Doctoral School in Mathematics

Variational and convex approximations of 1-dimensional optimal networks and hyperbolic obstacle problems

Mauro Bonafini

UNIVERSITÀ DEGLI STUDI DI TRENTO
Dipartimento di Matematica

UNIVERSITÀ DEGLI STUDI DI VERONA
Dipartimento di Informatica

2019

Doctoral thesis in Mathematics
Joint doctoral programme in Mathematics, **XXXI cycle**.

Department of Mathematics,
University of Trento.

Department of Computer Science,
University of Verona.

Supervisor: **Prof. Giandomenico Orlandi**

Introduction

In this thesis we investigate variational problems involving 1-dimensional sets (e.g., curves, networks) and variational inequalities related to obstacle-type dynamics from a twofold prospective. On one side, we provide variational approximations and convex relaxations of the relevant energies and dynamics, moving mainly within the framework of Γ-convergence and of convex analysis. On the other side, we thoroughly investigate the numerical optimization of the corresponding approximating energies, both to recover optimal 1-dimensional structures and to accurately simulate the actual dynamics.

Variational approximations and convex relaxation to the Gilbert–Steiner problem.
The starting point of our research is the Steiner tree problem: given N distinct terminal points $P_1, \dots, P_N$ in the Euclidean space $\mathbb{R}^d$, find the shortest connected graph joining them. In a more abstract way, what we are looking for is a minimizer of

$$\text{(STP)} \qquad \inf\{\mathcal{H}^1(L),\ L \text{ connected},\ \{P_1, \dots, P_N\} \subset L\}.$$

It is well known that an optimizer L for (STP) always exists, the solution being, in general, not unique for symmetric configurations of the terminal points. Any optimizer L turns out to be a tree, called Steiner Minimal Tree (SMT), and can be described as a union of segments connecting the given points, possibly meeting at 120° in at most $N-2$ further branched points, called Steiner points.

From a computational point of view, finding an SMT is known to be an NP-hard problem, even NP complete in certain cases: taking into account the upper bound on the number of possible Steiner points, the real problem, and thus the source of the high combinatorial complexity, is indeed to sort out the topology of an optimal graph. Many different approaches have been proposed to tackle the problem from a discrete point of view, both looking for exact algorithms in dimension 2 and 3 [110, 54] and providing more general PTAS approximation schemes [13, 14]. These algorithms are, up to date, the most efficient way to compute exact or approximated solutions to (STP), and it is not our purpose to battle in efficiency with them. Instead, we would like to look at the problem from the point of view of the Calculus of Variations, in order to provide a more robust variational approach to (STP) and, more generally, to problems involving 1-dimensional sets.

The prototypical problem we look at is the single sink Gilbert–Steiner problem, which fits within the realm of α-irrigation problems. Given again N distinct points $P_1, \dots, P_N$

in $\mathbb{R}^d$, we look for the optimal way to transport $N-1$ unit masses located at $P_1, \dots, P_{N-1}$ to P_N. Such a transportation is realized through a graph $L = \cup_{i=1}^{N-1} \lambda_i$, where each λ_i is a simple rectifiable curve that connects P_i to P_N and represents the path of the ith mass. Taking into account scaling effects, i.e., the more we transport the less we pay per unit mass, we fix $0 \leq \alpha \leq 1$ and look for an optimizer of

$$(I_\alpha) \qquad \inf \left\{ \int_L |\theta(x)|^\alpha d\mathcal{H}^1(x), \;\; \theta(x) = \sum_{i=1}^{N-1} \mathbf{1}_{\lambda_i}(x) \right\}.$$

The fact that $\theta(x) \mapsto |\theta(x)|^\alpha$ is a sublinear concave function of the transported mass density θ enforces aggregation of masses and the emergence of branching structures [21]. We notice that (I_1) corresponds to the Monge optimal transport problem, while (I_0) corresponds to (STP). As for (STP) a solution to (I_α) is known to exist and any optimal network turns out to be a tree.

In the recent years, as we outline in 1.1, many variational approaches for (I_α) have been proposed. Among them, the stepping stone of this thesis is the interpretation of (I_α) as a mass minimization problem in a cobordism class of integral currents with multiplicities in a suitable normed group as studied by Marchese and Massaccesi in [70, 69]. The underlying idea is to switch our focus from a problem involving families of $N-1$ curves $\{\lambda_1, \dots, \lambda_{N-1}\}$ each one connecting P_i to P_N, to a problem involving families of integral rectifiable 1-currents $\Lambda = (\Lambda_1, \dots, \Lambda_{N-1}) \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1}$ where each component has boundary $\partial \Lambda_i = \delta_{P_N} - \delta_{P_i}$. Within such a framework, given a norm Ψ on $\mathbb{R}^{N-1}$, we introduce the Ψ-mass of $\Lambda \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1}$ as

$$||\Lambda||_\Psi := \sup_{\substack{\omega \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d) \\ h \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^{N-1})}} \left\{ \sum_{i=1}^{N-1} \Lambda_i(h_i \omega), \quad |\omega(x)| \leq 1, \, \Psi^*(h(x)) \leq 1 \right\}$$

where Ψ^* is the dual norm to Ψ with respect to the scalar product on $\mathbb{R}^{N-1}$. When one chooses $\Psi = \Psi_\alpha$ as the $\ell_{1/\alpha}$-norm, it can be shown that (I_α) is equivalent to

$$(I_\alpha^c) \qquad \inf\{||\Lambda||_{\Psi_\alpha} \, : \, \Lambda = (\Lambda_1, \dots, \Lambda_{N-1}) \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1}, \, \partial \Lambda_i = \delta_{P_N} - \delta_{P_i}\}.$$

The equivalence, as we are going to discuss in the forthcoming chapters, is meant in the sense that any minimizer L of (I_α) describes a minimizer Λ_L of (I_α^c) with the same energy and, vice versa, the support of any minimizer of (I_α^c) describes a minimal graph for (I_α).

Moving from this equivalence, the first three chapters of this thesis focus on Ψ-mass minimization problems among suitably defined families of integral rectifiable 1-currents (or, equivalently, rectifiable vector valued measures), where Ψ is chosen in order to reproduce the desired behaviour of the optimal structures we are looking for. As our main results we provide a variational approximation of Ψ-mass minimization problems in $\mathbb{R}^d$ for any dimension $d \geq 2$ and we introduce a suitable notion of convex relaxation for (I_α) which can be extended to treat the surface Gilbert–Steiner problem (i.e., when our ambient space is assumed to be a manifold). In particular, the choice of a suitable "optimal"

restriction of (I^c_α) will be essential for the derivation of a sharp relaxed energy. The first three chapters are organized as follows.

Variational approximation in the planar case. In Chapter 1 we mainly focus on the planar case $\mathbb{R}^2$, where a Ψ-mass minimization problem can be further recast in terms of an optimal partition type problem involving integer valued vector functions. We provide a variational approximation of the arising limiting functional by means of Modica–Mortola type energies, and we prove that minimizers of regularized functionals identify, in the limit, a (local) minimizer of (I_α). We therefore address the numerical optimization of these regularized energies and provide examples for various α and different configurations of the endpoints. The content of Chapter 1 represents a joint work with G. Orlandi and É. Oudet, published in "SIAM Journal on Mathematical Analysis" [25]. The same results were previously announced in "Geometric Flows" [22] and in "Rendiconti Lincei - Matematica e Applicazioni" [24].

Variational approximation in arbitrary dimension. In Chapter 2 we switch our focus on the higher dimensional scenario, and provide a variational approximation of Ψ-mass energies by means of Ginzburg–Landau type functionals. The results build upon the work presented in [5, 6] and use as main technical tool the relationship between boundaries and Jacobians of vector valued Sobolev maps. The content of Chapter 2 represents a joint work with G. Orlandi and É. Oudet, currently submitted for publication [26].

Convex relaxation. One of the main issues in the direct numerical optimization of Ψ_α-mass problems in the form of (I^c_α) resides in the non convexity of the set of candidate minimizers, which are generally only rectifiable objects with integer multiplicities. In Section 1.4, to overcome this issue, we investigate possible convex relaxations of the problem, much in the spirit of [43]. We look at convex extensions of the energy on a wider class of objects, so as to include in the picture also diffused objects with real valued multiplicities. The sharpest of these relaxations is the main subject of Chapter 3, where we present an extensive numerical investigation of it both in two and three space dimensions. Furthermore, we propose to extend this convex framework to more general α-irrigation problems (with multiple sources/sinks) and to use the same approach to address Gilbert–Steiner problems on manifolds. The main advantage of the proposed framework relies in the possibility to introduce a corresponding notion of calibration, giving us an analytical tool to prove that minimizers of the relaxed functional are convex combinations of minimizers of (I_α). However, as discussed in Chapter 3, there exist configurations of the endpoints for which the relaxation fails to recover convex combinations of minimal graphs, proving in turn that for these configurations optima of (I_α) cannot be calibrated. The content of Chapter 3 represents a joint work with É. Oudet, currently submitted for publication [27] and partly announced in [22].

Hyperbolic obstacle problems.

Obstacle type problems are nowadays a very active field of research in the Calculus of Variations community. Loosely speaking, obstacle problems arise whenever we solve a partial differential equation or optimize an energy functional and we require at the same

time that the solution must lie above a given "obstacle" g. In case of an evolutive equation, one can think of the obstacle g as a physical obstruction to the movement. Consider, for example, the dynamic of a string having fixed extrema and oscillating above a table: as soon as the string reaches the table we have to take into account the collision, and such an interaction cannot be described by a classical PDE, but it is rather described by a variational inequality [107, 55].

Obstacle problems for the minimizers of classical energies and regularity of the arising free boundary have been extensively studied in the literature, together with the corresponding evolutive equations in a parabolic context (cf. Section 4.1). What seems to be missing in the picture is the hyperbolic scenario which, despite being in some cases as natural as the previous ones, has received little attention so far. In Chapter 4 we study the obstacle problem for the fractional wave equation

$$u_{tt} + (-\Delta)^s u = 0$$

where $(-\Delta)^s$ is the fractional Laplace operator of exponent s, with $s > 0$. For suitable initial data at time $t = 0$, we study the problem assuming homogeneous Dirichlet boundary conditions and under the additional constraint $u \geq g$ for a given profile g. The idea is to apply a convex minimization approach based on a semi-discrete approximation scheme: at each time step the subsequent approximation in time is obtained as the unique optimizer of a suitably defined convex energy depending on previous steps. Such an approach has been extensively exploited in the literature to address parabolic and hyperbolic evolution problems, and fits within the general framework of minimizing movements. As main result we prove existence of a suitably defined weak solution, together with the corresponding energy estimates. The approximating scheme allows to perform numerical simulations which give quite precise evidence of dynamical effects. In particular, based on our numerical experiments, we conjecture that this method is able to select, in cases of non uniqueness, the most dissipative solution, that is to say the one losing the maximum amount of energy at contact times. The content of Chapter 4 represents a joint work with M. Novaga and G. Orlandi, currently submitted for publication [23].

Summary of research outcome.

The thesis work led to the following publications and preprints, some of which constitute the content of this manuscript.

[22] Mauro Bonafini. Convex relaxation and variational approximation of the Steiner problem: theory and numerics. *Geom. Flows*, 3:19–27, 2018

[23] Mauro Bonafini, Matteo Novaga, and Giandomenico Orlandi. A variational scheme for hyperbolic obstacle problems. *Submitted, arXiv preprint arXiv:1901.06974*, 2019

[24] Mauro Bonafini, Giandomenico Orlandi, and Édouard Oudet. Convex relaxation and variational approximation of functionals defined on 1-dimensional connected sets. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 29(4):597–606, 2018

[25] Mauro Bonafini, Giandomenico Orlandi, and Édouard Oudet. Variational approximation of functionals defined on 1-dimensional connected sets: the planar case. *SIAM J. Math. Anal.*, 50(6):6307–6332, 2018

[26] Mauro Bonafini, Giandomenico Orlandi, and Édouard Oudet. Variational approximation of functionals defined on 1-dimensional connected sets in $\mathbb{R}^n$. *Submitted*, 2019

[27] Mauro Bonafini and Édouard Oudet. A convex approach to the Gilbert-Steiner problem. *Submitted, arXiv preprint arXiv:1810.05417*, 2018

Contents

Chapter 1

Variational approximation of functionals defined on 1-dimensional connected sets: the planar case

In this chapter we consider variational problems involving 1-dimensional connected sets in the Euclidean plane, such as the classical Steiner tree problem and the irrigation (Gilbert–Steiner) problem. We relate them to optimal partition problems and provide a variational approximation through Modica–Mortola type energies proving a Γ-convergence result. We also introduce a suitable convex relaxation and develop the corresponding numerical implementations. The proposed methods are quite general and the results we obtain can be extended to n-dimensional Euclidean space as shown in the next chapter.

1.1 Introduction

Connected 1-dimensional structures play a crucial role in very different areas like discrete geometry (graphs, networks, spanning and Steiner trees), structural mechanics (crack formation and propagation), inverse problems (defects identification, contour segmentation), etc. The modeling of these structures is a key problem both from the theoretical and the numerical point of view. Most of the difficulties encountered in studying such 1-dimensional objects are related to the fact that they are not canonically associated to standard mathematical quantities. In this article we plan to bridge the gap between the well-established methods of multiphase modeling and the world of one dimensional connected sets or networks. Whereas we strongly believe that our approach may lead to new points of view in quite different contexts, we restrict here our exposition to the study of two standard problems in the Calculus of Variations which are respectively the classical Steiner tree problem and the Gilbert–Steiner problem (also called the irrigation problem).

The Steiner Tree Problem (STP) [58] can be described as follows: given N points $P_1, \dots, P_N$ in a metric space X (e.g., X a graph, with P_i assigned vertices) find a connected graph $F \subset X$ containing the points P_i and having minimal length. Such an optimal graph F turns out to be a tree and is thus called a *Steiner Minimal Tree* (SMT). In case $X = \mathbb{R}^d$, $d \geq 2$ endowed with the Euclidean ℓ^2 metric, one refers often to the *Euclidean* or *geometric* STP, while for $X = \mathbb{R}^d$ endowed with the ℓ^1 (Manhattan) distance or for X contained in a fixed grid $\mathcal{G} \subset \mathbb{R}^d$ one refers to the *rectilinear* STP. Here we will adopt the general metric space formulation of [86]: given a metric space X, and given a compact (possibly infinite) set of terminal points $A \subset X$, find

$$\text{(STP)} \qquad \inf\{\mathcal{H}^1(S),\ S \text{ connected},\ S \supset A\},$$

where $\mathcal{H}^1$ indicates the 1-dimensional Hausdorff measure on X. Existence of solutions for (STP) relies on Golab's compactness theorem for compact connected sets, and it holds true also in generalized cases (e.g., $\inf \mathcal{H}^1(S)$, $S \cup A$ connected).

The Gilbert–Steiner problem, or α-irrigation problem [21, 112] consists of finding a network S along which to flow unit masses located at the sources $P_1, \dots, P_{N-1}$ to the target point P_N. Such a network S can be viewed as $S = \cup_{i=1}^{N-1} \gamma_i$, with γ_i a path connecting P_i to P_N, corresponding to the trajectory of the unit mass located at P_i. To favour branching, one is led to consider a cost to be minimized by S which is a sublinear (concave) function of the mass density $\theta(x) = \sum_{i=1}^{N-1} \mathbf{1}_{\gamma_i}(x)$: i.e., for $0 \leq \alpha \leq 1$, find

$$(I_\alpha) \qquad \inf \int_S |\theta(x)|^\alpha d\mathcal{H}^1(x).$$

Notice that (I_1) corresponds to the Monge optimal transport problem, while (I_0) corresponds to (STP). As for (STP) a solution to (I_α) is known to exist and the optimal network S turns out to be a tree [21].

Problems like (STP) or (I_α) are relevant for the design of optimal transport channels or networks connecting given endpoints, for example, the optimal design of net routing in VLSI circuits in the case $d = 2, 3$. The Steiner Tree Problem has been widely studied from the theoretical and numerical point of view in order to efficiently devise constructive solutions, mainly through combinatoric optimization techniques. Finding a Steiner Minimal Tree is known to be a NP hard problem (and even NP complete in certain cases), see, for instance, [13, 14] for a comprehensive survey on PTAS algorithms for (STP).

The situation in the Euclidean case for (STP) is theoretically well understood: given N points $P_i \in \mathbb{R}^d$ a SMT connecting them always exists, the solution being in general not unique (think, for instance, to symmetric configurations of the endpoints P_i). The SMT is a union of segments connecting the endpoints, possibly meeting at 120° in at most $N - 2$ further branch points, called *Steiner points*.

Nonetheless, the quest of computationally tractable approximating schemes for (STP) and for (I_α) has recently attracted a lot of attention in the Calculus of Variations community. In particular, (I_α) has been studied in the framework of optimal branched transport theory [21, 35], while (STP) has been interpreted as, respectively, a size minimization

problem for 1-dimensional connected sets [77, 49], or even a Plateau problem in a suitable class of vector distributions endowed with some algebraic structure [77, 70], to be solved by finding suitable *calibrations* [74]. Several authors have proposed different approximations of those problems, whose validity is essentially limited to the planar case, mainly using a phase field based approach together with some coercive regularization, see, e.g., [30, 44, 82, 29].

Our aim is to propose a variational approximation for (STP) and for the Gilbert–Steiner irrigation problem (in the equivalent formulations of [112, 69]) in the Euclidean case $X = \mathbb{R}^d$, $d \geq 2$. In this chapter we focus on the planar case $d = 2$ and prove a Γ-convergence result (see Theorem 1.3.12 and Proposition 1.3.11) by considering integral functionals of Modica–Mortola type [76]. In Chapter 2 we rigorously prove that certain integral functionals of Ginzburg-Landau type (see [6]) yield a variational approximation for (STP) and (I_α) valid in any dimension $d \geq 3$. This approach is related to the interpretation of (STP) and (I_α) as a mass minimization problem in a cobordism class of integral currents with multiplicities in a suitable normed group as studied by Marchese and Massaccesi in [70, 69] (see also [77] for the planar case). Our method is quite general and may be easily adapted to a variety of situations (e.g., in manifolds or more general metric space ambients, with densities or anisotropic norms, etc.).

The plan of the chapter is as follows: in Section 1.2 we reformulate (STP) and (I_α) as a suitable modification of the optimal partition problem in the planar case. In Section 1.3, we state and prove our main Γ-convergence results, respectively Proposition 1.3.11 and Theorem 1.3.12. Inspired by [43], we introduce in Section 1.4 a convex relaxation of the corresponding energies. In Section 1.5 we present our approximating scheme for (STP) and for the Gilbert-Steiner problem and illustrate its flexibility in different situations, showing how our convex formulation is able to recover multiple solutions whereas Γ-relaxation detects any locally minimizing configuration.

1.2 Steiner problem for Euclidean graphs and optimal partitions

In this section we describe some optimization problems on Euclidean graphs with fixed endpoints set A, like (STP) or irrigation-type problems, following the approach of [70, 69], and we rephrase them as optimal partition-type problems in the planar case $\mathbb{R}^2$.

1.2.1 Rank one tensor valued measures and acyclic graphs

For $M > 0$, we consider Radon measures Λ on $\mathbb{R}^d$ with values in the space of matrices $\mathbb{R}^{d\times M}$. For each $i = 1, \dots, M$ we define as Λ_i the vector measure representing the ith column of Λ, so that we can write $\Lambda = (\Lambda_1, \dots, \Lambda_M)$. The total variation measures $|\Lambda_i|$ are defined as usual with respect to the Euclidean structure on $\mathbb{R}^d$, while we set $\mu_\Lambda = \sum_{i=1}^M |\Lambda_i|$. Thanks to the Radon–Nikodym theorem we can find a matrix-valued density function $p(x) = (p_1(x), \dots, p_M(x))$, with entries $p_{ki} \in L^1(\mathbb{R}^d, \mu_\Lambda)$ for all $k = 1, \dots, d$

and $i = 1, \dots, M$, such that $\Lambda = p(x)\mu_\Lambda$ and $\sum_{i=1}^M |p_i(x)| = 1$ for μ_Λ-a.e $x \in \mathbb{R}^d$ (where on vectors of $\mathbb{R}^d$ $|\cdot|$ denotes the Euclidean norm). Whenever p is a rank one matrix μ_Λ-almost everywhere we say that Λ is a rank one tensor valued measure and we write it as $\Lambda = \tau \otimes g \cdot \mu_\Lambda$ for a μ_Λ-measurable unit vector field τ in $\mathbb{R}^d$ and $g\colon \mathbb{R}^d \to \mathbb{R}^M$ satisfying $\sum_{i=1}^M |g_i| = 1$.

Given $\Lambda \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^{d\times M})$ and a function $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^{d\times M})$, with $\varphi = (\varphi_1, \dots, \varphi_M)$, we have

$$\langle \Lambda, \varphi \rangle = \sum_{i=1}^M \langle \Lambda_i, \varphi_i \rangle = \sum_{i=1}^M \int_{\mathbb{R}^d} \varphi_i \, d\Lambda_i,$$

and fixing a norm Ψ on $\mathbb{R}^M$, one may define the Ψ-mass measure of Λ as

$$|\Lambda|_\Psi(B) := \sup_{\substack{\omega \in C_c^\infty(B;\mathbb{R}^d) \\ h \in C_c^\infty(B;\mathbb{R}^M)}} \{ \langle \Lambda, \omega \otimes h \rangle \,, \quad |\omega(x)| \le 1 \,, \ \Psi^*(h(x)) \le 1 \} \,, \tag{1.2.1}$$

for $B \subset \mathbb{R}^d$ open, where Ψ^* is the dual norm to Ψ w.r.t. the scalar product on $\mathbb{R}^M$, i.e.,

$$\Psi^*(y) = \sup_{x \in \mathbb{R}^M} \langle y, x \rangle - \Psi(x).$$

Denote $||\Lambda||_\Psi = |\Lambda|_\Psi(\mathbb{R}^d)$ the Ψ-mass norm of Λ. In particular, one can see that μ_Λ coincides with the measure $|\Lambda|_{\ell^1}$, which from now on will be denoted as $|\Lambda|_1$, and any rank one measure Λ may be written as $\Lambda = \tau \otimes g \cdot |\Lambda|_1$ so that $|\Lambda|_\Psi = \Psi(g)|\Lambda|_1$. Along the lines of [70] we will rephrase the Steiner and Gilbert–Steiner problem as the optimization of a suitable Ψ-mass norm over a given class of rank one tensor valued measures.

Let $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^d$, $d \ge 2$, be a given set of N distinct points, with $N > 2$. We define the class $\mathcal{G}(A)$ as the set of *acyclic graphs* L connecting the endpoints set A such that L can be described as the union $L = \cup_{i=1}^{N-1} \lambda_i$, where λ_i are simple rectifiable curves with finite length having P_i as initial point and P_N as final point, oriented by $\mathcal{H}^1$-measurable unit vector fields τ_i satisfying $\tau_i(x) = \tau_j(x)$ for $\mathcal{H}^1$-a.e. $x \in \lambda_i \cap \lambda_j$ (i.e., the orientation of λ_i is coherent with that of λ_j on their intersection).

For $L \in \mathcal{G}(A)$, if we identify the curves λ_i with the vector measures $\Lambda_i = \tau_i \cdot \mathcal{H}^1 \llcorner \lambda_i$, all the information concerning this acyclic graph L is encoded in the rank one tensor valued measure $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, where the $\mathcal{H}^1$-measurable vector field $\tau \in \mathbb{R}^d$ carrying the orientation of the graph L satisfies $\operatorname{spt} \tau = L$, $|\tau| = 1$, $\tau = \tau_i$ $\mathcal{H}^1$-a.e. on λ_i, and the $\mathcal{H}^1$-measurable vector map $g\colon \mathbb{R}^d \to \mathbb{R}^{N-1}$ has components g_i satisfying $g_i \cdot \mathcal{H}^1 \llcorner L = \mathcal{H}^1 \llcorner \lambda_i = |\Lambda_i|$, with $|\Lambda_i|$ the total variation measure of the vector measure $\Lambda_i = \tau \cdot \mathcal{H}^1 \llcorner \lambda_i$. Observe that $g_i \in \{0, 1\}$ a.e. for any $1 \le i \le N-1$ and, moreover, that each Λ_i verifies the property

$$\operatorname{div} \Lambda_i = \delta_{P_i} - \delta_{P_N}. \tag{1.2.2}$$

Definition 1.2.1. *Given any graph $L \in \mathcal{G}(A)$, we call the above constructed $\Lambda_L \equiv \Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ the canonical rank one tensor valued measure representation of the acyclic graph L.*

To any compact connected set $K \supset A$ with $\mathcal{H}^1(K) < +\infty$, i.e., to any candidate minimizer for (STP), we may associate in a canonical way an acyclic graph $L \in \mathcal{G}(A)$ connecting $\{P_1, \dots, P_N\}$ such that $\mathcal{H}^1(L) \leq \mathcal{H}^1(K)$ (see, e.g., Lemma 2.1 in [70]). Given such a graph $L \in \mathcal{G}(A)$ canonically represented by the tensor valued measure Λ, the measure $\mathcal{H}^1 \llcorner L$ corresponds to the smallest positive measure dominating $\mathcal{H}^1 \llcorner \lambda_i$ for $1 \leq i \leq N-1$. It is thus given by $\mathcal{H}^1 \llcorner L = \sup_i \mathcal{H}^1 \llcorner \lambda_i = \sup_i |\Lambda_i|$, the supremum of the total variation measures $|\Lambda_i|$. We recall that, for any nonnegative $\psi \in C_c^0(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \psi \, d\left(\sup_i |\Lambda_i|\right) = \sup\left\{ \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \varphi_i \, d|\Lambda_i| \,,\ \varphi_i \in C_c^0(\mathbb{R}^d),\ \sum_{i=1}^{N-1} \varphi_i(x) \leq \psi(x) \right\}.$$

Remark 1.2.2 (graphs as G-currents). In [70], the rank one tensor measure $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ identifying a graph in $\mathbb{R}^d$ is defined as a *current* with coefficients in the group $\mathbb{Z}^{N-1} \subset \mathbb{R}^{N-1}$. For $\omega \in \mathcal{D}^1(\mathbb{R}^d)$ a smooth compactly supported differential 1-form and $\vec{\varphi} = (\varphi_1, ..., \varphi_{N-1}) \in [\mathcal{D}(\mathbb{R}^d)]^{N-1}$ a smooth test (vector) function, one sets

$$\begin{aligned} \langle \Lambda, \omega \otimes \vec{\varphi} \rangle &:= \int_{\mathbb{R}^d} \langle \omega \otimes \vec{\varphi}, \tau \otimes g \rangle \ d\mathcal{H}^1 \llcorner L = \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \langle \omega, \tau \rangle \varphi_i g_i \, d\mathcal{H}^1 \llcorner L \\ &= \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \langle \omega, \tau \rangle \varphi_i \, d|\Lambda_i| \,. \end{aligned}$$

Moreover, fixing a norm Ψ on $\mathbb{R}^{N-1}$, one may define the Ψ-mass of the current Λ as it is done in (1.2.1). In [70] the authors show that classical integral currents, i.e., $G = \mathbb{Z}$, are not suited to describe (STP) as a mass minimization problem: for example, minimizers are not ensured to have connected support.

1.2.2 Irrigation-type functionals

In this section we consider functionals defined on acyclic graphs connecting a fixed set $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^d$, $d \geq 2$, by using their canonical representation as rank one tensor valued measures, in order to identify the graph with an *irrigation plan* from the point sources $\{P_1, \dots, P_{N-1}\}$ to the target point P_N. We focus here on suitable energies in order to describe the irrigation problem and the Steiner tree problem in a common framework as in [70, 69]. We observe, moreover, that the irrigation problem with one point source (I_α) introduced by Xia [112], in the equivalent formulation of [69], approximates the Steiner tree problem as $\alpha \to 0$ in the sense of Γ-convergence (see Proposition 1.2.4).

Consider on $\mathbb{R}^{N-1}$ the norms $\Psi_\alpha = |\cdot|_{\ell^{1/\alpha}}$ (for $0 < \alpha \leq 1$) and $\Psi_0 = |\cdot|_{\ell^\infty}$. Let $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be the canonical representation of an acyclic graph $L \in \mathcal{G}(A)$, so that we have $|\tau| = 1$, $g_i \in \{0, 1\}$ for $1 \leq i \leq N-1$ and hence $|g|_\infty = 1$ $\mathcal{H}^1$-a.e. on L. Let us define for such Λ and any $\alpha \in [0, 1]$ the functional

$$\mathcal{F}^\alpha(\Lambda) := ||\Lambda||_{\Psi_\alpha} = |\Lambda|_{\Psi_\alpha}(\mathbb{R}^d).$$

Observe that, by (1.2.1),

$$\mathcal{F}^0(\Lambda) = \int_{\mathbb{R}^d} |\tau||g|_\infty \, d\mathcal{H}^1 \llcorner L = \mathcal{H}^1(L)$$

and

$$\mathcal{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |\tau||g|_{1/\alpha} \, d\mathcal{H}^1 \llcorner L = \int_L |\theta|^\alpha d\mathcal{H}^1 \,, \tag{1.2.3}$$

where $\theta(x) = \sum_i g_i(x)^{1/\alpha} = \sum_i g_i(x) \in \mathbb{Z}$, and $0 \leq \theta(x) \leq N-1$. We thus recognize that minimizing the functional $\mathcal{F}^\alpha$ among graphs L connecting $P_1, \ldots, P_{N-1}$ to P_N solves the irrigation problem (I_α) with unit mass sources $P_1, \ldots, P_{N-1}$ and target P_N (see [69]), while minimizing $\mathcal{F}^0$ among graphs L with endpoints set $\{P_1, \ldots, P_N\}$ solves (STP) in $\mathbb{R}^d$.

Since both $\mathcal{F}^\alpha$ and $\mathcal{F}^0$ are mass-type functionals, minimizers do exist in the class of rank one tensor valued measures. The fact that the minimization problem within the class of canonical tensor valued measures representing acyclic graphs has a solution in that class is a consequence of compactness properties of Lipschitz maps (more generally by compactness theorem for G-currents [70]; in $\mathbb{R}^2$ it follows alternatively by the compactness theorem in the SBV class [12]). Actually, existence of minimizers in the canonically oriented graph class in $\mathbb{R}^2$ can be deduced as a byproduct of our convergence result (see Proposition 1.3.11 and Theorem 1.3.12) and in $\mathbb{R}^d$, for $d > 2$, by the parallel Γ-convergence analysis of Chapter 2.

Remark 1.2.3. A minimizer of $\mathcal{F}^0$ (resp., $\mathcal{F}^\alpha$) among tensor valued measures Λ representing admissible graphs corresponds necessarily to the canonical representation of a minimal graph, i.e., $g_i \in \{0,1\}$ $\forall\, 1 \leq i \leq N-1$. Indeed since $g_i \in \mathbb{Z}$, if $g_i \neq 0$, we have $|g_i| \geq 1$, hence $g_i \in \{-1, 0, 1\}$ for minimizers. Moreover, if $g_i = -g_j$ on a connected arc in $\lambda_i \cap \lambda_j$, with λ_i going from P_i to P_N and λ_j going from P_j to P_N, this implies that $\lambda_i \cup \lambda_j$ contains a cycle and Λ cannot be a minimizer. Hence, up to reversing the orientation of the graph, $g_i \in \{0,1\}$ for all $1 \leq i \leq N-1$.

We conclude this section by observing in the following proposition that the Steiner tree problem can be seen as the limit of irrigation problems.

Proposition 1.2.4. *The functional $\mathcal{F}^0$ is the Γ-limit, as $\alpha \to 0$, of the functionals $\mathcal{F}^\alpha$ with respect to the convergence of measures.*

Proof. Let $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be the canonical representation of an acyclic graph $L \in \mathcal{G}(A)$, so that $|\tau| = 1$ and $g_i \in \{0,1\}$ for all $i = 1, \ldots, N-1$. The functionals $\mathcal{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |g|_{1/\alpha} d\mathcal{H}^1 \llcorner L$ generates a monotonic decreasing sequence as $\alpha \to 0$, because $|g|_p \leq |g|_q$ for any $1 \leq q < p \leq +\infty$, and, moreover, $\mathcal{F}^\alpha(\Lambda) \to \mathcal{F}^0(\Lambda)$ because $|g|_q \to |g|_\infty$ as $q \to +\infty$. Then, by elementary properties of Γ-convergence (see, for instance, Remark 1.40 of [33]) we have $\mathcal{F}^\alpha \xrightarrow{\Gamma} \mathcal{F}^0$.

□

1.2.3 Acyclic graphs and partitions of $\mathbb{R}^2$

This section is dedicated to the two-dimensional case. The aim is to provide an equivalent formulation of (STP) and (I_α) in term of an optimal partition type problem. The equivalence of (STP) with an optimal partition problem has been already studied in the case $P_1, \dots, P_N$ lie on the boundary of a convex set, see, for instance, [10, 11] and Remark 1.2.10.

To begin we state a result saying that two acyclic graphs having the same endpoints set give rise to a partition of $\mathbb{R}^2$, in the sense that their oriented difference corresponds to the orthogonal distributional gradient of a piecewise integer valued function having bounded total variation, which in turn determines the partition (see [12]). This is actually an instance of the constancy theorem for currents or the Poincaré's lemma for distributions (see [56]).

Lemma 1.2.5. *Let $\{P, R\} \subset \mathbb{R}^2$ and let λ, γ be simple rectifiable curves from P to R oriented by $\mathcal{H}^1$-measurable unit vector fields τ', τ''. Define as above $\Lambda = \tau' \cdot \mathcal{H}^1 \llcorner \lambda$ and $\Gamma = \tau'' \cdot \mathcal{H}^1 \llcorner \gamma$.*

Then there exists a function $u \in SBV(\mathbb{R}^2; \mathbb{Z})$ such that, denoting Du and $Du^\perp$ respectively the measures representing the gradient and the orthogonal gradient of u, we have $Du^\perp = \Gamma - \Lambda$.

Proof. Consider simple oriented polygonal curves λ_k and γ_k connecting P to R such that the Hausdorff distance to, respectively, λ and γ is less than $\frac{1}{k}$ and the length of λ_k (resp., γ_k) converges to the length of λ (resp., γ). We can also assume without loss of generality that λ_k and γ_k intersect only transversally in a finite number of points $m_k \geq 2$. Let τ'_k, τ''_k be the $\mathcal{H}^1$-measurable unit vector fields orienting λ_k, γ_k and define the measures $\Lambda_k = \tau'_k \cdot \mathcal{H}^1 \llcorner \lambda_k$ and $\Gamma_k = \tau''_k \cdot \mathcal{H}^1 \llcorner \gamma_k$.

For a given $k \in \mathbb{N}$ consider the closed polyhedral curve $\sigma_k = \lambda_k \cup \gamma_k$ oriented by $\tau_k = \tau'_k - \tau''_k$ (i.e., we reverse the orientation of γ_k). For every $x \in \mathbb{R}^2 \setminus \sigma_k$ let us consider the index of x with respect to σ_k (or winding number) and denote it as

$$u_k(x) = \mathrm{Ind}_{\sigma_k}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\sigma_k} \frac{dz}{z - x}.$$

By Theorem 10.10 in [95], the function u_k is integer valued and constant in each connected component of $\mathbb{R}^2 \setminus \sigma_k$ and vanishes in the unbounded one. Furthermore, for a.e. $x \in \sigma_k$ we have

$$\lim_{\varepsilon \to 0^+} u_k(x + \varepsilon \tau_k(x)^\perp) - \lim_{\varepsilon \to 0^-} u_k(x + \varepsilon \tau_k(x)^\perp) = 1,$$

i.e., u_k has a jump of $+1$ whenever crossing σ_k from "right" to "left" (cf [91], Lemma 3.3.2). This means that

$$Du_k^\perp = -\tau_k \cdot \mathcal{H}^1 \llcorner \sigma_k = \Gamma_k - \Lambda_k.$$

Thus, $|Du_k|(\mathbb{R}^2) = \mathcal{H}^1(\sigma_k)$ and $\|u_k\|_{L^1(\mathbb{R}^2)} \leq C|Du_k|(\mathbb{R}^2)$ by Poincaré's inequality in BV. Hence $u_k \in SBV(\mathbb{R}^2; \mathbb{Z})$ is an equibounded sequence in norm, and by Rellich compactness theorem there exists a subsequence still denoted u_k converging in $L^1(\mathbb{R}^2)$ to a $u \in$

$SBV(\mathbb{R}^2;\mathbb{Z})$. Taking into account that we have $Du_k^\perp = \Gamma_k - \Lambda_k$, we deduce, in particular, that $Du^\perp = \Gamma - \Lambda$ as desired.

Remark 1.2.6. Let $A \subset \mathbb{R}^2$ as above. For $i = 1, ..., N-1$ let γ_i be the segment joining P_i to P_N, denote $\tau_i = \frac{P_N - P_i}{|P_N - P_i|}$ its orientation, and identify γ_i with the vector measure $\Gamma_i = \tau_i \cdot \mathcal{H}^1 \llcorner \gamma_i$. Then $G = \cup_{i=1}^{N-1} \gamma_i$ is an acyclic graph connecting the endpoints set A and $\mathcal{H}^1(G) = (\sup_i |\Gamma_i|)(\mathbb{R}^2)$.

Given the set of terminal points $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^2$ let us fix some $G \in \mathcal{G}(A)$ (for example, the one constructed in Remark 1.2.6). For any acyclic graph $L \in \mathcal{G}(A)$, denoting Γ (resp., Λ) the canonical tensor valued representation of G (resp., L), by means of Lemma 1.2.5 we have

$$\mathcal{H}^1(L) = \int_{\mathbb{R}^2} \sup_i |\Lambda_i| = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i| \tag{1.2.4}$$

for suitable $u_i \in SBV(\mathbb{R}^2;\mathbb{Z})$, $1 \le i \le N-1$. Thus, using the family of measures $\Gamma = (\Gamma_1, \dots, \Gamma_{N-1})$ of Remark 1.2.6, we are led to consider the minimization problem for $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ for the functional

$$F^0(U) = |DU^\perp - \Gamma|_{\Psi_0}(\mathbb{R}^2) = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i|. \tag{1.2.5}$$

Proposition 1.2.7. *There exists $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ such that*

$$F^0(U) = \inf_{V \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})} F^0(V).$$

Moreover, $\operatorname{spt} U \subset \Omega = \{x \in \mathbb{R}^2 : |x| < 10 \max_i |P_i|\}$.

Proof. Observe first that for any $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ with $F^0(U) < \infty$, we can find $\tilde{U}$ s.t. $F^0(\tilde{U}) \le F^0(U)$ and $\operatorname{spt} \tilde{U} \subset \Omega$. Indeed, consider $r = 8 \max_i |P_i|$, $\chi = \mathbf{1}_{B_r(0)}$ and $\tilde{U} = (\chi u_1, \dots, \chi u_{N-1})$. One has, for $1 \le i \le N-1$,

$$\int_{\mathbb{R}^2 \setminus B_r(0)} |D\tilde{u}_i| = \int_{\partial B_r(0)} |u_i^+|$$

where u_i^+ is the trace on $\partial B_r(0)$ of u_i restricted to $B_r(0)$, and

$$\begin{aligned}\int_{\mathbb{R}^2} |D\tilde{u}_i^\perp - \Gamma_i| &= \int_{B_r(0)} |Du_i^\perp - \Gamma_i| + \int_{\partial B_r(0)} |u_i^+| \\ &\le \int_{B_r(0)} |Du_i^\perp - \Gamma_i| + \int_{\mathbb{R}^2 \setminus B_r(0)} |Du_i| = \int_{\mathbb{R}^2} |Du_i^\perp - \Gamma_i|\end{aligned}$$

for any $i = 1, \dots, N-1$, i.e., $F^0(\tilde{U}) \le F^0(U)$.

Now consider a minimizing sequence $U^k \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ of F^0. We may suppose w.l.o.g. $\operatorname{spt}(U^k) \subset \Omega$, so that, for any $1 \le i \le N-1$,

$$|Du_i^k|(\Omega) \le |Du_i^k - \Gamma_i|(\Omega) + \mathcal{H}^1(G) \le F^0(U^k) + \mathcal{H}^1(G) \le 3\mathcal{H}^1(G)$$

for k sufficiently large. Hence U^k is uniformly bounded in BV by Poincaré inequality on Ω, so that it is compact in $L^1(\Omega;\mathbb{R}^{N-1})$ and, up to a subsequence, $U^k \to U$ a.e., whence $U \in SBV(\Omega;\mathbb{Z}^{N-1})$, $\operatorname{spt} U \subset \Omega$ and U minimizes F^0 by lower semicontinuity of the norm. □

We have already seen that to each acyclic graph $L \in \mathcal{G}(A)$ we can associate a function $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ such that $\mathcal{H}^1(L) = F^0(U)$. On the other hand, for minimizers of F^0, we have the following

Proposition 1.2.8. *Let $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ be a minimizer of F^0, then there exists an acyclic graph $L \in \mathcal{G}(A)$ connecting the terminal points $P_1,\dots,P_N$ and such that $F^0(U) = \mathcal{H}^1(L)$.*

Proof. Let $U = (u_1,\dots,u_{N-1})$ be a minimizer of F^0 in $SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$, and denote $\Lambda_i = \Gamma_i - Du_i^\perp$. Observe that each Du_i has no absolutely continuous part with respect to the Lebesgue measure (indeed u_i is piecewise constant being integer valued) and so $\Lambda_i = \tau_i \cdot \mathcal{H}^1 \llcorner \lambda_i$ for some 1-rectifiable set λ_i and $\mathcal{H}^1$-measurable vector field τ_i. Since we have $\operatorname{div}\Lambda_i = \delta_{P_i} - \delta_{P_N}$, λ_i necessarily contains a simple rectifiable curve λ_i' connecting P_i to P_N (use, for instance, the decomposition theorem for rectifiable 1-currents in cyclic and acyclic part, as it is done in [69], or the Smirnov decomposition of solenoidal vector fields [106]).

Now consider the canonical rank one tensor measure Λ' associated to the acyclic subgraph $L' = \lambda_1' \cup \dots \cup \lambda_{N-1}'$ connecting $P_1,\dots,P_{N-1}$ to P_N. Then by Lemma 1.2.5, there exists $U' = (u_1',\dots,u_{N-1}') \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ such that $Du_i'^\perp = \Gamma_i - \Lambda_i'$ and in particular $F^0(U') = \mathcal{H}^1(L') \le \mathcal{H}^1(L) \le F^0(U)$. We deduce $\mathcal{H}^1(L') = \mathcal{H}^1(L)$, hence $L' = L$, L is acyclic and $H^1(L) = F^0(U)$.

Remark 1.2.9. We have shown the relationship between (STP) and the minimization of F^0 over functions in $SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$, namely

$$\inf\{F^0(U) \,:\, U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})\} = \inf\{\mathcal{F}^0(\Lambda_L) \,:\, L \in \mathcal{G}(\{P_1,\dots,P_N\})\}.$$

A similar connection can be made between the α-irrigation problem (I_α) and minimization over $SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ of

$$F^\alpha(U) = |DU^\perp - \Gamma|_{\Psi_\alpha}(\mathbb{R}^2), \tag{1.2.6}$$

namely we have

$$\inf\{F^\alpha(U) \,:\, U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})\} = \inf\{\mathcal{F}^\alpha(\Lambda_L) \,:\, L \in \mathcal{G}(\{P_1,\dots,P_N\})\},$$

where $\mathcal{F}^\alpha$ is defined in equation (1.2.3). Indeed, given a norm Ψ on $\mathbb{R}^{N-1}$ and $F^\Psi(U) = |DU^\perp - \Gamma|_\Psi(\mathbb{R}^2)$ for $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$, the proofs of Propositions 1.2.7 and 1.2.8 carry over to this general context: there exists $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ realizing $\inf F^\Psi$, with $\operatorname{spt} U \subset \Omega$ and $DU^\perp - \Gamma = \Lambda_L$ with $\Lambda_L = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ the canonical representation of an acyclic graph $L \in \mathcal{G}(\{P_1,\dots,P_N\})$.

Remark 1.2.10. In the case $P_1, \dots, P_N \in \partial\Omega$ with $\Omega \subset \mathbb{R}^2$ a convex set, we may choose $G = \cup_{i=1}^{N-1} \gamma_i$ with γ_i connecting P_i to P_N and $\operatorname{spt} \gamma_i \subset \partial\Omega$. We deduce by (1.2.4) that for any acyclic graph $L \in \mathcal{G}(A)$

$$\mathcal{H}^1(L) = \int_\Omega \sup_i |Du_i^\perp|$$

for suitable $u_i \in SBV(\Omega; \mathbb{Z})$ such that (in the trace sense) $u_i = 1$ on $\gamma_i \subset \partial\Omega$ and $u_i = 0$ elsewhere in $\partial\Omega$, $1 \le i \le N-1$. We recover here an alternative formulation of the optimal partition problem in a convex planar set Ω as studied, for instance, in [10] and [11].

The aim of the next section is then to provide an approximation of minimizers of the functionals F^α (and more generally F^Ψ) through minimizers of more regular energies of Modica–Mortola type.

1.3 Variational approximation of F^α

In this section we state and prove our main results, namely Proposition 1.3.11 and Theorem 1.3.12, concerning the approximation of minimizers of F^α through minimizers of Modica–Mortola type functionals, in the spirit of Γ-convergence.

1.3.1 Modica–Mortola functionals for functions with prescribed jump

In this section we consider Modica–Mortola functionals for functions having a prescribed jump part along a fixed segment in $\mathbb{R}^2$ and we prove compactness and lower-bounds for sequences having a uniform energy bound. Let $P, Q \in \mathbb{R}^2$ and let s be the segment connecting P to Q. We denote by $\tau_s = \frac{Q-P}{|Q-P|}$ its orientation and define $\Sigma_s = \tau_s \cdot \mathcal{H}^1 \llcorner s$. Up to rescaling, suppose $\max(|P|, |Q|) = 1$ and let $\Omega = B_{10}(0)$ and $\Omega_\delta = \Omega \setminus (B_\delta(P) \cup B_\delta(Q))$ for $0 < \delta \ll |Q - P|$. We consider the Modica–Mortola type functionals

$$F_\varepsilon(u, \Omega_\delta) = \int_{\Omega_\delta} e_\varepsilon(u)\, dx = \int_{\Omega_\delta} \varepsilon |Du^\perp - \Sigma_s|^2 + \frac{1}{\varepsilon} W(u)\, dx, \tag{1.3.1}$$

defined for $u \in H_s = \{u \in W^{1,2}(\Omega_\delta \setminus s) \cap SBV(\Omega_\delta) \,:\, u|_{\partial\Omega} = 0\}$, where W is a smooth non negative 1-periodic potential vanishing on $\mathbb{Z}$ (e.g., $W(u) = \sin^2(\pi u)$). Define $H(t) = 2\int_0^t \sqrt{W(\tau)}\, d\tau$ and $c_0 = H(1)$.

Remark 1.3.1. Notice that any function $u \in H_s$ with $F_\varepsilon(u, \Omega_\delta) < \infty$ has necessarily a prescribed jump $u^+ - u^- = +1$ across $s \llcorner \Omega_\delta$ in the direction $\nu_s = -\tau_s^\perp$ in order to erase the contribution of the measure term Σ_s in the energy. We thus have the decomposition

$$Du^\perp = \nabla u^\perp \mathcal{L}^2 + Ju^\perp = \nabla u^\perp \mathcal{L}^2 + \Sigma_s \llcorner \Omega_\delta,$$

where $\nabla u \in L^2(\Omega_\delta)$ is the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^2$, and $Ju = (u^+ - u^-)\nu_s \cdot \mathcal{H}^1 \llcorner s = \nu_s \cdot \mathcal{H}^1 \llcorner s$.

Remark 1.3.2. Notice that we cannot work directly in Ω with F_ε due to summability issues around the points P and Q for the absolutely continuous part of the gradient, indeed there are no functions $u \in W^{1,2}(\Omega \setminus s)$ such that $u^+ - u^- = 1$ on s. To avoid this issue one could consider variants of the functionals $F_\varepsilon(\cdot, \Omega)$ by relying on suitable smoothings $\Sigma_{s,\varepsilon} = \Sigma_s * \eta_\varepsilon$ of the measure Σ_s, with η_ε a symmetric mollifier.

Proposition 1.3.3 (Compactness)**.** *For any sequence $\{u_\varepsilon\}_\varepsilon \subset H_s$ such that $F_\varepsilon(u_\varepsilon, \Omega_\delta) \leq C$, there exists $u \in SBV(\Omega_\delta; \mathbb{Z})$ such that (up to a subsequence) $u_\varepsilon \to u$ in $L^1(\Omega_\delta)$.*

Proof. By Remark 1.3.1 we have $Du_\varepsilon^\perp = \nabla u_\varepsilon^\perp \mathcal{L}^2 + \Sigma_s \llcorner \Omega_\delta$, and using the classical Modica–Mortola trick one has

$$\begin{aligned} C &\geq \int_{\Omega_\delta} \varepsilon |Du_\varepsilon^\perp - \Sigma_s|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, dx \\ &= \int_{\Omega_\delta} \varepsilon |\nabla u_\varepsilon^\perp|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, dx \geq 2 \int_{\Omega_\delta} \sqrt{W(u_\varepsilon)} |\nabla u_\varepsilon|\, dx. \end{aligned}$$

Recall that $H(t) = 2\int_0^t \sqrt{W(\tau)}\, d\tau$ and $c_0 = H(1)$. By the chain rule, we have

$$\begin{aligned} |D(H \circ u_\varepsilon)|(\Omega_\delta) &= 2 \int_{\Omega_\delta} \sqrt{W(u_\varepsilon)} |\nabla u_\varepsilon|\, dx + \int_s \big(H(u_\varepsilon^+) - H(u_\varepsilon^-)\big)\, d\mathcal{H}^1(x) \\ &\leq C + c_0 \mathcal{H}^1(s). \end{aligned}$$

We also have $(H \circ u_\varepsilon)|_{\partial\Omega} = 0$ since u_ε vanishes on $\partial\Omega$, so that, by the Poincaré inequality, $\{H \circ u_\varepsilon\}_\varepsilon$ is an equibounded sequence in $BV(\Omega_\delta)$, thus compact in $L^1(\Omega_\delta)$. In particular, there exists $v \in L^1(\Omega_\delta)$ such that, up to a subsequence, $H \circ u_\varepsilon \to v$ in $L^1(\Omega_\delta)$ and pointwise a.e. Since H is a strictly increasing continuous function with $c_0(t-1) \leq H(t) \leq c_0(t+1)$ for any $t \in \mathbb{R}$, then H^{-1} is uniformly continuous and $|H^{-1}(t)| \leq c_0^{-1}(|t|+1)$ for all $t \in \mathbb{R}$. Hence, up to a subsequence, the family $\{u_\varepsilon\}_\varepsilon \subset L^1(\Omega_\delta)$ is pointwise convergent a.e. to $u = H^{-1}(v) \in L^1(\Omega_\delta)$. By Egoroff's Theorem, for any $\sigma > 0$ there exists a measurable $E_\sigma \subset \Omega_\delta$, with $|E_\sigma| < \sigma$, such that $u_\varepsilon \to u$ uniformly in $\Omega_\delta \setminus E_\sigma$. Then, taking into account that $|t| \leq c_0^{-1}(|H(t)|+1)$ for all $t \in \mathbb{R}$, we have

$$\begin{aligned} ||u_\varepsilon - u||_{L^1(\Omega_\delta)} &\leq ||u_\varepsilon - u||_{L^1(\Omega_\delta \setminus E_\sigma)} + \int_{E_\sigma} (|u_\varepsilon| + |u|)\, dx \\ &\leq |\Omega|\, ||u_\varepsilon - u||_{L^\infty(\Omega_\delta \setminus E_\sigma)} + 2c_0^{-1}|E_\sigma| + c_0^{-1} \int_{E_\sigma} (|H \circ u_\varepsilon| + |v|)\, dx \end{aligned}$$

and for ε, σ small enough the right hand side can be made arbitrarily small thanks to the uniform integrability of the sequence $\{H \circ u_\varepsilon\}_\varepsilon$. Hence $u_\varepsilon \to u$ in $L^1(\Omega_\delta)$. Furthermore, by Fatou's lemma we have

$$\int_{\Omega_\delta} W(u)\, dx \leq \liminf_{\varepsilon \to 0} \int_{\Omega_\delta} W(u_\varepsilon)\, dx \leq \liminf_{\varepsilon \to 0} \varepsilon F_\varepsilon(u_\varepsilon, \Omega_\delta) = 0,$$

whence $u(x) \in \mathbb{Z}$ for a.e. $x \in \Omega_\delta$. Finally we have

$$c_0 |Du|(\Omega_\delta) = |D(H \circ u)|(\Omega_\delta) \leq \liminf_{\varepsilon \to 0} |D(H \circ u_\varepsilon)|(\Omega_\delta) \leq C + c_0 \mathcal{H}^1(s),$$

i.e. $u \in SBV(\Omega_\delta;\mathbb{Z})$.

□

Proposition 1.3.4 (Lower-bound inequality)**.** *Let $\{u_\varepsilon\}_\varepsilon \subset H_s$ and $u \in SBV(\Omega_\delta;\mathbb{Z})$ such that $u_\varepsilon \to u$ in $L^1(\Omega_\delta)$. Then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega_\delta) \geq c_0|Du^\perp - \Sigma_s|(\Omega_\delta). \tag{1.3.2}$$

Proof. Step 1. Let us prove first that for any open ball $B \subset \Omega_\delta$ we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B) \geq c_0|Du^\perp - \Sigma_s|(B). \tag{1.3.3}$$

We distinguish two cases, according to whether $B \cap s = \emptyset$ or not. In the first case we have

$$F_\varepsilon(u_\varepsilon,B) = \int_B \varepsilon|Du_\varepsilon^\perp|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,dx.$$

Reasoning as in the proof of Proposition 1.3.3,

$$c_0|Du|(B) = |D(H\circ u)|(B) \leq \liminf_{\varepsilon\to 0}|D(H\circ u_\varepsilon)|(B) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B),$$

and (1.3.3) follows.

In the case $B \cap s \neq \emptyset$ we follow the arguments of [15], and consider $u_0 = \mathbf{1}_{B^+}$, where $B^+ = \{z \in B \setminus s : (z-z_0)\cdot\nu_s > 0\}$, for $z_0 \in B \cap s$ and $\nu_s^\perp = \tau_s$, so that $Du_0^\perp = \Sigma_s \llcorner B$. Letting $v_\varepsilon = u_\varepsilon - u_0$ we have $Dv_\varepsilon^\perp = Du_\varepsilon^\perp - \Sigma_s = \nabla u_\varepsilon^\perp \mathcal{L}^2$, with $\nabla u_\varepsilon \in L^2(B)$ and $W(v_\varepsilon) = W(u_\varepsilon)$ on B by 1-periodicity of the potential W. Hence

$$F_\varepsilon(u_\varepsilon,B) = \int_B \varepsilon|Dv_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon)\,dx.$$

Let $v = u - u_0$, we have

$$c_0|Du^\perp - \Sigma_s|(B) = c_0|Dv|(B) \leq \liminf_{\varepsilon\to 0}\int_B \varepsilon|Dv_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon)\,dx = \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B)$$

and (1.3.3) follows.

Step 2. Since $|Du^\perp - \Sigma_s|$ is a Radon measure, one has

$$|Du^\perp - \Sigma_s|(\Omega_\delta) = \sup\left\{\sum_j |Du^\perp - \Sigma_s|(B_j)\right\} \tag{1.3.4}$$

where the supremum is taken among all finite collections $\{B_j\}_j$ of pairwise disjoint open balls such that $\cup_j B_j \subset \Omega_\delta$. Applying (1.3.3) to each B_j and summing over j we have

$$c_0\sum_j |Du^\perp - \Sigma_s|(B_j) \leq \sum_j \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,B_j) \leq \liminf_{\varepsilon\to 0}\sum_j F_\varepsilon(u_\varepsilon,B_j) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\Omega_\delta)$$

which gives (1.3.2) thanks to (1.3.4).

Remark 1.3.5. The proof of Proposition 1.3.4 can be easily adapted to prove a weighted version of (1.3.2): in the same hypothesis, for any non negative $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\delta} \varphi e_\varepsilon(u_\varepsilon)\,dx \geq c_0 \int_{\Omega_\delta} \varphi\, d|Du^\perp - \Sigma_s|.$$

Remark 1.3.6. Proposition 1.3.4 holds true also in case the measure Σ_s are associated to oriented simple polyhedral (or even rectifiable) finite length curves joining P to Q.

1.3.2 The approximating functionals F_ε^Ψ

We now consider Modica–Mortola approximations for Ψ-mass functionals such as F^α. Let $A = \{P_1, \dots, P_N\}$ be our set of terminal points and $\Psi\colon \mathbb{R}^{N-1} \to [0, +\infty)$ be a norm on $\mathbb{R}^{N-1}$. For any $i \in \{1, \dots, N-1\}$ let $\Gamma_i = \tau_i \cdot \mathcal{H}^1 \llcorner \gamma_i$ be the measure defined in Remark 1.2.6. Without loss of generality suppose $\max_i(|P_i|) = 1$ and define $\Omega = B_{10}(0)$ and $\Omega_\delta = \Omega \setminus \cup_i B_\delta(P_i)$ for $0 < \delta \ll \min_{ij} |P_i - P_j|$. Let

$$H_i = \{u \in W^{1,2}(\Omega \setminus \gamma_i) \cap SBV(\Omega) \,:\, u|_{\partial\Omega} = 0\}, \quad H = H_1 \times \cdots \times H_{N-1}, \tag{1.3.5}$$

and for $u \in H_i$ define

$$e_\varepsilon^i(u) = \varepsilon |Du^\perp - \Gamma_i|^2 + \frac{1}{\varepsilon} W(u). \tag{1.3.6}$$

Denote $\vec{e}_\varepsilon(U) = (e_\varepsilon^1(u_1), \dots, e_\varepsilon^{N-1}(u_{N-1}))$ and consider the functionals

$$F_\varepsilon^\Psi(U, \Omega_\delta) = |\vec{e}_\varepsilon(U)\,dx|_\Psi(\Omega_\delta), \tag{1.3.7}$$

or equivalently, thanks to (1.2.1),

$$F_\varepsilon^\Psi(U, \Omega_\delta) = \sup_{\varphi \in C_c^\infty(\Omega_\delta; \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \int_{\Omega_\delta} \varphi_i e_\varepsilon^i(u_i)\,dx, \quad \Psi^*(\varphi(x)) \leq 1 \right\}. \tag{1.3.8}$$

The previous compactness and lower-bound inequality for functionals with a single prescribed jump extend to F_ε^Ψ as follows.

Proposition 1.3.7 (Compactness)**.** *Given $\{U_\varepsilon\}_\varepsilon \subset H$ such that $F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq C$, there exists $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that (up to a subsequence) $U_\varepsilon \to U$ in $[L^1(\Omega_\delta)]^{N-1}$.*

Proof. For each $i = 1, \dots, N-1$, by definition of F_ε^Ψ we have

$$\int_{\Omega_\delta} e_\varepsilon^i(u_{\varepsilon,i})\,dx \leq \Psi^*(e_i) F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq C\Psi^*(e_i)$$

and the result follows applying Proposition 1.3.3 componentwise.

Proposition 1.3.8 (Lower-bound inequality)**.** *Let $\{U_\varepsilon\}_\varepsilon \subset H$ and $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that $U_\varepsilon \to U$ in $[L^1(\Omega_\delta)]^{N-1}$. Then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \geq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta). \tag{1.3.9}$$

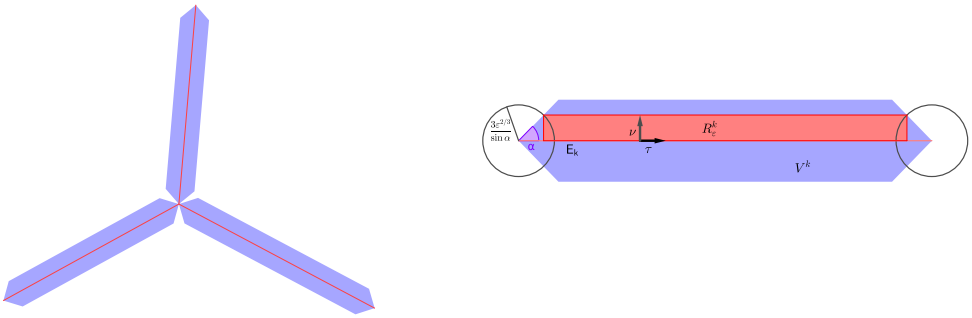

Figure 1.1: Typical shape of the sets V_k (left) and general construction involved in the definition of R^k_ε (right).

Proof. Fix $\varphi \in C^\infty_c(\Omega_\delta;\mathbb{R}^{N-1})$ with $\varphi_i \geq 0$ for any $i = 1,\dots,N-1$ and $\Psi^*(\varphi(x)) \leq 1$ for all $x \in \Omega_\delta$. By Remark 1.3.5 we have

$$c_0 \sum_{i=1}^{N-1} \int_{\Omega_\delta} \varphi_i \, d|Du_i^\perp - \Gamma_i| \leq \sum_{i=1}^{N-1} \liminf_{\varepsilon\to 0} \int_{\Omega_\delta} \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\,dx \leq \liminf_{\varepsilon\to 0} \sum_{i=1}^{N-1} \int_{\Omega_\delta} \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\,dx$$
$$\leq \liminf_{\varepsilon\to 0} F^\Psi_\varepsilon(U_\varepsilon,\Omega_\delta),$$

and taking the supremum over φ we get (1.3.9).

□

We now state and prove a version of an upper-bound inequality for the functionals F^Ψ_ε which will enable us to deduce the convergence of minimizers of F^Ψ_ε to minimizers of $F^\Psi(U,\Omega_\delta) = c_0|DU^\perp - \Gamma|_\Psi(\Omega_\delta)$, for $U \in SBV(\Omega_\delta;\mathbb{Z}^{N-1})$.

Proposition 1.3.9 (Upper-bound inequality)**.** *Let $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be a rank one tensor valued measure canonically representing an acyclic graph $L \in \mathcal{G}(A)$, and let $U = (u_1,\dots,u_{N-1}) \in SBV(\Omega_\delta;\mathbb{Z}^{N-1})$ such that $Du_i^\perp = \Gamma_i - \Lambda_i$ for any $i = 1,\dots,N-1$. Then there exists a sequence $\{U_\varepsilon\}_\varepsilon \subset H$ such that $U_\varepsilon \to U$ in $[L^1(\Omega_\delta)]^{N-1}$ and*

$$\limsup_{\varepsilon\to 0} F^\Psi_\varepsilon(U_\varepsilon,\Omega_\delta) \leq c_0|DU^\perp - \Gamma|_\Psi(\Omega_\delta). \tag{1.3.10}$$

Proof. Step 1. We consider first the case $\Lambda_i = \tau_i \cdot \mathcal{H}^1 \llcorner \lambda_i$ with λ_i a polyhedral curve transverse to γ_i for any $1 \leq i < N$. Then the support of the measure Λ is an acyclic polyhedral graph (oriented by τ and with normal $\nu = \tau^\perp$) with edges $E_0,\dots,E_M$ and vertices $\{S_0,\dots,S_\ell\} \not\subseteq (\cup_i \gamma_i) \cap \Omega_\delta$ such that $E_k = [S_{k_1}, S_{k_2}]$ for suitable indices $k_1, k_2 \in \{0,\dots,\ell\}$. Denote also $g^k = g|_{E_k} \in \mathbb{R}^{N-1}$ and recall $g^k_i \in \{0,1\}$ for all $1 \leq i < N$. By

finiteness there exist $\eta > 0$ and $\alpha \in (0, \pi/2)$ such that given any edge E_k of that graph the sets

$$V^k = \{x \in \mathbb{R}^2, \mathrm{dist}(x, E_k) < \min\{\eta,\, \cos(\alpha) \cdot \mathrm{dist}(x, S_{k_1}),\, \cos(\alpha) \cdot \mathrm{dist}(x, S_{k_2})\}\}$$

are disjoint and their union forms an open neighbourhood of $\cup_i \lambda_i \setminus \{S_0, \dots, S_\ell\}$ (choose, for instance, α such that 2α is smaller than the minimum angle realized by two edges and then pick η satisfying $2\eta \tan\alpha < \min_j \mathcal{H}^1(E_j)$).

For $0 < \varepsilon \ll \delta$, let $B_\varepsilon^m = \left\{x \in \mathbb{R}^2 : |x - S_m| < \frac{3\varepsilon^{2/3}}{\sin\alpha}\right\}$, $B_\varepsilon = \cup_m B_\varepsilon^m$ and define $R_\varepsilon^k \subset V^k$ as

$$R_\varepsilon^k = \{y + t\nu : y \in E_k,\, \min\{\mathrm{dist}(y, S_{k_1}), \mathrm{dist}(y, S_{k_2})\} > 3\varepsilon^{2/3}\cot(\alpha),\, 0 < t \le 3\varepsilon^{2/3}\}.$$

Let φ_0 be the optimal profile for the 1-dimensional Modica–Mortola functional, which solves $\varphi_0' = \sqrt{W(\varphi_0)}$ on $\mathbb{R}$ and satisfies $\lim_{\tau\to-\infty}\varphi_0(\tau) = 0$, $\lim_{\tau\to\infty}\varphi_0(\tau) = 1$ and $\varphi_0(0) = 1/2$. Let us define $\tau_\varepsilon = \varepsilon^{-1/3}$, $r_\varepsilon^+ = \varphi_0(\tau_\varepsilon)$, $r_\varepsilon^- = \varphi_0(-\tau_\varepsilon)$, and

$$\tilde\varphi_\varepsilon(\tau) = \begin{cases} 0 & \tau < -\tau_\varepsilon - r_\varepsilon^- \\ \tau + \tau_\varepsilon + r_\varepsilon^- & -\tau_\varepsilon - r_\varepsilon^- \le \tau \le -\tau_\varepsilon \\ \varphi_0(\tau) & |\tau| \le \tau_\varepsilon \\ \tau - \tau_\varepsilon + r_\varepsilon^+ & \tau_\varepsilon \le \tau \le \tau_\varepsilon + 1 - r_\varepsilon^+ \\ 1 & \tau > \tau_\varepsilon + 1 - r_\varepsilon^+ \end{cases}$$

Observe that $(1 - r_\varepsilon^+)$ and r_ε^- are $o(1)$ as $\varepsilon \to 0$. For $x = y + t\nu \in R_\varepsilon^k$ let us define $\varphi_\varepsilon(x) = \tilde\varphi_\varepsilon\left(\frac{t}{\varepsilon} - \tau_\varepsilon - r_\varepsilon^-\right)$, so that, as $\varepsilon \to 0$,

$$\begin{aligned}&\int_{R_\varepsilon^k \cap \Omega_\delta} \varepsilon|D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon)\,dx \le \mathcal{H}^1(E_k \cap \Omega_\delta)\int_{-\tau_\varepsilon - r_\varepsilon^-}^{2\tau_\varepsilon - r_\varepsilon^-} |D\tilde\varphi_\varepsilon(\tau)|^2 + W(\tilde\varphi_\varepsilon(\tau))\,d\tau + o(1)\\ &\le \mathcal{H}^1(E_k \cap \Omega_\delta)\int_{-\tau_\varepsilon}^{\tau_\varepsilon} 2\varphi_0'(\tau)\sqrt{W(\varphi_0(\tau))}\,d\tau + o(1) \le c_0\mathcal{H}^1(E_k \cap \Omega_\delta) + o(1).\end{aligned}$$

Define, for $x \in \Omega_\delta \setminus B_\varepsilon$,

$$u_{\varepsilon,i}(x) = \begin{cases} u_i(x) + \varphi_\varepsilon(x) - 1 & \text{if } x \in (R_\varepsilon^k \setminus B_\varepsilon) \cap \Omega_\delta \text{ whenever } E_k \subset \lambda_i \\ u_i(x) & \text{elsewhere on } \Omega_\delta \setminus B_\varepsilon \end{cases}$$

and on $B_\varepsilon \cap \Omega_\delta$ define $u_{\varepsilon,i}$ to be a Lipschitz extension of $u_{\varepsilon,i}|_{\partial(B_\varepsilon\cap\Omega_\delta)}$ with the same Lipschitz constant, which is of order $1/\varepsilon$. Remark that $u_{\varepsilon,i}$ has the same prescribed jump as u_i across γ_i, and thus $F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) < \infty$. Moreover, $u_{\varepsilon,i} \to u_i$ in $L^1(\Omega_\delta)$.

Observe now that if E_k is contained in $\lambda_i \cap \lambda_j$ then by construction

$$e_\varepsilon^i(u_{\varepsilon,i}) = e_\varepsilon^j(u_{\varepsilon,j}) = \varepsilon|D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon)$$

on $\tilde{R}^k_\varepsilon = (R^k_\varepsilon \cap \Omega_\delta) \setminus B_\varepsilon$. Let $\varphi = (\varphi_1, \dots, \varphi_{N-1})$, with $\varphi_i \geq 0$ and $\Psi^*(\varphi) \leq 1$, we deduce

$$\begin{aligned}
\int_{\Omega_\delta} \sum_i \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\, dx &\leq \sum_{k=1}^{\ell} \int_{\tilde{R}^k_\varepsilon} \sum_i \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\, dx + \int_{B_\varepsilon \cap \Omega_\delta} \sum_i \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\, dx \\
&\leq \sum_{k=1}^{\ell} \int_{\tilde{R}^k_\varepsilon} \sum_i \varphi_i g^k_i \left(\varepsilon |D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon) \right) dx + \int_{B_\varepsilon \cap \Omega_\delta} \Psi(\vec{e}_\varepsilon(U_\varepsilon))\, dx \\
&\leq \sum_{k=1}^{\ell} \int_{\tilde{R}^k_\varepsilon} \Psi(g^k) \left(\varepsilon |D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon) \right) dx + C\varepsilon^{1/3} \\
&\leq \sum_{k=1}^{\ell} \Psi(g^k)(c_0 \mathcal{H}^1(E_k \cap \Omega_\delta) + o(1)) + C\varepsilon^{1/3} \leq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta) + o(1)
\end{aligned}$$

as $\varepsilon \to 0$. In view of (1.3.8) we have

$$F^\Psi_\varepsilon(U_\varepsilon, \Omega_\delta) \leq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta) + o(1),$$

and conclusion (1.3.10) follows.

Step 2. Let us now consider the case $\Lambda_L \equiv \Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, $L = \cup_i \lambda_i$ and the λ_i are not necessarily polyhedral. Let $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that $DU^\perp = \Gamma - \Lambda_L$. We rely on Lemma 1.3.10 below to secure a sequence of acyclic polyhedral graphs $L_n = \cup_i \lambda^n_i$, λ^n_i transverse to γ_i, and s.t. the Hausdorff distance $d_H(\lambda^n_i, \lambda_i) < \frac{1}{n}$ for all $i = 1, \dots, N-1$, and $|\Lambda_{L_n}|_\Psi(\Omega_\delta) \leq |\Lambda_L|_\Psi(\Omega_\delta) + \frac{1}{n}$. Let $U^n \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that $(DU^n)^\perp = \Gamma - \Lambda_{L_n}$. In particular, $U^n \to U$ in $[L^1(\Omega_\delta)]^{N-1}$ and by step 1 we may construct a sequence U^n_ε s.t. $U^n_\varepsilon \to U^n$ in $[L^1(\Omega_\delta)]^{N-1}$ and

$$\begin{aligned}
\limsup_{\varepsilon \to 0} F^\Psi_\varepsilon(U^n_\varepsilon, \Omega_\delta) &\leq c_0 |(DU^n)^\perp - \Gamma|_\Psi(\Omega_\delta) = c_0 |\Lambda_{L_n}|_\Psi(\Omega_\delta) \\
&\leq c_0 |\Lambda_L|_\Psi(\Omega_\delta) + \frac{c_0}{n} = c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta) + \frac{c_0}{n}.
\end{aligned}$$

We deduce

$$\limsup_{n \to \infty} F^\Psi_{\varepsilon_n}(U^n_{\varepsilon_n}, \Omega_\delta) \leq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta)$$

for a subsequence $\varepsilon_n \to 0$ as $n \to +\infty$. Conclusion (1.3.10) follows.

□

Lemma 1.3.10. *Let $L \in \mathcal{G}(A)$, $L = \cup_{i=1}^{N-1} \lambda_i$, be an acyclic graph connecting $P_1, \dots, P_N$. Then for any $\eta > 0$ there exists $L' \in \mathcal{G}(A)$, $L' = \cup_{i=1}^{N-1} \lambda'_i$, with λ'_i a simple polyhedral curve of finite length connecting P_i to P_N and transverse to γ_i, such that the Hausdorff distance $d_H(\lambda_i, \lambda'_i) < \eta$ and $|\Lambda_{L'}|_\Psi(\mathbb{R}^2) \leq |\Lambda_L|_\Psi(\mathbb{R}^2) + \eta$, where Λ_L and $\Lambda_{L'}$ are the canonical tensor valued representations of L and L'.*

Proof. Since $L \in \mathcal{G}(A)$, we can write $L = \cup_{m=1}^{M} \zeta_m$, with ζ_m simple Lipschitz curves such that, for $m_i \neq m_j$, $\zeta_{m_i} \cap \zeta_{m_j}$ is either empty or reduces to one common endpoint.

Let $\Lambda_L = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be the rank one tensor valued measure canonically representing L, and let $d_m = \Psi(g(x))$ for $x \in \zeta_m$. The d_m are constants because by construction g is constant over each ζ_m. Now consider a polyhedral approximation $\tilde{\zeta}_m$ of ζ_m having its same endpoints, with $d_H(\tilde{\zeta}_m, \zeta_m) \leq \eta$, $\mathcal{H}^1(\tilde{\zeta}_m) \leq \mathcal{H}^1(\zeta_m) + \eta/(CM)$ (C to be fixed later) and, for $m_i \neq m_j$, $\tilde{\zeta}_{m_i} \cap \tilde{\zeta}_{m_j}$ is either empty or reduces to one common endpoint. Observe that whenever ζ_m intersects some γ_i, such a $\tilde{\zeta}_m$ can be constructed in order to intersect γ_i transversally in a finite number of points. Define $L' = \cup_{m=1}^M \tilde{\zeta}_m$ and let $\Lambda_{L'} = \tau' \otimes g' \cdot \mathcal{H} \llcorner L'$ be its canonical tensor valued measure representation. Then, by construction $\Psi(g'(x)) = d_m$ for any $x \in \tilde{\zeta}_m$, hence

$$|\Lambda_{L'}|_\Psi(\mathbb{R}^2) = \sum_{m=1}^M d_m \mathcal{H}^1(\tilde{\zeta}_m) \leq \sum_{m=1}^M d_m \left(\mathcal{H}^1(\zeta_m) + \frac{\eta}{CM} \right) \leq |\Lambda_L|_\Psi(\mathbb{R}^2) + \eta,$$

provided $C = \max\{\Psi(g) : g \in \mathbb{R}^{N-1}, g_i \in \{0,1\} \text{ for all } i = 1, \dots, N-1\}$. Finally, remark that $d_H(L, L') < \eta$ by construction.

Thanks to the previous propositions we are now able to prove the following

Proposition 1.3.11 (Convergence of minimizers)**.** *Let $\{U_\varepsilon\}_\varepsilon \subset H$ be a sequence of minimizers for F_ε^Ψ in H. Then (up to a subsequence) $U_\varepsilon \to U$ in $[L^1(\Omega_\delta)]^{N-1}$, and $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ is a minimizer of $F^\Psi(U, \Omega_\delta) = c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta)$ in $SBV(\Omega_\delta; \mathbb{Z}^{N-1})$.*

Proof. Let $V \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that $DV^\perp = \Gamma - \Lambda$, where Λ canonically represents an acyclic graph $L \in \mathcal{G}(A)$, and let $V_\varepsilon \in H$ such that $\limsup_{\varepsilon \to 0} F_\varepsilon^\Psi(V_\varepsilon, \Omega_\delta) \leq F^\Psi(V, \Omega_\delta)$. Since $F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq F_\varepsilon^\Psi(V_\varepsilon, \Omega_\delta)$, by Proposition 1.3.7 there exists $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ s.t. $U_\varepsilon \to U$ in $[L^1(\Omega_\delta)]^{N-1}$ and by Proposition 1.3.8 we have

$$F^\Psi(U, \Omega_\delta) \leq \liminf_{\varepsilon \to 0} F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq \limsup_{\varepsilon \to 0} F_\varepsilon^\Psi(V_\varepsilon, \Omega_\delta) \leq F^\Psi(V, \Omega_\delta)\,.$$

Given a general $V \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ we can proceed like in Remark 1.2.9 and find V' such that $DV'^\perp = \Gamma - \Lambda_{L'}$ with L' acyclic, and $F^\Psi(V', \Omega_\delta) \leq F^\Psi(V, \Omega_\delta)$. The conclusion follows.

□

Let us focus on the case $\Psi = \Psi_\alpha$, where $\Psi_\alpha(g) = |g|_{1/\alpha}$ for $0 < \alpha \leq 1$ and $\Psi_0(g) = |g|_\infty$, and denote $F_\varepsilon^0 \equiv F_\varepsilon^{\Psi_0}$ and $F_\varepsilon^\alpha \equiv F_\varepsilon^{\Psi_\alpha}$. For $U = (u_1, \dots, u_{N-1}) \in H$ we have

$$F_\varepsilon^0(U, \Omega_\delta) = \int_{\Omega_\delta} \sup_i e_\varepsilon^i(u_i)\, dx, \qquad F_\varepsilon^\alpha(U, \Omega_\delta) = \int_{\Omega_\delta} \left(\sum_{i=1}^{N-1} e_\varepsilon^i(u_i)^{1/\alpha} \right)^\alpha dx, \quad (1.3.11)$$

and

$$F^0(U, \Omega_\delta) := c_0 |DU^\perp - \Gamma|_{\Psi_0}(\Omega_\delta) \quad \text{and} \quad F^\alpha(U, \Omega_\delta) := c_0 |DU^\perp - \Gamma|_{\Psi_\alpha}(\Omega_\delta), \quad (1.3.12)$$

which are the localized versions of (1.2.5) and (1.2.6).

Theorem 1.3.12. *Let* $\{P_1, \dots, P_N\} \subset \mathbb{R}^2$ *such that* $\max_i |P_i| = 1$, $0 < \delta \ll \max_{ij} |P_i - P_j|$, $\Omega = B_{10}(0)$ *and* $\Omega_\delta = \Omega \setminus (\cup_i B_\delta(P_i))$. *For* $0 \le \alpha \le 1$ *and* $0 < \varepsilon \ll \delta$, *denote* $F_\varepsilon^{\alpha,\delta} \equiv F_\varepsilon^\alpha(\cdot, \Omega_\delta)$ *and* $F^{\alpha,\delta} \equiv F^\alpha(\cdot, \Omega_\delta)$, *with* $F_\varepsilon^\alpha(\cdot, \Omega_\delta)$ *(resp.,* $F^\alpha(\cdot, \Omega_\delta)$*) defined in* (1.3.11) *(resp.* (1.3.12)*).*

(i) Let $\{U_\varepsilon^{\alpha,\delta}\}_\varepsilon$ *be a sequence of minimizers for* $F_\varepsilon^{\alpha,\delta}$ *on* H, *with* H *defined in* (1.3.5). *Then, up to subsequences,* $U_\varepsilon^{\alpha,\delta} \to U^{\alpha,\delta}$ *in* $[L^1(\Omega_\delta)]^{N-1}$ *as* $\varepsilon \to 0$, *with* $U^{\alpha,\delta} \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ *a minimizer of* $F^{\alpha,\delta}$ *on* $SBV(\Omega_\delta; \mathbb{Z}^{N-1})$. *Furthermore,* $F_\varepsilon^{\alpha,\delta}(U_\varepsilon^{\alpha,\delta}) \to F^{\alpha,\delta}(U^{\alpha,\delta})$.

(ii) Let $\{U^{\alpha,\delta}\}_\delta$ *be a sequence of minimizers for* $F^{\alpha,\delta}$ *on* $SBV(\Omega_\delta; \mathbb{Z}^{N-1})$. *Up to subsequences we have* $U^{\alpha,\delta} \to U^\alpha|_{\Omega_\eta}$ *in* $[L^1(\Omega_\eta)]^{N-1}$ *as* $\delta \to 0$ *for every fixed* η *sufficiently small, with* $U^\alpha \in SBV(\Omega; \mathbb{Z}^{N-1})$ *a minimizer of* F^α *on* $SBV(\Omega; \mathbb{Z}^{N-1})$, *and* F^α *defined in* (1.2.5), (1.2.6). *Furthermore,* $F^{\alpha,\delta}(U^{\alpha,\delta}) \to F^\alpha(U^\alpha)$.

Proof. In view of Proposition 1.3.11 it remains to prove item (ii). The sequence $\{U^{\alpha,\delta}\}_\delta$ is equibounded in $BV(\Omega_\eta)$ uniformly in η, hence $U^{\alpha,\delta} \to U$ in $[L^1(\Omega_\eta)]^{N-1}$ for all $\eta > 0$ sufficiently small, with $U^\alpha \in SBV(\Omega; \mathbb{Z}^{N-1})$ and $F^{\alpha,\eta}(U^\alpha) \le \liminf_{\delta\to 0} F^{\alpha,\eta}(U^{\alpha,\delta})$ by lower semicontinuity of $F^{\alpha,\eta}$. On the other hand, let $\bar{U}^\alpha$ be a minimizer of F^α on $SBV(\Omega; \mathbb{Z}^{N-1})$. We have $F^{\alpha,\eta}(U^{\alpha,\delta}) \le F^{\alpha,\delta}(U^{\alpha,\delta})$ for any $\delta < \eta$, and by minimality, $F^{\alpha,\delta}(U^{\alpha,\delta}) \le F^{\alpha,\delta}(\bar{U}^\alpha) \le F^\alpha(\bar{U}^\alpha) \le F^\alpha(U^\alpha)$. This proves (ii).

1.4 Convex relaxation

In this section we propose convex positively 1-homogeneous relaxations of the irrigation-type functionals $\mathcal{F}^\alpha$ for $0 \le \alpha < 1$ so as to include the Steiner tree problem corresponding to $\alpha = 0$ (notice that the case $\alpha = 1$ corresponds to the well-known Monge-Kantorovich optimal transportation problem with respect to the Monge cost $c(x, y) = |x - y|$).

More precisely, we consider relaxations of the functional defined by

$$\mathcal{F}^\alpha(\Lambda) = \|\Lambda\|_{\Psi_\alpha} = \int_{\mathbb{R}^d} |g|_{1/\alpha} \, d\mathcal{H}^1 \llcorner L$$

if Λ is the canonical representation of an acyclic graph L with terminal points $\{P_1, \dots, P_N\} \subset \mathbb{R}^d$, so that in particular, according to Definition 1.2.1, we can write $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ with $|\tau| = 1$, $g_i \in \{0, 1\}$. For any other $d \times (N-1)$-matrix valued measure Λ on $\mathbb{R}^d$ we set $\mathcal{F}^\alpha(\Lambda) = +\infty$.

As a preliminary remark observe that, since we are looking for positively 1-homogeneous extensions, any candidate extension $\mathcal{R}^\alpha$ satisfies

$$\mathcal{R}^\alpha(c\Lambda) = |c|\mathcal{F}^\alpha(\Lambda)$$

for any $c \in \mathbb{R}$ and Λ of the form $\tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ as above. As a consequence we have that $\mathcal{R}^\alpha(-\Lambda) = \mathcal{R}^\alpha(\Lambda)$, where $-\Lambda$ represents the same graph L as Λ but only with reversed orientation.

1.4.1 Extension to rank one tensor measures

First of all let us discuss the possible positively 1-homogeneous convex relaxations of $\mathcal{F}^\alpha$ on the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes g \cdot |\Lambda|_1$, where $|\tau| = 1$, $g \in \mathbb{R}^{N-1}$ (cf. Section 1.2.1). For a generic rank one tensor valued measure $\Lambda = \tau \otimes g \cdot |\Lambda|_1$ we can consider extensions of the form

$$\mathcal{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} \Psi^\alpha(g)\, d|\Lambda|_1$$

for a convex positively 1-homogeneous Ψ^α on $\mathbb{R}^{N-1}$ (i.e., a norm) verifying

$$\begin{aligned} &\Psi^\alpha(g) = |g|_{1/\alpha} && \text{if } g_i \in \{0,1\} \text{ for all } i = 1, \dots, N-1, \\ &\Psi^\alpha(g) \geq |g|_{1/\alpha} && \text{for all } g \in \mathbb{R}^{N-1}. \end{aligned} \tag{1.4.1}$$

One possible choice is represented by $\Psi^\alpha(g) = |g|_{1/\alpha}$ for all $g \in \mathbb{R}^{N-1}$, while sharper relaxations are given by, for $\alpha > 0$,

$$\Psi^\alpha_*(g) = \left(\sum_{1 \leq i \leq N-1} |g_i^+|^{1/\alpha} \right)^\alpha + \left(\sum_{1 \leq i \leq N-1} |g_i^-|^{1/\alpha} \right)^\alpha, \tag{1.4.2}$$

and for $\alpha = 0$ by

$$\Psi^0_*(g) = \sup_{1 \leq i \leq N-1} g_i^+ \; - \inf_{1 \leq i \leq N-1} g_i^-, \tag{1.4.3}$$

with $g_i^+ = \max\{g_i, 0\}$ and $g_i^- = \min\{g_i, 0\}$. In particular, Ψ^α_* represents the maximal choice within the class of extensions Ψ^α satisfying

$$\Psi^\alpha(g) = |g|_{1/\alpha} \quad \text{if } g_i \geq 0 \text{ for all } i = 1, \dots, N-1.$$

Indeed, for $\alpha > 0$, $g \in \mathbb{R}^{N-1}$ and $g^\pm = (g_1^\pm, \dots, g_{N-1}^\pm)$, we have

$$\begin{aligned} \Psi^\alpha(g) &\leq \Psi^\alpha(g^+ + g^-) = 2\Psi^\alpha\left(\frac{1}{2}g^+ + \frac{1}{2}g^-\right) \leq 2\left(\frac{1}{2}\Psi^\alpha(g^+) + \frac{1}{2}\Psi^\alpha(g^-)\right) \\ &= \Psi^\alpha(g^+) + \Psi^\alpha(g^-) = |g^+|_{1/\alpha} + |g^-|_{1/\alpha} = \Psi^\alpha_*(g). \end{aligned}$$

The interest in optimal extensions Ψ^α on rank one tensor valued measures relies in the so-called calibration method as a minimality criterion for Ψ^α-mass functionals, as it is done, in particular, in [70] for (STP) using the (optimal) norm Ψ^0_*.

According to the convex extensions Ψ^α and Ψ^0 considered, when it comes to finding minimizers of, respectively, $\mathcal{R}^\alpha$ and $\mathcal{R}^0$ in suitable classes of weighted graphs with prescribed fluxes at their terminal points, or more generally in the class of rank one tensor valued measures having divergence prescribed by (1.2.2), the minimizer is not necessarily the canonical representation of an acyclic graph. Let us consider the following example, where the minimizer contains a cycle.

Example 1.4.1. Consider the Steiner tree problem for $\{P_1, P_2, P_3\} \subset \mathbb{R}^2$. We claim that a minimizer of $\mathcal{R}^0(\Lambda) = \int_{\mathbb{R}^2} |g|_\infty \, d|\Lambda|_1$ within the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes g \cdot |\Lambda|_1$ satisfying (1.2.2) is supported on the triangle $L = [P_1, P_2] \cup [P_2, P_3] \cup [P_1, P_3]$, hence its support is not acyclic and such a minimizer is not related to any optimal Steiner tree. Denoting τ the global orientation of L (i.e., from P_1 to P_2, P_1 to P_3 and P_2 to P_3) we actually have as minimizer

$$\Lambda = \tau \otimes \left(\left[\frac{1}{2}, -\frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_1, P_2] + \left[\frac{1}{2}, \frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_3, P_2] + \left[\frac{1}{2}, \frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_3, P_1] \right). \tag{1.4.4}$$

The proof of the claim follows from Remark 1.4.2 and Lemma 1.4.3.

Remark 1.4.2 (Calibrations). A way to prove the minimality of $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ within the class of rank one tensor valued Radon measures satisfying (1.2.2) is to exhibit a *calibration* for Λ, i.e., a matrix valued differential form $\omega = (\omega_1, \dots, \omega_{N-1})$, with $\omega_j = \sum_{i=1}^d \omega_{ij} dx_i$ for measurable coefficients ω_{ij}, such that

- $d\omega_j = 0$ for all $j = 1, \dots, N-1$;
- $\|\omega\|_* \leq 1$, where $\|\cdot\|_*$ is the dual norm to $\|\tau \otimes g\| = |\tau| \cdot |g|_\infty$, defined as
$$\|\omega\|_* = \sup\{\tau^t \, \omega \, g \ : \ |\tau| = 1, \ |g|_\infty \leq 1\};$$
- $\langle \omega, \Lambda \rangle = \sum_{i,j} \tau_i \omega_{ij} g_j = |g|_\infty$ pointwise, so that
$$\int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda).$$

In this way for any competitor $\Sigma = \tau' \otimes g' \cdot |\Sigma|_1$ we have $\langle \omega, \Sigma \rangle \leq |g'|_\infty$, and, moreover, $\Sigma - \Lambda = DU^\perp$, for $U \in BV(\mathbb{R}^2; \mathbb{R}^{N-1})$, hence

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda - \Sigma \rangle = \int_{\mathbb{R}^2} \langle \omega, DU^\perp \rangle = \int_{\mathbb{R}^2} \langle d\omega, U \rangle = 0 \, .$$

It follows

$$\mathcal{R}^0(\Sigma) \geq \int_{\mathbb{R}^2} \langle \omega, \Sigma \rangle = \int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda) \, ,$$

i.e. Λ is a minimizer within the given class of competitors.

Let us construct a calibration $\omega = (\omega_1, \omega_2)$ for Λ in the general case $P_1 \equiv (x_1, 0)$, $P_2 \equiv (x_2, 0)$ and $P_3 \equiv (0, x_3)$, with $x_1 < 0$, $x_1 < x_2$ and $x_3 > 0$.

Lemma 1.4.3. *Let P_1, P_2, P_3 defined as above and Λ as in* (1.4.4). *Consider $\omega = (\omega_1, \omega_2)$ defined as*

$$\omega_1 = \frac{1}{2a}[(x_1 + a)dx + x_3 dy], \quad \omega_2 = \frac{1}{2a}[(x_1 - a)dx + x_3 dy], \quad \textit{for } (x, y) \in B_L$$
$$\omega_1 = \frac{1}{2b}[(x_2 + b)dx + x_3 dy], \quad \omega_2 = \frac{1}{2b}[(x_2 - b)dx + x_3 dy], \quad \textit{for } (x, y) \in B_R$$

with B_L the left half-plane w.r.t. the line containing the bisector of vertex P_3, B_R the corresponding right half-plane and $a = \sqrt{x_1^2 + x_3^2}$, $b = \sqrt{x_2^2 + x_3^2}$. The matrix valued differential form ω is a calibration for Λ.

Proof. For simplicity we consider here the particular case $x_1 = -\frac{1}{2}$, $x_2 = \frac{1}{2}$ and $x_3 = \frac{\sqrt{3}}{2}$ (the general case is similar). For this choice of x_1, x_2, x_3 we have

$$\omega_1 = \frac{1}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \omega_2 = -\frac{3}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \text{for } (x,y) \in \mathbb{R}^2,\ x < 0,$$
$$\omega_1 = \frac{3}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \omega_2 = -\frac{1}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \text{for } (x,y) \in \mathbb{R}^2,\ x > 0.$$

The piecewise constant 1-forms ω_i for $i = 1, 2$ are globally closed in $\mathbb{R}^2$ (on the line $\{x = 0\}$ they have continuous tangential component), $\|\omega\|_* \leq 1$ (cf. Remark 1.4.2), and taking their scalar product with, respectively, $(1,0) \otimes (1/2, -1/2)$, $(-1/2, \sqrt{3}/2) \otimes (1/2, 1/2)$ for $x < 0$ and $(1/2, \sqrt{3}/2) \otimes (1/2, 1/2)$ for $x > 0$ we obtain in all cases $1/2$, i.e., $|g|_\infty$, so that

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda)\,.$$

Hence ω is a calibration for Λ.

□

Remark 1.4.4. A calibration always exists for minimizers in the class of rank one tensor valued measures as a consequence of Hahn-Banach theorem (see, e.g., [70]), while it may be not the case in general for graphs with integer or real weights. The classical minimal configuration for (STP) with 3 endpoints P_1, P_2 and P_3 admits a calibration with respect to the norm Ψ^0_* in $\mathbb{R}^{N-1}$ (see [70]) and hence it is a minimizer for the relaxed functional $\mathcal{R}^0(\Lambda) = ||\Lambda||_{\Psi^0_*}$ among all real weighted graphs (and all rank one tensor valued Radon measures satisfying (1.2.2)). It is an open problem to show whether or not a minimizer of the relaxed functional $\mathcal{R}^0(\Lambda) = ||\Lambda||_{\Psi^0_*}$ has integer weights.

1.4.2 Extension to general matrix valued measures

Let us turn next to the convex relaxation of $\mathcal{F}^\alpha$ for generic $d \times (N-1)$ matrix valued measures $\Lambda = (\Lambda_1, \ldots, \Lambda_{N-1})$, where Λ_i, for $1 \leq i \leq N-1$, are the vector measures corresponding to the columns of Λ. As a first step observe that, due to the positively 1-homogeneous request on $\mathcal{R}^\alpha$, whenever $\Lambda = p \cdot \mathcal{H}^1 \llcorner L = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, with $|\tau| = cte.$ and $g_i \in \{0, 1\}$, we must have

$$\mathcal{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |\tau| |g|_{1/\alpha}\, d\mathcal{H}^1 \llcorner L = \int_{\mathbb{R}^d} \Phi_\alpha(p)\, d\mathcal{H}^1 \llcorner L,$$

with $\Phi_\alpha(p) = |\tau||g|_{1/\alpha}$ defined only for matrices $p \in K_0$ ($+\infty$ otherwise), where

$$K_0 = \{\tau \otimes g \in \mathbb{R}^{d \times (N-1)},\ g_i \in \{0,1\},\ \ |\tau| = cte.\}.$$

Following [43], we look for Φ_α^{**}, the positively 1-homogeneous convex envelope on $\mathbb{R}^{d\times(N-1)}$ of Φ_α. Setting $q=(q_1,\dots,q_{N-1})$, with $q_i\in\mathbb{R}^d$ its columns, we have that the convex conjugate function $\Phi_\alpha^*(q)=\sup\{q\cdot p-\Phi_\alpha(p),\ p\in K_0\}$ is given by

$$\begin{aligned}\Phi_\alpha^*(q)&=\sup\left\{\tau^t\cdot q\cdot g-|\tau|\cdot|g|_{1/\alpha}\,,\quad |\tau|=cte.,\ g=\sum_{i\in J}e_i,\ J\subset\{1,\dots,N-1\}\right\}\\&=\sup\left\{c\left[\tau^t\cdot\left(\sum_{j\in J}q_j\right)-|J|^\alpha\right]\,,\quad c\geq 0,\,|\tau|=1,\,J\subset\{1,\dots,N-1\}\right\}.\end{aligned}$$

Hence Φ_α^* is the indicator function of the convex set

$$K^\alpha=\left\{q\in\mathbb{R}^{d\times(N-1)},\ \left|\sum_{j\in J}q_j\right|\leq|J|^\alpha\ \ \forall\,J\subset\{1,\dots,N-1\}\right\},$$

and, in particular, for $\alpha=0$, it holds (cf. [43]) that

$$K^0=\left\{q\in\mathbb{R}^{d\times(N-1)},\ \left|\sum_{j\in J}q_j\right|\leq 1\ \ \forall\,J\subset\{1,\dots,N-1\}\right\}.$$

It follows that Φ_α^{**} is the support function of K^α, i.e., for $p\in\mathbb{R}^{d\times(N-1)}$,

$$\Phi_\alpha^{**}(p)=\sup_{q\in K^\alpha}p\cdot q=\sup\left\{p\cdot q\,,\ \left|\sum_{j\in J}q_j\right|\leq|J|^\alpha\,,\ J\subset\{1,\dots,N-1\}\right\}.\tag{1.4.5}$$

We are then led to consider, for matrix valued test functions $\varphi=(\varphi_1,\dots,\varphi_{N-1})$, the relaxed functional

$$\mathcal{R}^\alpha(\Lambda)=\int_{\mathbb{R}^d}\Phi_\alpha^{**}(\Lambda)=\sup\left\{\sum_{i=1}^{N-1}\int_{\mathbb{R}^d}\varphi_i\,d\Lambda_i,\quad\varphi\in C_c^\infty(\mathbb{R}^d;K^\alpha)\right\}.$$

Observe that for Λ a rank one tensor valued measure and $\alpha=0$ the above expression coincides with the one obtained in the previous section choosing $\Psi^0=\Psi^0_*$.

In the planar case $d=2$, consider a $2\times(N-1)$-matrix valued measure $\Lambda=(\Lambda_1,\dots,\Lambda_{N-1})$ such that $\operatorname{div}\Lambda_i=\delta_{P_i}-\delta_{P_N}$. Fix a measure Γ as, for instance, in Remark 1.2.6. We have $\operatorname{div}(\Lambda-\Gamma)=0$ in $\mathbb{R}^2$ and by Poincaré's lemma there exists $U\in BV(\mathbb{R}^2;\mathbb{R}^{N-1})$ such that $\Lambda=\Gamma-DU^\perp$. So the relaxed functional reads

$$\mathcal{E}^\alpha(U)=\mathcal{R}^\alpha(\Lambda)\quad\text{for }\Lambda=\Gamma-DU^\perp,\,U\in BV(\mathbb{R}^2;\mathbb{R}^{N-1}).\tag{1.4.6}$$

The relaxed irrigation problem $(I^\alpha)\equiv\min_{BV}\mathcal{E}^\alpha(U)$ can thus be described in the following equivalent way, according to (1.4.5): let $q=\varphi$ be any matrix valued test function (with columns $q_i=\varphi_i$ for $1\leq i\leq N-1$), then we have

$$(I^\alpha)\equiv\min_{U\in BV(\mathbb{R}^2;\mathbb{R}^{N-1})}\sup\left\{\int_{\mathbb{R}^2}\sum_{i=1}^{N-1}(Du_i^\perp-\Gamma_i)\cdot\varphi_i\,,\quad\varphi\in C_c^\infty(\mathbb{R}^2;K^\alpha)\right\}.$$

Notice that with respect to the similar formulation proposed in [43], there is here the presence of an additional "drift" term; moreover, the constraints set K^α is somewhat different.

We compare now the functional $\mathcal{E}^\alpha(U)$ with the actual convex envelope $(F^\alpha)^{**}(U)$ in the space $BV(\mathbb{R}^2;\mathbb{R}^{N-1})$, where we set $F^\alpha(U) = |DU^\perp - \Gamma|_{\ell^{1/\alpha}}(\mathbb{R}^2)$ if $\Gamma - DU^\perp = \Lambda$ canonically represents an acyclic graph, and $F^\alpha(U) = +\infty$ elsewhere in $BV(\mathbb{R}^2;\mathbb{R}^{N-1})$. In the spirit of [43] (Proposition 3.1), we have

Lemma 1.4.5. *We have* $\mathcal{E}^\alpha(U) \le (F^\alpha)^{**}(U) \le (N-1)^{1-\alpha}\mathcal{E}^\alpha(U)$ *for any* $U \in BV(\mathbb{R}^2;\mathbb{R}^{N-1})$ *and any* $0 \le \alpha < 1$.

Proof. Observe that $\mathcal{E}^\alpha(U) \le (\mathcal{F}^\alpha)^{**}(U)$ by convexity of $\mathcal{E}^\alpha(U)$. Moreover, whenever $\Lambda = \Gamma - DU^\perp$ canonically represents a graph connecting $P_1,\dots,P_N$, we have $(F^\alpha)^{**}(U) \le (F^1)^{**}(U)$ since $F^\alpha(U) \le F^1(U)$. For $\alpha > 0$, denoting $\Lambda = \Gamma - DU^\perp$, we deduce

$$(F^1)^{**}(U) \le \sum_{i=1}^{N-1} |\Lambda_i|(\mathbb{R}^d) \le (N-1)^{1-\alpha}\left(\sum_{i=1}^{N-1}|\Lambda_i|^{1/\alpha}\right)^\alpha(\mathbb{R}^d) \le (N-1)^{1-\alpha}\mathcal{E}^\alpha(U),$$

and analogously we have $(F^1)^{**}(U) \le (N-1)\mathcal{E}^0(U)$.

1.5 Numerical identification of optimal structures

1.5.1 Local optimization by Γ-convergence

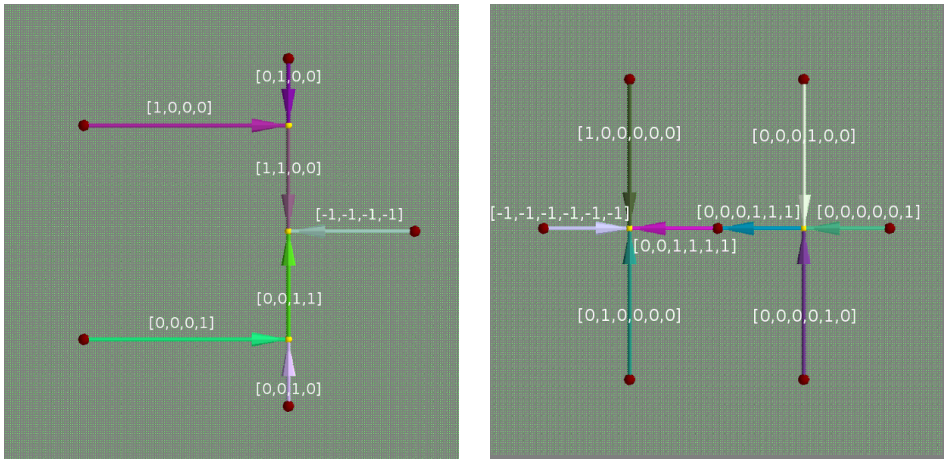

Figure 1.2: Rectilinear Steiner trees and associated vectorial drifts for five and seven points

In this section, we plan to illustrate the use of Theorem 1.3.12 to identify numerically local minima of the Steiner problem. We base our numerical approximation on a standard discretization of (1.3.11). Let $\Omega = (0,1)^2$ and assume $\{P_1,\dots,P_N\} \subset \Omega$; thus, as a

standard consequence, the associated Steiner tree is also contained in Ω. Consider a Cartesian grid covering Ω of step size $h = \frac{1}{S}$ where $S > 1$ is a fixed integer. Dividing every square cell of the grid into two triangles, we define a triangular mesh $\mathcal{T}$ associated to Ω and replace each point P_i with the closest grid point.

Fix now Γ_i an oriented vectorial measure absolutely continuous with respect to $\mathcal{H}^1$ as in Remark 1.2.6. Assume for simplicity that Γ_i is supported on γ_i a union of vertical and horizontal segments contained in Ω and covered by the grid associated to the discrete points $\{(kh, lh),\, 0 \leq k, l < S\}$. Notice that such a measure can be easily constructed by considering, for instance, the oriented ℓ^1-spanning tree of the given points.

To mimic the construction in Section 1.3.2, we define the function space

$$H_i^h \equiv P_1(\mathcal{T}, \Omega \setminus \gamma_i) \cap BV(\Omega)$$

to be the set of functions which are globally continuous on $\Omega \setminus \gamma_i$ and piecewise linear on every triangle of $\mathcal{T}$. Moreover, we require that every function of H_i^h has a jump through γ_i of amplitude -1 in the orthogonal direction of the orientation of Γ_i. Observe that H_i^h is a finite dimensional space of dimension S^2: one element u_i^h can be described by $S^2 + n_i$ parameters and n_i linear constraints describing the jump condition where n_i is the number of grid points covered by γ_i.

Then, we define

$$f_h^i(u_i^h) = h|Du_i^h|^2 + \frac{1}{h}W(u_i^h), \tag{1.5.1}$$

if $u \in L^1(\Omega)$ is in H_i^h and extend f_h^i by letting $f_h^i(u) = +\infty$ otherwise. Notice that these discrete energy densities do not contain the drift terms Γ_i because the information about the drift has been encoded within the discrete spaces H_i^h, leaving us to deal only with the absolutely continuous part of the gradient (see Remark 1.3.1). Then, for $U^h = (u_1^h, \dots, u_{N-1}^h) \in H_1^h \times \cdots \times H_{N-1}^h$ we define

$$G_h^0(U^h) = \int_\Omega \sup_{1 \leq i \leq N-1} f_h^i(u_i^h) \quad \text{and} \quad G_h^\alpha(U^h) = \int_\Omega \left(\sum_{i=1}^{N-1} f_h^i(u_i^h)^{1/\alpha} \right)^\alpha .$$

By a similar strategy we used to prove Theorem 1.3.12, we still also have convergence of minimizers of G_h^0 (resp., G_h^α) to minimizers of $c_0 F^0$ (resp., $c_0 F^\alpha$) with respect to the strong topology of $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$. Observe that an exact evaluation of the integrals involved in (1.5.1) is required to obtain this convergence result (an approximation formula can also be used but then a theoretical proof of convergence would require to study the interaction of the order of approximation with the convergence of minimizers). We point out that this constraint is not critical from a computational point of view since every function u_h^i of finite energy has a constant gradient on every triangle of the mesh. On the other hand, the potential integral can be evaluated formally to obtain an exact estimate of this term whith respect to the degrees of freedom which describe a function of H_i^h.

Based on these results we performed two different numerical experiments. We first approximated the optimal Steiner trees associated to the vertices of a triangle, a regular pentagon and a regular hexagon with its center. To obtain the results of figure 1.3 we

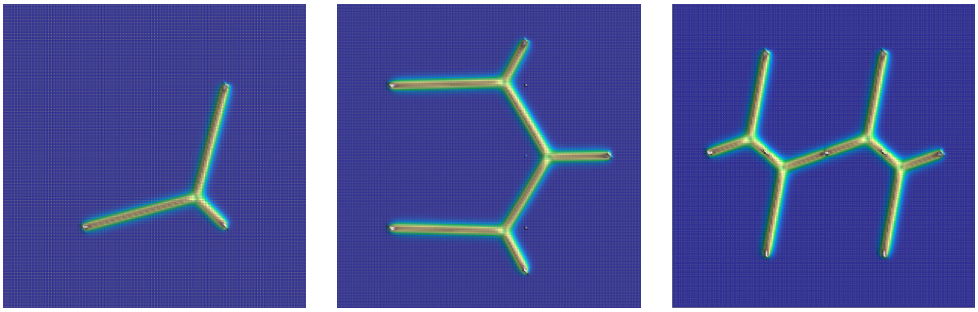

Figure 1.3: Local minimizers obtained by the Γ-convergence approach applied to 3, 5 and 7 points

discretized the problem on a grid of size 200×200. In the case of the triangle we used the associated spanning tree to define the measures $(\Gamma_i)_{i=1,2}$. In the case of the pentagon and of the hexagon we used the rectilinear Euclidean Steiner trees computed by the Geosteiner's library (see, for instance, [110]) to initiate the vectorial measures. We refer to figure 1.2 for an illustration of both singular vector fields. We solved the resulting finite dimensional problem using an interior point solver. Notice that in order to deal with the nonsmooth cost function G_h^0 we had to introduce standard gap variables to get a smooth nonconvex constrained optimization problem. Using [37], we have been able to recover the locally optimal solutions depicted in figure 1.3 in less than five minutes on a standard computer. Whereas the results obtained for the triangle and the pentagon describe globally optimal Steiner trees, the one obtained for the hexagon and its center is only a local minimizer.

In a second experiment we focus on simple irrigation problems to illustrate the versatility of our approach. We applied exactly the same approach to the pentagon setting minimizing the functional G_h^α. We illustrate our results in figure 1.4 in which we recover the solutions of Gilbert-Steiner problems for different values of α. Observe that for small values of α, as expected by Proposition 1.2.4, we recover an irrigation network close to an optimal Steiner tree.

1.5.2 Convex relaxation and multiple solutions

The convex relaxation of Steiner problem (I^0) obtained following [43] reads in our discrete setting as:

$$\min_{(u_i^h)_{1\leq i<N}} \sup_{(\varphi_i^h)_{1\leq i<N}\in K^0} \frac{h^2}{2} \sum_{t\in\mathcal{T}} \sum_{i=1}^{N-1} (\nabla u_i^h)_t \cdot (\varphi_i^h)_t \tag{1.5.2}$$

where

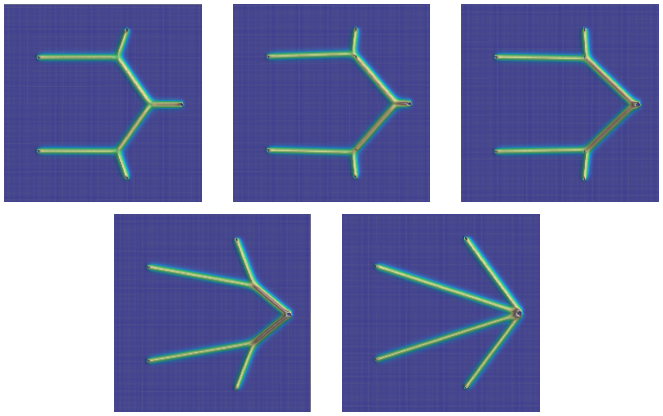

Figure 1.4: Gilbert–Steiner solutions associated to parameters $\alpha = 0.2$, 0.4, 0.6, 0.8 and 1 (from left to right)

$$K^0 = \left\{ (\varphi_i^h)_{1\leq i<N} \in (\mathbb{R}^{2\mathcal{T}})^{N-1} \,|\forall J \subset \{1,\dots,N-1\}, \forall t \in \mathcal{T}, \left|\sum_{j\in J} (\varphi_j^h)_t\right| \leq 1 \right\} \quad (1.5.3)$$

and $\forall 1 \leq i < N$, $u_i^h \in H_i^h$. Applying conic duality (see, for instance, Lecture 2 of [19]), we obtain that the optimal vector (u_i^h) solves the following minimization problem

$$\min_{(u_i^h)_{1\leq i<N} \in L,\, (\psi_J^h)_{J\subset\{1,\dots,N-1\}} \in (\mathbb{R}^{2\mathcal{T}})^{2^{N-1}}} \frac{h^2}{2} \sum_{t\in\mathcal{T}} \sum_{J\subset\{1,\dots,N-1\}} |(\psi_J^h)_t| \quad (1.5.4)$$

where L is the set of discrete vectors $(u_i^h)_{1\leq i<N}$ which satisfy $\forall i = 1,\dots,N-1$, $\forall t \in \mathcal{T}$:

$$(\nabla u_i^h)_t = \sum_{J\subset\{1,\dots,N-1\},\, i\in J} (\psi_J^h)_t. \quad (1.5.5)$$

We solved this convex linearly constrained minimization problem using the conic solver of the library Mosek [78] on a grid of dimension 300×300. Observe that this convex formulation is also well adapted to the, now standard, large scale algorithms of proximal type. We studied four different test cases: the vertices of an equilateral triangle, a square, a pentagon and finally an hexagon and its center as in previous section. As illustrated in the left picture of figure 1.5, the convex formulation is able to approximate the optimal structure in the case of the triangle. Due to the symmetries of the problems, the three last examples do not have unique solutions. Thus, the result of the optimization is expected to be a convex combination of all solutions whenever the relaxation is sharp, as it can be observed on the second and fourth case of figure 1.5. Notice that we do not expect this behaviour to hold for any configuration of points. Indeed the numerical solution in the third picture of figure 1.5 is not supported on a convex combination of global solutions since the density in the middle point is not 0. Whereas the local Γ-convergence approach

of previous section was only able to produce a local minimum in the case of the hexagon and its center, the convexified formulation gives a relatively precise idea of the set of optimal configurations (see the last picture of figure 1.5 where we can recognize within the figure the two global solutions).

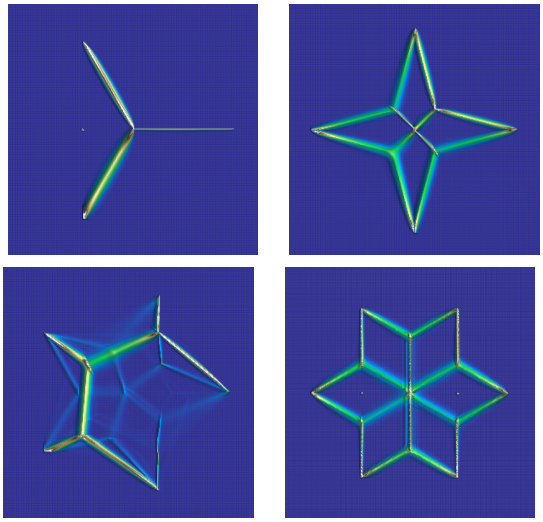

Figure 1.5: Results obtained by convex relaxation for $3, 4$, 5 and 7 given points

Chapter 2

Variational approximation of functionals defined on 1-dimensional connected sets in $\mathbb{R}^d$

In this chapter we consider the Euclidean Steiner tree problem and, more generally, (single sink) Gilbert–Steiner problems as prototypical examples of variational problems involving 1-dimensional connected sets in $\mathbb{R}^d$. Following the the analysis for the planar case presented in the previous chapter, we provide a variational approximation through Ginzburg–Landau type energies proving a Γ-convergence result for $d \geq 3$.

2.1 Introduction

The (single sink) Gilbert–Steiner problem, or α-irrigation problem [21, 112] consists of finding a network L along which to flow unit masses located at the sources $P_1, \dots, P_{N-1}$ to the target point P_N, and in choosing such a network in order to optimize a sublinear (concave) function of the transported mass density. Geometrically speaking, the network L can be viewed as $L = \cup_{i=1}^{N-1} \lambda_i$, with λ_i a path connecting P_i to P_N, describing the trajectory of the unit mass located at P_i. To favour branching, one considers a cost which is a sublinear function of the mass density $\theta(x) = \sum_{i=1}^{N-1} \mathbf{1}_{\lambda_i}(x)$, so that, for $0 \leq \alpha \leq 1$, we are led to study

$$\inf \int_L |\theta(x)|^\alpha d\mathcal{H}^1(x). \tag{I_α}$$

In particular, (I_0) reduces to the optimization of the total length of the graph L and thus corresponds to the classical Euclidean Steiner Tree Problem (STP), i.e., finding the shortest connected graph which contains the terminal points $P_1, \dots, P_N$. For any $\alpha \in [0, 1]$ a solution to (I_α) is known to exist and any optimal network turns out to be a tree [21].

As pointed out in the previous chapter, the Gilbert–Steiner problem represents the basic example of problems defined on 1-dimensional connected sets, and it has recently

received a renewed attention in the Calculus of Variations community. In the last years available results focused on variational approximations of the problem mainly in the planar case [30, 44, 82, 29], while higher dimensional approximations have been recently proposed in [45, 28].

In this chapter we extend to the higher dimensional context the two dimensional analysis developed in Chapter 1 and we propose a variational approximation for (I_α) in the Euclidean space $\mathbb{R}^d$, $d \geq 3$. We prove a result in the spirit of Γ-convergence (see Theorem 2.4.6 and Proposition 2.4.5) by considering integral functionals of Ginzburg–Landau type [5, 6] (see also [97]). This approach builds upon the interpretation of (I_α) as a mass minimization problem in a cobordism class of integral currents with multiplicities in a suitable normed group (as studied in [70, 69]). Thus, the relevant energy turns out to be a convex positively 1-homogeneous functional (a norm), for which one can use calibration type arguments to prove minimality of certain given configurations [69, 74]. The proposed method is quite flexible and can be adapted to a variety of situations, including manifold type ambients where a suitable formulation in vector bundles can be used (this will be treated in a forthcoming work).

Eventually, we remark that another way to approach the problem is to investigate possible convex relaxations of the limiting functional, as already pointed out in Section 1.4. In the next chapter we will further extend such an approach, so as to include more general irrigation-type problems (with multiple sources/sinks) and even problems for 1-dimensional structures on manifolds.

The plan of the chapter is as follows. In Section 2.2 we briefly review the main concepts needed in the subsequent sections and in Section 2.3 we recall the variational setting for (I_α) relying on the concept of Ψ-mass. We then provide in Section 2.4 a variational approximation of the problem in any dimension $d \geq 3$ by means of Ginzburg–Landau type energies.

2.2 Preliminaries and notations

In this section we fix the notation used in the rest of the chapter and some basic facts. We will follow closely [5, 6], to which we refer for a more detailed treatment.

For any $d \geq 2$, we denote by $\{e_1, \dots, e_d\}$ the standard basis of $\mathbb{R}^d$, B^d_r is the open ball in $\mathbb{R}^d$ with centre the origin and radius r, $\mathbb{S}^{d-1} = \partial B^d_1$ is the unit sphere in $\mathbb{R}^d$, and

$$\alpha_d = |B^d_1|, \quad \beta_d = (d-1)^{d/2}\alpha_d,$$

where $|\cdot|$ stands for the Lebesgue measure of the given set. For $0 \leq k \leq n$ we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. Furthermore, we assume we are given N distinct points $P_1, \dots, P_N$ in $\mathbb{R}^d$, for $n \geq 3$ and $N \geq 2$, and we denote $A = \{P_1, \dots, P_N\}$. We also assume, without loss of generality, that $A \subset B^d_1$.

Ginzburg–Landau functionals. We consider a continuous potential $W \colon \mathbb{R}^{d-1} \to \mathbb{R}$

which vanishes only on $\mathbb{S}^{d-2}$ and is strictly positive elsewhere, and we require

$$\liminf_{|y|\to 1} \frac{W(y)}{(1-|y|)^2} > 0 \quad \text{and} \quad \liminf_{|y|\to\infty} \frac{W(y)}{|y|^{d-1}} > 0.$$

Given $\varepsilon > 0$, $\Omega \subset \mathbb{R}^d$ open and $u \in W^{1,d-1}(\Omega;\mathbb{R}^{d-1})$, we set

$$F_\varepsilon(u,\Omega) := \int_\Omega e_\varepsilon(u)\,dx = \int_\Omega \frac{1}{d-1}|Du|^{d-1} + \frac{1}{\varepsilon^2}W(u)\,dx, \tag{2.2.1}$$

where $|Du|$ is the Euclidean norm of the matrix Du.

Currents. Given $k = 0,\dots,n$, let $\bigwedge^k(\mathbb{R}^d)$ be the space of k-covectors on $\mathbb{R}^d$ and $\bigwedge_k(\mathbb{R}^d)$ the space of k-vectors. The canonical basis of $\bigwedge^1(\mathbb{R}^d)$ will be denoted as $\{dx^1,\dots,dx^d\}$. For a k-covector ω we define its comass as

$$||\omega||^* = \sup\{\omega\cdot v \,:\, v \text{ is a simple } k\text{-vector with } |v| = 1\}.$$

For $\Omega \subset \mathbb{R}^d$, a k-form on Ω is a map from Ω into the space of k-covectors and a k-dimensional current is a distribution valued into the space of k-vectors. We denote as $\mathcal{D}^k(\Omega)$ the space of all smooth k-forms with compact support and as $\mathcal{D}_k(\Omega)$ the space of all k-currents. In particular, the space $\mathcal{D}_k(\Omega)$ can be identified with the dual of the space $\mathcal{D}^k(\Omega)$ and equipped with the corresponding weak* topology. Furthermore, for $T \in \mathcal{D}_k(\Omega)$ and an open subset $V \subset \Omega$, we define the mass of T in V as

$$||T||_V = \sup\{T(\omega) \,:\, \omega \in \mathcal{D}^k(V), ||\omega(x)||^* \le 1 \text{ for every } x\}$$

and we denote the mass of T as $||T|| = ||T||_\Omega$. The boundary of a k-current T is the $(k-1)$-current characterized as $\partial T(\omega) = T(d\omega)$ for every $\omega \in \mathcal{D}^{k-1}(\Omega)$, where $d\omega$ is the exterior differential of the form ω. Let $T \in \mathcal{D}_k(\Omega)$ be a current with locally finite mass, then there exist a positive finite measure μ_T on $\mathbb{R}^d$ and a Borel measurable map $\tau\colon \Omega \to \bigwedge_k(\mathbb{R}^d)$ with $||\tau|| \le 1$ μ_T-a.e., such that

$$T(\omega) = \int_{\mathbb{R}^d} \omega(x)\cdot\tau(x)\,d\mu_T(x) \quad \text{for every } \omega \in \mathcal{D}^k(\Omega). \tag{2.2.2}$$

We denote $|T| = |\mu_T|$ the variation of the measure μ_T, so that, given $V \subset \Omega$, one has $||T||_V := |T|(V)$. A k-current T is said to be normal whenever both T and ∂T have finite mass, and we denote as $\mathbb{N}_k(\Omega)$ such space.

Given a k-rectifiable set Σ oriented by τ and a real-valued function $\theta \in L^1_{loc}(\mathcal{H}^k \llcorner \Sigma)$, we define the current $T = [\![\Sigma,\tau,\theta]\!]$ as

$$T(\omega) = \int_\Sigma \theta(x)\omega(x)\cdot\tau(x)\,d\mathcal{H}^k(x),$$

and we refer to θ as the multiplicity of the current. A k-current T is called rectifiable if it can be represented as $T = [\![\Sigma,\tau,\theta]\!]$ for a k-rectifiable set Σ and an integer valued multiplicity θ. If both T and ∂T are rectifiable, we say T is an integral current and

denote as $\mathcal{I}_k(\Omega)$ the corresponding group. A polyhedral current in $\mathbb{R}^d$ is a finite sum of k-dimensional oriented simplexes S_i endowed with some constant integer multiplicities σ_i, and we generally assume that $S_i \cap S_j$ is either empty of consists of a common face of S_i and S_j. As it is done in [6], we introduce the following flat norm of a current $T \in \mathcal{D}_k(\Omega)$:

$$\mathbf{F}_\Omega(T) := \inf\{||S||_\Omega \,:\, S \in \mathcal{D}_{k+1}(\Omega) \text{ and } T = \partial S\}, \tag{2.2.3}$$

and the infimum is taken to be $+\infty$ if T is not a boundary.

Jacobians of Sobolev maps and boundaries. Given $\Omega \subset \mathbb{R}^d$ open and $u \in W^{1,d-2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{d-1}) \cap L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^{d-1})$, following [63], we define the $(d-2)$-form

$$j(u) = \sum_{i=1}^{d-1} (-1)^{i-1} u_i \cdot \bigwedge_{j\neq i} du_j$$

and we set the Jacobian of u to be

$$Ju := \frac{1}{d-1} d[j(u)]$$

in the sense of distributions. This means that for any $\omega \in \mathcal{D}^{d-1}(\Omega)$

$$Ju \cdot \omega = \frac{1}{d-1} \int_{\mathbb{R}^d} d^*\omega \cdot j(u)\, dx,$$

where d^* is the formal adjoint of d. By means of the $\star$ operator we can identify such a form with a 1-current $\star Ju$. In our specific context, the $\star$ operator can be defined, at the level of vectors/covectors, as follows: given a $(d-1)$-covector w, the vector $\star w$ is defined by the identity

$$v \cdot \star w = (v \wedge w)(e_1 \wedge \cdots \wedge e_d) \quad \text{for all } v \in \wedge^1(\mathbb{R}^d).$$

Jacobians turn out to be the main tool in our analysis due to their relation with boundaries. In order to highlight such a relation we need some additional notation: given any segment S in $\mathbb{R}^d$ and given $\delta, \gamma > 0$, let us define the set

$$U(S,\delta,\gamma) = \left\{ x \in \mathbb{R}^d \,:\, \mathrm{dist}(x,S) < \min\left\{ \delta, \frac{\gamma}{\sqrt{1+\gamma^2}} \mathrm{dist}(x,\partial S) \right\} \right\}.$$

If we identify the line spanned by S with $\mathbb{R}$, we can write each point $x \in U(S,\delta,\gamma)$ as $x = (x', x'') \in \mathbb{R} \times \mathbb{R}^{d-1}$, so that

$$U(S,\delta,\gamma) = \{x' \in S \,:\, |x''| \leq \min(\delta, \gamma \cdot \mathrm{dist}(x', \partial S))\}.$$

We can now recall the main result of [5] (rewritten in our specific context).

Theorem 2.2.1 (Theorem 5.10, [5]). *Let $M = [\![\Sigma, \tau, 1]\!]$ be the (polyhedral) boundary of a polyhedral current N of dimension 2 in $\mathbb{R}^d$, and let F_0 denote the union of the faces of N of dimension 0. Then there exists $u \in W^{1,d-2}(\mathbb{R}^d; \mathbb{S}^{d-2})$ such that $\star Ju = \alpha_{d-1} M$, with u locally Lipschitz in the complement of $\Sigma \cup F_0$ and constant outside a bounded neighbourhood of N, and Du belongs to L^p for every $p < d-1$ and satisfies $|Du(x)| = O(1/\mathrm{dist}(x, \Sigma \cup F_0))$. Moreover, there exist $\delta, \gamma > 0$ small enough such that, for each 1-simplex $S_k \subset \Sigma$, one has*

$$u(x) = \frac{x''}{|x''|} \quad \text{for all } x = (x', x'') \in U(S_k, \delta, \gamma).$$

2.3 Gilbert–Steiner problems and currents

In this section we briefly review (this time in terms of currents) the approach used in [25, 27], which is to say the framework introduced by Marchese and Massaccesi in [70, 69], and describe Gilbert–Steiner problems in terms of a minimum mass problem for a given family of rectifiable 1-currents in $\mathbb{R}^d$.

The set of possible minimizers for (I_α) can be reduced to the set of (connected) acyclic graphs L that are described as the superposition of $N-1$ curves.

Definition 2.3.1. *We define $\mathcal{G}(A)$ to be the set of acyclic graphs L of the form*

$$L = \bigcup_{i=1}^{N-1} \lambda_i,$$

where each λ_i is a simple rectifiable curve connecting P_i to P_N and oriented by an $\mathcal{H}^1$-measurable unit vector field τ_i, with $\tau_i(x) = \tau_j(x)$ for $\mathcal{H}^1$-a.e. $x \in \lambda_i \cap \lambda_j$, and we denote by τ the corresponding global orientation, i.e., $\tau(x) = \tau_i(x)$ for $\mathcal{H}^1$-a.e. $x \in \lambda_i$.

It can be shown (see, e.g., [70, Lemma 2.1]), that (I_α) is equivalent to

$$\min \left\{ \int_L |\theta(x)|^\alpha d\mathcal{H}^1, \quad L \in \mathcal{G}(A), \;\; \theta(x) = \sum_{i=1}^{N-1} \mathbf{1}_{\lambda_i}(x) \right\}. \tag{2.3.1}$$

Given now $L \in \mathcal{G}(A)$, we identify each component λ_i with the corresponding 1-current $\Lambda_i = [\![\lambda_i, \tau_i, 1]\!]$ and we consider $\Lambda = (\Lambda_1, \dots, \Lambda_{N-1}) \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1}$.

Definition 2.3.2. *We define $\mathcal{L}(A)$ to be the set $\Lambda \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1}$ such that each component is of the form $\Lambda_i = [\![\lambda_i, \tau_i, 1]\!]$ for some $L \in \mathcal{G}(A)$, and write $\Lambda \equiv \Lambda_L$ to highlight the supporting graph.*

Given $\Lambda = (\Lambda_1, \dots, \Lambda_{N-1}) \in [\mathbb{N}_1(\mathbb{R}^d)]^{N-1}$ and a function $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^{d \times N-1})$, with $\varphi = (\varphi_1, \dots, \varphi_{N-1})$, one sets

$$\langle \Lambda, \varphi \rangle = \sum_{i=1}^{N-1} \langle \Lambda_i, \varphi_i \rangle$$

and for a norm Ψ on $\mathbb{R}^{N-1}$, we define the Ψ-mass measure of Λ as

$$|\Lambda|_\Psi(\Omega) := \sup_{\substack{\omega\in C_c^\infty(\Omega;\mathbb{R}^d)\\ h\in C_c^\infty(\Omega;\mathbb{R}^{N-1})}} \{\langle \Lambda, \omega\otimes h\rangle\,, \quad |\omega(x)|\le 1\,,\ \Psi^*(h(x))\le 1\}\,, \tag{2.3.2}$$

for $\Omega\subset\mathbb{R}^d$ open, where $\Psi^*(y)=\sup_{x\in\mathbb{R}^{N-1}}\langle y,x\rangle-\Psi(x)$ is the dual norm to Ψ w.r.t. the scalar product on $\mathbb{R}^{N-1}$, and we let the Ψ-mass norm of Λ to be

$$||\Lambda||_\Psi = |\Lambda|_\Psi(\mathbb{R}^d). \tag{2.3.3}$$

As described in [70, 25, 27], the problem defined in (2.3.1) is equivalent to

$$\inf\{||\Lambda||_{\Psi_\alpha}\,:\,\Lambda=(\Lambda_1,\dots,\Lambda_{N-1})\in[\mathcal{I}_1(\mathbb{R}^d)]^{N-1},\ \partial\Lambda_i=\delta_{P_N}-\delta_{P_i}\}, \tag{2.3.4}$$

where Ψ_α is the $\ell^{1/\alpha}$ norm on $\mathbb{R}^{N-1}$ for $0<\alpha\le 1$, and the ℓ^∞ norm for $\alpha=0$. This means that any minimizer $\bar\Lambda$ of (2.3.4) is of the form $\bar\Lambda=\Lambda_{\bar L}$ for a minimizer $\bar L$ of (2.3.1), and given any minimizer $\bar L$ of (2.3.1) then the corresponding $\Lambda_{\bar L}$ minimizes (2.3.4).

Remark 2.3.3. In [70, 69] problem (2.3.4) is introduced in the context of a mass minimization problem for integral currents with coefficients in a suitable normed group. In that case, the Ψ-mass defined above is simply the mass of the current deriving from the particular choice of the norm for the coefficients group.

Calibrations. One of the main advantages of formulation (2.3.4) is the possibility to introduce calibration-type arguments for proving minimality of a given candidate. For a fixed $\bar\Lambda\in[\mathbb{N}_1(\mathbb{R}^d)]^{N-1}$, a (generalized) calibration associated to $\bar\Lambda$ is a linear and bounded functional $\varphi\colon[\mathbb{N}_1(\mathbb{R}^d)]^{N-1}\to\mathbb{R}$ such that

(i) $\varphi(\bar\Lambda)=||\bar\Lambda||_\Psi$,

(ii) $\varphi(\partial R)=0$ for any $R\in[\mathbb{N}_2(\mathbb{R}^d)]^{N-1}$,

(iii) $\varphi(\Lambda)\le||\Lambda||_\Psi$ for any $\Lambda\in[\mathbb{N}_1(\mathbb{R}^d)]^{N-1}$.

The existence of a calibration is a sufficient condition to prove minimality in (2.3.4). Indeed, let $\bar\Lambda$ be a competitor in (2.3.4) and φ be a calibration for $\bar\Lambda$. Consider any $\Lambda\in[\mathbb{N}_1(\mathbb{R}^d)]^{N-1}$, with $\partial\Lambda_i=\delta_{P_N}-\delta_{P_i}$. By assumption, for each $i=1,\dots,N-1$, one has $\partial(\bar\Lambda_i-\Lambda_i)=0$, so that there exists a 2-current R_i such that $\bar\Lambda_i=\Lambda_i+\partial R_i$. Hence,

$$||\bar\Lambda||_\Psi \overset{\text{(i)}}{=} \varphi(\bar\Lambda)=\varphi(\Lambda+\partial R)=\varphi(\Lambda)+\varphi(\partial R) \overset{\text{(iii), (ii)}}{\le} ||\Lambda||_\Psi$$

which proves the minimality of $\bar\Lambda$ in (2.3.4) (and, more generally, also minimality among normal currents). We also remark that once a calibration exists it must calibrate all minimizers.

A calibration-type argument. The general idea behind calibrations can be used to tackle minimality in suitable subclasses of currents, as long as the previous derivation can be proved to still hold true. Consider, as displayed in figure 2.1, the Steiner tree

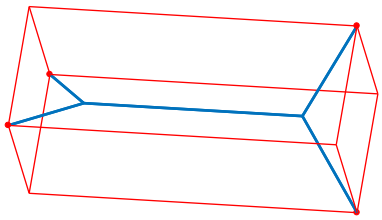

Figure 2.1: We consider the Steiner tree problem for 4 vertices obtained as "opposite" couples of vertices of a rectangular cuboid.

problem for four points in $\mathbb{R}^3$ with $P_1 = (-3/2, -\sqrt{3}/2, 0)$, $P_2 = (-3/2, \sqrt{3}/2, 0)$, $P_3 = (3/2, 0, \sqrt{3}/2)$ and $P_4 = (3/2, 0, -\sqrt{3}/2)$. Let us identify the two points $S_1 = (-1, 0, 0)$ and $S_2 = (1, 0, 0)$, and fix as norm Ψ the ℓ^∞ norm on the coefficients space $\mathbb{R}^3$. Given a list of points $Q_1, \dots, Q_k$, we write as $[Q_1, \dots, Q_k]$ the polyhedral current connecting them and oriented from Q_1 to Q_k. Our aim is to prove that

$$\bar{\Lambda} = ([P_1, S_1, S_2, P_4], [P_2, S_1, S_2, P_4], [P_3, S_2, P_4])$$

is a minimizer of the Ψ-mass $|| \cdot ||_\infty \equiv || \cdot ||_{\ell^\infty}$ among all currents $\Lambda \in \mathcal{B}$, where $\mathcal{B} \subset [\mathbb{N}_1(\mathbb{R}^3)]^3$ is the family of currents Λ satisfying the given boundary conditions $\partial \Lambda_i = \delta_{P_4} - \delta_{P_i}$, and such that there exist a positive finite measure μ_Λ on $\mathbb{R}^3$, a unit vector field τ_Λ and a function $g^\Lambda \colon \mathbb{R}^3 \to \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_2 + e_3\}$ such that $\Lambda_i(\omega) = \int_{\mathbb{R}^3} g_i^\Lambda(x)\omega \cdot \tau_\Lambda \, d\mu_\Lambda$. Let us formally identify any such object as $\Lambda = (\tau_\Lambda \otimes g^\Lambda)\mu_\Lambda$ (loosely speaking, we consider only the family of normal rank one currents with a prescribed superposition pattern for different flows). It can be easily seen that $\bar{\Lambda} \in \mathcal{B}$ and for any $\Lambda \in \mathcal{B}$ we have $||\Lambda||_\infty = \int_{\mathbb{R}^3} ||g^\Lambda(x)||_\infty \, d\mu_\Lambda(x)$. For proving minimality of $\bar{\Lambda}$ for the ℓ^∞-mass among all competitors in $\mathcal{B}$ we can use a calibration argument: let us consider $\varphi \colon [\mathbb{N}(\mathbb{R}^3)]^3 \to \mathbb{R}$ defined as

$$\varphi(\Lambda) = \sum_{i=1}^{3} \langle \Lambda_i, \omega_i \rangle$$

where ω_i are fixed to be

$$\omega_1 = \frac{1}{2}dx^1 + \frac{\sqrt{3}}{2}dx^2, \quad \omega_2 = \frac{1}{2}dx^1 - \frac{\sqrt{3}}{2}dx^2, \quad \omega_3 = -\frac{1}{2}dx^1 - \frac{\sqrt{3}}{2}dx^3.$$

One can show by direct computations that $\varphi(\bar{\Lambda}) = ||\bar{\Lambda}||_\infty$, so that given any other $\Lambda \in \mathcal{B}$ and $R \in [\mathbb{N}_2(\mathbb{R}^3)]^3$ such that $\bar{\Lambda} = \Lambda + \partial R$, we have $||\bar{\Lambda}||_\infty = \varphi(\bar{\Lambda}) = \varphi(\Lambda) + \varphi(\partial R)$, for

which

$$\varphi(\Lambda) = \sum_{i=1}^{3} \int_{\mathbb{R}^3} g_i^\Lambda(x)\omega_i \cdot \tau_\Lambda \, d\mu_\Lambda \leq \int_{\mathbb{R}^3} ||g^\Lambda||_\infty \, d\mu_\Lambda = ||\Lambda||_\infty$$

because $g^\Lambda \in \{e_1, e_2, e_3, e_1+e_2, e_1+e_2+e_3\}$ for μ_Λ-a.e. x, and

$$\varphi(\partial R) = \sum_{i=1}^{3} \langle R_i, d\omega_i \rangle = 0.$$

Hence, $||\bar{\Lambda}||_\infty \leq ||\Lambda||_\infty$ for any $\Lambda \in \mathcal{B}$. Up to permutations, the class $\mathcal{B}$ represents every possible acyclic graph $L \in \mathcal{G}(\{P_1, P_2, P_3, P_4\})$ with 2 additional Steiner points and thus the support of $\bar{\Lambda}$ is an optimal Steiner tree within that family of graphs. Remark that any minimal configuration cannot have 0 or 1 Steiner points because these configurations violate the 120° angle condition, so that we can conclude that the support of $\bar{\Lambda}$ is indeed an optimal Steiner tree. This extends for the first time to an higher dimensional context calibration-type arguments which up to now have been extensively used almost exclusively in the planar case (e.g., in [70, 69]).

In chapter 1, we investigate a variational approximation of (2.3.4) in the two dimensional case, relying on a further reformulation of the problem within a suitable family of SBV functions and then providing a variational approximation based on Modica–Mortola type energies. Here, instead, we work in dimension three and higher and address (2.3.4) directly by means of Ginzburg–Landau type energies.

2.4 Variational approximation of Ψ-masses

In this section we state and prove our main results, namely Proposition 2.4.5 and Theorem 2.4.6, concerning the approximation of minimizers of Ψ-masses functionals through Jacobians of minimizers of Ginzburg–Landau type functionals, much in the spirit of [6].

2.4.1 Ginzburg–Landau functionals with prescribed boundary data

In this section, following closely [6], we consider Ginzburg–Landau functionals for functions having a prescribed trace v on the boundary of a given open Lipschitz domain.

Domain and boundary datum. Fix two points $P, Q \in \mathbb{R}^d$, with $\max(|P|, |Q|) \leq 1$, and let Σ be a simple acyclic polyhedral curve joining P and Q, and oriented from Q to P. Let $S_1, \ldots, S_K$ be the K segments composing Σ and, for $\delta, \gamma > 0$ small enough define

$$U = \bigcup_{k=1}^{K} U(S_k, \delta, \gamma), \quad \text{and} \quad \Omega^{\delta,\gamma} = B_{10}^d \setminus \bar{U}. \tag{2.4.1}$$

Consider the boundary datum $v \in W^{1-1/(d-1),d-1}(\partial\Omega^{\delta,\gamma}; \mathbb{S}^{d-2})$ defined as

$$v(x) = \begin{cases} \dfrac{x''}{|x''|} & \text{for } x = (x', x'') \in \partial U \\ e_{d-1} & \text{for } x \in \partial B_{10}^d \end{cases} \tag{2.4.2}$$

By construction one has

$$\star Jv = \alpha_{d-1}(\delta_Q - \delta_P).$$

In this context, for only two points, the Ψ-mass reduces (up to a constant) to the usual mass, and thus we can directly rely on Corollary 1.2 of [6], which yields the following.

Theorem 2.4.1. *For $\delta, \gamma > 0$ small enough, consider the Lipschitz domain $\Omega^{\delta,\gamma}$ defined in* (2.4.1) *and let v be the boundary datum defined in* (2.4.2)*.*

(i) Consider a (countable) sequence $\{u_\varepsilon\}_\varepsilon \subset W^{1,d-1}(\Omega^{\delta,\gamma}; \mathbb{R}^{d-1})$ with trace v on $\partial\Omega^{\delta,\gamma}$ such that $F_\varepsilon(u_\varepsilon, \Omega^{\delta,\gamma}) = O(|\log\varepsilon|)$. Then, up to subsequences, there exists a rectifiable 1-current M supported in $\bar{\Omega}^{\delta,\gamma}$, with $\partial M = \delta_Q - \delta_P$, such that the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ to $\alpha_{d-1}M$ and

$$\liminf_{\varepsilon\to 0} \frac{F_\varepsilon(u_\varepsilon, \Omega^{\delta,\gamma})}{|\log\varepsilon|} \geq \beta_{d-1}||M|| \tag{2.4.3}$$

(ii) Given a rectifiable 1-current M supported in $\bar{\Omega}^{\delta,\gamma}$ such that $\partial M = \delta_Q - \delta_P$, for every $\varepsilon > 0$ we can find u_ε such that $u_\varepsilon = v$ on $\partial\Omega^{\delta,\gamma}$, $\mathbf{F}_{\mathbb{R}^d}(\star Ju_\varepsilon - \alpha_{d-1}M) \to 0$ and

$$\lim_{\varepsilon\to 0} \frac{F_\varepsilon(u_\varepsilon, \Omega^{\delta,\gamma})}{|\log\varepsilon|} = \beta_{d-1}||M||$$

In particular, given $\{u_\varepsilon\}_\varepsilon$ a sequence of minimizers of $F_\varepsilon(\cdot, \Omega^{\delta,\gamma})$ with trace v on $\partial\Omega^{\delta,\gamma}$, then $F_\varepsilon(u_\varepsilon, \Omega^{\delta,\gamma}) = O(|\log\varepsilon|)$ and, possibly passing to a subsequence, the Jacobians $\star Ju_\varepsilon$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ to $\alpha_{d-1}M$, where M minimizes the mass among all rectifiable 1-currents supported on $\bar{\Omega}^{\delta,\gamma}$ with boundary $\delta_Q - \delta_P$.

Point (i) of the previous theorem corresponds to the derivation of Section 1.3.1, where we consider Modica–Mortola functionals for maps with prescribed jump, and here the prescribed jump is somehow replaced by the prescribed boundary datum "around" the drift Σ. As before, the idea is now to extend the previous (single-component) result to problems involving Ψ-masses for $N \geq 3$.

2.4.2 The approximating functionals F_ε^Ψ

We now consider Ginzburg–Landau approximations for Ψ-masses whenever we are given $N \geq 3$ points. Fix then a norm $\Psi\colon \mathbb{R}^{N-1} \to [0, +\infty)$ on $\mathbb{R}^{N-1}$, and consider the Ψ-mass defined in (2.3.3).

Construction of the domain. Fix a family of $N-1$ simple polyhedral curves γ_i each one connecting P_i to P_N and denote by $\Gamma_i = [\![\gamma_i, \tau_i, 1]\!]$ the associated 1-current (oriented from P_N to P_i). Suppose, without loss of generality, that $\gamma_i \cap \gamma_j = \{P_N\}$ for any $i \neq j$, i.e., any two curves do not intersect each other. Every γ_i can then be viewed as the concatenation of m_i (oriented) segments $S_{i,1}, \dots, S_{i,m_i}$, for each of which we consider the neighbourhood

$$U_{i,j}^{\delta,\gamma} = U(S_{i,j}, \delta, \gamma)$$

for $\delta, \gamma > 0$. Define now $V_i^{\delta,\gamma} = \cup_j U_{i,j}^{\delta,\sigma}$ and observe that, by finiteness, we can fix δ, γ sufficiently small such that $\bar{V}_i^{\delta,\gamma} \cap \bar{V}_j^{\delta,\gamma} = \{P_N\}$ for any $i \neq j$. The domain we are going to work with is

$$\Omega^{\delta,\gamma} = B_{10}^d \setminus \left(\cup_i \bar{V}_i^{\delta,\gamma}\right) \tag{2.4.4}$$

Boundary datum and approximating functionals. Following the same idea used in the previous section, fix $N-1$ functions $v_i \in W^{1-1/(d-1),d-1}(\partial\Omega^{\delta,\gamma}; \mathbb{S}^{d-2})$ such that

$$v_i(x) = \begin{cases} \dfrac{x''}{|x''|} & \text{for } x = (x', x'') \in \partial U_{i,j}^{\delta,\gamma} \\ e_{d-1} & \text{for } x \in \partial\Omega^{\delta,\gamma} \setminus \partial V_i^{\delta,\gamma} \end{cases}$$

By construction v_i "winds around" γ_i and is constant on the rest of the given boundary. As such, one sees that $\star J v_i = \alpha_{d-1}(\delta_{P_N} - \delta_{P_i})$. As our functional space we consider

$$H_i^{\delta,\gamma} = \{u \in W^{1,d-1}(\Omega^{\delta,\gamma}; \mathbb{R}^{d-1}) : u|_{\partial\Omega^{\delta,\gamma}} = v_i\}, \quad H^{\delta,\gamma} = H_1^{\delta,\gamma} \times \cdots \times H_{N-1}^{\delta,\gamma}, \tag{2.4.5}$$

and for $U = (u_1, \dots, u_{N-1}) \in H^{\delta,\gamma}$ and $\vec{e}_\varepsilon(U) = (e_\varepsilon(u_1), \dots, e_\varepsilon(u_{N-1}))$, we define the approximating functionals

$$F_\varepsilon^\Psi(U, \Omega^{\delta,\gamma}) = |\vec{e}_\varepsilon(U)\, dx|_\Psi(\Omega^{\delta,\gamma}), \tag{2.4.6}$$

or equivalently, thanks to (2.3.2),

$$F_\varepsilon^\Psi(U, \Omega^{\delta,\gamma}) = \sup_{\varphi \in C_c^\infty(\Omega^{\delta,\gamma}; \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \int_{\Omega^{\delta,\gamma}} \varphi_i e_\varepsilon(u_i)\, dx, \quad \Psi^*(\varphi(x)) \leq 1 \right\}. \tag{2.4.7}$$

Lower-bound inequality Results on "compactness" and lower-bound inequality presented in the previous section extends to F_ε^Ψ as follows.

Proposition 2.4.2. *Consider a (countable) sequence $\{U_\varepsilon\}_\varepsilon \subset H^{\delta,\gamma}$ such that $F_\varepsilon^\Psi(U_\varepsilon, \Omega^{\delta,\gamma}) = O(|\log\varepsilon|)$. Then, up to subsequences, there exists a family $M = (M_1, \dots, M_{N-1})$ of rectifiable 1-currents supported in $\bar{\Omega}^{\delta,\gamma}$, with $\partial M_i = \delta_{P_N} - \delta_{P_i}$, such that the Jacobians $\star J u_{\varepsilon,i}$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ to $\alpha_{d-1} M_i$ and*

$$\liminf_{\varepsilon \to 0} \frac{F_\varepsilon^\Psi(U_\varepsilon, \Omega^{\delta,\gamma})}{|\log\varepsilon|} \geq \beta_{d-1} ||M||_\Psi. \tag{2.4.8}$$

Proof. For each $i = 1, \dots, N-1$, by definition of F_ε^Ψ we have

$$\int_{\Omega^{\delta,\gamma}} e_\varepsilon(u_{\varepsilon,i})\, dx \leq \Psi^*(e_i) F_\varepsilon^\Psi(U_\varepsilon, \Omega^{\delta,\gamma}) = O(|\log\varepsilon|)$$

and the first part of the statement follows applying Proposition 2.4.1 componentwise. Fix now $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^{N-1})$ with $\varphi_i \geq 0$ for any $i = 1, \dots, N-1$ and $\Psi^*(\varphi(x)) \leq 1$ for all x.

Then, thanks to (2.4.3), we have

$$\beta_{d-1}\sum_{i=1}^{N-1}\langle M_i,\varphi_i\rangle \le \frac{1}{|\log\varepsilon|}\sum_{i=1}^{N-1}\liminf_{\varepsilon\to 0}\int_{\Omega^{\delta,\gamma}}\varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx$$
$$\le \frac{1}{|\log\varepsilon|}\liminf_{\varepsilon\to 0}\sum_{i=1}^{N-1}\int_{\Omega^{\delta,\gamma}}\varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx \le \liminf_{\varepsilon\to 0}\frac{F_\varepsilon^\Psi(U_\varepsilon,\Omega^{\delta,\gamma})}{|\log\varepsilon|},$$

which yields (2.4.8) taking the supremum over φ.

□

Upper-bound inequality and behaviour of minimizers. We now state and prove a version of an upper-bound inequality for the functionals F_ε^Ψ which is tailored to investigate the behaviour of Jacobians of minimizers of F_ε^Ψ.

Proposition 2.4.3 (Upper-bound inequality)**.** *Let $\Lambda=\Lambda_L\in\mathcal{L}(A)$, with $L\in\mathcal{G}(A)$ an acyclic graph supported in $\bar\Omega^{\delta,\gamma}$. Then there exists a sequence $\{U_\varepsilon\}_\varepsilon\subset H^{\delta,\gamma}$ such that $\mathbf{F}_{\mathbb{R}^d}(\star Ju_{\varepsilon,i}-\alpha_{d-1}\Lambda_i)\to 0$, and*

$$\limsup_{\varepsilon\to 0}\frac{F_\varepsilon^\Psi(U_\varepsilon,\Omega^{\delta,\gamma})}{|\log\varepsilon|}\le\beta_{d-1}||\Lambda||_\Psi. \tag{2.4.9}$$

Proof. Step 1. We assume that $L=\cup_i\lambda_i\in\mathcal{G}(A)$ is an acyclic polyhedral graph fully contained in $\Omega^{\delta,\gamma}$, which is to say $\lambda_i\cap\partial\Omega^{\delta,\gamma}=\{P_i,P_N\}$, and let τ be its global orientation. Such a graph L can then be decomposed into a family of K oriented segments $S_1,\dots,S_K$, with orientation given by τ. For each segment S_k consider the set $U_k^{'}=U(S_k,\delta',\gamma')$, for parameters $0<\delta'<\delta$ and $0<\gamma'<\gamma$, and choose δ',γ' small enough so that sets U_k' are pairwise disjoint. Define as $V_i^{'}$ the union of the $U_k^{'}$ covering λ_i, and let $V'=\cup_iV_i'=\cup_kU_k'$. Eventually, define vectors $g^k\in\mathbb{R}^{N-1}$ as $g_i^k=1$ if $S_k\subset\lambda_i$ and $g_i^k=0$ otherwise. Collect these vectors in a function $g\colon V'\to\mathbb{R}^{N-1}$ defined as $g(x)=g^k$ for $x\in U_k'$.

For the construction of the approximating sequence we relay on the following fact, which is a direct consequence of Theorem 2.2.1: for each $i=1,\dots,N-1$ there exists $u_i\in W^{1,d-2}(\Omega^{\delta,\gamma};\mathbb{S}^{d-2})$ and a finite set of points F_0^i such that:

(i) $u_i|_{\partial\Omega^{\delta,\gamma}}=v_i$, which is to say u_i satisfies the given boundary conditions, and furthermore $\star Ju_i=\alpha_{d-1}\Lambda_i$;

(ii) u_i is locally Lipschitz in $\bar\Omega^{\delta,\gamma}\setminus(\lambda_i\cup F_0^i)$ and

$$|Du_i(x)|=O(1/\mathrm{dist}(x,\gamma_i\cup\lambda_i\cup F_0^i));$$

(iii) within the set V' every function behaves like

$$u_i(x)=\begin{cases}\dfrac{x''}{|x''|} & \text{for } x=(x',x'')\in V_i^{'}\\ e_{d-1} & \text{for } x\in\Omega^{\delta,\gamma}\setminus V_i^{'}\end{cases}$$

In particular, we observe that for any $k \in \{1, \dots, M\}$, if $S_k \subset \lambda_i$ and $S_k \subset \lambda_j$, then $u_i = u_j$ on $U_k^{'}$ by (iii). Thus, we can define a "global" function $u\colon V' \to S^{d-2}$ such that $u(x) = x''/|x''|$ for any $x \in V'$ and, consequently, $u_i|_{V'} = g_i(x)u(x)$.

Starting form each u_i we define our family of approximating maps: for any $\varepsilon \in (0, \delta')$ let $\Omega_\varepsilon^{\delta,\gamma} = \Omega^{\delta,\gamma} \setminus \cup_i B_{2\varepsilon}(P_i)$, and let $u_{\varepsilon,i}\colon \Omega_\varepsilon^{\delta,\gamma} \to \mathbb{R}^{d-1}$ be defined as

$$u_{\varepsilon,i}(x) = h_{\varepsilon,i}(x)u_i(x) \quad \text{where} \quad h_{\varepsilon,i}(x) = \min\left(1, \frac{\operatorname{dist}(x, \lambda_i \cup F_0^i)}{\varepsilon}\right). \tag{2.4.10}$$

Complete these maps on $B_{2\varepsilon}(P_i) \cap \Omega^{\delta,\gamma}$ by means of a Lipschitz extension of the function $u_{\varepsilon,i}$ with Lipschitz constant of the order of $1/\varepsilon$, using v_i as boundary value on $B_{2\varepsilon}(P_i) \cap \partial\Omega^{\delta,\gamma}$. The resulting maps are locally Lipschitz in the complement of $\cup_k \partial S_k$, belong to $W^{1,d-1}(\Omega^{\delta,\gamma}; \mathbb{R}^{d-1})$ and by construction $u_{\varepsilon,i}|_{\partial\Omega^{\delta,\gamma}} = v_i$, i.e., $u_{\varepsilon,i} \in H_i^{\delta,\gamma}$. Each $u_{\varepsilon,i}$ converges strongly to u_i in $W^{1,d-2}(\Omega^{\delta,\gamma}; \mathbb{R}^{d-1})$ and, in particular, the Jacobians $\star Ju_{\varepsilon,i}$ converge to $\star Ju_i = \alpha_{d-1}\Lambda_i$ in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ (see Remark 2.11 of [6]).

We now consider the energy behaviour, working locally on every $U_k^{'}$: for $\varepsilon \in (0, \delta')$, let us consider

$$\begin{aligned} U_{k,\varepsilon,1}^{'} &:= \{x \in U_k^{'} \,:\, \operatorname{dist}(x, S_k) \le \varepsilon\} \cap \Omega_\varepsilon^{\delta,\gamma} \\ U_{k,\varepsilon,2}^{'} &:= (U_k^{'} \setminus U_{k,\varepsilon,1}^{'}) \cap \Omega_\varepsilon^{\delta,\gamma} \\ V_{out} &:= \Omega_\varepsilon^{\delta,\gamma} \setminus V^{'} \end{aligned}$$

Let $\varphi = (\varphi_1, \dots, \varphi_{N-1})$, with $\varphi_i \ge 0$ and $\Psi^*(\varphi) \le 1$, we compute

$$\begin{aligned} \int_{\Omega^{\delta,\gamma}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx \le \sum_{k=1}^{K} \left[\int_{U'_{k,\varepsilon,1}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx + \int_{U'_{k,\varepsilon,2}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx \right] + \\ + \sum_{j=1}^{N} \int_{B_{2\varepsilon}(P_j)} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx + \int_{V_{out}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx. \end{aligned}$$

Fix $1 \le k \le K$ and consider the sets of indices $I_k = \{i \,:\, S_k \subset \gamma_i\}$ and $I_k^c = \{1, \dots, N-1\} \setminus I_k$. Let us analyse separately the four kinds of integrals appearing in the above expression.

- The first family of integrals on each $U'_{k,\varepsilon,1}$ splits as

$$\int_{U'_{k,\varepsilon,1}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx = \int_{U'_{k,\varepsilon,1}} \sum_{i\in I_k} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx + \int_{U'_{k,\varepsilon,1}} \sum_{i\in I_k^c} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx.$$

  We distinguish between two case.

  Case $i \in I_k$: we have $|Du_i(x)| \le C/\operatorname{dist}(x, S_k)$ thanks to (iii), and therefore

$$|Du_{\varepsilon,i}(x)| \le h_{\varepsilon,i}(x)|Du_i(x)| + |Dh_{\varepsilon,i}(x)||u_i(x)| \le \frac{C}{\varepsilon}.$$

Using that $W(u_{\varepsilon,i}) \le C$ and $|U'_{k,\varepsilon,1}| \le C\varepsilon^{d-1}$, we obtain

$$F_\varepsilon(u_{\varepsilon,i}, U'_{k,\varepsilon,1}) \le C \quad \text{for all } k,i \text{ such that } S_k \subset \lambda_i. \tag{2.4.11}$$

Case $i \in I_k^c$: in this situation we have $u_{\varepsilon,i} = u_i$ on $U'_{k,\varepsilon,1}$ and $\mathrm{dist}(x, F_0^i) \le C\mathrm{dist}(x, \gamma_i \cup \lambda_i)$. In particular, combining (ii) and (2.4.10), we have

$$|Du_{\varepsilon,i}(x)| \le C/\mathrm{dist}(x, F_0^i).$$

Using the fact that $W(u_{\varepsilon,i}) = 0$ in the complement of an ε-neighbourhood $(\lambda_i \cup F_0^i)_\varepsilon$ of $\lambda_i \cup F_0^i$, we get

$$\begin{aligned} F_\varepsilon(u_{\varepsilon,i}, U'_{k,\varepsilon,1}) &\le C \int_{U'_{k,\varepsilon,1}} \frac{dx}{\mathrm{dist}(x, F_0^i)^{d-1}} + \frac{C}{\varepsilon^2}|(\lambda_i \cup F_0^i)_\varepsilon| \\ &\le C \quad \text{for all } k,i \text{ such that } S_k \nsubseteq \lambda_i. \end{aligned} \tag{2.4.12}$$

Combining (2.4.11) and (2.4.12) we obtain

$$\int_{U'_{k,\varepsilon,1}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\, dx \le C \quad \text{for all } 1 \le k \le K, 1 \le i \le N-1. \tag{2.4.13}$$

- The second family of integrals on each $U'_{k,\varepsilon,2}$ splits analogously into

$$\int_{U'_{k,\varepsilon,2}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\, dx = \int_{U'_{k,\varepsilon,2}} \sum_{i \in I_k} \varphi_i e_\varepsilon(u_{\varepsilon,i})\, dx + \int_{U'_{k,\varepsilon,2}} \sum_{i \in I_k^c} \varphi_i e_\varepsilon(u_{\varepsilon,i})\, dx.$$

Let us distinguish the same two cases as above.

Case $i \in I_k$: here we have $u_{\varepsilon,i} = u_i$ within $U'_{k,\varepsilon,2}$ and so $u_{\varepsilon,i}$ takes values in S^{d-2}, reducing this way $e_\varepsilon(u_{\varepsilon,i})$ to $\frac{1}{d-1}|Du_i|^{d-1}$. For every $x \in U'_k$ one has

$$|Du_i(x)| = \left| D\frac{x''}{|x''|} \right| = \frac{(d-2)^{1/2}}{|x''|}.$$

Hence,

$$\begin{aligned} F_\varepsilon(u_{\varepsilon,i}, U'_{k,\varepsilon,2}) &\le \mathcal{H}^1(S_k) \frac{(d-2)^{(d-1)/2}}{d-1} \int_{B_{\delta'}^{d-1} \setminus B_\varepsilon^{d-1}} \frac{dx''}{|x''|^{d-1}} \\ &\le \mathcal{H}^1(S_k) \frac{(d-2)^{(d-1)/2}}{d-1} \int_\varepsilon^1 \frac{(d-1)\alpha_{d-1}\rho^{d-2}}{\rho^{d-1}}\, d\rho \\ &\le \beta_{d-1}|\log\varepsilon| \cdot \mathcal{H}^1(S_k) \quad \text{for all } k,i \text{ such that } S_k \subset \lambda_i. \end{aligned} \tag{2.4.14}$$

Case $i \in I_k^c$: the same derivation done for obtaining (2.4.12) applies, so that

$$F_\varepsilon(u_{\varepsilon,i}, U'_{k,\varepsilon,2}) \le C \quad \text{for all } k,i \text{ such that } S_k \nsubseteq \lambda_i. \tag{2.4.15}$$

Taking into account (2.4.14), (2.4.15), and that $\sum_{i\in I_k}\varphi_i(x) = \sum_{i=1}^{N-1} g_i^k\varphi_i(x) \leq \Psi(g^k)$, we have

$$\int_{U'_{k,\varepsilon,2}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx \leq C + \Psi(g^k)\beta_{d-1}|\log\varepsilon|\cdot\mathcal{H}^1(S_k) \tag{2.4.16}$$

for all $1\leq k\leq K$, $1\leq i\leq N-1$.

- For any given $j=1,\dots,N$ the contribution on $B_{2\varepsilon}(P_j)$ is of order ε, so that in particular

$$\int_{B_{2\varepsilon}(P_j)} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx \leq C. \tag{2.4.17}$$

- The last integral on V_{out} can be treated as in the derivation of (2.4.12) and (2.4.15), so that we have

$$\int_{V_{out}} \sum_{i=1}^{N-1} \varphi_i e_\varepsilon(u_{\varepsilon,i})\,dx \leq C \quad \text{for all } 1\leq k\leq K, 1\leq i\leq N-1. \tag{2.4.18}$$

If we combine (2.4.13), (2.4.16), (2.4.17), (2.4.18), divide by $|\log\varepsilon|$, take $\varepsilon\to 0$ and consider the supremum over φ in view of (2.4.7), we have

$$\limsup_{\varepsilon\to 0}\frac{F_\varepsilon^\Psi(U_\varepsilon,\Omega^{\delta,\gamma})}{|\log\varepsilon|} \leq \beta_{d-1}|\Lambda|_\Psi(\Omega^{\delta,\gamma}) = \beta_{d-1}||\Lambda||_\Psi,$$

which is the sought for conclusion.

Step 2. Let us now consider the case $\Lambda_L\equiv\Lambda=(\Lambda_1,\dots,\Lambda_{N-1})$, $L=\cup_i\lambda_i$ and the λ_i are not necessarily polyhedral and possibly lying on the boundary of $\Omega^{\delta,\gamma}$. We rely on Lemma 2.4.4 below to construct a sequence of acyclic polyhedral graphs $L_m=\cup_i\lambda_i^m$, λ_i^m contained in $\Omega^{\delta,\gamma}$, and s.t. the Hausdorff distance $d_H(\lambda_i^m,\lambda_i)<\frac{1}{m}$ for all $i=1,\dots,N-1$, and $||\Lambda_{L_m}||_\Psi\leq||\Lambda_L||_\Psi+\frac{1}{m}$. For $\Lambda_{L_m}=(\Lambda_1^m,\dots,\Lambda_{N-1}^m)$, by step 1 we may construct a sequences $\{U_\varepsilon^m\}_\varepsilon$ such that $\mathbf{F}_{\mathbb{R}^d}(\star Ju_{\varepsilon,i}^m-\alpha_{d-1}\Lambda_i^m)\to 0$ as $\varepsilon\to 0$ for each m and, in particular,

$$\limsup_{\varepsilon\to 0}\frac{F_\varepsilon^\Psi(U_\varepsilon^m,\Omega^{\delta,\gamma})}{|\log\varepsilon|} \leq \beta_{d-1}||\Lambda_{L_m}||_\Psi \leq \beta_{d-1}||\Lambda||_\Psi+\frac{C}{m}.$$

We deduce that $\mathbf{F}_{\mathbb{R}^d}(\star Ju_{\varepsilon_m,i}^m-\alpha_{d-1}\Lambda_i)\to 0$ and

$$\limsup_{m\to\infty}\frac{F_{\varepsilon_m}^\Psi(U_{\varepsilon_m}^m,\Omega^{\delta,\gamma})}{|\log\varepsilon_m|} \leq \beta_{d-1}||\Lambda||_\Psi$$

for a subsequence $\varepsilon_m\to 0$ as $m\to+\infty$. Conclusion (2.4.9) follows.

We recall Lemma 1.3.10, which provides the relevant approximation used above, where polyhedral approximations are here supposed to live within the set $\Omega^{\delta,\gamma}$ (i.e., with no relevant part on the boundary).

Lemma 2.4.4. *Let $L \in \mathcal{G}(A)$, $L = \cup_{i=1}^{N-1}\lambda_i$, be an acyclic graph connecting $P_1, \dots, P_N$ with $\lambda_i \subset \bar{\Omega}^{\delta,\gamma}$. Then for any $\eta > 0$ there exists $L' \in \mathcal{G}(A)$, $L' = \cup_{i=1}^{N-1}\lambda_i'$, with $\lambda_i' \subset \Omega^{\delta,\gamma} \cup \{P_i, P_N\}$ a simple polyhedral curve of finite length connecting P_i to P_N, such that the Hausdorff distance $d_H(\lambda_i, \lambda_i') < \eta$ and $||\Lambda_{L'}||_\Psi \leq ||\Lambda_L||_\Psi + \eta$.*

Thanks to the previous propositions we are now able to prove our main result on the behaviour of the Jacobians of the minimizers.

Proposition 2.4.5 (Behaviour of minimizers)**.** *Let $\{U_\varepsilon\}_\varepsilon \subset H^{\delta,\gamma}$ be a sequence of minimizers for F_ε^Ψ in $H^{\delta,\gamma}$. Then (up to a subsequence) the Jacobians $\star Ju_{\varepsilon,i}$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ to $\alpha_{d-1}M_i$, with $M = (M_1, \dots, M_{N-1})$ a minimizer of*

$$\inf\{||\Lambda||_\Psi \,:\, \Lambda = (\Lambda_1, \dots, \Lambda_{N-1}) \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1}, \operatorname{spt}\Lambda_i \subset \bar{\Omega}^{\delta,\gamma}, \partial\Lambda_i = \delta_{P_N} - \delta_{P_i}\}. \tag{2.4.19}$$

Proof. Let $\Lambda = \Lambda_L$ canonically representing an acyclic graph $L \subset \bar{\Omega}^{\delta,\gamma}$, and let $\{V_\varepsilon\}_\varepsilon \subset H^{\delta,\gamma}$ such that $\limsup_{\varepsilon\to 0} \frac{F_\varepsilon^\Psi(V_\varepsilon,\Omega^{\delta,\gamma})}{|\log\varepsilon|} \leq ||\Lambda||_\Psi$ and $\mathbf{F}_{\mathbb{R}^d}(\star Jv_{\varepsilon,i} - \alpha_{d-1}\Lambda_i) \to 0$. Since $F_\varepsilon^\Psi(U_\varepsilon, \Omega^{\delta,\gamma}) \leq F_\varepsilon^\Psi(V_\varepsilon, \Omega^{\delta,\gamma})$, by Proposition 2.4.2 there exists a family $M = (M_1, \dots, M_{N-1})$ of rectifiable 1-currents supported in $\bar{\Omega}^{\delta,\gamma}$, with $\partial M_i = \delta_{P_N} - \delta_{P_i}$, such that the Jacobians $\star Ju_{\varepsilon,i}$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ to $\alpha_{d-1}M_i$. Then, by (2.4.8), we have

$$\beta_{d-1}||M||_\Psi \leq \liminf_{\varepsilon\to 0} \frac{F_\varepsilon^\Psi(U_\varepsilon,\Omega^{\delta,\gamma})}{|\log\varepsilon|} \leq \limsup_{\varepsilon\to 0} \frac{F_\varepsilon^\Psi(V_\varepsilon,\Omega^{\delta,\gamma})}{|\log\varepsilon|} \leq \beta_{d-1}||\Lambda||_\Psi.$$

Given any other generic $\Lambda \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1}$ with $\operatorname{spt}\Lambda_i \subset \bar{\Omega}^{\delta,\gamma}$ and $\partial\Lambda_i = \delta_{P_N} - \delta_{P_i}$, as one does in the derivation of (2.3.1) (see, e.g., Lemma 2.1 in [70]), we can always find $\bar{L} \in \mathcal{G}(A)$ supported in $\bar{\Omega}^{\delta,\gamma}$ such that $||\Lambda_{\bar{L}}||_\Psi \leq ||\Lambda||_\Psi$, and thus M minimizes (2.4.19) as desired. □

Finally, let us highlight the case $\Psi = \Psi_\alpha$, where $\Psi_\alpha(g) = |g|_{1/\alpha}$ for $0 < \alpha \leq 1$ and $\Psi_0(g) = |g|_\infty$, and denote $F_\varepsilon^0 \equiv F_\varepsilon^{\Psi_0}$ and $F_\varepsilon^\alpha \equiv F_\varepsilon^{\Psi_\alpha}$. For $U = (u_1, \dots, u_{N-1}) \in H^{\delta,\gamma}$ we have

$$F_\varepsilon^0(U,\Omega^{\delta,\gamma}) = \int_{\Omega^{\delta,\gamma}} \sup_i e_\varepsilon(u_i)\,dx, \qquad F_\varepsilon^\alpha(U,\Omega^{\delta,\gamma}) = \int_{\Omega^{\delta,\gamma}} \left(\sum_{i=1}^{N-1} e_\varepsilon(u_i)^{1/\alpha}\right)^\alpha dx. \tag{2.4.20}$$

Theorem 2.4.6. *Let $\{P_1, \dots, P_N\} \subset \mathbb{R}^d$ such that $\max_i |P_i| = 1$, and let $\Omega^{\delta,\gamma}$ be defined as in (2.4.4) for δ, γ small enough, with $\gamma = \bar{c}\delta$. For $0 \leq \alpha \leq 1$ and $0 < \varepsilon \ll \delta$, denote $F_\varepsilon^{\alpha,\delta} \equiv F_\varepsilon^\alpha(\cdot, \Omega^{\delta,\gamma})$, with $F_\varepsilon^\alpha(\cdot,\Omega^{\delta,\gamma})$ defined in (2.4.20).*

(i) Let $\{U_\varepsilon^{\alpha,\delta}\}_\varepsilon$ be a sequence of minimizers for $F_\varepsilon^{\alpha,\delta}$ in $H^{\delta,\gamma}$, with $H^{\delta,\gamma}$ defined in (2.4.5). Then, up to subsequences, the Jacobians $\star Ju_{\varepsilon,i}^{\alpha,\delta}$ converge in the flat norm $\mathbf{F}_{\mathbb{R}^d}$ to $\alpha_{d-1}M_i^{\alpha,\delta}$, where $M^{\alpha,\delta} = (M_1^{\alpha,\delta}, \dots, M_{N-1}^{\alpha,\delta})$ minimizes (2.4.19).

(ii) Let $M^{\alpha,\delta} = (M_1^{\alpha,\delta}, \dots, M_{N-1}^{\alpha,\delta})$ be a sequence of minimizers for (2.4.19). Then, up to subsequences, we have $\mathbf{F}_{\mathbb{R}^d}(M_i^{\alpha,\delta} - M_i^{\alpha}) \to 0$ as $\delta \to 0$ for every $i = 1, \dots, N-1$, with $M^{\alpha} = (M_1^{\alpha}, \dots, M_{N-1}^{\alpha})$ a minimizer of

$$\inf\{||\Lambda||_{\Psi_\alpha} \,:\, \Lambda = (\Lambda_1, \dots, \Lambda_{N-1}) \in [\mathcal{I}_1(\mathbb{R}^d)]^{N-1},\ \partial\Lambda_i = \delta_{P_N} - \delta_{P_i}\} \tag{2.4.21}$$

and, in turn, $M^{\alpha} = \Lambda_{L_\alpha}$ for an optimizer L_α of the α-irrigation problem (I_α) with terminals $P_1, \dots, P_N$.

Proof. In view of Proposition 2.4.5 it remains to prove item (ii). For each $i = 1, \dots, N-1$, the sequence $\{M_i^{\alpha,\delta}\}_\delta$ is equibounded in mass, hence there exists a rectifiable 1-current M_i^{α}, with $\partial M_i^{\alpha} = \delta_{P_N} - \delta_{P_i}$, such that $M_i^{\alpha,\delta} \to M_i^{\alpha}$ in the flat norm. Let us call $M^{\alpha} = (M_1^{\alpha}, \dots, M_{N-1}^{\alpha})$ the limiting family and let $\bar{M}^{\alpha} = (\bar{M}_1^{\alpha}, \dots, \bar{M}_{N-1}^{\alpha})$ be a minimizer of (2.4.21). In the same spirit of Lemma 2.4.4, starting with our minimizer $\bar{M}^{\alpha}$, we can construct a new family $\tilde{M}^{\alpha,\delta} = (\tilde{M}_1^{\alpha,\delta}, \dots, \tilde{M}_{N-1}^{\alpha,\delta})$ supported in $\bar{\Omega}^{\delta,\gamma}$ such that $||\tilde{M}^{\alpha,\delta}||_{\Psi^\alpha} \leq ||\bar{M}^{\alpha,\delta}||_{\Psi^\alpha} + C\delta$. Hence,

$$\begin{aligned} ||\bar{M}^{\alpha}||_{\Psi^\alpha} \leq ||M^{\alpha}||_{\Psi^\alpha} &\leq \liminf_{\delta\to 0} ||M^{\alpha,\delta}||_{\Psi^\alpha} \leq \liminf_{\delta\to 0} ||\tilde{M}^{\alpha,\delta}||_{\Psi^\alpha} \\ &\leq \liminf_{\delta\to 0} ||\bar{M}^{\alpha,\delta}||_{\Psi^\alpha} + C\delta = ||\bar{M}^{\alpha}||_{\Psi^\alpha}, \end{aligned}$$

and so M^{α} has to be a minimizer of (2.4.21). The correspondence of minimizers of (2.4.21), which is to say of (2.3.4), with minimizers of (I_α) follows by the discussion of Section 2.3.

□

Chapter 3

A convex approach to the Gilbert–Steiner problem

We describe a convex relaxation for the Gilbert–Steiner problem both in $\mathbb{R}^d$ and on manifolds, extending the framework proposed in Chapter 1, and we discuss its sharpness by means of calibration type arguments. The minimization of the resulting problem is then tackled numerically and we present results for an extensive set of examples. In particular we are able to address the Steiner tree problem on surfaces.

3.1 Introduction

In the Steiner tree problem, at least in its classical Euclidean version, we are given N distinct points $P_1, \dots, P_N$ in $\mathbb{R}^d$ and we have to find the shortest connected graph containing the points P_i. From an abstract point of view this amounts to find a graph solving the variational problem

$$\text{(STP)} \qquad \inf\{\mathcal{H}^1(L),\ L \text{ connected},\ L \supset \{P_1, \dots, P_N\}\},$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbb{R}^d$. An optimal (not necessarily unique) graph L always exists and, by minimality, L is indeed a tree. Every optimal tree can be described as a union of segments connecting the endpoints and possibly meeting at 120° in at most $N-2$ further branch points, called Steiner points.

On the other hand, the (single sink) Gilbert–Steiner problem [59] consists of finding a network L along which to flow unit masses located at the sources $P_1, \dots, P_{N-1}$ to the unique target point P_N. Such a network L can be viewed as $L = \cup_{i=1}^{N-1} \lambda_i$, with λ_i a path connecting P_i to P_N, corresponding to the trajectory of the particle located at P_i. To favour branching, one is led to optimize a cost which is a sublinear (concave) function of the mass density $\theta(x) = \sum_{i=1}^{N-1} \mathbf{1}_{\lambda_i}(x)$: i.e., for $0 \le \alpha \le 1$, find

$$(I_\alpha) \qquad \inf\left\{ E^\alpha(L) = \int_L |\theta(x)|^\alpha d\mathcal{H}^1(x) \right\}.$$

Problem (I_α) can be seen as a particular instance of an α-irrigation problem [21, 112] involving the irrigation of the atomic measures $\sum_{i=1}^{N-1}\delta_{P_i}$ and $(N-1)\delta_{P_N}$, and we notice that (I_1) corresponds to the Monge optimal transport problem, while (I_0) corresponds to (STP) (the energy to be optimized reduces to the length of L). As for (STP) a solution to (I_α) is known to exist and any optimal network L turns out to be a tree [21].

The Steiner tree problem is known to be computationally hard (even NP complete in certain cases [65]), nonetheless in $\mathbb{R}^2$ and $\mathbb{R}^3$ we have efficient algorithms which allow us to obtain explicit solutions (see, for instance, [110, 54]), while a comprehensive survey on PTAS algorithms for (STP) can be found in [13, 14]. However, the general applicability of these schemes restricts somehow to the Steiner tree case. For this reason we stick here with a more abstract variational point of view, which allows us to treat in a unified way the Steiner and Gilbert–Steiner problems.

Many different variational approximations for (STP) and/or (I_α) have been proposed, starting form the simple situation where the points P_i lie on the boundary of a convex set: in this case (STP) is known to be an instance of an optimal partition problem [10, 11]. More recently several authors treated these problems within the framework of minimal networks on covering spaces [8, 41, 42] and in the spirit of Γ-convergence using approximating functionals modelled on Modica–Mortola or Ambrosio–Tortorelli type energies, initially focusing mainly on the two dimensional case [82, 30, 44], lately extending the same ideas also to higher dimensions [45, 28].

Within this sole we introduced in Chapter 1 a Γ-convergence type result in the planar case and at the same time we propose a convex framework for the Steiner and Gilbert–Steiner problem. The approach moves from the work of Marchese and Massaccesi [70] and considers ideas from [43] in order to obtain a convex relaxation of the energy we are dealing with. The aim of this chapter is then to provide an extensive numerical investigation of the relaxation proposed in Section 1.4.2, adapting it to the treatment of more general Gilbert–Steiner problems (with multiple sources/sinks) and addressing its validity and applicability to problems defined on manifolds. In contrast to classical Γ-convergence type approaches, which may numerically end up in local minima (unless carefully taking initial guesses), this convex formulation is able to identify (in many cases) convex combinations of optimal networks, allowing us to have an idea of their structure. Furthermore, up to our knowledge, this is the very first formulation leading to a numerical approximation of the Steiner tree problem on manifolds.

The chapter is organized as follows. In Section 3.2 we review the convex framework presented in Chapter 1 for the α-irrigation problem (I_α) and extend it to the treatment of more general situations with multiple sources/sinks, both in $\mathbb{R}^d$ and on manifolds. In Section 3.3 we see how the formulation simplifies for a network (STP) on graphs, with the relevant energy reducing to the norm introduced in [70]. We then proceed in Section 3.4 to describe our algorithmic scheme for the minimization of the proposed energy functional in the Euclidean setting and we present in Section 3.6 various results for (STP) and α-irrigation problems in two and three dimensions. In Section 3.7 we eventually detail our algorithmic approach on surfaces and present some results obtained on spheres, tori and other surfaces with boundaries.

3.2 Convex relaxation for irrigation type problems

In this section we first review the convex framework introduced in Section 1.4.2 for the α-irrigation problem (I_α) and then discuss how this same formulation can be extended to address more general Gilbert–Steiner problems with multiple sources/sinks in $\mathbb{R}^d$ or even on manifolds.

3.2.1 The Euclidean Gilbert–Steiner problem

Fix a set of N distinct points $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^d$, $d \geq 2$. A candidate minimizer for (I_α) is given as a family of simple rectifiable curves $(\gamma_i)_{i=1}^{N-1}$, each one connecting P_i to P_N. For optimality reasons we can choose these curves so that the resulting network $L = \cup_i \lambda_i$ contains no cycles (see Lemma 2.1 in [70]), restricting this way the set of possible minimizers to the set of (connected) *acyclic graphs* L that can be described as

$$L = \bigcup_{i=1}^{N-1} \lambda_i, \quad \text{s.t.} \quad \begin{array}{l} \cdot\ \lambda_i \text{ is a simple rectifiable curve connecting } P_i \text{ to } P_N, \\ \cdot\ \text{each } \lambda_i \text{ is oriented by an } \mathcal{H}^1\text{-measurable unit vector field } \tau_i, \\ \cdot\ \tau_i(x) = \tau_j(x) \text{ for } \mathcal{H}^1\text{-a.e. } x \in \lambda_i \cap \lambda_j, \end{array}$$

where the last condition requires the $N-1$ pieces composing L to share the same orientation on intersections. Let us call $\mathcal{G}(A)$ the set of acyclic graphs L having such a representation. Hence, we can reduce ourself to consider

$$\inf \left\{ \int_L |\theta(x)|^\alpha d\mathcal{H}^1, \quad L \in \mathcal{G}(A), \ \ \theta(x) = \sum_{i=1}^{N-1} \mathbf{1}_{\lambda_i}(x) \right\}.$$

To each $L \in \mathcal{G}(A)$ we now associate a measure taking values in $\mathbb{R}^{d\times(N-1)}$ as follows: identify the curves λ_i with the vector measures $\Lambda_i = \tau_i \cdot \mathcal{H}^1 \llcorner \lambda_i$, and consider the rank one tensor valued measure $\Lambda = (\Lambda_1, \dots, \Lambda_{N-1})$, which can be written as $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, with

- $\tau \colon \mathbb{R}^d \to \mathbb{R}^d$ a unit vector field providing a global orientation for L, satisfying $\operatorname{spt} \tau = L$ and $\tau = \tau_i$ $\mathcal{H}^1$-a.e. on λ_i,
- $g \colon \mathbb{R}^d \to \mathbb{R}^{N-1}$ a multiplicity function whose entries satisfy $g_i \cdot \mathcal{H}^1 \llcorner L = \mathcal{H}^1 \llcorner \lambda_i$.

Observe that $g_i \in \{0, 1\}$ a.e. for any $1 \leq i \leq N-1$ (in particular, $g_i(x) = 1$ if $x \in \lambda_i$), and by construction the measures Λ_i verify

$$\operatorname{div} \Lambda_i = \delta_{P_i} - \delta_{P_N}. \tag{3.2.1}$$

Definition 3.2.1. *Given any graph $L \in \mathcal{G}(A)$, we call the above constructed measure $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ the canonical (rank one) tensor valued measure representation of the acyclic graph L and denote the set of such measures as $\mathcal{L}(A)$.*

Let us define on the space of matrix valued Radon measures $\mathcal{M}(\mathbb{R}^d;\mathbb{R}^{d\times(N-1)})$ the functional

$$\mathcal{F}^\alpha(\Lambda)=\begin{cases}\displaystyle\int_{\mathbb{R}^d}||g||_{1/\alpha}\,d\mathcal{H}^1\llcorner L & \text{if } \Lambda=\tau\otimes g\cdot\mathcal{H}^1\llcorner L\in\mathcal{L}(A)\\ +\infty & \text{otherwise}\end{cases}$$

where we assume $1/0=\infty$. When $\Lambda=\tau\otimes g\cdot\mathcal{H}^1\llcorner L\in\mathcal{L}(A)$, since by construction $g_i\in\{0,1\}$ on L and $g_i(x)=1$ whenever $x\in\lambda_i$, one immediately gets

$$\mathcal{F}^\alpha(\Lambda)=\int_L\left(\sum_{i=1}^{N-1}g_i(x)^{1/\alpha}\right)^\alpha d\mathcal{H}^1=\int_L\left(\sum_{i=1}^{N-1}g_i(x)\right)^\alpha d\mathcal{H}^1=\int_L\left(\sum_{i=1}^{N-1}\mathbf{1}_{\lambda_i}(x)\right)^\alpha d\mathcal{H}^1,$$

which is exactly the cost E^α associated to L in (I_α). We recognize that minimizing $\mathcal{F}^\alpha$ among measures $\Lambda\in\mathcal{L}(A)$ corresponds to minimize E^α among graphs $L\in\mathcal{G}(A)$, and thus solves (I_α) in $\mathbb{R}^d$.

This reformulation of (I_α) involves the minimization of a convex energy, namely $\mathcal{F}^\alpha$, but the problem is still nonconvex due to the non convexity of $\mathcal{L}(A)$ (the domain of definition of $\mathcal{F}^\alpha$). In view of a convex formulation the optimal choice would be to consider the convex envelope $(\mathcal{F}^\alpha)^{**}$ of the energy, but such an object (up to our knowledge) has no explicit representation. Hence, following [43], we instead look for a "local" convex envelope of the form

$$\mathcal{R}^\alpha(\Lambda)=\int_{\mathbb{R}^d}\Psi_\alpha(\Lambda)\tag{3.2.2}$$

with $\Psi_\alpha\colon\mathbb{R}^{d\times(N-1)}\to[0,+\infty)$ a 1-homogeneous, convex, continuous function such that $\mathcal{R}^\alpha(\Lambda)=\mathcal{F}^\alpha(\Lambda)$ whenever $\Lambda\in\mathcal{L}(A)$. The integral in (3.2.2), as outlined in [32], can be defined as

$$\begin{aligned}&\int_{\mathbb{R}^d}\Psi_\alpha(\Lambda)=\int_{\mathbb{R}^d}\Psi_\alpha\left(\frac{d\Lambda_a}{d\mathcal{L}^d}\right)dx+\int_{\mathbb{R}^d}\Psi_\alpha\left(\frac{d\Lambda_s}{d|\Lambda_s|}\right)d|\Lambda_s|\\&=\sup_{\varphi\in C_c^\infty\left(\mathbb{R}^d;\mathbb{R}^{d\times(N-1)}\right)}\left\{\sum_{i=1}^{N-1}\int_{\mathbb{R}^d}\varphi_i\,d\Lambda_i-\int_{\mathbb{R}^d}\Psi_\alpha^*(\varphi)\,dx,\quad\Psi_\alpha^*(\varphi)\in L^1(\mathbb{R}^d)\right\}\end{aligned}\tag{3.2.3}$$

where $\Lambda=\Lambda_a+\Lambda_s$ is the Lebesgue decomposition of Λ w.r.t. the d-dimensional Lebesgue measure $\mathcal{L}^d$, $|\Lambda_s|$ is the total variation of Λ_s, φ_i are the columns of the function $\varphi(x)=(\varphi_1(x),\dots,\varphi_{N-1}(x))$ and Ψ_α^* is the Legendre-Fenchel conjugate of Ψ_α on $\mathbb{R}^{d\times(N-1)}$: for $p=(p_1,\dots,p_{N-1})\in\mathbb{R}^{d\times(N-1)}$ and $q=(q_1,\dots,q_{N-1})\in\mathbb{R}^{d\times(N-1)}$ we have

$$\Psi_\alpha^*(q)=\sup_p\left[\langle q,p\rangle-\Psi_\alpha(p)\right]=\sup_p\left[\sum_{i=1}^{N-1}q_i\cdot p_i-\Psi_\alpha(p)\right].$$

We immediately see that the evaluation of $\mathcal{R}^\alpha$ on any $\Lambda\in\mathcal{L}(A)$, i.e., $\Lambda=\tau\otimes g\cdot\mathcal{H}^1\llcorner L$ with $||\tau||_2=1$ and $g_i\in\{0,1\}$, only involves the singular part of the decomposition, so that

$$\mathcal{R}^\alpha(\Lambda)=\int_{\mathbb{R}^d}\Psi_\alpha(\tau\otimes g)d\mathcal{H}^1\llcorner L.$$

Since we require $\mathcal{R}^\alpha(\Lambda) = \mathcal{F}^\alpha(\Lambda)$ on these measures, we then look for a 1-homogeneous, convex, continuous function Ψ_α such that

$$\Psi_\alpha(p) = ||g||_{1/\alpha} \quad \text{whenever } p \in K_\alpha = \{\tau \otimes g, \quad ||\tau||_2 = 1,\ g_i \in \{0,1\}\}.$$

The maximal function satisfying this condition can be computed as the 1-homogeneous convex envelope of the function

$$\Phi_\alpha(p) = \begin{cases} ||g||_{1/\alpha} & \text{if } p \in K_\alpha \\ +\infty & \text{otherwise} \end{cases}$$

and, as show in Section 1.4.2, it turns out to be $\Phi_\alpha^{**}(p) = \sup_{q \in K^\alpha} \langle p, q \rangle$, which is to say the support function of the set

$$K^\alpha = \left\{ p \in \mathbb{R}^{d \times (N-1)}, \quad \left\| \sum_{j \in J} p_j \right\|_2 \leq |J|^\alpha \quad \forall\, J \subset \{1, ..., N-1\} \right\},$$

with $|J|$ the cardinality of the set J. Thanks to (3.2.3), setting $\Psi_\alpha = \Phi_\alpha^{**}$, we can finally define

$$\mathcal{R}^\alpha(\Lambda) = \sup \left\{ \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \varphi_i \, d\Lambda_i, \quad \varphi \in C_c^\infty(\mathbb{R}^d; K^\alpha) \right\},$$

and consider the relaxed problem

$$\inf \left\{ \mathcal{R}^\alpha(\Lambda), \quad \operatorname{div} \Lambda_i = \delta_{P_i} - \delta_{P_N} \text{ for all } i = 1, \ldots, N-1 \right\}. \tag{3.2.4}$$

This formulation provides the convex framework we were looking for: the problem is now defined on the whole space of matrix valued Radon measures and the energy is convex as it is a supremum of linear functionals.

However the functional $\mathcal{R}^\alpha$ is obtained only as a "local" convex envelope of $\mathcal{F}^\alpha$ and as such it is not expected to always coincide with the true convex envelope, as we will see in Example 3.2.2. Thus, given a minimizer $\bar{\Lambda}$ of (3.2.4) we can end up in three different situations:

1. $\bar{\Lambda} \in \mathcal{L}(A)$, then $\bar{\Lambda}$ is also a minimizer of $\mathcal{F}^\alpha$ and we have solved our original problem;
2. $\mathcal{R}^\alpha(\bar{\Lambda}) = \inf_\Lambda \mathcal{F}^\alpha(\Lambda)$, then $\bar{\Lambda}$ is a convex combination of minimizers of $\mathcal{F}^\alpha$;
3. $\mathcal{R}^\alpha(\bar{\Lambda}) < \inf_\Lambda \mathcal{F}^\alpha(\Lambda)$, which means that the relaxation is not tight and generally speaking minima of $\mathcal{R}^\alpha$ have no relation with minima of $\mathcal{F}^\alpha$.

For a given set of terminal points $A = \{P_1, \ldots, P_N\}$ we will then call the relaxation (3.2.4) to be tight (or sharp) whenever one of its minimizers satisfies 1. or 2., i.e., whenever its minimizers are related to the actual minimizers of $\mathcal{F}^\alpha$ as it is the case with real convex envelopes. Unfortunately, as the following counterexample shows, the relaxation is not always sharp.

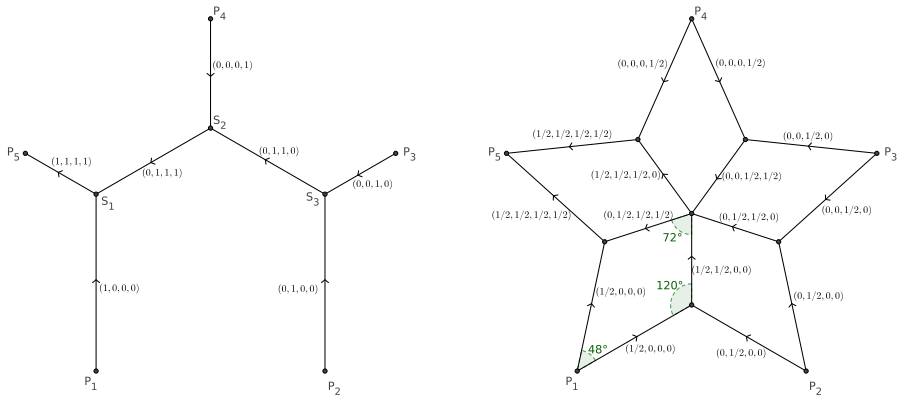

Figure 3.1: Left: an optimal Steiner tree viewed as its corresponding measure Λ. Right: a rank one tenor valued measure $\Sigma = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, with L the graph itself, τ and g as displayed.

Example 3.2.2. [Non sharpness for pentagon configurations] Consider as terminal points the five vertices of a regular pentagon of side $\ell > 0$ and let $\beta = \frac{3}{10}\pi$. In this situation (STP) has 5 minimizers which are the one in the left picture of figure 3.1 and its 4 rotations. The energy $\mathcal{R}^0$ of a Steiner tree, which corresponds by construction to its length, is equal to $\ell \tan\beta\,(1 + \sin\beta + \sqrt{3}\cos\beta) \approx 3.8911 \cdot \ell$. However none of the optimal Steiner trees is a minimizer for (3.2.4). Indeed we can exhibit an admissible tensor valued measure Σ with an energy strictly less than the energy of a Steiner tree: consider for example the rank one tensor valued measure Σ constructed in the right picture of figure 3.1. Such a measure satisfies the divergence constraints and its energy, which amounts to $1/2$ the length of its support, is equal to $\frac{5}{4}\ell(\sqrt{3} + \tan\beta) \approx 3.8855 \cdot \ell$. Hence we are in the third case of the previous list: the relaxation is not tight and as we already said there is in general no way of reconstructing an optimum for (STP) staring from a minimizer of $\mathcal{R}^0$ (in this case our numerical results suggest Σ as the actual minimizer of $\mathcal{R}^0$). Another example of non-sharpness can be obtained considering as terminal points the vertices of the pentagon plus the center: also in this case Σ has less energy than any optimal Steiner tree.

Despite the previous example, the proposed relaxation can be proved to be sharp in many situations. Indeed, thanks to the duality nature of $\mathcal{R}^\alpha$, we can prove minimality of certain given measures by means of calibration type arguments. This implies that whenever we are able to find a calibration for a given $\bar{\Lambda} \in \arg\min_\Lambda \mathcal{F}^\alpha(\Lambda)$ then the relaxation is sharp because $\bar{\Lambda}$ will also be a minimizer for $\mathcal{R}^\alpha$. A calibration, at least in the simple case of $\mathbb{R}^2$, can be defined as follows

Definition 3.2.3. *Fix a matrix valued Radon measure $\Lambda = (\Lambda_1, \dots, \Lambda_{N-1})$ and $\varphi \in C_c^\infty(\mathbb{R}^2; K^\alpha)$. We say that φ is a calibration for Λ if $\nabla \times \varphi_i = 0$ for all $i = 1, \dots, N-1$,*

and φ realizes the supremum in the definition of $\mathcal{R}^\alpha$, i.e.,

$$\sum_{i=1}^{N-1} \int_{\mathbb{R}^2} \varphi_i \, d\Lambda_i = \mathcal{R}^\alpha(\Lambda).$$

The only existence of such an object certifies the optimality of Λ in (3.2.4). Indeed, let $\Sigma = (\Sigma_1, \dots, \Sigma_{N-1})$ be another competitor, with $\mathcal{R}^\alpha(\Sigma) < \infty$ and $\operatorname{div} \Sigma_i = \delta_{P_i} - \delta_{P_N}$ for each $i = 1, \dots, N-1$. Hence $\operatorname{div}(\Lambda_i - \Sigma_i) = 0$ and we have[1]

$$\int_{\mathbb{R}^2} \varphi_i \, d(\Lambda_i - \Sigma_i) = 0, \tag{3.2.5}$$

so that

$$\begin{aligned} \mathcal{R}^\alpha(\Lambda) &= \sum_{i=1}^{N-1} \int_{\mathbb{R}^2} \varphi_i \, d\Lambda_i = \sum_{i=1}^{N-1} \left(\int_{\mathbb{R}^2} \varphi_i \, d(\Lambda_i - \Sigma_i) + \int_{\mathbb{R}^2} \varphi_i \, d\Sigma_i \right) \\ &\leq 0 + \mathcal{R}^\alpha(\Sigma) = \mathcal{R}^\alpha(\Sigma). \end{aligned}$$

In $\mathbb{R}^d$ with $d > 2$, the definition of a calibration extends as it is, where now $\nabla \times \varphi_i$ stands for the exterior derivative of the 1-form associated to the vector field φ_i. Also (3.2.5) generalizes and the proof carries over directly.

For the case $\alpha = 0$, which corresponds to (STP), we can take advantage of calibration arguments of [70] to justify sharpness of (3.2.4) for some classical choices of $\{P_1, \dots, P_N\}$. Indeed, as we will see in the next section, whenever Λ is a rank one tensor valued measure, for instance whenever it concentrates on a graph and has real-valued weights, $\mathcal{R}^\alpha$ coincides with the norm introduced in [70] to study (STP) as a mass-minimization problem for 1-dimensional currents with coefficients in a suitable normed group. Thus, every calibrated example in that context turns out to be a calibrated configuration in our framework, i.e., a situation where $\mathcal{R}^0$ is sharp (see [70, 74]).

3.2.2 Extensions: generic Gilbert–Steiner problems and manifolds

The same ideas developed in the previous paragraph can be extended beyond the (single sink) Gilbert–Steiner problem (I_α) in order to address problems with possibly multiple sources/sinks in an Euclidean setting or even problems formulated within manifolds.

Following the strategy introduced in [69] the energy $\mathcal{F}^\alpha$ can also be used to address the general (oriented version of) "who goes where" problem. In this context we do not have to move all the mass to a single sink but instead we are given a family of source/sink couples and we have to move a unit mass from each source to each given destination. Thus, letting $\{S_1, \dots, S_m\} \subset \mathbb{R}^d$ be the set of (unit) sources and $\{T_1, \dots, T_m\} \subset \mathbb{R}^d$ the corresponding set of (unit) sinks, we optimize the same energy E^α involved in the definition of (I_α) but this time among oriented networks of the form $L = \cup_{i=1}^m \lambda_i$, with λ_i

[1] This generalizes the "smooth" case: thinking to Λ_i and Σ_i as "regular" vector fields we have that $\Lambda_i - \Sigma_i$ is a gradient, whence integrating by parts and using that φ_i is curl-free we get zero.

a simple rectifiable curve connecting S_i to T_i. The same derivation as above can then be repeated, leading us to the relaxed formulation

$$\inf\{\mathcal{R}^\alpha(\Lambda), \quad \Lambda = (\Lambda_1, \dots, \Lambda_m), \operatorname{div} \Lambda_i = \delta_{S_i} - \delta_{T_i} \text{ for all } i = 1, \dots, m\}. \tag{3.2.6}$$

We remark that in the previous who goes where problem, differently to what happens in [21], we do not allow two paths λ_i, λ_j to have opposite orientation on intersections, i.e., particles have to go the same way when flowing in the same region.

The previous approach to the "who goes where" problem can now be used within the formulation of more general branched transportation problems, where we are just required to move mass from a set of (unit) sources $\{S_1, \dots, S_m\} \subset \mathbb{R}^d$ to a set of (unit) sinks $\{T_1, \dots, T_m\} \subset \mathbb{R}^d$, without prescribing the final destination of each particle. In this context the problem can be tackled as follows: for every possible coupling between sources and sinks, i.e., among all permutations $\sigma \in \mathcal{S}_m$, solve the corresponding "who goes where" problem with pairs $(S_i, T_{\sigma(i)})_{i=1}^m$, and then take the coupling realizing the minimal energy. Each "who goes where" can be relaxed as done in (3.2.6), providing this way a relaxed formulation also for the case of generic multiple sources/sinks.

We point out how the extension of the previous discussion to a manifold framework is direct: the derivation that led us to the energy $\mathcal{R}^\alpha$, together with problems (3.2.4) and (3.2.6), is still valid on surfaces embedded in the three dimensional space, with the only difference that divergence constraints have to be intended as involving the tangential divergence operator on the given surface.

3.3 A first simple approximation on graphs

In this section we first see how the previous formulation simplifies when we consider the Steiner tree problem in the context of graphs, in which case the energy reduces to the norm introduced in [70]. Then, once we are able to address (STP) on networks, we try to approximate the Euclidean (STP) by means of a discretization of the domain through an augmented graph.

3.3.1 The Steiner tree problem on graphs

Consider a connected graph $G = (V, E)$ in $\mathbb{R}^d$, where $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$ and $E = \{e_1, \dots, e_m\}$ is a set of m segments. Each $e_j = [v_j^1, v_j^2]$ connects two vertices v_j^1, v_j^2, has length $\ell(e_j) = ||v_j^2 - v_j^1||_2$ and is oriented by $\tau_{e_j} = (v_j^2 - v_j^1)/|v_j^2 - v_j^1|$. Furthermore, we can assume without loss of generality that edges intersect each other in at most 1 point. The Steiner Tree Problem within G can be formulated in the same fashion as its Euclidean counterpart: given a set of terminal points $A = \{P_1, \dots, P_N\} \subset V$ find the shortest connected sub-graph spanning A. As in the Euclidean case a solution always exists and optimal sub-graphs are indeed sub-trees (they contain no cycles).

Following what we did above in the Euclidean case, we can decompose any candidate sub-graph $L \subset G$ into the superposition of $N - 1$ paths λ_i within the graph, each one connecting P_i to P_N. Each path is identified as the support of a flow $V_i \colon E \to \{-1, 0, 1\}$

flowing a unit mass from P_i to P_N: we set $V_i(e) = 1$ if edge e is travelled in its own direction within path λ_i, -1 if it is travelled in the opposite way and 0 otherwise. By construction we satisfy the discrete version of (3.2.1), i.e., the classical Kirchhoff conditions: for all "interior" vertices $v \in V \setminus \{P_i, P_N\}$ we have

$$\sum_{e\in\delta^+(v)} V_i(e) - \sum_{e\in\delta^-(v)} V_i(e) = 0, \tag{3.3.1a}$$

with $\delta^{\pm}(v)$ the set of outgoing/incoming edges at vertex v, and (P_i, P_N) is the source/sink couple, meaning

$$\sum_{e\in\delta^+(P_i)} V_i(e) - \sum_{e\in\delta^-(P_i)} V_i(e) = 1, \qquad \sum_{e\in\delta^+(P_N)} V_i(e) - \sum_{e\in\delta^-(P_N)} V_i(e) = -1. \tag{3.3.1b}$$

Setting $V = (V_1, \dots, V_{N-1})$ and $L = \operatorname{supp} V = \cup\{e \in E : V(e) \neq 0\}$, we have

$$\mathcal{H}^1(L) = \sum_{e\in E} \ell(e) \cdot ||V(e)||_\infty =: \mathcal{F}(V),$$

and as before a solution to the network (STP) can be found minimizing $\mathcal{F}$ among vector valued flows $V\colon E \to \{-1, 0, 1\}^{N-1}$ satisfying the above flux conditions (3.3.1). Let us identify each family V with a tensor valued measure $\Lambda = (\Lambda_1, \dots, \Lambda_{N-1})$ defined on the whole $\mathbb{R}^d$ by setting

$$\Lambda_i = \sum_{e\in E} V_i(e)\, \tau_e \cdot \mathcal{H}^1 \llcorner e, \quad \text{i.e.,} \quad \Lambda = \sum_{e\in E} \tau_e \otimes V(e) \cdot \mathcal{H}^1 \llcorner e. \tag{3.3.2}$$

The idea is now to drop the integer constraint $\{-1, 0, 1\}$ on each V_i and optimize the previously defined energy $\mathcal{R}^0$ among tensor valued measures of the form (3.3.2), obtaining the relaxed energy

$$\mathcal{R}(V) = \mathcal{R}^0(\Lambda) = \sup_{\varphi\in C_c^\infty(\mathbb{R}^d;K^0)} \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \varphi_i \, d\Lambda_i = \sup_{\varphi\in C_c^\infty(\mathbb{R}^d;K^0)} \sum_{i=1}^{N-1} \sum_{e\in E} \left(V_i(e) \int_e \varphi_i \, ds \right).$$

Since edges intersect in at most 1 point it is possible to interpret the last supremum as a supremum over test functions entirely supported on the graph and of the form $\varphi = \sum_e \tau_e \otimes W(e)$ with $W\colon E \to \mathbb{R}^{N-1}$. By assumption, for almost every point x on the graph (except at intersections) there exists only one edge e containing x; hence, the pointwise constraint $\varphi(x) \in K^0$ translates into $\varphi \llcorner e \in K^0$ for all edges $e \in E$, i.e.,

$$\left\| \sum_{j\in J} W_j(e)\tau_e \right\|_2 = \left| \sum_{j\in J} W_j(e) \right| \leq 1 \quad \forall\, J \subset \{1, ..., N-1\}.$$

These new constraints involve only vectors $W(e)$ and are equivalent to the unique constraint

$$||W(e)||_* = \left[\sum_{j=1}^{N-1} (W_j(e) \vee 0) \right] \vee \left[-\sum_{j=1}^{N-1} (W_j(e) \wedge 0) \right] \leq 1,$$

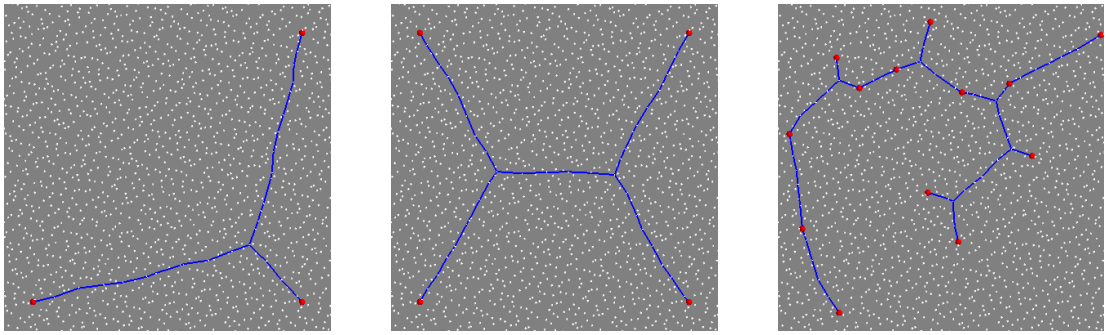

Figure 3.2: Approximations of (STP) for 3, 4 and 13 terminal points (red) using the augmented graph idea, $K = 1681$, $M = 30$. Edges carrying a non-zero flux are displayed.

which amounts to require that the maximum between the ℓ^1 norm of the positive part and the ℓ^1 norm of the negative part of $W(e)$ has to be less or equal to 1. The energy can be finally rewritten as

$$\begin{aligned}\mathcal{R}(V) &= \sup\left\{\sum_{e\in E}\ell(e)\,V(e)\cdot W(e),\quad ||W(e)||_* \le 1\ \forall e\in E\right\}\\ &= \sum_{e\in E}\ell(e)\left(\sup_i[V_i(e)\vee 0]-\inf_i[V_i(e)\wedge 0]\right)=\sum_{e\in E}\ell(e)||V(e)||.\end{aligned}$$

The norm $||\cdot||$ is exactly the norm used in [70] to study (STP) using currents with coefficients in normed groups and hence we can take advantage of calibration arguments of [70] to justify the sharpness of the relaxation for calibrated configurations of terminal points. Of course the counterexample 3.2.2 still applies to this discrete version of the problem using as graph G the union of the two graphs of picture 3.1: the minimizer concentrates on the star and not on the Steiner structure.

Optimization of $\mathcal{R}$ under the (linear) flux constraints (3.3.1) can then be performed solving a linear program: in order to linearize the objective we introduce two sets of variables $\{s_e\}_{e\in E}$, $\{i_e\}_{e\in E}$, and for each $e\in E$ we require $i_e\le 0$, $s_e\ge 0$ and

$$i_e \le V_i(e) \le s_e \quad \text{for all } i = 1,\dots,N-1,$$

so that the objective reduces to $\sum_e \ell(e)(s_e - i_e)$. Whenever the size of the resulting linear program is too big to be treated by standard interior point solvers we can alternatively apply the cheaper first order scheme proposed in [88] (see Section 3.4 for details).

3.3.2 Graphs and the Euclidean (STP)

Once we have a method to approximate (STP) on networks we can try to address the Euclidean (STP) through the use of an augmented graph. The core idea is the following: let $\{x_1,\dots,x_K\}$, $K\in\mathbb{N}$, be a set of scattered points that uniformly covers an open convex

domain Ω such that $\{P_1, \dots, P_N\} \subset \Omega$ and let $V = \{x_1, \dots, x_K\} \cup \{P_1, \dots, P_N\}$. Fix $M \in \mathbb{N}$ and construct the graph $G = (V, E)$ where each $v \in V$ is connected through segments to its M closest neighbours. For M sufficiently large the network G is connected and solving (STP) within G provides an approximation of the underlying Euclidean Steiner tree.

We see in figure 3.2 two examples with $K = 1681$ and $M = 30$. In both cases results are very close to the optimal Steiner tree and for obtaining them we simply solve a medium scale linear program. However the use of a fixed underlying graph has some drawbacks. For example, we cannot expect edges meeting at triple points to satisfy the $120°$ condition and what should be a straight piece in the optimal tree is only approximated by a sequence of (non-aligned) edges. A possible remedy for obtaining "straighter" solutions is to increase M, allowing this way longer edges, but this would increase the size of the problem. Furthermore obtaining convex combinations of minimizers is almost impossible because the underlying graph is not regular and having two sub-graphs with the exact same energy is very rare. On the other hand taking regularly distributed points generates many equivalent solutions even when there should be only one.

We also observe that this simplified framework is specific to the Euclidean Steiner tree case: the corresponding graph framework for (I_α) does not end up in a linear program and no direct extension to the manifold case is possible. This lack of generality, together with the intrinsic low precision of the approach as a consequence of working on a graph, leads us to switch our focus on the direct minimization of $\mathcal{R}^\alpha$ on the whole of $\mathbb{R}^2/\mathbb{R}^3$.

3.4 Generic Euclidean setting, the algorithmic approach

Motivated by the shortcomings of the previous simplified framework, we present in this section our approach for solving (3.2.4) in $\mathbb{R}^2$ (the same ideas extends to the three dimensional setting). Our resolution is based on a staggered grid for the discretization of the unknowns coupled with a conic solver (or a primal-dual scheme) for the optimization of the resulting finite dimensional problem.

3.4.1 Spatial discretization

Assume without loss of generality that $P_1, \dots, P_N$ are contained in the interior of $\Omega = [0,1] \times [0,1]$, which will be our computational domain. From a discrete standpoint we view the unknown vector measures $(\Lambda_1, \dots, \Lambda_{N-1})$ in (3.2.4) as a family $V = (V_1, \dots, V_{N-1})$ of vector fields in Ω and, due to the divergence constraints that we need to satisfy, we discretize these unknown fields on a staggered grid (this way our degrees of freedom are directly related to the flux of each vector field through the given grid interface). Fix then a regular Cartesian grid of size $M \times M$ over Ω and let $h = 1/M$. The first component $V_{i,1}$ of each vector field is placed on the midpoints of the vertical cells interfaces whereas

the second components $V_{i,2}$ on the horizontal ones, so that to have on each element (k,ℓ)

$$V_i|_{(k,\ell)} = \begin{pmatrix} (V_{i,1}^{k+1,\ell} - V_{i,1}^{k,\ell})(x-(k-1)h)/h + V_{i,1}^{k,\ell} \\ (V_{i,2}^{k,\ell+1} - V_{i,2}^{k,\ell})(y-(\ell-1)h)/h + V_{i,2}^{k,\ell} \end{pmatrix}.$$

The component $V_{i,1}$ is described by $(M+1)\times M$ unknowns whereas $V_{i,2}$ is described by $M\times(M+1)$ parameters. Regarding the test functions $\varphi = (\varphi_1,\dots,\varphi_{N-1})$ we define them to be piecewise constant on each element of the grid, i.e., for any cell (k,ℓ) we have $\varphi_i^{k,\ell} = (\varphi_{i,1}^{k,\ell}, \varphi_{i,2}^{k,\ell}) \in \mathbb{R}^2$.

Within this setting the optimization of the energy $\mathcal{R}^\alpha$ translates into

$$\min_{(V_{i,d}^{k,\ell})} \sup_{(\varphi_{i,d}^{k,\ell})\in K^\alpha} \sum_{k,\ell} \sum_{i=1}^{N-1} h^2 \left[\frac{V_{i,1}^{k,\ell} + V_{i,1}^{k+1,\ell}}{2}\,\varphi_{i,1}^{k,\ell} + \frac{V_{i,2}^{k,\ell} + V_{i,2}^{k,\ell+1}}{2}\,\varphi_{i,2}^{k,\ell} \right] \tag{3.4.1}$$

under the condition $\operatorname{div} V_i = \delta_{P_i} - \delta_{P_N}$ for all $i = 1,\dots,N-1$. Since the flux of each V_i over the generic cell (k,ℓ) is given by

$$F_i^{k,\ell} = h(V_{i,1}^{k+1,\ell} - V_{i,1}^{k,\ell}) + h(V_{i,2}^{k,\ell+1} - V_{i,2}^{k,\ell}),$$

the divergence constraints translate, at a discrete level, into

$$\begin{cases} F_i^{k,\ell} = 0 & \text{whenever cell } (k,\ell) \text{ does not contain } P_i \text{ or } P_N, \\ F_i^{k,\ell} = 1 & \text{if cell } (k,\ell) \text{ contains } P_i, \\ F_i^{k,\ell} = -1 & \text{if cell } (k,\ell) \text{ contains } P_N, \end{cases} \tag{3.4.2}$$

complemented with a "zero flux" condition at the boundary, i.e., we set $V_{i,d}^{k,\ell} = 0$ whenever it refers to a boundary interface. We finally observe that, by construction, $\varphi \in K^\alpha$ if for each cell (k,ℓ) in the grid the matrix $\varphi^{k,\ell} = (\varphi_1^{k,\ell},\dots,\varphi_{N-1}^{k,\ell})$ satisfies

$$\left\| \sum_{j\in J} \varphi_j^{k,\ell} \right\|_2 \le |J|^\alpha \text{ for all } J \subset \{1,\dots,N-1\}.$$

For the resolution of this finite dimensional optimization problem we then propose two different and somehow complementary approaches.

3.4.2 Optimization via conic duality

The inf-sup problem (3.4.1) can be written, thanks to conic duality (see, e.g., Lecture 2 of [19]), as a pure minimization problem involving the degrees of freedom $(V_{i,d}^{k,\ell})$ and a set of dual variables $(\psi_{J,d}^{k,\ell})$ indexed over subsets $J \subset \{1,\dots,N-1\}$. Indeed, for fixed $1 \le k,\ell \le M$ and $J \subset \{1,\dots,N-1\}$, one has

$$\inf_{\psi_J^{k,\ell}\in\mathbb{R}^2} \left(|J|^\alpha ||\psi_J^{k,\ell}||_2 - \langle\, \psi_J^{k,\ell}, \sum_{j\in J} \varphi_j^{k,\ell} \,\rangle \right) = \begin{cases} 0 & \text{if } \left\| \sum_{j\in J} \varphi_j^{k,\ell} \right\|_2 \le |J|^\alpha \\ -\infty & \text{otherwise,} \end{cases}$$

so that, if we denote $\tilde{V}_i^{k,\ell} = ((V_{i,1}^{k,\ell} + V_{i,1}^{k+1,\ell})/2, (V_{i,2}^{k,\ell} + V_{i,2}^{k,\ell+1})/2) \in \mathbb{R}^2$, (3.4.1) is equivalent to

$$\min_{(V_{i,d}^{k,\ell})} \sup_{(\varphi_{i,d}^{k,\ell})} \left[\sum_{k,\ell} \sum_{i=1}^{N-1} h^2 \langle \tilde{V}_i^{k,\ell}, \varphi_i^{k,\ell} \rangle + \inf_{(\psi_{J,d}^{k,\ell})} \sum_{k,\ell} \sum_{J} h^2 \left(|J|^\alpha ||\psi_J^{k,\ell}||_2 - \langle \psi_J^{k,\ell}, \sum_{j \in J} \varphi_j^{k,\ell} \rangle \right) \right].$$

Switching the sup over $(\varphi_{i,d}^{k,\ell})$ and the inf over $(\psi_{J,d}^{k,\ell})$ we obtain

$$\min_{(V_{i,d}^{k,\ell})} \inf_{(\psi_{J,d}^{k,\ell})} \left[\sum_{k,\ell} \sum_{J} h^2 |J|^\alpha ||\psi_J^{k,\ell}||_2 + h^2 \sup_{(\varphi_{i,d}^{k,\ell})} \sum_{k,\ell} \left(\sum_{i=1}^{N-1} \langle \tilde{V}_i^{k,\ell}, \varphi_i^{k,\ell} \rangle - \langle \psi_J^{k,\ell}, \sum_{j \in J} \varphi_j^{k,\ell} \rangle \right) \right].$$

Since the inner sup is either 0 if $\tilde{V}_i^{k,\ell} = \sum_{J \ni i} \psi_J^{k,\ell}$ for all $1 \le k, \ell \le M$ and $1 \le i \le N-1$ or $+\infty$ otherwise, the previous problem eventually leads to

$$\min_{(V_{i,d}^{k,\ell}), (\psi_{J,d}^{k,\ell})} \sum_{k,\ell} \sum_{J} h^2 |J|^\alpha \, \|\psi_J^{k,\ell}\|_2 \tag{3.4.3}$$

where each V_i satisfies the same flux constraints (3.4.2) and for all cells (k, ℓ) and all $i = 1, \dots, N-1$ we must satisfy

$$\frac{V_{i,1}^{k,\ell} + V_{i,1}^{k+1,\ell}}{2} = \sum_{J \ni i} \psi_{J,1}^{k,\ell} \quad \text{and} \quad \frac{V_{i,2}^{k,\ell} + V_{i,2}^{k,\ell+1}}{2} = \sum_{J \ni i} \psi_{J,2}^{k,\ell}. \tag{3.4.4}$$

Problem (3.4.3) under the set of linear constraints (3.4.2) and (3.4.4) can now be solved invoking the conic solver of the library MOSEK [78] within the framework provided by [52].

3.4.3 Optimization via primal-dual schemes

Collect all the $(V_{i,d}^{k,\ell})$ into a vector $\mathbf{v} \in \mathbb{R}^{n_v}$, $n_v = (N-1)(2M^2 + 2M)$, and all the $(\varphi_{i,d}^{k,\ell})$ into $\boldsymbol{\varphi} \in \mathbb{R}^{n_\varphi}$, $n_\varphi = (N-1)2M^2$. Moving the constraints on φ into the objective via the convex indicator function, the discrete energy (3.4.1) can be written down as

$$\min_{\mathbf{v}} \max_{\boldsymbol{\varphi}} \langle \boldsymbol{\varphi}, B\mathbf{v} \rangle - \chi_{K^\alpha}(\boldsymbol{\varphi})$$

for a suitable (sparse) matrix B of size $n_\varphi \times n_v$, while the divergence constraints reduce to $A\mathbf{v} = \mathbf{b}$ for a suitable (sparse) matrix A of size $n_\lambda \times n_v$ and a vector $\mathbf{b} \in \mathbb{R}^{n_\lambda}$. To the set of liner constraints $A\mathbf{v} - \mathbf{b} = 0$ we can now associate a dual variable $\boldsymbol{\lambda} \in \mathbb{R}^{n_\lambda}$ so that they can be incorporated into the objective as

$$\min_{\mathbf{v}} \max_{\boldsymbol{\varphi}, \boldsymbol{\lambda}} \langle \boldsymbol{\varphi}, B\mathbf{v} \rangle - \chi_{K^\alpha}(\boldsymbol{\varphi}) + \langle \boldsymbol{\lambda}, A\mathbf{v} - b \rangle.$$

The problem, written this way, turns into an instance of a general inf-sup problem of the form

$$\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} \langle y, Kx \rangle + G(x) - F^*(y) \tag{3.4.5}$$

with K an $m \times n$ matrix and $G\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $F^*\colon \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ convex lsc functions. Among the possible numerical schemes which have been developed in the literature for the resolution of (3.4.5) we choose here the preconditioned primal-dual scheme presented in [88]. The scheme can be summarized as follows: let $\gamma \in [0, 2]$, $T = \operatorname{diag}(\tau_1, \dots, \tau_n)$ and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_m)$, with

$$\tau_j = \frac{1}{\sum_{i=1}^m |K_{ij}|^{2-\gamma}} \quad \text{and} \quad \sigma_i = \frac{1}{\sum_{j=1}^n |K_{ij}|^{\gamma}},$$

fix $x^0 \in \mathbb{R}^n$, $y^0 \in \mathbb{R}^m$, and iterate for any $k > 0$

$$\begin{cases} x^{k+1} = (I + T\partial G)^{-1}(x^k - TK^T y^k) \\ y^{k+1} = (I + \Sigma \partial F^*)^{-1}(y^k + \Sigma K(2x^{k+1} - x^k)) \end{cases} \tag{3.4.6}$$

In this context the proximal mappings are defined as

$$(I + T\partial G)^{-1}(\hat{x}) = \arg\min_x \left[G(x) + \frac{1}{2}\langle T^{-1}(x - \hat{x}),\, x - \hat{x}\rangle \right]$$

and represent the extension of the classical definition with constant step size to this situation with "variable dependent" step sizes.

In our specific use case the scheme takes the following form: define $T = \operatorname{diag}(\tau_1, \dots, \tau_{n_v})$, $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_{n_\varphi})$ and $\tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1, \dots, \tilde{\sigma}_{n_\lambda})$, with

$$\tau_j = \frac{1}{\sum_{i=1}^{n_\varphi} |B_{ij}|^{2-\gamma} + \sum_{i=1}^{n_\lambda} |A_{ij}|^{2-\gamma}}, \quad \sigma_i = \frac{1}{\sum_{j=1}^{n_v} |B_{ij}|^{\gamma}}, \quad \tilde{\sigma}_i = \frac{1}{\sum_{j=1}^{n_v} |A_{ij}|^{\gamma}},$$

given $\mathbf{v}^0, \boldsymbol{\varphi}^0, \boldsymbol{\lambda}^0$ iterate for $k > 0$

$$\begin{cases} \mathbf{v}^{k+1} = \mathbf{v}^k - T(B^T \boldsymbol{\varphi}^k + A^T \boldsymbol{\lambda}^k) \\ \boldsymbol{\varphi}^{k+1} = \operatorname{proj}(\boldsymbol{\varphi}^k + \Sigma B(2\mathbf{v}^{k+1} - \mathbf{v}^k) \mid K^\alpha) \\ \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \tilde{\Sigma}(A(2\mathbf{v}^{k+1} - \mathbf{v}^k) - \mathbf{b}) \end{cases} \tag{3.4.7}$$

The computational bottleneck for this simple iterative procedure resides in the projection of a given vector $\bar{\boldsymbol{\varphi}} \in \mathbb{R}^{n_\varphi}$ onto the convex set K^α. By definition this operation reduces to the cell-wise projection on K^α of the matrices $\varphi^{k,\ell}$, and so we fix a $d \times (N-1)$ matrix $q = (q_1, \dots, q_{N-1})$ and split the discussion into two sub-steps.

Projection on individual sets: for each fixed subset $J \subset \{1, \dots, N-1\}$ we define the convex set

$$K_J^\alpha = \left\{ p \in \mathbb{R}^{d \times (N-1)}, \quad \left\| \sum_{j \in J} p_j \right\|_2 \leq |J|^\alpha \right\}.$$

The projection of q over K_J^α can be computed explicitly: define $v_J = \sum_{j \in J} q_j$, then the projection $p = \operatorname{proj}(q \mid K_J^\alpha) = (p_1, \dots, p_{N-1})$ has columns defined as $p_j = q_j$ if $j \notin J$ and

$$p_j = q_j - 1/|J| \left(\|v\|_2 - |J|^\alpha \right)^+ \frac{v}{\|v\|_2} \quad \text{if } j \in J.$$

Projection on the intersection: observe that $K^\alpha = \cap_J K^\alpha_J$, i.e., K^α is the intersection of a family of convex sets. In order to get an approximation of $\text{proj}(q \mid K^\alpha)$ we can apply the Dykstra's projection algorithm (see [53]). The scheme in our setting is the following: let $J_1, \dots, J_{2^{N-1}}$ be all the subsets of $\{1, \dots, N-1\}$, let $\{y_j^0\}_{j=1}^{2^{N-1}}$ be 2^{N-1} null matrices of size $d \times (N-1)$, $p^0 = q$, then for any $k \geq 1$ iterate

$$\begin{cases} p_0^k = p^{k-1} \\ \text{for } j = 1, \dots, 2^{N-1} \\ \qquad p_j^k = \text{proj}(p_{j-1}^k + y_j^{k-1} \mid K^\alpha_{J_j}) \\ \qquad y_j^k = y_j^{k-1} + p_{j-1}^k - p_j^k \\ \text{end for} \\ p^k = p_{2^{N-1}}^k \end{cases}$$

We then have $p^k \to \text{proj}(q \mid K^\alpha)$ as $k \to +\infty$.

Remark 3.4.1. Each step of the previous iterative projection procedure requires 2^{N-1} sub-projections and thus the scheme is intrinsically time-consuming. Up to our knowledge there seems to be no immediate simplifications to avoid some of the 2^{N-1} inner projections: for example, the restriction of the inner loop over sets $K^\alpha_{J_j}$ such that $q \notin K^\alpha_{J_j}$ is not going to work in general. At the same time we observe that established convergence rates for (3.4.6) do not apply in this case because our projection, which represents one of the two proximal mappings, is only approximated and not exact, making us falling back in a context like [102].

3.5 Numerical details

The two resolution paths presented above allow us to overcome some shortcomings of the simplified framework of Section 3.3 but introduces at the same time an higher computational cost, mainly depending on the combinatorial nature of the set K^α, which reflects in the high number of variables involved in (3.4.3) and in the complicated projection in (3.4.7).

Generally speaking the primal-dual scheme is the cheapest of the two in terms of computational resources: it can be implemented so that every operation is done in-place, reducing to almost zero any further memory requirement apart from initialization, while the interior point approach used by a conic solver is extremely demanding in terms of memory due to the 2^{N-1} additional variables needed to define (3.4.3). However, since we are looking for 1-dimensional structures, our solver also needs to be able to provide very localized optima, and with this regards the primal-dual approach is not very satisfactory. As we can see in figure 3.3, where we use the two schemes for the same regular 201×201 grid over $[0, 1]^2$, the solution provided by the primal-dual scheme is more diffused than the one obtained using the conic approach. For this reason we would like to use for our experiments the conic formulation (3.4.3) and to do so, in order to be able to treat medium

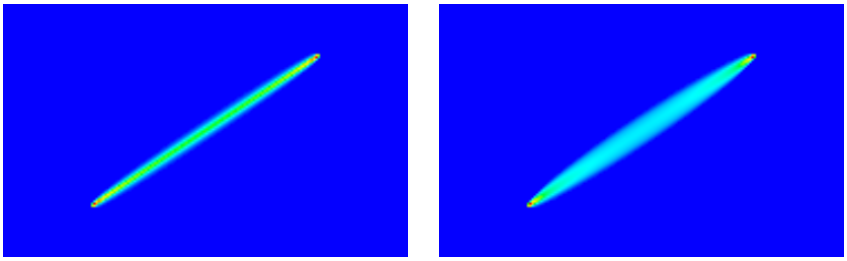

Figure 3.3: Energy concentration for the minimizer of $\mathcal{R}^0$ for $P_1 = (1/4, 1/3)$, $P_2 = (3/4, 2/3)$. Left: solution obtained via a conic solver, final energy ≈ 0.606307. Right: solution obtained after 200000 iterations of the primal-dual scheme, $\gamma = 0.6$, final energy ≈ 0.606765.

scale problems, we need to find a way to reduce a-priori the huge number of additional variables that are introduced: this can be done both via a classical grid refinement and via a variables "selection".

3.5.1 Grid refinement

The numerical solution is expected to concentrate on a 1-dimensional structure, and so the grid needs to be fine only on a relatively small region of the domain. This suggests the implementation of a refinement strategy able to localize in an automatic way the region of interest. For doing so we use non-conformal quadtree type meshes (see, e.g., [96, 17]), which are a particular class of grids where the domain is partitioned using M square cells as $\Omega = \cup_m S_m$ and each square cell S_m can be obtained by recursive subdivision of the box $[0, 1]^2$ (see figure 3.4 for examples of such grids). As in the case of uniform regular meshes we employ for the discretization a staggered approach: we set the degrees of freedom of vector fields on faces of each element, with the additional requirement that whenever a face is also a subsegment of another face then the two associated degrees of freedom are equal (this is to maintain continuity of the normal components of the discrete fields across edges and guarantees that fluxes are globally well behaved). The matrix valued function φ is again defined to be constant on each element of the grid so that the nature of the discrete problem we need to solve remains the same.

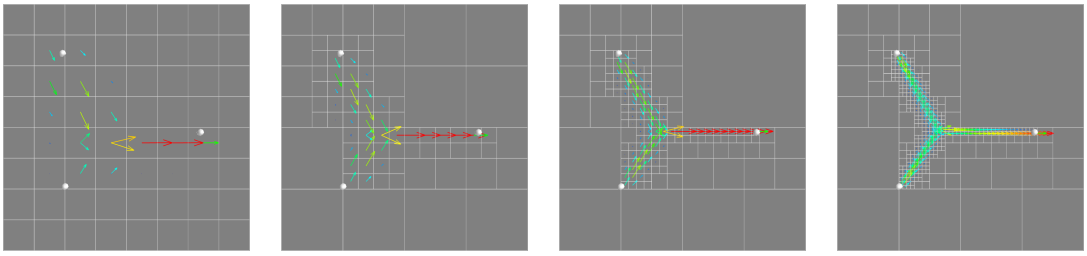

Figure 3.4: Refinement example for 3 points. At each iterate we plot the grid and the two fields V_1, V_2, which are then used to build the next grid.

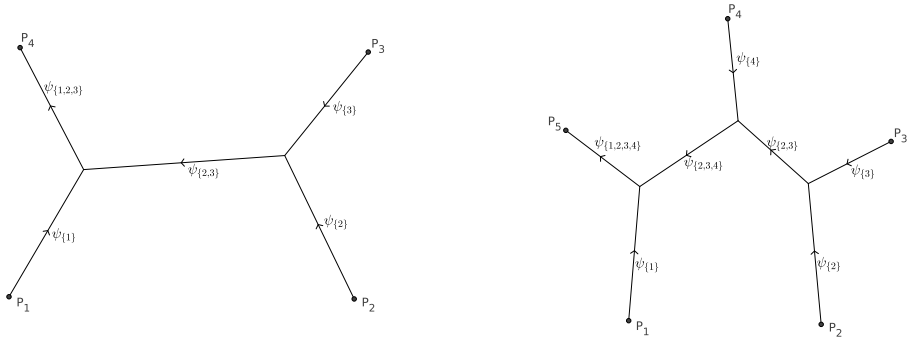

Figure 3.5: Expected behaviour of the variables $\{\psi_J\}_J$. We can see how each V_i can be reconstructed as the union of the ψ_J such that $i \in J$ and that only a subset of the ψ_J is used.

A refinement procedure can then be described as follows: fix a coarse quadtree grid $\mathcal{T}$, for example a regular 8×8 one, and then

- solve the problem on the given grid $\mathcal{T}$;
- identify elements of the grid where the solution concentrates the most and label them as "used", identify elements of the grid where the solution is almost zero and label them as "unused";
- refine the grid subdividing each "used" element into 4 equal sub-elements and try to merge "unused" elements into bigger ones (the merging will occur if four elements labelled as "unused" have the same father in the quadtree structure);
- repeat.

As we can see in figure 3.4 this procedure allows us to localize computations in a neighbourhood of the optimal structure we are looking for. This way we can attain a good level of fineness around the solution without being forced to employ a full grid which would require the introduction of a lot of useless degrees of freedom.

3.5.2 Variables selection

Generally speaking, in an optimum for (3.4.3) most of the variables ψ_J will turn out to be identically 0 while the ones that are not 0 everywhere will be concentrated on small regions of the domain. Indeed each ψ_J can be seen as a possible building block of the final solution because, due to formula (3.4.4), the vector field $\psi_{\{j_1,\dots,j_k\}}$, $\{j_1,\dots,j_k\} \subset \{1,\dots,N-1\}$, represents the portion of the graph where the fields $V_{j_1},\dots,V_{j_k}$ coincide (see, for example, figure 3.5 for a visual depiction in two cases). This means that we expect only a few ψ_J to be non zero on each element of the grid. With this in mind we can add the following selection procedure to the previous refinement scheme: given an approximate

solution on a grid $\mathcal{T}$, we identify for each square element S_m which are the non zero variables $\psi_{J_1^m}, \dots, \psi_{J_{km}^m}$ on that element and then, at the next step, we introduce only these variables in that particular region (in case the element S_m is one of those labelled as "used" this means that in the next optimization we will use only $\psi_{J_1^m}, \dots, \psi_{J_{km}^m}$ within its 4 children).

The main advantage of this procedure is clear: once we are able to identify the regions where each variable ψ_J concentrates (if any) we can dramatically reduce the number of unknowns we need to introduce, passing from 2^{N-1} vector fields to be defined on each element to only a few of them. Thanks to these two refinement procedures we are now in a position to efficiently tackle the optimization of $\mathcal{R}^\alpha$ using accurate conic solvers.

3.6 Results in flat cases

We present in this section different results obtained using the outlined scheme integrated with the two refinement procedures described above.

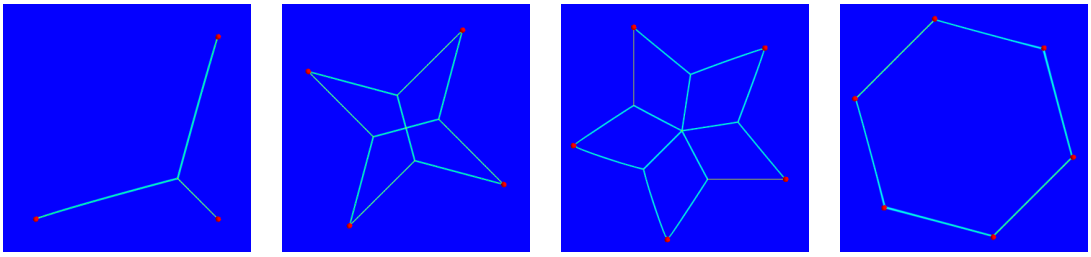

Figure 3.6: Optima of $\mathcal{R}^0$ in $\mathbb{R}^2$ for $3, 4, 5, 6$ terminal points on the vertices of regular polygons.

In figure 3.6 we compute minimizers of the relaxed energy $\mathcal{R}^0$ for regular configurations of terminal points placed on the vertices of a triangle, a square, a pentagon and an hexagon. In all cases we start with a regular 32×32 mesh and then apply the previous refinement procedures 5 times, ending up with a grid size of $1/1024$ around the optimal structure. In the first example we are able to retrieve the unique minimizer while in the second example we obtain a convex combination of the two possible minimizers for (STP). In the latter case this behaviour is expected because for this particular configuration of points the relaxation is sharp due to the calibration argument presented in [70]. In the third experiment we recover the star-shaped counterexample of figure 3.1 which seems to be the actual minimizer of the relaxed problem and in the last picture we get a convex combination of the six possible minimizers. We remark that the hexagon case is not a calibrated example in the work of Marchese–Massaccesi but our numerical result suggests the existence of a calibration because the relaxation seems to be sharp.

In figure 3.7 we first compute a minimizer for a 7 points configuration (6 vertices of the hexagon plus the center) and observe how we are able to obtain a convex combination of the two Steiner trees (again this was expected due to a calibration argument). We

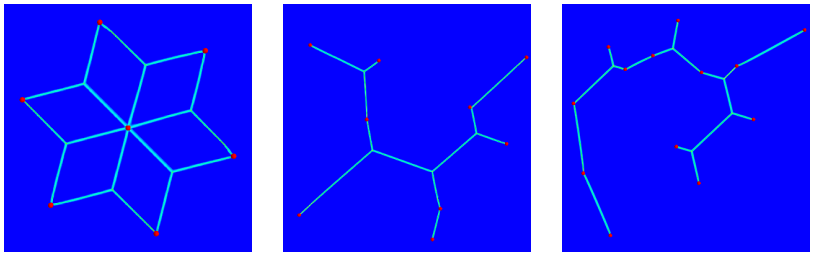

Figure 3.7: Optima of $\mathcal{R}^0$ in $\mathbb{R}^2$ for $7, 9$ and 13 terminal points.

observe that in this example the points do not lie on the boundary of a convex set, meaning that the problem cannot be simplified into an optimal partition problem as it is done, for example, in [43]. We then move to some non symmetric distributions of terminal points: in the second picture we see the result for 9 randomly selected points while in the third one we increase the number of terminals up to 13. In this last case an ad-hoc approach is necessary. Due to the high number of variables introduced in (3.4.1) a direct minimization using a conic solver is unfeasible even for very coarse grids (the amount of memory required to just set up the interior point solver is too much). To circumvent this problem we first compute a rough solution either optimizing $\mathcal{R}^0$ on a coarse grid using the primal-dual minimization scheme or applying the augmented graph idea presented in section 3.3 (see picture 3.2), and then we use this approximation for deducing which are the variables ψ_J active at a given point: for every cell (k, ℓ) of a uniform grid we introduce $\psi_{\{j_1,\dots,j_k\}}$ on that cell only if in the approximate solution every field $V_{j_1}, \dots, V_{j_k}$ is not identically zero in a suitable neighbourhood of the cell. This way we rule out a huge amount of the ψ_J obtaining a problem which is now tractable through interior point schemes.

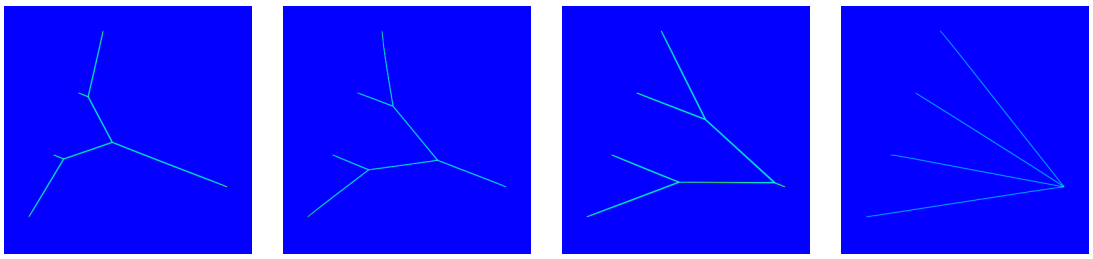

Figure 3.8: Irrigation networks minimizing $\mathcal{R}^\alpha$ and moving 4 masses to a unique sink, $\alpha = 0.6, 0.8, 0.95, 1$.

In figure 3.8 we test the relaxation $\mathcal{R}^\alpha$ for a simple irrigation problem where we approximate the shape of the optimal network moving 4 unit masses located at $S_1 = (0.4, 0.9)$, $S_2 = (0.3, 0.65)$, $S_3 = (0.2, 0.4)$, $S_4 = (0.1, 0.15)$, to the unique sink $T = (0.9, 0.27)$. We can see how for small α the optimal shape is close to the optimal Steiner tree while for higher values of α the network approaches more and more the configuration

for an optimal Monge–Kantorovitch transport attaining it for $\alpha = 1$ as expected.

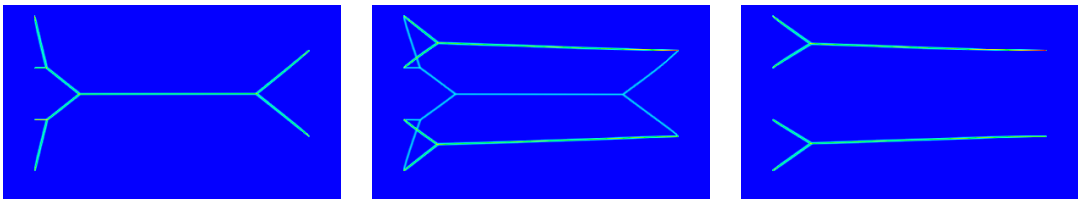

Figure 3.9: Optima of $\mathcal{R}^\alpha$ for moving 4 masses from left to right, $\alpha = 0.65, 0.7, 0.75$. The pairings realizing the first infimum and the third one are different.

We turn next in figure 3.9 to an example where 4 unit masses located at 4 sources on the left ($S_1 = (0.1, 0.55)$, $S_2 = (0.1, 0.4)$, $S_3 = (0.1, 0.25)$, $S_4 = (0.1, 0.1)$) has to be moved to 2 sinks of magnitude 2 on the right ($T_1 = (0.9, 0.2)$, $T_2 = (0.9, 0.45)$). Since for each mass we have two possible destinations we need to loop over all feasible combinations of source/sink couples, solve the corresponding "who goes where" problem and then choose the one giving the optimizer with less energy. In the examples the optimal couplings are $\{(S_1, T_1), (S_2, T_1), (S_3, T_2), (S_4, T_2)\}$ for $\alpha = 0.65$ and $\{(S_1, T_2), (S_2, T_2), (S_3, T_1), (S_4, T_1)\}$ for $\alpha = 0.75$. In the case $\alpha = 0.7$ we are at the switching point between a connected and a disconnected optimal structure and our relaxed optimum concentrates on both.

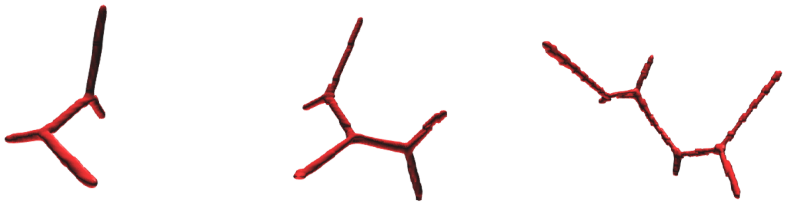

Figure 3.10: Optima of $\mathcal{R}^0$ for $4, 5$ and 7 points in $\mathbb{R}^3$.

The numerical scheme we have described for the two dimensional case can be extended directly to the three dimensional context for addressing the optimization of $\mathcal{R}^\alpha$ in $\mathbb{R}^3$. Non-conformal quadtree type grids are replaced by non-conformal octree type grids (see [96]) and a staggered approach is employed placing the degrees of freedom on faces of each cubic element composing the grid. The underlying structure of the discrete optimization we end up with remains the same and the two refinement procedures can be extended as they are, without any major change. We see in figure 3.10 the results for $4, 5$ and 7 points configurations. All the examples are purely 3-dimensional and in the first two cases we have a maximum number of Steiner points (respectively, 2 and 3), while in the last case the optimal structure consists of two "disjoint" optimal sub-trees connected through a central terminal point.

3.7 Extension to surfaces

As already observed in Section 3.2 the proposed relaxation can also be used to address (STP) and α-irrigation problems on surfaces. Up to our knowledge, even in the Steiner tree case, this is the first numerical approximation of these problems covering the manifold framework. Theoretically speaking what we need to do is to solve problem (3.2.4) on a manifold S embedded in $\mathbb{R}^3$, where now a candidate minimizer Λ is a matrix valued measure defined on the manifold and divergence constraints translate accordingly. From a numerical point of view our unknowns are again vector fields $(V_1, \dots, V_{N-1})$ living on the surface and the domain will be approximated by means of a triangulated surface $\mathcal{T}_h$. We first discuss the direct extension of the staggered grid idea to $\mathcal{T}_h$ and then present a more accurate discretization, eventually used in our experiments.

3.7.1 Raviart–Thomas approach

The staggered approach presented for quadrilateral grids can be extended to triangular meshes considering a discretization based on the so-called Raviart–Thomas basis functions, which are vector valued functions whose degrees of freedom are related to the flux of the given basis function across edges (see [36]).

Let $\mathcal{T}_h$ be a regular triangulation of S, with n vertices and m edges, and consider the lowest order Raviart–Thomas basis functions over $\mathcal{T}_h$: for each edge e in the triangulation we call K_- the "left" triangle adjacent to e and K_+ the "right" triangle adjacent to e (according to a given fixed orientation) and define the vector function

$$\Phi_e(x) = \begin{cases} \dfrac{\ell_e}{2A_e^+}(x - p_+) & \text{if } x \in K_+ \\ -\dfrac{\ell_e}{2A_e^-}(x - p_-) & \text{if } x \in K_- \\ (0,0,0) & \text{otherwise} \end{cases}$$

where ℓ_e is the length of the edge, $A_e^\pm = |K_\pm|$ are the areas of the triangles and p_+, p_- are the opposite corners (with the obvious modification for boundary edges). We then approximate each V_i, $i = 1, \dots, N-1$, as

$$V_i(x) = \sum_{e=1}^{m} V_i^e \, \Phi_e(x)$$

and as before matrix valued variables $\varphi = (\varphi_1, \dots, \varphi_{N-1})$ are considered to be piecewise constant over each element of the triangulation, i.e., $\varphi_i|_K = \varphi_i^K = (\varphi_{i,1}^K, \varphi_{i,2}^K, \varphi_{i,3}^K) \in \mathbb{R}^3$ for all $K \in \mathcal{T}_h$, $i = 1, \dots, N-1$. The unknowns are then the family of parameters (V_i^e) and $(\varphi_{i,d}^K)$. Looking at $\mathcal{R}^\alpha$ the integral we need to compute becomes

$$\sum_{K \in \mathcal{T}_h} \sum_{i=1}^{N-1} \int_K \left(\sum_{e=1}^{m} V_i^e \, \Phi_e(x) \right) \cdot \varphi_i^K \, dx, \tag{3.7.1}$$

and can be made explicit as follows: let e_j^K be the edge of triangle K opposite to point P_j^K (jth point of triangle K) and $s^{K,e_j^K} = \pm 1$ the position of that triangle with respect to the edge e_j^K, then (3.7.1) yields

$$\frac{1}{6}\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{N-1}\Big[s^{K,e_1^K}\ell_{e_1^K}V_i^{e_1^K}(P_2^K+P_3^K-2P_1^K)\varphi_{i,1}^K + s^{K,e_2^K}\ell_{e_2^K}V_i^{e_2^K}(P_1^K+P_3^K-2P_2^K)\varphi_{i,2}^K + s^{K,e_3^K}\ell_{e_3^K}V_i^{e_3^K}(P_1^K+P_2^K-2P_3^K)\varphi_{i,3}^K\Big].$$

The structure of the discrete energy is the same as the one obtained in the Euclidean setting (the conditions on φ translates again in the element-wise constraint $\varphi^K \in K^\alpha$ for all $K \in \mathcal{T}_h$). Furthermore within this Raviart–Thomas framework we have two advantages: fields V_i are by construction surface vector fields (i.e., they live in the tangent space to the surface) and divergence constraints translate into simple flux conditions of the form

$$s^{K,e_1^K}\ell_{e_1^K}V_i^{e_1^K} + s^{K,e_2^K}\ell_{e_2^K}V_i^{e_2^K} + s^{K,e_3^K}\ell_{e_3^K}V_i^{e_3^K} = 0 \text{ or } \pm 1$$

depending on K containing P_i, P_N or none of them, and $V_i^e = 0$ whenever e is a boundary edge. The price to pay for such simplicity resides in the fact that this Raviart–Thomas approximation is a low-order scheme. The objects we would like to approximate are singular vector fields concentrated on 1-dimensional structures but with this approach we generally obtain solutions that are quite diffused and can only give us an approximate idea of the underlying optimal set. At the same time this diffusivity prevents a good refinement because the refined region turns out to be too large. For this reason a better approximation space is needed, even if we will end up with a more complex discrete problem.

3.7.2 $\mathbb{P}_2$-based approach

Let $\mathcal{T}_h$ be a regular triangulation of S. We consider the standard discrete space

$$X_h^2 = \{v_h \in C^0(\mathcal{T}_h) \,:\, v_h|_K \in \mathbb{P}_2(K), \text{ for all } K \in \mathcal{T}_h\}$$

and take vector fields $V_i \in (X_h^2)^3$ for all $i = 1, \dots, N-1$. As in the staggered case matrix valued variables $\varphi = (\varphi_1, \dots, \varphi_{N-1})$ are defined to be piecewise constant over each element of the triangulation, i.e., $\varphi_i|_K = \varphi_i^K = (\varphi_{i,1}^K, \varphi_{i,2}^K, \varphi_{i,3}^K)$ for all $K \in \mathcal{T}_h$, $i = 1, \dots, N-1$. The energy $\mathcal{R}^\alpha$ is then

$$\sup\left\{\sum_{K\in\mathcal{T}_h}\sum_{i=1}^{N-1}\int_K V_i\cdot\varphi_i^K\,dx, \quad \varphi^K \in K^\alpha \text{ for all } K\in\mathcal{T}_h\right\}$$

and the integral over each triangle K can be computed explicitly in terms of the degrees of freedom associated to V_i and φ_i, $i = 1, \dots, N-1$ (the integrand reduces to a polynomial

of degree 2). We are left with the specification of how we impose divergence and tangency constraints on each V_i, $i = 1, \dots, N-1$.

Divergence constraints: for each vector field V_i we have to impose $\operatorname{div} V_i = \delta_{P_i} - \delta_{P_N}$, where this time the divergence has to be interpreted as the tangential divergence operator on surfaces (see, for instance, [92]). We observe that $\operatorname{div} V_i$ is piecewise linear over each element of the triangulation and thus, for $K \in \mathcal{T}_h$ not containing P_i or P_N we impose $(\operatorname{div} V_i)|_K = 0$ requiring it to be 0 at the three vertices of K. On the other hand, if $K \in \mathcal{T}_h$ contains P_i (or P_N) we require the flux of V_i over ∂K to be $+1$ (or -1). Eventually, for each boundary edge e_b of the triangulation we request the flux of V_i through e_b to be 0.

Tangency constraints: while for the Raviart–Thomas approach the approximate fields are surface vector fields by construction, for this $\mathbb{P}_2$ approach we need to impose this constraint as an additional condition. For doing so we require tangency of V_i at each node of the triangulation and at the mid-point of each edge. Normals at these points are approximated as a weighted average of normals of surrounding elements.

The above constraints, as it happens in the staggered case, translate into linear constraints over the degrees of freedom of $V_1, \dots, V_{N-1}$, and the discrete problem we end up with can be solved using the same strategies presented in Section 3.4. Eventually we observe that we can extend the refinement procedures of Section 3.5 also on triangulated surfaces taking advantage of the re-meshing functionalities of the Mmg Platform [1]: at each step we identify the region where the solution concentrates the most and then remesh the surface requiring the new mesh to be finer in that region and coarser elsewhere.

3.7.3 Results

In figure 3.11 we see the results obtained through the $\mathbb{P}_2$-based approach for 3 instances of (STP) on the sphere. In the first case (upper-left) we approximate the Steiner tree associated to the terminal points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and observe how we get a classical triple junction. In the second example (upper-middle and upper-right) we add a fourth point, $(0,-1,0)$, and obtain a convex combination of minimizers: in this case a possible minimizer can be constructed using the structure of the first picture completed with an geodesic arc connecting $(0,0,1)$ to $(0,-1,0)$. We also observe that due to the refinement steps energy concentrates only on two of the possible four minimizers, the two around which the mesh gets refined. In the third example (second row) we add a fifth point, $(-1,0,0)$, and obtain a convex combination of the two minimizers.

As we change the topological nature of the surface results become more interesting. We approximate in figure 3.12 minimizers of $\mathcal{R}^0$ for some points configurations on the torus. In the first example (upper-left) we fix two terminal points opposite to each other on the largest equator and observe an energy concentration on four different paths (each one a geodesic connecting the two points). For certain 3 points configurations we obtain a unique structure with a triple junction (upper-right), while for 3 points in a symmetric disposition on the largest equator we observe as solution a convex combination

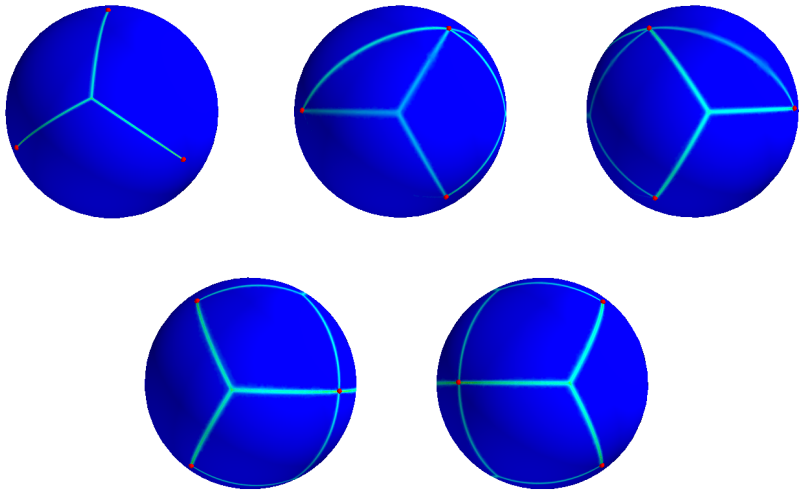

Figure 3.11: Optima of $\mathcal{R}^0$ for $3, 4, 5$ points on a sphere, single view for 3 terminals and different view angles for $4, 5$ terminals.

of the 6 possible minimizers (bottom-left). In the last example (bottom-right) we increase the number of holes of our torus and obtain for a symmetrical 3 points configuration a minimizer which cannot be seen as a convex combination of Steiner trees (i.e., another non sharpness example).

Finally, in figure 3.13, we test our relaxation on some surfaces with boundary. In the first example we connect three given points on the graph of a function while in the last two we use flat surfaces with holes, which can be seen as the flat version of the previous tori. In this case solutions can adhere to the interior boundary of the domain as long as this is energetically favourable. Observe that, similarly to counter example of figure 3.1, we obtain a profile which is not a convex combination of optimal trees. As in figure 3.1, we suspect this solution to illustrate the fact that our convexification may be not sharp in specific situations.

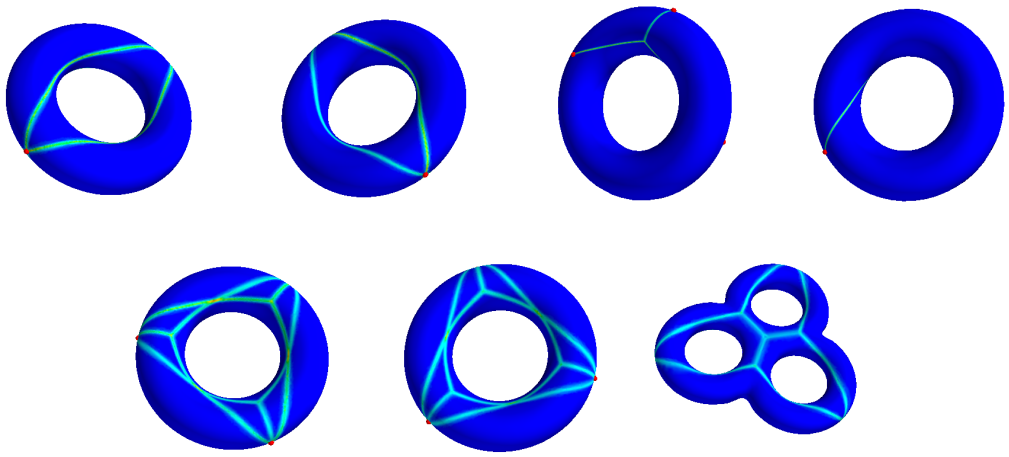

Figure 3.12: Optima of $\mathcal{R}^0$ for $2, 3$ points on different tori (front/back views).

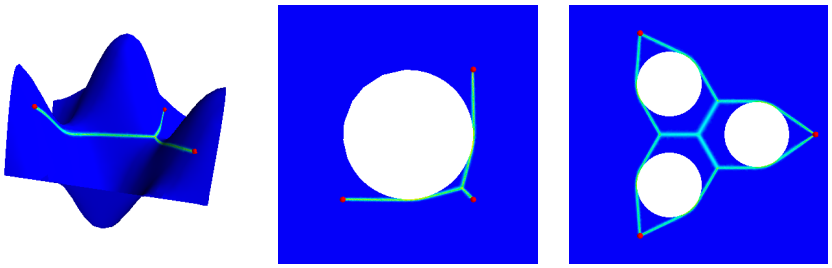

Figure 3.13: Optima of $\mathcal{R}^0$ for 3 points on the graph of a function and on some punctured domains in $\mathbb{R}^2$.

Chapter 4

A variational scheme for hyperbolic obstacle problems

We consider an obstacle problem for (possibly non-local) wave equations, and we prove existence of weak solutions through a convex minimization approach based on a time discrete approximation scheme. We provide the corresponding numerical implementation and raise some open questions.

4.1 Introduction

Obstacle type problems are nowadays a well established subject with many dedicated contributions in the recent literature. Obstacle problems for the minimizers of classical energies and regularity of the arising free boundary have been extensively studied, both for local operators (see, e.g., [38, 93] and references therein) and non-local fractional type operators (see, e.g. [105] and the review [93]). The corresponding evolutive equations have also been considered, mainly in the parabolic context [40, 39, 79, 16]. What seems to be missing in the picture is the hyperbolic scenario which, despite being in some cases as natural as the previous ones, has received little attention so far.

Among the available results for hyperbolic obstacle problems there is a series of works by Schatzman and collaborators [99, 100, 101, 84], where the existence of a solution is proved via penalty methods and, furthermore, existence of energy preserving solutions are proved in dimension 1 whenever the obstacle is concave [100]. The problem is also considered in [72], where the author proves the existence of a (possibly dissipative) solution within a more general framework but under technical hypotheses. More recently the 1d situation has been investigated in [66] through a minimization approach based on time discretization, see also [60, 81, 108, 48] for contributions on related problems using the same point of view. Another variational approach to hyperbolic problems, through an elliptic regularization suggested by De Giorgi, is given in [103] and subsequent papers (see, for instance, [47] for time dependent domains).

In this chapter we use a convex minimization approach, relying on a semi-discrete

approximation scheme (as in [66, 60, 48]), to deal with more general situations so as to include also non-local hyperbolic problems in the presence of obstacles, in arbitrary dimension. As main results we prove existence of a suitably defined weak solution to the wave equation involving the fractional Laplacian with or without an obstacle, together with the corresponding energy estimates. Those results are summarized in Theorem 4.3.2 and Theorem 4.4.2 (see Section 4.3 and 4.4). The approximating scheme allows to perform numerical simulations which give quite precise evidence of dynamical effects. In particular, based on our numerical experiments for the obstacle problem, we conjecture that this method is able to select, in cases of nonuniqueness, the most dissipative solution, that is to say the one losing the maximum amount of energy at contact times.

Eventually, we remark that this approach is quite robust and can be extended for instance to the case of adhesive phenomena: in these situations an elastic string interacts with a rigid substrate through an adhesive layer [46] and the potential energy governing the interaction can be easily incorporated in our variational scheme.

The chapter is organized as follows. We first recall the main properties of the fractional Laplace operator and fractional Sobolev spaces in Section 4.2 and then, in Section 4.3, we introduce the time-disretized variational scheme and apply it to the non-local wave equation (with the fractional Laplacian), proving Theorem 4.3.2. In Section 4.4 we adapt the scheme so as to include the obstacle problem, proving existence of weak solutions in Theorem 4.4.2. In the last section we describe the corresponding numerical implementation providing some examples and we conclude with some remarks and open questions.

4.2 Fractional Sobolev spaces and the fractional Laplacian operator

In this section we briefly review the main definitions and properties of the fractional setting and we fix the notation used in the rest of the chapter. For a more complete introduction to fractional Sobolev spaces we point to [51, 75] and references therein.

Fractional Sobolev spaces. Let $\Omega \subset \mathbb{R}^d$ be an open set. For $s \in \mathbb{R}$, we define the Sobolev spaces $H^s(\Omega)$ as follows:

- for $s \in (0,1)$ and $u \in L^2(\Omega)$, define the Gagliardo semi-norm of u as

$$[u]_{H^s(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

  The fractional Sobolev space $H^s(\Omega)$ is then defined as

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \, : \, [u]_{H^s(\Omega)} < \infty \right\},$$

  with norm $||u||_{H^s(\Omega)} = (||u||^2_{L^2(\Omega)} + [u]^2_{H^s(\Omega)})^{1/2}$;

- for $s \geq 1$ let us write $s = [s] + \{s\}$, with $[s]$ integer and $0 \leq \{s\} < 1$. The space $H^s(\Omega)$ is then defined as

$$H^s(\Omega) = \{u \in H^{[s]}(\Omega) \,:\, D^\alpha u \in H^{\{s\}}(\Omega) \text{ for any } \alpha \text{ s.t. } |\alpha| = [s]\},$$

  with norm $||u||_{H^s(\Omega)} = (||u||^2_{H^{[s]}(\Omega)} + \sum_{|\alpha|=[s]} ||D^\alpha u||^2_{H^{\{s\}}(\Omega)})^{1/2}$;

- for $s < 0$ we define $H^s(\Omega) = (H_0^{-s}(\Omega))^*$, where as usual the space $H_0^s(\Omega)$ is obtained as the closure of $C_c^\infty(\Omega)$ in the $||\cdot||_{H^s(\Omega)}$ norm.

Fractional Laplacian. For any $s > 0$, denote by $(-\Delta)^s$ the fractional Laplace operator, which (up to normalization factors) can be defined as follows:

- for $s \in (0,1)$, we set

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+2s}}\, dy, \quad x \in \mathbb{R}^d;$$

- for $s \geq 1$, $s = [s] + \{s\}$, we set $(-\Delta)^s = (-\Delta)^{\{s\}} \circ (-\Delta)^{[s]}$.

Let us define, for any $u, v \in H^s(\mathbb{R}^d)$, the bilinear form

$$[u,v]_s = \int_{\mathbb{R}^d} (-\Delta)^{s/2} u(x) \cdot (-\Delta)^{s/2} v(x)\, dx$$

and the corresponding semi-norm $[u]_s = \sqrt{[u,u]_s} = ||(-\Delta)^{s/2} u||_{L^2(\mathbb{R}^d)}$. Define on $H^s(\mathbb{R}^d)$ the norm $||u||_s = (||u||^2_{L^2(\mathbb{R}^d)} + [u]_s^2)^{1/2}$, which in turn is equivalent to the norm $||\cdot||_{H^s(\mathbb{R}^d)}$.

The spaces $\tilde{H}^s(\Omega)$. Let $s > 0$ and fix Ω to be an open bounded set with Lipschitz boundary. The space we are going to work with throughout this chapter is

$$\tilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^d) \,:\, u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega\},$$

endowed with the $||\cdot||_s$ norm. This space corresponds to the closure of $C_c^\infty(\Omega)$ with respect to the $||\cdot||_s$ norm. We have also $(\tilde{H}^s(\Omega))^* = H^{-s}(\Omega)$, see [75, Theorem 3.30].

We finally recall the following embedding results (see [51]).

Theorem 4.2.1. *Let $s > 0$. The following holds:*

- *if $2s < d$, then $\tilde{H}^s(\Omega)$ embeds in $L^q(\Omega)$ continuously for any $q \in [1, 2^*]$ and compactly for any $q \in [1, 2^*)$, with $2^* = 2d/(d-2s)$;*

- *if $2s = d$, then $\tilde{H}^s(\Omega)$ embeds in $L^q(\Omega)$ continuously for any $q \in [1, \infty)$ and compactly for any $q \in [1, 2]$;*

- *if $2s > d$, then $\tilde{H}^s(\Omega)$ embeds continuously in $C^{0,\alpha}(\Omega)$ with $\alpha = (2s-d)/2$.*

4.3 A variational scheme for the fractional wave equation

In this section, as a first step towards obstacle problems, we extend to the fractional wave equation a time-disretized variational scheme which traces back to Rothe [94] and since then has been extensively applied to many different hyperbolic type problems, see, e.g., [108, 80, 109, 48].

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. Given $u_0 \in \tilde{H}^s(\Omega)$ and $v_0 \in L^2(\Omega)$, the problem we are interested in is the following: find $u = u(t,x)$ such that

$$\begin{cases} u_{tt} + (-\Delta)^s u = 0 & \text{in } (0,T)\times\Omega \\ u(t,x) = 0 & \text{in } [0,T]\times(\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{4.3.1}$$

where the "boundary" condition is imposed on the complement of Ω due to the non-local nature of the fractional operator. In particular, we look for weak type solutions of (4.3.1).

Definition 4.3.1. *We say a function*

$$u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)), \quad u_{tt} \in L^\infty(0,T;H^{-s}(\Omega)),$$

is a weak solution of (4.3.1) *if*

$$\int_0^T \int_\Omega u_{tt}(t)\varphi(t)\,dxdt + \int_0^T [u(t),\varphi(t)]_s\,dt = 0 \tag{4.3.2}$$

for all $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$ *and the initial conditions are satisfied in the following sense:*

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \left(||u(t)-u_0||^2_{L^2(\Omega)} + [u(t)-u_0]^2_s \right) dt = 0 \tag{4.3.3}$$

and

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h ||u_t(t)-v_0||^2_{L^2(\Omega)}\,dt = 0. \tag{4.3.4}$$

The aim of this section is then to prove the next theorem.

Theorem 4.3.2. *There exists a weak solution of the fractional wave equation* (4.3.1).

The existence of a such a weak solution will be proved by means of an implicit variational scheme based on the idea of minimizing movements [9] introduced by De Giorgi, elsewhere known also as the discrete Morse semiflow approach or Rothe's scheme [94].

4.3.1 Approximating scheme

For any $n > 0$ let $\tau_n = T/n$, $u^n_{-1} = u_0 - \tau_n v_0$, and $u^n_0 = u_0$ (conventionally we intend $v_0(x) = 0$ for $x \in \mathbb{R}^d \setminus \Omega$). For any $0 < i \leq n$, given u^n_{i-2} and u^n_{i-1}, define

$$u^n_i = \arg\min_{u \in \tilde{H}^s(\Omega)} J^n_i(u) = \arg\min_{u \in \tilde{H}^s(\Omega)} \left[\int_\Omega \frac{|u - 2u^n_{i-1} + u^n_{i-2}|^2}{2\tau_n^2}\, dx + \frac{1}{2}[u]^2_s \right]. \tag{4.3.5}$$

Each u^i_n is well defined: indeed, existence of a minimizer can be obtained via the direct method of the calculus of variations while uniqueness follows from the strict convexity of the functional J^n_i. Each minimizer u^n_i can be characterize in the following way: take any test function $\varphi \in \tilde{H}^s(\Omega)$, then, by minimality of u^n_i in $\tilde{H}^s(\Omega)$, one has

$$\frac{d}{d\varepsilon} J^n_i(u^n_i + \varepsilon\varphi)|_{\varepsilon=0} = 0,$$

which rewrites as

$$\int_\Omega \frac{u^n_i - 2u^n_{i-1} + u^n_{i-2}}{\tau_n^2}\, \varphi\, dx + [u^n_i, \varphi]_s = 0 \quad \text{for all } \varphi \in \tilde{H}^s(\Omega). \tag{4.3.6}$$

We define the piecewise constant and piecewise linear interpolation in time of the sequence $\{u^n_i\}_i$ over $[-\tau_n, T]$ as follows: let $t^n_i = i\tau_n$, then the piecewise constant interpolant is given by

$$\bar{u}^n(t,x) = \begin{cases} u^n_{-1}(x) & t = -\tau_n \\ u^n_i(x) & t \in (t^n_{i-1}, t^n_i], \end{cases} \tag{4.3.7}$$

and the piecewise linear one by

$$u^n(t,x) = \begin{cases} u^n_{-1}(x) & t = -\tau_n \\ \dfrac{t - t^n_{i-1}}{\tau_n} u^n_i(x) + \dfrac{t^n_i - t}{\tau_n} u^n_{i-1}(x) & t \in (t^n_{i-1}, t^n_i]. \end{cases} \tag{4.3.8}$$

Define $v^n_i = (u^n_i - u^n_{i-1})/\tau_n$, $0 \leq i \leq n$, and let v^n be the piecewise linear interpolation over $[0,T]$ of the family $\{v^n_i\}_{i=0}^n$, defined similarly to (4.3.8). Taking the variational characterization (4.3.6) and integrating over $[0,T]$ we obtain

$$\int_0^T \int_\Omega \left(\frac{u^n_t(t) - u^n_t(t-\tau_n)}{\tau_n} \right) \varphi(t)\, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s\, dt = 0$$

for all $\varphi \in L^1(0,T; \tilde{H}^s(\Omega))$, or equivalently

$$\int_0^T \int_\Omega v^n_t(t)\varphi(t)\, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s\, dt = 0. \tag{4.3.9}$$

The idea is now to pass to the limit $n \to \infty$ and prove, using (4.3.9), that the approximations u^n and $\bar{u}^n$ converge to a weak solution u of (4.3.1). For doing so the main tool is the following estimate.

Proposition 4.3.3 (Key estimate)**.** *The approximate solutions $\bar{u}^n$ and u^n satisfy*

$$\|u_t^n(t)\|_{L^2(\Omega)}^2 + [\bar{u}^n(t)]_s^2 \leq C(u_0, v_0)$$

for all $t \in [0,T]$, with $C(u_0, v_0)$ a constant independent of n.

Proof. For each fixed $i \in \{1, \dots, n\}$ consider equation (4.3.6) with $\varphi = u_{i-1}^n - u_i^n$, so that we have

$$\begin{aligned} 0 &= \int_\Omega \frac{(u_i^n - 2u_{i-1}^n + u_{i-2}^n)(u_{i-1}^n - u_i^n)}{\tau_n^2}\, dx + [u_i^n, u_{i-1}^n - u_i^n]_s \\ &\leq \frac{1}{2\tau_n^2}\int_\Omega (u_{i-1}^n - u_{i-2}^n)^2 - (u_i^n - u_{i-1}^n)^2\, dx + \frac{1}{2}([u_{i-1}^n]_s^2 - [u_i^n]_s^2), \end{aligned}$$

where we use the fact that $b(a-b) \leq \frac{1}{2}(a^2-b^2)$. Summing for $i = 1, \dots, k$, with $1 \leq k \leq n$, we get

$$\begin{aligned} \left\| \frac{u_k^n - u_{k-1}^n}{\tau_n} \right\|_{L^2(\Omega)}^2 + [u_k^n]_s^2 &\leq \frac{1}{\tau_n^2}\|u_0 - u_{-1}^n\|_{L^2(\Omega)}^2 + [u_0]_s^2 \\ &= ||v_0||_{L^2(\Omega)}^2 + [u_0]_s^2. \end{aligned}$$

The result follows by the very definition of u^n and $\bar{u}^n$.

Remark 4.3.4. Given a weak solution u of (4.3.1) we can speak of the energy quantity

$$E(t) = ||u_t(t)||_{L^2(\Omega)}^2 + [u(t)]_s^2.$$

One can easily see by an approximation argument that E is conserved throughout the evolution and, as a by-product of the last proof, we see that also the energy of our approximations is at least non-increasing, i.e., $E_i^n \leq E_{i-1}^n$, where $E_i^n = E(u^n(t_i^n)) = ||v_i^n||_{L^2(\Omega)}^2 + [u_i^n]_s^2$. Furthermore we also remark that we cannot improve this estimate, meaning that generally speaking the given approximations u^n are not energy preserving.

Thanks to Proposition 4.3.3, we can now prove convergence of the u^n.

Proposition 4.3.5 (Convergence of u^n)**.** *There exists a subsequence of steps $\tau_n \to 0$ and a function $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$, with $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$, such that*

$$\begin{aligned} &u^n \to u && \text{in } C^0([0,T];L^2(\Omega)) \\ &u_t^n \rightharpoonup^* u_t && \text{in } L^\infty(0,T;L^2(\Omega)) \\ &u^n(t) \rightharpoonup u(t) && \text{in } \tilde{H}^s(\Omega) \text{ for any } t \in [0,T]. \end{aligned}$$

Proof. From Proposition 4.3.3 it follows that

$$u_t^n(t) \text{ and } v^n(t) \text{ are bounded in } L^2(\Omega) \text{ uniformly in } t \text{ and } n, \tag{4.3.10}$$

$$u^n(t) \text{ is bounded in the } [\cdot]_s \text{ semi-norm uniformly in } t \text{ and } n. \tag{4.3.11}$$

Observe now that $u^n(\cdot,x)$ is absolutely continuous on $[0,T]$; thus, for all $t_1,t_2\in[0,T]$ with $t_1<t_2$, we have

$$||u^n(t_2,\cdot)-u^n(t_1,\cdot)||_{L^2(\Omega)}=\left(\int_\Omega\left(\int_{t_1}^{t_2}u_t^n(t,x)\,dt\right)^2dx\right)^{\frac12}$$
$$\leq\left(\int_{t_1}^{t_2}||u_t^n(t,\cdot)||^2_{L^2(\Omega)}\,dt\right)^{\frac12}(t_2-t_1)^{\frac12}\leq C(t_2-t_1)^{\frac12},$$

where we made use of the Hölder's inequality and of Fubini's Theorem. This estimate yields

$$u^n(t)\text{ is bounded in }L^2(\Omega)\text{ uniformly in }t\text{ and }n,\tag{4.3.12}$$
$$u^n\text{ is equicontinuous in }C^0([0,T];L^2(\Omega)).\tag{4.3.13}$$

From (4.3.9), using (4.3.12) and (4.3.11), we can also deduce that $v_t^n(t)$ is bounded in $H^{-s}(\Omega)$ uniformly in t and n. All together we have

$$u^n\text{ is bounded in }W^{1,\infty}(0,T;L^2(\Omega))\text{ and in }L^\infty(0,T;\tilde H^s(\Omega)),\tag{4.3.14}$$
$$v^n\text{ is bounded in }L^\infty(0,T;L^2(\Omega))\text{ and in }W^{1,\infty}(0,T;H^{-s}(\Omega)).\tag{4.3.15}$$

Thanks to (4.3.13), (4.3.14) and (4.3.15) there exists a function $u\in L^\infty(0,T;\tilde H^s(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))\cap C^0([0,T];L^2(\Omega))$ such that

$$\begin{aligned}&u^n\to u && \text{in } C^0([0,T];L^2(\Omega))\\ &u_t^n\rightharpoonup^* u_t && \text{in } L^\infty(0,T;L^2(\Omega))\\ &u^n(t)\rightharpoonup u(t) && \text{in } \tilde H^s(\Omega)\text{ for any } t\in[0,T]\end{aligned}$$

and there exists $v\in W^{1,\infty}(0,T;H^{-s}(\Omega))$ such that

$$v^n\rightharpoonup^* v\text{ in }L^\infty(0,T;L^2(\Omega))\quad\text{and}\quad v^n\rightharpoonup^* v\text{ in }W^{1,\infty}(0,T;H^{-s}(\Omega)).$$

As one would expect $v(t)=u_t(t)$ as elements of $L^2(\Omega)$ for a.e. $t\in[0,T]$: indeed, for $t\in(t_{i-1}^n,t_i^n]$ and $\varphi\in\tilde H^s(\Omega)$, we have by construction $u_t^n(t)=v^n(t_i^n)$, and so

$$\int_\Omega(u_t^n(t)-v^n(t))\varphi\,dx=\int_\Omega(v^n(t_i^n)-v^n(t))\varphi\,dx=\int_\Omega\left(\int_t^{t_i^n}v_t^n(s)\,ds\right)\varphi\,dx$$
$$\leq\tau_n||v_t^n||_{L^\infty(0,T;H^{-s}(\Omega))}||\varphi||_{H^s(\mathbb{R}^d)}$$

which implies, for any $\psi(t,x)=\varphi(x)\eta(t)$ with $\varphi\in\tilde H^s(\Omega)$ and $\eta\in C_0^1([0,T])$, that

$$\int_0^T\left[\int_\Omega(u_t(t)-v(t))\varphi\,dx\right]\eta(t)\,dt=\int_0^T\int_\Omega(u_t(t)-v(t))\psi\,dxdt$$
$$=\lim_{n\to\infty}\int_0^T\int_\Omega(u_t^n(t)-v^n(t))\psi\,dxdt=\lim_{n\to\infty}\int_0^T\left[\int_\Omega(u_t^n(t)-v^n(t))\varphi\,dx\right]\eta(t)\,dt$$
$$\leq\lim_{n\to\infty}\tau_nT||v_t^n||_{L^\infty(0,T;H^{-s}(\Omega))}||\varphi||_{H^s(\mathbb{R}^d)}||\eta||_\infty=0.$$

Hence we have

$$\int_\Omega (u_t(t) - v(t))\varphi \, dx = 0 \quad \text{for all } \varphi \in \tilde{H}^s(\Omega) \text{ and a.e. } t \in [0,T],$$

which yields the sought for conclusion. Thus, $v_t = u_{tt}$ and $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$. □

Proposition 4.3.6 (Convergence of $\bar{u}^n$)**.** *Let u be the limit function obtained in Proposition 4.3.5, then*

$$\bar{u}^n \rightharpoonup^* u \text{ in } L^\infty(0,T;\tilde{H}^s(\Omega)).$$

Proof. By definition we have

$$\sup_{t\in[0,T]} \int_\Omega |u^n(t,x) - \bar{u}^n(t,x)|^2 \, dx = \sum_{i=1}^n \sup_{t\in[t_{i-1}^n, t_i^n]} (t - t_i^n)^2 \int_\Omega (v_i^n)^2 \, dx$$
$$\leq \tau_n^2 \sum_{i=1}^n ||v_i^n||^2_{L^2(\Omega)} \leq C\tau_n$$

which implies $\bar{u}^n \to u$ in $L^\infty(0,T;L^2(\Omega))$. Furthermore, taking into account Proposition 4.3.3, $\bar{u}^n(t)$ is bounded in $\tilde{H}^s(\Omega)$ uniformly in t and n, so that we have $\bar{u}^n \rightharpoonup^* u$ in $L^\infty(0,T;\tilde{H}^s(\Omega))$ and, as it happens for u^n, $\bar{u}^n(t) \rightharpoonup u(t)$ in $\tilde{H}^s(\Omega)$ for any $t \in [0,T]$. □

We can now pass to the limit in (4.3.9) to prove u to be a weak solution, thus proving Theorem 4.3.2.

Proof of Theorem 4.3.2. The limit function u obtained in Proposition 4.3.5 is a weak solution of (4.3.1). Indeed, for each $n > 0$, by (4.3.9) one has

$$\int_0^T \int_\Omega v_t^n(t)\varphi(t) \, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s \, dt = 0$$

for any $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$. Passing to the limit as $n \to \infty$, using Propositions 4.3.5 and 4.3.6, we immediately get

$$\int_0^T \int_\Omega u_{tt}(t)\varphi(t) \, dxdt + \int_0^T [u(t), \varphi(t)]_s \, dt = 0.$$

Regarding the initial conditions (4.3.3) and (4.3.4) it suffices to prove that, if $t_k \to 0$ are Lebesgue points for both $t \mapsto ||u_t(t)||^2_{L^2(\Omega)}$ and $t \mapsto [u(t)]_s^2$, then

$$[u(t_k)]_s^2 \to [u_0]_s^2 \quad \text{and} \quad ||u_t(t_k)||^2_{L^2(\Omega)} \to ||v_0||^2_{L^2(\Omega)}. \tag{4.3.16}$$

From the fact that $u_t \in W^{1,\infty}(0,T;H^{-s}(\Omega))$ we have $u_t(t_k) \to v_0$ in $H^{-s}(\Omega)$ and, since $u_t(t_k)$ is bounded in $L^2(\Omega)$ and $\tilde{H}^s(\Omega) \subset L^2(\Omega)$ is dense, we also have $u_t(t_k) \rightharpoonup v_0$

in $L^2(\Omega)$. On the other hand $u(t_k) \to u(0) = u_0$ strongly in $L^2(\Omega)$ because $u \in C^0([0,T]; L^2(\Omega))$ and, being $u(t_k)$ bounded in $\tilde{H}^s(\Omega)$, $u(t_k) \rightharpoonup u(0)$ in $\tilde{H}^s(\Omega)$ and $[u_0]_s \leq \liminf_k [u(t_k)]_s$. To prove (4.3.16) it suffices to observe that

$$\limsup_{k\to\infty} \left([u(t_k)]_s^2 + ||u_t(t_k)||^2_{L^2(\Omega)} \right) \leq [u_0]_s^2 + ||v_0||^2_{L^2(\Omega)}$$

by energy conservation.

□

4.4 The obstacle problem

In this section we switch our focus to hyperbolic obstacle problems for the fractional Laplacian. We will see how a weak solution can be obtained by means of a slight modification of the previously presented scheme, whose core idea has already been used in other obstacle type problems (for example, in [66, 79]).

As above, let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary and consider $g\colon \Omega \to \mathbb{R}$, with

$$g \in C^0(\bar{\Omega}), \quad g < 0 \text{ on } \partial\Omega.$$

We are still interested in a non-local wave type dynamic like the one of equation (4.3.1), where now we require the solution u to lay above g: this way g can be interpreted as a physical obstacle that our solution cannot go below. Consider then an initial datum

$$u_0 \in \tilde{H}^s(\Omega), \quad u_0 \geq g \text{ a.e. in } \Omega,$$

and $v_0 \in L^2(\Omega)$. Equation (4.3.1), with the addition of the obstacle g, reads as follows: find a function $u = u(t,x)$ such that

$$\begin{cases} u_{tt} + (-\Delta)^s u \geq 0 & \text{in } (0,T) \times \Omega \\ u(t,\cdot) \geq g & \text{in } [0,T] \times \Omega \\ (u_{tt} + (-\Delta)^s u)(u-g) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{4.4.1}$$

In this system the function u is required to be an obstacle-free solution whenever away from the obstacle, where $u - g > 0$, while we only require a variational inequality (first line) when u touches g. The main difficulty in (4.4.1) is the treatment of contact times: the previous system does not specify what kind of behaviour arises at contact times, leaving us free to choose between "bouncing" solutions, the profile hits the obstacle and bounces back with a fraction of the previous velocity (see, e.g., [84]), and an "adherent" solution, the profile hits the obstacle and stops (this way we dissipate energy). The definition of weak solution we are going to consider includes both of these cases.

Definition 4.4.1. *We say a function $u = u(t,x)$ is a weak solution of* (4.4.1) *if*

1. $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ *and* $u(t,x) \geq g(x)$ *for a.e.* $(t,x) \in (0,T)\times\Omega$*;*
2. *there exist weak left and right derivatives* $u_t^\pm$ *on* $[0,T]$ *(with appropriate modifications at endpoints);*
3. *for all* $\varphi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^1(0,T;\tilde{H}^s(\Omega))$ *with* $\varphi \geq 0$, $spt\,\varphi \subset [0,T)$, *we have*
$$-\int_0^T \int_\Omega u_t \varphi_t \, dxdt + \int_0^T [u,\varphi]_s \, dt - \int_\Omega v_0 \, \varphi(0) \, dx \geq 0$$
4. *the initial conditions are satisfied in the following sense*
$$u(0,\cdot) = u_0, \quad \int_\Omega (u_t^+(0) - v_0)(\varphi - u_0)\, dx \geq 0 \quad \forall \varphi \in \tilde{H}^s(\Omega), \varphi \geq g.$$

Within this framework we can partially extend the construction presented in the previous section so as to prove existence of a weak solution.

Theorem 4.4.2. *There exists a weak solution u of the hyperbolic obstacle problem* (4.4.1)*, and u satisfies the energy inequality*
$$||u_t^\pm(t)||^2_{L^2(\Omega)} + [u(t)]_s^2 \leq ||v_0||^2_{L^2(\Omega)} + [u_0]_s^2 \quad \textit{for a.e. } t \in [0,T]. \tag{4.4.2}$$

We remark here that this definition of weak solution is weaker than the one proposed in [72, 55], in which the authors construct a solution to (4.4.1) as a limit of (energy preserving) solutions u^n of regularized systems, where the constraint $u^n \geq g$ is turned into a penalization term in the equation. Furthermore, up to our knowledge, the problem of the existence of an energy preserving weak solution to (4.4.1) is still open: one would expect the limit function in [72, 55] to be the best known candidate, while a partial result for concave obstacles in 1d was provided by Schatzman in [100].

4.4.1 Approximating scheme

The idea is to replicate the scheme presented in Section 4.3 for the obstacle-free dynamic: define
$$K_g = \{u \in \tilde{H}^s(\Omega) \,|\, u \geq g \text{ a.e. in } \Omega\}$$
and, for any $n > 0$, let $\tau_n = T/n$. Define $u^n_{-1} = u_0 - \tau_n v_0$ and $u^n_0 = u_0$, and construct recursively the family of functions $\{u^n_i\}_{i=1}^n \subset \tilde{H}^s(\Omega)$ as
$$u^n_i = \arg\min_{u \in K_g} J^n_i(u),$$
with J^n_i defined as in (4.3.5). Notice how the minimization is now over functions $u \geq g$ in Ω so that to respect the additional constraint introduced by the obstacle. Since

$K_g \subset \tilde{H}^s(\Omega)$ is convex, existence and uniqueness of each u_i^n can be proved by means of standard arguments. Regarding the variational characterization of each minimizer u_i^n, we cannot take arbitrary variations $\varphi \in \tilde{H}^s(\Omega)$ (we may end up exiting the feasible set K_g), and so we need to be more careful: we take any test $\varphi \in K_g$ and consider the function $(1-\varepsilon)u_i^n + \varepsilon\varphi$, which belongs to K_g for any sufficiently small positive ε. Thus, since u_i^n minimizes J_i^n, we have the following inequality

$$\frac{d}{d\varepsilon} J_i^n(u_i^n + \varepsilon(\varphi - u_i^n))|_{\varepsilon=0} \geq 0,$$

which rewrites as

$$\int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}(\varphi - u_i^n)\,dx + [u_i^n, \varphi - u_i^n]_s \geq 0 \quad \text{for all } \varphi \in K_g. \tag{4.4.3}$$

In particular, since every $\varphi \geq u_i^n$ is an admissible test function, we also have

$$\int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}\varphi\,dx + [u_i^n, \varphi]_s \geq 0 \quad \text{for all } \varphi \in \tilde{H}^s(\Omega), \varphi \geq 0. \tag{4.4.4}$$

We define $\bar{u}^n$ and u^n as, respectively, the piecewise constant and the piecewise linear interpolation in time of $\{u_i^n\}_i$ (as in (4.3.8), (4.3.7)), and v^n as the piecewise linear interpolant of velocities $v_i^n = (u_i^n - u_{i-1}^n)/\tau_n$, $0 \leq i \leq n$. Using (4.4.4), the analogue of (4.3.6) takes the following form

$$\int_0^T \int_\Omega \left(\frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n} \right) \varphi(t)\,dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s\,dt \geq 0$$

for all $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$, $\varphi(t,x) \geq 0$ for a.e. $(t,x) \in (0,T)\times\Omega$.

In view of a convergence result, we observe that the same energy estimate of Proposition 4.3.3 extends to this new context: for any $n > 0$, we have

$$\|u_t^n(t)\|_{L^2(\Omega)}^2 + [\bar{u}^n(t)]_s^2 \leq C(u_0, v_0)$$

for all $t \in [0,T]$, with $C(u_0,v_0)$ a constant independent of n. The exact same proof of Proposition 4.3.3 applies: just observe that, taking $\varphi = u_{i-1}^n$ in (4.4.3), one gets

$$0 \leq \int_\Omega \frac{(u_i^n - 2u_{i-1}^n + u_{i-2}^n)(u_{i-1}^n - u_i^n)}{\tau_n^2}\,dx + [u_i^n, u_{i-1}^n - u_i^n]_s$$

and then the rest follows. Convergence of the interpolants is then a direct consequence.

Proposition 4.4.3 (Convergence of u^n and $\bar{u}^n$, obstacle case)**.** *There exists a subsequence of steps $\tau_n \to 0$ and a function $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ such that*

$$u^n \to u \text{ in } C^0([0,T];L^2(\Omega)), \qquad \bar{u}^n \rightharpoonup^* u \text{ in } L^\infty(0,T;\tilde{H}^s(\Omega)),$$
$$u_t^n \rightharpoonup^* u_t \text{ in } L^\infty(0,T;L^2(\Omega)), \qquad u^n(t) \rightharpoonup u(t) \text{ in } \tilde{H}^s(\Omega) \text{ for any } t \in [0,T],$$

and furthermore $u(t,x) \geq g(x)$ for a.e. $(t,x) \in [0,T]\times\Omega$.

Proof. To obtain the existence of u and all the convergences we can repeat the first half of the proof of Proposition 4.3.5 and the proof of Proposition 4.3.6. The fact that $u(t,x) \geq g(x)$ for a.e. $(t,x) \in [0,T] \times \Omega$ is a direct consequence of the fact that $u_i^n \in K_g$ for all n and $0 \leq i \leq n$.

□

The missing step with respect to the obstacle-free dynamic is that generally speaking $u_{tt} \notin L^\infty(0,T;H^{-s}(\Omega))$. The cause of such a behaviour is clear already in 1d: suppose the obstacle to be $g = 0$ and imagine a flat region of u moving downwards at a constant speed; when this region reaches the obstacle the motion cannot continue its way down (we need to stay above g) and so the velocity must display an instantaneous and sudden change in a region of non-zero measure (within our scheme the motion stops on the obstacle and velocity drops to 0 on the whole contact region). Due to this possible behaviour of u_t, we cannot expect u_{tt} to posses the same regularity as in the obstacle-free case. Nevertheless, such discontinuities in time of u_t are somehow controllable and we can still provide some sort of regularity results, which are collected in the following propositions.

Proposition 4.4.4. *Let u be the weak limit obtained in Proposition 4.4.3 and, for any fixed $0 \leq \varphi \in \tilde{H}^s(\Omega)$, let $F\colon [0,T] \to \mathbb{R}$ be defined as*

$$F(t) = \int_\Omega u_t(t)\varphi \, dx. \tag{4.4.5}$$

Then $F \in BV(0,T)$ and, in particular, $u_t^n(t) \rightharpoonup u_t(t)$ in $L^2(\Omega)$ for a.e. $t \in [0,T]$.

Proof. Let us fix $\varphi \in \tilde{H}^s(\Omega)$ with $\varphi \geq 0$, and consider the functions $F^n\colon [0,T] \to \mathbb{R}$ defined as

$$F^n(t) = \int_\Omega u_t^n(t)\varphi \, dx. \tag{4.4.6}$$

Observe that $||F^n||_{L^1(0,T)}$ is uniformly bounded because u_t^n is bounded in $L^2(\Omega)$ uniformly in n and t. Furthermore, for every fixed $n > 0$ and $0 \leq i \leq n$, we deduce from (4.4.4) that

$$\left| \int_\Omega (v_i^n - v_{i-1}^n)\varphi \, dx \right| - \int_\Omega (v_i^n - v_{i-1}^n)\varphi \, dx \leq \tau_n |[u_i^n, \varphi]_s| - \tau_n [u_i^n, \varphi]_s. \tag{4.4.7}$$

Summing over $i = 1, \dots, n$ and using Proposition 4.3.3, we get

$$\begin{aligned}
\sum_{i=1}^n \left| \int_\Omega (v_i^n - v_{i-1}^n)\varphi \, dx \right| &\leq \int_\Omega v_n^n \varphi \, dx - \int_\Omega v_0 \varphi \, dx + \sum_{i=1}^n \tau_n |[u_i^n, \varphi]_s| - \sum_{i=1}^n \tau_n [u_i^n, \varphi]_s \\
&\leq ||v_n^n||_{L^2(\Omega)} ||\varphi||_{L^2(\Omega)} + ||v_0||_{L^2(\Omega)} ||\varphi||_{L^2(\Omega)} + 2\tau_n \sum_{i=1}^n |[u_i^n, \varphi]_s| \\
&\leq ||v_n^n||_{L^2(\Omega)} ||\varphi||_{L^2(\Omega)} + ||v_0||_{L^2(\Omega)} ||\varphi||_{L^2(\Omega)} + 2\tau_n \sum_{i=1}^n [u_i^n]_s [\varphi]_s \\
&\leq C ||\varphi||_{H^s(\mathbb{R}^d)}
\end{aligned}$$

with C independent of n. Thus, $\{F^n\}_n$ is uniformly bounded in $BV(0,T)$ and by Helly's selection theorem there exists a function $\bar{F}$ of bounded variation such that $F^n(t) \to \bar{F}(t)$ for every $t \in (0,T)$.

Take now $\psi(t,x) = \varphi(x)\eta(t)$ for $\eta \in C_c^\infty(0,T)$, using that $u_t^n \rightharpoonup^* u_t$ in $L^\infty(0,T;L^2(\Omega))$, one has

$$\int_0^T \int_\Omega u_t(t)\psi \, dxdt = \lim_{n\to\infty} \int_0^T \int_\Omega u_t^n(t)\psi \, dxdt = \lim_{n\to\infty} \int_0^T \int_\Omega u_t^n(t)\varphi \, dx \, \eta(t)dt$$
$$= \int_0^T \lim_{n\to\infty} \int_\Omega u_t^n(t)\varphi \, dx \, \eta(t) \, dt = \int_0^T \bar{F}(t)\eta(t) \, dt$$

where the passage to the limit under the sign of integral is possible due to the pointwise convergence of F^n to $\bar{F}$ combined with the dominated convergence theorem. We conclude

$$\int_0^T \left(\int_\Omega u_t(t)\varphi \, dx - \bar{F}(t) \right) \eta(t) \, dt = 0$$

and, by the arbitrariness of η, we have $F = \bar{F}$ for a.e. $t \in (0,T)$, which is to say $F \in BV(0,T)$. In particular,

$$\int_\Omega u_t(t)\varphi \, dx = F(t) = \lim_{n\to\infty} \int_\Omega u_t^n(t)\varphi \, dx \quad \text{for a.e. } t \in (0,T),$$

meaning $u_t^n(t) \rightharpoonup u_t(t)$ in $L^2(\Omega)$ for almost every $t \in (0,T)$: indeed the last equality can first be extended to every $\varphi \in \tilde{H}^s(\Omega)$ (just decomposing $\varphi = \varphi^+ - \varphi^-$ in its positive and negative parts) and then to every $\varphi \in L^2(\Omega)$ being $\tilde{H}^s(\Omega) \subset L^2(\Omega)$ dense. □

Remark 4.4.5. In the rest of this section we choose to use the "precise representative" of u_t given by $u_t(t) =$ weak-L^2 limit of $u_t^n(t)$, which is then defined for all $t \in [0,T]$.

Proposition 4.4.6. *Fix $0 \le \varphi \in \tilde{H}^s(\Omega)$ and let F de defined as in* (4.4.5)*. Then, for any $t \in (0,T)$, we have*

$$\lim_{r\to t^-} F(r) \le \lim_{s\to t^+} F(s).$$

Proof. First of all we observe that the limits we are interested in exist because $F \in BV(0,T)$. Fix then $t \in (0,T)$ and let $0 < r < t < s < T$. For each n define r_n and s_n such that $r \in (t^n_{r_n-1}, t^n_{r_n}]$ and $s \in (t^n_{s_n-1}, t^n_{s_n}]$. If we consider the functions F^n defined in (4.4.6) and take into account (4.4.7), one can see that

$$F^n(s) - F^n(r) = \int_\Omega (u_t^n(s) - u_t^n(r))\varphi \, dx = \int_\Omega (v^n_{s_n} - v^n_{r_n})\varphi \, dx$$
$$= \sum_{i=r_n+1}^{s_n} \int_\Omega (v_i^n - v_{i-1}^n)\varphi \, dx \ge \tau_n \sum_{i=r_n+1}^{s_n} ([u_i^n, \varphi]_s - |[u_i^n, \varphi]_s|)$$
$$\ge -2C\tau_n(s_n - r_n)||\varphi||_{H^s(\mathbb{R}^d)}$$

for some positive constant C independent of n. Since $|s-r| \geq |t^n_{s_n-1} - t^n_{r_n}| = \tau_n(s_n - 1 - r_n)$ we can conclude

$$F^n(s) - F^n(r) \geq -2C|s-r| \cdot ||\varphi||_{H^s(\mathbb{R}^d)} - 2C\tau_n ||\varphi||_{H^s(\mathbb{R}^d)}.$$

Passing to the limit $n \to \infty$ we get $F(s) - F(r) \geq -2C|s-r| \cdot ||\varphi||_{H^s(\mathbb{R}^d)}$, which in turn implies the conclusion.

The last result tells us that the velocity u_t does not present sudden changes in regions where it is positive, accordingly with the fact that whenever we move upwards there are no obstacles to the dynamic and u_t is expected to have, at least locally in time and space, the same regularity it has in the obstacle-free case.

We eventually switch prove conditions 2, 3 and 4 of our definition of weak solution, thus proving Theorem 4.4.2.

Proof of Theorem 4.4.2. Let u be the limit function obtained in Proposition 4.4.3. We verify one by one the four conditions required in Definition 4.4.1.

(1.) The first condition is verified thanks to Proposition 4.4.3.

(2.) Existence of weak left and right derivatives $u_t^{\pm}$ on $[0,T]$ follows from Proposition 4.4.4: just observe that, for any fixed $\varphi \in \tilde{H}^s(\Omega)$, the function

$$F(t) = \int_\Omega u_t(t)\varphi \, dx$$

is $BV(0,T)$ and thus left and right limits of F are well defined for any $t \in [0,T]$. This, in turn, implies condition 2. in our definition of weak solution.

(3.) For $n > 0$ and any test function $\varphi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^1(0,T;\tilde{H}^s(\Omega))$, with $\varphi \geq 0$, $\operatorname{spt}\varphi \subset [0,T)$, we recall that

$$\int_0^T \int_\Omega \left(\frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n} \right) \varphi(t) \, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s \, dt \geq 0.$$

Thanks to Proposition 4.4.3, we have

$$\int_0^T [\bar{u}^n(t), \varphi(t)]_s \, dt \to \int_0^T [u(t), \varphi(t)]_s \, dt \quad \text{as } n \to \infty$$

while, on the other hand, we also have

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n} \varphi(t) \, dxdt = \int_0^{T-\tau_n} \int_\Omega u_t^n(t) \left(\frac{\varphi(t) - \varphi(t+\tau_n)}{\tau_n} \right) dxdt \\
&- \int_0^{\tau_n} \int_\Omega \frac{v_0}{\tau_n} \varphi(t) \, dxdt + \int_{T-\tau_n}^T \int_\Omega \frac{u_t^n(t)}{\tau_n} \varphi(t) \, dxdt \\
&\to \int_0^T \int_\Omega u_t(t)(-\varphi_t(t)) \, dxdt - \int_\Omega v_0 \, \varphi(0) \, dx + 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This proves condition 3. for weak solutions.

(4.) The fact that $u(0) = u_0$ is a direct consequence of $u^n(0) = u_0$ and of the convergence of u^n to u in $C^0([0,T];L^2(\Omega))$. We are left to check the initial condition on velocity. Suppose, without loss of generality, that the sequence u^n is constructed by taking $n \in \{2^m : m > 0\}$ (each successive time grid is obtained dividing the previous one). Fix then n and $\varphi \in K_g$, let $T^* = m\tau_n$ for $0 \le m \le n$ (i.e., T^* is a "grid point"). Let us evaluate

$$\int_0^{T^*} \int_\Omega \frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n}(\varphi - \bar{u}^n(t)) = \sum_{i=1}^{m} \int_{t_{i-1}^n}^{t_i^n} \int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}(\varphi - u_i^n)$$

$$= \int_\Omega \sum_{i=1}^{m} \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n}(\varphi - u_i^n) = \int_\Omega \sum_{i=1}^{m} (v_i^n - v_{i-1}^n)(\varphi - u_i^n)$$

$$= -\int_\Omega v_0^n(\varphi - u_1^n)\,dx + \int_\Omega v_m^n(\varphi - u_m^n)\,dx + \tau_n \sum_{i=1}^{m-1} \int_\Omega v_i^n v_{i-1}^n\,dx$$

$$= -\int_\Omega v_0(\varphi - u_n(\tau_n))\,dx + \int_\Omega u_t^n(T^*)(\varphi - u^n(T^*))\,dx + \tau_n \sum_{i=1}^{m-1} \int_\Omega v_i^n v_{i-1}^n\,dx.$$

Using (4.4.3) we observe that

$$\int_0^{T^*} \int_\Omega \frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n}(\varphi - \bar{u}^n(t))\,dxdt + \int_0^{T^*} [\bar{u}^n(t), \varphi - \bar{u}^n(t)]_s\,dt \ge 0,$$

which combined with the above expression and previous estimates on u_i^n and v_i^n leads to

$$-\int_\Omega v_0(\varphi - u_n(\tau_n))\,dx + \int_\Omega u_t^n(T^*)(\varphi - u^n(T^*))\,dx \ge$$
$$-\tau_n \sum_{i=1}^{m-1} \int_\Omega v_i^n v_{i-1}^n\,dx - \tau_n \sum_{i=1}^{m} [u_i^n, \varphi - u_i^n]_s \ge -CT^* - CT^* ||\varphi||_{H^s(\mathbb{R}^d)}.$$

Passing to the limit as $n \to \infty$, using $u^n(\tau_n) \to u(0)$ and $u_t^n(T^*) \rightharpoonup u_t(T^*)$ (due to the use of the precise representative), we get

$$-\int_\Omega v_0(\varphi - u(0))\,dx + \int_\Omega u_t(T^*)(\varphi - u(T^*))\,dx \ge -CT^* - C||\varphi||_{H^s(\mathbb{R}^d)} T^*.$$

Taking now $T^* \to 0$ along a sequence of "grid points" we have

$$\int_\Omega (u_t^+(0) - v_0)(\varphi - u(0))\,dx \ge 0.$$

And this completes the first part of the proof. We are left to prove the energy inequality (4.4.2). For this, recall that from Remark 4.3.4 it follows that, for all $n > 0$,

$$||v^n(t)||_{L^2(\Omega)}^2 + [u^n(t)]_s^2 \le ||v_0||_{L^2(\Omega)}^2 + [u_0]_s^2 \quad \text{for all } t \in [0,T].$$

Passing to the limit as $n \to \infty$ we immediately get (4.4.2).

We conclude this section with some remarks and observations about the solution u obtained through the proposed semi-discrete convex minimization scheme in the scenario $s = 1$. First of all we identify the weak solution u obtained above to be a more regular solution whenever approximations u^n stay strictly above g.

Proposition 4.4.7 (Regions without contact)**.** *Let $s = 1$ and, for $\delta > 0$, suppose there exists an open set $A_\delta \subset \Omega$ such that $u^n(t,x) > g(x) + \delta$ for a.e. $(t,x) \in (0,T) \times \Omega$ and for all $n > 0$. Then $u_{tt} \in L^\infty(0,T; H^{-1}(A_\delta))$ and u satisfies* (4.3.2) *for all $\varphi \in L^1(0,T; H^1_0(A_\delta))$.*

Proof. Take $\varphi \in H^1_0(\Omega)$ with $\operatorname{spt}\varphi \subset A_\delta$. Then, for every n and $0 \le i \le n$, the function $u^n_i + \varepsilon\varphi$ belongs to K_g for ε sufficiently small: indeed, for $x \in A_\delta$, we have $u^n_i(x) + \varepsilon\varphi(x) \ge g(x) + \delta + \varepsilon\varphi(x) \ge g(x)$ for small ε, regardless of the sign of $\varphi(x)$. In particular, equation (4.4.4) can be written as

$$\int_\Omega \frac{u^n_i - 2u^n_{i-1} + u^n_{i-2}}{\tau_n^2}\varphi\,dx + \int_\Omega \nabla u^n_i \cdot \nabla\varphi\,dx = 0 \quad \text{for all } \varphi \in H^1_0(\Omega), \operatorname{spt}\varphi \subset A_\delta.$$

This equality allows us to carry out the second part of the proof of Proposition 4.3.5, so that, in the same notation, we can prove $v^n_t(t)$ to be bounded in $H^{-1}(A_\delta)$ uniformly in t and n. Thus, $v \in W^{1,\infty}(0,T; H^{-1}(A_\delta))$ and

$$v^n \rightharpoonup^* v \text{ in } L^\infty(0,T; L^2(A_\delta)) \quad \text{and} \quad v^n \rightharpoonup^* v \text{ in } W^{1,\infty}(0,T; H^{-1}(A_\delta)).$$

Localizing everything on A_δ, we can prove $v_t = u_{tt}$ in A_δ so that

$$u_{tt} \in L^\infty(0,T; H^{-1}(A_\delta)),$$

and equation (4.3.2) follows by passing to the limit as done in the proof of Theorem 4.3.2 (cf. [109, 48]).

□

Remark 4.4.8 (One dimensional case with $s = 1$). In the one dimensional case and for $s = 1$ the analysis boils down to the problem considered by Kikuchi in [66]. In this particular situation a stronger version of Proposition 4.4.7 holds: suppose that $\Omega = [0,1]$, then for any $\varphi \in C^0_0([0,T), L^2(0,1)) \cap W^{1,2}_0((0,T) \times (0,1))$ with $\operatorname{spt}\varphi \subset \{(t,x) : u(t,x) > 0\}$,

$$-\int_0^T \int_0^1 u_t\varphi_t\,dxdt + \int_0^T \int_0^1 u_x\varphi_x\,dxdt - \int_0^1 v_0\,\varphi(0)\,dx = 0.$$

4.5 Numerical implementation and open questions

The constructive scheme presented in the previous sections can be easily used to provide a numerical simulation of the relevant dynamic, at least in the case $s = 1$ where we can employ a classical finite element discretization. However, we observe that a similar finite element approach can be extended to the fractional setting $s < 1$ following for example the pipeline described in [4, 3].

Minimization of energies J_i^n can be carried out by means of a piecewise linear finite element approximation in space: given a triangulation $\mathcal{T}_h$ of the domain Ω we introduce the classical space

$$X_h^1 = \{u_h \in C^0(\bar{\Omega}) \,:\, u_h|_K \in \mathbb{P}_1(K), \text{ for all } K \in \mathcal{T}_h\}.$$

For $n > 1$, $0 < i \leq n$, and given $u_{i-1}^n, u_{i-2}^n \in X_h^1$, we optimize J_i^n among functions in X_h^1 respecting the prescribed Dirichlet boundary conditions (which are local because $s = 1$). We get this way a finite dimensional optimization problem for the degrees of freedom of u_i^n and we solve it by a gradient descend method combined with a dynamic adaptation of the descend step size.

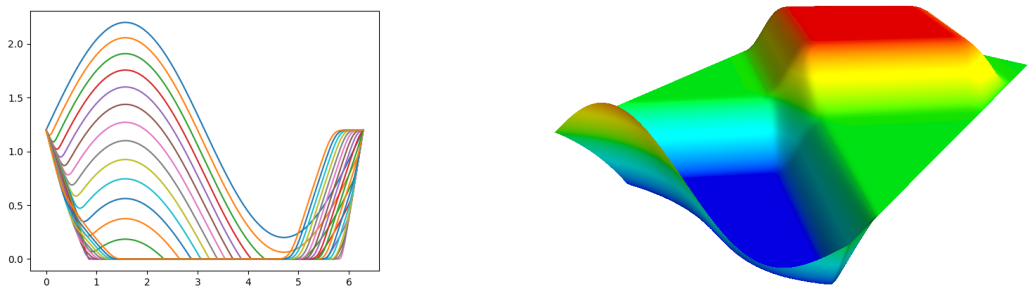

Figure 4.1: Time evolution of the solution till $t = 1.5$ (left) and space-time depiction of the same evolution till $T = 10$ (right).

In the simulation in figure 4.1 we take $\Omega = (0, 2\pi)$ and $u_0(x) = \sin(x) + 1.2$, with a constant initial velocity of -2 which pushes the string towards the obstacle $g = 0$. The boundary conditions are set to be $u(t, 0) = u(t, 2\pi) = 1.2$ and the simulation is performed up to $T = 10$ using a uniform grid with $h = 2\pi/200$ and a time step $\tau = 1/100$. We can see how the profile stops on the obstacle after impact (blue region in the right picture of figure 4.1) and how the impact causes the velocity to drop to 0 and thus a loss of energy (as displayed in figure 4.2). As soon as the profile leaves the obstacle the dynamic goes back to a classical wave dynamic and energy somehow stabilizes even if, as expected, it is not fully conserved from a discrete point of view. Due to energy dissipation at impact times, in the long run we expect the solution to never hit the obstacle again because the residual energy will only allow the profile to meet again the obstacle at 0 speed, i.e., without any loss of energy. Thus, also in higher dimension, we expect the solution u obtained through the proposed scheme to become an obstacle-free solution of the wave equation as soon as the energy of the system drops below a certain value, preventing this way future collisions. This can be roughly summarized in the following conjecture.

Conjecture 1 (Long time behaviour)**.** *Let $s = 1$ and, given an obstacle problem in the form of equation* (4.4.1)*, let u be the weak solution obtained through the convex minimiza-*

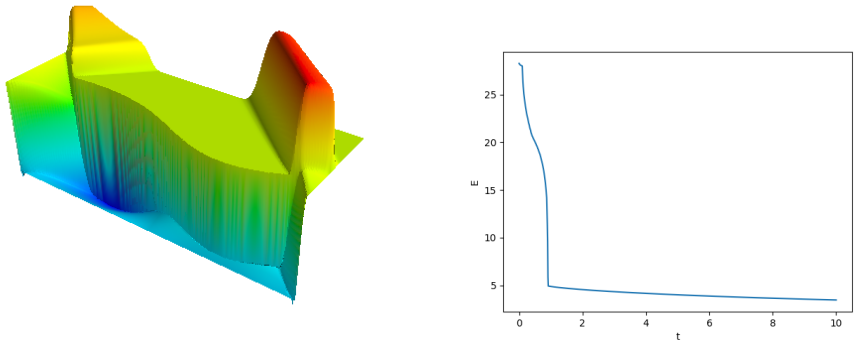

Figure 4.2: Time evolution of the velocity up to $t = 2$ (left) and energy (right).

tion approach of Section 4.4.1. Then, at least for sufficiently regular obstacles g, there exists $\bar{t} > 0$ such that $E(u(t))$ is constant for any $t > \bar{t}$.

Alongside the previous conjecture, we observe that the solution u obtained here seems to be, among all possible weak solutions, the one dissipating its kinetic energy at highest rate, when colliding with the obstacle g, and so the one realizing the "adherent" behaviour we mentioned before. At the same time, from the complete opposite perspective, one could ask if it is possible to revise the scheme so that to obtain energy preserving approximations u^n, and try to use these approximations to provide an energy preserving weak solution (maybe under suitable additional hypothesis on the obstacle).

As already observed in the introduction, the proposed method can be extended to the case of semi-linear wave equations of the type

$$u_{tt} + (-\Delta)^s u + f(u) = 0$$

with f a suitable function, possibly non-smooth. For example, one can consider f to be the (scaled) derivative of a balanced, double-well potential, e.g., $f(u) = \frac{1}{\varepsilon^2}(u^3 - u)$ for $\varepsilon > 0$: certain solutions of that equation are intimately related to timelike minimal hypersurfaces, i.e., with vanishing mean curvature with respect to Minkowski space-time metric [50, 62, 18]. On the other hand, as we said in the introduction, one could also manage adhesive type dynamics assuming f to be the (non-smooth) derivative of a smooth potential Φ, as it is done in [46].

We eventually observe that the proposed approximations u^n can be constructed, theoretically and numerically, also for a double obstacle problem, i.e., $g(x) \leq u(t,x) \leq f(x)$ for a suitable lower obstacle g and upper obstacle f. However, in this new context, the previous convergence analysis cannot be replicated because even the basic variational characterization (4.4.4) is generally false and a more localized analysis would be necessary. Anyhow, also in this situation one would expect the solution to behave like an obstacle-free solution after some time, as suggested in Conjecture 1.

Bibliography

[1] *Online at https://www.mmgtools.org/.*

[2] Jeongho Ahn. A vibrating string with dynamic frictionless impact. *Appl. Numer. Math.*, 57(8):861–884, 2007.

[3] Mark Ainsworth and Christian Glusa. Aspects of an adaptive finite element method for the fractional Laplacian: a priori and a posteriori error estimates, efficient implementation and multigrid solver. *Comput. Methods Appl. Mech. Engrg.*, 327:4–35, 2017.

[4] Mark Ainsworth and Christian Glusa. Towards an efficient finite element method for the integral fractional Laplacian on polygonal domains. In *Contemporary computational mathematics—a celebration of the 80th birthday of Ian Sloan. Vol. 1, 2*, pages 17–57. Springer, Cham, 2018.

[5] Giovanni Alberti, Sisto Baldo, and Giandomenico Orlandi. Functions with prescribed singularities. *Journal of the European Mathematical Society*, 5(3):275–311, 2003.

[6] Giovanni Alberti, Sisto Baldo, and Giandomenico Orlandi. Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J.*, 54(5):1411–1472, 2005.

[7] Micol Amar and Giovanni Bellettini. A notion of total variation depending on a metric with discontinuous coefficients. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11(1):91–133, 1994.

[8] Stefano Amato, Giovanni Bellettini, and Maurizio Paolini. Constrained BV functions on covering spaces for minimal networks and Plateau's type problems. *Adv. Calc. Var.*, 10(1):25–47, 2017.

[9] Luigi Ambrosio. Minimizing movements. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 19:191–246, 1995.

[10] Luigi Ambrosio and Andrea Braides. Functionals defined on partitions in sets of finite perimeter. I. Integral representation and Γ-convergence. *J. Math. Pures Appl. (9)*, 69(3):285–305, 1990.

[11] Luigi Ambrosio and Andrea Braides. Functionals defined on partitions in sets of finite perimeter. II. Semicontinuity, relaxation and homogenization. *J. Math. Pures Appl. (9)*, 69(3):307–333, 1990.

[12] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[13] Sanjeev Arora. Polynomial time approximation schemes for Euclidean traveling salesman and other geometric problems. *J. ACM*, 45(5):753–782, 1998.

[14] Sanjeev Arora. Approximation schemes for NP-hard geometric optimization problems: a survey. *Math. Program.*, 97(1-2, Ser. B):43–69, 2003. ISMP, 2003 (Copenhagen).

[15] Sisto Baldo and Giandomenico Orlandi. Codimension one minimal cycles with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_p$, and variational functionals on fibered spaces. *J. Geom. Anal.*, 9(4):547–568, 1999.

[16] Begoña Barrios, Alessio Figalli, and Xavier Ros-Oton. Free boundary regularity in the parabolic fractional obstacle problem. *Comm. Pure Appl. Math.*, 71(10):2129–2159, 2018.

[17] J. M. Bass and J. T. Oden. Adaptive finite element methods for a class of evolution problems in viscoplasticity. *Internat. J. Engrg. Sci.*, 25(6):623–653, 1987.

[18] Giovanni Bellettini, Matteo Novaga, and Giandomenico Orlandi. Time-like minimal submanifolds as singular limits of nonlinear wave equations. *Phys. D*, 239(6):335–339, 2010.

[19] Ahron Ben-Tal and Arkadi Nemirovski. *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*, volume 2. Siam, 2001.

[20] Marc Bernot, Vicent Caselles, and Jean-Michel Morel. Traffic plans. *Publ. Mat.*, 49(2):417–451, 2005.

[21] Marc Bernot, Vicent Caselles, and Jean-Michel Morel. *Optimal transportation networks: models and theory*, volume 1955. Springer Science & Business Media, 2009.

[22] Mauro Bonafini. Convex relaxation and variational approximation of the Steiner problem: theory and numerics. *Geom. Flows*, 3:19–27, 2018.

[23] Mauro Bonafini, Matteo Novaga, and Giandomenico Orlandi. A variational scheme for hyperbolic obstacle problems. *Submitted, arXiv preprint arXiv:1901.06974*, 2019.

[24] Mauro Bonafini, Giandomenico Orlandi, and Édouard Oudet. Convex relaxation and variational approximation of functionals defined on 1-dimensional connected sets. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 29(4):597–606, 2018.

[25] Mauro Bonafini, Giandomenico Orlandi, and Édouard Oudet. Variational approximation of functionals defined on 1-dimensional connected sets: the planar case. *SIAM J. Math. Anal.*, 50(6):6307–6332, 2018.

[26] Mauro Bonafini, Giandomenico Orlandi, and Édouard Oudet. Variational approximation of functionals defined on 1-dimensional connected sets in $\mathbb{R}^n$. *Submitted*, 2019.

[27] Mauro Bonafini and Édouard Oudet. A convex approach to the Gilbert-Steiner problem. *Submitted, arXiv preprint arXiv:1810.05417*, 2018.

[28] Matthieu Bonnivard, Elie Bretin, and Antoine Lemenant. Numerical approximation of the steiner problem in dimension 2 and 3. 2018.

[29] Matthieu Bonnivard, Antoine Lemenant, and Vincent Millot. On a phase field approximation of the planar Steiner problem: existence, regularity, and asymptotic of minimizers. *Interfaces Free Bound.*, 20(1):69–106, 2018.

[30] Matthieu Bonnivard, Antoine Lemenant, and Filippo Santambrogio. Approximation of length minimization problems among compact connected sets. *SIAM J. Math. Anal.*, 47(2):1489–1529, 2015.

[31] Guy Bouchitté. Singular perturbations of variational problems arising from a two-phase transition model. *Appl. Math. Optim.*, 21(3):289–314, 1990.

[32] Guy Bouchitté and Michel Valadier. Integral representation of convex functionals on a space of measures. *J. Funct. Anal.*, 80(2):398–420, 1988.

[33] Andrea Braides. Γ-*convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[34] Alessio Brancolini and Benedikt Wirth. Equivalent formulations for the branched transport and urban planning problems. *J. Math. Pures Appl. (9)*, 106(4):695–724, 2016.

[35] Lorenzo Brasco, Giuseppe Buttazzo, and Filippo Santambrogio. A Benamou-Brenier approach to branched transport. *SIAM J. Math. Anal.*, 43(2):1023–1040, 2011.

[36] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[37] Richard H. Byrd, Jorge Nocedal, and Richard A. Waltz. KNITRO: An integrated package for nonlinear optimization. In *Large-scale nonlinear optimization*, volume 83 of *Nonconvex Optim. Appl.*, pages 35–59. Springer, New York, 2006.

[38] L. A. Caffarelli. The obstacle problem revisited. *J. Fourier Anal. Appl.*, 4(4-5):383–402, 1998.

[39] Luis Caffarelli and Alessio Figalli. Regularity of solutions to the parabolic fractional obstacle problem. *J. Reine Angew. Math.*, 680:191–233, 2013.

[40] Luis Caffarelli, Arshak Petrosyan, and Henrik Shahgholian. Regularity of a free boundary in parabolic potential theory. *J. Amer. Math. Soc.*, 17(4):827–869, 2004.

[41] Marcello Carioni and Alessandra Pluda. Calibrations for minimal networks in a covering space setting. *arXiv preprint arXiv:1707.01448*, 2017.

[42] Marcello Carioni and Alessandra Pluda. On different notions of calibrations for minimal partitions and minimal networks in $\setminus^2$. *arXiv preprint arXiv:1805.11397*, 2018.

[43] Antonin Chambolle, Daniel Cremers, and Thomas Pock. A convex approach to minimal partitions. *SIAM J. Imaging Sci.*, 5(4):1113–1158, 2012.

[44] Antonin Chambolle, Luca Alberto Davide Ferrari, and Benoit Merlet. A phase-field approximation of the steiner problem in dimension two. *Advances in Calculus of Variations*, 2017.

[45] Antonin Chambolle, Luca Alberto Davide Ferrari, and Benoit Merlet. Variational approximation of size-mass energies for k-dimensional currents. *arXiv preprint arXiv:1710.08808*, 2017.

[46] G. M. Coclite, G. Florio, M. Ligabò, and F. Maddalena. Nonlinear waves in adhesive strings. *SIAM J. Appl. Math.*, 77(2):347–360, 2017.

[47] Gianni Dal Maso and Lucia De Luca. A minimization approach to the wave equation on time-dependent domains. *Advances in Calculus of Variations*, 2018.

[48] Gianni Dal Maso and Christopher J. Larsen. Existence for wave equations on domains with arbitrary growing cracks. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 22(3):387–408, 2011.

[49] Thierry De Pauw and Robert Hardt. Size minimization and approximating problems. *Calc. Var. Partial Differential Equations*, 17(4):405–442, 2003.

[50] Manuel del Pino, Robert Jerrard, and Monica Musso. Interface dynamics in semilinear wave equations. *arXiv preprint arXiv:1808.02471*, 2018.

[51] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[52] Iain Dunning, Joey Huchette, and Miles Lubin. JuMP: A Modeling Language for Mathematical Optimization. *SIAM Review*, 59(2):295–320, 2017.

[53] Richard L. Dykstra. An algorithm for restricted least squares regression. *J. Amer. Statist. Assoc.*, 78(384):837–842, 1983.

[54] Claudia D'Ambrosio, Marcia Fampa, Jon Lee, and Stefan Vigerske. On a nonconvex minlp formulation of the euclidean steiner tree problem in n-space. In *International Symposium on Experimental Algorithms*, pages 122–133. Springer, 2015.

[55] Christof Eck, Jiří Jarušek, and Miroslav Krbec. *Unilateral contact problems, variational methods and existence theorems*, volume 270 of *Pure and Applied Mathematics (Boca Raton)*. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[56] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[57] Marco Frontini and Laura Gotusso. Numerical study of the motion of a string vibrating against an obstacle by physical discretization. *Applied Mathematical Modelling*, 14(9):489–494, 1990.

[58] E. N. Gilbert and H. O. Pollak. Steiner minimal trees. *SIAM J. Appl. Math.*, 16:1–29, 1968.

[59] Edgar N Gilbert. Minimum cost communication networks. *Bell Labs Technical Journal*, 46(9):2209–2227, 1967.

[60] Elliott Ginder and Karel Švadlenka. A variational approach to a constrained hyperbolic free boundary problem. *Nonlinear Anal.*, 71(12):e1527–e1537, 2009.

[61] Yanghong Huang and Adam Oberman. Numerical methods for the fractional Laplacian: a finite difference–quadrature approach. *SIAM J. Numer. Anal.*, 52(6):3056–3084, 2014.

[62] Robert Jerrard. Defects in semilinear wave equations and timelike minimal surfaces in Minkowski space. *Anal. PDE*, 4(2):285–340, 2011.

[63] Robert L Jerrard and Halil Mete Soner. Functions of bounded higher variation. *Indiana University mathematics journal*, pages 645–677, 2002.

[64] Peter W. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.*, 102(1):1–15, 1990.

[65] Richard M. Karp. Reducibility among combinatorial problems. pages 85–103, 1972.

[66] Koji Kikuchi. Constructing a solution in time semidiscretization method to an equation of vibrating string with an obstacle. *Nonlinear Analysis: Theory, Methods & Applications*, 71(12):e1227–e1232, 2009.

[67] Jong Uhn Kim. A boundary thin obstacle problem for a wave equation. *Comm. Partial Differential Equations*, 14(8-9):1011–1026, 1989.

[68] Antoine Lemenant and Filippo Santambrogio. A Modica-Mortola approximation for the Steiner problem. *C. R. Math. Acad. Sci. Paris*, 352(5):451–454, 2014.

[69] Andrea Marchese and Annalisa Massaccesi. An optimal irrigation network with infinitely many branching points. *ESAIM Control Optim. Calc. Var.*, 22(2):543–561, 2016.

[70] Andrea Marchese and Annalisa Massaccesi. The Steiner tree problem revisited through rectifiable G-currents. *Adv. Calc. Var.*, 9(1):19–39, 2016.

[71] Andrea Marchese, Annalisa Massaccesi, and Riccardo Tione. A multi-material transport problem and its convex relaxation via rectifiable g-currents. *arXiv preprint arXiv:1712.08942*, 2017.

[72] Kenji Maruo. Existence of solutions of some nonlinear wave equations. *Osaka J. Math.*, 22(1):21–30, 1985.

[73] Annalisa Massaccesi. *Currents with coefficients in groups, applications and other problems in Geometric Measure Theory*. PhD thesis, Ph.D. thesis, Scuola Normale Superiore di Pisa, 2014.

[74] Annalisa Massaccesi, Édouard Oudet, and Bozhidar Velichkov. Numerical calibration of Steiner trees. *Applied Mathematics & Optimization*, pages 1–18, 2017.

[75] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[76] Luciano Modica and Stefano Mortola. Un esempio di Γ^- convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.

[77] Frank Morgan. Clusters with multiplicities in $\mathbb{R}^2$. *Pacific J. Math.*, 221(1):123–146, 2005.

[78] APS Mosek. The MOSEK optimization software. *Online at http://www. mosek. com*, 54, 2010.

[79] Matteo Novaga and Shinya Okabe. Regularity of the obstacle problem for the parabolic biharmonic equation. *Math. Ann.*, 363(3-4):1147–1186, 2015.

[80] Seiro Omata. A numerical method based on the discrete Morse semiflow related to parabolic and hyperbolic equation. *Nonlinear Anal.*, 30(4):2181–2187, 1997.

[81] Seiro Omata, Masaki Kazama, and Hideaki Nakagawa. Variational approach to evolutionary free boundary problems. *Nonlinear Anal.*, 71(12):e1547–e1552, 2009.

[82] Edouard Oudet and Filippo Santambrogio. A Modica-Mortola approximation for branched transport and applications. *Arch. Ration. Mech. Anal.*, 201(1):115–142, 2011.

[83] Nicholas C. Owen and Peter Sternberg. Nonconvex variational problems with anisotropic perturbations. *Nonlinear Anal.*, 16(7-8):705–719, 1991.

[84] Laetitia Paoli and Michelle Schatzman. A numerical scheme for impact problems. I. The one-dimensional case. *SIAM J. Numer. Anal.*, 40(2):702–733, 2002.

[85] Emanuele Paolini. Minimal connections: the classical Steiner problem and generalizations. In *Bruno Pini Mathematical Analysis Seminar 2012*, volume 2012 of *Bruno Pini Math. Anal. Semin.*, pages 72–87. Univ. Bologna, Alma Mater Stud., Bologna, 2012.

[86] Emanuele Paolini and Eugene Stepanov. Existence and regularity results for the Steiner problem. *Calc. Var. Partial Differential Equations*, 46(3-4):837–860, 2013.

[87] Emanuele Paolini and Eugene Stepanov. Structure of metric cycles and normal one-dimensional currents. *J. Funct. Anal.*, 264(6):1269–1295, 2013.

[88] Thomas Pock and Antonin Chambolle. Diagonal preconditioning for first order primal-dual algorithms in convex optimization. In *Computer Vision (ICCV), 2011 IEEE International Conference on*, pages 1762–1769. IEEE, 2011.

[89] Thomas Pock, Antonin Chambolle, Daniel Cremers, and Horst Bischof. A convex relaxation approach for computing minimal partitions. In *Computer Vision and Pattern Recognition, 2009. CVPR 2009. IEEE Conference on*, pages 810–817. IEEE, 2009.

[90] P. Ramachandran and G. Varoquaux. Mayavi: 3D Visualization of Scientific Data. *Computing in Science & Engineering*, 13(2):40–51, 2011.

[91] John Roe. *Winding around*, volume 76 of *Student Mathematical Library*. American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2015. The winding number in topology, geometry, and analysis.

[92] Marie E. Rognes, David A. Ham, Colin J. Cotter, and Andrew T.T. McRae. Automating the solution of PDEs on the sphere and other manifolds in FEniCS 1.2. *Geoscientific Model Development*, 6(6):2099–2119, 2013.

[93] Xavier Ros-Oton. Obstacle problems and free boundaries: an overview. *SeMA J.*, 75(3):399–419, 2018.

[94] Erich Rothe. Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben. *Math. Ann.*, 102(1):650–670, 1930.

[95] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[96] Hanan Samet. An overview of quadtrees, octrees, and related hierarchical data structures. In *Theoretical Foundations of Computer Graphics and CAD*, pages 51–68. Springer, 1988.

[97] Etienne Sandier. Ginzburg-Landau minimizers from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ and minimal connections. *Indiana Univ. Math. J.*, 50(4):1807–1844, 2001.

[98] Filippo Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[99] Michelle Schatzman. A class of nonlinear differential equations of second order in time. *Nonlinear Anal.*, 2(3):355–373, 1978.

[100] Michelle Schatzman. A hyperbolic problem of second order with unilateral constraints: the vibrating string with a concave obstacle. *J. Math. Anal. Appl.*, 73(1):138–191, 1980.

[101] Michelle Schatzman. The penalty method for the vibrating string with an obstacle. In *Analytical and numerical approaches to asymptotic problems in analysis (Proc. Conf., Univ. Nijmegen, Nijmegen, 1980)*, volume 47 of *North-Holland Math. Stud.*, pages 345–357. North-Holland, Amsterdam-New York, 1981.

[102] Mark Schmidt, Nicolas L Roux, and Francis R Bach. Convergence rates of inexact proximal-gradient methods for convex optimization. In *Advances in neural information processing systems*, pages 1458–1466, 2011.

[103] Enrico Serra and Paolo Tilli. Nonlinear wave equations as limits of convex minimization problems: proof of a conjecture by De Giorgi. *Ann. of Math. (2)*, 175(3):1551–1574, 2012.

[104] Raffaella Servadei and Enrico Valdinoci. The Brezis-Nirenberg result for the fractional Laplacian. *Trans. Amer. Math. Soc.*, 367(1):67–102, 2015.

[105] Luis Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[106] S. K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. *Algebra i Analiz*, 5(4):206–238, 1993.

[107] David E. Stewart. *Dynamics with inequalities*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Impacts and hard constraints.

[108] Atsushi Tachikawa. A variational approach to constructing weak solutions of semilinear hyperbolic systems. *Adv. Math. Sci. Appl.*, 4(1):93–103, 1994.

[109] Karel Švadlenka and Seiro Omata. Mathematical modelling of surface vibration with volume constraint and its analysis. *Nonlinear Anal.*, 69(9):3202–3212, 2008.

[110] DM Warme, Pawel Winter, and Martin Zachariasen. GeoSteiner 3.1. *Department of Computer Science, University of Copenhagen (DIKU)*, 2001.

[111] Brian White. Existence of least-energy configurations of immiscible fluids. *J. Geom. Anal.*, 6(1):151–161, 1996.

[112] Qinglan Xia. Optimal paths related to transport problems. *Commun. Contemp. Math.*, 5(2):251–279, 2003.