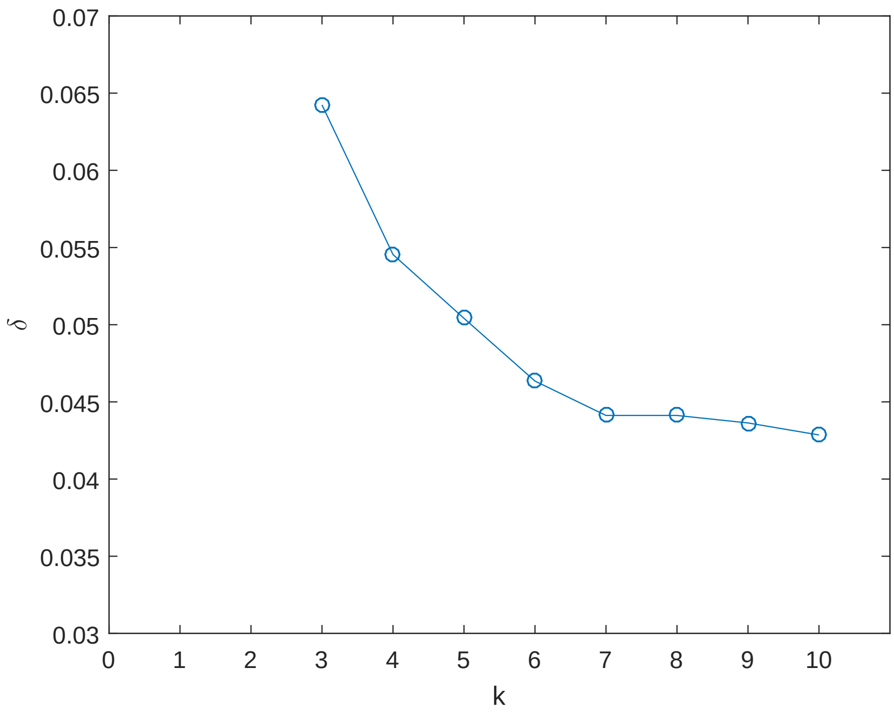


Figure 3.3: The objective value δ corresponding to the k-additivity cases (for the cases $k = 1, 2$ there is no solution of the coefficients α_j)

3.5 Discussion and conclusion

In this chapter we investigated the feasible region of the coefficients associated with the 2-additive and 3-additive cases in n dimensions. We found the analytical expression for the coefficients α_j, with $j = 1, \ldots, n$, of the binomial decomposition of OWA functions in terms of OWA weights. This result improves our understanding of the relationship between OWA weights and the associated coefficients of the binomial decomposition of OWA functions.

In the context of k-additivity, this result suggests us a new approach for solving research problems related to the determination of OWA weights in high dimensions. We transformed the original problems, expressed in terms of OWA weights, into problems in which the weights are substituted by a new set of coefficients. We introduced a new methodology to determine OWA weights in large-scale optimization problems by constraining the complexity of OWA weight distributions through the k-additivity of coefficients α_j, with $j = 1, \ldots, n$. Empirical results show that a small set of the coefficients in the binomial decomposition can efficiently model the full-dimensional set of the OWA weights. The time for solving the optimization problems is significantly reduced by our proposed model for dimensions $n = 20, 30, 40$. On one hand, our methodology can assist decision makers in finding OWA weights faster for some given degrees of orness. On the other hand, decision makers can exploit the flexibility of the model and choose k-additivity that can reduce the computation load and at the same time can derive the best approximated OWA weights.

Our approach has some limitations. The experiments are carried out in dimensions up to 40, which are relatively large with respect to the ones in the literature. However, those dimensions are still relatively small to be representative of very large-scale optimization problems. In addition, no sensitivity analysis has been done, to measures how a small change of the coefficients in the binomial decomposition obtained from our model affects the OWA weights. Moreover, our model was tested only on the minimax disparity model.

We suggest some possible future research directions: 1) the evaluation of our proposed method in higher dimensions for the minimax disparity model; 2) the application of our method to other existing models for determining OWA weights; 3) the development of an algorithm to identify which k-additive level in the set $\{1, \ldots, n\}$ gives the best trade-off between accuracy and computational complexity according to the specific applications.

In this chapter we provided an analytical expression for the coefficient of the binomial decomposition of OWA functions and suggested a useful approach to solve optimization problems for determining OWA weights when the dimension of the OWA weights is high. In the next chapter we consider the binomial decomposition of OWA functions for some families of welfare functions, with a particular focus on generalized Gini, the S-Gini and the Lorenzen welfare functions.

Chapter 4

Welfare functions and their binomial decomposition

In this chapter we recall a family of welfare functions whose expressions are seen as an instance of OWA functions. We consider generalized Gini welfare functions and two parametric families, namely the S-Gini and Lorenzen welfare functions, in the context of the binomial decomposition framework. We show the analogy between these parametric families of generalized Gini welfare functions and the binomial welfare functions. We then derive analytical expressions for the coefficients of the binomial decomposition of the S-Gini and Lorenzen welfare functions in terms of their respective parameters. The numerical results show that they follow interesting patterns.

4.1 Background

Efforts to raise global living standards have received considerable attention in recent decades. In particular, many academics and practitioners have focused on issues such as how to measure economic growth, how to determine economic welfare in relation to a particular society, whether the current income distribution is less unequal or more unequal than in the past, and whether current tax policies help to reduce the wealth gap between the rich and the poor. To address these concerns, income welfare and inequality measurement represent important indicators of the level of well-being.

Several income inequality measures have been introduced, such as Gini [56], Bonferonni [17], De Vergottini [37], Theil [102], Atkinson [9], Sen [99], see also [31, 32, 33, 100]. Among these available inequality indices, the Gini inequality index has attracted a great deal of interest since its computation can be geometrically seen in relation with the

Lorenz curve, see also [42, 55]. The connnection between the Gini index and the concept of social welfare has been analyzed and summarized by Lambert [78]. Other studies introduced several extensions of the Gini index [24, 31, 39, 43, 112, 113]. Weymark [108] proposed the generalized Gini inequality indices and the corresponding welfare functions. In the framework of Atkinson-Kolm-Sen, introduced in [9, 75, 98], the relationship between welfare functions and the associated absolute inequality indices is expressed by the Blackorby and Donalson's correspondence formula [15, 16].

Generalized Gini welfare functions [108] correspond to the S-concave OWA functions [115]. As introduced in Chapter 2, OWA functions are special cases of Choquet integrals in which the associated capacities are symmetric [50]. Moreover, symmetric Choquet integrals, i.e. OWA functions, have been studied in the binomial decomposition framework proposed by Calvo and De Baets [27]. Any OWA function can be formulated in terms of binomial OWA functions. Bortot and Marques Pereira [21] examined this framework in the context of generalized Gini welfare functions. Generalized Gini welfare functions can be expressed by two equivalent functional bases, the binomial welfare functions and the Atkinson-Kolm-Sen associated binomial inequality indices.

In this chapter our objective is to investigate the binomial decomposition of two parametric families of generalized Gini welfare functions, the S-Gini and Lorenzen welfare functions. We identify the relationship between the respective parameters of these welfare functions and the associated coefficients of the binomial decomposition framework. Moreover, we numerically compare two parametric welfare functions and the binomial welfare functions for verifying their similarities [20].

The remainder of this chapter is organized as follows. Section 4.2 reviews the basic definitions regarding generalized Gini welfare functions and the associated Gini inequality indices. This section concentrates on generalized Gini welfare functions in the binomial decomposition framework. In Sect. 4.3, the S-Gini family of welfare functions is discussed and compared with that of the binomial welfare functions. Moreover, we derive an analytical expression for the coefficients of the binomial decomposition of S-Gini welfare functions in terms of their inequality aversion parameter. Similarly, Sect. 4.4 examines the Lorenzen family of welfare functions. Finally, in Sect. 4.5 we present a summarizing discussion.

4.2 The binomial decomposition of generalized Gini welfare functions

This section reviews the fundamental definitions regarding welfare functions and their properties, with a particular focus on the description of generalized Gini welfare functions. In the binomial decomposition framework, generalized Gini welfare functions can be uniquely expressed in terms of two equivalent functional bases, the binomial welfare functions and the associated binomial inequality indices.

4.2.1 Generalized Gini welfare function and inequality indices

In this section we present the basic definitions of welfare functions and the associated inequality indices for the allocation of economic resources in a population. The economic resource we consider is the income of a population of $n \geq 2$ individuals, on the non-negative income domain $D = [0, \infty)$. Points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ denote the income distributions of two populations X and Y. Their arithmetic means are denoted by $\bar{x}$ and $\bar{y}$, respectively. In the following we begin by introducing fundamental notions of majorization relations, income transfers, Schur-convexity and Schur-concavity.

Definition 4.1. Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function.

1. A is *monotonic* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ whenever $\boldsymbol{x} \geq \boldsymbol{y}$ holds, then $A(\boldsymbol{x}) \geq A(\boldsymbol{y})$. Moreover, A is *strictly monotonic* whenever $\boldsymbol{x} > \boldsymbol{y}$ holds, then $A(\boldsymbol{x}) > A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.

2. A is *idempotent* if $A(x \cdot \mathbf{1}) = x$, for all $x \in \mathbb{D}$. On the other hand, A is *nilpotent* if $A(x \cdot \mathbf{1}) = 0$, for all $x \in \mathbb{D}$.

3. A is *symmetric* if $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$, for any permutation σ on $\{1, \ldots, n\}$ and all $\boldsymbol{x} \in \mathbb{D}^n$.

4. A is *invariant for translations* if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is called *stable for translations* if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.

5. A is *invariant for dilations* if $A(t \cdot \boldsymbol{x}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is called *stable for dilations* or *homogeneous* if $A(t \cdot \boldsymbol{x}) = t\, A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.

Definition 4.2. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, according to the theory of majorization by Marshall and Olkin [86], the *majorization relation* $\preceq$ on $\mathbb{D}^n$ is defined as follows:

$$\boldsymbol{x} \preceq \boldsymbol{y} \quad \text{if} \quad \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, \ldots, n \tag{4.1}$$

where the case $k = n$ is an equality due to $\bar{x} = \bar{y}$. We denote $\boldsymbol{x} \prec \boldsymbol{y}$ to express that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and not $\boldsymbol{y} \preceq \boldsymbol{x}$. We use the notion $\boldsymbol{x} \sim \boldsymbol{y}$ to express that $\boldsymbol{x}$ and $\boldsymbol{y}$ are indifferent if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{y} \preceq \boldsymbol{x}$.

The majorization relation can be interpreted also with reference to the concept of Lorenz dominance. By $\boldsymbol{x} \prec \boldsymbol{y}$ we mean that $\boldsymbol{x}$ is *Lorenz superior* to $\boldsymbol{y}$. On the other hand, $\boldsymbol{x} \sim \boldsymbol{y}$ means that $\boldsymbol{x}$ is *Lorenz indifferent* to $\boldsymbol{y}$.

To give an example, we consider an income distribution $\boldsymbol{x} \in \mathbb{D}^n$ with its arithmetic mean $\bar{x}$. As the majorization relation suggests, we derive $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ from the fact that $\sum_{i=1}^{k} \bar{x} \geq \sum_{i=1}^{k} x_{(i)}$ for $k = 1, \ldots, n$. Moreover, $\boldsymbol{x}$ majorizes $\bar{x} \cdot \mathbf{1}$ if the income in $\boldsymbol{x}$ is not equally distributed. In such case, $\bar{x} \cdot \mathbf{1}$ is Lorenz superior to $\boldsymbol{x}$.

Definition 4.3. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$ be the income distributions. The redistributed income $\boldsymbol{x}$ obtained by transferring an amount of income from the relatively richer individual y_j to the relatively poorer individual y_i in the original distribution $\boldsymbol{y}$, where $y_i \leq y_j$ for a pair of individuals $i, j \in \{1, \ldots, n\}$, is given as follows

$$x_i = (1 - \varepsilon)\, y_i + \varepsilon y_j \qquad\qquad x_j = \varepsilon y_i + (1 - \varepsilon)\, y_j \qquad\qquad i, j = 1, \ldots, n \tag{4.2}$$

for $\varepsilon \in [0, 1]$, and $x_k = y_k$ for $k \neq i, j$. The *Pigou-Dalton principle of transfer* says that a progressive transfer from a richer to a poorer individual without changing their relative positions reduces the level of inequality. This principle is a fundamental axiom of inequality measures in the theory of social welfare and it is linked to the concepts of S-concavity and S-convexity. According to Marshall and Olkin [86], strict S-convexity implies symmetry and requires that inequality measures decrease under the progressive transfers. In contrast, strict S-concavity implies symmetry and requires that inequality measures increase under progressive transfers.

Definition 4.4. Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function. The notions of *Schur-convexity* (S-convexity) and *Schur-concavity* (S-concavity) of the function A in relation with the majorization relation are specified as follows:

1. A is *S-convex* for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ holds, then $A(\boldsymbol{x}) \leq A(\boldsymbol{y})$. Moreover, A is said to be *strict S-convex* if $\boldsymbol{x} \prec \boldsymbol{y}$ holds, then $A(\boldsymbol{x}) < A(\boldsymbol{y})$.

2. A is *S-concave* for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ holds, then $A(\boldsymbol{x}) \geq A(\boldsymbol{y})$. Moreover, A is said to be *strict S-concave* if $\boldsymbol{x} \prec \boldsymbol{y}$ holds, then $A(\boldsymbol{x}) > A(\boldsymbol{y})$.

Note that the axioms of S-convexity and S-concavity imply symmetry, since $\boldsymbol{x} \sim \boldsymbol{x}_\sigma \Rightarrow A(\boldsymbol{x}) = A(\boldsymbol{x}_\sigma)$.

Proposition 4.5. *Let $A, B : \mathbb{D}^n \longrightarrow \mathbb{D}$ be two OWA functions with the weighting vectors $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0,1]^n$ where $\sum_{i=1}^n u_i = 1$ and $\boldsymbol{v} = (v_1, \ldots, v_n) \in [0,1]^n$ where $\sum_{i=1}^n v_i = 1$, respectively. For a given $\boldsymbol{x} \in \mathbb{D}^n$, it holds that $A(\boldsymbol{x}) \leq B(\boldsymbol{x})$ if and only if*

$$\sum_{i=1}^k u_i \geq \sum_{i=1}^k v_i \qquad \textit{for} \quad k = 1, \ldots, n \tag{4.3}$$

where the case $k = n$ is an equality since the sum of their weights is equal to 1.

In the following, we present the properties of S-convexity and S-concavity of OWA functions with respect to their weighting vectors.

Proposition 4.6. *Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be an OWA function associated with a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$ such that $\sum_{i=1}^n w_i = 1$. The following holds*

1. *A is S-convex if and only if the OWA weights are non-decreasing $w_1 \leq \ldots \leq w_n$. Moreover, A is strictly S-convex if and only if the weights are increasing $w_1 < \ldots < w_n$.*

2. *A is S-concave if and only if the OWA weights are non-increasing $w_1 \geq \ldots \geq w_n$. Moreover, A is strictly S-concave if and only if the weights are decreasing $w_1 > \ldots > w_n$.*

Definition 4.7. An averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is a *welfare function* if it is continuous and S-concave. The welfare function is said to be *strict* if it is a strict averaging function which is strictly S-concave.

According to the basic axioms of averaging functions, the welfare function A is monotonic and idempotent. The welfare function A is, therefore, non-decreasing on the domain $\mathbb{D}^n$. In particular, it is monotonically increasing on the diagonal where $\boldsymbol{x} = x \cdot \mathbf{1}$, with $x \in D$. Consider any two income distributions, the distribution which is more equally distributed than the other is said to be Lorenz superior. The welfare function of the Lorenz superior distribution is not less than the one of the other. In the case of a strict welfare function, the welfare function of the Lorenz superior distribution is greater than the one of the other.

Chisini [35] introduced the concept of Chisini mean which is more general than the notions of other means, such as the arithmetic mean. The Chisini mean, or the *uniform equivalent income*, of a distribution $(x_1, \ldots, x_n)$ with respect to the welfare function A is the income $\tilde{x}$ that makes two income distributions ethically indifferent in terms of the welfare functions $A(\tilde{x} \cdot \mathbf{1}) = A(\boldsymbol{x})$. Since the welfare function is idempotent, we have $A(\tilde{x} \cdot \mathbf{1}) = \tilde{x}$ and therefore $\tilde{x} = A(\boldsymbol{x})$. According to the majorization relation, the equally-distributed income $(\bar{x}, \ldots, \bar{x})$ is majorized by any income distribution $\boldsymbol{x} \in \mathbb{D}^n$, hence $A(\bar{x} \cdot \mathbf{1}) \geq A(\boldsymbol{x})$, or equivalently written as $A(\boldsymbol{x}) \leq \bar{x}$. The arithmetic mean $\bar{x}$ and the uniform equivalent income $\tilde{x}$ are, therefore, related by $0 \leq \tilde{x} \leq \bar{x}$.

In the following, we present the basic definitions of absolute and relative inequality indices. Kolm [76, 77] introduced the notions of absolute indices and proposed the transformation between absolute and relative indices. The relative indices are obtained by dividing their associated absolute indices by the arithmetic mean of the income distributions.

Definition 4.8. A function $G : \mathbb{D}^n \longrightarrow \mathbb{D}$ is an *absolute inequality index* if it is continuous, nilpotent, S-convex, and invariant for translations. The absolute inequality index is said to be *strict* if it is strictly S-convex.

Definition 4.9. A function $G_R : \mathbb{D}^n \longrightarrow \mathbb{D}$ is a *relative inequality index* if it is continuous, nilpotent, S-convex, and invariant for dilations. The relative inequality index is said to be *strict* if it is strictly S-convex.

In the work of Atkinson [9], Kolm [75], and Sen [98], the authors suggested the functional form expressing the relationship between the relative indices and the welfare functions. Blackorby and Donalson [16] revisited the absolute inequality indices and proved that for each absolute index, there exists the associated family of welfare functions. Moreover the study shows that an inequality index is absolute if and only if the corresponding welfare function is stable for translations. On the other hand, an inequality index is relative if and only if the corresponding welfare function is stable for dilations.

Definition 4.10. Given a welfare function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ which is stable for translations, the associated *Atkinson-Kolm-Sen (AKS) absolute inequality index* $G : \mathbb{D}^n \longrightarrow \mathbb{D}$ is defined as

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) \tag{4.4}$$

Definition 4.11. Given a welfare function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ which is stable for dilations, the associated *Atkinson-Kolm-Sen (AKS) relative inequality index* $G_R : \mathbb{D}^n \longrightarrow \mathbb{D}$ is defined as

$$G_R(\boldsymbol{x}) = 1 - \frac{A(\boldsymbol{x})}{\bar{\boldsymbol{x}}} \tag{4.5}$$

for $\boldsymbol{x} \neq \mathbf{0}$, and $G_R(\mathbf{0}) = 0$.

In the following, we present generalized Gini welfare functions, introduced by Weymark [108], and the associated generalized Gini inequality indices by means of the correspondence formula proposed by Blackorby and Donaldson [16].

Definition 4.12. A *generalized Gini welfare function* is an OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ with an associated non-increasing weighting vector $\boldsymbol{w} = (w_1, \dots, w_n) \in [0,1]^n$, such that $w_1 \geq \cdots \geq w_n \geq 0$ and $\sum_{i=1}^n w_i = 1$, defined as

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}. \tag{4.6}$$

The associated *generalized Gini absolute inequality index* is defined as

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) = -\sum_{i=1}^{n} (w_i - \frac{1}{n}) \, x_{(i)} \, . \tag{4.7}$$

Generalized Gini welfare functions are particular instances of OWA functions with non-increasing weighting vectors. Larger weights are given to poorer individuals while smaller weights are given to richer individuals. This behavior is related to the propery of S-concavity. If two income distributions have the same mean and one of them is more equally distributed than the other, the non-increasing weighting vector ensures that the former distribution ranked no worse than the other with respect to generalized Gini welfare functions. Generalized Gini welfare functions and the associated generalized Gini absolute inequality indices are stable for translations, as suggested by Blackorby and Donaldson [16], and stable for dilations, respectively. The weight normalization of generalized Gini welfare functions implies the sum of the coefficients of the associated generalized Gini absolute inequality index $G(\boldsymbol{x})$ is zero. As a result, the generalized Gini absolute inequality indices are not OWA functions.

One important instance of the generalized Gini AKS framework is the classical Gini welfare function and the associated classical Gini absolute inequality indices which are described in the following.

Definition 4.13. A *classical absolute Gini inequality index* is a function $G^c : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as

$$G^c(\boldsymbol{x}) = \frac{1}{2n^2} \sum_{i,j=1}^{n} |x_i - x_j| \tag{4.8}$$

which can be written equivalently as

$$G^c(\boldsymbol{x}) = -\sum_{i=1}^{n} \frac{n - 2i + 1}{n^2} \, x_{(i)}. \tag{4.9}$$

The associated *classical Gini welfare function* is defined as

$$A_G^c(\boldsymbol{x}) = \bar{\boldsymbol{x}} - G^c(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{2(n-i)+1}{n^2} x_{(i)}. \tag{4.10}$$

We notice that the sum of the weights of the classical Gini welfare functions is equal to 1 due to $\sum_{i=1}^{n} 2(n-i)+1 = n^2$.

4.2.2 The binomial decomposition of generalized Gini welfare functions

This section examines the binomial decomposition of OWA functions in the context of generalized Gini welfare functions. The binomial decomposition of generalized Gini welfare functions can be formulated in terms of two equivalent functional forms, the binomial welfare functions and the associated binomial inequality indices.

Definition 4.14. The *binomial welfare functions* $C_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, with $j = 1, \ldots, n$, are defined as

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} \qquad\qquad w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}} \qquad j = 1, \ldots, n \tag{4.11}$$

where w_{ji}, with $i, j = 1, \ldots, n$, are the binomial weights.

We notice that the binomial welfare functions and the binomial OWA functions, as in (3.39), are the same. The binomial weights satisfy the weight normalization, as proven in Proposition 3.9. According to the cumulative property of the binomial weights in Proposition 3.11, we have $\sum_{i=1}^{k} w_{j-1,i} \leq \sum_{i=1}^{k} w_{ji}$, where $k = 1, \ldots, n$. Therefore, the binomial welfare functions C_j, with $j = 1, \ldots, n$, satisfy the inequalities $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \ldots \geq C_n(\boldsymbol{x}) \geq 0$, for any $\boldsymbol{x} \in \mathbb{D}^n$, due to Proposition 4.5.

Moreover, $C_1(\boldsymbol{x})$ corresponds to the arithmetic mean $= \bar{x}$. The remaining binomial welfare functions C_j, with $j = 2, \ldots, n$, have the first $n - j + 1$ weights being positive and non-linear decreasing and the last $j - 1$ weights being null in correspondence with the richest individuals $x_{(n-j+2)}, \ldots, x_{(n)}$ of the population. This weighting structure demonstrates the analogous behavior between the binomial welfare functions and the poverty measures, which progressively focus their measures on the poorest sector of the population. In addition, the non-increasing weights w_{ji} ensure the S-concavity of the binomial welfare functions due to Proposition 4.6.

Proposition 4.15. *A generalized Gini welfare function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \ldots + \alpha_n C_n(\boldsymbol{x}) \tag{4.12}$$

where the coefficients α_j, $j = 1, \ldots, n$, are subject to the following conditions,

$$\alpha_1 = 1 - \sum_{j=2}^{n} \alpha_j \geq 0 \tag{4.13}$$

$$\sum_{j=2}^{n} \left[1 - n \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \right] \alpha_j \leq 1 \qquad i = 2, \ldots, n \tag{4.14}$$

$$\sum_{j=2}^{n} \frac{\binom{n-i}{j-2}}{\binom{n}{j}} \alpha_j \geq 0 \qquad i = 2, \ldots, n\,. \tag{4.15}$$

The constraints (4.13)-(4.14) correspond to the boundary conditions and monotonicity conditions for the coefficients α_j, with $j = 1, \ldots, n$, as they hold for the binomial decomposition of OWA functions (3.41)-(3.43). Generalized Gini welfare functions are particular instances of OWA functions when their weights are non-increasing. This weighting structure ensures the S-concavity of the generalized Gini welfare functions. The coefficients of the binomial decomposition of generalized Gini welfare functions are, thereby, subject to the S-concavity conditions (4.15). The detailed proof of the S-concavity condition is provided in Bortot and Marques Pereira [21]. The feasible region of the coefficients α_j, with $j = 1, \ldots, n$, is convex since it is obtained from the intersection of a set of halfspaces.

Any generalized Gini welfare function as shown in (4.12) can be expressed in terms of the binomial welfare functions C_j, with $j = 1, \ldots, n$. In the following we consider the transformation, as suggested by Blackorby and Donaldson [16], see also Bortot and Marques Pereira [21], from the binomial welfare function to the binomial inequality index and vice versa. We demonstrate that the original expression of the binomial decomposition can be formulated in terms of the binomial Gini absolute inequality index.

Definition 4.16. Consider the binomial welfare functions $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $C_j(\mathbf{x}) = \sum_{i=1}^{n} w_{ji} x_{(i)}$ for $j = 1, \ldots, n$. The associated *binomial inequality indices* $G_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $j = 1, \ldots, n$, are defined as

$$G_j(\mathbf{x}) = \bar{x} - C_j(\mathbf{x}) \qquad j = 1, \ldots, n \tag{4.16}$$

which can be written as

$$G_j(\mathbf{x}) = - \sum_{i=1}^{n} v_{ji} x_{(i)} = - \sum_{i=1}^{n} \left[w_{ji} - \frac{1}{n} \right] x_{(i)} \qquad j = 1, \ldots, n \tag{4.17}$$

where the coefficients $v_{ji} = w_{ji} - \frac{1}{n}$, with $i, j = 1, \ldots, n$, are equal to $-1/n$ when $i + j > n + 1$, since in such case the binomial weights w_{ji} are zero.

The binomial inequality indices assign the same coefficients $-v_{ji}$ to the $j-1$ richest individuals in the population. As a result, they are progressively insensitive to income transfers that take place among the richest individuals. In addition, the coefficients $-v_{ji}$ sum up to zero due to the weight normalization of the binomial weights w_{ji}. The binomial inequality indices G_j, with $j = 1, \ldots, n$, are, therefore, nilpotent and invariant for translations.

The S-concavity of the binomial welfare functions, as discussed in (4.14), ensures the S-convexity of the associated binomial inequality indices. As an example, given two distributions $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ having the same mean $\bar{x} = \bar{y}$, we assume that $\boldsymbol{x} \preceq \boldsymbol{y}$. Due to the S-concavity of the binomial welfare functions, we have $C_j(\boldsymbol{x}) \geq C_j(\boldsymbol{y})$. Accordingly, the S-convexity of the binomial inequality indices holds $G_j(\boldsymbol{x}) \leq G_j(\boldsymbol{y})$.

In the following, we give examples of the coefficients $-v_{ji} \in [-(n-1)/n, 1/n]$, with $i, j = 1, \ldots, n$, of the binomial inequality indices G_j, with $j = 1, \ldots, n$, for dimensions $n = 2, 3, 4, 5, 6$. In these examples, the coefficients $-v_{ji}$, with $i, j = 1, \ldots, n$, are simply denoted by $-\boldsymbol{v}_j$.

$n = 2$

$$\begin{aligned} -\boldsymbol{v}_1 &= (0, 0) \\ -\boldsymbol{v}_2 &= (-\tfrac{1}{2}, \tfrac{1}{2}) \end{aligned}$$

$n = 3$

$$\begin{aligned} -\boldsymbol{v}_1 &= (0, 0, 0) \\ -\boldsymbol{v}_2 &= (-\tfrac{1}{3}, 0, \tfrac{1}{3}) \\ -\boldsymbol{v}_3 &= (-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}) \end{aligned}$$

$n = 4$

$$\begin{aligned} -\boldsymbol{v}_1 &= (0, 0, 0, 0) \\ -\boldsymbol{v}_2 &= (-\tfrac{3}{12}, -\tfrac{1}{12}, \tfrac{1}{12}, \tfrac{3}{12}) \\ -\boldsymbol{v}_3 &= (-\tfrac{2}{4}, 0, \tfrac{1}{4}, \tfrac{1}{4}) \\ -\boldsymbol{v}_4 &= (-\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \end{aligned}$$

As concerns the S-concavity and S-convexity of the binomial welfare functions and the associated inequality indices, let us consider a population $\boldsymbol{x} \in \mathbb{D}^n$ with $n \geq 2$ individuals and its unit arithmetic mean $\bar{x} = 1$. The family of income distributions of the population $\boldsymbol{x}$ with respect to various levels of inequality is specified by

$$x_{(i)} = n\Big[f_\beta\Big(\frac{i}{n}\Big) - f_\beta\Big(\frac{i-1}{n}\Big)\Big] \qquad i = 1, \ldots, n \tag{4.18}$$

$n = 5$

$$\begin{aligned}
-\boldsymbol{v}_1 &= (0,0,0,0,0)\\
-\boldsymbol{v}_2 &= (-\tfrac{2}{10},-\tfrac{1}{10},0,\tfrac{1}{10},\tfrac{2}{10})\\
-\boldsymbol{v}_3 &= (-\tfrac{4}{10},-\tfrac{1}{10},\tfrac{1}{10},\tfrac{2}{10},\tfrac{2}{10})\\
-\boldsymbol{v}_4 &= (-\tfrac{3}{5},0,\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5})\\
-\boldsymbol{v}_5 &= (-\tfrac{4}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5})
\end{aligned}$$

$n = 6$

$$\begin{aligned}
-\boldsymbol{v}_1 &= (0,0,0,0,0,0)\\
-\boldsymbol{v}_2 &= (-\tfrac{5}{30},-\tfrac{3}{30},-\tfrac{1}{30},\tfrac{1}{30},\tfrac{3}{30},\tfrac{5}{30})\\
-\boldsymbol{v}_3 &= (-\tfrac{20}{60},-\tfrac{8}{60},\tfrac{1}{60},\tfrac{7}{60},\tfrac{10}{60},\tfrac{10}{60})\\
-\boldsymbol{v}_4 &= (-\tfrac{15}{30},-\tfrac{3}{30},\tfrac{3}{30},\tfrac{5}{30},\tfrac{5}{30},\tfrac{5}{30})\\
-\boldsymbol{v}_5 &= (-\tfrac{4}{6},0,\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6})\\
-\boldsymbol{v}_6 &= (-\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6})
\end{aligned}$$

where $f_\beta : [0,1] \to [0,1]$ is the parametric Lorenz curve defined as

$$f_\beta(r) = re^{-\beta(1-r)} \qquad r \in [0,1]. \tag{4.19}$$

Figure 4.1 illustrates the parametric Lorenz curve with respect to some parameter values $\beta = 0, 1, \ldots, 8$. The equidistribution line corresponds to the value $\beta = 0$ when all individuals receive the same amount of income. Moreover, the Lorenz curve exhibits the Lorenz dominance as β increases from 0 to 8. The values of the binomial welfare functions C_j, with $j = 1, \ldots, n$, in dimensions $n = 4, 6, 8$ are illustrated in Figs. 4.2 - 4.4. The data show that the dominance relations $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \ldots \geq C_n(\boldsymbol{x}) \geq 0$ hold between the binomial welfare functions in an individual distribution with respect to the fixed value of the inequality aversion. On the other hand, they highlight the S-concavity of the binomial welfare functions, expressed as $C_j(\boldsymbol{x}) \geq C_j(\boldsymbol{y})$ where $\boldsymbol{x} \preceq \boldsymbol{y}$, as the inequality parameter increases from 0 to 8.

Similarly, we compute the values of the binomial inequality indices in relation to various Lorenz curves. The result confirms the relations $0 = G_1(\boldsymbol{x}) \leq G_2(\boldsymbol{x}) \leq \ldots \leq G_n(\boldsymbol{x}) \leq 1$, as expected. Moreover, when the inequality parameter increases from 0 to 8, the binomial inequality indices of the Lorenz superior income distributions (above) are always equal to or less than those of the Lorenz inferior income distributions (below) due to the S-convexity of the binomial inequality indices.

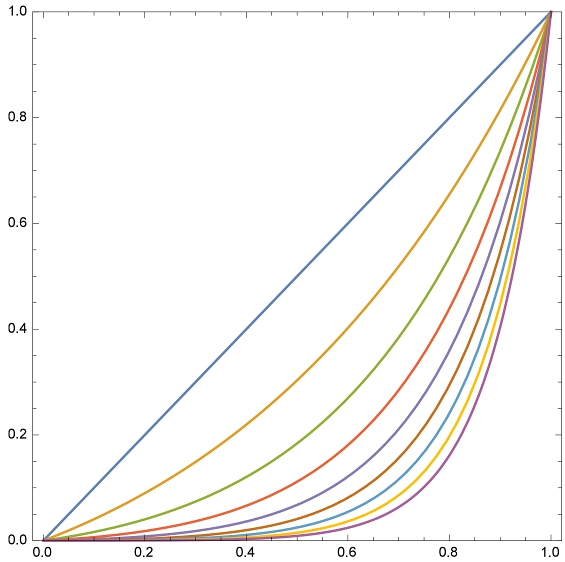


FIGURE 4.1: Parametric Lorenz curve for parameter values $\beta = 0, 1, \ldots, 8$

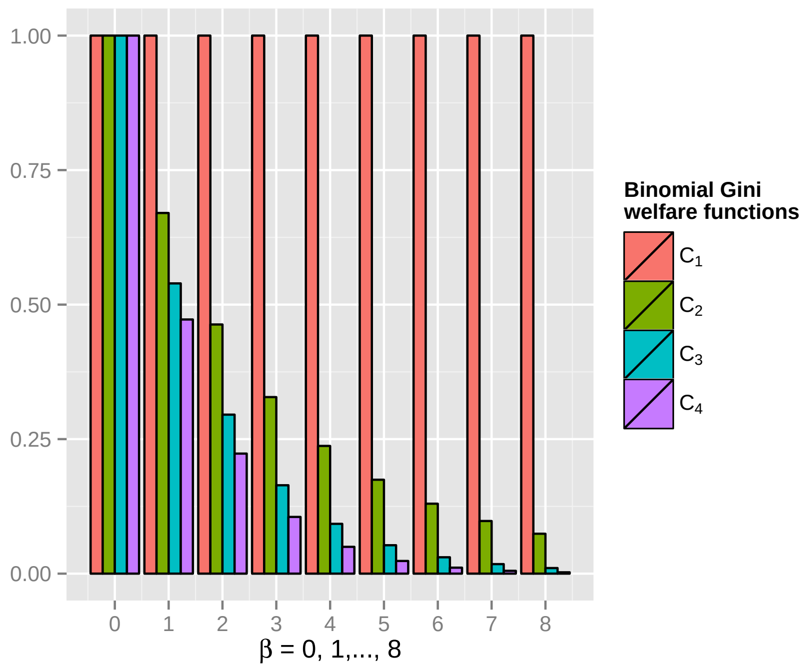


FIGURE 4.2: Values of the binomial welfare functions C_j, $j = 1, \ldots, n$ with $n = 4$ for Lorenz curve parameter $\beta = 0, 1, \ldots, 8$

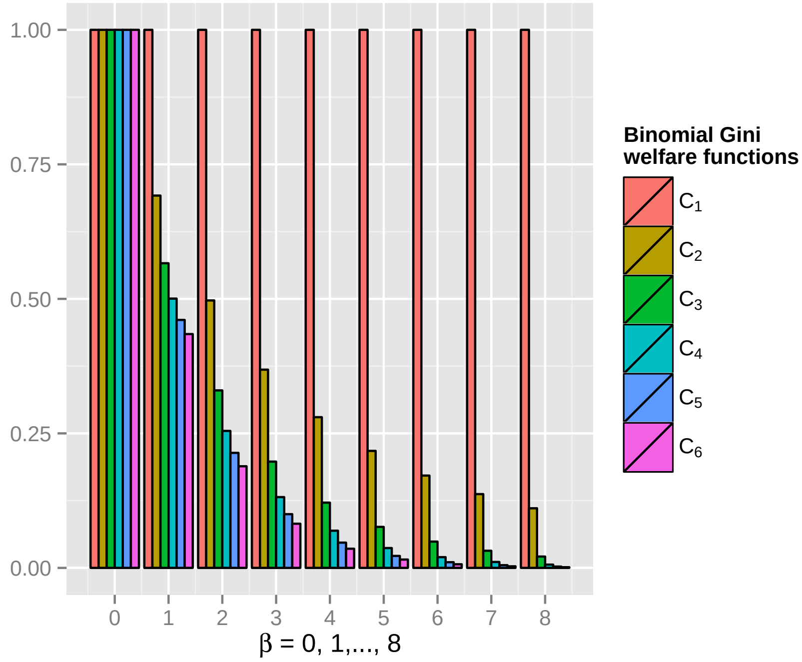


FIGURE 4.3: Values of the binomial welfare functions C_j, $j = 1, \ldots, n$ with $n = 6$ for Lorenz curve parameter $\beta = 0, 1, \ldots, 8$

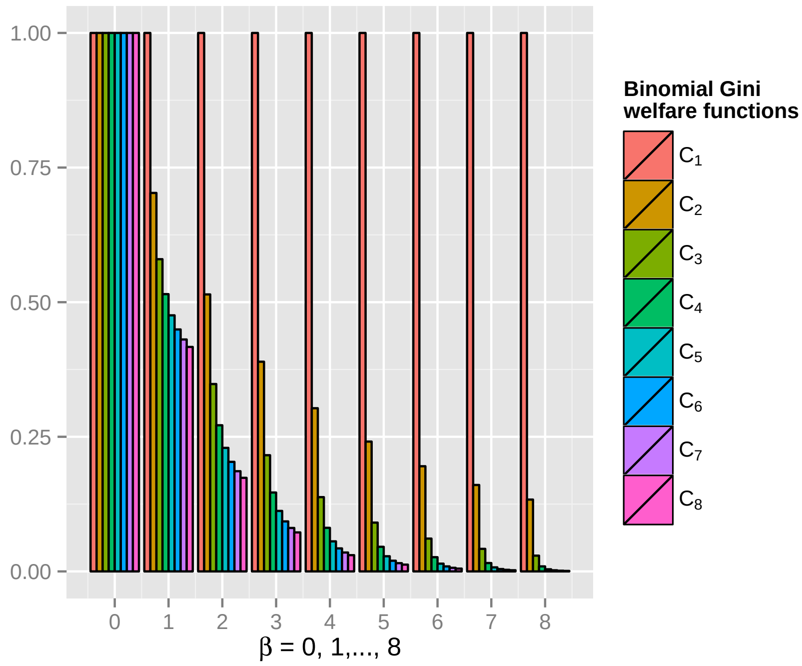


FIGURE 4.4: Values of the binomial welfare functions C_j, $j = 1, \ldots, n$ with $n = 8$ for Lorenz curve parameter $\beta = 0, 1, \ldots, 8$

4.3 The single parameter family of generalized Gini welfare functions

This section focuses on the single parameter family of generalized Gini welfare functions, called the S-Gini family, in the context of the binomial decomposition framework. We derive an analytical expression for the coefficients of the binomial decomposition of S-Gini welfare functions in terms of the inequality aversion parameter. We also analyze the analogy between the binomial welfare functions and the S-Gini family.

4.3.1 The S-Gini family in the binomial decomposition framework

The S-Gini family of welfare functions has been introduced by Donaldson and Weymark [39], see also [1, 2, 3, 24, 40, 74, 127].

Definition 4.17. The *S-Gini welfare function* associated with weigting vector $\boldsymbol{w}^S(\delta)$ is an OWA function $A^S_\delta : \mathbb{D}^n \to \mathbb{D}$ defined as

$$A^S_\delta(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^S(\delta)\, x_{(i)} \qquad w_i^S(\delta) = \left(\frac{n-i+1}{n}\right)^\delta - \left(\frac{n-i}{n}\right)^\delta \qquad \delta \in [1,\infty) \tag{4.20}$$

where $w_i^S(\delta)$ are the S-Gini weights and δ is an inequality aversion parameter.

In the context of the binomial decomposition (4.12), one expresses the S-Gini welfare functions A^S_δ as linear combinations with coefficients $\alpha_1,\ldots,\alpha_n$ of the binomial welfare functions $C_1,\ldots,C_n$.

Proposition 4.18. *The S-Gini welfare functions can be written uniquely as*

$$A^S_\delta(\mathbf{x}) = \alpha_1(\delta)C_1(\boldsymbol{x}) + \alpha_2(\delta)C_2(\boldsymbol{x}) + \ldots + \alpha_n(\delta)C_n(\boldsymbol{x}) \tag{4.21}$$

for all $\delta \in [1,\infty)$. The coefficients α_j, $j = 1,\ldots,n$, are subject to the conditions (4.13) - (4.15).

The binomial decomposition of the S-Gini welfare functions can be written as

$$\sum_{i=1}^{n} w_i^S(\delta)\, x_{(i)} = \alpha_1(\delta)\sum_{i=1}^{n} w_{1i}\, x_{(i)} + \alpha_2(\delta)\sum_{i=1}^{n} w_{2i}\, x_{(i)} + \cdots + \alpha_n(\delta)\sum_{i=1}^{n} w_{ni}\, x_{(i)} \tag{4.22}$$

for all $\delta \in [1,\infty)$.

The binomial weights $w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}}$, $i, j = 1, \ldots, n$, have $n-j+1$ positive non-linear decreasing weights and $j-1$ null last weights. As a result, for each value of the S-Gini parameter $\delta \in [1, \infty)$, we obtain a unique solution $\alpha_1(\delta), \ldots, \alpha_n(\delta)$ by solving the following triangle linear system (Proposition 3.12),

$$\begin{cases} w_1^S(\delta) &= w_{11}\alpha_1(\delta) + w_{21}\alpha_2(\delta) + \cdots + w_{n-1,1}\alpha_{n-1}(\delta) + w_{n,1}\alpha_n(\delta) \\ w_2^S(\delta) &= w_{12}\alpha_1(\delta) + w_{22}\alpha_2(\delta) + \cdots + w_{n-1,2}\alpha_{n-1}(\delta) \\ \ldots. & \\ w_n^S(\delta) &= w_{1n}\alpha_1(\delta)\,. \end{cases} \tag{4.23}$$

Since the S-Gini family is composed of OWA functions, the coefficients $\alpha_j(\delta)$, with $j = 1, \ldots, n$, take the form

$$\alpha_j(\delta) = \frac{1}{w_{j,n-j+1}} \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} w_{n-p}^S(\delta) \tag{4.24}$$

as proven in (3.66). Each $\alpha_j(\delta)$ is explicitly expanded as a function of the set of S-Gini weights, $\{w_{n-j+1}^S(\delta), \ldots, w_n^S\}$, as follows,

$$\begin{aligned} w_{j,n-j+1} \cdot \alpha_j(\delta) &= \sum_{p=j-1}^{0} (-1)^{j-p-1} \binom{j-1}{j-p-1} w_{n-p}^S(\delta) \\ &= (-1)^0 \binom{j-1}{0} w_{n-j+1}^S(\delta) + (-1)^1 \binom{j-1}{1} w_{n-j+2}^S(\delta) + \ldots + \\ &\quad + (-1)^{j-2} \binom{j-1}{j-2} w_{n-1}^S(\delta) + (-1)^{j-1} \binom{j-1}{j-1} w_n^S(\delta) \\ &= \binom{j-1}{0} w_{n-j+1}^S(\delta) - \binom{j-1}{1} w_{n-j+2}^S(\delta) + \ldots + \\ &\quad +(-1)^{j-2} \binom{j-1}{j-2} w_{n-1}^S(\delta) + (-1)^{j-1} \binom{j-1}{j-1} w_n^S(\delta). \end{aligned} \tag{4.25}$$

According to the definition of the S-Gini weights (4.20), we have

$$\begin{aligned} w_{n-j+1}^S(\delta) &= \left(\frac{j}{n}\right)^\delta - \left(\frac{j-1}{n}\right)^\delta \\ w_{n-j+2}^S(\delta) &= \left(\frac{j-1}{n}\right)^\delta - \left(\frac{j-2}{n}\right)^\delta \\ &\vdots \\ w_{n-1}^S(\delta) &= \left(\frac{2}{n}\right)^\delta - \left(\frac{1}{n}\right)^\delta \end{aligned}$$

$$w_n^S(\delta) = \left(\frac{1}{n}\right)^\delta - \left(\frac{0}{n}\right)^\delta.$$

By replacing the notations of the S-Gini weights with their values, the coefficients in (4.25) are equivalently written,

$$\begin{aligned}\frac{1}{\binom{n}{j}} \cdot \alpha_j(\delta) &= \binom{j-1}{0}\left[\left(\frac{j}{n}\right)^\delta - \left(\frac{j-1}{n}\right)^\delta\right] - \binom{j-1}{1}\left[\left(\frac{j-1}{n}\right)^\delta - \left(\frac{j-2}{n}\right)^\delta\right] \\ &+ \ldots + (-1)^{j-2}\binom{j-1}{j-2}\left[\left(\frac{2}{n}\right)^\delta - \left(\frac{1}{n}\right)^\delta\right] + (-1)^{j-1}\binom{j-1}{j-1}\left[\left(\frac{1}{n}\right)^\delta - \left(\frac{0}{n}\right)^\delta\right]\end{aligned} \tag{4.26}$$

for all $\delta \in [1, \infty)$. Notice that there is a common term $\frac{1}{n^\delta}$ on the right-hand side. We move this term to the left-hand side and group the functions of δth powers in the following order,

$$\begin{aligned}\frac{n^\delta}{\binom{n}{j}} \cdot \alpha_j(\delta) &= \binom{j-1}{0} j^\delta - \left[\binom{j-1}{0} + \binom{j-1}{1}\right](j-1)^\delta + \ldots + \\ &+ (-1)^{j-1}\left[\binom{j-1}{j-2} + \binom{j-1}{j-1}\right]1^\delta + (-1)^{j-1}\binom{j-1}{j-1}0^\delta.\end{aligned} \tag{4.27}$$

By applying Pascal's rule, we have

$$\begin{aligned}\binom{j-1}{0} &= \binom{j}{0} \\ \binom{j-1}{0} + \binom{j-1}{1} &= \binom{j}{1} \\ \binom{j-1}{1} + \binom{j-1}{2} &= \binom{j}{2} \\ \binom{j-1}{j-2} + \binom{j-1}{j-1} &= \binom{j}{j-1}.\end{aligned}$$

Therefore, we have

$$\frac{n^\delta}{\binom{n}{j}} \cdot \alpha_j(\delta) = \binom{j}{0} j^\delta - \binom{j}{1}(j-1)^\delta + \binom{j}{2}(j-2)^\delta - \ldots + (-1)^{j-1}\binom{j}{j-1}1^\delta. \tag{4.28}$$

Finally, the analytical solutions $\alpha_j(\delta)$ can be rewritten as the sum of δ powers of the set of integers $1, \ldots, j$ as follows

$$\alpha_j(\delta) = \frac{\binom{n}{j}}{n^\delta} \sum_{p=1}^{j} (-1)^{p-1}\binom{j}{p-1}(j-p+1)^\delta \tag{4.29}$$

for all $\delta \in [1, \infty)$.

In the following, we graphically illustrate the analytical expressions, as in (4.29), for the coefficients $\alpha_j(\delta)$, with $j = 1, \ldots, n$, for the cases $n = 4, 6, 8$.

Example 4.1. *In the case $n = 4$, the linear system (4.23) corresponds to*

$$\begin{cases} w_1^S(\delta) = w_{11}\alpha_1(\delta) + w_{21}\alpha_2(\delta) + w_{31}\alpha_3(\delta) + w_{41}\alpha_4(\delta) \\ w_2^S(\delta) = w_{12}\alpha_1(\delta) + w_{22}\alpha_2(\delta) + w_{32}\alpha_3(\delta) \\ w_3^S(\delta) = w_{13}\alpha_1(\delta) + w_{23}\alpha_2(\delta) \\ w_4^S(\delta) = w_{14}\alpha_1(\delta) \end{cases} \tag{4.30}$$

and admits the unique solution according to the analytical expression (4.29)

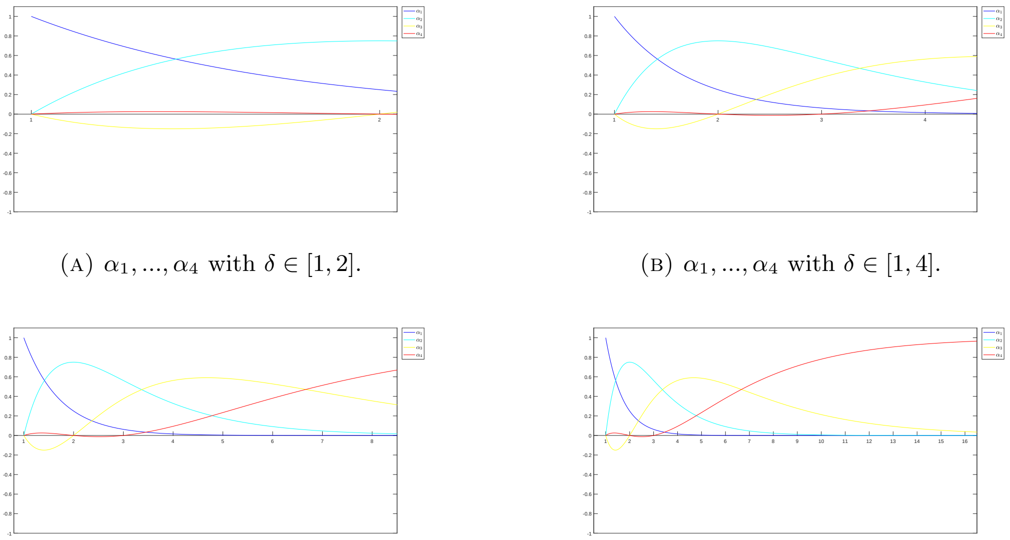
$$\begin{cases} \alpha_1(\delta) = 4 \cdot \left[1^\delta\right] / 4^\delta \\ \alpha_2(\delta) = 6 \cdot \left[2^\delta \ - 2 \cdot 1^\delta\right] / 4^\delta \\ \alpha_3(\delta) = 4 \cdot \left[3^\delta \ - 3 \cdot 2^\delta \ + 3 \cdot 1^\delta\right] / 4^\delta \\ \alpha_4(\delta) = \left[4^\delta - 4 \cdot 3^\delta \ + 6 \cdot 2^\delta \ - 4 \cdot 1^\delta\right] / 4^\delta \end{cases} \tag{4.31}$$

in which the coefficients $\alpha_j(\delta)$, with $j = 1, \ldots, 4$, are explicitly given as functions of the parameter $\delta \in [1, \infty)$, as illustrated in Fig. 4.5.

Example 4.2. *In the case $n = 6$, the linear system (4.23) corresponds to*

$$\begin{cases} w_1^S(\delta) = w_{11}\alpha_1(\delta) + w_{21}\alpha_2(\delta) + w_{31}\alpha_3(\delta) + w_{41}\alpha_4(\delta) + w_{51}\alpha_5(\delta) + w_{61}\alpha_6(\delta) \\ w_2^S(\delta) = w_{12}\alpha_1(\delta) + w_{22}\alpha_2(\delta) + w_{32}\alpha_3(\delta) + w_{42}\alpha_4(\delta) + w_{52}\alpha_5(\delta) \\ w_3^S(\delta) = w_{13}\alpha_1(\delta) + w_{23}\alpha_2(\delta) + w_{33}\alpha_3(\delta) + w_{43}\alpha_4(\delta) \\ w_4^S(\delta) = w_{14}\alpha_1(\delta) + w_{24}\alpha_2(\delta) + w_{34}\alpha_3(\delta) \\ w_5^S(\delta) = w_{15}\alpha_1(\delta) + w_{25}\alpha_2(\delta) \\ w_6^S(\delta) = w_{16}\alpha_1(\delta) \end{cases} \tag{4.32}$$

and admits the unique solution according to the analytical expression (4.29)



(A) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 2]$.

(B) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 4]$.

(C) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 8]$.

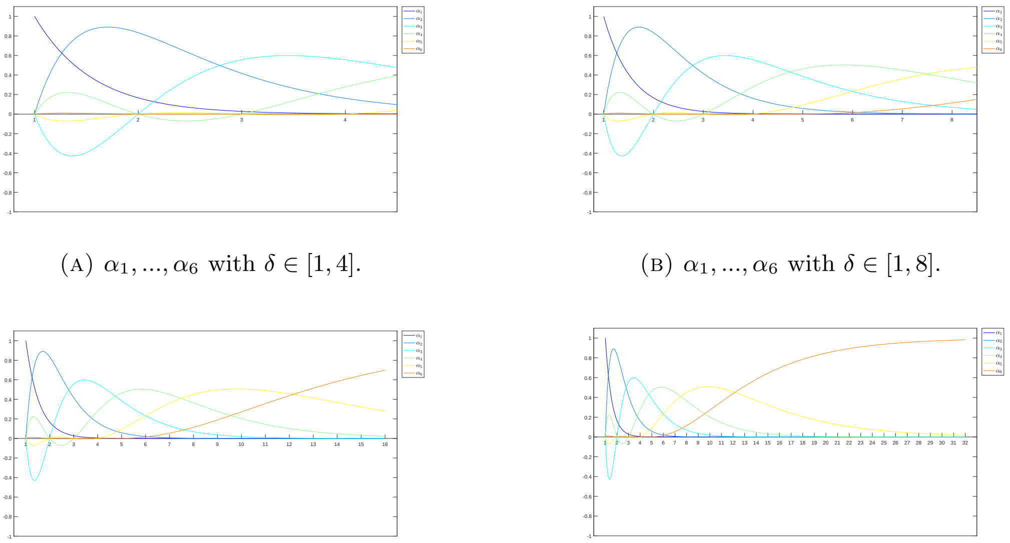
(D) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 32]$.

FIGURE 4.5: Coefficients of the binomial decomposition for $n = 4$

$$
\begin{cases}
\alpha_1(\delta) = 6 \cdot \left[1^\delta\right] / 6^\delta \\
\alpha_2(\delta) = 15 \cdot \left[2^\delta \ - 2 \cdot 1^\delta\right] / 6^\delta \\
\alpha_3(\delta) = 20 \cdot \left[3^\delta \ - 3 \cdot 2^\delta \ + 3 \cdot 1^\delta\right] / 6^\delta \\
\alpha_4(\delta) = 15 \cdot \left[4^\delta - 4 \cdot 3^\delta \ + 6 \cdot 2^\delta \ - 4 \cdot 1^\delta\right] / 6^\delta \\
\alpha_5(\delta) = 6 \cdot \left[5^\delta - 5 \cdot 4^\delta + 10 \cdot 3^\delta \ - 10 \cdot 2^\delta \ + 5 \cdot 1^\delta\right] / 6^\delta \\
\alpha_6(\delta) = \left[6^\delta - 6 \cdot 5^\delta + 15 \cdot 4^\delta - 20 \cdot 3^\delta \ + 15 \cdot 2^\delta \ - 6 \cdot 1^\delta\right] / 6^\delta
\end{cases}
\tag{4.33}
$$

in which the coefficients $\alpha_j(\delta)$, with $j = 1, \ldots, 6$, are explicitly given as functions of the parameter $\delta \in [1, \infty)$, as illustrated in Fig. 4.6.

Example 4.3. *In the case $n = 8$, the linear system (4.23) admits the unique solution according to the analytical expression (4.29)*



(A) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 4]$.

(B) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 8]$.

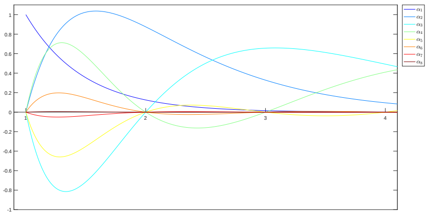
(C) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 16]$.

(D) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 32]$.

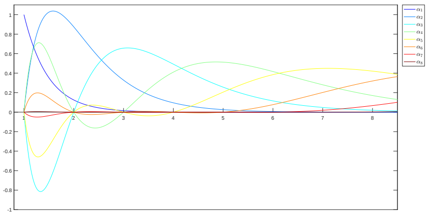
FIGURE 4.6: Coefficients of the binomial decomposition for $n = 6$

$$\begin{cases} \alpha_1(\delta) = 8 \cdot \left[1^\delta\right] / 8^\delta \\ \alpha_2(\delta) = 28 \cdot \left[2^\delta \ - 2 \cdot 1^\delta\right] / 8^\delta \\ \alpha_3(\delta) = 56 \cdot \left[3^\delta \ - 3 \cdot 2^\delta \ + 3 \cdot 1^\delta\right] / 8^\delta \\ \alpha_4(\delta) = 70 \cdot \left[4^\delta - 4 \cdot 3^\delta \ + 6 \cdot 2^\delta \ - 4 \cdot 1^\delta\right] / 8^\delta \\ \alpha_5(\delta) = 56 \cdot \left[8^\delta - 5 \cdot 4^\delta + 10 \cdot 3^\delta \ - 10 \cdot 2^\delta \ + 5 \cdot 1^\delta\right] / 8^\delta \\ \alpha_6(\delta) = 28 \cdot \left[6^\delta - 6 \cdot 5^\delta + 15 \cdot 4^\delta - 20 \cdot 3^\delta \ + 15 \cdot 2^\delta \ - 6 \cdot 1^\delta\right] / 8^\delta \\ \alpha_7(\delta) = 8 \cdot \left[7^\delta - 7 \cdot 6^\delta + 21 \cdot 5^\delta - 35 \cdot 4^\delta + 35 \cdot 3^\delta \ - 21 \cdot 2^\delta \ + 7 \cdot 1^\delta\right] / 8^\delta \\ \alpha_8(\delta) = \left[8^\delta - 8 \cdot 7^\delta + 28 \cdot 6^\delta - 56 \cdot 5^\delta + 70 \cdot 4^\delta - 56 \cdot 3^\delta \ + 28 \cdot 2^\delta \ - 8 \cdot 1^\delta\right] / 8^\delta \end{cases} \tag{4.34}$$

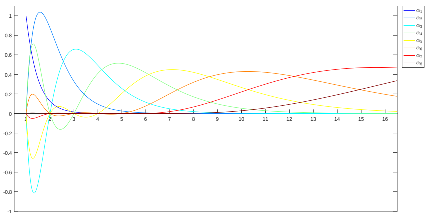
in which the coefficients $\alpha_j(\delta)$, with $j = 1, \ldots, 8$, are explicitly given as functions of the parameter $\delta \in [1, \infty)$, as illustrated in Fig. 4.7.



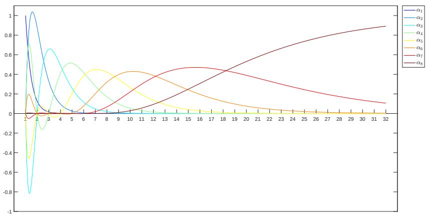
(A) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 4]$.



(B) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 8]$.



(C) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 16]$.



(D) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 32]$.

FIGURE 4.7: Coefficients of the binomial decomposition for $n = 8$

4.3.2 The analogy between the binomial welfare functions and the parametric S-Gini family

The S-Gini welfare functions, as defined in (4.20), take the form of $A_1^S(\boldsymbol{x}) = \bar{x} = C_1(\boldsymbol{x})$ and $A_\infty^S(\boldsymbol{x}) = x_{(1)} = C_n(\boldsymbol{x})$ with respect to the inequality aversion parameter values $\delta = 1$ and $\delta = \infty$, respectively. The S-Gini welfare function of a distribution $\boldsymbol{x} \in \mathbb{D}^n$ is, therefore, bounded in the interval $[x_{(1)}, \bar{x}]$, equivalently written as $C_n(\boldsymbol{x}) \leq A^S(\boldsymbol{x}) \leq C_1(\boldsymbol{x})$. In other words, the S-Gini welfare functions interpolate between the first and last binomial welfare functions as the inequality aversion parameter increases from $\delta = 1$ to $\delta = \infty$.

In the following, we carry out an experiment to identify the parameter values of the S-Gini welfare functions that can produce the weighting vectors most similar to the ones of the binomial welfare functions. The degrees of similarity are measured by mean square differences. Before computing, we first need to normalize the inequality aversion parameter from the infinite range to the definite one by the following transformation

$$\Delta = 1\Big(\frac{2}{\delta+1}\Big) + n\Big(\frac{\delta-1}{\delta+1}\Big) \tag{4.35}$$

where the normalized inequality aversion parameter Δ is equal to 1 and n when $\delta = 1$ and $\delta = \infty$, respectively.

Figures 4.8 and 4.9 show the normalized parameter Δ of the S-Gini welfare functions whose weights are most similar to the ones of the binomial welfare functions C_j associated with the binomial indices $j = 1, \ldots, n$. The normalized parameter Δ increases non-linearly very fast for the values of $j = 1, \ldots, n$. After the rapid changes, the normalized parameter gradually stabilizes and reaches its upper bound at $\Delta = n$.

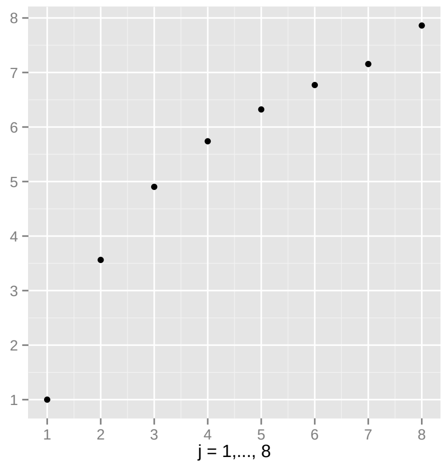
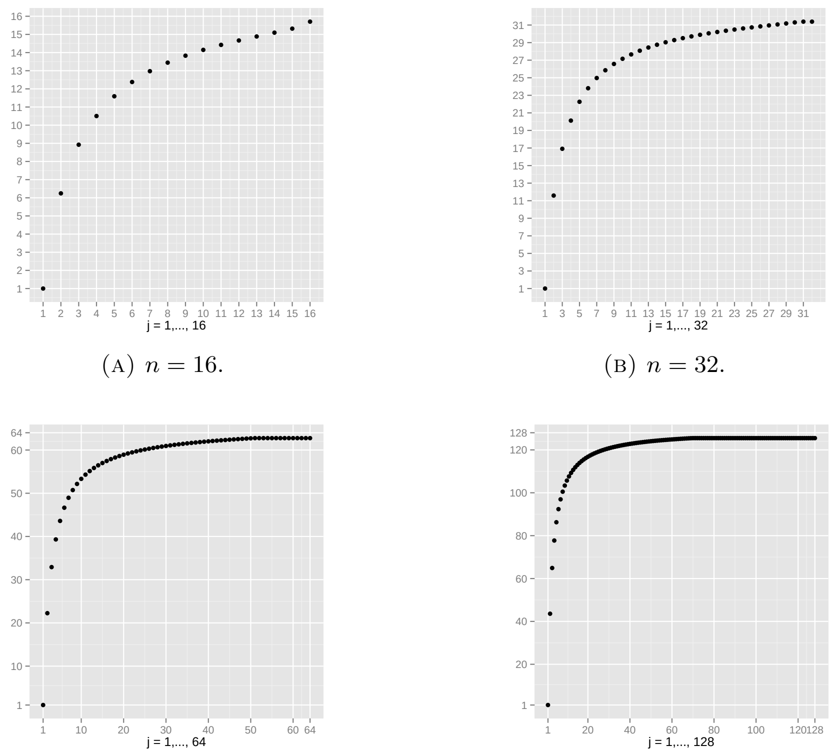


FIGURE 4.8: The normalized parameter Δ (as shown on the vertical axis) of the S-Gini welfare function whose weight distribution is the closest to the one of each binomial welfare function C_j, $j = 1, \ldots, n = 8$



(A) $n = 16$.

(B) $n = 32$.

(C) $n = 64$.

(D) $n = 128$.

FIGURE 4.9: As in Fig. 4.8 with $n = 16, 32, 64, 128$

4.4 The Lorenzen family of generalized Gini welfare functions

This section reviews the parametric Lorenzen family of generalized Gini welfare functions. We consider two equivalent expressions of this family with respect to the formal definition and the binomial decomposition framework. We then study the relationship between the Lorenzen weights and the associated coefficients of the binomial decomposition framework. In particular, we derive an analytical expression for the coefficients of the binomial decomposition of Lorenzen welfare functions in terms of the Lorenzen index. We also analyze the analogy between the binomial welfare functions and the Lorenzen family.

4.4.1 The Lorenzen family in the binomial decomposition framework

The Lorenzen family, proposed by Lorenzen [83], is a parametric family of generalized Gini welfare functions.

Definition 4.19. The *Lorenzen welfare function* associated with weigting vector $\boldsymbol{w}^L(l)$ is an OWA function $A_l^L : \mathbb{D}^n \to \mathbb{D}$ defined as

$$A_l^L(\boldsymbol{x}) = \sum_{i=1}^{l} w_i^L(l) x_{(i)} \qquad w_i^L(l) = \frac{l+n-2i+1}{nl} \qquad l = 1, \ldots, n \tag{4.36}$$

where $w_i^L(l)$ are the Lorenzen weights and l is the number of the poorest individuals that are taken into account in the Lorenzen welfare functions.

The Lorenzen weights $w_i^L(l)$, with $i = 1, \ldots, n$, can be equivalently written as

$$w_i^L(l) = \begin{cases} \frac{l+n-2i+1}{nl} = \frac{l+n+1}{nl} - \frac{2i}{nl} & \text{if } i \leq l \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.37}$$

For instance, the non-zero Lorenzen weights in dimension n are

$$\begin{aligned} w_1^L(l) &= \frac{l+n+1}{nl} - \frac{2}{nl} \\ w_2^L(l) &= \frac{l+n+1}{nl} - \frac{4}{nl} \\ w_3^L(l) &= \frac{l+n+1}{nl} - \frac{6}{nl} \\ &\vdots \end{aligned}$$

$$w_l^L(l) \quad = \quad \frac{l+n+1}{nl} - \frac{2l}{nl}. \tag{4.38}$$

We can prove that the Lorenzen weights have unit sum directly from (4.38) as follows

$$\sum_{i=1}^{l} w_i^L(l) \quad = \quad \sum_{i=1}^{l} \frac{l+n+1}{nl} - \sum_{i=1}^{l} \frac{2i}{nl} \tag{4.39}$$

The first summation of l times of the constant values $\frac{l+n+1}{nl}$ is reduced to $\frac{l+n+1}{n}$. The second summation is equal to $\frac{2}{nl}\sum_{i=1}^{l} i = \frac{l+1}{n}$. The Lorenzen weights are, therefore, sum up to 1.

In the context of the binomial decomposition (4.12), each Lorenzen welfare function A_j^L can be expressed in terms of the binomial Gini welfare functions $C_1, \ldots, C_n$ as follows,

$$A_j^L(\boldsymbol{x}) \quad = \quad \alpha_1(l)C_1(\boldsymbol{x}) + \alpha_2(l)C_2(\boldsymbol{x}) + \ldots + \alpha_n(l)C_n(\boldsymbol{x}) \tag{4.40}$$

for $j = 1, \ldots, n$, which can be written as

$$\sum_{i=1}^{n} w_{ji}^L(l)x_{(i)} \quad = \quad \alpha_1(l)\sum_{i=1}^{n} w_{1i}x_{(i)} + \alpha_2(l)\sum_{i=1}^{n} w_{2i}x_{(i)} + \ldots + \alpha_n(l)\sum_{i=1}^{n} w_{ni}x_{(i)}. \tag{4.41}$$

The binomial weights $w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}}, i, j = 1, \ldots, n$, have $n-j+1$ positive non-linear decreasing weights and $j-1$ null last weights. Therefore, for each value of the Lorenzen index $l = 1, \ldots, n$, we obtain a unique solution $\alpha_1(l), \ldots, \alpha_n(l)$ by solving the triangle linear system (Proposition 3.12),

$$\begin{cases} w_1^L(l) & = & w_{11}\alpha_1(l) + w_{21}\alpha_2(l) + \cdots + w_{n-1,1}\alpha_{n-1}(l) + w_{n,1}\alpha_n(l) \\ w_2^L(l) & = & w_{12}\alpha_1(l) + w_{22}\alpha_2 l(l) + \cdots + w_{n-1,2}\alpha_{n-1}(l) \\ \ldots. & & \\ w_{n-1}^L(l) & = & w_{1,n-1}\alpha_1(l) + w_{2,n-1}\alpha_2(l) \\ w_n^L(l) & = & w_{1n}\alpha_1(l) \end{cases} \tag{4.42}$$

Since the Lorenzen family is a particular case of the OWA functions, the coefficients $\alpha_j(l)$, with $j, l = 1, \ldots, n$, have the following analytical expression, similarly to (3.66),

$$\alpha_j(l) \quad = \quad \frac{1}{w_{j,n-j+1}} \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} w_{n-p}^L(l). \tag{4.43}$$

Each $\alpha_j(l)$ is expressed as a function of the set of Lorenzen weights $\{w^L_{n-j+1}(l), \ldots, w^L_n(l)\}$ where the Lorenzen weights are defined as (4.37), see also (4.38). The Lorenzen weights $w^L_i(l)$ are non-zero when the Lorenzen index $i \leq l$ for $l = 1, \ldots, n$. The coefficients $\alpha_j(l)$ are, therefore, explicitly determined by the set of the non-zero Lorenzen weights $\{w^L_{n-j+1}, \ldots, w^L_{l-1}, w^L_l\}$, where $l \geq n - j + 1$. The expression of the coefficients $\alpha_j(l)$ can be written shortly as follows

$$\alpha_j(l) = \begin{cases} \frac{1}{w_{j,n-j+1}} \sum_{p=n-l}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} w^L_{n-p}(l) & \text{if } l \geq n-j+1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.44}$$

By substituting the Lorenzen weights (4.37), the above expression can be written as

$$\alpha_j(l) = \begin{cases} \frac{1}{w_{j,n-j+1}} \sum_{p=n-l}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} \left[\frac{l-n+1}{nl} + \frac{2p}{nl}\right] & \text{if } l \geq n-j+1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.45}$$

In the two equivalent expressions (4.43)-(4.44) of the coefficients $\alpha_j(l)$, we observe some interesting properties of the coefficients $\alpha_j(l)$ in correspondence with different choices of the Lorenzen index $l = 1, \ldots, n$. The detailed proof is presented in Appendix A, Lemma 4.

(i) $\alpha_j(l) = 0$ where $j < n - l + 1$ for the Lorenzen index $l = 1, \ldots, n$

(ii) $\alpha_1(l) = \ldots = \alpha_{n-1}(l) = 0$ and $\alpha_n(l) = 1$ for the Lorenzen index $l = 1$

(iii) $\alpha_1(l) = 1/n$, $\alpha_2(l) = (n-1)/n$, and $\alpha_3(l) = \ldots = \alpha_n(l) = 0$ for the Lorenzen index $l = n$

(iv) $\alpha_1(l) = 0$, $\alpha_2(l) = 1$ and $\alpha_j(l) = 0$, where $j = 3, \ldots, n$, for the Lorenzen index $l = n - 1$.

In the following we graphically illustrate the analytical expressions, as in (4.45), of the coefficients $\alpha_j(l)$, with $j, l = 1, \ldots, n$, in the binomial decomposition of Lorenzen welfare functions for the cases $n = 4, 6, 8$.

Example 4.4. *In the case $n = 4$, the linear system (4.42) corresponds to*

$$\begin{cases} w_1^L(l) = w_{11}\alpha_1(l) + w_{21}\alpha_2(l) + w_{31}\alpha_3(l) + w_{41}\alpha_4(l) \\ w_2^L(l) = w_{12}\alpha_1(l) + w_{22}\alpha_2(l) + w_{32}\alpha_3(l) \\ w_3^L(l) = w_{13}\alpha_1(l) + w_{23}\alpha_2(l) \\ w_4^L(l) = w_{14}\alpha_1(l) \end{cases} \tag{4.46}$$

and admits the unique solution according to the analytical expression (4.45)

$$\begin{cases} \alpha_1(l = 1, \ldots, 4) = \left(0, 0, 0, \frac{1}{4}\right) \\ \alpha_2(l = 1, \ldots, 4) = \left(0, 0, 1, \frac{3}{4}\right) \\ \alpha_3(l = 1, \ldots, 4) = \left(0, \frac{3}{2}, 0, 0\right) \\ \alpha_4(l = 1, \ldots, 4) = \left(1, -\frac{1}{2}, 0, 0\right) \end{cases} \tag{4.47}$$

in which the coefficients $\alpha_j(l)$, with $j = 1, \ldots, 4$, are explicitly given as functions of the parameter $l = 1, \ldots, 4$, as illustrated in Fig. 4.10.

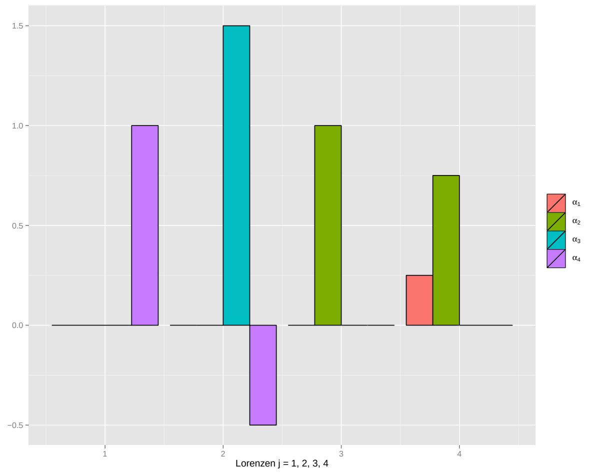


FIGURE 4.10: $\alpha_1(l), ..., \alpha_(l)$ with $l = 1, ..., 4$ for $n = 4$

Example 4.5. *In the case $n = 6$, the linear system (4.42) corresponds to*

$$\begin{cases} w_1^L(l) = w_{11}\alpha_1(l) + w_{21}\alpha_2(l) + w_{31}\alpha_3(l) + w_{41}\alpha_4(l) + w_{51}\alpha_5(l) + w_{61}\alpha_6(l) \\ w_2^L(l) = w_{12}\alpha_1(l) + w_{22}\alpha_2(l) + w_{32}\alpha_3(l) + w_{42}\alpha_4(l) + w_{52}\alpha_5(l) \\ w_3^L(l) = w_{13}\alpha_1(l) + w_{23}\alpha_2(l) + w_{33}\alpha_3(l) + w_{43}\alpha_4(l) \\ w_4^L(l) = w_{14}\alpha_1(l) + w_{24}\alpha_2(l) + w_{34}\alpha_3(l) \\ w_5^L(l) = w_{15}\alpha_1(l) + w_{25}\alpha_2(l) \\ w_6^L(l) = w_{16}\alpha_1(l) \end{cases} \tag{4.48}$$

and admits the unique solution according to the analytical expression (4.45)

$$\begin{cases} \alpha_1(l = 1, \dots, 6) = \left(0, 0, 0, 0, 0, \frac{1}{6}\right) \\ \alpha_2(l = 1, \dots, 6) = \left(0, 0, 0, 0, 1, \frac{5}{6}\right) \\ \alpha_3(l = 1, \dots, 6) = \left(0, 0, 0, \frac{5}{2}, 0, 0\right) \\ \alpha_4(l = 1, \dots, 6) = \left(0, 0, \frac{10}{3}, -\frac{5}{2}, 0, 0\right) \\ \alpha_5(l = 1, \dots, 6) = \left(0, \frac{5}{2}, -\frac{10}{3}, \frac{5}{4}, 0, 0\right) \\ \alpha_6(l = 1, \dots, 6) = \left(1, -\frac{3}{2}, 1, -\frac{1}{4}, 0, 0\right) \end{cases} \tag{4.49}$$

in which the coefficients $\alpha_j(l)$, with $j = 1, \dots, 6$, are explicitly given as functions of the parameter $l = 1, \dots, 6$, as illustrated in Fig. 4.11.

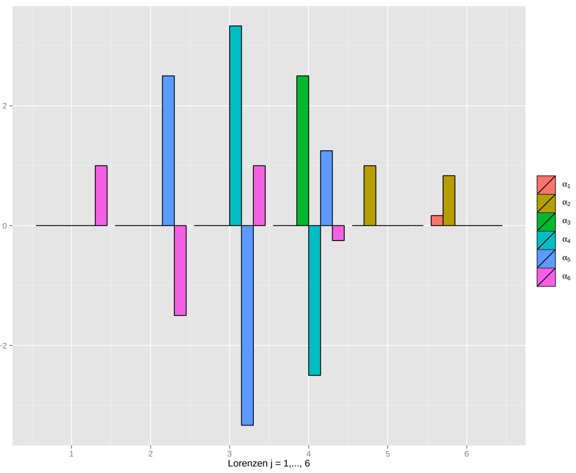


FIGURE 4.11: $\alpha_1(l), ..., \alpha_6(l)$ with $l = 1, ..., 6$ for $n = 6$

Example 4.6. *In the case $n = 8$, the linear system (4.42) admits the unique solution according to the analytical expression (4.45)*

$$\begin{cases} \alpha_1(l) = \left(0,0,0,0,0,0,0,\frac{1}{8}\right) \\ \alpha_2(l) = \left(0,0,0,0,0,0,1,\frac{7}{8}\right) \\ \alpha_3(l) = \left(0,0,0,0,0,\frac{7}{2},0,0\right) \\ \alpha_4(l) = \left(0,0,0,0,7,-\frac{35}{6},0,0\right) \\ \alpha_5(l) = \left(0,0,0,\frac{35}{4},-14,\frac{35}{6},0,0\right) \\ \alpha_6(l) = \left(0,0,7,-\frac{63}{4},\frac{63}{5},-\frac{7}{2},0,0\right) \\ \alpha_7(l) = \left(0,\frac{7}{2},-\frac{28}{3},\frac{21}{2},-\frac{28}{5},\frac{7}{6},0,0\right) \\ \alpha_8(l) = \left(1,-\frac{5}{2},\frac{10}{3},-\frac{5}{2},1,-\frac{1}{6},0,0\right) \end{cases} \tag{4.50}$$

in which the coefficients $\alpha_j(l)$, with $j = 1,\ldots,8$, are explicitly given as functions of the parameter $l = 1,\ldots,8$, as illustrated in Fig. 4.12.

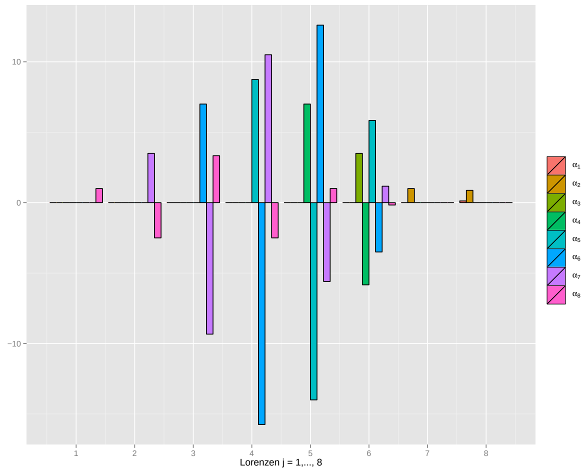
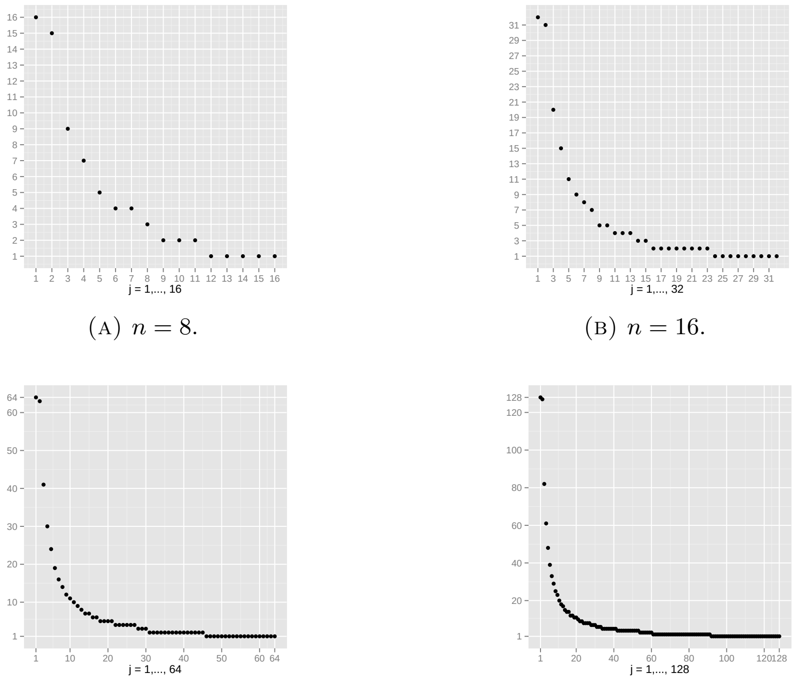


FIGURE 4.12: $\alpha_1(l),...,\alpha_8(l)$ with $l = 1,...,8$ for $n = 8$

4.4.2 The analogy between the binomial welfare functions and the parametric Lorenzen family

The Lorenzen welfare functions, as defined in (4.36), take the form of $A_1^L(\boldsymbol{x}) = x_{(1)} = C_n(\boldsymbol{x})$ and $A_n^L(\boldsymbol{x}) = \bar{x} - G^C(\boldsymbol{x}) = A^C(\boldsymbol{x})$ with respect to the Lorenzen parameter values $l = 1$ and $l = n$, respectively. The Lorenzen welfare function of a distribution $\boldsymbol{x} \in \mathbb{D}^n$ is, therefore, bounded in the interval $C_n(\boldsymbol{x}) \leq A^L(\boldsymbol{x}) \leq A^C(\boldsymbol{x})$. In other words, the Lorenzen welfare functions interpolate between the last binomial welfare functions and the classical Gini welfare function as the inequality aversion parameter increases from $l = 1$ to $l = n$.

In Fig. 4.13 we illustrate the Lorenzen index l of the Lorenzen welfare functions whose weights are most similar to the ones of the binomial welfare functions C_j associated with the binomial indices $j = 1, \ldots, n$. We notice that the Lorenzen index l reduces non-linearly very fast with respect to the increasing index $j = 1, \ldots, n$. After the rapid changes, the Lorenzen index gradually stabilizes and reaches its lower bound at $l = 1$.



(A) $n = 8$.

(B) $n = 16$.

(C) $n = 32$.

(D) $n = 64$.

FIGURE 4.13: The index value l (as shown on the vertical axis) of the Lorenzen welfare function whose weight distribution is the closest to the one of each binomial OWA function C_j, $j = 1, \ldots, n = 8, 16, 32, 64$

4.5 Discussion and conclusion

In this chapter we have discussed the binomial decomposition of OWA functions in the restricted context of generalized Gini welfare functions, including two parametric families of welfare functions, namely the S-Gini and Lorenzen families. Our study showed that the binomial welfare functions behave in analogy with the S-Gini and Lorenzen families. As the S-Gini inequality aversion parameter δ increases from 1 to infinity, the weights of the S-Gini welfare functions are more similar to the ones of the binomial welfare functions $C_1, \ldots, C_n$. The S-Gini welfare functions associated with the parameters $\delta = 1$ and $\delta = \infty$ correspond to the first and the last binomial welfare functions, respectively. On the other hand, the Lorenzen welfare functions interpolate between the last binomial welfare function C_n and the classical Gini welfare function, which combines the first two binomial welfare functions C_1 and C_2.

Furthermore, we derived the analytical expressions for the coefficients $\alpha_1, \ldots, \alpha_n$ in terms of the respective parameters of the S-Gini and Lorenzen families. From a mathematical point of view, these analytical expressions provide an alternative means to study the weight distributions of the two parametric families through the distributions of the associated coefficients in the binomial decomposition framework. Further research on this direction is, therefore, needed in the future.

Chapter 5

Conclusion

In this thesis we used the binomial decomposition framework to address several research issues related to OWA functions. We obtained the analytical expression for the coefficients of the binomial decomposition of OWA functions in terms of OWA weights. We also applied the binomial decomposition framework to two parametric families of OWA functions, namely the S-Gini and Lorenzen welfare functions. We studied the analogy between the binomial OWA functions and these parametric welfare functions. We also found the analytical formulations expressing the close relationships between the weights of the parametric welfare functions and the associated coefficients of the binomial decomposition framework. Further investigation on these relationships might help us to better understand the behavior of the coefficients of the binomial decomposition framework in the restricted context of welfare functions.

In addition, we used the binomial decomposition framework in combination with the concept of k-additivity to control the computational complexity of OWA weights when solving the minimax disparity model in large-scale optimization problems. Instead of considering an entire set of OWA weights, we used the limited set of the coefficients to reduce the dimension up to $\left(1 - \frac{k}{n}\right)\%$, where n is the number of variables in the original problem and k is the level of k-additivity. Our proposed model can be solved faster by optimizers in an approximated way for some levels of k-additivity and some values of orness, leading to a significant reduction in running time. Our experiments are carried out in dimensions up to 40, which are relatively large with respect to the state of the art in the literature ($n = 3, 4, 5, 6$). However, those dimensions are still relatively small if one wants to achieve a very statistically significant reduction in running time. We suggest to extend our experiments in higher dimensions as future research.

Moreover, our proposed model might have no feasible solution with respect to some levels of k-additivity for some values of orness, since the number of the first k coefficients is too small to satisfy a required set of constraints. This suggests us to develop an algorithm in our future research for identifying which k-additive level provides the best trade-off between the computational cost and the accuracy of the approximated, and possibly suboptimal, OWA weights.

Appendix A

In the Appendix we present the detailed proofs of some classical identities used in our thesis.

Lemma 1. For every $n \in \mathbb{N}$ and $n \geq 1$, the alternating sum of binomial coefficients

$$S_n = \sum_{i=0}^{n} (-1)^{-i} \binom{n}{i} \tag{1}$$

is equal to 0 (see also p.417 in [95]).

Proof.

According to the binomial theorem for any non-negative integer n, we have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i. \tag{2}$$

By choosing $x = -1$, the alternating sum S_n of binomial coefficients is equal to zero. □

Lemma 2. For every $n \in \mathbb{N}$ and $n \geq 1$, the summation

$$S_{n-1} = \sum_{i=0}^{n-1} (-1)^{-i} \binom{n}{i+1} \tag{3}$$

is equal to 1.

Proof.

$$\begin{aligned} S_{n-1} - 1 &= \sum_{i=0}^{n-1} (-1)^{-i} \binom{n}{i+1} - 1 \\ &= \sum_{i=0}^{n-1} (-1)^{-i} \binom{n}{i+1} + (-1)\binom{n}{0} \end{aligned}$$

$$= (-1)^0\binom{n}{1} + (-1)^1\binom{n}{2} + \ldots + (-1)^{n-1}\binom{n}{n} + (-1)^{-1}\binom{n}{0}$$

$$= \sum_{i=0}^{n}(-1)^{i-1}\binom{n}{i}. \tag{4}$$

By applying Lemma 1, we obtain $S_{n-1} - 1 = 0$ which can be rewritten as $S_{n-1} = 1$. □

Lemma 3. For every $n \in \mathbb{N}$ and $n \geq 2$, the summation $S_n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}i$ is equal to 0.

Proof.

We can prove this identity by using the strong induction rule.

1. *Base case:* Consider $n = 2$, the statement $S_2 = \sum_{i=0}^{2}(-1)^i\binom{2}{i} = (-1)^1\binom{2}{1}1 + (-)^2\binom{2}{2}2$ is equal to 0. Hence S_n holds for $n = 2$.

2. *Hypothesis:* Assume that S_n is true for some fixed k, it yields

$$S_k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}i. \tag{5}$$

3. *Induction:* We need to prove that $S_{k+1} = 0$ also holds where S_{k+1} is given as

$$\begin{aligned} S_{k+1} &= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}i \\ &= \sum_{i=0}^{k+1}(-1)^i\binom{k}{i-1}i + \sum_{i=0}^{k+1}(-1)^i\binom{k}{i}i. \end{aligned} \tag{6}$$

We have used Pascal's identity $\binom{j}{j-p} = \binom{j-1}{j-p-1} + \binom{j-1}{j-p}$ in order to split the sum into two summations. In the second summation, the upper limit when $i = k + 1$ yields $\binom{k}{i} = 0$, hence it is simplified as $\sum_{i=1}^{k}(-1)^i\binom{k}{i}i$. According to Lemma 1, this summation is equal to 0. At the end, the statement S_{k+1} is reduced to

$$S_{k+1} = \sum_{i=0}^{k+1}(-1)^i\binom{k}{i-1}i. \tag{7}$$

By substituting j for $i-1$, we can align the indices of the binomial coefficient with the upper and lower limits of the summation as follows

$$S_{k+1} = \sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}(j+1). \tag{8}$$

By splitting the sum into two groups, it can be rewritten as follows

$$S_{k+1} = \sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j} + \sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}j. \tag{9}$$

The first summation is recognized as the classical identity which has been proved in Lemma 1. The second summation can be written with the lower indice starting from $j = 1$ instead of $j = 0$ since $\binom{k}{j}j = 0$ when $j = 0$.

$$S_{k+1} = \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}j. \tag{10}$$

Due to the induction hypothesis, the remaining summation is equal to 0. Therefore, S_{k+1} is true for $n = k + 1$.

In summary, by mathematical induction, for all $n \geq 2$ the statement S_n is true. □

Lemma 4. In the following part we prove some interesting properties of the coefficients $\alpha_j(l)$, with $j, l = 1, \dots, n$, of the binomial decomposition of Lorenzen welfare functions.

(i) $\alpha_j(l) = 0$ where $j < n - l + 1$ for the Lorenzen index $l = 1, \dots, n$

(ii) $\alpha_1(l) = \ldots = \alpha_{n-1}(l) = 0$ and $\alpha_n(l) = 1$ for the Lorenzen index $l = 1$

(iii) $\alpha_1(l) = 1/n$, $\alpha_2(l) = (n-1)/n$, and $\alpha_3(l) = \ldots = \alpha_n(l) = 0$ for the Lorenzen index $l = n$

(iv) $\alpha_1(l) = 0$, $\alpha_2(l) = 1$ and $\alpha_j(l) = 0$, where $j = 3, \dots, n$, for the Lorenzen index $l = n - 1$.

Proof.

(i) $\alpha_j(l) = 0$ where $j < n - l + 1$ for the Lorenzen index $l = 1, \dots, n$.

The property (i) is evidently seen from the second conditional expression in (4.45). That means all $\alpha_j(l)$ are equal to 0 for $j + l < n + 1$. □

(ii) $\alpha_1(l) = \ldots = \alpha_{n-1}(l) = 0$ and $\alpha_n(l) = 1$ for the Lorenzen index $l = 1$.

Concerning the analytical expression (4.45), the coefficients α_j are non-zero if $l \geq n - j + 1$. By substituting $l = 1$, we get $j \geq n$. Therefore, the coefficients α_j are equal to 0 when $l = 1$ for every $j = 1, \ldots, n-1$, and the non-zero coefficient α_n is expressed as

$$\alpha_n(l=1) \quad = \quad \frac{1}{w_{n,1}}(-1)^0 \binom{n-1}{0} \left[\frac{1-n+1}{n} + \frac{2(n-1)}{n} \right] \tag{11}$$

which follows that $\alpha_n(l = 1) = 1$ due to $w_{n,1} = 1$. □

(iii) $\alpha_1 = 1/n$, $\alpha_2 = (n-1)/n$, and $\alpha_3 = \ldots = \alpha_n = 0$ for the Lorenzen index $l = n$.

From the expression (4.45), we have

$$\alpha_1(l=n) = \frac{1}{w_{1,n}}(-1)^0 \binom{0}{0} \left[\frac{n-n+1}{n \cdot n} + \frac{2 \cdot 0}{n \cdot n} \right] \tag{12}$$

which then yields $\alpha_1(l = n) = \frac{1}{n}$ since $w_{1,n} = \frac{1}{n}$.

Similarly, $\alpha_2(l)$ can be written as

$$\alpha_2(l) \quad = \quad \frac{1}{w_{2,n-1}} \sum_{p=0}^{1} (-1)^{1-p} \binom{1}{1-p} \left[\frac{l-n+1}{nl} + \frac{2p}{nl} \right]. \tag{13}$$

By substituting n for l into the previous formula, we have

$$\alpha_2(l=n) \quad = \quad \frac{1}{w_{2,n-1}} \left[-\binom{1}{1} \left(\frac{n-n+1}{n^2} + \frac{2 \cdot 0}{n^2} \right) + \binom{1}{0} \left(\frac{n-n+1}{n^2} + \frac{2 \cdot 1)}{n^2} \right) \right] \tag{14}$$

It follows that $\alpha_2(l = n) = \frac{n-1}{n}$ where we have used that $w_{2,n-1} = \frac{2}{n(n-1)}$.

We now prove that $\alpha_j(l = n) = 0$ for every $j = 3, \ldots, n$. In other words, the expression (4.45)

$$\alpha_j(l=n) = \frac{1}{w_{j,n-j+1}} \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} \left[\frac{l-n+1}{nl} + \frac{2p}{nl} \right] \tag{15}$$

is equal to 0 for the cases of $j = 3, \ldots, n$ when $l = n$.

Using Lemma 1, the expression $\frac{l-n+1}{nl} \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1}$ is equal to 0. The summation in (15) is, therefore, reduced to

$$\alpha_j(l=n) = \frac{1}{w_{j,n-j+1}} \left(\frac{2}{nl} \right) \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} p \tag{16}$$

where $j = 3, \dots, n$.

Our hypothesis is to prove the expression (16) is equal to 0 for every $j = 3, \dots, n$. To prove this we use the strong induction rule.

1. *Base case:* Consider the base case for $j = 3$

$$
\begin{aligned}
\alpha_3(l = n) &= \frac{1}{w_{3,n-2}} \left(\frac{2}{nl}\right) \sum_{p=0}^{2} (-1)^{2-p} \binom{2}{2-p} p \\
&= \frac{1}{w_{3,n-2}} \left(\frac{2}{nl}\right) \left[\binom{2}{2} \cdot 0 - \binom{2}{1} \cdot 1 + \binom{2}{0} \cdot 2 \right]. \qquad (17)
\end{aligned}
$$

By simplifying common terms in the summation, the coefficient $\alpha_3(l = n)$ is reduced to zero. Hence, the expression (16) is true for $j = 3$.

2. *Hypothesis:* Assume that the coefficients α_j in (16) are equal to 0 for some fixed $j \geq 3$, that is,

$$
\alpha_j(l = n) = \frac{1}{w_{j,n-j+1}} \left(\frac{2}{nl}\right) \sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} p. \qquad (18)
$$

3. *Induction:* We need to prove that $\alpha_{j+1}(l = n) = 0$ is true.

$$
\begin{aligned}
\alpha_{j+1} &= \frac{1}{w_{j+1,n-j}} \left(\frac{2}{nl}\right) \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{j-p} p \\
&= \frac{1}{w_{j+1,n-j}} \left(\frac{2}{nl}\right) \frac{1}{(-1)^{-1}} \sum_{p=0}^{j} (-1)^{j-p-1} \left[\binom{j-1}{j-p-1} + \binom{j-1}{j-p} \right] p. \qquad (19)
\end{aligned}
$$

We have used Pascal's identity $\binom{j}{j-p} = \binom{j-1}{j-p-1} + \binom{j-1}{j-p}$ in order to split the sum into two summations. In the first summation, the upper limit $p = j$ yields $\binom{j-1}{j-p-1} = 0$. We, therefore, can set the upper limit of the first summation to $j - 1$. In this way, the first summation is written shortly as $\sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} p$, which is proportional to coefficients α_j in (18). As the result of the hypothesis $\alpha_j = 0$ for some fixed $j \geq 3$, we obtain $\sum_{p=0}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} (n - p) = 0$. Therefore, the first summation in the expression of the coefficients α_{j+1} is equal to 0 and the remaining is simplified to

$$
\alpha_{j+1} = \frac{1}{w_{j+1,n-j}} \left(\frac{2}{nl}\right) \frac{1}{(-1)^{-1}} \sum_{p=0}^{j} (-1)^{j-p-1} \binom{j-1}{j-p} p. \qquad (20)
$$

Since the lower limit $p = 0$ makes $\binom{j-1}{j-p}p = 0$, we rewrite the above formula with the lower limits starting from $p = 1$ as below

$$\alpha_{j+1} = \frac{1}{w_{j+1,n-j}} \left(\frac{-2}{nl}\right) \sum_{p=1}^{j} (-1)^{j-p} \binom{j-1}{j-p} p. \tag{21}$$

By substituting $k + 1$ for p into the previous expression, we can align the indices of the binomial coefficient with the upper and lower limits of the summation,

$$\begin{aligned} \alpha_{j+1} &= \frac{1}{w_{j+1,n-j}} \frac{2}{nl} \sum_{k=0}^{j-1} (-1)^{j-k-1} \binom{j-1}{j-k-1} (1+k) \\ &= \frac{1}{w_{j+1,n-j}} \frac{2}{nl} \left[\sum_{k=0}^{j-1} (-1)^{j-k-1} \binom{j-1}{j-k-1} + \sum_{k=0}^{j-1} (-1)^{j-k-1} \binom{j-1}{j-k-1} k \right] \end{aligned} \tag{22}$$

From Pascal's rule, it follows that $\binom{j-1}{j-k-1} = \binom{j-1}{k}$. In this refinement, we notice the similarity of the first and second summations with Lemma 1 and Lemma 3, respectively, which have been proved being zero in Appendix A. As the result, we conclude that the coefficients α_{j+1} are equal to 0 for the case $j + 1$.

Hence, by mathematical induction, for each $j = 3, \ldots, n$ the coefficients α_j are equal to 0 when $l = n$. □

(iv) $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_j = 0$, where $j = 3, \ldots, n$, for $l = n - 1$.

We begin with (4.45) for the choice of $l = n - 1$, the coefficients α_j are expressed as follows

$$\alpha_j(l = n-1) = \frac{1}{w_{j,n-j+1}} \cdot \frac{2}{n(n-1)} \sum_{p=1}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} p. \tag{23}$$

In the case of $\alpha_1(l = n - 1)$ in which $j = 1$ yields $n - j + 1 = n > l$. Therefore, α_1 is equal to 0.

By analogy, we consider α_2 given as follow

$$\alpha_2(l) = \frac{1}{w_{2,n-1}} \cdot \frac{2}{n(n-1)} \sum_{p=1}^{1} (-1)^{1-p} \binom{1}{1-p} p. \tag{24}$$

Since $w_{2,n-1} = \frac{2}{n(n-1)}$, it follows that $\alpha_2(l = n - 1) = 1$.

In the following part we aim to prove the coefficients $\alpha_j(l = n-1)$ are equal to 0 for every $j = 3, \ldots, n$. In other words, the coefficients α_j, as indicated in (23),

$$\alpha_j(l = n-1) = \frac{1}{w_{j,n-j+1}} \cdot \frac{2}{n(n-1)} \sum_{p=1}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} p \tag{25}$$

are equal to 0 for the cases of $j = 3, \ldots, n$ when $l = n-1$.

1. *Base case:* Consider the base case for $j = 3$

$$\alpha_3(l = n-1) \quad = \quad \frac{1}{w_{3,n-2}} \cdot \frac{2}{n(n-p)} \sum_{p=1}^{2} (-1)^{2-p} \binom{2}{2-p} p. \tag{26}$$

It is evident to obtain that $\alpha_3(l = n-1) = 0$.

2. *Hypothesis:* Assume that the coefficients $\alpha_j(l = n-1)$, for $j = 3, \ldots, n$,

$$\alpha_j(l = n-1) = \frac{1}{w_{j,n-j+1}} \cdot \frac{2}{n(n-1)} \sum_{p=1}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} p \tag{27}$$

are equal to 0 for some fixed $j \geq 3$.

3. *Induction:* We need to prove that $\alpha_{j+1}(l = n-1)$ is equal to 0.

$$\begin{aligned} \alpha_{j+1}(l = n-1) &= \frac{1}{w_{j+1,n-j}} \cdot \frac{2}{n(n-1)} \sum_{p=1}^{j} (-1)^{j-p} \binom{j}{j-p} p \\ &= \frac{1}{w_{j+1,n-j}} \cdot \frac{2}{n(n-1)} \sum_{p=1}^{j} (-1)^{j-p} \left[\binom{j-1}{j-p-1} + \binom{j-1}{j-p} \right] p. \end{aligned} \tag{28}$$

We have used Pascal's identity $\binom{j}{j-p} = \binom{j-1}{j-p-1} + \binom{j-1}{j-p}$ in order to split the sum into two summations. In the first summation, the upper limit $p = j$ yields $\binom{j-1}{j-p-1} = 0$. We, therefore, can set the upper limit of the first summation to $j-1$. In this way, the first summation is written shortly as $-\sum_{p=1}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} (n-p)$, which is proportional to the coefficients α_j in (27). As the result of the hypothesis $\alpha_j = 0$ for some fixed $j \geq 3$, we obtain $\sum_{p=1}^{j-1} (-1)^{j-p-1} \binom{j-1}{j-p-1} (n-p) = 0$. Therefore, the first summation in the expression of the coefficient α_{j+1} is equal to 0 and the remaining is simplified to

$$\alpha_{j+1} \quad = \quad \frac{1}{w_{j+1,n-j}} \cdot \frac{2}{n(n-1)} \sum_{p=1}^{j} (-1)^{j-p} \binom{j-1}{j-p} p. \tag{29}$$

By substituting $k+1$ for p into the previous expression, we can align the indices of the binomial coefficient with the upper and lower limits of the summation,

$$\begin{aligned}\alpha_{j+1} &= \frac{1}{w_{j+1,n-j}}\left(\frac{2}{nl}\right)\sum_{k=0}^{j-1}(-1)^{j-k-1}\binom{j-1}{j-k-1}(1+k) \\ &= \frac{1}{w_{j+1,n-j}}\left(\frac{2}{nl}\right)\left[\sum_{k=0}^{j-1}(-1)^{j-k-1}\binom{j-1}{j-k-1}+\sum_{k=0}^{j-1}(-1)^{j-k-1}\binom{j-1}{j-k-1}k\right] \quad (30)\end{aligned}$$

From the Pascal identify, it follows that $\binom{j-1}{j-k-1} = \binom{j-1}{k}$. In this refinement, we notice the similarity of the first and second summations with Lemma 1 and Lemma 3, respectively. As the result, we conclude that the coefficient α_{j+1} is equal to 0 for the case $j+1$ when $l = n-1$.

Hence, by mathematical induction, the coefficients $\alpha_j(l = n-1) = 0$ has been proved for every $j = 3, \ldots, n$. $\square$

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