Regular Black Hole and Cosmological Spacetimes in Non-Polynomial Gravity Theories

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To Fede,

Abstract

General Relativity is known to suffer from singularities at short distances, which indicates the breakdown of its predictability, for instance at the center of black holes, and in the very early universe. This is one of the main reason to look for a Quantum Theory of Gravity, that would describe spacetime geometry as a quantum field, and possibly cure these classical singularities. However, no consensus on the topic has yet been reached, as many different approaches have been proposed, but none has yet received an experimental confirmation. This is in part due to the extraordinary small scale at which quantum gravitational effects are expected to become dominant, and to the technical difficulty to make unambiguous predictions.

For this reason, many works have focused on the so-called effective approaches in which the possible high energy corrections to General Relativity are classified, and their theoretical and observational predictions derived, with the idea that among these modifications, some could come as the semi-classical limits of quantum gravity theories. A way to discriminate between the different proposals is precisely the absence of singular geometries in their solutions.

In the first two Chapters of this thesis, we will present such an effective approach, in which the action of General Relativity is modified at high energy by non-polynomial curvature invariants, which are constructed in such a way that the dynamical spherically symmetric sector of these theories (which contain both cosmological and non-rotating black hole spacetimes) yield second order field equations. These properties of the non-polynomial invariants follow from a peculiar algebraic identity satisfied by the Cotton tensor in this class of geometries. As we will see in the last two Chapters, having second order dynamical spherically symmetric field equations is necessary in order to recover some quantum corrected geometries that have been found from more fundamental approaches like Loop Quantum Cosmology and Asymptotic Safety, within its Einstein-Hilbert truncation.

The existence of such gravitational models provides an interpretation of two-dimensional Horndeski theory as describing the dynamical spherically symmetric sector of specific higher dimensional non-polynomial gravity theories. Therefore, it allows to have some concrete d-dimensional formulations of the two-dimensional Einstein-Dilaton and Lovelock Designer effective approaches that have been studied extensively, in particular to find and study the properties of non-singular black holes. This enables us to propose two four-dimensional effective-like actions, which are constructed in such a way that their dynamical spherically symmetric sectors decompose in the same way as those of General Relativity and Gauss-Bonnet gravity.

In the remaining Chapters, we essentially investigate the solutions and properties of these theories. It is shown that the first one leads to regular (A)dS-core black hole solutions, with the correct quantum correction to their Newton potentials and logarithmic correction to their entropies. The charged generalization is considered, and a way to avoid the mass inflation instability of their inner horizons is found, provided that a bound between the mass and the charge is satisfied. In Chap. 4, we establish a reconstruction procedure able to find theories admitting as solutions the Modesto semi-polymeric black hole, as well as the D'Ambrosio-Rovelli and Visser-Hochberg geometries. All these black holes are regular and derived or inspired by quantum gravity results. They have many properties in common, as for example the fact that they automatically regularize the Coulomb singularity of a static electric field.

Finally, the last Chapter is devoted to the theory whose dynamical spherically symmetric sector is a generalization of the one of Gauss-Bonnet gravity. It is shown that the Loop quantum cosmology bounce universe and some Asymptotic Safety black holes can be reconstructed from two members of these theories. In particular, the associated black hole solutions of the first are regular, and the associated cosmological solution of the second is as well, and describe a universe which is eternal in the past, and behaves as de Sitter spacetime in the limit $t \to -\infty$. Some generalizations of these results are provided, and the Mimetic gravity formulations of the cosmological solutions are found.

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Contents

Introduction

General Relativity (GR) is an extremely successful physical theory. More than 100 years after its discovery, no experiment has yet found significant discrepancies between its predictions and the observations. On the contrary, the recent gravitational waves observations coming from the merger of two black holes (BH) [1], and two Neutron stars [2], were found in complete agreement with the general relativistic predictions of the wave-forms of gravitational waves. Since these first detections, many other BH-BH mergers have been detected, all of them in agreement with GR [3–6]. Even more recently, the first image of a black hole horizon was reconstructed from the observation of the supermassive black hole in the center of the giant elliptical galaxy M87 [7–12], confirming once again the prediction of General Relativity.

However, despite these great successes, General Relativity suffers from internal inconsistencies, as many of its solutions contain singularities at small scale [13–15]. In particular, this is what happens at the center of its black hole solutions and in the very early epoch of its cosmological solutions. Singularities can be generically characterized in two different ways : the curvature of spacetime becomes infinite at some point or in some regions, and/or the world-lines of particles end suddenly there, in which case the spacetime is called geodesically incomplete. In both cases, this indicates the breakdown of GR's predictability in these very small regions, and thus requires new theories that would avoid these inconsistencies and describe what happens at the center of physical black holes and in the very early universe.

Fortunately, such an ultraviolet (UV) catastrophe has already been encountered with the classical model of the hydrogen atom, what led to the discovery of quantum mechanics, which ended up solving the issue. Since then, it was discovered that at small scale, every dynamical degrees of freedom (that we know of) are quantum. For these reasons, it is widely expected that a complete theory of Quantum Gravity (QG) should solve the singularity issue of General Relativity, by describing the spacetime geometry as a quantum field. Many different QG theories have been proposed, among which Loop Quantum Gravity (LQG), Spin-Foams and Group field theories [16–18], Asymptotic Safety (AS) [19, 20], Causal Dynamical Triangulation [21, 22], String Theory [23, 24], Non-commutative geometry [25, 26] or Causal Sets [27]. Each of these theories present some interest and specific characteristics, but none has yet received an experimental confirmation. This is in part due to the extraordinary small scale at which quantum gravitational effects are expected to become dominant, and to the technical difficulty to make unambiguous predictions. However, theoretical results in favour of the absence of singularities have been established. In particular, singularity resolutions in cosmological spacetimes, replacing the big bang singularity by a dynamical bounce [28–34], by a de Sitter phase [35–40] ; and in black hole geometries, where their center is described by a bouncing dynamical region [41–47], eventually tunnelling into a white hole [48–53], or a static (Anti)-de Sitter ((A)dS) core [54–63].

Despite these results, there is no general agreement toward a particular QG candidate, and the wide variety of mathematical tools and concepts that each of these are using makes them quite difficult to compare with each other, although some general results regarding the behaviour of quantum geometry have been found from different approaches, see for example [64–69]. For this reason, many works have focused on the so-called effective approaches in which the possible high energy corrections to General Relativity are classified, and their theoretical and observational predictions derived, with the idea that among these modifications, some should come as the semi-classical limits of some quantum gravity theories. It would also be possible in principle for a single effective approach (EA) to cover the semi-classical limit of different QGTs. A way to discriminate between the different proposals would precisely be the absence of singular geometries. This is the topic of this thesis : we will consider Lagrangian high energy modifications to GR able to yield singularity-free black hole and cosmological solutions, and in particular some spacetimes inspired or derived from Quantum Gravity theories.

There is a huge bestiary of possible high energy corrections to General Relativity. First, one has to *choose* a formalism (i.e. a set of gravitational variables) to express GR, because very different formalisms lead exactly to the same predictions as its historic formulation, but their high energy modifications can break this equivalence. Indeed, GR is usually expressed in terms of a metric (or tetrad) field, with gravity being described by the curvature of a metric-compatible connection. But considering the metric and the connection as independent (i.e. the Palatini formalism), although equivalent for the Einstein-Hilbert action (EHA), might yield different high energy corrections [70]. Similarly for the Teleparallel equivalent of GR [71–74], in which the curvature is vanishing but the torsion is responsible for gravity. In many cases however, it is possible to reconstruct an effective metric field from the basic gravitational variables, so that finding regular metric fields can present some interests for a wide range of formalisms. Throughout this thesis, we will consider the metric formalism. Note that regular geometries have also been found in these alternative formalisms [75–82].

Then, once a formalism to write the EHA is chosen, one can consider different types of UV corrections, depending on the number of degrees of freedom that they add to classical gravity (scalar, tensor, vector, gauge field, etc). For example, there has been an extensive amount of work devoted to the study of scalar-tensor theories [83–88]. Interestingly, even higher order gravitational corrections to metric GR fall in this class, because despite the fact that they only involve the metric field, they necessarily yield higher order field equations, meaning that they contain additional degrees of freedom. This is due to one of the most fundamental theorem regarding geometrical gravitational theories, the Lovelock one [89], stating that in a torsion-free four-dimensional Lorentzian manifold $\mathcal{M}$ with a metric-compatible connection, the only (metric) action leading to (non-trivial) second order field equations *for any metric field* $g_{\mu\nu}$ is given by the Einstein-Hilbert action supplemented by a cosmological constant. Within these models, many regular solutions have been constructed, in particular using the so-called Non-Linear Electrodynamics (NLED) approach [90–97], which however possesses some limitations [98–100], but also from a $SU(2)$ Yang-Mills theory non-minimally coupled to gravity [101, 102] and other Non-minimal coupling theories [103] or from Phantom scalar fields [104, 105], Mimetic Gravity [108–115], higher order gravitational theories [116–118], and Horndeski theory [100]. Other ways to find or reconstruct regular solutions have also been studied like for example the fluid approach, see in particular [100, 119, 120], and many regular cosmological solutions have been uncover in these ways [109, 121–127]. This approach was also extensively used in the construction of wormhole solutions [128]. See also [129, 130] for reviews on cosmological singularities and regular black holes [131] within these approaches.

Nevertheless, it turns out that many Quantum Gravity non-singular metric fields describing cosmological and black hole geometries have been found not to contain additional integration constants (what would be the sign of new degrees of freedom in the semi-classical limit). For example, in the case of effective regular corrections to Schwarzschild spacetime from Loop Quantum Gravity techniques [41–47, 132], in which the absence of singularity is controlled by the minimal value of the (quantum) area operator ; Asymptotic Safety within the Einstein-Hilbert truncation [54–63], in which the regularity is obtained from the existence of a UV fixed point of the dimensionless running Newton constant ; but also from Non-Local [133] and Non-commutative geometry inspired (from a Generalized Uncertainty Principle) [134–136] or Conformal [137–139] black holes. Similar behaviour have been found in cosmology, see for instance [28–30, 140, 141] for LQC and [36, 37, 40] for AS. Although this might simply be due to the specific approximations needed to reach these results, the singularities are nonetheless cured from two very different ways : either by integration constants (in the case of new degrees of freedom), or by a quantization parameter (or a coupling constant in an effective action) in QG. It might therefore be interesting to see if these effective Quantum Gravity results can be obtain or not by adding high energy gravitational corrections to the Einstein-Hilbert action. As we mentioned previously, it will not be possible to find such new terms leading to second order field equations *for any metric*, but it might be possible for the class of metrics we will be interested in : non-rotating black hole and cosmological ones, which are part of the more general class of dynamical spherically symmetric spacetimes (DSSS). Gravitational theories of this kind have already been discovered. In alternative formalisms of GR, see for instance [142], where regular solutions were found from metric-affine theories leading to second order field equations. Concerning the metric formalism that we will be interested in, a notable example is the class of Quasi-Topological Gravities [143–150], consisting of polynomial corrections to EHA in the Riemann tensor and its covariant derivatives, which yield second order field equations for DSSS, and higher order otherwise. However, this last one generally do not lead to regular black hole solutions. Therefore, the subject of the first two Chapters of this dissertation will be to study other kind of (metric) gravitational actions, so-called Non-Polynomial Gravities [151–156], which are high energy corrections to General Relativity involving non-polynomial curvature invariants, but leading nonetheless to polynomial and second order field equations in DSSS. In particular, we will see that these models provide a d-dimensional interpretation for many two-dimensional

Einstein-Dilaton theories [41–43, 157–165] (and their Lovelock Designer generalizations [166–171]), which have been used to find and model key properties of non-singular black holes. Then from these models, we will find and study regular black hole and cosmological spacetimes.

To be more precise, we will concentrate in all this thesis on the regularity of the curvature invariants of the associated geometries, as opposed to the geodesic completeness. For most regular spacetimes, these two notions are concomitant, but it happens sometimes that one regularity criterion is satisfied while the other is not, see [78, 172–174]. Moreover, it was found in [175] that if one writes the Schwarzschild geometry in a more general coordinate system covering twice the usual ($r > 0$)-region, then it is possible to extend the geodesics ending at the singularity towards a white-hole region (time-reversal of the first), what makes (in this case) the concept of geodesic completeness weaker than the regularity of invariants, because easier to circumvent. However, this is not a settle issue, as many arguments also points toward the geodesic completeness as the most relevant characterization of singularities, see for example [106, 107]. Though, they will be equivalent for most part of the regular geometries that we will find, so that the regularity of curvature invariants is simply a more convenient notion.

In any case, many other criteria beside the regularity are usually used in order to constraint further non-singular geometries, and hopefully, to describe more realistic black holes. In particular we will mention multiple times the concept of Limiting Curvature [108, 109, 111, 113, 176], which in some sense is a generalization of the regularity of curvature invariants. It consists of having geometries whose curvature is bounded by a universal constant. For instance, when applied to black holes, it means that whatever its mass (charge or spin), the curvature should remain smaller that such a bound. Furthermore, another very important condition to satisfy for a high energy correction of Schwarzschild black hole is to have the correct one-loop quantum correction to Newton potential, see [177–179]. This correction is one of the few consistency check at one's disposal in order to check if a high energy modification of gravity is in accordance with what we know about low energy quantum gravity.

Depending on the type of non-singular black holes, other criterion might be important to satisfy as well. The regular black holes (RBHs) we will find in this work can be divided into two main categories. First, (A)dS-core ones, for which the so-called Sakharov criterion is satisfied [180] (see also [155] for its covariant formulation). These black holes possess a static inner region which behaves as (A)dS (or slight deformations) for $r \approx 0$. Because of this static core, they must possess an inner horizon, what makes them unstable due to the mass inflation phenomenon [181–191]. As we will see, this is also an important issue that has to be tackled when dealing with these kinds of RBHs. For more references regarding this kind of geometries, see for example [176, 192–202]. The other main class of non-singular black holes do not suffer from this issue, as they contain a single horizon. They usually describe a bouncing interior region, and have been found in particular from the (Loop) polymer quantization of the spherically symmetric sector of Einstein gravity [41–44].

Although we will not study these aspects of regular black holes here, remark that in all these cases, a side-effect of removing the singularity is expected to be a dissolution of the so-called information loss paradox (see [203–205] and references therein) : if matter is not destroyed by the singularity inside a black hole, either it can directly goes out (after a long time for the external observer) via a bounce [206–214], or via evaporation [194, 198] or finally be locked inside in the form of a regular stable extremal remnant [205, 215–220], that might wait for a non-perturbative effect to release its matter content. Note that this last possibility might also be a candidate for dark matter [220–224].

Outline

The thesis is organised as follows. In Chap.1 Sec.1, we introduce the covariant formalism of dynamical spherically symmetric spacetimes and their topological generalizations, and show how any d-dimensional (metric) gravitational actions can be interpreted in this sector as a two-dimensional scalar-tensor theory. We will see that this is the basic principle underlying the so-called Einstein-Dilaton and Lovelock-Designer effective approaches, that have been used to model and investigate the properties of non-singular black holes. However, these two-dimensional theories are usually taken to be of the Horndeski type, i.e. they yield second order field equations.

Therefore, in Sec.2, we will survey well-known d-dimensional theories admitting a second order DSS sector, in particular Lovelock-Lanczos (thus General Relativity), and Quasi-Topological

Gravities. The first result of this thesis regards this first class of theories. We will show that in arbitrary dimensions, there exists a particular regularization of the Critical and higher orders Lovelock-Lanczos (L-L) scalars which enables them to yield non-vanishing DSS spherically symmetric field equations, what we take as a hint for the existence of more general d-dimensional gravitational theories leading to second order DSS equations of motion.

In Sec.3, a detour is made to understand the structure of the Critical Order L-L scalars. In particular, we will discuss the very common claim that in the metric formalism, these invariants can be written as total divergences of some curvature vectors. It is shown that it is indeed possible, but at the price of considering non-polynomial curvature vectors. Then, we study the four dimensional non-polynomial theory constructed from the contraction with itself of the curvature vector whose divergence gives the Gauss-Bonnet scalar. It is shown to yield second order (and polynomial) static spherically symmetric and cosmological field equations. Moreover, the associated solutions share some similarities with (higher dimensional) L-L gravities, due to the existence of degenerate vacua.

Given that our aim is to find gravitational theories leading to second order DSS field equations, and eventually to Quantum Gravity inspired regular geometries, these results naturally lead us to consider Non-Polynomial Gravity theories as candidates for this task, which is the topic of Chap.2. In an introductory Sec.1, we review the Deser-Sarioglu-Tekin (DST) Action [151], which was the first NPG model leading to polynomial and second order field equations for any dynamical spherically symmetric spacetime. In particular, we explain the origin of this property, based on a special algebraic property of the DSS decomposition of the Weyl tensor, and check the equivalence between the symmetric reduction of the full EOMs and the ones derived from the symmetric reduction of this action. However, the DST action is not an high energy correction to General Relativity, so that we then use the DST scalar to generalize four dimensional quadratic gravity, and show that for a suitable choice of coupling constants, the theory reduces in DSS to a gravitational Pais-Uhlenbeck oscillator, and thus yields second order static spherically symmetric field equations and degenerate solutions.

Nevertheless, few new gravitational theories can be constructed in this way, and even fewer yield second order DSS field equations. Therefore, in Sec.2, we present a new algebraic relation, based on the DSS decomposition of the Cotton tensor. It allows to construct a d-dimensional non-polynomial rank-two curvature tensor, which looses its curvature dependence in DSS, as it reduces to a sum of the dynamical spherically symmetric metric and the degenerate metric of the horizon manifold. From this result, it becomes possible to show that any second order DSS sector (i.e. any 2D Horndeski theory) can be reached from specific d-dimensional (metric) gravitational theories constructed from this tensor. Consequently, once a specific two-dimensional Horndeski theory is shown to have regular solutions, say cosmological or black hole ones, it automatically proves that, in the theory space of d-dimensional metric gravity, there exists specific models whose DSS sector admit these solutions.

For this reason, in Sec.3, we start investigating specific two-dimensional Horndeski theories, with the aim of finding suitable criteria that single out specific subclasses. More specifically, we will consider two power-series effective-like actions that generalize the DSS sector of Gauss-Bonnet and Einstein gravity for any order of corrections. It amounts to require for each order of correction to preserve the DSS reductions of both the Lagrangian and field equations of these theories. In the first case, we end-up with a one set of dimensionless coupling constants (one for each order) deformation of the DSS sector of Lovelock-Lanczos gravity, which is non-trivial at and beyond the critical order ; while for the second case, we find a two sets of dimensionless coupling constants high energy corrections to GR. We show that this last theory present some similarities with Lovelock-Lanczos theory, as its critical order scalar is related in DSS to the Euler characteristic of the two-dimensional transverse manifold. However, it can be regularized in such a way that it contributes to the field equations.

The solutions and properties of this last family of theories are investigated in Chap. 3. In Sec.1 and 2, we show that they satisfy Birkhoff's theorem and admit all the (A)dS-core rational regular black holes satisfying $g_{tt}g_{rr} = -1$ in Schwarzschild gauge, and such that g_{tt} depends linearly on the mass. Thus, they constitute high energy corrections to the so-called Poisson-Israel black hole [225]. Moreover, it is shown that the effect of the critical order scalar is to yield the correct quantum correction to Newton potential, and a logarithmic correction to the Entropy (Sec.3, 4).

The Sec. 5 is devoted to the study of the vacua ($M = 0$) of these models, where we find that it is possible to have Minkowski spacetime for all the orders of corrections. Interestingly, extremal black hole vacua are also possible for some choices of the coupling constants. In this case, Minkowski space acquires a small scale structure described by an extremal horizon boundary, near which the spacetime behaves as a Bertotti-Robinson metric $AdS_2 \times S^2$.

The charged solutions are derived in Sec.6, and it is shown that a specific non-minimal coupling of the Maxwell Lagrangian with a non-polynomial curvature scalar allows to regularize the Coulomb singularity. The (minimally coupled) charged solutions are considered in Sec.7, where the issue of the mass inflation instability is tackled. We show that, within this model, providing one considers an infinite number of corrections to the EH action, so that the (non-perturbative) neutral solutions describe single-horizon regular black holes, it is possible for the resulting charged solutions to avoid mass inflation, provided that a specific bound relating the charge and the mass of the geometry is satisfied. As we will see in the next Chapter, this is a general property of the charged generalizations of single-horizon regular black holes. Finally, in Sec. 8, we briefly investigate the Friedmann-Lemaître-Robertson-Walker (FLRW) sector of the model, and comment more generally on the FLRW sector of NPG models.

The Chap. 4 starts our reconstructions of effective actions admitting Quantum Gravity (inspired) regular solutions, what will be continued in the last Chapter. We focus on NPG theories whose DSS sector belongs to the Einstein-Dilaton gravities and admit single-horizons regular black holes, i.e. dynamical interiors. To do so, we will first establish in Sec.1 the general way to reconstruct the Einstein-Dilaton actions that admit any static black hole solution for which g_{tt} and g_{rr}^{-1} are linear in the mass. This will enable us to find four dimensional Non-Polynomial Gravity theories admitting the Modesto semi-polymeric black hole (Sec.1) [44], the D'Ambrosio-Rovelli (Sec.2) [175, 226] and Visser-Hochberg (Sec.3) [227, 228] geometries.

In particular, this first was derived in [44] from the Loop quantization of the spherically symmetric sector of Einstein gravity. We will see that these three spacetimes describe similar corrections to Schwarzschild geometry in which, instead of ending at the singularity, the dynamical interior regions of these black holes can be continued to white hole interiors. Moreover, it will be shown that their charged generalizations automatically cure the Coulomb singularity.

Finally, we investigate in Chap. 5 some four dimensional non-singular cosmological and black hole solutions arising from the generalization of the DSS sector of Lovelock gravity found in Chap. 2. In Sec. 1, we first align with many recent works aiming at finding effective gravitational actions admitting the Loop Quantum Cosmology effective solution (see for example [109] for the Mimetic gravity case), in which the big bang singularity is replaced by a bounce of the universe. We present there a work carried out by Stefano Chinaglia, Sergio Zerbini and the author in [154], in which the LQC bounce is recovered from another class of NPG theories.

Then, in Sec.2 we derive the FLRW and static spherically symmetric sectors of the considered theory. It turns out that the one-parameter deformation of Lovelock gravity is topological in the cosmological sector, so that in this sector it is just a Lovelock-like theory which can be extended beyond the critical order. For a specific choice of the remaining set of coupling constants, it gives the LQC result. However, the deformation does affect the static spherically symmetric sector. We show in Sec.3 that for two very simple choices of the one-parameter deformation, the theory admits (in addition to the LQC bounce) regular black holes. In particular, we exhibit two examples, one with an (A)dS-core, and another one with a single-horizon (for very small values of the mass, it has two). In both cases, these geometries have Limiting Curvature and the correct quantum correction to the Newtonian potential (provided that they have a unique spherically symmetric vacuum given by Minkowski spacetime).

In Sec. 4, we turn to a Quantum Gravity effective black hole found in [54] and derived within the Einstein-Hilbert truncation of Asymptotic Safety. By considering another overall set of coupling constants in front of the Lagrangian terms, we show that one of the two previous choices of the one-parameter deformation leads to this black hole solution, while the other also yields to an (A)dS-core regular black hole which satisfies the asymptotic conditions of [54] to be another Asymptotic Safety BH candidate. Once again, these geometries have Limiting Curvature and the correct quantum correction to the Newtonian potential. Furthermore, we show that the cosmological solution associated with these two theories is non-singular and, while it reproduces GR at late times, it describes a past-eternal dS region that replaces the classical singularity. Interestingly, it shares some resemblance with AS cosmology, where a so-called Non-Gaussian fixed point driven inflationary era was found to arise without the need of the inflaton, see [229–233].

Finally, Sec. 5 is devoted to generalizing the previous results by finding other members of the considered class of theories admitting regular solutions. The Sec. 6 concludes this thesis by presenting the Mimetic gravity formulation of the cosmological solutions.

I. Second Order Gravity theories in Dynamical Spherically Symmetric spacetimes

One of the most fundamental theorem regarding geometrical gravitational theories is the Lovelock one [89], stating that in a torsion-free four-dimensional Lorentzian manifold $\mathcal{M}$ with a metric-compatible connection, the only (metric) action leading to second order field equations *for any metric field* $g_{\mu\nu}$ is given by the Einstein-Hilbert action supplemented by a cosmological constant and the Gauss-Bonnet scalar :

$$I[g_{\mu\nu}] = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(-2\Lambda + R + l^2\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu}{}^{\alpha\beta}R^{\mu\nu}{}_{\alpha\beta}\right)\right) \tag{1}$$

where l is a length scale. As we will see in the last section of this Chapter, this last term yields identically vanishing field equations in four dimensions. However, some solutions associated with this theory possess singularities, i.e. regions, curves or points in which geodesics cannot be extended, what is usually associated with some divergences of the curvature invariants. In particular, they usually arise in small regions of spacetime containing very high energy, like at the center of black holes and in the very early universe. The simplest examples being the Schwarzschild singularity of neutral and static black holes

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2 \tag{2}$$

which is located at time $r = 0$, and the big bang one.

Therefore, a natural idea to cure singularities at high energy, would be to add high energy corrections to the previous action, what would naturally lead to higher order field equations due to the Lovelock Theorem, i.e. to the apparition of new gravitational degrees of freedom. As we saw in the Introduction, this is a legitimate possibility, and many regular geometries have been found in this way. Due to the presence of new degrees of freedom, it will generically modify the classical geometries by adding integration constants, for example

$$ds^2 = -X_{(M,Q_p)}(r)dt^2 + \frac{dr^2}{Y_{(M,Q_p)}(r)} + r^2 d\Omega^2 \tag{3}$$

where X, Y are modified radial functions w.r.t. General Relativity, and Q_p is a set of integration constants coming from the new degrees of freedom, for example the charge of an electrostatic field, the initial density of a fluid, etc.

However, singularities are also importantly expected to be cured by a Quantum Theory of Gravity, and as it turns out, many results from different approaches to QG have lead to non-singular corrections to classical geometries which do not contain such additional integration constants. For example, in the case of effective regular corrections to Schwarzschild spacetime from Loop Quantum Gravity techniques [42–44], or Asymptotic Safety within the Einstein-Hilbert truncation [54–57]. Similar behaviour have been found in cosmology, see for instance [28–30, 140, 141] for LQC and [36, 37, 40] for AS.

Although this might simply be due to the specific approximations needed to reach these results, the singularities are nonetheless cured from two very different ways : either by integration constants, or by a quantization parameter (or a coupling constant in an effective action). It might therefore be interesting to see if these effective Quantum Gravity results can be obtain or not by adding high energy gravitational corrections to the Einstein-Hilbert action. As we mentioned in italic, it will not be possible to find such new terms leading to second order field equations for any metric, but it might be possible for the class of metrics we will be interested in : non-rotating black hole and cosmological ones, which are part of the more general class of dynamical spherically symmetric spacetimes (DSSS).

The aim of this Chapter is to briefly survey what this kind of spacetimes are, and to see some examples of high energy corrections to General Relativity which does lead to second order field equations in DSSS.

In Sec.1, after establishing the basic definition of a DSSS, we will see what can be the most general structure of a (metric) gravitational theory with such property, and make the connection with a popular effective approach to find and study regular solutions, which is to consider two-dimensional Einstein-Dilaton theories (and their Lovelock-Designer generalizations [166–169]), see for instance [41–43, 157–165, 234–237], as if they were coming from the dynamical spherically symmetric reduction of some gravitational theory.

In Sec.2, we apply this formalism to Lovelock-Lanczos (in particular General Relativity) theories and some Quasi-Topological gravities, which are some polynomial (in the curvature) high energy corrections to GR leading to second order field equations in DSSS. Moreover, we will show a particular regularization of the Lovelock scalars, what enables them to contribute to the DSS field equations, even at and beyond the critical order.

Finally, in Sec. 3, we will study a specific property of the Lovelock scalars in critical dimensions, what will enable us to have some hints towards more general gravitational theories, non-polynomial in the curvature, that lead to second order spherically symmetric and cosmological field equations.

A. Covariant formalism & Effective two-dimensional Einstein-Dilaton gravities

As we said in the introduction, we will be interested in this thesis by metric gravitational theories. For a general torsion-free d-dimensional Lorentzian manifold $\mathcal{M}$ with a metric-compatible connection, we can define the Christoffel Symbols by :

$$\Gamma_{\mu\nu}{}^{\alpha} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right)$$

so that the Curvature Riemann Tensor can be written in terms of it as :

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_{[\nu} \Gamma_{\mu]\alpha}{}^{\beta} + \Gamma_{\alpha[\mu}{}^{\lambda} \Gamma_{\nu]\lambda}{}^{\beta}$$

where $T_{[\mu\nu]} = T_{\mu\nu} - T_{\nu\mu}$. Then, any metric gravitational theory can be written in the form

$$I = \int_{\mathcal{M}} d^d x \sqrt{-g}\, \mathcal{L}\left(R_{\mu\nu}{}^{\alpha\beta}, \nabla_\sigma, g_{\gamma\delta} \right) \tag{4}$$

where the Lagrangian density $\mathcal{L}$ can be an arbitrary function of the curvature invariants, built from contractions of the metric, its associated covariant derivative and the Riemann tensor. We wish to explain how these kinds of action decompose when the metric field is restricted to Dynamical spherical symmetry. This is especially relevant for the following reason : due to Palais's "Principle of Symmetric Criticality" [238, 239], the DSS reduction of the field equations is equivalent with the field equations derived from the DSS reduction of the action by varying the components of the DSS metric field. Therefore, as we will be interested by theories admitting second order DSS field equations, we will usually check this property in this way.

Dynamical spherically symmetric spacetimes

In the following, we consider $(n + 2)$-dimensional spherically symmetric spacetimes and their topological generalizations (that we will usually call dynamical spherical symmetry (DSS) as well for simplicity), given by the warped spacetimes $\mathcal{M}_{(n,k)} = \Sigma \times \Omega_{(n,k,r)}$, defined by :

$$ds^2 = d\Sigma^2 + d\Omega^2_{n,k,r} \tag{5}$$

where r is a scalar field on the 2-dimensional manifold Σ, defined by the interval

$$d\Sigma^2 = \gamma_{ab} dx^a dx^b$$

with coordinates x^a, $a = 1, 2$. And $\Omega_{(n,k,r)}$ is the topological generalization of the n-dimensional sphere of radius $r(x)$, with

$$d\Omega^2_{n,k,1} = \begin{cases} d\theta_1^2 + \sum\limits_{i=2}^{n} \prod\limits_{j=1}^{i-1} \sin^2\theta_j d\theta_i^2 \ , \quad \text{for } \ k = 1 \, , \\[2ex] d\theta_1^2 + \sinh^2\theta_1 \left(d\theta_2^2 + \sum\limits_{i=3}^{n} \prod\limits_{j=2}^{i-1} \sin^2\theta_j d\theta_i^2 \right) \ , \quad \text{for } \ k = -1 \, , \\[2ex] \sum\limits_{i=1}^{n} d\phi_i^2 \ , \quad \text{for } \quad k = 0 \, , \end{cases}$$

$k = 0, 1, -1$ corresponding respectively to toroidal, spherical and compact hyperbolic horizon manifolds Ω. We note the n-dimensional metric of $\Omega_{(n,k,r)}$ as σ_{ik}, with $i, j = 1, ..., n$ and the d-dimensional metric $\bar{g}_{\mu\nu}$. Furthermore, we will often write the restriction to ansatz (5) of any tensor T, depending on the metric field and its derivatives, as : $T\rfloor := T\big|_{g=\bar{g}}$.

With this splitting of the metric $\bar{g}_{\mu\nu}$ comes naturally the associated Ricci scalars of the manifolds $\Omega_{(n,k,r)}$ and Σ that we denote respectively $R^{(n)}(\Omega) = k\left(\frac{n(n-1)}{r(x^a)^2}\right)$ and $R^{(2)}(\gamma)$ (or simply $R^{(2)}$), and the metric-compatible covariant derivative associated with γ_{ab} that we note $\mathcal{D}_a$.

Moreover, in this class of spacetimes, if we define,

$$
\begin{aligned}
\xi^a &:= \frac{\mathcal{D}^a r}{r} \ , \quad \chi := \xi^a \xi_a \ , \quad \mathcal{Z} := \frac{R^{(n)}}{n(n-1)} - \chi \ , \\
\mathcal{Y}^a_b &:= \frac{\mathcal{D}^a \mathcal{D}_b r}{r} \ , \quad \psi := -\left(\frac{1}{2}R^{(2)} + \mathcal{Y} + \mathcal{Z}\right),
\end{aligned}
\tag{6}
$$

it can be checked in a straightforward way that the non-vanishing components of the d-dimensional Christoffel symbols are :

$$
\begin{aligned}
&\Gamma_{ab}^{\ \ c} = {}^{(2)}\Gamma_{ab}^{\ \ c}(x^a) \ , \quad \Gamma_{ij}^{\ \ k} = {}^{(d-2)}\Gamma_{ij}^{\ \ k}(x^i) \ , \\
&\Gamma_{ij}^{\ \ a} = -\sigma_{ij}\xi^a \ , \quad \Gamma_{ai}^{\ \ j} = \xi_a \delta_i^j,
\end{aligned}
\tag{7}
$$

so that the Riemann and Ricci tensors as well as the Ricci scalar decompose as :

$$
\begin{aligned}
&R^{ai}_{\ \ bj} = -\mathcal{Y}^a_b \delta^j_i \ , \quad R^{ij}_{\ \ kl} = \mathcal{Z}\delta^i_{[k}\delta^j_{l]} \ , \\
&R^{ab}_{\ \ ce} = \frac{1}{2}R^{(2)}\delta^a_{[c}\delta^b_{e]} \ , \quad R^j_i = \delta^j_i\left(-\mathcal{Y} + (d-3)\mathcal{Z}\right) \ , \\
&R^b_a = \frac{1}{2}\delta^b_a R^{(2)} - (d-2)\mathcal{Y}^b_a \ , \quad R = R^{(2)} - 2(d-2)\mathcal{Y} + (d-2)(d-3)\mathcal{Z},
\end{aligned}
\tag{8}
$$

where we used the fact that any two-dimensional metric is conformally flat, so that all the components of its Riemann tensor are proportional to the Ricci scalar. We summarize the d-dimensional decomposition properties of the curvature tensors that we will use in this thesis in the first Appendix.

From d-dimensional DSS to Two-dimensional scalar-tensor theories

Therefore any d-dimensional metric theory will have its DSS decomposition spanned by

$$
\left(\gamma_{ab} \, , \, \mathcal{Y}^a_b \, , \, \chi \, , \, R^{(n)} \, , \, R^{(2)} \, , \, \mathcal{D}_a\right)
\tag{9}
$$

Moreover, considering the additional splitting $\sqrt{-g} = \sqrt{\sigma}\sqrt{-\gamma}r^{d-2}$, where σ and γ are the determinants of the metrics of respectively $\Omega_{(n,k,r)}$ and Σ, we can now reduce the general action (4) to DSS as follows :

$$
I\Big\rfloor = \mathcal{A}_{k,n,1}\int_\Sigma d^2x\sqrt{-\gamma}r^{d-2}\,\mathcal{L}\left(\gamma_{ab} \, , \, \frac{\mathcal{D}^a\mathcal{D}_b r}{r} \, , \, \frac{\mathcal{D}^a r \mathcal{D}_a r}{r^2} \, , \, k\left(\frac{n(n-1)}{r^2}\right) \, , \, R^{(2)}(\gamma) \, , \, \mathcal{D}_a\right)
\tag{10}
$$

where $\mathcal{A}_{k,n,1} = \int d^n x\sqrt{\sigma}$ is the volume of the n-dimensional manifold $\Omega_{(n,k,1)}$ of radius unity. Therefore, any spherically symmetry sector of a gravitational theory reduces to a particular two-dimensional scalar-tensor theory. As we said before, the aim of this thesis is to find gravitational theories (4), leading to second order DSS field equations, so that they will belong in this sector to the most general second order scalar-tensor theory.

Two-dimensional Horndeski theory & Lovelock-Designer Gravity

As it is well known, second order scalar-tensor theories have been classified by Horndeski in [83]. The procedure to construct second order field equations for any field theory and in any dimensions is essentially to make use of the totally antisymmetric tensors $\delta^{\mu_1 \ldots \mu_q}_{\nu_1 \ldots \nu_q}$ (as we will see in more details in the last section of this Chapter). In particular, in two dimensions, the only non-vanishing ones are δ^a_b and δ^{ce}_{ab}. For this reason, the usual terms of four-dimensional Horndeski theory given by $G_{\mu\nu}$ and $(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3$ are identically vanishing in two dimensions. See the last section of this Chapter for more details. Therefore, the most general second order scalar tensor theory in two dimensions is given by :

$$\mathscr{L}^{2D}_{\text{Horndeski}} = \sqrt{-\gamma}\left(\eta\left(r,\tilde{\chi}\right) - G_3\left(r,\tilde{\chi}\right)\tilde{\mathcal{Y}} + G_4\left(r,\tilde{\chi}\right)R^{(2)}\left(\gamma\right) - 2\frac{\partial G_4\left(r,\tilde{\chi}\right)}{\partial\tilde{\chi}}\left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right)\right) \tag{11}$$

where in all this thesis (unless otherwise specified), we define $\tilde{\chi}_a = \mathcal{D}_a r$, $\tilde{\chi} = \mathcal{D}_a r\mathcal{D}^a r$, $\tilde{\mathcal{Y}} = \mathcal{D}^2 r$ (with $\mathcal{D}^2 = \mathcal{D}_a\mathcal{D}^a$), and $\tilde{\mathcal{Y}}_2 = \mathcal{D}_a\mathcal{D}_b r\mathcal{D}^a\mathcal{D}^b r$.

As an aside note, remark that in [170], Kunstatter, Maeda and Taves have conjectured that the following theory, named “Lovelock-Designer gravity” is the most general two-dimensional scalar-tensor theory (see eq(3.2) of this paper and the comments that follow)

$$\mathscr{L} = \sqrt{-\gamma}\left(\zeta\left(r,\tilde{\chi}\right) + \mathcal{X}\left(r,\tilde{\chi}\right)\frac{\tilde{\chi}_a\mathcal{D}^a\tilde{\chi}}{\tilde{\chi}} + \phi\left(r\right)R^{(2)}\left(\gamma\right)\right) \tag{12}$$

To prove this, we just need to show that this Lagrangian differ from the previous one by a total derivative. Indeed, defining[1] $\Gamma_a := \mathcal{D}_{[b}r\mathcal{D}_{a]}\mathcal{D}^b r$, and using the notation $X_r := \partial_r X$ and $X_{\tilde{\chi}} := \partial_{\tilde{\chi}}X$, we have

$$\begin{aligned}\mathscr{L}^{2D}_{\text{Horndeski}} =& \sqrt{-\gamma}\Bigg(\left(\eta + \tilde{\chi}\left(2\phi_{rr} + G_{3,r} - 2G_{4,rr}\right)\right) + \left(\phi_r - G_{4,r} - \tilde{\chi}\left(2G_{4,r\tilde{\chi}} - G_{3,\tilde{\chi}}\right)\right)\frac{\tilde{\chi}_a\mathcal{D}^a\tilde{\chi}}{\tilde{\chi}} \\ &+ \phi R^{(2)} - \mathcal{D}_a\left(\tilde{\chi}^a\left(2\phi_r + G_3 - 2G_{4,r}\right) + \frac{2\Gamma^a}{\tilde{\chi}}\left(\phi - G_4\right)\right)\Bigg)\end{aligned} \tag{13}$$

for any *arbitrary* function $\phi(r)$. Therefore, identifying $\zeta = \eta + \tilde{\chi}\left(2\phi_{rr} - G_r\right)$, $G = 2G_{4,r} - G_3$ and $\mathcal{X} = \phi_r - \tilde{\chi}G_{\tilde{\chi}} - G_{4,r}$ lead to the result. This can be checked by expanding the total derivative in the last equation, and using that

$$R^{(2)}\left(\gamma\right) = 2\,\mathcal{D}_a\left(\frac{\Gamma^a}{\tilde{\chi}}\right) \tag{14}$$

what will be shown in the last section of this Chapter, see eq(36). Thus, even the Lovelock-Designer gravity contains a bit of redundancy, because we can set $\phi(r) = 0$ without loss of generality. This might seem strange for the most general scalar-tensor theory not to depend directly on the curvature, but as we see in the last equation, the two-dimensional Ricci scalar can be written in terms of covariant derivatives of any scalar field, and in particular $r(x^a)$. This will be made clearer in the last section of this Chapter, where we generalize this kind of identity for arbitrary dimension.

The field equations w.r.t. the metric and the scalar field have been found in [170] and reads :

$$\begin{aligned}\frac{\delta\mathscr{L}}{\delta\gamma_{ab}} &\propto \left(\mathcal{X} - \phi_r\right)\left(\tilde{\mathcal{Y}}_{ab} - \gamma_{ab}\tilde{\mathcal{Y}}\right) + \gamma_{ab}\left(\phi_{rr}\tilde{\chi} - \frac{1}{2}\zeta\right) + \left(\zeta_{\tilde{\chi}} - \mathcal{X}_r - \phi_{rr}\right)\tilde{\chi}_{ab} = 0 \\ \frac{\delta\mathscr{L}}{\delta r} &\propto -\left(\mathcal{X} - \phi_r\right)R^{(2)} + \left(\zeta_r + 2\left(\mathcal{X}_{rr} - \zeta_{r\tilde{\chi}}\right)\tilde{\chi}\right) + 2\left(2\mathcal{X}_r - \zeta_{\tilde{\chi}}\right)\tilde{\mathcal{Y}} \\ &\quad + 2\mathcal{X}_{,\tilde{\chi}}\left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right) + 2\left(\mathcal{X}_{r\tilde{\chi}} - \zeta_{\tilde{\chi}\tilde{\chi}}\right)\tilde{\chi}_a\mathcal{D}^a\tilde{\chi} = 0\end{aligned} \tag{15}$$

The spirit of the two-dimensional effective approach

Taking into account what we learnt from the previous paragraphs, we can now draw the general line of thought followed by studying two-dimensional scalar-tensor theories. As any d-dimensional gravitational theory reduces in DSS to some subclass of these models, one can directly work in two-dimensions, solves the theory by finding $(\gamma_{ab}(x^a), r(x^a))$, and assume that they describe a d-dimensional dynamical spherically symmetric spacetime given by :

[1] And we still use the convention $X_{[ab]} = X_{ab} - X_{ba}$.

$$ds^2 = \gamma_{ab}(x)dx^a dx^b + r^2(x)d\Omega^2_{n,k,1} \tag{16}$$

In particular, this effective approach has been used to study regular black hole geometries, see for example [41–43] for Einstein-Dilaton gravities (i.e. two-dimensional Horndeski theories for which $\partial_{\tilde{\chi}} G_4 (r, \tilde{\chi}) = 0$), [166–169] for Lovelock Designer (i.e. due to Eq(13), 2D Horndeski), while key properties of effective black holes have been studied within this framework, like their formation and evaporation [41, 157], thermodynamics and observables [158–162], quasinormal modes [234], quantum properties [163–165], and behaviour in the presence of matter fields [235–237].

A main obstacle in this road is the fact that the two-dimensional scalar-tensor theories are usually taken to be second order. However, we said before that in four dimensions, Lovelock Theorem prevents to find gravitational theories leading for any metric field to second order field equations. Therefore, if the 2D Horndeski theories usually studied in this effective approach can be at all lifted to four (or higher) dimensions by the identification (16), we can be sure that the resulting $4D$ theories will not yield second order field equations for all metric fields, even if they do for DSS ones.

One of the main result of this thesis (see Chap. 2) is precisely to prove that there exists such d-dimensional gravitational theories of the form (4), which indeed yield second order field equations for DSS, i.e. whose DSS decompositions belong to Lovelock-Designer gravities (i.e. 2D Horndeski theories).

Finally, before continuing by some classical examples of such theories, note that 2D effective actions have also been studied following other reasoning. For example the fact that Quantum Gravity results indicate that spacetime would be effectively two-dimensional at small scale, see for example [64]. One can also think about Liouville theory [240–246] or the non-local Polyakov action [247]. In these cases, there might be no need for the identification (16). The Chapter 3, 4 and 5 of this thesis are essentially based on 2D scalar-tensor theories admitting regular solutions, so that in principle they can be interpreted in either ways. However, we usually lift to higher dimensions, due to the results of Chapter 2, in particular to mimic some four-dimensional quantum gravity inspired geometries.

B. Lovelock-Lanczos theories & Quasi-Topological Gravity

Now we will see a few examples of d-dimensional gravitational theories (4) leading to second order field equations for dynamical spherically symmetric spacetimes. In particular General Relativity, Lovelock-Lanczos (LLG) and some Quasi-topological gravities (QTG). In the case of LLG, we will see a first hint that it is indeed possible to find d-dimensional actions leading to a general (2D) Lovelock-Designer gravity, while the QTG provides a first example of theories with second order DSS field equations, but higher order otherwise.

Decomposition of General Relativity

The first and simplest example is of course the case of General Relativity, given by the Einstein-Hilbert action :

$$I_{EH} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g}\, R \tag{17}$$

In DSS, the Ricci scalar decomposes as :

$$\begin{aligned} R \rfloor &= R^{(2)} - 2(d-2)\mathcal{Y} + (d-2)(d-3)\mathcal{Z} \\ &= R^{(2)}(\gamma) + R^{(n)}(\Omega) - 2\frac{\mathcal{D}^2 r^n}{r^n} + n(n-1)\frac{\mathcal{D}r.\mathcal{D}r}{r^2} \end{aligned} \tag{18}$$

while the symmetric reduction of the Einstein tensor $G_{\mu\nu}$ is given by

$$\begin{aligned} G_{AB} &= 2\Big(- r\left(\mathcal{D}_A\mathcal{D}_B - \gamma_{AB}\mathcal{D}^2\right) r + \frac{1}{2}(n-1)\gamma_{AB}\left(\mathcal{D}r.\mathcal{D}r - k\right)\Big) r^{n-3} \\ G_{\phi\phi} &= \frac{1}{2}\left((n-1)(n-2)\left(\mathcal{D}r.\mathcal{D}r - k\right) - r^2 R^{(2)}(\gamma) + 2(n-1) r \mathcal{D}^2 r\right) r^{n-4}\,. \end{aligned} \tag{19}$$

Of course it leads to second order field equations in DSS because it belongs to the Lovelock gravity theories which does so for any metric field.

Decomposition of Lovelock gravity

A more involved decomposition is the one of the general power-series Lovelock-Lanczos gravity, given by :

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^{\,n+2}x\,\sqrt{-g}\sum_{p=1}^{m} l^{2(p-1)}\,\frac{w_p}{2^p}\,\delta^{\mu_1\nu_1\ldots\mu_p\nu_p}_{\alpha_1\beta_1\ldots\alpha_p\beta_p}\,R^{\alpha_1\beta_1}_{\mu_1\nu_1}\ldots R^{\alpha_p\beta_p}_{\mu_p\nu_p} \tag{20}$$

Defining the Lagrangian density as :

$$\mathcal{L}_{(p)} = \frac{1}{2^p}\,\delta^{\mu_1\nu_1\ldots\mu_p\nu_p}_{\alpha_1\beta_1\ldots\alpha_p\beta_p}\,R^{\alpha_1\beta_1}_{\mu_1\nu_1}\ldots R^{\alpha_p\beta_p}_{\mu_p\nu_p} \tag{21}$$

Yields the following decomposition that was derived in [166] :

$$\mathcal{L}_{(p)}\Big\rfloor = \frac{(d-2)!}{(d-2p)!}\left(\alpha_p\mathcal{Z}^2 + \mathcal{Z}\left(\beta_p\mathcal{Y} + \gamma_p R^{(2)}\right) + \zeta_p\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right)\mathcal{Z}^{p-2} \tag{22}$$

where $\alpha_p = (d-2p)(d-2p-1)$, $\beta_p = -2(d-2p)p$, $\gamma_p = p$ and $\zeta_p = 2p(p-1)$. It can be written equivalently as :

$$\begin{aligned}\mathcal{L}_{(p)}\Big\rfloor = {} & (d-2p)\,\nu_p\left(\left((d-2p-1)\mathcal{Z} - 2p\mathcal{Y}\right)\mathcal{Z}^{p-1} + p\,r^{2(1-p)}\chi_a\left(2\Gamma^a\mathcal{J} - k^{p-1}v^a\right)\right)\\ & + p\,\nu_p\,r^{2-d}\mathcal{D}_a\left(r^{d-2p}\left(k^{p-1}v^a - 2\Gamma^a\mathcal{J}\right)\right)\end{aligned} \tag{23}$$

where $\nu_p = \frac{(d-2)!}{(d-2p)!}$, $\mathcal{J} = \frac{k^{p-1}-(k-\tilde{\chi})^{p-1}}{\tilde{\chi}}$ and $R^{(2)} = \nabla_a v^a$. This relation is shown in the first section of the Appendix, and follows from several results found in [171] (see the first Appendix of this paper).

Therefore, the DSS sector of LLG possesses a very curious property. To see which one, recall first that in critical dimension ($d = m/2$ in (20), see the next section), the Lovelock-Lanczos scalar does not contribute to the field equations (what can be seen here by the fact that for $d = 2p$, the bulk term of their DSS decomposition (23) vanishes identically) and also that the higher order (than critical) LL scalars are identically vanishing (indeed, for $p > 2d$, we have $\nu_p = 0$ for any dimension d). However, we see that if we consider the Lovelock-Lanczos Lagrangian density with the particular choice of the free set of dimensionless coupling constants given by

$$w_p = \frac{\sigma_p}{(d-2p)\,\nu_p} = \sigma_p\,\frac{(d-2p-1)!}{(d-2)!}, \tag{24}$$

then, in dynamical spherically symmetric spacetimes, all the Lovelock-Lanczos scalars of any order can contribute to the (DSS) field equations in any dimensions by considering the following theory :

$$\begin{aligned}\mathscr{L}_{(d,p)}\left[\gamma_{ab};r\right] := {} & \left(\sum_{p=1}^{m}\sigma_p\,\frac{(d-2p-1)!}{(d-2)!}\,\frac{1}{2^p}\,\delta^{\mu_1\nu_1\ldots\mu_p\nu_p}_{\alpha_1\beta_1\ldots\alpha_p\beta_p}\,R^{\alpha_1\beta_1}_{\mu_1\nu_1}\ldots R^{\alpha_p\beta_p}_{\mu_p\nu_p}\right)\Bigg|_{g_{\mu\nu}=\bar{g}_{\mu\nu}}\\ = {} & \sum_{p=1}^{m}\sigma_p\left(\left((d-2p-1)\mathcal{Z} - 2p\mathcal{Y}\right)\mathcal{Z}^{p-1} + p\,r^{2(1-p)}\chi_a\left(2\Gamma^a\mathcal{J} - k^{p-1}v^a\right)\right)\\ & + \sum_{p=1}^{m}\sigma_p\,\frac{(d-2p-1)!}{(d-2)!}p\,r^{2-d}\mathcal{D}_a\left(r^{d-2p}\left(k^{p-1}v^a - 2\Gamma^a\mathcal{J}\right)\right)\end{aligned} \tag{25}$$

where one has to specify the dimension d and the order of correction p *after* the symmetric reduction, in order to make the critical and higher order scalars contribute to DSS (providing that one considers $m > d/2$).

However we see that at the critical order $d = 2p$, the price of making the Critical Order Scalar contribute to the field equation is to produce a divergent boundary term. Indeed, *after* performing the DSS reduction (and neglecting the angular part) we obtain for the critical order

$$\begin{aligned}\lim_{d\to 2p}\left(\left(\beta_p\sqrt{-g}\,\mathcal{L}_{(p)}\right)\Big|\right) &= \sqrt{-\gamma}\,\sigma_p\left(-\left(\mathcal{Z}+2p\mathcal{Y}\right)\mathcal{Z}^{p-1}r^{2(p-1)} + p\,\chi_a\left(2\Gamma^a\mathcal{J} - k^{p-1}v^a\right)\right)\\ &\quad + p\,\frac{\sigma_p}{(2(p-1))!}\,\mathcal{D}_a\left(\left(k^{p-1}v^a - 2\Gamma^a\mathcal{J}\right)\lim_{p\to\frac{d}{2}}\frac{r^{d-2p}}{d-2p}\right)\\ &= \sqrt{-\gamma}\,\sigma_p\left(-\left(\mathcal{Z}+2p\mathcal{Y}\right)\mathcal{Z}^{p-1}r^{2(p-1)} + p\,\log r\left(k^{p-1}R^{(2)} - 2\mathcal{D}_a\left(\Gamma^a\mathcal{J}\right)\right)\right)\\ &\quad + p\,\frac{\sigma_p}{(2(p-1))!}\,\mathcal{D}_a\left(\left(k^{p-1}v^a - 2\Gamma^a\mathcal{J}\right)\left(\lim_{p\to\frac{d}{2}}\frac{r^{d-2p}}{d-2p} - \log r\right)\right)\end{aligned} \tag{26}$$

As far as we know, this procedure as been discovered for the Gauss-Bonnet scalar in four dimension by Y. Tomozawa in [248], and it has been further studied by Cognola, Myrzakulov, Sebastiani and Zerbini in [249] where they have found the associated black hole and cosmological solutions of the Einstein + $\lim\limits_{d\to 4}$(Gauss-Bonnet/$(d-4)$) theory. Therefore, the previous result establishes the generalization of this procedure for the critical order scalars in any dimensions, and also provides a way to consider scalars with orders larger than the critical one. In particular, it allows to consider non-perturbative corrections in the parameter l by taking $m \to \infty$, what we will use in Eq(454) of the last Chapter to find a regularized four-dimensional Lovelock-Lanczos theory admitting black hole and cosmological solutions that are both regular. Note that another way to make the Gauss-Bonnet scalar contributes to the four-dimensional field equations without non-minimal coupling with other fields has been found in [250].

We did not know if there would exist larger sets of metric fields $\bar{g}_{\mu\nu}$ than DSS ones for which such regularization procedure works. However, in a recent paper [251] that appeared during the review of this thesis, it was shown that also the first order perturbations (of the Einstein + $\lim\limits_{d\to 4}$(Gauss-Bonnet/$(d-4)$) theory) around (A)dS vacuum, spherically symmetric and FLRW spacetimes do possess this property. Since then, preliminary results have been found by Alessandro Casalino and ourself, showing that it also works for higher order Lovelock-Lanczos scalars, higher order perturbations, and also for other backgrounds than DSS ones, making these theories promising to be suitable quantum gravity effective actions.

Finally, the existence of these DSS sectors of specific cases of Lovelock-Lanczos theories (that might well be singular in some other sectors than DSS) can be a hint that more general gravitational theories can admit a DSS decomposition leading to 2D Horndeski theories, and more specifically to Lovelock-like ones.

Decomposition of a Quasi-topological gravity

In particular, a similar procedure was used in [143] (see Eq(54) of this paper) to construct Quasi-topological gravities. These theories are gravitational and metric actions that yield second order field equations in DSS, but usually in higher than four dimensions. In other sector, their field equations are higher order ones. See for example [143–150]. To see this by an example found in [144] (Eq(3.10) of this paper), consider the cubic polynomial invariants of the Riemann tensor :

$$\begin{aligned} S_1 &= R_{\mu\ \nu}^{\ \alpha\ \beta}R_{\alpha\ \beta}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \mu\ \nu}\\ S_2 &= R^{\mu\nu}_{\ \ \alpha\beta}R_{\mu\nu}^{\ \ \alpha\beta}R\\ S_3 &= R_{\mu\nu\alpha\rho}R^{\mu\nu\alpha}_{\ \ \ \ \sigma}R^{\rho\sigma}\\ S_4 &= R_{\mu\nu\alpha\beta}R^{\mu\alpha}R^{\nu\beta}\\ S_5 &= R_\mu^{\ \nu}R_\nu^{\ \alpha}R_\alpha^{\ \mu}\\ S_6 &= R_\mu^{\ \nu}R_\nu^{\ \mu}R\\ S_7 &= R^3\end{aligned} \tag{27}$$

Then, in DSS, they decomposes as

$$
\begin{aligned}
S_1 &= n(n-1)(n-2)\mathcal{Z}^3 + \frac{3}{2}nR_{(2)}\left(\mathcal{Y}^2 - \mathcal{Y}_2\right) + 3n(n-1)\mathcal{Z}\mathcal{Y}_2 \\
S_2 &= \left(R_{(2)}^2 + 4n\mathcal{Y}_2 + 2n(n-1)\mathcal{Z}^2\right)R \\
S_3 &= \frac{1}{2}R_{(2)}^2\left(R_{(2)} - n\mathcal{Y}\right) + n\left(R_{(2)}\mathcal{Y}_2 - 2n\mathcal{Y}_3\right) - 2n\mathcal{V}\left(\mathcal{Y}_2 + (n-1)\mathcal{Z}^2\right) \\
S_4 &= \frac{1}{4}R_{(2)}\left(R_{(2)}^2 - 2nR_{(2)}\mathcal{Y} + 2n^2\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right) + n\left(R_{(2)}\mathcal{Y} - 2n\mathcal{Y}_2\right)\mathcal{V} + n\,(n-1)\,\mathcal{Z}\mathcal{V}^2 \\
S_5 &= \frac{1}{4}\left(R_{(2)}^3 - 3nR_{(2)}^2\mathcal{Y} + 6n^2R_{(2)}\mathcal{Y}_2 - 4n^3\mathcal{Y}_3 - 4n\mathcal{V}^3\right) \\
S_6 &= \left(\frac{1}{2}R_{(2)}^2 - nR_{(2)}\mathcal{Y} + n^2\mathcal{Y}_2 + n\mathcal{V}^2\right)R
\end{aligned}
\tag{28}
$$

where $\mathcal{V} := (\mathcal{Y} - (n-1)\mathcal{Z})$. Therefore, one can see that the following theory

$$
\mathcal{U} = S_1 + \frac{3}{8(n-2)(2n+1)}\left((3n-2)S_2 - 4(3n+2)S_6 - 8n\left(S_3 - 2S_5\right) + (n+2)\left(8S_4 + S_7\right)\right) \tag{29}
$$

has a DSS decomposition which is contained within 2D Horndeski theories, as it is given by :

$$
\begin{aligned}
\mathcal{U} = \frac{3(n-1)}{2n+1}\Bigg(&\frac{n^3}{2-n}\left(\mathcal{Y}^3 - 3\mathcal{Y}\mathcal{Y}_2 + 2\mathcal{Y}_3\right) + 4p\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\mathcal{Z} + pR_{(2)}\mathcal{Z}^2 \\
&+ \frac{p(n-4)}{3}\mathcal{Z}^2\left((n-5)\mathcal{Z} - 6\mathcal{Y}\right)\Bigg)
\end{aligned}
\tag{30}
$$

with $p = n(3n^2 - 3n - 2)/8$. Note that it can be further reduced using that $\mathcal{Y}^3 - 3\mathcal{Y}\mathcal{Y}_2 + 2\mathcal{Y}_3 = 0$ in two dimensions.

These theories are very similar to Lovelock-Lanczos theories in DSS, and can be found in a straightforward way by summing all the scalars of a given order in the FKWC basis of [252, 253], decomposing the whole expression in DSS and then adjusting the parameter so that the DSS sector of the resulting combination belongs to 2D Horndeski theory. However only the basis for order up to eight were displayed in these papers.

Therefore, we see that already some gravitational theories can provide second order DSS field equations, so that one might hint that it will actually be possible to find some such theories leading to the regular quantum gravity effective geometries that we mentioned in the introduction of this Chapter.

C. Lovelock-Lanczos Scalars in Critical Dimensions

Now we will focus on a particular property of the Lovelock-Lanczos scalars, which will enable us to find new metric gravitational theories leading to second order field equations in (static) spherically symmetric and cosmological Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes.

In critical dimensions, the Lovelock-Lanczos scalars are known to be related to the Euler densities of the manifolds under consideration. In the tetrad formalism (in terms of differential forms), these invariants can be expressed locally as exact forms involving the curvature and the spin-connection, so that their integrals give the Euler characteristics of the associated manifolds. However, in the metric formalism, a similar covariant expression of the Euler densities in terms of the divergence of a vector field, depending only on the metric field and its derivatives, is sometimes considered not to exist. To see that it does, we have derived such an expression of the critical Lovelock-Lanczos invariants as the covariant derivative of a vector depending on the Riemann tensor and an arbitrary scalar field. In particular, if the scalar field is chosen to be a curvature scalar, the expression only involves non-polynomial curvature terms. We realized later that an identical result had already been found some time ago by Horndeski in [254] (although using a vector field instead of a scalar field), where the idea of replacing the arbitrary scalar field by a curvature scalar was also suggested. However, the construction works only for metric fields for which there exists a curvature invariant with non-vanishing kinetic term. Therefore having both covariance and dependence on the curvature implies some form of background dependence of the formula in the metric formalism. Finally, as these vectors are non-polynomial in the scalar field, we briefly study the invariant built by contracting the vector, whose divergence gives the Gauss-Bonnet scalar, with itself, and it turns out to belong to the so-called non-polynomial gravity models, as for specific backgrounds, this invariant becomes polynomial in the curvature and its dependence on the scalar field disappear. In a four dimensional FLRW spacetime, it gives the same H^6 correction to the Friedmann equation as the sixth order Lovelock scalar would. In static spherical symmetry, we find two degenerate vacua and a singular solution.

Introduction

Lovelock gravity is the most general metric and tetrad theory of gravity leading to second order field equations from a covariant action involving only the gravitational fields (metric or tetrad and associated connections if the Palatini formalism is considered) [89, 255]. The action reads :

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^d x\,\sqrt{-g}\sum_{i=1}^{\frac{\mathcal{O}}{2}} l^{2(i-1)}\,\frac{\beta_i}{2^i}\,\delta^{\mu_1\nu_1\ldots\mu_i\nu_i}_{\alpha_1\beta_1\ldots\alpha_i\beta_i}\,R^{\alpha_1\beta_1}_{\mu_1\nu_1}\ldots R^{\alpha_i\beta_i}_{\mu_i\nu_i} \tag{31}$$

where G is the Newton constant, $\delta^{\mu_1\nu_1\ldots\mu_i\nu_i}_{\alpha_1\beta_1\ldots\alpha_i\beta_i}$ is the generalized Kronecker delta, l is a length scale introduced for dimensional reasons, and the β_i are a set of dimensionless coupling constants. Due to its definition involving totally antisymmetric tensors, the series of actions truncates when the order $\mathcal{O}$ of correction, i.e. the number of derivatives in the action, is equal to the number of dimensions $\mathcal{O}_{\mathcal{C}} = d$. This order, or alternatively the dimension corresponding to the order d action, is usually called "critical". We are interested in this section by the critical order Lovelock-Lanczos Lagrangian densities given by :

$$\mathcal{L}_{(d)} = \frac{1}{2^n}\delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\prod_{k=1}^{d-1} R^{\nu_k\nu_{k+1}}_{\mu_k\mu_{k+1}} \tag{32}$$

where $d = 2n$. They have the remarkable property that their integrals over the manifold under consideration is proportional to the Euler characteristic of the manifold, which is a topological invariant integer. Therefore, the actions associated with these Lagrangian densities must have identically vanishing field equations, for any configuration of the gravitational field. A very natural question is therefore if it is possible to express the $\mathcal{L}_{(d)}$ as divergences of some vectors depending on the curvature, in such a way that the associated field equations would be trivially vanishing. In the tetrad formalism where differential forms are used, this amounts to express locally the critical order Lovelock-Lanczos Lagrangian form, given by,

$$L_{(d)} = \epsilon_{a_1b_1\ldots a_nb_n}\mathcal{R}^{a_1b_1}\wedge\ldots\wedge\mathcal{R}^{a_nb_n} \tag{33}$$

in terms of an exact form, where $\mathcal{R}$ is the curvature 2-form and ϵ is the Levi-Civita symbol. This is where a difference between metric and tetrad formalisms appears : it is well known that $L_{(d)}$ can be expressed as the exterior derivative of the Chern-Simons form (see for example [256]), but an expression in the metric formalism was, to our knowledge, still missing ; although, as we said previously, it turns out that Horndeski already settled this issue some time ago, in [254].

In the four dimensional case, corresponding to the Gauss-Bonnet scalar, the following elegant expression was proposed in [257] and mentioned in [258],

$$\begin{aligned}\sqrt{-g}\mathcal{L}_{(4)} =& \sqrt{-g}\left(R^2 - 4R^\nu_\mu R^\mu_\nu + R_{\mu\nu}{}^{\sigma\rho}R^{\mu\nu}{}_{\sigma\rho}\right)\\ =& \partial_\mu\left(\sqrt{-g}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\rho\sigma}{}^{\lambda\delta}\Gamma_{\nu\lambda}^{\ \ \rho}\left(\frac{1}{3}\Gamma_{\alpha\zeta}^{\ \ \sigma}\Gamma_{\beta\delta}^{\ \ \zeta} - \frac{1}{2}R^\sigma{}_{\delta\alpha\beta}\right)\right)\end{aligned} \tag{34}$$

However, it was noticed in [256] that it is in fact incorrect, although it curiously holds for a variety of spacetimes, like Schwarzschild, Kerr, Milne and Godel. In fact, it is not even generally covariant as the presence of Christoffel symbols, which are not the components of a tensor, can suggest. On the other hand, in terms of differential forms we have :

$$L_{(4)} = d\left(\epsilon_{abcd}\left(\omega^{ab}\wedge\mathcal{R}^{cd} - \frac{1}{3}\omega^{ab}\wedge\omega^c{}_f\wedge\omega^{fd}\right)\right) \tag{35}$$

where ω is the spin-connection. This difference between tetrad and metric formalisms has even led some authors to think that no expression of the critical order Lovelock-Lanczos scalars in terms of a divergence of a vector involving only the metric field and its derivatives in a covariant way exist (see [259], [260] and [261]). However, key steps toward such a formula have been achieved in [256] and [88]. In the first one, a covariant formula involving an arbitrary unit normalized vector field was found for the two dimensional case, while a covariant formula involving an arbitrary scalar field was found for the four dimensional case in the second one. One of the aim of this section is to generalize the results of these papers for general dimensions, using an arbitrary scalar field.

In section 1, we show that in two dimensions, the Lovelock-Lanczos scalar (which is simply the Ricci scalar) can be expressed as the total divergence of a vector depending on an arbitrary scalar field. This expression reduces to the one found in [256] by suitably choosing the arbitrary unit normalized vector that was used there. In section 2, we expose the result for the general case, which is proven in the appendix, by finding a general vector whose divergence gives the critical order Lovelock-Lanczos scalars. In particular, if the scalar field is chosen to be a curvature scalar, the expression only involves the metric field and its derivatives. As this vector is non-polynomial in the scalar field, in section 3, we briefly study the invariant built from contracting this vector with itself, and it turns out to belong to the so-called non-polynomial gravity models (see [151–156]), as for specific backgrounds, this invariant becomes polynomial in the curvature and its dependence on the scalar field disappears. In a four dimensional FLRW spacetime, it gives the same H^6 correction to the Friedmann equation as the sixth order Lovelock scalar would, while in static spherical symmetry the solutions are degenerate.

While looking for additional references in the later stage of this work, we realized that the mentioned result of [256] and [88] had actually already been generalized in [254], where Horndeski found that the Euler densities can be expressed as divergences depending on an arbitrary non-null vector field. Moreover, other constructions have been carried out in order to express the Euler densities in terms of divergences of purely (geo)metrical quantities, see [262]. However, the introduction of additional geometrical structures were needed to do so.

1. Ricci scalar in two dimensions

Let's start with the 2-dimensional case. For simplicity, we first define $\Gamma_\alpha := \nabla_{[\sigma}\phi\nabla_{\alpha]}\nabla^\sigma\phi$, $\phi_{;\alpha} := \nabla_\alpha\phi$ and we use the convention $X_{[\alpha\sigma]} = X_{\alpha\sigma} - X_{\sigma\alpha}$. Then we have :

$$\begin{aligned}
R &= 2\left(\frac{\phi^{;\alpha}\left(\Box\nabla_\alpha - \nabla_\alpha\Box\right)\phi}{\phi_{;\gamma}\phi^{;\gamma}}\right) = 2\left(\frac{\nabla^\alpha\Gamma_\alpha + \left(\phi^{;\alpha}_{;\sigma}\phi^{;\sigma}_{;\alpha} - \left(\phi^{;\alpha}_{;\alpha}\right)^2\right)}{\phi_{;\gamma}\phi^{;\gamma}}\right)\\
&= 2\left(\frac{\nabla^\alpha\Gamma_\alpha}{\phi_{;\gamma}\phi^{;\gamma}} - \frac{\Gamma^\alpha\nabla_\alpha\left(\phi_{;\gamma}\phi^{;\gamma}\right)}{\left(\phi_{;\gamma}\phi^{;\gamma}\right)^2}\right) = 2\,\nabla^\alpha\left(\frac{\Gamma_\alpha}{\phi_{;\gamma}\phi^{;\gamma}}\right)\\
&= 2\nabla^\alpha\left(\frac{\nabla_{[\sigma}\phi\nabla_{\alpha]}\nabla^\sigma\phi}{\phi_{;\gamma}\phi^{;\gamma}}\right)
\end{aligned} \tag{36}$$

The first line can be shown by first noting that $\gamma^{\alpha\sigma}\nabla_{[\alpha}\nabla_{\beta]}\nabla_\sigma = \gamma^{\alpha\sigma}R_{\alpha\beta\sigma\delta}\nabla^\delta = R_{\beta\delta}\nabla^\delta$. Then, as we said in the introduction, the field equations associated with the critical Lovelock-Lanczos scalars are identically vanishing. Therefore, in two dimensions, the Einstein tensor vanishes and we have $R_{\beta\delta} = \frac{1}{2}g_{\beta\delta}R$, in such a way that $\Box\nabla_\alpha - \nabla_\alpha\Box = \frac{1}{2}R\,\nabla_\alpha$. The second line can be shown by noting that in two dimensions, for any tensor X of rank greater or equal to two, we have $\delta^{[\alpha}_{[\sigma}X^{\beta]}_{\delta]} = \frac{1}{2}X_{\mu\nu}g^{\mu\nu}\delta^{[\alpha}_{[\sigma}\delta^{\beta]}_{\delta]}$, and applying this identity to $\delta^{[\alpha}_{[\sigma}\phi^{;\beta]}_{;\delta]}\phi^{;\sigma}_{;\alpha}\phi_{;\beta}\phi^{;\delta}$ leads to the result. Using the Levi-Civita tensor, it can be written as :

$$\mathcal{L}_{(2)} = 2\,\varepsilon^{\mu\nu}\varepsilon_{\sigma\rho}\nabla^\rho\left(\frac{\phi^{;\sigma}_{;\mu}\phi_{;\nu}}{\phi_{;\gamma}\phi^{;\gamma}}\right) \tag{37}$$

Note that this equation is valid for any scalar field ϕ whose kinetic term is non-vanishing. This is the price to pay to work in coordinates : because the Christoffel symbols are not the components of a tensor (contrary to the spin connection which is a one form), we need to adjust the order of derivative via the introduction of an additional field.

The answer to the question of whether the two dimensional Ricci scalar can be expressed as the divergence of a vector involving only the metric field and its derivatives is therefore positive, as the scalar field can be chosen to be any curvature invariant. For example, using the Kretschmann invariant $\mathcal{Q} := R^{\mu\nu}{}_{\sigma\rho} R_{\mu\nu}{}^{\sigma\rho}$ we have :

$$\mathcal{L}_{(2)} = 2\, \varepsilon^{\mu\nu} \varepsilon_{\sigma\rho} \nabla^{\rho} \left(\frac{\mathcal{Q}_{;\mu}^{;\sigma} \mathcal{Q}_{;\nu}}{\mathcal{Q}_{;\gamma} \mathcal{Q}^{;\gamma}} \right) \tag{38}$$

One has to be careful to choose a curvature invariant $\phi = \mathscr{R}$ such that $\mathscr{R}_{;\gamma} \mathscr{R}^{;\gamma} \neq 0$. This excludes constant curvature spaces for instance.

Note also that for all metric fields such that $R_{;\gamma} R^{;\gamma} \neq 0$, one could use the Ricci scalar itself and obtain a quite baroque identity relating this invariant with an infinite tower of derivatives of itself :

$$\begin{aligned} R &= \left(\frac{R_{[;\beta_1} R^{;\beta_1}_{;\alpha_1]}}{R_{;\gamma_1} R^{;\gamma_1}} \right)^{;\alpha_1} \\ &= \left(\frac{ \left(\frac{(\dots)_{[;\beta_2} (\dots)^{;\beta_2}_{;\alpha_2]}}{(\dots)_{;\gamma_2} (\dots)^{;\gamma_2}} \right)^{;\alpha_2}_{[;\beta_1} \left(\frac{(\dots)_{[;\beta_3} (\dots)^{;\beta_3}_{;\alpha_3]}}{(\dots)_{;\gamma_3} (\dots)^{;\gamma_3}} \right)^{;\alpha_3 ;\beta_1}_{;\alpha_1]} }{ \left(\frac{(\dots)_{[;\beta_4} (\dots)^{;\beta_4}_{;\alpha_4]}}{(\dots)_{;\gamma_4} (\dots)^{;\gamma_4}} \right)^{;\alpha_4}_{;\gamma_1} \left(\frac{(\dots)_{[;\beta_5} (\dots)^{;\beta_5}_{;\alpha_5]}}{(\dots)_{;\gamma_5} (\dots)^{;\gamma_5}} \right)^{;\alpha_5 ;\gamma_1} } \right)^{;\alpha_1} \end{aligned} \tag{39}$$

where we have considered here that $X_{[\alpha\sigma]} := \frac{1}{2} \left(X_{\alpha\sigma} - X_{\sigma\alpha} \right)$ in order to avoid a repetition of factor two. This kind of curious identities are possible to write for all the Euler densities.

Finally, it is worth emphasizing that the equation (37) has already been found in a slightly different form in [256], where a normalized vector field n^{α} was used instead of a scalar field :

$$R = 2\nabla_{\alpha} \left(n^{[\alpha} \nabla_{\sigma} n^{\sigma]} \right) \tag{40}$$

Here n^{α} is chosen to be time-like ($n^{\alpha} n_{\alpha} = -1$), so that setting $n^{\alpha} = \frac{\nabla^{\alpha}\phi}{\sqrt{-\nabla_{\sigma}\phi \nabla^{\sigma}\phi}}$, with $\nabla_{\sigma}\phi\nabla^{\sigma}\phi < 0$ leads to Eq(37).

2. Critical Dimensions Lovelock scalars as total derivatives

For general even dimensions, the critical Lovelock-Lanczos scalars can be found in terms of the divergence of a vector from a generalization of Eq(37). In order to find the expression, one needs to use the following $\frac{d}{2} + 1$ identities which hold in dimension d :

$$\chi\, \Omega_{1,p} - (d - 2p)\Omega_{2,p} - 2p\, \Omega_{3,p} = 0 \tag{41}$$

where

$$\begin{aligned} \Omega_{1,p} &= \delta^{\mu_1 \dots \mu_d}_{\nu_1 \dots \nu_d} \prod_{n=1}^{d-2p-1} R^{\nu_n \nu_{n+1}}_{\mu_n \mu_{n+1}} \prod_{m=d-2p+1}^{d} \phi^{\nu_m}_{\mu_m} \\ \Omega_{2,p} &= \delta^{\mu_1 \dots \mu_d}_{\nu_1 \dots \nu_d} \chi^{\nu_2}_{\bullet} R^{\nu_1 \bullet}_{\mu_1 \mu_2} \prod_{n=3}^{d-2p-1} R^{\nu_n \nu_{n+1}}_{\mu_n \mu_{n+1}} \prod_{m=d-2p+1}^{d} \phi^{\nu_m}_{\mu_m} \\ \Omega_{3,p} &= \delta^{\mu_1 \dots \mu_d}_{\nu_1 \dots \nu_d} \chi^{\nu_{d-2p+1}}_{\bullet} \phi^{\bullet}_{\mu_{d-2p+1}} \prod_{n=1}^{d-2p-1} R^{\nu_n \nu_{n+1}}_{\mu_n \mu_{n+1}} \prod_{m=d-2p+2}^{d} \phi^{\nu_m}_{\mu_m} \end{aligned} \tag{42}$$

using the notations : $\phi_{\alpha} := \phi_{;\alpha} := \nabla_{\alpha}\phi$, $\phi^{\beta}_{\alpha} := \phi^{;\beta}_{;\alpha}$, $\chi^{\beta}_{\alpha} := \phi_{\alpha}\phi^{\beta}$, $\chi := \phi_{\alpha}\phi^{\alpha}$ and $R^{\sigma\rho}_{\alpha\beta} := R_{\alpha\beta}{}^{\sigma\rho}$. Moreover, we note the repeated of indices that are not contracted with the totally antisymmetric tensor as $X_{\bullet} Y^{\bullet} := X_{\alpha} Y^{\alpha}$. A detailed proof of these dimensional identities can be found in the first section of the Appendix. They are the generalization to general dimensions of the identities (3.4), (3.5) and (3.6) of [83], which refers to [263, 264], in which these identities are established.

Using these identities, it can be shown that the d-dimensional Lovelock-Lanczos scalar can be expressed as the divergence of a vector as :

$$\mathcal{L}_{(d)} = \varepsilon^{\mu_1 \rho_1 \dots \mu_n \rho_n} \varepsilon_{\nu_1 \sigma_1 \dots \nu_n \sigma_n}\, \nabla^{\sigma_n} \left(\frac{\phi^{;\nu_n}_{;\mu_n} \phi_{;\rho_n}}{\phi_{;\alpha}\phi^{;\alpha}} \sum_{p=0}^{n-1} \alpha_p \prod_{k=1}^{n-p-1} R^{\nu_k \sigma_k}_{\mu_k \rho_k} \prod_{m=n-p}^{n-1} \frac{\phi^{;\nu_m}_{;\mu_m} \phi^{;\sigma_m}_{;\rho_m}}{\phi_{;\alpha}\phi^{;\alpha}} \right) \tag{43}$$

where the series of coefficients α_p is found by solving the equation

$$\alpha_{(p-1)}\,(d-2p)-\alpha_p\left(p+\frac{1}{2}\right)=0 \tag{44}$$

whose solution is given by :

$$\alpha_{(p)}=\frac{2^{3p-n+2}\,p!\,n!}{(n-p-1)!(2p+1)!}=2^{p-n+1}\,n\,\sqrt{\pi}\binom{n-1}{p}\frac{\Gamma\left(p+1\right)}{\Gamma\left(p+\frac{3}{2}\right)} \tag{45}$$

and ε is the Levi-Civita tensor that can be introduced by using the following relation which depends on the signature of the manifold : $\varepsilon^{\mu_1\rho_1\ldots\mu_n\rho_n}\varepsilon_{\nu_1\sigma_1\ldots\nu_n\sigma_n}=(-1)^s\,\delta^{\mu_1\rho_1\ldots\mu_n\rho_n}_{\nu_1\sigma_1\ldots\nu_n\sigma_n}$. The integer s counts the number of minus signs in the signature of the metric. Here we have chosen the Lorentzian signature given by $s=1$. A detailed proof of this identity is given in the second section of the Appendix. While looking for additional references in the later stage of this work, we realized that this expression involving an arbitrary scalar field was already known previously, but in a different form. Indeed, it can be derived from a more general result found by Horndeski in [254] using instead an arbitrary non-null vector field u. From Eq(43), the Horndeski formula follows from the replacement $\phi_{;\alpha}\to u_\alpha$. This is quite natural since looking at the proof of the Appendix, we actually never use the symmetry of $\phi_{;\mu;\nu}$, so that we could have used such a vector field instead.

Similarly to the two dimensional case, this relation being true for any scalar field ϕ (with non vanishing kinetic term), it is in particular true when ϕ is a curvature scalar (with the same property), the simplest one being the Ricci scalar. Therefore, the previous identity can also be written solely in terms of the metric field and its derivatives as for example :

$$\mathcal{L}_{(d)}=\delta^{\mu_1\rho_1\ldots\mu_n\rho_n}_{\nu_1\sigma_1\ldots\nu_n\sigma_n}\,\nabla^{\sigma_n}\left(\frac{R^{;\nu_n}_{;\mu_n}R_{;\rho_n}}{R_{;\alpha}R^{;\alpha}}\sum_{p=0}^{n-1}\left(\frac{-\,2^{3p-n+2}\,p!\,n!}{(n-p-1)!(2p+1)!}\right)\prod_{k=1}^{n-p-1}R^{\nu_k\sigma_k}_{\mu_k\rho_k}\prod_{m=n-p}^{n-1}\frac{R^{;\nu_m}_{;\mu_m}R^{;\sigma_m}_{;\rho_m}}{R_{;\alpha}R^{;\alpha}}\right) \tag{46}$$

This answer the question raised in the introduction whether, like in the tetrad formalism, it is possible or not to write the critical Lovelock-Lanczos scalars as the total derivatives of some vectors depending only on the metric field and its derivatives. It is indeed possible and there is actually an infinite number of ways to do so, one for each curvature invariant built from the Riemann tensor, the metric field and the covariant derivative compatible with it[2]. That being said, these kinds of identities are well-defined only when the chosen curvature scalar $\phi=\mathscr{R}$ is such that $\mathscr{R}_{;\gamma}\mathscr{R}^{;\gamma}\neq 0$, what excludes for instance constant curvature spaces, as we said in the previous section.

Therefore, the covariance of the equation and its dependence on the metric field only are obtained at the price of excluding some metric fields, that is, by introducing some background dependence[3].

Furthermore, in addition to this difference, there will always be a discrepancy between the two formalisms because the previous expressions are possible to write in the tetrad formalism (see Eq(50) below), while expressing the Lovelock-Lanczos Lagrangian form as the exterior derivative of a form involving the spin connection and the curvature is not possible in the metric formalism due the lack of covariance of the Christoffel symbols (except at the price of introducing additional structures [262]). Moreover, it is in any case well-known that the adequate formalism for physics should be the tetrad one, given that this is the usual way to couple fermions to gravity.

Finally, note that in order to express the critical Lovelock-Lanczos scalars in terms of curvature only, one needs to use very high order curvature invariants. Indeed the highest order polynomial curvature invariant of the sum is given by the term $p=n-1$, which contains $(n-1)$ scalars of the form $\phi^{;\nu_n}_{;\mu_n}\phi_{;\rho_n}\phi^{;\nu_m}_{;\mu_m}\phi^{;\sigma_m}_{;\rho_m}$. Thus, if we choose the scalar field ϕ to be equal to an order $2q$

[2] Moreover, in addition to this ambiguity in the choice of the scalar field, one can always add to the vector inside the total derivative any other one v^γ with vanishing divergence $\nabla_\gamma v^\gamma=0$.

[3] However, one could say that this is a "mild" background dependence, because the class of metric g for which all the curvature invariants $\mathscr{R}$ satisfy $\mathscr{R}_{;\gamma}\mathscr{R}^{;\gamma}=0$ is quite reduced. Furthermore, it is always possible to construct a deformed metric g_l, depending on a parameter l, for which there exists a $\mathscr{R}$ satisfying $\mathscr{R}_{;\gamma}\mathscr{R}^{;\gamma}\neq 0$, and such that $\lim_{l\to 0}g_l=g$, so that one could use g_l to calculate the right-hand-side of the formula, and then take the limit. It could be interesting to classify the metric for which all the curvature invariants satisfy $\mathscr{R}_{;\gamma}\mathscr{R}^{;\gamma}=0$, but for which the Euler density is non-vanishing, in order to understand better this issue.

curvature invariant we can estimate the number of derivatives of the metric $\mathcal{O}$ of this invariant as $\mathcal{O} = (14q+1)(n-1)$. For example, considering the four dimensional ($n=2$) Gauss-Bonnet invariant and choosing the scalar field to be equal to the Ricci scalar ($q=1$), already gives $\mathcal{O} = 15$.

Indeed, in four dimensions we obtain :

$$\mathcal{L}_{(4)} = 2\, \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\sigma\rho\lambda\delta} \nabla^{\delta} \left(\frac{\phi^{;\lambda}_{;\alpha} \phi_{;\beta}}{\phi_{;\gamma}\phi^{;\gamma}} \left\{ R_{\mu\nu}{}^{\sigma\rho} + \frac{4}{3} \frac{\phi^{;\sigma}_{;\mu} \phi^{;\rho}_{;\nu}}{\phi_{;\gamma}\phi^{;\gamma}} \right\} \right) \tag{47}$$

In order to see explicitly what vector has a divergence equal to the Gauss-Bonnet scalar, we can expand the totally antisymmetric tensor using that

$$\delta^{\mu_1 \dots \mu_d}_{\nu_1 \dots \nu_d} = \sum_{k=1}^{d} (-1)^{d+k} \delta^{\mu_d}_{\nu_k} \delta^{\mu_1 \dots \mu_k \dots \mu_{d-1}}_{\nu_1 \dots \bar{\nu}_k \dots \nu_d} \tag{48}$$

where the bar in $\bar{\nu}$ means that this index is omitted. Defining $\mathcal{Y} = \phi^{;\alpha}_{;\alpha}$, $\mathcal{Y}_2 = \phi^{;\beta}_{;\alpha}\phi^{;\alpha}_{;\beta}$ and $\mathcal{Y}_3 = \phi^{;\beta}_{;\alpha}\phi^{;\alpha}_{;\gamma}\phi^{;\gamma}_{;\beta}$ for simplicity, it gives :

$$\begin{aligned} \delta^{\mu\nu\alpha\beta}_{\sigma\rho\lambda\delta} \phi^{;\lambda}_{;\alpha}\phi_{;\beta} R_{\mu\nu}{}^{\sigma\rho} &= 4 \left(R^{\sigma\rho}{}_{\alpha\delta} \phi^{;\alpha}_{;\sigma}\phi_{;\rho} + G^{\rho}_{\delta} \left(\phi^{;\alpha}_{;\rho}\phi_{;\alpha} - \mathcal{Y}\phi_{;\rho} \right) + R^{\alpha}_{\beta} \phi^{;\beta}_{[;\delta}\phi_{;\alpha]} \right) \\ \delta^{\mu\nu\alpha\beta}_{\sigma\rho\lambda\delta} \phi^{;\sigma}_{;\mu}\phi^{;\rho}_{;\nu}\phi^{;\lambda}_{;\alpha}\phi_{;\beta} &= \phi_{;\delta} \left(\mathcal{Y}^3 - 3\mathcal{Y}\mathcal{Y}_2 + 2\mathcal{Y}_3 \right) + 3\phi_{;\alpha} \left(\left[\mathcal{Y}_2 - \mathcal{Y}^2 \right] \phi^{;\alpha}_{;\delta} + 2\phi^{;\alpha}_{;\rho} \left[\mathcal{Y}\phi^{;\rho}_{;\delta} - \phi^{;\gamma}_{;\delta}\phi^{;\rho}_{;\gamma} \right] \right) \end{aligned} \tag{49}$$

Note that although the result Eq(47) was not explicitly written in components, it can be found in [88] from Eq(98) and Eq(109), setting $G_5(\phi, X) = 1/X$, $E_6 = \frac{3}{2}$, with $X := -\frac{1}{2}\phi_{;\gamma}\phi^{;\gamma}$. Also in this case, these results were already found by Horndeski in [254], although he was using a vector field instead of a scalar, and the metric formalism instead of differential forms. Using the notations of [88], we can rewrite the general result Eq(43) in terms of differential forms as :

$$L_{(d)} = \mathcal{D} \left(\epsilon_{a_1 b_1 \dots a_n b_n} \frac{\nabla^{b_n}\phi}{\phi_{;\gamma}\phi^{;\gamma}} \sum_{p=0}^{n-1} \gamma_p \bigwedge_{k=1}^{n-p-1} \mathcal{R}^{a_k b_k} \bigwedge_{m=n-p}^{n-1} \frac{\Phi^{a_m} \wedge \Phi^{b_m}}{\phi_{;\gamma}\phi^{;\gamma}} \wedge \Phi^{a_n} \right) \tag{50}$$

where $\mathcal{D}$ is the exterior covariant derivative associated with the spin connection ω so that $\mathcal{D}\omega^a{}_b = \mathcal{R}^a{}_b$, $\Phi^a := \nabla^a \nabla_b \phi\ \theta^b$ and θ^a is a tetrad defined by $\theta^a := e^a_\mu dx^\mu$, with $\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b$. The series of coefficients γ_p can be found from the facts that $d^{2n}x\, \sqrt{-g}\, \mathcal{L}_n = L_n$, and $\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd} \theta^c \wedge \theta^d$ as $\gamma_p = \frac{4^{p+1}\, p!\, n!}{(n-p-1)!(2p+1)!}$.

As a last comment, note that the arbitrariness in the choice of ϕ makes seemingly unrelated theories, with different field contents, equivalent. For example if we define the vector whose divergence gives the Gauss-Bonnet scalar as :

$$\Omega^{(\phi)}_{\delta} = 2\, \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\sigma\rho\lambda\delta} \frac{\phi^{;\lambda}_{;\alpha}\phi_{;\beta}}{\phi_{;\gamma}\phi^{;\gamma}} \left\{ R_{\mu\nu}{}^{\sigma\rho} + \frac{4}{3} \frac{\phi^{;\sigma}_{;\mu}\phi^{;\rho}_{;\nu}}{\phi_{;\gamma}\phi^{;\gamma}} \right\} \tag{51}$$

Then, using an integration by part and choosing the arbitrary scalar ϕ to be respectively, the same scalar field as the one already present in the theory, another different scalar field ψ or even a curvature invariant like the Ricci scalar, the following examples of theories are equivalent :

$$\begin{aligned} I &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R + \phi \left(R^2 - 4R^{\nu}_{\mu}R^{\mu}_{\nu} + R_{\mu\nu}{}^{\sigma\rho}R^{\mu\nu}{}_{\sigma\rho} \right) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \right) \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R - \Omega^{(\phi)}_{\delta}\nabla^{\delta}\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \right) \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R - \Omega^{(\psi)}_{\delta}\nabla^{\delta}\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \right) \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R - \Omega^{(R)}_{\delta}\nabla^{\delta}\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \right) \\ &= \dots \end{aligned} \tag{52}$$

It would be interesting to study in detail how exactly the field equations of these seemingly different kind of theories end up describing the same physics, and also to see if some minimal modifications of each of these could break this equivalence. For example, introducing a constraint to remove the non-polynomiality of Ω by the replacement : $\frac{X}{\phi_{;\gamma}\phi^{;\gamma}} \to \lambda X + \sigma\left(\phi_{;\gamma}\phi^{;\gamma}\lambda - 1\right)$, where σ and λ are Lagrange multiplier fields, might end up producing different theories for different choice of the scalar field ϕ. Of course, this argument also concerns higher dimensional Euler densities, like for example the sixth dimensional one, given by :

$$\mathcal{L}_{(6)} = 2\,\varepsilon^{\mu\rho\alpha\beta\zeta\delta}\varepsilon_{\nu\sigma\iota\theta\kappa\tau}\nabla^{\tau}\left(\frac{\phi^{;\kappa}_{;\zeta}\phi_{;\delta}}{\phi_{;\gamma}\phi^{;\gamma}}\left\{\frac{3}{4}R^{\nu\sigma}_{\mu\rho}R^{\iota\theta}_{\alpha\beta} + 2\frac{R^{\nu\sigma}_{\mu\rho}\phi^{;\iota}_{;\alpha}\phi^{;\theta}_{;\beta}}{\phi_{;\gamma}\phi^{;\gamma}} + \frac{8}{5}\frac{\phi^{;\nu}_{;\mu}\phi^{;\sigma}_{;\rho}\phi^{;\iota}_{;\alpha}\phi^{;\theta}_{;\beta}}{\left(\phi_{;\gamma}\phi^{;\gamma}\right)^2}\right\}\right) \tag{53}$$

3. Non-Polynomial Gravity property of the Gauss-Bonnet Vector

Now we wish to focus on the non-polynomial vector Ω, whose divergence equal the Gauss-Bonnet scalar $\nabla^{\delta}\Omega_{\delta} = R^2 - 4R^{\nu}_{\mu}R^{\mu}_{\nu} + R_{\mu\nu}{}^{\sigma\rho}R^{\mu\nu}{}_{\sigma\rho}$.

In order to do so, consider a coordinate system x^{μ}, where $\mu = w_1, w_2, w_3, w$. Then, suppose that the scalar field ϕ only depends on the coordinate w. If the scalar field is chosen to be a curvature scalar or an additional degree of freedom of a gravitational theory, this will be the case for example in homogeneous spacetimes or in static spherical symmetry[4]. In all these cases, the non-polynomial vector collapses to :

$$\Omega_{\delta} = -2\,\varepsilon^{\mu\nu\alpha w}\varepsilon^{\sigma\rho\lambda}{}_{\delta}\,\frac{\Gamma_{\alpha\lambda}{}^{w}}{g^{ww}}\left\{R_{\mu\nu\sigma\rho} + \frac{4}{3}\frac{\Gamma_{\mu\sigma}{}^{w}\Gamma_{\nu\rho}{}^{w}}{g^{ww}}\right\} \tag{54}$$

Therefore, at least in these classes of spacetimes, the non-polynomiality and the dependence of this vector on the scalar field disappear, and it simply becomes a third order curvature vector. This kind of tensorial quantities whose non-polynomiality depends on the backgrounds under considerations have been studied for example in [151–156], where in the first paper by S. Deser, O. Sarioglu and B. Tekin, the gravitational theories of this kind were named "non-polynomial gravities". This will be the topic of the next Chapter.

These properties allow to construct gravitational theories admitting second order field equations in specific backgrounds like FLRW and spherically symmetric spacetimes. In this sense, it is very similar to the Quasi-Topological Gravities, that achieve the same goal but without non-polynomialities. As we saw before, all these models "circumvent" the Lovelock theorem in the following way : instead of demanding that the curvature invariant under consideration gives second order field equations for any metric field (what singles out uniquely the Lovelock-Lanczos scalars), it is allowed to have this property only for some subset of metric fields (for example in spherical symmetry). It is not known how large this subset can be. For example, if it is possible to construct new curvature invariants leading to second order field equations for a general axisymmetric spacetime ansatz.

As it turns out, in addition of being polynomial in these sectors, the following action leads to second order field equations (at least) in static spherically symmetric and Friedmann-Lemaître-Robertson-Walker spacetimes :

$$I = \frac{l^{4p}}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left(\Omega_{\delta}\,\Omega^{\delta}\right)^p \tag{55}$$

where l is a length scale introduced for dimensional reasons, and p is a positive integer. In the following, we set $l^{4p} = 16\pi G$ for simplicity. In these sectors, the minisuperspace field equations are equivalent to the symmetric reduction of the full field equations due to Palais's "Principle of Symmetric Criticality" [238, 239]. In static spherically symmetric spacetimes and their topological generalizations described by the metric :

$$ds^2 = -a(r)b(r)^2dt^2 + \frac{dr^2}{a(r)} + r^2\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2\phi^2\right) \tag{56}$$

[4]Otherwise, if the scalar field ϕ is not considered to be related to the degrees of freedom of the theory, it does not even need to follow the symmetries of the spacetimes in order to have $\nabla^{\delta}\Omega_{\delta} = \mathcal{L}_{(4)}$. This is because this equation follows from purely algebraic considerations.

where $k = 1, -1, 0$ correspond respectively to spherical, hyperbolic and planar symmetry, we have $\Omega_\delta\,\Omega^\delta = \frac{4(k-a)^2\left(ba'+2ab'\right)^2}{r^4ab^2}$. The minisuperspace field equations can then be written in the following way :

$$\begin{aligned}
\frac{\delta I}{\delta b} &= -\frac{4}{r^3ab^2}\,(k-a)\,\left(\Omega_\delta\,\Omega^\delta\right)^{p-1}\left(2p\mathcal{U} + r\left(k(2p-1)+(1-6p)a\right)\left(ba'+2ab'\right)^2\right) = 0 \\
\frac{\delta I}{\delta a} &= -\frac{4\,p}{r^3a^2b}\,(k-a)\,\left(\Omega_\delta\,\Omega^\delta\right)^{p-1}\,\mathcal{U} = 0
\end{aligned} \tag{57}$$

where $\mathcal{U} = (2p-1)\Omega'\,r^3a^{\frac{3}{2}}b^2 + 4\left(ba'+2ab'\right)ra^2b'$ and $\Omega = 2\,(k-a)\left(ba'+2ab'\right)/\left(r^2\sqrt{a}b\right)$.

There are three classes of solutions, two of which do not depend on p and are degenerate vacua, in the sense that they do not fix entirely the metric field (up to diffeomorphism). They are given by $a = k$, b arbitrary and $a = \frac{1}{b^2}$, b arbitrary, and can be made regular. For example choosing $b = (k - L\,r^2/(r^3+v))^{-1/2}$ in the first case, where v and L have dimensions of a volume and a length, or $b = 1 + r^2/L^2$ in the second. The third solution is given by $a = \frac{2p-1}{6p-1}k$, $b = \left(r^3+\alpha\right)^{2p}$, where α is an integration constant, and is singular at the origin. It is interesting to note that the degenerate solutions share some similarities with well-known solutions of Lovelock-Lanczos gravities, that are also degenerate vacua, see [166, 265, 266].

This parallel between Lovelock-Lanczos gravities and this action is also present in Friedmann-Lemaître-Robertson-Walker spacetimes given by the interval $ds^2 = -N(t)dt^2 + a(t)^2\left(dx^2+dy^2+dz^2\right)$. Indeed, in this case we have $\Omega_\delta\,\Omega^\delta = -16\frac{H^6}{N^3}$, where H is the Hubble parameter, and therefore

$$\begin{aligned}
\left.\frac{\delta I}{\delta N}\right|_{N=1} &= a^3(-1)^{p+1}2^{4p-1}\,(6p-1)\,H^{6p} = 0 \\
\left.\frac{\delta I}{\delta a}\right|_{N=1} &= -3(-1)^p16^p(6p-1)a^2H^{2(3p-1)}\left(H^2+2p\dot{H}\right) = 0
\end{aligned} \tag{58}$$

These are the same Friedmann equations that would give the Lovelock-Lanczos scalars of order $6p$. However, recall that in the present case we are working in four dimensions, so that these scalars are identically vanishing. It means that, at least in this sector, the model (55) can be used to mimic the effect of these invariants in four dimensions.

4. Discussion

We have presented in this section a general formula to express the critical order Lovelock-Lanczos scalars as the covariant divergence of a vector depending on the Riemann tensor and on an arbitrary scalar field Eq(43) in the metric formalism. Moreover, choosing the scalar field to be a curvature invariant makes the formula depending only on the metric and its derivatives in a covariant way. This provides a proof of a very common claim that in critical dimensions, the Lovelock-Lanczos invariants are total derivatives. However, this is at the price of involving a very high number of derivatives of the gravitational field, and also to introduce some implicit background dependence in the formula. This is because the kinetic term associated with the chosen curvature invariant has to be non vanishing, what is not possible to find for some classes of metric fields. When the scalar field is chosen to be the Euler density itself, quite baroque identities can be found, relating the Euler densities with infinite towers of derivatives of themselves, like Eq(39).

As we saw in the four dimensional case, Eq(43) enables to find some equivalences between seemingly different gravitational theories with different field contents. It would be interesting to understand better how this equivalence at the level of the action would echo at the level of the field equations. Moreover, finding the conformal transformation properties of the vectors whose divergences give the critical Lovelock-Lanczos scalars, would increase even more the number of models that can be related in this way. This might lead to some interesting phenomenological consequences when these models are slightly modified, breaking the equivalence. In four dimensions, it would amount to find minimal modifications to Einstein-Gauss-Bonnet types of gravities.

Finally, we studied the four dimensional vector whose divergence gives the Gauss-Bonnet scalar. It turns out that the curvature invariants built from this vector belong to the so-called Non-Polynomial Gravity models, as they are polynomial and lead to second order field equations in specific backgrounds. This property is similar to the so-called Quasi-Topological Gravities, and also to Mimetic gravities, in which, due to the presence of a constraint, the mimetic scalar field can be used to reach, on-shell, new second order curvature invariants, see for example [108, 109]. In FLRW and static spherically symmetric spacetimes, the model shares some similarities with Lovelock-Lanczos theories.

II. Non-polynomial gravity

In the previous Chapter, we saw a first instance of higher order (metric) gravitational theory leading to second order equations of motion for specific classes of spacetimes : FLRW and static spherical symmetry. Alongside this property, the considered theory was a Non-Polynomial (NP) combination of curvature invariants, which yet becomes polynomial in the curvature for these classes of spacetimes. Our aim now will be to find and study Non-Polynomial gravitational (NPG) theories whose whole dynamical spherically symmetric sectors lead to polynomial and second order field equations, so that these theories reduces in this sector to combinations of the 2D curvature scalars introduced in the first section of Chapter 1. In the introduction Sec.1 we present a brief overview of the Deser-Sarioglu-Tekin action, which was the first Non-Polynomial gravity theory discovered, and is based on a specific algebraic property of the Weyl tensor in DSS.

In Sec.2 we present another algebraic relation based on the DSS decomposition of the Cotton tensor, what enables us to lift all the 2D curvature scalars seen in the previous Chapter to higher dimensions. In the sense that from this property of the Cotton tensor, it is possible to construct gravitational theories whose DSS decomposition is given by these scalars. This result provides a new d-dimensional effective approach to (re)construct or find regular solutions from gravitational actions depending on the curvature only, what can be seen as an advantage compared to the usual fluid [100,119,120] and Non-Linear Electrodynamics [90–97] approaches, which are usually used to generate regular black holes and cosmologies. Of course, 2D effective theories have been extensively used to model and investigate the properties of black holes, see for example [41–43,157–165,170, 171,234–237,267,268], but up to now, no d-dimensional formulations of these theories in terms of gravitational corrections to the Einstein-Hilbert action was available, to our knowledge.

The Sec.3 is the starting point of the results of the Chapters 3 and 5. We present a way to find high energy corrections to the DSS sectors of both Gauss-Bonnet and Einstein gravities. The procedure essentially relies on the preservation of the DSS decomposition of these theories for all higher orders of corrections, and as we will see in the mentionned Chapters, it leads to a large variety of regular solutions. Due to the results of Sec.2, we can construct 4-dimensional gravitational theories admitting these DSS reduction and solutions. Finally, note that in the third section of the Appendix, we study some properties of NPG models, in particular the Wald entropy and some interpretations of these models.

Note that other use of non-polynomial curvature terms have been found in other contexts. For example in [269], NP scalars were used to locate the horizon of black holes, see also [270, 271] for polynomial curvature invariants characterizing the horizon locations ; while in [272], a modified Plebanski action with a non-polynomial additional term was considered, for a gauge group embedding the Euclidean Lorentz one, $SO(4)$ (see Eq(9) and (26) of this paper).

A. Introduction

1. Deser-Sarioglu-Tekin Action

It has been noticed some time ago that it is possible, in four dimensions, to construct purely gravitational actions (without additional fields) that lead to polynomial and first order spherically symmetric field equations [151,156], and eventually to black hole solutions [151]. The first model of this kind, presented in [151] by Deser, Sarioglu and Tekin, was to consider the $d = n + 2$ dimensional action :

$$I = \int_{\mathcal{M}} d^{n+2}x\sqrt{-g}\left(R + \sigma\sqrt{\frac{n+1}{n-1}}\,\mathcal{N}^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu}{}^{\alpha\beta}\right), \tag{59}$$

where $\mathcal{N}$ is a non-polynomial curvature tensor defined as

$$\mathcal{N}^{\mu\nu}{}_{\alpha\beta} := \frac{W^{\mu\nu}{}_{\alpha\beta}}{\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}}\,, \tag{60}$$

in term of the Weyl tensor W, defined in components by $W^{\mu\nu}{}_{\alpha\beta} := R^{\mu\nu}{}_{\alpha\beta} - 2\delta^{[\mu}_{[\alpha}A^{\nu]}_{\beta]}$, where A is the Schouten tensor, $A_{\mu\nu} := \frac{1}{n}\left(R_{\mu\nu} - \frac{R}{2(n+1)}\,g_{\mu\nu}\right)$ [5]. Note that because $R^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta} = W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta}$,

[5]For any a, b we use the convention $a^{[\mu}_{[\alpha}b^{\nu]}_{\beta]} = \frac{1}{2}\left(a^{\mu}_{\alpha}b^{\nu}_{\beta} - a^{\nu}_{\alpha}b^{\mu}_{\beta} - a^{\mu}_{\beta}b^{\nu}_{\alpha} + a^{\nu}_{\beta}b^{\mu}_{\alpha}\right)$.

the second term in the action becomes simply :

$$\mathcal{N}^{\mu\nu}_{\ \ \alpha\beta}R_{\mu\nu}^{\ \ \alpha\beta} = \sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}} \tag{61}$$

which is the form used in [151]. In general, one has to be careful that the square of the Weyl tensor be positive, what is not always the case [273, 274], but one could as well consider an absolute value. This action leads to first order field equations in spherical symmetry because the tensor $\mathcal{N}$ is independant of the metric in this class of spacetimes, as it decomposes as follows :

$$\mathcal{N}^{\mu\nu}_{\ \ \alpha\beta}\Big\rfloor = -\sqrt{\frac{n-1}{n+1}}\left(\tau^{[\mu}_{[\alpha}\rho^{\nu]}_{\beta]} - \frac{1}{n}\omega^{[\mu}_{[\alpha}\sigma^{\nu]}_{\beta]} + \frac{1}{n(n-1)}\sigma^{[\mu}_{[\alpha}\sigma^{\nu]}_{\beta]}\right) , \tag{62}$$

where following [151], we define the projectors :

$$\begin{aligned} &\tau^{\nu}_{\mu} = \delta^t_{\mu}\delta^{\nu}_t = \mathrm{diag}(1,0,0,...,0) \,, \quad \rho^{\nu}_{\mu} = \delta^r_{\mu}\delta^{\nu}_r = \mathrm{diag}(0,1,0,...,0) \,, \\ &\omega^{\nu}_{\mu} = \tau^{\nu}_{\mu} + \rho^{\nu}_{\mu} \,, \quad \sigma^{\nu}_{\mu} = \sum_{i=1}^{n}\delta^i_{\mu}\delta^{\nu}_i = \mathrm{diag}(0,0,1,...,1) \,. \end{aligned} \tag{63}$$

This property of $\mathcal{N}$ is equivalent to the fact that in spherically symmetric spacetimes, all the $\frac{(n-1)(n+2)(n+3)(n+4)}{12}$ independent components of the Weyl tensor collapse to a single one. Therefore, taking its square-root in (59) leads to a linear second order scalar in spherical symmetry. Indeed, from Eq(473) we get :

$$W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}\Big\rfloor = \frac{4(n-1)}{n+1}\left(\frac{1}{2}R^{(2)} + \frac{\mathcal{D}^2 r}{r} + \frac{R^{(n)}}{n(n-1)} - \frac{\mathcal{D}_a r\mathcal{D}^a r}{r^2}\right)^2 \tag{64}$$

And thus,

$$\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}\Big\rfloor = -2\sqrt{\frac{n-1}{n+1}}\left(\frac{1}{2}R^{(2)} + \frac{\mathcal{D}^2 r}{r} + \frac{R^{(n)}}{n(n-1)} - \frac{\mathcal{D}_a r\mathcal{D}^a r}{r^2}\right) \tag{65}$$

This theory was shown to have (singular) black hole solutions, given by

$$d\Sigma^2 = -r^{\frac{2(n+1)\sigma}{\sigma-1}}\left(\frac{k\,(2\sigma - n(n-1))}{n(n(2\sigma-1)+1)} + \mu r^{\frac{n(2\sigma-1)+1}{1-\sigma}}\right)dt^2 + \frac{dr^2}{\frac{k(2\sigma-n(n-1))}{n(n(2\sigma-1)+1)} + \mu r^{\frac{n(2\sigma-1)+1}{1-\sigma}}} \tag{66}$$

where μ is an integration constant. These solutions were further studied in [275–277]. However, there was a concern in [151] regarding the equivalence between the symmetry reduced field equations and the minisuperspace field equations when such a non-polynomial curvature scalar is considered. Given the complexity of the NPG actions that we will consider in this thesis, it is important to see once that indeed reducing the full field equations to DSS and deriving the minisuperspace field equations from the DSS reduction of the action are equivalent procedures. To address this issue, and to confirm Palais's Principle of Symmetric Criticality, [238, 239], note that the full field equations are :

$$G_{\mu\nu} + \sigma\sqrt{\frac{n+1}{n-1}}\left(-\frac{1}{2}g_{\mu\nu}\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}} + W_{\mu}^{\ \alpha\beta\gamma}\mathcal{N}_{\nu\alpha\beta\gamma} + 2\left(\frac{R_{\alpha\beta}}{n} + \nabla_\alpha\nabla_\beta\right)\mathcal{N}_{\mu\ \nu}^{\ \alpha\ \beta}\right) = 0 \tag{67}$$

In four dimensions, due to the relation

$$W_{\mu}^{\ \alpha\beta\gamma}W_{\nu\alpha\beta\gamma} = \frac{1}{4}g_{\mu\nu}W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma} \tag{68}$$

the field equations further simplify to :

$$G_{\mu\nu} + \sigma\sqrt{3}\left(-\frac{1}{4}g_{\mu\nu}\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}} + \left(R_{\alpha\beta} + 2\nabla_\alpha\nabla_\beta\right)\mathcal{N}_{\mu\ \nu}^{\ \alpha\ \beta}\right) = 0 \tag{69}$$

On the other hand, at the level of the symmetric reduction of the action, one can see that the Lagrangian $L := \sqrt{-g}\left(R + \sigma\sqrt{\frac{n+1}{n-1}}\mathcal{N}^{\mu\nu}_{\ \ \alpha\beta}R_{\mu\nu}^{\ \ \alpha\beta}\right)$ becomes

$$L\Big\rfloor = \sqrt{-\gamma}r^n\left((n(n-1) - 2\sigma)\frac{k - \mathcal{D}_a r\mathcal{D}^a r}{r^2} + (1-\sigma)R^{(2)} - 2(n+\sigma)\frac{\mathcal{D}^2 r}{r}\right) \tag{70}$$

where we have neglected the angular part for simplicity. This is nothing but a special case of two dimensional Horndeski theory, so that using Eq(15) with $\eta(r,\tilde{\chi}) = (n(n-1)-2\sigma)r^{n-2}(k-\tilde{\chi})$, $G_3(r) = 2(n+\sigma)r^{n-1}$ and $G_4(r) = (1-\sigma)r^n$, we get the following 2D field equations :

$$\begin{aligned}
&n(\sigma-1)\frac{\mathcal{D}_a\mathcal{D}_b r}{r} + n\sigma(n+1)\frac{\mathcal{D}_a r\mathcal{D}_b r}{r^2} + \frac{\gamma_{ab}}{2}\left(\frac{\alpha k}{r^2} - 2n(\sigma-1)\frac{\mathcal{D}^2 r}{r} + \beta\frac{\mathcal{D}_c r\mathcal{D}^c r}{r^2}\right) = 0 \\
&n(1-\sigma)R^{(2)} - 2n(n-1+2\sigma)\frac{\mathcal{D}^2 r}{r} - (n-2)\left(\frac{\alpha k}{r^2} + n(n-1+2\sigma)\frac{\mathcal{D}_c r\mathcal{D}^c r}{r^2}\right) = 0
\end{aligned} \tag{71}$$

where $\alpha := (2\sigma - n(n-1))$ and $\beta := n(n-1-2n\sigma)$. Then a simple way to confirm that the DSS reduction of the full field equations (67) does give the 2D ones (71) is to expand on a computer both equations for a general gauge of the interval Eq(5). For example, in four dimensions we can choose,

$$ds^2 = -a(t,x)b(t,x)^2dt^2 + \frac{dx^2}{a(t,x)} + r\,(t,x)^2\left(\frac{d\theta^2}{1-k\theta^2} + \theta^2 d\phi^2\right) \tag{72}$$

and see that both sets of equations are indeed equivalent for any DSS spacetimes.

As a general comment on Non-Polynomial gravity theories, note that the actions can also be written in a polynomial way by introducing Lagrange multiplier fields in the action, so that the non-polynomiality can be reabsorbed into a curvature constraint. In the present case, we can for example consider the alternative action :

$$I = \int_{\mathcal{M}} d^{n+2}x\sqrt{-g}\left(R + \sigma\sqrt{\frac{n+1}{n-1}}\left(\phi + \lambda\left(\phi^2 - W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}\right)\right)\right). \tag{73}$$

where ϕ and λ are Lagrange multipliers. This kind of polynomial constraints leading to non-polynomial actions are quite similar to what was found in [272], where a modified Plebanski action with a non-polynomial additional term (or equivalently a polynomial constraints leading to it) was considered, for a gauge group embedding the Euclidean Lorentz one, $SO(4)$ (see Eq(9) and (26) of this paper).

Finally, it is worth mentioning that this theory is not an high energy modification of General Relativity, in the sense that it does not contain an additional (small length) scale at which the correction would not be negligible. Another way of seeing this would be to observe that in DSS spacetimes, the NPG curvature invariant $\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}$ contains only linear powers of second derivatives of the metric field, just like General Relativity. Therefore, in the following section, we will use this NPG scalar to construct high energy modification to GR at quadratic order.

2. Generalized Quadratic Gravity & Gravitational Pais-Uhlenbeck oscillator

What we will be interested in the this thesis are gravitational (metric) high energy corrections to General Relativity, written as power series in a small length scale parameter. In this spirit, and following the results of the previous section, it is quite natural to investigate what effect the scalar $\mathcal{W} := \sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}$ would have on the most general quadratic (metric) gravity theory, that can be parametrized by the scalars : $(R^2, W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta}, R^2 - 4R_{\mu\nu}R^{\mu\nu} + R^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu}{}^{\alpha\beta})$. As it is well-known, the Gauss-Bonnet scalar does not contribute to the gravitational field equations, so we can neglect it from the beginning. Then we are naturally led to consider the following generalized quadratic gravity :

$$I = \int_{\mathcal{M}} d^{n+2}x\sqrt{-g}\left(\alpha R^2 + \beta W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta} + \gamma\sqrt{\frac{n+1}{n-1}}\mathcal{W}R\right) \tag{74}$$

In four dimensions only, it is possible to get second order field equations in spherically symmetric spacetimes from this action. In order to do so, we need to have $\alpha = -\frac{1}{2}$, $\beta = -\frac{3}{2}$ and $\gamma = 1$. Choosing the following gauge, $ds^2 = -a(r)b(r)^2dt^2 + \frac{dr^2}{a(r)} + r^2\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\phi^2\right)$, the lagrangian becomes :

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}R^2 - \frac{3}{2}W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta} + \sqrt{3}\mathcal{W}R\right) = -\frac{18\left((ab)'\right)^2}{b} \tag{75}$$

The minisuperspace field equations are : $\partial\mathcal{L}/\partial b = -\frac{\mathcal{L}}{b} + 36\frac{a}{b^2}\left(b(ab)'' - b'(ab)'\right) = 0$ and $\partial\mathcal{L}/\partial a = \frac{36}{b}\left(b(ab)'' - b'(ab)'\right) = 0$ so that combining them gives $\mathcal{L} = 0$ which is solved by $a = \text{cst}/b$. The constant can be fixed to unity by a redefinition of the time coordinate. This relation solves the remaining equation, so that any metric

$$ds^2 = b(r)\left(-dt^2 + dr^2\right) + r^2\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\phi^2\right) \tag{76}$$

is a solution of this theory, whatever k and $b(r)$. This kind of degenerate solutions are quite reminiscent of the solutions found in the last section of the previous Chapter, and it is interesting to note that they share some similarities with well-known solutions of Lovelock-Lanczos gravities, that are also degenerate vacua, see [166, 265, 266].

It is therefore possible to have regular solutions at the origin providing that, for spherical and hyperbolic topologies : $b(0) = 1/k$ and $b'(0) = 0$, meaning that $b(r \to 0) = k + b''(0)r^2$, or for planar topology, $b(r \to 0) = \mu/r^2$; what can be checked by computing the curvature scalars $R, R^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu}{}^{\alpha\beta}, R^{\mu\nu}R_{\mu\nu}$. For example, for $k = 0$, if $b(r) = \mu/r^2 + \nu/r^3$ the Ricci scalar and the Kretschmann invariant are :

$$R = \frac{r^3\mu^2}{(r\mu+\nu)^3} - \frac{5r}{r\mu+\nu} \quad , \quad R^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu}{}^{\alpha\beta} = \frac{r^2\left(24r^4\mu^4 + 120r^3\mu^3\nu + 220r^2\mu^2\nu^2 + 172r\mu\nu^3 + 49\nu^4\right)}{(r\mu+\nu)^6} \tag{77}$$

while for $k \neq 0$, if $b(r) = k + \mu r^2$, then

$$R = \frac{2r^2\mu^2\left(3+r^2\mu\right)}{\left(1+r^2\mu\right)^3} \quad , \quad R^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu}{}^{\alpha\beta} = \frac{4\mu^2\left(6 + r^2\mu\left(10 + r^2\mu\left(11 + r^2\mu\left(4 + r^2\mu\right)\right)\right)\right)}{\left(1+r^2\mu\right)^6} \tag{78}$$

Of course, to investigate if these kind of corrections can lead to regular *black hole* solutions, one would need to consider an action $I_{EH} + l^2 I$, where l is a coupling constant with dimension of a length, but it is nonetheless interesting to see that considering solely the quadratic theory, regular spherically symmetric spacetimes are indeed solutions. However, it seems difficult to find analytic black hole solutions when the GR term is considered.

Now, in order to understand the reason for the existence of such a strange four-dimensional theory, we can use the covariant formalism and check that :

$$\begin{aligned}\alpha R^2 + \beta W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta} + \gamma\sqrt{\frac{n+1}{n-1}}\mathcal{W}R &= \left(\alpha + \beta - \frac{2\beta}{n+1} + \gamma\right)R^2_{(2)} - 4\left(n(\gamma - n\alpha) - \frac{(n-1)\beta}{n+1}\right)\mathcal{Y}^2 \\ &+ (n-1)\left(\frac{4\beta}{n+1} + n((n-1)n\alpha + 2\gamma)\right)\mathcal{Z}^2 + \left(\frac{(8(n-1))\beta}{n+1} + 2n((n-3)\gamma - 2n(n-1)\alpha)\right)\mathcal{Y}\mathcal{Z} \\ &+ \left(2\left((n-1)\left(\frac{2\beta}{n+1} - \gamma\right) - 2n\alpha\right)\mathcal{Y} + \left(2(n-1)n\alpha + \frac{4(n-1)\beta}{n+1} + 2\gamma + (n-1)n\gamma\right)\mathcal{Z}\right)R_{(2)}\end{aligned} \tag{79}$$

In four dimensions it reduces to :

$$\alpha R^2 + \beta W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta} + \gamma\sqrt{\frac{n+1}{n-1}}\mathcal{W}R = -18\gamma\mathcal{Y}^2 + \frac{2}{3}\lambda\mathcal{Y}\left(-R_{(2)} + \mathcal{Y} - 2\mathcal{Z}\right) + \frac{1}{3}\sigma\left(8\mathcal{Y}^2 + \left(R^2_{(2)} + 2\mathcal{Z}\right)^2\right) \tag{80}$$

where $\lambda = 3(\alpha+\gamma) + \beta$ and $\sigma = 12\alpha - 2\beta + 3\gamma$. We see that for $\lambda = \sigma = 0$, the Ricci scalar of the two-dimensional manifold Σ disappears completely. It corresponds precisely to the present model, which is why the metric solutions are underdetermined. Moreover, expending $\mathcal{Y}$ we can see that :

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}R^2 - \frac{3}{2}W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta} + \sqrt{3}\,\mathcal{W}R\right) = -18\sqrt{-\gamma}\left(\Box r\right)^2 \tag{81}$$

So that this model is a kind of gravitational Pais-Uhlenbeck oscillator, see for example [278–282]. Therefore, it is not strictly speaking a second order theory in a general DSS spacetimes, but still reduces to one. Indeed, in this formalism, the field equations associated with the two-dimensional metric γ_{ab} are given by :

$$\begin{aligned}-\frac{\delta_\gamma\mathcal{L}}{18\sqrt{-\gamma}} &= \left(\gamma_{ab}\left(\frac{1}{2}\left(\Box r\right)^2 + \partial^e r\nabla_e\Box r\right) - 2\partial_{(a}r\nabla_{b)}\Box r\right)\delta\gamma^{ab} \\ &- \nabla_e\left(\gamma_{ab}\Box r\partial^e r\delta\gamma^{ab}\right) + 2\nabla_a\left(\Box r\partial_b r\delta\gamma^{ab}\right)\end{aligned} \tag{82}$$

while performing the variation of the Lagrangian with respect to the scalar field r gives :

$$-\frac{\delta_r \mathcal{L}}{18\sqrt{-\gamma}} = 2\Box^2 r\delta r + 2\nabla_a \left(\Box r \nabla^a \delta r - \nabla^a \Box r \delta r\right) \tag{83}$$

Therefore, the field equations reduce to $\Box r = 0$, which is just one second order scalar equation for the two unspecified functions of a gauge fixed spherically symmetric spacetime.

Despite the possible interest in this model and the DST action, they are not able to give regular black hole solutions, or even analytic black hole for the former (when coupled to GR). We considered these theories simply to familiarize the reader with NPG theories, and investigate some of the properties they might have. However, for our purpose, we will now see a much more efficient way to find NPG theories leading to second order field equations in DSS spacetimes, and eventually to regular solutions.

B. Algebraic Relations & Unified Approach for Effective Two Dimensional Theories

Up to now, by using the DSS decomposition of the Weyl tensor, we were able to consider the NPG gravitational scalar $\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}$ and study some models in which it appears. Alternatively, one could also use the Non-Polynomial curvature tensor $\mathcal{N}$ defined by Eq(60), and build from it many curvature invariants which would be, by construction (due to Eq(62)), polynomial and second order in DSS spacetimes, for example $\nabla_\alpha \mathcal{N}_{\mu\nu}{}^{\lambda\delta}\nabla^\alpha \mathcal{N}^{\mu\nu}{}_{\lambda\delta}$. This kind of constructions is the topic of this section.

1. Algebraic relations

First, recall that the property Eq(62) of the Weyl tensor follows from the DSS decomposition of this tensor :

$$\begin{aligned}
W^{ab}{}_{ce} &= \frac{d-3}{d-1}\delta^a_{[c}\delta^b_{e]}\psi\,, \\
W^{ij}{}_{kl} &= \frac{2}{(d-2)(d-1)}\delta^i_{[l}\delta^j_{k]}\psi\,, \\
W^{ai}{}_{bj} &= \frac{d-3}{(d-2)(d-1)}\delta^a_b\delta^i_j\psi.
\end{aligned} \tag{84}$$

where $\psi := -\left(\frac{1}{2}R^{(2)} + \mathcal{Y} + \mathcal{Z}\right)$, $\mathcal{Y} = \frac{\mathcal{D}^2 r}{r}$ and $\mathcal{Z} = \frac{R^{(n)}}{n(n-1)} - \frac{\mathcal{D}_a r \mathcal{D}^a r}{r^2}$. Therefore, considering the d-dimensional curvature invariant $\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}$ allows to get access to the differential structure of the Weyl tensor, entirely captured by the second order two dimensional scalar ψ, which otherwise would not possible to reach from polynomial combinations of the Riemann tensor and its covariant derivative $\mathcal{P}\left(R^{\mu\nu}_{\alpha\beta}, \nabla_\sigma, g_{\delta\rho}\right)$. Of course this comes associated with the price of having highly non-linear field equations for other backgrounds than Dynamical Spherical Symmetry and their topological generalizations, or alternatively, of having higher order (but polynomial) curvature constraints in the action, as we saw with Eq(73).

However, instead of directly using the scalar $\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}$, one could use the Non-polynomial curvature tensor $\mathcal{N}$, defined by Eq(60) as $\mathcal{N}^{\mu\nu}{}_{\alpha\beta} := \frac{W^{\mu\nu}{}_{\alpha\beta}}{\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}}$. As seen from Eq(62), considering this tensor allows to get access to the tensorial structure of the Weyl tensor in DSS spacetimes, and to construct new non-polynomial d-dimensional curvature scalars whose spherically symmetric sector is polynomial and second order. For example, one might consider a Lagrangian density :

$$\mathcal{L} = \nabla_\alpha \mathcal{N}_{\mu\nu}{}^{\lambda\delta}\nabla^\alpha \mathcal{N}^{\mu\nu}{}_{\lambda\delta} + \phi_{\mu\nu}{}^{\lambda\delta}\left(\sqrt{W_{\rho\sigma\delta\gamma}W^{\rho\sigma\delta\gamma}}\mathcal{N}^{\mu\nu}{}_{\lambda\delta} - W^{\mu\nu}{}_{\lambda\delta}\right) \tag{85}$$

where ϕ is a set of Lagrange multiplier fields, so that on-shell and for DSS spacetimes, one would obtain another new second order curvature invariant. Many new scalars can be constructed in this way. For example, at quadratic order in the curvature, we have $R_{\mu\nu}R^{\alpha\beta}\mathcal{N}^{\mu\nu}{}_{\alpha\beta}$, $R^{\alpha\beta}_{\mu\nu}\Box\mathcal{N}^{\mu\nu}{}_{\alpha\beta}$, etc... so that considering specific combinations of these scalars (polynomial in DSS) might lead to a second order DSS sector.

Similarly, we have found that another non-polynomial tensor, depending only on a quadratic combination of the Cotton tensor (which is proportional the divergence of the Weyl one), decomposes in a similar fashion as $\mathcal{N}$ in spacetimes (5). Therefore, it is also possible to build gravitational actions from it, leading to second order field equations and black hole solutions. However, we will see in this thesis that contrary to the actions built from $\mathcal{N}$, it is quite simple to find non-singular black hole solutions from this new non-polynomial tensor, under quite simple assumptions regarding the form of the action.

First, let us define the Cotton tensor C by its components as follows :

$$C_{\alpha\beta\gamma} = \nabla_\alpha R_{\beta\gamma} - \nabla_\beta R_{\alpha\gamma} + \frac{1}{2(n+1)} \left(g_{\alpha\gamma}\nabla_\beta R - g_{\beta\gamma}\nabla_\alpha R\right).$$

Then, in spacetimes (5), the following algebraic identity holds in dimension $d = n + 2 > 3$ (for $d = 3$, this is identically satisfied for any metric and so does not carry any new information),

$$\Big(- (n+1)C_\mu{}^{\alpha\beta} + C_\mu{}^{\beta\alpha}\Big)C^\nu{}_{\alpha\beta} = \frac{1}{2}\left(-n\,\delta^\nu_\mu + \frac{n^2-1}{n}\sigma^\nu_\mu\right) C^{\rho\sigma\gamma}C_{\rho\sigma\gamma}\,, \tag{86}$$

where σ is the projector on the horizon manifold $\Omega_{(n,k,r)}$, defined in Eq(63). Indeed, the decomposition of the Cotton tensor in DSS is :

$$\begin{aligned} C^{ec}{}_b &= (d-2)\left(\xi^{[c}\delta_b^{e]}\psi + \frac{1}{d-1}\delta_b^{[e}\partial^{c]}\psi\right)\,, \\ C^{jb}{}_i &= -\delta_i^j\left(\frac{1}{d-1}\partial^b\psi + \xi^b\psi\right)\,, \end{aligned} \tag{87}$$

what allows to prove Eq(475), which itself proves Eq(86). All the other components of the Cotton tensor are vanishing.

Similarly to the previous case with the Weyl tensor, this relation allows to define the following non-polynomial curvature tensor, which does not depend on the derivatives of the metric, and therefore is of "zeroth order" in spacetimes (5),

$$u^\nu_\mu := \frac{\Big(- (n+1)C_\mu{}^{\alpha\beta} + C_\mu{}^{\beta\alpha}\Big)C^\nu{}_{\alpha\beta}}{C_{\sigma\rho\delta}C^{\sigma\rho\delta}}\,, \quad \text{with} \quad u^\nu_\mu\Big\rfloor = \frac{1}{2}\left(-n\,\delta^\nu_\mu + \frac{n^2-1}{n}\sigma^\nu_\mu\right). \tag{88}$$

Finally, from this tensor, one can construct d-dimensional tensors that reduce to the degenerate metrics of respectively $\Omega_{(n,k,r)}$ and Σ in spacetimes (5) as follows :

$$\begin{aligned} \sigma_{\mu\nu} &:= \frac{n}{n^2-1}\left(2u_{\mu\nu} + n\,g_{\mu\nu}\right)\,, \\ \omega_{\mu\nu} &:= g_{\mu\nu} - \sigma_{\mu\nu} = -\frac{g_{\mu\nu} + 2\,n\,u_{\mu\nu}}{n^2-1}\,. \end{aligned} \tag{89}$$

As we will see in the next section, once these metrics are reached from d-dimensional curvature tensors, then all the 2-dimensional spherically symmetric curvature invariants can be obtain from d-dimensional ones. In particular those leading to second order spherically symmetric field equations.

It is not known to us at present if, for more general spacetimes than (5), for exemple for axisymmetric ones, there exist or not other identities like (62) and (86), of the form :

$$T_{\alpha_1\ldots\alpha_m}\left(R^{\mu\nu}_{\alpha\beta}, \nabla_\sigma, g_{\delta\rho}\right) = \eta_{\alpha_1\ldots\alpha_m}\,\mathcal{P}\left(R^{\mu\nu}_{\alpha\beta}, \nabla_\sigma, g_{\delta\rho}\right), \tag{90}$$

where T and $\mathcal{P}$ are respectively polynomial tensor and scalar in the Riemann tensor and its derivatives, while $\eta_{\alpha_1\ldots\alpha_m}$ is a sum of projectors in the class of spacetimes under consideration.

Once again, from such relations, one could construct the non-polynomial tensor $T_{\alpha_1\ldots\alpha_m}/\mathcal{P}$, such that for the specific subset of metric fields for which this identity holds, it would give the order-0 tensor $\eta_{\alpha_1\ldots\alpha_m}$. This might allow to find new second order scalars. If these types of identities emerge exclusively from the existence of Killing vectors, then there might be some also for Axisymmetric spacetimes, and it would be interesting to investigate if regular rotating black hole solutions can be found in this way. Moreover, note that in our work, we only investigated algebraic relations like Eq(90) emerging from (up to) third-order curvature tensors, like the Cotton tensor, or as we will see later from $\partial_\alpha R$. In principle, other such relations might exist from higher orders as well. For example, in [156], we attempted to classify some NPG relations arising in FLRW spacetimes up to order 8 in the so-called FKWC basis, which classifies order by order the independent (polynomial and metric) curvature scalars, see [252, 253].

On the contrary, it is not difficult to find these kind of relations for even more restricted or different classes of metrics than (5). We already saw such a decomposition with Eq(54), but this can be generalized as follows : Consider any curvature scalar $\mathcal{P}\left(R^{\mu\nu}_{\alpha\beta}, \nabla_\sigma, g_{\delta\rho}\right)$ (or even any non-gravitational scalar field), then for any spacetimes such that the scalars take only one argument (for example static spherical symmetry, FLRW, Bianchi I, etc), say x^q, we have :

$$\frac{\partial_\alpha \mathcal{P}}{\sqrt{\partial_\beta \mathcal{P} \partial^\beta \mathcal{P}}} \propto \delta^q_\alpha \tag{91}$$

This is the tensor used in [154] to construct an action reproducing the Loop quantum cosmology bounce solution, what will be reviewed in the last Chapter of this thesis. Note that this property is lost for DSS, while it holds for, say Bianchi I spacetimes, for which the properties Eq(62) and Eq(86) of the Weyl and Cotton tensors are lost.

In static spherical symmetry, many new Non-Polynomial tensors can be found, for example, to mention a few we have :

$$\frac{C_{\alpha\beta\gamma}}{\sqrt{C_{\sigma\rho\delta}C^{\sigma\rho\delta}}} \quad , \quad \frac{\Box^n W_{\mu\nu\alpha\beta}}{\sqrt{\Box^n W_{\gamma\delta\lambda\rho}\Box^n W^{\gamma\delta\lambda\rho}}} \tag{92}$$

for any integer n. Using these additional order-0 tensors allows to construct even more second order models than those found from (62) and (86), and in particular second order scalars that are not part of the 2D DSS scalars that we saw in the previous Chapter.

Conversely, and this is obviously the more interesting direction, as the metric becomes more general (for example from static to dynamical spherical symmetry), the population of such tensors and scalars decreases, what singles out some of them. Unfortunately, we have not found an efficient way to systematically find other NPG algebraic relations of the form Eq(90).

2. Non-polynomial second order curvature invariants & "Degeneracies"

The main result of the previous section is that we have found a way to get access to the two-dimensional degenerate metrics of the manifolds Σ and $\Omega_{(n,k,r)}$, Eq(89), from the DSS reduction of a non-polynomial combination of squares of the d-dimensional Cotton tensor. As we will see now, it follows that all the 2D second order scalar-tensor invariants in Horndeski theory, Eq(11), can be obtained from the DSS decomposition of suitably chosen d-dimensional curvature invariants, so that a large class of 2D Horndeski theory can be lifted to d dimensions.

To see this, first recall that in a theory of gravity which is torsionless, with a metric-compatible connection, all the information about the curvature is contained in the Riemann tensor, so that all possible second order curvature invariants are contained in (invariant) combinations of its components. In particular, for Dynamical Spherically Symmetric spacetimes, recall that defining $\chi := \frac{\gamma^{ab}\mathcal{D}_a r \mathcal{D}_b r}{r^2}$, $\mathcal{Z} := \frac{R^{(n)}}{n(n-1)} - \chi$ and $\mathcal{Y}^a_b := \frac{\mathcal{D}^a\mathcal{D}_b r}{r}$, the Riemann tensor decomposes as follows :

$$R^{ai}{}_{bj} = -\mathcal{Y}^a_b \delta^j_i \quad , \quad R^{ij}{}_{kl} = \mathcal{Z}\delta^i_{[k}\delta^j_{l]} \quad , \quad R^{ab}{}_{ce} = \frac{1}{2} R^{(2)}(\gamma)\, \delta^a_{[c}\delta^b_{e]} \tag{93}$$

so that, at best, one can hope to construct d-dimensional theories whose DSS decomposition are spanned by $\left(\gamma_{ab}, \mathcal{Y}^a_b, \chi, R^{(n)}, R^{(2)}\right)$, where $\frac{R^{(n)}}{n(n-1)} = \frac{k}{r^2}$ and k is the topological parameter. In order to see what combinations of these curvature quantities might lead to second order DSS field equations, recall that the most general two-dimensional (Horndeski) scalar-tensor theory is given by :

$$L^{2D}_{\text{Horndeski}} = \sqrt{-\gamma}\left(\eta(r,\tilde{\chi}) - G_3(r,\tilde{\chi})\,\tilde{\mathcal{Y}} + G_4(r,\tilde{\chi})\,R^{(2)}(\gamma) - 2\frac{\partial G_4(r,\tilde{\chi})}{\partial\tilde{\chi}}\left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right)\right) \tag{94}$$

where $\tilde{\chi} := \gamma^{ab}\mathcal{D}_a r\mathcal{D}_b r$, $\tilde{\mathcal{Y}}_{ab} := \mathcal{D}_a\mathcal{D}_b r$, so that $\tilde{\mathcal{Y}} := \tilde{\mathcal{Y}}_{ab}\gamma^{ab}$ and $\tilde{\mathcal{Y}}_2 = \tilde{\mathcal{Y}}_{ab}\tilde{\mathcal{Y}}^{ab}$. This is because in two dimensions, the usual additional terms of four-dimensional Horndeski theory are identically vanishing $G_{ab} = 0$ and $\tilde{\mathcal{Y}}^3 - 3\tilde{\mathcal{Y}}\tilde{\mathcal{Y}}_2 + 2\tilde{\mathcal{Y}}_3 = 0$, where G is the Einstein tensor associated with the metric γ and $\tilde{\mathcal{Y}}_3 := \tilde{\mathcal{Y}}^b_a\tilde{\mathcal{Y}}^a_c\tilde{\mathcal{Y}}^c_b$.

Therefore, given that, neglecting the angular part, the d-dimensional quantity $\sqrt{-g}$ decomposes in DSS as

$$\sqrt{-g} = \sqrt{-\gamma}r^{d-2} \tag{95}$$

the most general decomposition of a d-dimensional theory leading to second order DSS field equations would be

$$L = \sqrt{-\gamma}r^{d-2}\left(\mathscr{A} - \mathscr{B}\,\mathcal{Y} + \mathscr{C}\,R^{(2)} - 2\mathscr{C}_\chi\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right) \tag{96}$$

which is related to 2D Horndeski theory by $\eta\left(r,\tilde{\chi}\right) = r^{d-2}\mathscr{A}\left(\mathcal{R},\chi\right)$, $G_3\left(r,\tilde{\chi}\right) = r^{d-3}\mathscr{B}\left(\mathcal{R},\chi\right)$, $G_4\left(r,\tilde{\chi}\right) = r^{d-2}\mathscr{C}\left(\mathcal{R},\chi\right)$, where $\mathcal{R} := \frac{k}{r^2}$.

Therefore, among all the combinations of $\left(\gamma_{ab},\mathcal{Y}^a_b,\chi,R^{(n)},R^{(2)}\right)$, we need now to find d-dimensional Non-Polynomial Gravity curvature invariants whose DSS reduction is given by the scalars $\left(\chi,\mathcal{R},R^{(2)},\mathcal{Y},\mathcal{Y}_2\right)$. A quite simple way to do so is to consider the scalars

$$R^{\mu\nu}u_{\mu\nu}\quad,\quad\nabla^\alpha\nabla^\beta u_{\alpha\beta}\quad,\quad\nabla_\gamma u_{\alpha\beta}\nabla^\gamma u^{\alpha\beta}\quad,\quad\text{and}\quad R \tag{97}$$

which are all by construction second order in DSS spacetimes. However, reaching $\mathcal{Y}_2$ is somewhat more difficult given that it contains four derivatives of the scalar field r. To do so, we can consider the additional scalars $\nabla_\mu\nabla_\nu\omega_{\sigma\rho}\nabla^\mu\nabla^\nu\omega^{\sigma\rho}$ and $\nabla^\mu\omega_{\mu\nu}\nabla^\nu\left(\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\right)$. Indeed, define

$$\begin{aligned}
R_{(\Omega)} &:= \left(R^{\alpha\beta} - \nabla^\alpha\nabla^\beta\right)\sigma_{\alpha\beta}\,,\\
R_{(\Sigma)} &:= \left(R^{\alpha\beta} + \nabla^\alpha\nabla^\beta\right)\omega_{\alpha\beta} - \frac{n-1}{2}\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\,,
\end{aligned} \tag{98}$$

and in four dimensions ($n = 2$),

$$\mathcal{K} := \frac{1}{4}\nabla_\mu\nabla_\nu\omega_{\sigma\rho}\nabla^\mu\nabla^\nu\omega^{\sigma\rho} - \frac{5}{16}\left(\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\right)^2 + \frac{1}{8}\nabla^\mu\omega_{\mu\nu}\nabla^\nu\left(\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\right) \tag{99}$$

Then, in DSS spacetimes, using the decompositions of the Christoffel symbols and the Ricci tensor Eq(7,8), these scalars simply reduce to :

$$R_{(\Omega)}\Big\rfloor = R^{(n)}(\Omega) = k\left(\frac{n(n-1)}{r^2}\right)\;,\quad R_{(\Sigma)}\Big\rfloor = R^{(2)}\left(\gamma\right)\;,\quad \mathcal{K}\Big\rfloor = \mathcal{Y}_2 \tag{100}$$

For convenience for the following, we redefine the quantity $\mathcal{R}$ to be a NPG scalar instead of a 2D one, as $\mathcal{R} := \frac{R_{(\Omega)}}{n(n-1)}$. To complete our list of second order NPG invariants, one has also,

$$\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\Big\rfloor = 2n\,\frac{\gamma^{ab}\mathcal{D}_a r\,\mathcal{D}_b r}{r^2}\,,\qquad \nabla^\alpha\nabla^\beta\omega_{\alpha\beta}\Big\rfloor = \frac{\mathcal{D}^2 r^n}{r^n} = n(n-1)\chi + n\mathcal{Y}\,. \tag{101}$$

so that

$$\frac{1}{n}\left(\nabla^\alpha\nabla^\beta\omega_{\alpha\beta} - \frac{n-1}{2}\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\right)\Big\rfloor = \frac{\mathcal{D}^2 r}{r} \tag{102}$$

This conclude our construction of NPG invariants leading to second order DSS field equations. These results provide a unified approach to reinterpret in a precise way a very large number of two-dimensional effective scalar-tensor theories Eq(96) as being the Dynamical Spherically Symmetric sector of some d-dimensional, purely gravitational (metric) theories. This is the main result of this section, especially regarding regular black hole solutions, given the significant literature on effective 2D dimensional models, used to study these kind of geometry, see for example [41–43] for Einstein-Dilaton gravities, [166–169] for Lovelock Designer (i.e. due to Eq(13), 2D Horndeski), while key properties of effective black holes have been studied within this framework, like their formation and evaporation [41, 157], thermodynamics and observables [158–162], quasinormal modes [234], quantum properties [163–165], and behaviour in the presence of matter fields [235–237].

However, it is clear that Eq(96) is a reduced subset of 2D Horndeski theories. Even more if we are interested solely by polynomial 2D field equations for any given order (we will be, because it is the whole point of NPG models), in which case the functions $\mathscr{A}(\mathcal{R},\chi)$, $\mathscr{B}(\mathcal{R},\chi)$ and $\mathscr{C}(\mathcal{R},\chi)$ are given by :

$$\begin{aligned}\mathscr{A}(\mathcal{R},\chi) &= \sum_{i=0}^{p}\alpha_i\mathcal{R}^i\chi^{p-i}\\ \mathscr{B}(\mathcal{R},\chi) &= \sum_{i=0}^{p}\beta_i\mathcal{R}^i\chi^{p-i}\\ \mathscr{C}(\mathcal{R},\chi) &= \sum_{i=0}^{p}\delta_i\mathcal{R}^i\chi^{p-i}\end{aligned} \tag{103}$$

where α, β, δ are dimensionless sets of coupling constants and $2p$ is the number of derivatives (the order) of the DSS theory. In particular, for planar topology $k = 0$, the scalar $\mathcal{R} = 0$, so that the space of theories is even more reduced.

Degeneracies :

An important point about the previous constructions is the following. Note that we could also have used many other d-dimensional scalars in order to get the desired spherically symmetric invariants. Indeed, it is possible to find a large number of relations like :

$$-\frac{n+1}{2}\,\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\Big\rfloor = -\frac{n}{2}\,\nabla_\alpha\mathcal{N}_{\mu\nu}{}^{\lambda\delta}\nabla^\alpha\mathcal{N}^{\mu\nu}{}_{\lambda\delta}\Big\rfloor\,.$$

This means that, in order to obtain for exemple the 2D scalar $\frac{\gamma^{AB}\mathcal{D}_A r\,\mathcal{D}_B r}{r^2}$, we could also have used the scalar $\nabla_\alpha\mathcal{N}_{\mu\nu}{}^{\lambda\delta}\nabla^\alpha\mathcal{N}^{\mu\nu}{}_{\lambda\delta}$ defined in terms of the Weyl tensor by (62), instead of $\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}$, which depends on the Cotton tensor. These two scalars are completely different in general, and yet lead to the same spherically symmetric sector. Similarly, considering the very large number of inequivalent d-dimensional scalars whose DSS sector is by construction second order, i.e. of the form

$$u...u\mathcal{N}...\mathcal{N}\,\nabla\nabla\mathcal{N}\;\;,\;\;\; u...u\mathcal{N}...\mathcal{N}\,\nabla\nabla u\;\;,\;\;\; u...u\mathcal{N}...\mathcal{N}\,\mathrm{Riem}\;\;,\;\;\; u...u\mathcal{N}...\mathcal{N}\,\nabla u\nabla\mathcal{N}\;,\;\;\text{etc...} \tag{104}$$

where all the possible contractions of indices using the metric tensor are implied, we already know that they must all reduce to a combination of the five 2D invariants, $\big(\chi,\mathcal{R},R^{(2)},\mathcal{Y},\mathcal{Y}_2\big)$, so that there is a huge degeneracy in the choice of a specific set of five d-dimensional curvature invariants constituting a basis in DSS. However, outside of their DSS sector, these scalars become inequivalent, so that the degeneracy is broken.

It is also conceivable that there might exist simpler ways to reach the five spherically symmetric invariants $\big(\chi,\mathcal{R},R^{(2)},\mathcal{Y},\mathcal{Y}_2\big)$, than by using the Cotton tensor and the algebraic relation (86). More interestingly, there might exists more complex ways, involving higher (than sixth) order algebraic relations of the form Eq(90), which would hold for larger classes of spacetimes than solely Dynamical Spherical Symmetry. If so, it is also possible that only some specific combinations of the 2D scalars $\big(\chi,\mathcal{R},R^{(2)},\mathcal{Y},\mathcal{Y}_2\big)$ could be reached from these (would be) new d-dimensional NPG invariants. This depopulation would therefore restrict even more the class of accessible 2D Horndeski theories from which one can look for regular solutions.

These kind of degeneracies are quite omnipresent in NPG models, and they are the reason why one should not take too seriously any specific d-dimensional NPG model without a good reason for the choices of the *representatives* of the 2D invariants. What we find interesting with Non-Polynomial Gravity is that it provides a proof of existence of d-dimensional curvature invariant whose DSS sector are polynomial and second order. And as we will see in the last two Chapters of this thesis, it proves that, in the theory space of metric gravity, there exists many theories whose solutions are identical to some effective metric fields found from Quantum Gravity considerations, in particular the semi-polymeric Loop black hole of [44], the Loop Quantum Cosmology bounce

[28–31], the Asymptotic Safety black hole [54], as well as some Quantum gravity inspired solutions like the D'Ambrosio-Rovelli black-to-white-hole metric [175] and the Visser-Hochberg black bounce [227, 228].

However, additional informations should be used to choose among the representatives of a 2D scalar, for example by calculating the perturbations around DSS solutions, associated with two different NPG theories with the same DSS sector. It seems however quite difficult to do, even when the non-polynomiality is reabsorbed into curvature constraints like in Eq(73,85). Moreover, even disregarding the technical difficulty of deriving perturbations from theories involving such high order curvature constraints, it should also be noted that it seems very likely that many ghost-like degrees of freedom would appear. Minimizing the number of these ghost modes could be a useful criterion to select the representatives. Regarding the same kind of issues in the context of Quasi-topological gravities, which are polynomial in curvature for general metric but yield higher order field equations outside of DSS, see the discussion in the Sec. 8 of [143].

Finally, we also note for latter use that in the following we will distinguish the number of derivatives of the previous curvature scalars in the following way : as in spherical symmetry $R^{\mu\nu}u_{\mu\nu}$, $\nabla^\alpha\nabla^\beta u_{\alpha\beta}$ and $\nabla_\gamma u_{\alpha\beta}\nabla^\gamma u^{\alpha\beta}$ contain only two derivatives of the metric, we count these (and the previous scalars derived from these) as having the same order as the Ricci scalar. Moreover, as $\mathcal{R} = k/r^2$, we will also consider the scalar $\mathcal{R}^{i/2}$, where i is an integer, and count the order of this last as having i derivatives of the metric, even if i is odd. This is solely because it turns out that the Poisson-Israel regular black hole of [41, 225] can be found from a non-polynomial order-5 lagrangian in four dimensions (as we will see in the third section of the next Chapter), otherwise it is just a matter of useful convention to name properly some actions that we will be considering in the following.

C. Gravitational Actions from Generalizations of Einstein and Gauss-Bonnet gravities

From the results of the previous section, we can now study some effective 2D Horndeski models, like it is usually done to find regular solutions [41–43, 170, 171, 235–237], and lift the resulting theory to d dimensions.

Irrespectively of the specific effective approach one is using (Non-Linear Electrodynamics, exotic fluids, 2D Horndeski, gravitational theories with additional degrees of freedom like $f(R)$, d-dimensional Horndeski, scalar-vector-tensor theories, etc...) to study regular solutions, there are two general ways to do so : either one finds some criteria to construct a specific theory (i.e. select a theory among the huge number of possibilities) from which regular solutions are *found* (a notable example being the Non-Minimal Einstein-Yang-Mills-SU(2) black hole, found in [101, 102] by fixing two coupling constants among the three possible entering in the non-minimal coupling), or one starts with a regular spacetime ansatz and *reconstructs* the corresponding theory. Of course, the first way is the most interesting one, because once regular solutions are found, it gives some values to the criteria/hypothesis used to select a priori the corresponding theory (for example symmetry considerations in [101, 102]), whereas the second one usually amounts to imposing a solution to a differential equations with the freedom of fixing entire functions (infinite sets of coupling constants) to do so. That being said, it is not true that from any class of theory with entire free functions, it is possible to reconstruct any regular solutions. So that it is still interesting to understand, for a given regular metric field, what classes of theories are able to produce it.

These considerations are the topic of this last section of this Chapter. Within the effective two dimensional Horndeski approach (or equivalently, from NPG theories), we will establish some criteria to reduce the freedom of two dimensional Horndeski theories, in order to *find* or *reconstruct* regular solutions from more restricted classes of theories, what will be done in the remaining three Chapters of this thesis. Our criteria is quite simple and relies on generalizing the DSS sector of Einstein and Gauss-Bonnet gravity while preserving the DSS decompositions of both the theories and their associated 2D covariant field equations, for any order of corrections, so that similarly to Lovelock-Lanczos theories, we will obtain Non-Polynomial gravity theories in the form of power-series of invariants, in a length scale parameter l.

1. Generalizing the DSS sector of Gauss-Bonnet gravity

We first start by establishing a quite natural generalization of the DSS sector of Lovelock-Lanczos theory, that can be seen as the most general two dimensional Horndeski theory sharing with Gauss-Bonnet gravity the same DSS decomposition of its Lagrangian, and of its field equations.

To do so, recall that the dynamical spherically symmetric decomposition of the Lagrangian of Lovelock-Lanczos gravity is given by :

$$\mathcal{L}_{(p)} = \frac{(d-2)!}{(d-2p)!}\left(\alpha_p \mathcal{Z}^2 + \mathcal{Z}\left(\beta_p \mathcal{Y} + \gamma_p R^{(2)}\right) + \zeta_p\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right)\mathcal{Z}^{p-2} \tag{105}$$

where $\alpha_p = (d-2p)(d-2p-1)$, $\beta_p = -2(d-2p)p$, $\gamma_p = p$ and $\zeta_p = 2p(p-1)$. The associated minisuperspace covariant field equations are :

$$\begin{aligned} G^{(p)}_{ab} &= -\frac{(d-2)!}{(d-2p-1)!}\left(p\left(\mathcal{Y}_{ab} - \gamma_{ab}\mathcal{Y}\right) + \frac{1}{2}\left(d-2p-1\right)\mathcal{Z}\,\gamma_{ab}\right)\mathcal{Z}^{p-1} \\ G^{(p)}_{ij} &= -\frac{(d-3)!}{2(d-2p-1)!}\sigma_{ij}\left(\alpha_{(p+\frac{1}{2})}\mathcal{Z}^2 - p\left((d-2p-1)\mathcal{Y} - R^{(2)}\right)\mathcal{Z} + \zeta_p\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right)\mathcal{Z}^{p-2} \end{aligned} \tag{106}$$

Our intention is to see that the previous equations, based on an order $2p$ theory, can be recovered from a simple generalization of the DSS sector of Gauss-Bonnet gravity, of order 4, this later being given by :

$$\mathcal{L}_{(2)} = \frac{(d-2)!}{(d-4)!}\left(\alpha_2 \mathcal{Z}^2 + \mathcal{Z}\left(\beta_2 \mathcal{Y} + \gamma_2 R^{(2)}\right) + \zeta_2\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right) \tag{107}$$

where $\alpha_2 = (d-4)(d-5)$, $\beta_2 = -4(d-4)$, $\gamma_2 = 2$ and $\zeta_2 = 4$. The procedure is simply obtained by comparing the two Lagrangians (105) and (107), and is essentially based on demanding that both the higher order generalized Lagrangians and field equations have the same spherically symmetric decomposition as Gauss-Bonnet gravity.

First, regarding the generalization of the DSS of Gauss-Bonnet gravity to an order $2p$ theory, we impose that the spherically symmetric decomposition of this theory is preserved for any order $2p$, i.e. it is spanned by the four terms : $\left(\mathcal{Q}^2_{(p)}, \mathcal{Q}_{(p)}\mathcal{Y}, \mathcal{Q}_{(p)}R^{(2)}, \left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right)$, where $\mathcal{Q}_2\left(r, \gamma_{ab}, \nabla_a\right) = \mathcal{Z}$. Then, the higher orders are generated by multiplying all the terms by the common scalar factor $\mathcal{Q}_{(p)}$, leading to a Lagrangian term :

$$\mathcal{L}_{(p)} = \left(\alpha_p \mathcal{Q}^2_{(p)} + \mathcal{Q}_{(p)}\left(\beta_p \mathcal{Y} + \gamma_p R^{(2)}\right) + \zeta_p\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right)\mathcal{Q}^{p-2}_{(p)} \tag{108}$$

where $(\alpha_p, \beta_p, \gamma_p, \zeta_p)$ are arbitrary sets of dimensionless coupling constants.

Similarly, we demand that the field equations of the higher order generalizations are respectively spanned by : $\left(\left(\mathcal{Y}_{ab} - \gamma_{ab}\mathcal{Y}\right), \mathcal{Q}_{(p)}\,\gamma_{ab}\right)$ and $\left(\mathcal{Q}^2_{(p)}, \mathcal{Y}\mathcal{Q}_{(p)}, R^{(2)}\mathcal{Q}_{(p)}, \left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\right)$, so that any order of corrections to Gauss-Bonnet gravity preserves the spherically symmetric decomposition of this last.

A first condition to preserve the decomposition of the DSS Gauss-Bonnet field equations is to have second order field equations for all the higher order generalizations, i.e. we need the theory to be of the Horndeski type Eq(11) :

$$\mathcal{Q}_p\left(r, \tilde{\chi}\right) \quad \text{and} \quad -2\gamma_p \partial_{\tilde{\chi}}\left(\mathcal{Q}^{p-1}_p\right) = \zeta_p \frac{\mathcal{Q}^{p-2}_p}{r^2} \tag{109}$$

Moreover, the $\mathcal{Q}_p$ should be 2D scalars that can be reached from the DSS decomposition of some d-dimensional Non-Polynomial curvature invariant. Therefore, the specific dependence of $\mathcal{Q}$ on the scalar field r and its kinetic term $\tilde{\chi}$ must be of the form :

$$\mathcal{Q}_p\left(\frac{k}{r^2}, \frac{\tilde{\chi}}{r^2}\right) = \mathcal{Q}_p\left(\mathcal{R}, \chi\right) \tag{110}$$

Note also that the dimension of the $\mathcal{Q}_p$ being the inverse of an area, the most general such function can be written as :

$$\mathcal{Q}_p = \mathcal{R}\, f_p\left(\frac{\mathcal{R}}{\chi}\right) + \lambda_p \chi\, g_p\left(\frac{\mathcal{R}}{\chi}\right) \tag{111}$$

where f_p and g_p are arbitrary sets of functions and λ_p is an arbitrary set of dimensionless coupling constants. Lovelock gravity corresponds to $f_p = -g_p = \mathrm{id}$ and $\lambda_p = 1$, and in order to avoid non-polynomialities in the 2D scalars (what is the whole point of Non-Polynomial Gravity theories), we assume that

$$\mathcal{Q}_p = \frac{k - \lambda_p \tilde{\chi}}{r^2} \tag{112}$$

so that the Horndeski condition Eq(109) gives

$$\zeta = 2(p-1)\gamma\lambda \tag{113}$$

In order to preserve the decomposition of the Gauss-Bonnet DSS field equations for all orders, we need to derive the field equations associated with Eq(108), given the previous conditions on $\mathcal{Q}_p$ and ζ_p. To do so, note that the present theory is a related to 2D Horndeski theory Eq(11) by :

$$\eta\left(r,\tilde{\chi}\right) = \alpha_p r^{d-2}\mathcal{Q}^p_{(p)} \;\;,\;\; G_3\left(r,\tilde{\chi}\right) = -\beta_p r^{d-3}\mathcal{Q}^{p-1}_{(p)} \;\;,\;\; G_4\left(r,\tilde{\chi}\right) = \gamma_p r^{d-2}\mathcal{Q}^{p-1}_{(p)} \tag{114}$$

Using Eq(15), and defining $\mathcal{E}_{ab} := \frac{\delta\left(r^{d-2}\sqrt{-\gamma}\mathcal{L}_p\right)}{\delta\gamma_{ab}}$ and $\mathcal{E} := \frac{\delta\left(r^{d-2}\sqrt{-\gamma}\mathcal{L}_p\right)}{\delta r}$, the field equations associated with the 2D metric γ_{ab} become :

$$\begin{aligned}
&\mathcal{E}_{ab} = r^{d-2}\mathcal{Q}^{p-2}\left(\left(a_{(\gamma)}\mathcal{R} + b_{(\beta,\gamma,\lambda)}\chi\right)\left(\mathcal{Y}_{ab} - \gamma_{ab}\mathcal{Y}\right) + \frac{1}{2}\mathcal{Q}\left(c_{(\beta,\gamma)}\chi - \alpha_p\mathcal{Q}\right)\gamma_{ab} - d_{(\alpha,\beta,\gamma,\lambda)}\mathcal{Q}\chi_{ab}\right)\\
&a_{(\gamma)} := -\gamma_p(d-2p)\;,\;\; b_{(\beta,\gamma,\lambda)} := \lambda\left(\left(\beta_p(p-1) + \gamma_p(d-2p)(2p-1)\right)\right)\\
&c_{(\beta,\gamma)} := \left(\beta_p + 2\gamma_p(d-2p)\right)(d-2p-1)\;\;,\;\; d_{(\alpha,\beta,\gamma,\lambda)} := p\lambda_p\alpha_p + (\beta_p + \gamma_p(d-2p))(d-2p-1)
\end{aligned} \tag{115}$$

while the field equation associated with the 2D scalar field r is given by :

$$\begin{aligned}
\mathcal{E} = \Bigg(\Big(&-\left\{a_{(\gamma)}\mathcal{R} + b_{(\beta,\gamma,\lambda)}\chi\right\}R^{(2)} + 2\left\{\left(i_{(\alpha,\beta,\lambda)} + \alpha_p\lambda_p(p-1)\right)\mathcal{R} + \lambda_p j_{(\alpha,\beta,\gamma,\lambda)}\chi\right\}\mathcal{Y}\Big)\mathcal{Q}\\
&+ (d-2(p+1))\left\{\alpha_p\mathcal{R} + i_{(\alpha,\beta,\lambda)}\chi\right\}\mathcal{Q}^2 + 2(p-1)\lambda_p\left\{e_{(\beta,\gamma)}\mathcal{R} - b_{(\beta,\gamma,\lambda)}\chi\right\}\left(\mathcal{Y}^2 - \mathcal{Y}_2\right)\\
&- 4(p-1)\lambda_p d_{(\alpha,\beta,\gamma,\lambda)}\chi_{ab}\mathcal{Y}^{ab}\mathcal{Q}\Bigg)r^{d-3}\mathcal{Q}^{p-3}
\end{aligned} \tag{116}$$

where we used that $\tilde{\chi}_a\nabla^a\tilde{\chi} = 2r^3\chi_{ab}\mathcal{Y}^{ab}$, and for simplicity we have defined :

$$\begin{aligned}
&e_{(\beta,\gamma)} := \beta_p + 3\gamma_p(d-2p)\;,\;\; i_{(\alpha,\beta,\lambda)} := \alpha_p\lambda_p(2p-1) + \beta_p(d-2p-1)\\
&j_{(\alpha,\beta,\gamma,\lambda)} := 2(p-1)d_{(\alpha,\beta,\gamma,\lambda)} - p\, i_{(\alpha,\beta,\lambda)}
\end{aligned} \tag{117}$$

Comparing these field equations with the Lovelock-Lanczos ones, Eq(106), we see that the terms $\chi_{ab}\mathcal{Y}^{ab}$ and χ_{ab} are not present in this last, so that to preserve the DSS decomposition of Gauss-Bonnet gravity for all the higher order generalizations, we need to set

$$d_{(\alpha,\beta,\gamma,\lambda)} = 0 \tag{118}$$

meaning that

$$\begin{aligned}
\alpha_p &= -\frac{1}{p\lambda_p}(\beta_p + \gamma_p(d-2p))(d-2p-1)\\
i_{(\alpha,\beta,\lambda)} &= -\frac{j_{(\alpha,\beta,\gamma,\lambda)}}{p}\\
j_{(\alpha,\beta,\gamma,\lambda)} &= (d-2p-1)\Big((p-1)\beta_p + (d-2p)(2p-1)\gamma_p\Big)
\end{aligned} \tag{119}$$

At this point, it is interesting to see what this condition means for static spherical symmetry. To do so, consider for example a four dimensional metric given by :

$$ds^2 = -a(r)b^2(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\Omega_k^2 \tag{120}$$

Then, the minisuperspace field equations w.r.t. $a(r)$ and $b(r)$ associated with the theory Eq(108), while taking into account the conditions (112) and (113), are given by :

$$\begin{aligned}
\frac{\delta\left(r^2 b(r)\mathcal{L}_p\right)}{\delta b(r)} =& \left(\frac{r^{3-2p}}{2p-3}\left(k-\lambda_p a(r)\right)^{p-1}\Big(-\alpha_p k + (2\gamma_p(p-2)(2p-3)-(p-1)\alpha_p\lambda_p)a(r)\Big)\right)' \\
&+ d_{(\alpha,\beta,\gamma,\lambda)} r^{2(1-p)} a(r)\left(k-\lambda_p a(r)\right)^{p-2}\left(-k+\lambda_p a(r) - \frac{p-1}{2p-3}\lambda_p r a'(r)\right) \\
\frac{\delta\left(r^2 b(r)\mathcal{L}_p\right)}{\delta a(r)} =& \frac{r^{3-2p}}{2p-3}\left(k-\lambda_p a(r)\right)^{p-2}\left(p\lambda_p m_{(\alpha,\gamma,\lambda)} a(r) - 2(p-2)(2p-3)\gamma_p k\right) b'(r) \\
&+ d_{(\alpha,\beta,\gamma,\lambda)} \frac{r^{2(1-p)}}{2p-3}\Big((3-2p)(k-\lambda_p a(r))b(r) + (p-1)\lambda_p r a(r) b'(r)\Big)
\end{aligned} \tag{121}$$

where $m_{(\alpha,\gamma,\lambda)} = 2(p-2)(2p-3)\gamma_p - (p-1)\alpha_p\lambda_p$. They are related to the components of $\mathcal{E}_{ab}$ as follows :

$$\begin{aligned}
\frac{2}{a(r)b(r)^2}\mathcal{E}_{tt} &= \frac{\delta\left(r^2 b(r)\mathcal{L}_p\right)}{\delta b(r)} \\
a(r)b(r)\mathcal{E}_{rr} &= -\frac{1}{2}b(r)\frac{\delta\left(r^2 b(r)\mathcal{L}_p\right)}{\delta b(r)} + a(r)\frac{\delta\left(r^2 b(r)\mathcal{L}_p\right)}{\delta a(r)}
\end{aligned} \tag{122}$$

and $\mathcal{E}$ is redundant. Therefore, preserving the decomposition of GB gravity (setting $d_{(\alpha,\beta,\gamma,\lambda)} = 0$) means that $b = 1$ and $\frac{\delta\left(r^2 b(r)\mathcal{L}_p\right)}{\delta b(r)}$ is integrable, which are known to be very important properties of Lovelock-Lanczos gravities. In particular, the second one is related to the fact that black hole solutions in LLG are found by solving the so-called Wheeler polynomial [283]. Similar deformations of Lovelock-Lanczos gravity which respect this condition have been found from Quasi-Topological gravities [150], but only for specific order of corrections p and dimensions d.

Taking into account the conditions Eq(112,113,118), the Lagrangian becomes :

$$\mathcal{L}_{(p)} = \left(\beta\mathcal{Y}\mathcal{Q} + \gamma\left(2(p-1)\lambda\left(\mathcal{Y}^2 - \mathcal{Y}_2\right) + \mathcal{Q}R^{(2)}\right) - \frac{(\beta+(d-2p)\gamma)(d-2p-1)}{p\lambda}\mathcal{Q}^2\right)\mathcal{Q}^{p-2} \tag{123}$$

As we saw, all the models belonging to this class share some key properties with LL gravities. They constitute a two sets of dimensionless coupling constants deformation of the DSS sector of this theory, because we can redefine $(\lambda_p, \gamma_p, \beta_p)$, so that one of these is reabsorbed by an overall set of coupling constants which multiplies each $\mathcal{L}_{(p)}$.

In the last chapter of this thesis where this result will be used, we will reduce further the freedom in the coupling constant by considering a four-dimensional theory ($d = 4$), with $\lambda_p = 1$, so that $\mathcal{Q} = \mathcal{Z}$. We will also set $\beta_p = 2w_p\left(\zeta_p + \frac{p(\zeta_p-1)}{p-1}\right)$ and $\gamma_p = \frac{w_p\zeta_p}{p-2}$ where ζ_p and w_p are two sets of coupling constants replacing β_p and γ_p, so that $\zeta_p = p$ corresponds to the DSS sector of Lovelock-Lanczos gravity :

$$\begin{aligned}
\mathcal{L}_{(p)} = \frac{w_p}{(p-1)(p-2)}\Big(& 2(p-2)\Big((\zeta_p(2p-1)-p)\mathcal{Y} + (\zeta_p-1)(2p-3)\mathcal{Z}\Big)\mathcal{Z} \\
&+ \zeta_p(p-1)\left(2(p-1)\left(\mathcal{Y}^2-\mathcal{Y}_2\right) + \mathcal{Z}R^{(2)}\right)\Big)\mathcal{Z}^{p-2}
\end{aligned} \tag{124}$$

The overall normalization $\frac{w_p}{(p-1)(p-2)}$ is chosen so that the order $2p = 4$ scalar contributes to the field equations, by virtue of the decomposition Eq(23) reported in the first Chapter, at the price of implying a divergent boundary term (unless suitable counterterms are added to the total action).

By construction, the $p=1$ scalar is excluded but our aim being to find high energy corrections to GR, we will simply consider theories of the form

$$\mathcal{L} = R + \sum_{p=2}^{m} \mathcal{L}_p \tag{125}$$

As we will see in the last chapter, taking $m \to \infty$ and choosing suitably w_p and ζ_p leads to gravitational theories admitting both regular black hole and cosmological solutions. But for now, we only wish to express the Lagrangian (124) as a Non-polynomial gravity theory. Due to the equivalences Eq(100) found in the previous section, one possible way is to consider the following four dimensional theories :

$$I = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\left(R + \sum_{p=2}^{\infty} \frac{w_p\,\mathcal{Z}^{p-2}}{(p-1)(p-2)}\left(\sigma_p \mathcal{Z}^2 + \Big(\rho_p \mathcal{Y} + \delta_p R_{(\Sigma)}\Big)\mathcal{Z} + \nu_p\left(\mathcal{Y}^2 - \mathcal{K}\right)\Big)\right) \tag{126}$$

where

$$\begin{aligned} &\sigma_p = 2(p-2)(2p-3)(\zeta_p - 1) \ , \quad \rho_p = 2(p-2)\big(\zeta_p(2p-1) - p\big) \\ &\delta_p = \zeta_p(p-1) \qquad\qquad\qquad\quad , \quad \nu_p = 2(p-1)\delta_p \end{aligned} \tag{127}$$

while the Non-Polynomial curvature invariants $\big(\mathcal{Z}, \mathcal{Y}, R_{(\Sigma)}, \mathcal{K}\big)$, defined in Eq(98,99) can be expressed as :

$$\begin{aligned} &\mathcal{Z} = \mathcal{R} - \frac{\nabla_\gamma \omega_{\alpha\beta} \nabla^\gamma \omega^{\alpha\beta}}{4} \quad , \qquad \mathcal{R} = \frac{1}{2}\left(R - \left(R^{\alpha\beta} - \nabla^\alpha \nabla^\beta\right)\omega_{\alpha\beta}\right) \\ &\mathcal{Y} = \frac{1}{2}\left(\nabla^\alpha \nabla^\beta \omega_{\alpha\beta} - \frac{1}{2}\nabla_\gamma \omega_{\alpha\beta} \nabla^\gamma \omega^{\alpha\beta}\right) \quad , \qquad R_{(\Sigma)} = R^{\alpha\beta}\omega_{\alpha\beta} + 2\mathcal{Y} \ , \\ &\mathcal{K} = \frac{1}{4}\nabla_\mu \nabla_\nu \omega_{\sigma\rho} \nabla^\mu \nabla^\nu \omega^{\sigma\rho} - \frac{5}{16}\left(\nabla_\gamma \omega_{\alpha\beta} \nabla^\gamma \omega^{\alpha\beta}\right)^2 + \frac{1}{8}\nabla^\mu \omega_{\mu\nu} \nabla^\nu \left(\nabla_\gamma \omega_{\alpha\beta} \nabla^\gamma \omega^{\alpha\beta}\right) \end{aligned} \tag{128}$$

where the tensor $\omega_{\mu\nu}$ is itself defined by Eq(89) in terms of the non-polynomial curvature tensor $u_{\mu\nu}$ by : $\omega_{\mu\nu} := -\frac{1}{3}\left(g_{\mu\nu} + 4\,u_{\mu\nu}\right)$. Like in Eq(73,85), it is possible to reabsorb the non-polynomiality into a curvature constraint of the form

$$\lambda^{\mu\nu}\left(u_{\mu\nu}\, C_{\sigma\rho\delta} C^{\sigma\rho\delta} - \Big(-(n+1)C_\mu{}^{\alpha\beta} + C_\mu{}^{\beta\alpha}\Big)C_{\nu\beta\alpha}\right) \tag{129}$$

so that, on-shell, the tensor u is given by Eq(88), in such a way that the theory would be a bi-metric theory $(g_{\mu\nu}, u_{\mu\nu})$ with a set of Lagrange multiplier fields $\lambda^{\mu\nu}$. However, as we mentioned previously, this is a property following from the particular *choice* of *d-dimensional representatives* of the 2D Horndeski scalars, but other choices are possible, for example using the tensor $\mathcal{N}$, leading to other field contents for the four dimensional theory. Moreover, even when using solely the tensor u, it seems in principle possible to reach the higher order Lagrangian terms $\mathcal{L}_p$ by considering higher order combinations of u, spanned by NP curvature invariants of the form $\nabla^{(i)}u\nabla^{(j)}\mathscr{R}^{(\frac{p}{2}-i-j)}$, where $\mathscr{R}$ stands for an arbitrary polynomial second order curvature tensor.

2. Generalizing the DSS sector of Einstein gravity

Now we will construct a power series of high energy corrections to the Einstein-Hilbert action, and we want it to be as close to the spherically symmetric sector of General Relativity as possible. In order to do so, the main condition is, as before, to preserve the differential structures of the spherically symmetric reduction of both the Ricci scalar and the Einstein tensor. To be more precise, we impose the following requirements that all the corrections of order i have to satisfy :

1. The corrections must preserve the splitting of the Ricci scalar Eq(18) ;
2. Their field equations must preserve the splitting of the Einstein tensor Eq(19) ;
3. The spherically symmetric variational principle is well defined ;
4. The "critical order-d" scalars ($\mathcal{R}^{(d-2)/2}\,\mathcal{S}_{d-2}$, see below), where $d = n+2$ is the dimension of $\mathcal{M}$, contributes to the field equations (otherwise, it gives the Euler characteristics of Σ, as we will see).

Following the equations (100,101), the first requirement is satisfied by considering a linear combination of the four scalars : $\left(\mathcal{R}, \nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}, \nabla^\alpha\nabla^\beta\omega_{\alpha\beta}, R_{(\Sigma)}\right)$. Then, to produce high energy corrections, we just multiply these terms by powers of the curvature invariant $\mathcal{R}$. Because this term is, for spacetime (5), proportional to the Ricci scalar $R^{(n)}(\Omega)$ of the horizon manifold $\Omega_{(n,k,r)}$, this allows to obtain polynomial and second order actions for any order of corrections, without adding degrees of freedom. Moreover, it preserves the splitting of the Ricci scalar.

Therefore, we need to consider the following d-dimensional action :

$$\begin{aligned} I = &\frac{1}{16\pi G}\int_{\mathcal{M}} d^{n+2}x\,\sqrt{-g}\left(R - 2\Lambda + \sum_{i=1}^{m} l^i\,\mathcal{R}^{i/2}\,\left(\alpha_i\,\mathcal{R} + \beta_i\,\mathcal{S}_{(i)}\right)\right) \\ &+ \frac{1}{8\pi G}\oint_{\partial\mathcal{M}} d^{n+1}x\,\sqrt{-\lambda}\,\epsilon\,\left(1 + u\sum_{i=1}^{m} l^i\,\mathcal{R}^{i/2}\right)K\,. \end{aligned} \tag{130}$$

where K is the extrinsic curvature, introduced to have a well-defined variational principle, λ is the determinant of the induced metric on the boundary of spacetime, $\epsilon = 1\,(\text{or} -1)$ for space(time)-like boundary, l is a length scale introduced for dimensional reasons, α_i and β_i are dimensionless series of coupling constants and the non-polynomial scalars $\mathcal{S}_{(i)}$ are defined for any order of correction $\mathcal{O} = i + 2$ by :

$$\mathcal{S}_{(i)} := \frac{a}{2n}\,\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta} + b\,\nabla^\alpha\nabla^\beta\omega_{\alpha\beta} + c\,R_{(\Sigma)}\,, \tag{131}$$

such that, using Eq(100,101), its restriction to (5) has the same form as the one of the Ricci scalar (18) (the term $R^{(n)}(\Omega)$ of it being given by the scalar $\mathcal{R}$, multiplied by α_i in the previous action):

$$\mathcal{S}_{(i)}\rfloor = a\,\frac{\mathcal{D}r.\mathcal{D}r}{r^2} + b\,\frac{\mathcal{D}^2 r^n}{r^n} + c\,R^{(2)}(\gamma)\,, \tag{132}$$

where a, b, c and u are some constants depending on the dimension $d = n + 2$ and on the order of correction i. They will be fixed by imposing the remaining three conditions. After that, the action will be left with two series of free coupling constants α_i and β_i, which is why we have splitted the action in this way $(\alpha\,\mathcal{R} + \beta\,\mathcal{S})$. Some of them will be fixed by requiring for the black hole solutions of the model to be regular at radius $r = 0$ and to have a Schwarzschild-dS behaviour at infinity.

Two dimensional Spherically Symmetric Field Equations :

In order to fix the free constants a, b, c and u, we consider the "$\mathcal{S}$-part" of the action given by the order-$(i+2)$ scalar $\mathcal{R}^{i/2}\,\mathcal{S}_i$ plus the boundary term :

$$I_{\mathcal{S}} := \int_{\mathcal{M}} d^{n+2}x\,\sqrt{-g}\,\mathcal{R}^{i/2}\,\mathcal{S}_{(i)} + 2\,u\oint_{\partial\mathcal{M}} d^{n+1}x\,\sqrt{-\lambda}\,\mathcal{R}^{i/2}\,K\,. \tag{133}$$

Following [284], we choose the boundary to respect the symmetry of ansatz (5). The normal to it is chosen to be $n^\alpha = \{n^a(x), 0, ..., 0\}$, so that the $(n+2)$-dimensional extrinsic curvature K is splitted into the 2-dimensional one and the areal radius :

$$K\rfloor = K^{(2)} + n\frac{n^a\mathcal{D}_a r}{r}\,. \tag{134}$$

The determinents of the metrics also split into $\sqrt{-g} = \sqrt{-\gamma}\sqrt{\sigma}\,r^n$ and $\sqrt{-\lambda} = \sqrt{-h}\sqrt{\sigma}\,r^n$, where σ and h are respectively the determinants of the induced metrics on the horizon manifold of radius unity $\Omega_{(n,k,1)}$, and of the induced metric on the boundary of Σ, noted $\partial\Sigma$. Therefore, by integrated out the angular part (and setting for now $k = 1$ for simplicity), we are left with :

$$I_{\mathcal{S}} = \mathcal{A}_{k,n,1}\left(\int_\Sigma d^2x\,\sqrt{-\gamma}\,r^n\,\mathcal{R}^{i/2}\,\mathcal{S}_{(i)} + 2\,u\oint_{\partial\Sigma} dl\,\sqrt{h}\,r^n\,\mathcal{R}^{i/2}\,\left(K^{(2)} + n\frac{n^a\mathcal{D}_a r}{r}\right)\right).$$

where $\mathcal{A}_{k,n,1} = \int d^n x\sqrt{\sigma}$ is the volume of the n-dimensional manifold $\Omega_{(n,k,1)}$ of radius unity.

Expanding (using eq(100,132)) gives :

$$
\begin{aligned}
\frac{I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} &= \int_\Sigma d^2x\,\sqrt{-\gamma}\,\Big(a\,r^{n-i-2}\mathcal{D}r.\mathcal{D}r + b\,r^{-i}\mathcal{D}^2 r^n + c\,r^{n-i}R^{(2)}(\gamma)\Big)\\
&\qquad + 2\,u\oint_{\partial\Sigma} dl\,\sqrt{h}\,r^{n-i-1}\,\Big(r\,K^{(2)} + n\,\mathbf{n}.\mathcal{D}r\Big)\\
&= \int_\Sigma d^2x\,\sqrt{-\gamma}\,\Big(e\,r^{n-i-2}\gamma^{ab}\mathcal{D}_a r\mathcal{D}_b r + c\,r^{n-i}R^{(2)}(\gamma)\Big)\\
&\qquad + \oint_{\partial\Sigma} dl\,\sqrt{h}\,r^{n-i-1}\,\Big(2\,u\,r\,K^{(2)} + n(2\,u+b)\,n^a\mathcal{D}_a r\Big)\ ,
\end{aligned}
$$

where we have defined $e = a + b\,i\,n$. Now deriving the field equations from this action, while imposing that $\delta\gamma^{ab}\big|_{\partial\Sigma} = 0$ and $\delta r\big|_{\partial\Sigma} = 0$, gives :

$$
\begin{aligned}
\frac{\delta I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} = \int_\Sigma d^2x\,\sqrt{-\gamma}\Bigg[&\Big(c\,(n-i)\,r^2R^{(2)}(\gamma) -\ e\,(n-i-2)\,\mathcal{D}r.\mathcal{D}r - 2\,e\,r\mathcal{D}^2 r\Big)\delta r\\
&+ r\Big(q\,\mathcal{D}_a r\mathcal{D}_b r + p\,\gamma_{ab}\mathcal{D}r.\mathcal{D}r - c\,(n-i)\,r\,\big(\mathcal{D}_a\mathcal{D}_b - \gamma_{ab}\mathcal{D}^2\big)\,r\Big)\delta\gamma^{ab}\Bigg]\,r^{n-i-3}\\
+ \oint_{\partial\Sigma} dl\,\sqrt{h}\,r^{n-i-1}&\Big((c-u)\,r\,h_{ab}\,\mathbf{n}.\mathcal{D}\delta\gamma^{ab} + n\,(2u+b)\,n^a\delta\mathcal{D}_a r\Big)\,,
\end{aligned}
$$

where $q = \Big(e - c\,(n-i)(n-i-1)\Big)$ and $p = \Big(c\,(n-i)(n-i-1) - e/2\Big)$

We can now impose the third requirement of the previous section : in order to have a well-defined variational principle for any order i of correction and in any dimension $d = n+2$, we need to impose $c = u$ and $b = -2u$.

Moreover, following the second requirement, we want to preserve the decomposition of the Einstein tensor in spherical symmetry. As seen in Eq(19), the term proportional to $\mathcal{D}_a r\mathcal{D}_b r$ in the last equation is not present in G_{ab}, while all the others are. Therefore, we impose for any i that $e = c(n-i)(n-i-1)$, in order to cancel this term.

Now that we have fixed the constants a, b and c from these two conditions, the variation of the "$\mathcal{S}$-part" of the action (plus boundary term) becomes :

$$
\begin{aligned}
\delta I_{\mathcal{S}} = u(n-i)\int_\Sigma d^2x\,\sqrt{-\gamma}\Big(&\Big(-(n-i-1)(n-i-2)\mathcal{D}r.\mathcal{D}r + r^2R^{(2)}(\gamma) - 2(n-i-1)r\mathcal{D}^2 r\Big)\,\delta r\\
&+ r\Big(-r\,\big(\mathcal{D}_a\mathcal{D}_b - \gamma_{ab}\mathcal{D}^2\big)\,r + \frac{1}{2}(n-i-1)\gamma_{ab}\mathcal{D}r.\mathcal{D}r\Big)\delta\gamma^{ab}\Big)\,r^{n-i-3}\mathcal{A}_{k,n,1}
\end{aligned}
\tag{135}
$$

We see that when $i = n$, there is no contribution from this part of the action to the field equations. Given that the scalar $\mathcal{R}^{i/2}\,\mathcal{S}_i$ contains in general a number $\mathcal{O} = i+2$ of derivatives of the metric in DSS, this means that when the number of derivatives is equal to the critical order $\mathcal{O}_{\mathcal{C}} := d$, given by the dimension $d = n+2$ of the spacetime $\mathcal{M}$, the scalar $\mathcal{R}^{(d-2)/2}\,\mathcal{S}_{d-2}$ is topological. Interestingly, this is the same order of derivatives at which appears the d-dimensional Euler density associated with $\mathcal{M}$.

In our case however, when restricted to spherically symmetric spacetimes, this order $\mathcal{O} = d$ scalar is the Euler density of the 2-dimensional manifold Σ (which is $R^{(2)}(\gamma)$). Indeed from the relations we just wrote between the constants (a, b, c), the order of correction $\mathcal{O} = i+2$ and the dimension $d = n+2$, one gets :

$$
\frac{I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} = u\int_\Sigma d^2x\sqrt{-\gamma}\,\Big((n-i)(n-i-1)\mathcal{D}r.\mathcal{D}r + r^2R^{(2)}(\gamma)\Big)\,r^{n-i-2} + 2u\oint_{\partial\Sigma} dl\sqrt{h}\,r^{n-i}\,K^{(2)} \tag{136}
$$

Therefore, when $i = n$ (or $\mathcal{O} = d$), this part of our action reduces in spherical symmetry to the 2-dimensional Einstein-Hilbert action together with its Gibbons-Hawking-York boundary term :

$$
\begin{aligned}
&\int_{\Sigma\times\Omega} d^{\,d}x\,\sqrt{-g}\,\mathcal{R}^{(d-2)/2}\,\mathcal{S}_{(d-2)} + 2\oint_{\partial\mathcal{M}} d^{\,d-1}x\,\sqrt{-\lambda}\,\mathcal{R}^{(d-2)/2}\,K\\
&= u\,\mathcal{A}_{k,n,1}\,\Big(\int_\Sigma d^2x\,\sqrt{-\gamma}\,R^{(2)}(\gamma) + 2\oint_{\partial\Sigma} dl\,\sqrt{h}\,K^{(2)}\Big)\ .
\end{aligned}
\tag{137}
$$

So just like for Lovelock-Lanczos gravities, the scalar of critical order $\mathcal{O}_{\mathcal{C}} = d$ does not contribute to the equations of motion. However, it is possible to nonetheless have a non-vanishing contribution from it via the following procedure.

From [256], one sees that the two dimensional Ricci scalar can be expressed, for any unit normalized vector field ξ (that we choose to be space-like) as :

$$R^{(2)}(\gamma) = \mathcal{D}_a v^a = 2\,\mathcal{D}_a \left(-\xi^a\, K^{(2)} + \xi^b \mathcal{D}_b \xi^a\right) .$$

Making use of Stokes theorem, the action reduces to :

$$\begin{aligned}\frac{I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} &= u\,(n-i)\int_\Sigma d^2x\,\sqrt{-\gamma}\,\left(-v^a\, r\mathcal{D}_a r + (n-i-1)\mathcal{D}r.\mathcal{D}r\right) r^{n-i-2} \\ &\quad + u \oint_{\partial\Sigma} dl\,\sqrt{h}\,\left(2K^{(2)} + v^a\, n_a\right) r^{n-i}\,,\end{aligned} \tag{138}$$

Therefore, in order for our fourth and last requirement to be fulfilled (which was that this critical order scalar contributes to the field equations), we just need to choose $u = \frac{1}{n-i}$, the counterpart being that the boundary term is now singular for the critical order scalar[6]. This concludes the construction of this Non-Polynomial gravity theory, which will be extensively studied in the next Chapter.

D. Discussion

We studied in this Chapter the so-called Non-Polynomial Gravity theories, which are (non-polynomial) higher order metric theories of gravity with the following remarkable property : for some specific subsets of the space of metric fields, it yields field equations that are polynomial and second order. Therefore they bypass the Lovelock theorem, stating the uniqueness of Lovelock-Lanczos gravity as metric theories admitting second order field equations, by restricting the set of metric for which this property holds : if one considers the whole space of metrics, L-L gravities are singled out, otherwise, new theories complete the collection. As we will see in the remaining Chapters of this thesis, the interest of having such new second order theories comes from the fact that many Quantum Gravity effective geometries, like Black Holes and Cosmologies, do not contain additional integration constants, so that, if they can be derived from effective metric actions, they should yield second order effective field equations.

Being interested in this thesis by regular black hole and cosmological solutions, we focused mainly on Non-Polynomial Gravity theories whose entire Dynamical Spherically Symmetric sector is polynomial and second order. After having briefly reviewed in Sec.1 the Deser-Sarioglu-Tekin Action and its natural Quadratic Gravity extension, we established in Sec.2 a new algebraic property responsible for the existence of new NPG models, based on the decomposition of the Cotton tensor on DSS spacetimes. It allowed us to find d-dimensional gravitational actions whose DSS is given by two-dimensional Horndeski theory, i.e. the most general scalar-tensor theory leading to second order field equations. Many studies on black holes have followed the effective two-dimensional approach, so that this result provides a proof of existence, in the theory space of $d > 3$-dimensional metric gravity, of theories admitting these effective black holes.

Finally, in Sec.3, we attempted to reduce the large freedom of two-dimensional Horndeski theory by generalizing the DSS sectors of both Einstein and Gauss-Bonnet gravities for arbitrary orders of correction. In both cases, the resulting theories contains two arbitrary dimensionless coupling constants per higher order of correction. The generalization procedure relies on the preservation of the DSS decomposition of both the Lagrangian and the field equations of the mentioned theories, so that it can be viewed as minimal high energy corrections to these theories. In the case of the generalization of Gauss-Bonnet gravity, Lovelock-Lanczos gravity is recovered as a special case. Interestingly, we also saw that the minimal generalization of the DSS sector of Einstein gravity shares some similarities with L-L gravity, in the sense that the Critical order scalar reduces in DSS to a topological invariant, the Euler characteristic of the two-dimensional manifold Σ. Due to the results of Sec.2, we were able to find d-dimensional Non-Polynomial Gravity theories admitting these dynamical spherically symmetric sectors.

[6]However, if we choose the (arbitrary) vector field ξ to be orthonormal to the boundary, in such a way that $\xi = n$, then we see that the boundary term cancel identically, because ξ being a unit normalized vector field, $\xi_a \xi^b \mathcal{D}_b \xi^a = 0$. In this case, the divergence is no more an issue.

III. Minimal extension of General Relativity dynamical spherical symmetry

This Chapter will mainly focus on the study of the previously found gravitational action Eq(130). As we saw, it constitutes a two sets of coupling constants deformation of General Relativity, in the form of a power series of corrections in a length scale l. In dynamical spherical symmetry, it reduces to a minimal extension of GR, in the sense that it is the only class of corrections which preserves the DSS decomposition of both the Ricci scalar and Einstein tensor, for all the higher orders.

In the two-dimensional covariant formalism, this family of theories belongs to the subset of the so-called Einstein-Dilaton gravity, given that it does not contain the additional term $\mathcal{Y}^2 - \mathcal{Y}_2$, specific of the more general two dimensional Horndeski theory, while the functions (η, G_3, G_4) of Eq(94) depends on the scalar field r, but not on its kinetic term. As we said before, these theories have been extensively studied as effective approaches in the investigations of black holes properties, see for example [41–43]. In particular, one of the members of the family of theories that we will consider has already been found in [41], where the corresponding Poisson-Israel regular black hole,

$$ds^2 = -\left(1 - \frac{2\,r^2 M}{r^3 + l^3}\right) dt^2 + \frac{dr^2}{1 - \frac{2\,r^2 M}{r^3+l^3}} + r^2 d\Omega^2 \tag{139}$$

discovered in [225] from a semi-classical argument regarding the high energy corrections to Schwarzschild geometry coming from the vacuum energy-density, was recovered from a 2D Einstein-Dilaton gravity. Within our approach, it will correspond to an Einstein-Hilbert action supplemented by an order five non-polynomial correction. As we will see, it is the lower order of corrections able to cure the Schwarzschild singularity in four dimensions.

What is interesting with the series of actions Eq(130) is that it provides a natural generalization of this black hole to arbitrary high order of corrections. As we will see, it also possesses many interesting features that are expected from Quantum Gravity arguments. Indeed, after reviewing the action, we will see in Sec.1 that the spherically symmetric solutions of this model are all the rational (in the radius) black hole where g_{tt} is linear in the mass. Moreover, the models satisfy Birkhoff theorem for any order of corrections.

In Sec.2, we show that there exists some particular combinations of the free sets of dimensionless coupling constants (α, β) (i.e. particular subclasses of the Actions Eq(130)) for which the solutions

- Are Regular at $r = 0$, what requires for the minimal order of correction to E-H Action to be greater that the critical order (its number of derivatives of the metric is greater than the number of dimensions), in what case the geometry describes by an (A)dS-core regular black hole. This means that *perturbative* corrections in the coupling constant l are able to regularize the Schwarzschild singularity, what is very different from the other actions leading to regular black holes that we will study in the last two Chapters of this thesis. However, similar results have been found in [285, 286] where the singularity of the Newton potential was found to be cured from four dimensional *polynomial* gravitational models, as soon as the order of the correction is (strictly) greater than four ;
- Contain the correct one-loop Quantum Correction to Newton potential [177–179], providing that one identifies the length scale l in the action with the Planck length, and Schwarzschild asymptotic behaviour at $r \to \infty$. This quantum gravity inspired correction arises within the present approach from the Critical Order Scalar, i.e. in four dimensions, from the analogue of a Gauss-Bonnet scalar, what is in accordance with the "one-loop" character of the correction [177] ;
- Contain, together with the cosmological constant, an additional infrared correction which gives an exact linear correction to Schwarzschild asymptotic behaviour. These kind of corrections have already been considered in [287–292] to model the dark "matter" phenomenon on galactic scales, but within the framework of Conformal gravity.

Two specific examples of four dimensional solutions are presented in Sec.3. We focus there on the first two orders of correction to E-H action able to cure the singularity. For an order five correction[7],

[7] We refer the reader to the last paragraph of the second Section of the previous Chapter for what we mean precisely by an odd number of derivatives.

the model admits the Poisson-Israel regular black hole as we mentioned previously, while for sixth order of correction, the solution is very similar to the Non-Minimal Yang-Mills (A)dS-core black hole found in [101, 102] from a theory involving a $SU(2)$-Yang-Mills field, in a Wu-Yang monopole configuration [293], non-minimally coupled with gravity. Moreover, we briefly check in both these cases that the presence of the critical order-four correction (responsible to the one-loop Quantum Correction to Newton potential) also implies a logarithmic correction to the classical entropy of the outer horizons. This is a characteristic shared by many semi-classical quantum correction to the Bekenstein-Hawking entropy, see for example [65–68, 294–313]. However, providing that the sign of the correction to the Newton potential agrees with [177], then the factor in front of the logarithmic correction to the entropy is positive, which is not the most popular result. In Sec. 4, we derive some thermodynamical quantities of the full solution, but unfortunately, we had no time to analyse them in details for specific cases. Note however that it confirms the role of the Critical Order Scalar in the logarithmic correction of the entropy, from both the minisuperspace Wald entropy and the assumption of the validity of the first law of thermodynamics.

The Sec. 5 is devoted to the study of the vacua ($M = 0$) of these models, where we find that it is possible to have Minkowski spacetime for all the orders of corrections. Interestingly, extremal black hole vacua are also a possibility, what will be used in Sec.7 regarding the mass-inflation instability of (A)dS-core regular black holes. In this case, Minkowski space acquires a small scale structure described by an extremal horizon boundary, near which the spacetime behaves as a Bertotti-Robinson metric $AdS_2 \times S^2$, see [314–317]. The existence of such extremal vacuum states provides a realization of Nernst's third law of thermodynamics for this class of model, as indeed the entropy of the black hole solutions tends to a universal constant as their temperature goes to zero.

The Sec.6 shows that, providing one considers a non-minimal coupling between the Maxwell Lagrangian and powers of the non-polynomial scalar $\mathcal{R}$, responsible for the higher order corrections to E-H action[8], the Coulomb singularity can also be cured, while the curvature remains regular.

In Sec.7, we address the issue of the so-called Mass-Inflation instability [181–190], which affects any (A)dS-core regular black holes due to the fact that they contain an inner horizon. We show that within this approach, the only way to avoid mass-inflation in the neutral case is to consider an infinite number of corrections, what makes the associated actions non-perturbative in the coupling constant l. We present two distinct ways to do so within our setting : one based on an infinite series of corrections, leading at each order to (A)dS-core regular black holes sharing the same extremal vacuum, but in which a discontinuity arises at the inner horizon in the limit of infinite orders (without introducing a curvature singularity) ; another is simply to choose the coupling constants so that the resulting non-perturbative theory has a single-horizon regular black hole, see for example [41–44], in which such black holes were found in the context of the polymer quantization of spherically symmetric models. For the charged case, we show that it is still possible to avoid the mass-inflation, provided that a particular bound between the mass and the charge is respected. It is of the form

$$2M \geq l + \frac{Q^2}{l} \tag{140}$$

Finally, in the Sec.8 we briefly investigate the FLRW sector of the model, and comment more generally on the FLRW sector of NPG models.

A. Action & Rational Regular Black Holes

We start by a quick summary of the theory Eq(130) that we will use in this Chapter. From the conditions on the constants (a, b, c, u) of Eq(131) that we have previously found, we obtain the following action,

$$\begin{aligned} I =& \frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g} \sum_{i=-2}^{m} l^i \, \mathcal{R}^{i/2} \left(\alpha_i \, \mathcal{R} + \beta_i \, \mathcal{S}_{(i)} \right) \\ &+ \frac{1}{8\pi G} \oint_{\partial\mathcal{M}} d^{n+1}x \sqrt{-\lambda} \sum_{i=0}^{m} l^i \, \frac{\beta_i}{n-i} \, \mathcal{R}^{i/2} \, K \, . \end{aligned} \tag{141}$$

[8] We remind the reader that the DSS decomposition of this scalar makes it proportional to the Ricci scalar of the horizon manifold $\Omega_{n,k,r}$.

where, from the definitions (131), (100), (101) and the relations previously found between the constants (a, b, c, u) and (i, n), we have :

$$\begin{aligned} \mathcal{S}_{(i)} &= \frac{1}{(i-n)(n^2-1)} \left(-\frac{2i(i+1)n}{n^2-1} \nabla_\gamma u_{\alpha\beta} \nabla^\gamma u^{\alpha\beta} + R + 2n \left(R^{\alpha\beta} - \nabla^\alpha \nabla^\beta \right) u_{\alpha\beta} \right), \\ \mathcal{R} &= \frac{1}{(n-1)^2(n+1)} \Big(nR + 2 \left(R^{\alpha\beta} - \nabla^\alpha \nabla^\beta \right) u_{\alpha\beta} \Big), \end{aligned} \tag{142}$$

which only involves two new scalars $\left(R^{\alpha\beta} - \nabla^\alpha \nabla^\beta \right) u_{\alpha\beta}$ and $\nabla_\gamma u_{\alpha\beta} \nabla^\gamma u^{\alpha\beta}$, highly non-linear in curvature tensors, that are expressed in terms of $u_{\alpha\beta}$, defined by Eq(88).

Note that we have included the cosmological constant and the four dimensional Ricci scalar into the sum because, providing that $\alpha_{-2} = -2l^2\,\Lambda$, $\beta_{-2} = 0$, $\alpha_{-1} = \beta_{-1} = 0$, $\alpha_0 = n(n-1)$ and $\beta_0 = n$, the first two non-vanishing term of the sum are indeed the cosmological constant Λ and the Ricci scalar R. This is true for any backgrounds, not only for spherical symmetry, due to the previous definitions of $\mathcal{S}_{(i)}$ and $\mathcal{R}$, which give :

$$R = n(n-1)\mathcal{R} + n\,\mathcal{S}_0. \tag{143}$$

Remark that this background independent property is the result of our particular *choice* of *representatives* of the corresponding two-dimensional scalars. We will see later the role of the terms involving α_{-1} and β_{-1}, but as they are infrared corrections to General Relativity, we neglect them for now.

In spacetimes (5), we can integrate out the angular part of Lagrangian, so that the action simply reduces to :

$$\begin{aligned} I \Big\rfloor &= \frac{\mathcal{A}_{k,n,1}}{16\pi G} \sum_{i=-2}^{m} l^i\, k^{i/2} \left\{ \mathscr{I}_{(i)} + \frac{2\beta_i}{n-i} \oint_{\partial\Sigma} dl\, \sqrt{h}\, r^{n-i}\, K^{(2)} \right\}, \\ \mathscr{I}_{(i)} &:= \int_\Sigma d^2x\, \sqrt{-\gamma} \left(\alpha_i\, k + \beta_i \left((n-i-1)\mathcal{D}r.\mathcal{D}r + \frac{r^2}{n-i}\, R^{(2)}(\gamma) \right) \right) r^{n-i-2}\,. \end{aligned} \tag{144}$$

It is quite surprising that such a complicated action (141), which is highly non-linear in the curvature tensors, leads to such very simple spherically symmetric action (144). As we saw previously, this is essentially due to the algebraic property Eq(86), whose effect in the action can be reabsorbed by a curvature constraint of the form

$$\lambda^{\mu\nu} \Big(u_{\mu\nu}\, C_{\sigma\rho\delta} C^{\sigma\rho\delta} - \Big(-(n+1) C_\mu{}^{\alpha\beta} + C_\mu{}^{\beta\alpha} \Big) C_{\nu\beta\alpha} \Big)$$

in which case the theory is a bi-metric theory $(g_{\mu\nu}, u_{\mu\nu})$, with a set of Lagrange multipliers $\lambda^{\mu\nu}$.

Rational Regular Black Holes

The simpliest way to find the general solution for spacetimes (5) of this class of models is the Weyl approach, that also allows to prove without trouble Birkhoff theorem as we will see. Due to the Principle of Symmetric Criticality applied to spherical symmetry [238,239], the Weyl approach is strictly equivalent to the derivation of the full equations of motion and then performing the symmetric reduction, what makes this “Principle” very powerfull in the context of non-polynomial gravity where the full equations of motion can be very complicated.

Recall that the bulk action is :

$$\frac{\mathcal{A}_{k,n,1}}{16\pi G} \int_\Sigma d^2x\, \sqrt{-\gamma}\, r^n \sum_{i=-2}^{m} l^i\, k^{i/2}\, \mathcal{R}^{i/2} \Big(\alpha_i\, \mathcal{R} + \beta_i\, \mathcal{S}_{(i)} \Big) \tag{145}$$

To apply the Weyl method, we choose the gauge,

$$d\Sigma^2 = -a(t,r) b(t,r)^2 dt^2 + \frac{dr^2}{a(t,r)} + 2b(t,r) f(t,r) dt\, dr, \tag{146}$$

where we have chosen the scalar field $r(x^A)$ to be the radial coordinate, so that the $\mathcal{S}$-term of the action gives :

$$
\begin{aligned}
&\sqrt{-\gamma}\, r^n\, \mathcal{R}^{i/2}\, \mathcal{S}_{(i)}\Big\rfloor = r\, Z\left(ab' - f\frac{\dot{a}}{a} + \frac{abff'}{1+f^2}\right) - \frac{1}{n-i}\left(\partial_r \Psi + \partial_t \Phi\right)\,,\\
&\text{with}\,,\\
&Z = k^{i/2} r^{n-i-2}/\sqrt{1+f^2}\,,\quad \Psi = rZ\Big((n+i)ab + r(ba' + 2ab') - rf\frac{\dot{a}}{a}\Big)\,,\\
&\Phi = \frac{rZ}{ab}\Big(f\left(2nab + r(ba' + 2ab')\right) + 2rabf' + r\frac{\dot{a}}{a}\Big)\,.
\end{aligned}
\tag{147}
$$

what makes the bulk action proportional to :

$$
\int_\Sigma dr\, dt \sum_{i=-2}^{m} l^i\, k^{i/2}\, r^{n-i-2}\left(\alpha_i\, k\, b\sqrt{1+f^2} + \frac{\beta_i\, r}{\sqrt{1+f^2}}\left(ab' - f\frac{\dot{a}}{a} + \frac{abff'}{1+f^2}\right)\right) \tag{148}
$$

Following [318], in order to prove Birkhoff theorem we can derive the equations of motion with respect to a, b and f, and then evaluate the result for $f = 0$. This provides an additional equation which enforces the integration constant (related to the physical mass of the object) to be time-independent. Therefore, one can see that it will be equivalent to derive the equations of motion from the following integral :

$$
\int dr\, dt\left(\Delta_\alpha\, k\, b + \Delta_\beta\, r\left(ab' - f\frac{\dot{a}}{a}\right)\right) \tag{149}
$$

where we have defined $\Delta_\beta := \sum_{i=-2}^{m} l^i\, k^{i/2}\, r^{n-i-2}\, \beta_i$ and $\Delta_\alpha := \sum_{i=-2}^{m} l^i\, k^{i/2}\, r^{n-i-2}\, \alpha_i$. The equations of motion are then easy to derive and given by :

$$
\begin{aligned}
\left.\frac{\delta I}{\delta a}\right|_{f=0} &= (r\Delta_\beta)\, b' = 0\\
\left.\frac{\delta I}{\delta b}\right|_{f=0} &= k\Delta_\alpha - \Big(r\Delta_\beta\, a\Big)' = 0\\
\left.\frac{\delta I}{\delta f}\right|_{f=0} &= -\,(r\Delta_\beta)\,\frac{\dot{a}}{a} = 0
\end{aligned}
\tag{150}
$$

where primes and dots are respectively derivatives with respect to the radius r and time coordinate t. It has the following solutions,

$$
\begin{aligned}
a(r) = {}& \frac{1}{n\, r^m + \sum_{i=1}^{m} l^i\, k^{i/2}\, \beta_i\, r^{m-i}}\Bigg(\left(-\frac{2}{n+1}\Lambda r^2 + k\, n\right) r^m + k\sum_{i=1}^{n-2} l^i\, k^{i/2}\,\frac{\alpha_i}{n-i-1}\, r^{m-i}\\
&+\left(l^{n-1} k^{(n+1)/2}\alpha_{n-1}\log\left(\frac{r}{l}\right) - 4\, M\right) r^{m-n+1} + k\sum_{i=n}^{m} l^i\, k^{i/2}\,\frac{\alpha_i}{n-i-1}\, r^{m-i}\Bigg)\\
b = {}& b(t), \quad \text{and} \quad a(t,r) = a(r)
\end{aligned}
\tag{151}
$$

Where M is proportional to the ADM mass of this spacetime when the topology is spherical ($k = 1$), and as a consequence of the third equation, it is time-independant, what proves Birkhoff theorem.

Therefore, except for the logarithmic term, the unique solutions of models (141) in sector (5) are all the rational black holes with g_{tt} linear in the integration constant M, because it is clear that we can fix the coupling constants α_i in order to have $a(r_H) = 0$ for some positive radius r_H corresponding to the location of an event horizon.

Note that this general solution reduces to Schwarzschild-(A)dS geometry when the topology of the horizon is toroidal ($k = 0$). This is because of our use of the scalar $\mathcal{R}$ in the construction of higher order corrections to E-H action. In dynamical spherical symmetry, this scalar reduces to $\mathcal{R} = k/r(x^a)^2$, meaning that when $k = 0$, it identically vanishes. It means that the solutions with planar or toroidal topology are identical to those of GR, i.e. singular.

B. Regularity conditions & Asymptotic behaviour

Now we will show how all these solutions can have a Schwarzschild-(A)dS behaviour at spatial infinity ($r \to \infty$) by imposing conditions on the α_i for $1 \leq i \leq n-2$. They can be made regular at the center ($r \to 0$) by fixing properly α_m and α_{m-1}, replacing the classical singularity of the Schwarzschild geometry by an (A)dS regular core. It is also possible to have the proper quantum correction to Newton potential if $\beta_2 \neq 0$, i.e. if we consider the contribution of the critical order scalar $\mathcal{R}^{(d-2)/2}\mathcal{S}_{(d-2)}$. Finally a correction to the Newton gravitational potential that is linear in the radius can be found, by considering the infrared correction $\sqrt{\mathcal{R}}$, i.e. providing $\alpha_{-1} \neq 0$.

1. Behaviour at Spatial Infinity

First, let's see how it is possible for these black holes to have a behaviour at spatial infinity that is very close (or identical) to the usual Schwarzschild-(A)dS one. In order to do so, we need to impose some relations between $n-2$ dimensionless coupling constants (α, β), because otherwise, the asymptotic behaviour of the solution would be of the form $a(r \to \infty) = \sum_{p=-2}^{d-3} q_p r^{-p} + O(r^{2-d})$ with non vanishing coefficients q_p. So we need to cancel all the q_p from $p = 1$ to $p = d-4$ in order to have the usual Schwarzschild-(A)dS behaviour.

This can be done in a quite straightforward way by imposing the following relations for all $1 \leq i \leq n-2$: $\alpha_i = (n-i-1)\left(\beta_i + \frac{2l^2\Lambda}{n^3(n+1)}\left\{\beta_i\left(n\beta_2 - \beta_1^2\right) + n\left(\beta_{i+1}\beta_1 - n\beta_{i+2}\right)\right\}\right)$. In four dimensions, there is no relation to impose because $n = 2$. The behaviour of the general solution (151) at infinity then becomes :

$$a(r \to \infty) = -\frac{2\Lambda}{n(n+1)}\, r^2 + \frac{2\sqrt{k}\,\Lambda\, l\,\beta_1}{n^2(n+1)}\, r + k\left(1 - 2\,l^2\,\Lambda\,\frac{\beta_1^2 - n\,\beta_2}{n^3\,(n+1)}\right) - \frac{4M}{n\,r^{\,d-3}}$$
$$+\frac{k^{\frac{n+1}{2}}\,l^{n-1}}{n\,r^{\,d-3}}\left(\alpha_{n-1}\log\left(\frac{r}{l}\right) - \beta_{n-1} + 2l^2\Lambda\frac{\beta_{n-1}\left(\beta_1^2 - n\beta_2\right) + n\left(n\beta_{n+1} - \beta_1\beta_n\right)}{n^3(n+1)}\right) + O\left(\frac{1}{r^{d-2}}\right) \tag{152}$$

We see that when the cosmological constant is set to zero, $\Lambda = 0$, we just need to impose $\alpha_{n-1} = \beta_{n-1} = 0$ in order to have exactly the Schwarzschild asymptotics. Otherwise, we need in addition to set $\beta_1 = \beta_2 = \beta_{n+1} = 0$. That being said, the corrections present in the previous equation could have interesting observational consequences, although they are very small.

Remark also that in higher dimensions, the previous relations imply the presence of corrections in the action of the form $\left(\Lambda\, l^{i+2}\,\mathcal{R}^{(i+1)/2}\right)$ in order to have the previous asymptotic behaviour, meaning that the cosmological constant appears in the high energy corrections for $1 \leq i \leq d-4$ (expect if one just sets all the α_i and β_i involved in the previous relations to zero).

2. Infrared correction and Dark Matter

Until now, we have set the coupling constants $\alpha_{-1} = \beta_{-1} = 0$. Although we will only consider the first one, they both correspond to Infrared corrections to E-H action. Therefore, they might have interesting consequences on the asymptotic structure $r \to \infty$ of the solution of the field equations (150). To see this, like for the cosmological constant term, we have to redefine $\alpha_{-1} = 4\,\lambda\, l$, where λ is the inverse of a large length scale parameter. Restricting to four dimensions ($n = 2$) and spherical topology ($k = 1$) and setting $\alpha_1 = \beta_1 = 0$, we can consider the action :

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(-2\Lambda + 4\lambda\sqrt{\mathcal{R}} + R + \sum_{i=2}^{m} l^i\,\mathcal{R}^{i+2}\Big(\alpha_i\mathcal{R} + \beta_i\mathcal{S}_{(i)}\Big)\right). \tag{153}$$

Its unique static solution has the following behaviour at infinity :

$$a\,(r \to \infty) = -\frac{\Lambda\, r^2}{3} + \lambda\, r + \left(1 + \frac{1}{6}l^2\Lambda\beta_2\right) - \frac{2M_{\text{eff}}}{r} + O\left(\frac{1}{r^2}\right) \tag{154}$$

with $M_{\text{eff}} = M - \frac{l^2}{12}\left(-3\beta_2\lambda + \beta_3\, l\,\Lambda\right)$. This asymptotic behaviour is the same for all maximal order of correction $m+2$. If $m < 3$, we have $\beta_3 = 0$ and if $m < 2$, $\beta_2 = 0$.

Therefore this infrared correction $\sqrt{\mathcal{R}}$ contributes linearly in the radius r, which is the same kind of correction to Schwarzschild geometry that appears in Conformal Gravity where $a(r) = 1 - \frac{2M}{r} + \lambda\, r$ and $b(r) = 1$, see [287] and also [288] for a similar solution in $F(R)$ gravity.

In Conformal Gravity, the parameter λ is an integration constant and not a coupling constant, because in the metric formalism, the field equations of Conformal gravity are fourth order. However, it turns out that by considering it to be a universal constant (like a coupling constant) with value $\lambda = 5.42 \times 10^{-41}\text{cm}^{-1}$ [290], it was shown that this spherically symmetric solution allows to fit the shapes of the rotation curves of some galaxies (see [289–292]). In this setting, dark "matter" would be an effect of the curvature, not of some unknown kind of matter.

However, there is a main difference between Conformal Gravity and our action. In this first, there seems to be another contribution to the rotation curves that is specific to Conformal Gravity (or more generally higher order theories). It comes from large scale inhomogeneities and it is the combination of the two contributions that allows to fit the rotation curves. In our case we do not seem to have this last one, and therefore, even if the linear correction of Eq(154) can improve the behaviour of GR around galactic scales, it might not be able to fit completely the rotation curves of galaxies and some (but less) additional dark matter components should still be necessary, for example, dark matter has been proposed to be made of black [219–221] or white hole remnants [223, 224], new fundamental particles [319, 320], or superfluid matter [321–324], to cite some possibilities. An advantage of our approach to get a linear correction to the gravitational potential is that λ is a coupling constant which explains why it would have a universal value (at least at a given scale).

3. Quantum correction to Newton potential

A very important condition to satisfy for a high energy correction of Schwarzschild black hole is to have the correct one-loop quantum correction to Newton potential $\left(M\, l_P^2/r^3\right)$, see [177–179], where l_P is the Planck length. This correction is one of the few consistency check at one's disposal in order to check if a high energy modification of gravity is in accordance with what we know about low energy quantum gravity. To see that it is indeed possible for the solution (151), we restrict again to four dimensions ($n = 2$), spherical topology ($k = 1$) and set $\alpha_{-1} = \beta_{-1} = \alpha_1 = \beta_1 = 0$ and $\Lambda = 0$.

Then the behaviour of the general solution (151) at infinity is :

$$a\,(r \to \infty) = 1 - \frac{2M}{r} - (\alpha_2 + \beta_2)\,\frac{l^2}{r^2} + \beta_2 \frac{M\, l^2}{r^3} - (\alpha_3 + 2\beta_3)\,\frac{l^3}{4r^3} + O\left(\frac{1}{r^4}\right) \tag{155}$$

Like previously, if $m < 3$ we have in addition $\alpha_3 = \beta_3 = 0$. We see that the correct one-loop quantum correction to Newton potential is indeed present, providing that we identify our coupling constant length scale l with the Planck length $l = l_P$ and consider $\beta_2 \neq 0$. In order to have only this correction up to $O\left(\frac{1}{r^4}\right)$, we also need to set $\alpha_2 = (n - i - 1)\,\beta_2$, with $i = 2$, and $\alpha_3 = (n - i - 1)\beta_3$ with $i = 3$.

It is interesting to see that this correction comes from the critical order scalar $\mathcal{R}^{(d-2)/2}\mathcal{S}_{(d-2)}$. As we saw in Eq(137), this scalar is proportional to the Ricci scalar $R^{(2)}\,(\gamma_{ab})$ of Σ in spherical symmetry, making it a topological term if our fourth condition at the beginning of the corresponding section of the previous Chapter would not be satisfied. Therefore, the one-loop quantum correction to Newton potential is indeed coming from the first (even order) high energy correction to GR in the theory (141), what reinforces its interpretation as a would be effective field theory. We will see in the next section that this term is also responsible for the logarithmic correction to the classical entropy.

Moreover, the sign of β_2 was found to be positive in [177–179] which is good thing in our case because otherwise it could produce some true singularities at the denominator of $a(r)$. However, even if $\beta_2 < 0$, it remains possible to adjust accordingly the remaining constants such that the denominator is never vanishing for $r > 0$.

4. Central (A)dS-core

Finally, it turns out that these black hole solutions, that are in some sense the slightest corrections to Schwarzschild geometry, can be made regular at the origin, providing that the Sakharov criterion holds. It consists of imposing that the black hole has an (A)dS core close to $r = 0$, what ensures that the curvature scalars are regular at the origin, see [180] and also [155] for a discussion of a covariant formulation of this criterion.

The first thing to note is that the regularity is possible only when the maximal order of correction $\mathcal{O}_{\mathcal{M}} := m + 2 > d = n + 2$, i.e. the scalar $\mathcal{R}^{m/2}\left(\alpha_m \mathcal{R} + \beta_m \mathcal{S}_{(m)}\right)$, contain more derivatives than the number of dimensions, given by the critical order $\mathcal{O}_{\mathcal{C}} = d$. Taking into account this regularity condition and defining $\gamma_i := \beta_i - \frac{\alpha_i}{n-i-1}$, the Sakharov criterion implies the following relations between α and β: $\gamma_m = \gamma_{m-1} = 0$ [9].

This is the same class of relations $\alpha_i = (n-i-1)\,\beta_i$ that we need to have both the quantum correction to Newton potential and the Schwarzschild behaviour at $r \to \infty$ for higher dimensional solutions. As we will see later, it is also the necessary and sufficient condition to have Minkowski vacuum for any maximal order of correction $\mathcal{O}_{\mathcal{M}} = m+2$.

For all the cases where $\mathcal{O}_{\mathcal{M}} > \mathcal{O}_{\mathcal{C}} + 1$, it gives the following behaviour close to the origin :

$$a(r \to 0) = k - \left(\frac{\gamma_{m-2}}{\beta_m}\right)\frac{r^2}{l^2} + O(r^3) \tag{156}$$

Whereas when $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{C}} + 1$, we need the additional assumption $\alpha_{n-1} = 0$, corresponding to the vanishing of the logarithmic term, which otherwise would appear in the (A)dS term at this maximal order of correction. The central core then behaves as :

$$a(r \to 0) = k - \left(\frac{4M\,k^{-(n+1)/2}\,l^{1-n} + \beta_{n-1}}{\beta_{n+1}}\right)\frac{r^2}{l^2} + O(r^3) \tag{157}$$

It is interesting to note that despite some similarities of the action (141) with Lovelock-Lanczos gravity that we have seen in Eq(137), a main difference is that it is possible in our case to have scalars containing more derivatives than the critical order ($\mathcal{O} = d$), and yet contributing to the field equations. Whereas in L-L gravity, the critical order scalar is a topological term, and all the L-L scalars with more derivatives are identically vanishing.

As we can see from the previous regularity criterion, this is precisely the condition to avoid the classical singularity : we need to consider corrections to Einstein-Hilbert action containing (at least one) more derivatives than the critical order. For example, in a 4-dimensional spacetime, we need at least a sixth order correction (of the form $\mathscr{R}^3$, discarding "odd order" corrections), but there exists no such correction, polynomial in curvature, leading at the same time to second order field equations for any metric (by Lovelock theorem), and it is known that the usual sixth order polynomial higher derivatives gravities do not cure the singularity. However, in the recent papers [285, 286], it was reported that the Newtonian limit of polynomial higher order gravities becomes regular at the origin precisely when considering at least sixth order scalars, which is an interesting similarity with the result presented here.

Therefore, in the present case, it is possible to cure the Schwarzschild singularity from perturbative corrections in the parameter l, in the sense that the action (141) is a power series of l (not in the sense that we have an expansion in term of small perturbations of the metric around a background solution $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$), and we need a finite number of corrections in l to have a regular core. From this point of view, curing the classical singularity of the black hole solution (151) of the theory (141) is a perturbative problem. It is very different from the other actions leading to regular black holes that we will study in the last two Chapters of this thesis, which need to be non-perturbative in l.

Finally, remark that this parameter appears as a coupling constant in the Lagrangian, and therefore can never be vanishing, whatever the configuration (determined in the neutral case by the mass M) of the solutions. As mentioned previously, this is a property shared by a lot of quantum gravity inspired corrections to Schwarzschild geometry for example Polymer [41–47], Asymptotically Safe [54–63], Non-Local [133] and Non-commutative geometry inspired (from a Generalized Uncertainty Principle) [134–136] or Conformal [137–139] black holes. In these approaches, the parameter from which the singularity is cured usually arises as a quantization parameter or a coupling constant in an effective action.

This is quite different from the resolutions of the singularity problem via the presence of additional fields, for example in Einstein gravity minimally coupled with Non-Linear Electrodynamics (NLED) [90–95], and its $f(R)$ [96], Gauss-Bonnet [97] extensions, still coupled with NLED, in

[9] Note that, in higher dimensions and with a vanishing cosmological constant, these regularity conditions are curiously the same as the relations needed to have a Schwarzschild asymptotic behaviour : curing the central singularity requires the same structure in the coupling constants ($\alpha_i = (n-i-1)\,\beta_i$) than the one needed to recover Einstein gravity at infinity.

the Non-Minimal Yang-Mills theory [101, 102], in theories where a Phantom scalar field is introduced [104, 105], etc. In these cases, there exists singular configurations, those for which the integration constants Q, which come from the field equations associated with the additional fields, and regularize the singularity, are vanishing. Thus, if one intends to solve the singularity problem in this way, one also has to explain why this particular configuration $Q = 0$ is forbidden. For example, in the case of adding exotic fluids on the right-hand-side of Einstein equations to cure singularities, it should also be explained why such a fluid should always be present at high energy, or why all kind of matter should behave like such fluid at high energy, in order for the singularity to be absent in general.

These are possible reasons why the resolution of the singularity issue from second order gravitational corrections of the Einstein equations might be satisfactory, and thus, why gravitational theories like Lovelock-Lanczos, Quasi-Topological and Non-Polynomial gravities present some interests in this regard.

C. Examples

In the last section, we have found that the unique spherically symmetric solution of (141) which is regular at the origin and have the slightly modified Schwarzschild-(A)dS asymptotic behaviour (152) is given by :

$$\begin{aligned} a(r) = k + \frac{1}{n\, r^m + \sum_{i=1}^{m} l^i\, k^{i/2}\, \beta_i\, r^{m-i}} \Bigg(&-\frac{2\,\Lambda}{n+1} \left(r^{m+2} - \frac{1}{n^3} \sum_{i=1}^{n-2} l^{i+2}\, k^{(i+2)/2}\, \zeta_i\, r^{m-i} \right) \\ &+ \left(l^{n-1} k^{(n+1)/2} \left(\alpha_{n-1} \log\left(\frac{r}{l}\right) - \beta_{n-1} \right) - 4\,M \right) r^{m-n+1} - \sum_{i=n}^{m-2} l^i\, k^{(i+2)/2}\, \gamma_i\, r^{m-i} \Bigg) \end{aligned} \tag{158}$$

where $\zeta_i = \beta_i \left(n\beta_2 - \beta_1^2 \right) + n \left(\beta_{i+1}\beta_1 - n\beta_{i+2} \right)$. It depends on : the dimension of the considered spacetime $d = n + 2$; the maximal order of correction $\mathcal{O}_{\mathcal{M}} = m + 2 > \mathcal{O}_{\mathcal{C}} = d$; the cosmological constant Λ ; the length scale l ; the remaining dimensionless coupling constants (α, β) ; the mass of the system M ; the topological parameter k. When $\mathcal{O}_{\mathcal{M}} = d + 1$, we need to set $\alpha_{n-1} = 0$. For the quantum correction to Newton potential to be present we need $\beta_2 \neq 0$ and $l = l_P$.

Finally, in order for the solutions to be regular for all positive values of the radius r, one must also impose that the denominator is never vanishing for these values. As the coupling constants β_i are free, this is always possible to do for any m and n. However, if one wants that the theory gives regular solution for both spherical ($k = 1$) and hyperbolic ($k = -1$) topologies, a sufficient condition would be to consider the non-vanishing coupling constants to be all the $\alpha_{(4i)}$ and $\beta_{(4i)}$. Otherwise, the denominator could be regular for one topology and divergente for another. As we already said earlier, the case of totoidal topology ($k = 0$) just gives General Relativity because all the corrections vanish, and thus, it seems not possible to make these kind of solutions regular within our model.

Now we will see two four-dimensional examples of already known regular black holes that are subcases of this general solution (158).

1. Poisson-Israel regular black hole ($m = 3$)

In four dimensions, the lowest maximal order of correction $\mathcal{O}_{\mathcal{M}} = m + 2$ needed in order to obtain a regular black hole solution is $\mathcal{O}_{\mathcal{M}} = d + 1 = 5$, i.e., its highest order of correction is given by the scalar $R^{3/2} \left(\alpha_3 \mathcal{R} + \beta_3 \mathcal{S}_{(3)} \right)$. It turns out that a particular case of the action (141) with $m = 3$ gives the so-called Poisson-Israel regular black hole. This is why we have considered half integer powers of the scalar $\mathcal{R}$ in the action. The solution reads[10] :

$$a(r) = k - \frac{8Mr^2}{4r^3 - 2k\alpha_2 l^2 r - \alpha_3 k^{3/2} l^3} \tag{159}$$

where the regularity conditions have already been imposed and read $\beta_2 = -\alpha_2$ and $\beta_3 = -\alpha_3/2$. In order to have a non-vanishing denominator for $r > 0$ and $k = 1$ we also need $\alpha_2 \leq 0$ and $\alpha_3 \leq 0$

[10] Note that the solution in the presence of a cosmological constant and the term $\sqrt{\mathcal{R}}$ in the lagrangian, responsible for the linear correction at infinity, is $a(r) = k - 2r^2 \left(4M + \sqrt{k}\lambda r^2 + \frac{2}{3}\Lambda r^3 \right) / \left(4r^3 - 2k\alpha_2 l^2 r - \alpha_3 k^{3/2} l^3 \right)$.

or $\alpha_2 > 0$ and $\alpha_3 < -2\sqrt{6}\alpha_2^{3/2}/9$. Note that if we require the solution to be real in spherical symmetry $k = 1$, then it cannot be so for $k = -1$, corresponding to hyperbolic horizon geometry, which is real only if α_3 is imaginary. This is a general property of odd corrections within these models.

Setting $\alpha_2 = 0$ and $\alpha_3 = -4$ gives the Poisson-Israel (PI) regular black hole, which was first found in [225] from a semi- classical argument involving the vacuum energy-density, and from a 2-dimensional dilaton gravity model in [41], where it was also studied in details. However, contrary to the PI RBH, this modification has the proper quantum correction to Newton potential due to the condition $\alpha_2 = -\beta_2 \neq 0$.

As usual with (A)dS-core RBHs, the solution has two horizons. The outer one is located at

$$r_H = \frac{2M}{3} + \frac{l}{3}\left(z\left(y + \sqrt{-8z^3 + y}\right)^{-1/3} + \frac{1}{2}\left(y + \sqrt{-8z^3 + y}\right)^{1/3}\right) \tag{160}$$

where $z = 3\alpha_2 + 8M^2/l^2$ and $y = 27\alpha_3 + 36\alpha_2 M/l + 64M^3/l^3$. Then by defining the temperature associated with this horizon as $T := \kappa_+/2\pi$, where $\kappa_+ = a'(r_+)/2$ is its surface gravity, we obtain

$$T = \frac{4Mr_H^2 + 2\alpha_2 l^2 r_H + 3\alpha_3 l^3/2}{16\pi M r_H^3} \tag{161}$$

Therefore, by assuming that the first law of thermodynamics holds $dM = TdS$, we can calculate the semiclassical correction ($l/M \ll 1$) to the entropy by integrating the expansion $1/T = 8\pi M + \beta_2\pi l^2/M - \frac{\alpha_3\pi l^3}{2M^2} + \frac{3\pi\alpha_2^2 l^4}{8M^3} + O\left(l^5/M^4\right)$ which gives :

$$S\left(M \gg l\right) = 4\pi M^2 + \beta_2\pi l^2 \log\left(M\right) + \frac{\alpha_3\pi l^3}{2M} - \frac{3\pi\beta_2^2}{16M^2} + O\left(\frac{l^5}{M^3}\right) \tag{162}$$

Thus, the correction provides a logarithmic correction to the classical entropy, given by $S_{cl} = \mathcal{A}/4 = 4\pi M^2$, which is controlled by the Critical Order Scalar $\mathcal{RS}_{(2)}$.

2. Non-minimal-Yang-Mills-like regular black hole ($m = 4$)

The next correction corresponds to $m = 4$, i.e. to a scalar $\mathcal{R}^2\left(\alpha_4\mathcal{R} + \beta_4\mathcal{S}_{(4)}\right)$, where we discard the previous odd order correction. In this case, the solution in presence of a cosmological constant is :

$$a(r) = k - \frac{r^2\left(4Mr + \frac{2}{3}\Lambda r^4 + k^2 l^2\left(\alpha_2 + \beta_2\right)\right)}{2r^4 + k\beta_2 l^2 r^2 + k^2\beta_4 l^4} \tag{163}$$

For a particular choice of coupling constants and for spherical topology, it can be reduced to

$$a(r) = 1 - \frac{r^2\left(\frac{\Lambda}{3}r^3 + 2Mr - Q_m^2\right)}{r^4 + 2\,q_b\,Q_m^2}\,. \tag{164}$$

where q_b and Q_m have respectively the dimensions of an area and a length. This is almost the same regular black hole as the one found in [101, 102] by Baladin, Lemos and Zayats. It was found from an action involving a $SU(2)$ Yang-Mills field non-minimally coupled with gravity via the coupling term $\mathscr{R}^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta}$, where $\mathscr{R}$ is linear in the Riemann tensor. To do so, one needs for the Yang-Mills field to be in a Wu-Yang monopole configuration [293] of charge Q_m; and q_b is a coupling constant appearing in front of the non-minimal coupling. Therefore the difference between this black hole and the one found here is that, in our case, Q_m is a coupling constant, not an integration constant.

Returning to the general solution, its outer horizon is given by the largest positive solution of $a(r_H) = 0$ which can be written as

$$r_H = \frac{M}{2} + \frac{l}{2}\left(\sqrt{y + w} + \sqrt{2y - w + \frac{M}{l}\left(\alpha_2 + \frac{M^2}{l^2}\right)/\sqrt{y + w}}\right) \tag{165}$$

where $x = \alpha_2\left(\alpha_2^2 - 72\beta_4\right) - 216\beta_4\frac{M^2}{l^2}$, $z = \sqrt{x - \left(\alpha_2^2 + 24\beta_4\right)^3} - x$, $w = \frac{1}{6}\left(z^{1/3} + \left(\alpha_2^2 + 24\beta_4\right) z^{-1/3}\right)$ and $y = \frac{\alpha_2}{3} + \frac{M^2}{l^2}$. As before, we can calculate the temperature :

$$T = \frac{4Mr_H^3 + 2\alpha_2 r_H^2 l^2 - 4\beta_4 l^4}{4\pi r_H^3\left(4Mr_H + \left(\alpha_2 + \beta_2\right) l^2\right)} \tag{166}$$

and derive the entropy from $S := \int dM/T(M)$, which gives :

$$S\left(M \gg l\right) = 4\pi M^2 + \beta_2 \pi l^2 \log\left(M\right) - \frac{\pi l^4\left(\alpha_2^2 - 2\alpha_2\beta_2 + 6\beta_4\right)}{16M^2} + O\left(\frac{l^6}{M^5}\right) \tag{167}$$

The first correction to the classical entropy is the same logarithmic correction as in the $m = 3$ case, given by the critical order scalar $\mathcal{R}\mathcal{S}_2$. Therefore we can assume that this correction is preserved for all the higher order corrections $m > 4$.

D. Thermodynamics

1. Minisuperspace Wald Entropy

From our analysis of the minisuperspace Wald entropy presented in Appendix 5., we see that the only terms that contribute to the Wald entropy are those containing $R^{(2)}$ in the spherically decomposition of the gravitational theory. Therefore, we can restrict ourself to the "$\mathcal{S}$" part of the action, which contains such terms and is given by :

$$\frac{I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} = \int_\Sigma d^2x\sqrt{-\gamma}\left((n-i-1)\mathcal{D}r.\mathcal{D}r + \frac{r^2R^{(2)}}{n-i}\right)r^{n-i-2} + 2\oint_{\partial\Sigma} dl\sqrt{h}\frac{r^{n-i}}{n-i}K^{(2)} \tag{168}$$

Therefore, for any order of correction $i \neq n$, the minisuperspace Wald entropy Eq(512) becomes :

$$S_{msW} = \frac{r_H^{n-i}}{4(n-i)G} = \pi\left(\frac{r_H^{n-i}}{n-i}\right) \tag{169}$$

where we set $G = 1/4\pi$. There are two ways of dealing with the minisuperspace Wald entropy of the critical order scalar $\mathcal{R}^{(d-2)/2}\mathcal{S}_{(d-2)}$. As the entropy is defined up to a constant, one can just take the first convergent term of the series $\frac{r_H^{n-i}}{n-i} = \frac{1}{n-i} + \log\left(r_H\right) + O\left(n-i\right)$ by subtracting the infinite constant. Alternatively, one can integrate by part the action :

$$\begin{aligned}\frac{I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} = &\int_\Sigma d^2x\,\sqrt{-\gamma}\,\left(-v^A\,r\mathcal{D}_A r + (n-i-1)\mathcal{D}r.\mathcal{D}r\right)r^{n-i-2} \\ &+ \frac{1}{n-i}\oint_{\partial\Sigma} dl\,\sqrt{h}\,\left(2K^{(2)} + v^A\,n_A\right)r^{n-i}\,,\end{aligned} \tag{170}$$

where $v_a = \Gamma_a/\tilde{\chi}$ (see Eq(36)), and the spherically symmetric action of the critical order scalar becomes :

$$\begin{aligned}\frac{I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} &= \int_\Sigma d^2x\,\sqrt{-\gamma}\,\left(-v^A\mathcal{D}_A\log r - \frac{\mathcal{D}r.\mathcal{D}r}{r^2}\right) + \oint_{\partial\Sigma} dl\,\sqrt{h}\,\left(2K^{(2)} + v^A\,n_A\right)\lim_{i\to n}\left(\frac{r^{n-i}}{n-i}\right) \\ &= \int_\Sigma d^2x\,\sqrt{-\gamma}\,\left(R^{(2)}\log r - \frac{\mathcal{D}r.\mathcal{D}r}{r^2}\right) + \oint_{\partial\Sigma} dl\,\sqrt{h}\left(\lim_{i\to n}\frac{2K^{(2)} + v^A\,n_A}{n-i} + 2\,K^{(2)}\log r\right),\end{aligned} \tag{171}$$

In this case, the bulk part of the action has no more any divergence and the minisuperspace Wald entropy gives directly the logarithmic term.

Finally, the minisuperspace Wald entropy of the full theory is :

$$S = \pi r_H^n + \pi\sum_{i=1}^{n-1} k^{i/2}l^i\frac{\beta_i}{n-i}r_H^{n-i} + \beta_n\pi k^{n/2}l^n\log\left(r_H\right) + \pi\sum_{i=n+1}^{m} k^{i/2}l^i\frac{\beta_i}{n-i}r_H^{n-i} \tag{172}$$

In order for the lowest correction to the entropy to be the logarithmic one, as many quantum gravity results are suggesting [65–68, 294–313], we set $\beta_i = 0$ for $1 \le i \le n-1$ which gives :

$$S = \pi r_H^n + \beta_n\pi k^{n/2}l^n\log\left(r_H\right) + \pi\sum_{i=n+1}^{m} k^{i/2}l^i\frac{\beta_i}{n-i}r_H^{n-i} \tag{173}$$

2. Thermodynamical quantities

We consider now the class of solutions given by

$$a(r) = k - \frac{4Mr^{m-n+1} + \sum\limits_{i=n}^{m} k^{(i+2)/2} l^i r^{m-i} \left(\beta_i - \frac{\alpha_i}{n-i-1}\right)}{nr^m + \sum\limits_{i=n}^{m} k^{i/2} l^i r^{m-i} \beta_i}$$

The location of the horizons is defined by $a(r_H) = 0$, so that we can express the mass in terms of r_H as follows :

$$M = \frac{k}{4}\left(n\, r_H^{n-1} + \sum_{i=n}^{m} k^{i/2} l^i r_H^{n-i-1} \frac{\alpha_i}{n-i-1}\right) \tag{174}$$

Therefore, the temperature is

$$T = \frac{\kappa}{2\pi} = \frac{a'(r_H)}{4\pi} = \frac{k}{4\pi r_H}\left(\frac{n(n-1)r_H^m + \sum\limits_{i=n}^{m} k^{i/2} l^i r^{m-i} \alpha_i}{nr_H^m + \sum\limits_{i=n}^{m} k^{i/2} l^i r^{m-i} \beta_i}\right) \tag{175}$$

If we assume that the first law of thermodynamics holds $dM = TdS$, the entropy can be calculate as :

$$S = \int \frac{dM}{T(M)} = \int \frac{M'(r_H)}{T(r_H)} dr_H = \pi r_H^n + \pi \sum_{i=n}^{m} k^{i/2} l^i \frac{\beta_i}{n-i} r_H^{n-i} \tag{176}$$

Interestingly, the critical order $(n = i)$ scalar $\mathcal{R}^{(d-2)/2}\mathcal{S}_{(d-2)}$ gives rise to a divergent constant entropy at leading order, and to a logarithmic correction at the next one :

$$S\left(\mathcal{R}^{(d-2)/2}\mathcal{S}_{(d-2)}\right) = k^{i/2} l^i \frac{\beta_i}{n-i} r_H^{n-i} + \beta_n \pi k^{n/2} l^n \log(r_H) + O(n-i) \tag{177}$$

As the entropy is defined up to a constant, we might neglect the infinite constant term, in which case Eq(176) agrees with the minisuperspace Wald Entropy (172), because we end up with :

$$S = \int \frac{dM}{T(M)} = \int \frac{M'(r_H)}{T(r_H)} dr_H = \pi r_H^n + \beta_n \pi k^{n/2} l^n \log(r_H) + \pi \sum_{i=n+1}^{m} k^{i/2} l^i \frac{\beta_i}{n-i} r_H^{n-i} \tag{178}$$

In order to discuss the thermal stability of the black hole, we derive the heat capacity :

$$\begin{aligned} C_v &= \frac{\partial M}{\partial T} = \frac{M'(r_H)}{T'(r_H)} \\ &= \frac{\pi r_H^{n-m}\left(n(n-1)r_H^m + \sum\limits_{i=n}^{m} k^{i/2} l^i r^{m-i}\alpha_i\right)\left(nr_H^m + \sum\limits_{i=n}^{m} k^{i/2} l^i r^{m-i}\beta_i\right)^2}{(n-1)n^2 r_H^{2m} - n\sum\limits_{i=n}^{m} k^{i/2} l^i r_H^{2m-i}\xi_i - \sum\limits_{(i,j)=n}^{m} k^{(i+j)/2} l^{i+j} r_H^{2m-i-j}\sigma_{(ij)}} \end{aligned} \tag{179}$$

where $\xi_i = (i-1)(n-1)\beta_i - (i+1)\alpha_i$, $\sigma_{(ij)} = (i-j-1)\alpha_j\beta_i$ and $\sum\limits_{(i,j)=n}^{m} = \sum\limits_{i=n}^{m}\sum\limits_{j=n}^{m}$. Finally the free energy is given by :

$$\begin{aligned} F &= M - TS \\ &= \frac{k}{4r_H}\left(\frac{nr^{m+n} + \sum\limits_{(i,j)=n+1}^{m} k^{(i+j)/2} l^{i+j} r_H^{m+n-i-j}\lambda_{(ij)} - g(r_H) + \sum\limits_{i=n+1}^{m} k^{i/2} l^i r^{m-i} h_i(r_H)}{nr_H^m + \sum\limits_{i=n}^{m} k^{i/2} l^i r^{m-i}\beta_i}\right) \end{aligned} \tag{180}$$

where for simplicity we have defined $\lambda_{(ij)} = \frac{(i-j-1)\alpha_j\beta_i}{(n-i)(j+1-n)}$ and

$$h_i(r_H) = \frac{(i-1)n\beta_i r_H^n}{i-n} + \frac{\alpha_i}{1+i-n}\Big(-(i+1)r_H^n + k^{n/2}l^n\left(-1+(-1-i+n)\log\left(r_H\right)\right)\beta_n\Big)$$
$$g(r_H) = -nr_H^n\beta_n + \log(r_H)\Big(n(n-1)r_H^n + k^{n/2}l^n\alpha_n\Big)\beta_n + \alpha_n\Big((n+1)r_H^n + k^{n/2}l^n\beta_n\Big)$$

Unfortunately, we had no time to make a careful study of the thermodynamic properties one can extract from these quantities. This will be the topic of some future work.

E. Vacuum structure

Now, let's see what are the possible vacua of these theories. As it is sometimes the case from quantum gravity inspired corrections to Schwarzschild geometry (see for example the Modesto Polymer black holes eq(256) and eq(316) found in [44], the Polymer black hole of [41–43], the D'Ambrosio-Rovelli metric eq(285) for Black to White hole transition found in [175] and the Visser-Hochberg black bounce eq(308) found in [227]), in addition to Minkowski vacuum, the vacua of the action (141) can also be some "dressed" vacua that are not Minkowski when $M = 0$ and $\Lambda = 0$, but reduce to it when $l \to 0$ or $r \to \infty$, giving a small scale structure to Minkowski spacetime. More precisely, we will see that they can either be anisotropic horizonless spacetimes, like in the previous references, regular black holes with 2 horizons, or finally extremal ones. Remark that finding the quantum effective description of Minkowski spacetime has been attempted following other approaches than the (loop) quantization of spherically symmetric Einstein gravity, for example in non-commutative field theories one can consider non-commutative deformations of Minkowski spacetimes (so-called κ-Minkowski, see for example [325]) which have been linked with the Minkowski regime of LQG in [326,327]. Although these approaches tackle more directly the issue, it is nonetheless interesting to see what kind of effective deformation of Minkowski spacetime can be found in the $M = 0$ regime of some high energy corrected Schwarzschild geometry.

In order to simplify the analysis and stay as close as possible to standard high energy power series corrections to GR, like for example $F(R)$ and Lovelock-Lanczos gravities, we consider actions of the form

$$I_m = \int_{\mathcal{M}} d^{n+2}x\,\sqrt{-g}\left(-2\Lambda + R + \sum_{i=j}^{m} l^i\,\mathcal{R}^{i/2}\left(\alpha_i\,\mathcal{R} + \beta_i\,\mathcal{S}_{(i)}\right)\right)$$

with $n+1 \le m$ in order to have regular solutions, for which we require that all the truncations I_p to order $p+2$ with $n+1 \le p \le m$, share the same regular vacuum with the full I_m theory, while preserving regularity and Schwarzschild asymptotic behaviour for each p. In the context of power series $F(R)$ and L-L gravities, all the truncations have a Minkowski vacuum for example. In some sense, if one interprets the present model as an effective action, it amounts to say that we require the vacuum to be stable against loop correction.

By doing so, the spherically symmetric solution of the full theory can be written in four dimensions as : $a(r) = a_{\text{vac}}(r) + MZ(r)$ where $a_{\text{vac}}(r \to 0) = 1 + O\left(r^2\right)$, $a_{\text{vac}}(r \to \infty) = 1 + O\left(1/r^2\right)$, $Z(r \to \infty) = -2/r + O\left(1/r^2\right)$ and $Z(r \to 0) = O\left(r^2\right)$.

The first vacuum of interest is Minkowski spacetime. When $\Lambda = 0$ and $M = 0$, the general vacuum solution (158) is Minkowski provided that $\alpha_{n-1} = \beta_{n-1} = \gamma_i = 0$, which gives the following massive solutions :

$$a(r) = k - 4\,M\frac{r^{m-n+1}}{n\,r^m + \sum\limits_{i=1}^{m} l^i\,k^{i/2}\,\beta_i\,r^{m-i}} \tag{181}$$

As we said previously, these conditions $\gamma_i = 0$ are the same required to have a regular core, the quantum correction to Newton potential and the correct Schwarzschild behaviour at $r \to \infty$, what makes these choices of coupling constants (α, β) quite interesting.

In order to study the other kind of vacua, note that the vacuum $a_{\text{vac}}(r)$ being the vacuum of all the truncations of the considered theory, as we required, it is also in particular the vacuum of the lowest $p_{\min}$ truncation that is regular. Here we will restrict to four dimensions, which gives $p_{\min} \geq 3$, and by considering the first correction of even order $p_{\min} = 4$, all the vacua of these kind of models, for any m, will be contain in the truncation :

$$\int_{\mathcal{M}} d^4x \sqrt{-g} \left(-2\Lambda + R + \sum_{j=1}^{p_{\min}/2} l^{2j} \mathcal{R}^j \left(\alpha_{2j} \mathcal{R} + \beta_{2j} \mathcal{S}_{2j} \right) \right)$$

where we only consider even order corrections. The vacuum of this theory is :

$$a_{\text{vac}}(r) = k - \frac{r^2 \left(\frac{2}{3} \Lambda r^4 + k^2 l^2 \left(\alpha_2 + \beta_2 \right) \right)}{2r^4 + k\beta_2 l^2 r^2 + k^2 \beta_4 l^4} \tag{182}$$

An important criterion to satisfy would be that the theory admits regular solutions for both hyperbolic and spherical horizon topologies, what is actually possible providing that $\beta_4 > \beta_2^2/8$. These topological solutions might have their physical relevance when considering gravity at high energy, however it is usually difficult to make sense of these with a positive cosmological constant (because in this case when the “mass” parameter is switched on, they are usually not static black holes but dynamical spacetimes), despite the fact that in any case, they are present as solutions of the effective field equations, so that they should contribute to the quantum gravity path integral.

Concerning the cosmological constant, we see that, while it is possible to have exactly Minkowski vacuum ($\Lambda = 0$) and preserve the regularity of the massive solutions ($\beta_4 \neq 0$), as we just saw, it is not so for (A)dS vacua ($\Lambda \neq 0$). The vacua with non-vanishing cosmological constant have necessarily a small scale structure, and reduce to (A)dS vacua only at infinity (or when $l = 0$).

From now on, we consider a vanishing cosmological constant $\Lambda = 0$ in order to simplify the analysis. Then, the general vacuum can be rewritten as :

$$\begin{aligned} a(r) &= \frac{2k \left(r^2 - r_1^2 \right) \left(r^2 - r_2^2 \right)}{2r^4 + k\beta_2 l^2 r^2 + k^2 \beta_4 l^4} , \\ r_1 &= \frac{l}{2} \sqrt{k \left(\alpha_2 - \sqrt{\alpha_2^2 - 8\beta_4} \right)} , \\ r_2 &= \frac{l}{2} \sqrt{k \left(\alpha_2 + \sqrt{\alpha_2^2 - 8\beta_4} \right)} , \end{aligned} \tag{183}$$

In order to have horizons, we need $0 < \beta_4 \leq \alpha_2^2/8$. Then, $\alpha_2 > 0$ gives two horizons for spherical topology $k = 1$ with r_1 (r_2) the inner (outer) horizon and no horizon for $k = -1$, while $\alpha_2 < 0$ gives two horizons for hyperbolic topology with $r_2 \leq r_1$, and no horizon for $k = 1$. So it is not possible to have horizons for both topologies at the same time. In all the other cases, the vacuum is horizonless for both topologies. As mentionned previously, these kind of anisotropic deformations of Minkowski vacuum have already been found from quantum gravity inspired approaches, in particular the horizonless ones.

In the present models however, we saw that it is also possible to have vacuum ($M = 0$) regular black holes replacing Minkowski spacetime at small scale. Such strange vacua have already been found previously in the context of an Einstein-Gauss-Bonnet theory with a negative cosmological constant, [328], but contrary to the vacua of the present model, Eq(183), these Gauss-Bonnet black holes are singular.

These spherical vacuum BHs seem problematic when they possess a dynamical interior region (between two distinct horizons). Indeed, first from a thermodynamical point of view, one would be tempted to associate a temperature to the outer horizon r_+ of the massive solution $a(r) = a_{\text{vac}}(r) + MZ(r)$. Then, by defining the temperature of the massive solution as $T := \kappa_+/2\pi$, where $\kappa_+ = a'(r_+)/2$ is the surface gravity of the event horizon, we see that when $M \to 0$, we get a finite temperature,

$$T(M \to 0) = \frac{\alpha_2 \left(\alpha_2 + \sqrt{\alpha_2^2 - 8\beta_4} \right) - 8\beta_4}{\pi \, l \left(\alpha_2 + \beta_2 \right) \left(\alpha_2 + \sqrt{\alpha_2^2 - 8\beta_4} \right)^{3/2}} + o(M^0) \tag{184}$$

because as we saw, $r_+(M \to 0) = \frac{l}{2}\sqrt{\left(\alpha_2 + \sqrt{\alpha_2^2 - 8\beta_4}\right)}$. However, if there is a finite temperature emitted by the vacuum, there should be a backreaction on it, making it unstable. Of course a more careful study of the thermodynamical stability of this geometry would be needed to reach concrete conclusions, but in any case, a second issue would be the presence of a Cauchy horizon, which is also known to be unstable [181, 329]. In [328], it was found that the temperature of a particular class of vacuum hyperbolic topology ($k = -1$) black holes, embedded in AdS, does also have non-vanishing temperature for $M = 0$ (see eq(3.18) of the mentioned paper). However, as far as we know, the previous vacua eq(183) are the first example of spherical and regular black holes with such a property. Note that they can also be generalized with a positive cosmological constant and preserve this property.

The other possibility would be that the above temperature vanishes, which means to have an extremal regular vacuum, with $\beta_4 = \alpha_2^2/8$. In this case, the surface gravity κ vanishes, which prevent the usual mass inflation instability of Cauchy horizons, because it is controlled by κ_- as $m \propto \exp(-\kappa_- v)$, where v is the advanced time. It is possible for other instabilities to be present in these extremal vacua though.

The condition to have regular vacua for both topologies then becomes $\alpha_2^2 > \beta_2^2$ and recall that $\alpha_2 > 0$ is needed in order to have an extremal spherical ($k = 1$) vacuum. The resulting solution is therefore given by :

$$a(r) = \frac{2\,k\left(r^2 - \frac{\alpha_2}{4}\,k\,l^2\right)^2}{2\,r^4 + k\,\beta_2\,l^2\,r^2 + \frac{\alpha_2^2}{8}\,k^2\,l^4} \tag{185}$$

Remark that these extremal (ground) states are very different from the usual extremal states of (regular or not) two-horizons black holes (see for example [330] for the usual extremal limits of d-dimensional black holes). Indeed, in this case, even when generalized to the charged and rotating case, there will always be a zero-temperature state reached for the $M = Q = J = 0$, associated with a finite entropy, because the horizon of the vacuum being located at

$$r_H = \frac{l\,\sqrt{\alpha_2}}{2}\,, \tag{186}$$

the entropy Eq(178) for this configuration becomes constant. This is true for all order of corrections because we considered a series of corrections for which all the truncations share the same vacuum, as we said previously. Therefore, the existence of such extremal vacuum states provides a realization of Nernst's third law of thermodynamics, stating that the entropy S of a system must go to zero or to a universal constant as its temperature T goes to zero, see for example [331–333]. That being said, when considering $Q \neq 0$ and $J \neq 0$, there might be other extremal states for which the entropy is neither zero, nor a universal constant, as it is usually the case.

A possible interpretation of such vacua might be that Minkowski space acquires a small scale structure described by an extremal horizon boundary, near which the spacetime behaves as a Bertotti-Robinson metric $AdS_2 \times S^2$, see [314–317]. Indeed, the near-horizon geometry can be found from the transformation $t \to t/\epsilon$, $r \to \sqrt{2}\,\alpha_2\,l + r\epsilon$, followed by the limit $\epsilon \to 0$, which gives:

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + \frac{2\alpha_2}{\alpha_2 + \beta_2}L^2 d\Omega^2 \tag{187}$$

where $L = \sqrt{\alpha_2 + \beta_2}\,l/2\sqrt{2}$ and $\frac{2\alpha_2}{\alpha_2+\beta_2} > 1$. As this kind of vacua provide a natural AdS region which does not require a negative cosmological constant (actually it is even possible to find extremal vacua with $\Lambda > 0$), it might be relevant regarding the AdS/CFT correspondence, see for example [334] concerning AdS/CFT in the context of two-dimensional dilaton gravity theory.

F. Maxwell field non-minimally coupled with the non-polynomial curvature scalar $\mathcal{R}$

We will now consider the presence of an eletromagnetic field, in order to see that the regularity of the solutions is preserved when these black holes are charged. While doing so, we also add a non-minimal coupling in the form of terms $\mathcal{R}^{i/2}F^{\mu\nu}F_{\mu\nu}$, where $\mathcal{R}$ is defined by Eq(142). Recall that in spherically symmetric spacetimes, this scalar reduces to the Ricci scalar of the horizon manifold, and it is the same that we have used to produce higher order corrections to E-H action. As we will see, the presence of this simple term regularizes the Coulomb singularity of the eletric field in spherically symmetric spacetimes. We work here in four dimensions. The action reads,

$$I=\int_{\mathcal{M}} d^4x\,\sqrt{-g}\left(\frac{1}{16\pi G}\sum_{i=-2}^{m} l^i\,\mathcal{R}^{i/2}\left(\alpha_i\,\mathcal{R}+\beta_i\,\mathcal{S}_{(i)}\right)-\frac{1}{4\,\mu_0}F^{\mu\nu}F_{\mu\nu}\sum_{i=0}^{p}\lambda_i L^i\mathcal{R}^{i/2}\right) \tag{188}$$

where the λ_i and L are coupling constants that have respectively no dimension and the dimension of a length. $F_{\mu\nu}$ are the components of the Faraday tensor of the electromagnetic field A. In addition to $c=1$, we set $G=1/4\pi$ and $\mu_0=1$. Moreover, in order to recover Maxwell theory when $L=0$, we need $\lambda_0=1$.

In order to derive the spherically symmetric field equations, we choose the following gauges : $A=A_0(t,r)\,dt+A_r(t,r)\,dr$, and the gauge Eq(146). It gives :

$$\begin{aligned}
&\left.\frac{\delta I}{\delta a}\right|_{f=0}^{A_r=0}=r\Delta_\beta b'=0\,,\quad \left.\frac{\delta I}{\delta f}\right|_{f=0}^{A_r=0}=-r\Delta_\beta\frac{\dot a}{a}=0\,,\quad \left.\frac{\delta I}{\delta A_r}\right|_{f=0}^{A_r=0}=r^2\Delta_\lambda\partial_t\left(\frac{A_0'}{b}\right)=0\,,\\
&\left.\frac{\delta I}{\delta b}\right|_{f=0}^{A_r=0}=k\Delta_\alpha-\Big(r\Delta_\beta\,a\Big)'-2\,r^2\Delta_\lambda\frac{A_0'^2}{b^2}=0\,,\quad \left.\frac{\delta I}{\delta A_0}\right|_{f=0}^{A_r=0}=-\left(\frac{r^2\Delta_\lambda A_0'}{b}\right)'=0\,.
\end{aligned} \tag{189}$$

where $\Delta_\lambda=\sum\limits_{i=0}^{m}L^i\,k^{i/2}\,r^{-i}\,\lambda_i$. The first three equations give respectively $b=b(t)$, which can be set to $b=1$ by a time reparametrization, $a=a(r)$ and $A_0'=A_0'(r)$. This proves Birkhoff theorem for both the electromagnetic and gravitational fields. The last equation can be simplified to $A_0'=-\frac{q}{r^2\Delta_\lambda}$, where q is the charge of the black hole, what reduces the fourth one to :

$$k\int\Delta_\alpha-r\,\Delta_\beta\,a+2\,qA_0=4M \tag{190}$$

and M is its mass.

In order to simplify the analysis, we set the horizon topology to be spherical $k=1$ and the cosmological constant $\Lambda=0$. Moreover to see the effect of the electric field on the regularity of the spacetime, we consider the lower (even) order of correction at which the singularity disappear in the neutral case ($m=4$)[11] and set the odd order corrections to zero : $\alpha_1=\beta_1=\alpha_3=\beta_3=0$. To integrate easily the remaining field equation for A_0, we also restrict to $\lambda_i=\delta_i^p$ for $i>0$, where δ is the Kronecker delta, which means that we consider a single correction $\mathcal{R}^{p/2}F^{\mu\nu}F_{\mu\nu}$ to the Maxwell Lagrangian. Finally, we impose the regularity condition $\alpha_4=-3\beta_4$ and to have a Minkowski vacuum when the charge and mass are vanishing $\alpha_2=-\beta_2$. It gives :

$$a(r)=1+2\,r^3\left(\frac{qA_0(r)-2M}{2\,r^4+\beta_2\,l^2\,r^2+\beta_4\,l^4}\right). \tag{191}$$

For $p=0$, that is, for Maxwell theory without non-minimal coupling given by $A_0=q/r$, the gravitational field has the following structure close to the origin :

$$a(r\to 0)=1+\frac{2\,q^2\,r^2}{\beta_4\,l^4}-\frac{4\,M\,r^3}{\beta_4\,l^4}+O\left(r^4\right). \tag{192}$$

Therefore, the regularity of the gravitational field is indeed preserved by the addition of a Maxwell field.

[11] A fortiori, if the regularity of spacetime is preserved (when an electrostatic field is added) for lower order of corrections, it will be also preserved for higher orders, at least within our model.

However, remark that this property for a black hole solution to keep its regularity intact in the presence of a Maxwell field is not trivial. Indeed, if we had chosen $m = 3$ (which correspond to a Poisson-Israel-like regular black hole with the proper quantum correction to the Newtonian potential), with again for simplicity a Minkowski vacuum when $q = 0$, the solution would be : $a(r) = 1 + \frac{2\,r\left(q^2 - 2M\,r\right)}{2\,r^3 + \beta_2\,r\,l^2 + \beta_3\,l^3}$, and the behaviour close to zero : $a(r \to 0) = 1 + 2\,q^2\,r/\beta_3\,l^3 + O\left(r^2\right)$. Thus, the Sakharof criterium is not satisfied and the generalized Poisson-Israel solution looses its regularity when a Maxwell field is added.

Now let's see the effect of the non-minimal coupling on the regularity of the electrostatic solutions. We consider only the three first orders of corrections, $p = 1, 2, 3$, but a general solution can be found in terms of Gauss's hypergeometric function :

$$A_0(r) = c(q) + \frac{q}{r}\left(1 - {}_2F_1\left(1, -\frac{1}{p}; \frac{p-1}{p}; -\left(\frac{r}{L}\right)^p\right)\right), \tag{193}$$

where $c(q)$ is an integration constant linear in the charge q, that has to be fixed differently for each p in order to have the usual $A_0\,(r \to \infty) = q/r + O\left(1/r^2\right)$ behaviour. The first three cases give :

$$\begin{aligned}
&\text{For } p = 1 : \quad A_0(r) = \frac{q}{L}\log\left(\frac{r+L}{r}\right) \\
&\text{For } p = 2 : \quad A_0(r) = \frac{q}{L}\left(\frac{\pi}{2} - \arctan\left(\frac{r}{L}\right)\right) \\
&\text{For } p = 3 : \quad A_0(r) = \frac{q}{6\,L}\left(2\sqrt{3}\left(\frac{\pi}{2} + \arctan\left(\frac{1}{\sqrt{3}}\left(1 - \frac{2\,r}{L}\right)\right)\right) + \log\left(1 + \frac{3\,r\,L}{L^2 - r\,L + r^2}\right)\right).
\end{aligned} \tag{194}$$

These solutions lead respectively to the following scalars :

$$F^{\mu\nu}F_{\mu\nu} = \begin{cases} -\frac{2\,q^2}{r^2\,(L+r)^2} \\ \\ -\frac{2\,q^2}{(L^2+r^2)} \\ \\ -\frac{2\,q^2\,r^2}{(L^3+r^3)^2} \end{cases} \tag{195}$$

Looking at the regularity of $F^{\mu\nu}F_{\mu\nu}$ in the first case, we see that, even if it improves the behaviour of the electric field at the origin, by correcting the $1/r$ potential by a logarithmic one, the corresponding correction $\mathcal{R}^{1/2}F^{\mu\nu}F_{\mu\nu}$ is not enough to cure the Coulomb singularity, while the higher order corrections are.

Therefore, in four dimensions and within this framework, one needs at least a non-minimal coupling containing two derivatives of the metric (in spherical symmetry), and a gravitational correction containing six derivatives (without counting odd orders) in order to have both a regular electric and gravitational fields, that is, corrections to the Einstein-Hilbert-Maxwell action of the form $\mathscr{R}^3 + \mathscr{R}F^{\mu\nu}F_{\mu\nu}$.

This last term is similar to the non-minimal coupling found in [101, 102] to give rise to regular black hole in the presence of a Yang-Mills field in a Wu-Yang monopole configuration. We already mentioned this solution in the third section of this Chapter. In our case, the order four non-minimal coupling $\mathscr{R}F^{\mu\nu}F_{\mu\nu}$ cure the Maxwell singularity while the gravitational part of the action makes the gravitational field regular, whereas in the mentioned case the order four non-minimal coupling $\mathscr{R}^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\mu}$ cure the gravitational singularity, while the Wu-Yang ansatz solves the Yang-Mills equations and is regular.

G. Mass Inflation & Regular black holes

In this section, we want to address the question of the Mass Inflation (MI) instability of our solutions. It is a classical phenomenon associated with the presence of an inner (Cauchy) horizon inside a black hole, which is notably the case for charged and rotating black holes, and for our purpose, (A)dS-core regular black holes. It reveals that inner horizons are unstable when both outgoing and ingoing perturbations (usually modelled as spherically symmetric null shells) are taken into account, because null rays are exponentially focused there, what end up triggering an exponential growth of the mass in the region of the associated Penrose diagram enclosed by the shells and the inner horizon. This instability has been extensively studied in the past, see for example [181–190]. In particular, it has been applied to (A)dS-core regular black holes in [190]. In order to see more in detail how and why this phenomenon occur, we will follow closely this paper, and refer the reader to it for more details and references.

However, note that the debate on the subject does not seem settled, as for example in [191], it was noted that by distinguishing between global Cauchy horizon and local apparent inner horizon, the mass inflation does not occur in the usual drastic way.

We will nonetheless consider here the standard derivation of the phenomenon, and see that it is possible, within the present model, to avoid the instability by considering $m \to \infty$, i.e. an infinite number of corrections to GR, resulting in a non-perturbative action for the parameter l.

A brief overview of Mass Inflation

First, consider a static black hole given, as in our case, by a metric of the form

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\Omega^2 \tag{196}$$

We assume that it possesses both outer and inner horizons, located at $a(r_\pm) = 0$, so that it can be written as $a(r) = (r - r_-)(r - r_+)X(r)$, where $X(r) > 0$ for $r > 0$. Then, in the ingoing Eddington-Finkelstein coordinate defined by $dv = dt + \frac{dr}{a(r)}$, the metric becomes :

$$ds^2 = -a(r)dv^2 + 2drdv + r^2 d\Omega^2 \tag{197}$$

and outgoing radial null curves are defined by $\frac{dr}{dv} = \frac{a(r)}{2}$, so that close to the inner horizon, located at r_-, they behave as :

$$\frac{d\,(r - r_-)}{dv} = \frac{a'(r_-)}{2}(r - r_-) + o\,(r - r_-) = \kappa_-(r - r_-) + o\,(r - r_-) \tag{198}$$

because, as we saw previously, in the kind of black hole geometry studied here, the surface gravity is given by $\kappa_\pm = a'(r_\pm)/2$. In particular for the inner horizon surface gravity we have

$$\kappa_- = \left(\frac{r_- - r_+}{2}\right) X(r_-) < 0 \tag{199}$$

so that at the inner horizon these outgoing null rays are exponentially focused. Now, if one models the (continuous) perturbations produced in a collapse by two (ingoing and outgoing) spherically symmetric null shells, they divide the interior of the black hole into four regions with different geometries. We refer the reader to [190] for more details, but the idea to deduce the mass inflation in the region enclosed by the shells and the inner horizon is the following. The different masses of each regions (which determine the associated geometries) can be related, at the crossing points of the shells $r_0(v)$ via conservation laws, by the so-called DTR relations [335, 336]. On the four different masses, three can be related to the masses of the two shells and the mass of the black hole as seen from infinity. The mass of the ingoing shell is usually determined by Price's law [337, 338] describing the decay of the ingoing tails with respect to the coordinate v. Therefore, it is possible to deduce the mass $m(r_0(v))$ of the region enclosed by the shells and the inner horizon, which is given at late times ($v \gg -1/\kappa_-$) by :

$$m(r_0(v)) \propto f(v)e^{-\kappa_- v} \tag{200}$$

where $f(v) = v^{-\gamma}$, $\gamma > 0$, follows from Price's law. As we said previously $\kappa_- < 0$, so that there is indeed an exponential growth of the mass in this region, which is controlled by the value of the inner horizon surface gravity κ_-. This is the essential conclusion that we will be interested in.

Indeed, our aim is not to model differently the phenomenon, but rather to see how high energy corrections, within our model, can affect the inner horizon surface gravity κ_-. In particular, we will see two different ways to avoid mass inflation within our model. In both cases, one needs an infinite number of corrections, $m \to \infty$, because otherwise the inner horizon surface gravity is finite. The first one is simply to consider series of coupling constants (α, β), such that the resulting solution possesses a single horizon, but is nonetheless regular. In this case, the finite m "truncated" actions do not admit regular black holes, only the non-pertubative limit does. In the second case, we consider a non-pertubative series of corrections such that all the truncations admit an (A)dS-core regular black hole (and thus suffer from Mass Inflation but are regular), and share the same extremal black hole vacuum (which "dresses" Minkowski geometry), what provides a precise example of what was said in the fifth section of this Chapter. In this case, it is possible to fix the remaining coupling constants in such a way that in the limit $m \to \infty$, the inner horizon surface gravity $\kappa_- \to 0$. Thus, if considered as an effective field theory, it is always possible to consider a sufficiently high order of correction m, so that Mass Inflation is negligible.

Importantly, note that quite generally, resolving the mass inflation issue for neutral and non-rotating regular black hole is interesting but nonetheless very insufficient : in order to cure the singularity, one invokes the presence of a central (A)dS core, but in return, it requires the presence of an inner horizon which is not present in the classical case, and is unstable. Therefore resolving mass inflation for neutral regular black holes is just a necessary condition for the (A)dS-core regularity to be a solid proposal to cure BH singularities. The real issue with mass inflation arises in the charged and/or rotating cases, because the classical solutions do suffer from it. For this reason, after seeing how to cure the mass inflation of some neutral black holes of the present model, we will also consider the charged case, and see that the instability can still arise, but can also be avoided when specific bounds relating the charge and mass are satisfied.

In all this section we consider the four-dimensional theory $n = 2$, spherical topology $k = 1$, with $\alpha_1 = \beta_1 = 0$, no cosmological constant $\Lambda = 0$, so that the black hole solution Eq(151), for $m \to \infty$, can be written as :

$$\begin{aligned} &b(r) = 1 \ , \quad a(r) = 1 - \frac{4MX(r)}{r} + K(r) \ , \\ &X(r) = \frac{1}{2 + \sum\limits_{i=2}^{\infty} l^i \beta_i r^{-i}} \ , \quad K(r) = -\frac{\sum\limits_{i=2}^{\infty} l^i \gamma_i r^{-i}}{2 + \sum\limits_{i=2}^{\infty} l^i \beta_i r^{-i}} \end{aligned} \tag{201}$$

where $\gamma_i := \beta_i - \frac{\alpha_i}{1-i}$.

1. Single-horizon regular black holes

Let's start by the most obvious way to avoid mass inflation in the neutral case : to consider regular black holes with a single horizon, so that the geometry simply does not possess an inner horizon. Examples of such single-horizon RBHs have been found in [41–44] by applying the loop quantization techniques to spherically symmetric geometries, as well as in [175, 226] and [227, 228] and they usually describe bouncing inner regions. We will study these spacetimes in more details in the next Chapter. As we are going to see, it requires $m \to \infty$, so that the conditions (found in the second section) to have perturbative actions in l admitting (A)dS-core regular black holes are not satisfied. Therefore, both the regularity and the stability are reached only for $m \to \infty$, while for finite m, the solutions have two horizons, are unstable and singular. In this part, we will consider for simplicity that the solutions have Minkowski vacuum, so that $K(r) = 0$.

Roughly speaking, the way to construct regular black holes that have no (A)dS-core at $r = 0$ is the following (within the solutions (201) with $K = 0$) :

- Choose a function $V(r)$ which becomes complex for $r < l$, and consider that $X(V(r), r)$, so that the solution is defined only for $r \geq l$;

- Then derive the curvature scalars R, $R_{\mu\nu}R^{\mu\nu}$ associated with the metric (201), and expand them close to $r = l$ in order to see the conditions required on X to have a regular solution at $r = l$.

- Taking into account that we also want a Schwarzschild behaviour + quantum correction to Newton potential at infinity, $a(r \to \infty) = 1 - \frac{2M}{r} + \frac{\lambda l^2 M}{r^3} + o\left(\frac{1}{r^3}\right)$, we can then find an interpolating function X.

The single-horizon non-singular black holes belong to a subclass of the geometry constructed in this way. For example, following this recipe, one can construct the two-horizon RBH :

$$\begin{aligned} a(r) =&\ 1 - \frac{4M\left(r^2 - l^2\right)^2}{r^4\left(r + \sqrt{r^2 - l^2}\right)} \\ \underset{r\to\infty}{\longrightarrow}&\ 1 - \frac{2M}{r} + \frac{7Ml^2}{2r^3} + O\left(\frac{l^4 M}{r^5}\right) \\ \underset{r\to l}{\longrightarrow}&\ 1 - \frac{16M\left(r-l\right)^2}{l^3} + O\left(\frac{M(r-l)^{5/2}}{l^{7/2}}\right) \end{aligned} \tag{202}$$

whose regularity can be checked by computing its curvature invariants :

$$R = \frac{8M(3r^4 + l^4)}{l^2 r^5} - \frac{12M(2l^4 + l^2r^2 + 2r^4)\sqrt{r^2 - l^2}}{l^2 r^6} \tag{203}$$

and

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} =&\ \frac{8M^2}{r^{12}\left(r + \sqrt{r^2 - l^2}\right)^2}\Big(180l^8 - 208l^6r^2 + 77l^4r^4 + 5l^2r^6 + 10r^8 \\ &- 2r\sqrt{r^2 - l^2}\left(84l^6 - 38l^4r^2 + 5l^2r^4 + 5r^6\right)\Big) \end{aligned} \tag{204}$$

However, one can check that it admits two horizons, so that it suffers from mass inflation. Indeed, from the third line of eq(202), we see that in the inner region $r \approx l$, $a(r) \approx 1 > 0$. This is actually a kind of "modified" (A)dS-core RBH, for which the core is located at $r = l$ rather than $r = 0$.

In order to have only one horizon, we need the additional assumption that $a(l) < 0$ when the solution possesses an horizon. For example, consider the following metric :

$$\begin{aligned} a(r) =&\ 1 - \frac{6Mr^4}{2r^5 + (r^2 - l^2)^{5/2}} \\ \underset{r\to\infty}{\longrightarrow}&\ 1 - \frac{2M}{r} - \frac{5Ml^2}{3r^3} + O\left(\frac{l^4 M}{r^5}\right) \\ \underset{r\to l}{\longrightarrow}&\ 1 - \frac{3M}{l} + \frac{3M(r-l)}{l^2} + O\left(\frac{M(r-l)^2}{l^3}\right) \end{aligned} \tag{205}$$

which is regular because :

$$R = \frac{30M\,l^2r^2\left(2r^5\sqrt{r^2 - l^2}(r^2 - 4l^2) - (l^2 - r^2)^3(6l^2 + r^2)\right)}{\left(2r^5 + (l^2 - r^2)^2\sqrt{r^2 - l^2}\right)^3} \tag{206}$$

and

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} =&\ \Big(20l^{14} - 84l^{12}r^2 + 133l^{10}r^4 - 105l^8r^6 + 70l^6r^8 - 230l^4r^{10} + 225l^2r^{12} - 65r^{14} \\ &+ 4r^5\left(r^2 - l^2\right)^2\sqrt{r^2 - l^2}\left(20l^4 + 14l^2r^2 - 13r^4\right)\Big)\frac{450M^2l^4r^4(l^2 - r^2)}{\left(2r^5 + \left(r^2 - l^2\right)^2\sqrt{r^2 - l^2}\right)^6} \end{aligned} \tag{207}$$

It can be checked (for example by plotting the solution or using Mathematica, as we did) that it has a unique horizon determined by $a(r_H) = 0$. When $M = l/3$, the horizon is located at $r_H = l$, while for $M < l/3$, the geometry is horizonless. Note that from the change of coordinate $r = \sqrt{\tilde{r}^2 + l^2}$ where the range of $\tilde{r}$ is $0 \leq \tilde{r} < \infty$, the interval becomes :

$$ds^2 = -\left(1 - \frac{6M\left(l^2+\tilde{r}^2\right)^2}{\tilde{r}^5 + 2\left(l^2+\tilde{r}^2\right)^{5/2}}\right) dt^2 + \frac{\tilde{r}^2\, d\tilde{r}^2}{\left(l^2+\tilde{r}^2\right)\left(1 - \frac{6M\left(l^2+\tilde{r}^2\right)^2}{\tilde{r}^5+2\left(l^2+\tilde{r}^2\right)^{5/2}}\right)} + \left(\tilde{r}^2 + l^2\right) d\Omega^2 \tag{208}$$

Having just one horizon, these kind of geometries avoid naturally mass inflation. Now concerning the charged case, note first that from Eq(190), we see that the charged solutions can be found from the replacement $M \to M - QA_0(r)/2$. Thus, considering just a Maxwell field without non-minimal coupling leads to

$$a(r) = 1 - \frac{6\left(M - \frac{Q^2}{r}\right) r^4}{2r^5 + \left(r^2 - l^2\right)^{5/2}} \tag{209}$$

which is still regular because we have

$$\begin{aligned} R(r \to l) &= \frac{45(Q^2 - 2lM)\sqrt{r-l}}{2l^{9/2}\sqrt{2}} + o\left(\sqrt{r-l}\right) \\ R_{\mu\nu}R^{\mu\nu}(r \to l) &= \frac{9Q^4}{l^8} + O\left(\sqrt{r-l}\right) \end{aligned} \tag{210}$$

This charged geometry has two horizons when $a(l) \geq 0$, otherwise it behaves pretty much like the neutral case, with one single horizon and an interior dynamical region. Therefore there exists a condition to avoid the mass inflation instability which is given by $a(l) < 0$, i.e.

$$M > \frac{l}{3} + \frac{Q^2}{2l} \tag{211}$$

As long as this bound is satisfied, the black hole interior will be stable (at least against mass inflation), and otherwise an inner horizon would form inside the black hole. For example, while evaporating due to Hawking radiation, the mass would decrease without changing the charge, what can end-up triggering the instability.

Of course this is just an example (somewhat badly picked because it is quite difficult to have analytic expressions for the locations of the horizons) of single-horizon charged RBHs that can be constructed from the present model. Note also that these kinds of construction seem quite general, meaning that single-horizons regular black holes do avoid mass-inflation when some bounds like (211) are satisfied. Moreover, when the charge becomes too large w.r.t. the mass, the geometry becomes horizonless, so that the mass inflation regime of single-horizon RBHs is bounded from above and bellow. Therefore, if mass-inflation does not drive the geometry outside of spherical symmetry, it might end-up either producing an horizonless geometry or a single-horizon horizon RBH.

2. Discontinuous extremal inner horizon (A)dS-core black hole

Now let's turn to the mass-inflation instability of (A)dS-core regular black holes. As we will see, it seems possible within our model to avoid it in a similar way as in the previous case. However, the construction here is more involved. First, we need to consider an infinite series of corrections ($m \to \infty$) to EH action, otherwise the surface gravity κ_- of the inner horizon would be finite, triggering the mass inflation. But to make the case of (A)dS-cores, we still need the geometry to contain these.

Therefore, a first requirement on the non-perturbative theory ($m \to \infty$) is that for any finite truncation $m < \infty$, the solution is an (A)dS-core RBH with Schwarzschild asymptotic behaviour. This implies some relations between the coupling constants (α, β).

Then the way we have found to avoid mass inflation is the following. We consider in addition that for any m, the vacuum is the same and given by an extremal (A)dS-core RBH that "dresses" the usual Minkowski vacuum (which is recovered asymptotically when $l \to 0$ or $r \to \infty$). This is the kind of theories we discussed in the fifth section of this Chapter, and recall that imposing for all the truncations to share the same vacuum is a quite simple requirement satisfied by the usual effective actions of gravity like Lovelock-Lanczos or $F(R)$. This also requires some additional relations between (α, β). Finally, one can derive the surface gravity of the inner horizon κ_- for an arbitrary truncation m, and tune the remaining (α, β) so that in the limit $m \to \infty$, we have $\kappa_- \to 0$.

Like for the case of single-horizon RBHs, there are many examples that can be found within this model, but we will analyse only one here. First we choose the following (shared) vacuum :

$$a_{\text{vac}}(r) = \frac{\left(l^2 - r^2\right)^2}{\left(l^2 + r^2\right)^2} \tag{212}$$

describing an extremal black hole deformation of Minkowski spacetime, so that for any truncations m, the solutions can be written as

$$\begin{aligned} b(r) = 1 \ , \quad a_m(r) = (1 + K_m(r)) - \frac{4MX_m(r)}{r} = a_{\text{vac}}(r) - \frac{4MX_m(r)}{r} \ , \\ X_m(r) = \frac{1}{2 + \sum_{i=2}^{m} l^i \beta_i r^{-i}} \ , \quad K_m(r) = -\frac{\sum_{i=2}^{m} l^i \gamma_i r^{-i}}{2 + \sum_{i=2}^{m} l^i \beta_i r^{-i}} \end{aligned} \tag{213}$$

We are not interested here to reconstruct precisely the constants (α, β), but the way to see what relations this requirement implies on them is obviously to fix the constants so that $1 + K_m(r) = a_{\text{vac}}(r)$.

A solution which satisfies the previous requirements is given by :

$$\begin{aligned} a_m(r) &= \frac{\left(l^2 - r^2\right)^2}{\left(l^2 + r^2\right)^2} - \frac{2Mr^{2m-1}}{\left(l^2 + r^2\right)^2 \sum_{n=1}^{m} (n-1) l^{2(n-2)} r^{2(m-n)}} \\ &= \frac{\left(l^2 - r^2\right)^2}{\left(l^2 + r^2\right)^2} \left(1 - \frac{2M}{r\left(1 + \left(\frac{l}{r}\right)^{2m}\left((m-1) - m\left(\frac{r}{l}\right)^2\right)\right)}\right) \end{aligned} \tag{214}$$

First, it is obviously a solution of our model, being a rational black hole linear in the mass. Moreover, for any order of corrections m, i.e. for any truncation of the non-perturbative theory $m \to \infty$, the shared vacuum is indeed $a_{vac}(r)$, while the asymptotic behaviours are :

$$\begin{aligned} a_m(r \to 0) &= 1 - \frac{4r^2}{l^2} + o\left(\frac{r^2}{l^2}\right) \\ a_m(r \to \infty) &= 1 - \frac{2M}{r} - \frac{4l^2}{r^2} + O\left(\frac{Ml^2}{r^3}\right) \end{aligned} \tag{215}$$

Thus, for each m, the solution describes an (A)dS-core regular black hole with two horizons. We are interested in the value of the surface gravity of the inner horizon κ_- as $m \to \infty$. To do so, note first that the non-perturbative solution is given by :

$$a_{(m\to\infty)}(r) = \begin{cases} \left(\frac{l^2 - r^2}{l^2 + r^2}\right)^2 \ , & \text{if} \quad r \leq l, \\ \left(\frac{l^2 - r^2}{l^2 + r^2}\right)^2 \left(1 - \frac{2M}{r}\right) \ , & \text{if} \quad r > l, \end{cases} \tag{216}$$

Therefore, the geometry becomes discontinuous at $r = l$ in the non-perturbative limit, due to the distributional character of $a_{(m)}$ in this limit. However, neither $a_{(m\to\infty)}(r > l)$ nor $a_{(m\to\infty)}(r \leq l)$ have singularities at $r = l$, so that the curvature invariants of the full solution $a_{(m\to\infty)}(r)$ are discontinuous at this point, but finite[12] . This behaviour of the non-perturbative limit is quite

[12] Actually, it turns out that only the curvature invariant that are polynomials of the Riemann tensor are discontinuous, while the scalars involving covariant derivatives are continuous and vanishing at $r = l$.

interesting in itself, but before investigating it further, we want first to see how κ_- is affected by it. In this limit, the inner horizon is located at $r_- = l$, so that it is given by :

$$\kappa_- := \frac{a'_m(r_-)}{2} \underset{m\to\infty}{\approx} \frac{a'_m(l)}{2} = \frac{(5-4m)M}{6m(m-1)l^2} \underset{m\to\infty}{\approx} -\frac{2M}{3ml^2} + O\left(\frac{1}{m^2}\right) \tag{217}$$

We see that the effect of mass inflation, which is controlled by the value of κ_- in Eq(200), can be made arbitrarily small providing that one considers sufficiently high order of corrections m. More precisely, from Eq(200), we see that one needs not consider m such that $|\kappa_-|v \approx \frac{M}{ml^2}v \ll 1$. Moreover, for any large but finite values of m, the geometry is perfectly smooth, so that one can effectively avoid the issue of the discontinuity of the non-perturbative solution (216) and mass inflation by working with very large, but finite number of corrections to E-H action. This provide a possible mechanism to avoid this instability in the context of (A)dS-core regular black hole geometries.

Similarly to the previous case, the charged black hole solution is found from the replacement $M \to M - QA_0(r)/2$, which gives again $\left(\frac{l^2-r^2}{l^2+r^2}\right)^2$ for $r \leq l$ but $\left(\frac{l^2-r^2}{l^2+r^2}\right)^2 \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$ for $r > l$. This means that it is possible to avoid the mass inflation in this charged case providing that the smallest zero of the equation $1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0$ is smaller than l, i.e. if the following bound is satisfied :

$$M \geq \frac{l}{2} + \frac{Q^2}{2l} = \frac{1}{2}\sqrt{\frac{\hbar c}{G}}\left(n + \frac{\alpha Q^2}{ne^2}\right) \tag{218}$$

where we have restored the units and assumed that l is a multiple of the Planck length : $l = nl_P$. For stellar mass black holes $M \approx 10^{30}$kg (with radius $r \approx 1.5$km) and $n = 1$, the bound is already very hard to satisfy and gives $Q \lessapprox 18C$, which is the typical charge of a lightning on Earth. If the charge is larger, then the black hole becomes unstable. But note that the existence of this kind of bound would make sense given that, just like for angular momentum, a charge produces a repulsive potential that counterbalance the gravitational one, so that if a black hole is too charged or is rotating too fast, the end state of a collapse can be expected to be a bounce instead of a regular black hole (whose only instability would be due to Hawking radiation). Note also that this relation for $Q = 0$ gives a lower bound on the mass of such black holes $M \geq \frac{n}{2} M_P$, which is saturated when $2M = l$, that is when there is a single extremal horizon, stopping the evaporation, and resulting in a finite mass remnant.

To conclude this section, note that in order to interpret properly the non-perturbative solution $a_{(m\to\infty)}(r)$, one should study the behaviour of geodesics for an arbitrary truncation $a_m(r)$ and see what happens to the observables of test particles in the limit $m \to \infty$.

For example, the limit $m \to \infty$ might make the amount of proper-time/affine parameter needed to reach $r = l$ infinite (which seems possible given the discontinuity appearing in this limit). If this is the case, one might consider that the solution $a_{(m\to\infty)}(r)$ splits into two causally separated geometries, $a_{(m\to\infty)}(r > l) = a_{vac}(1 - 2M/r)$ and $a_{(m\to\infty)}(r \leq l) = a_{vac}$, describing respectively a single-horizon RBH with a dynamical interior, similarly to what we saw in the previous section, and a vacuum core, modelling the most inner region of the black hole.

This would have some similarities with the D'Ambrosio-Rovelli [175, 226] and Visser-Hochberg [227, 228] black holes. Indeed, we will see in the next Chapter that in Schwarzschild gauge, these geometries describe two distinct regions : for $r > l$, a Lorentzian single-horizon black hole, while for $r < l$, they describe an Euclidean region.

In this case, it might be more convenient to express the non-perturbative solution in the following way. For $r > l$, we perform the coordinate change $r = \sqrt{l^2 + \rho^2}$, for $-\infty < \rho < \infty$, so that the interval becomes :

$$ds^2_{(r>l)} = -\frac{\rho^4}{(2l^2+\rho^2)^2}\left(1 - \frac{2M}{l^2+\rho^2}\right)dt^2 + \frac{(2l^2+\rho^2)^2\, d\rho^2}{\rho^2\,(l^2+\rho^2)\left(1 - \frac{2M}{\sqrt{l^2+\rho^2}}\right)} + \left(l^2+\rho^2\right)d\Omega^2 \tag{219}$$

which describes a similar geometry as the Visser-Hochberg black bounce [227, 228]. Indeed, the radial null-curves are determined by :

$$\frac{d\rho}{dt} = \pm \frac{\sqrt{\rho^6 \left(l^2+\rho^2+4M\left(M-\sqrt{l^2+\rho^2}\right)\right)}}{\left(2l^2+\rho^2\right)^2} \tag{220}$$

so that the geometry possesses an extremal horizon at $\rho = 0$, and for $l < 2M$, it has in addition a black hole horizon at $\rho = \sqrt{4M^2 - l^2}$ and a white hole one at $\rho = -\sqrt{4M^2 - l^2}$.

Concerning the inner core solution $r < l$, one can perform the change of coordinate $r = l \sin\psi$ and take $-\infty < \psi < \infty$,

$$ds^2_{(r\le l)} = -\frac{\cos^4\psi \, dt^2}{\left(1+\sin^2\psi\right)^2} + l^2 \frac{\left(1+\sin^2\psi\right)^2 d\psi^2}{\cos^2\psi} + l^2 \sin^2\psi d\Omega^2 \tag{221}$$

so that the interval describes a periodic multi-horizons geometry that are determined by $\psi = \pm\frac{\pi}{2} + 2n\pi$, where n is an arbitrary integer.

H. FLRW sector and alternative actions

1. FLRW sector of the Non-polynomial curvature invariants

To conclude this Chapter, we will briefly see how one can find a Friedmann-Lemaître-Robertson-Walker sector to the non-polynomial curvature invariants that we are using. Recall that they are constructed from the non-polynomial tensor Eq(88) :

$$u^\nu_\mu := \frac{\Big(-(n+1)C_\mu{}^{\alpha\beta} + C_\mu{}^{\beta\alpha}\Big) C^\nu{}_{\alpha\beta}}{C_{\sigma\rho\delta}C^{\sigma\rho\delta}} \tag{222}$$

So that, naively evaluating this tensor for any conformally flat spacetime leads for its components to indeterminate forms $\frac{0}{0}$. Note that it is not strictly a requirement to study cosmological geometries to be able to have an FLRW sector, as one could directly work with anisotropic cosmological backgrounds (with two scale factors, in order to stay within dynamical spherical symmetry) for which the previous tensor is well-defined.

FLRW from Dynamical Spherical Symmetry

However, as it turns out, it seems nonetheless possible to assign an FLRW sector to the non-polynomial scalars built from this tensor, and in the last Chapter of this thesis, we will study regular cosmological solutions from this result. In order to do so, first recall that from a general four dimensional dynamical spherically symmetric spacetime, in diagonal gauge, given by

$$ds^2 = -e(t,w)b^2(t,w)dt^2 + \frac{dw^2}{e(t,w)} + r^2(t,w)\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\phi^2\right) \tag{223}$$

the Friedmann-Lemaître-Robertson-Walker metric can be found from two different ways :

1. From $k = 0$: set $r(t,w) = a(t)$, $e(t,w) = 1/a^2(t)$ and $b(t,w) = a(t)\sqrt{N(t)}$, so that the interval becomes

$$ds^2 = -N(t)dt^2 + a^2(t)\left(dw^2 + d\rho^2 + \rho^2 d\phi^2\right) \tag{224}$$

so that from the change of coordinates $y = \rho\cos\phi$, $x = \rho \sin\phi$ and $w = z$ we obtain FLRW metric :

$$ds^2 = -N(t)dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{225}$$

2. From $k = 1$: set $r(t,w) = a(t)w$, $e(t,w) = 1/a^2(t)$ and $b(t,w) = a(t)\sqrt{N(t)}$, so that the interval becomes

$$ds^2 = -N(t)dt^2 + a^2(t)\left(dw^2 + w^2\left(\frac{d\rho^2}{1-\rho^2} + \rho^2 d\phi^2\right)\right) \tag{226}$$

then by setting first $\rho = \sin\theta$, and then performing the usual change of coordinates $x = w\sin\theta\cos\phi$, $y = w\sin\theta\sin\phi$ and $z = w\cos\theta$ leads again to (225).

Therefore, FLRW metric being a subsector of a general DSS spacetimes, a way to evaluate the tensor (222) in this sector would be to evaluate it on a general DSS ansatz ; as we saw in Eq(88) it gives

$$u^{\nu}_{\mu}\Big\rfloor = \frac{1}{2}\left(-n\,\delta^{\nu}_{\mu} + \frac{n^2-1}{n}\sigma^{\nu}_{\mu}\right) \tag{227}$$

where $\sigma^{\nu}_{\mu} = \sum_{i=1}^{n} \delta^{i}_{\mu}\delta^{\nu}_{i} = \mathrm{diag}(0,0,1,...,1)$, as we saw in Eq(63).

Reminder of the construction of non-polynomial scalars

Now we need to see how the non-polynomial scalars that we are using behave when after the evaluation on DSS, we further restrict to FLRW by one of the two ways that we just mentioned. First recall that we are using the scalars of Eq(98,99,101) defined by :

$$\begin{aligned}
R_{(\Omega)} &:= \left(R^{\alpha\beta} - \nabla^{\alpha}\nabla^{\beta}\right)\sigma_{\alpha\beta}\,, \\
R_{(\Sigma)} &:= \left(R^{\alpha\beta} + \nabla^{\alpha}\nabla^{\beta}\right)\omega_{\alpha\beta} - \frac{n-1}{2}\nabla_{\gamma}\omega_{\alpha\beta}\nabla^{\gamma}\omega^{\alpha\beta}\,, \\
\mathcal{K} &:= \frac{1}{4}\nabla_{\mu}\nabla_{\nu}\omega_{\sigma\rho}\nabla^{\mu}\nabla^{\nu}\omega^{\sigma\rho} - \frac{5}{16}\left(\nabla_{\gamma}\omega_{\alpha\beta}\nabla^{\gamma}\omega^{\alpha\beta}\right)^2 + \frac{1}{8}\nabla^{\mu}\omega_{\mu\nu}\nabla^{\nu}\left(\nabla_{\gamma}\omega_{\alpha\beta}\nabla^{\gamma}\omega^{\alpha\beta}\right)\,,
\end{aligned} \tag{228}$$

as well as $\nabla_{\gamma}\omega_{\alpha\beta}\nabla^{\gamma}\omega^{\alpha\beta}$ and $\frac{1}{n}\left(\nabla^{\alpha}\nabla^{\beta}\omega_{\alpha\beta} - \frac{n-1}{2}\nabla_{\gamma}\omega_{\alpha\beta}\nabla^{\gamma}\omega^{\alpha\beta}\right)$. Then, in DSS spacetimes, we saw that these scalars simply reduce to :

$$R_{(\Omega)}\Big\rfloor = R^{(n)}(\Omega) = k\left(\frac{n(n-1)}{r^2}\right)\;,\quad R_{(\Sigma)}\Big\rfloor = R^{(2)}(\gamma)\;,\quad \mathcal{K}\Big\rfloor = \frac{\mathcal{D}_a\mathcal{D}_b r\mathcal{D}^a\mathcal{D}^b r}{r^2}$$

and

$$\nabla_{\gamma}\omega_{\alpha\beta}\nabla^{\gamma}\omega^{\alpha\beta}\Big\rfloor = 2n\,\frac{\gamma^{ab}\mathcal{D}_a r\,\mathcal{D}_b r}{r^2}\quad,\quad \frac{1}{n}\left(\nabla^{\alpha}\nabla^{\beta}\omega_{\alpha\beta} - \frac{n-1}{2}\nabla_{\gamma}\omega_{\alpha\beta}\nabla^{\gamma}\omega^{\alpha\beta}\right)\Big\rfloor = \frac{\mathcal{D}^2 r}{r}$$

where from Eq(89), we can define the tensors σ and ω as :

$$\begin{aligned}
\sigma_{\mu\nu} &:= \frac{n}{n^2-1}\left(2u_{\mu\nu} + n\,g_{\mu\nu}\right)\,, \\
\omega_{\mu\nu} &:= g_{\mu\nu} - \sigma_{\mu\nu} = -\frac{g_{\mu\nu} + 2\,n\,u_{\mu\nu}}{n^2-1}\,.
\end{aligned}$$

Therefore, by first evaluating the tensor u on DSS, we obtain the two-dimensional scalars

$$\left(\mathcal{R} := \frac{k}{r^2}\;,\quad R^{(2)}(\gamma)\;,\quad \frac{\mathcal{D}_a\mathcal{D}_b r\mathcal{D}^a\mathcal{D}^b r}{r^2}\;,\quad \frac{\mathcal{D}_a r\,\mathcal{D}^a r}{r^2}\;,\quad \frac{\mathcal{D}^2 r}{r}\right) \tag{229}$$

FLRW sector of the two-dimensional scalars

Now we can simply evaluate the previous two-dimensional scalars following the procedure 1. and 2. of the last paragraph :

1. From $k = 0$, we obtain the following results

$$\begin{aligned}
&\mathcal{R} = 0\;,\quad R^{(2)}(\gamma) = \frac{-\dot{a}\dot{N} + 2N\ddot{a}}{aN^2}\;,\quad \frac{\mathcal{D}_a r\,\mathcal{D}^a r}{r^2} = -\frac{\dot{a}^2}{a^2 N}\;, \\
&\frac{\mathcal{D}^2 r}{r} = \frac{a\dot{a}\dot{N} - 2N\left(\dot{a}^2 + a\ddot{a}\right)}{2a^2N^2}\;,\quad \frac{\mathcal{D}_a\mathcal{D}_b r\mathcal{D}^a\mathcal{D}^b r}{r^2} = \frac{a^2\dot{a}^2\dot{N}^2 - 4a^2N\dot{a}\dot{N}\ddot{a} + 4N^2\left(\dot{a}^4 + a^2\ddot{a}^2\right)}{4a^4N^4}
\end{aligned} \tag{230}$$

2. While from $k = 1$, it gives the same results for $R^{(2)}(\gamma)$, $\frac{\mathcal{D}_a\mathcal{D}_b r\mathcal{D}^a\mathcal{D}^b r}{r^2}$ and $\frac{\mathcal{D}^2 r}{r}$, but not for the remaining ones given by :

$$\mathcal{R} = \frac{1}{r^2(x,y,z)a^2}\;,\quad \frac{\mathcal{D}_a r\,\mathcal{D}^a r}{r^2} = \frac{1}{r^2(x,y,z)a^2} - \frac{\dot{a}^2}{a^2 N} \tag{231}$$

which are obviously not invariant scalars (because scalars cannot depend on the coordinates (x, y, z) in FLRW), except if one only considers the following combination of the two

$$\mathcal{Z} := \mathcal{R} - \frac{\mathcal{D}_a r\, \mathcal{D}^a r}{r^2} \tag{232}$$

In which case the results for the "$k = 0$" and "$k = 1$" procedures agree with each other.

Note that this is not very surprising because as we saw in the previous Chapter Eq(93), the Riemann tensor decomposes in DSS as

$$R^{ai}{}_{bj} = -\mathcal{Y}^a_b \delta^j_i \quad , \quad R^{ij}{}_{kl} = \mathcal{Z} \delta^i_{[k} \delta^j_{l]} \quad , \quad R^{ab}{}_{ce} = \frac{1}{2} R^{(2)}(\gamma)\, \delta^a_{[c} \delta^b_{e]} \tag{233}$$

where $\mathcal{Y}^a_b := \frac{\mathcal{D}^a \mathcal{D}_b r}{r}$, and thus only involves the specific combination $\mathcal{Z}$ of the two 2D-scalars $\mathcal{R}$ and $\frac{\mathcal{D}_a r\, \mathcal{D}^a r}{r^2}$.

Therefore, it seems that it is indeed possible to assign a FLRW sector to the non-polynomial curvature invariants (228), providing that we consider the combination $\mathcal{Z}$ of the NP scalars $R_{(\Omega)}$ and $\nabla_\gamma \omega_{\alpha\beta} \nabla^\gamma \omega^{\alpha\beta}$.

However, one might still wonder if it would be possible to assign directly an FLRW sector to these two scalars by the "$k = 0$" procedure which seems to give healthy FLRW scalars

$$\mathcal{R} \;\; = 0 \;\; , \;\; \tfrac{\mathcal{D}_a r\, \mathcal{D}^a r}{r^2} = -\tfrac{\dot{a}^2}{a^2 N} \tag{234}$$

If so, then the action Eq(141) that we considered in this Chapter would give identically vanishing corrections to the E-H action in this sector, because the higher orders are generated by powers of $\mathcal{R}$, which would be identically vanishing in this sector. Therefore, we will now see a way to still get some non-vanishing corrections for FLRW.

2. Alternative action admitting a non-trivial FLRW sector

Recall that the action Eq(141) that we have studied in this Chapter was given by

$$\begin{aligned} I &= \frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g}\, (-2\Lambda + R) + I_{(m,p)} \\ I_{(m,p)} &:= \frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g} \sum_{i=p}^{m} l^i\, \mathcal{R}^{i/2} \left(\alpha_i\, \mathcal{R} + \beta_i\, \mathcal{S}_{(i)} \right) \end{aligned} \tag{235}$$

where we have discarded the boundary term for simplicity. From there, a possible way to have a non-trivial FLRW sector (*if we restrict to the "$k = 0$" procedure only*) is to consider the alternative action, setting $G = 16\pi$:

$$I_{(m,p)} = \frac{1}{2} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g} \sum_{i=p}^{m} l^i\, \phi^{i/2} \Big(\alpha_i\, \left((i+2)\mathcal{R} - i\phi\right) + \beta_i\, \mathcal{S}_{(i)}\, \phi^{-1}\, \left(i\,(\mathcal{R} - \phi) + 2\phi\right) \Big) \tag{236}$$

where ϕ is a Lagrange multiplier field, whose field equation is :

$$\frac{\delta I}{\delta \phi} = \frac{1}{4} (\mathcal{R} - \phi) \sum_{i=-2}^{m} i\phi^{i/2} \left(\frac{(i+2)\alpha_i}{\phi} + \frac{(i-2)\beta_i \mathcal{S}_{(i)}}{\phi^2} \right) = 0 \tag{237}$$

The branch $\phi = \mathcal{R}$ leads back to the action (235), while the other one provides a new branch of solutions, which yields non-trivial cosmological field equations. For example, considering only one UV correction gives $\phi = \frac{(-m+2)\beta_m}{(m+2)\alpha_m} \mathcal{S}_{(m)}$, and by restricting to FLRW by using the definition of $\mathcal{S}_i$, eq(142), and the eq(230) that we established previously, as well as $\mathcal{R} = 0$, the four dimensional action reduces to :

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left(\frac{1}{m+2} \right)^{1+m/2} \left(\frac{\alpha_m}{\beta_m} \right)^{m/2} \beta_m \left((3-m)\,(m+4)\, \frac{H^2}{N} + 6\frac{\dot{H}}{N} - 3\frac{H\dot{N}}{N^2} \right)^{1+m/2} \tag{238}$$

where $H = \dot{a}/a$ is the Hubble parameter. The solutions of this model are quite difficult to find and the cosmological field equations are higher order ones, so that we have not investigated further the cosmological sector of this theory.

However, note that this introduction of a Lagrange multiplier field can be used to give a non-vanishing cosmological sector to any NPG theory which reduces in DSS to 2D Einstein-Dilaton gravities.

IV. Quantum gravity inspired solutions from Einstein-Dilaton gravities

In the previous Chapter, we saw a first example of a family of two-dimensional Einstein-Dilaton gravities leading to regular black hole solutions. In particular, in the gauge (146), the solution were such that $b = 1$ and a was linear in the mass. Moreover, it turned out that a simple way to avoid the mass-inflation instability of the charged solutions was to consider non-perturbative effective actions (in the parameter l), such that their neutral solutions possess a single horizon, while still being fully regular.

In this Chapter, we will pursue the investigation of such single-horizon RBH geometries, by considering effective-like actions leading to $b \neq 1$. Although the family of actions studied in the previous Chapter was motivated by preserving the DSS decompositions of both the Ricci scalar and Einstein tensor, from which we *found* RBH solutions, we will follow the other path in this Chapter : from specific RBHs ansatz, we will *reconstruct* the two-dimensional Einstein-Dilaton (ED) theories that admit such solutions.

To do so, we will first establish in Sec.1 the general way to reconstruct any static black hole solution for which a is linear in the mass parameter M, from 2D ED.

Then, we will be interested by the reconstruction of specific Quantum Gravity inspired single-horizon RBHs. In Sec.2, we will reconstruct the Einstein-Dilaton action associated with the Modesto semi-polymeric black hole, found in [44] and further studied for example in [45–47]. This geometry has been derived from the Loop quantization of spherically symmetric Einstein gravity, is regular, and in particular, the singularity of Schwarzschild geometry is replaced by a bouncing region. We will see that only a part of this geometry (including a part of the interior) can be a solution of ED gravity, so that in order to have a complete spacetime, we need to extend the coordinate in such a way that the geometry describes an interior black hole solution evolving into a white hole.

Such geometries have also been found in the past. In particular the D'Ambrosio-Rovelli metric [175, 226] and the Visser-Hochberg black bounce [227, 228] both describe the same kind of interior, where for some extended "radial" coordinate x (which is time-like in these regions), the $0 < x < x_H$ region is a black hole interior, while the $-x_H < x < 0$ region is a white hole one. In Sec.3 and 4. we reconstruct the Einstein-Dilaton actions leading to these metric fields. This class of non-singular modifications to Schwarzschild spacetime have been proposed in [175, 226] to provide a semi-classical description of a purely quantum phenomena : the transition from a black hole state to a white hole one. Interpreted in this way, they provide a continuous description of the inner regions of the so-called Planck stars [211–214]). In this scenario, after having formed an horizon, the collapsing matter would quickly (in its proper-time) bounce back, ending up releasing the information contained within the horizon, due to quantum gravity effect. From the point of view of a far-away observer, the huge time-dilation would make this bounce very slow, explaining the large life-time of observed black holes.

Curiously, we will see that, in Schwarzschild gauge, these two solutions are very similar and share a peculiar feature : while the regions $r > l$ describe the Lorentzian solutions, the $r < l$ regions, that are attached smoothly to the first ones, are Euclidean.

Moreover, an other interesting characteristic of the three geometries of Sec.2,3,4 is that contrary to the previous Einstein-Dilaton black holes that we found in the previous Chapter, the curvature corrections in the field equations affect the matter part (because $b \neq 1$). In particular, considering a Maxwell theory minimally coupled with the ED theories that we will consider leads naturally to a gravitational modification of the electrostatic field, making it regular. Therefore, in these particular types of single-horizon regular black hole, the Coulomb singularity is also cured.

A. Reconstruction and Birkhoff theorem

As a first case of study, we want to be able to reconstruct the two dimensional spherically symmetric action associated with any (spherically symmetric) metric whose dependence on the mass is linear.

As we will see, the following two dimensional Einstein-Dilaton gravity is the simplest one with this property :

$$I = \int_\Sigma = \sqrt{-\gamma}\, r^2 \left(f_0 \mathcal{R} + f_1 R^{(2)} + f_2 \chi + f_3 \mathcal{Y} \right) \tag{239}$$

where r is a scalar field on the two dimensional manifold Σ, $(f_0(r), f_1(r), f_2(r), f_3(r))$ are arbitrary functions of this scalar field r, and

$$\mathcal{R} := \frac{1}{r^2} \quad , \qquad \chi := \frac{\mathcal{D}_a r \mathcal{D}^a r}{r^2} \quad , \qquad \mathcal{Y} := \frac{\mathcal{D}^2 r}{r} \tag{240}$$

An important observation, to proceed is that in the following gauge

$$ds^2 = -a(t,r) b(t,r)^2 dt^2 + \frac{dr^2}{a(t,r)} + r^2 d\Omega^2 \tag{241}$$

the field equations associated with the previous action with respect to $a(t,r)$ and $b(t,r)$ decouple, so that $\frac{\delta I}{\delta a}$ and $\frac{\delta I}{\delta b}$ depend, respectively, only on $b(t,r)$ and $a(t,r)$. This gauge was used in several papers where the minisuperspace "Weyl" approach to spherical symmetry was studied, for example in [318, 339, 340].

As we want to reconstruct the actions that give well-known static spherically symmetric effective metric, we restrict to $a(r)$ and $b(r)$. The dynamical sector can be studied once the action is found. Then we have :

$$\sqrt{-\gamma} = b \quad , \qquad \chi = \frac{a}{r^2} \quad , \qquad \sqrt{-\gamma} R^{(2)} = -\left(2ab' - ba'\right)' \quad , \qquad \sqrt{-\gamma}\mathcal{Y} = (ab)' \tag{242}$$

It is then straightforward to show that :

$$\begin{aligned} \frac{\delta I}{\delta a} &= h_1(r)\, b + h_2(r)\, r b' = 0 \\ \frac{\delta I}{\delta b} &= f_0 + h_3(r)\, a - h_2(r)\, r a' = 0 \end{aligned} \tag{243}$$

where $h_1 = -2f_1 + f_2 - f_3 - r\left(4f_1' + f_3' + r f_1''\right)$, $h_2 = 2f_1 + r f_1'$ and $h_3 = -4f_1 + f_2 - f_3 - r\left(8f_1' + f_3' + 2r f_1''\right)$. Note that, indeed, due to the linearity of these differential equations on a, a', b, b', the integration constant (the mass) can only appear linearly in the solution. Moreover, it is only present in a, while the integration constant appearing when solving the field equation for b can always be reabsorbed by a redefinition of the time coordinate. In order to simplify the reconstruction procedure, it can be useful to use

$$\begin{aligned} &f_0(r) = g_0(r) Y(r) \quad , \qquad f_1(r) = \frac{\alpha}{r^2} - \frac{\int^r g_1(x) Y(x) dx}{r^2} \quad , \\ &f_2(r) = f_3(r) + r f_3'(r) + Y(r)\left(g_2(r) - 2g_1'(r)\right) - 2g_1(r) Y'(r) \end{aligned} \tag{244}$$

where α is an arbitrary constant appearing because $R^{(2)}$ is a total derivative, see Eq(36), so that we can set $\alpha = 0$. The field equations simplify to :

$$\begin{aligned} \frac{\delta I}{\delta a} &= \left(Y\left(g_2 - g_1'\right) - g_1 Y'\right) b - g_1 Y\, b' = 0 \\ \frac{\delta I}{\delta b} &= Y\left(g_0 + g_2\, a + g_1\, a'\right) = 0 \end{aligned} \tag{245}$$

Then the procedure to reconstruct the action (239) from a given metric g is the following. In the previous gauge, the metric only depends on (a, b) and a is linear in the mass. Therefore $a(r) = s(r) + M u(r)$, and so this metric function is a solution of the following differential equation:

$$(su' - us') - u'\, a + u\, a' = 0 \tag{246}$$

This fixes

$$g_0 = su' - us' \quad , \qquad g_2 = -u' \quad , \qquad g_1 = u \tag{247}$$

and the field equations for b reduces to

$$\frac{\delta I}{\delta a} = -2Yu'\,b - u\,(Y\,b' + Y'\,b) = 0 \tag{248}$$

where u and b are given by the ansatz metric from which we want to find an action. The solution for Y is simply

$$Y = \frac{\beta}{b\,u^2} \tag{249}$$

with β an integration constant that amounts to a normalization of the action, so we can set it to unity. Therefore, given a static spherically symmetric ansatz $ds^2 = -a(r)b(r)^2dt^2 + \frac{dr^2}{a(r)} + r^2d\Omega^2$ with $a(r) = s(r) + Mu(r)$, we showed that it can be derived from the two-dimensional Einstein-Dilaton action :

$$\begin{aligned} I = \int \sqrt{-\gamma}\, r^2 \Bigg(\left(\frac{(su' - s'u)}{b\,u^2}\right)\mathcal{R} - \left(\frac{1}{r^2}\int \frac{dr}{b(r)u(r)}\right) R^{(2)} + \left(\frac{2ub' + bu'}{b^2\,u^2}\right)\chi \\ + (f_3 + r f_3')\,\chi + f_3\mathcal{Y}\Bigg) \end{aligned} \tag{250}$$

where the terms involving f_3 are boundary terms that can be chosen to simplify the action and have a well-defined spherically symmetric variational principle. Moreover, we see that the action is automatically written in covariant form, so that we can consider r as a scalar field and relax the choice that it is the spatial coordinate of Σ. It is also possible to introduce a time dependence on the metric γ, so that we can look for dynamical solutions, Birkhoff theorem, etc. Finally, provided that we replace all the terms[13] involving the scalar field r by $r = 1/\sqrt{\mathcal{R}}$, where $\mathcal{R}$ is proportional to the Ricci scalar of the horizon manifold as previously, we can reach these $2D$ actions via the spherically symmetric reduction of four dimensional Non-Polynomial Gravities from the Eq(98,100, 101). This allows to investigate topological solutions as well, because in general we have $\mathcal{R} = k/r^2$ where k is the topological parameter. As we will see later, it also enables one to find the charged solutions of the model.

Note that in the process, we have discarded many actions that could have the same solutions. First, still considering second order field equations, we see that from equation (246) to the identification (247), we could also have considered that the differential equation satisfied by a was

$$H\,(r, a, a')\,((su' - us') - u'\,a + u\,a')\,Y = 0 \tag{251}$$

For any given functional $H\,(r, a, a')$. So that there would exists two branches of solutions $H\,(r, a, a') = 0$ and $(su' - us') - u'\,a + u\,a' = 0$.

Secondly, it is also possible that the actual effective quantum gravity corrections to the spherically symmetric sector of General Relativity are fundamentally higher order ones. In this case, still assuming that the effective solution given by quantum gravity is $a = s + Mu$ (but this time, only when the additional "quantum degrees of freedom" are vanishing), it is very much unclear what the differential equation might look like, as a can a priori depend on the additional integration constants q_p in an arbitrary manner, for example $a = s(r)(1 + J(q_p, M, r)) + Mu(r)(1 + K(q_p, M, r))$, so that $J(q_p, M, r) \to 0$ and $K(q_p, M, r) \to 0$ when $q_p = 0$. Moreover, an action leading to higher order field equations for a and b will likely break the property present in our case that the gauge (241) produces two decoupled differential equations on a and b from action (239).

Before concluding this section, let's see that Birkhoff's theorem holds for all the class of action (239). In order to see this, we follow [318] in which a minisuperspace approach to prove Birkhoff's theorem is presented. We first choose the gauge,

$$ds^2 = -a(t, r)b(t, r)^2dt^2 + \frac{dr^2}{a(t, r)} + 2b(t, r)f(t, r)dt\,dr + r^2d\Omega^2 \tag{252}$$

[13]Except the common factor r^2 which comes from the spherically symmetric reduction of the determinant of the four dimensional metric g as : $\sqrt{-g} = r^2\sqrt{-\gamma}$.

where $f(t,r)$ can be reabsorbed by a coordinate transformation. Then Birkhoff theorem can be checked from the equation $\frac{\delta I}{\delta f}\big|_{f=0}=0$. In this field equation, only the non-vanishing terms when $f=0$ will contribute. In our case, the useful terms of the action (239) are :

$$I=\int d^2x\, r^2\sqrt{-\gamma}\left(\frac{f_1}{a\left(1+f^2\right)^2}\left(f'\dot{a}-a'\dot{f}\right)+...\right) \tag{253}$$

It gives the following field equation :

$$\left.\frac{\delta I}{\delta f}\right|_{f=0}=-r\left(2f_1+rf_1'\right)\dot{a}=0 \tag{254}$$

Therefore $\dot{a}=0$, what proves Birkhoff theorem. Note that the equation $2f_1+rf_1'=0$ simply corresponds to the term $\int\sqrt{-\gamma}\,R^{(2)}$ in the action, which is topological. It means that in these kind of actions, it is the linearity of the two dimensional Ricci scalar which allows to have Bikhoff theorem.

There are many regular black holes that one can reconstruct from this technique. However, in many cases, the integrals in Eq(250) are difficult to find. For example, consider the Non-Commutative geometry inspired (A)dS-core regular black hole [134–136], given by :

$$ds^2=-\left(1-\frac{4M}{\sqrt{\pi}\,r}\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right)\right)dt^2+\left(1-\frac{4M}{\sqrt{\pi}\,r}\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right)\right)^{-1}dr^2+r^2d\Omega^2 \tag{255}$$

where $\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right):=\int_0^{r^2/4\theta}\sqrt{t}e^{-t}dt$. In this case we have, $b(r)=s(r)=1$ and $u(r)=\frac{4}{\sqrt{\pi}\,r}\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right)$. Therefore we should integrate $1/u(r)$ to find the exact form of the Lagrangian.

B. Modesto semi-polymeric black hole

Instead, in the following sections we will focus on regular black holes for which the integrals can be found, and more specifically, we will concentrate on the class of regular black holes that possess only one horizon, making the interior region solely dynamical. They differ quite drastically from the usual (A)dS-core ones as we saw before, and some have been found in the context of Einstein-Dilaton gravity before, for example in [41–43]. The single-horizon regular black hole found in these papers was based on a polymer quantization of spherically symmetric Einstein-gravity. There are many inequivalent polymerization schemes one can use, depending on the choice of geometrical variables, and therefore different resulting black hole interiors, correcting Schwarzschild geometry. In addition to the previous references, see for example [44, 341–344].

In this section, we will focus on a particular result by Modesto [44], in which a two-parameter family of regular black holes was found, one parameter being related to the minimal area in LQG [16] (which is the lowest eigenvalue of the area operator), while the other is the so-called polymeric-parameter δ. To understand the meaning of this parameter, recall that in LQG, the geometric variable which is quantized is not directly the (Ashtekar) connection $A=\Gamma+\gamma K$, where Γ, K and γ are respectively the spin-connection, the extrinsic curvature and the Barbero-Immirzi parameter, but rather the $SU(2)$ group elements reached from the holonomy of the connection along the edge l of a graph, $h=\exp\int_l A$. Then, in order to find effective corrections to Einstein equations, one has to approximate the gravitational field strength (the curvature) F in terms of the holonomies around closed loops. The polymeric-parameter is precisely what controls this approximation. When $\delta\to 0$, the approximate field strength reduces to the classical one $F_\delta(h)\to F(A)$.

As it was shown in [44], it turns out that this parameter does not play a role in the singularity resolution of the effective solution, and the physical observable are not affected by it. Moreover, the $\delta\to 0$ limit of the solution is a single-horizon regular black hole, while for $\delta\neq 0$, it has two. For these reasons and following our previous analysis on mass-inflation, it seems tempting to study this $\delta=0$ solution, which was called "semi-polymeric".

1. Action for semi-polymeric black hole

Let's apply our previous reconstruction technique to the Modesto semi-polymeric black hole given by :

$$ds^2=-\frac{\rho^3\left(\rho-2m\right)}{\mathfrak{a}^2+\rho^4}dt^2+\frac{\mathfrak{a}^2+\rho^4}{\rho^3\left(\rho-2m\right)}d\rho^2+\frac{\mathfrak{a}^2+\rho^4}{\rho^2}d\Omega^2 \tag{256}$$

where $\mathfrak{a}$ was considered to be the minimal area in LQG. In our case, it will just be a free coupling constant in an effective-like action.

In order to apply the previous reconstruction technique, one needs to put the metric in Schwarzschild gauge via the following change of coordinates :

$$r^2 = \frac{\mathfrak{a}^2 + \rho^4}{\rho^2} \tag{257}$$

which has two distinct branches

$$\rho = \pm \frac{1}{\sqrt{2}} \sqrt{r^2 - \sqrt{r^4 - 4\mathfrak{a}^2}} \;\; , \quad \rho = \pm \frac{1}{\sqrt{2}} \sqrt{r^2 + \sqrt{r^4 - 4\mathfrak{a}^2}} \tag{258}$$

The first branch corresponds to the values $\rho^2 \leq \mathfrak{a}$ of the original coordinate while the second one to $\mathfrak{a} \leq \rho^2$. In both cases, $2\,\mathfrak{a} \leq r^2$. Keeping track of these ranges, we get the following two metrics for respectively the first and second branches :

$$\begin{aligned} ds^2_{(\rho^2 \leq \mathfrak{a})} &= -\frac{1}{2}\left(1 - \frac{\sqrt{X} + 2\sqrt{2}\,m\sqrt{r^2 - \sqrt{X}}}{r^2}\right) dt^2 + \frac{\sqrt{2}r^4\sqrt{r^2 - \sqrt{X}}}{X\left(-4m + \sqrt{2}\sqrt{r^2 - \sqrt{X}}\right)} dr^2 + r^2 d\Omega^2 \\ ds^2_{(\mathfrak{a} \leq \rho^2)} &= -\frac{1}{2}\left(1 + \frac{\sqrt{X} - 2\sqrt{2}\,m\sqrt{r^2 + \sqrt{X}}}{r^2}\right) dt^2 + \frac{\sqrt{2}r^4\sqrt{r^2 + \sqrt{X}}}{X\left(-4m + \sqrt{2}\sqrt{r^2 + \sqrt{X}}\right)} dr^2 + r^2 d\Omega^2 \end{aligned} \tag{259}$$

where $X := r^4 - 4\mathfrak{a}^2$. Note that we have chosen $\rho > 0$ here because it is the location of the Schwarzschild black hole, but $\rho < 0$ is included and only flip the sign in front of the mass. Given the previous reconstruction technique, we will get different Lagrangians for the different branches. Therefore, it will not be possible to cover all the semi-polymeric metric (256) in this way. The portion of interest to us is the second one, because it has the classical Schwarzschild asymptotic behaviour either when $r \to \infty$ or $\mathfrak{a} \to 0$. On the contrary, the semi-classical limit ($\mathfrak{a} \to 0$) of the first branch gives $ds^2 = r^2 d\Omega^2$. Interestingly, this kind of degenerate metric are actually solutions of Einstein Equations in the first order formalism, when torsion is present, see [345]. It means that, in this formalism, the first branch could also be seen as an effective geometry with a well-defined semi-classical limit.

In the gauge (241), it gives :

$$\begin{aligned} b_{(\mathfrak{a} \leq \rho^2)} &= \frac{r}{\sqrt{2}} \sqrt{\frac{r^2 + \sqrt{X}}{X}} \\ b_{(\rho^2 \leq \mathfrak{a})} &= \frac{r}{\sqrt{2}} \sqrt{\frac{r^2 - \sqrt{X}}{X}} \end{aligned} \tag{260}$$

and

$$\begin{aligned} a_{(\mathfrak{a} \leq \rho^2)} &= \frac{X}{2\mathfrak{a}^2\, r^4} \left(2\mathfrak{a}^2 + \sqrt{2}m\left(\sqrt{X} - r^2\right)\sqrt{r^2 + \sqrt{X}}\right) \\ a_{(\rho^2 \leq \mathfrak{a})} &= \frac{X}{2\mathfrak{a}^2\, r^4} \left(2\mathfrak{a}^2 - \sqrt{2}m\left(\sqrt{X} + r^2\right)\sqrt{r^2 - \sqrt{X}}\right) \end{aligned} \tag{261}$$

which leads to the Lagrangians :

$$\begin{aligned} \mathcal{L}_{(\mathfrak{a} \leq \rho^2)} &= \frac{\mathcal{R}}{2} + \frac{R^{(2)}}{4}\sqrt{1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2} - \left(\frac{1 - 2\sqrt{1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2}}{2\,(1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2)}\right)\chi + \mathcal{X} \\ \mathcal{L}_{(\rho^2 \leq \mathfrak{a})} &= -\frac{\mathcal{R}}{2} + \frac{R^{(2)}}{4}\sqrt{1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2} - \left(\frac{1 + 2\sqrt{1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2}}{2\,(1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2)}\right)\chi + \mathcal{X} \end{aligned} \tag{262}$$

where $\mathcal{X} := f_3\mathcal{Y} + (f_3 + r f_3')\,\chi$ is just a boundary term that can be chosen so that the variational principle is well-defined. It is clear that only one of the branches can be obtained from a given Lagrangian, so that in order to find the complete Semi-Polymeric metric (256), one would need to consider a more general types of actions than (239) for which the field equations would admit the two solutions (259) that could then be glued together. However, as we will see later, the fact that only one region of the Modesto metric can be reached imposes to use coordinates system in which the $\rho \geq \mathfrak{a}$ region is covered twice in order to have a complete and singularity-free spacetime, what end up describing a black hole whose interior evolves into a white hole, like in [175]. This will be made clearer later.

Now let's focus on the Einstein branch. Recall first that in a four dimensional spherically symmetric spacetime, the Ricci scalar decomposes as $R = R^{(2)} + 2\mathcal{R} - 4\mathcal{Y} - 2\chi$. Then, using the fact that $\mathcal{X}$ does not contribute to the field equations, we can expand the functions

$$\begin{aligned}\sqrt{1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2} &= \sum_{i=0}^{\infty}(-4)^i\mathfrak{a}^{2i}\binom{\frac{1}{2}}{i}\mathcal{R}^{2i}\\ \frac{1 + 2\sqrt{1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2}}{2\,(1 - 4\,\mathfrak{a}^2\,\mathcal{R}^2)} &= \sum_{i=0}^{\infty}4^i\mathfrak{a}^{2i}\left(-1 + \frac{2\Gamma\left(i + \frac{1}{2}\right)}{i!\sqrt{\pi}}\right)\mathcal{R}^{2i}\end{aligned} \tag{263}$$

and use the zeroth order terms to introduce the Ricci scalar as :

$$\mathcal{L}_{SP} := \frac{R}{4} + \frac{R^{(2)}}{4}\sum_{i=1}^{\infty}(-4)^i\mathfrak{a}^{2i}\binom{\frac{1}{2}}{i}\mathcal{R}^{2i} + \frac{\chi}{2}\sum_{i=1}^{\infty}4^i\mathfrak{a}^{2i}\left(-1 + \frac{2\Gamma\left(i + \frac{1}{2}\right)}{i!\sqrt{\pi}}\right)\mathcal{R}^{2i} + \mathcal{X} \tag{264}$$

We can observe that contrary to the models (141) which removed the singularity of Schwarzschild spacetime from perturbative corrections in the coupling constant l, here the corrections to the Einstein-Hilbert action are non-perturbative in $\mathfrak{a}$. This will also be the case for the D'Ambrosio-Rovelli [175] and the Visser-Hochberg black bounce metric [227] that we will reconstruct in the other sections of this Chapter. As these three theories describes non-singular black hole with one horizon (in the $r > 0$ region), contrary to (141) which lead to (A)dS-core regular black holes, it might be tempting to conjecture that these firsts can be found only from non-perturbative corrections in the parameter l (or $\mathfrak{a} =: l^2$ here).

Finally, using the correspondence Eq(98, 100, 101) between 2D Horndeski and NPG models, we can write an action involving only curvature invariants whose spherically symmetric sector is given by Eq(264), keeping in mind the degeneracy that we talked about in the same section of the same Chapter.

2. Verification & horizon structure

We want to check that indeed, this action admits the metric $ds^2_{(\mathfrak{a}\leq\rho^2)}$ (259) as a solution. In order to do so, consider the following gauge :

$$ds^2 = -a(x)b(x)^2dt^2 + \frac{dx^2}{a(x)} + r(x)^2d\Omega^2 \tag{265}$$

Then we can develop the Lagrangian to find :

$$\begin{aligned}\sqrt{-g}\mathcal{L}_{SP} =& N\left(r'(r^4 - 4\mathfrak{a}^2)\,(2r\,a\,b' + b(r\,a' + 2a\,r')) + bX\left(-4\mathfrak{a}^2 + r^4\left(1 - a\,r'^2\right)\right)\right)\\ &- \left(r\left(abr' + \frac{r}{4}X\left(ba' + 2ab'\right)\right)\right)'\end{aligned} \tag{266}$$

where $N = \frac{r^2}{2(r^4 - 4\mathfrak{a}^2)^{\frac{3}{2}}}$ and $X = \sqrt{1 - \frac{4\mathfrak{a}^2}{r^4}}$. Among the three functions a, b, r, only two are required in static spherical symmetry. Therefore one of these can be reabsorbed by a gauge transformation. This provides a quite simple way to find the solution of this system, which is to derive $\frac{\delta\mathcal{I}_{\mathcal{SP}}}{\delta a}\Big|_{b=1}$ and $\frac{\delta\mathcal{I}_{\mathcal{SP}}}{\delta b}\Big|_{b=1}$. It can be checked that $\frac{\delta\mathcal{I}_{\mathcal{SP}}}{\delta r}\Big|_{b=1}$ is redundant. It gives :

$$\begin{aligned}\mathcal{E}_a := \frac{\delta\mathcal{I}_{\mathcal{SP}}}{\delta a}\bigg|_{b=1} &= N\left(\left(4\mathfrak{a}^2 + (1 - X)\,r^4\right)r'^2 - r\left(r^4 - 4\mathfrak{a}^2\right)r''\right) = 0\\ \mathcal{E}_b := \frac{\delta\mathcal{I}_{\mathcal{SP}}}{\delta b}\bigg|_{b=1} &= -NX\left(4\,a(4\mathfrak{a}^2 - r^4)\mathcal{E}_a + \left(4\mathfrak{a}^2 + r^4Xr'\left(ra' + 2ar'\right) - r^4\left(1 + ar'^2\right)\right)\right) = 0\end{aligned} \tag{267}$$

We see that in order to have real field equations, we need to have $r(x)^2 \geq 2\mathfrak{a}$, therefore we can set $r(x) = \frac{\sqrt{\mathfrak{a}^2+S(x)^4}}{S(x)}$, where $-\infty < S(x) < \infty$, so that the first equation becomes

$$\begin{aligned} \mathcal{E}_a &= -\frac{\mathfrak{a}^2 + S^4}{2S^3} S'' = 0 \quad ; \quad \text{if } S^2 \geq \mathfrak{a} \\ \mathcal{E}_a &= \frac{\mathfrak{a}^2 + S^4}{2S^4} (-2S'^2 + SS'') = 0 \quad ; \quad \text{if } S^2 \leq \mathfrak{a} \end{aligned} \tag{268}$$

The solutions are respectively $S(x) = \alpha_1 + \alpha_2 r$ for $S^2 \geq \mathfrak{a}$ and $S(x) = \frac{1}{\alpha_3 r + \alpha_4}$ for $S^2 \leq \mathfrak{a}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are integration constants. Setting $\alpha_3 = \alpha_2/\mathfrak{a}$ and $\alpha_4 = \alpha_1/\mathfrak{a}$, both solutions lead to

$$r(x) = \frac{\sqrt{a^2 + (\alpha_1 + \alpha_2 x)^4}}{\alpha_1 + \alpha_2 x} \tag{269}$$

Moreover, the condition $S^2 \leq \mathfrak{a}$ of the second solution becomes the same as the condition for the first one. Therefore, one of the solution is redundant. From the expression of $r(x)$ and the fact that $(\alpha_1 + \alpha_2 x)^2 \geq a$, the second equation of motion becomes :

$$1 + \alpha_1^2 \left(-1 + \frac{3\mathfrak{a}}{(\alpha_1 + \alpha_2 x)^4} a \right) - \frac{\alpha_2 \left(\mathfrak{a}^2 + (\alpha_1 + \alpha_2 x)^4 \right) a'}{(\alpha_1 + \alpha_2 x)^3} = 0 \tag{270}$$

whose solution is given by

$$a(x) = \frac{(\alpha_1 + \alpha_2 x)^3 (x + \alpha_2 \beta)}{\alpha_2 \left(\mathfrak{a}^2 + (\alpha_1 + \alpha_2 x)^4 \right)} \tag{271}$$

where β is an integration constant. Now, in order to obtain the Modesto metric, we change coordinate as follows :

$$\rho = \alpha_1 + \alpha_2 x \ , \quad \text{and} \qquad t = \alpha_2 T \tag{272}$$

Then renaming $T \to t$ and setting $\beta = \frac{\alpha_1 - 2m}{\alpha_1^2}$ lead to the semi-polymeric metric (256) for $\rho^2 \geq \mathfrak{a}$ due to the condition $S(x)^2 = (\alpha_1 + \alpha_2 x)^2 \geq \mathfrak{a}$. This shows that the integration constants were only gauge parameters that we could have fixed directly to $\alpha_1 = 0$ and $\alpha_2 = 1$. From now on, when we will solve $\delta I/\delta a$ with $b(x)$ fixed, we will directly fixed these parameters without showing how to reabsorbed these by gauge transformations.

It is worth noting that writing this metric in the gauge (256) is not very satisfactory in our case because only the region $\mathfrak{a} \leq \rho^2$ is a solution of the field equation. However, if we do not consider the region $\rho \leq -\sqrt{\mathfrak{a}}$ because it does not correspond to the Schwarzschild black hole[14], it is possible to change coordinate in order to cover twice the $\rho \geq \sqrt{\mathfrak{a}}$ region, so that a geodesic ending at $\rho = \sqrt{\mathfrak{a}}$ is actually continued to a white hole and another asymptotically flat region. This procedure is very similar to the "doubling" of the $r > 0$ region of Schwarzschild spacetime, used in [175] in order to see that the geodesics of the Schwarzschild black hole can be continued beyond the singularity when another (time-reverse) identical spacetime is attached at the singularity. Interestingly, we see in this paper that this procedure simply amounts to perform the coordinate change $r \to x^2$, where r is the radial coordinate of Schwarzschild spacetime. In our case, from the metric (259) in Schwarzschild coordinates, one can perform the following coordinate transformation :

$$r = \pm \left(4\mathfrak{a}^2 + R^4 \right)^{\frac{1}{4}} \tag{273}$$

where the range of the new coordinate is $-\infty < R < \infty$, what corresponds to the Modesto metric

[14]Indeed, when $\mathfrak{a} \to 0$, the region $\rho \leq -\sqrt{\mathfrak{a}}$ corresponds to the $r < 0$ region of the Schwarzschild solution, which do not contain an horizon for positive mass.

for all $\sqrt{\mathfrak{a}} \leq \rho$. Renaming $R \to r$, we obtain the following metric :

$$\begin{aligned} ds^2 = &-\frac{1}{2}\left(1+\frac{r^2}{\sqrt{4\mathfrak{a}^2+r^4}}\right)\left(1-\frac{2\sqrt{2}m}{\sqrt{r^2+\sqrt{4\mathfrak{a}^2+r^4}}}\right)dt^2 \\ &+\left(\frac{r^2}{\sqrt{4\mathfrak{a}^2+r^4}\left(1-\frac{2\sqrt{2}m}{\sqrt{r^2+\sqrt{4\mathfrak{a}^2+r^4}}}\right)}\right)dr^2+\sqrt{4\mathfrak{a}^2+r^4}\,d\Omega^2 \end{aligned} \tag{274}$$

Although at $r = 0$ the metric is degenerate, the curvature invariants are all regular everywhere. For example, the Ricci and Kretschmann scalars are given by :

$$\begin{aligned} R &= -\frac{8a^2\left(8a^2+3r^2\left(r^2+X\right)-2\sqrt{2}m\sqrt{r^2+X}\left(r^2+2X\right)\right)}{X^3\left(r^2+X\right)^2} \\ R^{\alpha\beta}_{\mu\nu}R^{\mu\nu}_{\alpha\beta} &= 64\Bigg(\frac{4a^2\left(10a^4+48a^2m^2r^2-9m^2r^6\right)}{X^5\left(r^2+X\right)^3}+\frac{6a^2\left(64a^4m^2+24a^4r^2+24a^2m^2r^4-5m^2r^8\right)}{X^6\left(r^2+X\right)^3} \\ &\quad+\frac{3r^4\left(21a^4+m^2r^6\right)}{X^6\left(r^2+X\right)^2}-\frac{4\sqrt{2}a^2m\left(40a^4+r^2\left(25a^2-3r^4\right)\left(r^2+X\right)\right)}{X^6\left(r^2+X\right)^{5/2}}\Bigg) \end{aligned}$$

where $X = \sqrt{r^4 + 4\mathfrak{a}^2}$, so that at $r = 0$ we have $R = -\frac{2}{a} + \frac{4m}{a^{3/2}}$ and $R^{\alpha\beta}_{\mu\nu}R^{\mu\nu}_{\alpha\beta} = \frac{10}{a^2} - \frac{40m}{a^{5/2}} + \frac{48m^2}{a^3}$.

This metric is quite similar to both the D'Ambrosio-Rovelli [175] and the Visser-Hochberg black bounce metric found in [227] and studied recently in [228] as it describes a bouncing non-singular black hole, where after crossing the horizon of the black hole, an observer end up traversing a white hole horizon toward another asymptotically flat region. We can look for the radial null curves to locate the horizons :

$$\frac{dr}{dt} = \pm\frac{1}{\sqrt{2}\,r}\left(-2\sqrt{2}m+\sqrt{r^2+\sqrt{4\mathfrak{a}^2+r^4}}\right) \tag{275}$$

so that if $\mathfrak{a} < 4m^2$, there are two horizons located at $r = \pm\frac{\sqrt{16m^4-\mathfrak{a}^2}}{2m}$, if $\mathfrak{a} = 4m^2$, there is only one extremal horizon located at $r = 0$, and finally, if $\mathfrak{a} > 4m^2$ the spacetime is horizonless.

3. Charged black hole & regularization of Coulomb singularity

Now we want to couple the theory (264) with an electromagnetic field by considering the following theory :

$$\begin{aligned} I = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\Bigg(&\frac{R}{4}+\sum_{i=1}^{\infty}\left(\frac{R_{(\Sigma)}}{4}(-4)^i\mathfrak{a}^{2i}\binom{\frac{1}{2}}{i}\mathcal{R}^{2i}+\frac{\chi}{2}4^i\mathfrak{a}^{2i}\left(-1+\frac{2\Gamma\left(i+\frac{1}{2}\right)}{i!\sqrt{\pi}}\right)\mathcal{R}^{2i}\right) \\ &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Bigg) \end{aligned} \tag{276}$$

where we have set $c = 1$, $16\pi G = 4$ and $\mu_0 = 1$. Considering an electrostatic field, given by $A = A_0(r)dt$, gives $F^{\mu\nu}F_{\mu\nu} = -2\frac{A_0'^2}{b^2}$. Therefore, in the gauge (265), the field equation $\frac{\delta I}{a}\big|_{b=1}$ will remain unchanged with respect to the uncharged case given by Eq(267). We can therefore derive the field equations with respect to b and A_0 and then impose that $r(x)$ is given by Eq(269)[15]. Moreover, as we saw in the previous section, the integration constant α_1 and α_2 are gauge parameters, so that we can directly consider :

$$r(x) = \pm\frac{\sqrt{\mathfrak{a}^2+x^4}}{x} =: r_0(x) \tag{277}$$

Finally, we also consider the more general case where instead of having a spherical horizon topology $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, we take the angular part of the metric to be $d\Omega_k^2 := \frac{d\theta^2}{1-k\theta^2} + \theta^2 d\phi^2$, where $k = 1, -1, 0$ correspond respectively to spherical, hyperbolic and planar horizon topologies.

[15] Recall that the field equation associated with the scalar field r is redundant.

However, note that the topological parameter k comes in the action from the scalar $\mathcal{R}$ whose spherically symmetric decomposition is $\mathcal{R} = k/r^2$. Therefore, the planar topology $k = 0$ actually cancel the higher order terms in the action, and do not give any new solutions.

For $k = 1$ and $k = -1$, the minisuperspace field equations of this system are given by :

$$\begin{aligned} \left.\frac{\delta I}{\delta b}\right|_{b=1}^{r=r_0} &= \frac{kx^4 + \left(3\mathfrak{a}^2 - x^4\right) a - x\left(\mathfrak{a}^2 + x^4\right)\left(a' + xA_0'^2\right)}{2x^4} = 0 \\ \left.\frac{\delta I}{\delta A_0}\right|_{b=1}^{r=r_0} &= \frac{2\left(\mathfrak{a}^2 - x^4\right)A_0' - x\left(\mathfrak{a}^2 + x^4\right)A_0''}{x^3} = 0 \end{aligned} \tag{278}$$

and the associated solutions are :

$$\begin{aligned} a &= \frac{x^3\left(k\,x - 2m - QA_0\right)}{\mathfrak{a}^2 + x^4} \\ A_0 &= -\frac{Q}{2\sqrt{2\mathfrak{a}}}\left(\arctan\left(\frac{x\sqrt{2\mathfrak{a}}}{x^2 - \mathfrak{a}}\right) + \operatorname{arctanh}\left(\frac{x\sqrt{2\mathfrak{a}}}{x^2 + \mathfrak{a}}\right)\right) \end{aligned} \tag{279}$$

As before, the range of the coordinate x is $x^2 \geq \mathfrak{a}$. Therefore, we see that, contrary to the substitution $2mr \to 2mr - Q^2$ that was used in Eq(66) of [346], where rotating and charged generalizations of the polymeric black holes where studied, in our case the specific gravitational corrections that we consider do change the electric field and the metric in a more drastic way.

Indeed, the effect of these geometrical high energy corrections is to smooth out the Coulomb singularity, as it can be seen from the fact that :

$$F^{\mu\nu}F_{\mu\nu} = -\frac{2Q^2x^4}{\left(\mathfrak{a}^2 + x^4\right)^2} \tag{280}$$

At $x = \sqrt{\mathfrak{a}}$, the electric potential is constant (or vanishing if we apply a gauge transformation to it), $A_0(x \to \sqrt{\mathfrak{a}}) = -Q\left(\frac{\pi + 2\operatorname{arctanh}(1/\sqrt{2})}{4\sqrt{2\mathfrak{a}}}\right)$, while at infinity, where x becomes the radial coordinate of Schwarzschild geometry, it gives the usual Coulomb potential with some effective corrections :

$$A_0(x \to \infty) = -\frac{Q}{x} + \frac{\mathfrak{a}^2 Q}{5x^5} + o\left(1/x^5\right) \tag{281}$$

Usually, charged metric fields possess an inner and an outer horizon, what is the cause of the mass inflation instability as we saw previously. However, similarly to Eq(209), the previous solution possesses only one horizon (for $x \geq \sqrt{\mathfrak{a}}$) as long as the following bound is satisfied :

$$m \geq \frac{8\mathfrak{a} + Q^2\sqrt{2}\left(\pi + 2\operatorname{arccoth}\sqrt{2}\right)}{16\sqrt{\mathfrak{a}}} \tag{282}$$

with $\operatorname{arccoth}\sqrt{2} \approx 0.88.....$ Thus, under this condition, this charged spacetime does not suffer from the mass-inflation instability. When the bound is saturated, there is one outer horizon and one extremal horizon at $x = \sqrt{\mathfrak{a}}$ (what corresponds to $r = 0$ in the same coordinate system as Eq(274)). As extremal horizons have vanishing surface gravity, mass-inflation is not present in this case neither, because as seen in Eq(200), it is controlled by the surface gravity of the inner horizon.

Like for the neutral case, we can change the gauge in order to have a complete metric by gluing two $x \geq \sqrt{\mathfrak{a}}$ regions together, what can be done by first putting the metric to Schwarzschild gauge (choosing the correct branch as we saw in the reconstruction) and then to perform the additional transformation (273), what amounts to :

$$x = \frac{1}{\sqrt{2}}\sqrt{r^2 + \sqrt{r^4 + 4\mathfrak{a}^2}} \tag{283}$$

where $-\infty < r < \infty$, giving

$$\begin{aligned} ds^2 = &-\frac{1}{2}\left(1 + \frac{r^2}{\sqrt{4\mathfrak{a}^2 + r^4}}\right)\left(k - \sqrt{2}\frac{2m + QA_0}{\sqrt{r^2 + \sqrt{4\mathfrak{a}^2 + r^4}}}\right)dt^2 \\ &+ \left(\frac{r^2}{\sqrt{4\mathfrak{a}^2 + r^4}\left(k - \sqrt{2}\frac{2m + QA_0}{\sqrt{r^2 + \sqrt{4\mathfrak{a}^2 + r^4}}}\right)}\right)dr^2 + \sqrt{4\mathfrak{a}^2 + r^4}\,d\Omega^2 \end{aligned} \tag{284}$$

C. D'Ambrosio-Rovelli metric for Black to White hole transition

Following the same procedure as we did for the Modesto metric, we can reconstruct an action that gives the D'Ambrosio-Rovelli metric [175, 226] describing the continuous transition between a black hole and a white hole as its unique (because Birkhoff theorem holds) spherically symmetric solution. The metric is :

$$ds^2 = -\frac{x^2-2m}{x^2+l}dt^2 + \frac{4\left(x^2+l\right)^2}{x^2-2m}dx^2 + \left(x^2+l\right)^2 d\Omega^2 \tag{285}$$

As we will see, it can be derived from the following action :

$$\begin{aligned} I = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\Bigg(& \frac{R_{(\Sigma)}}{8}\left(\sqrt{1-l\sqrt{\mathcal{R}}}\left(2+3l\sqrt{\mathcal{R}}\right)+3l^2\mathcal{R}\log\left(\frac{1+\sqrt{1-l\sqrt{\mathcal{R}}}}{l\mathcal{R}^{1/4}}\right)\right) \\ & + \frac{\mathcal{R}}{2}\sqrt{1-l\sqrt{\mathcal{R}}} + \frac{\chi}{2\sqrt{1-l\sqrt{\mathcal{R}}}}\Bigg) \end{aligned} \tag{286}$$

Alternatively, introducing the Ricci scalar by expanding the functions of $\mathcal{R}$ at zeroth order in l and by performing an integration by part, the Lagrangian can also be written as :

$$\begin{aligned} \mathcal{L} =& \frac{1}{8}\sum_{n=1}^{\infty}(-1)^n\, l^n\, \mathcal{R}^{\frac{n}{2}}\left(4\binom{-1/2}{n}\chi - 3\binom{\frac{1}{2}}{n-1}R_{(\Sigma)} + 2\binom{\frac{1}{2}}{n}\left(2\mathcal{R}+R_{(\Sigma)}\right)\right) \\ & + \frac{R}{4} + \frac{R_{(\Sigma)}}{8}\left(3l^2\mathcal{R}\log\left(\frac{1+\sqrt{1-l\sqrt{\mathcal{R}}}}{l\mathcal{R}^{1/4}}\right)\right) \end{aligned} \tag{287}$$

because the term $l^2\mathcal{R}\log\left(\frac{1+\sqrt{1-l\sqrt{\mathcal{R}}}}{l\mathcal{R}^{1/4}}\right)$ vanishes when $l=0$. As we see, the action consists of a powerseries of half-integer powers of the curvature invariant $\mathcal{R}$, which in spherical symmetry is given by $\mathcal{R}=k/r^2$, where k is the topological parameter and r is a scalar field on the manifold Σ. Therefore, there will be two branches depending on $r>0$ or $r<0$, and the hyperbolic topology ($k=-1$) is not considered. Here we consider $r>0$ because it is the case for the D'Ambrosio-Rovelli metric and is the region where the Schwarzschild black hole lies. Under this condition, the minisuperspace Lagrangian of (286) in gauge (265) is given by :

$$\begin{aligned} \mathcal{L} \equiv \Bigg(& -\frac{\sqrt{r}}{b}\left(2r\sqrt{r(r-l)}+3l\left(l\log\left(\frac{\sqrt{r}+\sqrt{r-l}}{l}\right)+\sqrt{r(r-l)}\right)\right)\left(3a'b'+ba''+2ab''\right) \\ & + 4\sqrt{r-l} + \frac{4arr'^2}{\sqrt{r-l}}\Bigg)\frac{1}{8r^{5/2}} \end{aligned}$$

which gives the following minisuperspace field equations :

$$\begin{aligned} \frac{\delta I}{\delta a} =& \frac{\sqrt{r}\left(blr'^2+2r(r-l)\left(b'r'-br''\right)\right)}{4(r-l)^{3/2}} = 0 \\ \frac{\delta I}{\delta b} =& \frac{(r-l)^2 - a(r-2l)rr'^2 - (r-l)r^2\left(a'r'+2ar''\right)}{2\sqrt{r}(r-l)^{3/2}} = 0 \end{aligned} \tag{288}$$

Once again, $\frac{\delta I}{\delta r}$ is redundant. Choosing a gauge where $b(x)=2\sqrt{r(x)}$ in order to put the metric (265) in the same form as (285), we obtain :

$$\left.\frac{\delta I}{\delta a}\right|_{b=2\sqrt{r}} = \frac{r^2}{2\left(r-l\right)^{\frac{3}{2}}}\left(r'^2+2(l-r)r''\right) = 0 \tag{289}$$

The solution is

$$r(x) = \left(1+\frac{\gamma^2}{4\alpha}\right)l + \alpha x^2 + \gamma x\sqrt{l} \tag{290}$$

where α and γ are dimensionless integration constants. As we saw before for the semi-polymeric metric, there remains some gauge freedom in the solution for r, so that we can directly choose $\gamma = 0$ and $\alpha = 1$, which gives $r(x) = x^2 + l =: r_0(x)$. Then the second equation reduces to :

$$\left.\frac{\delta I}{\delta b}\right|_{b=2\sqrt{r}}^{r=r_0} = \frac{x}{2|x|\sqrt{x^2+l}} \left(x - 8x\left(x^2+l\right)a - 2\left(x^2+l\right)^2 a' \right) = 0 \tag{291}$$

and its solution is

$$a(x) = \frac{x^2 - 2m}{4\left(x^2+l\right)^2} \tag{292}$$

where m is an integration constant related to the mass of the solution. Therefore, we see that the action (286) indeed has the metric (285) as its spherically symmetric solution.

Interestingly, when the system (288) is solved in Schwarzschild gauge ($r(x) = x$), one can uncover other regions of the spacetime solution which have different signatures. Indeed in this case the field equations becomes :

$$\begin{aligned}
\left.\frac{\delta I}{\delta a}\right|_{r(x)=x} &= \frac{\sqrt{x\left(x-l\right)}}{4\left(x-l\right)^2}\left(lb + 2x\left(x-l\right)b'\right) = 0 \\
\left.\frac{\delta I}{\delta b}\right|_{r(x)=x} &= \frac{1}{2\sqrt{x}\left(x-l\right)^{\frac{3}{2}}}\left((2l-x)xa + (l-x)\left(l - x + x^2 a'\right)\right) = 0
\end{aligned} \tag{293}$$

we rename $x \to r$ in order to have the usual notations of the Schwarzschild gauge, and it gives

$$\begin{aligned}
b(r) &= \frac{\sqrt{r}}{\sqrt{r-l}} \\
a(r) &= \frac{(r-l)(r-2M)}{r^2}
\end{aligned} \tag{294}$$

where M is the mass of the spacetime as we will show later. Therefore, $b(r)$ and the previous field equations are real only for $r > l$, but recall that our metric is given by (265), in which the square of b appears, so that even if the previous field equations becomes complex when $r < l$, the metric remains nonetheless real, as it is given by :

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{l}{r}\right)\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2 \tag{295}$$

The metric admits two horizons located at $r = 2M$ and $r = l$, when $l = 2M$ the horizon is extremal and located at $r = l$.

However, we see that if $2M > l$, then the regions $r > 2M$, $l < r < 2M$ and $r < l$ have respectively the signatures $(-+++)$, $(+-++)$ and $(++++)$, meaning that the last region is Euclidean. For $l > 2M$, we have the signatures $(-+++)$, $(--++)$ and $(++++)$ for the regions $r > l$, $2M < r < l$ and $r < 2M$. Therefore, we see in this gauge that it is possible to glue smoothly at $r = l$ the Lorentzian region with the Euclidean one, without the appearance of a curvature singularity at $r = l$, as it can be seen from the curvature invariants :

$$R = \frac{3lM}{r^4}\;, \quad R_{\mu\nu}R^{\mu\nu} = \frac{l^2\left(9M^2 - 2Mr + 3r^2\right)}{2r^8} \tag{296}$$

This is very peculiar, but also interesting given that the metric (285) was proposed in order to model with a metric field some (loop) quantum gravity results suggesting that a black hole state can tunnel to a white hole one when quantum gravity is taken into account [48–52]. Moreover, other works have suggested that signature changes might occur in Planckian regime, see for example [69, 347–354], where in particular, [348] showed that some class of effective 2D Einstein-Dilaton gravity can produce signature change when holonomy corrections are considered.

It would therefore be interesting to see if other Lorentzian metric like (285), describing the continuous transition between a black hole and a white hole would also possess some signature changes in other regions of the solution. As we will see in the next section, this is the case of the

Visser-Hochberg black bounce [227], which is surprisingly very similar to the D'Ambrosio-Rovelli metric in Schwarzschild gauge, as the only difference between the two is that the factor $r/(r-l)$ in g_{rr} is replaced by $r^2/(r^2-l^2)$, thus implying the same signature changes.

It can be seen from the previous curvature invariants that this metric has a singularity at $r=0$. As in both cases ($l>2M$ and $2M>l$) the most inner region is Euclidean, this singularity is space-like.

Finally, note that from this metric, one can recover the D'Ambrosio-Rovelli metric (285) from the coordinate transformation $r=l+x^2$ with the additional substitution $M=\frac{1}{2}(2m+l)$. This enables to cover only the Lorentzian part of the solution, which is singularity-free, by "doubling" the $r\geq l$ Lorentzian region of the full solution, what produces a geometry describing a black hole evolving into a white hole. As it was noticed in [226], this metric cannot by itself describes a so-called "Planck star" (see [211–214]), what would be a black hole whose end-state is to bounce into a white hole, releasing the information that was trapped inside. Here however, the asymptotic region external to the black hole, and the one external to the white hole, are disconnected, and the black-white holes are eternal. So that the inner bouncing region only describes the region of a Planck star that replaces the Schwarzschild singularity. In order to describe a whole Planck star, one needs in addition to have a causal connection between the two asymptotically flat regions that does not pass through the black-white holes (so that the collapse and bounce happen in the "same" universe), together with a model of formation of a black hole and disappearance of the resulting white hole, see the Fig(5) of [226]. It could therefore be interesting, now that we have 2D covariant field equations associated with this static solution, to model the formation and evaporation of this solution (and similar metrics, as the "doubled" semi-polymeric metric (274) and Visser-Hochberg one, that we will see in the next section) by studying the collapse and evaporation in this theory.

Concerning the different choices of integration constant m and M, although the D'Ambrosio-Rovelli metric was written using m, it turns out that the physical mass of the solution is given by M, as we will see briefly. Consider the Komar mass of a static spacetime (see for example [355] Eq(11.2.9)),

$$M_K=-\frac{1}{8\pi}\int_\Omega *d\xi \tag{297}$$

where ξ is the Killing one-form. Then, following [356], we consider a metric (265) in Schwarzschild gauge (setting $r(x)=x$ and renaming $x\to r$), and introduce the orthonormal frame $\{\theta^\mu\}$,

$$\theta^0=a^{1/2}bdt\,,\quad \theta^1=a^{-1/2}dr\,,\quad \theta^2=rd\theta\,,\quad \theta^3=r\sin\theta d\phi \tag{298}$$

so that $dt\wedge dr=b^{-1}\theta^0\wedge\theta^1$ and in terms of which the Killing one form becomes $\xi=-ab^2dt=-a^{1/2}b\theta^0$. Applying an exterior derivative and taking the Hodge dual gives

$$*d\xi=*\left(-\left(ab^2\right)'dr\wedge dt\right)=*\left(b^{-1}\left(ab^2\right)'\theta^0\wedge\theta^1\right)=-r^2b^{-1}\left(ab^2\right)'d\Omega \tag{299}$$

because $*\left(\theta^0\wedge\theta^1\right)=-\theta^2\wedge\theta^3=-r^2d\Omega$. We can now integrate out the angular part, which gives for (294),

$$M_K(r)=\frac{1}{2}r^2b^{-1}\left(ab^2\right)'=M\sqrt{1-\frac{l}{r}}\underset{r\to\infty}{\longrightarrow}M. \tag{300}$$

As the Komar mass (297) is gauge invariant, it indicates that the correct mass of the spacetime (in particular its Lorentzian part given by (285)) should be M rather than m.

We will now consider a charged generalization of the previous solution, and see that once again the effect of the curvature corrections is to make the electrostatic field regular in the Lorentzian region.

Like in the case of the semi-polymeric black hole, the field equation $\delta I/\delta a$ is unchanged, so, as before, if we choose the gauge $b(x)=2\sqrt{r(x)}$, the solution is $r(x)=x^2+l$, while if we choose the Schwarzschild gauge $r(x)=x$ (and rename $x\to r$), the solution is $b(r)=\sqrt{r}/\sqrt{r-l}$. The other two independent field equations with respect to b and A_0 are modified w.r.t. the neutral case as :

$$\begin{aligned}\frac{\delta I}{\delta b}&=\frac{(r-l)^2-a(r-2l)rr'^2-(r-l)r^2\left(a'r'+2ar''\right)}{2\sqrt{r}(r-l)^{3/2}}-\frac{r^2A_0'^2}{2b^2}=0\\ \frac{\delta I}{\delta A_0}&=\frac{r}{b^2}\left(A_0'\left(rb'-2br'\right)-brA_0''\right)=0\end{aligned} \tag{301}$$

In Schwarzschild gauge ($b(x) = \sqrt{x}/\sqrt{x-l}$, $x \to r$), the solution for the electric potential is :

$$A_0(r) = \frac{2\alpha\sqrt{r-l}}{l\sqrt{r}} + \gamma \tag{302}$$

where α and γ are integration constants, the last one being a gauge parameter. In order to have the Coulomb potential $A_0 = -Q/r$ at $r \to \infty$, we need $\alpha = Q$ and $\gamma = -\frac{2Q}{l}$, where Q is the charge of the black hole, so that

$$A_0(r) = \frac{2Q}{l}\left(\frac{\sqrt{r-l}}{\sqrt{r}} - 1\right) \underset{r\to\infty}{\longrightarrow} -\frac{Q}{r} - \frac{Q\,l}{4r^2} + o\left(\frac{l}{r^2}\right) \tag{303}$$

Then the solution of the last equation is :

$$a(r) = \left(1 - \frac{l}{r}\right)\left(1 - \frac{2M - QA_0}{r}\right) \tag{304}$$

Therefore, changing gauge, $r = x^2 + l$ (and setting $M = m + l/2$ for simplicity, as in (285)), in order to select only the Lorentzian part of the solution as we did before gives :

$$\begin{aligned} ds^2 &= -\left(\frac{x^2 - 2m - QA_0}{x^2 + l}\right)dt^2 + \left(\frac{4\left(x^2+l\right)^2}{x^2 - 2m - QA_0}\right)dx^2 + \left(x^2+l\right)^2 d\Omega^2 \\ A_0(x) &= \frac{2Q}{l}\left(\frac{|x|}{\sqrt{x^2+l}} - 1\right) \underset{x\to\pm\infty}{\longrightarrow} -\frac{Q}{x^2} \end{aligned} \tag{305}$$

which are not smooth functions. This discontinuity might be viewed as a "wrong" choice of gauge for the electric field, and a wrong definition of the charge in the white hole region ($x < 0$). Indeed, we can consider instead $A_0(x) = \frac{2Q_{bh/wh}}{l}\left(\frac{x}{\sqrt{x^2+l}} \mp 1\right) \underset{x\to\pm\infty}{\longrightarrow} \mp\frac{Q_{bh/wh}}{x^2}$ where $Q_{bh} = -Q_{wh} \geq 0$, and Q_{bh}, Q_{wh} corresponding respectively to the charge as seen from $x \to \infty$ and $x \to -\infty$. Alternatively, the difference of sign of the charge can interpreted if we describe the geometry with two coordinate systems, (t, x) when $x > 0$ and $(-t, x)$ when $x < 0$ and considering $A = A_0(x)dT = \frac{2Q}{l}\left(\frac{x}{\sqrt{x^2+l}} \mp 1\right)dT \underset{x\to\pm\infty}{\longrightarrow} \mp\frac{Q}{x^2}dT$. This can be understood by the fact that the white hole region is the time reverse of the black hole one, so that, in this coordinate system where the metric does not have any pathology at $x = 0$, if we want to have a Coulomb potential at both $x \to \infty$ and $x \to -\infty$, we need this absolute value in the electric potential, or the alternative descriptions that we mentioned.

That being said, from the Schwarzschild gauge, one can also choose another coordinate transformation $r = l + x^4$, which provides smooth solutions but at the price of having a degenerate metric at $x = 0$, like in the case of the "doubled" semi-polymeric black hole Eq(274) :

$$\begin{aligned} ds^2 &= -\left(\frac{x^4 - 2m - QA_0}{x^4 + l}\right)dt^2 + \left(\frac{16x^2\left(x^4+l\right)^2}{x^4 - 2m - QA_0}\right)dx^2 + \left(x^4+l\right)^2 d\Omega^2 \\ A_0 &= \frac{2Q}{l}\left(\frac{x^2}{\sqrt{x^4+l}} - 1\right) \end{aligned} \tag{306}$$

Then the curvature and electric invariants can be shown to be smooth and regular everywhere :

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= -\frac{2Q^2}{\left(x^4+l\right)^4} \quad , \quad R = \frac{3l\left(l+2m\right)}{2\left(x^4+l\right)^4} + 3Q^2\left(\frac{x^2}{\left(x^4+l\right)^{9/2}} - \frac{1}{\left(x^4+l\right)^4}\right) \\ R_{\mu\nu}R^{\mu\nu} &= \frac{3Q^4\left(10x^4+3l\right)}{2\left(x^4+l\right)^9} + \frac{Q^2x^2\left(-22Q^2 + l\left(17l + 22m + 6x^4\right)\right)}{2\left(x^4+l\right)^{17/2}} \\ &\quad + l\frac{17l^3 + 8Q^2\left(x^4 - 9m\right) + 4l^2\left(5x^4 + 7m\right) + 4l\left(9m^2 - 7Q^2 + 3x^8\right)}{8\left(x^4+l\right)^8} \end{aligned} \tag{307}$$

In particular, they are regular at $x = 0$, which corresponds to the point $x = 0$ in the previous coordinate system, where the discontinuity in the derivatives of the electric potential and metric lied. Thus, from the point of view of curvature singularities, nothing problematic happens in the region $x \approx 0$. A careful study of the geodesics of this geometry remains important to be sure that it is geodesically complete, although it seems to be the case. Indeed, even if they usually arise in other formalisms of gravity where metric and connection are independent, it was shown in [78] that some singularities of geodesics can still be present, even if the curvature invariants are finite.

D. Visser-Hochberg black bounce

To conclude this section, we will just expose the same results for the Visser-Hochberg black bounce [227, 228] given by

$$ds^2 = -\left(1 - \frac{2M}{\sqrt{x^2+l^2}}\right) dt^2 + \frac{dx^2}{1-\frac{2M}{\sqrt{x^2+l^2}}} + \left(x^2+l^2\right) d\Omega^2 \tag{308}$$

As we will see, the action and solutions are very similar to previous case with the D'Ambrosio-Rovelli metric. First, the Lagrangian reads :

$$\begin{aligned}\mathcal{L} &= \frac{R}{4} + \frac{R_{(\Sigma)}}{4}\left(l^2\mathcal{R}\log\left(\frac{1+\sqrt{1-l^2\mathcal{R}}}{l\sqrt{\mathcal{R}}}\right)\right) \\ &+ \sum_{n=1}^{\infty}\frac{(-1)^n}{4} l^{2n}\,\mathcal{R}^n\left(2\binom{-1/2}{n}\chi + \binom{\frac{1}{2}}{n}\left(2\mathcal{R}+R_{(\Sigma)}\right)\right)\end{aligned} \tag{309}$$

It is surprisingly quite the same structure as (286), what might be a hint to understand the general structure of the 2D actions that admit such kind of solutions. In spherical symmetry, choosing the general gauge (265), the Lagrangian reduces to :

$$\mathcal{L} = \frac{\sqrt{r^2-l^2}}{2r^3} + \frac{ar'^2}{2r\sqrt{r^2-l^2}} - \frac{3a'b' + ba'' + 2ab''}{4br^2}\left(l^2\log\left(\frac{r+\sqrt{r^2-l^2}}{l}\right) + r\sqrt{r^2-l^2}\right) \tag{310}$$

Considering directly the charged case gives the following system of equations :

$$\begin{aligned}\frac{\delta I}{\delta a} &= \frac{r}{2\left(r^2-l^2\right)^{3/2}}\left(l^2br'^2 + r\left(r^2-l^2\right)\left(b'r'-br''\right)\right) = 0 \\ \frac{\delta I}{\delta A_0} &= \frac{r}{b^2}\left(A_0'\left(rb'-2br'\right) - brA_0''\right) = 0 \\ \frac{\delta I}{\delta b} &= \frac{1}{2}\left(-\frac{r^2A_0'^2}{b^2} + \frac{\sqrt{r^2-l^2}}{r} - \frac{ar\left(r^2-3l^2\right)r'^2}{\left(r^2-l^2\right)^{3/2}} - \frac{r^2\left(a'r'+2ar''\right)}{\sqrt{r^2-l^2}}\right) = 0\end{aligned} \tag{311}$$

In Schwarzschild gauge, the solutions are :

$$\begin{aligned}ds^2 &= -\left(1-\frac{2M+QA_0}{r}\right) dt^2 + \frac{dr^2}{\left(1-\frac{l^2}{r^2}\right)\left(1-\frac{2M+QA_0}{r}\right)} + r^2 d\Omega^2 \\ A &= -\frac{Q}{l}\text{arccot}\left(\frac{\sqrt{r^2-l^2}}{l}\right) dt\end{aligned} \tag{312}$$

In the neutral case, it has exactly the same structure as the D'Ambrosio-Rovelli metric in Schwarzschild gauge (295). In particular, they have the same signature changes for the same values of the mass, and the same range of the radial coordinate r. Furthermore, we see that the charged electric field is also regular.

Finally, note that just like for the semi-polymeric metric with Eq(282), there is a condition for both charged D'Ambrosio-Rovelli metric and Visser-Hochberg black bounce to have only one black hole horizon, what ensures that they do not suffer from the mass-inflation instability in this regime. From the change of coordinate $r = \sqrt{x^2+l^2}$, the interval and electric potential become :

$$\begin{aligned}ds^2 &= -\left(1-\frac{2M+QA_0}{\sqrt{x^2+l^2}}\right) dt^2 + \frac{dr^2}{\left(1-\frac{2M+QA_0}{\sqrt{x^2+l^2}}\right)} + \left(x^2+l^2\right) d\Omega^2 \\ A &= -\frac{Q}{l}\ \text{arccot}\frac{|x|}{l}\ dt\end{aligned} \tag{313}$$

Therefore, the radial null curves are given by :

$$\frac{dx}{dt} = \pm\left(1-\frac{2M+QA_0}{\sqrt{x^2+l^2}}\right) \tag{314}$$

Besides the black and white hole horizons that are also present in the neutral case, there is the possibility now to have an inner horizon at $x = 0$ for $M = \frac{l}{2} + \frac{\pi Q^2}{4l}$. Thus, when

$$M > \frac{l}{2} + \frac{Q^2}{l} \tag{315}$$

the spacetime still possesses no inner horizons, what makes it stable against mass inflation. However, when $M < \frac{l}{2} + \frac{Q^2}{l}$, the spacetime will have inner horizons, until the charge is large enough to render the metric horizonless.

E. Discussion

All the models studied here admit solutions representing one-horizon regular black holes, that can be extended to describe, in their interior dynamical region, the transition between a black hole and a white hole. This is very different from the usual maximal extension of, say, an (A)dS-core regular black hole, that possesses a static inner region, even if white holes are also present in these cases. However, their charged counterparts do possess an inner horizon in some ranges of the parameters (M, Q), and therefore are quite similar to (A)dS core black holes. As we also saw in the previous Chapter, Eq(209) where a similar object was derived, this property to have only one horizon arises usually when, once written in Schwarzschild gauge, the spacetime stops being Lorentzian for some values of the radial coordinate, becoming either complex for eq(209) and the semi-polymeric black hole eq(256), or Euclidean for eq(285) and eq(308). Note that the discontinuous (A)dS-core black hole eq(216) could be an example of another way to have single-horizon RBHs.

Despite their interests, the black holes studied in this section do not have the proper quantum correction to the Newton potential and do not satisfy the Limiting curvature conjecture, therefore it would be interesting to see if they can be generalized in order to admit both these properties.

Furthermore, we saw that the "doubled" semi-polymeric black hole, that was originally derived from the minisuperspace loop quantization of the spherically symmetric sector of Einstein gravity is actually describing a very similar object as the D'Ambrosio-Rovelli metric that was itself introduced to model with a metric field the effect of loop quantum gravity and spin-foams on the quantum region replacing the classical singularity. Thus, it seems that there might already be in the loop quantization of SS GR the way to describe semi-classically this quantum tunnelling region.

Possible next works concerning the Modesto black hole would be to find a 2D-Horndeski theory (and thus, from the correspondence provided by Eq(98, 100 , 101), a 4D Non-Polynomial gravity with this corresponding spherically symmetric sector) admitting the full polymeric solution, given by

$$ds^2 = -\frac{(r - r_-)(r - r_+)(r + r_x)^2}{\mathfrak{a}^2 + r^4} dt^2 + \frac{(\mathfrak{a}^2 + r^4)(r + r_x)^2}{r^4 (r - r_-)(r - r_+)} dr^2 + \left(\frac{r^4 + \mathfrak{a}^2}{r^2}\right) d\Omega^2 \tag{316}$$

where $r_- = 2M\frac{P^2}{(1+P)^2}$, $r_+ = 2M\frac{1}{(1+P)^2}$, $r_x = 2M\frac{P}{(1+P)^2}$, and P is the polymeric function which depends on the Immirzi (γ) and polymeric (δ) parameters. However, we see that this metric possesses 2 horizons for any non-vanishing values of the mass, and therefore, it is unstable against the mass-inflation instability. Moreover, it does not have quite the proper quantum correction to the Newton potential as predicted by [177–179]. On the other hand, it has the advantage that it satisfies the Limiting Curvature Conjecture as it can be checked easily :

$$R \underset{M\to\infty}{\longrightarrow} -\frac{4\left(3\mathfrak{a}^2 r^6 + r^{10}\right)}{\left(\mathfrak{a}^2 + r^4\right)^3} + O(1/M) \quad , \quad R_{\mu\nu}R^{\mu\nu} \underset{M\to\infty}{\longrightarrow} \frac{48\left(3\mathfrak{a}^4 r^{12} + r^{20}\right)}{\left(\mathfrak{a}^2 + r^4\right)^6} + O(1/M) \tag{317}$$

In the next section, we will present two NPG theories with a Lovelock-type spherically symmetric sector whose cosmological solution gives the LQC bounce, while their black hole solutions are regular and also satisfying the Limiting Curvature. One of these is an (A)dS-core RBH, while the second has one or two horizons depending on the value of the mass.

Note that another effective black hole interior was found in a recent paper [357] from the full Loop Quantum Gravity theory, contrary to the Modesto black hole which comes from a minisuperspace approach. Thus, it could be very interesting to see if the kind of 2D effective actions that we are studying here are able or not to admit such quantum gravity effective solution.

Other continuations would be to study these 2D theories in the context of the formation and evaporation of the horizons of the solutions, that is when collapsing matter is present and when the backreaction of the Hawking radiation is taken into account. This is possible because we saw that the reconstruction provides the full dynamical (and possibly topological) generalization of the "vacuum" (point-like) sector that admit the static solutions that we discussed. It is also possible to study the generalization of the previous charged or neutral solutions when a cosmological constant is present.

Moreover we saw that interesting common features of these "black-white hole" solutions and the associated 2D theories, are that some of these admit non-Lorentzian signature regions that can be join smoothly to the Lorentzian ones, which would deserve some interpretations. Moreover, the three high energy curvature corrections that we studied automatically cure the Coulomb singularity of the electric field, which is a very nice feature of these models that differs from the theory (141) studied in the previous Chapter. It would therefore be interesting to see in more details what class of 2D models that admit such kind of regular gravitational solutions implies this regularization of the static electric field. A first observation on this point is that if in Schwarzschild gauge, we have $b(r) = 1$, then it is not possible to do so as seen from Eq(301).

As a final remark, we emphasize that, despite the fact that they describe metric fields which resemble to what is expected from some semi-classical limit of some quantum gravity theory, these models are purely classical as they are, and in order to give them a quantum interpretation, one would need to understand how the DSS sector of GR, given by $R = R^{(2)} + 2(\mathcal{R} - \chi) - 4\mathcal{Y}$, could be quantized to find the effective DSS sector of the Lagrangians (264), (286) or (309).

V. Quantum gravity inspired solutions from Lovelock-Designer gravities

Similarly to the previous one, this chapter is devoted to the reconstruction of some effective quantum gravity regular black holes and cosmologies. Although we saw that the simplest spherically symmetric models, of the Einstein-Dilaton type, can already cover a part of the metrics found in the literature, it is nonetheless very insufficient regarding cosmological solutions (because the scalar $\mathcal{R}$ identically vanishes in FLRW), and also black holes with Limiting Curvature (because the metric must be non-perturbative in the mass to have this property).

We will therefore consider here a more advanced kind of effective DSS theory based on the DSS sector of Lovelock-Lanczos gravity. More precisely, we will investigate the theory given by eq(126), that was found to be the most general 2D Horndeski theory which preserves the DSS decomposition of the Lagrangian and field equations of Gauss-Bonnet gravity, for any order of corrections. Moreover, due to the NPG expression of the theory, the critical order scalar and the higher order ones do contribute to the dynamics, what is very different from the Lovelock-Lanczos case. This kind of effective "Lovelock Designer" theories have been studied for example in [170, 171], while the DSS sector of Lovelock-Lanczos gravity has been investigated in details in [166–169].

Concerning the specific model eq(126) that we will be interested in, recall that it was found from the more general theory Eq(123) by choosing $\lambda_p = 1$, which can now be reinterpreted from the result (232) of the last section of the Chapter 3, as the only choice leading to a well-defined FLRW sector for both procedures (1. & 2.) used to deduce FLRW from DSS. Modulo an overall set of coupling constants which is also present in Lovelock-Lanczos gravity, it constitutes a one set of coupling constants (one for each order of correction) deformation of the dynamical spherically symmetric sector of L-L gravity. However, we will see that this new terms do not contribute to the cosmological field equations, because the deformation becomes a total derivative in this sector. Therefore, it will be possible to find classes of theories admitting the same (Lovelock Designer) cosmological solutions, but different black holes counterparts.

As we will see, reconstructing a one-parameter family of actions leading to the Loop Quantum cosmology bounce [28–31] will allow us to investigate the black hole sectors of these models and to discover regular ones. Similarly we will see that a very similar family of theories admit an Asymptotic Safety regular black hole [54], and looking at the associated cosmological sector will uncover a regular universe in which the big bang singularity is replaced by a past-eternal de Sitter spacetime. Based on these results, it is then straightforward to find many other theories admitting regular cosmologies.

Actually, in a very recent paper by Chamseddine, Mukhanov and Russ [358][16], they have found a very similar non-singular cosmological solution from a Mimetic model, where the mimetic field was interpreted as producing a covariant running of both the Newton and cosmological constant, thus also making a link between these kinds of past-eternal de Sitter spacetimes and Asymptotic Safety-like approaches, where quantum effects are captured by the running of the coupling constants of the theory. In particular, the modification of Friedmann equation given in eq(28) of their paper is just a special case of the more general one Eq(441, 467) that we have found here.

The goals of this chapter are therefore twofold. First, as we said, we will use already known non-singular solutions to find new ones by a suitable choice of theory admitting the formers. Secondly, a more ambitious aim is to find adequate semi-classical formulations of quantum gravity results, from both Loop Quantum Cosmology and Asymptotic Safety, in the sense discussed in [110–115]. In these papers, the possible equivalence between LQC and the Mimetic gravity theory admitting the LQC bounce was discussed. However, it was shown in [113] that despite the similarity for cosmological backgrounds, the mimetic and polymer black holes are nonetheless very different, as it can be seen by comparing Eq(4.18) and Eq(4.25) of this paper. See also [74–77, 140, 141, 154] for similar papers reconstructing the LQC bounce at the background level.

Regarding these possible equivalences, expressing the LQC bounce and AS black hole as solutions of Lovelock-like theories is therefore tempting, given the very peculiar status of Lovelock-Lanczos models as the "most natural" generalizations of Einstein gravity.

[16] Note that this paper appeared *after* the submission of our thesis for review.

A. Motivation : LQC bounce from a NPG model

Some time ago, Helling [140] and independently Date and Sengupta [141], suggested a modification of GR Lagrangian which gives the same correction to the Friedmann equation than LQC, with therefore the same bounce. This approach was intended to be an effective action formulation of the loop quantization procedure of FLRW space-times. Helling showed that a formulation in terms of an infinite sum of curvature invariants is possible, but it was not possible to write it explicitly.

More recently, Chamseddine and Mukhanov working within the so called "mimetic approach" [359,360] (see also the similar construction in [361]) followed this idea, and in two papers [108,109] made use of a non-polynomial function of the mimetic field in a simple manner, and were able to reproduce the LQC result. Note that, within the mimetic approach, but including in the action a suitable potential for the mimetic field, it is possible to find other cosmological bounces, see for examples [360, 362]. Furthermore, the bounce mimetic approach has recently been generalized in [112]. Other recent papers on bounce loop cosmology are [32,33], while mimetic modified gravity is discussed in [363].

In this section, we propose to implement the Helling construction by finding an explicit Lagrangian built only from the metric field and that leads to the LQC corrections. This Lagrangian is constructed via non-polynomial gravities. Once again, the NPG approach is intended to mimic a specific sector of a fundamental (i.e. background independent) effective theory, in which only gravitational metric corrections with no additional derivatives are present. In this way, invariants built making use of non-polynomial terms in the metric become polynomials in the FLRW sector, becoming candidates to build an effective action there.

This part is intended as a motivation in order to see that a specific cosmological sector can be reached from many different DSS actions, but that the associated static spherically symmetric sectors of these actions can differ drastically from one to the other. Within this approach (and more generally), it is therefore of crucial importance to study *both* sectors, and see if reconstructing a regular cosmology implies that the SSS sector is also regular, or reciprocally. As we will see here, although the FLRW sector of the theory leads to the LQC bounce, the SSS sector is very complicated and thus, does not seem interesting to investigate further.

However, all the results concerning the cosmological sector will be possible to extend to any theory admitting the same Friedmann equations, for example the mimetic gravities that we mentionned previously, and the Lovelock-like gravities that we will study after this section.

This section is based on a conjoint work carried out with Stefano Chinaglia and Sergio Zerbini in [154].

1. Action & equations of motion

To begin with, let us consider a flat Friedmann-Lemaître-Robertson-Walker metric (FLRW) $\bar{g}$ defined by the following space-time interval :

$$ds^2 = -N(t)dt^2 + a(t)^2 d\vec{x}^{\,2}. \tag{318}$$

Here $N(t)$ is an arbitrary function which implements the time reparametrization invariance.

We want to build an effective action that reproduces some quantum geometry corrections. For this reason, we will be interested by scalars that are built from a particular geometric property of FLRW space-times, namely that the following projector :

$$\tau^{\alpha}_{\beta} = \delta^{\alpha}_{t}\delta^{t}_{\beta} = \mathrm{diag}(1,0,0,0), \tag{319}$$

is actually a true tensor, and that the quantity $\sqrt{N}\delta^{t}_{\alpha}$ is a true vector in FLRW. In order to see why, we can provide explicit tensorial forms to these objets. Consider the following vector and tensor :

$$V_{\alpha} := \frac{\partial_{\alpha}R}{\sqrt{-\partial_{\sigma}R\partial^{\sigma}R}} \qquad \text{and} \qquad V_{\alpha\beta} := V_{\alpha}V_{\beta}\,. \tag{320}$$

For the considered metric, these geometric tensors are of order-0, namely, they do not depend on the derivatives of the metric (here the scale factor). Indeed denoting the restrictions of the tensors

(320) on (318) by $V\big\rfloor = V\big|_{g=\bar{g}}$, etc..., one can see that they are indeed order-0 tensors with the claimed geometrical interpretation :

$$V_\alpha\big\rfloor = -\sqrt{N}\delta^t_\alpha \qquad \text{and} \qquad V_{\alpha\beta}\big\rfloor = -\tau_{\alpha\beta} \tag{321}$$

This property follows from the fact that for any scalar Q, $\partial_\alpha Q$ has only one component when evaluated on (318). We have chosen $Q = R$ here for simplicity. It is exactly the same type of property that the Weyl and Cotton tensors have in spherical symmetry [151, 156, 275], except that here the property is quite trivial. See [153] for more details in the case of spherical symmetry.

In all classes of space-times that share this property, like spherical symmetric space-times or Bianchi type I, one can build scalars from these tensors that will be second order in these classes, in particular in (318), but higher order otherwise. The two second order invariants we shall be interested in are

$$K := \frac{1}{9}\big(\nabla^\alpha\nabla^\beta V_{\alpha\beta} - V_\alpha\nabla^\alpha\nabla^\beta V_\beta\big) \qquad \text{and} \qquad \Omega := \frac{R}{6} - 2K\,. \tag{322}$$

Gravity models based on such kind of properties have been called "non-polynomial gravities" in [151], and so we will use this name here. These scalars are chosen so that their restrictions to (318) are :

$$K\Big\rfloor = \frac{H^2}{N} \qquad \text{and} \qquad \Omega\Big\rfloor = \frac{\dot{H}}{N} - \frac{H\,\dot{N}}{2N^2}\,, \tag{323}$$

because, together with the Ricci scalar, they form a basis of order-2 scalars in FLRW space-times, and setting $N(t) = 1$, they are actually the simpliest ones. Here, H is the Hubble parameter, and $\dot{H} = \frac{dH}{dt}$. Note that working in flat FLRW space-times, there exists other invariants, which have similar properties, see for example [152, 156], but the ones we have chosen are also relevant in spherically symmetric space-times.

We recall that, in principle, it is possible to reproduce the loop quantum cosmology modification of Friedmann equation, and therefore the bounce that replaces the big bang, via higher order corrections to Einstein-Hilbert action [140, 141]. These corrections have to lead to second order equations of motion, as shown by Helling, and so are truly geometrical corrections, in the sense that, unlike a generic modified gravity model, they do not involve additional fields with no direct geometrical meaning compared to the metric, or, for example, compared to the scalar field responsible to the local rescaling invariance in some models of conformal gravity.

In the paper [140], it was also shown that such corrections are possible to write as an infinite series of polynomials of contractions of Ricci tensors, even though it was not possible to write this effective action explicitly. In our approach, making use of the two scalars (322) defined above, a possible way to achieve this task is to start with the following action :

$$I = \int d^4x\sqrt{-g}\left(\frac{R - 2\Lambda + \left[\mathscr{L}^\infty_{NPG}\right]}{2\kappa} + \mathscr{L}_m\right), \tag{324}$$

where $\kappa = 8\pi$, with the Newton constant $G = 1$, Λ is the cosmological constant, $\mathscr{L}_m$ is the Lagrangian density of matter, and

$$\mathscr{L}^\infty_{NPG} = -2\Omega + \frac{4\Omega}{S}\left(1 - \sqrt{1 - S}\right). \tag{325}$$

Here we have introduced the dimensionless scalar $S = \frac{3}{2\pi\rho_c}K$, with ρ_c playing the role of critical density, which in our approach is a *free* dimensional parameter.

Some comments are in order. This contribution, which modifies the GR term, may also be thought of as

$$\mathscr{L}^\infty_{NPG} = -4\sum_{i=0}^{\infty}(-1)^{i+1}\binom{1/2}{1+i}S^i\,\Omega\,, \tag{326}$$

where $\binom{n}{m}$ is the generalized binomial coefficient defined by $\binom{n}{m} := \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}$. The bracket in equation (324) are used in order to emphasize that, within this kind of minisuperspace approach, one can only hope to find the desired scalar up to scalars that vanish or are boundary terms (at least) in the class of space-times in which the reconstruction is done, in our case in flat FLRW (318). For example, one could add scalars involving the Weyl tensor or background dependent boundary terms (as those of [156]) in the action without modifying the dynamics of (324) for FLRW space-time. Therefore, $\mathscr{L}^{\infty}_{NPG}$ is only a particular NPG representative of an infinite class of scalars (that includes polynomial ones like in [140]) with equivalent contributions to the equations of motion in (318).

Furthermore, for this specific space-time, the additional term, despite its non-polynomiality, may be considered in (326) as an infinite sum of polynomials in the metric, and therefore (324) constitutes a suitable effective action, whose coupling constants are fixed in order to reproduce the LQC modification of Friedmann equations. Note also that the $i = 0$ term of the sum (326), namely 2Ω, is equivalent to the Ricci scalar in FLRW, since they differ from each other by a total derivative. Moreover, in its present form, the correction $\mathscr{L}^{\infty}_{NPG}$ seems of higher order, but it is in fact equivalent, up to (background dependent) boundary terms, to the correction of [140, 141]. Indeed, they differ from each other by a total derivative :

$$\left(R - 2\Omega + \frac{4\,\Omega}{S}\left(1-\sqrt{1-S}\right)\right)\Big\rfloor = 8\pi\rho_c\left(1-\sqrt{1-S}-\sqrt{S}\arcsin(\sqrt{S})\right)\Big\rfloor + \frac{4}{\sqrt{-g}}\sqrt{\frac{2\pi\rho_c}{3}}\dot{B}\Big\rfloor \tag{327}$$

with

$$B = \frac{\sqrt{-g}}{\sqrt{N}}\left(\csc^{-1}\left(\frac{1}{\sqrt{S}}\right) - \frac{1-S-\sqrt{1-S}}{\sqrt{S}}\right). \tag{328}$$

Note that in both cases, the GR contribution is cancelled, because $\sqrt{-g}\big(R-2\Omega\big)\rfloor = \frac{d}{dt}\left(\frac{4\,a^3 H}{\sqrt{N}}\right)$, and what is left is only a non-polynomial effective action and an effective cosmological constant $8\pi\rho_c$ in the first order form of the right-hand-side. Therefore, in FLRW and up to boundary terms, the series (326) of polynomial curvature scalars is the only one that gives the LQC modification of Friedmann equation, as we will see now.

Making use of a minisuperspace approach (Weyl method), from ansatz (318) and action (324) we can derive the Euler-Lagrange (EL) equations of motion by making the variation with respect to Lagrangian coordinates $N(t)$ and $a(t)$. The Principle of Symmetric Criticality applied to the isometry group of an homogeneous and isotropic universe assures that the reverse process (the right one) will give the same results [238, 239].

We also assume that the matter is a perfect fluid, with equation of state $p = w\rho$, ρ and p being the density and the pressure. Making the variation with respect to $N(t)$, one gets the Friedmann equation, and by setting $N(t) = 1$ after the variation, one has

$$4\pi\rho_c\left(1-\sqrt{1-\frac{3H^2}{2\pi\rho_c}}\right) = 8\pi\rho + \Lambda\,. \tag{329}$$

As a first check, when $\frac{H^2}{\rho_c} \ll 1$, one recovers the Friedmann equation of GR.

Defining $\bar{\rho} := \frac{\Lambda}{8\pi} + \rho$, one gets the standard form of the LQC corrected Friedmann equation :

$$H^2 = \frac{8\pi\bar{\rho}}{3}\Big(1-\frac{\bar{\rho}}{\rho_c}\Big). \tag{330}$$

Making the variation with respect to $a(t)$, one gets the other Friedmann equation, which contains the acceleration. For our purposes, we do not need it since it can derived from (329) and the energy conservation equation

$$\frac{d\rho}{dt} + 3H\big(\rho+p\big) = 0\,, \tag{331}$$

consequence of the diffeomorphism invariance of our invariant action. Thus, only two of these three equations are independent, and one may use only equations (329) and (331).

2. Exact solutions for general equation of state parameter w and cosmological constant Λ

We recall the equation of state for the perfect fluid $p = w\rho$. Then, introducing for the sake of simplicity $\tilde{\rho} = 8\pi\rho$ and $\mu = \frac{1}{8\pi\rho_c}$, one has

$$
\begin{aligned}
3H^2 &= (\tilde{\rho} + \Lambda) - \mu(\tilde{\rho} + \Lambda)^2, \\
\frac{d\tilde{\rho}}{\tilde{\rho}} &= -3(1+w)H dt.
\end{aligned} \tag{332}
$$

First we note that, without solving the differential equation, it is possible to show that a bounce solution is present, namely there exists $a_* > 0$ such that $H_* = 0$ and $\dot{H}_* > 0$. In fact the first equation on the bounce $H_* = 0$ gives the condition $1 - \mu\Lambda = \mu\tilde{\rho}_*$, namely $\mu\Lambda < 1$, which is therefore a necessary condition.

We now derive the exact solution. Inserting the first equation into the second one leads to :

$$
\frac{dX}{(X - \Lambda)\sqrt{X - \mu X^2}} = \pm\sqrt{3}(1+w)dt\,. \tag{333}
$$

where $X = \rho + \Lambda$. Thus,

$$
\frac{2\tanh^{-1}\left(\frac{\sqrt{\Lambda}\sqrt{1-X\mu}}{\sqrt{X}\sqrt{1-\Lambda\mu}}\right)}{\sqrt{\Lambda}\sqrt{1-\Lambda\mu}} = \pm\sqrt{3}(1+w)t + c\,, \tag{334}
$$

where c is the integration constant. In the following, we may put $c = 0$ without any problem. Solving in X and thus in $\tilde{\rho}$ gives :

$$
\tilde{\rho}(t) = -\frac{2\Lambda(-1+\Lambda\mu)}{-1 + 2\Lambda\mu + \cosh\left(\left(\pm\sqrt{3}t(1+w)\right)\sqrt{\Lambda}\sqrt{1-\Lambda\mu}\right)}. \tag{335}
$$

The second equation of (332) admits the usual well known solution $a = a_0\tilde{\rho}^{\frac{-1}{3(1+w)}}$. As a consequence, one has

$$
a(t) = a_0\left(\frac{-1 + 2\Lambda\mu + \cosh\left(\left(\sqrt{3}(1+w)t\right)\sqrt{\Lambda}\sqrt{1-\Lambda\mu}\right)}{2\Lambda(1-\Lambda\mu)}\right)^{\frac{1}{3(1+w)}}. \tag{336}
$$

Here, we recover the condition $1 - \mu\Lambda > 0$, $\Lambda > 0$. Given this solution, one can check that the scalar $\partial_\sigma R\partial^\sigma R$ is not vanishing everywhere, and the scalars (322) are indeed well defined.

As a further check of the solution, we can study the two limits $\mu \to 0$ and $\Lambda \to 0$. First, the GR limit, namely

$$
\begin{aligned}
\lim_{\mu\to 0}\tilde{\rho}(t) &= \Lambda\ \text{csch}^2\Big(\frac{1}{2}\sqrt{\Lambda}(\sqrt{3}t(1+w)\Big), \\
\lim_{\mu\to 0} a(t) &= a_0\Big(\frac{1 - \cosh(\sqrt{\Lambda}(\sqrt{3}t(1+w))}{2\Lambda}\Big)^{\frac{1}{3(1+w)}},
\end{aligned} \tag{337}
$$

This is the solution of GR with non vanishing cosmological constant, and one recovers the Big Bang solution at $t = 0$.

In the other limit, one has

$$
\begin{aligned}
\lim_{\Lambda\to 0}\tilde{\rho}(t) &= \frac{4}{\left(\sqrt{3}t(1+w)\right)^2 + 4\mu}, \\
\lim_{\Lambda\to 0} a(t) &= a_0\Big(\mu + \frac{1}{4}(\sqrt{3}t(1+w))^2\Big)^{\frac{1}{3(1+w)}}.
\end{aligned} \tag{338}
$$

and one recovers the original LQG bounce solution in absence of cosmological constant.

Now we study our exact solution with respect to the coordinate time t. We already have shown the existence of the bounce. In particular, for t small, one has,

$$a(t\to 0) = a_0 \left(\frac{\mu}{1-\mu\Lambda}\right)^{1/3(1+w)} \left(1+\frac{(1-\mu\Lambda)(1+w)}{4\mu}t^2+...\right) \ . \tag{339}$$

We see that the minimal value is $a(0) = a_0\left(\frac{\mu}{1-\mu\Lambda}\right)^{1/3(1+w)}$, corresponding to the bounce. Moreover, already eq. (336) shows that there $a(t)$ is never vanishing: indeed, the hyperbolic cosine is always greater than 1, so $\cosh x - 1 \geq 0$; and since μ and Λ are both positive, the scale factor is always positive and never vanishing.

The other interesting limit is the one for t very large. Since we already have taken the cosmological constant into account, we take $w > -1$. We remind that $\cosh x \to e^{|x|}$, for $x \to \pm\infty$, and one has

$$a(t\to\infty) = \frac{a_0}{(2\Lambda(1-\mu\Lambda))^{1/3(1+w)}}\left(2\mu\Lambda - 1 + \exp\left(\sqrt{3\Lambda(1-\mu\Lambda)}(1+w)t\right)\right)^{1/3(1+w)} , \tag{340}$$

the exponential becomes dominant corresponding to an accelerating universe. Thus, our solution may represent dark energy (DE), with a chosen suitable scale, and for large t [364], [365].

We conclude this Section discussing the limits μ and Λ large. We have seen that the product $\mu\Lambda$ must be $\mu\Lambda < 1$. This is not a problem for DE issue because $\mu = \frac{1}{8\pi\rho_c}$ mimics a quantum correction and thus it can be taken small, because ρ_c is very large, and for DE Λ is small.

Finally, concerning the scalars used in the construction, given their non-polynomial forms, one could wonder if they are regular at the bounce, like polynomial scalars. One can check that given the solution (336), their behaviours are :

$$\begin{aligned} &\lim_{t\to 0} \partial_\sigma R \partial^\sigma R = \lim_{t\to 0} K = 0, \\ &\lim_{t\to 0} \nabla^\alpha\nabla^\beta V_{\alpha\beta} = \lim_{t\to 0} V_\alpha \nabla^\alpha\nabla^\beta V_\beta = 3\lim_{t\to 0}\Omega = \frac{-3(1+w)(-1+\Lambda\mu)}{2\mu}, \end{aligned} \tag{341}$$

namely no problem when $\mu \neq 0$.

3. Static spherically symmetric sector

Now we will briefly consider the static spherically symmetric sector of this theory, just enough to see that, contrary to the cosmological sector, it is very complicated and it would be quite hard to find any exact solution. In this sector, given by,

$$ds^2 = -ab^2dt^2 + \frac{dr^2}{a} + r^2 d\Omega_k^2 \tag{342}$$

the scalars K and Ω become :

$$\begin{aligned} K &= -\frac{1}{36r^2ab^2}\left(rba' + 2a(2b+rb')\right)^2 \\ \Omega &= \frac{1}{18r^2b^2}\left(4r^2ab'^2 + b^2\left(6k + 10a - 4ra' + \frac{r^2a'^2}{a} - 3r^2a''\right) + rb\left((4a-5ra')b' - 6rab''\right)\right) \end{aligned} \tag{343}$$

Therefore, even though the minisuperspace field equations associated with a Lagrangian density $L = K^i\Omega$ are second order, as one can check, they are involving terms like $a'b''$ or $b'a''$, making these very difficult to solve.

In order to obtain both analytic regular black hole and cosmological solutions, we therefore need a better choice of theory, what will be the topic of remaining sections of this chapter.

B. Modified Lovelock-Designer gravities

In the following sections, we will investigate the theory eq(126), that we recall is given by

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(R + z\sum_{p=2}^{\infty}\frac{w_p\,\mathcal{Z}^{p-2}}{(p-1)(p-2)}\left(\sigma_p\mathcal{Z}^2 + \left(\rho_p\mathcal{Y} + \delta_p R_{(\Sigma)}\right)\mathcal{Z} + \nu_p\left(\mathcal{Y}^2 - \mathcal{K}\right)\right)\right) \tag{344}$$

where

$$\begin{aligned}\sigma_p &= 2(p-2)(2p-3)(\zeta_p-1) \ \ , \ \ \rho_p = 2(p-2)\big(\zeta_p(2p-1)-p\big)\\ \delta_p &= \zeta_p(p-1) \qquad\qquad\qquad\qquad , \ \ \nu_p = 2(p-1)\delta_p\end{aligned} \tag{345}$$

while the Non-Polynomial curvature invariants $\big(\mathcal{Z},\mathcal{Y},R_{(\Sigma)},\mathcal{K}\big)$, defined in Eq(98,99) can be expressed as :

$$\begin{aligned}\mathcal{Z} &= \mathcal{R} - \frac{\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}}{4} \quad , \qquad \mathcal{R} = \frac{1}{2}\left(R - \left(R^{\alpha\beta}-\nabla^\alpha\nabla^\beta\right)\omega_{\alpha\beta}\right)\\ \mathcal{Y} &= \frac{1}{2}\left(\nabla^\alpha\nabla^\beta\omega_{\alpha\beta} - \frac{1}{2}\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\right) \quad , \qquad R_{(\Sigma)} = R^{\alpha\beta}\omega_{\alpha\beta} + 2\mathcal{Y}\ ,\\ \mathcal{K} &= \frac{1}{4}\nabla_\mu\nabla_\nu\omega_{\sigma\rho}\nabla^\mu\nabla^\nu\omega^{\sigma\rho} - \frac{5}{16}\left(\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\right)^2 + \frac{1}{8}\nabla^\mu\omega_{\mu\nu}\nabla^\nu\left(\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}\right)\end{aligned} \tag{346}$$

where the tensor $\omega_{\mu\nu}$ is itself defined by Eq(89) in terms of the non-polynomial curvature tensor $u_{\mu\nu}$ by : $\omega_{\mu\nu} := -\frac{1}{3}\left(g_{\mu\nu} + 4\,u_{\mu\nu}\right)$. The case $\zeta_p = p$ corresponds to Lovelock Designer.

Then the covariant DSS decomposition of the Lagrangian $\mathcal{L}_p$ used in the previous action

$$I = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\left(R + z\sum_{p=2}^{\infty}\mathcal{L}_p\right)$$

is given by :

$$\begin{aligned}\mathcal{L}_{(p)} = \frac{w_p}{(p-1)(p-2)}\Bigg(&2(p-2)\Big(\big(\zeta_p(2p-1)-p\big)\mathcal{Y} + (\zeta_p-1)(2p-3)\mathcal{Z}\Big)\mathcal{Z}\\ &+\zeta_p(p-1)\left(2(p-1)\left(\mathcal{Y}^2-\mathcal{Y}_2\right) + \mathcal{Z}R^{(2)}\right)\Bigg)\mathcal{Z}^{p-2}\end{aligned} \tag{347}$$

where $\mathcal{Z} = \frac{k-\mathcal{D}_a r\mathcal{D}^a r}{r^2}$, $\mathcal{Y} = \frac{\mathcal{D}^2 r}{r}$ and $\mathcal{Y}_2 = \frac{\mathcal{D}_a\mathcal{D}_b r\mathcal{D}^a\mathcal{D}^b r}{r^2}$ and $R^{(2)}$ is the Ricci scalar of the manifold Σ, i.e. associated with the metric γ_{ab}. We will use the minisuperspace approaches for both cosmological and black hole solutions, so that we can consider the general DSS gauge :

$$ds^2 = -a(t,r)b(t,r)^2dt^2 + \frac{dr^2}{a(t,r)} + \rho\,(t,r)^2\left(\frac{d\theta^2}{1-k\theta^2} + \theta^2 d\phi^2\right) \tag{348}$$

1. Spherical Symmetry

Let's start with the spherically symmetric sector (and its topological generalizations) of the previous modification of (beyond the critical order) Lovelock gravity. From the metric (348), it amounts to restrict the metric variables to :

$$\rho\,(t,r) = r \ \ ; \quad a(t,r) = a(r) \ \ ; \quad b(t,r) = b(r) \tag{349}$$

Thus, we have $\sqrt{-g} = r^2 b$, and the minisuperspace Lagrangian becomes :

$$\begin{aligned}\sqrt{-g}\mathcal{L}_p \ &= \frac{2w_p}{p-1}\left(\frac{k-a}{r^2}\right)^{p-1}\Big((\zeta-1)b\Big((2p-3)\,(k-a) + pra'\Big) + (\zeta-p)rab'\Big)\\ &\quad - \frac{\zeta\,w_p}{p-2}\left(r^2\left(\frac{k-a}{r^2}\right)^{p-1}(ba'+2ab')\right)'\end{aligned} \tag{350}$$

Using the Weyl approach, the minisuperspace field equations are :

$$\begin{aligned}\frac{\delta I}{\delta b} &= \left(2\,r\,(k-a)\left(1 - z\sum_{p=2}^{m} w_p r^{2(1-p)}\,(k-a)^{p-2}\left(\frac{(\zeta-1)k}{p-1} - a\right)\right)\right)' = 0\\ \frac{\delta I}{\delta a} &= 2r\left(1 - z\sum_{p=2}^{m} w_p r^{2(1-p)}\,(k-a)^{p-2}\,(\zeta\,k - p\,a)\right)b' = 0\end{aligned} \tag{351}$$

Therefore, $b = 1$ in all these classes of models. Now note that like in the case of the spherically symmetric sector of Lovelock gravity, the first equation can directly be integrated to give :

$$2\,r\,(k-a)\left(1 - z\sum_{p=2}^{m} w_p r^{2(1-p)}\,(k-a)^{p-2}\left(\frac{(\zeta-1)k}{p-1} - a\right)\right) = 4M \quad ; \quad b = 1 \tag{352}$$

where M is the mass of the solution, as it can be seen by setting $z = 0$, what gives the Schwarzschild black hole.

In order to find the spherically symmetric solutions of any specific theories determined by some given ζ_p and w_p, one simply has to solve the previous algebraic equation for a.

2. FLRW

Similarly, one can investigate the cosmological sector of the theory by setting :

$$\rho\,(t,r) = a(t)r \quad ; \quad a(t,r) = \frac{1}{a(t)^2} \quad ; \quad b(t,r) = a(t)\sqrt{N(t)} \quad ; \quad k = 1 \tag{353}$$

In this case we have $\sqrt{-g} = \sqrt{N}a^3$, and the minisuperspace Lagrangian becomes :

$$\sqrt{-g}\mathcal{L}_p = \frac{6w_p}{2p-1}a^3 N^{\frac{1}{2}-p}H^{2p} + w_p\frac{2\left((p-2)p+(2p-1)\zeta\right)}{(p-2)(p-1)(2p-1)}\partial_t\left(a^3 N^{\frac{1}{2}-p}H^{2p-1}\right) \tag{354}$$

where the Hubble parameter H is defined by $H := \frac{\dot{a}}{a}$. This makes clear what we intended previously by saying that this modification of Lovelock gravity is topological in FLRW spacetimes, because as we see the modification (multiplied by ζ) becomes a boundary term in this sector. As N can be reabsorbed by a gauge transformation, we can derive the minisuperspace field equations and then set $N = 1$,

$$\begin{aligned} \left.\frac{\delta I}{\delta N}\right|_{N=1} &= 3a^3\left(H^2 - z\sum_{p=2}^{m} w_p H^{2p}\right) \\ \left.\frac{\delta I}{\delta a}\right|_{N=1} &= 6a^2\left(\left(3H^2 + 2\dot{H}\right) - z\sum_{p=2}^{m} w_p H^{2(p-1)}\left(3H^2 + 2p\dot{H}\right)\right) \end{aligned} \tag{355}$$

what enables to have directly the modifications of Friedmann equations by consider a perfect fluid with equation of state $p = w\rho$, where ρ and p are its density and pressure. They are given by :

$$\begin{aligned} \frac{8\pi\rho}{3} &= H^2 - z\sum_{p=2}^{m} w_p H^{2p} =: J\left(H^2\right) \\ -8p &= \left(3H^2 + 2\dot{H}\right) - z\sum_{p=2}^{m} w_p H^{2(p-1)}\left(3H^2 + 2p\dot{H}\right) \end{aligned} \tag{356}$$

In order to solve the equation for the scale factor, one can therefore use the following equation :

$$\frac{\dot{a}}{a} = \sqrt{J^{-1}\left(\rho\right)} \tag{357}$$

Because from the acceleration equation (the second one) or equivalently the conservation of energy density, we have

$$\rho\,(a) = \left(\frac{a}{a_0}\right)^{-3(w+1)} \tag{358}$$

Therefore, in order to find the scale factor as a function of time, one needs to integrate the following equation :

$$\frac{da}{a\sqrt{J^{-1}\left(a\right)}} = dt \tag{359}$$

And then to solve the equation for a.

3. Closed and hyperbolic FLRW sector

A possible follow up of the study of these theories would be to study the effect of the previous one-parameter modifications in a non-flat FLRW spacetime. This sector is reached from :

$$\rho\,(t,r) = a(t)r \;\; ; \;\; a(t,r) = \frac{(1-\bar{k}r^2)}{a(t)^2} \;\; ; \;\; b(t,r) = a(t)\sqrt{\frac{N(t)}{1-\bar{k}r^2}} \;\; ; \;\; k=1 \tag{360}$$

we rename $\bar{k}$ as k. Then the minisuperspace Lagrangian reduces to :

$$\begin{aligned}\sqrt{-g}\mathcal{L}_p \;=\; \frac{w_p a^3 N^{-(p-1/2)}}{(p-1)(p-2)}\left(H^2+\frac{kN}{a^2}\right)^{p-1}\Big(&-2(p-2)(p+2\zeta-3)\frac{kN^2}{a^2}+6(p-\zeta-2)H^2N \\ &+\left(p(p-2)+\zeta(2p-1)\right)\left(2N\dot{H}-H\dot{N}\right)\Big)\end{aligned} \tag{361}$$

The question is therefore to understand if the corrections, parametrized by ζ, do contribute to this sector (like in spherical symmetry), or not (like in flat FLRW). To see this, note that the minisuperspace field equations are :

$$\begin{aligned}\left.\frac{\delta I}{\delta N}\right|_{N=1} &= 3a^3\left(\frac{k}{a^2}+H^2\right)\left(1-\frac{z\,w_p}{p-1}\left(\frac{k}{a^2}+H^2\right)^{p-2}\left((p-1)H^2+\left(\frac{2\zeta+p}{3}-1\right)\frac{k}{a^2}\right)\right)\\ \left.\frac{\delta I}{\delta a}\right|_{N=1} &= 6a^2\left(\frac{k}{a^2}+3H^2+2\dot{H}+\frac{zw_p}{p-1}\left(\frac{k}{a^2}+H^2\right)^{p-2}(X+Y)\right)\end{aligned} \tag{362}$$

where $X=(2p-3)\left(\frac{k}{a^2}+H^2\right)\left(\left(\frac{2\zeta+p}{3}-1\right)\frac{k}{a^2}+(p-1)H^2\right)$ and $Y=-2(p-1)\left(\frac{2\zeta+p}{3}\frac{k}{a^2}+pH^2\right)\left(H^2+\dot{H}\right)$. Therefore, we see that for non-flat topology FLRW universe, the modifications parametrized by ζ do contribute to the high energy corrections. It would be interesting to see if the solutions of this system for a closed universe $k=1$, or open hyperbolic one $k=-1$, in the presence of a perfect fluid, would be regular or not for some choices of (w_p,ζ_p). However, it seems quite difficult to find analytic solutions for the choices that we will study in the following sections.

C. LQC bounce & regular black holes

1. LQC bounce

As we did in the first section of this Chapter, it is then possible to reconstruct the action leading to the Loop Quantum Cosmology corrections Eq(330) by choosing the following shape factor w_p :

$$w_p = -2\binom{1/2}{p}\left(-\frac{3}{2\pi}\right)^{p-1} l^{2(p-1)} \quad , \quad z=1 \quad , \quad \zeta = p \tag{363}$$

where $\zeta=p$ corresponds to (the spherically symmetric reduction of) Lovelock gravity. Indeed, we can check that Eq(356) becomes in this case :

$$H^2-\sum_{p=2}^{\infty}w_pH^{2p} = \frac{4\pi}{3l^2}\left(1-\sqrt{1-\frac{3l^2H^2}{2\pi}}\right) = \frac{8\pi\rho}{3} \tag{364}$$

So that solving for H^2 gives the LQC modification of Friedmann equation :

$$H^2 = \frac{8\pi\rho}{3}\left(1-l^2\rho\right) =: X\left(\rho\right) \tag{365}$$

In order to derive the scale factor, we follow the previous steps by just integrating the equation $da/\left(a\sqrt{X\left(\rho\left(a\right)\right)}\right)=dt$. Solving for a gives the regular bounce :

$$a(t) = a_0\left(l^2+6\pi\left(1+w\right)^2t^2\right)^{\frac{1}{3(w+1)}} \tag{366}$$

The question is now if there is a regular black hole solution associated with the static spherically sector of the same theory. From the same choices (363), but this time using a static interval given by (349), the Eq(352) becomes :

$$\frac{8\pi r^3}{3l^2}\left(1-\sqrt{1-\frac{3l^2}{2\pi r^2}(k-a)}\right)=4M \tag{367}$$

whose general solution for arbitrary horizon topology k is given by :

$$a(r)=k+\frac{M}{2r^4}\left(\frac{3l^2M}{\pi}-4r^3\right) \tag{368}$$

It is clearly not regular, but the singularity is time-like instead of the space-like one of the Schwarzschild case, so that the causal structure of this solution is similar to a Reissner-Nordstrom black hole.

This is where the fact that the parameter ζ does not contribute to FLRW dynamics can play a role, because it is therefore possible to find theories that automatically possesses the LQC bounce, but have different black hole solutions than the previous singular one. This is what we are going to see now.

2. First modification of Lovelock gravity & (A)dS-core black hole :

As it turns out, there are very simple choices of ζ that leads to regular solutions. In the two following sections, we will consider some 1-parameter deformations of the spherically symmetric sector of Lovelock-Lanczos gravity. First, we choose the following one :

$$w_p=-2\binom{1/2}{p}\left(-\frac{3}{2\pi}\right)^{p-1}l^{2(p-1)}\quad,\quad z=1\quad,\quad \zeta=(1-\sigma)\,p^2+\sigma p \tag{369}$$

where we kept the same w_p in order to have the LQC bounce in the FLRW sector of the model, and we see that $\sigma=1$ corresponds to the Lovelock-Designer theory. In this case, Eq(352) becomes[17] :

$$\frac{8\pi r^3}{3\,l^2}\left(1+\frac{3l^2}{4\pi}\left(\frac{k(\sigma-1)}{r^2}\right)-\frac{1+\frac{3l^2}{4\pi}\left(\frac{k(\sigma-3)+2a}{r^2}\right)}{\sqrt{1+\frac{3l^2}{2\pi}\left(\frac{-k+a}{r^2}\right)}}\right)=4M \tag{370}$$

Therefore, the condition to have real equations of motion $1+\frac{3l^2}{2\pi}\left(\frac{-k+a}{r^2}\right)\geq 0$ implies that $a(r)\geq k-\frac{2\pi r^2}{3l^2}$, meaning that the solution is at least de Sitter everywhere, what prevents to have a space-like singularity at the time $r=0$. However, it is still in principle possible to have time-like ones, like in the previous case of Lovelock gravity $\zeta=p$. As we will see, this modification actually cure completely the Schwarzschild singularity. In order to derive the solutions, let's define

$$\alpha(r)^2=1-\frac{3l^2}{2\pi}\left(\frac{k-a(r)}{r^2}\right) \tag{371}$$

and set $\mathfrak{a}=\frac{3l^2}{4\pi}$ to simplify the notations. Using α instead of a, the Eq(370) becomes :

$$2M+\frac{r}{\mathfrak{a}|\alpha|}\left(|\alpha|-1\right)\left(\mathfrak{a}k(1-\sigma)+r^2|\alpha|\right)=0 \tag{372}$$

We first wish to consider the vacua of the theory given by the solutions of the previous equation when $M=0$. There are in general two solutions $|\alpha|=1$ and, for spherical and hyperbolic topologies $(k\neq 0)$, $|\alpha|=\mathfrak{a}(\sigma-1)\frac{k}{r^2}$, what corresponds to :

$$a=k\quad,\quad a=k-\frac{2\pi r^2}{3l^2}+\frac{3k^2l^2\,(\sigma-1)^2}{8\pi r^2} \tag{373}$$

where the second one exists if $\sigma\neq 1$. As we see, the second solution is singular, but note that if we consider $\sigma<1$, then, for spherical topology $(k=1)$, the unique vacuum is Minkowski spacetime

[17] In practice, we performed these kind of summations using Wolfram Mathematica.

because the second factor of Eq(372) is strictly positive. For this reason, we choose this range for σ. However, in this case, for hyperbolic topology ($k=-1$), the model admits the second solution as vacuum :

$$ds^2 = -\left(-1 - \frac{r^2}{2\mathfrak{a}} + \frac{\mathfrak{a}\,(\sigma-1)^2}{2r^2}\right)dt^2 + \frac{dr^2}{-1 - \frac{r^2}{2\mathfrak{a}} + \frac{\mathfrak{a}(\sigma-1)^2}{2r^2}} + r^2\left(\frac{d\theta^2}{1+\theta^2} + \theta^2 d\phi^2\right) \tag{374}$$

This vacuum describes a static region surrounded by two hyperbolic topology (cosmological) horizons located at $r = \pm\sqrt{\mathfrak{a}}\sqrt{-1+\sqrt{2+\sigma(\sigma-2)}}$ in which a naked time-like singularity is present at $r=0$. It is very similar to the solution of Einstein-Maxwell theory with a positive cosmological constant $\Lambda_{\text{eff}} = \frac{3}{2\mathfrak{a}}$, an effective charge $Q_{\text{eff}} = \frac{\mathfrak{a}(\sigma-1)^2}{2}$ and $k=-1$, $M=0$. At infinity, $r\to\infty$, the solution describes an anisotropic cosmological vacuum. Indeed, changing coordinates as follows $t=z$, $\theta=\sinh\psi$, $r=\sqrt{2\mathfrak{a}}\sinh\left(\frac{\tau}{\sqrt{2\mathfrak{a}}}\right)$, gives the following metric :

$$ds^2 = -d\tau^2 + \cosh^2\left(\frac{\tau}{\sqrt{2\mathfrak{a}}}\right)dz^2 + 2\mathfrak{a}\sinh^2\left(\frac{\tau}{\sqrt{2\mathfrak{a}}}\right)\left(d\psi^2 + \sinh^2\psi d\phi^2\right) \tag{375}$$

This solution (374,375) being singular, we see that it is not possible, for any σ, to have only regular solutions from the associated action given by the choices (369). We will also see in the next sections that, for another choice of ζ_p, there are again some singular topological solutions, even if in both cases the Schwarzschild black hole is regularized. It would be interesting to understand if it is possible at all to find a theory ζ_p which admits only regular solutions for the three topologies $k=0,-1,1$, when w_p is chosen to give the LQC bounce.

In the present case, note that for planar topology $k=0$, $|\alpha|=0$ is not allowed because $|\alpha|$ is also present at the denominator of the field equation. Let's just say, en passant, that it would correspond to a de Sitter vacuum, as it is given by

$$ds^2 = \frac{r^2}{2\mathfrak{a}}dt^2 - \frac{2\mathfrak{a}}{r^2}dr^2 + r^2\left(d\theta^2 + \theta^2 d\phi^2\right) \tag{376}$$

and applying the following coordinate transformations : $z = \frac{t}{\sqrt{2\mathfrak{a}}}$, $x=\theta\cos\phi$, $y=\theta\sin\phi$ and $r = e^{-\frac{\tau}{\sqrt{2\mathfrak{a}}}}$, one gets the cosmological patch of de Sitter spacetime with an effective cosmological constant $\Lambda_{\text{eff}} = \frac{3}{2\mathfrak{a}}$:

$$ds^2 = -d\tau^2 + e^{-\frac{\sqrt{2}\tau}{\sqrt{\mathfrak{a}}}}\left(dx^2+dy^2+dz^2\right) \tag{377}$$

Now that the vacua of the theory are found, we can consider systems with a point-like mass by re-writing the Eq(372) as :

$$\frac{r^3}{\mathfrak{a}|\alpha|}\left(|\alpha| - \frac{e-\sqrt{e^2-4\mathfrak{a}k\,r^4(\sigma-1)}}{2r^3}\right)\left(|\alpha| - \frac{e+\sqrt{e^2-4\mathfrak{a}k\,r^4(\sigma-1)}}{2r^3}\right) = 0 \tag{378}$$

where $e := r^3 + \mathfrak{a}\left(-2M + k\,r(\sigma-1)\right)$. Before discussing in more details the conditions to have real solutions, we can see that their general form is :

$$a_\pm = k - \frac{r^2}{2\mathfrak{a}} + \frac{1}{8\mathfrak{a}r^4}\left(e \pm \sqrt{e^2 - 4\mathfrak{a}k\,r^4(\sigma-1)}\right)^2 \tag{379}$$

We first consider spherical topology $k=1$. As we chose $\sigma<1$, we see that the interior of the square root is positive for all values of the radial coordinate. Using the triangle inequality, we can see that the first factor of (378) gives the solution a_- for all $r<0$ (because $\frac{e-\sqrt{e^2-4\mathfrak{a}k\,r^4(\sigma-1)}}{2r^3} > 0$ in this case), while the second solution gives a_+ for all $r\geq 0$. As it turns out, a_- describes a naked singularity in an asymptotically flat spacetime, for any value of the parameter M, because we have $a_- > 0$ and

$$a_-\left(r\to 0^-\right) = \frac{2\mathfrak{a}M^2}{r^4} + O\left(\frac{\mathfrak{a}M}{r^3}\right) \tag{380}$$

This is not that worst than the General Relativistic case where in this region $r<0$, the metric is given by $a = 1-2M/r$ which is also a naked singularity.

However in our case, an interesting consequence of the present modification is that for $r \geq 0$, the singularity of the Schwarzschild black hole is regularized, because the metric function a_+ satisfies the Sakharov criterion as it possesses a de Sitter core :

$$a_+ \left(r \longrightarrow 0^+\right) = 1 - \frac{r^2}{2\mathfrak{a}} + \frac{(\sigma - 1)^2 r^4}{8\mathfrak{a}M^2} + O\left(\frac{r^5}{\mathfrak{a}M^3}\right) \tag{381}$$

This proves that this black hole solution belongs to the (A)dS-core regular black holes. As we already said previously, they possess an outer as well as an inner horizon, what makes them unstable against the mass-inflation phenomenon, unless they are extremal.

As an additional interesting property, note that this class of black hole satisfies the principle of Limiting Curvature because the limit $a_+ (M \longrightarrow \infty)$ is also given by the previous series, so that the curvature of this solution is bounded by the one a de Sitter spacetime $a = 1 - \frac{r^2}{2\mathfrak{a}}$. Indeed, one can also see that when $r \to 0$ or $M \to \infty$, the Ricci scalar becomes $R = 6/\mathfrak{a}$ and we also have $R_{\mu\nu}R^{\mu\nu} = 9/\mathfrak{a}^2$. Other (A)dS-core regular black holes satisfying this property have been found and studied, but the point here is that this black hole follows from a theory that admit also the LQC bounce solution in its FLRW sector, making these two physical solutions regular.

Another curious property of this solution is worth mentioning. It is the fact that, at infinity, the modification provides precisely the analogue of the Quantum correction to the Newton potential that was derived in [177, 178] :

$$a_+(r \longrightarrow \infty) = 1 - \frac{2M}{r} - \frac{2\mathfrak{a}M(\sigma - 1)}{r^3} + o\left(\frac{1}{r^3}\right) \tag{382}$$

Moreover, comparing with the mentioned papers, we see that the condition $\sigma < 1$, that we imposed in order to have Minkowski spacetime as the unique spherical vacuum of the theory, is precisely what provides the right sign (positive) in front of this correction. In order to know if a solution can a candidate to be an effective black hole that could be derived from a full quantum gravity theory, this condition is crucial because it relies on one of the few results that is usually not contested in quantum gravity, as it is derived from a semi-classical approximation.

Finally, let's just note that among the topological solutions of this model, among which many are singular, the spacetime characterized by a_+ with $k = -1$ and $r < 0$ is a regular dynamical spacetime which is "de Sitter"-like for both $r \to 0^-$ and $r \to -\infty$:

$$\begin{aligned} a_{(+\,;\,k=-1)} \left(r \to 0^-\right) &= -1 - \frac{r^2}{2\mathfrak{a}} + \frac{(\sigma - 1)^2 r^4}{8\mathfrak{a}M^2} + O\left(\frac{r^5}{\mathfrak{a}M^3}\right) \\ a_{(+\,;\,k=-1)} \left(r \to -\infty\right) &= -\frac{r^2}{2\mathfrak{a}} - 1 + \frac{\mathfrak{a}(\sigma - 1)^2}{2r^2} + O\left(\frac{\mathfrak{a}^2 M}{r^5}\right) \end{aligned} \tag{383}$$

This is a solution because for any $r < 0$ and for $k = -1$, $\frac{e + \sqrt{e^2 - 4\mathfrak{a}k\, r^4(\sigma - 1)}}{2r^3} > 0$. In both these limits, this geometry is given by Eq(375), but note that its limit of vanishing "mass" parameter $M \to 0$ reduces to the vacuum (374) only for $r^3 + \mathfrak{a}r(\sigma - 1) < 0$, while the branch $r^3 + \mathfrak{a}r(\sigma - 1) > 0$ of this solution gives $a_{(+\,;\,k=-1)} (M \to 0) = -1 - \frac{2rM}{r^2 + \mathfrak{a}(\sigma - 1)}$. Therefore, this solution is curiously discontinuous for $M = 0$, because it "intersects" both hyperbolic vacua $a = -1$ and (374).

3. Second modification of Lovelock gravity & bouncing black hole :

In the previous section, we saw that when the overall constant w_p, that determines the "shape" of the series of high-energy corrections to GR, is chosen so that the FLRW sector of the theory gives the LQC bounce, there is a very simple one-parameter family of deformations of this spherically symmetric sector of Lovelock-Lanczos gravity, given by $\zeta = (1 - \sigma)\, p^2 + \sigma p$, which admits also an (A)dS-core regular black hole solution which satisfies the principle of Limiting curvature and contains the correct quantum correction to the Newton potential provided that the unique spherically symmetric vacuum is Minkowski spacetime.

However, it is not very clear if this is a general properties of these ($\zeta \neq p$) modifications of Lovelock-Lanczos gravity. In this section, we will see that another very simple modification also provides a regular black hole counterpart to the LQC bounce, but this time, the geometry of the correction to the Schwarzschild black hole will be a "single-horizon" regular black hole for large mass (similar to the RBHs eq(209, 256, 285, 308) that we studied previously), and a two horizon regular black hole for low mass. Once again, it will have Limiting Curvature and the condition to have a unique spherically symmetric vacuum given by Minkowski spacetime will be equivalent to have the correct sign in the Quantum correction to Newton potential.

We choose here the following deformation ζ, keeping once again the same w_p, and we will proceed similarly to the previous section :

$$w_p = -2\binom{1/2}{p}\left(-\frac{3}{2\pi}\right)^{p-1} l^{2(p-1)} \quad , \quad z = 1 \quad , \quad \zeta = (1-\sigma) + \sigma p \tag{384}$$

With these choices, the Eq(352) becomes :

$$-2kr(\sigma - 1) + \frac{2r^3}{\mathfrak{a}}\left(\frac{\sigma k - a}{k - a}\right)\left(1 - \sqrt{1 - \frac{2\mathfrak{a}}{r^2}(k - a)}\right) = 4M \tag{385}$$

where $\sigma = 1$ still corresponds to the spherically symmetric sector of Lovelock-Lanczos gravity with our choice of w_p. In order to ensure that the equation is real, we set once again

$$a = k - \frac{r^2}{2\mathfrak{a}} + \frac{r^2\alpha(r)^2}{2\mathfrak{a}} \tag{386}$$

what allows to rewrite the previous equation in terms of α as :

$$\frac{2r}{\mathfrak{a}\,(1+|\alpha|)}\,(1-|\alpha|)\left(r^2(1+|\alpha|) + \mathfrak{a}k(\sigma - 1)\right) = 4M \tag{387}$$

We first focus on the vacua of this theory $M = 0$. Considering a spherical horizon topology $k = 1$, we see that, provided $\sigma > 1$, the only spherical vacuum is Minkowski spacetime given by $\alpha = 1$. Thus, we choose this range for σ. For $k = 0$, there is no solution and for $k = -1$ we have the usual $a = -1$ topological solution of GR and the additional singular vacuum $a = -\sigma + \frac{\mathfrak{a}(\sigma-1)^2}{2r^2}$ for $-\sqrt{\mathfrak{a}(\sigma - 1)} < r < \sqrt{\mathfrak{a}(\sigma - 1)}$, because in order to be solution we need $|\alpha| = -1 + \frac{\mathfrak{a}(\sigma-1)}{r^2} > 0$. From now on, we will only discuss the spherical massive solutions, because as we see, this topological solution is quite pathological.

The previous field equation can be written in the following form, in which the solutions are simple to find :

$$\frac{-2r^3}{\mathfrak{a}(1+|\alpha|)}\left(|\alpha| - \frac{e - 2r^3 - \sqrt{e^2 + 8k\mathfrak{a}r^4(\sigma - 1)}}{2r^3}\right)\left(|\alpha| - \frac{e - 2r^3 + \sqrt{e^2 + 8k\mathfrak{a}r^4(\sigma - 1)}}{2r^3}\right) = 0 \tag{388}$$

where $e(r) := 2r^3 - \mathfrak{a}\,(2M + kr\,(\sigma - 1))$. In this form, we see directly that for spherical topology and $\sigma > 1$ there is no issue regarding the square-root. Then, in order to have access easily to the values of r such that $\frac{e-2r^3-\sqrt{e^2+8k\mathfrak{a}r^4(\sigma-1)}}{2r^3} > 0$ or $\frac{e-2r^3+\sqrt{e^2+8k\mathfrak{a}r^4(\sigma-1)}}{2r^3} > 0$, note that :

$$\frac{e - 2r^3 \pm \sqrt{e^2 + 8k\mathfrak{a}r^4(\sigma - 1)}}{2r^3} = \frac{-(2\mathfrak{a}v + u) \pm \sqrt{(2\mathfrak{a}v + u)^2 - 8\mathfrak{a}r^3 v}}{2r^3} \tag{389}$$

where $u(r) = r\left(r^2 + 2\mathfrak{a}(\sigma - 1)\right)$, $v(r) = M + \frac{r}{2}\left((1-\sigma) - \frac{r^2}{\mathfrak{a}}\right)$ therefore when $k = 1$, we need $v \le 0$ in order to have the second solution, while the first one is a solution for all $r < 0$, but describes a naked singularity like in the previous section. We therefore focus on the $r > 0$ region. One can check that $v \le 0$ corresponds to the following bound for the radial coordinate :

$$r \ge l := \frac{\mathfrak{a}^{1/3}}{\sqrt{3}}\left(\left(3\sqrt{3}M + \sqrt{27M^2 + \mathfrak{a}(\sigma - 1)^3}\right)^{1/3} - \frac{a^{1/3}(\sigma - 1)}{\left(3\sqrt{3}M + \sqrt{27M^2 + \mathfrak{a}(\sigma - 1)^3}\right)^{1/3}}\right) \tag{390}$$

For this range, we get the following solution as a modification of Schwarzschild geometry :

$$a(r) = 1 - \frac{r^2}{2\mathfrak{a}} + \frac{\left(-2r^3 + e + \sqrt{e^2 + 8\mathfrak{a}r^4(\sigma - 1)}\right)^2}{8\mathfrak{a}r^4} \tag{391}$$

We see that it is quite similar to the regular black hole of the previous section. For example, it also shares the properties to have the correct (sign of the) quantum correction to the Newton potential, provided that the unique spherical vacuum is Minkowski spacetime (as we saw, it amounts to have $\sigma > 1$) :

$$a(r \longrightarrow \infty) = 1 - \frac{2M}{r} + \frac{\mathfrak{a}M\,(\sigma-1)}{r^3} + o\left(1/r^3\right) \tag{392}$$

Let's consider for now this spacetime for all $r > 0$ (even if the region $r < l$ is not solution of the theory). Interestingly, as in the previous section, it is also an (A)dS-core regular black hole because it satisfies the Sakharov criterion :

$$a(r \longrightarrow 0^+) = 1 + \frac{(1-\sigma)\,r^3}{\mathfrak{a}M} + o\left(r^3\right) \tag{393}$$

Therefore, this is very similar to what happened for the Modesto semi-polymeric black hole (256) : even it is regular, only a part of the metric was actually solution of the considered theory. In order to preserve geodesic completeness, we therefore needed to "double" the allowed region, what forced us to consider the metric (274). In our case, we need to "double" the $r \geq l$ region, what will end-up describing a bouncing black hole interior.

But first we want to note that the principle of Limiting Curvature is satisfied for this metric. First, it is satisfied for the whole geometry $r > 0$. Indeed, the Ricci scalar can be written as :

$$\begin{aligned} R = &\frac{-4r^9 n_1 + r^6 n_2 + 2r^3(2av+u)n_3 - 10r^{12}(2av+5u) + (2av+u)^3(4av+u)(12av+5u)}{8ar^6\left((2av+u)^2 - 8ar^3v\right)^{3/2}} \\ &+ \frac{r^3-u}{2ar^3} - \frac{\left(4av+r^3+u\right)\left(12av+r^3+5u\right)}{8ar^6} \end{aligned} \tag{394}$$

with $n_1 = \left(46a^2v^2 + 57auv + 5u^2\right)$, $n_2 = \left(168a^3v^3 + 268a^2uv^2 + 194au^2v + 43u^3\right)$, and $n_3 = \left(-256a^3v^3 - 140a^2uv^2 - 2au^2v + 5u^3\right)$. We do not write the exact expression of the scalar $R_{\mu\nu}R^{\mu\nu}$ because it is quite long, but it can be derived using the general formula of these invariants. Then, one can check that

$$\begin{aligned} R(r/M \longrightarrow 0^+/\infty) &= \frac{20(\sigma-1)r}{\mathfrak{a}M} + O\left(\frac{r^2}{\mathfrak{a}M^2}\right) \\ R_{\mu\nu}R^{\mu\nu}(r/M \longrightarrow 0^+/\infty) &= \frac{104(\sigma-1)^2r^2}{\mathfrak{a}^2M^2} + O\left(\frac{r^3}{\mathfrak{a}^2M^3}\right) \end{aligned} \tag{395}$$

where the notation means that the both limits $r \longrightarrow 0^+$ and $M \longrightarrow \infty$ are equivalent. Therefore the full spacetime satisfies the principle of Limiting Curvature. The question is therefore if the actual solution $r \geq l$ does as well. This is somewhat unclear because the point $r = l$ of the full geometry at which it becomes a solution of the theory depends on the value of the mass. In order to see that the solution does have Limiting curvature, note that

$$l\,(M \longrightarrow \infty) = (2\mathfrak{a}M)^{1/3} + O\left(\frac{\mathfrak{a}^{2/3}}{M^{1/3}}\right) \tag{396}$$

To evaluate the scalars at $r = l$, a simple way is to replace the mass by l using that $v(l) = 0 = M + \frac{l}{2}\left((1-\sigma) - \frac{l^2}{\mathfrak{a}}\right)$. Moreover, due to the previous equation, instead of taking the limit $M \to \infty$ we can as well consider $l \to \infty$. Doing so gives :

$$\begin{aligned} R(r=l) &= \frac{-3l^4 + 18\mathfrak{a}l^2(\sigma-1) + 23\mathfrak{a}^2(\sigma-1)^2}{\mathfrak{a}\left(l^2 + 2\mathfrak{a}(\sigma-1)\right)^2} \underset{l\to\infty}{\longrightarrow} -\frac{3}{\mathfrak{a}} \\ R_{\mu\nu}R^{\mu\nu}(r=l) &= \frac{5\left(53\mathfrak{a}^4(\sigma-1)^4 + 84\mathfrak{a}^3l^2(\sigma-1)^3 + 24\mathfrak{a}^2l^4(\sigma-1)^2 + 9l^8\right)}{2\mathfrak{a}^2\left(2\mathfrak{a}(\sigma-1)+l^2\right)^4} \underset{l\to\infty}{\longrightarrow} \frac{45}{2\mathfrak{a}^2} \end{aligned} \tag{397}$$

This proves that this black hole geometry is such that its curvature invariants are bounded by a universal constant, thus satisfying the principle of Limiting Curvature.

As we said previously, in order to have a geodesically complete spacetime, we need to extend the regular solution (391), what can be done by applying the following change of coordinate :

$$r = \sqrt{x^2 + l^2} \tag{398}$$

where the point $r = l$ is now located by $x = 0$. Then the spacetime interval becomes :

$$ds^2 = -a\left(\sqrt{x^2 + l^2}\right) dt^2 + \frac{x^2\, dx^2}{\left(x^2 + l^2\right) a\left(\sqrt{x^2 + l^2}\right)} + \left(x^2 + l^2\right) d\Omega^2 \tag{399}$$

This coordinate system is useful to understand what happens at the point $x = 0$, while in order to locate the horizons for $x > 0$ and $x < 0$, we will use the previous radial coordinate r.

The radial null curves are characterized by

$$\frac{dx}{dt} = \pm\frac{a\left(\sqrt{x^2 + l^2}\right)\sqrt{x^2 + l^2}}{x} \tag{400}$$

so that the horizons are located when this expression vanishes. At $x = 0$, it becomes :

$$\frac{a\left(\sqrt{x^2 + l^2}\right)\sqrt{x^2 + l^2}}{x} = \frac{a(l)l}{x} + O\left(x\right) \tag{401}$$

with $a(l) = 1 - \frac{l^2}{2\mathfrak{a}}$, because $v(l) = 0$. Therefore, if $l = \sqrt{2\mathfrak{a}}$, what corresponds to a mass $M = \frac{\sqrt{\mathfrak{a}}(\sigma+1)}{\sqrt{2}}$, we have around $x \approx 0$, $\frac{dx}{dt} = \mp\frac{x}{\sqrt{2\mathfrak{a}}} + O\left(x^2\right)$, so that there is an horizon at $x = 0$. Otherwise, the horizons are located at r_H, the zeros of $a(r)$ for $r > l$, corresponding to $x_H = \pm\left(r_H^2 - l^2\right)$ in the $x > 0$ and $x < 0$ regions.

When $M > \frac{\sqrt{\mathfrak{a}}(\sigma+1)}{\sqrt{2}}$, there is only one horizon which for large mass is located at the Schwarzschild radius $r = 2M$. In this case the whole interior region of the black hole is dynamical. As far as we know, this is the first example of a "one"-horizon regular black hole satisfying the principle of Limiting Curvature, and having the proper quantum correction to the Newton potential. In [108], another non-singular black hole with a dynamical inner region and having Limited curvature, was found. Also in the case of the Mimetic gravity theory studied in this paper, the cosmological counterpart of the black hole is the LQC bounce.

Though, in the present case, there exists a narrow range of mass values,

$$M_0 := \frac{\sqrt{\mathfrak{a}}}{6\sqrt{3\sigma}}\sqrt{-1 - \sigma^3 + 33\sigma(1+\sigma) + (1 + \sigma(14+\sigma))^{3/2}} < M < \frac{\sqrt{\mathfrak{a}}(\sigma+1)}{\sqrt{2}} \tag{402}$$

for which the geometry also contains an inner horizon, meaning that the spacetime suffers from mass-inflation and is unstable in this case. When the lower bound is reached by the mass $M = M_0$, there is one single extremal horizon with static regions on each side, while for $M < M_0$, the spacetime is horizonless. Note that regarding mass inflation, it might be an improvement to have two stable regimes (one corresponding to a single dynamical interior region and the other to an extremal/horizonless spacetime) of the mass bounding the unstable one because if spherical symmetry is preserved during the instability, the unstable regime should drive the dynamics towards one of the two stable ones.

D. Asymptotic Safety black holes & regular past de Sitter cosmology

In this section, we will study another "shape factor" w_p, and we will see that it can provide an effective Action for the Asymptotic Safety black hole derived in [54]. In this paper, the idea was to use the Asymptotic Safety techniques to find an effective correction to Schwarzschild geometry, and it was found to be regular at the origin, replacing the singularity at time $r = 0$ by a de Sitter core at radius $r = 0$. In substance, within an Einstein-Hilbert truncation of the gravitational action (where for $n > 1$, the R^n-types curvature scalars are not taken into account), one solves the renormalization group equation in order to find the dependence of the Newton constant G_0 on the momentum scale k at which an experiment is performed, what gives a running Newton constant $G(k)$:

$$G(k) = \frac{G_0}{1 + \omega G_0 k^2} \tag{403}$$

Two attractive fixed points are found, one perturbative for $k^2 \to 0$ and a non-perturbative one for $k^2 \to \infty$ so that $G(k \to 0) = G_0$ and $G(k \to \infty) \approx \frac{1}{\omega k^2}$.

Then, the idea is to mimic what is done in Quantum Electrodynamics to find the effective correction to the Coulomb potential : after finding the momentum dependence of the effective electric charge $e(k/k_0)$, one identifies the high energy momentum scale with the inverse of the distance $e(r_0/r)$. However, General Relativity being invariant under change of coordinates, one should use an invariant notion of the distance. Thus, instead of the identification $k \to 1/r$, one can use the proper distance of the Schwarzschild geometry, what gives $k \to \xi/\left(\int_{\mathcal{C}} \sqrt{|ds^2|}\right) =: \xi/d(r)$:

$$G(r) = \frac{G_0 d(r)^2}{d(r)^2 + \omega\xi^2 G_0} \tag{404}$$

where ξ is a numerical factor. The exact result depends on the curve, but the important asymptotic behaviours $r \to 0$ and $r \to \infty$ seem independent of this choice and have to be given by

$$d(r \to \infty) = r + O\left(\frac{1}{r}\right) \quad , \quad d(r \to 0) = \frac{r^{3/2}}{\sqrt{\nu G_0 M}} + o\left(r^{3/2}\right) \tag{405}$$

where $\nu > 0$ is a numerical factor. Therefore, the following choice was made

$$d(r) = \sqrt{\frac{r^3}{r + \nu G_0 M}} \tag{406}$$

to satisfy the previous conditions. It gives the following correction to Schwarzschild geometry :

$$a = 1 - \frac{2G_0 M r^2}{r^3 + \omega\xi^2 G_0 \left(r + \nu G_0 M\right)} \tag{407}$$

For additional references concerning AS black holes, see for example [55–63].

We will now see that, if we consider the same 1-parameter deformations of the DSS of Lovelock-Lanczos theories as before, i.e. considering the action (344) with the choices : $z = 1$, $\zeta = (1-\sigma) + \sigma p$ or $\zeta = (1-\sigma)\,p^2 + \sigma p$, but with a different "shape" factor w_p that the one that gave the LQC bounce, it is possible to obtain the previous regular black hole solution for the first choice of ζ, while for any ζ, the associated cosmological solution describes a non-singular universe where the big bang singularity is replaced by a past eternal de Sitter region $a(t \to -\infty) = e^{\frac{2\sqrt{2\pi}t}{l}}$. For late times, it gives the GR results. Moreover, we show that the second choice of ζ, given by $\zeta = (1-\sigma)\,p^2 + \sigma p$ has two regular spherically symmetric solutions : one gives a RBH solution satisfying the "Asymptotic Safety conditions" (405), while the other is an anisotropic quasi-de Sitter spacetime.

1. Asymptotic Safety black hole

Consider the following theory, which is the same one-parameter modification of the DSS of each Lovelock-Lanczos terms as (384), but with a different "shape factor" w_p :

$$w_p = -\left(\frac{2}{\gamma}\right)^{1-p} l^{2(p-1)} \quad , \quad z = 1 \quad , \quad \zeta = (1-\sigma) + \sigma p = p + \frac{2\delta(p-1)}{\gamma} \quad , \quad \sigma = 1 + \frac{2\delta}{\gamma} \tag{408}$$

Then, in spherical symmetry, performing the summation in Eq(352) gives the following integrated field equation for $a(r)$:

$$\frac{4r\left(r^2 + \delta k l^2\right)(k-a)}{2r^2 - \gamma l^2\,(k-a)} = 4M \tag{409}$$

The solution is given by :

$$a = k - \frac{2Mr^2}{r^3 + l^2\left(\delta k r + \gamma M\right)} \tag{410}$$

Therefore, it gives precisely the Asymptotic Safety black hole (407), providing that we identify $\delta = \omega\xi^2 G_0/l^2$, $\gamma = \omega\nu\xi^2 G_0/l^2$ and $M \to G_0 M$. Note that this black hole is a slightly modified

Hayward one [194], which was found to be a solution of a Lovelock Designer theory in [170, 171]. This is how we realized that the AS black hole could also be found from Lovelock-like gravities.

We will not study in details this black hole because it was already done in [54], but in order to compare with our previous results, note that it is an (A)dS-core regular black hole with Limiting Curvature and, provided that there is no singularity at the denominator for spherical topology ($\delta > 0$, $\gamma > 0$), it has the correct quantum correction to the Newton potential :

$$
\begin{aligned}
a\left(r/M \to 0/\infty\right) &= k - \frac{2r^2}{\gamma l^2} + \frac{2k\delta r^3}{\gamma^2 M l^2} + O\left(\frac{r^4}{\gamma^3 l^2 M^2}\right) \\
a\left(r \to \infty\right) &= k - \frac{2M}{r} + \frac{2k\delta l^2 M}{r^3} + O\left(\frac{l^2 M^2}{r^4}\right)
\end{aligned}
\tag{411}
$$

Note that considering positive γ and δ here means that $\sigma = 1 + \frac{2\delta}{\gamma} > 1$, what is the same condition that we had in the "LQC black hole" case Eq(384). Moreover, as long as

$$
M > \frac{2\sqrt{3}\delta^{3/2} l}{9\gamma} \tag{412}
$$

then $r^3 + l^2\left(-\delta r + \gamma M\right) > 0$, so that the hyperbolic topology solution ($k = -1$) is also regular for all $r > 0$. Note that M is not interpreted as the mass of the system in this case. Finally, whatever the mass, the $k = 0$ planar solution is always regular and given by :

$$
a = -\frac{2Mr^2}{r^3 + l^2 \gamma M} \tag{413}
$$

This is quite in contrast with the topological solutions found previously in the case where w_p is chosen, so that the cosmological solution gives the LQC bounce, because many of these were singular. Finally, the condition to have horizons in the spherical case is :

$$
M \geq \frac{l}{8}\sqrt{9(4\delta + 3\gamma) + \frac{\sqrt{2\delta + \gamma}(2\delta + 9\gamma)^{3/2} - 4\delta^2}{\gamma}} \tag{414}
$$

When the bound is saturated, the black hole describes an extremal remnant.

2. Regular past de Sitter universe

We will now study the cosmological FLRW sector of this theory. As we already know, the specific choice of ζ does not influence this sector, so that the following results are also valid for a theory admitting exactly the LL spherically symmetric decomposition ($\zeta = p$ or $\delta = 0$). Performing the summation in Eq(356) with our choice (408) of w_p leads to :

$$
\frac{2H^2}{2 - \gamma l^2 H^2} = \frac{8\pi\rho}{3} \tag{415}
$$

We see that the limit $\rho \to \infty$ corresponds to a geometry satisfying $H^2 = \frac{2}{\gamma l^2}$, i.e. a de Sitter spacetime with an effective cosmological constant $\Lambda_{\text{eff}} = \frac{6}{\gamma l^2}$. Thus, we can already hint that the full solution will be de Sitter-like at $t \to -\infty$, geodesically complete, but will have unbounded energy density ρ. Let's see this in more details. Solving for the Hubble parameter gives :

$$
H^2 = \frac{8\pi\rho}{3 + 4\pi\gamma l^2 \rho} \tag{416}
$$

For simplicity, we define $\mathfrak{l} = 2\sqrt{\pi\gamma}\, l$. Then we can integrate Eq(359) to find the relation between the scale factor and the time coordinate :

$$
\frac{\mathfrak{l}}{3\sqrt{2\pi}(w+1)}\left(\sqrt{1 + \frac{3a^{3(w+1)}}{\mathfrak{l}^2}} + \log\left(\frac{2a^{3(w+1)/2}}{\mathfrak{l}\left(1 + \sqrt{1 + \frac{3a^{3(w+1)}}{\mathfrak{l}^2}}\right)}\right)\right) = t + t_0 \tag{417}
$$

Therefore, we see that the scale factor goes to zero only asymptotically when $t \to -\infty$, and behaves as an exponential of the time, what indicates that the big bang is indeed replaced by a past-eternal de Sitter spacetime, that suddenly enters the GR regime. For example, for respectively dust $w = 0$ and radiation $w = 1/3$, we can expand for a scale factor close to zero :

$$
\begin{aligned}
\frac{\mathfrak{l}\left(2+\log\left(\frac{a^3}{\mathfrak{l}^2}\right)\right)}{6\sqrt{2\pi}} + O\left(\frac{a^3}{\mathfrak{l}}\right) &= t \\
\frac{\mathfrak{l}\left(1+\log\left(\frac{a^2}{\mathfrak{l}}\right)\right)}{4\sqrt{2\pi}} + O\left(\frac{a^4}{\mathfrak{l}}\right) &= t
\end{aligned}
\tag{418}
$$

Solving for a gives

$$
\begin{aligned}
a_{w=0}\left(t \to -\infty\right) &= \mathfrak{l}^{2/3}\, e^{\frac{2\sqrt{2\pi}t}{\mathfrak{l}}} \\
a_{w=1/3}\left(t \to -\infty\right) &= \sqrt{\mathfrak{l}}\, e^{\frac{2\sqrt{2\pi}t}{\mathfrak{l}}}
\end{aligned}
\tag{419}
$$

As de Sitter spacetime is geodesically complete, this metric is as well. At late time, we have

$$
\frac{a^{3(w+1)/2}}{(w+1)\sqrt{6\pi}} - \frac{a^{-3(w+1)/2}\mathfrak{l}^2}{6(w+1)\sqrt{6\pi}} + O(\mathfrak{l}^3) = t \tag{420}
$$

where we have reabsorbed the constant by shifting the time coordinate. At zeroth order, it gives the well known $a \propto t^{2/(3(w+1))}$ behaviour of General Relativity, what shows that this spacetime indeed interpolates between a past-eternal de Sitter phase and the GR one, for any state parameter $w > -1$.

With this kind of solutions, we see that what happens at the Planck scale is very different than for a bounce. In this second case, the Planck scale indicates the minimal value of the scale factor and the maximal value of the curvature. Understood in terms of the LQG results, this can be related to the fundamental discreteness of area and volume in this quantum theory [16]. However, in the first case, there is no minimal value for the scale factor and the Planck scale indicates when the geometry enters a dS phase, for which the curvature reaches its maximal and fixed value given by an effective cosmological constant

$$
\Lambda_{\text{eff}} = \frac{6}{\gamma l^2} = \frac{6}{\nu\xi^2\omega G_0} \tag{421}
$$

where in the last equality we made the same identification as we did in the previous section, so that the black hole solution is exactly the same as the AS one of [54].

Finally, it might be interesting to put this kind of metrics in perspective with inflation, in order to understand if they could have similar features with some models of this phenomenon, at the background level. Note however that in the present case, it is not the presence of additional fields like the inflaton that produces the dS expansion and transition toward the GR phase, but rather an high energy curvature modification of the FLRW sector of GR, that leads to second order corrections to Friedmann equations.

This is once again very similar to what was found in the case of Asymptotically Safe cosmology, where a so-called Non-Gaussian fixed point driven inflationary era was found to arise without the need of the inflaton [229–233]. See also [36–40] for additional references on AS cosmology.

Finally, note that in a very recent paper by Chamseddine, Mukhanov and Russ [358], it was found a very similar non-singular cosmological solution from a Mimetic model, that was considered to be a (covariant) model for the running of the cosmological and Newton constants. Therefore, it provides another link between these kinds of past-eternal de Sitter spacetimes and Asymptotic Safety-like approaches, where quantum effects are captured by the running of the coupling constants of the theory.

Interestingly, both in our case and in [358], the running is restricted to the Einstein-Hilbert truncation of the quantum effective action. Indeed, the AS black hole of [54] from which we have reconstructed the theory Eq(408) was found within this specific truncation, and from our results, we can then conjecture that the associated cosmological solution Eq(417) should come from the same truncation as well.

However, it is to be expected that contrary to our case, in which both the black hole and cosmological solutions of the theory are similar (or identical for the BH) to Asymptotic Safety results, the mimetic theory proposed in [358] probably does not admit an AS-like black hole, just like the mimetic theory developed in [359, 360] by the same authors admit the LQC bounce, but its associated mimetic regular black hole is very different from polymer-like ones, see [113].

3. Second modification, Asymptotic Safety black hole & anisotropic quasi-de Sitter solution

Now we wish to consider the same 1-parameter deformation of the DSS sector of the Lovelock scalars given by (369), and we will see that also for the present choice of w_p, it gives regular solutions. We have :

$$w_p = -\left(\frac{2}{\gamma}\right)^{1-p} l^{2(p-1)} \quad , \quad z = 1 \quad , \quad \zeta = (1-\sigma)\, p^2 + \sigma p = p - \frac{2\delta(p-1)p}{\gamma} \quad , \quad \sigma = 1 + \frac{2\delta}{\gamma} \tag{422}$$

So that Eq(352) becomes :

$$\frac{4\,(k-a)\left(2r^4 + l^2\left(k\left(\delta\gamma k l^2 - r^2(4\delta+\gamma)\right) + \gamma\left(r^2 - \delta k l^2\right)a\right)\right)}{\left(2r^2 - \gamma l^2(k-a)\right)^2} = 4M \tag{423}$$

First focusing on the vacua ($M = 0$), we see that they belong to two classes :

$$a = k \quad , \quad a = k - \frac{2r^2}{\gamma l^2}\left(\frac{r^2 - 2\delta k l^2}{r^2 - \delta k l^2}\right) \tag{424}$$

Interestingly, the second vacuum, which exists for $\delta \neq 0$, is also regular for spherical topology $k = 1$, providing that $\delta < 0$. We therefore choose this range for this parameter. In the following, we will only consider spherical topology. In this case, for respectively $\gamma > 0$ or $\gamma < 0$, this vacuum describes quasi de Sitter or anti-de Sitter metrics because :

$$\begin{aligned} a\,(r\to 0) &= k - \frac{4r^2}{\gamma l^2} + O\left(\frac{r^4}{l^4}\right) \\ a\,(r\to\infty) &= -\frac{2r^2}{\gamma l^2} + k\left(1 + \frac{2\delta}{\gamma}\right) + O\left(\frac{l^2}{r^2}\right) \end{aligned} \tag{425}$$

So while the static region is (A)dS near $r \approx 0$, there is a slight deformation to (A)dS for $r \to \infty$, parametrized by δ/γ. In the de Sitter case, there is a cosmological horizon, that we can locate at $a(r_H) = 0$, what gives :

$$r_H = \frac{l}{2}\sqrt{4\delta + \gamma + \sqrt{16\delta^2 + \gamma^2}} \tag{426}$$

In both cases, the Ricci and Kretschmann invariants are given by :

$$\begin{aligned} R &= \frac{4\left(-12\delta^3 l^6 + 21\delta^2 l^4 r^2 - 19\delta l^2 r^4 + 6r^6\right)}{\gamma l^2\left(r^2 - \delta l^2\right)^3} \\ R^{\alpha\beta}_{\mu\nu} R^{\mu\nu}_{\alpha\beta} &= \frac{16\left(24\delta^6 l^{12} - 84\delta^5 l^{10} r^2 + 165\delta^4 l^8 r^4 - 160\delta^3 l^6 r^6 + 103\delta^2 l^4 r^8 - 38\delta l^2 r^{10} + 6r^{12}\right)}{\gamma^2 l^4\left(r^2 - \delta l^2\right)^6} \end{aligned} \tag{427}$$

Similarly, we can check that the massive solutions are also regular. They are given by :

$$a_\pm = k - \frac{r^2}{\gamma l^2} + \frac{r^3}{\gamma}\left(\frac{\delta k - \frac{\gamma M}{r} \pm \sqrt{\left(2\delta k - \frac{r^2}{l^2}\right)^2 - 4\delta\gamma k \frac{M}{r}}}{r^3 + l^2(\gamma M - \delta k r)}\right) \tag{428}$$

Therefore, in order not to have any issue with the square-root and denominator for spherical topology and $r > 0$, we can choose $\gamma > 0$, meaning that the previous vacuum is quasi dS (instead of quasi AdS). For $k = 1$, the previous solutions satisfy the Sakharov criterion at $r = 0$ and has Limiting curvature :

$$a_\pm\,(r\to 0) = 1 - \frac{2r^2}{\gamma l^2} \pm \frac{2\sqrt{\delta\gamma}\, r^{5/2}}{\gamma^2 l^2\sqrt{M}} + O\left(\frac{r^{7/2}}{l^2 M^{3/2}}\right) \tag{429}$$

Moreover, like the previous solutions that we have found, this black hole solution a_+ has the correct correction to Newton potential,

$$a_+(r \to \infty) = 1 - \frac{2M}{r} - \frac{4\delta l^2 M}{r^3} + O\left(\frac{l^2 M^2}{r^4}\right) \tag{430}$$

while the massive quasi dS solution behaves as :

$$a_-(r \to \infty) = -\frac{2r^2}{\gamma l^2} + \left(1 + \frac{2\delta}{\gamma}\right) + O\left(\frac{l^2}{r^2}\right) \tag{431}$$

We see that the mass does not affect the quasi dS vacuum at this order. Actually the first correction which does is $O\left(Ml^4/r^5\right)$, so that this vacuum is barely affected by the presence of mass.

To conclude this section, we will establish that the black hole a_+ is an "Asymptotic Safety black hole" in the same sense as the previous one Eq(407), as it satisfies the conditions (405). Indeed, consider the following equation

$$a_+(r) = 1 - \frac{2MG(r)}{r} \tag{432}$$

where $G(r)$ is given by the AS results (404). Then, solving the previous equation for the distance function $d(r)$, one can check that with the identifications $\delta = -\frac{\omega\xi^2 G_0}{2l^2}$, $\gamma = \frac{\omega\xi^2\nu G_0}{l^2}$ and $M \to G_0 M$, it is given by :

$$d(r) = \sqrt{\frac{2\omega\xi^2 r^3 G_0}{-r^3 + \omega\xi^2 G_0\left(2\nu M G_0 + r\right) + \sqrt{r}\sqrt{r^5 + \omega\xi^2 G_0\left(2r^3 + \omega\xi^2 G_0\left(2\nu G_0 M + r\right)\right)}}} \tag{433}$$

which indeed satisfies the conditions (405). In this case the solution can be written as :

$$a_+ = 1 - \frac{r^2}{\nu\tilde{\omega}G_0}\left(1 + \frac{\tilde{\omega}G_0\left(2\nu G_0 M + r\right) - 2\sqrt{r^6 + \tilde{\omega}G_0 r\left(2r^3 + \tilde{\omega}G_0\left(2\nu G_0 M + r\right)\right)}}{2r^3 + \tilde{\omega}G_0\left(2\nu G_0 M + r\right)}\right) \tag{434}$$

where $\tilde{\omega} := \omega\xi^2$.

Concerning the second quasi dS solution, note that it describes a non-perturbative solution in the length scale l (or in G_0 with the previous identifications) that does not have a General Relativity counterpart, because when $l \to 0$ the solution behaves as (431). Therefore, it could be interesting to put it in perspective with a particular point, that was noted in [54], concerning the fact that there are renormalization group trajectories, those for which $g(k) > g_*^{\mathrm{UV}}$, where $g(k)$ is the dimensionless running Newton constant ($g(k) := k^{d-2}G(k)$) and $g_*^{\mathrm{UV}} = 1/(\omega + 2/(3\pi))$ the UV fixed point, that precisely describes strong coupling solutions that "do not connect to a perturbative large distance regime"[18]. The solution a_- could therefore be a candidate for such strong regime metric.

E. Other regular solutions

Given the previous results reproducing some effective quantum gravity results, it might be interesting to understand better the general choices of w_p and ζ that could lead to regular solutions, in order to see if the regularity is specific to our choices or more general. A point in favour of the second possibility is that it seems actually quite simple to find choices of w_p that lead to regular cosmologies.

1. Regular bounce & past-eternal de Sitter cosmologies

As a first case of study, we consider theories admitting bouncing solutions. For example, the following theory :

$$w_p = -4\binom{1/4}{p}\left(-\frac{3}{2\pi}\right)^{p-1} l^{2(p-1)} \quad , \quad z = 1 \tag{435}$$

[18]This is a quote from [54].

which gives the following modification to Friedmann equation :

$$\frac{8\pi}{l^2}\left(1-\left(1-\frac{3}{2\pi}l^2H^2\right)^{1/4}\right)=8\pi\rho \tag{436}$$

Therefore, solving the H^2 as we did previously leads to the following modification :

$$H^2=\frac{2\pi\rho}{3}\left(2-l^2\rho\right)\left(1+\left(1-l^2\rho\right)^2\right)\underset{l/\rho\to 0}{\longrightarrow}\frac{8\pi\rho}{3}-4\pi l^2\rho^2+O\left(l^4\rho^3\right) \tag{437}$$

Although it might be more difficult to find analytic solutions in this case, it is actually very similar to the LQC case, because here as well the energy density has a maximal value $\rho=2/l^2$ at which the Hubble factor vanishes, what is the sign that the GR singularity is replaced by a bounce. Of course, it also reduces to the FLRW sector of General Relativity for $l=0$.

Another interesting thing to try would be to reconstruct the w_p that lead to already known (second order) corrections to Friedmann equations coming from LQG, in order to derive their black hole counterparts within this approach. For example, in [366], some refinements of the LQC results were found when the symmetric reduction is applied after the LQG quantization, what results with a different Hamiltonian constraint than LQC. Interestingly, two branches were found :

$$\begin{aligned}
H^2&=\frac{8\pi G\rho}{3}\left(1-\frac{\rho}{\rho_c}\right)\left(1+\frac{\gamma^2}{\gamma^2+1}\left(\frac{\sqrt{\rho/\rho_c}}{1+\sqrt{1-\rho/\rho_c}}\right)^2\right)\\
H^2&=\frac{1}{\lambda^2\left(1+\gamma^2\right)^2}\left(1-\frac{\rho}{\rho_c}\right)\left(1+\left(\frac{1-2\gamma^2+\sqrt{1-\rho/\rho_c}}{4\gamma^2\left(1+\sqrt{\rho/\rho_c}\right)}\right)\frac{\rho}{\rho_c}\right)
\end{aligned} \tag{438}$$

where $\lambda^2=4\sqrt{3}\pi\gamma l_{\text{Planck}}^2$ and γ is the Barbero-Immirzi parameter. Therefore, it should be possible in principle to inverse the relations $X\left(H^2\right)=\frac{8\pi\rho}{3}$ and find the w_p such that $H^2-z\sum_{p=2}^{\infty}w_pH^{2p}=X\left(H^2\right)$. However, in the case of the previous equation the result seems non-analytic, but a numerical study could still be possible.

Let's turn now to the past-eternal dS cosmologies, similar to the one we found previously. Also in this case we can find quite easily some generalizations. For instance, consider

$$w_p=\frac{1}{\alpha+\beta}\left(\gamma+\beta+\frac{\beta(\gamma-\alpha)}{\alpha}\left(-\frac{\alpha}{\beta}\right)^p\right)l^{2(p-1)}\quad,\quad z=1 \tag{439}$$

for which the Friedmann equation is :

$$H^2+\sum_{p=2}^{\infty}w_pH^{2p}=\frac{H^2\left(\beta+\gamma l^2H^2\right)}{\left(1-l^2H^2\right)\left(\beta+\alpha l^2H^2\right)}=\frac{8\pi\rho}{3} \tag{440}$$

Considering $\alpha>0$, $\beta>0$ and $\gamma>0$ to stay as close to the previous case as possible, and setting $\mathfrak{a}=4\pi l^2$, we get :

$$H^2=\frac{2\pi}{\mathfrak{a}(3\gamma+2\alpha\mathfrak{a}\rho)}\left(2(\alpha-\beta)\mathfrak{a}\rho-3\beta+\sqrt{9\beta^2+12\beta\mathfrak{a}(2\gamma-\alpha+\beta)\rho+4(\alpha+\beta)^2\mathfrak{a}^2\rho^2}\right) \tag{441}$$

It is then possible to find the relation between the scale factor and the time coordinate. As a simple case, setting $\alpha=\beta=\gamma/2=1$ gives

$$\begin{aligned}
&\frac{1}{6\sqrt{\pi}(w+1)}\left(Y+\sqrt{\mathfrak{a}}\left(2\left(\sqrt{2}\arctan\left(\frac{Y}{2\sqrt{2\mathfrak{a}}}\right)-\arctan\left(\frac{Y}{2\sqrt{\mathfrak{a}}}\right)\right)+\log\left(\frac{Y-2\sqrt{\mathfrak{a}}}{Y+2\sqrt{\mathfrak{a}}}\right)\right)\right)\\
&=t+t_0
\end{aligned} \tag{442}$$

where $Y=\sqrt{3a^{3(w+1)}+\sqrt{16\mathfrak{a}^2+48\mathfrak{a}a^{3(w+1)}+9a^{6(w+1)}}}$. Once again, this describes a de Sitter phase for small values of the scale factor and GR otherwise.

Therefore, there are whole families of models admitting this kind of solutions, and one could generalize even more Eq(440) with general ratio of polynomials of H^2.

2. Fully regular DSS Lovelock Designer theory from KMT regular black hole

To conclude our survey of the regular solutions of these kinds of theories, note that there also exist some of these for which all the spherically symmetric solutions and the flat FLRW sector are regular. We do not know for the closed and hyperbolic FLRW. In order to see this, we will use a result that was found in [170] by G. Kunstatter, H. Maeda and T. Taves, in which they used a Lovelock Designer theory ($\zeta = p$) to find the following (A)dS-core black hole solution :

$$a = 1 + \frac{r^2}{l^4 M}\left(r^3 - \sqrt{4l^4M^2 + r^6}\right) \tag{443}$$

To get this solution, the choice of theory is

$$w_p = 2^{-p}\left((-1)^p - 1\right) l^{2(p-1)} \quad , \quad z = 1 \quad , \quad \zeta = p \tag{444}$$

which gives for static spherically symmetric spacetimes (and their topological generalizations) :

$$\frac{8r^5(k-a)}{l^4\left(k + \frac{2r^2}{l^2} - a\right)\left(-k + \frac{2r^2}{l^2} + a\right)} = 4M \tag{445}$$

so that the general solutions are given by :

$$a_\pm = k + \frac{r^2}{l^4 M}\left(r^3 \pm \sqrt{4l^4M^2 + r^6}\right) \tag{446}$$

where a_- corresponds to the previous black hole, because we have

$$a_\pm\left(r \to \mp\infty\right) = k - \frac{2M}{r} + O\left(\frac{l^4M^3}{r^7}\right) \tag{447}$$

We already see from this equation that it does not possess the quantum correction to Newton potential. However, all its topological solutions $k = 1, -1, 0$ for $-\infty < r < \infty$ are regular, whereas in the other cases we usually had naked singularities. Indeed, the behaviour of these solutions is :

$$\begin{aligned} a_\pm\left(r \to 0\right) &= k \pm \frac{2r^2}{l^2} + O\left(\frac{r^5}{Ml^4}\right) \\ a_\pm\left(r \to \pm\infty\right) &= \frac{2r^5}{l^4 M} + k + \frac{2M}{r} + O\left(\frac{l^4M^3}{r^7}\right) \end{aligned} \tag{448}$$

Let's now focus on the solution a^+ for $-\infty < r < \infty$ and hyperbolic topology ($k = -1$), that we denote a_{Hyp}. As we saw, $a_{Hyp}(r \to +\infty) > 0$, while $a_{Hyp}(0) = -1 < 0$, meaning that this solution describes a four dimensional regular black hole with an hyperbolic horizon at some $r_+ > 0$. Instead of being embedded into AdS (corresponding to $a(r \to \infty) = k + \frac{r^2}{l^2} + O\left(\frac{1}{r^3}\right)$) as this kind of solutions usually are, here we have a different power of the radius involved, but the horizon structure remains similar to usual topological black holes. The location of the horizons is given by $a_{Hyp}(r_H) = 0$, and related to the parameter M by

$$M = \frac{2r_H^5}{l^4 - 4r_H^4} \tag{449}$$

Therefore, we have $0 \leq r_+ \leq \frac{l}{\sqrt{2}}$, meaning that this kind of black holes are very small, their sizes being bounded by the value of the coupling constant l which is expected to be Planckian. In addition, due to the asymptote at $r_H = \frac{l}{\sqrt{2}}$, the radius of the object r_+ is almost insensitive to the value of the parameter M. If it is to be related with the physical mass of the object as $M = \alpha M_{\text{phys}} + M^*$, where M^* and α are some constants (as it is usually the case, see for example [367–369]), then it would mean that the location of the outer horizon has almost no dependence on the mass, which is very different from usual spherical or axisymmetric black holes. It might indicate a thermodynamical stability of these objects, but a detailed analysis would be needed.

This solution can have three different horizon structures depending on the value of the parameter M. For $M < \frac{5^{5/4} l}{8\sqrt{2}}$, there is only an outer horizon, so that the interior solution is dynamical and behaves as $a_+(r \to -\infty) = -\left(1 + \frac{2M}{r}\right) + O\left(\frac{l^4 M^3}{r^7}\right)$. For $M = \frac{5^{5/4} l}{8\sqrt{2}}$, the geometry of the interior solution ($r < r_+$) also contains an extremal horizon at $r_- = -\frac{5^{1/4} l}{\sqrt{2}}$, while for $M > \frac{5^{5/4} l}{8\sqrt{2}}$, two inner horizons are present, so that the interior is divided into sequences of static and dynamical regions.

Concerning the flat cosmological solution, it is once again an eternal past-dS regular spacetime. Performing the summation gives,

$$\frac{4H^2}{4 - l^4 H^4} = \frac{8\pi\rho}{3} \tag{450}$$

so that

$$H^2 = \frac{-3 + \sqrt{9 + 64\pi^2 l^4 \rho^2}}{4\pi l^4 \rho} \tag{451}$$

After using the conversation of energy density so that we have a relation $\rho(a)$, one can integrate and find that :

$$\frac{1}{3(w+1)}\left(\frac{4\sqrt{\pi} l^2}{X} + \frac{l}{\sqrt{2}}\left(2\text{arccot}\left(\frac{2\sqrt{2\pi} l}{X}\right) + \log\left(\frac{4\sqrt{\pi} l - \sqrt{2} X}{4\sqrt{\pi} l + \sqrt{2} X}\right)\right)\right) = t + t_0 \tag{452}$$

where $X = \sqrt{-3a^{3(w+1)} + \sqrt{9a^{6(w+1)} + 64\pi^2 l^4}}$. Similarly to the previous cases,

$$a\left(t \to -\infty\right) = l^{\frac{2}{3(w+1)}} e^{\frac{\sqrt{2}t}{l}} = a_0 e^{\frac{\sqrt{2}t}{l}} \tag{453}$$

so that this solution also describes a past-eternal dS that suddenly enters the General Relativity regime.

Therefore, as we saw, all the solutions that we derived from this Lovelock Designer model are regular, what is a notable property given our previous survey of other solutions and the apparent difficulty to find topological regular solutions. Non-flat cosmological solutions should also be checked, and if they turn out to be also regular, it would make all the DSS sector of this model avoid singularities in the presence of classical matter. Of course, in order to be a good candidate for being the DSS sector of a (would be) semi-classical limit of quantum gravity, its black hole solution should have the correct quantum correction to Newton potential, what is not the case here. It remains to be seen if both properties can come together for some models, but we do not see any obstruction to it.

Finally, recall that this theory corresponds to $\zeta_p = p$, which is the Lovelock-Lanczos gravity case. Therefore the previous DSS sector of the 4-dimensional NPG theory Eq(344, 444) can also be found from the four dimensional regularization of (the dynamical spherically symmetric sector of) Lovelock-Lanczos gravity, Eq(25), about which we discussed in Chap.I Sec.B. The corresponding theory reads :

$$I_{(d;m)} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^{\,d}x\,\sqrt{-g}\sum_{p=1}^{m} l^{2(p-1)}\left(\frac{1 + (-1)^{p-1}}{2^{2p-1}}\,\frac{(d-2p-1)!}{(d-2)!}\right)\delta^{\mu_1\nu_1\ldots\mu_p\nu_p}_{\alpha_1\beta_1\ldots\alpha_p\beta_p}\prod_{k=1}^{p} R^{\alpha_k\beta_k}_{\mu_k\nu_k} \tag{454}$$

So that, at least once the DSS reduction has been performed, the following action is well-defined,

$$\mathfrak{I} = \lim_{m\to\infty}\left(\lim_{d\to 4}\left(I_{(d;m)}\Big\rfloor\right)\right), \tag{455}$$

and yields the previous regular solutions.

F. Mimetic Gravity formulation for the cosmological solutions

As an aside to conclude this chapter, we will now see that all the previous cosmological solutions can be found from Mimetic gravity [359–363, 370, 371]. It could also be done from the kind of models we studied in the first section of this chapter, but we saw that analytic black hole solutions seem difficult to find in this case. On the contrary, it was shown in [108] and [109] that it is possible to construct a Mimetic gravity model admitting both the LQC bounce solution and a non-singular black hole. Reconstructing different Mimetic models admitting the cosmological solutions of the past sections could therefore allow to find other non-singular mimetic gravity black hole.

To start with, the kind of models in question are the following :

$$I=\int d^4x\sqrt{-g}\left(-\frac{1}{2}R+\lambda\left(\partial_\alpha\phi\partial^\alpha\phi-1\right)+f\left(\chi\right)\right) \tag{456}$$

where ϕ is the mimetic field, λ is a Lagrange multiplier and $\chi=\Box\phi$. The Euler-Lagrange equation w.r.t. λ is therefore a constraint on the scalar field ϕ :

$$\partial_\alpha\phi\partial^\alpha\phi=1 \tag{457}$$

Following closely [108], it can be shown that in synchronous coordinates,

$$ds^2=-dt^2+\gamma_{ij}\left(t,x^l\right)dx^idx^j \tag{458}$$

the solution of the previous equation is $\phi=\pm t+c$, where c is an integration constant. Therefore,

$$\chi=\Box\phi=\frac{\dot{\gamma}}{2\gamma} \tag{459}$$

where γ is the determinant of the spatial metric γ_{ij}. This shows that similarly to Non-Polynomial gravities, the Mimetic models (456) also allow to reach more curvature invariants than the usual polynomials of the Riemann tensor and its covariant derivatives do. Furthermore, the curvature invariant it gives access to is first order in the metric field, what is an additional similarity between the two kind of theories.

In particular, in FLRW spacetimes given by,

$$ds^2=-dt^2+a^2(t)\delta_{ij}dx^idx^j \tag{460}$$

the mimetic field becomes related to the Hubble parameter in the following way,

$$\chi=\Box\phi=3H \tag{461}$$

so that these Mimetic theories also shares the same FLRW sector as Lovelock Designer gravities, as it can be checked from the first Friedmann equation, given by :

$$\frac{\chi^2}{3}+f\left(\chi\right)-\chi\partial_\chi f\left(\chi\right)=8\pi\rho \tag{462}$$

where we have neglected the mimetic cold matter that should contribute to the right-hand side as $e/(2a^3)$, where e is an integration constant. Indeed, the left-hand-side is just a general function of the Hubble parameter, so that if it can be expanded in the following way,

$$\frac{\chi^2}{3}+f\left(\chi\right)-\chi\partial_\chi f\left(\chi\right)=\sum_{p=1}^{\infty}H^{2p}v_p=:X\left(H\right) \tag{463}$$

each terms of the sum do correspond to the FLRW sector of an order $2p$ Lovelock term. From this result, one can reconstruct the previous cosmological solutions as follows :

- For the LQC bounce, given by Eq(364), we have :

$$X\left(H\right)=\frac{4\pi}{3l^2}\left(1-\sqrt{1-\frac{3l^2H^2}{2\pi}}\right) \tag{464}$$

  so that setting $l=\frac{2\sqrt{\pi}}{\chi_m}$ and expressing H in terms of χ due to Eq(461) gives the Eq(21) of [108] by solving the differential Eq(463) :

$$f\left(\chi\right)=\chi_m^2\left(1+\frac{\chi^2}{3\chi_m^2}-\sqrt{1-\frac{2\chi^2}{3\chi_m^2}}-\sqrt{\frac{2}{3}}\frac{\chi}{\chi_m}\arcsin\left(\sqrt{\frac{2}{3}}\frac{\chi}{\chi_m}\right)\right) \tag{465}$$

- Similarly, for the bounce Eq(436), setting $l = \frac{1}{\chi_m}\sqrt{\frac{2\pi}{3}}$ we get :

$$\begin{aligned} X(H) &= \frac{8\pi}{l^2}\left(1-\left(1-\frac{3}{2\pi}l^2H^2\right)^{1/4}\right) \\ f(\chi) &= 4\chi_m^2\left(1-\left(1-\frac{\chi^2}{9\chi_m^2}\right)^{1/4}\right)+\frac{\chi^2}{9}\left(3-2\,{}_2F_1\left(\frac{1}{2},\frac{3}{4};\frac{3}{2};\frac{\chi^2}{9\chi_m^2}\right)\right) \end{aligned} \tag{466}$$

  We performed the integral with Mathematica, and it is written in terms of the hypergeometric function.

- And finally, for the Past de Sitter cosmologies Eq(440) :

$$\begin{aligned} X(H) &= \frac{H^2\left(\beta+\gamma l^2H^2\right)}{\left(1-l^2H^2\right)\left(\beta+\alpha l^2H^2\right)} \\ f(\chi) &= \frac{\chi^2}{3}-\frac{\chi\left((\alpha-\gamma)\sqrt{\beta}\arctan\left(\frac{\sqrt{\alpha}\,l\chi}{3\sqrt{\beta}}\right)+(\beta+\gamma)\sqrt{\alpha}\,\mathrm{arctanh}\left(\frac{l\chi}{3}\right)\right)}{3(\alpha+\beta)\sqrt{\alpha}\,l} \end{aligned} \tag{467}$$

It would then be interesting to investigate if the black hole solutions of the previous Mimetic gravity theories are regular, similarly to the first choice of $f(\chi)$, studied in [108], where it was shown that indeed the singularity of Schwarzschild geometry is cured. Actually, as we noted previously, a special case of the previous modification of Friedmann equation Eq(467) was also found in the recent paper [358], but from a slightly different Mimetic theory than Eq(456).

Moreover, an advantage of the mimetic formulation of these regular solutions is that it is possible to calculate the perturbations around the regular background solutions. Especially for the past-eternal de Sitter solutions that might be used to model inflation, given that they describe an exponential phase of expansion in the very early universe that is then replaced by the usual GR cosmology.

On the contrary, when using Non-Polynomial gravities, the perturbations seem very difficult to find, and given the degeneracy of the NPG scalars that we exemplified by the Eq(104), it might be better to first have an idea of what representatives of a specific 2D scalar should be chosen (for example choosing $\nabla_\gamma\omega_{\alpha\beta}\nabla^\gamma\omega^{\alpha\beta}$ or $\nabla_\alpha\mathcal{N}_{\mu\nu}^{\ \ \lambda\delta}\nabla^\alpha\mathcal{N}^{\mu\nu}_{\ \ \lambda\delta}$ to reach $\frac{\gamma^{AB}\mathcal{D}_A r\,\mathcal{D}_B r}{r^2}$) before calculating perturbations. A possible way of choosing would be to find $4D$ NPG scalars that have the DSS sector given by eq(100, 101), and are also polynomial and second order for some Bianchi models, so that anisotropic perturbations could be calculated without going outside the range of validity of the DSS decomposition properties of the NPG scalars.

Concerning the Mimetic formulation, note also that this kind of models are contained within the so-called DHOST gravities, see for example [84–87], so that similarly to the case of NPG, a large number of DHOST theories can produce the LQC bounce. However, the observation [372] has indicated that the speed of gravitational waves is extremely close to that of light, as it is the case in General Relativity, and this puts some constraints on the viability of these models, see [373–375].

G. Discussion

The primary aim of this chapter was to present different kind of four dimensional gravitational theories, Mimetic and Non-polynomial, whose FLRW sectors are the same as those of Lovelock-Lanczos theories, in the sense that they provide corrections to the first Friedmann equation in the form of powers of the Hubble parameter $H = \dot{a}/a$. The interest in these models comes from the fact that, contrary to the usual Lovelock-Lanczos theories (except in infinite dimensions), the general correction to GR involves all the powers of H, in the form of a general function $f(H)$. It is then possible to reconstruct the function f so that the flat cosmological solution is non-singular. In particular, the Loop Quantum cosmology bounce can be obtain from these effective theories.

We saw three completely different ways to obtain the same cosmological corrections, and there is possibly an infinite number of gravitational theories with this property. Therefore, it is quite important to study other sectors of these models, in order to see if non-singular solutions can also be found, and if not, to discard the associated models. For example, the Mimetic formulation of the LQC bounce also provides a non-singular black hole solution.

Other ways to reach $f(H)$-type corrections to Friedmann equation have been found within modified Teleparallel Gravity [75–77], where ${R^\mu}_{\nu\alpha\beta} = 0$, $\nabla_\alpha g_{\mu\nu} = 0$ while the torsion is non vanishing and captures entirely gravitational effects, and modified Coincident Gravity [74], where ${R^\mu}_{\nu\alpha\beta} = 0$, $\Gamma^\alpha_{[\mu\nu]} = 0$, while the non-metricity plays the role of gravitational field strength. However, if one's aim is to find a semi-classical formulation of some quantum gravity results, in particular LQC and AS black holes, it seems odd to use a different formalism of General Relativity to reach these semi-classical corrections than the one used in these quantum gravity approaches, in which the geometry is torsionless and the connection and metric are compatible. Other approaches reaching the same solution have been to construct infinite series of polynomials of the Ricci tensor in [140], to use the NPG property holding for FLRW spacetimes, $R - \sqrt{R^2 - 6\mathcal{G}} = 12H^2$, where $\mathcal{G}$ is the Gauss-Bonnet scalar, in [152]. It has also been tried to find the LQC bounce from $f(R)$ [116] and $R + f(\mathcal{G})$ [117,118] gravities, but an additional degree of freedom generically appears in these theories.

In this chapter, we focused mainly on another formulation of the LQC bounce, consisting in a theory which shares all its dynamical spherically symmetric sector with Lovelock-Lanczos gravity, up to a one-parameter deformation which is topological for FLRW spacetimes. This theory can be written, using the Ricci and Cotton tensors, in the form of the Non-Polynomial gravity action (344).

As we saw, it is possible for two different choices of the "shape factor" w_p to reach the LQC bounce $w_p = -2\binom{1/2}{p}\left(-\frac{3}{2\pi}\right)^{p-1} l^{2(p-1)}$ and the AS black hole $w_p = -\left(\frac{2}{\gamma}\right)^{1-p} l^{2(p-1)}$. Then, for the two choices of ζ_p that we studied, the solutions were :

- For $\zeta = (1-\sigma) + \sigma p$; for the "LQC choice" of w_p, a "one-horizon" regular black hole ; for the "AS choice", it gives the RBH found in [54].
- For $\zeta = (1-\sigma)\, p^2 + \sigma p$; for the "LQC choice" of w_p, an (A)dS-core regular black hole ; for the "AS choice", an (A)dS-core regular black hole together with a quasi dS non-perturbative solution.

In the "LQC case", an interesting feature of the choice $\zeta = (1-\sigma) + \sigma p$ is that the resulting black hole solution is quite similar to the Modesto semi-polymeric black hole Eq(256), with the positive differences that it has limiting curvature and the correct quantum correction to the Newton potential. As we saw, all the black hole solutions that we derived from the models (w_p, ζ_p) have these properties. For low values of the mass, it possesses two horizons, what makes it closer to the fully polymeric Modesto black hole, Eq(316).

In the models mimicking the AS black holes, the associated cosmological solution was found to be a geodesically complete spacetime where the GR singularity is replaced by a past-eternal dS phase which connects to the GR regime. As we said previously, it turns out that this kind of cosmological singularity resolution via an inflationary phase is what has been predicted from Asymptotic Safety cosmology, see for example [229–233]. Moreover, in [54], the AS black hole was also accompanied by a non-perturbative solution with no semi-classical limit. Given the previous analogy between AS results and the present NPG models, it might be tempted to consider the quasi-dS solution coming from the model $\zeta = (1-\sigma)\, p^2 + \sigma p$ as a candidate for such AS non-perturbative solution. Based on these results, it would be interesting to investigate further the relation between the dynamical spherically symmetric corrections to GR that Asymptotic Safety (within the Einstein-Hilbert truncation) implies, and the DSS sector of our one-parameter deformation of Lovelock-Lanczos gravity.

It is also interesting to see that effective solutions coming from two completely different Quantum Gravity theories are possible to reach from a single curvature scalar $\mathcal{L}_{(p,\zeta)}$, from which the theories $R + \sum_p w_p \mathcal{L}_{(p,\zeta)}$ are constructed (see (344)). The difference between the theories admitting LQC or AS effective solutions being entirely contained in the choice of the overall set of coupling constants w_p.

Finally, we saw that in many cases, curing the Schwarzschild singularity introduces new singular solutions for planar and hyperbolic topologies ($k = 0$ and $k = -1$), although the theory (408) has a better behaviour than the others in this respect. We do not know if it is possible to find a couple (w_p, ζ_p) admitting only regular solutions for cosmological backgrounds (spherical, flat, and hyperbolic) and "spherical" symmetry with $k = 0, 1, -1$, but it would be interesting to see. A possible way to start would be to see if regular solutions are also possible to find for more general choices of ζ_p, for example with

$$\zeta_p = \sum_{i=1}^{j} a_{(i)} p^i \tag{468}$$

However, analytic solutions seem difficult to find in this way, at least for the previous choices of w_p, although some can be written in terms of hypergeometric functions.

Conclusion

This thesis was mainly divided into two parts. In the first two Chapters, we focused on d-dimensional (metric) gravitational theories admitting second order dynamical spherically symmetric field equations. The idea behind it being that many Quantum Gravity results provide non-singular corrections to the classical Schwarzschild and cosmological geometries, in which the regularity is controlled by the quantization parameters. This is very different from many regularizations found in Modified Gravity theories, where the field equations are usually higher order ones, or containing additional degrees of freedom, thus introducing new integration constants to cure the singularities. More specifically, we focused on the so-called Non-Polynomial Gravity theories. Despite their name, they admit the peculiar property that for some specific classes of metric fields (which depend on the model), their field equations are polynomial, and for our purpose, second order. This is due to specific algebraic decomposition properties of the non-polynomial curvature invariants and tensors (in these classes of metric), that are used in the construction of the theories. In this regard, our main result is to have found the decomposition properties of the Cotton tensor in dynamical spherically symmetric spacetimes, from which one can construct a non-polynomial tensor, ratio of two quadratic combinations of the Cotton tensor, which in turn can be used to construct d-dimensional gravitational theories with second order DSS field equations. As we saw, it provides a higher dimensional formulation of the Einstein-Dilaton and Lovelock-Designer effective approaches, and thus, if these lasts admit regular solutions, it provides a proof of existence, in the theory space of metric gravity, of theories admitting them.

However, many issues come with such theories. In particular, if considered outside of the range of validity of the previous algebraic properties, they probably yield additional ghost-like degrees of freedom. A first step toward an understanding of this issue would be to calculate the perturbations of these theories around some of their dynamical spherically symmetric solutions. Nevertheless, even without mentioning the technical difficulty of such a task, there is already a conceptual problem to be resolved before investigating the behaviour of NPG theories in these regimes. As we saw in Chap.2 Sec.2, there is a huge degeneracy of NPG actions admitting the exact same DSS sector, but being entirely different outside of DSS. In this regard, a characteristic illustration would be Eq(503) found in the third Appendix, which establishes the equivalence at the DSS level of General Relativity and a Non-Polynomial Gravity theory constructed from non-polynomial combinations of the Weyl tensor. Of course in this case, we well know what *representative* of the given DSS class one should derive perturbations and other solutions from. But in general this is not so clear, and some ghost-like degrees of freedom could well appear in some NPG model, but not in some other ones, with the same DSS sector.

A possible way to circumvent this issue and break these degeneracies would be to find other algebraic decomposition properties of curvature tensors holding for larger classes of metric fields. At present, we do know that other properties (than the ones related to DSS), like those used in Eq(54,321), cover other classes of metric fields (for example Bianchi I), but not a general DSS ansatz. Therefore, it would be interesting to find the largest classes of metric fields such that particular NP curvature invariants yield second order field equations[19]. For example, we do not know if such theory exists for axisymmetric spacetime, but if so, it could lead to new rotating black hole solutions, different from those found from polynomial gravities, and potentially regular, as it happened for DSS. However, note that in this case Palais's "Principle of Symmetric Criticality" would probably not hold any more, so that no short-cut would be available to derive the solutions.

As a final remark regarding Non-Polynomial theories and their special properties, it could be interesting to investigate if they are preserved when considering alternative formalisms than the metric one, for example (for curvature gravity) the Palatini or Purely affine formalisms, or even the Teleparallel ("torsion gravity") and Coincident ("non-metricity gravity") ones. Furthermore, it is also possible that completely other kinds of gravitational theories exist and admit the same DSS sector as some NPG theories. As a more restricted example, recall that we saw in Chapter 5. that different Non-Polynomial Gravity theories, regularized 4-dimensional Lovelock-Lanczos

[19] Similarly, one might wonder what is the more general class of metric fields for which the regularization of the Lovelock-Lanczos scalars beyond the critical order, found in Eq(25), would hold. For example, does it still work if one uses an axisymmetric ansatz ? A possibility, to stay as close to DSS as possible and have a chance to preserve this property, would be to perform a covariant $2+n$ decomposition of Lovelock-Lanczos gravity, and see under which conditions such regularization would still possible, the more restrictive which does hold being when the n-dimensional manifold is maximally symmetric, as we showed.

gravity and the Mimetic gravity (456) all agree on their FLRW sector. As we said there, this can allow to calculate the perturbations around some regular cosmological solutions from their mimetic formulation. However, the black hole sector would be changed in this case as the Mimetic theory probably do not admit the NPG regular black hole solutions. It remains to be seen what mimetic black hole solutions are associated with the regular cosmological solutions found in this Chapter. In particular, it would be interesting to see if they are also regular, and what kind of mimetic black holes are associated with the non-singular past eternal dS cosmologies. As seen as solutions of Mimetic Gravities, it would be interesting to investigate further these last class of geometries in the context of inflation, because in this case calculating the perturbations would not be such an issue.

The second part of this thesis was mainly focused on some effective Einstein-Dilaton and Lovelock-Designer gravities (interpreted in terms of their NPG formulations) and their associated solutions. In Chap.3 and Chap.5, we investigated two theories whose DSS sectors are in some sense the most minimal power-series generalizations of the DSS sectors of Einstein and Gauss-Bonnet gravities, because for any order of corrections, they preserve the DSS decompositions of both their Lagrangians and field equations. It turned out that they admit many regular black hole and cosmological solutions. In Chap.4, another class of Einstein-Dilaton gravities was used to reconstruct regular solutions. More precisely, we showed in Chap.4 and Chap.5 that many RBHs and non-singular cosmologies, originally derived (or inspired) from different Quantum Gravity approaches (Asymptotic Safety and Loop Quantum Cosmology), were solutions of the corresponding Non-Polynomial Gravity theories, thus proving that the theory space of metric gravity contains some classes of theory able to reach these solutions. It would be interesting to understand better the possible link (if any other exists) between these 2D effective theories (and more ambitiously, the associated NPGs) and the precise behaviour of these Quantum Gravity approaches in DSS. For example, we noted many resemblance between the solutions of the Lovelock-Designer theory Eq(422) and the DSS solutions predicted by Asymptotic Safety. Indeed, we saw in particular three solutions : an AS regular black hole, a non-perturbative quasi-dS anisotropic solution and a past-eternal dS cosmology. Although the first can be checked to be a legitimate AS BH candidate (as it satisfies the conditions (405)), it is not that obvious for the other two. But as we noted, even these solutions share some resemblance with AS results. Concerning the regular black holes associated with the LQC bounce, the more interesting one, derived from the theory (384) was curiously belonging to the class of single-horizon RBHs, which is what is usually predicted from the Loop quantization procedure, as we also saw with the semi-polymeric black hole in Chap.4. This also would deserve further clarifications.

Irrespectively of some highly speculative relations with Quantum Gravity, we have also learnt some general lessons from the 2D effective theories that we studied. For example, we saw that the theories leading to the Modesto, D'Ambrosio-Rovelli and Visser-Hochberg non-singular single-horizon black holes automatically cure the Coulomb singularity of the electric potential. A more general understanding of this property would be desirable, as it could provide an interesting classification of regular Einstein-Dilaton theories. Moreover, one might wonder if this kind of property would also apply to other (external) fields. As we saw, a necessary condition to have this property is that in Schwarzschild gauge, $b(r) \neq 1$. But we also found out in Chap.3 that when $b(r) = 1$, it is still possible to cure the Coulomb singularity by considering a non-minimal coupling between the Maxwell action and a non-polynomial curvature invariant. Using the same procedure with other fields could also be interesting in this case.

Furthermore, a useful aspect of 2D effective approaches is that one can also investigate (within spherical symmetry) dynamical phenomena, not just static black holes. Therefore, a logical continuation for the different black hole solutions that we have found or reconstructed would be to investigate their dynamical formations when collapsing matter is considered, and to model their behaviour under Hawking evaporation, as it is usually done in 2D effective approaches.

Another aspect of the 2D Einstein-Dilaton gravities of Chap. 3 and Chap. 4 is that they apparently do not possess a (isotropic) cosmological sector. Under the assumption (seen in Chap.3 Sec.8) that the NPG invariant $\mathcal{R}$ can be regularized (to zero) in FLRW, we saw a possible way to have non trivial corrections to the cosmological sector of GR by using a trick involving a Lagrange Multiplier field. Although the procedure seems quite artificial, it might still be interesting to find what cosmological solutions the theories of Chap.4 would provide.

Other 2D Lovelock-Designer that we left aside should also be investigated, for example the more general theory given by Eq(123) without setting $d_{(\alpha,\beta,\gamma,\lambda)} = 0$. They are a kind of hybrid between Einstein-Dilaton models (with either trivial or ill-defined cosmological sectors) and the Lovelock-like ones given by Eq(344). Indeed, if the cosmological sector of $\mathcal{R}$ and $\nabla_\gamma \omega_{\alpha\beta} \nabla^\gamma \omega^{\alpha\beta}$ can be defined (by the "$k = 0$" procedure), then these theories would have a similar FLRW sector as Lovelock-Lanczos gravities (because in this case $\mathcal{R} = 0$ and $\nabla_\gamma \omega_{\alpha\beta} \nabla^\gamma \omega^{\alpha\beta} \propto H^2$), but in static spherical symmetry they satisfy $b(r) \neq 1$, like the Einstein-Dilaton theories of Chap. 4. However, even if the FLRW sector is not well-defined, it is still possible, for any 2D effective theory to consider cosmological models with two scale factors in the presence of usual cosmological matter.

Finally, at the level of the solutions, we saw different general behaviours emerging from our study. An important one was that all the charged single-horizon RBHs can avoid the mass-inflation instability (Chap.3 Sec.7 and Chap.4), provided that a bound relating the mass and the charge is satisfied. This is quite in contrast with the case of the class of (A)dS-core regular black holes, which already in the neutral case suffer from this instability. A quite logical follow-up on this issue would be to apply the Newman-Janis algorithm in order to generalize our black hole solutions to rotating ones, and see if some bounds between charge, mass and angular momentum can still guaranty the stability of their interiors.

Moreover, we also saw a peculiar type of (A)dS-core BH with a discontinuous and extremal inner horizon in Chap.3 Sec.7 that also avoids mass inflation in this way. However, one would need to study the geodesics of an arbitrary truncation (for which the geometry is smooth and regular) in the limit of infinite order of corrections (at which the discontinuity appears) in order to understand better this kind of solution. More generally, it might be interesting to study the behaviour of geodesics in all the solutions, and maybe more specifically for the "doubled" Modesto semi-polymeric black hole Eq(274) and for the single-horizon one associated with the LQC bounce Eq(399). Indeed, these solutions have the special property that by changing the coordinate system, they are found to be contained in specific regions of other regular solutions Eq(256) and Eq(391), but the field equations prevent to extend the coordinates to these regions. Therefore, these regular (from the point of view of curvature invariants) geometries might well be geodesically incomplete, as it is known that these two properties are not implying one another.

As a final remark, note that it would also be interesting to investigate the thermodynamical properties of the different black hole solutions, and see if common properties can be obtained for the different classes of models we studied in this thesis. The case of topological (A)dS-core and single-horizon regular black holes might also present some interesting surprises. Concerning topological black holes, see for instance [367–369].

Appendix

Dynamical Spherical Symmetry relations

1. Decomposition of d-dimensional curvature tensors

In order to derive the spherically symmetric decompositions of respectively the Weyl Eq(62) and Cotton tensors Eq(86), the Ricci scalar Eq(18) and Einstein tensor Eq(19), and the NPG scalars Eq(100), Eq(101), we first define :

$$\begin{aligned} \xi^a &:= \frac{\nabla^a r}{r} \ , \quad \chi := \xi^a \xi_a \ , \quad \mathcal{Z} := \frac{R^{(n)}}{n(n-1)} - \chi \ , \\ \mathcal{Y}^a_b &:= \frac{\nabla^a \nabla_b r}{r} \ , \quad \psi := -\left(\frac{1}{2} R^{(2)} + \mathcal{Y} + \mathcal{Z}\right) . \end{aligned} \tag{469}$$

The non-vanishing components of the Christoffel symbols are :

$$\begin{aligned} \Gamma_{ab}^{\ \ c} &=^{(2)} \Gamma_{ab}^{\ \ c}\,(x^a) \ , \quad \Gamma_{ij}^{\ \ k} =^{(d-2)} \Gamma_{ij}^{\ \ k}\,(x^i) \ , \\ \Gamma_{ij}^{\ \ a} &= -\sigma_{ij}\xi^a \ , \quad \Gamma_{ai}^{\ \ j} = \xi_a \delta^j_i . \end{aligned} \tag{470}$$

It gives the following Riemann and Ricci tensors as well as the Ricci scalar :

$$\begin{aligned} R^{ai}_{\ \ bj} &= -\mathcal{Y}^a_b \delta^j_i \ , \quad R^{ij}_{\ \ kl} = \mathcal{Z}\delta^i_{[k}\delta^j_{l]} \, , \\ R^{ab}_{\ \ ce} &= \frac{1}{2} R^{(2)} \delta^a_{[c}\delta^b_{e]} \ , \quad R^j_i = \delta^j_i \left(-\mathcal{Y} + (d-3)\mathcal{Z}\right) \, , \\ R^b_a &= \frac{1}{2}\delta^b_a \, R^{(2)} - (d-2)\mathcal{Y}^b_a \ , \quad R = R^{(2)} - 2(d-2)\mathcal{Y} + (d-2)(d-3)\mathcal{Z}, \end{aligned} \tag{471}$$

So that the Schouten tensor becomes,

$$\begin{aligned} A^j_i &= -\frac{\delta^j_i}{(d-2)(d-1)}\left(\frac{1}{2}R^{(2)} + \mathcal{Y} - \frac{d(d-3)}{2}\mathcal{Z}\right) , \\ A^b_a &= \frac{\delta^b_a}{d-1}\left(\frac{1}{2}R^{(2)} + \mathcal{Y} - \frac{d-3}{2}\mathcal{Z}\right) - \mathcal{Y}^b_a \, , \end{aligned} \tag{472}$$

where we use $\delta^{[a}_{[c}\mathcal{Y}^{b]}_{e]} - \mathcal{Y}\delta^{[a}_{[c}\delta^{b]}_{e]} = 0$ in 2-dimensions for any tensor $\mathcal{Y}$. Then the non-vanishing components of the Weyl tensor are :

$$\begin{aligned} W^{ab}_{\ \ ce} &= \frac{d-3}{d-1}\delta^a_{[c}\delta^b_{e]}\psi \, , \\ W^{ij}_{\ \ kl} &= \frac{2}{(d-2)(d-1)}\delta^i_{[l}\delta^j_{k]}\psi \, , \\ W^{ai}_{\ \ bj} &= \frac{d-3}{(d-2)(d-1)}\delta^a_b\delta^i_j\psi . \end{aligned} \tag{473}$$

Written in terms of four dimensional quantities, this proves Eq(62). Finally, using the identity $C^{\mu\nu}_{\ \ \alpha} = -\frac{d-2}{d-3}\nabla^\beta W^{\mu\nu}_{\ \ \alpha\beta}$ leads to the decomposition of the d-dimensional Cotton tensor :

$$\begin{aligned} C^{ec}_{\ \ b} &= (d-2)\left(\xi^{[c}\delta^{e]}_b\psi + \frac{1}{d-1}\delta^{[e}_b\partial^{c]}\psi\right) , \\ C^{jb}_{\ \ i} &= -\delta^j_i\left(\frac{1}{d-1}\partial^b\psi + \xi^b\psi\right) , \end{aligned} \tag{474}$$

and defining $\Omega_a := \frac{\partial_a\psi}{d-1} + \xi_a\psi$ gives the following relations which proves Eq(86) :

$$\begin{aligned} C^{\mu\nu}_{\ \ \alpha}C_{\mu\nu}^{\ \ \alpha} &= 2(d-2)(d-1)\Omega_a\Omega^a \, , \\ C^{\mu\ \beta}_{\ \alpha}C^{\nu\alpha}_{\ \ \beta} &= (d-2)\delta^\mu_a\delta^\nu_b\Omega^a\Omega^b + \left(\delta^\mu_i\delta^\nu_j\sigma^{ij} + (d-2)^2\delta^\mu_a\delta^\nu_b\gamma^{ab}\right)\Omega^c\Omega_c \, , \\ C^{\mu\alpha}_{\ \ \beta}C^{\nu\beta}_{\ \ \alpha} &= (d-2)(d-1)\delta^\mu_a\delta^\nu_b\Omega^a\Omega^b \, . \end{aligned} \tag{475}$$

2. Alternative form of the DSS sector of Lovelock-Lanczos gravity

In this section, we wish to prove Eq(23). Define $\Gamma_a := \nabla_{[b} r \nabla_{a]} \nabla^b r$, $\tilde{\chi}_a := \nabla_a r$, $\tilde{\chi}_{ab} := \tilde{\chi}_a \tilde{\chi}_b$, $\tilde{\chi} := \tilde{\chi}_a \tilde{\chi}^a$, $\tilde{\mathcal{Y}}_{ab} := \nabla_a \nabla_b r$, $\tilde{\mathcal{Y}} := \tilde{\mathcal{Y}}_{ab}\gamma^{ab}$ and $\tilde{\mathcal{Y}}_2 := \tilde{\mathcal{Y}}_{ab}\tilde{\mathcal{Y}}^{ab}$. Then we consider only the part of the Lovelock-Lanczos Lagrangian given by :

$$\begin{aligned}\mathcal{L} &= r^{-2(p-1)}\left(2(p-1)\left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right)(k-\tilde{\chi})^{p-2} + R^{(2)}(k-\tilde{\chi})^{p-1}\right)\\ &= r^{-2(p-1)}\left(R^{(2)}k^{p-1} + \sum_{i=0}^{p-2}\frac{(p-1)!(-1)^i k^{p-2-i}\tilde{\chi}^i}{(i+1)!(p-i-2)!}\left(2(i+1)\left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right) - \tilde{\chi}R^{(2)}\right)\right)\end{aligned} \tag{476}$$

Using that $\tilde{\chi}R^{(2)} = 2\tilde{\chi}^a\left(\Box\nabla_a - \nabla_a\Box\right)r$, which comes from the commutation of covariant derivatives and the vanishing of the Einstein tensor in 2D, and that $\tilde{\chi}^a\left(\Box\nabla_a - \nabla_a\Box\right)r = \left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right) + \nabla_a\Gamma^a$, the lagrangian can be written as :

$$\begin{aligned}\mathcal{L} &= r^{-2(p-1)}\left(R^{(2)}k^{p-1} + 2\sum_{i=0}^{p-2}\alpha_i\tilde{\chi}^i\left(i\left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right) - \nabla_a\Gamma^a\right)\right)\\ &= r^{-2(p-1)}R^{(2)}k^{p-1} + 2\,r^{2-d}\sum_{i=0}^{p-2}\alpha_i\left(\nabla_a\left(r^{d-2p}\right)\tilde{\chi}^i\Gamma^a - \nabla_a\left(\tilde{\chi}^i r^{d-2p}\Gamma_a\right)\right)\end{aligned} \tag{477}$$

where $\alpha_i = \frac{(p-1)!(-1)^i k^{p-2-i}}{(i+1)!(p-i-2)!}$ and in the last line we used that in two dimensions, for any tensor T, we have $\delta^{[a}_{[c}T^{b]}_{e]} = \frac{1}{2}T_{dh}\gamma^{dh}\delta^{[a}_{[c}\delta^{b]}_{e]}$. Applying this formula to $\delta^{[a}_{[c}\tilde{\mathcal{Y}}^{b]}_{e]}\tilde{\mathcal{Y}}^c_a\tilde{\chi}^e_b$ leads to $\Gamma_a\nabla^a\tilde{\chi} = -\tilde{\chi}\left(\tilde{\mathcal{Y}}^2 - \tilde{\mathcal{Y}}_2\right)$. Finally, we obtain

$$\begin{aligned}\mathcal{L} &= r^{-2(p-1)}R^{(2)}k^{p-1} + 2(d-2p)r^{-2p+1}\tilde{\chi}_a\Gamma^a\mathcal{J} - 2r^{2-d}\nabla_a\left(r^{d-2p}\Gamma^a\mathcal{J}\right)\\ &= (d-2p)r^{-2p+1}\tilde{\chi}_a\left(2\Gamma^a\mathcal{J} - k^{p-1}v^a\right) + r^{2-d}\nabla_a\left(r^{d-2p}\left(k^{p-1}v^a - 2\Gamma^a\mathcal{J}\right)\right)\end{aligned} \tag{478}$$

where $\mathcal{J} = \sum_{i=0}^{p-2}\alpha_i\tilde{\chi}^i = \frac{k^{p-1}-(k-\tilde{\chi})^{p-1}}{\tilde{\chi}}$ and in the last line we have used the fact that in 2 dimensions, the Ricci scalar can be written as a total derivative : $R^{(2)} = \nabla_a v^a$.

The spherically symmetric sector of Lovelock Gravity can therefore be written as :

$$\begin{aligned}\mathcal{L}_{(p)} = (d-2p)\,\nu_p &\left(\left((d-2p-1)\mathcal{Z} - 2p\frac{\tilde{\mathcal{Y}}}{r}\right)\mathcal{Z}^{p-1} + p\,r^{-2p+1}\tilde{\chi}_a\left(2\Gamma^a\mathcal{J} - k^{p-1}v^a\right)\right)\\ &+ p\,r^{2-d}\nabla_a\left(r^{d-2p}\left(k^{p-1}v^a - 2\Gamma^a\mathcal{J}\right)\right)\end{aligned} \tag{479}$$

where $\nu_p = \frac{(d-2)!}{(d-2p)!}$.

Critical Order Lovelock-Lanczos scalars

3. Dimensional Identities

We use the notations : $\phi_\alpha := \phi_{;\alpha} := \nabla_\alpha\phi$, $\phi^\beta_\alpha := \phi^{;\beta}_{;\alpha}$, $\chi^\beta_\alpha := \phi_\alpha\phi^\beta$ and $R^{\sigma\rho}_{\alpha\beta} := R_{\alpha\beta}{}^{\sigma\rho}$. Moreover, as all the indices are labelled by integer, we usually write : $R_{(k)(k+1)} := R^{\nu_k\nu_{k+1}}_{\mu_k\mu_{k+1}}$ and $\phi_{(k)} := \phi^{\nu_k}_{\mu_k}$. For the following proof, there will be few free indices and few indices that are not contracted with the totally antisymmetric tensor, thus in order to make things clearer we note $T_\bullet$ such free index or such contracted ones $T_\bullet K^\bullet$.

We are working in d dimensions, therefore,

$$I^\bullet_{\left(0\le p\le\frac{d}{2}\right)} := \delta^{\mu_1\ldots\mu_d\,\bullet}_{\nu_1\ldots\nu_d\nu_{d+1}}\phi^{\nu_{d+1}}\prod_{n=1}^{d-2p-1}R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}}\prod_{m=d-2p+1}^{d}\phi^{\nu_m}_{\mu_m} = 0. \tag{480}$$

Expanding the totally antisymmetric tensor in its free index gives :

$$\delta^{\mu_1\ldots\mu_d\,\bullet}_{\nu_1\ldots\nu_d\nu_{d+1}} = \sum_{k=1}^{d+1}(-1)^{d+1+k}\delta^\bullet_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\bar{\nu}_k\ldots\nu_d\nu_{d+1}} \tag{481}$$

where the bar in $\bar{\nu}$ means that this index is omitted. Therefore,

$$
\begin{aligned}
I^{\bullet}_{\left(0\leq p\leq \frac{d}{2}\right)} =& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\,\phi^{\bullet}\prod_{n=1}^{d-2p-1} R_{(n)(n+1)}\prod_{m=d-2p+1}^{d}\phi_{(m)} \\
&+\sum_{k=1}^{d}(-1)^{k+1}\delta^{\bullet}_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\bar{\nu}_k\ldots\nu_d\nu_{d+1}}\,\phi^{\nu_{d+1}}\prod R\prod\phi
\end{aligned}
\tag{482}
$$

Let's call $I^{\bullet}$ the second term. Then, we split the sum into odd and even k and put the index ν_{d+1} in the empty slot left by the ν_k index. As we consider an even dimensional manifold, we have for k odd, $\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\bar{\nu}_k\ldots\nu_d\nu_{d+1}} = -\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}$ and for k even $\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\bar{\nu}_k\ldots\nu_d\nu_{d+1}} = \delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}$. Therefore,

$$
\begin{aligned}
I^{\bullet} = -\Bigg(&\sum_{k=1\,;\,\text{odd}}^{d-1}\delta^{\bullet}_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}\phi^{\nu_{d+1}}\prod R\prod\phi \\
&+\sum_{k=2\,;\,\text{even}}^{d}\delta^{\bullet}_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}\phi^{\nu_{d+1}}\prod R\prod\phi\Bigg)
\end{aligned}
\tag{483}
$$

Then, we split further the sum depending on whether $1\leq k\leq d-2p$ or $d-2p+1\leq k\leq d$:

$$
\begin{aligned}
I^{\bullet} = -\prod\phi\Bigg(&\sum_{k=1\,;\,\text{odd}}^{d-1}\delta^{\bullet}_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}\phi^{\nu_{d+1}}R_{(12)}\ldots R^{\nu_k\nu_{k+1}}_{\mu_k\mu_{k+1}}\ldots R_{(d-2p-1)(d-2p)} \\
&+\sum_{k=2\,;\,\text{even}}^{d}\delta^{\bullet}_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}\phi^{\nu_{d+1}}R_{(12)}\ldots R^{\nu_{k-1}\nu_k}_{\mu_{k-1}\mu_k}\ldots R_{(d-2p-1)(d-2p)}\Bigg) \\
-\prod R\Bigg(&\sum_{k=1\,;\,\text{odd}}^{d-1}\delta^{\bullet}_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}\phi^{\nu_{d+1}}\phi_{(d-2p+1)}\ldots\phi^{\nu_k}_{\mu_k}\ldots\phi_{(d)} \\
&+\sum_{k=2\,;\,\text{even}}^{d}\delta^{\bullet}_{\nu_k}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_{d+1}\ldots\nu_d}\phi^{\nu_{d+1}}\phi_{(d-2p+1)}\ldots\phi^{\nu_k}_{\mu_k}\ldots\phi_{(d)}\Bigg)
\end{aligned}
\tag{484}
$$

The index ν_k disappear as it is contracted, and we rename ν_{d+1} as ν_k which gives :

$$
\begin{aligned}
I^{\bullet} = -\prod\phi\Bigg(&\sum_{k=1\,;\,\text{odd}}^{d-1}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_k\ldots\nu_d}\phi^{\nu_k}R_{(12)}\ldots R^{\bullet\;\nu_{k+1}}_{\mu_k\mu_{k+1}}\ldots R_{(d-2p-1)(d-2p)} \\
&+\sum_{k=2\,;\,\text{even}}^{d}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_k\ldots\nu_d}\phi^{\nu_k}R_{(12)}\ldots R^{\nu_{k-1}\;\bullet}_{\mu_{k-1}\mu_k}\ldots R_{(d-2p-1)(d-2p)}\Bigg) \\
-\prod R\Bigg(&\sum_{k=1\,;\,\text{odd}}^{d-1}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_k\ldots\nu_d}\phi^{\nu_k}\phi_{(d-2p+1)}\ldots\phi^{\bullet}_{\mu_k}\ldots\phi_{(d)} \\
&+\sum_{k=2\,;\,\text{even}}^{d}\delta^{\mu_1\ldots\mu_k\ldots\mu_d}_{\nu_1\ldots\nu_k\ldots\nu_d}\phi^{\nu_k}\phi_{(d-2p+1)}\ldots\phi^{\bullet}_{\mu_k}\ldots\phi_{(d)}\Bigg)
\end{aligned}
\tag{485}
$$

Then we relabel the indices of the first sum as : $\mu_{k+1}\to\mu_1\to\mu_3\to\ldots\to\mu_{k-2}\to\mu_{k+1}$ and $\mu_k\to\mu_2\to\mu_4\to\ldots\to\mu_{k-1}\to\mu_k$ and the same for ν. For the second sum, $\mu_{k-1}\to\mu_1\to\mu_3\to\ldots\to\mu_{k-3}\to\mu_{k-1}$ and $\mu_k\to\mu_2\to\mu_4\to\ldots\to\mu_{k-2}\to\mu_k$ and the same for ν. For the last two sums, $\mu_{d-2p+1}\to\mu_{d-2p+2}\to\ldots\to\mu_k\to\mu_{d-2p+1}$. This renaming of indices mixes the indices of the delta but as we do the same for μ and ν, the overall permutation is even. Therefore, (using in addition the antisymmetry of the Riemann tensor in its two couples of indices),

$$\begin{aligned}I^{\bullet} = &-\prod \phi \Bigg(\sum_{k=1\,;\,\mathrm{odd}}^{d-1} \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \phi^{\nu_2} R^{\nu_1\,\bullet}_{\mu_1\mu_2} \prod_{n=3}^{d-2p-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \\ &+ \sum_{k=2\,;\,\mathrm{even}}^{d} \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \phi^{\nu_2} R^{\nu_1\,\bullet}_{\mu_1\mu_2} \prod_{n=3}^{d-2p-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \Bigg) \\ &-\prod R \Bigg(\sum_{k=1\,;\,\mathrm{odd}}^{d-1} \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \phi^{\nu_{d-2p+1}} \phi^{\bullet}_{\mu_{d-2p+1}} \prod_{m=d-2p+2}^{d} \phi^{\nu_m}_{\mu_m} \\ &+ \sum_{k=2\,;\,\mathrm{even}}^{d} \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \phi^{\nu_{d-2p+1}} \phi^{\bullet}_{\mu_{d-2p+1}} \prod_{m=d-2p+2}^{d} \phi^{\nu_m}_{\mu_m} \Bigg)\end{aligned} \tag{486}$$

Therefore the general term of the sums do not depend on k any more and so

$$\begin{aligned}I^{\bullet} = &-(d-2p)\, \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\, \phi^{\nu_2}\, R^{\nu_1\,\bullet}_{\mu_1\mu_2} \prod_{n=3}^{d-2p-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \prod_{m=d-2p+1}^{d} \phi^{\nu_m}_{\mu_m} \\ &- 2p\, \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\, \phi^{\nu_{d-2p+1}}\, \phi^{\bullet}_{\mu_{d-2p+1}} \prod_{n=1}^{d-2p-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \prod_{m=d-2p+2}^{d} \phi^{\nu_m}_{\mu_m}\end{aligned} \tag{487}$$

Finally, we contract $I^{\bullet}_{\left(0\leq p\leq \frac{d}{2}\right)}$ with $\phi_{\bullet}$ to get the following relations :

$$I^{\bullet}_{\left(0\leq p\leq \frac{d}{2}\right)}\phi_{\bullet} = \chi\Omega_{1,p} - (d-2p)\Omega_{2,p} - 2p\Omega_{3,p} = 0 \tag{488}$$

where

$$\begin{aligned}\Omega_{1,p} =& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \prod_{n=1}^{d-2p-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \prod_{m=d-2p+1}^{d} \phi^{\nu_m}_{\mu_m} \\ \Omega_{2,p} =& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \chi^{\nu_2}_{\bullet} R^{\nu_1\,\bullet}_{\mu_1\mu_2} \prod_{n=3}^{d-2p-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \prod_{m=d-2p+1}^{d} \phi^{\nu_m}_{\mu_m} \\ \Omega_{3,p} =& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \chi^{\nu_{d-2p+1}}_{\bullet} \phi^{\bullet}_{\mu_{d-2p+1}} \prod_{n=1}^{d-2p-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \prod_{m=d-2p+2}^{d} \phi^{\nu_m}_{\mu_m}\end{aligned} \tag{489}$$

4. Proof of Equation (43)

Now consider the following scalar

$$\begin{aligned}\mathscr{E} :=& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \nabla^{\nu_d} \left(\frac{\phi^{;\nu_{d-1}}_{;\mu_{d-1}} \phi_{;\mu_d}}{\phi_{;\alpha}\phi^{;\alpha}} \sum_{p=0}^{\frac{d-2}{2}} \alpha_{(p)} \left\{ \prod_{n=1}^{d-2p-3} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \prod_{m=d-2p-1}^{d-2} \phi^{;\nu_m}_{;\mu_m} \right\} \frac{1}{(\phi_{;\alpha}\phi^{;\alpha})^p} \right) \\ =& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \nabla^{\nu_d} \left(\frac{\phi^{;\nu_{d-1}}_{;\mu_{d-1}} \phi_{;\mu_d}}{\phi_{;\alpha}\phi^{;\alpha}} \sum_{p=0}^{\frac{d-2}{2}} \alpha_{(p)} \left\{ \frac{R^{\nu_1\nu_2}_{\mu_1\mu_2} \ldots R^{\nu_{d-3-2p}\nu_{d-2(p+1)}}_{\mu_{d-3-2p}\mu_{d-2(p+1)}} \phi^{;\nu_{d-1-2p}}_{;\mu_{d-1-2p}} \ldots \phi^{;\nu_{d-2}}_{;\mu_{d-2}}}{(\phi_{;\alpha}\phi^{;\alpha})^p} \right\} \right) \\ =& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \sum_{p=0}^{\frac{d-2}{2}} \alpha_{(p)} \nabla^{\nu_d} \left(\frac{1}{(\phi_{;\alpha}\phi^{;\alpha})^{p+1}} \phi^{;\nu_{d-1}}_{;\mu_{d-1}} \phi_{;\mu_d} \prod_{1}^{d-2p-3} R \prod_{d-2p-1}^{d-2} \phi \right)\end{aligned} \tag{490}$$

Note that $\nabla^{\nu_d}\left(\frac{1}{(\phi_{;\alpha}\phi^{;\alpha})^{p+1}}\right) = -2(p+1)\phi^{\bullet}\phi^{\nu_d}_{\bullet}\frac{1}{(\phi_{;\alpha}\phi^{;\alpha})^{p+2}}$ and that by antisymmetric commutation of covariant derivatives : $\delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\nabla^{\nu_d}\left(\phi^{;\nu_{d-1}}_{;\mu_{d-1}}\right) = -\frac{1}{2}R^{\nu_{d-1}\nu_d}_{\mu_{d-1}\,\bullet}\phi^{\bullet}\,\delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}$. Moreover, by Bianchi identity together with antisymmetry in all the indices, the covariant derivative of the product of

Riemann tensors vanishes. Therefore,

$$
\begin{aligned}
\mathscr{E} = \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \sum_{p=0}^{\frac{d-2}{2}} \alpha_{(p)} \Bigg(& -2(p+1) \frac{\chi^{\bullet}_{\mu_d}\phi^{\nu_d}_{\bullet}}{(\phi_{;\alpha}\phi^{;\alpha})^{p+2}} \prod_{1}^{d-2p-3} R \prod_{d-2p-1}^{d-1} \phi \\
& + \frac{1}{(\phi_{;\alpha}\phi^{;\alpha})^{p+1}} \Bigg[-\frac{1}{2}\chi^{\bullet}_{\mu_d} R^{\nu_{d-1}\nu_d}_{\mu_{d-1}\bullet} \prod_{1}^{d-2p-3} R \prod_{d-2p-1}^{d-1} \phi + \prod_{1}^{d-2p-3} R \prod_{d-2p-1}^{d} \phi \\
& - \frac{1}{2}\chi^{\bullet}_{\mu_d}\phi^{\nu_{d-1}}_{\mu_{d-1}} \prod_{1}^{d-2p-3} R \sum_{i=d-2p-1}^{d-2} R^{\nu_i\nu_d}_{\mu_i\bullet}\, \phi_{(d-2p-1)}\ldots\phi_{(i-1)}\phi_{(i+1)}\ldots\phi_{(d-2)} \Bigg] \Bigg)
\end{aligned}
\tag{491}
$$

Now, as before, we can rewrite the terms with contractions on the Riemann tensor by putting the contracted index on a term $R^{\nu_1\,\bullet}_{\mu_1\mu_2}$. In the first such term, it can be done by the following permutations $(d-1) \to (1) \to (3) \to \ldots \to (d-2p-3) \to (d-2p-1) \to (d-2p+1) \to \ldots \to (d-1)$ and $(d) \to (2) \to (4) \to \ldots \to (d-2) \to (d)$ of both μ and ν (so that it does not change the order of indices of the δ), which gives :

$$
-\frac{1}{2}\delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\chi^{\bullet}_{\mu_d} R^{\nu_{d-1}\nu_d}_{\mu_{d-1}\bullet} \prod_{1}^{d-2p-3} R \prod_{d-2p-1}^{d-1} \phi = -\frac{1}{2}\,\Omega_{2,p}
\tag{492}
$$

while for the second term we do $(i) \to (1) \to (3) \to \ldots \to (d-2p-3) \to (d-2p-1) \to (d) \to (2) \to (4) \to \ldots \to (d-2p-2) \to (d-2p) \to (d-2p+1) \to \ldots \to (i-1) \to (i)$, for both μ and ν. This makes the general term of the sum independent from i, so that :

$$
-\frac{1}{2}\delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\chi^{\bullet}_{\mu_d}\phi^{\nu_{d-1}}_{\mu_{d-1}} \prod_{1}^{d-2p-3} R \sum_{i=d-2p-1}^{d-2} R^{\nu_i\nu_d}_{\mu_i\bullet}\, \phi_{(d-2p-1)}\ldots\phi_{(i-1)}\phi_{(i+1)}\ldots\phi_{(d-2)} = -p\,\Omega_{2,p}
\tag{493}
$$

Concerning the remaining two terms in $\mathscr{E}$, note that :

$$
\begin{aligned}
& \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \sum_{p=0}^{\frac{d-2}{2}} \alpha_{(p)} \frac{1}{\chi^{p+2}} \left(\chi \prod_{d-2p-1}^{d} \phi - 2(p+1)\chi^{\bullet}_{\mu_d}\phi^{\nu_d}_{\bullet} \prod_{d-2p-1}^{d-1} \phi \right) \prod_{1}^{d-2p-3} R \\
&= \delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d} \sum_{q=0}^{\frac{d}{2}} \alpha_{(q-1)} \frac{1}{\chi^{q+1}} \left(\chi \prod_{d-2q+1}^{d} \phi - 2q\chi^{\bullet}_{\mu_d}\phi^{\nu_d}_{\bullet} \prod_{d-2q+1}^{d-1} \phi \right) \prod_{1}^{d-2q-1} R \\
&= \sum_{q=0}^{\frac{d}{2}} \alpha_{(q-1)}(d-2q)\Omega_{2,q}
\end{aligned}
\tag{494}
$$

where in the last line we used that $\chi\Omega_{1,p} - (d-2p)\Omega_{2,p} - 2p\Omega_{3,p} = 0$. Thus, we can rewrite the total derivative $\mathscr{E}$ as :

$$
\begin{aligned}
\mathscr{E} &= -\frac{\alpha_0}{2}\frac{1}{\chi}\Omega_{2,0} + \sum_{p=1}^{\frac{d-2}{2}} \left(\alpha_{(p-1)}\,(d-2p) - \alpha_p \left(p + \frac{1}{2} \right) \right) \frac{\Omega_{2,p}}{\chi^{p+1}} \\
&= -\frac{\alpha_0}{2\,d}\,\Omega_{1,0} + \sum_{p=1}^{\frac{d-2}{2}} \left(\alpha_{(p-1)}\,(d-2p) - \alpha_p \left(p + \frac{1}{2} \right) \right) \frac{\Omega_{2,p}}{\chi^{p+1}}
\end{aligned}
\tag{495}
$$

If we choose the series $\alpha_{(p)}$ to be such that :

$$
\alpha_{(p-1)}\,(d-2p) - \alpha_p \left(p + \frac{1}{2} \right) = 0
\tag{496}
$$

for all $0 \leq p \leq \frac{d-2}{2}$, then

$$
\alpha_{(p)} = c\,\frac{2^{3p}p!}{(\frac{d}{2}-p-1)!(2p+1)!}
\tag{497}
$$

where c does not depend on p. It gives $\alpha_0 = \frac{c}{\left(\frac{d}{2}-1\right)!}$, and therefore, by definition of $\Omega_{1,0}$:

$$\begin{aligned}\mathscr{E} &= -\frac{\alpha_0}{2\,d}\,\delta^{\mu_1\ldots\mu_d}_{\nu_1\ldots\nu_d}\prod_{n=1}^{d-1} R^{\nu_n\nu_{n+1}}_{\mu_n\mu_{n+1}} \\ &= -\frac{\alpha_0\,2^{\frac{d}{2}-1}}{d}\,\mathcal{L}_{(d)} = -\frac{c\,2^{\frac{d}{2}-1}}{d\,\left(\frac{d}{2}-1\right)!}\,\mathcal{L}_{(d)}\end{aligned} \tag{498}$$

Therefore, choosing $c = \frac{d\left(\frac{d}{2}-1\right)!}{2^{\frac{d}{2}-1}}$ leads to

$$\mathcal{L}_{(d)} = \varepsilon^{\mu_1\rho_1\ldots\mu_n\rho_n}\varepsilon_{\nu_1\sigma_1\ldots\nu_n\sigma_n}\ \nabla^{\sigma_n}\left(\frac{\phi^{;\nu_n}_{;\mu_n}\phi_{;\rho_n}}{\phi_{;\alpha}\phi^{;\alpha}}\sum_{p=0}^{n-1}\alpha_p\prod_{k=1}^{n-p-1}R^{\nu_k\sigma_k}_{\mu_k\rho_k}\prod_{m=n-p}^{n-1}\frac{\phi^{;\nu_m}_{;\mu_m}\phi^{;\sigma_m}_{;\rho_m}}{\phi_{;\alpha}\phi^{;\alpha}}\right) \tag{499}$$

with $n = \frac{d}{2}$, and for Lorentzian signature, $\varepsilon^{\mu_1\rho_1\ldots\mu_n\rho_n}\varepsilon_{\nu_1\sigma_1\ldots\nu_n\sigma_n} = -\,\delta^{\mu_1\rho_1\ldots\mu_n\rho_n}_{\nu_1\sigma_1\ldots\nu_n\sigma_n}$, where

$$\alpha_{(p)} = d\,\frac{2^{3p-\frac{d}{2}+1}p!\left(\frac{d}{2}-1\right)!}{\left(\frac{d}{2}-p-1\right)!(2p+1)!} = 2^{p-\frac{d}{2}}\,d\sqrt{\pi}\binom{\frac{d}{2}-1}{p}\frac{\Gamma\left(p+1\right)}{\Gamma\left(p+\frac{3}{2}\right)} \tag{500}$$

Supplements on Non-Polynomial Gravity theories

5. Wald Entropy

In this Appendix, we will consider the following point. Even if we saw for the DST action in Eq(67, 71) that the minisuperspace field equations are identical to the symmetric reduction of the full EOMs (which has to be true due to the principle of symmetric criticality [211]), as far as we know, it has not been shown for the Wald entropy (which similarly to the derivation of the full field equations, also involves the full Lagrangian) that the minisuperspace Wald Entropy is equivalent to the full one. Therefore, we cannot a priori (without a formal proof) consider the Wald entropy of the Non-Polynomial gravity models that we are studying, in the same way as the field equations ; i.e. instead of deriving the Wald entropy from the full Lagrange, derive only the one associated with its spherically symmetric reduction.

For this reason, we will briefly check that for a quite simple example of Non-Polynomial Gravity theory which has the same DSS sector as General Relativity, the full Wald entropy associated with the (NPG) Schwarzschild black hole does reproduce the GR result, so that one could have instead consider the minisuperspace Wald entropy without changing the result.

The Wald entropy [377], see also [378], is defined by :

$$S_W = -2\pi\int_\Sigma\left(\frac{\partial L}{\partial R_{\mu\nu\alpha\beta}}\right)^0\epsilon_{\mu\nu}\epsilon_{\alpha\beta}\sqrt{h}d^2x \tag{501}$$

with $\epsilon_{\mu\nu}$ the binormal to the horizon Σ, so that $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$, $\epsilon_{01} = -\epsilon_{10} = 1$ while the other components vanish, $L = \frac{1}{16\pi G}\mathscr{L}$, the "0" means that the tensorial quantity is evaluated on-shell and h is the induced metric on the horizon. It can be applied for asymptotically flat spacetime admitting a bifurcate Killing horizon.

A four-dimensional example of NPG Wald entropy

Consider the non-polynomial scalar[20] :

$$\mathcal{R} := R_{\mu\nu\alpha\beta}\mathcal{N}^{\mu\nu\sigma\rho}\mathcal{N}^{\alpha\beta}{}_{\sigma\rho}. \tag{502}$$

Then, one can check that at the level of the DSS decomposition, we have

$$R\rfloor = \left(6\mathcal{R} - \sqrt{3}\mathcal{W}\right)\rfloor,^{21} \tag{503}$$

[20] Not to be confused with the scalar $\mathcal{R}$ that we used to reach the Ricci scalar of the horizon manifold in Eq(100).

Therefore, they admit the same Schwarzschild black hole, so that the Wald entropy can be calculate from the theory

$$I = \int d^4x\sqrt{-g}\big(6\mathcal{R} - \sqrt{3}\mathcal{W}\big) \tag{504}$$

and should yield the same result as General Relativity. To check this, note that

$$\frac{\delta \mathcal{W}}{\delta R_{\mu\nu\alpha\beta}} = \mathcal{N}^{\mu\nu\alpha\beta}, \tag{505}$$

which is in agreement with the derivation of [275] but expressed in a more compact form, and

$$\frac{\delta \mathcal{R}}{\delta R_{\mu\nu\alpha\beta}} = \mathcal{N}^{\mu\nu}{}_{\bullet}{}^{\bullet}\mathcal{N}^{\alpha\beta}{}_{\bullet}{}^{\bullet} + \frac{2}{3}g^{\mu\alpha}g^{\nu\beta} + \frac{2}{\mathcal{W}}\big(R^{\mu\nu}{}_{\bullet}{}^{\bullet}\mathcal{N}^{\alpha\beta}{}_{\bullet}{}^{\bullet} - 2R^{\mu}{}_{\bullet}{}^{\bullet}\mathcal{N}^{\alpha}{}_{\bullet}{}^{\bullet}g^{\nu\beta} - \mathcal{R}\mathcal{N}^{\mu\nu\alpha\beta}\big). \tag{506}$$

We see that it is non-polynomial in general, and in fact even $\frac{\delta\mathcal{R}}{\delta R_{\mu\nu\alpha\beta}}\rfloor$ is. However, it turns out that the contraction with the binormals makes these terms cancelled, giving the right result :

$$\Big(\frac{\delta \mathcal{R}}{\delta R_{\mu\nu\alpha\beta}}\Big)^0\epsilon_{\mu\nu}\epsilon_{\alpha\beta} = -\frac{2}{3}, \quad \text{and we know that} \quad \Big(\frac{\delta R}{\delta R_{\mu\nu\alpha\beta}}\Big)^0\epsilon_{\mu\nu}\epsilon_{\alpha\beta} = -2 = \Big(\frac{\delta\sqrt{3}\mathcal{W}}{\delta R_{\mu\nu\alpha\beta}}\Big)^0\epsilon_{\mu\nu}\epsilon_{\alpha\beta} \tag{507}$$

Therefore :

$$S_W^R = 6S_W^{\mathcal{R}} - \sqrt{3}S_W^{\mathcal{W}}, \tag{508}$$

what shows that the entropies of the same BH solution of these two completely different theories (for general metric fields other than DSS ones) is indeed the same which is a good thing considering that the entropy should be a property of the black hole itself, irrespectively of the dynamical theory from which it comes from.

Minisuperspace Wald entropy

Now that we have checked for a non-polynomial gravity theory, we will consider the following minisuperspace Wald entropy :

$$S_{msW} := -2\pi\int_\Sigma \Big(\frac{\partial L\rfloor}{\partial R_{abcd}}\Big)^0\epsilon_{ab}\epsilon_{cd}\sqrt{h}d^2x \tag{509}$$

and check that for many gravitational theories admitting second order DSS field equations (Lovelock & Quasi-Topological), we have

$$S_{msW} = S_W \tag{510}$$

The only 2D spherically symmetric scalar containing R_{abcd} are the powers of the 2D Ricci scalar $R^{(2)}(\gamma)$. Therefore, if the previous equation is true, the only contribution to the Wald entropy of any gravitational theory $F\,(\mathrm{Riem})$ is contained in the first term of the DSS decomposition of the theory : $F\,(\mathrm{Riem}) = G\left(R^{(2)}, \mathcal{Z}, \mathcal{Y}, \mathcal{Y}_i\right) + O\left(\mathcal{Z}, \mathcal{Y}, \mathcal{Y}_i\right)$.

$$S_{msW} = -\frac{1}{8G}\int_\Sigma \left(\gamma^{ac}\gamma^{bd}\epsilon_{ab}\epsilon_{cd}\right)^0 \left(\frac{\partial G\left(R^{(2)}, \mathcal{Z}, \mathcal{Y}, \mathcal{Y}_i\right)}{\partial R^{(2)}}\right)^0 r_H^n dx^2 \tag{511}$$

[21] Here one should stop and wonder if this equality isn't just an hidden background independant one like it can happen within this approach (if it would be the case, of course the full EOMs of GR (R) and this theory would be trivially the same). It can be rewritten as $R\rfloor = \big(6\frac{R_{\mu\nu\alpha\beta}W^{\mu\nu\sigma\rho}W^{\alpha\beta}{}_{\sigma\rho}}{W_{\lambda\delta\iota\kappa}W^{\lambda\delta\iota\kappa}} - \sqrt{3}\sqrt{W_{\mu\nu\alpha\beta}W^{\mu\nu\alpha\beta}}\big)\rfloor$, and if it would be background independant, then by rearanging, any metric should satisfies :

$$36\Big(R_{\mu\nu\alpha\beta}W^{\mu\nu\sigma\rho}W^{\alpha\beta}{}_{\sigma\rho}\Big)^2 - 12R\Big(R_{\mu\nu\alpha\beta}W^{\mu\nu\sigma\rho}W^{\alpha\beta}{}_{\sigma\rho}\Big)W^{\lambda\delta}{}_{\iota\kappa}W_{\lambda\delta}{}^{\iota\kappa} + R^2\big(W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta}\big)^2$$
$$- 3\big(W^{\mu\nu}{}_{\alpha\beta}W_{\mu\nu}{}^{\alpha\beta}\big)^3 = 0.$$

However, $g = \mathrm{diag}\big(-1, A(x_2,x_3), 1, 1\big)$ with coordinates (x_0, x_1, x_2, x_3) does not.

For $\gamma^{00}\gamma^{11} = -1$, which is the case for Einstein, Lovelock and Quasi-Topological black holes, $\left(\gamma^{ac}\gamma^{bd}\epsilon_{ab}\epsilon_{cd}\right)^0 = -2$ and the minisuperspace Wald entropy reduces to :

$$S_{msW} = \frac{1}{4G}\int_\Sigma \left(\frac{\partial G\left(R^{(2)}, \mathcal{Z}, \mathcal{Y}, \mathcal{Y}_i\right)}{\partial R^{(2)}}\right)^0 r_H^n dx^2 \tag{512}$$

For black holes of the form $ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\Omega^2_{k,n}$, the 2D scalars become : $\mathcal{Z} = (k-a)/r^2$, $R^{(2)} = -a''$ and $\mathcal{Y}_a^b = (a'/2r)\,\mathrm{diag}\,(-a, 1/a)$ gives $\mathcal{Y} = a'/r$, $\mathcal{Y}_2 = a'^2/2r^2$ and $\mathcal{Y}_3 = a'^3/4r^3$.

Because we assume that the spacetime has an horizon at r_H, it can be written as $a(r) = (r-r_H)c(r)$, and the scalars, when evaluated at the horizon becomes : $\mathcal{Z} = k/r_H^2$, $R^{(2)} = -2c'(r_H)$, $\mathcal{Y} = c(r_H)/r_H$.

Applications : It gives the usual result for the Schwarzschild black hole in General Relativity and the black hole solutions of Lovelock and Quasi-Topological gravities. Indeed, for Lovelock black holes each term of the Lagrangian contributes as :

$$G\left(R^{(2)}, \mathcal{Z}, \mathcal{Y}, \mathcal{Y}_i\right) = l^{2(p-1)}\beta_p \frac{(d-2)!}{(d-2p)!}\left(\gamma_p R^{(2)}\mathcal{Z}\right)\mathcal{Z}^{p-2} \tag{513}$$

or using the integrated by part lagrangian :

$$G\left(R^{(2)}, \mathcal{Z}, \mathcal{Y}, \mathcal{Y}_i\right) = l^{2(p-1)}\beta_p \frac{(d-2)!}{(d-2p)!}\left(pR^{(2)}r^{2-2p}\right) \tag{514}$$

Therefore, the minisuperspace Wald entropy is in this case given by :

$$\begin{aligned} S_{msW} &= \frac{1}{4G}\sum_{p=1}^{m}\int_\Sigma \left(l^{2(p-1)}\beta_p \frac{(d-2)!}{(d-2p)!}\left(\gamma_p\mathcal{Z}\right)\mathcal{Z}^{p-2}\right)^0 r_H^n dx^2 \\ &= \frac{1}{4G}\sum_{p=1}^{m}\mu_p l^{2(p-1)}k^{p-1}r_H^{n-2(p-1)}\int_\Sigma dx^2 \\ &= \frac{\mathcal{A}_{n,k}}{4G}\sum_{p=1}^{m}\mu_p l^{2(p-1)}k^{p-1}\left(\frac{\mathcal{A}_{n,k}}{x_{n,k}}\right)^{2(1-p)/n} \end{aligned} \tag{515}$$

where $\mu_p = \beta_p\gamma_p\frac{(d-2)!}{(d-2p)!}$ and $x_{n,k}$ is a constant depending on the topology and dimension of the horizon manifold, for example for spherical topology it is given by $x_{n,1} = 2\pi^{(n+1)/2}/\Gamma\left((n+1)/2\right)$. This results are in agreement with [379].

As for the the case of the Quasi-Topological theory given by the scalar $\mathcal{U}$, the results is again in agreement with the standard result [144] and the calculation is very similar to the Lovelock case. More interestingly, consider the theory

$$\mathcal{T} = 14S_1 + 2S_3 + \frac{1}{n(2n+3)}\left(-\frac{p_1}{4}S_2 - 2p_2S_4 - \frac{4p_3}{3}S_5 + p_4S_6 - \frac{p_5}{12}S_7\right) \tag{516}$$

where $p_1 = 4n^2 - 13n - 4$, $p_2 = 4n^2 + 25n + 4$, $p_3 = 2n^2 - 27n + 4$, $p_4 = 4n^2 - 5n + 8$ and $p_5 = 4n^2 + 3n + 20$. In four dimensions, it gives :

$$\mathcal{T} = 12\left(R_{(2)}\mathcal{Y}_2 + 4\left(\mathcal{Y}\mathcal{Y}_2 - \mathcal{Y}_3\right) + 2\mathcal{Z}\left(R_{(2)}\mathcal{Y} + \mathcal{Y}_2\right)\right) \tag{517}$$

The decomposition of $\mathcal{T}$ in arbitrary dimension is :

$$\begin{aligned} \mathcal{T} =& \frac{n-1}{4(2n+3)}R_{(2)}\left((2-n)\left(4n^3 - 9n^2 + 43n - 12\right)\mathcal{Z}^2 + 16\left(n^2 + 9n - 1\right)\left(\mathcal{Y}_2 + 2\mathcal{Y}\mathcal{Z}\right)\right) \\ &+ O\left(F\left(\mathcal{Z}, \mathcal{Y}, \mathcal{Y}_2, \mathcal{Y}_3\right)\right) \end{aligned} \tag{518}$$

This QTG theory $\mathcal{T}$ differs from the previous ones by the fact that, although still linear in the Ricci scalar $R^{(2)}(\gamma)$ of the manifold Σ, the factor multipliying this scalar is non-linear in $\mathcal{Y}_i$ and $\mathcal{Z}$, so that it does not belong to 2D Horndeski theory. Nonetheless, the result agree again with [146], as it gives :

$$\frac{\partial G}{\partial R^{(2)}} = \frac{n-1}{4(2n+3)} \left((2-n) \left(4n^3 - 9n^2 + 43n - 12\right) \mathcal{Z}^2 + 16 \left(n^2 + 9n - 1\right) \left(\mathcal{Y}_2 + 2\mathcal{Y}\mathcal{Z}\right) \right) \quad (519)$$

$$S_{msW}\left(\mathcal{T}\right) = \frac{4(n-1)}{2n+3} \frac{1}{r_H^4} \left(-k^2 \frac{n-2}{16} \lambda + 8\pi r_H \left(n^2 + 9n - 1\right) T \left(k + \pi r T\right) \right) \quad (520)$$

where $T = \frac{c(r_H)}{4\pi}$ is the temperature of the black hole, $\lambda = \left(4n^3 - 9n^2 + 43n - 12\right)$.

6. Equivalence of NPG and Non-Minimal Coupling at the Field Equations level

Now we wish to present a possible interpretation of NPG theories, which is a bit baroque. First note that any NPG theory of order $\mathcal{O}$ constructed from two scalars $\mathcal{S}^{(p+\mathcal{O})}$ and $\mathcal{U}^{(p)}$ of respective orders $p + \mathcal{O}$ and p of the form :

$$\mathcal{L} = \frac{\mathcal{S}^{(p+\mathcal{O})}\left(\text{Riem}, \nabla\right)}{\mathcal{U}^{(p)}\left(\text{Riem}, \nabla\right)} \quad (521)$$

can be expressed at the level of the field equations as a polynomial higher order theory with a non-minimal coupling on the matter side. Indeed, the variation of the action with respect to the metric is of the form :

$$\begin{aligned} \delta_g \mathcal{L} &= \left(\frac{1}{\mathcal{U}}\right) \delta\mathcal{S} - \left(\frac{\mathcal{S}}{\mathcal{U}^2}\right) \delta\mathcal{U} \\ &= \sum_{n=0}^{\mathcal{O}+p} \left(\frac{1}{\mathcal{U}}\right) \mathcal{S}^{(p+\mathcal{O}-n)}_{\alpha\beta\mu_1\ldots\mu_n} \nabla_{\mu_1} \ldots \nabla_{\mu_n} \delta g^{\alpha\beta} - \sum_{n=0}^{p} \left(\frac{\mathcal{S}}{\mathcal{U}^2}\right) \mathcal{U}^{(p-n)}_{\alpha\beta\mu_1\ldots\mu_n} \nabla_{\mu_1} \ldots \nabla_{\mu_n} \delta g^{\alpha\beta} \\ &= \left(\sum_{n=0}^{\mathcal{O}+p} (-1)^n \nabla_{\mu_1} \ldots \nabla_{\mu_n} \left(\frac{1}{\mathcal{U}} \mathcal{S}^{(p+\mathcal{O}-n)}_{\alpha\beta\mu_1\ldots\mu_n} \right) - \sum_{n=0}^{p} (-1)^n \nabla_{\mu_1} \ldots \nabla_{\mu_n} \left(\frac{\mathcal{S}}{\mathcal{U}^2} \mathcal{U}^{(p-n)}_{\alpha\beta\mu_1\ldots\mu_n} \right) \right) \delta g^{\alpha\beta} \end{aligned} \quad (522)$$

where $\mathcal{S}^{(p+\mathcal{O}-n)}_{\alpha\beta\mu_1\ldots\mu_n}$ and $\mathcal{U}^{(p-n)}_{\alpha\beta\mu_1\ldots\mu_n}$ are two sets of rank n tensors of respective order $(p + \mathcal{O} - n)$, $(p-n)$, for respectively $0 \leq n \leq \mathcal{O}+p$ and $0 \leq n \leq p$. Moreover, the denominator with the largest order comes from the following terms :

$$\begin{aligned} \nabla_{\mu_1} \ldots \nabla_{\mu_{p+\mathcal{O}}} \left(1/\mathcal{U}\right) &= (-1)^{p+\mathcal{O}} (p+\mathcal{O})! \, \mathcal{U}^{-(p+\mathcal{O}+1)} \nabla_{\mu_1}\mathcal{U} \ldots \nabla_{\mu_{p+\mathcal{O}}}\mathcal{U} + \ldots \\ \nabla_{\mu_1} \ldots \nabla_{\mu_p} \left(1/\mathcal{U}^2\right) &= (-1)^p (p+1)! \, \mathcal{U}^{-(p+2)} \nabla_{\mu_1}\mathcal{U} \ldots \nabla_{\mu_p}\mathcal{U} + \ldots \end{aligned}$$

So that the variation of the Lagrangian with respect to the metric can be put in the form :

$$\begin{aligned} \frac{\delta_g \mathcal{L}}{\delta g^{\alpha\beta}} &= (-1)^{p+\mathcal{O}} (p+\mathcal{O})! \, \mathcal{U}^{-(p+\mathcal{O}+1)} \mathcal{S}^{(0)}_{\alpha\beta\mu_1\ldots\mu_{p+\mathcal{O}}} \nabla_{\mu_1}\mathcal{U} \ldots \nabla_{\mu_{p+\mathcal{O}}}\mathcal{U} + \mathcal{K}^{[\mathcal{O}]}_{\alpha\beta} \\ &= \mathcal{E}^{[(p+\mathcal{O})(p+1)]}_{\alpha\beta} \mathcal{U}^{-(p+\mathcal{O}+1)} \end{aligned} \quad (523)$$

Finally, the polynomial field equations of the theory, with a non-minimal coupling, and order $(p + \mathcal{O})(p + 1)$, can be written in the following form

$$l^{\mathcal{O}-2} \mathcal{E}^{[(p+\mathcal{O})(p+1)]}_{\alpha\beta} = \kappa \, \mathcal{U}^{(p+\mathcal{O}+1)} \, T_{\alpha\beta} \quad (524)$$

where l is a length scale introduced for dimensional reasons. As an example, consider the theory

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R + l^2 \mathcal{N}^{\mu\nu}_{\ \ \alpha\beta} \mathcal{N}^{\sigma\rho}_{\ \ \lambda\delta} R_{\sigma\rho}^{\ \ \alpha\beta} R_{\mu\nu}^{\ \ \lambda\delta} \right) \quad (525)$$

In this case, $\mathcal{N}^{\mu\nu}_{\ \ \alpha\beta} \mathcal{N}^{\sigma\rho}_{\ \ \lambda\delta} = W^{\mu\nu}_{\ \ \alpha\beta} W^{\sigma\rho}_{\ \ \lambda\delta} / W^{\kappa\iota}_{\ \ \pi\epsilon} W_{\kappa\iota}^{\ \ \pi\epsilon}$, therefore we have $\mathcal{O} = p = 4$. Thus, the polynomial field equations with non-minimal coupling are of order 40, i.e. of the form $\left(\mathscr{R}\left(\partial g \partial g, \partial\partial g\right)\right)^{20}$. This establishes that considering NPG theories allows to investigate the very high order of corrections in the curvature.

In the case of effective-like NPG actions, for example Eq(141), this is even more drastic. For example, consider the following theory :

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\sum_{i=-1}^{m} l^{2i}\mathcal{R}^i\left(R + \text{Riem}.u + \nabla u\nabla u + \nabla\nabla u\right) \tag{526}$$

where u is a non-polynomial tensor that can be expressed as a quotient of an order $2j$ tensor and an order $2j$ scalar as $u = \frac{T^{(2j)}}{\mathcal{K}^{(2j)}}$. Of course, in order to fall in the class of NPG that we are studying here, this tensor must be of zeroth order for a subset of the space of metric fields. In order to have the proper General Relativity limit when $l \to 0$, we have to choose $l^{-2}\alpha_{-1}\mathcal{L}_{-1} = -2\Lambda$, $\alpha_0 = 1$ and $\mathcal{L}_0 = R$. Then, expanding the terms inside the derivatives and putting all the terms of the action on the same denominator gives a ratio of two sets of polynomial higher order curvature scalars $\mathcal{X}$ and $\mathcal{K}^{4+3i}$ of respective order $2((i+1)+j(4+3i))$ and $2j(4+3i)$:

$$\begin{aligned}
I &= \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\sum_{i=-1}^{m} l^{2i}\frac{\mathcal{X}^{[2((i+1)+j(4+3i))]}}{\mathcal{K}^{4+3i}} \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\sum_{i=-1}^{m} l^{2i}\frac{\mathcal{K}^{3(m-i)}\mathcal{X}^{[2((i+1)+j(4+3i))]}}{\mathcal{K}^{4+3m}} \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\sum_{i=-1}^{m} l^{2i}\frac{\mathcal{S}^{[2(i+1+j(4+3m))]}}{\mathcal{U}^{[2j(4+3m)]}}
\end{aligned} \tag{527}$$

In the second line, we just put all the term of the sum on the highest order denominator in order to end up with two polynomial higher derivative scalars $\mathcal{S}^{[2(i+1+j(4+3m))]} := \mathcal{K}^{3(m-i)}\mathcal{X}^{[2((i+1)+j(4+3i))]}$ and $\mathcal{U}^{[2j(4+3m)]} := \mathcal{K}^{4+3m}$ of respective order $[2(i+1+j(4+3m))]$ and $[2j(4+3m)]$. We can now apply the previous result, with $p = 2j(4+3m)$ and $\mathcal{O} = 2(m+1)$.

For Eq(141), the order of the tensor T and scalar $\mathcal{K}$ is six because they are related to the square of the Cotton tensor, which gives $j = 3$. Now recall that the order of the polynomial field equations with non-minimal coupling is $(p+\mathcal{O})(p+1)$ so in this case it gives field equations of order $2(1+m+j(4+3m))(1+2j(4+3m))$. Thus, for the critical order theory of Eq(141) ($m = 1$), it gives 1978, while for the theory $m = 2$ admitting a non-minimal-Yang-Mills like black hole, the order is 4026. This is very much baroque.

7. Critical Order theory

Finally, to come full circle, we will see that the ($\mathcal{S}$-part of the) Critical Order theory of the models Eq(141), given by Eq(171), gives degenerate SS solutions, just like Eq(57) and Eq(76). Indeed, consider the theory

$$\frac{I_{\mathcal{S}}}{\mathcal{A}_{k,n,1}} = \int_{\Sigma} d^2x\,\sqrt{-\gamma}\,\left(R^{(2)}\log r - \frac{\mathcal{D}r.\mathcal{D}r}{r^2}\right) + \oint_{\partial\Sigma} dl\,\sqrt{h}\,\left(\lim_{i\to n}\frac{2K^{(2)} + v^A\,n_A}{n-i} + 2\,K^{(2)}\log r\right) \tag{528}$$

The field equations are

$$\begin{aligned}
0 &= +\,r^2R^{(2)} + 2(r\Box r - \nabla r.\nabla r) =: \mathscr{E} \\
0 &= -\,r\left(\nabla_a\nabla_b - \gamma_{ab}\Box\right)r - \frac{1}{2}\gamma_{ab}\nabla r\nabla r =: \mathcal{E}_{ab}
\end{aligned} \tag{529}$$

Then $\mathcal{E}_{ab}\gamma^{ab} = r\Box r - \nabla r.\nabla r = 0$, so the first equation implies $R^{(2)} = 0$ meaning that $\gamma_{ab} = \eta_{ab}$. The general solutions of the second equation are therefore all the functions $r(x^0+x^1)$ and $r(x^0-x^1)$, i.e. depending on the advanced and retarded time. The spherically symmetric solutions of this theory are therefore given by the intervals :

$$\begin{aligned}
ds^2 &= -dudv + r(u)^2 d\Omega^2_{n,k,1} \\
ds^2 &= -dudv + r(v)^2 d\Omega^2_{n,k,1}
\end{aligned} \tag{530}$$

The scalars are proportional to power of $k/r(u)^2$ and $k/r(v)^2$, so in order to have regular solutions, we just need $r(u) \neq 0$ for all u. Once again, we see that this theory has a degenerate spherically symmetric sector as it is not able to fix the metric entirely.

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