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Buckling of thin-walled cylinders from three dimensional nonlinear elasticity

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Introduction

The buckling of axially-compressed thin-walled cylinders is one of the most famous problems in mechanics and probably represents the most fascinating question in bifurcation theory after the Euler rod. Early calculations go back to Lorenz (1908), Timoshenko (1910), Southwell (1914), and von Mises (1914). Flügge (1932) obtained the currently available solution (through the introduction of hypotheses on the material behaviour and approximations on the thickness of the cylinder wall), which has been commented on and developed in detail in several important works (Flügge 1981; Donnell 1934; von Kármán and Tsien 1941; Wilkes 1955; see also the review by Simitses 1986).

Although the solution to this buckling problem was already known before World War Two, the strong imperfection-sensitivity of the mechanical system, discovered by Koiter (1945) and explaining the discrepancy between measured and predicted critical loads, has created a strong research focus, which also involves the akin structural response of thin spherical shells and still continues to attract attention (selected references are Calladine 1988; Calladine 2001; Tsien 2012; Elishakoff 2014; Jiménez et al. 2017).

It is well-known that the solution provided by Flügge is based on two main approximations, namely, (i.) that incremental constitutive equations are used, which relate the Oldroyd increment of the Kirchhoff stress to the incremental Eulerian strain, so that these do not follow from a finite strain formulation of a hyperelastic material and (ii.) on the smallness of the thickness of the cylinder wall, so that often statements are motivated invoking that 'plane stress' prevails. It would be therefore important for both 'theoretical and practical reasons' to derive the Flügge formulation from a three-dimensional finite elasticity context, including the calculations of the bifurcation loads and the determination of the famous formula for buckling of a 'mid-long' cylindrical shell.

Theoretically, a derivation of the bifurcation stress for an axially compressed thin-walled cylinder from the laws of nonlinear elastic deformations would allow to rigorously confirm the validity of the theory developed by Flügge and considered the reference in the field. From a 'practical point of view', imagine the calculation of the buckling of a cylindrical shell made up of an Ogden or a neo-Hookean compressible elastic material (Levinson and Burgess 1971; Ogden 1972b), or the bifurcation analysis of an artery obeying the Holzapfel et al. (2000) constitutive law. Currently, these calculations can only be done trying to reduce (which is not always possible) the nonlinear elastic constitutive laws to a small-strain version based on Lamé constants λ and μ and after this use of the Flügge formulation.

In this direction, to the author's knowledge, the only available work is that by Ciarlet and Paumier (1986), who used asymptotic expansion techniques, assuming a small thickness of the cylinder wall, to derive the equilibrium equations of a nonlinear elastic shell. They obtained equilibrium equations equivalent to the Marguerre-von Kármán equations and showed that the displacement field is of the Kirchhoff-Love type, with stresses displaying polynomial variations with respect to the cylinder thickness.

The present work addresses the rigorous derivation of the Flügge treatment of the buckling of a thin cylinder. A brief introduction of continuum mechanics is offered in Ch. 1, while Ch. 2 offers an overview of the main elements of incremental hyperelasticity. The incremental equilibrium equations in terms of generalized stresses are rigorously derived in Ch. 3 in terms of mean quantities (holding true regardless of the thickness of the cylinder), through a generalization of the approach introduced by Biot (1965) for rectangular plates. The incremental kinematics is postulated in Ch. 4 through a novel deduction from the deformation of a two-dimensional surface, thus generalizing an approach introduced to derive the incremental kinematics of a plate. The nonlinear elastic constitutive equations proposed by Pence and Gou (2015), describing a nearly incompressible neo-Hookean material, are used in a rigorous way. While the employed kinematics coincides with that used by Flügge, the incremental equilibrium and constitutive equations derived in this work are different from those given by Flügge, but are shown to reduce to the latter by invoking the smallness of the cylinder wall.

The equations derived for the incremental deformation of prestressed thin cylindrical shells are general and can be used for different purposes. The study of the bifurcation problem of a thin-walled circular cylinder subject to compressive load is offered in Ch. 6. When compared, the bifurcation landscape obtained from the formulation developed in this work and that given by Flügge are numerically shown to coincide and be consistent with results obtained by a fully three-dimensional theory of nonlinear elasticity (Ch. 8). In Ch. 7 the formula for the axial buckling stress of a 'mid-long' cylindrical shell made of a nearly incompressible neo-Hookean material and of a Mooney-Rivlin material are rigorously obtained from the presented formulation.

1 Elements of continuum mechanics

1.1 Introduction and notation

A brief review of the theory of continuum mechanics is presented in this chapter. The interested reader may refer to the monographs of Truesdell and Toupin (1960), Green and Zerna (1963), Gurtin (1981), Ciarlet (1988), Basar and Weichert (2000), Holzapfel (2000), Liu (2002), Truesdell and Noll (2004), Antman (2005), and Bigoni (2012).
Throughout the thesis, the notation defined in Gurtin (1981) will be used.

The Euclidean point space and its associated vector space are denoted by $\mathcal{E}$ and $\mathcal{V}$, respectively. Vectors will be denoted by boldface minuscule letters ($\mathbf{a}$, $\mathbf{b}$, ...), second-order tensors, i.e. linear transformations from $\mathcal{V}$ into itself, by boldface majuscule letters ($\mathbf{A}$, $\mathbf{B}$, ...) and fourth-order tensors, i.e. linear mappings from Lin into Lin, by a blackboard letter ($\mathbb{A}$, $\mathbb{B}$, ...). Lin is the set of second-order tensors, Sym and Skw its symmetric and skew-symmetric restrictions respectively, and Orth the set of the orthogonal second-order tensors.

A tensor $\mathbf{S}$ belongs to Sym if $\mathbf{S}^\top = \mathbf{S}$, where $\mathbf{S}^\top$ denotes the transpose of $\mathbf{S}$, and to Skw if $\mathbf{S}^\top = -\mathbf{S}$. Tensors that satisfy the orthogonality condition $\mathbf{Q}^\top \mathbf{Q} = \mathbf{Q}\,\mathbf{Q}^\top = \mathbf{I}$, build the subset Orth of Lin, where $\mathbf{I}$ is the identity tensor, defined by $\mathbf{I}\,\mathbf{v} = \mathbf{v}$ for every vector $\mathbf{v} \in \mathcal{V}$. Einstein summation convention over any repeated index will be used; in particular, Latin indices can take three different values (e.g. 1, 2, 3 or r, θ, z), while Greek indices can take two different values (e.g. 1, 2 or θ, z).

In the following, several products are defined by means of the generic vectors

$\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathcal{V}$ and tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathrm{Lin}$. The inner product between vectors will be denoted by $\mathbf{v} \cdot \mathbf{u}$, whilst the inner product between tensors by $\mathbf{A} \cdot \mathbf{B}$ or, equivalently by $\mathbf{A} : \mathbf{B}$. If a Cartesian coordinate frame consisting of an orthonormal basis $\{\mathbf{e}_i\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined, then $\mathbf{u} \cdot \mathbf{v} = u_i\, v_i$ and $\mathbf{A} \cdot \mathbf{B} = A_{ij}\, B_{ij}$, where $u_i = \mathbf{e}_i \cdot \mathbf{u}$ and $A_{ij} = \mathbf{e}_i \cdot (\mathbf{A}\, \mathbf{e}_j)$ are the Cartesian components of $\mathbf{v}$ and $\mathbf{A}$ respectively. The product of two tensors, $\mathbf{AB}$, is defined as the composition $\mathbf{A} \circ \mathbf{B}$ such that $(\mathbf{AB})\mathbf{u} = \mathbf{A}(\mathbf{Bu})$ for every vector $\mathbf{u} \in \mathrm{Lin}$; the product between n identical tensors $\mathbf{A}$ will be represented by $\mathbf{A}^n$. The dyadic product among vectors is defined as $(\mathbf{u} \otimes \mathbf{v})\, \mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\, \mathbf{u}$ while among tensors as $(\mathbf{A} \otimes \mathbf{B})\, \mathbf{C} = (\mathbf{B} \cdot \mathbf{C})\, \mathbf{A}$. Furthermore: $(\mathbf{A} \boxtimes \mathbf{B})\, \mathbf{C} = \mathbf{ACB}^\top$ and $(\mathbf{A} \underline{\overline{\otimes}} \mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{C} + \mathbf{C}^\top)\mathbf{B}^\top/2$. In particular, the latter product specializes into the fourth order symmetrizer $\mathbb{S}$ if $\mathbf{A} = \mathbf{B} = \mathbf{I}$.

The trace operator acting on a second-order tensor $\mathbf{A}$ is defined as $\mathrm{tr}\,(\mathbf{A}) = A_{ii}$, and it is usually referred to as the first *invariant* denoted by the symbol $\mathcal{I}_1(\mathbf{A})$. Furthermore, the inner product of two second-order tensors relates to the trace operator as $\mathbf{A} : \mathbf{B} = \mathrm{tr}\,(\mathbf{A}\, \mathbf{B}^\top)$. The determinant of a tensor $\mathbf{A}$ is defined as the determinant of its associated matrix representation and is independent of the coordinate system chosen to express its coefficients; the determinant is also called the third *invariant* and denoted by the symbol $\mathcal{I}_3(\mathbf{A})$. For the sake of completeness, the 'second' invariant of a generic second-order tensor $\mathbf{A}$ is defined as

$$\mathcal{I}_2(\mathbf{A}) = (\mathcal{I}_1(\mathbf{A})^2 - \mathcal{I}_1(\mathbf{A}^2))/2\,.$$

The divergence of a vector is defined as

$$\mathrm{div}\,\mathbf{u} = \mathrm{tr}\,(\mathrm{grad}\,\mathbf{u})\,, \tag{1.1}$$

while, for any constant vector $\mathbf{u}$, the divergence of a second-order tensor $\mathbf{A}$ reads

$$\mathrm{div}\,(\mathbf{A}) \cdot \mathbf{u} = \mathrm{div}\,(\mathbf{A}^\top \mathbf{u})\,. \tag{1.2}$$

The operator grad is the gradient, see §1.2.

1.2 Kinematics

A continuum body $\mathcal{B}$ is considered, which occupies a connected, open subset of the Euclidean space $\mathcal{E}$. Such a region at the time t is called a *configuration* of the body and will be denoted by B_t. Let B_0 be a fixed reference configuration of the body at a certain time t_0. Hence, the *motion* of the body is a vector-valued function $\boldsymbol{\chi}$ defined as

$$\boldsymbol{\chi} : B_0 \times \mathcal{I} \to \mathcal{E}\,, \quad (\mathbf{X}, t) \mapsto \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)\,, \tag{1.3}$$

where $\mathcal{I}$ is a time interval and vectors $\mathbf{X}$ and $\mathbf{x}$ represents material points of B_0 and B_t. Such a function transforms material point belonging to the reference configuration B_0 into *spatial* points belonging to the *actual* configuration B_t. The inverse motion, denoted with $\boldsymbol{\chi}^{-1}$ is the unique mapping such that $\mathbf{X} = \boldsymbol{\chi}(\mathbf{x}, t)$. To satisfy the existence of such a function, it is required that $\boldsymbol{\chi}$ is bijective and of class C^2. The function $\boldsymbol{\chi}$, for a fixed times, defines what is called the *deformation* of the body under investigation from its reference configuration B_0 into the current configuration B_t. Conversely, for a fixed material point $\mathbf{X}$, the motion describes the *trajectory* followed by the chosen material point.

Problems in continuum mechanics can be tackled either in the *Lagrangian* (also *material*) formulation or in the *Eulerian* (also *spatial*) formulation. The first is characterized by the use of the pair $(\mathbf{X}, t)$ as independent variables, whilst the latter by the use of the pair $(\mathbf{x}, t)$ as independent variables. The transformation from one approach to the other can be done by means of the function $\boldsymbol{\chi}$ and its inverse $\boldsymbol{\chi}^{-1}$. For example, denoting with f_m a material field, and with f_s a spatial field, the following relations between the two formulations can be established

$$f_\mathrm{s}(\mathbf{x}, t) = f_\mathrm{m}(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t) \quad \text{and} \quad f_\mathrm{m}(\mathbf{X}, t) = f_\mathrm{s}(\boldsymbol{\chi}(\mathbf{X}, t), t)\,. \tag{1.4}$$

The evolution in time of some field f can be described by means of the so-called *material time derivative*, i.e. the time derivative performed holding the material point $\mathbf{X}$ fixed. Usually, operators referring to the material formulation are denoted with capital letters, such that the definition of the material time derivative reads

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t}\bigg|_{\mathbf{X}} = \frac{\partial f_{\mathrm{m}}}{\partial t}(\mathbf{X}, t)\,. \tag{1.5}$$

It is common practice to equivalently refer to the material time derivative of a field f also with the notation $\dot{f}$. The *spatial* time derivative of a spatial field f_{s} is simply $\partial f_{\mathrm{s}}/\partial t$ holding the spatial point $\mathbf{x}$ fixed.

Under *velocity* and *acceleration* the first and second time derivative of the *motion* are understood, namely

$$\dot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t}\,, \quad \text{and} \quad \ddot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2}\,. \tag{1.6}$$

The material and spatial description of the above fields are related through the motion $\boldsymbol{\chi}$ as

$$\dot{\mathbf{x}}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)\,, \quad \text{and} \quad \ddot{\mathbf{x}}(\mathbf{X}, t) = \mathbf{a}(\mathbf{x}, t)\,. \tag{1.7}$$

The derivation of relations (1.7), is offered in what follows. To calculate the material time derivative of a spatial field f_{s}, the following steps have to be taken. Firstly, the spatial field has to be expressed as a function of the material point $\mathbf{X}$

$$f_{\mathrm{s}}(\mathbf{x}, t) = f_{\mathrm{s}}(\boldsymbol{\chi}(\mathbf{X}, t), t)\,, \tag{1.8}$$

and then the material time derivative can be computed using the chain rule as

$$\begin{aligned} \dot{f}_{\mathrm{s}} &= \frac{\partial f_{\mathrm{s}}(\mathbf{x}, t)}{\partial t} + \frac{\partial f_{\mathrm{s}}}{\partial \mathbf{x}} \cdot \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t}\bigg|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)} \\ &= \frac{\partial f_{\mathrm{s}(\mathbf{x},t)}}{\partial t} + \mathrm{grad}\,(f_{\mathrm{s}}) \cdot \mathbf{v}(\mathbf{x}, \mathbf{t})\,, \end{aligned} \tag{1.9}$$

where 'grad' is the spatial gradient and $\mathbf{v}$ is the spatial velocity field $(1.7)_1$. The acceleration then reads

$$\mathbf{a}_{\mathrm{m}}(\mathbf{X}, t) = \frac{D^2\boldsymbol{\chi}}{Dt^2}(\mathbf{X}, t) = \mathbf{a}_{\mathrm{s}}(\mathbf{x}, t)\,, \tag{1.10}$$

Throughout the thesis, the subscripts m or s will be dropped. When necessary, explicit reference to the independent variables of the specific problem will be made.

The introduction of two orthonormal bases, namely $\{\mathbf{E}_i\}$ for the reference configuration B_0 and $\{\mathbf{e}_i\}$ for the current configuration B_t, where $i = 1, 2, 3$, allows for the component-wise representation of vectors and tensors. For example, the vector position of the material point $\mathbf{X}$ can be represented as

$$\mathbf{X} - \mathbf{O}_0 = X_i\,\mathbf{E}_i\,, \tag{1.11}$$

where $\mathbf{O}_0$ is the (arbitrary) origin of the coordinate system in the reference configuration. The same applies to the current configuration.

The deformation gradient is introduced as the linear mapping that transforms elements from the reference configuration B_0 into the actual configuration B_t

$$\mathbf{F}(\mathbf{X}) = \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X})\,. \tag{1.12}$$

From its definition, it is clear that the deformation gradient is a two-point tensor, a feature that becomes evident by observing its components:

$$\mathbf{F} = F_{ij}\,\mathbf{e}_i \otimes \mathbf{E}_j\,, \quad F_{ij} = \frac{\partial \chi_i(\mathbf{X}, t)}{\partial X_j}\,. \tag{1.13}$$

To avoid non-physical effects, e.g. annihilation of fibers, it is required that the determinant of the deformation gradient $\det(\mathbf{F}) \equiv J > 0$, i.e. the mapping $\mathbf{F}$ has to be non-singular, hence invertible, allowing for the definition of the gradient of the inverse motion as

$$\mathbf{F}^{-1}(\mathbf{x}, t) = \mathrm{grad}\,\boldsymbol{\chi}^{-1}(\mathbf{x}, t)\,. \tag{1.14}$$

Because $\mathbf{F} \in \mathrm{Lin}^+$, it can be uniquely decomposed according to the so-called polar decomposition, that splits the amount of rotation ($\mathbf{R}$) and the amount of stretch ($\mathbf{U}$ or $\mathbf{V}$) induced during the deformation

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R}\,, \tag{1.15}$$

where $\mathbf{R} \in \mathrm{Orth}^+$ is a proper orthogonal tensor (rotation), whilst $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left* stretch tensors, defined as follows

$$\mathbf{U} = \sqrt{\mathbf{F}^\top\mathbf{F}}\,, \quad \mathbf{V} = \sqrt{\mathbf{F}\,\mathbf{F}^\top}\,. \tag{1.16}$$

From their definition it is clear that both tensors are symmetric and positive-definite, i.e. they belong to Sym^+. Furthermore, their spectral representations read

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i\,\mathbf{u}_i \otimes \mathbf{u}_i\,, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i\,\mathbf{v}_i \otimes \mathbf{v}_i\,, \tag{1.17}$$

where λ_i are the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$ and are usually referred to as *principal stretches* of the deformation, $\mathbf{u}_i$ and $\mathbf{v}_i$ the Lagrangian and Eulerian principal axes, respectively. Note that the Lagrangian and Eulerian principal axes are related through $\mathbf{v}_i = \mathbf{R}\,\mathbf{u}_i$. Furthermore, from Eq. (1.15)

$$\mathbf{V} = \mathbf{R}\,\mathbf{U}\,\mathbf{R}^\top . \tag{1.18}$$

Hence, the right and left stretch tensors share the same eigenvalues, whilst their eigenvectors are rotated through the tensor $\mathbf{R}$. Therefore, the set of invariants $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ is the same for the pairs $\{\mathbf{U},\mathbf{V}\}$ and $\{\mathbf{C},\mathbf{B}\}$. Another interesting relation following from (1.15) is

$$J \equiv \det\mathbf{F} = \det\mathbf{U} = \det\mathbf{V} = \lambda_1\,\lambda_2\,\lambda_3\,. \tag{1.19}$$

A deformation is called *isochoric*, i.e. volume-preserving, if $J = 1$.

The right and left Cauchy-Green deformation tensors are introduced as

$$\mathbf{C} = \mathbf{F}^\top\,\mathbf{F} = \mathbf{U}^2\,, \quad \mathbf{B} = \mathbf{F}\,\mathbf{F}^\top = \mathbf{V}^2\,, \tag{1.20}$$

respectively. Both tensors are positive definite and their spectral representations are

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2\,\mathbf{u}_i \otimes \mathbf{u}_i\,, \quad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2\,\mathbf{v}_i \otimes \mathbf{v}_i\,. \tag{1.21}$$

Furthermore, $\mathbf{B} = \mathbf{R}\,\mathbf{C}\,\mathbf{R}^\top$.

1.3 Stress

Let $\mathrm{d}\mathbf{f}$ be the vector representing the contact force that acts on an infinitesimal surface element $\mathrm{d}a$ characterized by the unit vector $\mathbf{n}$ in the current configuration B_t. The *traction vector* $\mathbf{t}$ is defined as the limit of the ratio $\mathrm{d}\mathbf{f}/\mathrm{d}a$ as the infinitesimal surface tends to vanish. Cauchy's theorem states that the dependence between the traction vector and the unit normal to the current surface element is linear. Hence, a second-order tensor field $\mathbf{T}$ can be defined, namely the Cauchy stress, independent of $\mathbf{n}$ such that

$$\mathbf{t}(\mathbf{x},\mathbf{n}) = \mathbf{T}(\mathbf{x})\,\mathbf{n}\,, \tag{1.22}$$

where the dependence on time was neglected. Defining $\mathbf{b}$ as the body forces per unit volume in the current configuration and $\mathbf{r}$ the position vector of the considered spatial point $\mathbf{x}$, the Euler axioms define the equilibrium conditions (translational and rotational) of a generic part $\mathcal{P}$ of the continuum body in its current configuration B_t

$$\begin{aligned} \int_{\mathcal{P}} \mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{t}\,\mathrm{d}a = 0\,, \\ \int_{\mathcal{P}} \mathbf{r}\times\mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{r}\times\mathbf{t}\,\mathrm{d}a = 0\,. \end{aligned} \tag{1.23}$$

Eq. (1.23) are automatically satisfied if a Cauchy stress tensor $\mathbf{T}$ exists such that: i) it is a symmetric tensor field, $\mathbf{T}^{\top} = \mathbf{T}$, and it satisfies the local equilibrium condition

$$\operatorname{div}\mathbf{T} + \mathbf{b} = 0\,, \tag{1.24}$$

Usually, the current configuration is unknown, so that the equilibrium condition (1.24) is conveniently rewritten in terms of material quantities. The current contact force $\mathrm{d}\mathbf{f}$ is transformed by means of the Nanson's formula as

$$\mathrm{d}\mathbf{f} = \mathbf{t}\,\mathrm{d}a = \mathbf{S}\,\mathbf{N}\,\mathrm{d}A\,. \tag{1.25}$$

where the definition of the *first* Piola-Kirchhoff stress tensor was introduced

$$\mathbf{S} = \mathbf{K}\,\mathbf{F}^{-\top}\,, \tag{1.26}$$

and $\mathbf{K} = J\mathbf{T}$ is the Kirchhoff stress tensor. The local equilibrium condition (1.24) in the reference configuration B_0 reads

$$\operatorname{Div}\mathbf{S} + \mathbf{b}_0 = 0\,, \tag{1.27}$$

where $\mathbf{b}_0$ is the vector representing the body forces per unit volume in the reference configuration.

1.4 Constitutive equations for isotropic elastic materials

From a mechanical point of view, a mechanical problem is described by Cauchy's first law of motion (1.24) and the mass conservation, i.e. by a set of ten unknowns (six stress components due to the symmetry of $\mathbf{T}$ and the three components of the position vector $\mathbf{x}$). However, only four scalar equations are available. Further conditions, namely the constitutive laws, are introduced to make such a problem solvable. Constitutive relations usually satisfies three principles: *i*) they should be invariant under changes of observer; *ii*) the stress in a body is determined by the history of the motion of that body (so-called *principle of determinism for the stress*); *iii*) the stress at a point depends on the history if the motion in the neighborhood of the same point (so-called *principle of local action*). In the following, a brief introduction to the class of isotropic materials is given, followed by its subset of elastic and finally hyperelastic materials.

A material is called Cauchy elastic if the state of stress in the current configuration B depends only on the state of the (current) deformation

$$\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t), \mathbf{X}) . \tag{1.28}$$

To satisfy the principle of invariance under changes of observer, Eq. (1.28) must also satisfy

$$\mathbf{T}^* = \hat{\mathbf{T}}(\mathbf{F}^*) , \tag{1.29}$$

where $\mathbf{T}^* = \mathbf{Q}\,\hat{\mathbf{T}} * (\mathbf{F})\,\mathbf{Q}^\top$ and $\mathbf{F}^* = \mathbf{Q}\,\mathbf{F}$, $\mathbf{Q}$ being an arbitrary proper rotation. Therefore the following relation has to hold true

$$\hat{\mathbf{T}}(\mathbf{Q}\,\mathbf{F}) = \mathbf{Q}\,\hat{\mathbf{T}}(\mathbf{F})\,\mathbf{Q}^\top . \tag{1.30}$$

Let B be the current configuration of a body which deforms according to the deformation tensor $\mathbf{F}$. As a thought experiment, if the same body experienced a rotation $\mathbf{Q}$ and only after the deformation $\mathbf{F}$, its total gradient of deformation would be $\mathbf{F}\,\mathbf{Q}$. A material is called *isotropic* if its response is invariant to any rotation

$$\hat{\mathbf{T}}(\mathbf{F}) = \hat{\mathbf{T}}(\mathbf{F}\,\mathbf{Q}) , \quad \forall \mathbf{Q} \in \mathrm{Orth}^+ , \tag{1.31}$$

for at least one reference configuration. By means of the polar decomposition, setting with $\mathbf{Q} = \mathbf{R}^\top$, it follows

$$\hat{\mathbf{T}}(\mathbf{F}) = \hat{\mathbf{T}}(\mathbf{F}\,\mathbf{R}^\top) = \hat{\mathbf{T}}(\mathbf{V}\,\mathbf{R}\mathbf{R}^\top) = \overline{\mathbf{T}}(\mathbf{B}) . \tag{1.32}$$

Since $\mathbf{B}$ is symmetric, use can be made of the representation theorem for isotropic functions (Truesdell and Noll 2004; Wang 1970; Zheng 1994), yielding, for the case of the isotropic response function $\overline{\mathbf{T}}$

$$\overline{\mathbf{T}} = \beta_0 \, \mathbf{I} + \beta_1 \, \mathbf{B} + \beta_{-1} \, \mathbf{B}^{-1} \,, \tag{1.33}$$

where the coefficients β_i are functions of the three invariants of the left Cauchy-Green deformation tensor $\mathbf{B}$. In case of incompressibility, i.e. no change in volume is allowed during the deformation and therefore $\mathcal{I}_3(\mathbf{B}) = 1$, Eq. (1.33) becomes (Truesdell and Noll 2004)

$$\overline{\mathbf{T}} = -\pi \, \mathbf{I} + \beta_1 \, \mathbf{B} + \beta_{-1} \, \mathbf{B}^{-1} \,. \tag{1.34}$$

1.5 Hyperelastic materials

A material is classified as *hyperelastic* (or equivalently *Green elastic*) if there exists a strain energy density function W that depends on some strain measure $\mathbf{E}^{(\mathrm{m})}$ such that the work-conjugate stress measure of the latter, $\mathbf{T}^{(\mathrm{m})}$, is obtained by means of

$$\mathbf{T}^{(\mathrm{m})} = \frac{\partial W(\mathbf{E}^{(\mathrm{m})})}{\partial \mathbf{E}^{(\mathrm{m})}} \,. \tag{1.35}$$

The quantities $\mathbf{T}^{(\mathrm{m})}$ and $\mathbf{E}^{(\mathrm{m})}$ are work-related stress and strain measures such that $\mathbf{K} \cdot \mathbf{D} = \mathbf{T}^{(\mathrm{m})} \cdot \dot{\mathbf{E}}^{(\mathrm{m})}$, where $\mathbf{K} = J \, \mathbf{T}$ is the Kirchhoff stress and $\mathbf{D} = \mathrm{sym}\,(\mathrm{grad}\, \mathbf{u})$. This general definition implies that the stress power, written with respect to the Lagrangian formulation, can be expressed by means of an exact differential of a scalar function W of the strain measure $\mathbf{E}^{(\mathrm{m})}$

$$\int_{P_0} \mathbf{S} \cdot \dot{\mathbf{F}} \, \mathrm{d}V = \int_{P_0} \mathbf{T}^{(\mathrm{m})} \cdot \dot{\mathbf{E}}^{(\mathrm{m})} \, \mathrm{d}V = \int_{P_0} \frac{\partial W}{\partial \mathbf{E}^{(\mathrm{m})}} \cdot \dot{\mathbf{E}}^{(\mathrm{m})} \, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_{P_0} W \, \mathrm{d}V \,. \tag{1.36}$$

Furthermore, the first Piola-Kirchhoff stress tensor can be obtained by conveniently rewriting the strain energy function W as a function of $\mathbf{F}$, $\tilde{W}(\mathbf{F})$, to yield

$$\mathbf{S} = \frac{\partial \tilde{W}(\mathbf{F})}{\partial \mathbf{F}} \,. \tag{1.37}$$

1.5.1 Compressible hyperelastic materials

The strain energy function chosen for the compressible material investigated in the present work is assumed in the form proposed by Pence and Gou (2015, their eq. (2.11)) for a nearly incompressible neo-Hookean material

$$\begin{aligned} W = \mu/2\left[\mathcal{I}_1(\mathbf{B}) - 3 - \ln\left(\mathcal{I}_3(\mathbf{B})\right)\right] + \\ + 1/2\ (\kappa - 2/3\,\mu)\left(\sqrt{\mathcal{I}_3(\mathbf{B})} - 1\right)^2, \end{aligned} \tag{1.38}$$

where $\mathcal{I}_1(\mathbf{B}) = \operatorname{tr}\mathbf{B}$, $\mathcal{I}_3(\mathbf{B}) = \det\mathbf{B}$, while μ and κ are the shear and bulk moduli of the material in the reference configuration, related to the equivalents of the Young modulus E and the Poisson's ratio ν through $\mu = E/(2\,(1+\nu))$ and $\kappa = E/(3\,(1-2\nu))$. Equation (1.38) describes a nearly incompressible neo-Hookean strain energy which, as a variant of the model used by Liu and Bertoldi (2015) and proposed by Rivlin and Thomas (1951), satisfies the stress-free condition of the undeformed state, Eq. $(1.39)_1$, and the consistency condition with the classical linearized elasticity theory, Eqs. $(1.39)_2$ and $(1.39)_3$ (Horgan and Saccomandi 2004)

$$\begin{cases} \overline{W}_{,1} + 2\overline{W}_{,2} + \overline{W}_{,3} = 0\,, \\ \overline{W}_{,1} + \overline{W}_{,2} = -(\overline{W}_{,2} + \overline{W}_{,3}) = \mu/2\,, \\ \overline{W}_{,11} + 4\overline{W}_{,12} + 4\overline{W}_{,22} + 2\overline{W}_{,13} + 4\overline{W}_{,23} + \overline{W}_{,33} = \kappa/4 + \mu/3\,, \end{cases} \tag{1.39}$$

where $\overline{W}_{,i} = \partial W/\partial\mathcal{I}_i(\mathcal{I}_1 = \mathcal{I}_2 = 3, \mathcal{I}_3 = 1)$. Therefore, the expression for the Kirchhoff stress becomes

$$\mathbf{K} = \mu\,(\mathbf{B} - \mathbf{I}) + J\,(\kappa - 2/3\mu)(J-1)\,\mathbf{I}\,. \tag{1.40}$$

The first Piola-Kirchhoff can be calculated as $\mathbf{S} = \mathbf{K}\,\mathbf{F}^{-\top}$.

1.5.2 Incompressible hyperelastic materials

In this section, the relations governing the behavior of some classes of incompressible materials are presented. A material is called *incompressible* if each and every motion of its material particles is isochoric. Hence, the internal constraint that has to be taken into account in the constitutive law is

$$\det \mathbf{F} = 1\,. \tag{1.41}$$

Condition (1.41) enters the constitutive law by means of a Lagrange multiplier π, such that the stress-deformation relation can be expressed as (Gurtin 1981)

$$\mathbf{T} = -\pi\,\mathbf{I} + \beta_1\,\mathbf{B} + \beta_{-1}\,\mathbf{B}^{-1}\,, \tag{1.42}$$

where β_1 and β_{-1} are the so-called *response coefficients* and are functions either of the two invariants $\mathcal{I}_1 = \operatorname{tr}\mathbf{B}$, $\mathcal{I}_2 = \operatorname{tr}\mathbf{B}^2$ or, equivalently, of the principal stretches λ_i. It is worth highlighting the fact that the response coefficient cannot take arbitrary values, but are subject to *empirical inequalities* on the basis of experimental evidence (Truesdell and Noll 2004)

$$\beta_1 > 0\,, \quad \beta_{-1} \leq 0\,. \tag{1.43}$$

Furthermore, it is interesting to note that the coefficient β_i can be explicitly determined once the current state of stress and deformation is known. To obtain this minor proof, it is sufficient to express (1.42) with respect to the Eulerian principal axes, yielding the following relations

$$T_{ii} = -\pi + \beta_1\,\lambda_i + \beta_{-1}\,\lambda_i^{-1}\,, \quad i = 1, 2, 3\,, \tag{1.44}$$

no sum over i. Hence, taking the difference of the first ($i = 1$) and the second ($i = 2$) equation with the third ($i = 3$) allows for the elimination of the unknown Lagrange multiplier π, leading to

$$\begin{cases} T_{11} - T_{33} = \beta_1(\lambda_1^2 - \lambda_3^2) - \beta_{-1}(\lambda_1^{-2} - \lambda_3^{-2})\,, \\ T_{22} - T_{33} = \beta_{-1}(\lambda_2^2 - \lambda_3^2) - \beta_{-1}(\lambda_2^{-2} - \lambda_3^{-2})\,. \end{cases} \tag{1.45}$$

Solving (1.45) for the coefficients β_1 and β_{-1} yields

$$\begin{aligned} \beta_1 &= \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{\lambda_1^2}{\lambda_1^2 - \lambda_3^2}\,(T_{11} - T_{33}) - \frac{\lambda_2^2}{\lambda_2^2 - \lambda_3^2}\,(T_{22} - T_{33})\right], \\ \beta_{-1} &= \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{1}{\lambda_1^2 - \lambda_3^2}\,(T_{11} - T_{33}) - \frac{1}{\lambda_2^2 - \lambda_3^2}\,(T_{22} - T_{33})\right], \end{aligned} \tag{1.46}$$

under the incompressibility constraint $\lambda_3 = f(\lambda_1^{-1}, \lambda_2^{-1})$, equivalent to $\det \mathbf{F} = 1$.

1.5.3 Some strain energy functions for incompressbile materials

In this section, two classes of strain energy functions for isotropic hyperlelastic solids are presented, namely the class of functions (Ogden 1972a) and the finite strain generalization of the J_2-deformation theory of plasticity (Hutchinson and Neale 1978).

The family of strain energy functions proposed by Ogden (1972a) is

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{s=1}^{M} \left[\frac{\beta_s}{\gamma_s} \left(\lambda_1^{\gamma_s} + \lambda_2^{\gamma_s} + \lambda_3^{\gamma_s} - 3 \right) \right], \tag{1.47}$$

and its results were shown to be in agreement with the experimental data offered in Treloar (1944) for the case of simple tension, pure shear and equibiaxial tension (Treloar 2009). For consistency with the linearized elasticity it is required that

$$2\mu = \beta_1 \sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1} \gamma_s \right], \tag{1.48}$$

where $\mu = E/3$ is the conventional shear modulus and E is the Young modulus. The above relation may be manipulated as

$$\beta_1 = 2\,\mu \left(\sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1} \gamma_s \right] \right)^{-1}, \tag{1.49}$$

and used to express the constitutive law (1.42) as

$$\mathbf{T} + \pi\, \mathbf{I} = 2\,\mu \left(\sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1} \gamma_s \right] \right)^{-1} \sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1} \mathbf{B}^{\gamma_s/2} \right], \tag{1.50}$$

where the left Green-Lagrange strain tensor is expressed in its spectral representation (1.21).

For later use (cp. §2.3), in what follows, the constitutive law of the class of Ogden materials (1.50) is further manipulated under the hypothesis of axisymmetric pre-buckling state, where the load is assumed to act in the direction of the axis of revolution only. The only nonzero stress component of the Cauchy stress is T_{zz}, such that

$$\mathbf{T} = T_{zz}\, \mathbf{G}\,, \quad \mathbf{G} = \mathbf{e}_z \otimes \mathbf{e}_z\,, \tag{1.51}$$

and condition $T_{rr} = T_{\theta\theta} = 0$ allows for the determination of the Lagrange multiplier π

$$\pi = 2\,\mu \left(\sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1}\,\gamma_s \right] \right)^{-1} \sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1}\,\lambda_z^{-\gamma_s/2} \right]. \tag{1.52}$$

Hence

$$T_{zz} = 2\,\mu \left(\sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1}\,\gamma_s \right] \right)^{-1} \sum_{s=1}^{M} \left[\frac{\beta_s}{\beta_1}\,\lambda_z^{-\gamma_s/2} \left(\lambda_z^{3\gamma_s/2} - 1 \right) \right]. \tag{1.53}$$

As a particular case of the Ogden material, the Mooney-Rivlin material model (Mooney 1940; Rivlin 1948) can be recovered setting $M = 2$, $\gamma_1 = 2$, $\gamma_2 = -2$ in Eq. (1.47). Note that the coefficients β_i are assumed to be constant, thus independent of the amount of stretch described by the quantities λ_i. Under the hypothesis (1.51), the first response coefficient reads $\beta_1 = \mu/(1-\beta)$, so that

$$\mathbf{T}_{\mathrm{MR}} = \mu\,(1-\beta)^{-1} \left[\lambda_z^{-1}\,(\lambda_z^3 - 1) + \beta\,\lambda_z \left(\lambda_z^{-3} - 1 \right) \right] \mathbf{G}\,, \tag{1.54}$$

where $\mu = E/3$ is the shear modulus, $\beta = \beta_{-1}/\beta_1$ is the ratio between the two constant response coefficients and $\mathbf{G} = \mathbf{e}_z \otimes \mathbf{e}_z$. For swollen sulfur-vulcanized natural rubber, Treloar (2009) found very good agreements with the experimental dataset (Treloar 1948) by setting $\beta = -1/10$.

The neo-Hookean material model is recovered by setting $M = 2$, $\gamma_1 = 2$, $\gamma_2 = -2$ in Eq. (1.47). Hence $\beta_1 = \mu$, $\beta_{-1} = 0$ and the Cauchy stress relates to the axial stretch under the hypothesis (1.51) as

$$\mathbf{T}_{\mathrm{nH}} = \mu\,\lambda_z^{-1}\,(\lambda_z^3 - 1)\,\mathbf{G}\,, \quad \mathbf{G} = \mathbf{e}_z \otimes \mathbf{e}_z\,, \tag{1.55}$$

where $\mu = E/3$ is the shear modulus.

The second class of incompressible materials considered in what follows is the so-called J_2-deformation theory of plasticity (Hutchinson and Neale 1980; Hutchinson and Tvergaard 1980; Neale 1981). The constitutive law can be expressed by

$$T_{ii} = -\pi + \frac{2}{3}\,E_s\,\epsilon_{ii}\,, \quad \text{no sum over } i,\ i = 1,2,3\,, \tag{1.56}$$

where $\pi = \operatorname{vol}\mathbf{T}$ is the Lagrange multiplier characterizing the incompressibility constraint, $\operatorname{vol}\mathbf{T}$ being the volumetric part of $\mathbf{T}$ defined as $\operatorname{vol}\mathbf{T} = \operatorname{tr}\mathbf{T}/3$.

$E_s = K\,\epsilon_e^{N-1}$ is the secant modulus, $\epsilon_{ii} = \log\lambda_i$ are the logarithmic strains, K is a positive constitutive parameter, $N \in \,]0,1]$ is the strain hardening exponent and ϵ_e is the effective strain defined as

$$\epsilon_e = \sqrt{\frac{2}{3}\,(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)}\,. \tag{1.57}$$

Therefore, the strain energy function W_{J2} can be expressed as

$$W_{\mathrm{J2}} = \frac{K}{N+1}\,\epsilon_e^{N+1}\,, \quad \epsilon_1 + \epsilon_2 + \epsilon_3 = 0\,, \tag{1.58}$$

and the Cauchy stress for a J_2-deformation theory material reads

$$\mathbf{T}_{\mathrm{J2}} + \pi\,\mathbf{I} = \frac{2}{3}\,K\epsilon_e^{N-1}\sum_{i=1}^{3}\left[\log\left(\lambda_i\right)\mathbf{v}_i \otimes \mathbf{v}_i\right]. \tag{1.59}$$

From Eq. (1.59), it can be immediately seen that the tangent stiffness approaches infinity in the vicinity of the unloaded state and, therefore, the stress response is discontinuous. Consequently, the isotropy that characterizes the unloaded state is immediately lost, discontinuously, whenever a loading path is prescribed starting from the unloaded configuration.

Specializing the above theory to the case of axisymmetric deformations is necessary for the problem tackled in Ch. 6.1. By identifying λ_1 with the stretch aligned with the axis of symmetry, say λ_z, and due to the incompressibility constraint (1.41), one obtains $\lambda_r = \lambda_\theta = \sqrt{\lambda_z}$. Hence the effective strain ϵ_e reads

$$\epsilon_e = \left|\log\lambda_z\right|\,, \tag{1.60}$$

and the expression for the Cauchy stress reads

$$\mathbf{T}_{\mathrm{J2}} + \pi\,\mathbf{I} = \frac{1}{3}\,K\left|\log\lambda_z\right|^{N-1}\,\log\left(\lambda_z\right)\left(-\mathbf{I} + 3\,\mathbf{G}\right), \tag{1.61}$$

where $\mathbf{G} = \mathbf{e}_z \otimes \mathbf{e}_z$, being $\mathbf{e}_z$ the unit vector aligned with the z-axis, i.e. the axis of symmetry. Next, the expressions for the response coefficients β_i are specialized to the axisymmetric case for J_2-deformation theory too. Noting that

$$T_{ii} - T_{33} = \frac{2}{3}\,K\,\epsilon_e^{N-1}\,\log\left(\lambda_i/\lambda_3\right), \tag{1.62}$$

and

$$\lim_{\lambda_2 \to \lambda_3} \left[\frac{\lambda_2^2}{\lambda_2^2 - \lambda_3^2} \left(T_{22} - T_{33} \right) \right] = \frac{1}{3} \, K \epsilon_e^{N-1} \,, \tag{1.63}$$

the response coefficients for the axisymmetric case read

$$\begin{aligned} \beta_1 &= K \, \epsilon_e^{N-1} \, \frac{\lambda_z}{\lambda_z^3 - 1} \left(\frac{\lambda_z^3}{\lambda_z^3 - 1} \, \log \lambda_z - \frac{1}{3} \right) , \\ \beta_{-1} &= K \, \epsilon_e^{N-1} \, \frac{\lambda_z^2}{\lambda_z^3 - 1} \left(\frac{1}{\lambda_z^3 - 1} \, \log \lambda_z - \frac{1}{3} \right) . \end{aligned} \tag{1.64}$$

Moreover, if the only load acting on the system is prescribed along the axis of symmetry, from the fact that the only non-zero component of the Cauchy stress is T_{zz}, it follows

$$\pi = \frac{1}{3} \, K \epsilon_e^{N-1} \, \log \lambda_z \,, \tag{1.65}$$

and, therefore

$$T_{zz} = K \, \left| \log \lambda_z \right|^{N-1} \, \log \lambda_z \, . \tag{1.66}$$

2 Incremental constitutive equations for isotropic, hyperelastic materials

The following sections give a brief overview of the fundamental equations governing the mechanics of incremental deformations superimposed on a given state of strain for isotropic, hyperelastic materials.

2.1 Incremental elastic constitutive equations

The increment in the generic Eulerian stress measure, say $\Delta\mathbf{T}^{(m)}$ is calculated as a Taylor series expansion of the consitutive law (1.28) conveniently expressed as a function of the generic strain measure $\mathbf{E}^{(\mathrm{m})}$

$$\mathbf{T}^{(\mathrm{m})} = \hat{\mathbf{T}}^{(\mathrm{m})}(\mathbf{E}^{(\mathrm{m})}) \,, \tag{2.1}$$

such that, for a small increment in the strain measure $\Delta\mathbf{E}^{(\mathrm{m})}$, the increment reads

$$\Delta\mathbf{T}^{(\mathrm{m})} \propto \frac{\partial\,\hat{\mathbf{T}}^{(\mathrm{m})}}{\partial\,\mathbf{E}^{(\mathrm{m})}} : \Delta\mathbf{E}^{(\mathrm{m})} \,. \tag{2.2}$$

Furthermore, assuming that the stress increment is state-dependent only, Eq. (2.2) can be equivalently obtained taking the material time derivative of the constitutive relation (2.1), such that

$$\dot{\mathbf{T}}^{(\mathrm{m})} = \mathbb{E} : \dot{\mathbf{E}}^{(\mathrm{m})} \,, \tag{2.3}$$

where the fourth-order tensor $\mathbb{E}$ is the so-called *elastic tensor* defined as

$$\mathbb{E} = \frac{\partial\,\hat{\mathbf{T}}^{(\mathrm{m})}}{\partial\,\mathbf{E}^{(\mathrm{m})}} \,, \quad \text{or} \quad \mathbb{E} = \frac{\partial^2 W(\mathbf{E}^{(\mathrm{m})})}{\partial^2\mathbf{E}^{(\mathrm{m})}} \,, \tag{2.4}$$

if a strain energy function (1.35) exists, i.e. for a hyperelastic material. It is common practice (e.g. Hutchinson and Tvergaard 1980; Marsden and Hughes 1983; Christoffersen 1991; Bigoni and Gei 2001; Bower 2010) to express the incremental constitutive relations by means of incremental objective stress quantities such as the Oldroyd (1950) and Jaumann (1905) increments, defined as

$$\overset{\circ}{\mathbf{K}} = \dot{\mathbf{K}} - \mathbf{L}\mathbf{K} - \mathbf{K}\mathbf{L}^{\top}\,, \quad \overset{\triangledown}{\mathbf{K}} = \dot{\mathbf{K}} - \mathbf{W}\mathbf{K} + \mathbf{K}\mathbf{W}\,, \tag{2.5}$$

where $\mathbf{W} = (\mathbf{L} - \mathbf{L}^{\top})/2$ is the skew-symmetric part of $\mathbf{L}$. Note that the above definitions hold true also for the Cauchy stress.

As an example, the calculation of the incremental constitutive relation between the second Piola-Kirchhoff stress tensor $\mathbf{T}^{(2)}$ and the Green-Lagrange strain tensor is offered in what follows. The increment $\dot{\mathbf{T}}^{(2)} = \left(\mathbf{T}^{(2)}\right)^{\bullet}$ reads

$$\dot{\mathbf{T}}^{(2)} = \left(\mathbf{F}^{-1}\,\mathbf{K}\,\mathbf{F}^{-\top}\right)^{\bullet}\,, \tag{2.6}$$

and recalling that $\left(\mathbf{F}^{-1}\right)^{\bullet} = -\mathbf{F}^{-1}\,\mathbf{L}$, Eq. (2.6) transforms to

$$\dot{\mathbf{T}}^{(2)} = -\mathbf{F}^{-1}\,\mathbf{L}\,\mathbf{K}\,\mathbf{F}^{-\top} + \mathbf{F}^{-1}\,\dot{\mathbf{K}}\,\mathbf{F}^{-\top} - \mathbf{F}^{-1}\,\mathbf{K}\,\mathbf{L}^{\top}\,\mathbf{F}^{-\top}\,. \tag{2.7}$$

Solving Eq. (2.7) for $\dot{\mathbf{K}}$ yields

$$\dot{\mathbf{K}} = \mathbf{F}\,\dot{\mathbf{T}}^{(2)}\,\mathbf{F}^{\top} + \mathbf{L}\,\mathbf{K} + \mathbf{K}\,\mathbf{L}^{\top}\,. \tag{2.8}$$

Eq. (2.8) significantly simplifies if the Oldroyd rate of the Kirchhoff stress $(2.5)_1$ is used, yielding

$$\overset{\circ}{\mathbf{K}} = \mathbf{F}\,\dot{\mathbf{T}}^{(2)}\,\mathbf{F}^{\top}\,. \tag{2.9}$$

Analogously, the material time derivative of the Green-Lagrange strain tensor reads

$$\dot{\mathbf{E}}^{(2)} = \mathbf{F}^{\top}\,\mathbf{D}\,\mathbf{F}\,, \tag{2.10}$$

such that the incremental constitutive law (2.4) can be manipulated to yield

$$\overset{\circ}{\mathbf{K}} = \mathbf{F}\left(\frac{\partial \mathbf{T}^{(2)}}{\partial \mathbf{E}^{(2)}} : \left(\mathbf{F}^{\top}\,\mathbf{D}\,\mathbf{F}\right)\right)\mathbf{F}^{\top}\,. \tag{2.11}$$

By means of the products defined in Ch. 1, Eq. (2.11) transforms into

$$\overset{\circ}{\mathbf{K}} = \mathbb{H} : \mathbf{D}\,, \quad \text{where} \quad \mathbb{H} = \mathbf{F}\boxtimes\mathbf{F} \circ \frac{\partial \mathbf{T}^{(2)}}{\partial \mathbf{E}^{(2)}} \circ \left(\mathbf{F}\boxtimes\mathbf{F}\right)^{\top}\,. \tag{2.12}$$

For a hyperelastic material, the fourth-order incremental constitutive tensor $\mathbb{H}$ in $(2.12)_2$ can be derived directly from the strain energy density function W as

$$\mathbb{H} = \mathbf{F} \boxtimes \mathbf{F} \circ \frac{\partial^2 W(\mathbf{E}^{(2)})}{\partial \mathbf{E}^{(2)} \, \partial \mathbf{E}^{(2)}} \circ (\mathbf{F} \boxtimes \mathbf{F})^{\top} . \tag{2.13}$$

2.2 Relative Lagrangian description

The results of §2.1 were obtained assuming that the constitutive law (2.1), in general nonlinear, is always known for each state of deformation. In doing this, the increment is calculated starting from a reference configuration $\mathcal{B}_0$ that continuously evolves with the deformation and at each step is identified with the current configuration $\mathcal{B}$. However, in practical problems, the constitutive relation between stress and strain measures is usually known for particular configurations only, where the system shows peculiar properties: for example the body is unstressed, or isotropic. Such configurations are denoted with $\mathcal{B}_\square$. Hence, the relations of §2.1 need further manipulations so that it is possible to express the increment of a stress measure at a specific strain level based on the knowledge of the constitutive law in the 'special' configuration $\mathcal{B}_\square$.

The strategy to tackle the problem consists of three key-steps (Bigoni 2012). Firstly, three different configurations $\mathcal{B}_\square$, $\mathcal{B}_0$ and $\mathcal{B}$ are introduced, see Fig. 2.1. The motion of a material point $\mathbf{x}_\square \in \mathcal{B}_\square$ into its transformed $\mathbf{x}_0 \in \mathcal{B}_0$ is described by means of the vector-valued mapping $\boldsymbol{\chi}_\star$, assumed to be independent from the time-parameter t, whereas the motion of a material point $\mathbf{x}_0$ into $\mathcal{B}$ is described by the time-dependent vector-valued function $\boldsymbol{\chi}$. Consequently:

$$\mathbf{x} = \boldsymbol{\chi}\left(\boldsymbol{\chi}_\star(\mathbf{x}_\square), t\right) . \tag{2.14}$$

Hence, the deformation gradients for each motion can be defined as

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}_0} , \quad \mathbf{F}_\star = \frac{\partial \boldsymbol{\chi}_\star}{\partial \mathbf{x}_\square} , \tag{2.15}$$

whereas their composition yields $\mathbf{F}_\square = \mathbf{F}\,\mathbf{F}_\star$. Note that $\mathbf{F}_\star$ is time-independent.

The constitutive law is introduced in the *special* configuration $\mathcal{B}_\square$ and is expressed by means of the second Piola-Kirchhoff stress tensor reads, under the

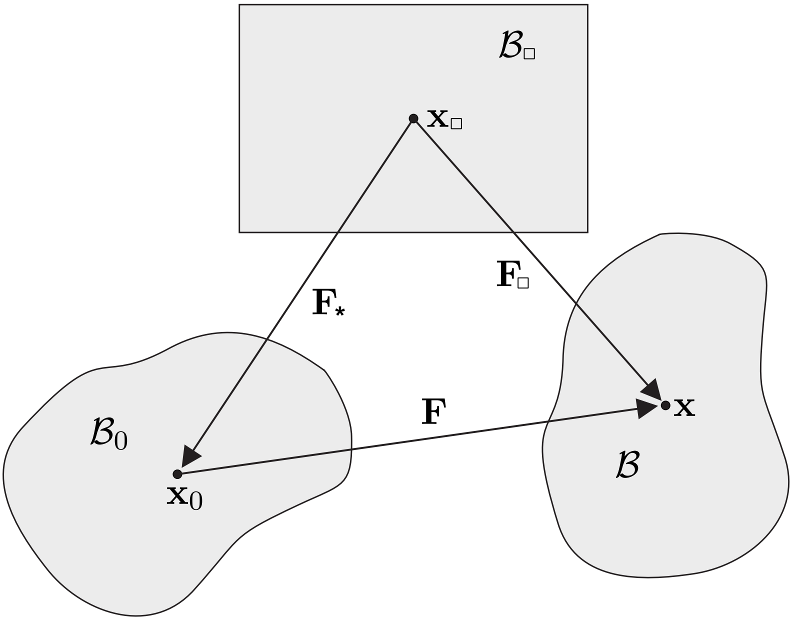

Fig. 2.1: *Definition of the different configurations involved in the computation of the incremental constitutive relations in the relative Lagrangian description.*

notation introduced so far

$$\mathbf{T}_\square^{(2)} = J_\square\, \mathbf{F}_\square^{-1}\, \mathbf{T}\, \mathbf{F}_\square^{-\top} \,. \tag{2.16}$$

Next, the material time derivative of $\mathbf{T}_\square^{(2)}$, $\dot{\mathbf{T}}_\square^{(2)}$, can formally be computed as

$$\dot{\mathbf{T}}_\square^{(2)} = \left(J_\square\, \mathbf{F}_\square^{-1}\, \mathbf{T}\, \mathbf{F}_\square^{-\top}\right)^{\bullet} \,. \tag{2.17}$$

However, the final expression of (2.17) is obtained after a third step is performed, namely the two configurations $\mathcal{B}_0$ and $\mathcal{B}$ are taken to coincide, i.e. the limit $\mathbf{F} \to \mathbf{I}$ is calculated. The following intermediate results are obtained

$$\dot{J}_\square = \frac{\partial \det\left(\mathbf{F}_\square\right)}{\partial \mathbf{F}_\square} : \frac{\partial \mathbf{F}_\square}{\partial t} = J_\square\, \operatorname{tr}\mathbf{D} \,, \quad \left(\mathbf{F}_\square^{-1}\right)^{\bullet} = -\mathbf{F}_\square^{-1}\,\mathbf{L} \,,$$

such that

$$\dot{\mathbf{T}}_\square^{(2)} = J_\square\, \mathbf{F}_\square^{-1} \left(\dot{\mathbf{T}} + (\operatorname{tr}\mathbf{D})\mathbf{T} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^\top\right) \mathbf{F}_\square^{-\top} \,. \tag{2.18}$$

Rearranging the above relation to obtain $\dot{\mathbf{T}}$ as a function of $\dot{\mathbf{T}}^{(2)}_{\square}$ allows to express the Jaumann derivative of the Cauchy stress as follows

$$\overset{\triangledown}{\mathbf{T}} = \frac{1}{J_{\square}}\, \mathbf{F}_{\square}\, \dot{\mathbf{T}}^{(2)}_{\square}\, \mathbf{F}^{\top}_{\square} - (\operatorname{tr}\mathbf{D})\, \mathbf{T} + \mathbf{D}\,\mathbf{T} + \mathbf{T}\,\mathbf{D}\,, \tag{2.19}$$

or, equivalently

$$\overset{\triangledown}{\mathbf{T}} = \mathbb{H} : \mathbf{D} + (-\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{T} + \mathbf{T} \boxtimes \mathbf{I}) : \mathbf{D}\,, \tag{2.20}$$

where, for a hyperelastic material $(2.4)_2$, the fourth-order incremental elasticity tensor $\mathbb{H}$ (2.13) reads

$$\mathbb{H} = \frac{1}{J_{\square}}\, (\mathbf{F}_{\square} \boxtimes \mathbf{F}_{\square})\, \frac{\partial^2 W}{\partial \mathbf{E}^{(2)}_{\square}\, \partial \mathbf{E}^{(2)}_{\square}} (\mathbf{F}_{\square} \boxtimes \mathbf{F}_{\square})^{\top}\,. \tag{2.21}$$

The same incremental relation can be expressed in a more compact way using the definition of the Oldroyd derivative of the Kirchhoff stress, which leads to

$$\overset{\circ}{\mathbf{K}} = \mathbb{H} : \mathbf{D}\,, \tag{2.22}$$

where $\mathbf{D} = (\mathbf{L} + \mathbf{L}^{\top})/2$ is the rate of strain.

2.3 Stress increments for compressible hyperelasticity

In this section, reference is made to a hyperelastic, isotropic material, for which the Oldroyd increment of the Kirchhoff stress can be expressed through a strain energy density W defined in the reference configuration by means of Eq. (2.22).

Inserting the definition (1.38) for the strain energy function into Eq. (2.22) yields the following form for the elastic fourth-order tensor $\mathbb{H}$ defined in (2.13)

$$\begin{aligned}\mathbb{H} = {} & (\kappa - 2/3\mu)(2J_{\square} - 1)\, \mathbf{I} \otimes \mathbf{I}\, (\mathbf{F}^{-1}_{\square} \boxtimes \mathbf{F}^{\top}_{\square}) + \\ & + 2\left[\mu/J_{\square} - (\kappa - 2/3\mu)(J_{\square} - 1)\right] \mathbb{S}\,.\end{aligned}$$

Hence, it is an easy task to derive the Oldroyd increment of the Kirchhoff stress (1.40), which reads as

$$\begin{aligned}\overset{\circ}{\mathbf{K}} = {} & (\kappa - 2/3\,\mu)(2J_{\square} - 1)\ \operatorname{tr}(\mathbf{F}^{-1}_{\square}\, \mathbf{D}\, \mathbf{F}_{\square})\, \mathbf{I} + \\ & + 2\left[\mu/J_{\square} - (\kappa - 2/3\mu)(J_{\square} - 1)\right] \mathbf{D}\,.\end{aligned} \tag{2.23}$$

The increment of the first Piola-Kirchhoff stress tensor can be calculated by means of Eq. (3.5).

2.4 Stress increments for incompressible hyperelasticity

In the following, an overview is given of the derivation of incremental constitutive relations for incompressible materials. The equations are then specialized to the case of the Mooney-Rivlin material model and the J_2-deformation theory material model.

Formally, the increment of Eq. (1.42) is calculated as

$$\dot{\mathbf{T}} = -\dot{\pi}\,\mathbf{I} + \beta_1\,\dot{\mathbf{B}} + \beta_{-1}\left(\mathbf{B}^{-1}\right)^{\bullet} + \dot{\beta}_1\,\mathbf{B} + \dot{\beta}_{-1}\,\mathbf{B}^{-1}\,, \tag{2.24}$$

where

$$\dot{\mathbf{B}} = \mathbf{L}\,\mathbf{B} + \mathbf{B}\,\mathbf{L}^{\top}\,, \quad \text{and}\left(\mathbf{B}^{-1}\right)^{\bullet} = -(\mathbf{B}^{-1}\,\mathbf{L} + \mathbf{L}^{\top}\,\mathbf{B}^{-1})\,. \tag{2.25}$$

Since the response coefficients $\dot{\beta}_i$ are functions of the invariants $\mathcal{I}_1$, $\mathcal{I}_2$, their increments can be calculated as

$$\dot{\beta}_i = \frac{\partial\beta_i}{\partial\mathcal{I}_1}\,\dot{\mathcal{I}}_1 + \frac{\partial\beta_i}{\partial\mathcal{I}_2}\,\dot{\mathcal{I}}_2\,, \tag{2.26}$$

and noting that $\mathrm{tr}\,(\mathbf{D}\,\mathbf{B}) = \mathrm{tr}\,(\mathbf{B}\,\mathbf{D})$ and that $\mathrm{tr}\,(\mathbf{W}\,\mathbf{B}) = 0$, the following results are obtained

$$\dot{\mathcal{I}}_1 = 2\,\mathbf{B} : \mathbf{D}\,, \quad \text{and} \quad \dot{\mathcal{I}}_2 = 2\,\mathbf{B}^2 : \mathbf{D}\,. \tag{2.27}$$

Hence, the final form of the increments of the response coefficient reads

$$\dot{\beta}_i = 2\,\frac{\partial\beta_i}{\partial\mathcal{I}_1}\,\mathbf{B} : \mathbf{D} + 4\,\frac{\partial\beta_i}{\partial\mathcal{I}_2}\,\mathbf{B}^2 : \mathbf{D}\,. \tag{2.28}$$

The above calculations were performed assuming that the coefficients β_i were functions of the (non-standard) invariants $\mathcal{I}_1$ and $\mathcal{I}_2$. However, as mentioned in §1.4, such coefficient can also be expressed by means of the principal stretches λ_i; under this hypothesis:

$$\dot{\beta}_i = \frac{\partial\beta_i}{\partial\lambda_1}\,\dot{\lambda}_1 + \frac{\partial\beta_i}{\partial\lambda_2}\,\dot{\lambda}_2\,.$$

The expressions for the increments $\dot{\lambda}_i$ can be obtained by means of the spectral representation of the left Cauchy-Green deformation tensor $\mathbf{B}$ $(1.21)_2$

$$\dot{\mathbf{B}} = \sum_{i=1}^{3} \left[2\,\lambda_i\,\dot{\lambda}_i\,\mathbf{b}_i \otimes \mathbf{b}_i + \lambda_i^2\,\dot{\mathbf{b}}_i \otimes \mathbf{b}_i + \lambda_i^2\,\mathbf{b}_i \otimes \dot{\mathbf{b}}_i \right], \tag{2.29}$$

where $\mathbf{b}_i$ are the principal directions of $\mathbf{B}$. To further manipulate Eq. (2.29), the following double contraction between $\dot{\mathbf{B}}$ and the dyad $\mathbf{b}_1 \otimes \mathbf{b}_1$ is calculated

$$\dot{\mathbf{B}} : \mathbf{b}_1 \otimes \mathbf{b}_1 = 2\,\lambda_1 \dot{\lambda}_1\,, \tag{2.30}$$

showing that the increment of the principal stretch λ_i can be expressed as

$$\dot{\lambda}_i = \frac{1}{2\,\lambda_i}\,\dot{\mathbf{B}} : \mathbf{b}_i \otimes \mathbf{b}_i\,, \tag{2.31}$$

no sum over i, $i = 1, 2$. Noting that $\dot{\mathbf{B}} : \mathbf{b}_i \otimes \mathbf{b}_i = \dot{B}_{ii}$ and using $(2.25)_1$, the final expression for the increment of the principal stretches is obtained

$$\dot{\lambda}_i = \lambda_i\,D_{ii}\,, \tag{2.32}$$

no sum over i, $i = 1, 2$. Therefore Eq. (2.28) reads

$$\dot{\beta}_i = \left(\frac{\partial \beta_i}{\partial \lambda_1}\,\lambda_1\,\mathbf{b}_1 \otimes \mathbf{b}_1 + \frac{\partial \beta_i}{\partial \lambda_2}\,\lambda_2\,\mathbf{b}_2 \otimes \mathbf{b}_2 \right) : \mathbf{D}\,, \quad i = 1, -1\,. \tag{2.33}$$

The use of (2.33) into Eq. (2.24) yields

$$\begin{aligned} \dot{\mathbf{T}} = & -\dot{\pi}\,\mathbf{I} + \beta_1 \left(\mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^{\top} \right) - \beta_{-1} \left(\mathbf{B}^{-1}\mathbf{L} + \mathbf{L}^{\top}\mathbf{B}^{-1} \right) + \\ & + \left(\frac{\partial \beta_1}{\partial \lambda_1}\,\lambda_1\,\mathbf{b}_1 \otimes \mathbf{b}_1 + \frac{\partial \beta_1}{\partial \lambda_2}\,\lambda_2\,\mathbf{b}_2 \otimes \mathbf{b}_2 \right) : \mathbf{D}\,\mathbf{B} + \\ & + \left(\frac{\partial \beta_{-1}}{\partial \lambda_1}\,\lambda_1\,\mathbf{b}_1 \otimes \mathbf{b}_1 + \frac{\partial \beta_{-1}}{\partial \lambda_2}\,\lambda_2\,\mathbf{b}_2 \otimes \mathbf{b}_2 \right) : \mathbf{D}\,\mathbf{B}^{-1}\,. \end{aligned} \tag{2.34}$$

As done for Eq. (2.8) in §2.1, the use of the Oldroyd or Jaumann rates yields the following equivalent expressions for the incremental constitutive law for the class of incompressible isotropic materials

$$\overset{\circ}{\mathbf{T}} = -\dot{\pi}\,\mathbf{I} + \mathbb{O} : \mathbf{D} \quad \text{and} \quad \overset{\triangledown}{\mathbf{T}} = -\dot{\pi}\,\mathbf{I} + \mathbb{J} : \mathbf{D}\,, \tag{2.35}$$

where $\mathbb{O}$ ($\mathbb{J}$) represents the fourth-order incremental elastic tensor relating the Oldroyd (Jaumann) rate of the Cauchy stress with the strain rate $\mathbf{D}$

$$\begin{aligned}\mathbb{O} = {} & \frac{\partial\beta_1}{\partial\lambda_1}\,\lambda_1\,\mathbf{B}\otimes\mathbf{b}_1\otimes\mathbf{b}_1 + \frac{\partial\beta_1}{\partial\lambda_2}\,\lambda_2\,\mathbf{B}\otimes\mathbf{b}_2\otimes\mathbf{b}_2 + \\ & + \frac{\partial\beta_{-1}}{\partial\lambda_1}\,\lambda_1\,\mathbf{B}^{-1}\otimes\mathbf{b}_1\otimes\mathbf{b}_1 + \frac{\partial\beta_{-1}}{\partial\lambda_2}\,\lambda_2\,\mathbf{B}^{-1}\otimes\mathbf{b}_2\otimes\mathbf{b}_2 + \\ & + 2\pi\,\mathbf{I}\boxtimes\mathbf{I} - 2\beta_{-1}\left(\mathbf{B}^{-1}\boxtimes\mathbf{I}^{\top} + \mathbf{I}\boxtimes\mathbf{B}^{-\top}\right),\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}\mathbb{J} = {} & \frac{\partial\beta_1}{\partial\lambda_1}\,\lambda_1\,\mathbf{B}\otimes\mathbf{b}_1\otimes\mathbf{b}_1 + \frac{\partial\beta_1}{\partial\lambda_2}\,\lambda_2\,\mathbf{B}\otimes\mathbf{b}_2\otimes\mathbf{b}_2 + \\ & + \frac{\partial\beta_{-1}}{\partial\lambda_1}\,\lambda_1\,\mathbf{B}^{-1}\otimes\mathbf{b}_1\otimes\mathbf{b}_1 + \frac{\partial\beta_{-1}}{\partial\lambda_2}\,\lambda_2\,\mathbf{B}^{-1}\otimes\mathbf{b}_2\otimes\mathbf{b}_2 + \\ & + \beta_1\,(\mathbf{I}\boxtimes\mathbf{B}^{\top} + \mathbf{B}\boxtimes\mathbf{I}^{\top}) - \beta_{-1}\,(\mathbf{I}\boxtimes\mathbf{B}^{-\top} + \mathbf{B}^{-1}\boxtimes\mathbf{I}^{\top})\,.\end{aligned} \tag{2.37}$$

For the particular case of axisymmetric geometry and deformation, say about the z-axis, the problem is governed by the axial stretch only, denoted by λ_z and identified with the stretch λ_1. Due to incompressibility and axisymmetry the explicit dependance on the amount of stretch in the axial direction can be written as $\lambda_2 = \lambda_3 = \lambda_z^{-1/2}$. Under such hypothesis, the response coefficients too are functions of λ_z only, $\beta_i = \beta_i(\lambda_z)$, and the following relations hold true

$$\mathbf{B} = \lambda_z^{-1}\left(\mathbf{I} + (\lambda_z^3 - 1)\,\mathbf{G}\right), \quad \dot{\beta}_i = \frac{\partial\beta_i}{\partial\lambda_z}\,\lambda_z\,\mathbf{G}:\mathbf{D}\,, \tag{2.38}$$

where $\mathbf{G} = \mathbf{e}_z\otimes\mathbf{e}_z$. Therefore, Eqs. (2.36) and (2.37) simplify as

$$\begin{aligned}\mathbb{O} = {} & 2(\pi - 2\beta_{-1}\lambda_z)\,\mathbf{I}\boxtimes\mathbf{I} + 2\beta_{-1}\,\frac{\lambda_z^3-1}{\lambda_z^2}(\mathbf{G}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{G}) + \\ & + \left(\frac{\partial\beta_1}{\partial\lambda_z} + \frac{\partial\beta_{-1}}{\partial\lambda_z}\lambda_z^2\right)\mathbf{I}\otimes\mathbf{G} + \frac{\lambda_z^3-1}{\lambda_z}\left(\frac{\partial\beta_1}{\partial\lambda_z}\lambda_z - \frac{\partial\beta_{-1}}{\partial\lambda_z}\right)\mathbf{G}\otimes\mathbf{G}\,,\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}\mathbb{J} = {} & 2\left(\frac{1}{\lambda_z}\,\beta_1 - \beta_{-1}\lambda_z\right)\mathbf{I}\boxtimes\mathbf{I} + \frac{\lambda_z^3-1}{\lambda_z}\left(\beta_1 + \frac{1}{\lambda_z}\beta_{-1}\right)(\mathbf{G}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{G}) + \\ & + \left(\frac{\partial\beta_1}{\partial\lambda_z} + \frac{\partial\beta_{-1}}{\partial\lambda_z}\lambda_z^2\right)\mathbf{I}\otimes\mathbf{G} + \frac{\lambda_z^3-1}{\lambda_z}\left(\frac{\partial\beta_1}{\partial\lambda_z}\lambda_z - \frac{\partial\beta_{-1}}{\partial\lambda_z}\right)\mathbf{G}\otimes\mathbf{G}\,,\end{aligned} \tag{2.40}$$

respectively.

As an example, we specialize the incremental constitutive tensor (2.36) for the Mooney-Rivlin material model under the axisymmetric hypothesis in the pre-buckling state

$$\begin{aligned} \mathring{\mathbf{T}} + \dot{\pi}\mathbf{I} &= 2\,\pi\mathbf{D} - 2\beta_1\,\beta\left(\mathbf{B}^{-1}\mathbf{D} + \mathbf{D}\,\mathbf{B}^{-1}\right) \\ &= \frac{2\,\mu}{1-\beta}\left[2\,\mathbf{D} - \beta\left(\mathbf{B}^{-1}\mathbf{D} + \mathbf{D}\mathbf{B}^{-1}\right)\right], \end{aligned} \tag{2.41}$$

where $\beta = \beta_{-1}/\beta_1$.

3 Incremental field equations in terms of generalized stresses

The incremental equilibrium equations are derived, which govern the buckling of a cylindrical shell of current length l, external radius r_e and internal radius r_i. The cylinder, whose thickness is denoted by $t = r_e - r_i$, is not assumed to be thin for the moment; therefore all results presented in this Section are rigorous in terms of mean values of the incremental field quantities. The adopted geometrical descriptors are the 'mid-surface' of the shell, defined through the mid-radius $a = (r_e + r_i)/2$ and the so-called 'reduced radius' $\overline{r} = r - a$, being r the radial coordinate.

Neglecting body forces, the incremental equilibrium of a pre-stressed solid can be expressed by means of the increment of the first Piola-Kirchhoff stress tensor $\mathbf{S}$, as

$$\mathrm{Div}\,\dot{\mathbf{S}} = \mathbf{0}\,, \tag{3.1}$$

In a polar coordinate system $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, Eq. (3.1) becomes

$$\begin{cases} \dot{S}_{rr} - \dot{S}_{\theta\theta} + \dot{S}_{r\theta,\theta} + (a + \overline{r}) \left(\dot{S}_{rz,z} + \dot{S}_{rr,r}\right) = 0\,, \\ \dot{S}_{r\theta} + \dot{S}_{\theta r} + \dot{S}_{\theta\theta,\theta} + (a + \overline{r}) \left(\dot{S}_{\theta z,z} + \dot{S}_{\theta r,r}\right) = 0\,, \\ \dot{S}_{zr} + \dot{S}_{z\theta,\theta} + (a + \overline{r}) \left(\dot{S}_{zz,z} + \dot{S}_{zr,r}\right) = 0. \end{cases} \tag{3.2}$$

Finally, the cylindrical shell is subject to traction-free surface boundary conditions on its lateral surface, so that

$$\dot{S}_{ir}(\overline{r} = \pm t/2) = 0\,, \quad i = r, \theta, z. \tag{3.3}$$

As $\mathbf{S} = \mathbf{K}\mathbf{F}^{-\top}$, $\dot{\mathbf{S}}$ and $\dot{\mathbf{K}}$ are related through

$$\dot{\mathbf{S}} = (\dot{\mathbf{K}} - \mathbf{K}\mathbf{L}^{\top})\,\mathbf{F}^{-\top}\,, \tag{3.4}$$

where $\mathbf{L} = \operatorname{grad} \mathbf{v}$ is the gradient of the incremental displacement field, $\mathbf{v}$. In a relative Lagrangean description, briefly described in §2.2, equation (3.4) becomes

$$\dot{\mathbf{S}} = (\operatorname{div} \mathbf{v})\, \mathbf{T} + \dot{\mathbf{T}} - \mathbf{K}\mathbf{L}^{\top} . \tag{3.5}$$

The (undeformed, stress-free) reference configuration may be described by means of the cylindrical coordinates (r_0, θ_0, z_0), where the z_0-axis is aligned with the axis of revolution of the shell.

Before bifurcation, the shell is assumed to undergo a homogeneous, axisymmetric compression in the longitudinal direction z, so that a uniaxial stress field is generated, which can be written as

$$\mathbf{K} = K_{zz}\, \mathbf{G} , \quad \mathbf{G} = \mathbf{e}_z \otimes \mathbf{e}_z . \tag{3.6}$$

The current configuration (r, θ, z) is fully described by means of the principal stretches $\{\lambda_r, \lambda_\theta, \lambda_z\}$

$$r = \lambda_r\, r_0, \quad \theta = \lambda_\theta\, \theta_0, \quad z = \lambda_z\, z_0 ,$$

with $\lambda_r = \lambda_\theta$ in case of axial symmetry. Therefore, the deformation gradient and the left Cauchy-Green deformation tensor read

$$\mathbf{F} = \operatorname{diag}\{\lambda_r;\, \lambda_\theta;\, \lambda_z\} \quad \text{and} \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\top} = \operatorname{diag}\{\lambda_r^2; \lambda_\theta^2; \lambda_z^2\} , \tag{3.7}$$

respectively.

Inserting the applied uniaxial stress in Eq. (3.6) into Eq. (3.5), the following relations between the components of the incremental first Piola-Kirchhoff stress $\dot{\mathbf{S}}$ are derived:

$$\dot{S}_{\theta r} = \dot{S}_{r\theta} , \quad \dot{S}_{zr} = \dot{S}_{rz} - v_{r,z}\, K_{zz} , \quad \dot{S}_{z\theta} = \dot{S}_{\theta z} - v_{\theta,z}\, K_{zz} . \tag{3.8}$$

The role played by the assumed pre-buckling state is twofold and depends on the constitutive law analyzed. For the nearly incompressible neo-Hookean material (1.38), such an assumption enforces the equality between the radial and circumferential stretches $\lambda_\theta = \lambda_r$ and it requires that $K_{rr} = K_{\theta\theta} = 0$. The latter condition reads

$$\mu\, (\lambda_r^2 - 1) + (\kappa - 2/3\mu)\, \lambda_r^2\, \lambda_z\, (\lambda_r^2\, \lambda_z - 1) = 0 , \tag{3.9}$$

and solving Eq. (3.9) in terms of the radial stretch λ_r yields

$$\lambda_r = \left[\frac{2\,(\lambda_z + 1)\,\nu + \delta - 1}{4\nu\,\lambda_z^2} \right]^{1/2} , \tag{3.10}$$

where

$$\delta = \sqrt{1 - 4\nu(\lambda_z - 1)\,[\lambda_z(3\nu - 2) + \nu - 1]} \,. \tag{3.11}$$

The use of Eq. (3.10) substantially simplifies Eqs. (1.40), so that its only non-zero components read as

$$S_{zz} = -\frac{E\,(2\,\nu\,(-2\lambda_z^4 + \lambda_z + 1) + \delta - 1)}{8\,\lambda_z^3\,\nu\,(\nu + 1)} , \tag{3.12}$$

and $K_{zz} = \lambda_z\, S_{zz}$.

Note that Flügge has used a constitutive equation as (2.23), but with $\mathbf{F}_\square = \mathbf{I}$ and $J_\square = 1$, namely

$$\mathring{\mathbf{K}} = (\kappa - 2/3\,\mu)\;(\mathrm{tr}\,\mathbf{D})\;\mathbf{I} + 2\mu\,\mathbf{D}. \tag{3.13}$$

3.1 Exact formulation

3.1.1 Generalized stresses

In the shell theory, it is a common practice to introduce the so-called *generalized stresses*, namely, the stress resultants defined per unit length acting on the mid-surface of the shell. For a cylinder of current (constant) wall thickness $t = \lambda_r\, t_0$, four averages are introduced, namely, the longitudinal and radial averages for resultant forces defined as

$$\begin{aligned} \overset{\star}{n}_{\cdot\theta} &= \int_{-t/2}^{t/2} [\overset{\star}{\text{stress}}]_{\cdot\theta}\; \mathrm{d}\overline{r} \,, \\ \overset{\star}{n}_{\cdot z} &= \int_{-t/2}^{t/2} [\overset{\star}{\text{stress}}]_{\cdot z}\; (1 + \overline{r}/a)\; \mathrm{d}\overline{r} \,, \end{aligned} \tag{3.14}$$

and for resultant moments, namely

$$\begin{aligned} \overset{\star}{m}_{\cdot\theta} &= -\int_{-t/2}^{t/2} [\overset{\star}{\text{stress}}]_{\cdot\theta}\, \overline{r}\; \mathrm{d}\overline{r} \,, \\ \overset{\star}{m}_{\cdot z} &= -\int_{-t/2}^{t/2} [\overset{\star}{\text{stress}}]_{\cdot z}\; (1 + \overline{r}/a)\, \overline{r}\; \mathrm{d}\overline{r} \,, \end{aligned} \tag{3.15}$$

where the subscript $\cdot$ stands for r, θ, or z in turn, the superimposed '$\star$' identifies a suitable increment and [stress] a generic Eulerian stress measure, whose average resultant force and moment are n and m, respectively. The minus signs in the above definitions are introduced such that to a positive resultant moment corresponds a tensile stress distribution on the inner surface of the cylinder. Note that the term $1+\overline{r}/a$ arises from the integration over a circular sector. In particular, the following generalized normal stresses play a role in what follows

n_{zz} : normal longitudinal force;
$n_{z\theta}$: longitudinal shear force;
$n_{\theta\theta}$: normal hoop force;
$n_{\theta z}$: circumferential shear force;
$n_{r\theta}$: transverse shear force;
n_{rz} : transverse shear force;

whereas for the generalized moments, the following quantities are introduced

m_{zz} : longitudinal bending moment;
$m_{z\theta}$: longitudinal twisting moment;
$m_{\theta\theta}$: hoop moment;
$m_{\theta z}$: circumferential twisting moment.

3.1.2 Incremental equilibrium equations in the referential description

Let us focus on Eq. $(3.2)_2$ and perform a through-thickness integration, after a multiplication by the reduced radius $\overline{r}$; this yields

$$\begin{aligned}\int_{-t/2}^{t/2}(\dot{S}_{r\theta}+\dot{S}_{\theta r})\,\overline{r}\ \mathrm{d}\overline{r}+\int_{-t/2}^{t/2}\dot{S}_{\theta\theta,\theta}\,\overline{r}\ \mathrm{d}\overline{r}\,+&\\ +\,a\int_{-t/2}^{t/2}\left(\dot{S}_{\theta z,z}+\dot{S}_{\theta r,r}\right)(1+\overline{r}/a)\,\overline{r}\ \mathrm{d}\overline{r}=0\,,&\end{aligned}\tag{3.16}$$

where the generalized moments $\dot{m}_{\theta\theta}$ and $\dot{m}_{\theta z}$ defined according to Eq. (3.15) can be easily recognized. The term

$$\int_{-t/2}^{t/2}\dot{S}_{\theta r,r}\,(1+\overline{r}/a)\,\overline{r}\ \mathrm{d}\overline{r}$$

can be transformed through an integration by parts as

$$-a \int_{-t/2}^{t/2} \dot{S}_{\theta r} \left(1 + 2\,\overline{r}/a\right) \mathrm{d}\overline{r} + a \left.\left[\dot{S}_{\theta r}\, \overline{r} \left(1 + \overline{r}/a\right)\right]\right|_{-t/2}^{t/2} ,$$

so that, exploiting Eq. $(3.8)_1$, Eq. (3.16) becomes

$$\dot{m}_{\theta\theta,\theta} + a\,\dot{m}_{\theta z,z} - a\,\dot{n}_{r\theta} - a \left.\left[\dot{S}_{\theta r}\, \overline{r} \left(1 + \overline{r}/a\right)\right]\right|_{-t/2}^{t/2} = 0\,. \tag{3.17}$$

The procedure leading to equation (3.17) can be repeated to yield the other rotational equilibrium equation

$$\begin{aligned} &a\,\dot{m}_{zz,z} + \dot{m}_{z\theta,\theta} - a\,\dot{n}_{rz} - P\,a/t \int_{-t/2}^{t/2} v_{r,z} \left(1 + \overline{r}/a\right) \mathrm{d}\overline{r} + \\ &\qquad + a \left.\left[\dot{S}_{zr}\, \overline{r} \left(1 + \overline{r}/a\right)\right]\right|_{-t/2}^{t/2} = 0\,. \end{aligned} \tag{3.18}$$

From a mechanical point of view, Eqs. (3.17) and (3.18) correspond to the moment equilibrium about the θ- and z-axis, respectively.

The three translational equilibrium equations for the generalized stresses can be obtained performing a through-thickness integration in a similar vein, with no need to premultiply Eqs. (3.2) by $\overline{r}$, leading to

$$\begin{cases} a\,\dot{n}_{zz,z} + \dot{n}_{z\theta,\theta} + a \left.\left[\dot{S}_{zr} \left(1 + \overline{r}/a\right)\right]\right|_{-t/2}^{t/2} = 0\,, \\ \dot{n}_{\theta\theta,\theta} + a\,\dot{n}_{\theta z,z} + \dot{n}_{r\theta} + a \left.\left[\dot{S}_{\theta r} \left(1 + \overline{r}/a\right)\right]\right|_{-t/2}^{t/2} = 0\,, \\ \dot{n}_{r\theta,\theta} + a\,\dot{n}_{rz,z} - \dot{n}_{\theta\theta} + a \left.\left[\dot{S}_{rr} \left(1 + \overline{r}/a\right)\right]\right|_{-t/2}^{t/2} = 0\,. \end{cases} \tag{3.19}$$

Imposing the traction-free surface boundary conditions (3.3) on Eqs. (3.19) and (3.18), the translational and rotational equilibrium equations are finally obtained

$$\begin{cases} a\,\dot{n}_{zz,z} + \dot{n}_{z\theta,\theta} = 0\,, \\ \dot{n}_{\theta\theta,\theta} + a\,\dot{n}_{\theta z,z} + \dot{n}_{r\theta} = 0\,, \\ \dot{n}_{r\theta,\theta} + a\,\dot{n}_{rz,z} - \dot{n}_{\theta\theta} = 0\,, \\ \dot{m}_{\theta\theta,\theta} + a\,\dot{m}_{\theta z,z} + a\,\dot{n}_{r\theta} = 0\,, \\ a\,\dot{m}_{zz,z} + \dot{m}_{z\theta,\theta} + a\,\dot{n}_{rz} - P\,a/t \int_{-t/2}^{t/2} v_{r,z} \left(1 + \overline{r}/a\right) \mathrm{d}\overline{r} = 0\,, \end{cases} \tag{3.20}$$

where P is the load per unit length along the mid circular surface, $P = K_{zz}\, t$.

A substitution of Eq. $(3.20)_4$ and Eq. $(3.20)_5$ into Eq. $(3.20)_2$ and Eq. $(3.20)_3$ allows the elimination of the transverse shear forces, so that the following equations are found

$$\begin{cases} a\,\dot{n}_{zz,z} + \dot{n}_{z\theta,\theta} = 0\,, \\ a\,\dot{n}_{\theta\theta,\theta} + a^2\,\dot{n}_{\theta z,z} - \dot{m}_{\theta\theta,\theta} - a\,\dot{m}_{\theta z,z} = 0\,, \\ \dot{m}_{\theta\theta,\theta\theta} + a\,\dot{m}_{\theta z,z\theta} + a^2\,\dot{m}_{zz,zz} + a\,\dot{m}_{z\theta,\theta z} + \\ \qquad - P\,a^2/t \displaystyle\int_{-t/2}^{t/2} v_{r,zz}\,(1 + \overline{r}/a)\,\mathrm{d}\overline{r} = 0\,, \end{cases}, \tag{3.21}$$

where the calculation rules for the transverse shear forces are

$$\begin{cases} \dot{n}_{r\theta} = -\dot{m}_{\theta\theta,\theta}/a - \dot{m}_{\theta z,z}\,, \\ \dot{n}_{rz} = -\dot{m}_{zz,z} - \dot{m}_{z\theta,\theta}/a + P/t \displaystyle\int_{-t/2}^{t/2} v_{r,z}\,(1 + \overline{r}/a)\,\mathrm{d}\overline{r}\,. \end{cases} \tag{3.22}$$

Note that all derived equations, and in particular Eqs. (3.21), are exact, i.e. no approximations were introduced so far.

3.1.3 Spatial formulation in terms of Oldroyd increment

The incremental equilibrium equations Eqs. (3.21) can be equivalently expressed by means of a new set of generalized stresses defined by means of the Oldroyd derivative of the Kirchhoff stress defined by means of $(2.5)_1$

$$\mathring{\mathbf{K}} = \dot{\mathbf{S}} - \mathbf{L}\mathbf{K}\,. \tag{3.23}$$

A new set of generalized stresses is obtained from the specialization of the generic definitions given in Eqs. (3.14) and (3.15), employing now the Oldroyd rate of the Kirchhoff stress.

Hence, the traction-free boundary condition (3.3) transforms in terms of the Oldroyd increment of the Kirchhoff stress as

$$\mathring{K}_{ir}(\overline{r} = \pm t/2) = 0\,, \quad i = r, \theta, z\,. \tag{3.24}$$

Using the symmetry of $\mathring{\mathbf{K}}$, Eqs. (3.21) become

$$\begin{cases} a^2\, \mathring{n}_{zz,z} + a\, \mathring{n}_{z\theta,\theta} + P\, a^2/t \displaystyle\int_{-t/2}^{t/2} v_{z,zz}\, (1+\overline{r}/a)\, \mathrm{d}\overline{r} = 0\,, \\ a\, \mathring{n}_{\theta\theta,\theta} + a^2\, \mathring{n}_{\theta z,z} - \mathring{m}_{\theta\theta,\theta} - a\, \mathring{m}_{\theta z,z} + \\ \qquad + P\, a^2/t \displaystyle\int_{-t/2}^{t/2} v_{\theta,zz}\, (1+\overline{r}/a)^2\, \mathrm{d}\overline{r} = 0\,, \\ \mathring{m}_{\theta\theta,\theta\theta} + a\, \mathring{m}_{\theta z,\theta z} + a^2\, \mathring{m}_{zz,zz} + a\, \mathring{m}_{z\theta,z\theta} + a\, \mathring{n}_{\theta\theta} + \\ \qquad - P\, a^2/t \displaystyle\int_{-t/2}^{t/2} \left(v_{\theta,\theta zz}\, \frac{\overline{r}}{a} + v_{z,zzz}\overline{r} + v_{r,zz} \right) \left(1 + \frac{\overline{r}}{a} \right) \mathrm{d}\overline{r} = 0\,. \end{cases} \tag{3.25}$$

3.2 Rotational equilibrium about the r-axis

A sixth incremental equilibrium equation, namely the rotational equilibrium about the r-axis, can be obtained from a through thickness integration of Eq. $(3.8)_3$ after multiplication by $(1+\overline{r}/a)$

$$\begin{aligned} \int_{-t/2}^{t/2} \dot{S}_{\theta z}\, (1+\overline{r}/a)\, \mathrm{d}\overline{r} = &\int_{-t/2}^{t/2} \dot{S}_{z\theta}\, (1+\overline{r}/a)\, \mathrm{d}\overline{r} + \\ &+ \int_{-t/2}^{t/2} K_{zz}\, v_{\theta,z}\, (1+\overline{r}/a)\, \mathrm{d}\overline{r}\,. \end{aligned} \tag{3.26}$$

The definitions of the generalized stresses according to (3.14) lead to the referential expression

$$a\, \dot{n}_{\theta z} - a\, \dot{n}_{z\theta} + \dot{m}_{z\theta} = P\, a/t \int_{-t/2}^{t/2} v_{\theta,z}\, (1+\overline{r}/a)\, \mathrm{d}\overline{r}\,, \tag{3.27}$$

which can be rewritten in a spatial formulation in terms of Oldroyd derivative as

$$a\, \mathring{n}_{\theta z} - a\, \mathring{n}_{z\theta} + \mathring{m}_{z\theta} = 0\,. \tag{3.28}$$

3.3 The Flügge approximation

Equations (3.25) *are exact* and will be used in the following. However, these equations do not coincide with the corresponding equations provided by Flügge

(1981), which can be recovered as an approximation of Eqs. (3.25) as follows. If a Taylor series approximation is introduced for small thickness t of the cylinder wall

$$\begin{aligned}
&\frac{1}{t}\int_{-t/2}^{t/2} v_i \left(1+\frac{\overline{r}}{a}\right)^2 \mathrm{d}\overline{r} = \frac{1}{t}\int_{-t/2}^{t/2} v_i \left(1+\frac{\overline{r}}{a}\right) \mathrm{d}\overline{r} + \mathcal{O}((t/a)^2)\,,\\
&\frac{1}{t}\int_{-t/2}^{t/2} v_i \,\frac{\overline{r}}{a}\left(1+\frac{\overline{r}}{a}\right) \mathrm{d}\overline{r} = \mathcal{O}((t/a)^2)\,,
\end{aligned} \tag{3.29}$$

the equations given in Flügge (1981) are recovered

$$\begin{cases}
a^2\,\mathring{n}_{zz,z} + a\,\mathring{n}_{z\theta,\theta} + P\,a^2/t \displaystyle\int_{-t/2}^{t/2} v_{z,zz}\;(1+\overline{r}/a)\;\mathrm{d}\overline{r} = 0\,,\\
a\,\mathring{n}_{\theta\theta,\theta} + a^2\,\mathring{n}_{\theta z,z} - \mathring{m}_{\theta\theta,\theta} - a\,\mathring{m}_{\theta z,z} + \\
\qquad + P\,a^2/t \displaystyle\int_{-t/2}^{t/2} v_{\theta,zz}\;(1+\overline{r}/a)\;\mathrm{d}\overline{r} = 0\,,\\
\mathring{m}_{\theta\theta,\theta\theta} + a\,\mathring{m}_{\theta z,\theta z} + a^2\,\mathring{m}_{zz,zz} + a\,\mathring{m}_{z\theta,z\theta} + a\,\mathring{n}_{\theta\theta} + \\
\qquad - P\,a^2/t \displaystyle\int_{-t/2}^{t/2} v_{r,zz}\;(1+\overline{r}/a)\;\mathrm{d}\overline{r} = 0\,.
\end{cases} \tag{3.30}$$

In addition to the above equations, Flügge adds Eq. (3.28) as a sixth equation. Note that Flügge never explicitly mentions the Oldroyd increment nor the Kirchhoff stress.

3.4 The plane stress approximation

When the wall of the cylinder is sufficiently thin, the plane stress assumption is introduced, so that the increment in the radial stress is usually assumed to be negligible. In the following we assume that this hypothesis holds true not only for the ground state, but also when bifurcation occurs, so that

$$\dot{S}_{rr}(\overline{r},\theta,z) = 0\,,\quad \overline{r}\in[-t/2;\,t/2]\,. \tag{3.31}$$

Assumption (3.31) affects significantly the constitutive relations of the material models considered in the present work. More precisely, it serves as condition to: *i)* link the radial stretch λ_r with the longitudinal stretch λ_z for the compressible material described by Eq. (1.38); *ii)* determine the incompressibility constraint π for the incompressible material models described by Eq. (1.42).

Starting with the compressible material (1.38), since $\dot{S}_{rr} = \mathring{K}_{rr}$, condition (3.31) may be expressed in terms of D_{rr}, using the increment in the first Piola-Kirchhoff stress obtained through Eq. (2.23), yielding

$$D_{rr} = -\frac{(D_{\theta\theta} + D_{zz})\,(3\kappa - 2\mu)(2\lambda_z\nu + \xi)\,\xi}{2\lambda_z\nu\left[(3\kappa - 2\mu)(2\lambda_z\,\nu + \xi) + 24\,\lambda_z\,\mu\,\nu\right]}\,, \tag{3.32}$$

where $\xi = \delta + 2\nu - 1$. Hence, substituting Eq. (3.32) back into the definition of $\mathring{\mathbf{K}}$ (2.22) yields the following expression for the out-of-diagonal components:

$$\mathring{K}_{\alpha\beta} = \frac{2\mu}{\lambda_z}\,D_{\alpha\beta}, \quad \{\alpha, \beta\} = \{r, \theta, z\},\ \alpha \neq \beta\,. \tag{3.33}$$

For the class of incompressible materials, due to incompressibility, the components of the symmetric part of the velocity gradient $\mathbf{D}$ are already linked by condition $\operatorname{tr}(\mathbf{D}) = 0$, or, equivalently

$$D_{rr} = -(D_{\theta\theta} + D_{zz})\,. \tag{3.34}$$

Therefore, it is condition (3.31) that allows for the elimination of the incremental Lagrange multiplier $\dot{\pi}$ yielding

$$\dot{\pi}_{\mathrm{nH}} = (2\mu\,D_{rr})/\lambda_z\,,$$

for the neo-Hookean material model,

$$\dot{\pi}_{\mathrm{MR}} = (2\mu\,(-1 + \beta\lambda_z^2)D_{rr}/((-1 + \beta)\lambda_z)\,,$$

for the Mooney-Rivlin material model, and

$$\dot{\pi}_{\mathrm{J}_2} = D_{zz}\frac{\partial\beta_1}{\partial\lambda_z} + \frac{2D_{rr}\beta_1}{\lambda_z} - 2\,D_{rr}\,\beta_{-1}\lambda_z + D_{zz}\,\frac{\partial\beta_{-1}}{\partial\lambda_z}\,\lambda_z^2 \tag{3.35}$$

for the J_2-deformation theory material model, respectively.

4 Kinematics of incremental deformation for a prestressed shell

4.1 A premise on the plane strain kinematics of bending

As a premise to the introduction of the incremental kinematics of the thin-walled cylinder, the Euler-Bernoulli kinematics of a beam, subject to small displacement gradients, is briefly derived, with the usual assumptions, see for instance (Love 1906).

The kinematics of the beam is chiefly described by the displacement $\bar{\mathbf{u}}(x_{01})$ of its axis, which is assumed to behave as the Euler's elastica, governing the kinematics of an extensible line. Employing this theory, the unit vector $\bar{\mathbf{n}}$, normal (counterclockwise rotated by $\pi/2$ with respect to the tangent) to the deformed line at the coordinate x_{01}, which singles out a point of the same line in the straight reference configuration is (Bigoni 2012; Bigoni 2019)

$$\bar{\mathbf{n}}(x_{01}) = \frac{1}{\sqrt{(1+\bar{u}_1')^2 + \bar{u}_2'^2}} \left(-\bar{u}_2'\, \mathbf{e}_1 + (1+\bar{u}_1')\, \mathbf{e}_2\right). \tag{4.1}$$

To introduce the thickness of a beam, the following displacement field is postulated (see Fig. 4.1)

$$\mathbf{u}(x_{01}, x_{02}) = \bar{\mathbf{u}}(x_{01}) + x_{02}\, \bar{\mathbf{n}}(x_{01}) - x_{02}\, \mathbf{e}_2\,, \tag{4.2}$$

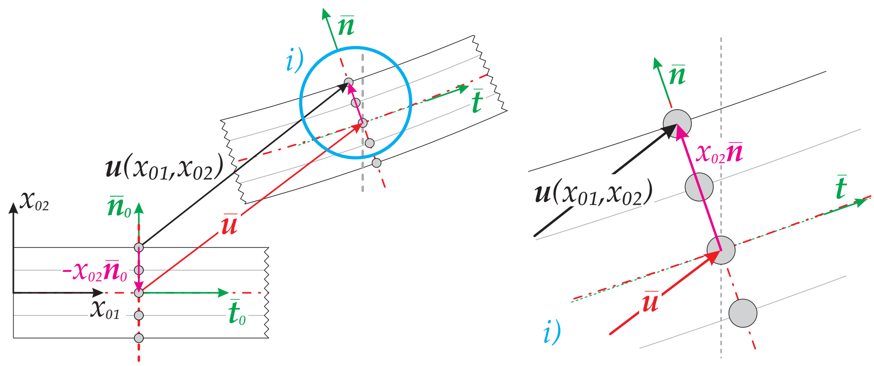

Fig. 4.1: *The kinematics of plane strain bending of a beam under the hypothesis of small displacement gradient*

where $x_{02} \in [-t/2, +t/2]$ and $\bar{\mathbf{u}}(x_{01})$ represents now the displacement of the beam's centroid, with unit normal $\bar{\mathbf{n}}(x_{01})$ given by Eq. (4.1). Component-wise the displacement field (4.2) reads as

$$
\begin{aligned}
u_1(x_{01}, x_{02}) &= \bar{u}_1 - x_{02} \frac{\bar{u}_2'}{\sqrt{(1+\bar{u}_1')^2 + \bar{u}_2'^2}}\,, \\
u_2(x_{01}, x_{02}) &= \bar{u}_2 + x_{02} \frac{1 + \bar{u}_1' - \sqrt{(1+\bar{u}_1')^2 + \bar{u}_2'^2}}{\sqrt{(1+\bar{u}_1')^2 + \bar{u}_2'^2}}\,,
\end{aligned}
\tag{4.3}
$$

so that where if the quantities $(\bar{u}_2')^2$ and $(\bar{u}_1')^2$ and $\bar{u}_2'\bar{u}_1'$ are assumed to be negligible compared to the unity, the linearized kinematics of beams is recovered

$$
u_1(x_{01}, x_{02}) \approx \bar{u}_1 - \bar{u}_2'\, x_{02}, \qquad u_2(x_{01}) \approx \bar{u}_2\,. \tag{4.4}
$$

4.2 The kinematics of the shell

The kinematics of a prestressed shell is introduced as an extension of the above-introduced rules for the beam, in other words, following the usual assumptions, discussed, among many others, by Love (1906), Flügge (1932), Podio-Guidugli (1989), Steigmann and Ogden (2014), and Geymonat et al. (2007). In particular, in a polar coordinate system, the undeformed and deformed shell geometry is expressed respectively as (Malvern 1969; Ogden 1984; Chapelle and Bathe 2011)

$$\mathbf{x}_0 = a\,\mathbf{e}_r + z\,\mathbf{e}_z\,, \quad \mathbf{x} = \left(a + \bar{v}_r\right)\mathbf{e}_r + \bar{v}_\theta\,\mathbf{e}_\theta + \left(z + \bar{v}_z\right)\mathbf{e}_z\,, \tag{4.5}$$

where a is the radius of the undeformed cylindrical surface, $\bar{v}_r$, $\bar{v}_\theta$ and $\bar{v}_z$ are its displacement components. Note that unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of the angular coordinate θ.

The unit normal to the deformed surface is defined as

$$\mathbf{n} = \frac{\mathbf{x}_{,\theta} \times \mathbf{x}_{,z}}{\left|\mathbf{x}_{,\theta} \times \mathbf{x}_{,z}\right|}\,, \tag{4.6}$$

where

$$\begin{aligned} \mathbf{x}_{,\theta} &= a\left[\frac{\bar{v}_{r,\theta} - \bar{v}_\theta}{a}\,\mathbf{e}_r + \left(1 + \frac{\bar{v}_r + \bar{v}_{\theta,\theta}}{a}\right)\mathbf{e}_\theta + \frac{\bar{v}_{z,\theta}}{a}\,\mathbf{e}_z\right], \\ \mathbf{x}_{,z} &= \bar{v}_{r,z}\,\mathbf{e}_r + \bar{v}_{\theta,z}\,\mathbf{e}_\theta + \left(1 + \bar{v}_{z,z}\right)\mathbf{e}_z\,. \end{aligned} \tag{4.7}$$

The vectors defined in (4.7) are parallel to the vectors

$$\begin{aligned} \hat{\mathbf{x}}_{,\theta} &= \frac{\left(\bar{v}_{r,\theta} - \bar{v}_\theta\right)/a}{1 + \left(\bar{v}_r + \bar{v}_{\theta,\theta}\right)/a}\,\mathbf{e}_r + \mathbf{e}_\theta + \frac{\bar{v}_{z,\theta}/a}{1 + \left(\bar{v}_r + \bar{v}_{\theta,\theta}\right)/a}\,\mathbf{e}_z\,, \\ \hat{\mathbf{x}}_{,z} &= \frac{\bar{v}_{r,z}}{1 + \bar{v}_{z,z}}\,\mathbf{e}_r + \frac{\bar{v}_{\theta,z}}{1 + \bar{v}_{z,z}}\,\mathbf{e}_\theta + \mathbf{e}_z\,, \end{aligned} \tag{4.8}$$

which, to the leading-order in the components of the incremental displacement gradient, are approximated by

$$\tilde{\mathbf{x}}_{,\theta} = \frac{\left(\bar{v}_{r,\theta} - \bar{v}_\theta\right)}{a}\,\mathbf{e}_r + \mathbf{e}_\theta + \frac{\bar{v}_{z,\theta}}{a}\,\mathbf{e}_z, \quad \tilde{\mathbf{x}}_{,z} = \bar{v}_{r,z}\,\mathbf{e}_r + \bar{v}_{\theta,z}\,\mathbf{e}_\theta + \mathbf{e}_z. \tag{4.9}$$

Note that equations (4.8) are approximated to (4.9), because the gradient of incremental displacement is assumed to be small. The parallelism (to the leading order) of vectors (4.9) with $\hat{\mathbf{x}}_{,\theta}$ and $\hat{\mathbf{x}}_{,z}$ can be proven by taking the cross product. Hence, the leading order approximation of the unit normal to the deformed cylindrical surface follows from $\tilde{\mathbf{x}}_{,\theta} \times \tilde{\mathbf{x}}_{,z}$ as

$$\mathbf{n} = \mathbf{e}_r + \frac{\bar{v}_\theta - \bar{v}_{r,\theta}}{a}\,\mathbf{e}_\theta - \bar{v}_{r,z}\,\mathbf{e}_z\,. \tag{4.10}$$

From the knowledge of the unit normal (4.10) and analogously to the beam theory approach, Eq. (4.2), the kinematics of a cylindrical shell can be assumed as (Chapelle and Bathe 2011)

$$\mathbf{v}(\bar{r}, \theta, z) = \bar{\mathbf{v}}(\theta, z) + \bar{r}\,\mathbf{n} - \bar{r}\,\mathbf{e}_r\,, \tag{4.11}$$

where $\bar{\mathbf{v}} = \mathbf{v}(\bar{r} = 0, \theta, z)$.

Based on the above simplified kinematics, the gradient of incremental displacements for the deformed shell is

$$\nabla\mathbf{v} = \begin{bmatrix} 0 & (\bar{v}_{r,\theta} - \bar{v}_\theta)/a & \bar{v}_{r,z} \\ \dfrac{\bar{v}_\theta - \bar{v}_{r,\theta}}{a} & \dfrac{1}{a+\bar{r}}\left(\bar{v}_r - \dfrac{\bar{r}}{a}\,\bar{v}_{r,\theta\theta} + \left(1 + \dfrac{\bar{r}}{a}\right)\bar{v}_{\theta,\theta}\right) & \left(1 + \dfrac{\bar{r}}{a}\right)\bar{v}_{\theta,z} - \dfrac{\bar{r}}{a}\,\bar{v}_{r,\theta z} \\ -\bar{v}_{r,z} & \dfrac{1}{a+\bar{r}}\left(\bar{v}_{z,\theta} - \bar{r}\,\bar{v}_{r,\theta z}\right) & \bar{v}_{z,z} - \bar{r}\,\bar{v}_{r,zz} \end{bmatrix},$$

so that the components of the Eulerian incremental strain $\mathbf{D}$ follow

$$\begin{aligned} &D_{rr} = D_{r\theta} = D_{rz} = 0\,, \\ &D_{\theta\theta} = \frac{1}{a+\bar{r}}\left(\bar{v}_r + (1+\bar{r}/a)\,\bar{v}_{\theta,\theta}\right) - \frac{\bar{r}}{a\,(a+\bar{r})}\,\bar{v}_{r,\theta\theta}\,, \quad D_{zz} = \bar{v}_{z,z} - \bar{r}\,\bar{v}_{r,zz}\,, \\ &D_{\theta z} = \frac{a}{2\,(a+\bar{r})}\left[\left(1+\frac{\bar{r}}{a}\right)^2 \bar{v}_{\theta,z} + \frac{\bar{v}_{z,\theta}}{a} - \frac{\bar{r}}{a}\left(1+\frac{\bar{r}}{a}\right)\bar{v}_{r,\theta z}\right]. \end{aligned} \tag{4.12}$$

An use of the constitutive equations (3.33) yields the non-zero components of $\mathring{\mathbf{K}}$, obtained under the hypotheses (3.10) and (3.32)

$$\begin{aligned} &\mathring{K}_{\theta z} = E^* D_{\theta z}/\lambda_z\,, \quad \mathring{K}_{\theta\theta} = E^*\,\left(\zeta_1\,D_{\theta\theta} + \zeta_2\,D_{zz}\right)/\gamma\,, \\ &\mathring{K}_{zz} = E^*\,\left(\zeta_2\,D_{\theta\theta} + \zeta_1\,D_{zz}\right)/\gamma\,, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} &E^* = E/(1+\nu)\,, \\ &\delta = \left(4\,\lambda_z^2\,\nu\,(2-3\nu) + 4\,\lambda_z\,\nu\,(2\nu-1) + (1-2\nu)^2\right)^{1/2}, \\ &\gamma = 2\,\lambda_z^2\,\nu\,\left[\,\lambda_z(3\nu-2) - 2\nu + 1\right], \\ &\zeta_1 = 4\,\lambda_z^2\,\nu(3\nu-2) + 4\,\lambda_z\,\nu(1-2\nu) - (2\nu-1)(\delta+2\nu-1)\,, \\ &\zeta_2 = 2\,\lambda_z^2\,\nu(3\nu-2) + 2\,\lambda_z\,\nu(1-2\nu) - (2\nu-1)(\delta+2\nu-1)\,. \end{aligned}$$

Similar reasoning are applied to the other material models considered in this work.

5 Summary of the equations governing the incremental deformation of an axially-stretched, thin-walled cylinder

The results obtained so far are briefly summarized in this chapter. It has been shown how to derive generalized (incremental) equilibrium equations starting from the local (incremental) equilibrium conditions. This method applies also to finite equilibrium equations: for example, in a Cartesian coordinate systems, the equilibrium equations for a beam can be recovered (for a similar approach see Janečka et al. 2016). Once the generalized stresses are defined either in terms of the increment of the first Piola-Kirchhoff stress tensor or by means of another suitable stress increment, the next step consists in the choice of a strain energy function to relate (incremental) stresses and (incremental) strains: the procedure shown in the previous chapter has been applied to different hyperelastic materials and as such can be defined as *material-independent*. To reduce the complexity of the problem, a suitable kinematics was chosen in order to express the rates of deformation at each material point as functions of their corresponding projections onto the mid-surface of the continuum. The plane stress approximations was then assumed to be valid as the shell thickness was assumed to be thin.

Next, the final form of the incremental equilibrium equations for the nearly incompressible neo-Hookean material (1.38) is offered, as a result of the above described steps

$$\begin{cases} 12\,\nu\,\tau\,\zeta_3\,\bar{v}_{r,z} - a^2\,(\tilde{p}_z + \zeta_6)\,\tau^3\,\bar{v}_{r,zzz} + 6\,(\zeta_5 + 1)\,\zeta_4\,\bar{v}_{r,\theta\theta z} + \\ \quad + 12\,a\,(\tilde{p}_z + \zeta_6)\,\tau\,\bar{v}_{z,zz} + 6\,(\zeta_4 + \tau)(\zeta_5 + 1)\,\bar{v}_{z,\theta\theta}/a + \\ \quad + 6\,\left((3\,\zeta_3 - 2)\,\nu + 1\right)\,\tau\,\bar{v}_{\theta,\theta z} = 0\,, \\ 24\,\zeta_7\,\bar{v}_{r,\theta}/a + a\,(1-\nu)(4\,\tilde{p}_z + (5\,\zeta_3 - 6)\,\nu + 3)\,\tau^2\,\bar{v}_{r,\theta zz} + \\ \quad + 12\,(1-\nu)((3\,\zeta_3 - 2)\,\nu + 1)\,\bar{v}_{z,\theta z} + \\ \quad + 3\,a\,(2\,\tilde{p}_z - \nu + 1)((\zeta_3 - 2)\,\nu + 1)\left(\tau^2 + 4\right)\,\bar{v}_{\theta,zz} + \\ \quad + 24\,\zeta_7\,\bar{v}_{\theta,\theta\theta}/a = 0\,, \\ 24\,(\zeta_4 + \tau)\,\zeta_6\,\bar{v}_r/a - 24\,a\,\tilde{p}_z\,\tau\,\bar{v}_{r,zz} + \\ \quad + 2\,a^3\,(\tilde{p}_z + \zeta_6)\,\tau^3\,\bar{v}_{r,zzzz} + 48\,\zeta_4\,\zeta_6\,\bar{v}_{r,\theta\theta}/a + \\ \quad + a\,\left((2\tilde{p}_z + (7\,\zeta_3 - 6)\,\nu + 3)\tau^3 + 12\,\zeta_4\,\zeta_5\right)\,\bar{v}_{r,\theta zz} + \\ \quad + 24\,\zeta_4\,\zeta_6\,\bar{v}_{r,\theta\theta\theta}/a + 24\,\zeta_3\,\nu\,\tau\,\bar{v}_{z,z} + \\ \quad - 2\,a^2\,(\tilde{p}_z + \zeta_6)\,\tau^3\,\bar{v}_{z,zzz} + 12\,\zeta_4\,\zeta_5\,\bar{v}_{z,\theta\theta z} + 24\,\tau\,\zeta_6\,\bar{v}_{\theta,\theta}/a + \\ \quad - a\,(4\,\tilde{p}_z + (5\,\zeta_3 - 6)\,\nu + 3)\,\tau^3\,\bar{v}_{\theta,\theta zz} = 0\,, \end{cases} \tag{5.1}$$

where

$$\zeta_3 = \lambda_r^2\,\lambda_z\,, \quad \zeta_4 = \ln\left(\frac{2+\tau}{2-\tau}\right) - \tau\,, \quad \zeta_5 = (-2 + \zeta_3)\,\nu\,,$$
$$\zeta_6 = 2\,\nu(\zeta_3 - 1) + 1\,, \quad \zeta_7 = (1-\nu)\,\zeta_6\,, \quad \tilde{p}_z = \zeta_3\,p_z\,,$$

and λ_r satisfies condition (3.10).

Similar equations can be written for the neo-Hookean, Mooney-Rivlin and J_2-deformation theory material model too.

6 Bifurcation of an axially-compressed, thin-walled cylinder

The equations summarized in Ch. 5 allow one to solve various problems involving axially prestressed circular cylinders. For instance, the solution for the bending of a long cylindrical shell by a load uniformely distributed along a circular section (§115 of Timoshenko and Woinowsky-Krieger 1959). Here the equations are used to solve for the bifurcation of an axially-compressed, thin-walled cylinder.

6.1 *Ansatz* for the velocity field at bifurcation

Introducing the longitudinal wave-number $\eta = m\pi a/l$, where a is the shell mid-radius, l is the shell length and m is the number of half-waves in the longitudinal direction, the following *ansatz* is assumed for the velocity field at bifurcation

$$\begin{cases} \bar{v}_r(\theta, z) = c_1 \cos(n\,\theta) \cos(\eta\, z/a)\,, \\ \bar{v}_\theta(\theta, z) = c_2 \sin(n\,\theta) \cos(\eta\, z/a)\,, \\ \bar{v}_z(\theta, z) = c_3 \cos(n\,\theta) \sin(\eta\, z/a)\,, \end{cases} \tag{6.1}$$

where n is the number of full-waves along the circumferential direction. We highlight that each component of the velocity field is assumed to be constant throughout the shell thickness. The final form of the three incremental equilibrium equations (3.25) can be obtained by substituting Eq. (6.1) into the kinematics (4.12) and then into the constitutive relations (4.13). Definitions (3.14) and (3.15) are then used to calculate the generalized stress forces and moments. Eventually, the bifurcation problem assumes the standard form

$$\mathbf{M} \cdot \mathbf{c} = \mathbf{0}\,, \tag{6.2}$$

where the matrix $\mathbf{M}$ depends on the wave numbers n and η, on the ratio $\tau = t/a$, on λ_z, the axial stretch and on the dimensionless axial load $p_z = P/D$, where $D = Et/(1-\nu^2)$ is the shell extensional rigidity. The latter quantity may vary depending on the assumed material model: for the incompressible neo-Hookean and the Mooney-Rivlin materials, $D = 4/(3\,E\,t)$, i.e. the limit of the shell axial rigidity for the case $\nu = 1/2$, whilst for the J_2-deformation theory material $D = K\,t$, where K is the constitutive parameter of the model. Finally, $\mathbf{c} = \{c_1, c_2, c_3\}$ is the vector of the bifurcation mode amplitudes.

The *ansatz* is such that the incremental displacement field automatically satisfies the boundary conditions of null incremental moments at the ends $z = 0$ and $z = l$. These boundary conditions also correspond to the free sliding conditions against a smooth rigid constraint, standard in incremental bifurcation of elastic solids (Hill and Hutchinson 1975). Bifurcation occurs when determinant of the coefficient matrix vanishes

$$\det \mathbf{M} = 0\,, \tag{6.3}$$

a relation defining the critical stretch λ_z (and from this also the dimensionless load p_z) for bifurcation as a function of the wave numbers n and η, and the geometrical parameter τ.

6.2 Bifurcation diagrams

We start by illustrating some results for the nearly incompressible neo-Hookean material. Fig. 6.1 shows an example of a buckling diagram obtained for $\nu = 0.3$ and $r_e/r_i = 1.05$. The critical axial stretch is plotted against the longitudinal wave-number η for different values of the circumferential wave-number n. The dashed lines represent modes at high axial stress that cannot spontaneously be reached by the system. Critical modes are represented by the continuous lines.

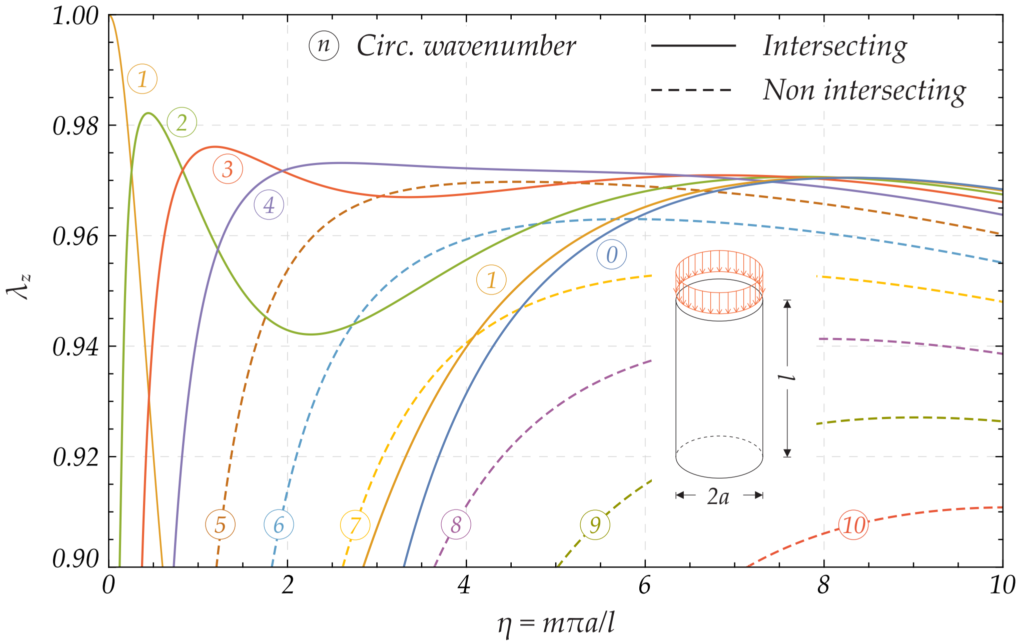

Fig. 6.1: *Critical stretch λ_z for bifurcation of an axially compressed thin-walled cylinder, as a function of the longitudinal wave-number η, for different values of the circumferential wave number n. Nearly incompressible neo-Hookean material ($\nu = 0.3$), $r_e/r_i = 1.05$. Continuous lines: intersecting critical modes used to build the buckling envelope; dashed lines: higher load modes. The mode with $n = 1$ corresponds to the Euler mode.*

As expected, for very slender cylinders, i.e. for small values of η, the mode with $n = 1$ corresponding to Euler buckling becomes dominant.

The lower envelopes of the dimensionless buckling load of the intersecting modes is shown in figure Fig. 6.2 for different ratios r_e/r_i. The bilogarithmic plot shows the existence of three different behaviours. On the right side of the plot, i.e. for very slender cylinders, the Euler's buckling load is clearly approached: a deeper investigation on this issue is offered in §7.2. On the left side of the plot, the critical load of a plate provided by Flügge (1973) and Timoshenko and Gere (1961) as

$$p_{z,\text{plate}} = k\pi^2 \left(\frac{m\,a}{l}\right)^2, \tag{6.4}$$

where $k = E\,t^3/(12\,(1-\nu^2))$, has been used in Fig. 6.2. The figure shows that the plate theory (6.4) provides a nice approximation to the critical loads in a range at large η. In particular, the gray region represents the intersection points between the plate buckling load and the solution to the shell buckling problem, a region where the plate approximation prevails; it is noted that the thinner is the shell, the wider is the range for which the plate approximation is valid. A third regime can be recognized in Fig. 6.2, namely, the intermediate region where the buckling load is constant and independent of both the circumferential wave-number n and the longitudinal wave-number η. The geometrical parameters characterizing this region identify the so-called 'mid-long' cylinders and will be analyzed in §7.1. The presented solution refers to the equations (5.1) introduced in the

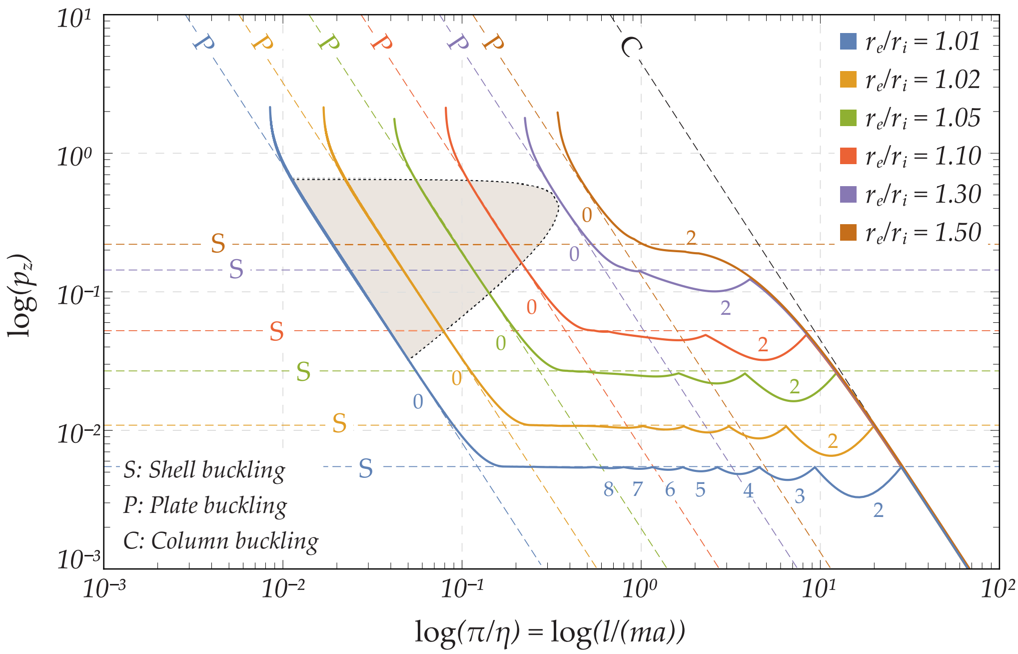

Fig. 6.2: *Lower envelopes of the dimensionless critical load p_z for the buckling of a cylindrical thin-walled cylinder, as functions of the longitudinal wave-number η for different ratios r_e/r_i. Nearly incompressible neo-Hookean material ($\nu = 0.3$). Dashed lines: asymptotic buckling load for 'mid-long' shells, plates, Euler's column. The small numbers represents the circumferential number n.*

previous chapter. The same bifurcation problem can be solved using Flügge's equations presented in §3.3 in combination to the incremental constitutive law (3.13) (cp. Flügge 1973) and a comparison between the two (not reported) shows that the results are practically coincident.

The accuracy of the method so far presented can be assessed through a comparison with the 3D full-field solution available for the incompressible materials (Bigoni and Gei 2001). However, for the Pence and Gou constitutive model (1.38) an *ad-hoc* full-field solution will be presented in Ch. 8.

Diagrams similar to Fig. 6.2 can be obtained for the Mooney-Rivlin material and for the J_2-deformation theory material too. For the case of incompressible materials, the accuracy of the reduced model solution proposed in the present work is validated by superimposing the results on the 3D full field solution offered in Bigoni and Gei (2001). Their results refer to the bucking of a coated cylinder subjected to axial compression. The 3D solution shown in the next figures has been calculated for the case in which the rigidity of the core tends to vanish, i.e. the coating can be interpreted as a hollow shell made of an incompressible hyperelastic material. Fig. 6.3 shows the upper envelope of the critical logarithmic strain λ_z as a function of the wave-number η for a Mooney-Rivlin material characterized by a constitutive parameter $\beta = -0.1$. For very thin cylinders ($r_e/r_i = 1.01$), the difference between the 2D and the 3D solution cannot be seen. Furthermore, the diagram clearly shows that for very thin shells, except when the slenderness approaches infinity, the critical stretch remains almost constant and independent of the circumferential wave-number n, Such behavior will be investigated in a more systematic way in §7.1. As the thickness increases, the difference in the critical stretch becomes bigger. Interestingly enough, though, for very slender shells, i.e. for very small values of η, mode 1 (Euler buckling) and mode 2 are practically superimposed to the 3D full-field solution. Similar results with the same peculiarities can be obtained for any value of the constitutive parameter β.The particular case for $\beta = 0$ corresponding to the case of an incompressible neo-Hookean material yields similar results and, therefore, the corresponding buckling diagram is not reported here.

Fig. 6.4 pertains to the other incompressible material analyzed in this work,

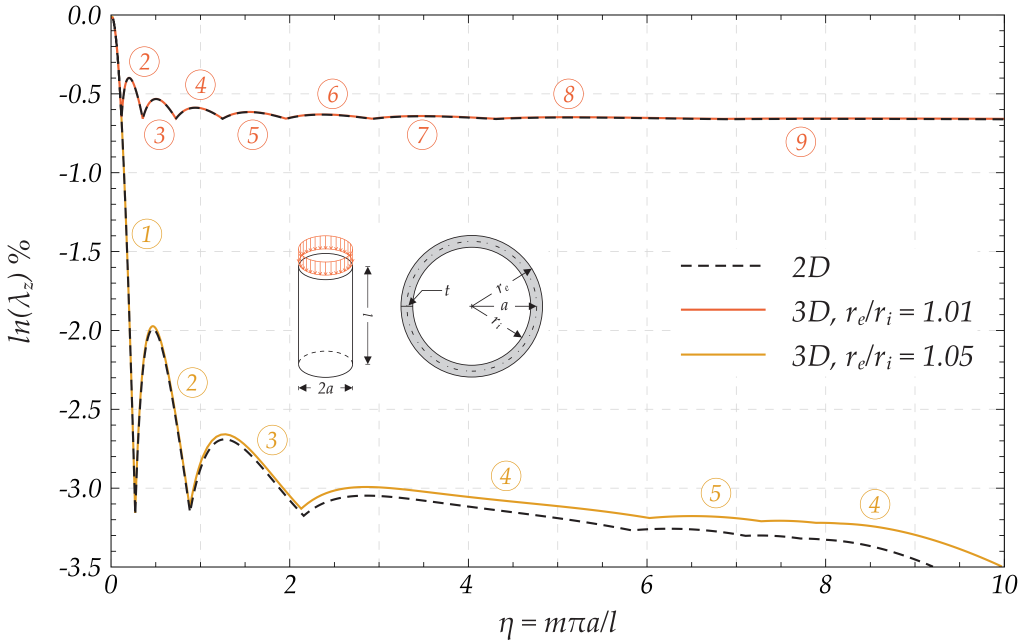

Fig. 6.3: *Upper envelopes of the critical logarithmic strain λ_z as functions of the longitudinal wave-number η for different ratios r_e/r_i. Comparison between 3D solution and the proposed reduced model solution for different. Mooney-Rivlin material, $\beta = -0.1$. The circled numbers corresponds to the circumferential wave-number n.*

namely the J_2-deformation theory material. Similar features as for the Mooney-Rivlin material can be seen: again, for very thin cylinders, the reduced and full-field solutions are indistinguishable. Interestingly enough, the reduced model is also able to catch bifurcation in tension, as shown in Fig. 6.5. As visible from the diagram, the 2D solution behaves as if the critical stretch obtained by means of the reduced model was unaffected by the ratio between outer and inner radius, i.e. from the shell-thickness. On the contrary, the critical stretch of the full-field solution increases as the the shell becomes thicker.

Fig. 6.6 shows a selection of eigenmodes arising at buckling for different values of circumferential wave-number n and different values of longitudinal half-waves m. The second column offers a 3D view of the eigenmode, the third column

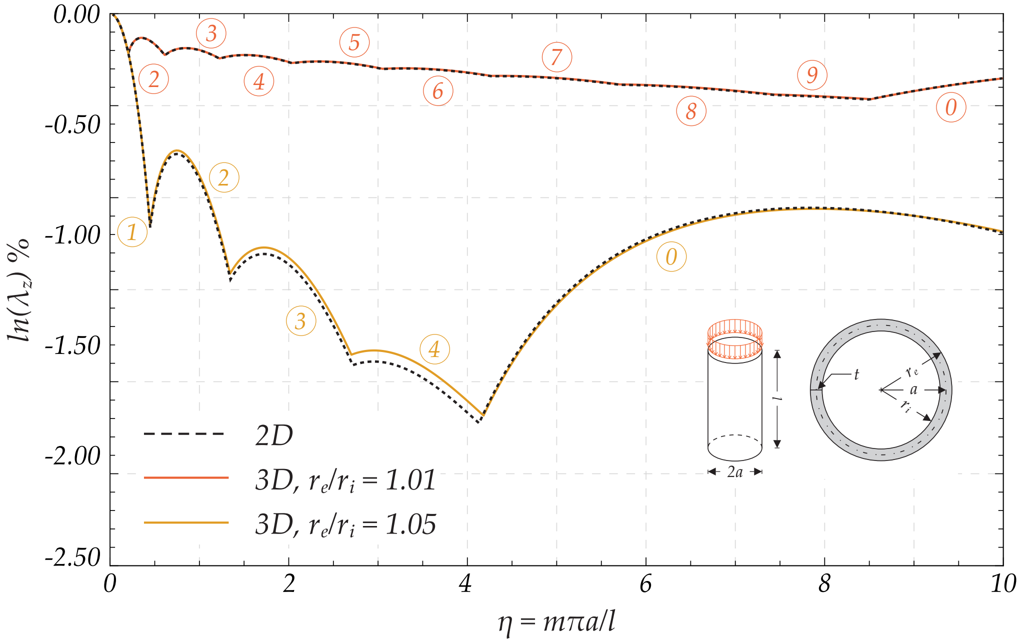

Fig. 6.4: *Upper envelopes of the critical logarithmic strain λ_z as functions of the longitudinal wave-number η for different ratios r_e/r_i. Comparison between 3D solution and the proposed reduced model solution for different. J_2-deformation theory material, N = 0.1, compression. The circled numbers corresponds to the circumferential wave-number n.*

reports 2D sections of the deformed shell and the fourth column a 2D section of the undeformed and deformed geometry. The color map helps visualizing the peculiar 'bulges' of a buckled shell.

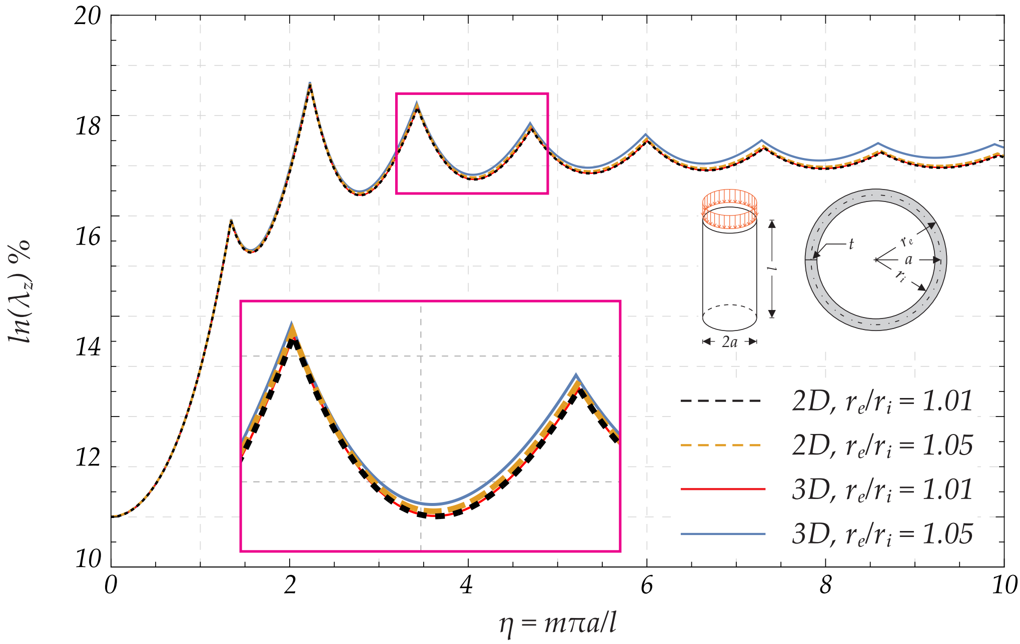

Fig. 6.5: *Lower envelopes of the critical logarithmic strain λ_z as functions of the longitudinal wave-number η for different ratios r_e/r_i. Comparison between 3D solution and the proposed reduced model solution for different. J_2-deformation theory material, $N = 0.1$, tension. The circled numbers corresponds to the circumferential wave-number n.*

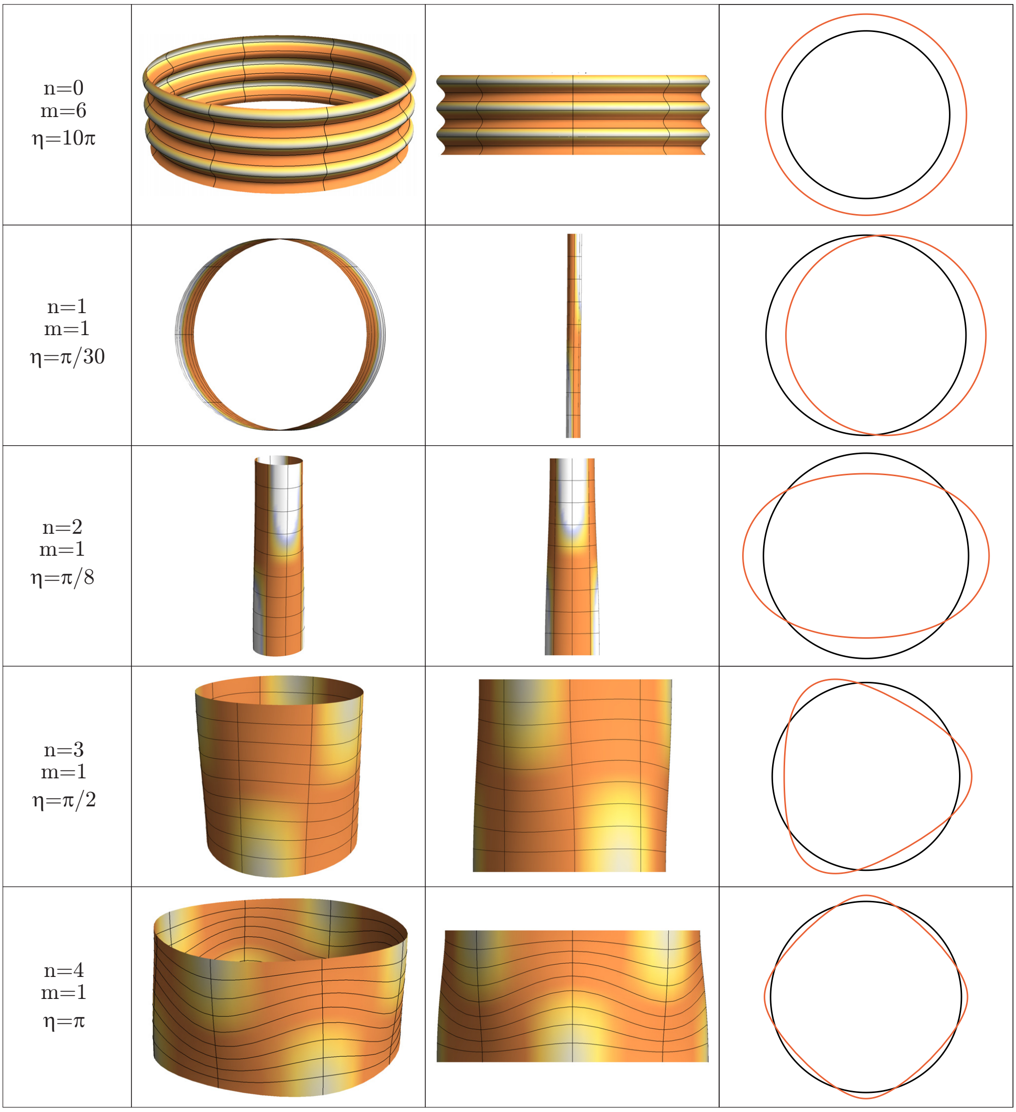

Fig. 6.6: *Example of buckling eigenmodes. Nearly incompressible neo-Hookean material ($\nu = 0.3$), $r_e/r_i = 1.05$.*

7 Asymptotic analyses for mid long, infinitely long, and infinitesimally short cylinders

Three important limit cases are analyzed in this Section, namely, the case of 'mid-long', infinitely long, and infinitesimally short cylindrical shells. Here the famous formula for mid-long cylinders obtained by Flügge and also the buckling of a cylindrical Euler beam are rigorously derived from the finite elasticity solution developed in the present work. The limit solution for the rectangular plate is obtained only numerically.

7.1 Buckling load of a 'mid-long' shell

Flügge noticed from bifurcation diagrams similar to those reported in Fig. 6.2 that the lower envelopes display a range of buckling load almost independent of the values of both wave-numbers n and η. This portion of the bifurcation diagram can be approximated by a straight line, with an error decreasing when the shell-thickness decreases, so that Flügge (1973) derived the formula

$$p_{z,\text{Flügge}} = \sqrt{\frac{1-\nu^2}{3}}\,\tau_0\,, \tag{7.1}$$

where $\tau_0 = (t/a)_0$, which approximates the critical load of 'mid-long' shells, made up of a linear elastic isotropic material. Now the same formula is rigorously obtained from the theory of finite elasticity so far implemented applied to the nearly incompressible neo-Hookean material model (1.38).

A Taylor-series expansion of Eq. (6.3) for $n = 0$ about $\tau = 0$ up to the order 4 and an expansion about $\lambda_z = 1$, truncated at the linear term, are performed. The expansions lead to an approximate expression for the determinant (6.3)

$$d(\tau, \nu, \tau, \eta, \lambda_z) = c_2\,\tau^2 + c_4\,\tau^4 + \mathcal{O}(\tau^5)\,, \tag{7.2}$$

where $d = \det(\mathbf{M})$ and

$$\begin{aligned}
c_2 &= -576\,\zeta_3\,(-1+\nu^2+(\lambda_z-1)\,(-3-4\nu+25\nu^2+\\
&\quad -16\nu^3+\eta^2(-1+\nu^2)))+\mathcal{O}(\lambda_z^2)\,,\\
c_4 &= -48\,\zeta_3\,(-4-\eta^4+2\eta^2\nu+3\nu^2+(\lambda_z-1)(2\eta^4\,(-2-2\nu+9\nu^2)+\\
&\quad -3\eta^2(1-4\nu+3\nu^2+6\nu^3)-4(3+4\nu-23\nu^2+12\nu^3)))+\\
&\quad +\mathcal{O}(\lambda_z^2)\,,\\
\zeta_3 &= \eta^4\,(2\nu^2-3\nu+1)^3\,.
\end{aligned}$$

Eq. (7.2) can be explicitly solved in λ_z as a function of the wave-number η and the other parameters ν and τ yielding

$$\lambda_z = \zeta_{14}/\zeta_{15}\,, \tag{7.3}$$

where

$$\begin{aligned}
\zeta_{14} &= 12(-2+\eta^2(-1+\nu^2)-4\,\nu\,(1-6\,\nu+4\,\nu^2))+\\
&\quad +(-8+\nu(-16+(89-48\,\nu)\nu)+\eta^4\,(-3+2\,\nu(-2+9\,\nu))+\\
&\quad -\eta^2\,(3+\nu\,(-10+9\,\nu\,(1+2\,\nu))))\,\tau^2\,,\\
\zeta_{15} &= 12(-3+\nu\,(-4+(25-16\,\nu)\nu)+\eta^2\,(-1+\nu^2))+\\
&\quad +(2\eta^4\,(-2+\nu\,(-2+9\,\nu))-3\,\eta^2(1+\nu\,(-4+3\,\nu+6\,\nu^2))+\\
&\quad -4(3+\nu\,(4+\nu\,(-23+12\,\nu))))\,\tau^2\,.
\end{aligned}$$

The asymptotic buckling load of a 'mid-long' shell can be calculated by minimizing (7.3) with respect to the variable η. To do this, the first derivative of (7.3) is calculated with respect to η and set equal to zero, which leads to 5 solutions:

one is trivial, two are purely imaginary conjugated roots and two are real and of opposite sign. From the latter pair, the following positive real root is selected

$$\bar{\eta} = \left(\frac{2\,\zeta_{17} + \sqrt{\zeta_{18}}}{2\,\zeta_{16}} \right)^{1/2}, \tag{7.4}$$

where

$$\begin{aligned}
\zeta_{16} &= 12\,(1-\nu^2)\,\tau^2 - (18\,\nu^3 - 17\,\nu^2 + 4\,\nu - 3)\,\tau^4\,,\\
\zeta_{17} &= 12\,(1 + 3\,\nu^2 - 20\,\nu^3 + 18\,\nu^4)\,\tau^2 + (4 + 8\,\nu^2 - 60\,\nu^3 + 54\,\nu^4)\,\tau^4\,,\\
\zeta_{18} &= 4\,(4\,\tau^4\,(2\,(3+\tau^2) + \nu^2\,(18 + 3\,(4+\tau^2)\,\nu(-10+9\,\nu) + 4\,\tau^2))^2 +\\
&\qquad + \zeta_{16}\,(24\,(3+\tau^2)\,\nu\,\tau^2 + 54\,(4+\tau^2)\,\nu^5\,\tau^2 + 4\,\nu^3\,\tau^2(60+19\,\tau^2) +\\
&\qquad + \nu^2\,(288 - 84\,\tau^2 - 59\,\tau^4) - 12\,(12 + 7\,\tau^2 + \tau^4) +\\
&\qquad - 3\,\nu^4(48 + 104\,\tau^2 + 23\,\tau^4)))\,.
\end{aligned}$$

A substitution of equation (7.4) into equation (7.3) yields the asymptotic dimensionless buckling load

$$p_z = p_z(\nu, \tau, n = 0, \eta = \bar{\eta}, \lambda_z = \lambda_z(\bar{\eta}))$$

for a 'mid-long' shell (positive if compressive)

$$p_z = \sqrt{\frac{1-\nu^2}{3}}\,\tau - \frac{\nu\,(10\nu^3 + 23\nu^2 - 49\nu + 22)}{6\,(1-\nu^2)}\,\tau^2 + \mathcal{O}(\tau^3)\,. \tag{7.5}$$

To the leading order in τ, Eq. (7.5) obtained for the nearly incompressible neo-Hookean material model (1.38), provides Eq. (7.1), which is now rigorously determined.

The same method has been applied to the other constitutive models, yielding

$$p_{z,\mathrm{nH}} = \frac{\tau}{2} \tag{7.6}$$

for the incompressible neo-Hookean material model. This result is consistent with Eq. (7.5) in the limit for ν that tends to 1/2. For the case of the Mooney-Rivlin material, the following result is obtained

$$p_{z,\mathrm{MR}} = \frac{\tau}{2}\left(1 - \frac{1+\beta}{1-\beta}\,\tau\right), \tag{7.7}$$

which shows the dependence of the dimensionless critical load also on the constitutive parameter β. For $\beta = 0$, the asymptotic critical load (7.6) for the neo-Hookean material is recovered.

For the J_2-deformation theory material, no explicit formula for the buckling of a 'mid-long' shell was found.

7.2 Infinitely long cylinders, the Euler buckling

The Euler buckling load of a thin cylindrical shell is obtained in this Section using perturbative techniques (Dingle 1973; Golubitsky and Schaeffer 1985; Simmonds and Mann 1998; Kokotović et al. 1999; Hunter 2004; Holmes 2013; Shchepakina et al. 2014), in the limits of large bifurcation wavelength, $1/\eta \approx 0$, and small thickness of the shell, $t/a \approx 0$.

For a hollow cylinder made up of a linear elastic isotropic material (Young's modulus E), the Euler buckling load is

$$N_{\mathrm{z,Euler}} = E J \frac{\pi^2}{{l_0}^2} \,, \tag{7.8}$$

where $J = \pi(R_e^4 - R_i^4)/4$ is the moment of area of the cross section area about an axis and l_0 is the shell height, so that, denoting with the suffix '0' quantities referred to the undeformed geometry, $R_e = a_0 + t_0/2$, $R_i = a_0 - t_0/2$ and $A_0 = \pi(R_e^2 - R_i^2) = 2\pi\, a_0^2\, \tau_0$ the following expression for (7.8) is obtained

$$N_{\mathrm{z,Euler}} = E\, A_0\, \frac{\pi^2}{4}\, \alpha_0^2 \left(2 + \frac{\tau_0^2}{2}\right) , \tag{7.9}$$

where $\alpha_0 = a_0/l_0$ is the stubbiness ratio and $\tau_0 = t_0/a_0$, both quantities referred to the undeformed configuration.

Referring now to the theory given in the present work, the Euler buckling mode corresponds to $n = 1$ and to slender cylinders, when the wave-number η tends to vanish. Since (7.9) is expressed in terms of the underformed geometry, all the quantities in (6.3) are transformed to be referred to the undeformed configuration,

$$\tau = \frac{t}{a} = \frac{\lambda_r\, t_0}{\lambda_r\, a_0} = \tau_0 \,, \qquad \eta = \pi \frac{a}{l} = \pi\, \frac{\lambda_r}{\lambda_z}\, \alpha_0 \,,$$

and the circumferential wave-number is assumed equal to the unity, $m = 1$. The bifurcation condition $\det(\mathbf{M})$ becomes a function of ν, τ_0, α_0, and λ_z, so that α_0 is identified as the small parameter related to the shell-stubbiness, while the asymptotic analysis is developed about $\lambda_z = 1$. An asymptotic expansion of $\lambda(\alpha_0)$ is sought in the power series form (Goriely et al. 2008)

$$\lambda_z(\alpha_0) = 1 + \sum_{k=1}^{M} \lambda_z^{(k)}\, \alpha_0^k + \mathcal{O}(\alpha_0^M)\,. \tag{7.10}$$

It is worth recalling that the condition $\{\lambda_z, \alpha_0\} = \{1, 0\}$ corresponds to the critical load of an infinite slender shell, that buckles at vanishing load. A substitution of Eq. (7.10) into (6.3) and an expansion of $d(\lambda_z, \alpha_0) = \det(\mathbf{M})$ in a Taylor series yields

$$d(\lambda_z, \alpha_0) = \sum_{k=1}^{N} d^{(k)}\, \alpha_0^k + \mathcal{O}(\alpha_0^N) \tag{7.11}$$

To satisfy the buckling condition $d(\alpha_0) = 0$ and due to the arbitrariness of the perturbation parameter, each coefficient $d^{(k)}$ corresponding to the k-th order of α_0 must vanish, a condition yielding a system of linear equations for the unknown coefficients $\lambda_z^{(k)}$. As noticed by Goriely et al. (2008), it is found in solving the linear system of equations that the first two are identically satisfied and $\lambda_z^{(k)} = 0$ for each odd value of k. Hence, an expression for $\lambda_z(\alpha_0)$ can be generated at least of order 2 in the parameter α_0 by assuming $n = 5$. In this way, $\lambda_z^{(1)} = \lambda_z^{(3)} = 0$ is obtained, so that

$$\lambda_z^{(2)} = -\frac{\pi^2}{2}\left(1 + \frac{(3 - 2\nu^2)}{12\,(1 - \nu^2)}\,\tau_0^2\right).$$

Therefore, the asymptotic expansion (7.10), up to third order of the critical stretch λ_z becomes

$$\lambda_z \approx 1 - \frac{\pi^2}{2}\left(1 + \frac{(3 - 2\nu^2)}{12\,(1 - \nu^2)}\,\tau_0^2\right)\alpha_0^2\,. \tag{7.12}$$

Eq. (7.12) provides

$$N_z = -E\,A_0\,\frac{\pi^2}{4}\,\alpha_0^2\left(2 + \frac{\tau_0^2}{2}\,\frac{3 - 2\nu^2}{3(1 - \nu^2)}\right) + \mathcal{O}(\tau_0^4) \tag{7.13}$$

for the critical load of the shell, asymptotically derived from the finite elasticity solution.

At this point, a comparison between equation (7.9) and (7.13) (note that in the former equation the load is positive when compressive and in the latter when tensile) shows coincidence when $\nu = 0$ or when the term τ_0^2 is neglected.

8 Bifurcation of a thick cylinder

The fully three-dimensional solution for the bifurcation of a thick cylinder made up of the hyperelastic material (1.38) is derived in this Section following Chau (1995) (see also Chau (1993), Chau (1995), and Chau and Choi (1998)). The main idea of this approach is to uncouple the three equations of incremental equilibrium for the linearized bifurcation problem through the introduction of two velocity potentials, namely Φ and Ψ. Chau (1995) developed his discussion to a general class of materials, whose incremental constitutive law is expressed by a relation between the Jaumann derivative of the Cauchy stress and the symmetric part of the Eulerian incremental strain:

$$\overset{\triangledown}{\mathbf{T}} = \dot{\mathbf{S}} - (\mathrm{tr}\,\mathbf{D})\,\mathbf{T} + \mathbf{T}\,\mathbf{D} - \mathbf{W}\,\mathbf{T}\,,$$

having the following structure

$$\begin{aligned}
\overset{\triangledown}{T}_{rr} &= C_{11}\, D_{rr} + C_{12}\, D_{\theta\theta} + C_{13}\, D_{zz}\,,\\
\overset{\triangledown}{T}_{\theta\theta} &= C_{12}\, D_{rr} + C_{11}\, D_{\theta\theta} + C_{13}\, D_{zz}\,,\\
\overset{\triangledown}{T}_{zz} &= C_{31}\,(D_{rr} + D_{\theta\theta}) + C_{33}\, D_{zz}\,,
\end{aligned} \tag{8.1}$$

for the diagonal components and

$$\begin{aligned}
\overset{\triangledown}{T}_{r\theta} = \overset{\triangledown}{T}_{\theta r} &= (C_{11} - C_{12})\, D_{r\theta}\,,\\
\overset{\triangledown}{T}_{\alpha z} = \overset{\triangledown}{T}_{z\alpha} &= 2\, C_{44}\, D_{\alpha z} \quad (\alpha = r, \theta)\,,
\end{aligned} \tag{8.2}$$

for the out-of-diagonal components. For a material with constitutive equation (2.22), it can be checked that the coefficients C_{ij} have the form

$$
\begin{aligned}
C_{11} &= \kappa\left(2\lambda_r^2\lambda_z - 1\right) + \frac{\mu}{3}\left(-4\,\lambda_r^2\,\lambda_z + 3\lambda_z^{-1}\,(\lambda_r^{-2}+1) + 2\right),\\
C_{12} = C_{13} &= \kappa\,\lambda_r^2\,\lambda_z + \frac{\mu}{3\lambda_z}\left(-2\,\lambda_r^2\,\lambda_z^2 + 3(\lambda_r^{-2}-1)\right),\\
C_{44} &= \frac{\kappa}{2}\left(\lambda_r^2\,\lambda_z - 1\right) + \frac{\mu}{6}\left(3\,\lambda_z^{-1}\left(1+\lambda_r^{-2}\,\lambda_z^2\right) + 2\left(1-\lambda_r^2\,\lambda_z\right)\right),\\
C_{31} &= C_{13} + \mu\,\lambda_z^{-1}\left(1-\lambda_r^{-2}\,\lambda_z^2\right),\\
C_{33} &= \kappa\left(2\,\lambda_r^2\,\lambda_z - 1\right) + \frac{1}{3}\mu\left(3\lambda_r^{-2}\lambda_z^{-1}\left(\lambda_z^2+1\right) - 4\,\lambda_r^2\,\lambda_z + 2\right).
\end{aligned}
\tag{8.3}
$$

Note that the coefficients (8.3) depend on the amount of stretch λ_r and λ_z in the prebifurcation state, and on the material parameters κ and μ. On the basis of the incremental equilibrium equations (3.1), the following velocity potentials are introduced

$$
\begin{aligned}
v_r &= \Phi_{,rz} + \Psi_{,\theta}/r\,, \quad v_\theta = \Phi_{,\theta z}/r - \Psi_{,r}\,,\\
v_z &= -\frac{C_{11}}{C_{13}+C_{44}(1-s)}\nabla_1\Phi - \frac{C_{44}\,(1+s)}{C_{13}+C_{44}\,(1-s)}\Phi_{,zz}\,,
\end{aligned}
\tag{8.4}
$$

where

$$
\nabla_1(\,\cdot\,) = r^{-1}(r\,(\,\cdot\,)_{,r})_{,r} + r^{-2}(\,\cdot\,)_{,\theta\theta}\,, \quad \text{and} \quad s = (2\,C_{44})^{-1}\,T_{zz}, \tag{8.5}
$$

so that the incremental equilibrium equations (3.1) can be rewritten as

$$
\begin{cases}
\left(\nabla_1 - {\nu_1}^2\dfrac{\partial}{\partial z^2}\right)\left(\nabla_1 - {\nu_2}^2\dfrac{\partial}{\partial z^2}\right)\Phi = 0\,,\\
\left(\nabla_1 + {\nu_3}^2\dfrac{\partial}{\partial z^2}\right)\Psi = 0\,,
\end{cases}
\tag{8.6}
$$

where ν_1 and ν_2 are the roots of the characteristic equation

$$
A\,\nu^4 + B\,\nu^2 + C = 0\,, \tag{8.7}
$$

and

$$
\nu_3^2 = 2\,C_{44}(C_{11}-C_{12})^{-1}(1+s)\,. \tag{8.8}
$$

The coefficients A, B and C are defined as

$$
\begin{aligned}
A &= C_{11}\,C_{44}\,(1-s)\,,\\
B &= C_{11}\,C_{33} - C_{13}\,C_{31} - C_{44}\left(2\,C_{13} + (1+s)\,(C_{31}-C_{13})\right),\\
C &= C_{33}\,C_{44}\,(1+s)\,.
\end{aligned}
\tag{8.9}
$$

Depending on the the nature of the roots ν_1, ν_2, a regime classification can be introduced. For the purposes of the present study, only diffuse bifurcation modes are investigated, hence failure of ellipticity will not be considered. The elliptic regime, to which the following calculations are referred, is described by the condition (Chau 1995): $B^2 - 4\,A\,C > 0$, $A\,C > 0$, $B > 0$. Fig. 8.1 shows the dependency of the stretch level λ_z at which ellipticity is lost on the Poisson's ratio ν.

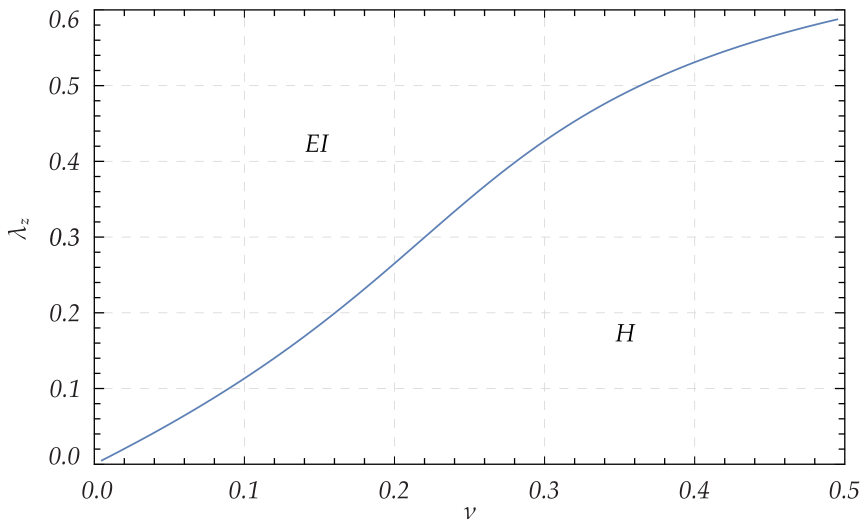

Fig. 8.1: *The EI region as a function of the Poisson's ratio ν.*

The following ansatz for the bifurcated velocity field is introduced:

$$\begin{aligned} \Phi(r,\theta,z) &= \phi(r)\,\sin(\eta\,z)\cos(n\,\theta)\,,\\ \Psi(r,\theta,z) &= \psi(r)\,\cos(\eta\,z)\sin(n\,\theta)\,, \end{aligned} \tag{8.10}$$

where the symbols n and η have the same meaning as in eqs. (6.1). Note that by means of the specific choice of the velocity potentials (8.10), the boundary conditions listed in Eq. (3.3) are automatically satisfied. Substitution of eqs. (8.10) into eqs. (8.6) yields

$$\begin{cases} \left(\nabla_2 + \eta^2\,\nu_1^2\right)\left(\nabla_2 + \eta^2\,\nu_2^2\right)\right)\phi = 0\,,\\ \left(\nabla_2 - \eta^2\,\nu_3^2\right)\psi = 0\,, \end{cases} \tag{8.11}$$

where

$$\nabla_2(\cdot) = r^{-1}(\cdot)_{,r} + (\cdot)_{,rr} - n^2\, r^{-2}(\cdot)\,. \tag{8.12}$$

The solutions to eqs. (8.11) are

$$\begin{aligned}\phi(r) &= b_1\, H_n^{(1)}(\eta\,\nu_1\, r) + b_2\, H_n^{(1)}(\eta\,\nu_2\, r) + \\ &\quad + b_3\, H_n^{(2)}(\eta\,\nu_1\, r) + b_4\, H_n^{(2)}(\eta\,\nu_2\, r)\,, \\ \psi(r) &= a_1\, I_n(\eta\,\nu_3\, r) + a_2\, K_n(\eta\,\nu_3\, r)\,,\end{aligned} \tag{8.13}$$

where $H_n^{(1)}$ and $H_n^{(2)}$ are the Hankel functions of order n, I_n and K_n are the modified Bessel functions of order n and the coefficients a_i and b_i are complex. The resulting velocity field reads

$$\begin{aligned}v_r(r,\theta,z) &= (\eta\,\phi_{,r} + n\,\psi/r)\,\cos(n\,\theta)\cos(\eta\, z)\,, \\ v_\theta(r,\theta,z) &= -\,(n\,\eta\,\phi/r + \psi_{,r})\,\sin(n\,\theta)\cos(\eta\, z)\,, \\ v_z &= \frac{C_{11}\,(n^2\,\phi - r\,\phi_{,r} - \phi_{,rr})\,/r^2 + \eta^2(C_{44} + T_{zz}/2)\,\phi}{(C_{13} + C_{44}) - T_{zz}}\cos(n\,\theta)\sin(\eta\, z)\,.\end{aligned} \tag{8.14}$$

The components of the rate of the first Piola-Kirchhoff stress tensor involved in the definition of the boundary conditions (3.3) read

$$\begin{aligned}\dot{S}_{rr} &= (C_{13}\,\eta\, f_z + C_{12}\,(f_r + n\, f_\theta)/r + C_{11}\, f_{r,r})\cos(n\,\theta)\cos(\eta\, z)\,, \\ \dot{S}_{\theta r} &= (C_{12} - C_{11})(n\, f_r + f_\theta - r\, f_{\theta,r})\,(2\, r)^{-1}\,\sin(n\,\theta)\cos(\eta\, z)\,, \\ \dot{S}_{zr} &= (C_{44} - T_{zz}/2)(f_{z,r} - \eta\, f_r)\,\cos(n\,\theta)\sin(\eta\, z)\,,\end{aligned} \tag{8.15}$$

where

$$\begin{aligned}f_r(r) &= (n\,\psi + \eta\, r\,\phi_{,r})/r\,, \\ f_\theta(r) &= -(n\,\eta\,\phi + r\,\psi_{,r})/r\,, \\ f_z(r) &= \frac{C_{11}\,(n^2\,\phi - r^2\,\phi_{,rr} - \phi_{,r})\,/r^2 + \eta^2\,(C_{44} + T_{zz}/2)\,\phi}{(C_{13} + C_{44} - T_{zz}/2)}\,.\end{aligned} \tag{8.16}$$

Enforcing the boundary conditions (3.3), namely, the null-traction condition on both the inner and the outer surface of the pre-stressed cylinder, yields an eigenvalue problem in the form

$$\mathbf{M}\,\mathbf{c} = \mathbf{0}\,, \quad \text{where} \quad \mathbf{c}^\top = \{a_1, a_2, b_1, b_2, b_3, b_4\}\,, \tag{8.17}$$

for which non-trivial solutions are found only if $\det \mathbf{M} = 0$. It can be shown that the latter condition only involves the material parameter ν, the ratio between the outer and inner radius r_e/r_i, the circumferential wave-number n, the longitudinal wave-number η, and the amount of pre-bifurcation axial stretch λ_z. For a given set of parameters ν, r_e/r_i, n and η, the critical axial stretch at which bifurcation occurs can be found numerically.

The accuracy of the approximated method presented in this work is compared in Fig. 8.2 with the just obtained three-dimensional solution. It is noticed from Fig. 8.2 that there is a poor convergence of the three-dimensional solution to the bifurcation load when the thickness ratio τ is becoming small. This is the reason why an approximate solution such as that given by Flügge or that presented in this work are useful.

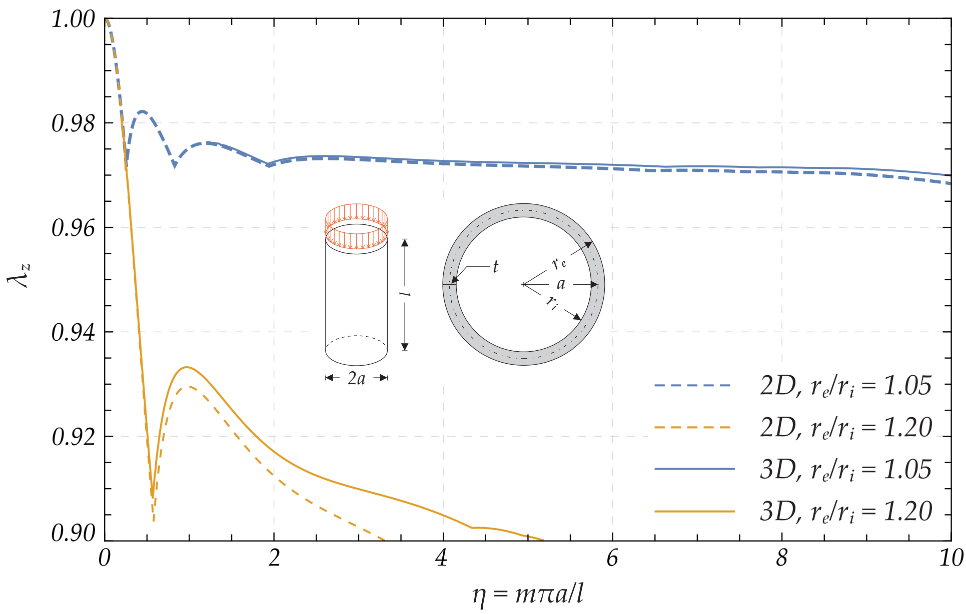

Fig. 8.2: *Bifurcation axial stretch of a thin cylinder: comparison between the three-dimensional (continuous line) and approximated (dashed line) solution.*

Conclusions

The buckling of thin cylindrical shell under axial compression has been investigated. In the first part of the thesis, the setup of a general framework that allows for the derivation of incremental equilibrium equations in terms of *generalized stresses* starting from the local incremental equilibrium condition has been presented. The results obtained are rigorous and don't introduce any approximations in the model. The second part of the thesis has been focused to a novel derivation of the incremental kinematics of a thin shell, namely the calculation of the deformed normal unit vector of a two-dimensional surface later used in the kinematical assumption. The incremental equilibrium equations for a nearly incompressible neo-Hookean material (Pence and Gou 2015) have been derived. The same method applies also to incompressible materials such as neo-Hookean, Mooney-Rivlin and J_2-deformation theory materials. The set of equations have been used to solve the buckling problem, showing that the reduced model developed in this work offers accurate results compared to the corresponding 3D full-field solutions as the shell thickness decreases. The 2D solution allows also for the onset of buckling in tension for the J_2 material. Furthermore, a 3D full-field solution for the particular strain-energy function chosen for the compressible material model has been developed to check the validity of the 2D solution. However, the poor numerical convergence of the 3D solution when the slenderness of the shell increases needs to be further investigated. Finally, the Euler buckling for a slender column as well as the dimensionless critical load of a 'mid-long' were recovered. For the latter, a new expression for the dimensionless buckling load of a cylindrical shell made of a Mooney-Rivlin material has been derived.

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