

From the Hamiltonian formalism to the Spin-Foams The final step in LQG?

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Chapter 1

Introduction

Already in ancient Greece, the pre-Socratic philosophers thought that natural phenomena, although different, were homogeneous, of the same fundamental nature. In their theories can be found the search for a common reference point (arché) that puts order in the chaotic multiplicity of phenomena. After Albert Einstein's theory of gravitation (General Relativity -GR-) was published in 1915, the search for a unified field theory that combines gravity with electromagnetism began to become serious. It seemed plausible that there were no other fundamental forces. The main contributors were Gunnar Nordstrom, Hermann Weyl, Arthur Eddington, Theodor Kaluza, Oskar Klein (See Theory of Kaluza-Klein, 1921) and most notably the many attempts by Einstein and his collaborators. No attempt went through. In the first half of the twentieth century quantum mechanics was consolidated, an instrument capable of overcoming the inadequacy of classical mechanics to explain phenomena and properties such as blackbody radiation, the photoelectric effect, the specific heat of solids, the atomic spectra, the stability of atoms, the Compton effect, When in the thirties Fermi and Yukawa's studies led to the discovery of nuclear forces, the quantum formalism proved to be appropriate for the description of the new phenomena and, in 1967-68, Sheldon Glashow, Steven Weinberg and Abdus Salam showed how the weak nuclear force and the electromagnetism were simply different manifestations of the same force (electroweak). Since then, proposals have been done to include in a single grand unification theory also the strong nuclear force, and some of them (GTU $SU(5)$ and $SO(10)$) have provided testable predictions as the quantization of electric charge. At classical level there is an extension of the Kaluza-Klein theory on a 11-dimensional space $M^4 \times S^1 \times S^2 \times CP^2$. It corresponds to Einstein's General Relativity with 7 extra dimensions, and considers all four forces as different expressions of a "mega" gravitational field. The forces are unified at the classical level but, once quantized, the theory turns out to be inconsistent and therefore unusable. This is because the nuclear forces have range of 10^{-15} m for strong force and of 10^{-18} m for weak force, distances at which classical physics loses its meaning. Ultimately, it seems that quantum mechanics is compatible with electroweak and strong interactions only if we limit ourselves to spaces of dimensionality less than or equal to 4. In addition, it is inconsistent with General Relativity for spaces with more of 3 dimensions. For these reasons,

the theory of Kaluza-Klein fails doubly. Really, the incompatibility is not between general relativity and quantum mechanics in its entirety, but rather between General Relativity and the method of calculation used in quantum mechanics: perturbative expansion whose terms, in the cases indicted above, become ∞ . To get around this problem two different approaches have been taken: String Theory and Loop Quantum Gravity. The first has completely changed the wording of quantum theory, from considering local interactions, where the phenomena occur at specific points (of Feynman graphs), to interactions “extended”, where the phenomena are distributed along one limited dimension (string), open or closed. This system has eliminated the divergences in the terms of perturbative expansion, but has developed other anomalies, eliminated only by building up the theory on a space of 11 dimensions. Unfortunately, the extra dimensions introduce a huge number of arbitrariness, such as the theory can predict everything and nothing. The scientific community hopes to identify some potential whose minimum make a selection between these arbitrariness, but we are still far from such a result. The alternative discussed in this thesis is the Loop Quantum Gravity. This is simply the union of GR and quantum mechanics, without modifying the basic axioms of both. It can be made only in spaces of dimensionality equal to 4 and it surrenders completely the perturbative expansion. This produces fascinating predictions, such as the inflation of early universe, and the lack of singularities in the black holes and in the big bang. It also provides the picture of a “combinatorial” universe, described by nodes connected by paths, whose only necessary variables are integer numbers associated with nodes and paths. This last point in particular escapes the string theory which, whilst losing the locality, is however concentrated within the “very small”. The Loop Quantum Gravity, by contrast, is able to describe the universe as a whole, and to deal with transitions between universes of different topology. The downside is that the calculations are so complex that they are impracticable. Strategies have been developed to introduce a different perturbative expansion that makes the calculations feasible, but this introduces important changes to the initial structure of the theory, in a way that eliminate the beautiful cosmological predictions. Nevertheless, we tried to calculate the graviton propagator in this new “modified framework”, and the result is compatible with linearized quantum GR¹. For this reason, this framework has not been abandoned. It also seems that this formalism can easily be extended to include extra-dimensions and adapted to the unified theory of Kaluza Klein.

This thesis has been developed in an attempt to contribute to the desire for simplification and connection to the essence that has always been in the natural sciences. In particular, it was given a demonstration of how the “modified framework” of Loop Quantum Gravity is derivable from a classical formulation of the GR of Palatini type. Finally, we give suggestions for extending the model to 11 dimensions, because 11 is the number suggested by String theory, by the classical theory of Kaluza Klein, and by the GTU $SO(10)$. Probably the truth lies somewhere in between, maybe an action of a 4-dimensional brane immersed in a 11-dimensional universe would be the right compromise between String Theory and Loop Quantum Gravity. A 4-dimensional brane represents our universe, and any contact with other branes of a much smaller

¹This is the theory that describes a non-interacting particle of spin 2. It is the approximation of general relativity used to describe gravitational waves. Although deriving from a theory that is incompatible with quantum mechanics, it is instead suitable to the formalism of the latter

scale put small pieces of it in vibration. Depending on the number of dimensions in which contact is, the part could be a vibrating string or a two- or three-brane (with probability decreasing rapidly moving from string to the three-brane). So, we even lose the distinction between the notions of particles and universes, making the first totally unnecessary. The action of a 4-brane is equivalent to the action of Loop Quantum Gravity, with the coordinate-fields which assume the role of gauge fields, and the indexes in the 11-dimensional space that would become similar to the indexes of internal gauge. This thesis focuses on two specific problems: the calculation of the graviton propagator in Loop Quantum Gravity and the derivation of the “modified framework” from the Palatini formulation of GR (Chapter 8). While the first it was simply supported with a minimum contribution, the second is a problem undertaken by the student in a completely independent way that, while waiting for more in-depth audits, has not yet shown any inconsistency and for now can be hailed a success. A small space is reserved for some inedited consideration undertaken by the student on the “physical” projector. This operator is intended to project the Hilbert space of kinematic states in the subspace of physical states. The conclusion of the argument is simple and somewhat disturbing: the Loop Quantum Gravity is not an unitary theory!

Chapter 2

Classical Theory

2.1 The ADM formulation

In this section we focus on the classical Hamiltonian metrical formulation of general relativity due to Arnowitt, Deser and Misner. They start from the Einstein-Hilbert action for metric tensor fields $g_{\mu\nu}$ of lorentzian ($s = -1$) or euclidean ($s = +1$) signature which propagate on a $(D + 1)$ -dimensional manifold M .

$$\frac{1}{\kappa} \int_M d^{D+1}X \sqrt{|\det(g)|} R^{(D+1)} \quad (2.1)$$

Our signature convention is “mostly plus”, that is $(-, +, \dots, +)$ or $(+, +, \dots, +)$ in the lorentzian or euclidean case respectively so that timelike vectors have negative norm in the lorentzian case. Here $\mu, \nu, \rho, \dots = 0, 1, \dots, D$ are indices for the components of spacetime tensors and X^μ are the coordinates of M in local trivializations¹. $R^{(D+1)}$ is the curvature scalar associated with $g_{\mu\nu}$ and $\kappa = 8\pi G$ where G is Newton’s constant in units where $c = 1$. The definition of the Riemann curvature tensor is in terms of one-forms given by

$$[\nabla_\mu, \nabla_\nu]u_\rho = {}^{(D+1)}R_{\mu\nu\rho}{}^\sigma u_\sigma \quad (2.2)$$

where ∇ denotes the unique torsion free, metric-compatible, covariant differential associated with $g_{\mu\nu}$. In order to cast (2.1) into canonical form one makes the assumption

¹The concept of local trivialization is linked to that of **fiber bundle**. A fiber bundle over a differential manifold M with atlas $\{U_I, \varphi_I\}$ is a quintuple (P, M, π, F, G) consisting of a differentiable manifold P (called the total space), a differentiable manifold M (called the base space), a differentiable surjection $\pi : P \rightarrow M$, a differentiable manifold F (called the typical fibre) which is diffeomorphic to every fibre $\pi^{-1}(x)$, $x \in M$ and a Lie group G (called the structure group) which acts on F on the left, $\lambda : G \times F \rightarrow F$; $(h, f) \mapsto \lambda(h, f) =: \lambda_h(f)$, $\lambda_h \circ \lambda_{h'} = \lambda_{hh'}$, $\lambda_{h^{-1}} = (\lambda_h)^{-1}$. Furthermore, for every U_I there exist diffeomorphisms $\phi_I : U_I \times F \rightarrow \pi^{-1}(U_I)$, called **local trivializations**, such that $\phi_{Ix} : F \rightarrow F_x := \pi^{-1}(x)$; $f \mapsto \phi_{Ix}(f) := \phi_I(x, f)$ is a diffeomorphism for every $x \in U_I$. Finally, we require that there exist maps $h_{IJ} : U_I \cap U_J \neq \emptyset \rightarrow G$, called transition functions, such that for every $x \in U_I \cap U_J \neq \emptyset$ we have $\phi_{Jx} = \phi_{Ix} \circ \lambda_{h_{IJ}(x)}$. In our case $\pi^{-1}(x)$ is the tangent space in x , F is \mathbf{R}^{D+1} and h_{IJ} is the jacobian matrix of the change of coordinates.

that M has the special topology $M = \mathbf{R} \times \sigma$ where σ is a fixed three-dimensional, compact manifold without boundary. By a theorem due to Geroch, if the spacetime is globally hyperbolic² then it is necessarily of this kind of topology. Therefore, for classical physics our assumptions about the topology of M seems to be no restriction at all, at least in the lorentzian signature case. In quantum gravity, however, different kinds of topologies and, in particular, topology changes are conceivable. Our philosophy will be first to construct the quantum theory of the gravitational field based on the classical assumption that $M = \mathbf{R} \times \sigma$ and then to lift this restriction in quantum theory. Having made this assumption, one knows that M foliates into hypersurfaces $\Sigma_t := X_t(\sigma)$, that is, for each fixed $t \in \mathbf{R}$ we have an embedding (a globally injective immersion) $X_t : \sigma \rightarrow M$ defined by $X_t(x) := X(t, x)$ where x^a , $a, b, c, \dots = 1, 2, \dots, D$ are local coordinate of σ . Likewise we have a diffeomorphism $X : \mathbf{R} \times \sigma \mapsto M$; $(t, x) \mapsto X(t, x) := X_t(x)$, in other words, a one-parameter family of embedding is equivalent to a diffeomorphism. Now, since the action (2.1) is invariant under all diffeomorphisms of M , the families of embeddings X_t are completely arbitrary. A useful parametrization of the embedding and its arbitrariness can be given through its deformation vector field

$$T^\mu(X) := \left. \frac{\partial X^\mu(t, x)}{\partial t} \right|_{X=X(t, x)} =: N(X)n^\mu(X) + N^\mu(X) \quad (2.3)$$

Here n^μ is a unit normal vector to Σ_t , that is $g_{\mu\nu}n^\mu n^\nu = s$, $g_{\mu\nu}n^\mu X_{,a}^\nu = 0$, and N^μ is tangential. The coefficients of proportionality N and N^μ respectively are called lapse function and shift vector field. Notice that implicit information about the metric $g_{\mu\nu}$ has been invoked into (2.3), namely we are dealing with spacelike embeddings and metrics of the above specified signature. The lapse is nowhere vanishing since for a foliation T must be timelike everywhere. Moreover, we take it to be positive everywhere as we want a future directed foliation. We need one more property of n : by the inverse function theorem, the surface Σ_t can be defined by an equation of the form $f(X) = t = \text{const.}$. Thus $0 = \lim_{\epsilon \rightarrow 0} [f(X_t(x + \epsilon b)) - f(X_t(x))]/\epsilon = b^a X_{,a}^\nu(f, \mu)_{X=X_t(x)}$ for any tangential vector b of σ in x . It follow that up to a normalization the normal vector is proportional to an exact one-form, $n_\mu = Ff_{, \mu}$. We now have the choice to work either on σ or on its image in M , $\Sigma = X(\sigma)$ when developing the tensor calculus of so-called spatial tensor fields. To work on Σ has the advantage that we can compare

²A spacetime manifold M is said to be globally hyperbolic if the following two conditions hold: (1) For every pair of points $p, q \in M$, $D^-(p) \cap D^+(q)$ is compact. Here $D^\pm(\sigma)$ is the future (past) of a subset σ of M , that is, the set of all points which can be reached from σ along future (past) timelike or null curves. (2) ‘‘Causality’’ holds on M (no closed timelike curves exist). Classically, a more restrictive and technical assumption is required, named strong causality (no ‘‘almost closed’’ timelike curves exist); but a recent result [91] shows that causality suffices.

In other words, a manifold M is globally hyperbolic if exists a Cauchy surface for M . If there are no closed timelike curves then, given σ a spacelike surface, if $D^+(\sigma) \cup \sigma \cup D^-(\sigma) = M$ the entire manifold, then σ is a Cauchy surface. Any surface of constant t in Minkowski space-time is a Cauchy surface.

A globally hyperbolic spacetime M is topologically isomorphic to $I \times \sigma$, for some Cauchy surface σ and some interval $I \subset \mathbf{R}$; the metric structure need not respect this decomposition, however. Essentially, it means that everything that happens on M is determined by the equations of motion, together with initial data specified on a surface.

spatial tensor fields with arbitrary tensor fields restricted to Σ because they are both tensor fields on this subset of M . Moreover, once we have developed tensor calculus on Σ we immediately have the one on σ by just pulling back tensor fields on Σ to σ via the embedding. Consider then the following tensor fields, called the first and second fundamental form of Σ :

$$q_{\mu\nu} := g_{\mu\nu} - sn_\mu n_\nu, \quad \text{and} \quad K_{\mu\nu} := q_\mu^\rho q_\nu^\sigma \nabla_\rho n_\sigma \quad (2.4)$$

where all indices are moved with respect to $g_{\mu\nu}$. The second form is also said ‘‘Extrinsic Curvature’’. Both the tensors in (2.4) are ‘‘spatial’’, i.e. they vanish when either of their indices is contracted with n^μ . A crucial property of $K_{\mu\nu}$ is its symmetry: we have $K_{[\mu\nu]} = q_\mu^\rho q_\nu^\sigma ((\nabla_{[\rho} l n(F))n_{\sigma]} + F \nabla_{[\mu} \nabla_{\nu]} f) = 0$ since ∇ is torsion free. From this fact one derives another useful differential geometric identity by employing the relation between the covariant differential and the Lie derivative:

$$\begin{aligned} 2K_{\mu\nu} &= q_\mu^\rho q_\nu^\sigma (2\nabla_{(\rho} n_{\sigma)}) \\ &= q_\mu^\rho q_\nu^\sigma (L_n g)_{\rho\sigma} = q_\mu^\rho q_\nu^\sigma (L_n q + sL_n n \otimes n)_{\rho\sigma} \\ &= q_\mu^\rho q_\nu^\sigma (L_n q)_{\rho\sigma} = (L_n q)_{\mu\nu} \end{aligned} \quad (2.5)$$

since $n^\mu L_n q_{\mu\nu} = -q_{\mu\nu} L_n n^\mu = 0$. Using $n^\mu = (T^\mu - N^\mu)/N$ we can write (2.5) in the form

$$2K_{\mu\nu} = \frac{1}{N} (L_{T-N} q)_{\mu\nu} - 2n^\rho q_{\rho(\mu} l n(N)_{,\nu)} = \frac{1}{N} (L_{T-N} q)_{\mu\nu} \quad (2.6)$$

Next we would like to construct a covariant differential associated with the metric $q_{\mu\nu}$. We would like to stress that this metric is non-degenerate as a bijection between spatial tensors only and not as a bijection between arbitrary tensors defined on Σ . Recall that, by definition, a differential ∇ is said to be covariant with respect to a metric g (of any signature) on a manifold M if it is 1) metric compatible, $\nabla g = 0$ and 2) torsion free, $[\nabla_\mu, \nabla_\nu]f = 0, \forall C^\infty(M)$. According to a classical theorem, these two conditions fix ∇ uniquely in terms of the Christoffel connection which in turn is defined by its action on one-forms through $\nabla_\mu u_\nu := \partial_\mu u_\nu - \Gamma_{\mu\nu}^\rho u_\rho$. Since the tensor q is a metric of euclidean signature on Σ we can thus apply these two conditions to q and we are looking for a covariant differential D on spatial tensor only such that 1) $D_\mu q_{\nu\rho} = 0$ and 2) $D_{[\mu} D_{\nu]} f = 0$ for scalars f . Of course the operator D should preserve the set of spatial tensor fields. It is easy to verify that $D_\mu f := q_\mu^\nu \nabla_\nu \tilde{f}$ and $D_\mu u_\nu := q_\mu^\rho q_\nu^\sigma \nabla_\rho \tilde{u}_\sigma$ for $u_\mu n^\mu = 0$ and extended to arbitrary tensors by linearity and Leibniz rule, does the job and thus, by the above mentioned theorem, is the unique choice. Here \tilde{f} and \tilde{u} denote arbitrary smooth extensions of f and u respectively into a neighbourhood of Σ in M , necessary in order to perform the ∇ operation. The covariant differential is independent of that extension as derivatives not tangential to Σ are projected out by the q tensor and we will drop the tilde again. We now ask what the Riemann curvature $R_{\mu\nu\rho}^{(D)\sigma}$ of D is in terms of that of ∇ . To answer this question we need the second covariant differential of a spatial co-vector u_ρ which when carefully using the

definition of D is given by

$$\begin{aligned} D_\mu D_\nu u_\rho &= q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} \nabla_{\mu'} D_{\nu'} u_{\rho'} \\ &= q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} \nabla_{\mu'} q_{\nu'}^{\nu''} q_{\rho'}^{\rho''} \nabla_{\nu''} u_{\rho''} \end{aligned} \quad (2.7)$$

The outer derivative hits either a q tensor or ∇u , the latter of which will give rise to a curvature term. Consider then the ∇q terms. Since ∇ is g compatible we have $\nabla q = s\nabla n \otimes n = s[(\nabla n) \otimes n + n \otimes (\nabla n)]$. Since all of these terms are contracted with q tensors and q annihilates n , the only terms that survive are proportional to terms either in the form

$$(\nabla_{\mu'} n_{\nu'}) n^{\rho''} (\nabla_{\nu''} u_{\rho''}) = -(\nabla_{\mu'} n_{\nu'}) (\nabla_{\nu''} n^{\rho''}) u_{\rho''} \quad (2.8)$$

where $n^\mu u_\mu = 0 \rightarrow \nabla_\nu (n^\mu u_\mu) = 0$ was exploited, or of the form $(\nabla_{\mu'} n_{\nu'}) (\nabla_{\nu''} u_{\rho''})$. Concluding, the only terms that survive from ∇q terms can be transformed into terms proportional to $\nabla n \otimes \nabla n$ or $\nabla n \otimes \nabla u$ where the ∇n factors, since contracted with q tensors, can be traded for extrinsic curvature terms (use $u_\mu = q_\mu^{\nu'} u_{\nu'}$ to do that). It turns out that the terms proportional to ∇u cancel each other when computing the antisymmetrized second D derivative of u due to the symmetry of K and we are thus left with the famous Gauss equation

$$\begin{aligned} R_{\mu\nu\rho}^{(D)\sigma} u_\rho &: = 2D_{[\mu} D_{\nu]} u_\rho \\ &= [2sK_{\rho[\mu} K_{\nu]}^\sigma + q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} q_\sigma^{\sigma'} R_{\mu'\nu'\rho'}^{(D+1)\sigma'}] u_\sigma \\ R_{\mu\nu\rho\sigma}^{(D)} &= 2sK_{\rho[\mu} K_{\nu]\sigma} + q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} q_\sigma^{\sigma'} R_{\mu'\nu'\rho'\sigma'}^{(D+1)} \end{aligned} \quad (2.9)$$

Using this general formula we can specialize to the Riemann curvature scalar which is our ultimate concern in view of the Einstein-Hilbert action. Employing the standard abbreviations $K := K_{\mu\nu} q^{\mu\nu}$ and $K^{\mu\nu} = q^{\mu\rho} q^{\nu\sigma} K_{\rho\sigma}$ (notice that indices for spatial tensors can be moved either with q or with g) we obtain

$$\begin{aligned} R^{(D)} &= R_{\mu\nu\rho\sigma}^{(D)} q^{\mu\rho} q^{\nu\sigma} \\ &= s[K^2 - K_{\mu\nu} K^{\mu\nu}] + q^{\mu\rho} q^{\nu\sigma} R_{\mu\nu\rho\sigma}^{(D+1)} \end{aligned} \quad (2.10)$$

Equation (2.10) is not yet quite what we want since it is not yet purely expressed in terms of $R^{(D+1)}$ alone. However, we can eliminate the second term in (2.10) by using $g = q + sn \otimes n$ and the definition of curvature $R_{\mu\nu\rho\sigma}^{(D+1)} n^\sigma = 2\nabla_{[\mu} \nabla_{\nu]} n_\rho$ as follows

$$\begin{aligned} R^{(D+1)} &= R_{\mu\nu\rho\sigma}^{(D+1)} g^{\mu\rho} g^{\nu\sigma} \\ &= q^{\mu\rho} q^{\nu\sigma} R_{\mu\nu\rho\sigma}^{(D+1)} + 2sq^{\rho\mu} n^\nu [\nabla_\mu, \nabla_\nu] n_\rho \\ &= q^{\mu\rho} q^{\nu\sigma} R_{\mu\nu\rho\sigma}^{(D+1)} + 2sn^\nu [\nabla_\mu, \nabla_\nu] n^\mu \end{aligned} \quad (2.11)$$

where in the first step we used the antisymmetry of the Riemann tensor to eliminate the term quartic in n and in the second step we used again $g = q - sn \otimes n$ and the

antisymmetry in the $\mu\nu$ indices. Now

$$n^\nu([\nabla_\mu, \nabla_\nu]n^\mu) = -(\nabla_\mu n^\nu)(\nabla_\nu n^\mu) + (\nabla_\nu n^\mu)(\nabla_\mu n^\nu) + \nabla_\mu(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu)$$

and using $\nabla_\mu s = 2n^\nu \nabla_\mu n_\nu = 0$ we have

$$\begin{aligned} \nabla_\mu n^\mu &= g^{\mu\nu} \nabla_\nu n_\mu = q^{\mu\nu} \nabla_\nu n_\mu = K \\ (\nabla_\mu n^\nu)(\nabla_\nu n^\mu) &= g^{\nu\sigma} g^{\rho\mu} (\nabla_\mu n_\sigma)(\nabla_\nu n_\rho) = q^{\nu\sigma} q^{\rho\mu} (\nabla_\mu n_\sigma)(\nabla_\nu n_\rho) = K_{\mu\nu} K^{\mu\nu} \end{aligned} \quad (2.12)$$

Combining (2.10), (2.11) and (2.12) we obtain the *Codacci equation*

$$R^{(D+1)} = R^{(D)} - s[K_{\mu\nu} K^{\mu\nu} - K^2] + 2s \nabla_\mu (n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu). \quad (2.13)$$

Inserting this differential geometric identity back into the action, the third term in (2.13) is a total differential which we drop for the time being as one can rederive it later on when making the variational principle well-defined.

At this point it is useful to pull back various quantities to σ . Consider the D spatial vector fields on Σ_t defined by

$$X_a^\mu(X) := X_{,a}^\mu(x, t)|_{X(x,t)=X} \quad (2.14)$$

Then we have due to $n_\mu X_a^\mu = 0$ that

$$q_{ab}(t, x) := (X_{,a}^\mu X_{,b}^\nu q_{\mu\nu})(X(x, t)) = g_{\mu\nu}(X(t, x)) X_{,a}^\mu(t, x) X_{,b}^\nu(t, x) \quad (2.15)$$

and

$$K_{ab}(t, x) := (X_{,a}^\mu X_{,b}^\nu K_{\mu\nu})(X(x, t)) = (X_{,a}^\mu X_{,b}^\nu \nabla_\mu n_\nu)(t, x) \quad (2.16)$$

Using q_{ab} and its inverse $q^{ab} = \epsilon^{aa_1 \dots a_{D-1}} \epsilon^{bb_1 \dots b_{D-1}} q_{a_1 b_1} \dots q_{a_{D-1} b_{D-1}} / [\det((q_{cd})) (D-1)!]$ we can express $q_{\mu\nu}, q^{\mu\nu}, q_\mu^\nu$ as

$$\begin{aligned} q^{\mu\nu}(X) &= [q^{ab}(x, t) X_{,a}^\mu X_{,b}^\nu](x, t)|_{X(x,t)=X} \\ q_\mu^\nu(X) &= g_{\mu\rho}(X) q^{\rho\nu}(X) \\ q_{\mu\nu}(X) &= g_{\nu\rho}(X) q_\mu^\rho(X) \end{aligned} \quad (2.17)$$

To verify that this coincides with our previous definition $q = g - sn \otimes n$ it is sufficient to check the matrix elements in the basis given by the vector fields n, X_a . Since for both definitions n is annihilated we just need to verify that (2.17) when contracted with $X_a \otimes X_b$ reproduces (2.15) which is indeed the case.

Next we define $N(x, t) := N(X(x, t)), \vec{N}^a(x, t) := q^{ab}(x, t) (X_b^\mu g_{\mu\nu} N^\nu)(X(x, t))$. Then it is easy to verify that

$$K_{ab}(x, t) = \frac{1}{2N} (\dot{q}_{ab} - (\mathcal{L}_{\vec{N}} q)_{ab})(x, t) \quad (2.18)$$

where $\dot{q}_{ab} = \mathcal{L}_{\vec{T}} q_{ab}$. We can now pull back the expressions quadratic in $K_{\mu\nu}$ that appear in (2.13) using (2.17) and find

$$\begin{aligned} K(x, t) &= (q^{\mu\nu} K_{\mu\nu})(X(x, t)) = (q^{ab} K_{ab})(x, t) \\ (K_{\mu\nu} K^{\mu\nu})(x, t) &= (K_{\mu\nu} K_{\rho\sigma} q^{\mu\rho} q^{\nu\sigma})(X(x, t)) = (K_{ab} K_{cd} q^{ac} q^{bd})(x, t) \end{aligned} \quad (2.19)$$

Likewise we can pull back the curvature scalar $R^{(D)}$. We have

$$\begin{aligned} R^{(D)}(x, t) &= (R_{\mu\nu\rho\sigma}^{(D)} q^{\mu\rho} q^{\nu\sigma})(X(x, t)) \\ &= (R_{\mu\nu\rho\sigma}^{(D)} X_a^\mu X_b^\nu X_c^\rho X_d^\sigma)(X(x, t)) q^{ac}(x, t) q^{bd}(x, t) \end{aligned} \quad (2.20)$$

We would like to show that this expression equals the curvature scalar R as defined in terms of the Christoffel connection for q_{ab} . To see this it is sufficient to compute $(X_a^\mu D_\mu F)(X(x, t)) = \partial_a F(X(x, t)) =: (D_a f)(x, t)$ with $f(x, t) := F(X(x, t))$ and with $u_a(x, t) := (X_a^\mu u_\mu)(X(x, t))$, $u^a(x, t) = q^{ab}(x, t) u_b(x, t)$

$$\begin{aligned} (D_a u_b)(x, t) &:= (X_a^\mu X_b^\nu D_\mu u_\nu)(X(x, t)) \\ &= X_{,a}^\mu(x, t) X_{,b}^\nu(x, t) (\nabla_\mu u_\nu)(X(x, t)) \\ &= (\partial_a u_b)(x, t) - X_{,ab}^\mu u_\mu(X(x, t)) \\ &\quad - u^c(x, t) \Gamma_{\rho\mu\nu}^{(D+1)}(X(x, t)) X_{,c}^\rho(x, t) (X_{,a}^\mu(x, t) X_{,b}^\nu(x, t)) \\ &= (\partial_a u_b)(x, t) - \Gamma_{cab}^{(D)}(x, t) u^c(x, t) \end{aligned} \quad (2.21)$$

where in the last step we have used the explicit expressions of the Christoffel connections $\Gamma^{(D+1)}$ and $\Gamma^{(D)}$ in terms of $g_{\mu\nu}$ and q_{ab} respectively. Now since every tensor field W is a linear combination of tensor products of one forms and since D_μ satisfies the Leibniz rule we easily find $(X_a^\mu X_b^\nu \dots D_\mu W_{\nu\dots})(X(x, t)) =: (D_a W_{b\dots})(x, t)$ where now D_a denotes the unique torsion-free covariant differential associated with q_{ab} and $W_{a\dots}$ is the pull-back of $W_{\mu\dots}$. In particular, we have $X_a^\mu X_b^\nu X_c^\rho D_\mu D_\nu u_\rho = D_a X_b^\mu X_c^\nu D_\mu u_\rho = D_a D_b u_c$ from which our assertion follows since

$$\begin{aligned} (R_{abcd} u_d)(x, t) &:= ([D_a, D_b] u_c)(x, t) = (X_a^\mu X_b^\nu X_c^\rho [D_\mu, D_\nu] u_\rho)(X(x, t)) \\ &= (X_a^\mu X_b^\nu X_c^\rho X_d^\sigma R_{\mu\nu\rho\sigma}^{(D)})(X(x, t)) u^d(x, t) \end{aligned} \quad (2.22)$$

From now on we will move indices with the metric q_{ab} only.

One now expresses the line element in the new system of coordinates x, t using the quantities q_{ab}, N, N^a (we refrain from displaying the arguments of the components of the metric)

$$\begin{aligned} ds^2 &= g_{\mu\nu} dX^\mu \otimes dX^\nu \quad (2.23) \\ &= g_{\mu\nu}(X(t, x)) [X_{,t}^\mu dt + X_{,a}^\mu dx^a] \otimes [X_{,t}^\nu dt + X_{,b}^\nu dx^b] \\ &= g_{\mu\nu}(X(t, x)) [N n^\mu dt + X_{,a}^\mu (dx^a + N^a dt)] \otimes [N n^\nu dt + X_{,b}^\nu (dx^b + N^b dt)] \\ &= [s N^2 + q_{ab} N^a N^b] dt \otimes dt + q_{ab} N^b [dt \otimes dx^a + dx^a \otimes dt] + q_{ab} dx^a \otimes dx^b \end{aligned}$$

and reads off the components g_{tt}, g_{ta}, g_{ab} of X^*g in this frame. Since the volume form $\Omega(X) := \sqrt{|\det(g)|} d^{D+1}X$ is covariant, i.e., $(X^*\Omega)(x, t) = \sqrt{|\det(X^*g)|} dt d^Dx$ we just need to compute $\det(X^*g) = s N^2 \det(q_{ab})$ in order to finally cast the action (2.1) into $D+1$ form. The result is (dropping the total differential in (2.13))

$$S = \frac{1}{\kappa} \int_{\mathbf{R}} dt \int_{\sigma} d^Dx \sqrt{\det(q)} |N| (R - s [K_{ab} K^{ab} - (K_a^a)^2]) \quad (2.24)$$

We could drop the absolute sign for N in (2.24) since we took N positive but we will keep it for the moment to see what happens if we allow arbitrary sign. Notice that (2.24) *vanishes identically* for $D = 1$, indeed in two spacetime dimensions the Einstein action is proportional to a topological charge, the so-called Euler characteristic of M and in what follows we concentrate on $D > 1$.

We now wish to cast this action into canonical form, that is, we would like to perform the Legendre transform from the Lagrangean density appearing in (2.24) to the corresponding Hamiltonian density. The action (2.24) depends on the velocities \dot{q}_{ab} of q_{ab} but not on those of N and N^a . Therefore we obtain for the conjugate momenta (use (2.18) and the fact that R does not contain time derivatives)

$$\begin{aligned} \frac{1}{\kappa} P^{ab}(t, x) &:= \frac{\delta S}{\delta \dot{q}_{ab}(t, x)} = -s \frac{|N|}{N\kappa} \sqrt{\det(q)} [K^{ab} - q^{ab}(K^c_c)] \\ P_N(t, x) &:= \frac{\delta S}{\delta \dot{N}(t, x)} = 0 \\ P_{N^a}(t, x) &:= \frac{\delta S}{\delta \dot{N}^a(t, x)} = 0 \end{aligned} \quad (2.25)$$

The Lagrangean in (2.24) is therefore a *singular* Lagrangean, one cannot solve all velocities for momenta. We can solve \dot{q}_{ab} in terms of q_{ab}, N, N^a and P^{ab} using (2.18) but this is not possible for \dot{N}, \dot{N}^a , rather we have the so-called *primary constraints*

$$P_N(t, x) = 0 \text{ and } P_{N^a}(t, x) = 0 \quad (2.26)$$

According to Dirac's theory, we are supposing to introduce Lagrange multiplier fields $\lambda(t, x), \lambda_a(t, x)$ for the primary constraints and to perform the Legendre transform as usual with respect to the remaining velocities which can be solved for. We have

$$\begin{aligned} \dot{q}_{ab} &= 2NK_{ab} + (\mathcal{L}_{\vec{N}}q)_{ab} \\ \dot{q}_{ab}P^{ab} &= (\mathcal{L}_{\vec{N}}q)_{ab}P^{ab} - 2s|N|\sqrt{\det(q)}[K_{ab}K^{ab} - K^2] \\ P_{ab}P^{ab} &= \det(q)(K_{ab}K^{ab} + (D-2)K^2) \\ P^2 &:= (P^a_a)^2 = (1-D)^2 \det(q)K^2 \end{aligned} \quad (2.27)$$

and by means of these formulae we obtain the canonical form of the action (2.24)

$$\begin{aligned} \kappa S &= \int_{\mathbf{R}} dt \int_{\sigma} d^D x \{ \dot{q}_{ab}P^{ab} + \lambda P_N + \lambda^a P_{N^a} - \\ &\quad - [\dot{q}_{ab}(P, q, N, \vec{N})P^{ab} - \sqrt{\det(q)}|N|(R - s[K_{ab}K^{ab} - K^2])(P, q, N, \vec{N})] \} \\ \kappa S &= \int_{\mathbf{R}} dt \int_{\sigma} d^D x \{ \dot{q}_{ab}P^{ab} - \\ &\quad - [\dot{q}_{ab}(P, q, N, \vec{N})P^{ab} + \lambda P_N + \lambda^a P_{N^a} - \\ &\quad \quad - \sqrt{\det(q)}|N|(R - s[K_{ab}K^{ab} - K^2])(P, q, N, \vec{N})] \} \\ &= \int_{\mathbf{R}} dt \int_{\sigma} d^D x \{ \dot{q}_{ab}P^{ab} - \end{aligned}$$

$$\begin{aligned}
& -[(\mathcal{L}_{\vec{N}}q)_{ab}P^{ab} + \lambda P_N + \lambda^a P_{Na} - \\
& \quad - \sqrt{\det(q)}|N|(R + s[K_{ab}K^{ab} - K^2])(P, q, N, \vec{N})] \} \\
= & \int_{\mathbf{R}} dt \int_{\sigma} d^D x \{ \dot{q}_{ab}P^{ab} - \\
& -[(\mathcal{L}_{\vec{N}}q)_{ab}P^{ab} + \lambda P_N + \lambda^a P_{Na} + \\
& \quad + |N|(-\frac{s}{\sqrt{\det(q)}}[P_{ab}P^{ab} - \frac{1}{D-1}P^2] - \sqrt{\det(q)}R) \}
\end{aligned}$$

Upon performing a spatial integration by parts (whose boundary term vanishes since $\partial\sigma = \emptyset$) one can cast it into the following more compact form

$$S = \frac{1}{\kappa} \int_{\mathbf{R}} dt \int_{\sigma} d^D x \{ \dot{q}_{ab}P^{ab} - [\lambda P_N + \lambda^a P_{Na} + N^a H_a + |N|H] \} \quad (2.28)$$

where

$$\begin{aligned}
H_a & := -2q_{ac}D_b P^{bc} \\
H & := -(\frac{s}{\sqrt{\det(q)}}[q_{ac}q_{bd} - \frac{1}{D-1}q_{ab}q_{cd}]P^{ab}P^{cd} + \sqrt{\det(q)}R) \quad (2.29)
\end{aligned}$$

are called the (*spatial*) *Diffeomorphism constraint* and *Hamiltonian constraint* respectively, for reasons which we will derive below.

The geometrical meaning of these quantities is as follows :

At fixed t the fields $(q_{ab}(t, x), N^a(t, x), N(t, x); P^{ab}(x, t), P_{Na}(t, x), P_N(t, x))$ label points (configuration; canonically conjugate momenta) in an infinite dimensional phase space \mathcal{M} (or symplectic manifold). Strictly speaking, we should now specify on what Banach space this manifold is modelled, however, we will be brief here as we are primarily not interested in the metric formulation of this section but rather in the connection formulation of the next section where we will give more details. For the purpose of this subsection it is sufficient to say that we can choose the model space to be the direct product of the space $T_2(\sigma) \times T_1(\sigma) \times T_0(\sigma)$ of smooth symmetric covariant tensor fields of rank 2, 1, 0 on σ respectively and the space $\tilde{T}^2(\sigma) \times \tilde{T}^1(\sigma) \times \tilde{T}^0(\sigma)$ of smooth symmetric contravariant tensor density fields of weight one and of rank 2, 1, 0 on σ respectively, equipped with some Sobolev norm. (The precise functional analytic description is somewhat more complicated in case that σ has boundary, but can also be treated). In particular, one shows that the action (2.28) is differentiable in this topology.

The phase space carries the strong symplectic structure Ω or Poisson bracket

$$\{P(f^2), F_2(q)\} = \kappa F_2(f^2), \{\vec{P}_N(\vec{f}^1), \vec{F}_1(\vec{N})\} = \kappa \vec{F}_1(\vec{f}^1), \{P_N(f), F(N)\} = \kappa F(f) \quad (2.30)$$

(all other brackets vanishing) where we have defined the following pairing, invariant under diffeomorphisms of σ , e.g.

$$\tilde{T}^2(\sigma) \times T_2(\sigma) \rightarrow \mathbf{R}; (F_2, f^2) \rightarrow F^2(f_2) := \int_{\sigma} d^D x F_2^{ab}(x) f_{ab}^2(x) \quad (2.31)$$

and similar for the other fields. Physicists use the following short-hand notation for (2.30)

$$\{P^{ab}(t, x), q_{cd}(t, x')\} = \kappa \delta_c^a \delta_d^b \delta^{(D)}(x, y). \quad (2.32)$$

In the language of symplectic geometry, the first term in the action (2.28) is a symplectic potential for the symplectic structure (2.30). We now turn to the meaning of the term in the square bracket in (2.28), that is, the ‘‘Hamiltonian’’

$$\kappa \mathbf{H} := \int_{\sigma} d^D x [\lambda P_N + \lambda^a P_{N_a} + N^a H_a + |N|H] =: P_N(\vec{\lambda}) + P_N(\lambda) + \vec{H}(\vec{N}) + H(|N|) \quad (2.33)$$

of the action and the associated equations of motion.

The variation of the action with respect to the Lagrange multiplier fields $\vec{\lambda}, \lambda$ reproduces the primary constraints (2.26). If the dynamics of the system is to be consistent, then these constraints must be preserved under the evolution of the system, that is, we should have e.g. $\dot{P}_N(t, x) := \{\mathbf{H}, P_N(t, x)\} = 0$ for all $x \in \sigma$, or equivalently, $\dot{P}_N(f) := \{\mathbf{H}, P_N(f)\} = 0$ for all (t -independent) smearing fields $f \in T_0(\sigma)$. However, we do not get zero but rather

$$\{\vec{P}_N(\vec{f}), \mathbf{H}\} = \vec{H}(\vec{f}) \text{ and } \{P_N(f), \mathbf{H}\} = H\left(\frac{N}{|N|}f\right) \quad (2.34)$$

which is supposed to vanish for all f, \vec{f} . Thus, consistency of the equations of motion ask us to impose the *secondary constraints*

$$H(x, t) = 0 \text{ and } H_a(x, t) = 0 \quad (2.35)$$

for all $x \in \sigma$. Since these two functions appear next to the P, P_{N_a} in (2.33), in general relativity the ‘‘Hamiltonian’’ is constrained to vanish! General relativity is an example of a so-called constrained Hamiltonian system with no true Hamiltonian. The reason for this will become evident in a moment.

Now one might worry that imposing consistency of the secondary constraints under evolution results in tertiary constraints etc., but fortunately, this is not the case. Consider the smeared quantities $H(f), \vec{H}(\vec{f})$, we obtain

$$\begin{aligned} \{\mathbf{H}, \vec{H}(\vec{f})\} &= \vec{H}(\mathcal{L}_{\vec{N}}\vec{f}) - H(\mathcal{L}_{\vec{f}}|N|) \\ \{\mathbf{H}, H(f)\} &= H(\mathcal{L}_{\vec{N}}f) + \vec{H}(\vec{W}(|N|, f, q)) \end{aligned} \quad (2.36)$$

where $\vec{W}(f, f', q)^a = q^{ab}(ff'_b - f'f_b)$. Equations (2.36) are equivalent to the *Dirac algebra*:

$$\begin{aligned} \{\vec{H}(\vec{f}), \vec{H}(\vec{f}')\} &= \kappa \vec{H}(\mathcal{L}_{\vec{f}}\vec{f}') \\ \{\vec{H}(\vec{f}), H(f)\} &= \kappa H(\mathcal{L}_{\vec{f}}f) \\ \{H(f), H(f')\} &= \kappa \vec{H}(\vec{N}(f, f', q)) \end{aligned} \quad (2.37)$$

also called the *hypersurface deformation algebra*. The meaning of (2.34,2.37) is that the constraint surface $\bar{\mathcal{M}}$ of \mathcal{M} , the submanifold of \mathcal{M} where the constraints hold,

is preserved under the motions generated by the constraints. In the terminology of Dirac, all constraints are of first class (determine coisotropic constraint submanifolds of \mathcal{M}) rather than of second class (determine symplectic constraint submanifolds of \mathcal{M}).

It remains to study the equations of motion of the canonical coordinates on the phase space themselves. Since $\dot{P}_N = H \frac{|N|}{N}$, $\dot{P}_{Na} = H_a$, it remains to study those of N, N^a, q_{ab}, P^{ab} . For shift and lapse we obtain $\dot{N}^a = \lambda^a, \dot{N} = \lambda$. Since λ^a, λ are *arbitrary, unspecified functions* we see that also the trajectory of lapse and shift is completely arbitrary. Moreover, the equations of motion of q_{ab}, P^{ab} are completely unaffected by the term $\vec{P}_N(\vec{\lambda}) + P_N(\lambda)$ in \mathbf{H} . It is therefore completely straightforward to solve the equations of motion as far as N, N^a are concerned : Simply treat N, N^a as *Lagrange multipliers* and drop all terms proportional to P_N, P_{Na} from the action (2.28). The result is the reduced action

$$S = \frac{1}{\kappa} \int_{\mathbf{R}} dt \int_{\sigma} d^D x \{ \dot{q}_{ab} P^{ab} - [N^a H_a + |N|H] \} \quad (2.38)$$

called the *Arnowitz – Deser – Misner action*. It is straightforward to check that as far as q_{ab}, P^{ab} are concerned, the actions (2.28) and (2.38) are completely equivalent.

The equations of motion of q_{ab}, P^{ab} then finally allow us to interpret the motions that the constraints generate on \mathcal{M} geometrically. Since the reduced Hamiltonian (using the same symbol as before)

$$\mathbf{H} = \frac{1}{\kappa} \int_{\sigma} d^D x [N^a H_a + |N|H] \quad (2.39)$$

is a linear combination of constraints we obtain the equations of motion once we know the Hamiltonian flow of the functions $\vec{H}(\vec{f}), H(f)$ for any \vec{f}, f separately. Denoting, for any function J on \mathcal{M} ,

$$\delta_{\vec{f}} J := \{ \vec{H}(\vec{f}), J \} \text{ and } \delta_f J := \{ H(f), J \} \quad (2.40)$$

it is easiest to begin with the corresponding equations for $J = F_2(q)$ since upon integration by parts we have $\vec{H}(\vec{f}) = \int d^D x P^{ab} (\mathcal{L}_{\vec{f}} q)_{ab}$ so that both constraint functions are simple polynomials in P^{ab} not involving their derivatives. We then readily find

$$\begin{aligned} \delta_{\vec{f}} F_2(q) &= \kappa F_2(\mathcal{L}_{\vec{f}} q) \\ \delta_f F_2(q) &= -2s\kappa \int_{\sigma} d^D x \frac{P_{ab} - P q_{ab}/(D-1)}{\sqrt{\det(q)}} F_2^{ab} f \end{aligned} \quad (2.41)$$

Using the relations (2.25), (2.18) the second identity in (2.41) can be written as

$$\delta_{|N|} q_{ab} = 2N\kappa K_{ab} = \kappa(\dot{q}_{ab} - (\mathcal{L}_{\vec{N}} q)_{ab})$$

In order to interpret this quantity, notice that the components of n_{μ} in the frame t, x^a are given by $n_t = n_{\mu} X_{,t}^{\mu} = sN$, $n_a = n_{\mu} X_{,a}^{\mu} = 0$. In order to compute the contravariant components n^{μ} in that frame we need the corresponding contravariant

metric components. From (2.23) we find the covariant components to be $g_{tt} = sN^2 + q_{ab}N^aN^b$, $g_{ta} = q_{ab}N^b$, $g_{ab} = q_{ab}$ so that the inverse metric has components $g^{tt} = s/N^2$, $g^{ta} = -sN^a/N^2$, $g^{ab} = q^{ab} + sN^aN^b/N^2$. Thus $n^t = 1/N$, $n^a = -N^a/N$ and since $q_{at} = q_{tt} = 0$ we finally obtain

$$\delta_{|N|}F_2(q) = \kappa F_2(\mathcal{L}_{Nn}q) \quad (2.42)$$

which of course we guessed immediately from the $D + 1$ dimensional identity (2.6). Concluding, as far as q_{ab} is concerned, H_a generates *on all of* \mathcal{M} diffeomorphisms of M that preserve Σ_t while H generates diffeomorphisms of M orthogonal to Σ_t .

The corresponding computation for $P(f^2)$ is harder due to the curvature term involved in H and due to the fact that the identity corresponding to (2.42) holds *only on shell*, that is, when the (vacuum) Einstein equations $G_{\mu\nu}^{(D+1)} := R_{\mu\nu}^{(D+1)} - \frac{g_{\mu\nu}}{2}R^{(D+1)} = 0$ hold. The variation with respect to $\vec{H}(\vec{f}) = -\int_{\sigma} d^Dx q_{ab}(\mathcal{L}_{\vec{f}}P)^{ab}$ (notice that P^{ab} carries density weight one to verify this identity) is still easy and yields the expected result

$$\delta_{\vec{f}}P(f^2) = \kappa(\mathcal{L}_{\vec{f}}P)(f^2) \quad (2.43)$$

We will now describe the essential steps for the analog of (2.42). The ambitious reader who wants to fill in the missing steps should expect to perform at least one A4 page of calculation in between each of the subsequent formulae.

We start from formula (2.29). Then

$$\begin{aligned} \{H(|N|), P^{ab}\} &= \frac{\delta H(|N|)}{\delta q_{ab}} \\ &= \frac{s|N|}{\sqrt{\det(q)}} [2(P^{ac}P_c^b - P^{ab}P/(D-1)) - \frac{q^{ab}}{2}(P^{cd}P_{cd} - P^2/(D-1))] \\ &\quad + \frac{\delta}{\delta q_{ab}} \int d^Dx |N| \sqrt{\det(q)} R \end{aligned} \quad (2.44)$$

where the second term comes from the $\sqrt{\det(q)}^{-1}$ factor and we used the well-known formula $\delta \det(q) = \det(q)q^{ab}\delta q_{ab}$. To perform the remaining variation in (2.44) we write

$$\delta \sqrt{\det(q)} R = [\delta \sqrt{\det(q)}] R + \sqrt{\det(q)} [\delta q^{ab}] R_{ab} + \sqrt{\det(q)} q^{ab} [\delta R_{ab}]$$

use $\delta \delta_b^a = \delta[q^{ac}q_{cb}] = 0$ in the second variation and can simplify (2.44)

$$\begin{aligned} \{H(|N|), P^{ab}\} &= \frac{2s|N|}{\sqrt{\det(q)}} [2(P^{ac}P_c^b - P^{ab}P/(D-1))] + \frac{q^{ab}|N|H}{2} + \\ &\quad + |N| \sqrt{\det(q)} (q^{ab}R - R^{ab}) + \\ &\quad + \int d^Dx |N| \sqrt{\det(q)} q^{cd} \frac{\delta}{\delta q_{ab}} R_{cd} \end{aligned} \quad (2.45)$$

The final variation is the most difficult one since R_{cd} contains second derivatives of q_{ab} . Using the explicit expression of R_{abcd} in terms of the Christoffel connection Γ_{ab}^c

and observing that, while the connection itself is not a tensor, its variation in fact is a tensor, we find after careful use of the definition of the covariant derivative

$$q^{cd}\delta R_{cd} = q^{cd}[-D_c\delta\Gamma_{ed}^e + D_e\delta\Gamma_{cd}^e] \quad (2.46)$$

We now use the explicit expression of Γ_{bc}^a in terms of q_{ab} and find

$$\delta\Gamma_{bc}^a = \frac{q^{ad}}{2}[D_c\delta q_{bd} + D_b\delta q_{cd} - D_d\delta q_{bc}] \quad (2.47)$$

Next we insert (2.46) and (2.47) into the integral appearing in (2.45) and integrate by parts two times using the fact that for the divergence of a vector v^a we have $\sqrt{\det(q)}D_a v^a = D_a(\sqrt{\det(q)}v^a) = \partial_a(\sqrt{\det(q)}v^a)$ (no boundary terms due to $\partial\sigma = \emptyset$) and find

$$\begin{aligned} & \int d^D x |N| \sqrt{\det(q)} q^{cd} \delta R_{cd} = \int d^D x \sqrt{\det(q)} q^{cd} [(D_c |N|) \delta\Gamma_{ed}^e - (D_e |N|) \delta\Gamma_{cd}^e] \\ &= \int d^D x \sqrt{\det(q)} q^{cd} q^{ef} [(D_c |N|) (D_d \delta q_{ef}) - (D_e |N|) (D_c \delta q_{df})] \\ &= \int d^D x \sqrt{\det(q)} [-(D_c D^c |N|) q^{ab} + (D^a D^b |N|)] \delta q_{ab} \end{aligned} \quad (2.48)$$

Collecting all contributions we obtain the desired result

$$\begin{aligned} \{H(|N|), P^{ab}\} &= \frac{2s|N|}{\sqrt{\det(q)}} [2(P^{ac}P_c^b - P^{ab}P/(D-1))] + \frac{q^{ab}|N|H}{2} \\ &+ |N| \sqrt{\det(q)} (q^{ab}R - R^{ab}) + \sqrt{\det(q)} [-(D_c D^c |N|) q^{ab} - (D^a D^b |N|)] \end{aligned} \quad (2.49)$$

which does not look at all as $\mathcal{L}_{Nn} P^{ab}$!

In order to compute $\mathcal{L}_{Nn} P^{ab}$ we need an identity for $\mathcal{L}_{Nn} K_{\mu\nu} = N \mathcal{L}_n K_{\mu\nu}$ which we now derive. Using the definition of the Lie derivative in terms of the covariant derivative ∇_μ and using $g = q + sn \otimes n$ one finds first of all

$$\mathcal{L}_n K_{\mu\nu} = -K K_{\mu\nu} + 2K_{\rho\mu} K_\nu^\rho + [\nabla_\rho (n^\rho K_{\mu\nu}) + 2s K_{\rho(\mu} n_{\nu)} \nabla_n n^\rho] \quad (2.50)$$

Using the Gauss equation (2.9) we find for the Ricci tensor $R_{\mu\nu}^{(D)}$ the following equation (use again $g = q + sn \otimes n$ and the definition of curvature as $R = [\nabla, \nabla]$)

$$R_{\rho\sigma}^{(D+1)} q_\mu^\rho q_\nu^\sigma - R_{\mu\nu}^{(D)} = s[-K_{\mu\nu} K + K_{\mu\rho} K_\nu^\rho + q_\mu^\rho q_\nu^\sigma n^\lambda [\nabla_\rho, \nabla_\lambda] n_\sigma] \quad (2.51)$$

We claim that the term in square brackets on the right hand side of (2.50) equals $(-s)$ times the sum of the left hand side of (2.51) and the term $-s(D_\mu D_\nu N)/N$. In order to prove this we manipulate the commutator of covariant derivatives appearing in (2.51) making use of the definition of the extrinsic curvature. One finds

$$\begin{aligned} & q_\mu^\rho q_\nu^\sigma n^\lambda [\nabla_\rho, \nabla_\lambda] n_\sigma \\ &= q_\mu^\rho q_\nu^\sigma n^\lambda (\nabla_\rho \nabla_\lambda n_\sigma) + K K_{\mu\nu} - \nabla_\rho (n^\rho K_{\mu\nu}) \\ &\quad - s(\nabla_n n^\rho) n_\nu K_{\mu\rho} - s(\nabla_n (n_\mu n^\rho)) (\nabla_\rho n_\nu) \end{aligned} \quad (2.52)$$

Using this identity we find for the sum of the term in square brackets on the right hand side of (2.50) and s times the sum of the right hand side of (2.51) the expression (dropping the obvious cancellations)

$$\begin{aligned}
& K_{\mu\rho}K_{\nu}^{\rho} + q_{\mu}^{\rho}q_{\nu}^{\sigma}n^{\lambda}(\nabla_{\rho}\nabla_{\lambda}n_{\sigma}) + s[K_{\rho\nu}n_{\mu}(\nabla_n n^{\rho}) - (\nabla_n(n_{\mu}n^{\rho}))(\nabla_{\rho}n_{\nu})] \\
= & K_{\mu\rho}K_{\nu}^{\rho} + q_{\mu}^{\rho}q_{\nu}^{\sigma}n^{\lambda}(\nabla_{\rho}\nabla_{\lambda}n_{\sigma}) + s[n_{\mu}(\nabla_n n^{\rho})\{q_{\rho}^{\sigma} - \delta_{\rho}^{\sigma}\}(\nabla_{\sigma}n_{\nu}) - (\nabla_n n_{\mu})(\nabla_n n_{\nu})] \\
= & K_{\mu\rho}K_{\nu}^{\rho} + q_{\mu}^{\rho}q_{\nu}^{\sigma}(\nabla_{\rho}\nabla_n n_{\sigma}) - q_{\mu}^{\rho}q_{\nu}^{\sigma}(\nabla_{\rho}n^{\lambda})(\nabla_{\lambda}n_{\sigma}) - s(\nabla_n n_{\mu})(\nabla_n n_{\nu}) \\
= & +q_{\mu}^{\rho}q_{\nu}^{\sigma}(\nabla_{\rho}\nabla_n n_{\sigma}) - s(\nabla_n n_{\mu})(\nabla_n n_{\nu}) \tag{2.53}
\end{aligned}$$

where in the second step it has been used that the curly bracket vanishes since it is proportional to n_{ρ} and contracted with the spatial vector $\nabla_n n^{\rho}$, in the third step we moved ∇^{λ} inside a covariant derivative and picked up a correction term and in the fourth step one realizes that this correction term is just the negative of the first term using that $K_{\mu\nu} = q_{\mu}^{\rho}\nabla_{\rho}n_{\nu}$. Our claim is equivalent to showing that the last line of (2.53) is indeed given by $-s(D_{\mu}D_{\nu}N)/N$.

To see this notice that if the surface Σ_t is defined by $t(X) = t = \text{const.}$ then $1 = T^{\mu}\nabla_{\mu}t$. Since $\nabla_{\mu}t$ is orthogonal to Σ_t we have $n_{\mu} = sN\nabla_{\mu}t$ as one verifies by contracting with T^{μ} and thus $N = 1/(\nabla_n t)$. Thus

$$\begin{aligned}
D_{\mu}N &= -N^2 D_{\mu}(\nabla_n t) = -N^2 q_{\mu}^{\nu}n^{\rho}(\nabla_{\rho}\nabla_{\nu}t) \\
&= -sN(\nabla_n n_{\nu}) = -sN\nabla_n n_{\mu} \tag{2.54}
\end{aligned}$$

where in the first step we interchanged the second derivative due to torsion freeness and could pull n^{ρ} out of the second derivative because the correction term is proportional to $n_{\rho}\nabla n^{\rho} = 0$ and in the second we have pulled in a factor of N , observed that the correction is annihilated by the projection, used once more $sN\nabla t = n$ and finally used that $\nabla_n n_{\nu}$ is already spatial. The second derivative then gives simply

$$\begin{aligned}
D_{\mu}D_{\nu}N &= -s(D_{\mu}N)\nabla_n n_{\nu} - sNq_{\mu}^{\rho}q_{\nu}^{\sigma}\nabla_{\rho}\nabla_n n_{\sigma} \\
&= N(\nabla_n n_{\mu})(\nabla_n n_{\nu}) - sNq_{\mu}^{\rho}q_{\nu}^{\sigma}\nabla_{\rho}\nabla_n n_{\sigma} \tag{2.55}
\end{aligned}$$

which is indeed N times (2.53) as claimed. Notice that in (2.55) we cannot replace N by $|N|$ if N is not everywhere positive so *the interpretation that we are driving at would not hold if we would not set $N = |N|$ everywhere.* It is at this point that we must take N positive in all that follows.

We have thus established the key result

$$\begin{aligned}
\mathcal{L}_{Nn}K_{\mu\nu} &= N(-K_{\mu\nu} + 2K_{\rho\mu}K_{\nu}^{\rho}) - s[D_{\mu}D_{\nu}N \\
&\quad + N(R_{\rho\sigma}^{(D+1)}q_{\mu}^{\rho}q_{\nu}^{\sigma} - R_{\mu\nu}^{(D)})] \tag{2.56}
\end{aligned}$$

In order to finish the calculation for $\mathcal{L}_{Nn}P^{\mu\nu}$ we need to know $\mathcal{L}_{Nn}\sqrt{\det(q)}$, $\mathcal{L}_{Nn}q^{\mu\nu}$. So far we have defined $\det(q)$ in the ADM frame only, its generalization to an arbitrary frame is given by

$$\det((q_{\mu\nu})(X)) := \frac{1}{D!}[(\nabla_{\mu_0}t)(X)\epsilon^{\mu_0\cdots\mu_D}][(\nabla_{\nu_0}t)(X)\epsilon^{\nu_0\cdots\nu_D}]q_{\mu_1\nu_1}(X)\cdots q_{\mu_D\nu_D}(X) \tag{2.57}$$

as one can check by specializing to the ADM coordinates $X^\mu = t, x^a$. Here $\epsilon^{\mu_0 \dots \mu_D}$ is the metric independent, totally skew Levi-Civita tensor density of weight one. One can verify that with this definition we have $\det(g) = sN^2 \det(q)$ by simply expanding $g = q + sn \otimes n$. It is important to see that $\mathcal{L}_T \nabla_\mu t = \mathcal{L}_N \nabla_\mu t = 0$ from which then follows immediately that

$$\mathcal{L}_{Nn} \sqrt{\det(q)} = \frac{1}{2} \sqrt{\det(q)} q^{\mu\nu} \mathcal{L}_{Nn} q_{\mu\nu} = N \sqrt{\det(q)} K \quad (2.58)$$

where (2.6) has been used. Finally, using once more (2.54) we find indeed

$$\mathcal{L}_{Nn} q^{\mu\nu} = -q^{\mu\rho} q^{\nu\sigma} \mathcal{L}_{Nn} q_{\rho\sigma} = -2NK^{\mu\nu} \quad (2.59)$$

We are now in position to compute the Lie derivative of $P^{\mu\nu} = -s\sqrt{\det(q)}[q^{\mu\rho}q^{\nu\sigma} - q^{\mu\nu}q^{\rho\sigma}]K_{\rho\sigma}$. Putting all six contributions carefully together and comparing with (2.49) one finds the non-trivial result

$$\begin{aligned} \{H(N), P^{\mu\nu}\} &= \frac{q^{\mu\nu}NH}{2} - N\sqrt{\det(q)}[q^{\mu\rho}q^{\nu\sigma} - q^{\mu\nu}q^{\rho\sigma}]R_{\rho\sigma}^{(D+1)} \\ &\quad + \mathcal{L}_{Nn}P^{\mu\nu} \end{aligned} \quad (2.60)$$

that is, only on the constraint surface and only when the (vacuum) equations of motion hold, can the Hamiltonian flow of $P^{\mu\nu}$ with respect to $H(N)$ be interpreted as the action of a diffeomorphism in the direction perpendicular to Σ_t . Now, using again the definition of curvature as the commutator of covariant derivatives it is not difficult to check that

$$\begin{aligned} G_{\mu\nu}n^\mu n^\nu &= \frac{sH}{2\sqrt{\det(q)}} \\ G_{\mu\nu}n^\mu q_\rho^\nu &= -\frac{sH_\rho}{2\sqrt{\det(q)}} \end{aligned} \quad (2.61)$$

so that the constraint equations actually are equivalent to $D+1$ of the Einstein equations. Since (2.60) contains besides H all the spatial projections of $G_{\mu\nu}$ we see that our interpretation of $\{H(N), P^{\mu\nu}\}$ holds only on shell, $G_{\mu\nu} = 0$.

This finishes our geometric analysis of the Hamiltonian flow of the constraints which shows that the symmetry group of spacetime diffeomorphisms $\text{Diff}(M)$ of Einstein's action is faithfully implemented in the canonical framework, although in a not very manifest way (more precisely, it is only a subset of those symmetries generated by the Lie algebra of the theory symmetry group). The importance of this result cannot be stressed enough: It is often said that every $(D+1)$ -diffeomorphism invariant quantity should be a Dirac observable since $\text{Diff}(M)$ is the symmetry of the Einstein-Hilbert action. But this would mean that any higher derivative theory (containing arbitrary scalars built from polynomials of the curvature tensor) would also have the same Dirac observables, meaning that to be an observable would be theory independent. The catch is that $(D+1)$ -dimensional diffeomorphism invariance is not only a kinematical statement but involves the theory dependent dynamics. The fact that the motions generated by the constraints can be interpreted as spacetime diffeomorphisms only on (the theory dependent) shell spells this out in a precise way.

What do these considerations tell us? The Hamiltonian of general relativity is not a true Hamiltonian but a linear combination of constraints. Rather than generating time translations it generates spacetime diffeomorphisms. Since the parameters of these diffeomorphisms, N, N^a are completely arbitrary unspecified functions, the corresponding motions on the phase space have to be interpreted as *gauge transformations*. This is quite similar to the gauge motions generated by the Gauss constraint in Maxwell theory. The basic variables of the theory, q_{ab}, P^{ab} are not observables of the theory because they are not gauge invariant. Let us count the number of kinematical and dynamical (true) degrees of freedom : The basic variables are both symmetric tensors of rank two and thus have $D(D+1)/2$ independent components per spatial point. There are $D+1$ independent constraints so that $D+1$ of these phase space variables can be eliminated. $D+1$ of the remaining degrees of freedom can be gauged away by a gauge transformation leaving us with $D(D+1) - 2(D+1) = (D-2)(D+1)$ phase space degrees of freedom or $(D-2)(D+1)/2$ configuration space degrees of freedom per spatial point. For $D=3$ we thus recover the two graviton degrees of freedom.

The further classical analysis of this system could now proceed as follows :

- 1) One determines a complete set of gauge invariant observables on the constraint surface $\bar{\mathcal{M}}$ and computes the induced symplectic structure $\bar{\Omega}$ on the so reduced symplectic manifold $\hat{\mathcal{M}}$. Equivalently, one obtains the full set of solutions to the equations of motion, the set of Cauchy data are then searched for observables. This programme of “symplectic reduction” could never be completed due to the complicated appearance of the Hamiltonian constraint. In fact, until today one does not know any observable for full general relativity.
- 2) One fixes a gauge and solves the constraints. Years of research in the field of solving the Cauchy problem for general relativity reveal that such a procedure works at most locally, that is, there do not exist, in general, global gauge conditions. This is reminiscent of the Gribov problem in non-Abelian Yang-Mills theories.

In summary, general relativity can be cast into Hamiltonian form, however, its equations of motion are complicated non-linear partial differential equations of second order and very difficult to solve. Nevertheless, the Cauchy problem is well-posed and the classical theory is consistent up to the point where singularities (e.g. black holes) appear. This is one instance where it is expected that the classical theory is unable to describe the system appropriately any longer and that the more exact theory of quantum gravity must take over in order to remove the singularity. This is expected to be quite in analogy to the case of the hydrogenium atom whose stability was a miracle to classical electrodynamics but was easily explained by quantum physics. Of course, the quantum theory of gravity is expected to be even harder to handle mathematically than the classical theory, however, as a zeroth step an existence proof would already be a triumph. Notice that up to date a similar existence proof for, say, QCD is lacking as well.

2.2 Estension of the ADM phase space

We would like to consider the phase space described in section 2.1 as the symplectic reduction of a larger symplectic manifold with coisotropic constraint surface. One defines a so-called co-D-bein field e_a^i on σ where the indices i, j, k, \dots take values $1, 2, \dots, D$. The D-metric is expressed in terms of e_a^i as

$$q_{ab} := \delta_{jk} e_a^j e_b^k. \quad (2.62)$$

Notice that this relation is invariant under local $SO(D)$ rotations $e_a^i \rightarrow O_j^i e_a^j$ and we therefore can view e_a^i , for $D = 3$, as an $su(2)$ -valued one-form (recall that the adjoint representation of $SU(2)$ on its Lie algebra is isomorphic with the defining representation of $SO(3)$ on \mathbf{R}^3 under the isomorphism $\mathbf{R}^3 \rightarrow su(2)$; $v^i \rightarrow v^i \tau_i$ where τ_i is a basis of $su(2)$). This observation makes it already obvious that we have to get rid of the $D(D-1)/2$ rotational degrees of freedom sitting in e_a^i but not in q_{ab} . Since the Cartan-Killing metric of $so(D)$ is just the Euclidean one we will in the sequel drop the δ_{ij} and also do not need to care about index positions.

Next we introduce yet another, independent one form K_a^i on σ which for $D = 3$ we also consider as $su(2)$ valued and from which the extrinsic curvature is derived as

$$-2sK_{ab} := \text{sgn}(\det((e_a^i))) K_{(a}^i e_{b)}^i. \quad (2.63)$$

We see immediately that K_a^i cannot be an arbitrary $D \times D$ matrix but must satisfy the constraint

$$G_{ab} := K_{[a}^j e_{b]}^j = 0 \quad (2.64)$$

since K_{ab} was a symmetric tensor field. With the help of the quantity

$$E_j^a := \frac{1}{(D-1)!} \epsilon^{aa_1 \dots a_{D-1}} \epsilon_{jj_1 \dots j_{D-1}} e_{a_1}^{j_1} \dots e_{a_{D-1}}^{j_{D-1}} \quad (2.65)$$

one can equivalently write (2.64) in the form

$$G_{jk} := K_{a[j} E_{k]}^a = 0 \quad (2.66)$$

Consider now the following functions on the extended phase space

$$q_{ab} := E_a^j E_b^j |\det((E_i^c))|^{2/(D-1)}, \quad P^{ab} := |\det((E_i^c))|^{-2/(D-1)} E_k^a E_k^d K_{[d}^j \delta_c^b] E_j^c \quad (2.67)$$

where E_a^j is the inverse of E_j^a . It is easy to see that when $G_{jk} = 0$, the functions (2.67) precisely reduce to the ADM coordinates. Inserting (2.67) into (2.29) we can also write the diffeomorphism and Hamiltonian constraint as functions on the extended phase space which one can check to be explicitly given by

$$H_a := -D_b [K_a^j E_j^b - \delta_a^b K_c^j E_j^c] \\ H := -\frac{s}{4\sqrt{\det(q)}} (K_a^l K_b^j - K_a^j K_b^l) E_j^a E_l^b - \sqrt{\det(q)} R \quad (2.68)$$

where $\sqrt{\det(q)} := |\det((E_j^a))|^{1/(D-1)}$ and $q^{ab} = E_j^a E_j^b / \det(q)$ by which $R = R(q)$ is considered as a function of E_j^a . Notice that, using (2.63), (2.65), expressions (2.68) indeed reduce to (2.29) up to terms proportional to G_{jk} .

Let us equip the extended phase space coordinatized by (K_a^i, E_i^a) with the symplectic structure (formally, that is without smearing) defined by

$$\{E_j^a(x), E_k^b(y)\} = \{K_a^j(x), K_b^k(y)\} = 0, \{E_i^a(x), K_b^j(y)\} = \kappa \delta_b^a \delta_i^j \delta(x, y) \quad (2.69)$$

We claim now that the symplectic reduction with respect to the constraint G_{jk} of the constrained Hamiltonian system subject to the constraints (2.66), (2.67) results precisely in the ADM phase space of section 2.1 together with the original diffeomorphism and Hamiltonian constraint.

To prove this statement we first of all define the smeared ‘‘rotation constraints’’

$$G(\Lambda) := \int_{\sigma} d^D x \Lambda^{jk} K_{aj} E_k^a \quad (2.70)$$

where $\Lambda^T = -\Lambda$ is an arbitrary antisymmetric matrix, that is, an $so(D)$ valued scalar on σ . They satisfy the Poisson algebra, using (2.69)

$$\{G(\Lambda), G(\Lambda')\} = G([\Lambda, \Lambda']) \quad (2.71)$$

in other words, $G(\Lambda)$ generates infinitesimal $SO(D)$ rotations as expected. Since the functions (2.67) are manifestly $SO(D)$ invariant by inspection they Poisson commute with $G(\Lambda)$, that is, they comprise a complete set of rotational invariant Dirac observables with respect to $G(\Lambda)$ for any Λ . As the constraints defined in (2.68) are in turn functions of these, $G(\Lambda)$ also Poisson commutes with the constraints (2.68) whence the total system of constraints consisting of (2.70), (2.68) is of first class.

Finally we must check that Poisson brackets among the q_{ab}, P^{cd} , considered as the functions (2.67) on the extended phase space with symplectic structure (2.69), is equal to the Poisson brackets of the ADM phase space (2.30, at least when $G_{jk} = 0$). Since q_{ab} is a function of E_j^a only it is clear that $\{q_{ab}(x), q_{cd}(y)\} = 0$. Next we have

$$\begin{aligned} \kappa\{P^{ab}(x), q_{cd}(y)\} &= \left(\frac{1}{2}[q^{a(e} q^{bf)} - q^{ab} q^{ef}]E_f^j\right)(x) \{K_e^j(x), (|\det(E)|^{2/(D-1)} E_c^k E_d^k)(y)\} \\ &= \left(\frac{1}{2}[q^{a(e} q^{bf)} - q^{ab} q^{ef}]E_f^j\right)(x) \left[\frac{2}{D-1} q_{cd}(x) \frac{\{K_e^j(x), |\det(E)|(y)\}}{|\det(E)|(x)} \right. \\ &\quad \left. + 2(\det(q) E_{(c}^k(x) \{K_e^j(x), E_{d)}^k(y)\})\right] \\ &= ([q^{a(e} q^{bf)} - q^{ab} q^{ef}] [-\frac{1}{D-1} q_{cd} q_{ef} + q_{e(c} q_{d)f}]) (x) \delta(x, y) \\ &= \delta_{(c}^a \delta_{d)}^b \delta(x, y) \end{aligned} \quad (2.72)$$

where we used $\delta E^{-1} = -E^{-1} \delta E E^{-1}$, $[\delta |\det(E)|] / |\det(E)| = [\delta \det(E)] / \det(E) = E_a^j \delta E_j^a$. The final Poisson bracket is the most difficult one. By carefully inserting the definitions, making use of the relations $E_j^a = \det(e) e_j^a$, $E_a^j = e_a^j \det(e)$, $e_j^a = q^{ab} e_b^j$ at

various steps one finds after two pages of simple but tedious algebraic manipulations that

$$\{P^{ab}(x), P^{cd}(y)\} = -\frac{\det(e)}{8}[q^{bc}G^{ad} + q^{bd}G^{ac} + q^{ac}G^{bd} + q^{ad}G^{bc}](x)\delta(x, y) \quad (2.73)$$

where $G^{ab} = q^{ac}q^{bd}G_{cd}$ and so (2.73) vanishes only at $G_{ab} = 0$.

Let us summarize : The functions (2.67) and (2.68) reduce at $G_{jk} = 0$ to the corresponding functions on the ADM phase space, moreover, their Poisson brackets among each other reduce at $G_{jk} = 0$ to those of the ADM phase space. Thus, as far as rotationally invariant observables are concerned, the only ones we are interested in, both the ADM system and the extended one are completely equivalent and we can as well work with the latter. This can be compactly described by saying that the symplectic reduction with respect to G_{jk} of the constrained Hamiltonian system described by the action

$$S := \frac{1}{\kappa} \int_{\mathbf{R}} dt \int_{\sigma} d^D x (\dot{K}_a^j E_j^a - [-\Lambda^{jk} G_{jk} + N^a H_a + NH]) \quad (2.74)$$

is given by the system described by the ADM action of section (2.1). Notice that, in accordance with what we said before, there is no claim that the Hamiltonian flow of K_a^j, E_j^a with respect to H_a, H is a spacetime diffeomorphism. However, since the Hamiltonian flow of H, H_a on the constraint surface $G_{jk} = 0$ is the same as on the ADM phase space for the gauge invariant observables q_{ab}, P^{ab} , a representation of $\text{Diff}(M)$ is still given on the constraint surface of $G_{jk} = 0$.

Canonical Transformation on the Extended Phase Space

Up to now we could work with arbitrary $D \geq 2$, however, what follows works only for $D = 3$. First we introduce the notion of the *spin connection* which is defined as an extension of the spatial covariant derivative D_a from tensors to generalized tensors with $so(D)$ indices. One defines

$$D_a u_{b..} v_j := (D_a u_b).. v_j + .. + u_{b..} (D_a v_j) \text{ where } D_a v_j := \partial_a v_j + \Gamma_{ajk} v^k \quad (2.75)$$

extends by linearity and requires that D_a is compatible with e_a^j , that is

$$D_a e_b^j = 0 \Rightarrow \Gamma_{ajk} = -e_k^b [\partial_a e_b^j - \Gamma_{ab}^c e_c^j] \quad (2.76)$$

Obviously Γ_a takes values in $so(D)$, that is, (2.76) defines an antisymmetric matrix.

Our aim is now to write the constraint G_{jk} in such a form that it becomes the Gauss constraint of an $SO(D)$ gauge theory, that is, we would like to write it in the form $G_{jk} = (\partial_a E^a + [A_a, E^a])_{jk}$ for some $so(D)$ connection A . It is here where $D = 3$ is singled out : What we have is an object of the form E_j^a which transforms in the defining representation of $SO(D)$ while E_{jk}^a transforms in the adjoint representation of $SO(D)$. It is only for $D = 3$ that these two are equivalent. Thus from now on we take $D = 3$.

The canonical transformation that we have in mind consists of two parts : 1) A constant Weyl (rescaling) transformation and 2) an affine transformation.

Constant Weyl Transformation

Observe that for any finite complex number $\beta \neq 0$, called the *Immirzi parameter*, the following rescaling $(K_a^j, E_j^a) \mapsto ({}^{(\beta)}K_a^j := \beta K_a^j, {}^{(\beta)}E_j^a := E_j^a/\beta)$ is a canonical transformation (the Poisson brackets (2.69) are obviously invariant under this map). We will use the notation $K = K^{(1)}, E = E^{(1)}$. In particular, for the rotational constraint (which we write in $D = 3$ in the equivalent form)

$$G_j = \epsilon_{jkl} K_{ak} E_l^a = \epsilon_{jkl} ({}^{(\beta)}K_a^k) ({}^{(\beta)}E_l^a) \quad (2.77)$$

is invariant under this rescaling transformation. We will consider the other two constraints (2.68) in a moment.

Affine Transformation

We notice from (2.76) that $D_a E_j^b = 0$. In particular, we have

$$D_a E_j^a = [D_a E^a]_j + \Gamma_{aj}{}^k E_k^a = \partial_a E_j^a + \epsilon_{jkl} \Gamma_a^k E_l^a = 0 \quad (2.78)$$

where the square bracket in the first identity means that D acts only on tensorial indices which is why we could replace D by ∂ as E_j^a is an $su(2)$ valued vector density of weight one. We also used the isomorphism between antisymmetric tensors of second rank and vectors in Euclidean space to define $\Gamma_a =: \Gamma_a^l T_l$ where $(T_l)_{jk} = \epsilon_{jlk}$ are the generators of $so(3)$ in the defining – or, equivalently, of $su(2)$ in the adjoint representation if the structure constants are chosen to be ϵ_{ijk} . Next we explicitly solve the spin connection in terms of E_j^a from (2.76) by using the explicit formula for Γ_{bc}^a and find

$$\begin{aligned} \Gamma_a^i &= \frac{1}{2} \epsilon^{ijk} e_k^b [e_{a,b}^j - e_{b,a}^j + e_j^c e_a^l e_{c,b}^l] \\ &= \frac{1}{2} \epsilon^{ijk} E_k^b [E_{a,b}^j - E_{b,a}^j + E_j^c E_a^l E_{c,b}^l] + \frac{1}{4} \epsilon^{ijk} E_k^b [2E_a^j \frac{(\det(E))_{,b}}{\det(E)} - E_b^j \frac{(\det(E))_{,a}}{\det(E)}] \end{aligned} \quad (2.79)$$

where in the second line we used that $\det(E) = [\det(e)]^2$ in $D = 3$. Notice that the second line in (2.79) explicitly shows that Γ_a^j is a homogenous rational function of degree zero of E_j^a and its derivatives. Therefore we arrive at the important conclusion that

$$({}^{(\beta)}\Gamma_a^j) := \Gamma_a^j ({}^{(\beta)}E) = \Gamma_a^j = \Gamma_a^j ({}^{(1)}E) \quad (2.80)$$

is itself invariant under the rescaling transformation. This is obviously also true for the Christoffel connection Γ_{bc}^a since it is a homogenous rational function of degree zero in q_{ab} and its derivatives and $q_{ab} = \det(E) E_a^j E_b^j \mapsto ({}^{(\beta)}q_{ab}) = \beta ({}^{(1)}q_{ab})$. Thus the derivative D_a is, in fact, independent of β and we therefore have in particular $D_a ({}^{(\beta)}E_j^a) = 0$. We can then write the rotational constraint in the form

$$G_j = 0 + \epsilon_{jkl} ({}^{(\beta)}K_a^k) ({}^{(\beta)}E_l^a) = \partial_a ({}^{(\beta)}E_j^a) + \epsilon_{jkl} [\Gamma_a^j + ({}^{(\beta)}K_a^k)] ({}^{(\beta)}E_l^a) =: ({}^{(\beta)}D_a ({}^{(\beta)}E_j^a) \quad (2.81)$$

This equation suggests to introduce the new connection

$$({}^{(\beta)}A_a^j) := \Gamma_a^j + ({}^{(\beta)}K_a^j) \quad (2.82)$$

This connection could be called the Sen – Ashtekar – Immirzi – Barbero connection (names in historical order) for the historical reasons mentioned in the beginning of this section. More precisely the Sen connection arises for $\beta = \pm i$, $G_j = 0$, the Ashtekar connection for $\beta = \pm i$, the Immirzi connection for complex β and the Barbero connection for real β . For simplicity we will refer to it as the *new* connection which now replaces the spin-connection Γ_a^j and gives rise to a new derivative $({}^{(\beta)}D_a$ acting on generalized tensors as the extension by linearity of the basic rules $({}^{(\beta)}D_a v_j := \partial_a v_j + \epsilon_{jkl} ({}^{(\beta)}A_a^k) v_l$ and $({}^{(\beta)}D_a u_b := D_a u_b$. Notice that (2.81) has *precisely* the structure of a Gauss law constraint for an $SU(2)$ gauge theory although $({}^{(\beta)}A$ qualifies as the pull-back to σ by local sections of a connection on an $SU(2)$ fibre bundle over σ only when β is real. Henceforth we will call G_j the *Gauss constraint*.

Given the complicated structure of (2.79) it is quite surprising that the variables $({}^{(\beta)}A, {}^{(\beta)}E)$ form a canonically conjugate pair, that is

$$\{({}^{(\beta)}A_a^j(x), {}^{(\beta)}A_b^k(y)\} = \{({}^{(\beta)}E_j^a(x), {}^{(\beta)}E_k^b(y)\} = 0, \{({}^{(\beta)}E_j^a(x), {}^{(\beta)}A_b^j(y)\} = \kappa \delta_b^a \delta_j^k \delta(x, y) \quad (2.83)$$

This is the key feature for why these variables are at all useful in quantum theory : If we would not have such a simple bracket structure classically then it would be very hard to find Hilbert space representations that turn these Poisson bracket relations into canonical commutation relations.

To prove (2.83) by means of (2.69) (which is invariant under replacing K, E by $({}^{(\beta)}K, {}^{(\beta)}E)$) we notice that the only non-trivial relation is the first one since $\{E_j^a(x), \Gamma_b^k(y)\} = 0$. That relation is explicitly given as

$$\beta [\{\Gamma_a^j(x), K_b^k(y)\} - \{\Gamma_b^k(y), K_a^j(x)\}] = \beta \kappa \left[\frac{\delta \Gamma_a^j(x)}{\delta E_b^k(y)} - \frac{\delta \Gamma_b^k(y)}{\delta E_j^a(x)} \right] = 0 \quad (2.84)$$

which is just the integrability condition for Γ_a^j to have a generating potential F . A promising candidate for F is given by the functional

$$F = \int_{\sigma} d^3x E_j^a(x) \Gamma_j^a(x) \quad (2.85)$$

since if (2.84) holds we have

$$\begin{aligned} \frac{\delta F}{\delta E_j^a(x)} - \Gamma_a^j(x) &= \int d^3y E_k^b(y) \frac{\delta \Gamma_b^k(y)}{\delta E_j^a(x)} = \int d^3y E_k^b(y) \frac{\delta \Gamma_a^j(x)}{\delta E_k^b(y)} \\ &= \frac{1}{\kappa} \{ \Gamma_a^j(x), \int d^3y K_b^k(y) E_k^b(y) \} = 0 \end{aligned} \quad (2.86)$$

because the function $\int d^3y K_b^k(y) E_k^b(y)$ is the canonical generator of *constant* scale transformations under which Γ_a^j is invariant as already remarked above. To show that

F is indeed a potential for Γ_a^j we demonstrate (2.86) in the form $\int d^3x E_j^a(x) \delta \Gamma_a^j(x) = 0$. Starting from (2.79) we have (using $\delta e_a^j e_j^b = \delta e_b^j e_k^b = 0$ repeatedly)

$$\begin{aligned}
e_i^a \delta \Gamma_a^i &= \frac{1}{2} \epsilon^{ijk} \det(e) e_i^a \delta(e_k^b [e_{a,b}^j - e_{b,a}^j + e_j^c e_a^l e_{c,b}^l]) \\
&= \frac{1}{2} \epsilon^{ijk} \det(e) [e_i^a \delta(e_k^b (e_{a,b}^j - e_{b,a}^j)) + \delta(e_k^b e_j^c e_{c,b}^i) - (\delta e_i^a) e_j^c e_a^l e_k^b e_{c,b}^l] \\
&= \frac{1}{2} \epsilon^{ijk} \det(e) [e_i^a \delta(e_k^b (e_{a,b}^j - e_{b,a}^j)) + \delta(e_k^b e_j^a e_{a,b}^i) + (\delta e_a^l) e_i^a e_j^c e_k^b e_{c,b}^l] \\
&= \frac{1}{2} \epsilon^{ijk} \det(e) [\delta(e_i^a e_k^b (e_{a,b}^j - e_{b,a}^j) + e_k^b e_j^a e_{a,b}^i) - (\delta e_i^a) e_k^b (e_{a,b}^j - e_{b,a}^j) + \\
&\quad + (\delta e_a^l) e_i^a e_j^c e_k^b e_{c,b}^l] \\
&= \frac{1}{2} \epsilon^{ijk} \det(e) [\delta(e_k^b (e_j^a e_{a,b}^i + e_i^a e_{a,b}^j) - e_i^a e_k^b e_{b,a}^j) + (\delta e_k^b) e_i^a e_{b,a}^j + (\delta e_i^a) e_k^b e_{b,a}^j \\
&\quad + (\delta e_a^l) e_i^a e_j^c e_k^b e_{c,b}^l] \\
&= -\frac{1}{2} \epsilon^{abc} [e_c^j \delta e_{b,a}^j - (\delta e_a^j) e_{c,b}^j] \\
&= -\frac{1}{2} \epsilon^{abc} \partial_a [(\delta e_b^j) e_c^j] \tag{2.87}
\end{aligned}$$

From the first to the second line we pulled e_i^a into the variation of the the third term of $\delta \Gamma_a^i$ resulting in a correction proportional to δe_a^i , in the next line we relabelled the summation index c into a in the third term and traded the variation of e_i^a for that of e_a^l in the fourth term, in the next line we pulled again e_i^a inside a variation resulting in altogether six terms, in the next line we collected the total variation terms and reordered them and in the fourth term we relabelled the summation indices a, b into b, a and i, k into k, i resulting in a minus sign from the ϵ^{ijk} , in the next line we realized that the first two terms are symmetric in i, j which thus drop out due to the ϵ^{ijk} and that the e_i^a and e_k^b variation pieces of the third term cancel against the fourth and fifth term, in the next line we made use of the relations $\det(e) \epsilon^{ijk} e_j^a e_k^b e_c^c = \epsilon^{abc} e_a^i$, $\det(e) \epsilon^{ijk} e_i^a e_j^b e_k^c = \epsilon^{abc}$ and relabelled j for l and in the last line finally we relabelled a for b in the second term resulting in a minus sign and allows us to write the whole thing as a derivative. It follows that

$$\int_{\sigma} d^3x E_j^a \delta \Gamma_a^j = -\frac{1}{2} \int_{\sigma} d^3x \partial_a (\epsilon^{abc} \delta e_b^j e_c^j) = \frac{1}{2} \int_{\partial \sigma} dS_a \epsilon^{abc} e_b^j \delta e_c^j \tag{2.88}$$

which vanishes since $\partial \sigma = \emptyset$. If σ has a boundary such as spatial infinity then the boundary conditions such as imposing e_a^j to be an even function on the asymptotic sphere under Cartesian coordinate reflection guarantee vanishing of (2.88) as well.

It remains to write the constraints (2.68) in terms of the variables $^{(\beta)}A, ^{(\beta)}E$. To that end we introduce the curvatures

$$\begin{aligned}
R_{ab}^j &:= 2\partial_{[a} \Gamma_{b]}^j + \epsilon_{jkl} \Gamma_a^k \Gamma_b^l \\
^{(\beta)}F_{ab}^j &:= 2\partial_{[a} ^{(\beta)}A_{b]}^j + \epsilon_{jkl} ^{(\beta)}A_a^k ^{(\beta)}A_b^l \tag{2.89}
\end{aligned}$$

whose relation with the covariant derivatives is given by $[D_a, D_b]v_j = R_{abjl}v^l = \epsilon_{jkl}R_{ab}^k v^l$ and $[(^{(\beta)}D_a, ^{(\beta)}D_b)v_j = (^{(\beta)}F_{abjl}v^l = \epsilon_{jkl}^{(\beta)}F_{ab}^k v^l$. Let us expand $(^{(\beta)}F$ in terms of Γ and $(^{(\beta)}K$

$$(^{(\beta)}F_{ab}^j = R_{ab}^j + 2\beta D_{[a}K_{b]}^j + \beta^2 \epsilon_{jkl}K_a^j K_b^k \quad (2.90)$$

Contracting with $(^{(\beta)}E$ yields

$$(^{(\beta)}F_{ab}^j (^{(\beta)}E_j^b = \frac{R_{ab}^j E_j^b}{\beta} + 2D_{[a}(K_{b]}^j E_j^b) + \beta K_a^j G_j \quad (2.91)$$

where we have used the Gauss constraint in the form (2.77). We claim that the first term on the right hand side of (2.91) vanishes identically. To see this we first derive from (2.76), due to torsion freeness of the Christoffel connection in the language of forms, the *algebraic Bianchi identity*

$$\begin{aligned} dx^a \wedge dx^b D_a e_b^j &= de^j + \Gamma_k^j \wedge e^k = 0 \\ \Rightarrow 0 &= -d^2 e^j = d\Gamma_k^j \wedge e^k - \Gamma_l^j \wedge de^l = [d\Gamma_k^j + \Gamma_l^j \wedge \Gamma_k^l] \wedge e^k = \Omega_k^j \wedge e^k \end{aligned} \quad (2.92)$$

Now $\Omega_k^j = \Omega^i(T_i)_{jk} =: (\Omega)_{jk}$ and we see that

$$\Omega = d\Gamma + \Gamma \wedge \Gamma = d\Gamma^i T_i + \frac{1}{2}[T_j, T_k]\Gamma^j \wedge \Gamma^k = \frac{1}{2}dx^a \wedge dx^b R_{ab}^i T_i$$

Thus the Bianchi identity can be rewritten in the form

$$\begin{aligned} \epsilon_{ijk}\epsilon^{efc}R_{ef}^j e_c^k &= 0 \Rightarrow \\ \frac{1}{2}\epsilon_{ijk}\epsilon^{efc}R_{ef}^j e_c^k e_a^i &= \frac{1}{2}E_j^b \epsilon_{cab}\epsilon^{efc}R_{ae}^j \\ &= R_{ab}^j E_j^b = 0 \end{aligned} \quad (2.93)$$

as claimed. Now we compare with the first line of (2.68) and thus arrive at the conclusion

$$(^{(\beta)}F_{ab}^j (^{(\beta)}E_j^b = H_a + (^{(\beta)}K_a^j G_j \quad (2.94)$$

Next we contract (2.90) with $\epsilon_{jkl} (^{(\beta)}E_k^a (^{(\beta)}E_l^b$ and find

$$\begin{aligned} &(^{(\beta)}F_{ab}^j \epsilon_{jkl} (^{(\beta)}E_k^a (^{(\beta)}E_l^b \\ &= \det(q) \frac{R_{abkl} e_k^a e_l^b}{\beta^2} - 2 \frac{E_j^a D_a G_j}{\beta} + (K_a^j E_j^a)^2 - (K_b^j E_j^a)(K_a^k E_k^b) \end{aligned} \quad (2.95)$$

Expanding $v_j = e_j^a v_a$, $v_a = e_a^j v_j$, using $D_a e_b^j = 0$ and comparing $[D_a, D_b]v_j$ with $[D_a, D_b]v_c$ for any v_j we find $R_{abij} = R_{abcd}e_i^c e_j^d$ and so (2.95) can be rewritten as

$$\begin{aligned} &(^{(\beta)}F_{ab}^j \epsilon_{jkl} (^{(\beta)}E_k^a (^{(\beta)}E_l^b \\ &= -\det(q) \frac{R}{\beta^2} - 2 (^{(\beta)}E_j^a D_a G_j + (K_a^j E_j^a)^2 - (K_b^j E_j^a)(K_a^k E_k^b) \end{aligned} \quad (2.96)$$

and comparing with the second line of (2.68) we conclude

$$\begin{aligned}
& {}^{(\beta)}F_{ab}^j \epsilon_{jkl} {}^{(\beta)}E_k^a {}^{(\beta)}E_l^b + 2 {}^{(\beta)}E_j^a D_a G_j \\
= & \sqrt{\det(q)} \left[-\sqrt{\det(q)} \frac{R}{\beta^2} - \frac{(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2}{\sqrt{\det(q)}} \right] \\
= & \frac{\sqrt{\det(q)}}{\beta^2} \left[-\sqrt{\det(q)} R - \beta^2 \frac{(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2}{\sqrt{\det(q)}} \right] \\
= & \frac{\sqrt{\det(q)}}{\beta^2} \left[H + \left(\frac{s}{4} - \beta^2 \right) \frac{(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2}{\sqrt{\det(q)}} \right] \\
= & 4s \sqrt{\det(q)} \left[-\frac{s}{4\sqrt{\det(q)}} [(K_b^j E_j^a)(K_a^k E_k^b) - (K_a^j E_j^a)^2] - \frac{s}{4\beta^2} \sqrt{\det(q)} R \right] \\
= & 4s \sqrt{\det(q)} \left[H - \left(1 + \frac{s}{4\beta^2} \right) \sqrt{\det(q)} R \right] \tag{2.97}
\end{aligned}$$

We see that the left hand side of (2.97) is proportional to H if and only if $\beta = \pm\sqrt{s}/2$, that is, imaginary (real) for Lorentzian (Euclidean) signature. We prefer, for reasons that become obvious only in a later section, to solve (2.97) for H as follows

$$\begin{aligned}
H = & \frac{\beta^2}{\sqrt{\det({}^{(\beta)}q\beta)}} [{}^{(\beta)}F_{ab}^j \epsilon_{jkl} {}^{(\beta)}E_k^a {}^{(\beta)}E_l^b + 2 {}^{(\beta)}E_j^a D_a G_j] \\
& + \left(\beta^2 - \frac{s}{4} \right) \frac{({}^{(\beta)}K_b^j {}^{(\beta)}E_j^a)({}^{(\beta)}K_a^k {}^{(\beta)}E_k^b) - ({}^{(\beta)}K_c^j {}^{(\beta)}E_j^c)^2}{\sqrt{\det({}^{(\beta)}q\beta)}} \tag{2.98}
\end{aligned}$$

In formula (2.98) we wrote everything in terms of ${}^{(\beta)}A$, ${}^{(\beta)}E$ if we understand ${}^{(\beta)}K = ({}^{(\beta)}A - \Gamma)$ and we used ${}^{(\beta)}q_{ab} = \beta^{-1} q_{ab} = ({}^{(\beta)}E_a^j {}^{(\beta)}E_b^j) \det({}^{(\beta)}E)$.

We notice that both (2.94) and (2.98) still involve the Gauss constraint. Since the transformation $K_a^j \mapsto ({}^{(\beta)}A_a^j, E_j^a \mapsto ({}^{(\beta)}A_a^j)$ is a canonical one, the Poisson brackets among the set of first class constraints given by G_j, H_a, H are unchanged. Let us write symbolically $H_a = H'_a + f_a^j G_j, H = H' + f^j G_j$ where H'_a, H' are the pieces of H_a, H respectively not proportional to the Gauss constraint. Since G_j generates a subalgebra of the constraint algebra it follows that the modified system of constraints given by G_j, H'_a, H' not only defines the same constraint surface of the phase space but also gives a first class system again, of course, with somewhat modified algebra which however coincides with the Dirac algebra on the submanifold $G_j = 0$ of the phase space. In other words, it is completely equivalent to work with the set of constraints G_j, H'_a, H' which we write once more, dropping the prime, as

$$\begin{aligned}
G_j &= {}^{(\beta)}D_a {}^{(\beta)}E_j^a = \partial_a {}^{(\beta)}E_j^a + \epsilon_{jkl} {}^{(\beta)}A_a^j {}^{(\beta)}E_j^a \\
H_a &= {}^{(\beta)}F_{ab}^j {}^{(\beta)}E_j^b \\
H &= \left[\beta^2 {}^{(\beta)}F_{ab}^j + \left(\beta^2 - \frac{s}{4} \right) \epsilon_{jmn} {}^{(\beta)}K_a^m {}^{(\beta)}K_b^n \right] \frac{\epsilon_{jkl} {}^{(\beta)}E_k^a {}^{(\beta)}E_l^b}{\sqrt{\det({}^{(\beta)}q\beta^3)}} \tag{2.99}
\end{aligned}$$

For easier comparison with the literature we also write (2.99) in terms of ${}^{(\beta)}A_a^j, K_a^j, E_j^a$

which gives

$$\begin{aligned}
G_j &= {}^{(\beta)}D_a E_j^a / \beta = (\partial_a {}^{(\beta)}E_j^a + \epsilon_{jkl} {}^{(\beta)}A_a^j E_l^k) / \beta \\
H_a &= {}^{(\beta)}F_{ab}^j E_j^b / \beta \\
H &= [{}^{(\beta)}F_{ab}^j + (\beta^2 - \frac{s}{4}) \epsilon_{jmn} K_a^m K_b^n] \frac{\epsilon_{jkl} E_k^a E_l^b}{\sqrt{\det(q)}} \quad (2.100)
\end{aligned}$$

At this point we should say that our conventions differ slightly from those in the literature : There one writes the constraint in terms of $\tilde{K}_a^j := K_a^j/2$ and one defines ${}^{(\beta)}\tilde{K} := \beta\tilde{K} = {}^{(\beta)}K/2 = {}^{(\beta/2)}K$ and ${}^{(\beta)}\tilde{A} := \Gamma + \beta\tilde{K} = \Gamma + \beta/2K = {}^{(\beta/2)}A$ at the price of $2 {}^{(\beta)}E$ being conjugate to ${}^{(\beta)}\tilde{A}$ instead of ${}^{(\beta)}E$ being conjugate to ${}^{(\beta)}A$. Thus ${}^{(\beta)}A = {}^{(2\beta)}\tilde{A} = {}^{(\tilde{\beta})}\tilde{A}$ with $\tilde{\beta} = 2\beta$. When writing H in terms of these quantities we find

$$H = [{}^{(\tilde{\beta})}F_{ab}^j + (\tilde{\beta}^2 - s) \epsilon_{jmn} \tilde{K}_a^m \tilde{K}_b^n] \frac{\epsilon_{jkl} E_k^a E_l^b}{\sqrt{\det(q)}} \quad (2.101)$$

where now $\tilde{\beta}^2 = s$ is the preferred value.

Summarizing, we have rewritten the Einstein Hilbert action in the following equivalent form

$$S = \frac{1}{\kappa} \int_{\mathbf{R}} dt \int_{\sigma} d^3x ({}^{(\beta)}\dot{A}_a^i {}^{(\beta)}E_i^a - [\Lambda^i G_i + N^a V_a + NH]) \quad (2.102)$$

where the appearing constraints are the ones given by either of (2.101), (2.100) or (2.99).

Several remarks are in order :

- *Four-dimensional Interpretation*

Let us try to give a four-dimensional meaning to ${}^{(\beta)}A$. To that end we must complete the 3-bein e_i^a to a 4-bein e_α^μ where μ is a spacetime tensor index and $\alpha = 0, 1, 2, 3$ an index for the defining representation of the Lorentz (Euclidean) group for $s = -1(+1)$. By definition $g_{\mu\nu} e_\alpha^\mu e_\beta^\nu = \eta_{\alpha\beta}$ is the flat Minkowski (Euclidean) metric. Thus e_0^μ, e_i^μ are orthogonal vectors and we thus choose $e_0^\mu = n^\mu$ and in the ADM frame with $\mu = t, a$ we choose $(e_i^\mu)_{\mu=a} = e_i^a$. Using the defining properties of a tetrad basis and the explicit form of $n^\mu, g_{\mu\nu}$ in the ADM frame derived earlier, above choices are sufficient to fix the tetrad components completely to be $e_0^t = 1/N, e_0^a = -N^a/N, e_i^t = 0, e_i^a$. Inversion gives (notice that $e_\mu^0 = s e_{\mu 0} = s g_{\mu\nu} e_0^\nu = s g_{\mu\nu} n^\nu = s n_\mu$) $e_t^0 = N, e_a^0 = 0, e_t^i = N^a e_a^i, e_a^i$. Finally we have for $q_\nu^\mu = \delta_\nu^\mu - s n^\mu n_\nu = \delta_\nu^\mu - e_0^\mu e_\nu^0$ in the ADM frame $q_t^t = 0, q_a^t = 0, q_t^a = N^a, q_b^a = \delta_b^a$. Thus we obtain, modulo $G_j = 0$

$$\begin{aligned}
K_a^j &= -2s e_j^b K_{ab} = -2s e_j^b q_a^\mu q_b^\nu \nabla_\mu n_\nu = -2e_j^b (\nabla_a e_b)^0 = 2e_j^b (\omega_a)^0{}_\alpha e_b^\alpha \\
&= 2e_j^b (\omega_a)^0{}_k e_b^k = 2(\omega_a)^0{}_j \quad (2.103)
\end{aligned}$$

where in the second identity the bracket denotes that ∇ only acts on the tensorial index and in the third we used the definition of the four dimensional spin

connection $\nabla_\mu e_\nu^\alpha = (\nabla_\mu e_\nu)^\alpha + (\omega_\mu)_\beta^\alpha e_\nu^\beta = 0$. On the other hand we have

$$(\Gamma_a)^j{}_k e_b^k = -(D_a e_b)^j = -g_a^\mu g_b^\nu (\nabla_\mu e_\nu)^j = -(\nabla_a e_b)^j = (\omega_a)^j{}_k e_b^k \quad (2.104)$$

whence $\omega_{ajk} = \Gamma_{ajk}$. It follows that

$${}^{(\beta)}A_{ajk} = \omega_{ajk} - 2\beta s \omega_{a0l} \epsilon_{jkl} \quad (2.105)$$

The Hodge dual of an antisymmetric tensor $T_{\alpha\beta}$ is defined by

$$*T_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \eta^{\gamma\gamma'} \eta^{\delta\delta'} T_{\gamma'\delta'}$$

Since $\epsilon_{0ijk} = \epsilon_{ijk}$ we can write (2.105) in the form

$${}^{(\beta)}A_{ajk} = \omega_{ajk} - 2\beta * \omega_{ajk} \quad (2.106)$$

Now an antisymmetric tensor is called (anti)self-dual provided that $*T_{\alpha\beta} = \pm\sqrt{s}T$ and the (anti)self-dual piece of any $T_{\alpha\beta}$ is defined by $T^\pm = \frac{1}{2}[T \pm *T/\sqrt{s}]$ since $*\circ* = s \text{ id}$. An (anti)self-dual tensor therefore has only three linearly independent components. This case happens for (2.106) provided that either $s = 1, \beta = \mp 1/2$ or $s = -1, \beta = \pm i/2$ and *in this case the new connection is just (twice) the (anti)self-dual piece of the pull-back to σ of the four-dimensional spin-connection*. In all other cases (2.106) is only half of the information needed in order to build a four-dimensional connection and therefore we do not know how it transforms under internal boosts. This is, from this perspective, the reason why one has to gauge fix the boost symmetry of the action

$$S = \int_M \text{tr} (F \wedge [* - \beta^{-1}](e \wedge e)),$$

by the time gauge $e_\mu^\alpha n^\mu = \delta_0^\alpha$, in order to remove the then present second class constraints and to arrive at the present formulation. Obviously, this is no obstacle, first, since there *does exist* a four-dimensional interpretation even in that case as we just showed and more explicitly from (2.69) and, secondly, since we are not interested in the transformation properties under spacetime diffeomorphisms and internal Lorentz transformations of non-gauge-invariant objects anyway, although from an aesthetic point of view it would be desirable to have such an interpretation.

- *Reality Conditions*

When β is real valued ${}^{(\beta)}A, {}^{(\beta)}E$ are both real valued and can directly be interpreted as the canonical pair for the phase space of an $SU(2)$ Yang-Mills theory. If β is complex then these variables are complex valued. However, they cannot be arbitrary complex functions on σ but are subject to the following reality conditions

$${}^{(\beta)}E/\beta = \overline{{}^{(\beta)}E/\beta}, \quad [{}^{(\beta)}A - \Gamma]/\beta = \overline{[{}^{(\beta)}A - \Gamma]/\beta} \quad (2.107)$$

where $\Gamma = \Gamma^{(\beta)}$ is a non-polynomial, not even analytic function. These reality conditions guarantee that there is no doubling of the number of degrees of freedom and one can check explicitly that they are preserved by the Hamiltonian flow of the constraints provided that Λ^j , the Lagrange multiplier of the Gauss constraint, is real valued. Thus, only $SU(2)$ gauge transformations are allowed but not general $SL(2, \mathbf{C})$ transformations. The reality conditions are difficult to implement in the quantum theory directly as already mentioned above.

- *Simplification of the Hamiltonian Constraint*

The original motivation to introduce the new variables was that for the quantization of general relativity it seemed mandatory to simplify the algebraic structure of the Hamiltonian constraint which for $s = -1$ requires $\beta = \pm i/2$ since then the constraint becomes polynomial after multiplying by a factor proportional to $\sqrt{\det(q)}$. On the other hand, then the reality conditions become non-polynomial. Finally, if one wants polynomial reality conditions then one must have β real and then the Hamiltonian constraint is still complicated. Thus it becomes questionable what has been gained. The answer is the following : For any choice of β one can actually make both the Hamiltonian constraint *and* the reality conditions polynomial by multiplying by a sufficiently high power of $\det(q)$. But the real question is whether the associated classical functions will become well-defined operator-valued distributions in quantum theory while keeping background independence. As we will see in later sections, the Hilbert space that we will choose does not support any quantum versions of these functions rescaled by powers of $\det(q)$ and there are abstract arguments that suggest that this is a representation independent statement. The requirement seems to be that the Hamiltonian constraint is a scalar density of weight one and thus we must keep the factor of $1/\sqrt{\det(q)}$ in (2.100) whatever the choice of β and therefore the motivation for polynomiality is lost completely. The motivation to have a connection formulation rather than a metric formulation is then that *that one can go much farther in the background independent quantization programme provided that β is real*. For instance, a connection formulation enables us to employ the powerful arsenal of techniques that have been developed for the canonical quantization of Yang-Mills theories, specifically Wilson loop techniques.

- *Choice of Fibre Bundle*

In the whole exposition so far we have assumed that we have a trivial principal $SU(2)$ bundle over σ so that we can work with a globally defined connection potential and globally defined electric field $^{(\beta)}A$, $^{(\beta)}E$ respectively. What about different bundle choices ?

Our situation is that we are dealing with a principal $SU(2)$ bundle over σ with pull-backs $^{(\beta)}A_I$ by local sections of a connection and local sections $^{(\beta)}E_I$ of an associated (under the adjoint representation) vector bundle of two forms and would like to know whether these bundles are trivial. Since the latter is built out of the 3-beins we can equivalently look also at the frame bundle of orthonormal frames in order to decide for triviality. Triviality of the frame bundle is equivalent to the triviality of its associated principal bundle and in turn to σ being

parallelizable. But this is automatically the case for any compact, orientable three manifold provided that $G = SU(2)$. More generally, in order to prove that a principal fibre bundle is trivial one has to show that the cocycle h_{IJ} of transition functions between charts of an atlas of σ is a coboundary, that is, its (non-Abelian) Čech cohomology class is trivial. So far we did not make the assumption that σ is compact or orientable. If σ is not compact but orientable then one usually requires that there is a compact subset B of σ such that $\sigma - B$ has the topology of the complement of a ball in \mathbf{R}^3 . Then the result holds in B and trivially in $\sigma - B$ and thus all over σ . Thus, compactness is not essential. If σ is not orientable then a smooth nowhere singular frame cannot exist and the above quoted result does not hold, there are no smooth 3-bein fields in this case. In that case we allow non-smooth 3-bein fields, that is, we allow that $\det(e)$ has finite jumps between $\pm|\det(e)|$ on subsets of σ of Lebesgue measure zero (two surfaces) due to change of sign of one of the three forms e^j . This requires that one works with a fixed trivialization at the gauge variant level classically. At the gauge invariant level the dependence on that trivialization disappears, so there is no problem. More specifically, the constraints H, H_a as well as the symplectic structure are gauge invariant while G_j is gauge covariant so that we have independence of the choice of trivialization again on the constraint surface G_j as expected, we get equivalence with the ADM formulation.. As we will see, the choice of the bundle will become completely irrelevant anyway in the quantum theory.

- *Orientation*

So far we did not need to impose any restriction on the orientation of the e_a^j . However, from $E_j^a = e_j^a \det(e)$ we easily obtain in $D = 3$ that $\det(E) = [\det(e)]^2 = \det(q) > 0$. Thus, classically the E_j^a are not arbitrary Lie algebra valued vector densities but rather are subject to the *anholonomic* constraint

$$\det(E) > 0 \tag{2.108}$$

One can remove this constraint by multiplying the basic variables by $\text{sgn}(\det(e))$: $E_j^a := \sqrt{\det(q)} e_j^a$, $K_a^j = -2s K_{ab} e_j^b$ (modulo $G_j = 0$) so that in fact $\det(E) = \det(q) \text{sgn}(\det(e))$ but then the result (2.87) fails to hold (the symplectic structure remains, surprisingly, unchanged), one would get instead

$$\int d^3x E_j^a \delta \Gamma_a^j = -\frac{1}{2} \int \text{sgn}(\det(e)) \epsilon^{abc} \partial_a (\delta e_b^j e_c^j) = \frac{1}{4} \int d^3x \partial_a [\text{sgn}(\det(e))] \epsilon^{abc} \delta q_{bc}$$

which is ill-defined since $0 = \epsilon^{abc} \delta q_{bc}$ is multiplied by the distribution $\partial_a [\text{sgn}(\det(e))]$ unless one makes further assumptions classically such as that this distributional one form has support on a set of measure zero (motivated by the fact that q_{ab} is smooth).

In view of these considerations we will from now on only consider positive β unless otherwise specified.

Chapter 3

Quantum space

3.1 Structure of quantum gravity

From the results in the previous chapter we see that General Relativity can be formulated as the dynamical system defined by the Hamilton-Jacobi equation:

$$\left[{}^{(\tilde{\beta})}F_{ab}^{ij} + (\tilde{\beta}^2 - s)\tilde{K}_a^i\tilde{K}_b^j \right] \frac{\delta S[A]}{\delta A_a^i} \frac{\delta S[A]}{\delta A_b^j} = 0 \quad \frac{\delta S[A]}{\delta A_a^i} = \frac{1}{\kappa} E_i^a \quad (3.1)$$

where the functional $S[A]$ is defined on the space \mathbf{G} of the 3d $SU(2)$ connections A_a^i , and is invariant under internal gauge transformations and 3d diffeomorphisms. From now we drop the symbols “ \sim ” and “ $(\tilde{\beta})$ ”, according with the literature. A quantization of the theory can be obtained in terms of complex-valued Schrodinger wave functionals $\psi[A]$ on \mathbf{G} . The quantum dynamics is inferred from the classical dynamics by interpreting $S[A]$ as \hbar times the phase of $\psi[A]$. Namely, interpreting the classical Hamilton-Jacobi theory as the Eikonal approximation (i.e. neglecting the second derivatives of the phase) of a quantum wave equation; semiclassical “wave packets” will then behave according to classical theory. This can be obtained defining the quantum dynamics by replacing derivatives of the Hamilton-Jacobi functional $S[A]$ with derivative operators. The hamiltonian flows of the functions \vec{H} , H , \vec{G} respect to $S[A]$ contain only first order derivative, so they remain unchanged and simply force $\psi[A]$ to be invariant under $SU(2)$ gauge transformations and 3d diffeomorphisms. Equation (3.1) gives

$$H\psi[A] = \left[F_{ab}^{ij} + (\beta^2 - s)K_a^iK_b^j \right] \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} \psi[A] = 0. \quad (3.2)$$

This is the Wheeler-De Witt or Einstein-Schrodinger equation. It governs the quantum dynamics of spacetime. In other words, the dynamics is defined by the hamiltonian operator $H = H[A, -i\hbar\delta/\delta A]$. More precisely, we want a rigged Hilbert space $S \subset K \subset S'$, where S is a suitable space of functional $\psi[A]$. Partial observables are represented by self-adjoint operators on K . Their eigenvalues describe the quantization of physical

quantities. The projector P , formally given by

$$P \sim \delta(H) \sim \int [DN] e^{-i \int d^3x N(x) H(x)}, \quad (3.3)$$

sends S to the space of solutions of (3.2). Its matrix elements between eigenstates of partial observables define the transition amplitudes of quantum gravity. These determine all probabilistic dynamical relations between any measurement that we can perform. A preferred state in K is $|0\rangle$, the eigenstate of the geometry with zero volume and zero area. The covariant vacuum is given by $|0_c\rangle = P|0\rangle$. The physical scalar product (i.e. the scalar product between the physical projections of kinematical states) can be written as

$$\sum_{\Gamma_1, \Gamma_2} \overline{\psi_1(\Gamma_1)} W(\Gamma_1, \Gamma_2) \psi_2(\Gamma_2) \quad \psi_1(\Gamma_1) = \langle \Gamma_1 | 1 \rangle \quad \psi_2(\Gamma_2) = \langle \Gamma_2 | 2 \rangle, \quad (3.4)$$

with Γ a suitable basis for the kinematical initial (2) and final (1) states, and

$$W(\Gamma_1, \Gamma_2) = \langle 1 | P | 2 \rangle.$$

We can rewrite the (3.4) as

$$\langle W_{12} | \psi_{12} \rangle = \langle 0 | P (|1\rangle \otimes |2\rangle) \rangle \quad (3.5)$$

in terms of the state $\psi_{12} = \bar{\psi}_1 \psi_2$ living on the Hilbert space $K_{12} = K_1^* \times K_2$. So, in general, we can consider a state ψ_3 which describe the fluctuations of the entire boundary of a spacetime region. We can write the correlation probability amplitude associated with a measurement of partial observables on the boundary surface as

$$\langle W_3 | \psi_3 \rangle \quad W_3 = \langle 0 | P | 3 \rangle,$$

where $|3\rangle$ will be the eigenstate of the partial observables corresponding to the measured eigenvalues.

3.2 The kinematical state space K

We limit ourselves here to case where the quantum state space is defined by the real connections. In this way we avoid technical complication arising from the reality conditions and we can take advantage of the loop quantization techniques of the lattice quantum field theory. The downside is a more complicated form of the Hamiltonian operator.

Let \mathbf{G} be the space of the smooth 3d real connections A defined everywhere on a 3d surface σ , except, possibly, at isolated points. Fix the topology of σ , say to a 3-sphere. We now define a space S of functional on \mathbf{G} . We are now going to make use of the geometrical interpretation of the field A as a connection. The $so(3)$ Lie algebra is the same as the $su(2)$ Lie algebra, and it is convenient to view A as an $su(2)$ connection.

Let τ_i be a fixed basis in the $su(2)$ Lie Algebra. We choose $\tau_i = -\frac{i}{2}\sigma_i$, where σ_i are the Pauli matrices. Write

$$A(\vec{x}) = A_a^i(\vec{x})\tau_i dx^a.$$

We know that an oriented path γ in σ and a connection A determine a group element $U(A, \gamma) = \text{Pexp} \int_{\gamma} A$, called the holonomy of the connection along the path. For a given γ , the holonomy $U(\gamma, A)$ is a functional on \mathbf{G} . Consider an ordered collection Γ of smooth oriented paths γ_l with $l = 1, \dots, L$ and a smooth function $f(U_1, \dots, U_L)$ of L group elements. A couple (Γ, f) defines a functional of A .

$$\psi_{\Gamma, f}[A] = f(U(A, \gamma_1), \dots, U(A, \gamma_L)). \quad (3.6)$$

S is defined as the linear space of all functionals $\psi_{\Gamma, f}[A]$, for all Γ and f . We call these functionals ‘‘cylindrical functions’’. In a suitable topology, which is not important to detail here, S is dense in the space of all continuous functionals of A , obviously only for real connections.

We call Γ an ‘‘ordered oriented graph’’ embedded in σ . We call simply ‘‘graph’’ an ordered oriented graph up to ordering and orientation, and denote it by the same letter Γ . Clearly, as far as cylindrical functions are concerned, changing the ordering and the orientation of a graph is just the same as changing the order of the arguments of the function f , or replacing arguments with their inverse.

We now define a scalar product on the space S . If two functionals $\psi_{\Gamma, f}$ and $\psi_{\Gamma, g}$ are defined with the same ordered oriented graph Γ , define

$$\langle \psi_{\Gamma, f} | \psi_{\Gamma, g} \rangle = \int dU_1, \dots, \int dU_L \overline{f(U_1, \dots, U_L)} g(U_1, \dots, U_L), \quad (3.7)$$

where dU is the Haar measure on $SU(2)$. Equation (3.7) correspond at the scalar product of a Yang-Mills theory on the lattice Γ . The extension of this scalar product to functionals defined on the same graph, but with different ordering or orientation, is obvious. But also the extensions to functionals defined on different graphs Γ is simple. In fact, observe that different couples (Γ, f) and (Γ', f') may define the same functional. For instance, say Γ is the union of the L' curves in Γ' and L'' other curves, and that $f(U_1, \dots, U_{L'}, U_{L'+1}, \dots, U_{L'+L''}) = f'(U_1, \dots, U_{L'})$; then, clearly, $\psi_{\Gamma, f} = \psi_{\Gamma', f'}$. Using this fact, it is clear that we can rewrite any two given functional $\psi_{\Gamma', f'}$ and $\psi_{\Gamma'', f''}$ as functionals $\psi_{\Gamma, f}$ and $\psi_{\Gamma, g}$ having the same graph Γ , where Γ is the union of Γ' and Γ'' . Using this fact, (3.7) becomes a definition valid for any two functionals in S :

$$\langle \psi_{\Gamma', f'} | \psi_{\Gamma'', f''} \rangle = \langle \psi_{\Gamma, f} | \psi_{\Gamma, g} \rangle.$$

Notice that even if (3.7) is similar to the scalar product of a lattice Yang-Mills theory, the difference is profound. Here we are dealing with a genuinely continuous theory, in which the states do not live on a single lattice Γ , but on all possible lattices in σ . There is no cut-off on the degrees of freedom, as in lattice Yang-Mills theory.

3.2.1 Loop states and loop transform (1)

An important example of a finite norm state is provided by the case $(\Gamma, f) = (\alpha, tr)$. That is, Γ is formed by a single closed curve α , or a “loop”, and f is the trace function on the group. We can write this state as ψ_α , or simply in Dirac notation as $|\alpha\rangle$. That is,

$$\psi_\alpha[A] = \psi_{\alpha, tr}[A] = \langle A|\alpha\rangle = tr U(A, \alpha) = tr Pe^{\oint_\alpha A}. \quad (3.8)$$

The very peculiar properties that these states have in quantum gravity, which will be illustrated later on, have motivated the entire LQG approach and its name. The norm of ψ_α is easily computed from (3.7):

$$|\psi_\alpha|^2 = \int dU |tr U|^2 = 1. \quad (3.9)$$

A “multiloop” is a collection $[\alpha] = (\alpha_1, \dots, \alpha_n)$ of a finite number n of (possibly overlapping) loops. A “multiloop” state is defined as

$$\psi_{[\alpha]}[A] = \psi_{\alpha_1}[A] \dots \psi_{\alpha_n}[A] = tr U(A, \alpha_1) \dots tr U(A, \alpha_n). \quad (3.10)$$

Multiloop states form a generalized (uncountable) basis for the space of cylindrical functions and the functional on loop space

$$\psi[\alpha] = \langle \psi_\alpha | \psi \rangle \quad (3.11)$$

is called “loop transform” of the state $\psi[A]$. The functional $\psi[\alpha]$ represent the quantum state as a functional on a space of loops. This formula, called the “loop transform”, is the formula through which LQG was originally constructed. Using the measure $d\mu_0[A]$ induced by the de Haar measure, this can be written as

$$\psi[\alpha] = \int d\mu_0[A] tr Pe^{\oint_\alpha A} \psi[A]. \quad (3.12)$$

Intuitively, this is a sort of infinite-dimensional Fourier transform from the A space to the α space.

3.2.2 Kinematical Hilbert space

Define the kinematical Hilbert space K of quantum gravity as the completion of S in the norm defined by the scalar product (3.7), and S' as the completion of S in the weak topology defined by (3.7). This completes the definition of the kinematical rigged Hilbert space $S \subset K \subset S'$.

The main reason for this definition is that the scalar product (3.7) is invariant under diffeomorphism and local gauge transformations and it is such that real classical observables become selfadjoint operators. These very strict conditions are the ones that the scalar product must satisfy in order to give a consistent theory with the correct classical limit. Furthermore, the main feature of this definition is that the loop states ψ_α are normalizable. As we shall see later on, loop states are natural objects in quantum

gravity. They diagonalize geometric observables and they are solutions of the Wheeler-DeWitt equations. Hence the kinematics as well as the dynamics select this space of states as the natural ones in gravity.

There are two objections that can be raised against the definition of K we have given. First, K , is **nonseparable**. This objection would be fatal in the context of flat-space quantum field theory, but it turns out to be harmless in a general-relativistic context, because of diffeomorphism invariance. Indeed, the “excessive size” of the nonseparable Hilbert space will turn out to be just gauge. It will be sufficient to factor away the diffeomorphism gauge to obtain a separable Hilbert space K_{diff} .

Second, loop states are normalizable in lattice Yang-Mills theory, but they are non-normalizable in continuous Yang-Mills theory. By analogy, one might object that they should not be normalizable states in continuous quantum gravity either. As we shall see, however, this analogy is misleading, again precisely because of the great structural difference between a diffeomorphism-invariant QFT and a QFT on a background. As we shall see, in continuous Yang-Mills theory a loop describes an unphysical excitation that has infinitesimal transversal physical size. In gravity, on the other hand, a loop state describes a physical excitation that has finite (planckian) transversal physical size. This will be clear in the subsection (3.2.6).

3.2.3 Boundary Hilbert space

There are two natural ways of defining the boundary space \mathbf{K} . We can either define $\mathbf{K} = K^* \otimes K$ and describe the quantum geometry of a spacetime region bounded by an initial and a final surface; or simply define $\mathbf{K} = K$ interpreting the closed connected surface σ as the boundary of a finite 4d spacetime region.

The space \mathbf{K} has a rich and beautiful structure. We mentioned here only a few aspects of this structure which are important for what follows.

- Graph subspace: The cylindrical functions with support on a given graph Γ form a finite-dimensional subspace $\tilde{\mathbf{K}}_\Gamma$ of \mathbf{K} . By definition, $\tilde{\mathbf{K}}_\Gamma = L_2[SU(2)^L]$, where L is the number of paths in Γ . The space $\tilde{\mathbf{K}}_\Gamma$ is the (unconstrained) Hilbert space of a lattice gauge theory with spatial lattice Γ . If the graph Γ is contained in the graph Γ' , the Hilbert space $\tilde{\mathbf{K}}_\Gamma$ is a proper subspace of the Hilbert space $\tilde{\mathbf{K}}_{\Gamma'}$. This nested structure of Hilbert spaces is called a projective family of Hilbert spaces. \mathbf{K} can be defined to be the projective limit of this family.
- An orthonormal basis: The tool for finding a basis in \mathbf{K} is the Peter-Weyl theorem, which states that a basis on the Hilbert space of L_2 functions on $SU(2)$ is given by the matrix elements of the irreducible representations of the group. Irreducible representations of $SU(2)$ are labelled by half-integer spin j . Call \mathbf{H}_j the Hilbert space on which the representation j is defined and v^α its vectors. Write the matrix elements of the representation j , which lives in $\mathbf{H}_j^* \otimes \mathbf{H}_j$, as

$$R_\beta^{(j)\alpha}(U) = \langle U | j, \alpha, \beta \rangle. \quad (3.13)$$

For each graph Γ choose an ordering and an orientation. Then a basis

$$|\Gamma, j_l, \alpha_l, \beta_l \rangle = |\Gamma, j_1, \dots, j_L, \alpha_1, \dots, \alpha_L, \beta_1, \dots, \beta_L \rangle \quad (3.14)$$

in $\tilde{\mathbf{K}}_\Gamma$ is simply obtained by tensoring the basis (3.13). That is,

$$\langle A | \Gamma, j_l, \alpha_l, \beta_l \rangle = R_{\beta_1}^{(j_1)\alpha_1}(U(A, \gamma_1)) \dots R_{\beta_L}^{(j_L)\alpha_L}(U(A, \gamma_L)). \quad (3.15)$$

This set of vectors in \mathbf{K} is not a basis because the same vector appears in $\tilde{\mathbf{K}}_\Gamma$ and $\tilde{\mathbf{K}}_{\Gamma'}$ if Γ is contained in the graph Γ' . However is very easy to get rid of the redundancy, because all $\tilde{\mathbf{K}}_\Gamma$ belong to the trivial representation of the paths that are in Γ' but not in Γ . Therefore, an orthonormal basis of \mathbf{K} is simply given by the states $|\Gamma, j_l, \alpha_l, \beta_l \rangle$ defined in (3.14) where the spins $j_l = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ never take the value zero.

- Proper graph subspaces. For each graph Γ , the proper graph subspace \mathbf{K}_Γ is the subset of $\tilde{\mathbf{K}}_\Gamma$ spanned by the basis state with $j_l > 0$. It is easy to see that all proper subspaces \mathbf{K}_Γ are orthogonal to each other, and they span \mathbf{K} ; we can write this as

$$\mathbf{K} = \bigoplus_{\Gamma} \mathbf{K}_\Gamma.$$

The “null” graph $\Gamma = 0$ is included in the sum; the corresponding Hilbert space is the one dimensional space spanned by the state $\psi[A] = 1$. This state is denoted $|0 \rangle$; thus $\langle A | 0 \rangle = 1$.

- $*\mathbf{K}$ as an L_2 space. We have defined \mathbf{K} as the completion of S in the scalar product defined by the bilinear form (3.7). Can this space be viewed as a space of square integrable functionals in some measure? The answer is yes, and it involves a beautiful mathematical construction that we will not describe here, since it is not needed for what follows. Very briefly, $\mathbf{K} \sim L_2[\mathbf{A}, d\mu_0]$, where \mathbf{A} is an extension of the space of the smooth connection. The extension includes distributional connections. The measure $d\mu_0$ is defined on this space and is called the Ashtekar-Lewandowski measure. The construction is analogous to the definition of the gaussian measure $d\mu_G[\phi] \sim e^{-\int dx dy \phi(x) G(x, y) \phi(y)} [d\phi]$, which, as is well known, needs to be defined on a space of distributions $\phi(x)$. The space \mathbf{A} has the beautiful property of being the Gelfand spectrum of the abelian C^* algebra formed by the smooth holonomies of the connection A .

3.2.4 Invariances of the scalar product

The kinematical state space $S \subset \mathbf{K} \subset S'$ carries a natural representation of local $SU(2)$, and $Diff(\sigma)$, simply realized by the transformations of the argument A . The scalar product defined above is invariant under these transformations. Therefore \mathbf{K} carries a unitary representation of local $SU(2)$ and $Diff(\sigma)$. Let us look at this in some detail.

Under (smooth) local $SU(2)$ gauge transformations $\lambda : \sigma \rightarrow SU(2)$ the connection A transform inhomogeneously like a gauge potential, i.e.,

$$A \rightarrow A_\lambda = \lambda A \lambda^{-1} + \lambda d\lambda^{-1}. \quad (3.16)$$

This transformation of A induces a natural representation of local gauge transformations $\psi(A) \rightarrow \psi(A_{\lambda^{-1}})$ on \mathbf{K} . Despite the inhomogeneous transformation rule (3.16) of the connection, the holonomy transforms homogeneously as

$$U[A, \gamma] \rightarrow U[A_\lambda, \gamma] = \lambda(x_f^\gamma) U[A, \gamma] \lambda^{-1}(x_i^\gamma), \quad (3.17)$$

where $x_i^\gamma, x_f^\gamma \in \sigma$ are the initial and final points of the path γ . For a given (Γ, f) define:

$$f_\lambda(U_1, \dots, U_L) = f(\lambda(x_f^{\gamma_1}) U_1 \lambda^{-1}(x_i^{\gamma_1}), \dots, \lambda(x_f^{\gamma_L}) U_L \lambda^{-1}(x_i^{\gamma_L})).$$

It is easy to see that the transformation of the quantum states is

$$\psi_{\Gamma, f}(A) \rightarrow [U_\lambda \psi_{\Gamma, f}](A) = \psi_{\Gamma, f}(A_{\lambda^{-1}}) = \psi_{\Gamma, f_{\lambda^{-1}}}(A). \quad (3.18)$$

Since the Haar measure is invariant under right and left group transformations, it follows immediately that (3.7) is invariant. From (3.18) and from their definition it is easy to see that basis states $|\Gamma, j_l, \alpha_l, \beta_l\rangle$ transform as

$$\begin{aligned} U_\lambda |\Gamma, j_l, \alpha_l, \beta_l\rangle &= R_{\alpha'_1}^{(j_1)\alpha_1}(\lambda^{-1}(x_{f_1})) R_{\beta_1}^{(j_1)\beta'_1}(\gamma(x_{i_1})) \dots \\ &\quad R_{\alpha'_L}^{(j_L)\alpha_L}(\lambda^{-1}(x_{f_L})) R_{\beta_L}^{(j_L)\beta'_L}(\gamma(x_{i_L})) \\ &|\Gamma, j_l, \alpha'_l, \beta'_l\rangle \end{aligned} \quad (3.19)$$

where i_l and f_l are the points where the link l begins and ends.

Consider now maps $\phi : \sigma \rightarrow \sigma$ that are continuous, invertible, and such that the map and its inverse are smooth everywhere, except, possibly, at a finite number of isolated points. Call these maps “extended diffeomorphisms”. Call the group formed by these maps $Diff^*$. Under an extended diffeomorphism, the transformation of the connection is well defined (recall $A \in \mathbf{G}$ is defined everywhere on σ except on a finite number of isolated points.): A transforms as a one-form,

$$A \rightarrow \phi^* A.$$

Hence, S carries the representation U_ϕ of $Diff^*$ defined by $U_\phi \psi(A) = \psi((\phi^*)^{-1} A)$. The holonomy transforms as

$$U[A, \gamma] \rightarrow U[\phi^* A, \gamma] = U[A, \phi^{-1} \gamma],$$

where $(\phi\gamma)(s) \equiv (\phi(\gamma(s)))$. That is, dragging A by a diffeomorphism ϕ is equivalent to dragging the curve γ . (Notice that if ϕ is not a proper diffeomorphism, the curve $\phi\gamma$ may fail to be smooth, at a finite number of points at most.) In turn, a cylindrical function $\psi_{\Gamma, f}[A]$ is sent to a cylindrical function $\psi_{\phi\Gamma, f}[A]$, namely one which is based on the shifted graph. Since the right-hand side of (3.7) does not depend explicitly on the graph, the diffeomorphism invariance of the inner product is immediate.

3.2.5 Internal gauge invariance. The space \mathbf{K}_0

We define \mathbf{K}_0 as the space of states in \mathbf{K} invariant under local $SU(2)$ gauge transformations. We call S_0 the gauge-invariant subspace of S and S'_0 its dual. It is not difficult to see that \mathbf{K}_0 is a proper subspace of \mathbf{K} . Examples of finite norm $SU(2)$ invariant states are provided by the loop states defined in section (3.2.1). In fact we will see that multiloop states are sufficient to span \mathbf{K}_0 . In the first years of the development of LQG, multiloop states were used as a basis for \mathbf{K}_0 ; however, this basis is overcomplete, and this fact complicates the formalism. Nowadays we have a much better control of \mathbf{K}_0 , thanks to the introduction of the spin-network states, which can be seen as finite linear combinations of multiloop states forming a genuine orthonormal basis. As we shall see in the next subsection, diffeomorphism invariance connect the quantum gravity spin-network basis to the Penrose's old "spin-network" idea that quantum states of the geometry can be described as abstract graphs carrying spins.

Denote "nodes" the end points of the oriented curves in Γ . Without loss of generality, assume that each set of curves Γ is formed by curves γ that, if they overlap at all, overlap only at nodes. Viewed in this way, Γ is in fact a graph immersed in the manifold, that is, a collection of nodes n , which are points in σ , joined by links l , which are curves in σ . The "outgoing multiplicity" m_{out} of a node is the number of links that begin at the node. The "ingoing multiplicity" m_{in} of a node is the number of links that end at the node. The multiplicity, or valence $m = m_{out} + m_{in}$ of a node is the sum of the two.

Given a graph Γ , for which an ordering and an orientation have been chosen, let j_l be an assignment of an irreducible representation, different from the trivial one, to each link l . Let i_n be an assignment of an intertwiner i_n to each node n . The intertwiner i_n associated with a node is between the representations associated with the links adjacent to the node. The triplet $S = (\Gamma, j_l, i_n)$ is called a "spin network embedded in σ ". A choice of j_l and i_n is called a "coloring" of the links, and the nodes, respectively.

Consider a spin-network $S = (\Gamma, j_l, i_n)$, with L links and N nodes. The state $|\Gamma, j_l, \alpha_l, \beta_l\rangle$ has L indices α_l and L indices β_l . The N intertwiners i_n have, altogether, precisely a set of indices dual to these. The contraction of the two

$$|S\rangle = \sum_{\alpha_l, \beta_l} v_{i_1}^{\beta_1 \dots \beta_{n_1}}_{\alpha_1 \dots \alpha_{n_1}} v_{i_2}^{\beta_{n_1+1} \dots \beta_{n_2}}_{\alpha_{n_1+1} \dots \alpha_{n_2}} \dots v_{i_N}^{\beta_{n_{N-1}+1} \dots \beta_L}_{\alpha_{n_{N-1}+1} \dots \alpha_L} |\Gamma, j_l, \alpha_l, \beta_l\rangle \quad (3.20)$$

defines the spin network state $|S\rangle$. The pattern of the contraction of the indices is dictated by the topology of the graph itself: the index α_l (resp. β_l) of the link l is contracted with the corresponding index of the intertwiner v_{i_n} of the node n where the link l starts (resp. ends). The gauge invariance of this state follows immediately from the transformation properties (3.19) of the basis states and the invariance of the intertwiners. As a functional of the connection, this state is

$$\psi_S[A] = \langle A | S \rangle \equiv \left(\bigotimes_l R^{(j_l)}(U[A, \gamma_l]) \right) \cdot \left(\bigotimes_n i_n \right). \quad (3.21)$$

The raised dot notation indicates the contraction between dual spaces: on the left, the tensor product of the matrices lives in the space $\otimes_l(\mathbf{H}_{j_l}^* \otimes \mathbf{H}_{j_l})$. On the right, the tensor product of all intertwiners lives precisely in the dual of this space.

Let us now enunciate the main fact concerning the spin network states: the ensemble of the spin network states $|S\rangle$ form an orthonormal basis in \mathbf{K}_0 . Orthonormality can be checked by a direct calculation. The basis is labeled by spin networks, namely graphs Γ and colorings (j, l, i_n) . Some comments. First, we have assumed the spins j_l to be all different from zero (a spin network containing a link l with $j_l = 0$ is identified with the spin network obtained by removing the link l). Second, this result is a simple consequence of the Peter-Weyl theorem, namely of the fact that the states $|\Gamma, j_l, \alpha_l, \beta_l\rangle$ form a basis in \mathbf{K} , and the very definition of the intertwiners. Third, the spin network basis is not unique, as it depends on the (arbitrary) choice of a basis in each space of intertwiners at each node. Notice also that, in the basis $|\Gamma, j_l, \alpha_l, \beta_l\rangle$, the label Γ runs over all nonoriented and nonordered graphs. However, for the definition of the coloring, an orientation and an ordering has to be chosen for each Γ . The space S_0 is the space of the finite linear combinations of spin network states, which is dense in \mathbf{K}_0 , and S'_0 is its dual.

3.2.6 Diffeomorphism invariance. The space \mathbf{K}_{diff}

Let us now turn to the second and far more crucial invariance: 3d diffeomorphism invariance. We have to find the diffeomorphism-invariant states. The spin network states $|S\rangle$ are not invariant under diffeomorphisms. A diffeomorphism moves the graph around on the (image induced by the atlas from the) manifold, and therefore changes the states. Notice, however, that a diffeomorphism may change more than the graph of a spin network, that is, the equation $U_\phi|\Gamma, j_l, i_n\rangle = |(\phi\Gamma), j_l, i_n\rangle$ is not always correct. In particular, a diffeomorphism that leaves the graph Γ invariant may still affect a spin network state $|\Gamma, j_l, i_n\rangle$. This is because, for each graph, the definition of the spin network state requires the choice of an orientation and ordering of the links, and these can be changed by a diffeomorphism. Here is an example. Let Γ be an “eyeglasses” graph formed by two loops α and β in the $j = 1/2$ representation connect by a path γ in the $j = 1$ representation. The space of the intertwiners at each node is one-dimensional, but this does not imply that there is no choice to be made for the basis, since if i is a normalized intertwiner, so is $-i$. With one choice, the state is

$$\psi_S[A] = (U(A, \alpha))_B^A \sigma_i^B (R^{(1)}(U(A, \gamma)))^i_j \sigma^j_C (U(A, \beta))_D^C$$

Using elementary $SU(2)$ representation theory (it will be clear in the next chapter) this can be rewritten (up to a normalization factor) as

$$\psi_S[A] \sim \text{tr} U(A, \alpha\gamma\beta\gamma^{-1}) - \text{tr} U(A, \alpha\gamma\beta^{-1}\gamma^{-1}).$$

Now consider a diffeomorphism ϕ that turns the loop β around, namely it reverse its orientation, while leaving α and γ as they are. Clearly this diffeomorphism will send the two terms of the last equation into each other, giving:

$$U_\phi \psi_S[A] = -\psi_S[A],$$

while $\phi\Gamma = \Gamma$.

Given an oriented and ordered graph Γ , there is a finite discrete group G_Γ of maps g_k , such as the one of the example, that change its order or orientation and that can be obtained as a diffeomorphism. The elements g_k of this group act on \mathbf{K}_Γ .

A moment of reflection will convince that the diff-invariant states are not in K_0 , they are in S'_0 . We are therefore in the common situation in which the solutions of a quantum equation must be searched in the extension of the Hilbert space, and the scalar product must be appropriately extended to the space of the solutions. The elements of S'_0 are linear functionals Φ on the functionals $\psi \in S_0$. The requirement of diff invariance makes sense in S'_0 because the action of the diffeomorphism group is well defined in S'_0 by duality

$$(U_\phi\Phi)(\psi) \equiv \Phi(U_{\phi^{-1}}\psi).$$

Therefore a diff-invariant element Φ of S'_0 is a linear functional such that

$$\Phi(U_\phi\psi) \equiv \Phi(\psi).$$

The space \mathbf{K}_{diff} is the space of these diff-invariant elements of S'_0 . Remarkably, we have a quite good understanding of this space, whose elements can be viewed as the quantum states of physical space.

We now define a map $P_{diff} : S_0 \rightarrow S'_0$, and show that the image of this map is precisely \mathbf{K}_{diff} . Let the state $P_{diff}\psi$ be the element of S'_0 defined by

$$(P_{diff}\psi)(\psi') = \sum_{\psi''=U_\phi\psi} \langle \psi'' | \psi' \rangle. \quad (3.22)$$

The sum is over all states ψ'' in S_0 for which there exist a $\phi \in Diff^*$ such that $\psi'' = U_\phi\psi$. The key point is that this sum is always finite, and therefore well defined. To see this, notice that since ψ and ψ' are in S_0 , they can be expanded in a finite linear combination of spin network states. If a diffeomorphism changes the graph of a spin network state ψ_S , then it takes it to a state orthogonal to itself. If it doesn't change the graph, then either it leaves the state invariant, so that no multiplicity appears in (3.22), or it changes the ordering or the orientation of the links; but these are discrete operations, giving at most a discrete multiplicity in the sum in (3.22). Therefore, the sum in (3.22) is always well defined. Clearly $P_{diff}\psi$ is diff invariant. Furthermore, it is not difficult to convince oneself that the functionals of the form (3.22) span the space of the diff-invariant states. Therefore, the image of P_{diff} is \mathbf{K}_{diff} . States related by a diffeomorphism are projected by P_{diff} to the same element of \mathbf{K}_{diff} :

$$P_{diff}\psi_S = P_{diff}(U_\phi\psi_S).$$

Finally, the scalar product on \mathbf{K}_{diff} is naturally defined by

$$\langle P_{diff}\psi_S, P_{diff}\psi_{S'} \rangle_{\mathbf{K}_{diff}} \equiv (P_{diff}\psi_S)(\psi_{S'})$$

This completely defines \mathbf{K}_{diff} . Equivalently, \mathbf{K}_{diff} is defined by the bilinear form

$$\langle \psi, \psi' \rangle_{\mathbf{K}_{diff}} \equiv \langle \psi | P_{diff} | \psi' \rangle \equiv \sum_{\psi'' = \phi\psi} \langle \psi'', \psi' \rangle$$

in S_0 .

To understand the structure of \mathbf{K}_{diff} , consider the action of P_{diff} on the states of the spin network basis. To this aim, observe that a diffeomorphism sends a spin network state $|S\rangle$ to an orthogonal state, or to a state obtained by a change in the order or the orientation of the links. Denote $g_k|S_k\rangle$ the states that are obtained from $|S\rangle$ by a change of orientation or ordering, and that can be obtained via a diffeomorphism, as in the example above. The maps g_k form the finite discrete group G_Γ , therefore the range of the discrete index k is finite. Then it is easy to see that

$$\langle S | P_{diff} | S' \rangle = \begin{cases} 0 & \text{if } \Gamma \neq \phi\Gamma' \\ \sum_k \langle S_k | g_k | S' \rangle & \text{if } \Gamma = \phi\Gamma' \end{cases}. \quad (3.23)$$

An equivalence class \mathbf{K} of unoriented graphs Γ under diffeomorphisms is called a “knot”. Knots without nodes have been widely studied by the branch of mathematics called knot theory, with the aim of classifying them. Knots with nodes have also been studied in knot theory, but to a lesser extent. From the first line of (3.23) we see that two spin-networks S and S' define orthogonal states in \mathbf{K}_{diff} unless they are knotted in the same way. That is, unless they are defined on graphs Γ and Γ' belonging to the same knot class \mathbf{K} . We call \mathbf{K}_K the subspace of \mathbf{K}_{diff} spanned by basis states labeled by the knot \mathbf{K} . That is,

$$\mathbf{K}_K = P_{diff} \mathbf{K}_\Gamma$$

for any $\Gamma \in \mathbf{K}$.

The states in \mathbf{K}_K are then undistinguished only by the coloring of links and nodes, like the abstract graphs of the Penrose’s idea. As observed before, the colorings are not necessarily orthonormal, due to the nontrivial action of the discrete symmetry group G_Γ . To find an orthonormal basis in \mathbf{K}_K we have therefore to further diagonalize the quadratic form defined by the second line of (3.23). Denote $|s\rangle = |\mathbf{K}, c\rangle$ the resulting states. The discrete label c is called the coloring of the knot \mathbf{K} . Up to the complications due to the discrete symmetry G_Γ , it corresponds to the coloring of the links and the nodes of Γ . The states $|s\rangle = |\mathbf{K}, c\rangle$ are called spin-knot states, or s-knot states.

The key property of knots is that they form a discrete set. Therefore, the label \mathbf{K} is discrete. It follows that \mathbf{K}_{diff} admits a discrete orthonormal basis $|s\rangle = |\mathbf{K}, c\rangle$. Thus, \mathbf{K}_{diff} is a separable Hilbert space. The “excessive size” of the kinematical Hilbert space \mathbf{K} , reflected in its nonseparability, turns out to be just a gauge artifact.

Chapter 4

Loop states and loop transform (2): Geometry Eigenvalues

We describe here a way to define a (overcomplete) basis for the space of Gauge-invariant states, starting from multiloops. In this framework we calculate the eigenvalues of the quantum operators corresponding to the classical Area and Volume. In the next subsection we compare the two basis that we call respectively ‘basis from representations’ and ‘basis from multiloops’.

4.1 Loop variables in classical GR

Summarizing what has been seen in the previous chapter, we consider a three-dimensional manifold σ and two real (smooth) $SO(3)$ fields $A_a^i(x)$ and $E_i^a(x)$ on σ . We use $a, b, \dots = 1, 2, 3$ for (abstract) spatial indices and $i, j, \dots = 1, 2, 3$ for internal $SO(3)$ indices. We indicate coordinates on σ with x . The relation between these fields and conventional metric gravitational variables is as follows: $E_i^a(x)$ is the (densitized) inverse triad, related to the three-dimensional metric $g_{ab}(x)$ of the constant-time surface by

$$g g^{ab} = E_i^a E_i^b, \quad (4.1)$$

where g is the determinant of g_{ab} ; and

$$A_a^i(x) = \Gamma_a^i(x) + \beta K_a^i(x), \quad (4.2)$$

where $\Gamma_a^i(x)$ is the $SU(2)$ spin connection associated to the triad and $K_a^i(x)$ is the extrinsic curvature of the three surface (up to indices’ position).

It is useful for what follows to consider the dimensional character of the field with care. We set the dimension of the fields as follow:

$$[g_{ab}] = L^2, \quad [E_i^a] = L^2,$$

$$[A_a^i] = \text{dimensionless.} \quad (4.3)$$

The popular choice of taking the metric dimensionless is not very sensible in GR. It forces coordinates to have dimensions of a length; but the freedom of arbitrary transformations on the coordinates is hardly compatible with dimensional coordinates. Coordinates, for instance, can be angles, and assigning to angles dimension of a length makes no sense. The Einstein action can be rewritten as

$$\begin{aligned} S &= \frac{1}{\kappa} \int d^4x \sqrt{g} R \\ &= \frac{1}{\kappa} \int dx^0 \int d^3x \left[-\frac{1}{\beta} \dot{A}_a^i E_i^a + \dot{A}_0^i G_i + N^a H_a + NH \right], \end{aligned} \quad (4.4)$$

where we have set

$$\kappa = \frac{16\pi G_N}{c^3} \quad (4.5)$$

G_N being Newton's gravitational constant, and G_i , H_a , H the Gauss, diffeomorphism and Hamiltonian constraints¹. It follows that the momentum canonically conjugate to A_a^i is

$$p_i^a(x) = \frac{\delta S}{\delta \dot{A}_a^i(x)} = -\frac{1}{\beta\kappa} E_i^a. \quad (4.6)$$

and therefore the fundamental Poisson bracket of the Hamiltonian theory is

$$\{A_a^i(x), E_j^b(y)\} = \beta\kappa \delta_a^b \delta_j^i \delta^3(x, y) \quad (4.7)$$

The spinorial version of the variables is given in terms of the Pauli matrices σ_i , $i = 1, 2, 3$, or the $su(2)$ generators $\tau_i = -\frac{i}{2} \sigma_i$, by

$$E^a(x) = -i E_i^a(x) \sigma_i = 2E_i^a(x) \tau_i \quad (4.8)$$

$$A_a(x) = -\frac{i}{2} A_a^i(x) \sigma_i = A_a^i(x) \tau_i \dots \quad (4.9)$$

$A_a(x)$ and $E^a(x)$ are 2×2 complex matrices. We use upper case indices $A, B, \dots = 1, 2$ for the spinor space on which the Pauli matrices act. Thus, the components of the gravitational fields are $A_{aA}{}^B(x)$ and $E^a{}_A{}^B(x)$. In order to construct the loop variables, we start from some definitions.

Segment. A segment γ is a continuous and piecewise smooth map from the closed interval $[0, 1]$ into σ . We write: $\gamma : s \mapsto \gamma^\alpha(s)$.

Loop. A loop α is a segment such that $\alpha^\alpha(0) = \alpha^\alpha(1)$. Equivalently, it is a continuous, piecewise smooth, map from the circle S_1 into σ .

Free Loop Algebra. We consider (formal) linear combinations Φ of (formal) products of loops, as in:

$$\Phi = c_0 + \sum_i c_i [\alpha_i] + \sum_{jk} c_{jk} [\alpha_j][\alpha_k] + \dots, \quad (4.10)$$

¹Here we have written \dot{A}_0 in place of Λ . This is what result starting from the four dimensional theory in terms of $SO(4)$ or $SO(1,3)$ connection and tetrad.

where the c 's are arbitrary complex number and the α 's are loops; we denote the space of such objects as the Free Loop Algebra $\mathcal{A}^f[\mathcal{L}]$.

Multiloop. We denote the monomials in $\mathcal{A}^f[\mathcal{L}]$, namely the elements of the form $\Phi = [\alpha_1] \dots [\alpha_n]$ as multiloops. We indicate multiloops by a Greek letter, in the same manner as (single) loops: $[\alpha] = [\alpha_1] \dots [\alpha_n]$.

We can now define the fundamental loop variables. Given a loop α and the points $s_1, s_2, \dots, s_n \in \alpha$ we define:

$$\mathcal{T}[\alpha] = -\text{Tr}[U_\alpha], \quad (4.11)$$

$$\mathcal{T}^a[\alpha](s) = -\text{Tr}[U_\alpha(s, s)E^a(s)] \quad (4.12)$$

and, in general

$$\mathcal{T}^{a_1 a_2}[\alpha](s_1, s_2) = -\text{Tr}[U_\alpha(s_1, s_2)E^{a_2}(s_2)U_\alpha(s_2, s_1)E^{a_1}(s_1)], \quad (4.13)$$

$$\begin{aligned} \mathcal{T}^{a_1 \dots a_N}[\alpha](s_1, \dots, s_N) &= -\text{Tr}[U_\alpha(s_1, s_N)E^{a_N}(s_N)U_\alpha(s_N, s_{N-1}) \dots \\ &\quad \dots U_\alpha(s_2, s_1)E^{a_1}(s_1)]. \end{aligned} \quad (4.14)$$

The function $\mathcal{T}[\alpha]$ defined in (4.11) for a single loop, can be defined over the whole free loop algebra $\mathcal{A}^f[\mathcal{L}]$: given the generic element $\Phi \in \mathcal{A}^f[\mathcal{L}]$ in (4.10), we pose

$$\mathcal{T}[\Phi] = -2c_0 + \sum_i c_i \mathcal{T}[\alpha_i] + \sum_{ij} c_{ij} \mathcal{T}[\alpha_i] \mathcal{T}[\alpha_j] + \dots \quad (4.15)$$

The reason for the -2 in the first term is the following. We may think of the first term of the sum as corresponding to the ‘‘point loop’’, or a loop whose image is a point. For this loop, the exponent in the corresponding holonomy is zero, so the holonomy is the identity (in $sl(2, C)$, namely in 2d) and \mathcal{T} is therefore -2 . We have

$$\text{Tr}[U_\alpha] \text{Tr}[U_\beta] - \text{Tr}[U_\alpha U_\beta] - \text{Tr}[U_\alpha U_{\beta^{-1}}] = 0, \quad (4.16)$$

$$\mathcal{T}[\alpha] \mathcal{T}[\beta] + \mathcal{T}[\alpha \#_s \beta] + \mathcal{T}[\alpha \#_s \beta^{-1}] = 0, \quad (4.17)$$

$$\mathcal{T}[\alpha[\beta]] + \mathcal{T}[\alpha \#_s \beta] + \mathcal{T}[\alpha \#_s \beta^{-1}] = 0. \quad (4.18)$$

We recall here, for later use, the retracing identity. For all loops α and segments γ , we have

$$\mathcal{T}[\alpha] = \mathcal{T}[\alpha \circ \gamma \circ \gamma^{-1}]. \quad (4.19)$$

The Poisson bracket algebra of these loop variables is easily computed. We give here the Poisson bracket of the \mathcal{T} variables of order 0 and 1.

$$\{\mathcal{T}[\alpha], \mathcal{T}[\gamma]\} = 0, \quad (4.20)$$

$$\{\mathcal{T}^a[\alpha](s), \mathcal{T}[\gamma]\} = -\beta \kappa \Delta^a[\gamma, s] \cdot \frac{1}{2} \{\mathcal{T}[\alpha \#_s \gamma] - \mathcal{T}[\alpha \#_s \gamma^{-1}]\}, \quad (4.21)$$

where we have defined:

$$\Delta^a[\gamma, s] = \int_\beta d\tau \dot{\gamma}^a(\tau) \delta^3[\gamma(\tau), s]. \quad (4.22)$$

4.2 The loop representation of quantum gravity

We now define the loop representation of quantum gravity as a linear representation of the Poisson algebra of the \mathcal{T} variables. First, we define the carrier space of the representation. To this aim, we consider the linear subspace \mathcal{K} of the free loop algebra defined by

$$\mathcal{K} = \{\Phi \in \mathcal{A}^f[\mathcal{L}] \mid \mathcal{T}[\Phi] = 0\}, \quad (4.23)$$

and we define the carrier space \mathcal{V} of the representation by

$$\mathcal{V} = \mathcal{A}^f[\mathcal{L}]/\mathcal{K}. \quad (4.24)$$

In other words, the state space of the loop representation is defined as the space of the equivalence classes of linear combinations of multiloops, under the equivalence defined by the Mandelstam relations

$$\Phi \sim \Psi \quad \text{if} \quad \mathcal{T}[\Phi] = \mathcal{T}[\Psi], \quad (4.25)$$

namely by the equality of the corresponding holonomies². We denote the equivalence classes defined in his way, namely the elements of the quantum state space of the theory as Mandelstam classes, and we indicate them in Dirac notation as $\langle \Phi |$. Clearly, the multiloop states $\langle \alpha |$ span (actually, overspan) the state space \mathcal{V} . Later we will define a scalar product on \mathcal{V} , and promote it to a Hilbert space. The reason for preferring a bra notation over a ket notation is just historical at this point. We recall that the loop representation was originally defined in terms of kets $|\psi\rangle$ in the dual of \mathcal{V} . These are represented on the (overcomplete) basis $\langle \alpha |$ by loop functionals

$$\psi(\alpha) = \langle \alpha | \psi \rangle. \quad (4.26)$$

The principal consequences of the Mandelstam relations are the following.

1. The element $\langle \alpha |$ does not depends on the orientation of α : $[\alpha] \sim [\alpha^{-1}]$.
2. The element $\langle \alpha |$ does not depend on the parameterization of α : $[\alpha] \sim [\beta]$ if $\beta^a(\tau) = \alpha^a(f(\tau))$.
3. Retracing: if γ is a *segment* starting in a point of α then.

$$[\alpha \circ \gamma \circ \gamma^{-1}] \sim [\alpha]. \quad (4.27)$$

4. Binor identity:

$$[\alpha] \cdot [\gamma] \sim -[\alpha \#_s \gamma] - [\alpha \#_s \gamma^{-1}]. \quad (4.28)$$

It has been conjectured that all Mandelstam relations can be derived by repeated use of these identities. We expect that the methods described below may allow to prove this conjecture, but we do not discuss this issue here.

² $\mathcal{T}[\Phi]$ is a function on configuration space, namely a function over the space of smooth connections. Equality between functions means of course having the same value for any value of the independent variable; here, for all (smooth) connections.

Next, we define the quantum operators corresponding to the \mathcal{T} -variables as linear operators on \mathcal{V} . These form a representation of the loop variables Poisson algebra. We define the loop operators as acting on the bra states $\langle \Phi |$ from the right. (Since they act on the right, they define, more precisely, an *anti*-representation of the Poisson algebra.) We define the $\hat{\mathcal{T}}[\alpha]$ operator by

$$\begin{aligned} \left\langle c_0 + \sum_i c_i [\alpha_i] + \sum_{ij} c_{ij} [\alpha_i][\alpha_j] + \dots \right| \hat{\mathcal{T}}[\alpha] &= \\ = \left\langle c_0[\alpha] + \sum_i c_i [\alpha_i][\alpha] + \sum_{ij} c_{ij} [\alpha_i][\alpha_j][\alpha] + \dots \right| . \end{aligned} \quad (4.29)$$

Next, we define the $\hat{\mathcal{T}}^a[\alpha](s)$ operator. This is a derivative operator (i.e. it satisfies Leibniz rule) over the free loop algebra such that

$$\langle [\gamma] | \hat{\mathcal{T}}^a[\alpha](s) = -il_0^2 \Delta^a[\gamma, s] \frac{1}{2} (\langle [\alpha \#_s \gamma] | - \langle [\alpha \#_s \gamma^{-1}] |), \quad (4.30)$$

where we have introduced the elementary length l_0 by

$$l_0^2 = \beta \hbar \kappa = \frac{16\pi \hbar \beta G_N}{c^3} = 16\pi l_{Planck}^2. \quad (4.31)$$

The definition extends on the entire free loop algebra by Leibniz rule and linearity. The two operators commute with the Mandelstam relations and are therefore well defined on \mathcal{V} .

Notice that the factor $\Delta^a[\gamma, s]$ in (4.30) depends on the orientation of the loop γ : it changes sign if the orientation of γ is reversed. So does the difference in the parentheses, therefore the r.h.s of (4.30) is independent from the orientation of γ , as the l.h.s.. On the other hand, both the r.h.s and the l.h.s of (4.30) change sign if we reverse the orientation of α .

The action of the $\hat{\mathcal{T}}^a[\alpha](s)$ operator on a state $\langle [\beta] |$ can be visualized graphically. The graphical action is denoted a “grasp”, and it can be described as follows: i. Disjoin the two edges of the loop β and the two edges of the loop α , that enter the intersection point s . ii. Pairwise join the four open ends of α and β in the two possible alternative ways. This defines two new states. Consider the difference between these two states (arbitrarily choosing one of the two as positive). iii. Multiply this difference by the factor $-il_0^2 \Delta^a[\beta, s]$, where the direction of β (which determines the sign of $\Delta^a[\beta, s]$) is determined as follows: it is the direction induced on β by α (which *is* oriented) in the term chosen as positive. A moment of reflection shows that the definition is consistent, and independent from the choice of the positive term. An explicit computation shows that the operators defined realize a linear representation of the Poisson algebra of the corresponding classical observables. The grasping rule generalizes to higher order \mathcal{T} -variables.

The action of $\hat{\mathcal{T}}^{a_1 \dots a_n}[\alpha](s_1, \dots, s_n)$, over a single loop-state $[\beta]$ is given as follows. First the result vanishes unless β crosses all the n points s_i . If it does, the action of $\hat{\mathcal{T}}^{a_1 \dots a_n}[\alpha](s_1, \dots, s_n)$ is given by the simultaneous grasp on all intersection points. This action produces 2^n terms. These terms are summed algebraically with

alternate signs, and the result is multiplied by a factor $-il_0^2 \Delta^a[\beta, s_1]$ for each grasp, where the sign of each coefficient $\Delta^a[\beta, s_i]$ is determined assuming that β is oriented consistently with α in the term chosen as positive. Again, a moment of reflection shows that the definition is consistent, and independent from the choice of the positive terms. The generalization to arbitrary states, using linearity and the Leibnitz rule, is straightforward. This concludes the construction of the linear ingredients of the loop representation.

4.3 Loop states and recoupling theory

A quantum state $\langle \Phi |$ in the state space \mathcal{V} is a Mandelstam equivalence class of elements of the form (4.10). We now show that because of the equivalence relation, these states obey the formal identities that define the Temperley-Lieb-Kauffman recoupling theory described in Ref.[8]. This fact yields two results. First, we can write a basis in \mathcal{V} . This basis is constructed in the next section. Second, recoupling theory becomes a powerful calculus in loop quantum gravity.

Consider the element Φ , given in (4.10), of the vector space $\mathcal{A}^f[\mathcal{L}]$. We need some definitions.

Graph of a state. We denote the union in σ of the images of all the loops in the r.h.s of (4.10) as the “graph of Φ ”, and we indicate it as Γ_Φ .

Vertex. We denote the points i where Γ_Φ fails to be a smooth submanifold of σ as “vertices”.

Edge. We denote the lines e of the graph connecting the vertices as “edges”.

Valence. We say that a vertex i has valence n , or is n -valent, if n edges are adjacent to it. A vertex can have any positive integer valence, including 1 and 2.

Clearly, Φ is not uniquely determined by its graph Γ_Φ . If our only information about a state is its graph, then we do not know how the state is decomposed into multiloops, nor how many single loops run along each edge, nor how the single loops are rooted through the vertices. We now introduce a graphical technique to represent this missing information. The technique is based on the idea of “blowing up” the graph -as if viewed through an infinite magnifying glass- and representing the additional information in terms of “planar” tangles on the blown up graph. As we will see, these tangles obey recoupling theory.

First, draw a graph isomorphic to Γ_Φ in the sense of graph theory (that is, the isomorphism preserves only adjacency relations between vertices and edges), on a two dimensional surface. As usual in graph theory, we must distinguish points representing vertices from accidental intersections between edges generated by the fact that we are representing a non-planar graph on a plane. Denote these accidental intersections as “false intersections”. Next, replace each vertex (not the false intersections) by (the interior of) a circle in the plane, and each edge by a ribbon connecting two circles. (At false intersections, ribbons bridge each other without merging.) In this way, we construct a “thickened out” graph: a two-dimensional oriented surface which (loosely speaking) has the topology of the graph Γ_Φ times the $[0, 1]$ interval.

Ribbon-net. We call this two-dimensional surface the “ribbon-net” (or simply the ribbon) of the graph Γ_Φ , and we denote it as R_Φ . Notice that the graph Γ_Φ is embedded in σ , while its ribbon-net R_Φ is not.

Now we can represent the missing information needed to reconstruct Φ from Γ_Φ as (a formal linear combination of) tangles drawn on the surface R_Φ . First, we represent each multiloop in (4.10) by means of a closed line over R_Φ :

Planar (representation of a) multiloop. For each loop α_i in a given multiloop α we draw a loop α_i over the ribbon-net R_Φ , wrapping around R_Φ in the same way in which α_i wraps around Γ_Φ . We denote the drawing (over R_Φ) of all the loops of a multiloop as “the planar representation” of the multiloop α , or simply as the “planar multiloop”. We indicate it as P_α .

For technical reasons, we allow edges and vertices of the ribbon-net to be empty of loops as well. Thus, we identify a ribbon-net containing a planar multiloop, with a second one obtained from the first by adding edges and vertices empty of loops. Finally:

Planar (representation of a) state. Every state $\langle \Phi |$ is a formal linear combination of multiloops: $\langle \Phi | = \sum_j c_j [\alpha_j]$ (up to equivalence). We denote the corresponding formal linear combination $P_\Phi = \sum_j c_j P_{\alpha_j}$ of planar multiloops on the ribbon-net R_Φ (up to equivalence), as a planar representation of $\langle \Phi |$.

We have split the information contained in Φ in two parts: Φ determines a graph Γ_Φ embedded in σ and a planar state P_Φ . P_Φ is a linear combinations of drawings of loops over a surface (the ribbon-net R_Φ) and codes the information on which loops are present and how they are rooted through intersections. This information is *purely combinatorial*. On the other hand, Γ_Φ contains the information on how the loops are embedded into σ .

Notice that a multiloop determines its planar representation only up to smooth planar deformations of the lines within the circles and the ribbons of the ribbon-net. In other words, we can arbitrarily deform the lines within each circle and within each ribbon, without changing Φ . In particular, the lines of the planar representation will intersect in points of R_Φ , and we can disentangle them. Under- and over-crossings of loops within R_Φ are not distinguished. Let us come to the key observation on which the possibility of using recoupling theory relies. Consider an element Φ of the free vector algebra. For simplicity, let us momentarily assume that Φ is formed by a single loop $\Phi = [\alpha]$ (which may self-intersect and run over itself). Thus $\Phi = (\Gamma_\alpha, P_\alpha)$. Consider an intersection of two lines (two segments of P_α) in R_Φ . Break the two lines meeting at this intersection, and pairwise rejoin the four legs, in the two alternative possible ways, as in Figure 4.1.

We obtain two new loops on R_Φ , which we denote as $P_{[\beta]}$ and $P_{[\gamma]}$. Consider the element Ψ of the free vector algebra uniquely determined by the graph $\Gamma_\Psi = \Gamma_\Phi$, and by the linear combination of planar representations $P_\Psi = -P_\beta - P_\gamma$. Notice that Ψ is different than Φ as an element of the free vector algebra; however, the two are in the same Mandelstam equivalence class because of the binor relation (4.28), and therefore they define the same element of the quantum state space \mathcal{V} . Namely $\langle \Psi | = \langle \Phi |$. We

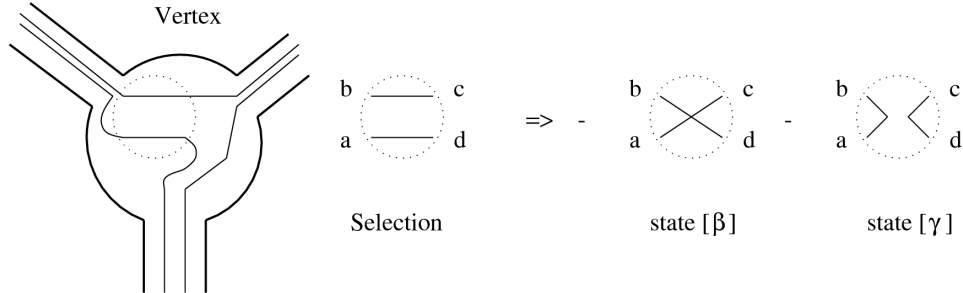


Figure 4.1: The binor identity.

say that two planar representations P_Φ and P_Ψ are “equivalent” if $\langle \Psi | = \langle \Phi |$. Thus, in dealing with planar representations of a quantum state $\langle \Phi |$, we can freely use the identity

$$\times = - \left| \right| - \smile \tag{4.32}$$

on P_Φ without changing the quantum state. This identity is the key axiom of recoupling theory.

An easy to derive consequence is that every closed line entirely contained within a circle, or within a ribbon, can be replaced by a factor $d = -2$. Furthermore, it is easy to see that the retracing identity (4.27) implies that the loops of P_Φ can be arbitrarily deformed within the *entire* ribbon-net, without changing the state $\langle \Phi |$. In particular, every loop contractible in R_Φ can be replaced by a factor $d = -2$. This is the second axiom of recoupling theory.

Thus P_Φ can be interpreted as a linear combination of tangles in the sense of reference [8]. The tangles obey the axioms of recoupling theory. They are confined inside the oriented surface R_Φ with has a highly nontrivial topology. This is the key result of this section.

An immediate consequence of the result is that we can write a basis in \mathcal{V} following [8]. Given a state $\langle \Psi |$, and its ribbon-net R_Φ , we can use (4.32) to eliminate all intersections from the P_α of each multiloop. Next, we can retrace each single line that returns over itself, and eliminate every loop contractible in R_Φ . We obtain parallel lines without intersections along each ribbon and routings without intersections at each vertex. No further use of the retracing or binor identity is then possible without altering this form. This procedure defines a basis of independent states, labeled by the graph, the number of lines along each edge, and elementary routings at each node. An elementary routing is a planar rooting of loops through the vertex of the ribbon-net, having no intersections. This basis is not very practical for calculations. In the next section, we use the technology of [8] to define a more useful basis.

4.4 The spin network basis

The representation (Γ_Φ, P_Φ) of a state $\langle \Phi |$ can be expanded in terms of a “virtual” trivalent representation as follows.

Virtual graph. To every graph Γ , we can associate a trivalent graph Γ^v as follows. For each n -valent vertex v of Γ , (arbitrarily) label the adjacent edges as $e_0 \dots e_{(n-1)}$, and disjoint them from v . Then, replace v with $n - 2$ trivalent vertices $N_1 \dots N_{n-2}$, denoted “virtual” vertices. Join the virtual vertices with $n - 3$ “virtual” edges $E_2 \dots E_{n-2}$, where E_i joins N_{i-1} and N_i . Prolong the edges $e_1 \dots e_{(n-2)}$ to reach the corresponding virtual vertices $N_1 \dots N_{n-2}$, and the edges e_0 and $e_{(n-1)}$ to reach the virtual vertices N_1 and N_{n-2} . Denote the resulting trivalent graph Γ^v as the virtual graph associated to Γ (for the chosen ordering of edges).

Virtual ribbon-net. We denote the ribbon-net of Γ_Φ^v as the virtual ribbon-net R_Φ^v of Φ . We view it as a subset of R_Φ , namely we view the virtual circles $N_1 \dots N_{n-2}$ and the virtual ribbons $E_2 \dots E_{n-2}$ as drawn inside the circle c representing v . This circle c indicates that the virtual vertices $N_1 \dots N_{n-2}$ correspond all to the same point of σ . (Thus, a virtual ribbon-net is a trivalent ribbon-net with strings of adjacent intersections specified.)

Virtual representation. Finally, deform P_Φ so that it lies entirely inside R_Φ^v . We indicate the deformed P_Φ as P_Φ^v , and call it the “virtual” planar representation of Φ . The virtual representation P_Φ^v of a state is not unique, due to the arbitrariness of assigning the ordering $e_0 \dots e_{(n-1)}$ to the edges of n -valent intersections.

The above construction is more difficult to describe in words than to visualize, and is illustrated in Figure 4.2.

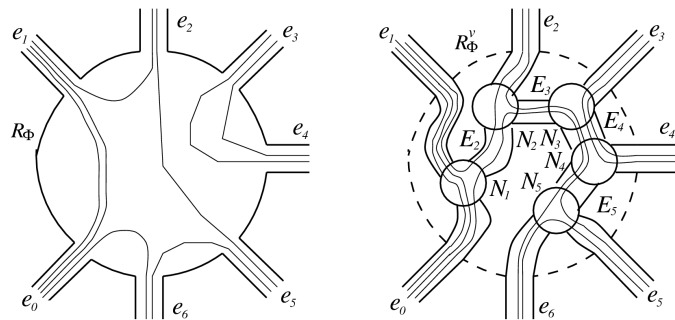


Figure 4.2: Construction of “virtual” vertices and “virtual” strips over an n -valent vertex.

Consider now deformations of the tangle P_Φ^v within R_Φ^v – a subset of the deformations within the full R_Φ . We can move all intersections of deform P_Φ^v away from

the vertices (to the virtual or real ribbons), leaving trivalent vertices free from intersections. Next, we can use the binor relation to remove all intersections from the ribbons, leaving non-intersecting tangles with n inputs and n outputs along each single ribbon e . As described in Sec 2.2 of [8], tangles of this kind can be described as elements of the (tangle-theoretic) Temperley-Lieb algebras $T_n^{(e)}$. A basis of this algebra is obtained by using the Jones-Wenzl projectors $\Pi_n^{(e)}$, which are just normalized antisymmetrizers. More precisely, given the multiloop P_α with n lines along the ribbon e , call $P_\alpha^{(p)}$, $p = 1 \dots n!$ the multiloops obtained by all possible permutations p in the way the n lines entering e are connected to the n outgoing lines, and $|p|$ the parity of the permutation, then

$$\Pi_n^{(e)} P_\alpha = \frac{1}{n!} \sum_p (-1)^{|p|} P_\alpha^{(p)}. \quad (4.33)$$

It follows from the completeness of the Jones-Wenzel projectors that a basis for all planar loops over a given R_Φ^v is given by the linear combination of loops in which the lines along each (virtual and real) edge are fully antisymmetrized. We can therefore expand every state in states in which lines are fully antisymmetrized along each ribbon. A state in which the lines along each (virtual or real) ribbon are fully antisymmetrized is a spin network state.

A spin network state is characterized by a graph Γ in σ , by the assignment of an ordering to the edges adjacent to each vertex, and by the number p_e of (antisymmetrized) lines in each virtual or real edge e . We denote the integer p_e as the ‘‘color’’ of the corresponding edge e of Γ^v . We will use also the ‘‘spin’’ j_e of the edge, defined as half its color: $j_e = \frac{1}{2} p_e$. At each vertex, the colors p_1 , p_2 and p_3 of the three adjacent edges satisfy a compatibility condition: there must exist three positive integers a , b and c (the number of lines rooted through each pair of edges) such that

$$p_1 = a + b, \quad p_2 = b + c, \quad p_3 = c + a, \quad (4.34)$$

It is easy to see that this condition is equivalent to the Clebsh-Gordon condition that each of the three $su(2)$ representations of spin $j_i = 1/2 p_i$ is contained in the tensor product of the other two. This construction is equivalent to the construction from representations. We can freely move the antisymmetrizers near the nodes. In this way every ribbon contains p equal holonomies in the $1/2$ representation. Because the equality of the holonomies, they can be seen as completely symmetrized. The symmetrization of p elements of $SU(2)$ in the fundamental representation is the $p/2$ representation of $SU(2)$. Every node, with the nearest antisymmetrizers, defines an intertwiner between the $p_1/2$, $p_2/2$, $p_3/2$ representations.

The spin network states form a basis in \mathcal{V} . The basis elements are given as follows. For every graph Γ embedded in σ , choose an ordering of the edges at each node. This choice associates an oriented trivalent virtual graph Γ^v (non-embedded) to every Γ .

Spin network. A spin network S is given by a graph Γ_S in σ , and by a compatible coloring $\{p_e\}$ of the associated oriented trivalent virtual graph Γ^v . Thus $S = (\Gamma_S, \{p_e\})$.

Spin network state. For every spin network S , the spin network quantum state $\langle S | = (\Gamma_S, P_S)$ is the element of \mathcal{V} determined by the graph Γ_S and by the linear

combination P_S of planar multiloops obtained as follows. Draw p_e lines on each ribbon e of the ribbon-net R_S^v ; connect lines at intersections without crossings; this gives a planar multiloop $P_S^{(0)}$; then

$$P_S = \prod_{e \in \Gamma} \Pi_{p_e}^{(e)} P_S^{(0)}. \quad (4.35)$$

We can then represent a spin network state as a colored trivalent graph over the ribbon-net R_S^v (with a single edge along each ribbon). We give here the formula that allows one to express the basis elements of a 4-valent intersection in terms of the basis elements of a different trivalent expansion. Using the recoupling theorem of [8] (pg. 60), we have immediately

$$\begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} j \\ \diagup \\ \bullet \\ \diagdown \\ d \end{array} \begin{array}{c} c \\ \diagup \\ \bullet \\ \diagdown \\ d \end{array} = \sum_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} \begin{array}{c} b \\ \diagdown \\ \bullet \\ \diagup \\ a \end{array} \begin{array}{c} i \\ \diagup \\ \bullet \\ \diagdown \\ d \end{array} \begin{array}{c} c \\ \diagup \\ \bullet \\ \diagdown \\ d \end{array} \quad (4.36)$$

where the quantities $\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}$ are $su(2)$ six-j symbols (normalized as in [8]).

4.4.1 The Action of the operators in the spin-network basis

We now describe how the \hat{T} operators act on the spin network states. From Eq. (4.29), the operator $\hat{T}[\alpha]$, acting on a state $\langle \Phi |$ simply adds a loop to $\langle \Phi |$. Consider the graph Γ formed by the union (in σ) of the graphs of Φ and α . Since we admit empty edges, we can represent Φ over the ribbon-net R associated to Γ . In this representation, the action of $\hat{T}[\alpha]$ consists in adding the draw of α over R . Using the expression for the Jones-Wenzl projectors in [8] (pg. 96), one can expand the non-antisymmetrized lines, if any, in combinations of antisymmetrized ones.

Higher order loop operators are expressed in terms of the elementary grasp operation, Eq. (4.30). The ribbon construction allows us to represent the grasp operation in a simpler form. Indeed, one easily sees that Eq. (4.30) is equivalent to the following: acting on an edge with color 1, the grasp creates two virtual trivalent vertices (inside the same circle, corresponding to the intersection point) – one on the spin-network state and one on the loop of the operator. The two vertices are joined by a virtual strip of color 2, and the overall multiplicative factor is determined as follows. The sign of the tangent of γ in $\Delta^a[\gamma, s]$ is determined by the orientation of γ consistent with the positive-terms of the loop expansion of the spin network. The equivalence between the old definition of the grasp and the new one is illustrated in Figure 4.3.

A straightforward computation, using Leibnitz rule, shows that acting on an edge with color p , the grasp has the very same action, with the multiplicative factor multiplied by p . Finally, notice that the two antisymmetrized loops form a (virtual) spin network edge of color 2. Therefore, we can express the action of the grasp in the spin

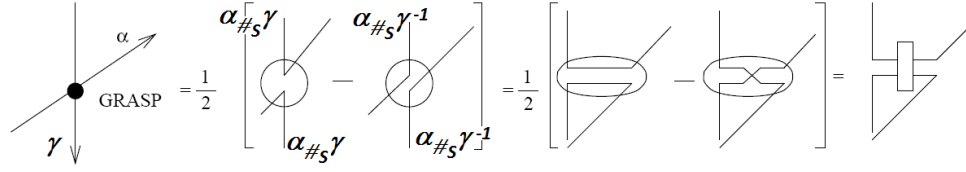


Figure 4.3: Action of the grasp.

network basis by the following equation

(4.37)

This simple form of the action of the loop operators on the spin-network basis is the reason that enables us to use recoupling-theory in actual calculations involving quantum gravity operators. Notice that it is the ribbon-net construction that allows us to “open up” the intersection point and represent it by means of two vertices (one over α and one over γ) and a (“zero length”) edge connecting the two vertices. These two vertices and this edge are all in the same point of the three-manifold σ .

Higher order loop operators act similarly, as sketched in Figure 4.4.

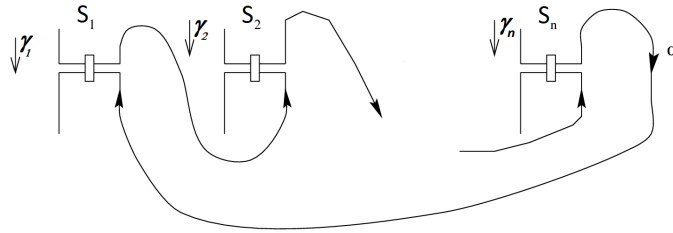


Figure 4.4: Representation of the n grasp of the $\mathcal{T}^{a_1 \dots a_n}[\alpha](s_1, \dots, s_n)$ operator.

4.5 The area operator

A surface Σ in σ is an embedding of a 2-dimensional manifold Σ , with coordinates $\sigma^u = (\sigma^1, \sigma^2)$, $u, v = 1, 2$, into σ . We write $S : \Sigma \rightarrow \sigma, \sigma^u \rightarrow x^u(\sigma)$. The metric

and the normal one form on Σ are given by

$$g^\Sigma = S^* g, \quad g_{uv}^\Sigma = \frac{\partial x^a}{\partial \sigma^u} \frac{\partial x^b}{\partial \sigma^v} g_{ab}; \quad (4.38)$$

$$n_a = \frac{1}{2} \epsilon^{uv} \epsilon_{abc} \frac{\partial x^b}{\partial \sigma^u} \frac{\partial x^c}{\partial \sigma^v}. \quad (4.39)$$

The area of Σ is

$$\begin{aligned} A[\Sigma] &= \int_\Sigma d^2 \sigma \sqrt{\det g^\Sigma} \\ &= \int_\Sigma d^2 \sigma \sqrt{\frac{1}{2} \epsilon^{u\bar{u}} \epsilon^{v\bar{v}} g_{uv}^\Sigma g_{\bar{u}\bar{v}}^\Sigma} = \int_\Sigma d^2 \sigma \sqrt{n_a n_b E^{ai} E_i^b}, \end{aligned} \quad (4.40)$$

where we have used

$$\begin{aligned} \epsilon^{u\bar{u}} \epsilon^{v\bar{v}} g_{uv}^\Sigma g_{\bar{u}\bar{v}}^\Sigma &= \epsilon^{u\bar{u}} \epsilon^{v\bar{v}} \frac{\partial x^a}{\partial \sigma^u} \frac{\partial x^b}{\partial \sigma^v} g_{ab} \frac{\partial x^{\bar{a}}}{\partial \sigma^{\bar{u}}} \frac{\partial x^{\bar{b}}}{\partial \sigma^{\bar{v}}} g_{\bar{a}\bar{b}}, \\ \epsilon^{u\bar{u}} \frac{\partial x^a}{\partial \sigma^u} \frac{\partial x^{\bar{a}}}{\partial \sigma^{\bar{u}}} &= \frac{1}{2} \epsilon^{u\bar{u}} \frac{\partial x^{a'}}{\partial \sigma^u} \frac{\partial x^{\bar{a}'}}{\partial \sigma^{\bar{u}}} \epsilon_{a'\bar{a}'c} \epsilon^{a\bar{a}c} = n_c \epsilon^{a\bar{a}c}, \\ gg^{c\bar{c}} &= \frac{1}{2} \epsilon^{a\bar{a}c} \epsilon^{b\bar{b}c} g_{ab} g_{\bar{a}\bar{b}}. \end{aligned}$$

We want to construct the quantum area operator $\hat{A}[\Sigma]$, namely a function of the loop representation operators whose classical limit is $A[\Sigma]$. Following conventional quantum field theoretical techniques, we deal with operator products by defining $\hat{A}[\Sigma]$ as a limit of regularized operators $\hat{A}_\epsilon[\Sigma]$ that do not contain operator products. The difficulty in the present context is to find a regularization that does not break general covariance. This can be achieved by a geometrical regularization.

We begin by constructing a classical regularized expression for the area, namely a one parameter family of classical functions of the loop variables $A_\epsilon[\Sigma]$ which converges to the area as ϵ approaches zero. Consider a small region Σ_ϵ of the surface Σ , whose coordinate area goes to zero with ϵ^2 . For every s in Σ , the smoothness of the classical fields implies that $E^a(s) = E^a(x_I) + O(\epsilon)$, where x_I is an arbitrary fixed point in Σ_ϵ . Also, $U_\alpha(s, t)_A^B = \mathbf{1}_A^B + O(\epsilon)$ for any $s, t \in \Sigma_\epsilon$ and α a (coordinate straight) segment joining s and t . It follows that to zeroth order in ϵ

$$\mathcal{T}^{ab}[\alpha_{st}](s, t) = -\text{Tr} [E^a(s) U_\alpha(s, t) E^b(t) U_\alpha(t, s)] = 2E^{ai}(x_I) E_i^b(x_I). \quad (4.41)$$

Using this, we can write

$$\begin{aligned} \epsilon^4 n_a(x_I) n_b(x_I) E^{ai}(x_I) E_i^b(x_I) &= \frac{1}{2} \int_{\Sigma_\epsilon} d^2 \sigma n_a(\sigma) \int_{\Sigma_\epsilon} d^2 \tau n_b(\tau) \mathcal{T}^{ab}[\alpha_{\sigma\tau}](\sigma, \tau) + \\ &+ O(\epsilon), \end{aligned} \quad (4.42)$$

where $\alpha_{\sigma\tau}$ is, say, a (coordinate) circular loop with the two points σ and τ on antipodal points. Next, consider the area of the full surface Σ . By the very definition of Riemann

integral, (4.40) can be written as

$$\begin{aligned} A[\Sigma] &= \int_{\Sigma} d^2\sigma \sqrt{n_a n_b E^{ai} E_i^b} \\ &= \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0}} \sum_{I_\epsilon} \epsilon^2 \sqrt{n_a(x_I) n_b(x_I) E^{ai}(x_I) E_i^b(x_I)} \end{aligned} \quad (4.43)$$

where, following Riemann, we have partitioned the surface Σ in N small surfaces Σ_{I_ϵ} of coordinate area ϵ^2 and x_I is an arbitrary point in Σ_{I_ϵ} . The convergence of the limit to the integral, and its independence from the details of the construction, are assured by the Riemann theorem for all bounded smooth fields. Inserting (4.42) in (4.43), we obtain the desired regularized expression for the classical area, suitable to be promoted to a quantum loop operator

$$A[\Sigma] = \lim_{\epsilon \rightarrow 0} A_\epsilon[\Sigma] , \quad (4.44)$$

$$A_\epsilon[\Sigma] = \sum_{I_\epsilon} \sqrt{A_{I_\epsilon}^2} , \quad (4.45)$$

$$A_{I_\epsilon}^2 = \frac{1}{2} \int_{\Sigma_{I_\epsilon} \otimes \Sigma_{I_\epsilon}} d^2\sigma d^2\tau n_a(\sigma) n_b(\tau) \mathcal{T}^{ab}[\alpha_{\sigma\tau}](\sigma, \tau). \quad (4.46)$$

Notice that the powers of the regulator ϵ in (4.42) and (4.43) combine nicely, so that ϵ appears in (4.44) only in the integration domains.

We are now ready to define the area operator:

$$\hat{A}[\Sigma] = \lim_{\epsilon \rightarrow 0} \hat{A}_\epsilon[\Sigma], \quad (4.47)$$

$$A_\epsilon[\Sigma] = \sum_{I_\epsilon} \sqrt{\hat{A}_{I_\epsilon}^2}, \quad (4.48)$$

$$\hat{A}_{I_\epsilon}^2 = \frac{1}{2} \int_{\Sigma_{I_\epsilon} \otimes \Sigma_{I_\epsilon}} d^2\sigma d^2\tau n_a(\sigma) n_b(\tau) \hat{\mathcal{T}}^{ab}[\alpha_{\sigma\tau}](\sigma, \tau). \quad (4.49)$$

The meaning of the limit in (4.47) needs to be specified. The specification of the topology in which the limit is taken is an integral part of the definition of the operator. As it is usual for limits involved in the regularization of quantum field theoretical operators, the limit cannot be taken in the Hilbert space topology where, in general, it does not exist. The limit must be taken in a topology that “remembers” the topology in which the corresponding classical limit (4.44) is taken. This is easy to do in the present context. We say that a sequence of (multi) loops α_ϵ converges to α if α_ϵ converges pointwise to α ; we say that a sequence of quantum states $\langle \alpha_\epsilon |$ converges to the state $\langle \alpha |$ if $\alpha_\epsilon \rightarrow \alpha$ for at least one α_ϵ in (the equivalence class) $\langle \alpha_\epsilon |$ ($\forall \epsilon$) and one $\alpha \in \langle \alpha |$. This definition extends immediately to general states $\langle \Phi |$ by linearity, and defines a topology on the state space, and the corresponding operator topology: $\hat{O}_\epsilon \rightarrow \hat{O}$ iff $\langle \Phi | \hat{O}_\epsilon \rightarrow \langle \Phi | \hat{O}$, $\forall \langle \Phi |$. Notice that the above is equivalent to say that $\langle \Phi_\epsilon |$ converges to $\langle \Phi |$ if $\mathcal{T}[\Phi_\epsilon]$ converges pointwise to $\mathcal{T}[\Phi]$.

An important consequence of the use of this topology is the following. Let $\langle \Phi_\epsilon |$ converge to $\langle \Phi |$. Then the graphs Γ_{Φ_ϵ} converge to Γ_Φ in the topology of σ . In other

words, given a δ -neighborhood of Γ_Φ , there exists an ϵ such that Γ_{Φ_ϵ} is included in the δ -neighborhood for all $\epsilon' < \epsilon$. Visually, we can imagine that the ribbon-nets R_{Φ_ϵ} “merge” into the ribbon-net R_Φ as ϵ approaches zero. In addition, the representations P_{Φ_ϵ} go to P_Φ , up to equivalence. This fact allows us to separate the study of a limit in two steps. First, we study the graph of the limit state. In this process, the representations P_{Φ_ϵ} are merged into the ribbon-net R of the limit state. Second, we can use recoupling theory on R , in order to express the limit representation in terms of the spin network basis.

We now study the action of the area operator $\hat{A}[\Sigma]$ given in (4.47) on a spin network state $\langle S|$. Namely, we compute $\langle S|\hat{A}[\Sigma]$. Let $S \cap \Sigma$ be the set of the points i in the intersection of Γ_S and Σ . In other words, we label by an index i the points where the spin network graph Γ_S and the surface Σ intersect. Generically $S \cap \Sigma$ is numerable, and does not include vertices of S . Here we disregard spin networks that have a vertex lying on Σ or a continuous number of intersection points with Σ . It was pointed out by A. Ashtekar that spin networks with a vertex *and* one -or more- of its adjacent edges lying on Σ are eigenstates of the area with eigenvalues that are not included in the spectrum of the operator derived below³.

For small enough ϵ , each intersection i will lie inside a distinct Σ_{I_ϵ} surface.⁴ Let us call Σ_{i_ϵ} the surface containing the intersection i (at every fixed ϵ), and e_i the edge through the intersection i . Notice that $\langle S|\hat{A}_{\Sigma_{I_\epsilon}}^2$ vanishes for all surfaces I_ϵ except the ones containing intersections. Thus the sum over surfaces \sum_{I_ϵ} reduces to a sum over intersections. Bringing the limit inside the sum and the square root, we can write

$$\langle S|\hat{A}[\Sigma] = \sum_{i \in \{S \cap \Sigma\}} \langle S|\sqrt{\hat{A}_i^2} \quad (4.50)$$

$$\hat{A}_i^2 = \lim_{\epsilon \rightarrow 0} \hat{A}_{i_\epsilon}^2 \quad (4.51)$$

For finite ϵ , the state $\langle S|\hat{A}_{i_\epsilon}^2$ has support on the union of the graphs of S and the graph of the loop $\alpha_{\sigma\tau}$ in the argument of the operator (4.49). But the last converges to a point on Γ_S as ϵ goes to zero. Therefore

$$\lim_{\epsilon \rightarrow 0} \Gamma_{\langle S|\hat{A}_{i_\epsilon}^2} = \Gamma_S. \quad (4.52)$$

The operator $\hat{A}[\Sigma]$ does not affect the graph of $\langle S|$. Next, we have to compute the planar representation of $\Gamma_{\langle S|\hat{A}[\Sigma]}$, which is a tangle on $R_{\langle S|\hat{A}[\Sigma]}$, namely a tangle on R_S . By equation (4.50), this is given by a sum of terms, one for each $i \in \{S \cap \Sigma\}$. Consider one of these terms. By definition of the \hat{T} loop operators and of the grasp

³Note added: the complete spectrum of the area has been obtained in the meanwhile in [7], and then reobtained in [5] using the methods developed in this paper.

⁴The (perhaps cavilling) issue that an intersection may fall on the *boundary* between two I_ϵ surfaces has been raised. This eventuality, however, does not generate difficulties for the following reason. The integrals we are using are not Lebesgue integrals, because, due to the presence of the δ 's, regions of zero measure of the integration domain cannot be neglected – nor doubly counted. Therefore in selecting the partition of Σ in the I_ϵ surfaces one must include each boundary in one and only one of the two surfaces (which are therefore partially open and partially closed). Boundary points are then normal points that fall inside one and only one integration domain.

operation (Section 3), this is obtained by inserting two trivalent intersections on the spin network edge e_i (inside its ribbon), connected by a new edge of color 2. This is because the circle $\Gamma_{\alpha\sigma\tau}$ has converged to a point on e_i ; in turn, this point is then expanded inside the ribbon as a degenerate loop following back and forward a segment connecting the two intersections. By indicating the representation of the spin network simply by means of its e_i edge, we thus have

$$\begin{aligned} \langle |^{p_e} | \hat{A}_{i_\epsilon}^2 &= \frac{1}{2} \int_{\Sigma_{i_\epsilon} \otimes \Sigma_{i_\epsilon}} d^2\sigma d^2\tau n_a(\sigma) n_b(\tau) \langle |^{p_e} | \hat{T}^{ab}[\alpha_{\sigma\tau}](\sigma, \tau) \\ &= -\frac{l_0^4}{2} \int_{\Sigma_{i_\epsilon} \otimes \Sigma_{i_\epsilon}} d^2\sigma d^2\tau n_a(\sigma) \Delta^a[\beta_e, \sigma] n_b(\tau) \Delta^b[\beta_e, \tau] p_e^2 \left\langle \begin{array}{c} p_e \\ \bullet \\ p_e \\ \bullet \\ p_e \end{array} \right\rangle^2, \end{aligned} \quad (4.53)$$

where we have already taken the limit (inside the integral) in the state enclosed in the brackets $\langle |$. Notice that this does not depend on the integration variables anymore, because the loop it contains does not represent the grasped loop for a finite ϵ , but the a ribbon expansion of the limit state. Notice also that the two integrals are independent, and equal. Thus, we can write

$$\langle |^{p_e} | \hat{A}_{i_\epsilon}^2 = -\frac{l_0^4}{2} \left(\int_{\Sigma_{i_\epsilon}} d^2\sigma n_a(\sigma) \Delta^a[\beta_e, \sigma] \right)^2 p_e^2 \left\langle \begin{array}{c} p_e \\ \bullet \\ p_e \\ \bullet \\ p_e \end{array} \right\rangle^2 \quad (4.54)$$

The parenthesis is easy to compute. Using (4.22), it becomes the analytic form of the intersection number between the edge and the surface

$$\int_{\Sigma_{i_\epsilon}} d^2\sigma n_a(\sigma) \Delta^a[\beta_e, \sigma] = \int_{\Sigma_{i_\epsilon}} d^2\sigma n_a(\sigma) \int_{\beta_e} d\tau \dot{\beta}_e^a(\tau) \delta^3[\beta_e(\tau), s] = \pm 1, \quad (4.55)$$

where the sign, which depends on the relative orientation of the loop and the surface, becomes then irrelevant because of the square. Thus

$$\langle |^{p_e} | \hat{A}_i^2 = -\frac{l_0^4}{2} p_e^2 \left\langle \begin{array}{c} p_e \\ \bullet \\ p_e \\ \bullet \\ p_e \end{array} \right\rangle^2, \quad (4.56)$$

where we have trivially taken the limit (4.51), since there is no residual dependence on ϵ . We have now to express the tangle inside the bracket in terms of (an edge of) a spin network state. But tangles inside ribbons satisfy recoupling theory, and we can therefore use the relative formulas, obtaining

$$\langle |^{p_e} | \hat{A}_{i_\epsilon}^2 = = l_0^4 \frac{p_e}{2} \left(\frac{p_e}{2} + 1 \right) \langle |^{p_e} |. \quad (4.57)$$

The square root in (4.50) is now easy to take because the operator \hat{A}_i^2 is diagonal.

$$\langle |^{p_e} | \hat{A}_i = \langle |^{p_e} | \sqrt{\hat{A}_i^2} = = \sqrt{l_0^4 \frac{p_e}{2} \left(\frac{p_e}{2} + 1 \right)} \langle |^{p_e} |. \quad (4.58)$$

Inserting in the sum (4.50), and shifting from color to spin notation, we obtain the final result

$$\langle S | \hat{A}[\Sigma] = \left(l_0^2 \sum_{i \in \{S \cap \Sigma\}} \sqrt{j_i(j_i + 1)} \right) \langle S | \quad (4.59)$$

where j_i is the spin of the edge crossing Σ in i . This result shows that the spin network states (with a finite number of intersection points with the surface and no vertices on the surface) are eigenstates of the area operator. The corresponding spectrum is labeled by multiplets $\vec{j} = (j_1, \dots, j_n)$ of positive half integers, with arbitrary n , and given by

$$A_{\vec{j}}[\Sigma] = l_0^2 \sum_i \sqrt{j_i(j_i + 1)}. \quad (4.60)$$

4.6 The Volume Operator

Consider a three dimensional region \mathcal{R} . The volume of \mathcal{R} is given by

$$V[\mathcal{R}] = \int_{\mathcal{R}} d^3x \sqrt{\det g} = \int_{\mathcal{R}} d^3x \sqrt{\frac{1}{3!} \left| \epsilon_{abc} \epsilon_{ijk} E^{ai} E^{bj} E^{ck} \right|}, \quad (4.61)$$

In order to construct a regularized form of this expression, consider the three index (three hands) loop variable:

$$\mathcal{T}^{abc}[\alpha](s, t, r) = -\text{Tr}[E^a(s)U_\alpha(s, t) E^b(t)U_\alpha(t, r)E^c(r)U_\alpha(r, s)]. \quad (4.62)$$

In the limit of the loop $[\alpha]$ shrinking to a point x we have:

$$\mathcal{T}^{abc}[\alpha](s, t, r) \rightarrow 2\epsilon_{ijk} E^{ai} E^{bj} E^{ck} = 2 \epsilon^{abc} \det(E). \quad (4.63)$$

Fix an arbitrary chart of M , and consider a small cubic region \mathcal{R}_I of coordinate volume ϵ^3 . Let x_I be an arbitrary but fixed point in \mathcal{R}_I . Since classical fields are smooth we have $E(s) = E(x_I) + O(\epsilon)$ for every $s \in \mathcal{R}_I$, and $U_\alpha(s, t)_A^B = \mathbf{1}_A^B + O(\epsilon)$ for any $s, t \in \mathcal{R}_I$ and straight segment α joining s and t . Consider the quantity

$$W_I = \frac{1}{16 \cdot 3! \epsilon^6} \int_{\partial \mathcal{R}_I} d^2\sigma \int_{\partial \mathcal{R}_I} d^2\tau \int_{\partial \mathcal{R}_I} d^2\rho \left| n_a(\sigma) n_b(\tau) n_c(\rho) \mathcal{T}^{abc}[\alpha_{\sigma\tau\rho}](\sigma, \tau, \rho) \right|, \quad (4.64)$$

where $\alpha_{\sigma\tau\rho}$ is a triangular loop joining the points σ , τ and ρ . Because of (4.63), we have, to lowest order in ϵ

$$\begin{aligned} W_I &= \frac{1}{8 \cdot 3! \epsilon^6} \left| \det(E(x_I)) \int_{\partial \mathcal{R}_I} d^2\sigma \int_{\partial \mathcal{R}_I} d^2\tau \int_{\partial \mathcal{R}_I} d^2\rho \cdot |n_a(\sigma) n_b(\tau) n_c(\rho) \epsilon^{abc}| \right. \\ &= \left. \left| \det(E(x_I)) \right|, \end{aligned} \quad (4.65)$$

Thus, W_I is a non-local quantity that approximates $\det g(x_I)$ for small ϵ . Using the Riemann theorem as in the case of the area, we can then write the volume $V[\mathcal{R}]$ of the

region \mathcal{R} as follows. For every ϵ , we partition of \mathcal{R} in cubes \mathcal{R}_{I_ϵ} of coordinate volume ϵ^3 . Then

$$V[\mathcal{R}] = \lim_{\epsilon \rightarrow 0} V_\epsilon[\mathcal{R}]; \quad (4.66)$$

$$V_\epsilon[\mathcal{R}] = \sum_{I_\epsilon} \epsilon^3 W_{I_\epsilon}^{1/2}. \quad (4.67)$$

4.6.1 Quantum volume operator

We have then immediately a definition of the quantum volume operator:

$$\hat{V}[\mathcal{R}] = \lim_{\epsilon \rightarrow 0} \hat{V}_\epsilon[\mathcal{R}]; \quad (4.68)$$

$$\hat{V}_\epsilon[\mathcal{R}] = \sum_{I_\epsilon} \epsilon^3 \hat{W}_{I_\epsilon}^{1/2}; \quad (4.69)$$

$$\hat{W}_{I_\epsilon} = \frac{1}{16 \cdot 3! \epsilon^6} \int_{\partial \mathcal{R}_I} d^2 \sigma \int_{\partial \mathcal{R}_I} d^2 \tau \int_{\partial \mathcal{R}_I} d^2 \rho \cdot |n_a(\sigma) n_b(\tau) n_c(\rho) \hat{T}^{abc}[\alpha_{\sigma\tau\rho}](\sigma, \tau, \rho)|. \quad (4.70)$$

Notice the crucial cancellation of the ϵ^6 factor. We refer to the previous section on the area operator for the discussion on the meaning of the limit and the split of the action of the operator in the computation of the graph and the representation. We will discuss the meaning of the square root later.

Let us now begin to compute the action of this operator on a spin network state. The three surface integrals on the surface of the cube and the line integrals along the loops combine –as in the case of the area– to give three intersection numbers, which select three intersection points between the spin network and the boundary of the cube. In these three points, which we denote as r , s and t , the loop $\alpha_{\sigma\tau\rho}$ of the operator grasps the spin network.

Notice that the integration domain of the (three) surface integrals is a six dimensional space –the space of the possible positions of three points on the surface of a cube. Let us denote this integration domain as D^6 . The absolute value in (4.70) plays a crucial role here: contributions from different points of D^6 have to be taken in their absolute value, while contributions from the same point of D^6 have to be summed algebraically before taking the absolute value. The position of each hand of the operator is integrated over the surface, and therefore each hand grasps each of the three points r , s and t , producing 3^3 distinct terms. However, because of the absolute value, a term in which two hands grasp the same point, say r , vanishes. This happens because the result of the grasp is symmetric, but the operator is antisymmetric, in the two hands – as follows from the antisymmetry of the trace of three sigma matrices. Thus, only terms in which each hand grasps a distinct point give non vanishing contributions. For each triple of points of intersection between spin network and cube's surface r , s and t , there are $3!$ ways in which the three hands can grasp the three points. These $3!$ terms have alternating signs because of the antisymmetry of the operator, but the absolute value prevents the sum from vanishing, and yields the same contribution for each of the $3!$ terms.

If there are only two intersection points between the boundary of the cube and the spin network, then there are always two hands grasping in the same point; contributions have to be summed before taking the absolute value, and thus they cancel. Thus, the sum in (4.69) reduces to a sum over the cubes I_ϵ^i whose boundary has at least three distinct intersections with the spin network, and the surface integration reduces to a sum over the triple-grasplings in *distinct* points. For ϵ small enough, the only cubes whose surface has at least three intersections with the spin network are the cubes containing a vertex i of the spin network. Therefore, the sum over cubes reduces to a sum over the vertices $i \in \{S \cap \mathcal{R}\}$ of the spin network, contained inside \mathcal{R} . Let us denote by I_ϵ^i the cube containing the vertex i . We then have

$$\begin{aligned} \langle S | \hat{V}[\mathcal{R}] &= \lim_{\epsilon \rightarrow 0} \sum_{i \in \{S \cap \mathcal{V}\}} \epsilon^3 \langle S | \sqrt{|\hat{W}_{I_\epsilon^i}|} \\ \langle S | \hat{W}_{I_\epsilon^i} &= \frac{i l_0^6}{16 \cdot 3! \epsilon^6} \sum_{s,t,r} \left\langle S \tilde{\#}_{s,t,r} \alpha_{s,t,r} \right\rangle, \end{aligned} \quad (4.71)$$

where s, t and r are three *distinct* intersections between the spin network and the boundary of the box, and we have indicated by $\left\langle S \tilde{\#}_s \tilde{\#}_t \tilde{\#}_r \alpha_{str} \right\rangle$ the result of the triple grasp of the three hands operator with loop α_{str} on S .

Let us compute one of the terms above, corresponding to a given triple of grasps, over an n -valent intersections. First of all, in the limit $\epsilon \rightarrow 0$ the operator does not change the graph of the quantum state, for the same reason the area operator doesn't. Thus, the computation reduces to a combinatorial computation of the action of the operator on the representation of the planar state, involving recoupling theory.

Let us represent a spin network state simply by means of the portion of its virtual net containing the vertex on which the operator is acting. We have

$$\begin{aligned} \left\langle \begin{array}{c} P_2 \\ P_1 \\ P_0 \end{array} \begin{array}{c} \dots \\ i_2 \\ i_1 \end{array} \begin{array}{c} \dots \\ i_3 \\ i_1 \end{array} \begin{array}{c} P_{n-3} \\ P_{n-2} \\ P_{n-1} \end{array} \right| \hat{W}_{I_\epsilon^i} &= \\ = \frac{i l_0^6}{16 \cdot 3!} \sum_{\substack{r=0, \dots, n-1 \\ t=0, \dots, n-1 \\ s=0, \dots, n-1}} \int_{\partial \mathcal{V}_I \otimes \partial \mathcal{V}_I \otimes \partial \mathcal{V}_I} d^2 \sigma d^2 \tau d^2 \rho \left| n_a(\sigma) \Delta^a[\gamma, \sigma] n_b(\tau) \Delta^b[\gamma, \tau] n_c(\rho) \Delta^c[\gamma, \rho] \right| \end{aligned} \quad (4.72)$$

$$\cdot \left\langle \begin{array}{c} P_2 \\ P_1 \\ P_0 \end{array} \begin{array}{c} \dots \\ i_2 \\ i_1 \end{array} \begin{array}{c} \dots \\ i_{n-2} \\ i_{n-1} \end{array} \begin{array}{c} P_{n-3} \\ P_{n-2} \\ P_{n-1} \end{array} \right| \left| \hat{W}_{[rts]}^{(n)} \right|$$

where $\hat{W}_{[rts]}^{(n)}$ is the operator that grasp the r, t and s edge of the the n -valent vertex

as follow:

$$\begin{aligned}
& \left\langle \begin{array}{c} P_2 \quad \cdots \quad P_{n-3} \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ P_1 \quad \cdots \quad P_{n-2} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ P_0 \quad \cdots \quad P_{n-1} \end{array} \middle| \hat{W}_{[rts]}^{(n)} = \\
& = P_r P_t P_s \left\langle \begin{array}{c} P_r \quad P_t \quad P_s \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ P_0 \quad \cdots \quad P_{n-1} \end{array} \middle| \right. \quad (4.73) \\
& = \sum_{k_2, \dots, k_{n-2}} W_{[rts]i_2 \dots i_{n-2}}^{(n) k_2 \dots k_{n-2}}(P_0, \dots, P_{n-1}) \cdot \\
& \quad \left\langle \begin{array}{c} P_2 \quad \cdots \quad P_{n-3} \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ P_1 \quad \cdots \quad P_{n-2} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ P_0 \quad \cdots \quad P_{n-1} \end{array} \middle| \right. ,
\end{aligned}$$

Notice that we have replaced the triangular loop with vertices r , s and t by three edges of color 2 joining the three points r , s and t to a trivalent vertex. This can be done as follows. First we deform the triangle over the ribbon-net. Indeed, as remarked for the case of the area, the tangle above does not represent a tangle extended in M , but just the expansion over the ribbon net of a rooting of lines in a single point of M . Second, we notice that we can antisymmetrize the two lines that exit from the hand of an operator by using the binor identity, because tracing a hand with a zero length loop gives a vanishing quantity.

The last equality in the last equation follows from the fact that trivalent spin network form a basis. From eq. (4.72) we see that the action of $\hat{W}_{I_i}^{(n)}$ splits into a multiplication by a numerical prefactor and a recoupling part given by eq. (4.73), which does not depend on the integration variables. Using eq. (4.55) we can perform the integration in eq. (4.72). This yields the intersection number between the edges r , s and t and the surface of the cube \mathcal{V}_I . The sign of the intersection number, coming from the relative orientation of the loop and the surface, is irrelevant, because of the presence of the absolute value.

Because of the symmetry properties of the 3-valent node (222), the $3!$ terms in eq. (4.73) are related by:

$$\hat{W}_{[i_1 i_2 i_3]}^{(n)} = (-1)^p \hat{W}_{[i_{p_1} i_{p_2} i_{p_3}]}^{(n)} \quad (4.74)$$

where p_i it is a permutation of 123, and p it is the order of the permutation. Thus,

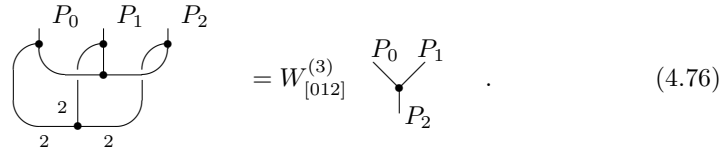
the action the volume operator on a generic spin network state $\langle S|$ is given by:

$$\hat{V}[\mathcal{V}] = l_0^3 \sum_{i \in \{S \cap \mathcal{V}\}} \sqrt{\sum_{\substack{r=0, \dots, n-3 \\ t=r+1, \dots, n-2 \\ s=t+1, \dots, n-1}} \left| \frac{i}{16} \hat{W}_{[rst]}^{(n_i)} \right|} \quad (4.75)$$

where n_i is the valence of the i -th intersection. Equations (4.73) and (4.75) completely define the volume operator. There are two remaining tasks: to find the explicit expression for the matrix $iW_{[rst]k_n-2 \dots k_3 k_2}^{(n) i_{n-2} \dots i_3 i_2}(P_{n-1}, \dots, P_0)$, which is defined in eq. (4.73) only implicitly; and to show that the absolute value and the square root in equation (4.75) are well defined. Below, we complete both tasks: we provide an explicit expression for $iW_{[rst]k_n-2 \dots k_3 k_2}^{(n) i_{n-2} \dots i_3 i_2}(P_{n-1}, \dots, P_0)$, and we prove that the argument of the absolute value is a diagonalizable finite dimensional matrix with real eigenvalues, and the argument of the square root is a finite dimensional diagonalizable matrix with positive real eigenvalues.

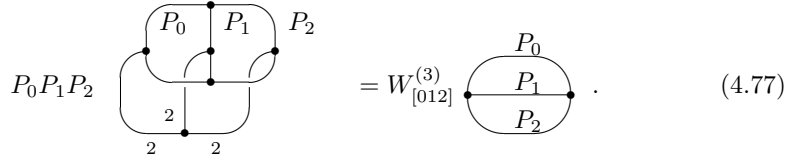
4.6.2 Trivalent vertices

We begin studying the case $n = 3$. It is easy to see that $W_{[012]}^{(3)} = 0$ from the relation



$$= W_{[012]}^{(3)} \begin{array}{c} P_0 \quad P_1 \\ \diagdown \quad \diagup \\ \cdot \\ \diagup \quad \diagdown \\ P_2 \end{array} \quad (4.76)$$

In fact, by closing the generic 3-valent node with itself we have



$$P_0 P_1 P_2 \begin{array}{c} P_0 \quad P_1 \quad P_2 \\ \diagdown \quad \diagup \\ \cdot \\ \diagup \quad \diagdown \\ P_2 \end{array} = W_{[012]}^{(3)} \begin{array}{c} P_0 \\ \circ \\ P_1 \\ \circ \\ P_2 \end{array} \quad (4.77)$$

Thus $W_{[012]}^{(3)}$ its determined by the Wigner $9J$ -symbol (the evaluation of the hexagonal net) as:

$$W_{[012]}^{(3)} = \frac{P_0 P_1 P_2 \left\{ \begin{array}{ccc} P_0 & P_1 & P_2 \\ P_0 & P_1 & P_2 \\ 2 & 2 & 2 \end{array} \right\}}{\theta(P_0, P_1, P_2)} \quad (4.78)$$

But the hexagonal net is antisymmetric for the exchange of two columns or of two rows. Therefore the matrix W^3 vanishes, and the trivalent vertices give no contribution to the volume.

4.6.3 Four-valent vertices

Next, we study the $n = 4$ case.

$$\hat{W}_{[012]}^{(4)} \begin{array}{c} P_1 \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ P_0 \quad P_3 \end{array} \begin{array}{c} i \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ P_2 \quad P_3 \end{array} = \sum_j W_{[012]i}^{(4)j} \begin{array}{c} P_1 \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ P_0 \quad P_3 \end{array} \begin{array}{c} j \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ P_2 \quad P_3 \end{array} \quad (4.79)$$

Using the same technique of the 3-valent node we can compute the matrix $W_{[012]i}^{(4)j}$ for a 4-valent node as follows

$$P_0 P_1 P_2 \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \end{array} P_3 = \sum_k W_{[012]i}^{(4)k} \delta_k^j \frac{\theta(P_0, P_1, j)\theta(P_2, P_3, j)}{\Delta_j}. \quad (4.80)$$

Using the relation

$$\begin{array}{c} 2 \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ P_2 \quad P_3 \end{array} \begin{array}{c} j \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ P_2 \quad P_3 \end{array} = \frac{Tet \begin{bmatrix} i & j & P_3 \\ P_2 & P_2 & 2 \end{bmatrix}}{\theta(2, j, i)} \begin{array}{c} j \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ P_2 \quad P_3 \end{array}, \quad (4.81)$$

we obtain:

$$W_{[012]i}^{(4)j} = \frac{P_0 P_1 P_2 \begin{Bmatrix} P_0 & P_1 & j \\ P_0 & P_1 & i \\ 2 & 2 & 2 \end{Bmatrix} Tet \begin{bmatrix} i & j & P_3 \\ P_2 & P_2 & 2 \end{bmatrix}}{\theta(2, j, i)} \cdot \frac{\Delta_j}{\theta(P_0, P_1, j)\theta(P_2, P_3, j)}. \quad (4.82)$$

We now prove that the matrix $i \cdot W_{[012]i}^{(4)j}$ is diagonalizable with real eigenvalues and, as a consequence, that its absolute values is well defined. To this aim, let us define the notation:

$$A_i^j = \frac{P_0 P_1 P_2 \begin{Bmatrix} P_0 & P_1 & j \\ P_0 & P_1 & i \\ 2 & 2 & 2 \end{Bmatrix} Tet \begin{bmatrix} i & j & P_3 \\ P_2 & P_2 & 2 \end{bmatrix}}{\theta(2, j, i)} \quad (4.83)$$

$$M(i) = \sqrt{\frac{\Delta_i}{\theta(P_0, P_1, i)\theta(P_2, P_3, i)}} \quad (4.84)$$

$$\tilde{W}_i^j = M(i) M(j) A_i^j \quad (4.85)$$

$$S_i^j = \delta_i^j M(i). \quad (4.86)$$

The matrix S_i^j can be consider as a change of basis in the space of the 4-valent vertices and the matrix $i \cdot W_{[012]i}^{(4)j}$ can be rewritten as:

$$iW_{[012]i}^{(4)j} = (S^{-1})_i^k \cdot (i\tilde{W}_k^l) \cdot S_l^j, \quad (4.87)$$

where, because of the antisymmetry properties of the $9J$ -symbol under exchange of two rows and the symmetry property of the Tet symbol⁵, the matrix \tilde{W}_k^l is antisymmetric. We have shown that in the basis shown in figure (4.5),

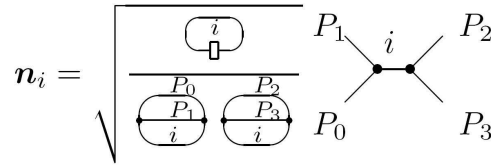


Figure 4.5: A base for the intertwiners where $\tilde{W}_{[012]i}^{(4)j}$ is an antisymmetric matrix

where $\tilde{W}_{[012]i}^{(4)j}$ is a *real antisymmetric* matrix. Moreover, from the admissibility condition for the 3-valent node of eq. (4.81), we see that \tilde{W}_k^l vanishes unless $k = l$ or $k = l \pm 2$. Thus, we have show that the operator $i \hat{W}_{[012]}^{(4)}$ may be represented by a purely imaginary antisymmetric matrix $i\tilde{W}_k^l$ with non-vanishing matrix elements only for $k = l \pm 2$. Such matrix is diagonalizable and has real eigenvalues.

Furthermore, notice the following. We write the dependence on the coloring of the external edges explicitly; namely we write $W_{[012]i}^{(4)j}(P_0, P_1, P_2, P_3)$. Using eq. (4.80), it is easy to see that the following relations hold between the matrices $W_{[i_1 i_2 i_3]i}^{(4)j}(P_0, P_1, P_2, P_3)$

$$\begin{aligned} W_{[013]i}^{(4)j}(P_0, P_1, P_2, P_3) &= W_{[012]i}^{(4)j}(P_0, P_1, P_3, P_2), \\ W_{[023]i}^{(4)j}(P_0, P_1, P_2, P_3) &= -W_{[123]i}^{(4)j}(P_3, P_2, P_1, P_0), \\ W_{[123]i}^{(4)j}(P_0, P_1, P_2, P_3) &= -W_{[012]i}^{(4)j}(P_3, P_2, P_0, P_1). \end{aligned} \quad (4.88)$$

We have shown that there exists a basis in which the four operators $i\hat{W}_{[i_1 i_2 i_3]}^{(4)}$ that define the action of the volume on four valent vertices, are purely imaginary antisymmetric matrices. The eigenvalues of the four operators $i\hat{W}_{[i_1 i_2 i_3]}^{(4)}$ are real and, if x is an eigenvalue, so is $-x$. Therefore, the absolute value of the matrices $i\hat{W}_{[i_1 i_2 i_3]}^{(4)}$ is well defined. It is given by a non-negative (i.e., having real eigenvalues equal or greater than zero) antisymmetric matrix. But the sum of non-negative matrices is a non-negative matrix. Therefore the sum of the the absolute values of the four matrices $i\hat{W}_{[i_1 i_2 i_3]}^{(4)}$ is a non-negative antisymmetric matrix as well. Thus, the volume operator

⁵For a discussion of the symmetry properties of the $9J$ -symbol and related quantities, see for instance [24]

is diagonalizable on the spin network basis, *with positive real eigenvalues*, if all the vertices have valence 3, 4. Below, we show that these results extend to vertices of arbitrary valence.

4.6.4 The case of an n -vertex

We now shown that there exists a basis in which all the operators

$$i\hat{W}_{[i_1 i_2 i_3]}^{(n)}(P_0, \dots, P_{n-1})$$

are represented by a purely imaginary antisymmetric matrix. Consider eq. (4.73). By repeated application of the recoupling theorem, eq. (4.73) can be rewritten as

$$\begin{aligned}
 P_r P_t P_s \left[\begin{array}{c} P_0 \quad P_r \quad P_t \quad P_s \quad \dots \quad P_{n-1} \\ \curvearrowright \quad \curvearrowright \quad \curvearrowright \quad \vdots \quad \curvearrowright \\ \hat{i}_2 \quad \hat{i}_3 \quad \hat{i}_4 \quad \vdots \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ 2 \quad \quad \quad 2 \quad \quad \quad \end{array} \right] &= \\
 &= \sum_{\hat{k}_2 \dots \hat{k}_{n-2}} W_{[rst] \hat{i}_2 \dots \hat{i}_{n-2}}^{(n) \hat{k}_2 \dots \hat{k}_{n-2}} \cdot \left[\begin{array}{c} P_0 \quad P_r \quad P_t \quad P_s \quad \dots \quad P_{n-1} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \hat{k}_2 \quad \hat{k}_3 \quad \hat{k}_4 \quad \vdots \end{array} \right] \quad (4.89)
 \end{aligned}$$

(we have assumed, without loss of generality, that there is no grasp on the P_0 or P_{n-1} edge). Closing the vertex with itself and using the relations

$$\left[\begin{array}{c} a \\ \curvearrowright \\ b \quad c \\ \curvearrowright \\ a \end{array} \right] = \frac{\left[\begin{array}{c} a \\ \curvearrowright \\ b \\ \curvearrowright \\ c \\ \square \\ a \end{array} \right]}{\left[\begin{array}{c} \square \\ a \end{array} \right]} \cdot \left| a \right. \quad (4.90)$$

$$\left[\begin{array}{c} a \\ \curvearrowright \\ b \quad d \\ \curvearrowright \\ c \quad e \\ \square \\ a \end{array} \right] = \frac{\left[\begin{array}{c} c \quad d \\ \curvearrowright \\ b \quad e \\ \square \\ a \end{array} \right]}{\left[\begin{array}{c} \square \\ a \end{array} \right]} \cdot \left| a \right. \quad (4.91)$$

we find

$$\begin{aligned}
 W_{[rst] \hat{i}_2 \dots \hat{i}_{n-2}}^{(n) \hat{k}_2 \dots \hat{k}_{n-2}} &= P_r P_t P_s \left\{ \begin{array}{c} \hat{k}_2 \quad P_t \quad \hat{k}_3 \\ \hat{i}_2 \quad P_t \quad \hat{i}_3 \\ 2 \quad 2 \quad 2 \end{array} \right\} \cdot \\
 & \cdot \frac{-1 \lambda_{\hat{k}_2}^{\hat{i}_2} 2 \delta_{\hat{i}_4}^{\hat{k}_4} \dots \delta_{\hat{i}_{n-2}}^{\hat{k}_{n-2}} \cdot \text{Tet} \left[\begin{array}{ccc} P_r & P_r & P_0 \\ \hat{k}_2 & \hat{i}_2 & 2 \end{array} \right] \text{Tet} \left[\begin{array}{ccc} \hat{i}_3 & \hat{k}_3 & \hat{k}_4 \\ P_s & P_s & 2 \end{array} \right] \Delta_{\hat{k}_2} \Delta_{\hat{k}_3}}{\theta(\hat{k}_2, 2, \hat{i}_2) \theta(\hat{k}_3, 2, \hat{i}_3) \theta(P_0, P_r, \hat{k}_2) \theta(\hat{k}_2, P_t, \hat{k}_3) \theta(\hat{k}_3, P_s, \hat{k}_4)} \quad (4.92)
 \end{aligned}$$

We now change basis in the same fashion as we did for the 4-valent vertex, [see Figure (4.5)]. We define a new basis in which any edge (real or virtual) is multiplied by $\sqrt{\Delta_i}$ (i coloring of the edge) and any vertex is divided by $\sqrt{\theta(a, b, c)}$ (a, b and c the coloring of the edges adjacent to the vertex). It is then easy to see that in this new basis the matrix on eq. (4.92) becomes real antisymmetric. Indeed, we have simply reduced the general problem to the case of four valent vertices. Now, the key result, that we shall prove in the next section is that, in the basis we have defined, the recoupling theorem is a *unitary* transformation. A unitary transformation preserves the property of a matrix of being diagonalizable and having real eigenvalues. It follows that the results we have obtained for the four-valent vertices hold in general. We are now ready to find an explicit expression for the recoupling matrix $iW_{[rst]k_{n-2}\dots k_3 k_2}^{(n) i_{n-2}\dots i_3 i_2}(P_{n-1}, \dots, P_0)$ of eq. (4.73) for a general valence n of the vertex.

Let us begin by sketching the procedure that we follow. First, the recoupling theorem allows us to move one of the three grasps from the external edge, say P_r , of eq. (4.73), and bring it to a virtual vertex. We denote this operation as Move 1:

$$\begin{aligned} \begin{array}{c} P_r \\ | \\ \bullet \\ / \quad \backslash \\ i_r \quad i_{r+1} \\ | \quad | \\ 2 \end{array} &= \sum_{k_r} \left\{ \begin{array}{ccc} i_{r+1} & i_r & k_r \\ 2 & P_r & P_r \end{array} \right\} \begin{array}{c} P_r \\ | \\ \bullet \\ / \quad \backslash \\ i_r \quad i_{r+1} \\ | \quad | \\ 2 \end{array} \\ &= \sum_{k_r} \left\{ \begin{array}{ccc} i_{r+1} & i_r & k_r \\ 2 & P_r & P_r \end{array} \right\} [\lambda_{k_r}^{2i_r}]^{-1} \begin{array}{c} P_r \\ | \\ \bullet \\ / \quad \backslash \\ i_r \quad i_{r+1} \\ | \quad | \\ 2 \end{array} . \end{aligned} \quad (4.93)$$

Second, we can use recoupling theorem repeatedly to move the grasp all the way to the edge P_0 . We denote this operation as Move 2:

$$\begin{array}{c} P_{r-1} \\ | \\ \bullet \\ / \quad \backslash \\ i_r \quad k_r \\ | \quad | \\ 2 \end{array} = \sum_{k_r} \left\{ \begin{array}{ccc} k_r & 2 & k_{r-1} \\ i_{r-1} & P_{r-1} & i_r \end{array} \right\} \begin{array}{c} P_{r-1} \\ | \\ \bullet \\ / \quad \backslash \\ i_{r-1} \quad k_r \\ | \quad | \\ 2 \end{array} . \quad (4.94)$$

In this way we can bring all three grasps to the edge P_0 . The final step is just given by recognizing that we have Tet structure on the edge P_0 .

Let us begin by applying Move 1 to the node r . We obtain

$$\left(\begin{array}{c} P_1 \\ | \\ \bullet \\ / \quad \backslash \\ i_{r-1} \quad i_r \\ | \quad | \\ 2 \end{array} \begin{array}{c} P_{r-1} \\ | \\ \bullet \\ / \quad \backslash \\ i_r \quad k_r \\ | \quad | \\ 2 \end{array} \begin{array}{c} P_r \\ | \\ \bullet \\ / \quad \backslash \\ i_r \quad i_{r+1} \\ | \quad | \\ 2 \end{array} \begin{array}{c} P_t \\ | \\ \bullet \\ / \quad \backslash \\ i_t \quad i_s \\ | \quad | \\ 2 \end{array} \begin{array}{c} P_s \\ | \\ \bullet \\ / \quad \backslash \\ i_s \quad i_{s+1} \\ | \quad | \\ 2 \end{array} \right) \begin{array}{c} P_0 \\ | \\ \bullet \\ / \quad \backslash \\ i_1 \quad i_2 \\ | \quad | \\ 2 \end{array} \begin{array}{c} P_{n-1} \\ | \\ \bullet \\ / \quad \backslash \\ i_{n-1} \quad i_n \\ | \quad | \\ 2 \end{array} \quad (4.95)$$

Then, using Move 2 we can move the $(i_r, k_r, 2)$ node to the left of the node (i_{r-1}, P_{r-1}, i_t) :

$$(4.96)$$

We repeat move 2 until the first node with the 2 edge is coupled to the P_0 edge. In this way, after a finite number of moves 2, we have transformed the original network to

$$(4.97)$$

Before repeating this procedure for each of the three grasps, it is convenient to rename the colors k_a of the virtual edges as \bar{k}_a (and to replace the remaining i_a by k_a as well; this can be done by inserting a sum over a \bar{k}_a multiplied by a $\delta_{i_a}^{\bar{k}_a}$).

Repeating the sequence of moves for the two grasps over the edges r and s , we transform the grasped vertex to the following final form:

$$(4.98)$$

This is equal to the original n -valent vertex with the i_a replaced by k_a and multiplied by $Tet[k_1, \bar{k}_1, \bar{k}_1; 2, 2, 2]$ (see eq. (4.91)). Bringing all together, we have shown that the action of the volume operator is described by the sum (4.75) extended over all vertices of the spin network, where the explicit form for the recoupling matrix (4.73) is given by

$$\begin{aligned}
W_{[rst]i_2 \dots i_{n-2}}^{(n) k_2 \dots k_{n-2}}(P_0, \dots, P_{n-1}) &= \sum_{\bar{k}_1, \dots, \bar{k}_{n-2}} \sum_{\tilde{k}_1, \dots, \tilde{k}_{n-2}} P_t P_r P_s \cdot \frac{Tet \left[\begin{array}{ccc} \bar{k}_1 & \tilde{k}_1 & k_1 \\ 2 & 2 & 2 \end{array} \right]}{\Delta_{P_0}} \\
&\cdot \left[\prod_{a=r+1}^{n-2} \delta_{i_a}^{\bar{k}_a} \right] \cdot M \left[\begin{array}{ccc} i_{r+1} & i_r & \bar{k}_r \\ 2 & r & P_r \end{array} \right] \cdot \left[\prod_{a=1}^{r-1} \left\{ \begin{array}{ccc} \bar{k}_{a+1} & 2 & \bar{k}_a \\ i_a & P_a & i_{a+1} \end{array} \right\} \right] \\
&\cdot \left[\prod_{b=t+1}^{n-2} \delta_{i_b}^{\tilde{k}_b} \right] \cdot M \left[\begin{array}{ccc} \bar{k}_{r+1} & \bar{k}_r & \tilde{k}_r \\ 2 & t & P_t \end{array} \right] \cdot \left[\prod_{b=1}^{t-1} \left\{ \begin{array}{ccc} \tilde{k}_{b+1} & 2 & \tilde{k}_b \\ \bar{k}_b & P_a & \bar{k}_{b+1} \end{array} \right\} \right] \\
&\cdot \left[\prod_{c=s+1}^{n-2} \delta_{i_c}^{k_c} \right] \cdot M \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} \left[\begin{array}{ccc} \tilde{k}_{s+1} & \tilde{k}_s & k_s \\ 2 & s & P_s \end{array} \right] \cdot \left[\prod_{c=1}^{s-1} \left\{ \begin{array}{ccc} k_{c+1} & 2 & k_c \\ \bar{k}_c & P_a & \tilde{k}_{c+1} \end{array} \right\} \right]
\end{aligned} \tag{4.99}$$

and

$$M \begin{bmatrix} i_{r+1} & i_r & k_r \\ 2 & r & P_r \end{bmatrix} = \begin{cases} 1, & r = 0; \\ [\lambda_{k_r}^{2i_r}]^{-1} \begin{Bmatrix} i_{r+1} & i_r & k_r \\ 2 & P_r & P_r \end{Bmatrix}, & 0 < r < n-1; \\ \lambda_{P_{n-1}}^{2P_{n-1}} = -1, & r = n-1. \end{cases} \quad (4.100)$$

where $i_1 = k_1 = P_0$ and $i_{n-1} = k_{n-1} = \bar{k}_{n-1} = \tilde{k}_{n-1} = P_{-1}$. (We have used the fact that $\lambda_a^{2a} = -1$.)

This formula can be specialized to the case of three-vertex ($n = 3$) and four-vertex ($n = 4$). In the case of three-vertex we have:

$$W^{(3)}(P_0, P_1, P_2) = \left| \sum_{\tilde{k}_1} P_0 P_1 P_2 [\lambda_{P_0}^{2\tilde{k}_1}]^{-1} \begin{Bmatrix} P_2 & 2 & P_0 \\ \tilde{k}_1 & P_1 & P_2 \end{Bmatrix} \cdot \begin{Bmatrix} P_2 & P_0 & \tilde{k}_1 \\ 2 & P_1 & P_1 \end{Bmatrix} \frac{\begin{bmatrix} P_0 & \tilde{k}_1 & P_0 \\ 2 & 2 & 2 \end{bmatrix}}{\Delta_{P_0}} \right|. \quad (4.101)$$

and a direct computation confirms that the volume of any three-vertex is 0. For the case of four-valent vertex, we obtain the formula:

$$W_{[013]i}^{(4)k} = \sum_{\tilde{k}_1} P_0 P_1 P_3 (-1) [\lambda_{P_0}^{2\tilde{k}_1}]^{-1} \begin{Bmatrix} i & P_0 & \tilde{k}_1 \\ 2 & P_1 & P_1 \end{Bmatrix} \begin{Bmatrix} P_3 & 2 & k \\ i & P_2 & P_3 \end{Bmatrix} \cdot \begin{Bmatrix} k & 2 & P_0 \\ \tilde{k}_1 & P_1 & i \end{Bmatrix} \cdot \frac{\text{Det} \begin{bmatrix} P_0 & \tilde{k}_1 & P_0 \\ 2 & 2 & 2 \end{bmatrix}}{\Delta_{P_0}}$$

and the other 3 matrix that appear in the definition of the action of the volume operator are easily deduced from the identities (4.88).

4.6.5 Summary of the volume's action

Finally, let us summarize the procedure for computing the eigenvalues and eigenvectors of the volume. Consider the spin-network states $\langle S \rangle$ with a fixed graph and a fixed coloring of the real edges, but with arbitrary intersections. The set of these spin networks forms a finite dimensional subspace V of the quantum state space. The subspace V is invariant under the action of the volume operator. We denote the valence of the real vertex i by n_i . Fix a trivalent decomposition of each vertex $i \in \{S \cup \mathcal{R}\}$. Consider all compatible colorings of the virtual edges. For every vertex, the number of the compatible colorings depends on the valence of the vertex, as well as on the coloring of the external edges. Let N_i be the number of compatible colorings of the vertex n_i . The dimension N of the subspace V we are considering is $N = \prod_i N_i$. Our aim is to diagonalize the volume operator in V .

We indicate a basis in V as follows. Given a vertex i with valence n_i , we have previously denoted compatible colorings of the internal edges by (i_2, \dots, i_{n_i-2}) . It is

more convenient here to simplify the notation by introducing a single index $K_i = 1, N_i$, which labels all compatible internal colorings of the vertex i .

We now recall the basic expression we have obtained for the volume, namely eq. (4.75):

$$\begin{aligned}\hat{V}[\mathcal{V}] &= l_0^3 \sum_{i \in \{S \cap \mathcal{V}\}} \hat{V}_i, \\ \hat{V}_i &= \sqrt{\sum_{\substack{r=0, \dots, n-3 \\ t=r+1, \dots, n-2 \\ s=t+1, \dots, n-1}} \left| \frac{i}{16} \hat{W}_{[rts]}^{(n_i)} \right|^2},\end{aligned}\quad (4.102)$$

where the first sum is over the vertices and the second sum is over the triples of edges adjacent to the vertex. We have shown that the operators $i\hat{W}_{[rts]}^{(n_i)}$ are diagonalizable matrices with real eigenvalues. These matrices have components

Since the matrices $i\hat{W}_{[rts]}^{(n_i)K_i}$ are diagonalizable with real eigenvalues, from the spectral theorem we can write them as:

$$i\hat{W}_{[rts]}^{(n_i)} = \sum_{\alpha} \alpha \lambda_{[rts]}^{(n_i)} \alpha \hat{P}_{[rts]}^{(i)}, \quad (4.103)$$

where $\alpha \lambda_{[rts]}^{(n_i)}$ are real quantities and the $\alpha \hat{P}_{[rts]}^{(i)}$ are the spectral projectors of the finite dimensional matrix operator $\hat{W}_{[rts]}^{(n_i)}$, acting on the i -th vertex's basis.

From (4.102), we have then

$$\hat{V}_i^2 = \sum_{\substack{r=0, \dots, n_i-3 \\ t=r+1, \dots, n_i-2 \\ s=t+1, \dots, n_i-1}} \sum_{\alpha} \frac{|\lambda_{\alpha}^{[rts]}|}{16} \alpha \hat{P}_{[rts]}^{(n_i)}. \quad (4.104)$$

Being the sum of hermitian non-negative matrices, \hat{V}_i^2 as well is diagonalizable with real non-negative eigenvalues, which we denote as $\lambda_{\beta_i}^2$, and spectral projectors P_{β_i} :

$$\hat{V}_i^2 = \sum_{\beta_i} \lambda_{\beta_i}^2 \hat{P}_{\beta_i}. \quad (4.105)$$

with $\lambda_{\beta_i} \geq 0$. Therefore we have

$$\hat{V}_i = \sum_{\beta_i} \lambda_{\beta_i} \hat{P}_{\beta_i} \quad (4.106)$$

and the volume is given by

$$\hat{V}[\mathcal{V}] = l_0^3 \sum_{i \in \{S \cap \mathcal{V}\}} \sum_{\beta_i} \lambda_{\beta_i} \hat{P}_{\beta_i}. \quad (4.107)$$

Now, the projectors acting on different vertices commute among themselves: $\hat{P}_{\beta_i} \hat{P}_{\beta_j} = \hat{P}_{\beta_j} \hat{P}_{\beta_i}$ if $i \neq j$. Therefore the eigenvectors of \hat{V} are the common eigenvectors of all \hat{V}_i . They are labeled by one β_i for every vertex i , namely by a multi-index $\vec{\beta} = (\beta_1 \dots \beta_p)$, where p is the number of vertices in the region. The corresponding spectral projectors $\hat{P}_{\vec{\beta}}$ of \hat{V} are the products over the vertices of the spectral projectors of the vertex volume operators \hat{V}_i

$$\hat{P}_{\vec{\beta}} = \prod_i \hat{P}_{\beta_i}. \quad (4.108)$$

It is immediate to conclude that

$$\hat{V} = l_0^3 \sum_{\vec{\beta}} \lambda_{\vec{\beta}} \hat{P}_{\vec{\beta}}, \quad (4.109)$$

where the eigenvalues of the volume are the sums of the eigenvalues of the volume of each intersection:

$$\lambda_{\vec{\beta}} = \sum_i \lambda_{\beta_i}. \quad (4.110)$$

The problem of the determination of the spectrum of the volume is reduced to a well defined calculation of the eigenvalues λ_{β_i} , which depend on the valence and coloring of adjacent vertices of the vertex i . Let us summarize the various steps of this computation. Given an arbitrary real vertex i with coloring of adjacent edges P_0, \dots, P_{n_i-1} : **(i)** determine the set of the possible colorings of its virtual edges, and label them by an index K_i ; **(ii)** using eq. (4.99) compute the matrix elements $\hat{W}_{[rts]K_i}^{(n_i)K_i}$; **(iii)** for each of this matrices, compute its spectral decomposition, i.e. the eigenvalues ${}^\alpha \lambda_{[rts]}^{(n_i)}$ and the spectral projectors ${}^\alpha \hat{P}_{[rts]}^{(n_i)}$; **(iv)** compute the matrix \hat{V}_i from eq. (4.104); **(v)** compute the eigenvalues of the matrix \hat{V}_i . The square root of these give the λ_{β_i} 's. All these steps can be fully performed using an algebraic manipulation program such as *Mathematica*. We have written a *Mathematica* program that performs these calculations, and we will give free access to this program on line. In Appendix F we give the values of the quantities $\lambda_{\beta_i}(P_0, \dots, P_{n_i-1})$ for some 4-valent and 5-valent vertex, computed using this program.

4.7 Comparison between ‘basis from representations’ and ‘basis from multiloops’

$SU(2)$ is the group of the unitary 2×2 complex matrices with determinant 1. We write these matrices as U_B^A where the indices A and B takes the values $A, B = 0, 1$. The fundamental representation of the group is defined by the natural action of these matrices on C^2 . The representation space is therefore the space of complex vectors with two components. These are called spinors and denoted:

$$\psi^A = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}. \quad (4.111)$$

Consider the space formed by completely symmetric spinors with n indices $\psi^{A_1 \dots A_n}$. This space transforms into itself under the action of $SU(2)$ on all the indices. Therefore, it defines a representation of $SU(2)$

$$\psi^{A_1 \dots A_n} \rightarrow U_{A'_1}^{A_1} \dots U_{A'_n}^{A_n} \psi^{A'_1 \dots A'_n}. \quad (4.112)$$

This representation is irreducible, has dimension $2j + 1$ and is called the spin- j representation of $SU(2)$, where $j = \frac{1}{2}n$. All unitary irreducible representations have this form. The antisymmetric tensor ϵ^{AB} (defined with $\epsilon^{01} = 1$) is invariant under the action of $SU(2)$

$$U_C^A U_D^B \epsilon^{CD} = \epsilon^{AB}. \quad (4.113)$$

Contracting this equation with ϵ_{AB} (defined with $\epsilon_{01} = 1$) we obtain the condition that the determinant of U is 1

$$\det U = \frac{1}{2} \epsilon_{AC} \epsilon^{BD} U_B^A U_D^C = 1 \quad (4.114)$$

since

$$\epsilon_{AB} \epsilon^{AB} = 2. \quad (4.115)$$

The inverse of an $SU(2)$ matrix can be written simply as

$$(U^{-1})_B^A = -\epsilon_{BD} U_C^D \epsilon^{CA}. \quad (4.116)$$

Most of $SU(2)$ representation theory follows directly from the invariance of ϵ_{AB} . For instance, consider the tensor product of the fundamental representation $j = 1/2$ with itself. This defines a reducible representation on the space of the two-index spinors ψ^{AB}

$$(\psi \otimes \phi)^{AB} = \psi^A \phi^B. \quad (4.117)$$

We can decompose any two-index spinor ψ^{AB} into its symmetric and its antisymmetric part

$$\psi^{AB} = \psi_0 \epsilon^{AB} + \psi_1^{AB}, \quad (4.118)$$

where

$$\psi_0 = \frac{1}{2} \epsilon_{AB} \psi^{AB} \quad (4.119)$$

and ψ_1^{AB} is symmetric. Because of the invariance of ϵ_{AB} , this decomposition is $SU(2)$ invariant. The one-dimensional invariant subspace formed by the scalars ψ_0 defines the trivial representation $j = 0$. The three-dimensional invariant subspace formed by the symmetric spinors ψ_1^{AB} defines the adjoint representation $j = 1$. Hence the tensor product of two spin-1/2 representations is the sum of a spin-0 and a spin-1 representation: $1/2 \otimes 1/2 = 0 \oplus 1$.

In general, if we tensor a representation of spin j_1 with a representation of j_2 we obtain the space of spinors with $2j_1 + 2j_2$ indices, symmetric in the first $2j_1$ and in the last $2j_2$ indices. By symmetrizing all the indices, we obtain an invariant subspace transforming in the representation $j_1 + j_2$. Alternatively, we can contract k indices of the first group with k indices of the second, using k times the tensor ϵ_{AB} , and then symmetrize the remaining $2(j_1 + j_2 - k)$ indices. This defines an invariant subspace of dimension $2(j_1 + j_2 - k) + 1$. The maximum value of k is clearly the smallest between $2j_1$ and $2j_2$. Hence, the tensor product of the representations j_1 and j_2 gives the sum of the representations $|j_1 - j_2|, |j_1 - j_2| + 1, \dots, (j_1 + j_2)$.

Thus, each irreducible j_3 appears in the product of two representations at most once and if and only if

$$j_1 + j_2 + j_3 = N \tag{4.120}$$

is integer and

$$|j_1 - j_2| \leq j_3 \leq (j_1 + j_2). \tag{4.121}$$

The two conditions are called the Clebsh-Gordon conditions. They are equivalent to the requirement that there exist three nonnegative integers a, b and c such that

$$2j_1 = a + c, \quad 2j_2 = a + b, \quad 2j_3 = b + c. \tag{4.122}$$

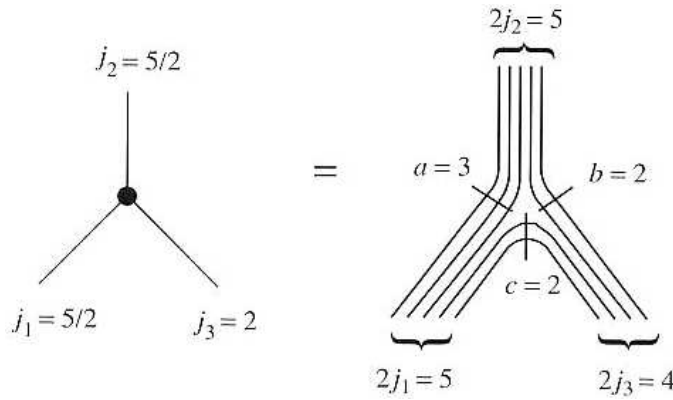


Figure 4.6: Clebsh-Gordon condition.

If we have three representations j_1, j_2, j_3 , the tensor product of the three contains the trivial representation if and only if one is in the product of the other two, namely, only if the Clebsh-Gordon conditions are satisfied. The invariant subspace of the product of the three is formed by invariant tensors with $2(j_1 + j_2 + j_3)$ indices, symmetric in the first $2j_1$, in the second $2j_2$ and in the last $2j_3$ indices. There is only one such tensor up to scaling, because it must be formed by combinations of the sole invariant tensor ϵ^{AB} . It is given by simply taking a tensors ϵ^{AB} , b tensors ϵ^{BC} , c tensors ϵ^{CD}

and symmetrizing separately the A , B and C indices. We can choose a preferred intertwiner by demanding that the intertwiner is normalized, namely multiplying it by a normalization factor. The normalized intertwiner is called the Wigner $3j$ -symbol.

There is a simple graphical interpretation to the tensor algebra of the $SU(2)$ irreducibles, suggested by the existence of the three integers a, b, c , see Figure (4.6). A representation of spin j is the symmetrized product of $2j$ fundamentals. When three representations come together, all fundamentals must be contracted among themselves. There will be a fundamentals contacted between j_1 and j_2 , and so on. Let us represent each irreducible of spin j as a line formed by $2j$ strands. An invariant tensor is a trivalent node where three such lines meet and all strands are connected across the node: a strands flow from j_1 to j_2 and so on. The meaning of the Clebsh-Gordon conditions is the readily apparent: (4.120) simply demands that the total number of strands is even, so they can pair; (4.121) demands that j_3 is neither larger than $j_1 + j_2$, because then some strands of j_3 would remain unmatched, nor smaller than $|j_1 - j_2|$, because then the largest among j_1 and j_2 would remain unmatched. Indeed, this relation between the lines and the strands reproduces precisely the relation between spin network ('basis from representations') and 'basis from multiloops'. We can represent the vertex as in figure (4.7) where the empty rectangles represent the, already seen, antisymmetrization of the strands, which reproduces the ϵ^{AB} tensors. Clearly, the antisymmetrizers can be translate from the vertex into the lines. The labels indicate the number of strands or twice the spins.

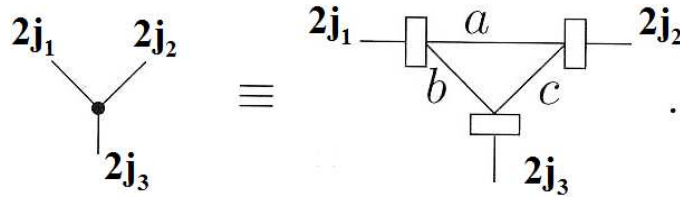


Figure 4.7: Trivalent vertex.

4.8 Black-hole thermodynamics - The value of β

The first hint that a black hole can have thermal properties came from classical GR. In 1972, Hawking proved a theorem stating that the Einstein equations imply that the area of the event horizon of a black hole cannot decrease. Shortly after, Bardeen, Carter and Hawking showed that in GR black holes obey a set of laws that strongly resembles the principles of thermodynamics; impressed by this analogy, Bekestein suggested that we should associate an entropy

$$S_{BH} = a \frac{k_B}{\hbar G_N} A \quad (4.123)$$

to a Schwarzschild black hole of surface area A . Here a is a constant of the order of unity, k_B the Boltzmann constant, and the speed of light is taken to be 1. The reason for the appearance of \hbar in this formula is essentially to get dimensions right. Bekenstein's suggestion was that the second law of thermodynamics should be extended in the presence of black holes: the total entropy that not decrease in time is the sum of the ordinary entropy with the black hole entropy S_{BH} . Bekenstein presented several physical arguments supporting this idea, but the reaction of the physics community was very cold, mainly for the following reason. The area A of a Schwarzschild black hole is related to its energy M by

$$M = \sqrt{\frac{A}{16\pi G_N^2}}. \quad (4.124)$$

If the (4.123) was correct, the standard thermodynamical relation $T^{-1} = dS/dE$ would imply the existence of a black-temperature

$$T = \frac{\hbar}{a32\pi k_B G_N M} \quad (4.125)$$

and therefore a black hole would emit thermal radiation at this temperature: a consequence difficult to believe. However, shortly after Bekenstein's suggestion, Hawking derived precisely such a black-hole thermal emission, from a completely different perspective. Using conventional methods of quantum field theory in curved spacetime, Hawking studied a quantum field in a gravitational background in which a black-hole forms (say a star collapse), and found that if the quantum field is initially in the vacuum state, after the star collapse we find it in a state that has properties of a thermal state. This can be interpreted by saying that the black-hole emits thermal radiation. Hawking computed the emission temperature to be

$$T = \frac{\hbar}{8\pi k_B G_N M}, \quad (4.126)$$

which beautifully supports Bekenstein's speculation, and fixes the constant a at

$$a = \frac{1}{4}, \quad (4.127)$$

so that (4.123) becomes

$$S_{BH} = \frac{k_B A}{4\hbar G_N}. \quad (4.128)$$

Since then, the subscript BH in S_{BH} does not mean "black-hole": it means "Bekenstein-Hawking." Hawking's theoretical discovery of black-hole emission has since been re-derived in a number of different ways, and is today generally accepted as very credible⁶.

⁶Although perhaps doubts remain about its interpretation. One can write a *pure* quantum state in which the energy distribution of the quanta is planckian. Is the state of the quantum field after the collapse truly a thermal state, or a pure state that has the energy distribution of a thermal state? Namely, are the relative phases of the different energy components truly random, or are they fixed deterministically by the initial state? Do the components of the planckian distribution form a thermal

Hawking's beautiful result raises a number of questions. First, in Hawking's derivation the quantum properties of gravity are neglected. Are these going to affect the result? Second, we understand macroscopical entropy in statistical mechanical terms as an effect of the microscopical degrees of freedom. What are the microscopical degrees of freedom responsible for the entropy (4.123)? Can we derive (4.123) from first principles? Because of the appearance of \hbar in (4.123), it is clear that the answer to these questions requires a quantum theory of gravity. The capability of answering these questions has since become a standard benchmark against which a quantum theory of gravity can be tested.

A detailed description of black-hole thermodynamics has been developed using LQG, and research is active in this direction. The major result is the derivation of (4.123) from first principles, for Schwarzschild and for other black-holes, with a well-defined calculation where no infinities appear. As far as we know, LQG is the only detailed quantum theory of gravity where this result can be achieved⁷.

As we illustrate below, the result of LQG calculations gives (4.123) with

$$a = \frac{\ln 2}{4\pi\sqrt{3}\beta}, \quad (4.129)$$

where β is the Immirzi parameter. This agrees with Hawking's value (4.127) provided that the Immirzi parameter has the value

$$\beta = \frac{\ln 2}{\pi\sqrt{3}} < 1. \quad (4.130)$$

In fact, this is the way the value of β is fixed in the theory nowadays. It is very important that $\beta < 1$: in fact we could see that a value $\beta > 1$ restricts the physical intertwiners to the Barret-Crane type. Barret-Crane intertwiners give a wrong semiclassical limit for the graviton propagator [90]. The calculation of β can be performed for different kinds of black holes, and the same value of β is found, assuring consistency. An independent way of determining β would make the result much stronger.

In what follows, we present the main ideas that underlie the derivation of this result.

4.8.1 The statistical ensemble

The degrees of freedom responsible for the entropy. Consider a black-hole with no charge and no angular momentum. Its entropy (4.123) can originate from horizon microstates, corresponding to a macrostate described by the Schwarzschild metric. Intuitively, we can think of this as an effect of fluctuations of the shape of the horizon.

One can raise an immediate objection to this idea: a black hole has "no hair", namely a black hole with no charge and no angular momentum is necessarily a spherically symmetric Schwarzschild black hole, leaving no free degrees of freedom to fluctuate.

or a quantum superposition? In the second case, the transition to a mixed state is just the normal result of the difficulty of measuring hidden correlations.

⁷So far, string theory can only deal with the highly unphysical extreme or nearly extreme black holes.

This objection, however, is not correct. It is the consequence of a common confusion about the meaning of the term “black-hole”. The confusion derives from the fact that the expression “black-hole” is used with two different meanings in the literature. In its first meaning a “black-hole” is a region of spacetime hidden beyond an horizon, such as a collapsed star. In its second meaning “black hole” is used as a synonym of ‘*stationary black hole*’. When one says that “a black hole is uniquely characterized by mass, angular momentum and charge”, one refers to *stationary* black holes, not to arbitrary black holes. In particular, a black hole with no charge and no angular momentum is *not* necessarily a Schwarzschild black hole and is *not* necessarily spherical. Its rich dynamics is illustrated, for instance, by the beautiful images of the rapidly varying shapes of the horizon obtained in numerical calculations of, say, the merging of two holes. Generally a black holes has a large number of degrees of freedom and its event horizon can take arbitrary shapes. These degrees of freedom of the horizon can be the origin of the entropy.

To be sure, in the classical theory a realistic black-hole with vanishing charge and vanishing angular momentum evolves very rapidly towards the Schwarzschild solution, by rapidly radiating away all excess energy. Its oscillations are strongly damped by the emission of gravitational radiation. But we cannot infer from this fact that the same is true in the quantum theory, or in a thermal context. In the quantum theory, the Heisenberg principle prevents the hole from converging exactly to a Schwarzschild metric, and fluctuations may remain. In fact, we will see that this is the case.

Recall that in the context of statistical mechanics, we must distinguish between the macroscopic state of a system and its microstates. Obviously the symmetry of the macrostate does not imply that the relevant microstates are symmetric. For instance, in the statistical mechanics of a sphere of gas, the individual motions of the gas molecules are certainly not confined by spherical symmetry. When the macrostate is spherically symmetric and stationary, the microstates are not necessarily spherically symmetric or stationary.

When we study the thermodynamical behavior of a Schwarzschild black-hole, it is therefore important to remember that the Schwarzschild solution is just the macrostate. Microstates can be nonstationary and non-spherically symmetric. Indeed, trying to explain black-hole thermodynamics from properties of stationary or spherically symmetric metrics alone is nonsense such as trying to derive the thermodynamics of an ideal gas in a spherical box just from spherically symmetric motions of the molecules.

Thermal fluctuations of the geometry. To make the case concrete, consider a realistic physical system containing a nonrotating and noncharged black hole as well as other physical components such as dust, gas or radiation, which we denote collectively as “matter”. We are interested in the statistical thermodynamics of such a system. Because of Einstein’s equations, at finite temperature the microscopic time-dependent inhomogeneities of the matter distribution due to its thermal motion must generate time-dependent microscopic thermal inhomogeneities in the gravitational field as well. One usually safely disregards these ripples of the geometry. For instance, we say that the geometry over the Earth’s surface is given by the Minkowski metric (or the Schwarzschild metric, due to the Earth’s gravitational field), disregarding the inhomogeneous time-dependent gravitational field generated by each individual fast-moving

air molecule. The Minkowski geometry is therefore a “macroscopic” coarse-grained average of the microscopic gravitational field surrounding us. These thermal fluctuations of the gravitational field are small and can be disregarded for most purposes, but not when we are interested in the statistical-thermodynamical properties of gravity: these fluctuations are precisely the sources of the thermal behavior of the gravitational field, as is the case for any other thermal behavior.

In a thermal context, the Schwarzschild metric represents therefore only the coarse grained description of a microscopically fluctuating geometry. Microscopically, the gravitational field is nonstationary (because it interacts with nonstationary matter) and nonspherically symmetric (because matter distribution is spherically symmetric on average only, and not on individual microstates). Its microstate, therefore, is *not* given by the Schwarzschild metric, but by some complicated time-dependent nonsymmetric metric.

Horizon fluctuations. Let us make the considerations above slightly more precise. Consider first the classical description of a system at finite temperature in which there is matter, the gravitational field and a black hole. Foliate spacetime into a family of spacelike surfaces Σ_t , labeled with a time coordinate t . The intersection h_t between the spacelike surface Σ_t and the event horizon (the boundary of the past or future null-infinity) defines the instantaneous microscopic configuration of the event horizon at coordinate time t . We loosely call h_t the surface of the hole, or the horizon. Thus, h_t is a closed $2d$ surface in Σ_t . As argued above, generally this microscopic configuration of the event horizon is not spherically symmetric. Denote by g_t the intrinsic and extrinsic geometry of the horizon h_t . Let \mathcal{M} be the space of all possible (intrinsic and extrinsic) geometries of a $2d$ surface. As t changes, the (microscopic) geometry of the horizon changes. Thus, g_t wanders in \mathcal{M} as t changes.

Since the Einstein evolution drives the black-hole towards the Schwarzschild solution, (we can choose the foliation in such way that) g_t will converge towards a point g_A of \mathcal{M} representing a sphere of a given radius A . However, as mentioned before, exact convergence may be forbidden by quantum theory, and quantum effects may keep g_t oscillating in a finite region around g_A .

Which microstates are responsible for S_{BH} ? Let us assume that (4.123) represent a true thermodynamical entropy associated with the black hole. That is, let us assume that heat exchanges between the hole and the exterior are governed by S_{BH} . Where are the microscopical degrees of freedom responsible for this entropy located? The microstates that are relevant for the entropy are only the ones that can affect energy exchanges with the exterior. That is, only the ones that can be distinguished from the exterior. If we have a system containing a perfectly isolated box, the internal states of the box do not contribute to the entropy of the system, as far as the heat exchange of the system with the exterior is concerned. The state of matter and gravity inside a black hole has no effect on the exterior. Therefore the states of the interior of the black hole are irrelevant for S_{BH} . To put it vividly, the black-hole interior may be in one out of an infinite number of states indistinguishable from the outside. For instance, the black-hole interior may, in principle, be given by an infinite Kruskal spacetime: on the other side of the hole there may be billions of galaxies that do not affect the side

detectable by us. The potentially infinite number of internal states does not affect the interaction of the hole with its surroundings and is irrelevant here, because it cannot affect the energetic exchanges between the hole and its exterior which are the ones that determine the entropy. We are only interested in configurations of the hole that have distinct effects on the exterior of the hole.

Observed from outside, the hole is completely determined by the geometric properties of its surface. Therefore, the entropy (relevant for the thermodynamical interaction of the hole with its surroundings) is entirely determined by the geometry of the black-hole surface, namely by g_t .

The statistical ensemble. We have to determine the ensemble of the microstates g_t over which the hole may fluctuate. In conventional statistical thermodynamics, the statistical ensemble is the region of phase space over which the system could wander if it were isolated, namely if it did not exchange energy with its surroundings. Can we translate this condition to the case of a black hole? The answer is yes, because we know that in GR energy exchanges of the black hole are accompanied by a change in its area. Therefore, we must define the statistical ensemble as the ensemble of g_t with a given value A of the area.

To support the choice of this ensemble consider the following⁸. The ensemble must contain reversible paths only. In the classical theory, reversible paths conserve the area, because of the Hawking's theorem. Quantum theory does not change this, because it allows area decrease only by emitting energy (Hawking radiance), namely violating the (counterfactual) assumption that defines the statistical ensemble: that the system does not exchange energy.

We can conclude that the entropy of a black-hole is given by the number of $N(A)$ of states of the geometry g_t of a $2d$ surface h_t of area A . The quantity $S(A) = k_B \ln N(A)$ is the entropy we should associate with the horizon in order to describe its thermal interactions with its surroundings.

Quantum theory. This number $N(A)$ is obviously infinite in the classical theory. But not in the quantum theory. The situation is similar to the case of the entropy of the electromagnetic field in a cavity, which is infinite classically and finite in quantum theory. To compute it, we have to count the number of (orthogonal) quantum states of the geometry of a two-dimensional surface, with total area A . The problem is now well defined, and can be translated into a direct computation.

Two objections. We have concluded that the entropy of a black-hole is determined by the number of the possible states of a $2d$ surface with area A . The reader may wonder if something has got lost in the argument: does this imply that *any* surface has an associated entropy, just because it has an area? Where has the information about the fact that this is a black-hole gone? These objections have often been raised to the argument above. Here is the answer.

The first objection can be answered as follows. Given any arbitrary surface, we can

⁸In this context, it perhaps worthwhile recalling that difficulties to rigorously justifying a priori the choice of the ensemble plague conventional thermodynamics anyway.

of course ask the mathematical question of how many states exist that have a given area. But there is no reason, generally, to say that there is an entropy associated with the surface. In a general situation, energy, or, more generally, information, can flow across a surface. The surface may emit heat without changing its geometry. Therefore, in general, the geometry of the surface and the number of its states have nothing to do with heat exchange or with entropy. But, in the special case of a black-hole the horizon screens us from the interior and any heat exchange that we can have with the hole must be entirely determined by the geometry of the surface. It is only in this case that the counting is meaningful, because it is only in this case that the number of states of a geometry of a given area corresponds precisely to the number of states of a region which are distinguishable from the exterior. To put it more precisely, the future evolution of the surface of a black-hole is completely determined by its geometry and by the exterior; this is not true for an arbitrary surface. It is because of these special properties of the horizon that the number of states of its geometry determine an entropy.

You can find out how much money you own by summing up the numbers written on your bank account. This does not imply that if you sum up the number written on an arbitrary piece of paper you get the amount of money you own. The calculation may be the same, but an arbitrary piece of paper is not a bank account, and only for a bank account does the result of the calculation have that meaning. Similarly, you can make the same calculation for any surface, but only for a black-hole, because of its special properties, is the result of the calculation an entropy.

The second objection concerns the role of the Einstein equations that is, the role of the dynamics. This objection has been raised often, but us and our collaborators have never understood it. The role of the Einstein equations is precisely the usual role that the dynamical equations always play in statistical mechanics. Generally, the only role of the dynamics is that of defining the energy of the system, which is the quantity which is conserved if the system is isolated, and exchanged when heat is exchanged. The statistical ensemble is then determined by the value of the energy. In the case of a black-hole, it is the Einstein equations that determine the fact that the *area* governs heat exchange with the interior of the hole. If it wasn't for the specific dynamics of general relativity, the area would not increase for an energy inflow or decrease for energy loss. Thus, it is the Einstein equations that determine the statistical ensemble.

4.8.2 Derivation of the Bekenstein-Hawking entropy

Above we have found, on physical grounds, what the entropy of a black-hole should be. It is given by

$$S_{BH} = k_B \ln N(A) \quad (4.131)$$

where $N(A)$ is the number of states that the geometry of a surface with area A can assume. It is now time to compute it.

Let the quantum state of the geometry of an equal-time spacelike $3d$ Σ_t be given by a state $|s\rangle$ determined by an s -knot s . The horizon is a $2d$ surface \mathcal{S} immersed in Σ_t . Its geometry is determined by its intersection with the s -knot s .

Intersections can be of three types: (a) an edge crosses the surface; (b) a vertex lies on the surface; (c) a finite part of the s -knot lies on the surface. Here we are interested in the geometry as seen from the exterior of the surface, therefore the geometry we consider is, more precisely, the limit of the geometry of a surface surrounding \mathcal{S} , as this approaches \mathcal{S} . This limit cannot detect intersections of the type (b) and (c), and we therefore disregard such intersections.

Let $i = 1, \dots, n$ label the intersections of the s -knot with the horizon \mathcal{S} . Let j_1, \dots, j_n be the spins of the links intersecting the surface. The area of the horizon is

$$A = 8\pi\beta\hbar G_N \sum_i \sqrt{j_i(j_i + 1)}. \quad (4.132)$$

The s -knot is cut into two parts by the horizon \mathcal{S} . Call s_{ext} the external part. The s -knot s_{ext} has n open ends, that end on the horizon. From the point of view of an external observer, a possible geometry of the surface is a possible way of “ending” the s -knot. A possible “end” of a link with spin j is simply a vector in the representation space \mathcal{H}_j . Therefore, a possible end of the external s -knot is a vector in $\otimes_i \mathcal{H}_{j_i}$. Thus, seen from the exterior, the degrees of freedom of the hole appear as a vector in this space. In this limit in which the area is large, any further constraint on these vectors becomes irrelevant. The possible states are obtained by considering all sets of j_i that give the area A and, for each set, the dimension of $\otimes_i \mathcal{H}_{j_i}$. It is not difficult to see that the number of possible states is dominated by the case $j_i = 1/2$. In this case, the area of a single link is

$$A_0 = 4\pi\beta\hbar G_N \sqrt{3}. \quad (4.133)$$

Hence, there are

$$n = \frac{A}{A_0} = \frac{A}{4\pi\beta\hbar G_N \sqrt{3}} \quad (4.134)$$

intersections, and the dimension of $\mathcal{H}_{1/2}$ is 2; so the number of states of the black-hole is

$$N = 2^n = 2^{A/4\pi\beta\hbar G_N \sqrt{3}}, \quad (4.135)$$

and the entropy is

$$S_{BH} = k_B \ln N = \frac{1}{\beta} \frac{\ln 2}{4\pi\sqrt{3}} \frac{k_B}{\hbar G_N} A. \quad (4.136)$$

This is the Bekenstein-Hawking entropy (4.123). The numerical factor agrees with the Hawking value (4.127), and we get (4.128), if the Immirzi parameter is fixed at the value

$$\beta = \frac{\ln 2}{\pi\sqrt{3}}. \quad (4.137)$$

One remark before concluding. The reader may object to the derivation above as follow: the states that we have counted are transformed into each other by a gauge

transformation. Why, then, do I consider them distinct in the entropy counting? The answer to this objection is the following. When we break the system into components, gauge degrees of freedom may become physical degrees of freedom on the boundary. The reason is that if we let the gauge group act independently on the two components, it will act *twice* on the boundary. A holonomy of a connection across the boundary, for instance, will become ill defined. Therefore, there are degrees of freedom on the boundary that are not gauge; they tie the two sides to each other, so to say.

To illustrate this point, let us consider two sets A and B , and a group G that acts (freely) on A and on B . Then G acts on $A \times B$. What is the space $(A \times B)/G$? One might be tempted to say that is (isomorphic to) $A/G \times B/G$, but a moment of reflection shows that this is not correct and the correct answer is

$$\frac{A \times B}{G} \sim \frac{A}{G} \times B. \quad (4.138)$$

(If G does not act freely over A , we have to divide B by the stability groups of the elements of A .) Now, imagine that A is the space of the states of the exterior of the black-hole, B the space of the states of the black-hole, and G the gauge group of the theory. Then we see that we must not divide B by the gauge group of the surface, but only by those internal gauges and diffeomorphisms that leave the rest of the spin network invariant⁹.

⁹Here only the boundary degrees of freedom due to internal gauge invariance are taken into account. Perhaps by taking also the boundary degrees of freedom due to diff invariance, into account, (4.137) could change.

Chapter 5

Dynamics

5.1 Hamiltonian operator

For the moment we consider the case $\beta = \sqrt{s}$ so that the piece proportional to $(\beta^2 - s)$ disappears from the Hamiltonian. The form of the Hamiltonian H which is most convenient for quantum theory is the following. Consider the Poisson bracket between the volume:

$$V = \int d^3x \sqrt{\det E(x)}$$

and the connection:

$$\{V, A_a^i(x)\} = (8\pi\gamma G) \frac{3E_j^b(x)E_k^c(x)\epsilon_{abc}\epsilon^{ijk}}{2\det E(x)}.$$

Using this, the Hamiltonian constraint become:

$$H[N] = \int N \text{tr} (F \wedge \{V, A\}) = 0. \quad (5.1)$$

In this form, all the operator can be defined as limits of holonomy operators of small paths, while the classical Poisson bracket can readily realized in the quantum theory as a quantum commutator. Fix a point x and a tangent vector u at x ; consider a path $\gamma_{x,u}$ of coordinate length ϵ that starts at x tangent to u . Then the holonomy can be expanded as:

$$U(A, \gamma_{x,u}) = 1 + \epsilon u^a A_a(x) + O(\epsilon^2) \quad (5.2)$$

Similarly, fix a point x and two tangent vectors u and v at x , and consider a small triangular loop $\alpha_{x,uv}$ with one vertex at x and two sides tangent to u and v at x , each of length ϵ . Then

$$U(A, \alpha_{x,uv}) = 1 + \frac{1}{2}\epsilon^2 u^a v^b F_{ab}(x) + O(\epsilon^3). \quad (5.3)$$

Using this, we can regularize the expression of the hamiltonian by writing it as

$$H = -\lim_{\epsilon \rightarrow 0} \frac{i}{\epsilon^3} \int N \epsilon^{ijk} \left((U_{\gamma^{-1}, u_k})^A_B (U_{\alpha, u_i u_j})^B_C [V, (U_{\gamma, u_k})^C_A] \right) d^3 x \quad (5.4)$$

Here (u_1, u_2, u_3) are any three tangent vectors at x whose triple product is equal to unity. The path drawn by the three holonomies has the shape in figure (5.1):

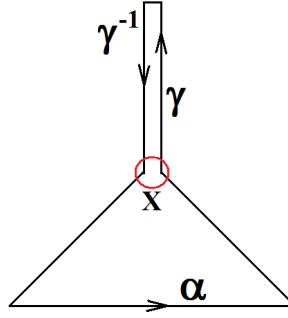


Figure 5.1: The shape of the path drawn by the three holonomies in the Hamiltonian operator.

Let us partition the 3d coordinate space in small regions R_m of coordinate volume ϵ^3 . We can then write the integral as a Riemann sum and write

$$H = -i \lim_{\epsilon \rightarrow 0} \sum_m N_m \epsilon^{ijk} \left((U_{\gamma^{-1}, u_k})^A_B (U_{\alpha, u_i u_j})^B_C [V(R_m), (U_{\gamma, u_k})^C_A] \right), \quad (5.5)$$

Now the paths start in a point x_m in the corresponding region. The limit is independent from the choice of this point for the Riemann theorem. Notice that the ϵ^3 factors cancel, and can therefore be dropped. When acting on a spin network state, this operator acts only on the nodes, because of the presence of the volume. (This is not changed by the presence of the term U_γ in the commutator for the following reason. The volume operator vanishes on trivalent nodes. The operator U_γ can at most increase the valence of a node by one. Therefore, there must be at least a trivalent node in the state for H not to vanish.) Therefore in the sum (5.5) only the regions R_m in which there is a node n give a nonvanishing contribution. Call R_n a region in which only the node n of a spin network S is located. Then

$$H|S\rangle = \lim_{\epsilon \rightarrow 0} H_\epsilon|S\rangle, \quad (5.6)$$

where

$$H_\epsilon|S\rangle = -i \sum_{n \in S} N_n \epsilon^{ijk} \left((U_{\gamma^{-1}, u_k})^A_B (U_{\alpha, u_i u_j})^B_C [V(R_n), (U_{\gamma, u_k})^C_A] \right) |S\rangle. \quad (5.7)$$

The sum is now on the nodes. The only possibility to have a nontrivial commutator is if the path γ itself touches the node. We therefore demand this. This can be obtained by requiring that x_n is precisely the location of the node. Finally there is a natural choice of the three vectors (u_1, u_2, u_3) and for the paths γ and α ; take (u_1, u_2, u_3) tangent to three links l, l', l'' emerging from the node n . (The condition that their triple product is unity can be satisfied by adjusting the length.) Take γ to be a path of coordinate length ϵ along the link l . Take α to be the triangle formed by two side of coordinate length ϵ along the other two links l' and l'' , and take the third side as a straight line (in the coordinate x) connecting the two end points. The sum over i, j, k is a sum over all permutation of the three links. If the node has valence higher than three, that is, if there are more than three links at the node n , we preserve covariance summing over all ordered triplets of distinct links. Thus we pose

$$H_\epsilon |S\rangle = -i \sum_{n \in S} N_n \sum_{\{l, l', l''\}} \epsilon^{ll'l''} \left((U_{\gamma^{-1}, l})^A_B (U_{\alpha, l'l''})^B_C [V(R_n), (U_{\gamma, l})^C_A] \right) |S\rangle. \quad (5.8)$$

5.1.1 Finiteness

In general the limit (5.6) does not exist, but it exists on a subclass of states: diffeomorphism-invariant states! As these are the physical states, this is what we need and it is sufficient to define the theory. Here is where the intimate interplay between diffeomorphism invariance and quantum field theoretical short-scale behavior begins to shine. To compute H on diff-invariant states, recall these are in the dual space S' . So far, we have only considered H on spin network state, or, by linearity, on S . The action of H on S' is immediately defined by duality

$$(H\Phi)(\psi) \equiv (\Phi)(H\psi). \quad (5.9)$$

The key point is that we want to consider the regularized operator on S' and take the limit there:

$$(H\Phi)(|S\rangle) = \lim_{\epsilon \rightarrow 0} \Phi(H_\epsilon |S\rangle). \quad (5.10)$$

Notice that the limit is now a limit of a sequence of numbers (not a limit of a sequence of Hilbert space vectors). We now show that the limit exists if $\Phi \in \mathbf{K}_{diff}$, namely if Φ is a diffeomorphism invariant state. The key to see this is the following crucial observation. Given a spin network S , the operator in the parentheses modifies the state $|S\rangle$ in two ways: by changing its graph Γ as well as its coloring. The volume operator does not change the graph. The graph is modified by the two operators U_γ and U_α . The first superimpose a path of length ϵ to the link l of Γ . The second superimpose a triangle with two side of length proportional to ϵ along the links l' and l'' of Γ , and a third side that is not on Γ . The fundamental observation is that for ϵ sufficiently small, changing ϵ in the operator changes the resulting states but not its diffeomorphism class: adding a smaller triangle is the same as adding a larger triangle

and then reducing it with a diffeomorphism. Φ is invariant under diffeomorphism and therefore the dependence on ϵ of the argument of the limit drops out. Therefore:

$$(H\Phi)(|S\rangle) = \lim_{\epsilon \rightarrow 0} \Phi(H_\epsilon |S\rangle) = \Phi(H|S\rangle), \quad (5.11)$$

where

$$H|S\rangle = -i \sum_{n \in S} N_n \sum_{\{l, l', l''\}} \epsilon^{ll''} \left((U_{\gamma^{-1}, l})^A_B (U_{\alpha, l'l''})^B_C [V(R_n), (U_{\gamma, l})^C_A] \right) |S\rangle. \quad (5.12)$$

and the size ϵ of the regularizing paths is taken to be small enough so that the added arc does not run over other nodes or links of S . The finiteness of the limit is the immediate.

5.1.2 Matrix elements

The resulting action of H on s -knot states is simple to derive and to illustrate. (i) The action gives a sum of terms, one for each node n of the state. (ii) for each node, H gives a further sum of terms, one for each triplet of links arriving at the node, and, for every triplet, one term for every permutation of the three links l, l', l'' . Each of these terms act as follow on the s -knot state (see figure 5.2). (iii) It creates two new nodes n' and n'' at a finite distance from n along the links l' and l'' . The exact location of these nodes is irrelevant for the s -knot states. (iv) It creates a new link of spin-1/2 connecting n' and n'' . (v) It changes the coloring j' of the link connecting n and n' and the coloring j'' of the link connecting n and n'' . These turn out to be the colors of the links l' and l'' increased or decreased by 1/2. (vi) It changes the intertwiner at the node n ; the new intertwiner is between the representations corresponding to new colorings of the adjacent links.

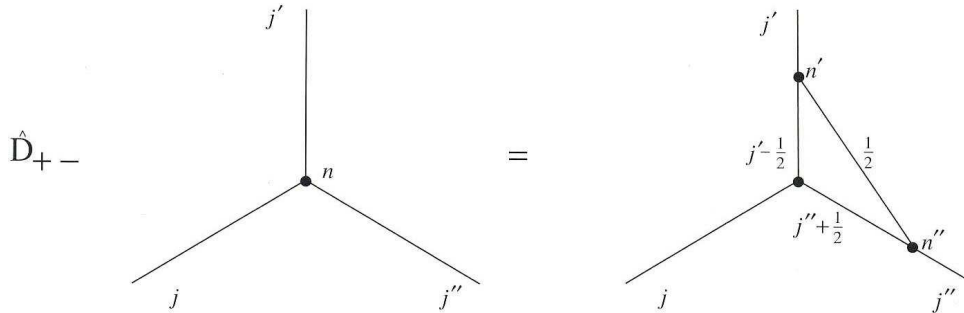


Figure 5.2: Action of $D_{n,l',l'',r,\epsilon,\epsilon'}$.

Call $D_{n,l',l'',r,\pm,\pm}$ an operator that acts around the node n by acting as described in (iii), (iv), (v). This is illustrated in figure (5.2). Then

$$H|S\rangle = \sum_{n \in S} N_n \sum_{\{l, l', l''\}, r} \sum_{\epsilon', \epsilon'' = \pm} H_{n, l', l'', \epsilon', \epsilon''} D_{n, l', l'', r, \epsilon', \epsilon''} |S\rangle. \quad (5.13)$$

The operator $H_{n, l', l'', \epsilon', \epsilon''}$ acts as a finite matrix on the space of the intertwiners at the node n . Recalling the definition of diffeomorphisms-invariant state, we can write the matrix elements of H among spin network states

$$\langle S' | H | S \rangle = \sum_{n \in S} N_n \sum_{|\psi\rangle = U_\phi | S' \rangle} \sum_{\{l, l', l''\}, r} \sum_{\epsilon', \epsilon'' = \pm} \langle \psi | H_{n, l', l'', \epsilon', \epsilon''} D_{n, l', l'', r, \epsilon', \epsilon''} | S \rangle. \quad (5.14)$$

The operator H is not symmetric. Professor C.Rovelli said that this is evident from the fact that it adds arcs but does not remove them¹. Its adjoint H^\dagger can be defined simply by the complex conjugate of the transpose of its matrix elements

$$\langle S' | H^\dagger | S \rangle = \overline{\langle S | H | S' \rangle}, \quad (5.15)$$

and a symmetric operator is defined by

$$H_s = \frac{1}{2}(H + H^\dagger). \quad (5.16)$$

This is an operator that adds as well as removes arcs. It is reasonable to expect that this operator be better behaved for the classical limit. Therefore we take this operator as the basic operator defining the theory.

The explicit computation of the matrix elements $H_{n, l', l'', \epsilon', \epsilon''}$ is a very hard (but well posed) problem in $SU(2)$ representation theory. It is discussed in detail in [18], where its matrix elements are explicitly given for simple nodes. For the same nodes, in [18] the values of H_s are given. The complexity of the calculation of $H_{n, l', l'', \epsilon', \epsilon''}$ for most nodes, has led researchers to a new questionable formulation of LQG, which will be exposed in the next chapters.

Despite the major simplicity of the calculations, this new road doesn't reproduce the beautiful results of the first version of LQG in cosmology, as the prediction of the inflation [93] and the removal of black-holes [92] and big bang singularities [94]. Maybe it will be necessary to step back and improve our computing ability.

5.2 The case $\beta \neq \sqrt{s}$

The case $\beta \neq \sqrt{s}$ can be obtained considering that

$$K_a^i(x) = \frac{\delta K}{\delta E_i^a(x)} = \{K, A_a^i(x)\} / \kappa \quad (5.17)$$

$$K = \{H_{\beta=\sqrt{s}, N \equiv 1}, V(R)\} \quad (5.18)$$

¹In our opinion this isn't a good explanation. This is because the added arcs can be compressed ($\epsilon \rightarrow 0$) until the effect of H will be only the substitution of the intertwiners. They are fictitious arcs, because in a small enough open set ($\epsilon \rightarrow 0$), all the spaces are flat and so the holonomies are trivial.

with

$$K := \int_R d^3x K_a^i E_i^a \quad (5.19)$$

and R some small region. In this way we can express $H_{\beta \neq \sqrt{s}}$ in terms of $H_{\beta = \sqrt{s}}$. Using the usual regularization in terms of infinitesimal holonomies we obtain

$$H_{\beta \neq \sqrt{s}} = H_{\beta = \sqrt{s}} + i \frac{8}{3\kappa^4} \sum_n \epsilon^{ijk} N_n \text{tr} (U_{u_i} [U_{u_i}^{-1}, K] U_{u_j} [U_{u_j}^{-1}, K] U_{u_k} [U_{u_k}^{-1}, V(R_n)]) \quad (5.20)$$

with the usual meaning for $u_i, u_j, u_k, V, N_n, R_n$. See [11].

Chapter 6

An Alternative way: BF theory and Spin-Foam

6.1 *BF* Theory: Classical Field Equations

To set up *BF* theory, we take as our gauge group any Lie group G whose Lie algebra \mathfrak{g} is equipped with an invariant nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. We take as our spacetime any n -dimensional oriented smooth manifold M , and choose a principal G -bundle¹ P over M . The basic fields in the theory are then:

- a connection A on P ,
- an $\text{ad}(P)$ -valued $(n - 2)$ -form B on M .

Here $\text{ad}(P)$ is the vector bundle² associated to P via the adjoint action of G on its Lie algebra. The curvature of A is an $\text{ad}(P)$ -valued 2-form F on M . If we pick a local

¹A principal G bundle is a fibre bundle where typical fibre and structure group coincide with G . On a principal fibre bundle we may define a right action $\rho : G \times P \rightarrow P$; $\rho_h(p) := \phi_I(\pi(p), h_I(p)h)$ for $p \in \pi^{-1}(U_I)$ where $h_I : P \rightarrow G$ is uniquely defined by $(x_I(p), h_I(p)) := \phi_I^{-1}(p)$. Since G acts transitively on itself from the right, this right action is obviously transitive in every fibre and fibre preserving. $s_I^\phi(x) := \phi_I(x, 1_G)$ is called the canonical local section. Conversely, given a system of local sections s_I one can construct local trivializations $\phi_I^\phi(x, h) := \rho_h(s_I(x))$, called canonical local trivializations.

Notice the identity $p = \rho_{h_I(p)}(s_I^\phi(\pi(p))) = \phi_I(\pi(p), h_I(p)) = \phi_{I\pi(p)}(h_I(p))$ for any $p \in \pi^{-1}(U_I)$. If $U_I \cap U_J \neq \emptyset$ and $p \in \pi^{-1}(U_I \cap U_J)$ this leads to $\rho_{h_I(p)}(s_I^\phi(\pi(p))) = \rho_{h_J(p)}^\phi(s_J^\phi(\pi(p)))$. Using the fact that ρ is a right action we conclude $s_J^\phi(\pi(p)) = \rho_{h_I(p)h_J(p)^{-1}}(s_I^\phi(\pi(p)))$. Since the left hand side does not depend any longer on the point p in the fibre above $x = \pi(p)$ we conclude that we have a G -valued functions $h_{IJ} : U_I \cap U_J \rightarrow G$, $x \mapsto [h_J(p)^{-1}h_I(p)]_{p \in \pi^{-1}(x)}$ where the right hand side is independent of the point in the fibre. The functions h_{IJ} are actually the structure functions of P : By definition we have $\phi_{Ix}(h_I(p)) = \phi_{Jx}(h_J(p))$, thus $h_I(p) = (\phi_{Ix}^{-1} \circ \phi_{Jx})(h_J(p)) = \lambda_{h_{IJ}(x)}(h_J(p)) = h_{IJ}(x)h_J(p)$ which also shows that the left action in P reduces to left translation in the fibre coordinate.

In a principal G bundle it is easy to see, using transitivity of the right action of G , that triviality is equivalent with the existence of a global section. This is not the case for vector bundles which always have the global section $s_I(x) = \phi_I(x, 0)$ but may have non-trivial transition functions.

²A vector bundle E is a fibre bundle whose typical fibre F is a vector space. The vector bundle associated with a principal G bundle P (where G is the structure group of E) under the left

trivialization we can think of A as a g -valued 1-form on M , F as a g -valued 2-form, and B as a g -valued $(n-2)$ -form.

The Lagrangian for BF theory is:

$$\mathcal{L} = \text{tr}(B \wedge F).$$

Here $\text{tr}(B \wedge F)$ is the n -form constructed by taking the wedge product of the differential form parts of B and F and using the bilinear form $\langle \cdot, \cdot \rangle$ to pair their g -valued parts. The notation ‘tr’ refers to the fact that when G is semisimple we can take this bilinear form to be the Killing form $\langle x, y \rangle = \text{tr}(xy)$, where the trace is taken in the adjoint representation.

We obtain the field equations by setting the variation of the action to zero:

$$0 = \delta \int_M \mathcal{L} \tag{6.1}$$

$$= \int_M \text{tr}(\delta B \wedge F + B \wedge \delta F) \tag{6.2}$$

$$= \int_M \text{tr}(\delta B \wedge F + B \wedge d_A \delta A) \tag{6.3}$$

$$= \int_M \text{tr}(\delta B \wedge F + (-1)^{n-1} d_A B \wedge \delta A) \tag{6.4}$$

where d_A stands for the exterior covariant derivative. Here in the second step we used the identity $\delta F = d_A \delta A$, while in the final step we did an integration by parts. We see that the variation of the action vanishes for all δB and δA if and only if the following field equations hold:

$$F = 0, \quad d_A B = 0.$$

These equations are rather dull. But this is exactly what we want, since it suggests that BF theory is a topological field theory! In fact, all solutions of these equations look the same locally, so BF theory describes a world with no local degrees of freedom. To see this, first note that the equation $F = 0$ says the connection A is flat. Indeed, all flat connections are locally the same up to gauge transformations. The equation $d_A B = 0$ is a bit subtler. It is not true that all solutions of this are locally the same up to a gauge transformation in the usual sense. However, BF theory has another sort of symmetry. Suppose we define a transformation of the A and B fields by

$$A \mapsto A, \quad B \mapsto B + d_A \eta$$

representation τ of G on F , denoted $E = P \times_\tau F$, is given by the set of equivalence classes $[(p, f)] = \{(\rho_h(p), \tau(h^{-1})f); h \in G\}$ for $(p, f) \in P \times F$. The projection is given by $\pi_E([(p, f)]) := \pi(p)$ and local trivializations are given by $\psi(x, f) = [(s_I(x), f)]$ since $[(\rho_h(s_I(x)), f)] = [(s_I(x), \tau(h)f)] = [(s_I(x), f)]$. Transition functions result from $u = [s_J(\pi(u)), f_J(u)] = [\rho_{h_{IJ}(\pi(u))}(s_I(\pi(u))), f_J(u)] = [(s_I(\pi(u)), \tau(h_{IJ}(\pi(u)))f_J(u))] = [(s_I(\pi(u)), f_I(u))]$ and are thus given by $\tau(\rho_{IJ}(x))$.

Conversely, given any vector bundle E we can construct a principal G bundle P such that E is associated with it by going through the above mentioned reconstruction process and by using the same structure group (with τ as the defining representation) acting on the fibre G by left translations and the same transition functions. A vector bundle is then called trivial if its associated principal fibre bundle is trivial.

for some $\text{ad}(P)$ -valued $(n-3)$ -form η . This transformation leaves the action unchanged:

$$\int_M \text{tr}((B + d_A \eta) \wedge F) = \int_M \text{tr}(B \wedge F + d_A \eta \wedge F) \quad (6.5)$$

$$= \int_M \text{tr}(B \wedge F + (-1)^n \eta \wedge d_A F) \quad (6.6)$$

$$= \int_M \text{tr}(B \wedge F) \quad (6.7)$$

where we used integration by parts and the Bianchi identity $d_A F = 0$. In the next section we shall see that this transformation is a ‘gauge symmetry’ of BF theory, in the more general sense of the term, meaning that two solutions differing by this transformation should be counted as physically equivalent. Moreover, when A is flat, any B field with $d_A B = 0$ can be written locally as $d_A \eta$ for some η ; this is an easy consequence of the fact that locally all closed forms are exact. Thus locally, all solutions of the BF theory field equations are equal modulo gauge transformations and transformations of the above sort.

Why is general relativity in 3 dimensions a special case of BF theory? To see this, take $n = 3$, let $G = \text{SO}(2, 1)$, and let $\langle \cdot, \cdot \rangle$ be minus the Killing form. Suppose first that $B: TM \rightarrow \text{ad}(P)$ is one-to-one. Then we can use it to define a Lorentzian metric on M as follows:

$$g(v, w) = \langle Bv, Bw \rangle$$

for any tangent vectors $v, w \in T_x M$. We can also use B to pull back the connection A to a metric-preserving connection Γ on the tangent bundle of M . The equation $d_A B = 0$ then says precisely that Γ is torsion-free, so that Γ is the Levi-Civita connection on M . Similarly, the equation $F = 0$ implies that Γ is flat. Thus the metric g is flat.

In 3 dimensional spacetime, the vacuum Einstein equations simply say that the metric is flat. Of course, many different A and B fields correspond to the same metric, but they all differ by gauge transformations. So in 3 dimensions, BF theory with gauge group $\text{SO}(2, 1)$ is really just an alternate formulation of Lorentzian general relativity without matter fields — at least when B is one-to-one. When B is not one-to-one, the metric g defined above will be degenerate, but the field equations of BF theory still make perfect sense. Thus 3d BF theory with gauge group $\text{SO}(2, 1)$ may be thought of as an extension of the vacuum Einstein equations to the case of degenerate metrics.

If instead we take $G = \text{SO}(3)$, all these remarks still hold except that the metric g is Riemannian rather than Lorentzian when B is one-to-one. We call this theory ‘Riemannian general relativity’. We study this theory extensively in what follows, because it is easier to quantize than 3-dimensional Lorentzian general relativity. However, it is really just a warmup exercise for the Lorentzian case — which in turn is a warmup for 4-dimensional Lorentzian quantum gravity.

We conclude with a word about double covers. We can also express general relativity in 3 dimensions as a BF theory by taking the double cover $\text{Spin}(2, 1) \cong \text{SL}(2, R)$ or $\text{Spin}(3) \cong \text{SU}(2)$ as gauge group and letting P be the spin bundle³. This does

³A spin bundle S is a fibre bundle whose typical fibre F is a spinor space. Concepts similar to vectorial case apply.

not affect the classical theory. As we shall see, it does affect the quantum theory. Nonetheless, it is very popular to take these groups as gauge groups in 3-dimensional quantum gravity. The question whether it is ‘correct’ to use these double covers as gauge groups seems to have no answer — until we couple quantum gravity to spinors, at which point the double cover is necessary.

6.2 Classical Phase Space

To determine the classical phase space of BF theory we assume spacetime has the form

$$M = R \times \sigma$$

where the real line R represents time and σ is an oriented smooth $(n-1)$ -dimensional manifold representing space. This is no real loss of generality, since any oriented hypersurface in any oriented n -dimensional manifold has a neighborhood of this form. We can thus use the results of canonical quantization to study the dynamics of BF theory on quite general spacetimes.

If we work in temporal gauge, where the time component of the connection A vanishes, we see the momentum canonically conjugate to A is

$$\frac{\partial \mathcal{L}}{\partial \dot{A}} = B.$$

This is reminiscent of the situation in electromagnetism, where the electric field is canonically conjugate to the vector potential. To understand the physical meaning of the theory, it is useful to think of this field as analogous to the electric field. Of course, the analogy is best when $G = U(1)$.

Let $P|_\sigma$ be the restriction of the bundle P to the ‘time-zero’ slice $\{0\} \times \sigma$, which we identify with σ . Before we take into account the constraints imposed by the field equations, the configuration space of BF theory is the space \mathcal{A} of connections on $P|_\sigma$. The corresponding classical phase space, which we call the ‘kinematical phase space’, is the cotangent bundle $T^*\mathcal{A}$. A point in this phase space consists of a connection A on $P|_\sigma$ and an $\text{ad}(P|_\sigma)$ -valued $(n-2)$ -form B on σ . The symplectic structure on this phase space is given by

$$\omega((\delta A, \delta B), (\delta A', \delta B')) = \int_S \text{tr}(\delta A \wedge \delta B' - \delta A' \wedge \delta B).$$

This reflects the fact that A and B are canonically conjugate variables. However, the field equations of BF theory put constraints on the initial data A and B :

$$F = 0, \quad d_A B = 0$$

where F is the curvature of the connection $A \in \mathcal{A}$, analogous to the magnetic field in electromagnetism. To deal with these constraints, we should apply symplectic reduction to $T^*\mathcal{A}$ to obtain the physical phase space.

The constraint $d_A B = 0$, called the Gauss law, is analogous to the equation in vacuum electromagnetism saying that the divergence of the electric field vanishes.

This constraint generates the action of gauge transformations on $T^*\mathcal{A}$. To see this, consider a scalar function f ; the condition $d_A B = 0$ implies

$$\begin{aligned}
\int_M d_A f \wedge d_A B &= 0 \\
&= \int_M d_A f \wedge * \left(\frac{\delta S[A]}{\delta A} \right) \\
&= \int_M d_A f \left(\frac{\delta S[A]}{\delta A} \right) d^4 x \\
&= \int_M \delta_f A \left(\frac{\delta S[A]}{\delta A} \right) d^4 x \\
&= \delta_f S[A] = 0
\end{aligned} \tag{6.8}$$

Doing symplectic reduction with respect to this constraint, we thus obtain the ‘gauge-invariant phase space’ $T^*(\mathcal{A}/G)$, where G is the group of gauge transformations of the bundle $P|_\sigma$.

The constraint $F = 0$ is analogous to an equation requiring the magnetic field to vanish. Of course, no such equation exists in electromagnetism; this constraint is special to BF theory. It generates transformations of the form

$$A \mapsto A, \quad B \mapsto B + d_A \eta,$$

which include diffeomorphisms. These transformations, discussed in the previous section, really are gauge symmetries as claimed. To see this, consider a g -valued $(n-3)$ -form η ; the condition $F = d_A A = 0$ implies

$$\begin{aligned}
\int_M F \wedge d_A \eta &= 0 \\
&= \int_M * \left(\frac{\delta S[B]}{\delta B} \right) \wedge d_A \eta \\
&= \int_M d_A \eta \left(\frac{\delta S[B]}{\delta B} \right) d^4 x \\
&= \int_M \delta_\eta B \left(\frac{\delta S[B]}{\delta B} \right) d^4 x \\
&= \delta_\eta S = 0
\end{aligned} \tag{6.9}$$

Doing symplectic reduction with respect to this constraint, we obtain the ‘physical phase space’ $T^*(\mathcal{A}_0/G)$, where \mathcal{A}_0 is the space of flat connections on $P|_\sigma$. Points in this phase space correspond to physical states of classical BF theory.

Remarks

1. The space \mathcal{A} is an infinite-dimensional vector space, and if we give it an appropriate topology, an open dense set of \mathcal{A}/G becomes an infinite-dimensional smooth manifold.

The simplest way to precisely define $T^*(\mathcal{A}/G)$ is as the cotangent bundle of this open dense set. The remaining points correspond to connections with more symmetry than the rest under gauge transformations. These are called ‘reducible’ connections. A more careful definition of the physical phase space would have to take these points into account.

2. The space \mathcal{A}_0/G is called the ‘moduli space of flat connections on $P|_\sigma$ ’. We can understand it better as follows. Since the holonomy of a flat connection around a loop does not change when we apply a homotopy to the loop, a connection $A \in \mathcal{A}_0$ determines a homomorphism from the fundamental group $\pi_1(\sigma)$ to G after we trivialize P at the basepoint $p \in \sigma$ that we use to define the fundamental group. If we apply a gauge transformation to A , this homomorphism is conjugated by the value of this gauge transformation at p . This gives us a map from \mathcal{A}_0/G to $\text{hom}(\pi_1(S), G)/G$, where $\text{hom}(\pi_1(\sigma), G)$ is the space of homomorphisms from $\pi_1(\sigma)$ to G , and G acts on this space by conjugation. When S is connected this map is one-to-one, so we have

$$\mathcal{A}_0/G \subseteq \text{hom}(\pi_1(\sigma), G)/G.$$

The space $\text{hom}(\pi_1(\sigma), G)/G$ is called the ‘moduli space of flat G -bundles over S ’. When $\pi_1(\sigma)$ is finitely generated (e.g. when σ is compact) this space is a real algebraic variety, and \mathcal{A}_0/G is a subvariety. Usually \mathcal{A}_0/G has singularities, but each component has an open dense set that is a smooth manifold. When we speak of $T^*(\mathcal{A}_0/G)$ above, we really mean the cotangent bundle of this open dense set, though again a more careful treatment would deal with the singularities.

We can describe \mathcal{A}_0/G much more explicitly in particular cases. For example, suppose that σ is a compact oriented surface of genus n . Then the group $\pi_1(S)$ has a presentation with $2n$ generators $x_1, y_1, \dots, x_n, y_n$ satisfying the relation

$$R(x_i, y_i) := (x_1 y_1 x_1^{-1} y_1^{-1}) \cdots (x_n y_n x_n^{-1} y_n^{-1}) = 1.$$

A point in $\text{hom}(\pi_1(S), G)$ may thus be identified with a collection $g_1, h_1, \dots, g_n, h_n$ of elements of G satisfying

$$R(g_i, h_i) = 1,$$

and a point in $\text{hom}(\pi_1(S), G)/G$ is an equivalence class $[g_i, h_i]$ of such collections.

The cases $G = \text{SU}(2)$ and $G = \text{SO}(3)$ are particularly interesting for their applications to 3-dimensional Riemannian general relativity. When $G = \text{SU}(2)$, all G -bundles over a compact oriented surface S are isomorphic, and $\mathcal{A}_0/G = \text{hom}(\pi_1(S), G)/G$. When $G = \text{SO}(3)$, there are two isomorphism classes of G -bundles over S , distinguished by their second Stiefel-Whitney number $w_2 \in \mathbf{Z}_2$. For each of these bundles, the points $[g_i, h_i]$ that lie in \mathcal{A}_0/G can be described as follows. Choose representatives $g_i, h_i \in \text{SO}(3)$ and choose elements \tilde{g}_i, \tilde{h}_i that map down to these representatives via the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$. Then $[g_i, h_i]$ lies in \mathcal{A}_0/G if and only if

$$(-1)^{w_2} = R(\tilde{g}_i, \tilde{h}_i).$$

For 3-dimensional Riemannian general relativity with gauge group $\text{SO}(3)$, the relevant bundle is the frame bundle of σ , which has $w_2 = 0$. For both $\text{SU}(2)$ and $\text{SO}(3)$, the

space \mathcal{A}_0/G has dimension $6n - 6$ for $n \geq 2$. For the torus \mathcal{A}_0/G has dimension 2, and for the sphere it is a single point.

We can now construct the gauge invariant quantum space S_0 as in LQG, generalizing cylindrical functions to n -dimensional spaces and Lie Group G .

6.3 Canonical Quantization via Triangulations

The constraint $F = 0$, which simultaneously generates the dynamics and imposes diffeomorphisms-invariance, restricts \mathbf{K}_0 (the space of continuous G -invariant functionals of the connection) to the space of G -invariant continuous functionals of the **flat** connection, named $Fun(\mathcal{A}_0/G)$.

Let us start with an n -dimensional real-analytic manifold M representing space. Given any triangulation of M we can choose a graph in M called the ‘dual 1-skeleton’, having one vertex at the center of each $(n - 1)$ -simplex and one edge intersecting each $(n - 2)$ -simplex. Using homotopies and skein relations, we can express any state in $Fun(\mathcal{A}_0/G)$ as a linear combination of states coming from spin networks whose underlying graph is this dual 1-skeleton. So at least for BF theory (where the connection is flat), there is no loss in working with spin networks of this special form.

It turns out that the working with a triangulation this way sheds new light on the observables discussed in the previous section. Moreover, the dynamics of BF theory is easiest to describe using triangulations. Thus it pays to formalize the setup a bit more. To do so, we borrow some ideas from lattice gauge theory.

Given a graph Γ , define a ‘connection’ on Γ to be an assignment of an element of G to each edge of Γ , and denote the space of such connections by \mathcal{A}_Γ . As in lattice gauge theory, these group elements represent the holonomies along the edges of the graph. Similarly, define a ‘gauge transformation’ on Γ to be an assignment of a group element to each vertex, and denote the group of gauge transformations by G_Γ . This group acts on \mathcal{A}_Γ in a natural way that mimics the usual action of gauge transformations on holonomies. Since \mathcal{A}_Γ is just a product of copies of G , we can use normalized Haar measure on G to put a measure on \mathcal{A}_Γ , and this in turn pushes down to a measure on the quotient space $\mathcal{A}_\Gamma/G_\Gamma$. Using these we can define Hilbert spaces $L^2(\mathcal{A}_\Gamma)$ and $L^2(\mathcal{A}_\Gamma/G_\Gamma)$.

In Sections 3.2.3, 3.2.4, 3.2.5, we saw how to extract a gauge-invariant function on the space of connections from any spin network embedded in space. The same trick works in the present context: any spin network Ψ with Γ as its underlying graph defines a function $\Psi \in L^2(\mathcal{A}_\Gamma/G_\Gamma)$. For example, if Ψ is the spin network in figure (6.1) and the connection A assigns the group elements g_1, g_2, g_3 to the three edges of Ψ , we have

$$\Psi(A) = R_{e_1}(g_1)_a^b R_{e_2}(g_2)_d^c R_{e_3}(g_3)_f^e (\iota_{v_1})_{ac}^f (\iota_{v_2})_{db}^e.$$

We again call such functions ‘spin network states’. Not only do these span $L^2(\mathcal{A}_\Gamma/G_\Gamma)$, it is easy again to choose an orthonormal basis of spin network states. Let $\text{Irrep}(G)$ be a complete set of irreducible unitary representations of G . To obtain spin networks $\Psi = (\Gamma, R, \iota)$ giving an orthonormal basis of $L^2(\mathcal{A}_\Gamma/G_\Gamma)$, let R range over all labellings

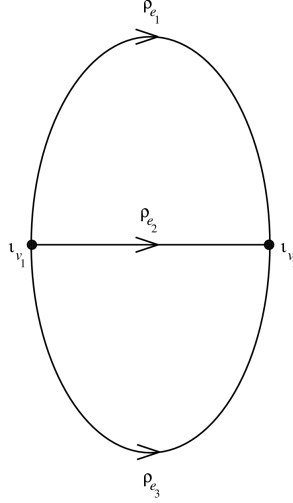


Figure 6.1:

of the edges of Γ by representations in $\text{Irrep}(G)$, and for each R and each vertex v , let the intertwiners ι_v range over an orthonormal basis of the space of intertwiners

$$\iota: R_{e_1} \otimes \cdots \otimes R_{e_n} \rightarrow R_{e'_1} \otimes \cdots \otimes R_{e'_m}$$

where the e_i are incoming to v and the e'_i are outgoing from v .

How do these purely combinatorial constructions relate to our previous setup where space is described by a real-analytic manifold σ equipped with a principal G -bundle? Quite simply: whenever Γ is a graph in σ , trivializing the bundle at the vertices of this graph gives a map from \mathcal{A} onto \mathcal{A}_Γ , and also a homomorphism from G onto G_Γ . Thus we have inclusions

$$L^2(\mathcal{A}_\Gamma) \hookrightarrow L^2(\mathcal{A})$$

and

$$L^2(\mathcal{A}_\Gamma/G_\Gamma) \hookrightarrow L^2(\mathcal{A}/G).$$

These constructions are particularly nice when Γ is the dual 1-skeleton of a triangulation of σ . Consider 3-dimensional Riemannian quantum gravity, for example. In this case Γ is always trivalent, see figure (6.2).

Since the representations of $\text{SU}(2)$ satisfy

$$j_1 \otimes j_2 \cong |j_1 - j_2| \oplus \cdots \oplus (j_1 + j_2),$$

each basis of intertwiners $\iota: j_1 \otimes j_2 \rightarrow j_3$ contains at most one element. Thus we do not need to explicitly label the vertices of trivalent $\text{SU}(2)$ spin networks with intertwiners; we only need to label the edges with spins. We can dually think of these spins as labelling the edges of the original triangulation. For example, the spin network state

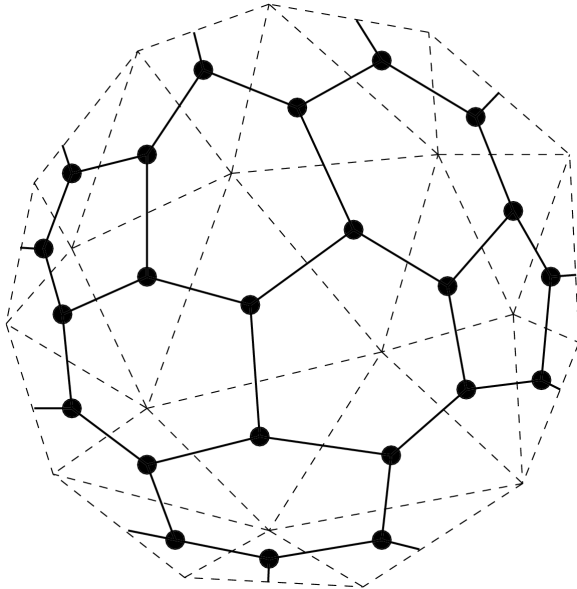


Figure 6.2:

in figure (6.3) corresponds to a triangulation with edges labelled by spins as in figure (6.4).

By the results of the previous sections, these spins specify the *lengths* of the edges, with spin j corresponding to length $\sqrt{j(j+1)}$. Note that for there to be an intertwiner $\iota: j_1 \otimes j_2 \rightarrow j_3$, the spins j_1, j_2, j_3 labelling the three edges of a given triangle must satisfy two constraints. First, the triangle inequality must hold:

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2.$$

This has an obvious geometrical interpretation. Second, the spins must sum to an integer. This rather peculiar constraint would hold automatically if we had used the gauge group $\text{SO}(3)$ instead of $\text{SU}(2)$. If we consider all labellings satisfying these constraints, we obtain spin network states forming a basis of $L^2(\mathcal{A}_\Gamma/G_\Gamma)$.

The situation is similar but a bit more complicated for 4-dimensional BF theory with gauge group $\text{SU}(2)$. Let S be a triangulated 3-dimensional manifold and let Γ be its dual 1-skeleton. Now Γ is a 4-valent graph with one vertex in the center of each tetrahedron and one edge intersecting each triangle. To specify a spin network state in $L^2(\mathcal{A}_\Gamma/G_\Gamma)$, we need to label each edge of Γ with a spin and each vertex with an intertwiner as in figure (6.5).

For each vertex there is a basis of intertwiners. We can draw such an intertwiner by formally ‘splitting’ the vertex into two trivalent ones and labelling the new edge with a spin j_5 , as in figure (6.6). In the triangulation picture, this splitting corresponds to chopping the tetrahedron in half along a parallelogram, see figure (6.7).

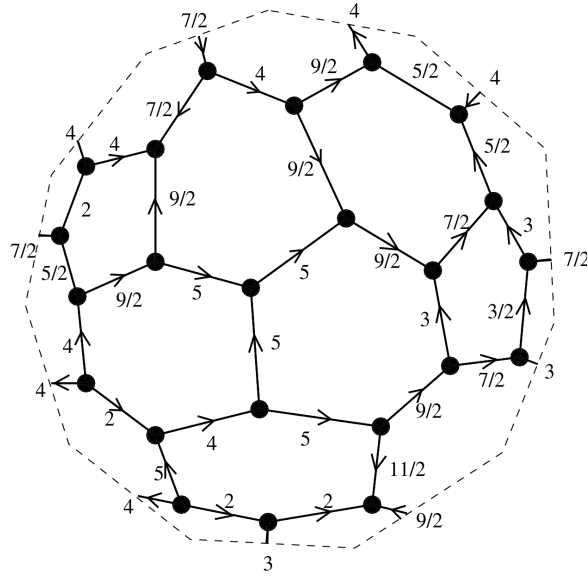


Figure 6.3:

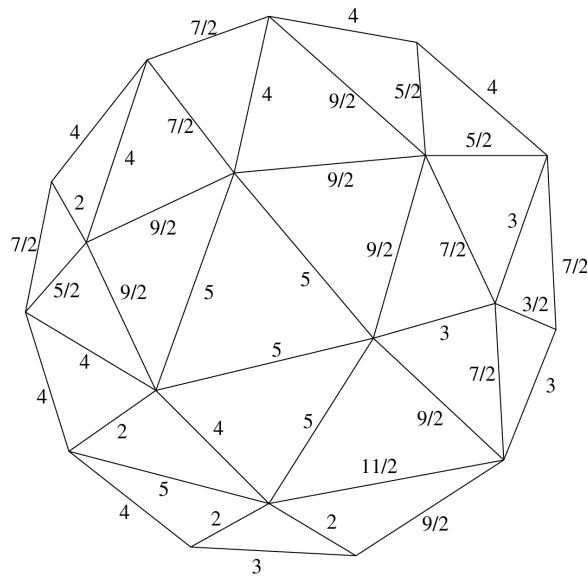


Figure 6.4:

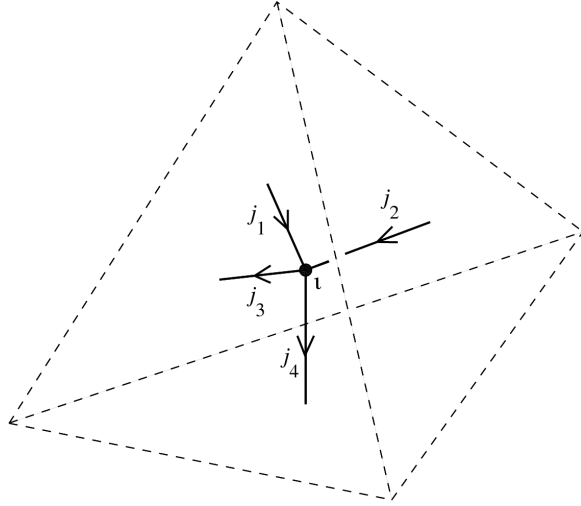


Figure 6.5:

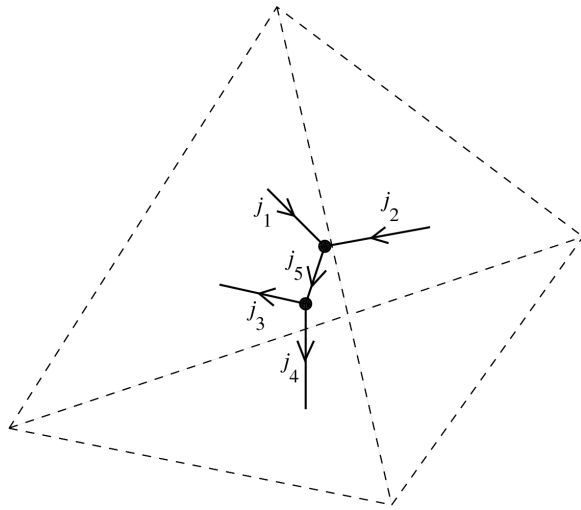


Figure 6.6:

We can thus describe a spin network state in $L^2(\mathcal{A}_\Gamma/G_\Gamma)$ by chopping each tetrahedron in half and labelling all the resulting parallelograms, along with all the triangles, by spins. These spins specify the *areas* of the parallelograms and triangles.

It may seem odd that in this picture the geometry of each tetrahedron is described by 5 spins, since classically it takes 6 numbers to specify the geometry of a tetrahedron.

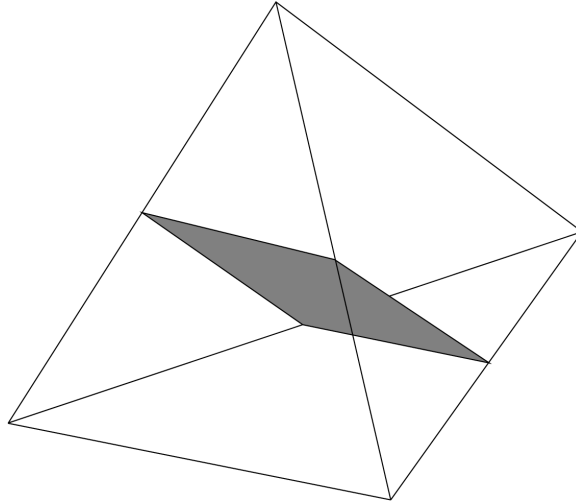


Figure 6.7:

In fact, this is a consequence of the uncertainty principle. The area operators for surfaces do not commute when the surfaces intersect. There are three ways to chop a tetrahedron in half using a parallelogram, but we cannot simultaneously diagonalize the areas of these parallelograms, since they intersect. We can describe a basis of states for the quantum tetrahedron using 5 numbers: the areas of its 4 faces and any *one* of these parallelograms. Different ways of chopping tetrahedron in half gives us different bases of this sort, and the matrix relating these bases goes by the name of the ‘ $6j$ symbols’:

$$\begin{array}{c}
 \begin{array}{ccc}
 & \swarrow & \searrow \\
 & j_1 & j_2 \\
 & \downarrow & \downarrow \\
 & \bullet & \\
 & | & \\
 & \downarrow & \\
 & j_5 & \\
 & \downarrow & \\
 & \bullet & \\
 \swarrow & & \searrow \\
 j_3 & & j_4
 \end{array}
 & = &
 \sum_{j_5} \begin{pmatrix} j_1 & j_2 & j_6 \\ j_4 & j_3 & j_5 \end{pmatrix}
 & \begin{array}{ccc}
 & \swarrow & \searrow \\
 & j_1 & j_2 \\
 & \downarrow & \downarrow \\
 & \bullet & \bullet \\
 & \swarrow & \searrow \\
 j_3 & & j_4
 \end{array}
 \end{array}$$

Remarks

1. For a deeper understanding of BF theory with gauge group $SU(2)$, it is helpful to start with a classical phase space describing tetrahedron geometries and apply geometric quantization to obtain a Hilbert space of quantum states. We can describe a tetrahedron in R^3 by specifying vectors B_1, \dots, B_4 normal to its faces, with lengths equal to the faces' areas. We can think of these vectors as elements of $so(3)$, which has a Poisson structure familiar from the quantum mechanics of angular momentum:

$$\{J^a, J^b\} = \epsilon^{abc} J^c.$$

The space of 4-tuples (B_1, \dots, B_4) thus becomes a Poisson manifold. However, a 4-tuple coming from a tetrahedron must satisfy the constraint $B_1 + \dots + B_4 = 0$. This constraint is the discrete analogue of the Gauss law $d_A B = 0$. In particular, it generates rotations, so if we take $so(3)^4$ and do Poisson reduction with respect to this constraint, we obtain a phase space whose points correspond to tetrahedron geometries modulo rotations. If we geometrically quantize this phase space, we obtain the 'Hilbert space of the quantum tetrahedron'.

Since this Hilbert space \mathcal{H} is a target space representation of $SU(2)$, it has operators \hat{J}^a on it satisfying the usual angular momentum commutation relations:

$$[\hat{J}^a, \hat{J}^b] = i\epsilon^{abc} \hat{J}^c.$$

We can think of \mathcal{H} as the 'Hilbert space of a quantum vector' and the operators \hat{J}^a as measuring the components of this vector. If we geometrically quantize $so(3)^{\otimes 4}$, we obtain $\mathcal{H}^{\otimes 4}$, which is the Hilbert space for 4 quantum vectors. There are operators on this Hilbert space corresponding to the components of these vectors:

$$\begin{aligned} \hat{B}_1^a &= \hat{J}^a \otimes 1 \otimes 1 \otimes 1 \\ \hat{B}_2^a &= 1 \otimes \hat{J}^a \otimes 1 \otimes 1 \\ \hat{B}_3^a &= 1 \otimes 1 \otimes \hat{J}^a \otimes 1 \\ \hat{B}_4^a &= 1 \otimes 1 \otimes 1 \otimes \hat{J}^a. \end{aligned} \tag{6.10}$$

One can show that the Hilbert space of the quantum tetrahedron is isomorphic to

$$\mathbf{T} = \{\psi \in \mathcal{H}^{\otimes 4}: (\hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \hat{B}_4)\psi = 0\}.$$

On the Hilbert space of the quantum tetrahedron there are operators

$$\hat{A}_i = (\hat{B}_i \cdot \hat{B}_i)^{\frac{1}{2}}$$

corresponding to the areas of the 4 faces of the tetrahedron, and also operators

$$\hat{A}_{ij} = ((\hat{B}_i + \hat{B}_j) \cdot (\hat{B}_i + \hat{B}_j))^{\frac{1}{2}}$$

corresponding to the areas of the parallelograms. Since $\hat{A}_{ij} = \hat{A}_{kl}$ whenever $(ijkl)$ is some permutation of the numbers (1234) , there are really just 3 different parallelogram area operators. The face area operators commute with each other and with

the parallelogram area operators, but the parallelogram areas do not commute with each other. There is a basis of \mathbf{T} consisting of states that are eigenvectors of all the face area operators together with any one of the parallelogram area operators. If for example we pick \hat{A}_{12} as our preferred parallelogram area operator, any basis vector ψ is determined by 5 spins:

$$\begin{aligned}\hat{A}_i\psi &= \sqrt{j_i(j_i+1)} & 1 \leq i \leq 4, \\ \hat{A}_{12}\psi &= \sqrt{j_5(j_5+1)}.\end{aligned}\tag{6.11}$$

This is because $\hat{J}^a\hat{J}^a = j(j+1)\mathbf{1}$ with $j = 0, \frac{1}{2}, 1, \dots$ when acts on its eigenstates. This basis vector corresponds to the intertwiner $\iota_j: j_1 \otimes j_2 \rightarrow j_3 \otimes j_4$ that factors through the representation j_5 .

In 4d BF theory with gauge group $SU(2)$, the Hilbert space $L^2(\mathcal{A}_\Gamma/G_\Gamma)$ described by taking the tensor product of copies of \mathbf{T} , one for each tetrahedron in the 3-manifold S , and imposing constraints saying that when two tetrahedra share a face their face areas must agree. This gives a clearer picture of the ‘quantum geometry of space’ in this theory. For example, we can define observables corresponding to the volumes of tetrahedra. The results nicely match those of loop quantum gravity, where it has been shown that spin network vertices give volume to the regions of space in which they lie. In loop quantum gravity these results were derived not from BF theory, but from Lorentzian quantum gravity formulated in terms of the real Ashtekar variables. However, these theories differ only in their dynamics.

6.4 Dynamics

We now turn from the spin network description of the kinematics of BF theory to the spin foam description of its dynamics. Our experience with quantum field theory suggests that we can compute transition amplitudes in BF theory using path integrals. To keep life simple, consider the most basic example: the partition function of a closed manifold representing spacetime. Heuristically, if M is a compact oriented n -manifold we expect that

$$\begin{aligned}Z(M) &= \int \int \mathcal{D}A \mathcal{D}B e^{i \int_M \text{tr}(B \wedge F)} \\ &= \int \mathcal{D}A \delta(F),\end{aligned}\tag{6.12}$$

where formally integrating out the B field gives a Dirac delta measure on the space of flat connections on the G -bundle P over M . The final result should be the ‘volume of the space of flat connections’, but of course this is ill-defined without some choice of measure.

To try to make this calculation more precise, we can *discretize* it by choosing a triangulation for M and working, not with flat connections on P , but instead with flat connections on the dual 2-skeleton. By definition, the ‘dual 2-skeleton’ of a triangulation has one vertex in the center of each n -simplex, one edge intersecting each $(n-1)$ -simplex, and one polygonal face intersecting each $(n-2)$ -simplex. We call

these ‘dual vertices’, ‘dual edges’, and ‘dual faces’, respectively. For example, when M is 3-dimensional, the intersection of the dual 2-skeleton with any tetrahedron looks like in figure (6.8).

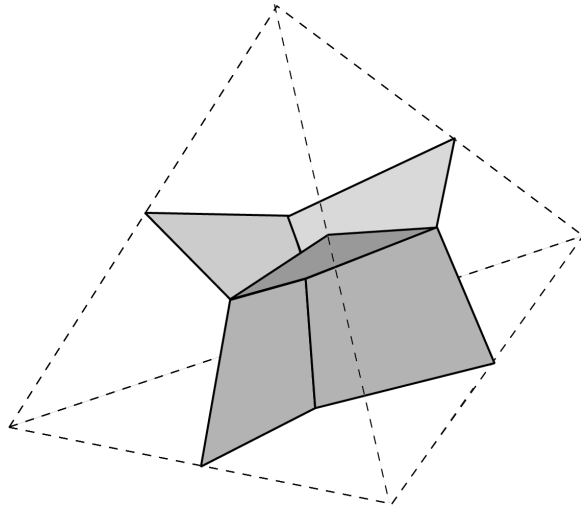


Figure 6.8:

while a typical dual face looks like in figure (6.9). Note that the dual faces can have any number of edges. To keep track of these edges, we fix an orientation and distinguished vertex for each face f and call its edges e_1f, \dots, e_Nf , taken in cyclic order starting from the distinguished vertex. Similarly, we call its vertices v_1f, \dots, v_Nf , see figure (6.10).

A ‘connection’ on the dual 2-skeleton is an object assigning a group element g_e to each dual edge e . For this to make sense we should fix an orientation for each dual edge. However, we can safely reverse our choice of the orientation as long as we remember to replace g_e by g_e^{-1} when we do so. We say that a connection on the dual 2-skeleton is ‘flat’ if that the holonomy around each dual face f is the identity:

$$g_{e_1f} \cdots g_{e_Nf} = 1$$

where we use the orientation of f to induce orientations of its edges. To make sense of our earlier formula for the partition function of BF theory, we can try defining

$$Z(M) = \int \prod_{e \in \mathcal{E}} dg_e \prod_{f \in \mathcal{F}} \delta(g_{e_1f} \cdots g_{e_Nf}),$$

where \mathcal{V} is the set of dual vertices, \mathcal{E} is the set of dual edges, \mathcal{F} is the set of dual faces, and the integrals are done using normalized Haar measure on G . Of course, since we are taking a product of Dirac deltas here, we run the danger that this expression will

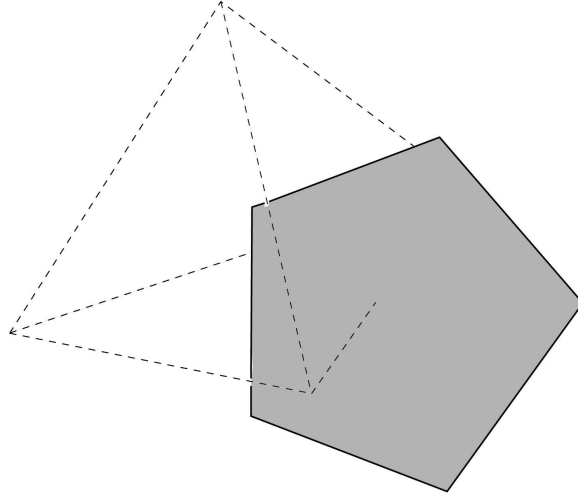


Figure 6.9:

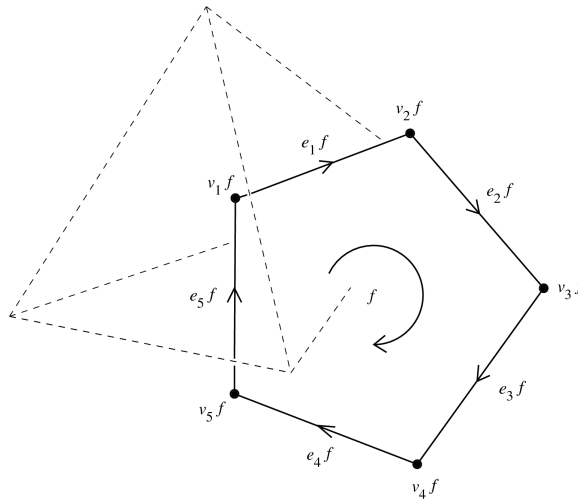


Figure 6.10:

not make sense. Nonetheless we proceed and see what happens! We begin by using the identity

$$\delta(g) = \sum_{R \in \text{Irrep}(G)} \dim(R) \text{tr}(R(g)),$$

obtaining

$$Z(M) = \sum_{R: \mathcal{F} \rightarrow \text{Irrep}(G)} \int \prod_{e \in \mathcal{E}} dg_e \prod_{f \in \mathcal{F}} \dim(R_f) \text{tr}(R_f(g_{e_1 f} \cdots g_{e_N f})).$$

This formula is really a discretized version of

$$Z(M) = \int \int \mathcal{D}A \mathcal{D}B e^{i \int_M \text{tr}(B \wedge F)}.$$

The analogue of A is the labelling of dual edges by group elements. The analogue of F is the labelling of dual faces by holonomies around these faces. These analogies make geometrical sense because A is like a 1-form and F is like a 2-form. What is the analogue of B ? It is the labelling of dual faces by representations! Since each dual face intersects one $(n-2)$ -simplex in the triangulation, we may dually think of these representations as labelling $(n-2)$ -simplices. This is nice because B is an $(n-2)$ -form. The analogue of the pairing $\text{tr}(B \wedge F)$ is the pairing of a representation R_f and the holonomy around the face f to obtain the number $\dim(R_f) \text{tr}(R_f(g_{e_1 f} \cdots g_{e_N f}))$.

Next we do the integrals over group elements in the formula for $Z(M)$. The details depend on the dimension of spacetime, and it is easiest to understand them with the aid of some graphical notation. In the previous section we saw how an abstract spin network Ψ together with a connection A on the underlying graph of Ψ give a number $\Psi(A)$. Since the connection A assigns a group element g_e to each edge of Ψ , our notation for the number $\Psi(A)$ will be a picture of Ψ together with a little circle containing the group element g_e on each edge e . When the g_e is the identity we will not bother drawing it. Also, when two or more parallel edges share the same group element g we use one little circle for both edges. For example, we take the definition in figure (6.11). This is just the graphical analogue of the equation $(R_1 \otimes R_2)(g) = R_1(g) \otimes R_2(g)$.

Now suppose M is 2-dimensional. Since each dual edge is the edge of two dual faces, each group element appears twice in the expression

$$\prod_{f \in \mathcal{F}} \text{tr}(R_f(g_{e_1 f} \cdots g_{e_N f})).$$

In our graphical notation, this expression corresponds to a spin network with one loop running around each dual face, as in figure (6.12). Here we have only drawn a small portion of the spin network. We can do the integral

$$\int \prod_{e \in E} dg_e \prod_{f \in F} \dim(R_f) \text{tr}(R_f(g_{e_1 f} \cdots g_{e_N f}))$$

by repeatedly using the formula

$$\int dg R_1(g) \otimes R_2(g) = \begin{cases} \frac{\iota^*}{\dim(R_1)} & \text{if } R_1 \cong R_2^* \\ 0 & \text{otherwise} \end{cases}$$

where $\iota: R_1 \otimes R_2 \rightarrow \mathbf{C}$ is the dual pairing when R_1 is the dual of R_2 . This formula holds because both sides describe the projection from $R_1 \otimes R_2$ onto the subspace of

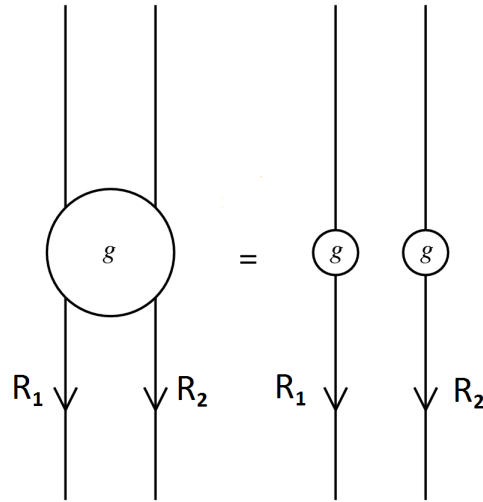


Figure 6.11:

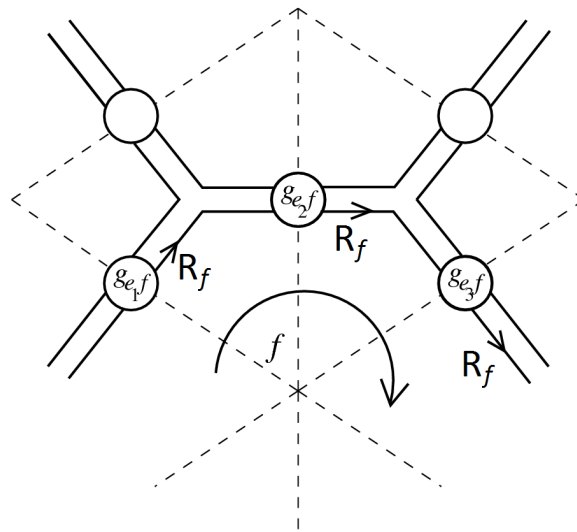


Figure 6.12:

vectors transforming in the trivial representation. Graphically, this formula can be written as the skein relation in figure (6.13).

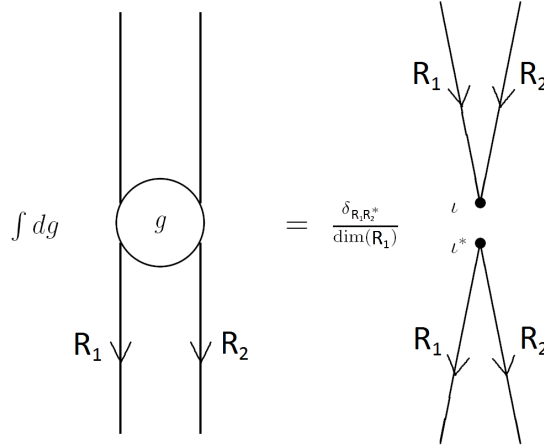


Figure 6.13:

Applying this to every dual edge, we see that when M is connected the integral

$$\int \prod_{e \in E} dg_e \prod_{f \in F} \dim(R_f) \text{tr}(R_f(g_{e_1 f} \cdots g_{e_N f}))$$

vanishes unless all the representations R_f are the same representation R , in which case it equals $\dim(R)^{|\mathcal{V}| - |\mathcal{E}| + |\mathcal{F}|}$. The quantity $|\mathcal{V}| - |\mathcal{E}| + |\mathcal{F}|$ is a topological invariant of M , namely the Euler characteristic $\chi(M)$. Summing over all labellings of dual faces, we thus obtain

$$Z(M) = \sum_{R \in \text{Irrep}(G)} \dim(R)^{\chi(M)}$$

The Euler characteristic of a compact oriented surface of genus n is $2 - 2n$. When $\chi(M) < 0$, the sum converges for any compact Lie group G , and we see that the partition function of our discretized BF theory is well-defined and *independent of the triangulation!* This is precisely what we would expect in a topological quantum field theory. For $\chi(M) \geq 0$, that is, for the sphere and torus, the partition function typically does not converge.

In the 3-dimensional case each group element shows up in 3 factors of the product over dual faces, since 3 dual faces share each dual edge:

We can do the integral over each group element using the formula

$$\int dg R_1(g) \otimes R_2(g) \otimes R_3(g) = \sum_{\iota} \iota^*$$

where the sum ranges over a basis of intertwiners $\iota: R_1 \otimes R_2 \otimes R_3 \rightarrow \mathbf{C}$, normalized so that $\text{tr}(\iota_1 \iota_2^*) = \delta_{\iota_1 \iota_2}$ for any two intertwiners ι_1, ι_2 in the basis. In our graphical notation this formula is written as in figure (6.15). Both sides represent intertwiners

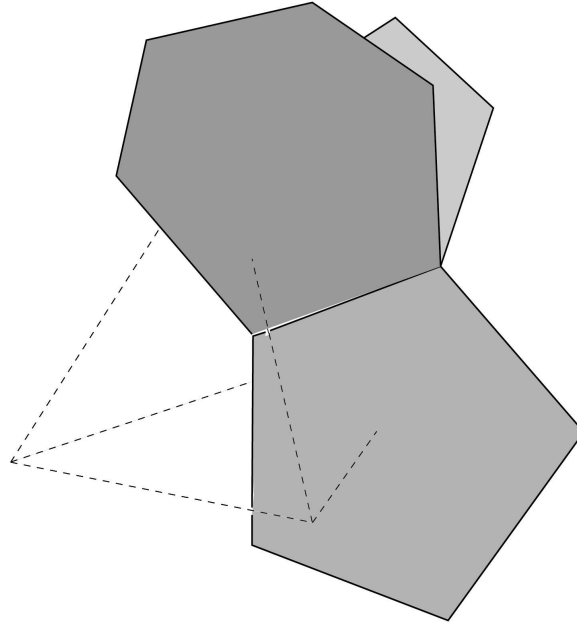


Figure 6.14:

from $R_1 \otimes R_2 \otimes R_3$ to itself. Again, the formula is true because both sides are different ways of describing the projection from $R_1 \otimes R_2 \otimes R_3$ onto the subspace of vectors that transform trivially under G . Using this formula once for each dual edge — or equivalently, once for each triangle in the triangulation — we can integrate out all the group elements g_e . Graphically, each time we do this, an integral over expressions like in figure (6.16), is replaced by a sum of expressions like in figure (6.17).

(We have not bothered to show the orientation of the edges in these pictures, since they depend on how we orient the edges of the dual 2-skeleton.) When we do this for all the triangular faces of a given tetrahedron, we obtain a little tetrahedral spin network like in figure (6.18) which we can evaluate in the usual way. This tetrahedral spin network is ‘dual’ to the original tetrahedron in the triangulation of M : its vertices (resp. edges, faces) correspond to faces (resp. edges, vertices) of the original tetrahedron. We thus obtain the following formula for the partition function in 3-dimensional BF theory:

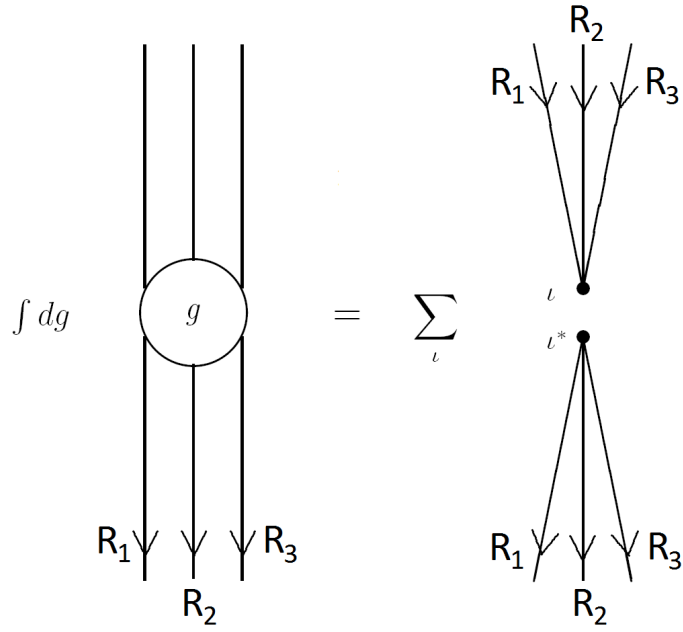


Figure 6.15:

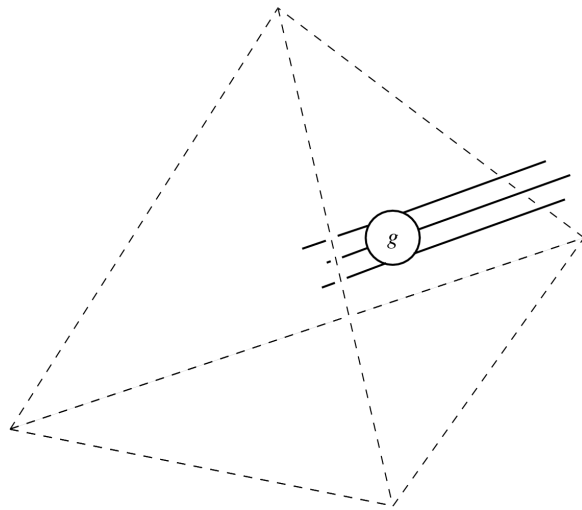


Figure 6.16:

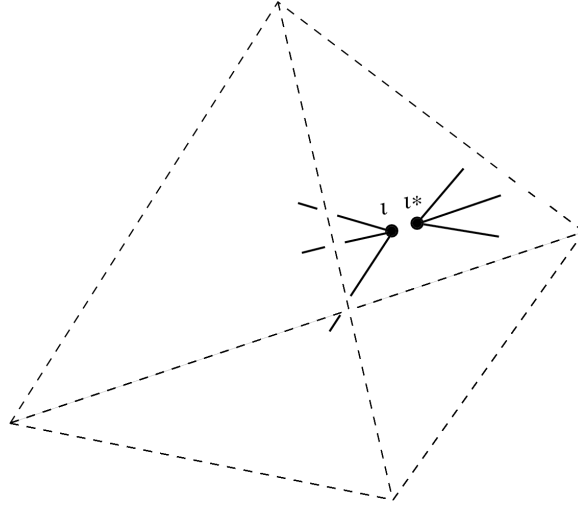
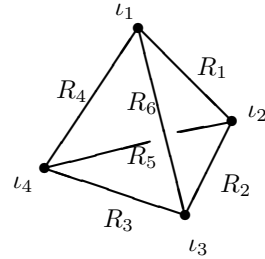


Figure 6.17:

$$Z(M) = \sum_{R: \mathcal{F} \rightarrow \text{Irrep}(G)} \sum_{\iota} \prod_{f \in \mathcal{F}} \dim(R_f) \prod_{v \in \mathcal{V}}$$



Here for each labelling $R: \mathcal{F} \rightarrow \text{Irrep}(G)$, we take a sum over labellings ι of dual edges by intertwiners taken from the appropriate bases. For each dual vertex v , the tetrahedral spin network shown above is built using the representations R_i labelling the 6 dual faces incident to v and the intertwiners ι_i labelling the 4 dual edges incident to v . When $G = \text{SU}(2)$ or $\text{SO}(3)$, the labelling by intertwiners is trivial, so the tetrahedral spin network depends only on 6 spins. Using our graphical notation, it is not hard to express the value of this spin network in terms of the $6j$ symbols described in the previous section. We leave this as an exercise for the reader.

The calculation in 4 dimensions is similar, but now 4 dual faces share each dual edge, so we need to use the formula

$$\int dg R_1(g) \otimes R_2(g) \otimes R_3(g) \otimes R_4(g) = \sum_{\iota} \iota^*$$

where now the sum ranges over a basis of intertwiners $\iota: R_1 \otimes R_2 \otimes R_3 \otimes R_4 \rightarrow \mathbf{C}$, normalized so that $\text{tr}(\iota_1 \iota_2^*) = \delta_{\iota_1 \iota_2}$ for any intertwiners ι_1, ι_2 in the basis. Again both

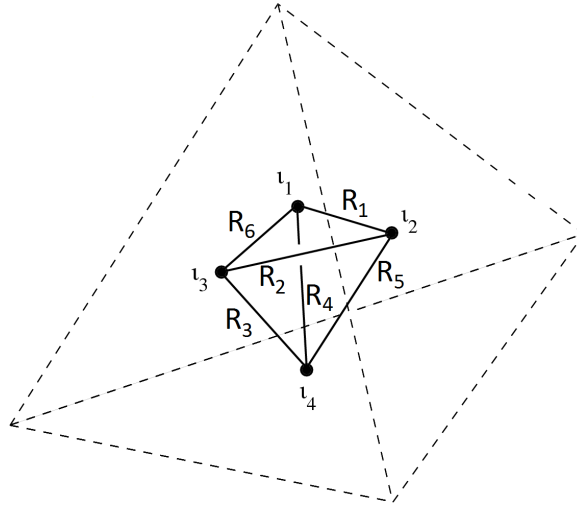


Figure 6.18:

sides describe the projection on the subspace of vectors that transform in the trivial representation, and again we can write the formula as a generalized skein relation, see figure (6.19). We use this formula once for each dual edge — or equivalently, once for each tetrahedron in the triangulation — to do the integral over all group elements in the partition function. Each time we do so, we introduce an intertwiner labelling the dual edge in question. We obtain the formula:

$$Z(M) = \sum_{R:\mathcal{F}\rightarrow\text{Irrep}(G)} \sum_{\iota} \prod_{f\in\mathcal{F}} \dim(R_f) \prod_{v\in\mathcal{V}}$$

The 4-simplex in this formula is dual to the 4-simplex in the original triangulation that contains $v \in \mathcal{V}$. Its edges are labelled by the representations labelling the 10 dual faces incident to v , and its vertices are labelled by the intertwiners labelling the 5 dual edges incident to v .

People often rewrite this formula for the partition function by splitting each 4-valent vertex into two trivalent vertices: the resulting equation involves a trivalent

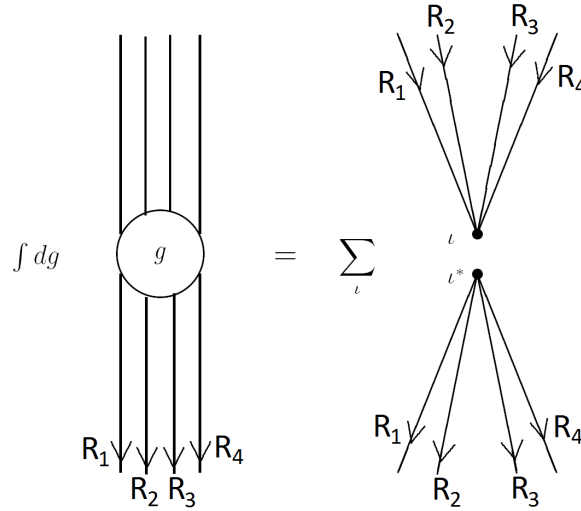


Figure 6.19:

spin network with 15 edges. In the $SU(2)$ case this trivalent spin network is called a ‘15j symbol’, since it depends on 15 spins.

Having computed the BF theory partition function in 2, 3, and 4 dimensions, it should be clear that the same basic idea works in all higher dimensions, too. We always get a formula for the partition function as a sum over ways of labelling dual faces by representations and dual edges by intertwiners. There is, however, a problem. The sum usually diverges! The only cases we know where it converges are when G is a finite group (see Remark 2 below), when M is 0- or 1-dimensional, or when M is 2-dimensional with $\chi(M) < 0$. Not surprisingly, these are a subset of the cases when the moduli space of flat connections on M has a natural measure. In other cases, it seems there are too many delta functions in the expression

$$Z(M) = \int \prod_{e \in \mathcal{E}} dg_e \prod_{f \in \mathcal{F}} \delta(g_{e_1 f} \cdots g_{e_N f})$$

to extract a meaningful answer.

Of course, there is more to dynamics than the partition function. For example, we also want to compute vacuum expectation values of observables, and transition amplitudes between states. It is not hard to generalize the formulas above to handle these more complicated calculations. However, at this point it helps to explicitly introduce the concept of a ‘spin foam’.

Remarks

1. Ponzano and Regge gave a formula for a discretized version of the action in 3-dimensional Riemannian general relativity. In their approach the spacetime manifold

M is triangulated and each edge is assigned a length. The Ponzano-Regge action is the sum over all tetrahedra of the quantity:

$$S = \sum_e \ell_e \theta_e$$

where the sum is taken over all 6 edges, ℓ_e is the length of the edge e , and θ_e is the dihedral angle of the edge e , that is, the angle between the outward normals of the two faces incident to this edge. One can show that in a certain precise their action is an approximation to the integral of the Ricci scalar curvature. In the limit of large spins, the value of the tetrahedral spin network described above is asymptotic to

$$\sqrt{\frac{2}{3\pi V}} \cos(S + \frac{\pi}{4}).$$

where the lengths ℓ_e are related to the spins j_e labelling the tetrahedron's edges by $\ell = j + 1/2$, and V is the volume of the tetrahedron. Naively one might have hoped to get $\exp(iS)$. That one gets a cosine instead can be traced back to the fact that the lengths of the edges of a tetrahedron only determine its geometry modulo rotation and reflection. The phase $\frac{\pi}{4}$ shows up because calculating the asymptotics of the tetrahedral spin network involves a stationary phase approximation.

2. Until now we have been assuming that G is connected. The main reason for this is that it ensures the map from \mathcal{A} to \mathcal{A}_Γ is onto for any graph Γ in σ , so that we have inclusions $L^2(\mathcal{A}_\Gamma) \hookrightarrow L^2(\mathcal{A})$ and $L^2(\mathcal{A}_\Gamma/G_\Gamma) \hookrightarrow L^2(\mathcal{A}/G)$. When G is not connected, these maps are usually not one-to-one.

Requiring that G be connected rules out all nontrivial finite groups. However, our formula for the BF theory partition function makes equally good sense for groups that are not connected. In fact, when G is finite, the partition function is convergent regardless of the dimension of M , and when a suitable normalization factor is included it becomes triangulation-independent.

In this model, the path integral is not an integral over flat connections on a fixed G -bundle over M , but rather a sum over isomorphism classes of G -bundles. In fact, our discretized formula for the path integral in BF theory always implicitly includes a sum over isomorphism classes of G -bundles, because it corresponds to an integral over the whole moduli space of flat G -bundles over M , rather than the moduli space of flat connections on a fixed G -bundle. (For the relation between these spaces, see Remark 2 in Section 6.2) When G is a finite group, the moduli space of flat G -bundles is discrete, with one point for each isomorphism class of G -bundle.

6.5 Spin Foams

We have seen that in BF theory the partition function can be computed by triangulating spacetime and considering all ways of labelling dual faces by irreducible representations and dual edges by intertwiners. For each such labelling, we compute an 'amplitude' as a product of amplitudes for dual faces, dual edges, and dual vertices. (By cleverly normalizing our intertwiners we were able to make the edge amplitudes

equal 1, rendering them invisible, but this was really just a cheap trick.) We then take a sum over all labellings to obtain the partition function.

To formalize this idea we introduce the concept of a ‘spin foam’. A spin foam is the 2-dimensional analog of a spin network. Just as a spin network is a graph with edges labelled by irreducible representations and vertices labelled by intertwiners, a spin foam is a 2-dimensional complex with faces labelled by irreducible representations and edges labelled by intertwiners. Of course, to make this precise we need a formal definition of ‘2-dimensional complex’. Loosely, such a thing should consist of vertices, edges, and polygonal faces. There is some flexibility about the details. However, we certainly want the dual 2-skeleton of a triangulated manifold to qualify. Since topologists have already studied such things, this suggests that we take a 2-dimensional complex to be what they call a ‘2-dimensional piecewise linear cell complex’.

The precise definition of this concept is somewhat technical, so we banish it to the Appendix and only state what we need here. A 2-dimensional complex has a finite set \mathcal{V} of vertices, a finite set \mathcal{E} of edges, and a finite set \mathcal{F}_N of N -sided faces for each $N \geq 3$, with only finitely many \mathcal{F}_N being nonempty. In fact, we shall work with ‘oriented’ 2-dimensional complexes, where each edge and each face has an orientation. The orientations of the edges give maps

$$s, t: \mathcal{E} \rightarrow \mathcal{V}$$

assigning to each edge its source and target. The orientation of each face gives a cyclic ordering to its edges and vertices. Suppose we arbitrarily choose a distinguished vertex for each face $f \in \mathcal{F}_N$. Then we may number all its vertices and edges from 1 to N . If we think of these numbers as lying in \mathbf{Z}_N , we obtain maps

$$e_i: \mathcal{F}_N \rightarrow \mathcal{E}, \quad v_i: \mathcal{F}_N \rightarrow \mathcal{V} \quad i \in \mathbf{Z}_N.$$

We say f is ‘incoming’ to e when the orientation of e agrees with the orientation it inherits from f , and ‘outgoing’ when these orientations do not agree, see figure (6.20).

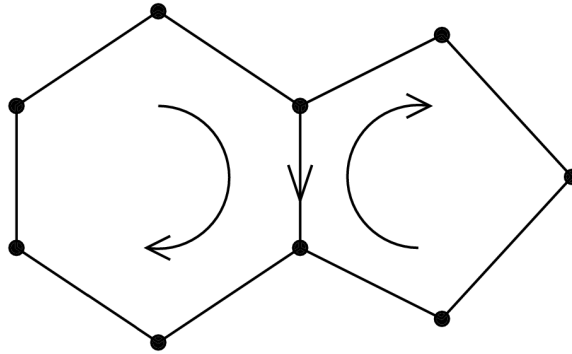


Figure 6.20:

With this business taken care of, we can define spin foams. The simplest kind is a

‘closed’ spin foam. This is the sort we sum over when computing partition functions in BF theory.

- A closed spin foam F is a triple (κ, R, ι) consisting of:
 1. a 2-dimensional oriented complex κ ,
 2. a labelling R of each face f of κ by an irreducible representation R_f of G ,
 3. a labelling ι of each edge e of κ by an intertwiner

$$\iota_e: R_{f_1} \otimes \cdots \otimes R_{f_n} \rightarrow R_{f'_1} \otimes \cdots \otimes R_{f'_m}$$

where f_1, \dots, f_n are the faces incoming to e and f'_1, \dots, f'_m are the faces outgoing from e .

Note that this definition is exactly like that of a spin network, but with everything one dimension higher! This is why **a generic slice of a spin foam is a spin network**. We can formalize this using the notion of a spin foam $F: \Psi \rightarrow \Psi'$ going from a spin network Ψ to a spin network Ψ' , see figure (6.21).

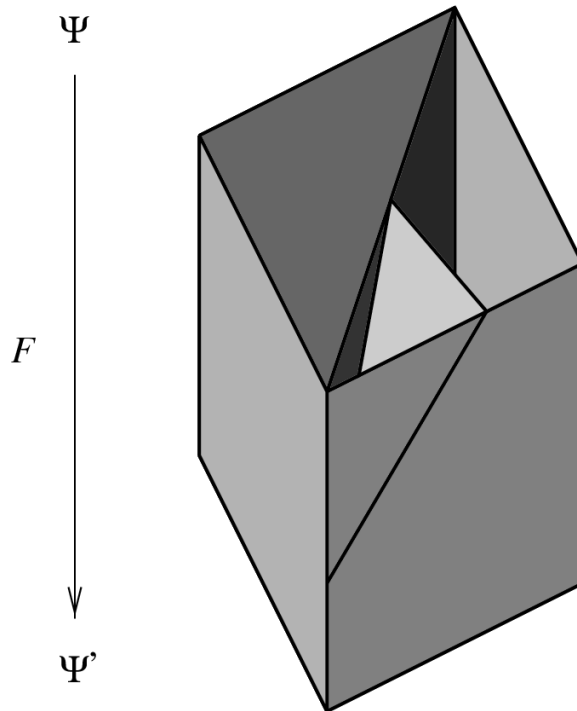


Figure 6.21:

This is the sort we sum over when computing transition amplitudes in BF theory. (To reduce clutter, we have not drawn the labellings of edges and faces in this spin foam.) In this sort of spin foam, the edges that lie in Ψ and Ψ' are not labelled by intertwiners. Also, the edges ending at spin network vertices must be labelled by intertwiners that match those labelling the spin network vertices. These extra requirements are lacking for closed spin foams, because a closed spin foam is just one of the form $F: \emptyset \rightarrow \emptyset$, where \emptyset is the ‘empty spin network’: the spin network with no vertices and no edges.

To make this more precise, we need to define what it means for a graph Γ to ‘border’ a 2-dimensional oriented complex κ . The reader can find this definition in Appendix A. What matters here is that if Γ borders κ , then each vertex v of Γ is the source or target of a unique edge \tilde{v} of κ , and each edge e of Γ is the edge of a unique face \tilde{e} of κ . Using these ideas, we first define spin foams of the form $F: \emptyset \rightarrow \Psi$:

- Suppose that $\Psi = (\Gamma, R, \iota)$ is a spin network. A spin foam $F: \emptyset \rightarrow \Psi$ is a triple $(\kappa, \tilde{R}, \tilde{\iota})$ consisting of:
 1. a 2-dimensional oriented complex κ such that Γ borders κ ,
 2. a labeling \tilde{R} of each face f of κ by an irreducible representation \tilde{R}_f of G ,
 3. a labeling $\tilde{\iota}$ of each edge e of κ not lying in Γ by an intertwiner

$$\tilde{\iota}_e: R_{f_1} \otimes \cdots \otimes R_{f_n} \rightarrow R_{f'_1} \otimes \cdots \otimes R_{f'_m}$$

where f_1, \dots, f_n are the faces incoming to e and f'_1, \dots, f'_m are the faces outgoing from e ,

such that the following hold:

1. For any edge e of Γ , $\tilde{R}_{\tilde{e}} = R_e$ if \tilde{e} is incoming to e , while $\tilde{R}_{\tilde{e}} = (R_e)^*$ if \tilde{e} is outgoing from e .
2. For any vertex v of Γ , $\tilde{\iota}_{\tilde{e}}$ equals ι_e after appropriate dualizations.

Finally, to define general spin foams, we need the notions of ‘dual’ and ‘tensor product’ for spin networks. The dual of a spin network $\Psi = (\Gamma, R, \iota)$ is the spin network Ψ^* with the same underlying graph, but with each edge e labelled by the dual representation R_e^* , and each vertex v labelled by the appropriately dualized form of the intertwining operator ι_v . Given spin networks $\Psi = (\Gamma, R, \iota)$ and $\Psi' = (\Gamma', R', \iota')$, their tensor product $\Psi \otimes \Psi'$ is defined to be the spin network whose underlying graph is the disjoint union of Γ and Γ' , with edges and vertices labelled by representations and intertwiners using R, R' and ι, ι' . As usual, duality allows us to think of an input as an output:

- Given spin networks Ψ and Ψ' , a spin foam $F: \Psi \rightarrow \Psi'$ is defined to be a spin foam $F: \emptyset \rightarrow \Psi^* \otimes \Psi'$.

Here is how we compute transition amplitudes in BF theory as a sum over spin foams. Suppose spacetime is given by a compact oriented cobordism $M: \sigma \rightarrow \sigma'$, where σ and σ' are compact oriented manifolds of dimension $n - 1$, see figure (6.22).

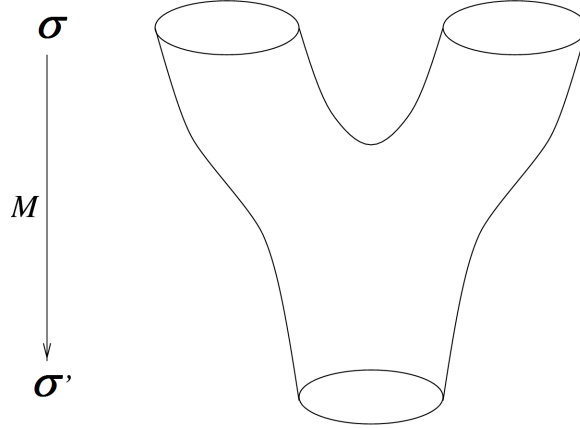


Figure 6.22:

Choose a triangulation of M . This induces triangulations of σ and σ' with dual 1-skeletons Γ and Γ' , respectively. As described in Section 6.3, in this triangulated context we can use $L^2(\mathcal{A}_\Gamma/G_\Gamma)$ as the gauge-invariant Hilbert space for σ . This Hilbert space has a basis given by spin networks whose underlying graph is the dual 1-skeleton of σ . Similarly, we use $L^2(\mathcal{A}_{\Gamma'}/G_{\Gamma'})$ as the space of gauge-invariant states on σ' . We describe time evolution as an operator

$$Z(M): L^2(\mathcal{A}_\Gamma/G_\Gamma) \rightarrow L^2(\mathcal{A}_{\Gamma'}/G_{\Gamma'}).$$

To specify this operator, it suffices to describe the transition amplitudes $\langle \Psi', Z(M)\Psi \rangle$ when Ψ, Ψ' are spin network states. We write this transition amplitude as a sum over spin foams going from Ψ to Ψ' :

$$\langle \Psi', Z(M)\Psi \rangle = \sum_{F: \Psi \rightarrow \Psi'} Z(F)$$

Since we are working with a fixed triangulation of M , we restrict the sum to spin foams whose underlying complex is the dual 2-skeleton of M . The crucial thing is the formula for the amplitude $Z(F)$ of a given spin foam F .

We have already given a formula for the amplitude of a closed spin foam in the previous section: it is computed as a product of amplitudes for spin foam faces, edges and vertices. A similar formula works for any spin foam $F: \Psi \rightarrow \Psi'$, but we need to make a few adjustments. First, when we take the product over faces, edges and vertices, we exclude edges and vertices that lie in Ψ and Ψ' . Second, we use the square root of the usual edge amplitude for edges of the form \tilde{v} , where v is a vertex of Ψ or Ψ' . Third, we use the square root of the usual face amplitudes for faces of the form \tilde{e} , where e is an edge of Ψ or Ψ' . The reason for these adjustments is that we want to have

$$Z(M')Z(M) = Z(M'M)$$

when $M: \sigma \rightarrow \sigma'$ and $M': \sigma' \rightarrow \sigma''$ are composable cobordisms and $M'M: \sigma \rightarrow \sigma''$ is their composite, see figure (6.23)

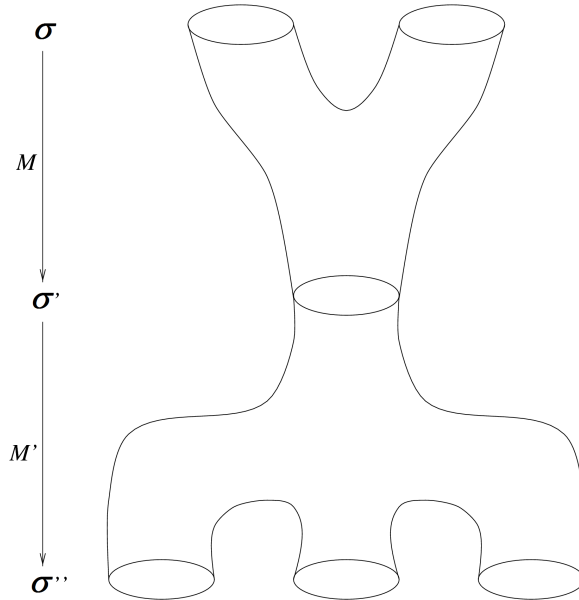


Figure 6.23:

For this to hold, we want

$$Z(F')Z(F) = Z(F'F)$$

whenever $F'F: \Psi \rightarrow \Psi''$ is the spin foam formed by gluing together $F: \Psi \rightarrow \Psi'$ and $F': \Psi' \rightarrow \Psi''$ along their common border Ψ' and erasing the vertices and edges that lie in Ψ' . The adjustments described above make this equation true. Of course, the argument that $Z(F')Z(F) = Z(F'F)$ implies $Z(M')Z(M) = Z(M'M)$ is merely formal unless the sums over spin foams used to define these time evolution operators converge in a sufficiently nice way.

Let us conclude with some general remarks on the meaning of the spin foam formalism. Just as spin networks are designed to merge the concepts of *quantum state* and the *geometry of space*, spin foams are designed to merge the concepts of *quantum history* and the *geometry of spacetime*. However, the concept of ‘quantum history’ is a bit less familiar than the concept of ‘quantum state’, so it deserves some comment. Perhaps the most familiar example of a quantum history is a Feynman diagram. A Feynman diagram determines an operator on Fock space, but there is more information in the diagram than this operator, since besides telling us transition amplitudes between states, the diagram also tells a story of ‘how the transition happened’. In other words, the internal edges and vertices of the diagram describe a ‘quantum history’ in which various virtual particles are created and annihilated.

Similarly, spin foams can be used to describe operators, but they contain extra information. If Ψ and Ψ' are spin networks with underlying graphs Γ and Γ' , respectively, then any spin foam $F: \Psi \rightarrow \Psi'$ determines an operator from $L^2(\mathcal{A}_\Gamma/G_\Gamma)$ to $L^2(\mathcal{A}_{\Gamma'}/G_{\Gamma'})$, which we also denote by F , such that

$$\langle \Phi', F\Phi \rangle = \langle \Phi', \Psi' \rangle \langle \Psi, \Phi \rangle$$

for any states Φ, Φ' . The time evolution operator $Z(M)$ is a linear combination of these operators weighted by the amplitudes $Z(F)$. But a spin foam contains more information than the operator it determines, since the operator depends only on the initial state Ψ and the final state Ψ' , not on the details of the spin foam at intermediate times. This extra information is what we call a ‘quantum history’.

How exactly does a spin foam describe the geometry of spacetime? In part, this follows from how spin networks describe the geometry of space. Consider, for example, 4d BF theory with gauge group $SU(2)$. Spin network edges give area to surfaces they puncture, while spin network vertices give volume to regions of space in which they lie. But a spin network edge is really just a slice of a spin foam face, and a spin network vertex is a slice of a spin foam edge. Thus in the spacetime context, spin foam faces give area to surfaces they intersect, while spin foam edges give 3-volume to 3-dimensional submanifolds they intersect. Continuing the pattern, one expects that spin foam vertices give 4-volume to regions of spacetime in which they lie. However, calculations have not yet been done to confirm this, in part because a thorough picture of the metric geometry of spacetime in 4 dimensions requires that one impose constraints on the E field. We discuss this a bit more in Section 6.6.

A similar story holds for 3d BF theory with gauge group $SU(2)$, or in other words, Riemannian quantum gravity in 3 dimensions. In this case, spin foam faces give length to curves they intersect and spin foam edges give area to surfaces they intersect. We expect that spin foam vertices give volume to regions of spacetime in which they lie, but so far the calculations remain a bit problematic.

Remarks

The physical meaning of the time evolution operators

$$Z(M): L^2(\mathcal{A}_\Gamma/G_\Gamma) \rightarrow L^2(\mathcal{A}_{\Gamma'}/G_{\Gamma'})$$

is somewhat subtle in a background-independent theory. For example, when $M = \sigma \times [0, 1]$ is a cylinder cobordism from S to itself, we should have $Z(M)^2 = Z(M)$. In this case $Z(M)$ should represent the projection from the gauge-invariant Hilbert space to the space of physical states.

6.6 4-Dimensional Quantum Gravity

We finally turn to theory that really motivates the interest in spin foam models: quantum gravity in 4 dimensions. Various competing spin foam models have been proposed for 4-dimensional quantum gravity — mainly in the Riemannian case so far.

While some of these models are very elegant, their physical meaning has not really been unravelled, and some basic problems remain unsolved. The main reason is that, unlike BF theory, general relativity in 4 dimensions has local degrees of freedom.

We begin by describing the Palatini formulation of general relativity in 4 dimensions. Let spacetime be given by a 4-dimensional oriented smooth manifold M . We choose a bundle \mathbf{T} over M that is isomorphic to the tangent bundle, but not in any canonical way. This bundle, or any of its fibers, is called the ‘internal space’. We equip it with an orientation and a metric η , either Lorentzian or Riemannian. Let P denote the oriented orthonormal frame bundle of M . This is a principal G -bundle, where G is either $SO(3, 1)$ or $SO(4)$ depending on the signature of η . The basic fields in the Palatini formalism are:

- a connection A on P ,
- a \mathbf{T} -valued 1-form e on M .

The curvature of A is an $\text{ad}(P)$ -valued 2-form which, as usual, we call F . Note however that the bundle $\text{ad}(P)$ is isomorphic to the second exterior power $\Lambda^2\mathbf{T}$. Thus we are free to switch between thinking of F as an $\text{ad}(P)$ -valued 2-form and a $\Lambda^2\mathbf{T}$ -valued 2-form. The same is true for the field $e \wedge e$.

The Lagrangian of the theory is

$$\mathcal{L} = \text{tr}(e \wedge e \wedge F).$$

Here we first take the wedge products of the differential form parts of $e \wedge e$ and F while simultaneously taking the wedge products of their ‘internal’ parts, obtaining the $\Lambda^4\mathbf{T}$ -valued 4-form $e \wedge e \wedge F$. The metric and orientation on \mathbf{T} give us an ‘internal volume form’, that is, a nowhere vanishing section of $\Lambda^4\mathbf{T}$. We can write $e \wedge e \wedge F$ as this volume form times an ordinary 4-form, which we call $\text{tr}(e \wedge e \wedge F)$.

To obtain the field equations, we set the variation of the action to zero:

$$\begin{aligned} 0 &= \delta \int_M \mathcal{L} \\ &= \int_M \text{tr}(\delta e \wedge e \wedge F + e \wedge \delta e \wedge F + e \wedge e \wedge \delta F) \end{aligned} \quad (6.13)$$

$$\begin{aligned} &= \int_M \text{tr}(2\delta e \wedge e \wedge F + e \wedge e \wedge d_A \delta A) \\ &= \int_M \text{tr}(2\delta e \wedge e \wedge F - d_A(e \wedge e) \wedge \delta A). \end{aligned} \quad (6.14)$$

The field equations are thus

$$e \wedge F = 0, \quad d_A(e \wedge e) = 0.$$

These equations are really just an extension of the vacuum Einstein equation to the case of degenerate metrics. To see this, first define a metric g on M by

$$g(v, w) = \eta(ev, ew).$$

When $e: TM \rightarrow \mathbf{T}$ is one-to-one, g is nondegenerate, with the same signature as η . The equation $d_A(e \wedge e) = 0$ is equivalent to $e \wedge d_A e = 0$, and when e is one-to-one this implies $d_A e = 0$. If we use e to pull back A to a metric-preserving connection Γ on the tangent bundle, the equation $d_A e = 0$ says that Γ is torsion-free, so Γ is the Levi-Civita connection of g . This lets us rewrite $e \wedge F$ in terms of the Riemann tensor. In fact, $e \wedge F$ is proportional to the Einstein tensor, so $e \wedge F = 0$ is equivalent to the vacuum Einstein equation.

There are a number of important variants of the Palatini formulation which give the same classical physics (at least for nondegenerate metrics) but suggest different approaches to quantization. Most simply, we can pick a spin structure on M and use the double cover $\text{Spin}(3, 1) \cong \text{SL}(2, \mathbf{C})$ or $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$ as gauge group. A subtler trick is to work with the ‘self-dual’ or ‘left-handed’ part of the spin connection. In the Riemannian case this amounts to using only one of the $\text{SU}(2)$ factors of $\text{Spin}(4)$ as gauge group; in the Lorentzian case we need to complexify $\text{Spin}(3, 1)$ first, obtaining $\text{SL}(2, \mathbf{C}) \times \text{SL}(2, \mathbf{C})$, and then use one of these $\text{SL}(2, \mathbf{C})$ factors. It is not immediately obvious that one can formulate general relativity using only the left-handed part of the connection, but the great discovery of Plebanski and Ashtekar is that one can. A further refinement of this trick allows one to formulate the canonical quantization of Lorentzian general relativity in terms of the e field and an $\text{SU}(2)$ connection. These so-called ‘real Ashtekar variables’ play a crucial role in most work on loop quantum gravity.

The Palatini formulation of general relativity brings out its similarity to BF theory. In fact, if we set $B = e \wedge e$, the Palatini Lagrangian looks exactly like the BF Lagrangian. The big difference, of course, is that not every $\text{ad}(P)$ -valued 2-form B is of the form $e \wedge e$. This restricts the allowed variations of the B field when we compute the variation of the action in general relativity. As a result, the equations of general relativity in 4 dimensions:

$$e \wedge F = 0, \quad d_A B = 0$$

are weaker than the BF theory equations:

$$F = 0, \quad d_A B = 0.$$

Another, subtler difference is that, even when B is of the form $e \wedge e$, we cannot uniquely recover e from B . In the nondegenerate case there is only a sign ambiguity: both e and $-e$ give the same B . Luckily, changing the sign of e does not affect the metric. In the degenerate case the ambiguity is greater, but we need not be unduly concerned about it, since we do not really know the ‘correct’ generalization of Einstein’s equation to degenerate metrics.

The relation between the Palatini formalism and BF theory suggests that one develop a spin foam model of quantum gravity by taking the spin foam model for BF theory and imposing extra constraints: quantum analogues of the constraint that B be of the form $e \wedge e$. However, there are some obstacles to doing this.

First, when computing transition amplitudes in BF theory, we only summed over spin foams living in the dual 2-skeleton of a fixed triangulation of spacetime. This was acceptable because we could later show triangulation-independence. But triangulation-independence is closely related to the fact that BF theory lacks local degrees of free-

dom: if we study BF theory on a triangulated manifold, subdividing the triangulation changes the gauge-invariant Hilbert space, but it does not increase the number of physical degrees of freedom. There is no particular reason to expect something like this to hold in 4d quantum gravity, since general relativity in 4 dimensions *does* have local degrees of freedom. So what should we do? Nobody knows! This problem requires careful thought and perhaps some really new ideas. In what follows, we simply ignore it and restrict attention to spin foams lying in the dual 2-skeleton of a fixed triangulation, for no particular good reason.

We begin by considering at the classical level the constraints that must hold for the B field to be of the form $e \wedge e$. We pick a spin structure for spacetime and take the double cover $\text{Spin}(4)$ as our gauge group. Locally we may think of the B field as taking values in the Lie algebra $\mathfrak{so}(4)$, but the splitting

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

lets us write B as the sum of left-handed and right-handed parts B^\pm taking values in $\mathfrak{so}(3)$. If $B = e \wedge e$, the following constraint holds for all vector fields v, w on M :

$$|B^+(v, w)| = |B^-(v, w)|$$

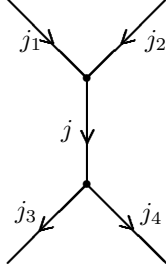
where $|\cdot|$ is the norm on $\mathfrak{so}(3)$ coming from the Killing form. In fact, this constraint is almost sufficient to guarantee that B is of the form $e \wedge e$. Unfortunately, in addition to solutions of the desired form, there are also solutions of the form $-e \wedge e$, $*(e \wedge e)$, and $-*(e \wedge e)$, where $*$ is the Hodge star operator on $\Lambda^2 \mathbf{T}$.

If we momentarily ignore this problem and work with the constraint as described, we must next decide how to impose this constraint in a spin foam model. First recall some facts about 4d BF theory with gauge group $\text{SU}(2)$. In this theory, a spin foam in the dual 2-skeleton of a triangulated 4-manifold is given by labelling each dual face with a spin and each dual edge with an intertwiner. This is equivalent to labelling each triangle with a spin and each tetrahedron with an intertwiner. We can describe these intertwiners by chopping each tetrahedra in half with a parallelogram and labelling all these parallelograms with spins. Then all the data are expressed in terms of spins labelling surfaces, and each spin describes the integral of $|B|$ over the surface it labels.

Now we are trying to describe 4-dimensional Riemannian quantum gravity as a BF theory with extra constraints, but now the gauge group is $\text{Spin}(4)$. Since $\text{Spin}(4)$ is isomorphic to $\text{SU}(2) \times \text{SU}(2)$, irreducible representation of this group are of the form $j^+ \otimes j^-$ for arbitrary spins j^+, j^- . Thus, before we take the constraints into account, a spin foam with gauge group $\text{Spin}(4)$ can be given by labelling each triangle and parallelogram with a *pair* of spins. These spins describe the integrals of $|B^+|$ and $|B^-|$, respectively, over the surface in question. Thus, to impose the constraint

$$|B^+(v, w)| = |B^-(v, w)|$$

at the quantum level, it is natural to restrict ourselves to labellings for which these spins are equal. This amounts to labelling each triangle with a representation of the form $j \otimes j$ and each tetrahedron with an intertwiner of the form $\iota_j \otimes \iota_j$, where $\iota_j: j_1 \otimes j_2 \rightarrow j_3 \otimes j_4$ is given in our graphical notation by:



and j_1, \dots, j_4 are the spins labelling the 4 triangular faces of the tetrahedron. More generally, we can label the tetrahedron by any intertwiner of the form $\sum_j c_j(\iota_j \otimes \iota_j)$.

However, there is a subtlety. There are three ways to split a tetrahedron in half with a parallelogram P , and we really want the constraint

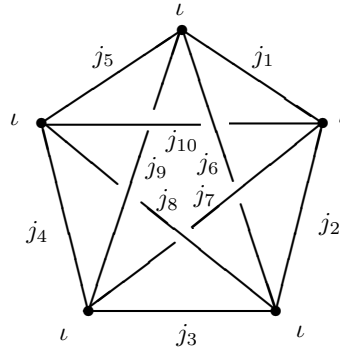
$$\int_P |B^+| = \int_P |B^-|$$

to hold for all three. To achieve this, we must label tetrahedra with intertwiners of the form $\sum_j c_j(\iota_j \otimes \iota_j)$ that *remain* of this form when we switch to a different splitting using the $6j$ symbols. Barrett and Crane found an intertwiner with this property:

$$\iota = \sum_j (2j + 1)(\iota_j \otimes \iota_j).$$

Later, Reisenberger proved that this was the unique solution. Thus, in this spin foam model for 4-dimensional Riemannian quantum gravity, seems fair to take as partition function:

$$Z(M) = \sum_{j:\mathcal{F} \rightarrow \{0, \frac{1}{2}, 1, \dots\}} \prod_{f \in \mathcal{F}} (2j_f + 1) \prod_{v \in \mathcal{V}} \iota$$



Here j_1, \dots, j_{10} are the spins labelling the dual faces meeting at the dual vertex in question, and ι is the Barrett-Crane intertwiner. One can also write down a similar formula for transition amplitudes.

There is however a serious problem: the three quantum operators corresponding to $\int_P |B^+|$ in the three different splits don't commute each other. The same holds for $\int_P |B^-|$. So, we can't impose simultaneously

$$O_1 |phys\rangle = \int_{P, split 1} |B^+| - |B^-| |phys\rangle = 0$$

$$O_2|phys\rangle = \int_{P, split 2} |B^+| - |B^-||phys\rangle = 0, \quad (6.15)$$

because $[O_1, O_2]|phys\rangle \neq 0$. This force us to do extra work to obtain a new proper vertex. We define

$$B_t = \int_t B$$

with t a triangle of the triangulation. We have

$$B = e \wedge e \Rightarrow \begin{cases} *B_t \cdot B_t = 0 & t \text{ triangle of the triangulation} \\ *B_t \cdot B_{t'} = 0 & t, t' \text{ triangles with one edge in common} \end{cases} \quad (6.16)$$

where $*$ acts on the internal indices. The second condition is substitutable with

$$n_i(*B_t)^{ij} = 0 \quad (6.17)$$

for some internal vector n_i . Introducing the Immirzi parameter

$$\begin{aligned} B \wedge F &\rightarrow \left(B + \frac{1}{\beta} *B \right) \wedge F \\ B &\rightarrow B + \frac{1}{\beta} *B := \hat{J}. \end{aligned} \quad (6.18)$$

In the hamiltonian BF-theory we have the correspondence⁴

$$\hat{J}_{ij}^{\mu 0}(x) \rightarrow \frac{\delta}{\delta A_{\mu}^{ij}(x)}$$

where the derivative operator, acting on a spin-network, gives the eigenvalue

$$\int_{\gamma} dt \dot{\gamma}(t) \delta^3(\gamma(t) - x) J^{ij}$$

with J^{ij} generator of $so(4)$. The integral in dt combines with the 2-dimensional integral in $|B_t|$, returning simply an intersection number. The constraints on B become

$$\begin{aligned} C_{tt} &= *J \cdot J \left(1 + \frac{1}{\beta^2} \right) - \frac{2}{\beta} J \cdot J = 0 \\ C_t^j &= n_j ((*J)^{ij} - \frac{1}{\beta} J^{ij}) = 0. \end{aligned} \quad (6.19)$$

The second one, for $n_i = \delta_i^0$, is rewritten as

$$C_t^j = \frac{1}{2} \epsilon_{kl}^i J^{kl} - \frac{1}{\beta} J^{0i} = L^j - \frac{1}{\beta} K^j = 0 \quad (6.20)$$

⁴ $\hat{J}_{ij}^{\mu\nu} = \epsilon_{ijkl} \epsilon^{\mu\nu\rho\sigma} \hat{j}_{\rho\sigma}^{kl}$

where L^j are generators of the subgroup $SO(3)$ that leaves n_i invariant, and K^j are generators of the corresponding boosts. The constraints C_t^j , for different j , don't commute each other: the solution is to replace them with a 'MASTER CONSTRAINT' obtained by summing their squares:

$$M_t = \sum_j (C_t^j)^2 = \sum_j (L^j - \frac{1}{\beta} K^j)^2 = 0. \quad (6.21)$$

In terms of the Casimir C_1 and of the pseudo-Casimir C_2 of $SO(4)$

$$\begin{aligned} C_1 &= J \cdot J = 2(L^2 + K^2) \\ C_2 &= *J \cdot J = 4L \cdot K, \end{aligned} \quad (6.22)$$

with L^2 Casimir of $SO(3)$, we obtain

$$M_t = L^2(1 - \frac{1}{\beta}) + \frac{1}{2\beta^2} C_1 - \frac{1}{2\beta} C_2 = 0 \quad (6.23)$$

$$C_{tt} = C_2(1 + \frac{1}{\beta^2}) - \frac{2}{\beta} C_1 = 0. \quad (6.24)$$

Replacing the second one in the first one

$$C_2 = 4\beta L^2. \quad (6.25)$$

We obtain the following conditions on the spins (j^+ , j^-) of the irreducible representations of $SO(4)$:

$$(j^+)^2 = \left(\frac{\beta + 1}{\beta - 1} \right)^2 (j^-)^2 \quad (6.26)$$

$$l = j^+ + j^- \quad (6.27)$$

where l is the quantum number associated to L . In this way the constraint selects the highest irreducible representation in the decomposition of the target space $\mathcal{H}_{(j^+, j^-)}$ seen as a reducible representation of the subgroup $SO(3)$:

$$\mathcal{H}_{(j^+, j^-)} = \mathcal{H}_{|j^+ - j^-|} \oplus \dots \oplus \mathcal{H}_{j^+ + j^-}. \quad (6.28)$$

This implies a particular form for the intertwiners I_n which enter in the $\{15j\}$ -symbol:

$$\begin{aligned} I_n \begin{matrix} q_1^+ q_1^- \\ q_2^+ q_2^- \\ q_3^+ q_3^- \\ q_4^+ q_4^- \end{matrix} &= \int_{SO(4)} dg \, c_n^{k_1 k_2 k_3 k_4} \times \bigotimes_{i=1}^4 R_{q_i^+ q_i^-, q_i^{+'} q_i^{-'}}^{\frac{(1+\gamma)l_i}{2}, \frac{|1-\gamma|l_i}{2}}(g) \, c_{k_i}^{q_i^{+'} q_i^{-'}} \\ &= \sum_i Q_i \, i_{q_1^+ q_1^-, q_2^+ q_2^-, q_3^+ q_3^-, q_4^+ q_4^-}^{\frac{(1+\gamma)l_1}{2}, \frac{|1-\gamma|l_1}{2}, \frac{(1+\gamma)l_2}{2}, \frac{|1-\gamma|l_2}{2}, \frac{(1+\gamma)l_3}{2}, \frac{|1-\gamma|l_3}{2}, \frac{(1+\gamma)l_4}{2}, \frac{|1-\gamma|l_4}{2}}. \end{aligned} \quad (6.29)$$

The weights Q_i are

$$Q_i = c_n^{k_1 k_2 k_3 k_4} \bigotimes_{i=1}^4 c_{k_i}^{q_i^+ q_i^-} i_{q_1^+ q_1^-, q_2^+ q_2^-, q_3^+ q_3^-, q_4^+ q_4^-}^{\frac{(1+\gamma)l_1}{2}, \frac{|1-\gamma|l_1}{2}, \frac{(1+\gamma)l_2}{2}, \frac{|1-\gamma|l_2}{2}, \frac{(1+\gamma)l_3}{2}, \frac{|1-\gamma|l_3}{2}, \frac{(1+\gamma)l_4}{2}, \frac{|1-\gamma|l_4}{2}}. \quad (6.30)$$

The c are intertwiners of $SO(3)$ and the indices k_i are in the representation l_i . We see as the intertwiner, in this case, isn't completely fixed, leaving open the choice of the quadrivalent intertwiners of $SO(3)$.

We write now the $15j$ -symbol of $SO(4)$ as the product of the two $15j$ -symbols of $SU(2)$. So, the amplitude of a single vertex bounded by ten $SU(2)$ spins $l_{ab}, a, b = 1, \dots, 5$ and five $SU(2)$ intertwiners i_a is given by

$$A(l_{ab}, i_a) = \sum_{i_a^+ i_a^-} 15j\left(\frac{(1+\beta)l_{ab}}{2}; i_a^+\right) 15j\left(\frac{(1-\beta)l_{ab}}{2}; i_a^-\right) \bigotimes_a f_{i_a^+ i_a^-}^{i_a}(l_{ab}) \quad (6.31)$$

where the $15j$ are the standard $SU(2)$ Wigner symbols, and

$$f_{i^+ i^-}^i := i_{(q_1^+ q_1^-) \dots (q_4^+ q_4^-)}^{i^+ i^-} c^{m_1 \dots m_4} \bigotimes_{i=1 \dots 4} c_{m_i}^{q_i^+ q_i^-}, \quad (6.32)$$

with the indices m_i in representation l_i . The partition function for an arbitrary triangulation, is given by gluing these amplitudes together with suitable edge and face amplitudes. It can be written as:

$$Z = \sum_{j_f, i_e} \prod_f d_f \prod_v A(l_f, i_e), \quad (6.33)$$

where

$$d_f := (|1 - \beta|l_f + 1) ((1 + \beta)l_f + 1). \quad (6.34)$$

6.7 Lorentzian theory

The unitary representations in the principal series are labelled by (n, ρ) , where n is a positive integer and ρ real [26, 27]. The Casimir operators for the representation (n, ρ) , are given by

$$C_1 = \frac{1}{2} (n^2 - \rho^2 - 4), \quad (6.35)$$

$$C_2 = n\rho. \quad (6.36)$$

Up to ordering ambiguities, equation (6.24) reads now

$$n\rho \left(\beta - \frac{1}{\beta} \right) = \rho^2 - n^2. \quad (6.37)$$

Solutions are given by either $\rho = \beta n$ or $\rho = -n/\beta$. The existence of these two solutions reflects the two sectors mentioned earlier with Immirzi parameter β and $-1/\beta$. BF theory can not a priori distinguish between these two sectors (see e.g. [25]). However, in our framework, the second constraint (6.25) breaks this symmetry and select the first branch $\rho = \beta n$. It further imposes that $l = n/2$, where l again labels the subspaces diagonalizing L^2 . Therefore the constraints select the lowest $SU(2)$ irreducible representation in the decomposition of $\mathcal{H}_{(n,\rho)} = \bigoplus_{l \geq n/2} \mathcal{H}_k$. Notice that the continuous label ρ becomes quantized, because n is discrete. It is because of this fact that any continuous spectrum depending on ρ (for example, Area and Volume) comes out effectively discrete on the subspace satisfying the simplicity constraints.

This construction defines the projection from the $SL(2, \mathbf{C})$ boundary Hilbert space to the $SU(2)$ space. For a single D matrix, this projection reads (see the [28]):

$$\begin{aligned} \pi & : L^2(SL(2, \mathbf{C})) \longrightarrow L^2(SU(2)) \\ D_{jqj'q'}^{n,\rho}(g) & \longmapsto D_{qq'}^{n/2}(u) \end{aligned} \quad (6.38)$$

This also defines an embedding from the $SU(2)$ Hilbert space to the $SL(2, \mathbf{C})$ space, given by inclusion followed by group averaging over the Lorentz group.

As before, in order to extend this result to the complete space \mathcal{H} we have to define the projection for the intertwiners. Consider four links meeting at a given node e of Γ , carrying representations $(n_1, \rho_1) \dots (n_4, \rho_4)$, satisfying the diagonal constraints. Consider the Hilbert space of tensors between these representations: $\mathcal{H}_e := \mathcal{H}_{(n_1, \rho_1)} \otimes \dots \otimes \mathcal{H}_{(n_4, \rho_4)}$. Construct the constraint $C_e := \sum_i M_{f_i}$. Imposing $C_e = 0$ strongly selects in each link the lowest $SU(2)$ along with the representations of the form $\rho = n\beta$. The last step is group averaging over $SL(2, \mathbf{C})$ which defines the physical intertwiner space for this node. The projection is then given by:

$$\begin{aligned} \pi & : Inv_{SL(2, \mathbf{C})}(\mathcal{H}_e) \longrightarrow Inv_{SU(2)}\left(\mathcal{H}_{\frac{n_1}{2}} \otimes \dots \otimes \mathcal{H}_{\frac{n_4}{2}}\right), \\ i_{(j_1, q_1) \dots (j_4, q_4)}^{(n_e, \rho_e)} & \longmapsto i_{(\frac{n_1}{2}, q_1) \dots (\frac{n_4}{2}, q_4)}^{(n_e, \rho_e)} \bigotimes_{i=1 \dots 4} c_{m_i}^{\frac{n_i}{2} q_i}. \end{aligned} \quad (6.39)$$

The embedding is given by:

$$\begin{aligned} f & : Inv_{SU(2)}(\mathcal{H}_{j_1} \otimes \dots \otimes \mathcal{H}_{j_4}) \longrightarrow Inv_{SL(2, \mathbf{C})}(\mathcal{H}_e), \\ c^{m_1 \dots m_4} & \longmapsto \int_{SL(2, \mathbf{C})} dg c^{m_1 \dots m_4} \bigotimes_{i=1}^{i=4} D_{(j'_i, m'_i)(j_i, m_i)}^{(2j_i, 2j_i\beta)}(g). \end{aligned}$$

The boundary space is once again just given by the $SU(2)$ spin networks.

We are now ready to define the vertex. Similarly as before, we obtain

$$\begin{aligned} A(l_{ab}, i_a) & = \sum_{n_a} \int d\rho_a (n_a^2 + \rho_a^2) \left(\bigotimes_a f_{n_a \rho_a}^{i_a}(l_{ab}) \right) \\ & \quad 15j_{SL(2, \mathbf{C})}((2l_{ab}, 2l_{ab}\beta); (n_a, \rho_a)) \end{aligned} \quad (6.40)$$

where we are now using the 15j of $SL(2, \mathbf{C})$ and

$$f_{n\rho}^i := i^{m_1 \dots m_4} \bar{C}_{(j_1, m_1) \dots (j_4, m_4)}^{m\rho}, \quad (6.41)$$

where $j_1 \dots j_4$ are the representations meeting at the node. The final partition function, for an arbitrary triangulation, is given by gluing these amplitudes together with suitable edge and face amplitudes:

$$Z = \sum_{l_f, i_e} \prod_f (2l_f)^2 (1 + \beta^2) \prod_v A(l_f, i_e). \quad (6.42)$$

Remarks

1. Regge gave a formula for a discretized version of the action in 4-dimensional Riemannian general relativity. In his approach, spacetime is triangulated and each edge is assigned a length. The Regge action is the sum over all 4-simplices of:

$$S = \sum_t A_t \theta_t$$

where the sum is taken over the 10 triangular faces t , A_t is the area of the face t , and θ_t is the dihedral angle of t , that is, the angle between the outward normals of the two tetrahedra incident to this face. Calculations suggest that the spin foam vertex amplitudes in the EPR theory are related to the Regge action by a formula very much like the one relating vertex amplitudes in 3d Riemannian quantum gravity to the Ponzano-Regge action (see Remark 1 of Section 8.103 and the beautiful work of J.W.Barrett and collaborators [73] where we see that the partition function of the EPR theory corresponds, in the limit of large scale $j \rightarrow \infty$, to the exponential of the Regge action).

Appendix: Piecewise linear cell complexes

Here we give the precise definition of ‘piecewise linear cell complex’. A subset $X \subseteq R^n$ is said to be a ‘polyhedron’ if every point $x \in X$ has a neighborhood in X of the form

$$\{\alpha x + \beta y : \alpha, \beta \geq 0, \alpha + \beta = 1, y \in Y\}$$

where $Y \subseteq X$ is compact. A compact convex polyhedron X for which the smallest affine space containing X is of dimension k is called a ‘ k -cell’. The term ‘polyhedron’ may be somewhat misleading to the uninitiated; for example, R^n is a polyhedron, and any open subset of a polyhedron is a polyhedron. Cells, on the other hand, are more special. For example, every 0-cell is a point, every 1-cell is a compact interval affinely embedded in R^n , and every 2-cell is a convex compact polygon affinely embedded in R^n .

The ‘vertices’ and ‘faces’ of a cell X are defined as follows. Given a point $x \in X$, let $\langle x, X \rangle$ be the union of lines L through x such that $L \cap X$ is an interval with x in its interior. If there are no such lines, we define $\langle x, X \rangle$ to be $\{x\}$ and call x a ‘vertex’ of X . One can show that $\langle x, X \rangle \cap X$ is a cell, and such a cell is called a ‘face’ of X . (In the body of this paper we use the words ‘vertex’, ‘edge’ and ‘face’ to stand for 0-cells, 1-cells and 2-cells, respectively. This should not be confused with the present use of these terms.)

One can show that any cell X has finitely many vertices v_i and that X is the convex hull of these vertices, meaning that:

$$X = \left\{ \sum \alpha_i v_i : \alpha_i \geq 0, \sum \alpha_i = 1 \right\}.$$

Similarly, any face of X is the convex hull of some subset of the vertices of X . However, not every subset of the vertices of X has a face of X as its convex hull. If the cell Y is a face of X we write $Y \leq X$. This relation is transitive, and if $Y, Y' \leq X$ we have $Y \cap Y' \leq X$.

Finally, one defines a ‘piecewise linear cell complex’, or ‘complex’ for short, to be a collection κ of cells in some R^n such that:

1. If $X \in \kappa$ and $Y \leq X$ then $Y \in \kappa$.
2. If $X, Y \in \kappa$ then $X \cap Y \leq X, Y$.

In this paper we restrict our attention to complexes with finitely many cells.

A complex is ‘ k -dimensional’ if it has cells of dimension k but no higher. A complex is ‘oriented’ if every cell is equipped with an orientation, with all 0-cells being equipped with the positive orientation. The union of the cells of a complex κ is a polyhedron which we denote by $|\kappa|$.

When discussing spin foams we should really work with spin networks whose underlying graph is a 1-dimensional oriented complex. Suppose Γ is a 1-dimensional oriented complex and κ is a 2-dimensional oriented complex. Note that the product $\Gamma \times [0, 1]$ becomes a 2-dimensional oriented complex in a natural way. We say Γ ‘borders’ κ if there is a one-to-one affine map $c: |\Gamma| \times [0, 1] \rightarrow |\kappa|$ mapping each cell of $\Gamma \times [0, 1]$ onto a unique cell of κ in an orientation-preserving way, such that c maps $\Gamma \times [0, 1]$ onto an open subset of $|\kappa|$. Note that in this case, c lets us regard each k -cell of Γ as the face of a unique $(k + 1)$ -cell of κ .

Chapter 7

Group Field Theory

7.1 The case of BF-Theory

There is a surprising duality between the Regge models on the one hand, and certain peculiar QFTs defined over a group (Group Field Theory, or GFT) on the other. This duality will play an important role in what follows. Consider a real field $\phi(g_1, g_2, g_3, g_4)$ over the cartesian product of four copies of $G = SO(4)$. Require that ϕ is $SO(4)$ invariant, in the sense

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_1 g, g_2 g, g_3 g, g_4 g), \quad (\forall g \in SO(4)). \quad (7.1)$$

Consider the QFT defined by the action

$$\begin{aligned} S[\phi] &= \frac{1}{2} \int \prod_{i=1}^4 dg_i \phi^2(g_1, g_2, g_3, g_4) + \\ &+ \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \phi(g_1, g_2, g_3, g_4) \phi(g_4, g_5, g_6, g_7) \phi(g_7, g_3, g_8, g_9) \times \\ &\times \phi(g_9, g_6, g_2, g_{10}) \phi(g_{10}, g_8, g_5, g_1). \end{aligned} \quad (7.2)$$

The potential (fifth-order) term has the structure of a 4-simplex: if we represent each of the five fields in the product as a node with 4 legs - one for each g_i - and connect pairs of legs corresponding to the same argument, we obtain (the one-skeleton of) a 4-simplex, see figure (7.1).

The remarkable fact about this field theory is the following. The Feynman expansion of the partition function of the GFT

$$Z = \int D\phi e^{-S[\phi]} \quad (7.3)$$

turns out to be given by a sum over Feynman graphs

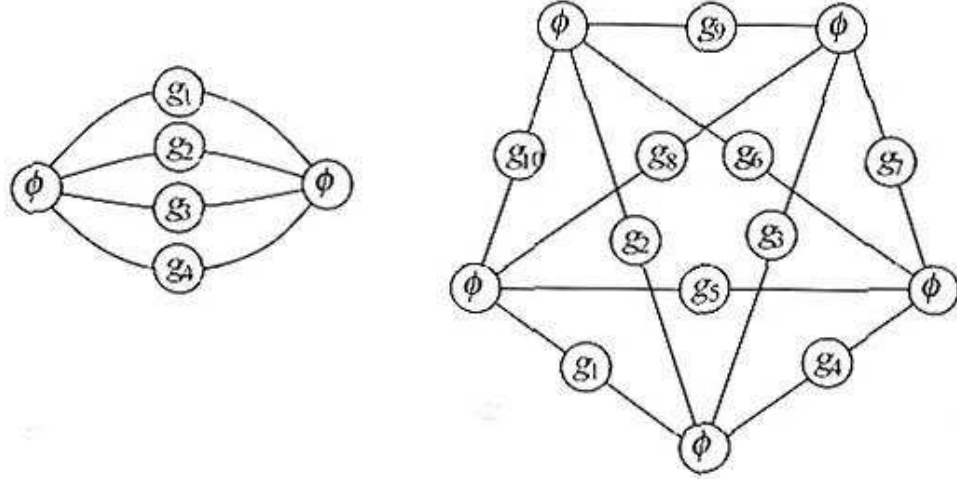


Figure 7.1: The structure of the kinetic and potential terms in the action.

$$Z = \sum_{\Gamma} \frac{\lambda^{v[\Gamma]}}{\text{sym}[\Gamma]} Z[\Gamma], \tag{7.4}$$

where the amplitude of a Feynman graph is

$$Z[\Gamma] = \sum_{j_f, i_e} \prod_f \text{dim}(j_f) \prod_v \{15j\}_v, \tag{7.5}$$

that is the BF amplitude on a triangulate manifold with group $SO(4)$. Here Γ is a Feynman graph, $v[\Gamma]$ the number of its vertices and $\text{sym}[\Gamma]$ its symmetry factor. The Feynman graphs Γ of the theory have a natural additional structure as two complexes. The Feynman integrals over momenta are discrete sums (because the space on which the QFT is defined is compact) over $SO(4)$ representations j_f and over intertwiners i_e associated with faces and edges of the two-complex. The proof of these results is a straightforward application of perturbative expansion methods in QFT, and the use of the Peter-Weyl theorem that allows us to mode-expand functions on a group in terms of a basis given by the unitary irreducible representations of the group.

Mode expansion. First, expand the field $\phi(g_1, g_2, g_3, g_4)$ into modes and rewrite the action in terms of these modes (in “momentum space”). Consider a square integrable function $\phi(g)$ over $SO(4)$. The Peter-Weyl theorem tells us that we can expand this function in the matrix elements $R_{\alpha\beta}^{(j)}(g)$ of the unitary irreducible representations j

$$\phi(g) = \phi_{\alpha\beta}^j R_{\alpha\beta}^{(j)}(g) \tag{7.6}$$

The indices α, β label basis vectors in the corresponding representation space. Accordingly, the field can be expanded into modes as

$$\phi(g_1, \dots, g_4) = \phi_{\alpha_1 \beta_1 \dots \alpha_4 \beta_4}^{j_1 \dots j_4} R_{\alpha_1 \beta_1}^{(j_1)}(g_1) \dots R_{\alpha_4 \beta_4}^{(j_4)}(g_4). \quad (7.7)$$

Using the invariance (7.1) under the left group action we can write

$$\phi(g_1, \dots, g_4) = \int_{SO(4)} dg \phi(gg_1, \dots, gg_4), \quad (7.8)$$

and exploit

$$\int dU R^{j_1}(U)_{\alpha'}^{\alpha} R^{j_2}(U)_{\beta'}^{\beta} \dots R^{j_n}(U)_{\delta'}^{\delta} = \sum_i v_i^{\alpha \beta \dots \delta} v_{\alpha' \beta' \dots \delta'}^i \quad (7.9)$$

with the index i labels the orthonormal basis $v_i^{\alpha \beta \dots \delta}$ in the space of the intertwiners between the representations of spin j_1, j_2, \dots, j_n . Substituting the mode expansion (7.6) we can write

$$\phi(g_1, g_2, g_3, g_4) = \phi_{\alpha_1 \dots \alpha_4 i}^{j_1 \dots j_4} R_{\alpha_1 \beta_1}^{j_1}(g_1) \dots R_{\alpha_4 \beta_4}^{j_4}(g_4) v_{\beta_1 \dots \beta_4}^i, \quad (7.10)$$

where we have defined

$$\phi_{\alpha_1 \dots \alpha_4 i}^{j_1 \dots j_4} = \phi_{\alpha_1 \beta_1 \dots \alpha_4 \beta_4}^{j_1 \dots j_4} v_{\beta_1 \dots \beta_4}^i. \quad (7.11)$$

We use the quantities $\phi_{\alpha_1 \dots \alpha_4 i}^{j_1 \dots j_4}$ as the Fourier components of the field. Written in terms of these, the kinetic term of the action reads

$$\frac{1}{2} \int \prod_{i=1}^4 dg_i \phi^2(g_1, \dots, g_4) = \frac{1}{2} \sum_{j_1 \dots j_4} \sum_i \phi^{j_1 \dots j_4 i} \phi^{j_1 \dots j_4 i}. \quad (7.12)$$

The interaction term becomes

$$\begin{aligned} & \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \phi(g_1, g_2, g_3, g_4) \phi(g_4, g_5, g_6, g_7) \phi(g_7, g_3, g_8, g_9) \times \\ & \times \phi(g_9, g_6, g_2, g_{10}) \phi(g_{10}, g_8, g_5, g_1) \\ & = \frac{\lambda}{5!} \sum_{j_1 \dots j_{10}} \sum_{i_1 \dots i_5} \phi^{j_1 j_2 j_3 j_4 i_1} \phi^{j_4 j_5 j_6 j_7 i_2} \phi^{j_7 j_3 j_8 j_9 i_3} \phi^{j_9 j_6 j_2 j_{10} i_4} \phi^{j_{10} j_8 j_5 j_1 i_5} \times \\ & \times \{15j\}(j_1 \dots j_{10}, i_1, \dots, i_5). \end{aligned} \quad (7.13)$$

Feynman graphs. The partition function is given by the integral over modes

$$Z = \int [D\phi^{j_1 \dots j_4 i}] e^{-S[\phi]}. \quad (7.14)$$

We expand Z in powers of λ . The gaussian integrals are easily computed, giving the propagator

$$P^{j_1 \dots j_4 i, j'_1 \dots j'_4 i'} \equiv \langle \phi^{j_1 \dots j_4 i} \phi^{j'_1 \dots j'_4 i'} \rangle = \frac{1}{4!} \sum_{\sigma} \delta_{i'}^i \delta^{j_1 j'_{\sigma(1)}} \dots \delta^{j_4 j'_{\sigma(4)}} \delta_{\alpha_1 \alpha'_{\sigma(1)}} \dots \delta_{\alpha_4 \alpha'_{\sigma(4)}}, \quad (7.15)$$

where σ are permutations of $\{1, 2, 3, 4\}$. There is a single vertex, of order five, which is:

$$\langle \phi^{j_1 j_2 j_3 j_4 i_1} \dots \phi^{j_{10} j_8 j_5 j_1 i_5} \rangle = \lambda \{15j\}(j_1, \dots, j_{10}, i_1, \dots, i_5) \prod_{n < m}^{10} \delta^{j_n j_m} \delta_{\alpha_n \alpha_m}. \quad (7.16)$$

The set of Feynman rules one gets is as follows. First, we obtain the usual overall factor $\lambda^{v[\Gamma]}/\text{sim}[\Gamma]$. Second, we represent each of the terms in the right-hand side of the definition (7.15) of the propagator by four parallel strands, as in figure (7.2), carrying the indices at their ends. We can represent the propagator itself by the symmetrization of the four strands. In addition, propagators are labeled by a representation i that in the dual picture is the spin of the corresponding edge, bordering four faces.



Figure 7.2: The propagator can be represented by a collection of four strands, each carrying a representation.

The Feynman graphs we get are all possible “4-strand” five-valent graphs, where a “4-strand graph” is a graph whose edges are collections of four strands, and whose vertices are those shown in figure (7.3). Each strand of the propagator can be connected to a single strand in each of the five “openings” of the vertex. Orientations in the vertex and in the propagators should match (this can always be achieved by changing a representation to its conjugate). Each strand of the 4-strand graph goes through several vertices and several propagators, and then closes, forming a cycle. A particular strand can go through a particular edge of the 4-strand graph more than once. Cycles get labeled by the simple representations of the indices. For each graph, the abstract set formed by the vertices, the edges, and the cycles form a two-complex, in which the faces are the cycles. The labeling of the cycles by simple representations of $SO(4)$ determines a coloring of the faces by spins. Thus, we obtain a colored two-simplex, namely a spinfoam.

7.2 Transition amplitudes

Next, consider $SO(4)$ -invariant transition amplitudes in the GFT. That is, let $f[\phi]$ be an $SO(4)$ -invariant polynomial functional of the field, and consider the amplitude

$$W(f) = \int D\phi f[\phi] e^{-S[\phi]} \quad (7.17)$$

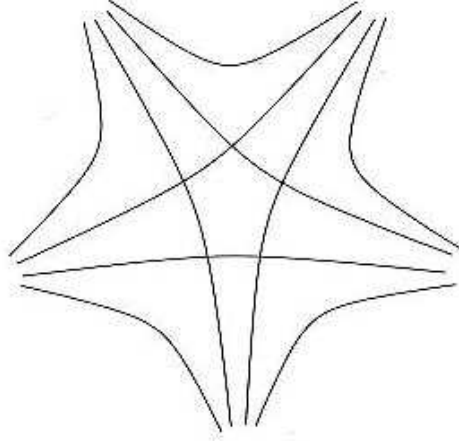


Figure 7.3: The structure of the vertex generated by the Feynman expansion.

and its expansion in Feynman graphs

$$W(f) = \sum_{\Gamma} \frac{\lambda^{v[\Gamma]}}{\text{sym}[\Gamma]} Z_f[\Gamma]. \quad (7.18)$$

It is simplest to construct all $SO(4)$ -invariant polynomial functionals of the field in momentum space, namely as functions of the Fourier modes $\phi_{\alpha_1 \dots \alpha_4 i}^{j_1 \dots j_4}$. To obtain an $SO(4)$ scalar, we must contract the indices α_n . We start with n field variables $\phi_{\alpha_1 \dots \alpha_4 i}^{j_1 \dots j_4}$ and contract the indices pairwise in all possible manners. The resulting functional is determined by a four-valent graph Γ giving the pattern of the indices contraction, colored with representations j_l on the links and the intertwiners i_n on the nodes. The set of data $s = (\Gamma, j_l, i_n)$ forms precisely a spin network. In other words, the $SO(4)$ -invariant observables of the GFT are labeled by spin networks!

Writing n_1, \dots, n_4 to indicate four links adjacent to the node n , we have

$$f_s[\phi] = \prod_n \phi_{\alpha_{n_1} \dots \alpha_{n_4} i_n}^{j_{n_1} \dots j_{n_4}} \prod_l \delta_{\alpha_{l_1} \alpha_{l_2}}, \quad (7.19)$$

where $n_i = l_1$ (or $n_i = l_2$) if the i th link of the node n is the outgoing (or ingoing) link l .

For instance the spin network $s = (\Gamma, j_1, \dots, j_4, i_1, i_2)$ on a graph with two nodes connected by four links determines the function of the field

$$f_s[\phi] = \phi_{\alpha_1 \dots \alpha_4 i_1}^{j_1 \dots j_4} \phi_{\alpha_1 \dots \alpha_4 i_2}^{j_1 \dots j_4}. \quad (7.20)$$

In the coordinate space it correspond to

$$f_s[\phi] = \int \prod_{l=1}^4 dg_l \prod_{n=1}^2 \phi(g_{n_1}, \dots, g_{n_4}) f_s(g_{n_i}), \quad (7.21)$$

where the spin network function is

$$f_s(g_{n_i}) = \prod_{n=1}^2 v_i^{\alpha_{n_1} \dots \alpha_{n_4}} \prod_{l=1}^4 R^{j_l}(g_l)_{\alpha_{l_1} \alpha_{l_2}}. \quad (7.22)$$

All transition amplitudes of the GFT can therefore be expressed in terms of the spin network amplitude

$$W(s) = \int D\phi f_s[\phi] e^{-S[\phi]}. \quad (7.23)$$

Consider the Feynman expansion of these. As usual in QFTs, the expectation value of a polynomial of order n in the fields has n external legs. In the Feynman expansion of the GFT we have, in addition, to consider the faces. These turn out to be bounded precisely by the links of the spin network. In other words, the Feynman expansion of $W(s)$ is given by

$$W(s) = \sum_{\partial\Gamma=s} \frac{\lambda^{v[\Gamma]}}{\text{sym}[\Gamma]} Z_s[\Gamma], \quad (7.24)$$

where the sum is over all two-complexes bounded by s and the amplitude of the Feynman graph is

$$Z_s[\Gamma] = \sum_{j_f, i_e} \prod_f \dim(j_f) \prod_v \{15j\}_v. \quad (7.25)$$

The coloring of the external nodes and links is determined by s and not summed over. Expressing this the other way around: the spinfoam sum at a fixed spin network boundary s is determined by the GFT expectation value (7.25)! As far as the BF-theory is concerned, the duality is not particularly useful. BF-theory has a large invariance group that implies that the theory is topological. This implies that the corresponding spinfoam model is triangulation invariant, up to a divergent factor. Therefore, the GFT amplitudes are given by divergent sums of equal terms. On the other hand, the spinfoam/GFT duality will play a crucial role in the context of the quantum gravity. Before this we must learn what is a special type of intertwiner called ‘‘coherent state’’.

7.3 Coherent States

The input data for the 4-simplex amplitude is a spin $l \in \{0, \frac{1}{2}, 1, \dots\}$ for each triangle of the 4-simplex and an $SU(2)$ intertwiner $\hat{\iota}$ for each tetrahedron. From $\hat{\iota}$, a $\text{Spin}(4)$ intertwiner ι is constructed, and then these $\text{Spin}(4)$ intertwiners are glued together in the standard fashion to construct an amplitude (a complex number) for this data. The only other input required is the Immirzi parameter β , which is a constant.

Firstly, a precise definition of $\hat{\iota}$ is required. For a given tetrahedron, one has to choose an ordering of its four faces, e.g., by numbering them with 1, 2, 3, 4. Then the $SU(2)$ intertwiner $\hat{\iota}$ is an element of

$$\mathrm{Hom}_{SU(2)}(\mathbf{C}, \bigotimes_{i=1}^4 V_{l_i}),$$

where the spaces are tensored together in the order $V_{l_1} \otimes V_{l_2} \otimes V_{l_3} \otimes V_{l_4}$. This ordering convention is used throughout.

Of course the spaces constructed using different orderings are easily related by an action of the permutation group. We use the *binor* category of representations [29, 30, 31] throughout the paper. In this category the crossing diagram is fermionic, which means that the crossing of two lines of odd spin gives a factor of -1 . For example, the map $V_{l_1} \otimes V_{l_2} \rightarrow V_{l_2} \otimes V_{l_1}$ is $x \otimes y \rightarrow (-1)^{4l_1 l_2} y \otimes x$. Spin network diagrams in this category can be evaluated using the Kauffman bracket [32] specialised to Kauffman's parameter $A = -1$. The binor calculus has the convenient feature that the framing of a curve does not affect the evaluation.

The ι are constructed as follows. Let (R_l, V_l) and $(R_{(j^-, j^+)}, V_{(j^-, j^+)})$ respectively denote the unitary, irreducible representations of $SU(2)$ and $\mathrm{Spin}(4) = SU(2)_- \times SU(2)_+$. There exists an injection

$$\phi : \mathrm{Hom}_{SU(2)}(\mathbf{C}, \bigotimes_{i=1}^4 V_{l_i}) \rightarrow \mathrm{Hom}_{\mathrm{Spin}(4)}(\mathbf{C}, \bigotimes_{i=1}^4 V_{(j_i^-, j_i^+)}) \quad (7.26)$$

$$\hat{\iota} \mapsto \phi(\hat{\iota}) := \iota, \quad (7.27)$$

embedding the vector space of $SU(2)$ intertwiners into the vector space of $\mathrm{Spin}(4)$ intertwiners. Explicitly, ϕ is constructed by using the Clebsch-Gordan intertwining maps $C_l^{j^- j^+} : V_l \rightarrow V_{j^-} \otimes V_{j^+}$ injecting the $SU(2)$ representation V_l into the highest diagonal $SU(2)$ subgroup factor $l = j^+ + j^-$ of $V_{(j^-, j^+)} \cong V_{j^-} \otimes V_{j^+}$ in the $\beta < 1$ case. The labels j^\pm and l are related via the Immirzi parameter by

$$j^\pm = \frac{1}{2}|1 \pm \beta|l. \quad (7.28)$$

These relations of course constrain the values of l so that the j^\pm are always half integers; specifically if $\beta = p/q$ is written in lowest terms, then l has to be a multiple of either $q/2$, or q in some cases.

The $\mathrm{Spin}(4)$ intertwiner ι is then obtained as follows

$$\iota := \phi(\hat{\iota}) = \int_{\mathrm{Spin}(4)} dG \bigotimes_{i=1}^4 (R_{j_i^-} \otimes R_{j_i^+})(G) \circ C_{l_i}^{j_i^- j_i^+} \circ \hat{\iota}^{l_1 l_2 l_3 l_4}, \quad (7.29)$$

where the notation $G = (X^-, X^+)$ is used (see figure 7.4). The group integration ensures that the resulting object is $\mathrm{Spin}(4)$ -invariant, i.e., is an element of

$$\mathrm{Hom}_{\mathrm{Spin}(4)}(\mathbf{C}, \bigotimes_{i=1}^4 V_{(j_i^-, j_i^+)}).$$

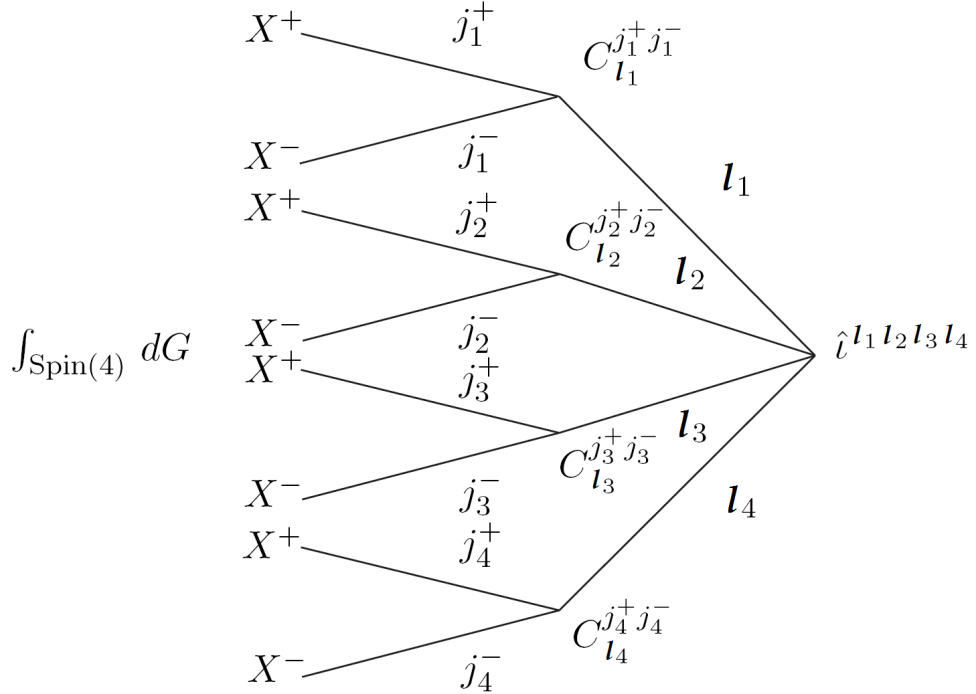


Figure 7.4: The $Spin(4)$ intertwiner ν .

The dynamics of the EPRL model is encoded in the four-simplex, or vertex amplitude f_4 constructed by contracting the specified intertwining operators associated to each of the five tetrahedra Δ_3 of the four-simplex Δ_4 . Labelling these tetrahedra by $a = 1, \dots, 5$, the ten triangles Δ_2 of Δ_4 are then indexed by the pair ab of tetrahedra which intersect on the triangle. There are two $SU(2)$ group elements (X_a^-, X_a^+) and one $SU(2)$ intertwiner $\hat{\iota}_a$ for each tetrahedron, and three (β -related) $SU(2)$ representations l_{ab} and (j_{ab}^-, j_{ab}^+) for each triangle. The intertwiner $\hat{\iota}_a$ lies in the space

$$\text{Hom}_{SU(2)}(\mathbf{C}, \bigotimes_{b:b \neq a} V_{l_{ab}}),$$

with the tensor product ordered by the numerical order of b , and $l_{ab} = l_{ba}$.

The amplitude $f_4 \in \mathbf{C}$ is defined by forming a closed spin network diagram from this data. The five intertwiners (vertices) ι_a are tensored together and then the free ends are joined pairwise according to the combinatorics, i.e. the edge a of vertex b is joined to edge b of vertex a . This is done using the standard ‘ ϵ inner product’ of irreducible representations of $SU(2)$, denoted

$$\epsilon_l: V_l \otimes V_l \rightarrow \mathbf{C}.$$

This is defined by a choice of the two-dimensional antisymmetric tensor ϵ for $SU(2)$ spin 1/2, and extended to arbitrary spin by tensor products of ϵ . This inner product

is represented in the spin network diagram as a semicircular arc. To combine the Spin(4) intertwiners ι_a , one regards each vertex as an $SU(2)$ spin network and uses one ϵ inner product to connect the j^+ edges and a second ϵ inner product to connect the j^- edges. The resulting closed diagram is evaluated using the binor calculus rules for all crossings. (Note: there is no sign for a crossing of a + line with a -. This makes at most a difference of an overall sign to f_4 .)

This yields a formula

$$f_4 = (-1)^x \int_{\text{Spin}(4)^5} \prod_a dX_a^+ dX_a^- \left(\bigotimes_{a < b} \mathcal{K}_{ab} \right) \circ \left(\bigotimes_a \hat{\iota}_a \right) \quad (7.30)$$

where the propagators $\mathcal{K}_{ab}: V^{l_{ab}} \otimes V^{l_{ab}} \rightarrow \mathbf{C}$ are defined by

$$\begin{aligned} \mathcal{K}_{ab} = & \epsilon_{j_{ab}^-} \otimes \epsilon_{j_{ab}^+} \circ \left(\left(R_{j_{ab}^-}(X_a^-) \otimes R_{j_{ab}^+}(X_a^+) \circ C_{l_{ab}}^{j_{ab}^- j_{ab}^+} \right) \otimes \right. \\ & \left. \otimes \left(R_{j_{ab}^-}(X_b^-) \otimes R_{j_{ab}^+}(X_b^+) \circ C_{l_{ab}}^{j_{ab}^- j_{ab}^+} \right) \right) \end{aligned} \quad (7.31)$$

and $(-1)^x$ is the sign defined by the diagrammatic calculus of spin networks. Notice that as f_4 is linear in the $\hat{\iota}$ we can use unnormalized intertwiners and push the normalization into $f_3(\hat{\iota})^2 = \frac{1}{\hat{\iota} \cdot \hat{\iota}}$, the asymptotic behaviour of which can then be analysed independently.

As a final remark, using the $SU(2)$ -invariance of the Clebsch-Gordan maps, one can set one of the two group arguments of the propagator, say the left handed part, to the identity. In this case, the amplitude (7.30) becomes the Feynman evaluation associated to a tensor field theory over $S^3 \cong SU(2)$. The ‘matter fields’ are identified as sections of the vector bundle $E = P \times_l V_l$ associated to the trivial principal bundle $P = \text{Spin}(4) \cong S^3 \times SU(2)$ with base manifold S^3 and structure group $SU(2)$ via the representation l . The internal indices of the propagators of the matter fields are contracted at the vertices of the diagram using the $SU(2)$ intertwiners $\hat{\iota}$, and the amplitude (7.30), with $X^- = \mathbb{1}$, is the Feynman evaluation associated to the complete graph with five vertices.

7.3.1 Coherent States

Let $\mathbf{n} \in S^2$ be a unit 3-vector. Then a coherent state $\alpha \in V_l$ in direction \mathbf{n} is a unit vector satisfying

$$(\mathbf{L} \cdot \mathbf{n}) \alpha = i l \alpha,$$

with \mathbf{L} the standard anti-hermitian rotation generators in the l representation and the dot ‘ \cdot ’ denoting the 3d (Euclidean) scalar product. The coherent state has the maximal spin projection along the \mathbf{n} axis.

One way of thinking of coherent states is to pick an arbitrary coherent state $\Gamma(\mathbf{n})$ for each unit vector \mathbf{n} . Then any other coherent state is a phase factor $e^{i\theta}$ times one of these standard ones, $\alpha = e^{i\theta} \Gamma(\mathbf{n})$. This information is displayed in bra-ket notation as

$$\alpha = |l, \mathbf{n}, \theta\rangle.$$

In fact the choice of section Γ does not play any role.

The notation is shortened in the following ways. If the parameter l is omitted, then the fundamental representation is used; thus $|\frac{1}{2}, \mathbf{n}, \theta\rangle = |\mathbf{n}, \theta\rangle$. Further, the parameter θ will be omitted later when the phase is obvious from the context.

Embedding V_l in $\otimes^{2l} V_{1/2}$ shows that

$$|l, \mathbf{n}, \theta\rangle = \otimes^{2l} |\mathbf{n}, \theta\rangle$$

is a coherent state for \mathbf{n} and V_l . This will be clearer in the next sections. Therefore the Hermitian inner product of coherent states obeys

$$|\langle l, \mathbf{n}_1, \theta_1 | l, \mathbf{n}_2, \theta_2 \rangle|^2 = \left(\frac{1}{2} (1 + \mathbf{n}_1 \cdot \mathbf{n}_2) \right)^{2l}, \quad (7.32)$$

where $|\cdot|^2$ denotes modulus square.

7.3.2 Coherent tetrahedra

To construct a coherent intertwiner for a given tetrahedron, the idea is to associate a coherent state to each one of its triangles. The geometrical picture is that the coherent state $|l, \mathbf{n}, \theta\rangle$ then carries the interpretation of the normal of length l and direction \mathbf{n} to the associated triangle, plus a phase factor.

Furthermore, we want to describe tetrahedra with three-dimensional rotational symmetry and the coherent intertwiners are thus constructed by integrating over all spatial directions the tensor product of four coherent states

$$\hat{\tau}_{l_1 l_2 l_3 l_4}^{m_1 m_2 m_3 m_4}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4) = \int_{\text{SU}(2)} dh \prod_{a=1}^4 \langle l_a m_a | \bigotimes_{i=1}^4 h | l_i, \mathbf{n}_i, \theta_i \rangle. \quad (7.33)$$

These intertwiners were introduced by Livine and Speziale [35], who gave an asymptotic formula for their normalisation.

According to the ‘quantization commutes with reduction’ theorem of Guillemin and Sternberg [34], the space of intertwiners is spanned by the $\hat{\tau}$ determined by vectors satisfying the *closure constraint*

$$l_1 \mathbf{n}_1 + l_2 \mathbf{n}_2 + l_3 \mathbf{n}_3 + l_4 \mathbf{n}_4 = 0 \quad (7.34)$$

Therefore in this paper the coherent tetrahedron states are always taken to satisfy this condition. The condition also implies there is a tetrahedron $t \in R^3$ in Euclidean space, with the standard metric, which has these four vectors as the outward face normals and triangle areas equal to l_i [33].

The tetrahedron t is uniquely specified by the four vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4$, up to parallel translation. Therefore the tetrahedron has a definite orientation. The coherent state $\hat{\tau}$ is averaged by the action of $\text{SU}(2)$ on the coherent states, which amounts to an action of $\text{SO}(3)$ on the tetrahedron t , i.e. rigid rotations which preserve the orientation. The geometry of the tetrahedron which is invariant under these rotations is a metric and an orientation. The coherent state is a quantum version of the geometry of this tetrahedron.

7.3.3 Exponentiated expression for the four-simplex amplitude

The general considerations of the previous sections are now applied to the case of a 4-simplex σ . The boundary data is specified on the simplicial 3-manifold $\Sigma = \partial\sigma$. Using the coherent states framework, the four-simplex amplitude is

$$f_4 = (-1)^{\chi'} \int_{\text{Spin}(4)^5} \prod_a dG_a \int_{\text{SU}(2)^5} \prod_a dh_a \prod_{a<b} \mathcal{P}_{ab}, \quad (7.35)$$

where the coherent propagator \mathcal{P}_{ab} is now given by

$$\mathcal{P}_{ab} = \langle l_{ab}, -\mathbf{n}_{ab} | R_{l_{ab}}(h_a^{-1}) C_{j_{ab}^- j_{ab}^+}^{l_{ab}} R_{j_{ab}^-}(X_{ab}^-) R_{j_{ab}^+}(X_{ab}^+) C_{l_{ab}}^{j_{ab}^- j_{ab}^+} R_{l_{ab}}(h_b) | l_{ba}, \mathbf{n}_{ba} \rangle, \quad (7.36)$$

using the notation $X_{ab}^\pm := (X_a^\pm)^{-1} X_b^\pm$, and $C_{j_- j_+}^l: V_l \rightarrow V_{j_-} \otimes V_{j_+}$ the reflected spin network, as shown in figure (7.5). This is proved starting from (7.30), rotating some

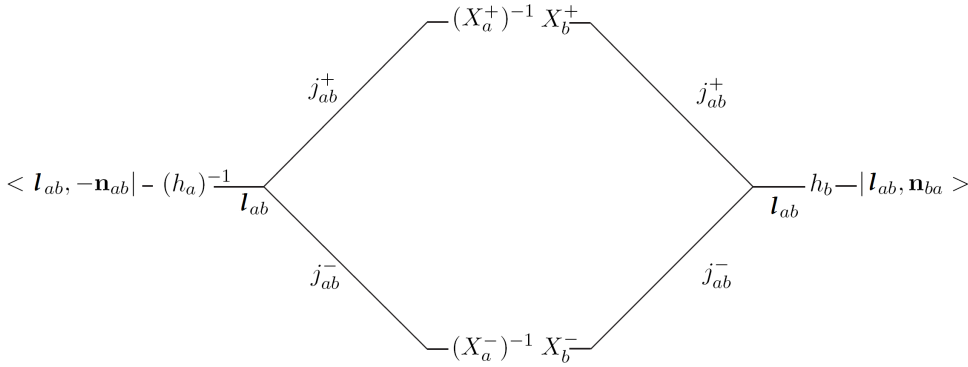


Figure 7.5: The propagator \mathcal{P}_{ab} for a single edge.

of the Clebsch-Gordan coefficients using the epsilon inner products, then converting these inner products to the Hermitian inner product¹, and also flipping the order of + and - lines, obtaining further factors of -1 which are absorbed into the definition of χ' .

The next step is to obtain an exponentiated version of the amplitude as a means to enter the framework of (extended) stationary phase. This is realized through a reformulation of the propagators.

¹The standard antilinear structure map for representations of $\text{SU}(2)$, $J: V_l \rightarrow V_l$ is defined by

$$\epsilon_l(\alpha, \alpha') = \langle J\alpha | \alpha' \rangle,$$

the left-hand side being the epsilon-inner product and the right hand side the Hermitian inner product. It obeys $Jg = gJ$ for all $g \in \text{SU}(2)$, $J^2 = (-1)^{2l}$ and $\langle J\alpha | J\alpha' \rangle = \overline{\langle \alpha | \alpha' \rangle}$. Since

$$J(i\mathbf{n} \cdot \mathbf{L}) = -(i\mathbf{n} \cdot \mathbf{L})J,$$

the map J takes a coherent state for \mathbf{n} to a coherent state for $-\mathbf{n}$.

The first remark in order is that the integration over $SU(2)$ in equation (7.35) at each vertex can be absorbed into the $Spin(4)$ integration because of the invariance of the Clebsch-Gordan maps. Then, the idea is to use the exponentiating property of the coherent states

$$|l, \mathbf{n}\rangle = |\mathbf{n}\rangle^{\otimes 2l}, \quad (7.37)$$

to reduce the propagator to a product of two propagators in the fundamental representation of $SU(2)$ to the power $2j_{\pm}$ respectively. This property is obvious as the product of $2l$ equal states is automatically symmetrized.

For $\beta < 1$, the Clebsch-Gordan coefficient $C_{l_{ab}}^{j_{ab}^{-}j_{ab}^{+}}$ injects into the highest spin subspace $l_{ab} = j_{ab}^{+} + j_{ab}^{-}$. Considered as a spin network, is easy to see that the symmetrizers on the j_{ab}^{+} and j_{ab}^{-} edges can be absorbed into the symmetrizer on the l_{ab} edge because of the stacking property of symmetrizers, see figure 7.6. The remaining

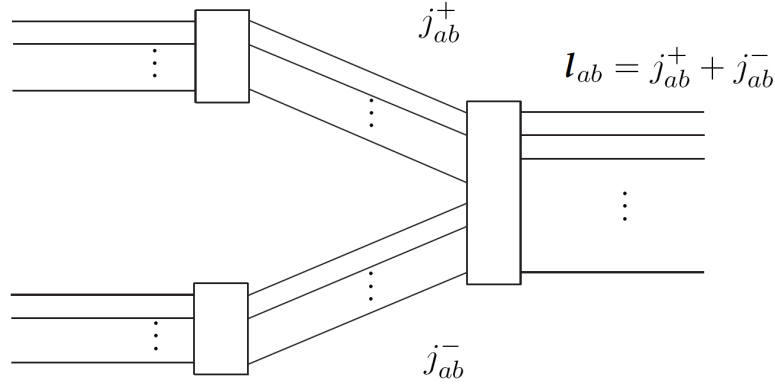


Figure 7.6: Expansion of a Clebsch-Gordan coefficient in terms of strands. The three-valent intertwiner $C_{l_{ab}}^{j_{ab}^{+}j_{ab}^{-}}$ for the case $\beta < 1$ shows how the projection to highest spin subspace l_{ab} makes two of the symmetrizers redundant.

symmetrizer now acts on the coherent states $|l_{ab}, \mathbf{n}_{ab}\rangle$ but since these are defined as the symmetrized tensor product of spin half coherent states $|\mathbf{n}_{ab}\rangle$ this final symmetrization can also be ignored. So we can remove the Clebsch-Gordan coefficients for the whole. We can now use the exponentiating property of the coherent states to further split the propagator into a product of terms in the fundamental representation. We obtain the following expression for the propagator

$$\mathcal{P}_{ab}^{\beta < 1} = \langle -\mathbf{n}_{ab} | X_{ab}^{-} | \mathbf{n}_{ba} \rangle^{2j_{ab}^{-}} \langle -\mathbf{n}_{ab} | X_{ab}^{+} | \mathbf{n}_{ba} \rangle^{2j_{ab}^{+}}. \quad (7.38)$$

The four-simplex amplitude can thus be re-expressed as

$$f_4 = (-1)^{\chi'} \int_{Spin(4)^5} \prod_a dG_a e^{S_{\beta < 1}}, \quad (7.39)$$

with the action given by

$$S_{\beta < 1} = \sum_{a < b} 2j_{ab}^- \ln \langle -\mathbf{n}_{ab} | X_{ab}^- | \mathbf{n}_{ba} \rangle + 2j_{ab}^+ \ln \langle -\mathbf{n}_{ab} | X_{ab}^+ | \mathbf{n}_{ba} \rangle. \quad (7.40)$$

Notice that this action is in general complex. The logarithm of a complex number is only defined up to a multiple of $2\pi i$, we can safely neglect this factor as it will not affect the stationary points and when it appears in the action it is exponentiated.

7.4 EPRL-GFT

The remaining step to arrive at a model with some chance of describing quantum GR is to implement a sum over two-complexes, so that the infinite number of degrees of freedom of the theory could be captured. How do we sum over two complexes? The problem is to select a class of two-complexes to sum over, and to fix the relative weight. Now, the duality between GFT and spin-foam models provides precisely a prescription for summing over two-complexes. It is therefore natural to take a dual formulation of the EPRL models as a natural ansatz for the complete sum over two-complexes. But is there a dual formulation of the EPRL models, or is duality a feature of the much simpler topological BF model?

Remarkably, a dual formulation of every spin-foam model can be obtained. EPRL models are obtained from the BF models (also called TOCY models) by restricting representations. This restriction implements the constraints that transform BF theory into GR. In the dual picture, the sum over representations is obtained as an expansion of the field over the group in modes. A generic field can be expanded in a sum over all unitary irreducible representations. How we can pick a field whose expansion contains only $((1 + \beta)l; (1 - \beta)l)$ representations? Pick a function

$$\begin{aligned} f^\beta(g) &= \left(\dim R^{(1+\beta)l; (1-\beta)l} \right) \text{tr} R^{(1+\beta)l; (1-\beta)l}(g) \\ &= \left(\dim R^{(1+\beta)l; (1-\beta)l} \right) R^{(1+\beta)l; (1-\beta)l}(g)^{\alpha\alpha} \end{aligned} \quad (7.41)$$

and define the projector P_β

$$\begin{aligned} P_\beta \phi(g) &= \int dh \phi(gh) f^\beta(h) \\ &= \int dh \phi_{\delta\epsilon}^{(j^+; j^-)} [R^{j^+; j^-}(gh)]^{\delta\epsilon} \left(\dim R^{(1+\beta)l; (1-\beta)l} \right) [R^{(1+\beta)l; (1-\beta)l}(h)]^{\alpha\alpha} \\ &= \left(\dim R^{(1+\beta)l; (1-\beta)l} \right) \phi_{\delta\epsilon}^{(j^+; j^-)} [R^{j^+; j^-}(g)]^{\delta\tau} \times \\ &\quad \times \int dh [R^{j^+; j^-}(h)]^{\tau\epsilon} [R^{(1+\beta)l; (1-\beta)l}(h)]^{\alpha\alpha} \\ &= \left(\dim R^{(1+\beta)l; (1-\beta)l} \right) \phi_{\delta\epsilon}^{(j^+; j^-)} [R^{j^+; j^-}(g)]^{\delta\tau} \delta^{\tau\alpha} \delta^{\epsilon\alpha} \frac{\delta^{j^+ (1+\beta)l} \delta^{j^- (1-\beta)l}}{\left(\dim R^{(1+\beta)l; (1-\beta)l} \right)} \\ &= \phi_{\delta\alpha}^{(1+\beta)l; (1-\beta)l} [R^{(1+\beta)l; (1-\beta)l}(g)]^{\delta\alpha}. \end{aligned} \quad (7.42)$$

We see that the field $P_\beta\phi$ contains only the desired representations. We take the previous field theory that we rewrite in the shorthand notation

$$S[\phi] = \frac{1}{2} \int \phi^2 + \frac{\lambda}{5!} \int \phi^5. \quad (7.43)$$

Second, instead of demanding the field to satisfy (7.1), we can take an arbitrary field ϕ and use the projection operator P_G defined by

$$P_G\phi(g_1, g_2, g_3, g_4) = \int_{SO(4)} dg\phi(g_1g, g_2g, g_3g, g_4g). \quad (7.44)$$

Define now the projection P_β for ϕ defined on $SO(4)^4$

$$\begin{aligned} P_\beta\phi(g_1, g_2, g_3, g_4) &= \\ &= \int_{H^4} dh_1 \dots dh_4 \phi(g_1h_1, g_2h_2, g_3h_3, g_4h_4) f_\beta(h_1) f_\beta(h_2) f_\beta(h_3) f_\beta(h_4). \end{aligned} \quad (7.45)$$

The action

$$S_B[\phi] = \frac{1}{2} \int (P_G P_\beta \phi)^2 + \frac{\lambda}{5!} \int (P_G P_\beta \phi)^5, \quad (7.46)$$

defines the partition function of the EPRL model for quantum gravity. For viewing the coherent states is sufficient to insert the identity operator

$$T_l := \int dn |(1 + \beta)l, n\rangle \otimes |(1 - \beta)l, n\rangle \langle (1 - \beta)l, n| \otimes \langle (1 + \beta)l, n| \quad (7.47)$$

between the matrix elements of $SO(4)$. For further investigations see [37, 84, 38]. The Lorentzian model group field theory is constructed similarly: we have only to substitute the characteristic invariants and representations of $SO(4)$ with those of $SO(1, 3)$.

Chapter 8

Graviton Propagator

An open problem in quantum gravity is to compute particle scattering amplitudes from the full background-independent theory, and recover low-energy physics. Here we define the dynamics by means of a spinfoam model dual to the EPRL-GFT, and we develop the calculation up to terms of first order in (the GFT coupling) λ . We compute a term in the (connected) two-point function, starting from full non-perturbative quantum general relativity, in an appropriate large distance limit. Only a few components of the boundary states contribute to low order on λ . The associated boundary amplitude can be read as the creation, interaction and annihilation of few “atoms of space”, in the sense in which Feynman diagrams in conventional quantum field theory expansion can be viewed as creation, interaction and annihilation of particles. Using a natural gaussian form of the vacuum state, peaked on the intrinsic *as well as the* extrinsic geometry of the boundary, we derive an expression for the graviton propagator. At large distance, this agrees with the conventional graviton propagator in the linearized theory.

Previous attempts to derive the graviton propagator from LQG adopted the Barrett-Crane spin foam vertex [59] as model for the dynamics [45, 46, 47, 48, 49, 50, 51, 52, 53] (see also [60, 61, 62] for investigations in the three-dimensional case). The analysis of [50, 51] shows that the Barrett-Crane model fails to give the correct scaling behavior for off-diagonal components of the graviton propagator. The problem can be traced back to a missing coherent cancellation of phases between the intertwiner wave function of the semiclassical boundary state and the intertwiner dependence of the model.

We briefly describe the quantity we want to compute. We consider a manifold \mathcal{R} with the topology of a 4-ball. Its boundary is a 3-manifold Σ with the topology of a 3-sphere S^3 . We associate to Σ a boundary Hilbert space of states: the LQG Hilbert space \mathcal{H}_Σ spanned by (abstract) spin networks. We call $|\Psi\rangle$ a generic state in \mathcal{H}_Σ . A spin foam model for the region \mathcal{R} provides a map from the boundary Hilbert space to \mathbf{C} . We call this map $\langle W|$. It provides a sum over the bulk geometries with a weight that defines our model for quantum gravity. The dynamical expectation value of an

operator \mathcal{O} on the state $|\Psi\rangle$ is defined via the following expression¹

$$\langle \mathcal{O} \rangle = \frac{\langle W|\mathcal{O}|\Psi \rangle}{\langle W|\Psi \rangle}. \quad (8.2)$$

The operator \mathcal{O} can be a geometric operator as the area, the volume or the length [74, 75, 76, 77, 78, 79, 80]. The geometric operator we are interested in here is the (density-two inverse-) metric operator $q^{ab}(x) = \delta^{ij} E_i^a(x) E_j^b(x)$. We focus on the *connected* two-point correlation function $G^{abcd}(x, y)$ on a semiclassical boundary state $|\Psi_0\rangle$. It is defined as

$$G^{abcd}(x, y) = \langle q^{ab}(x) q^{cd}(y) \rangle - \langle q^{ab}(x) \rangle \langle q^{cd}(y) \rangle. \quad (8.3)$$

The boundary state $|\Psi_0\rangle$ is semiclassical in the following sense: it is peaked on a given configuration of the intrinsic and the extrinsic geometry of the boundary manifold Σ . In terms of Ashtekar-Barbero variables these boundary data correspond to a couple (E_0, A_0) . The boundary data are chosen so that there is a solution of Einstein equations in the bulk which induces them on the boundary. A spin foam model has good semiclassical properties if the dominant contribution to the amplitude $\langle W|\Psi_0\rangle$ comes from the bulk configurations close to the classical 4-geometries compatible with the boundary data (E_0, A_0) . By *classical* we mean that they satisfy Einstein equations.

The classical bulk configuration we focus on is flat space. The boundary configuration that we consider is the following: we decompose the boundary manifold S^3 in five tetrahedral regions with the same connectivity as the boundary of a 4-simplex; then we choose the intrinsic and the extrinsic geometry to be the ones proper of the boundary of a Euclidean 4-simplex. By construction, these boundary data are compatible with flat space being a classical solution in the bulk.

For our choice of boundary configuration, the dominant contribution to the amplitude $\langle W|\Psi_0\rangle$ is required to come from bulk configurations close to flat space. The connected two-point correlation function $G^{abcd}(x, y)$ probes the fluctuations of the geometry around the classical configuration given by flat space. As a result it can be compared to the graviton propagator computed in perturbative quantum gravity.

8.1 The strategy: two-point function from the boundary amplitude

We begin by illustrating the quantities and some techniques that we are going to use in quantum gravity within a simple context.

¹This expression corresponds to the standard definition in (perturbative) quantum field theory where the *vacuum* expectation value of a product of local observables is defined as

$$\langle O(x_1) \cdots O(x_n) \rangle_0 = \frac{\int D[\varphi] O(x_1) \cdots O(x_n) e^{iS[\varphi]}}{\int D[\varphi] e^{iS[\varphi]}} \equiv \frac{\int D[\phi] W[\phi] O(x_1) \cdots O(x_n) \Psi_0[\phi]}{\int D[\phi] W[\phi] \Psi_0[\phi]}. \quad (8.1)$$

The vacuum state $\Psi_0[\phi]$ codes the boundary conditions at infinity. See the next sections.

8.1.1 A single degree of freedom

Consider the two-point function of a single harmonic oscillator with mass m and angular frequency ω . This is given by

$$G_0(t_1, t_2) = \langle 0|x(t_1)x(t_2)|0\rangle = \langle 0|x e^{-\frac{i}{\hbar}H(t_1-t_2)} x|0\rangle \quad (8.4)$$

where $|0\rangle$ is the vacuum state, $x(t)$ is the Heisenberg position operator at time t and H the hamiltonian. We write a subscript $_0$ in $G_0(t_1, t_2)$ to remind us that this is an expectation value computed on the vacuum state. Later we will also consider similar expectation values computed on other classes of states, as for instance in

$$G_\psi(t_1, t_2) = \langle \psi|x(t_1)x(t_2)|\psi\rangle. \quad (8.5)$$

Elementary creation and annihilation operator techniques give

$$G_0(t_1, t_2) = \frac{\hbar}{2m\omega} e^{-\frac{3}{2}i\omega(t_1-t_2)}. \quad (8.6)$$

In the Schrödinger picture, the r.h.s. of (8.4) reads

$$G_0(t_1, t_2) = \int dx_1 dx_2 \overline{\psi_0(x_1)} x_1 W[x_1, x_2; t_1, t_2] x_2 \psi_0(x_2) \quad (8.7)$$

where $\psi_0(x) = \langle x|0\rangle$ is the vacuum state and $W[x_1, x_2; t_1, t_2]$ is the propagator, namely the matrix element of the evolution operator

$$W[x_1, x_2; t_1, t_2] = \langle x_1|e^{-iH(t_1-t_2)}|x_2\rangle. \quad (8.8)$$

Recalling that

$$\psi_0(x) = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega}{2\hbar}x^2} \quad (8.9)$$

and

$$W(x_1, x_2; T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} e^{i\frac{m\omega}{2\hbar} \frac{(x_1^2+x_2^2) \cos \omega T - 2x_1 x_2}{\sin \omega T}} \quad (8.10)$$

are two gaussian expressions, we obtain the two-point function (8.4) as the second momentum of a gaussian

$$G_0(t_1, t_2) = \frac{m\omega}{\pi\hbar} \sqrt{\frac{1}{2i \sin \omega T}} \int dx_1 dx_2 x_1 x_2 e^{i\frac{m\omega}{2\hbar} \frac{(x_1^2+x_2^2) \cos \omega T - 2x_1 x_2}{\sin \omega T}} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}, \quad (8.11)$$

where the gaussian is the product of a “bulk” gaussian term and a “boundary” gaussian term. Using

$$\int dx_1 dx_2 x_1 x_2 e^{-\frac{1}{2}(xAx)} = \frac{2\pi}{\sqrt{\det A}} A_{12}^{-1}, \quad (8.12)$$

the evaluation of the integral in (8.11) is straightforward. It gives

$$G_0(t_1, t_2) = \frac{m\omega}{\pi\hbar} \sqrt{\frac{1}{2i \sin \omega(t_1 - t_2)}} \frac{2\pi}{\sqrt{\det A}} A_{12}^{-1} \quad (8.13)$$

in terms of the inverse of the covariance matrix of the gaussian

$$\begin{aligned} A &= \frac{m\omega}{\hbar} \begin{pmatrix} 1 - i \frac{\cos \omega(t_1 - t_2)}{\sin \omega(t_1 - t_2)} & \frac{i}{\sin \omega(t_1 - t_2)} \\ \frac{i}{\sin \omega(t_1 - t_2)} & 1 - i \frac{\cos \omega(t_1 - t_2)}{\sin \omega(t_1 - t_2)} \end{pmatrix} \\ &= \frac{-im\omega}{\hbar \sin \omega(t_1 - t_2)} \begin{pmatrix} e^{i\omega(t_1 - t_2)} & -1 \\ -1 & e^{i\omega(t_1 - t_2)} \end{pmatrix}. \end{aligned} \quad (8.14)$$

The matrix A is easy to invert and (8.13) gives precisely (8.6). We will find precisely this structure of a similar matrix to invert at the end of the calculation of this paper.

Notice that the two-point function (8.4) can also be written as the (analytic continuation of the euclidean version of) the functional integral

$$G_0(t_1, t_2) = \int Dx(t) x(t_1)x(t_2) e^{i \int_{-\infty}^{\infty} L(x, dx/dt)}. \quad (8.15)$$

where L is the harmonic oscillator lagrangian, and the measure is appropriately normalized. Let us break the (infinite number of) integration variables $x(t)$ in various groups: those where t is less, equal or larger than, respectively, t_1 and t_2 . Using this, and writing the integration variable $x(t_1)$ as x_1 and the integration variable $x(t_2)$ as x_2 , we can rewrite (8.15) as

$$G_0(t_1, t_2) = \int dx_1 dx_2 \overline{\psi_0(x_1)} x_1 W[x_1, x_2; t_1, t_2] x_2 \psi_0(x_2) \quad (8.16)$$

where

$$W[x_1, x_2; t_1, t_2] = \int_{x(t_2)=x_2}^{x(t_1)=x_1} Dx(t) e^{i \int_{t_2}^{t_1} L(x, dx/dt)} \quad (8.17)$$

is the functional integral restricted to the open interval (t_1, t_2) integrated over the paths that start at x_2 and end at x_1 ; while

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(t_1)=x} Dx(t) e^{i \int_{-\infty}^{t_1} L(x, dx/dt)} \quad (8.18)$$

is the functional integral restricted to the interval $(-\infty, t_1)$. As well known, in the euclidean theory this gives the vacuum state. Thus, we recover again the form (8.7) of the two-point function, with the additional information that the “bulk” propagator term can be viewed as the result of the functional integral in the interior of the (t_1, t_2) interval, while the “boundary” term can be viewed as the result of the functional integral in the exterior. In this language the specification of the particular state $|0\rangle$ on which the expectation value of $x(t_1)x(t_2)$ is computed, is coded in the boundary behavior of the functional integration variable at infinity: $x(t) \rightarrow 0$ for $t \rightarrow \pm\infty$.

The normalization of the functional measure in (8.15) is determined by

$$1 = \int Dx(t) e^{i \int_{-\infty}^{\infty} L(x, dx/dt)}. \quad (8.19)$$

Breaking this functional integral in the same manner as the above one gives

$$1 = \int dx_1 dx_2 \overline{\psi_0(x_1)} W[x_1, x_2; t_1, t_2] \psi_0(x_2) \quad (8.20)$$

or equivalently

$$1 = \langle 0 | e^{-\frac{i}{\hbar} H(t_1 - t_2)} | 0 \rangle. \quad (8.21)$$

Let us comment on the interpretation of (8.16) and (8.20), since analogues of these equations will play a major role below. Observe that (8.16) can be written in the form

$$G_0(t_1, t_2) = \langle W_{t_1, t_2} | \hat{x}_1 \hat{x}_2 \Psi_0 \rangle, \quad (8.22)$$

in terms of states and operators living in the Hilbert space $\mathcal{K}_{t_1, t_2} = \mathcal{H}_{t_1}^* \times \mathcal{H}_{t_2}$ (the tensor product of the space of states at time t_1 and the space of states at time t_2) formed by functions $\psi(x_1, x_2)$. (See Section 5.1.4 of [12] for details on \mathcal{K}_{t_1, t_2} .) Using the relativistic formulation of quantum mechanics developed in [12], this expression can be directly re-interpreted as follows. (i) The “boundary state” $\Psi_0(x_1, x_2) = \overline{\psi_0(x_1)} \psi_0(x_2)$ represents the joint boundary configuration of the system at the two times t_1 and t_2 , if no excitation of the oscillator is present; it describes the joint outcome of a measurement at t_1 and a measurement at t_2 , both of them detecting no excitations. (ii) The two operators x_1 and x_2 create a (“incoming”) excitation at $t = t_2$ and a (“outgoing”) excitation at $t = t_1$; thus the state $\hat{x}_1 \hat{x}_2 \Psi_0$ can be interpreted as a boundary state representing the joint outcome of a measurement at t_1 and a measurement at t_2 , both of them detecting a single excitation. (iii) The bra $W_{t_1, t_2}(x_1, x_2) = W[x_1, x_2; t_1, t_2]$ is the linear functional coding the dynamics, whose action on the two-excitation state associates it an amplitude, which can be compared with other similar amplitudes. For instance, observe that

$$\langle W_{t_1, t_2} | \hat{x}_2 \Psi_{t_1, t_2} \rangle = 0; \quad (8.23)$$

that is, the probability amplitude of measuring a single excitation at t_2 and no excitation at t_1 is zero. Finally, the normalization condition (8.20) reads

$$1 = \langle W_{t_1, t_2} | \Psi_0 \rangle; \quad (8.24)$$

which requires that the boundary state Ψ_0 is a solution of the dynamics, in the sense that its projection on t_1 is precisely the time evolution of its projection to t_2 . As we shall see below, this condition generalizes to the case of interest for general relativity. We call (8.24) the “Wheeler-deWitt” (WdW) condition. This condition satisfied by the boundary state should not be confused with the normalization condition,

$$1 = \langle \Psi_0 | \Psi_0 \rangle, \quad (8.25)$$

which is also true, and which follows immediately from the fact that $|0\rangle$ is normalized in \mathcal{H}_t .

In general, given a state $\Psi \in \mathcal{K}_{t_1, t_2}$, the equations

$$\langle W_{t_1, t_2} | \Psi \rangle = 1; \quad (8.26)$$

and

$$\langle \Psi | \Psi \rangle = 1, \quad (8.27)$$

are equivalent to the full quantum dynamics, in the following sense. If the state is of the form $\Psi = \psi_f \otimes \psi_i$, then (8.26) and (8.27) imply that

$$\psi_f = e^{-iHt} \psi_i. \quad (8.28)$$

Finally, recall that a coherent (semiclassical) state $\psi_{\mathbf{q}}(x) \sim e^{-\frac{\alpha}{2}(x-q)^2 + \frac{i}{\hbar}px}$ is peaked on values q and p of position and momentum. In particular, the vacuum state of the harmonic oscillator is the coherent state peaked on the values $q = 0$ and $p = 0$, with $\alpha = m\omega/\hbar$. Thus we can write $\psi_0 = \psi_{(q=0,p=0)}$. In the same manner, the boundary state $\Psi_0 = \overline{\psi_0(x_1)}\psi_0(x_2)$ can be viewed as a coherent *boundary* state, associated with the values $q_1 = 0$ and $p_1 = 0$ at t_1 and $q_2 = 0$ and $p_2 = 0$ at t_2 . We can write a *generic* coherent boundary state as

$$\Psi_{q_1,p_1,q_2,p_2}(x_1,x_2) = \overline{\psi_{(q_1,p_1)}(x_1)} \psi_{(q_2,p_2)}(x_2). \quad (8.29)$$

A special case of these coherent boundary states is obtained when (q_1,p_1) are the classical evolution at time $t_1 - t_2$ of the initial conditions (q_2,p_2) . That is, when in the $t_1 - t_2$ interval there exists a solution $q(t), p(t)$ of the classical equations of motion precisely bounded by q_1, p_1, q_2, p_2 , namely such that $q_1 = q(t_1), p_1 = p(t_1)$ and $q_2 = q(t_2), p_2 = p(t_2)$. If such a classical solution exists, we say that the quadruplet (q_1, p_1, q_2, p_2) is *physical*. As well known the harmonic oscillator dynamics gives in this case $e^{-iH(t_1-t_2)}\Psi_{q_2,p_2} = \Psi_{q_1,p_1}$, or

$$\langle W_{t_1,t_2} | \Psi_{q_1,p_1,q_2,p_2} \rangle = 1. \quad (8.30)$$

That is, it satisfies the WdW condition (8.26). In this case, we denote the semiclassical boundary state a *physical* semiclassical boundary states. The vacuum boundary state Ψ_0 is a particular case of this: it is the physical semiclassical boundary state determined by the classical solution $q(t) = 0$ of the equations of motion, which is the one with minimal energy. Given a physical boundary state, we can consider a two-point function describing the propagation of a quantum excitation “over” the semiclassical trajectory $q(t), p(t)$ as

$$G_{q_1,p_1,q_2,p_2}(t_1,t_2) = \langle \psi_{(q_1,p_1)} | x(t_1)x(t_2) | \psi_{(q_2,p_2)} \rangle = \langle W_{t_1,t_2} | \hat{x}_1 \hat{x}_2 \Psi_{q_1,p_1,q_2,p_2} \rangle. \quad (8.31)$$

This quantity will play a considerable role below. Indeed, the main idea here is to compute quantum-gravity n -point functions using states that describe the boundary value of the gravitatonal field on given boundary surfaces.

There is an interesting phenomenon regarding the *phases* of the boundary state $\Psi_{q_1,p_1,q_2,p_2}(x_1,x_2)$ and of the propagator $W_{t_1,t_2}(x_1,x_2)$ that should be noticed. If p_1 and p_2 are different from zero, they give rise to a phase factor $e^{-\frac{i}{\hbar}(p_1x_1-p_2x_2)}$, in the boundary state. In turn, it is easy to see that $W_{t_1,t_2}(x_1,x_2)$ contains precisely the inverse of this same phase factor, when expanded around (q_1,q_2) . In fact, the phase of the propagator is the classical Hamilton function $S_{t_1,t_2}(x_1,x_2)$ (the value of the action, as a function of the boundary values). Expanding the Hamilton function around q_1 and q_2 gives to first order

$$S_{t_1,t_2}(x_1,x_2) = S_{t_1,t_2}(q_1,q_2) + \frac{\partial S}{\partial x_1}(x_1 - q_1) + \frac{\partial S}{\partial x_2}(x_2 - q_2), \quad (8.32)$$

but

$$\frac{\partial S}{\partial x_1} = p_1 \quad \text{and} \quad \frac{\partial S}{\partial x_2} = -p_2. \quad (8.33)$$

Giving a phase factor $e^{\frac{i}{\hbar}(p_1x_1 - ip_2x_2)}$, which is precisely the inverse of the one in the boundary state. In the Schrödinger representation of (8.31), the gaussian factor in the boundary state peaks the integration around (q_1, q_2) ; in this region, we have that *the phase of the boundary state is determined by the classical value of the momentum, and is cancelled by a corresponding phase factor in the propagator W* . In particular, the rapidly oscillating phase in the boundary state fails to suppress the integral precisely because it is compensated by a corresponding rapidly oscillating phase in W . This, of course, is nothing that the realization, in this language, of the well-known emergence of classical trajectories from the constructive coherence of the quantum amplitudes. This phenomenon plays a major role below.

8.1.2 Field theory

Let us now go over to field theory. The two-point function (or particle propagator) is defined by the (analytic continuation of the euclidean version of the) path integral ($\hbar = 1$ from now on)

$$\begin{aligned} G_0(x, y) &= \langle 0 | \phi(x) \phi(y) | 0 \rangle = \langle 0 | \phi(\vec{x}) e^{-iH(x_0 - y_0)} \phi(\vec{y}) | 0 \rangle \\ &= \int D\phi(x) \phi(x) \phi(y) e^{iS[\phi]}, \end{aligned} \quad (8.34)$$

where the normalization of the measure is determined by

$$1 = \int D\phi(x) e^{iS[\phi]} \quad (8.35)$$

and the $_0$ subscript reminds that these are expectation values of products of field operators in the particular state $|0\rangle$. These equations generalize (8.15) and (8.19) to field theory.² As before, we can break the integration variables of the path integral in various groups. For instance, in the values of the field in the five spacetime-regions identified by t being less, equal or larger than, respectively, x_0 and y_0 . This gives a Schrödinger representation of the two-point function of the form

$$G_0(x, y) = \int D\varphi_1 D\varphi_2 \overline{\psi_0(\varphi_1)} \varphi_1(\vec{x}) W[\varphi_1, \varphi_2; (x_0 - y_0)] \varphi_2(\vec{y}) \psi_0(\varphi_2). \quad (8.36)$$

where φ_1 is the three-dimensional field at time t_1 , and φ_2 is the three-dimensional field at time t_2 . For a free field, the field propagator (or propagation kernel)

$$W(\varphi_1, \varphi_2; T) = \langle \varphi_1 | e^{-iHT} | \varphi_2 \rangle. \quad (8.37)$$

and the boundary vacuum state are gaussian expression in the boundary field $\varphi = (\varphi_1, \varphi_2)$. In a free theory, the boundary vacuum state can be written as a physical semiclassical state peaked on vanishing field and momentum π , as in (8.29):

$$\Psi_0(\varphi_1, \varphi_2) \equiv \Psi_{\varphi_1=0, \pi_1=0, \varphi_2=0, \pi_2=0}(\varphi_1, \varphi_2) = \overline{\psi_0(\varphi_1)} \psi_0(\varphi_2). \quad (8.38)$$

²A well-known source of confusion is of course given by the fact that in the case of a free particle the propagator (8.8) coincides with the 2-point function of the free field theory.

Notice that the momentum $\pi = \frac{d\varphi_1}{dt}$ is the derivative of the classical field normal to Σ .

More interesting for what follows, we can choose a compact finite region \mathcal{R} in spacetime, bounded by a closed 3d surface Σ , such that the two points x and y lie on Σ . Then we can separate the integration variables in (8.34) into those inside \mathcal{R} , those on Σ and those outside \mathcal{R} , and thus write the two-point function (8.34) in the form

$$G_0(x, y) = \int D\varphi \varphi(x) \varphi(y) W[\varphi; \Sigma] \Psi_0(\varphi), \quad (8.39)$$

where φ is the field on Σ ,

$$W[\varphi; \Sigma] = \int_{\partial\phi=\varphi} D\phi_{\mathcal{R}} e^{-iS_{\mathcal{R}}[\phi_{\mathcal{R}}]} \quad (8.40)$$

is the functional integral restricted to the region \mathcal{R} , and integrated over the interior fields $\phi_{\mathcal{R}}$ bounded by the given boundary field φ . The boundary state $\Psi_0(\varphi)$ is given by the integral restricted to the outside region, $\overline{\mathcal{R}}$. The boundary conditions on the functional integration variable

$$\phi_{\overline{\mathcal{R}}}(x) \rightarrow 0, \quad \text{for} \quad |x| \rightarrow \infty \quad (8.41)$$

determine the vacuum state. In a free theory, this is still a gaussian expression in φ , but the covariance matrix is non-trivial and is determined by the shape of Σ . The state Ψ_0 can nevertheless be still viewed as a semiclassical boundary state associated to the compact boundary, peaked on the value $\varphi = 0$ of the field and the value $\pi = 0$ of a (generalized) momentum (the derivative of the field normal to the surface) [12]. Equation (8.39) will be our main tool in the following.

In analogy with (8.22), equation (8.39) can be written in the form

$$G_0(x, y) = \langle W_{\Sigma} | \hat{\varphi}(x) \hat{\varphi}(y) \Psi_0 \rangle. \quad (8.42)$$

in terms of states and operators living in a boundary Hilbert space \mathcal{K}_{Σ} associated with the 3d surface Σ . In terms of the relativistic formulation of quantum mechanics, this expression can be interpreted as follows. (i) The “boundary state” Ψ_0 represents the boundary configuration of a quantum field on a surface Σ , when no particles are present; it represents the joint outcome of measurements on the entire surface Σ , showing no presence of particles. (ii) The two operators $\hat{\varphi}(x) \hat{\varphi}(y)$ create a (“incoming”) particle at y and a (“outgoing”) particle at x ; so that the boundary state $\varphi(x) \varphi(y) \Psi_0$ represents the joint outcome of measurements on Σ , detecting a (“incoming”) particle at y and a (“outgoing”) particle at x . (iii) Finally, the bra W_{Σ} is the linear functional coding the dynamics, whose action on the two-particle boundary state associates it an amplitude, which can be compared with other analogous amplitudes. The normalization condition for the measure, equation (8.35), becomes the WdW condition

$$1 = \langle W_{\Sigma} | \Psi_0 \rangle, \quad (8.43)$$

which singles out the physical boundary states.

Finally, as before, let $\mathbf{q} = (q, p)$ be a given couple of boundary values of the field φ and its generalized momentum on Σ . If there exists a classical solution ϕ of the

equations of motion whose restriction to Σ is q and whose normal derivative to Σ is p , then we say that $\mathbf{q} = (q, p)$ are *physical* boundary data. Let $\Psi_{\mathbf{q}}$ be a boundary state in \mathcal{K}_{Σ} peaked on these values: schematically

$$\Psi_{\mathbf{q}}(\varphi) \sim e^{-\int(\varphi-q)^2+i\int p\phi}. \quad (8.44)$$

If $\mathbf{q} = (q, p)$ are physical boundary data, we say that $\Psi_{\mathbf{q}}$ is a *physical* semiclassical state. In this case, we can consider the two-point function

$$G_{\mathbf{q}}(x, y) = \langle W_{\Sigma} | \hat{\varphi}(x) \hat{\varphi}(y) \Psi_{\mathbf{q}} \rangle \quad (8.45)$$

describing the propagation of a quantum, from y to x , over the classical field configuration ϕ giving the boundary data $\mathbf{q} = (q, p)$. In the Schrödinger representation of this expression, there is a cancellation of the phase of the boundary state $\Psi_{\mathbf{q}}$ with the phase of the propagation kernel W_{Σ} , analogous to the one we have seen in the case of a single degree of freedom.

8.1.3 Quantum gravity

Let us formally write (8.39) for pure general relativity, ignoring for the moment problems such as the definition of the integration measure, or ultraviolet divergences. Given a surface Σ , we can choose a generalized temporal gauge in which the degrees of freedom of gravity are expressed by the 3-metric γ induced on Σ , with components $\gamma_{ab}(x)$ $a, b = 1, 2, 3$. That is, if the surface is locally given by $x^4 = 0$, we gauge fix the 4d gravitational metric field $g_{\mu\nu}(x)$ by $g_{44} = 1, g_{40} = 0$, and $\gamma_{ab} = g_{ab}$. Then the graviton two-point function (8.39) reads in this gauge

$$G_0^{abcd}(x, y) = \int [D\gamma] h^{ab}(x) h^{cd}(y) W[\gamma; \Sigma] \Psi_0(\gamma), \quad (8.46)$$

where $h^{ab}(x) = \gamma^{ab}(x) - \delta^{ab}$. If we assume that $W[\gamma; \Sigma]$ is given by a functional integration on the bulk, as in (8.40), where measure and action are generally covariant, then we have immediately that $W[\gamma; \Sigma]$ is independent from (smooth deformations of) Σ . Hence, at fixed topology of Σ (say, the surface of a 3-sphere), we have $W[\gamma; \Sigma] = W[\gamma]$, that is

$$G_0^{abcd}(x, y) = \int [D\gamma] h^{ab}(x) h^{cd}(y) W[\gamma] \Psi_0(\gamma). \quad (8.47)$$

What is the interpretation of the boundary state $\Psi_0(\gamma)$ in a general covariant theory? In the case of the harmonic oscillator, the vacuum state $|0\rangle$ is the state that minimizes the energy. In the case of a free theory on a background, in addition, it is the sole Poincaré invariant state. In both cases the vacuum state can also be obtained from a functional integral by fixing the behavior of the fields at infinity. But in background-independent quantum gravity, there is no energy to minimize and no global Poincaré invariance. Furthermore, there is no background metric with respect to which to demand the gravitational field to vanish at infinity. In fact, it is well known that the unicity and the very definition of the vacuum state is highly problematic in nonperturbative quantum gravity, a phenomenon that begins to manifest itself already in QFT

on a curved background. Thus, in quantum gravity there is a multiplicity of possible states that we can consider as boundary states, instead of a single preferred one.

Linearized quantum gravity gives us a crucial hint, and provides us with a straightforward way to *interpret* semiclassical boundary states. Indeed, consider linearized quantum gravity, namely the well-defined theory of a noninteracting spin-2 graviton field $h_{\mu\nu}(x)$ on a flat spacetime with background metric $g_{\mu\nu}^0$. This theory has a preferred vacuum state $|0\rangle$. Now, choose a boundary surface Σ and denote $\mathbf{q} = (q, p)$ its three-geometry, formed by the 3-metric q_{ab} and extrinsic curvature field p^{ab} , induced on Σ by the flat background metric of spacetime. The vacuum state defines a gaussian boundary state on Σ , peaked around $h = 0$. We can schematically write this state as $\Psi_\Sigma(h) \sim e^{-\int h^2}$. Now, on Σ there are two metrics: the metric q induced by the background spacetime metric, and the metric $\gamma = q + h$, induced by the true physical metric $g_{\mu\nu} = g_{\mu\nu}^0 + h_{\mu\nu}$, which is the sum of the background metric and the dynamical linearized gravitational field. Therefore the vacuum functional $\Psi_0(h)$ defines a functional $\Psi_{\mathbf{q}}(\gamma)$ of the physical metric γ of Σ as follows

$$\Psi_{\mathbf{q}}(\gamma) = \Psi_{\mathbf{q}}(q + h) \equiv \Psi_0(h). \quad (8.48)$$

Schematically

$$\Psi_{\mathbf{q}}(\gamma) = \Psi_0(h) = \Psi_0(\gamma - q) \sim e^{-\int(\gamma-q)^2}. \quad (8.49)$$

A bit more precisely, we must also take into account a phase term, generated by the fact that the normal derivative of the induced metric does not vanish (q changes if we deform Σ). This gives, again very schematically

$$\Psi_{\mathbf{q}}(\gamma) \sim e^{-\int(\gamma-q)^2 + i \int p\gamma} \quad (8.50)$$

as in (8.44). Recall indeed that in general relativity the intrinsic and extrinsic geometry play the role of canonical variable and conjugate variable. A semiclassical boundary state must be peaked on both quantities, as coherent states of the harmonic oscillator are equally peaked on q and p . The functional $\Psi_{\mathbf{q}}$ of the metric can immediately be interpreted as a boundary state of quantum gravity, as determined by the linearized theory. Observe that it depends on the background geometry of Σ , because q and p do: the form of this state is determined by the location of Σ with respect to the background metric of space. Therefore (when seen as a function of the true metric γ) there are different possible boundary states in the linearized theory, depending on where is the boundary surface. Equivalently, there are different boundary states depending on what is the mean boundary geometry q on Σ .

Now, in full quantum gravity we must expect, accordingly, to have many possible distinct semiclassical boundary states $\Psi_{\mathbf{q}}(\gamma)$ that are peaked on distinct 3-geometries $\mathbf{q} = (q, p)$. In the background-independent theory they cannot be *anymore* interpreted as determined by the location of Σ with respect to the background (because there is no background!). But they can *still* be interpreted as determined by the mean boundary geometry \mathbf{q} on Σ . Their interpretation is therefore immediate: they represent coherent semiclassical states of the boundary geometry. The multiplicity of the possible locations of Σ with respect to the background geometry in the background-dependent theory, translates into a multiplicity of possible coherent boundary states in the background-independent formalism.

In fact, this conclusion follows immediately from the core physical assumption of general relativity: the identification of the gravitational field with the spacetime metric. A coherent boundary state of the gravitational field is peaked, in particular, on a given classical value of the metric. In the background-dependent picture, this can be interpreted as information about the location of Σ in spacetime. In a background-independent picture, there is no location in spacetime: the geometrical properties of anything is solely determined by the local value of the gravitational field. In a background-independent theory, the dependence on a boundary geometry is not in the location of Σ with respect to a background geometry, but rather in the boundary state of the gravitation field on the surface Σ itself.

Having understood this, it is clear that the two-point function of a background-independent theory can be *defined* as a function of the mean boundary geometry, instead of a function of the background metric. If $\mathbf{q} = (q, p)$ is a given geometry of a closed surface Σ with the topology of a 3-sphere, and $\Psi_{\mathbf{q}}$ is a coherent state peaked on this geometry, consider the expression

$$G_{\mathbf{q}}^{abcd}(x, y) = \int [D\gamma] h^{ab}(x) h^{cd}(y) W[\gamma] \Psi_{\mathbf{q}}(\gamma). \quad (8.51)$$

At first sight, this expression appears to be meaningless. The r.h.s. is completely independent from the location of Σ on the spacetime manifold. What is then the meaning of the 4d coordinates x and y in the l.h.s.? In fact, this is nothing than the usual well-known problem of the conventional definition of n -point functions in generally covariant theories: if action and measure are generally covariant, equation (8.34) is independent from x and y (as long as $x \neq y$); because a diffeomorphism on the integration variable can change x and y , leaving all the rest invariant. We seem to have hit the usual stumbling block that makes n -point functions useless in generally covariant theories.

In fact, we have not, because the very dependence of $G_{\mathbf{q}}^{abcd}(x, y)$ on \mathbf{q} provides the obvious solution to this problem: let us *define* a “generally covariant 2-point function” $\mathbf{G}_{\mathbf{q}}^{abcd}(x, y)$ as follows. Given a three-manifold S_3 with the topology of a 3-sphere, equipped with given fields $\mathbf{q} = (q_{ab}(\mathbf{x}), p^{ab}(\mathbf{y}))$, and given two points \mathbf{x} and \mathbf{y} on *this metric manifold*, we define

$$\mathbf{G}_{\mathbf{q}}^{abcd}(\mathbf{x}, \mathbf{y}) = \int [D\gamma] h^{ab}(\mathbf{x}) h^{cd}(\mathbf{y}) W[\gamma] \Psi_{\mathbf{q}}(\gamma). \quad (8.52)$$

The difference between (8.51) and (8.52) is that in the first expression x and y are coordinates in the background 4d *differential* manifold, while in the second \mathbf{x} and \mathbf{y} are points in the 3d *metric* manifold (S_3, q) . It is clear that with this definition the dependence of the 2-point function on \mathbf{x} and \mathbf{y} is non trivial: metric relations between \mathbf{x} and \mathbf{y} are determined by \mathbf{q} . In particular, a 3d active diffeomorphism on the integration variable g changes \mathbf{x} and \mathbf{y} , but also \mathbf{q} , leaving the metric relations between \mathbf{x} and \mathbf{y} invariant.

The physically interesting case is when $\mathbf{q} = (q, p)$ are a set of *physical* boundary conditions. Since we are considering here pure general relativity without matter, this means that there exists a Ricci flat spacetime with 4d metric g and an imbedding

$\Sigma : S_3 \rightarrow M$, such that g induces the three metric q and the extrinsic curvature p on S_3 . In this case, the semiclassical boundary state $\Psi_{\mathbf{q}}$ is a physical state. Measure and boundary states must be normalized in such a way that

$$\int [D\gamma] W[\gamma] \Psi_{\mathbf{q}}(\gamma) = 1. \quad (8.53)$$

Then the two point function (8.52) is a non-trivial and invariant function of the physical 4d distance

$$L = d_g(\Sigma(\mathbf{x}), \Sigma(\mathbf{y})). \quad (8.54)$$

It is clear that if g is the flat metric this function must reduce immediately to the conventional 2-point function of the linearized theory, in the appropriate large distance limit. This is the definition of a generally covariant two-point function which we use here.

Finally, the physical interpretation of (8.52) is transparent: it defines an amplitude associated to a joint set of measurements performed on a surface Σ bounding a finite spacetime region, where the measurements include: (i) the average geometry of Σ itself, namely the physical distance between detectors, the time lapse between measurements, and so on; as well as (ii) the detection of a (“outgoing”) particle (a graviton) at x and the detection of a (“incoming”) particle (a graviton) at y . The two kinds of measurements, that are considered of different nature in non-generally-relativistic physics, are on equal footing in general relativistic physics (see [12], pg. 152-153). In generally covariant quantum field theory, the single boundary state $\hat{h}^{ab}(x)\hat{h}^{cd}(y)\Psi_{\mathbf{q}}$ codes the two. Notice that the quantum geometry in the *interior* of the region \mathcal{R} is free to fluctuate. In fact, W can be interpreted as the sum over all interior 4-geometries. What is determined is a boundary geometry as measured by the physical apparatus that surrounds a potential interaction region.

Equation (8.52) can be realized concretely in LQG by identifying (i) the boundary Hilbert space associated to Σ with the separable Hilbert space spanned by the (abstract) spin network states s , namely the s -knot states; (ii) the linearized gravitational field operators $\hat{h}^{ab}(x)$ and $\hat{h}^{cd}(y)$ with the corresponding LQG operators; (iii) the boundary state $\Psi_{\mathbf{q}}$ with a suitable spin network functional $\Psi_{\mathbf{q}}[s]$ peaked on the geometry q ; and finally, (iv) the boundary functional $W[s]$, representing the functional integral on the interior geometries bounded by the boundary geometry s , with the $W[s]$ defined by a spin foam model. This, indeed, is given by a sum over interior spinfoams, interpreted as quantized geometries. This gives the expression

$$\mathbf{G}_{\mathbf{q}}^{abcd}(\mathbf{x}, \mathbf{y}) = \sum_s W[s] \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) \Psi_{\mathbf{q}}[s]. \quad (8.55)$$

which we analyze in detail in rest of the paper. The WdW condition reads

$$1 = \sum_s W[s] \Psi_{\mathbf{q}}[s]. \quad (8.56)$$

Using these two equations together, we can write

$$\mathbf{G}_{\mathbf{q}}^{abcd}(\mathbf{x}, \mathbf{y}) = \frac{\sum_s W[s] \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) \Psi_{\mathbf{q}}[s]}{\sum_s W[s] \Psi_{\mathbf{q}}[s]}, \quad (8.57)$$

a form that allows us to disregard the overall normalization of W and $\Psi_{\mathbf{q}}$. We analyze these ingredients in detail in the next section.

8.2 Graviton propagator: definition and ingredients

Equation (8.55) is well-defined if we choose a dynamical model giving $W[s]$, a boundary state $\Psi_{\mathbf{q}}[s]$ and a form for the operator $\hat{h}^{ab}(x)$. We choose the boundary functional $W[s]$ defined by the group field theory *EPRL-GFT*. We consider here only the lowest order terms in the expansion of $W[s]$ in the *GFT* coupling constant λ . Furthermore, we consider only the first order in a large distance expansion. Our aim is to recover the 2-point function of the linearized theory, namely the graviton propagator, in this limit.

8.2.1 The boundary functional $W[s]$

We recall the definition of $W[s]$ in the context of the spinfoam *GFT* where the theory is defined for a field $\phi : (SO(4))^4 \rightarrow R$ by an action of the form

$$S[\phi] = S_{\text{kin}}[\phi] + \frac{\lambda}{5!} S_{\text{int}}[\phi]. \quad (8.58)$$

$SO(4)$ -invariant observables of the theory are computed as the expectation values

$$W[s] = \frac{1}{Z} \int \text{D}\phi f_s(\phi) e^{-\int (P_G P_\beta \phi)^2 - \frac{\lambda}{5!} \int (P_G P_\beta \phi)^5} \quad (8.59)$$

where the normalization Z is the functional integral without $f_s(\phi)$, and $f_s(\phi)$ is the function of the field determined by the spin network $s = (\Gamma, j_l, i_n)$. Recall that a spin network is a graph Γ formed by nodes n connected by links l , colored with representations $j_l = (j_l^+, j_l^-) = ((1 + \beta)l, (1 - \beta)l)$ associated to the links and intertwiners i_n associated to the nodes. We note l_{nm} a link connecting the nodes n and m , and $j_{nm} \equiv j_{mn}$ the corresponding color. The spin network function is defined in terms of the modes (see chapter 7) by

$$f_s(\phi) = \sum_{\alpha_{nm}} \prod_n \phi_{\alpha_{nm} i_n}^{j_{nm}}. \quad (8.60)$$

Here n runs over the nodes and, for each n , the index m runs over the four nodes that bound the four links l_{nm} joining at n . Notice that each index $\alpha_{nm} \equiv \alpha_{mn}$ appears exactly twice in the sum, and are thus contracted.

Fixed a spin network s , (8.59) can be treated by a perturbative expansion in λ , which leads to a sum over Feynman diagrams. Expanding both numerator and denominator, we have

$$\begin{aligned} W[s] &= \frac{1}{Z_0} \int \text{D}\phi f_s(\phi) e^{-\int (P_G P_\beta \phi)^2} - \\ &+ \frac{1}{Z_0} \frac{\lambda}{5!} \left[\int \text{D}\phi f_s(\phi) \left(\int (P_G P_\beta \phi)^5 \right) e^{-\int (P_G P_\beta \phi)^2} - \right. \end{aligned} \quad (8.61)$$

Table 8.1: Terminology

	0d	1d	2d	3d	4d
Spin networks:	<i>node</i> ,	<i>link</i> ;			
Spinfoams:	<i>vertex</i> ,	<i>edge</i> ,	<i>face</i> ;		
Triangulation:	<i>point</i> ,	<i>segment</i> ,	<i>triangle</i> ,	<i>tetrahedron</i> ,	<i>four-simplex</i> .

$$\begin{aligned}
& - \frac{\int \mathcal{D}\phi \left(\int (P_G P_\beta \phi)^5 \right) e^{-\int (P_G P_\beta \phi)^2}}{Z_0} \int \mathcal{D}\phi f_s(\phi) e^{-\int (P_G P_\beta \phi)^2} \Bigg] + \\
& + \frac{1}{Z_0} \frac{\lambda^2}{2(5!)^2} \left[\int \mathcal{D}\phi f_s(\phi) \left(\int (P_G P_\beta \phi)^5 \right)^2 e^{-\int (P_G P_\beta \phi)^2} + \dots \right],
\end{aligned}$$

where $Z_0 = \int \mathcal{D}\phi e^{-\int (P_G P_\beta \phi)^2}$. As usual in QFT, the normalization Z gives rise to all vacuum–vacuum transition amplitudes, and its role is to eliminate disconnected diagrams.

Recall that this Feynman sum can be expressed as a sum over all connected spinfoams $\sigma = (\Sigma, j_f, i_e)$ bounded by the spin network s . A spinfoam is a two-complex Σ , namely an ensemble of faces f bounded by edges e , in turn bounded by vertices v , colored with representations j_f associated to the faces and intertwiners i_e associated to the edges.

The boundary of a spinfoam $\sigma = (\Sigma, j_f, i_e)$ is a spin network $s = (\Gamma, j_l, i_n)$, where the graph Γ is the boundary of the two-complex Σ , $j_l = j_f$ anytime the link l of the spin network bounds a face f of the spinfoam and $i_n = i_e$ anytime the node n of the spin network bounds an edge e of the spinfoam. See the Table 8.1 for a summary of the terminology.

The amplitudes can be reconstructed from the following Feynman rules; the propagator

$$\mathcal{P}_{\alpha_n i \alpha'_n i'}^{j_n j'_n} = \delta_{i, i'} \sum_{\pi(n)} \prod_n \delta_{j_n^+, j'^+_{\pi(n)}} \delta_{\alpha_n^+ \alpha'^+_{\pi(n)}} \delta_{j_n^-, j'^-_{\pi(n)}} \delta_{\alpha_n^- \alpha'^-_{\pi(n)}} \quad (8.62)$$

where $\pi(n)$ are the permutations of the four numbers $n = 1, 2, 3, 4$; and the vertex amplitude

$$\mathcal{V}_{j_{nm}}^{\alpha_{nm} i_n} = \sum_{\{i_{EPR}\}} \left(\prod_n \delta_{i_n i_{EPR, n}} \right) \left(\prod_{n \neq m} \delta_{\alpha_{nm}^+ \alpha_{mn}^+} \delta_{\alpha_{nm}^- \alpha_{mn}^-} \right) \mathcal{A}_{EPR}(j_{nm}, i_{EPR, n}), \quad (8.63)$$

where the index $n = 1, \dots, 5$ labels the five legs of the five-valent vertex; while the index $m \neq n$ labels the four indices on each leg. The sum on $\{i_{EPR}\}$ runs on all the

possible choices of *EPR*-intertwiners for every node n . We obtain in this manner the amplitude

$$W[s] = \frac{1}{Z} \sum_{\{i_{EPR}\}} \sum_{\sigma, \partial\sigma=s} \prod_{f \in \sigma} \dim[(1+\beta)l] \dim[(1-\beta)l] \cdot \prod_{v \in \sigma} \lambda \mathcal{A}_{EPR}(j_{nm}^v, i_{EPR,n}) \left(\prod_{n \in s} \langle i_n | i_{EPR,n} \rangle \right). \quad (8.64)$$

Here σ are spinfoams with vertices v dual to a four-simplex, bounded the spin network s . f are the faces of σ ; the spins j_{nm}^v label the representations associated to the ten faces adjacent to the vertex v , where $n \neq m = 1, \dots, 5$; $\dim(j)$ is the dimension of the representation j . The colors of a faces f of σ bounded by a link l of s is restricted to match the color of the link: $j_f = j_l$. The expression is written for arbitrary boundary spin-network intertwiners i_n : the scalar product is in the intertwiner space and derives from the fact that the vertex amplitude projects on the sole EPRL intertwiner. The relation between the different elements is summarized in Table 8.2.

The sum (8.64) can be written as a power series in λ

$$W[s] = \sum_{k=0}^{\infty} \lambda^k W_k[s] \quad (8.65)$$

with

$$W_k[s] = \frac{1}{Z} \sum_{\{i_{EPR}\}} \sum_{\sigma^k, \partial\sigma^k=s} \prod_{f \in \sigma} \dim[(1+\beta)l] \dim[(1-\beta)l] \cdot \prod_{v \in \sigma} \mathcal{A}_{EPR}(j_{nm}^v, i_{EPR,n}) \left(\prod_{n \in s} \langle i_n | i_{EPR,n} \rangle \right), \quad (8.66)$$

where σ^k is a spinfoam with k vertices.

Finally, recall that the last expression can be interpreted as the quantum gravity boundary amplitude associated to the boundary state defined by the spin network s . The individual spin foams σ appearing in the sum can be interpreted as (discretized) spacetimes bounded by a 3-geometry determined by s . That is, (8.64) can be interpreted as a concrete definition of the formal functional integral

$$\Psi[q] = \int_{\partial g=q} Dg e^{iS_{GR}[g]} \quad (8.67)$$

where q is a 3-geometry and the integral of the exponent of the general relativity action is over the 4-geometries g bounded by q . Indeed, (8.64) can also be derived from a discretization of a suitable formulation of this functional integral.

8.3 A comment on the “Order Zero”

The general covariant 2-point function is, to the order zero in λ ,

Table 8.2: Relation between a triangulation and its dual, in the 4d bulk and in its 3d boundary. In parenthesis: adjacent elements. In italic, the two-complex and the spin-network's graph. The spinfoam is $\sigma = (\Delta_4^*, j_f, i_e)$. The spin network is $s = (\Delta_3^*, j_l, i_n)$.

Δ_4	Δ_4^*		coloring
4-simplex	<i>vertex</i>	(5 edg, 10 fac)	
tetrahedron	<i>edge</i>	(4 faces)	i_e
triangle	<i>face</i>		j_f
segment			
point			

Δ_3	Δ_3^*		coloring
tetrahedron	<i>node</i>	(4 links)	$i_n = i_e$
triangle	<i>link</i>		$j_l = j_f$
segment			
point			

$$W_0[s] = Z_0^{-1} \int D\phi f_s(\phi) e^{-\int (P_G P_\beta \phi)^2}. \quad (8.68)$$

The Wick expansion of this integral gives non-vanishing contributions for all s with an even number of nodes. Since there are no vertices, each of these contributions is simply given by products of face contributions, namely products of dimensions of representations. The 2-point function (8.55) reads

$$\mathbf{G}_{\mathbf{q}}^{abcd}(\mathbf{x}, \mathbf{y}) = \sum_s W_0[s] \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) \Psi_{\mathbf{q}}[s]. \quad (8.69)$$

We are interested at the large $j_l^{(0)}$ regime, where the gaussian effectively restricts the sum over (a large region of) spins of order $j_l^{(0)}$. Over this region, the phase factors fluctuate widely, and suppress the sum, unless they are compensated by similar phase factors. But $W_0[s]$ contains only powers of j_l 's, and cannot provide this compensation. Hence we do not expect a contribution of zero order to the sum. The only exception can be the null spin network $s = \emptyset$, which gives $W[\emptyset] = 1$ because of the normalization. Hence, to order zero

$$\mathbf{G}_{\mathbf{q}}^{abcd}(\mathbf{x}, \mathbf{y}) = W_0[\emptyset] \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) C_\emptyset. \quad (8.70)$$

But is reasonable to assume that the semiclassical boundary state on a macroscopic geometry \mathbf{q} has vanishing component on $s = \emptyset$, whose interpretation is that of a quantum state without any volume. Hence we take $C_\emptyset = 0$, and we conclude that the 2-point function has no zero order component in λ .

This result has a compelling geometrical interpretation. The sum over spinfoams can be interpreted as a sum over 4-geometries. The boundary state $\Psi_{\mathbf{q}}[s]$ describes a boundary geometry which has a nontrivial extrinsic curvature, described by the phase

of the state. In the large distance limit, we expect semiclassical histories to dominate the path integral. These must be close to a classical solution of the equations of motion, fitting the boundary data. Because of the extrinsic curvature of the boundary data, it is necessary that the internal geometry has non vanishing 4-volume. A round soccer ball must have volume inside. But the four-volume of a spinfoam is given by its vertices, which are dual to the four-simplices of the triangulation. Absence of vertices means absence of four-volume. It is therefore to be expected that the zero order contribution, which has no vertices, and therefore zero volume, is suppressed by the phases of the boundary state, representing the extrinsic curvature. Let us therefore go over to the first order in λ .

8.4 First order: Semiclassical boundary state

Semiclassical boundary states are a key ingredient in the definition of boundary amplitudes. Here we describe in detail the construction of a boundary state peaked on the intrinsic and the extrinsic geometry of the boundary of a Euclidean 4-simplex. The construction is new: it uses the coherent intertwiners of Livine and Speziale [65] (see also [72]) together with a superposition over spins as done in [45, 46]. It can be considered as an improvement of the boundary state used in [50, 51, 52] where Rovelli-Speziale gaussian states [81] for intertwiners were used.

We consider a simplicial decomposition Δ_5 of S^3 . The decomposition Δ_5 is homeomorphic to the boundary of a 4-simplex: it consists of five cells t_a which meet at ten faces f_{ab} ($a, b = 1, \dots, 5$ and $a < b$). Then we consider the sector of the Hilbert space \mathcal{H}_Σ spanned by spin network states with graph Γ_5 dual to the decomposition Δ_5 , see figure 8.1.

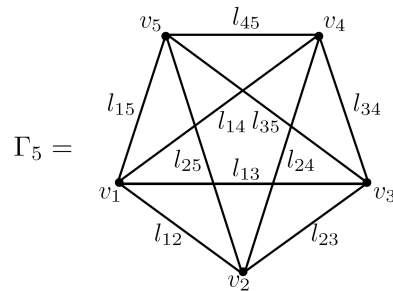


Figure 8.1: The complete graph Γ_5 with five nodes

Γ_5 is a complete graph with five nodes. We call v_a its nodes and l_{ab} ($a < b$) its ten links. Spin network states supported on this graph are labelled by ten spins l_{ab} ($a < b$) and five intertwiners i_a . We denote them by $|\Gamma_5, l_{ab}, i_a\rangle$ and call \mathcal{H}_{Γ_5} the Hilbert space they span. On \mathcal{H}_{Γ_5} we can introduce a *metric operator* smearing the electric field on surfaces dual to links, i.e. considering scalar products of fluxes. We focus on the node n and consider a surface f_{na} which cuts the link from the node n to the node a . The

flux operator through the surface f_{na} , parallel transported in the node n , is denoted³ $(E_n^a)_i$. It has the following three non-trivial properties:

(i) the flux operators $(E_n^a)_i$ and $(E_a^n)_i$ are related by a $SU(2)$ parallel transport g_{an} from the node a to the node n together with a change of sign which takes into account the different orientation of the face f_{an} ,

$$(E_n^a)_i = -(R_{an})_i^j (E_a^n)_j, \quad (8.71)$$

where R_{an} is the rotation which corresponds to the group element g_{an} associated to the link l_{an} , i.e. $R_{an} = D^{(1)}(g_{ab})$;

(ii) the commutator of two flux operators for the same face f_{na} is⁴

$$[(E_n^a)_i, (E_n^a)_j] = i\beta\varepsilon_{ij}^k (E_n^a)_k; \quad (8.72)$$

(iii) a spin network state is annihilated by the sum of the flux operators over the faces bounding a node

$$\sum_{c \neq n} (E_n^c)_i |\Gamma_5, l_{ab}, i_a\rangle = 0. \quad (8.73)$$

This last property follows from the $SU(2)$ gauge invariance of the spin network node.

Using the flux operator we can introduce the density-two inverse-metric operator at the node n , projected in the directions normal to the faces f_{na} and f_{nb} . It is defined as $E_n^a \cdot E_n^b = \delta^{ij} (E_n^a)_i (E_n^b)_j$. Its diagonal components $E_n^a \cdot E_n^a$ measure the area square of the face f_{na} ,

$$E_n^a \cdot E_n^a |\Gamma_5, l_{ab}, i_a\rangle = \left(\beta \sqrt{l_{na}(l_{na} + 1)} \right)^2 |\Gamma_5, l_{ab}, i_a\rangle. \quad (8.74)$$

Spin network states are eigenstates of the diagonal components of the metric operator. On the other hand, the off-diagonal components $E_n^a \cdot E_n^b$ with $a \neq b$ measure the dihedral angle between the faces f_{na} and f_{nb} (weighted with their areas). It reproduces the angle operator [78]. Using the recoupling basis for intertwiner space, we have that in general the off-diagonal components of the metric operator have non-trivial matrix elements

$$E_n^a \cdot E_n^b |\Gamma_5, l_{ab}, i_a\rangle = \sum_{i'_c} (E_n^a \cdot E_n^b)_{i_c}^{i'_c} |\Gamma_5, l_{ab}, i'_a\rangle. \quad (8.75)$$

We refer to [50, 51] for a detailed discussion. In particular, from property (ii), we have that some off-diagonal components of the metric operator at a node do not commute [77]

$$[E_n^a \cdot E_n^b, E_n^a \cdot E_n^c] \neq 0. \quad (8.76)$$

From this non-commutativity an Heisenberg inequality for dispersions of metric operators follows. Here we are interested in states which are peaked on a given value of *all* the off-diagonal components of the metric operator and which have dispersion of the order of Heisenberg's bound. Such states can be introduced using the technique

³Throughout the chapter $i, j, k \dots = 1, 2, 3$ are indices for vectors in \mathbf{R}^3 .

⁴Throughout the chapter we put $c = \hbar = G_{Newton} = 1$.

of coherent intertwiners [65, 72]. A coherent intertwiner between the representations l_1, \dots, l_4 is defined as⁵

$$\Phi^{m_1 \dots m_4}(\vec{n}_1, \dots, \vec{n}_4) = \frac{1}{\sqrt{\Omega(\vec{n}_1, \dots, \vec{n}_4)}} \int_{SU(2)} dh \prod_{a=1}^4 \langle l_a, m_a | D^{(l_a)}(h) | l_a, \vec{n}_a \rangle \quad (8.77)$$

and is labelled by four unit vectors $\vec{n}_1, \dots, \vec{n}_4$ satisfying the closure condition

$$l_1 \vec{n}_1 + \dots + l_4 \vec{n}_4 = 0. \quad (8.78)$$

The function $\Omega(\vec{n}_1, \dots, \vec{n}_4)$ provides normalization to one of the intertwiner. The function $\Phi^{m_1 \dots m_4}$ is invariant under rotations of the four vectors $\vec{n}_1, \dots, \vec{n}_4$. In the following we always assume that this invariance has been fixed with a given choice of orientation⁶.

Nodes of the spin network can be labelled with coherent intertwiners. In fact such states provide an overcomplete basis of \mathcal{H}_{Γ_5} . Calling $v_i^{m_1 \dots m_4}$ the standard recoupling basis for intertwiners, we can define the coefficients

$$\Phi_i(\vec{n}_1, \dots, \vec{n}_4) = v_i^{m_1 \dots m_4} \Phi_{m_1 \dots m_4}(\vec{n}_1, \dots, \vec{n}_4). \quad (8.79)$$

We define a *coherent spin network* $|\Gamma_5, l_{ab}, \Phi_a\rangle$ as the state labelled by ten spins l_{ab} and 4×5 normals \vec{n}_{ab} and given by the superposition

$$|\Gamma_5, l_{ab}, \Phi_a(\vec{n})\rangle = \sum_{i_1 \dots i_5} \left(\prod_{a=1}^5 \Phi_{i_a}(\vec{n}_{ab}) \right) |\Gamma_5, l_{ab}, i_a\rangle. \quad (8.80)$$

The expectation value of the metric operator on a coherent spin network is simply

$$\langle \Gamma_5, l_{ab}, \Phi_a | E_c^a \cdot E_c^b | \Gamma_5, l_{ab}, \Phi_a \rangle \simeq \beta^2 l_{ca} l_{cb} \vec{n}_{ca} \cdot \vec{n}_{cb} \quad (8.81)$$

in the large spin limit. As a result we can choose the normals \vec{n}_{ab} so that the coherent spin network state is peaked on a given intrinsic geometry of Σ .

Normals in different tetrahedra cannot be chosen independently if we want to peak on a Regge geometry [82]. The relation between normals is provided by the requirement that they are computed from the lengths of the edges of the triangulation Δ_5 . In fact, a state with generic normals (satisfying the closure condition (8.78)) is peaked on a discontinuous geometry. This fact can be seen in the following way: let us consider

⁵The state $|l, \vec{n}\rangle$ is a spin coherent state. It is labelled by a unit vector \vec{n} or equivalently by a point on the unit sphere. Given a $SU(2)$ transformation g which acts on the vector $+\vec{e}_z$ sending it to the vector $\vec{n} = Re_z$, a spin coherent state is given by $|l, \vec{n}\rangle = D^{(l)}(g)|l, +\rangle$. As a result, it is defined up to a phase $e^{i\alpha l}$ corresponding to a transformation $\exp(i\alpha \vec{n} \cdot \vec{J})|l, \vec{n}\rangle = e^{i\alpha l}|l, \vec{n}\rangle$. A phase ambiguity in the definition of the coherent intertwiner (8.77) follows. Such ambiguity becomes observable when a superposition over l is considered.

⁶For instance we can fix this redundancy assuming that the sum $l_1 \vec{n}_1 + l_2 \vec{n}_2$ is in the positive z direction while the vector $\vec{n}_1 \times \vec{n}_2$ in the positive y direction. Once chosen this orientation, the four unit-vectors $\vec{n}_1, \dots, \vec{n}_4$ (which satisfying the closure condition) depend only on two parameters. These two parameters can be chosen to be the dihedral angle $\cos \theta_{12} = \vec{n}_1 \cdot \vec{n}_2$ and the twisting angle $\tan \phi_{(12)(34)} = \frac{(\vec{n}_1 \times \vec{n}_2) \cdot (\vec{n}_3 \times \vec{n}_4)}{|\vec{n}_1 \times \vec{n}_2| |\vec{n}_3 \times \vec{n}_4|}$.

an edge of the triangulation Δ_5 ; this edge is shared by three tetrahedra; for each tetrahedron we can compute the expectation value of the length operator for an edge in its boundary [80]; however in general the expectation value of the length of an edge seen from different tetrahedra will not be the same; this fact shows that the geometry is *discontinuous*. The requirement that the semiclassical state is peaked on a Regge geometry amounts to a number of relations between the labels \vec{n}_{ab} . In the case of the boundary of a Euclidean 4-simplex (excluding the ‘rectangular’ cases discussed in [83]), the normals turn out to be completely fixed once we give the areas of the ten triangles or equivalently the ten spins l_{ab} ,

$$\vec{n}_{ab} = \vec{n}_{ab}(l_{cd}) . \quad (8.82)$$

This assignment of normals guarantees that the geometry we are peaking on is Regge-like. In particular, in this chapter we are interested in the case of a 4-simplex which is approximately regular. In this case the spins labelling the links are of the form $l_{ab} = l_0 + \delta l_{ab}$ with $\frac{\delta l_{ab}}{l_0} \ll 1$ and a perturbative expression for the normals solving the continuity condition is available:

$$\vec{n}_{ab}(l_0 + \delta l) = \vec{n}_{ab}(l_0) + \sum_{cd} v^{(ab)(cd)} \delta l_{cd} . \quad (8.83)$$

The coefficients $v^{(ab)(cd)}$ can be computed in terms of the derivative of the normals \vec{n}_{ab} with respect to the ten edge lengths, using the Jacobian of the transformation from the ten areas to the ten edge lengths of the 4-simplex⁷.

In the following we are interested in superpositions over spins of coherent spin networks. As coherent intertwiners are defined only up to a spin-dependent arbitrary phase, a choice is in order. We make the canonical choice of phases described in [73]. We briefly recall it here. Consider a non-degenerate Euclidean 4-simplex; two tetrahedra t_a and t_b are glued at the triangle $f_{ab} \equiv f_{ba}$. Now, two congruent triangles f_{ab} and f_{ba} in \mathbf{R}^3 can be made to coincide via a unique rotation $R_{ab} \in SO(3)$ which, together with a translation, takes one outward-pointing normal to minus the other one,

$$R_{ab} \vec{n}_{ab} = -\vec{n}_{ba} . \quad (8.84)$$

The canonical choice of phase for the spin coherent states $|l_{ab}, \vec{n}_{ab}\rangle$ and $|l_{ab}, \vec{n}_{ba}\rangle$ entering the coherent intertwiners Φ_a and Φ_b is given by lifting the rotation R_{ab} to a $SU(2)$ transformation g_{ab} and requiring that

$$|l_{ab}, \vec{n}_{ba}\rangle = D^{(l_{ab})}(g_{ab}) J |l_{ab}, \vec{n}_{ab}\rangle \quad (8.85)$$

where $J : \mathcal{H}_l \rightarrow \mathcal{H}_l$ is the standard antilinear map for $SU(2)$ representations defined by

$$\langle \epsilon | (|\alpha\rangle \otimes J|\beta\rangle) = \langle \beta | \alpha \rangle \quad \text{with } |\alpha\rangle, |\beta\rangle \in \mathcal{H}_l \quad (8.86)$$

and $\langle \epsilon |$ is the unique intertwiner in $\mathcal{H}_l \otimes \mathcal{H}_l$. In the following we will always work with coherent spin networks $|\Gamma_5, l_{ab}, \Phi_a(\vec{n}(l))\rangle$ satisfying the continuity condition, and with

⁷For a given choice of orientation, see footnote 5 in ‘‘LQG propagator from the new spin-foams’’, ArXiv 0905.4082v1.

the canonical choice for the arbitrary phases of coherent states. From now on we use the shorter notation $|l, \Phi(\vec{n})\rangle$.

Coherent spin networks are eigenstates of the diagonal components of the metric operator, namely the area operator for the triangles of Δ_5 . The extrinsic curvature to the manifold Σ measures the amount of change of the 4-normal to Σ , parallel transporting it along Σ . In a piecewise-flat context, the extrinsic curvature has support on triangles, that is it is zero everywhere except that on triangles. For a triangle f_{ab} , the extrinsic curvature K_{ab} is given by the angle between the 4-normals N_a^μ and N_b^μ to two tetrahedra t_a and t_b sharing the face f_{ab} . As the extrinsic curvature is the momentum conjugate to the intrinsic geometry, we have that a semiclassical state cannot be an eigenstate of the area as it would not be peaked on a given extrinsic curvature. In order to define a state peaked both on intrinsic and extrinsic geometry, we consider a superposition of coherent spin networks,

$$|\Psi_0\rangle = \sum_{l_{ab}} \psi_{l_0, \phi_0}(l) |l, \Phi(\vec{n})\rangle, \quad (8.87)$$

with coefficients $\psi_{l_0, \phi_0}(l)$ given by a gaussian times a phase,

$$\begin{aligned} \psi_{l_0, \phi_0}(l) &= \frac{1}{N} \exp\left(-\sum_{ab, cd} \alpha^{(ab)(cd)} \frac{l_{ab} - l_{0ab}}{\sqrt{l_{0ab}}} \frac{l_{cd} - l_{0cd}}{\sqrt{l_{0cd}}}\right) \times \\ &\times \exp\left(-i \sum_{ab} \phi_0^{ab} (l_{ab} - l_{0ab})\right). \end{aligned} \quad (8.88)$$

As we are interested in a boundary configuration peaked on the geometry of a regular 4-simplex, we choose all the background spins to be equal, $l_{0ab} \equiv l_0$. Later we will consider an asymptotic expansion for large l_0 . The phases ϕ_0^{ab} are also chosen to be equal. The extrinsic curvature at the face f_{ab} in a regular 4-simplex is $K_{ab} = \arccos N_a \cdot N_b = \arccos(-\frac{1}{4})$. In Ashtekar-Barbero variables (E_0, A_0) we have

$$\phi_0 \equiv \phi_0^{ab} = \beta K_{ab} = \beta \arccos(-1/4). \quad (8.89)$$

The 10×10 matrix $\alpha^{(ab)(cd)}$ is assumed to be complex with positive definite real part. Moreover we require that it has the symmetries of a regular 4-simplex. We introduce the matrices $P_k^{(ab)(cd)}$ with $k = 0, 1, 2$ defined as

$$P_0^{(ab)(cd)} = 1 \quad \text{if } (ab) = (cd) \text{ and zero otherwise,} \quad (8.90)$$

$$P_1^{(ab)(cd)} = 1 \quad \text{if } \{a = c, b \neq d\} \text{ or a permutation of it} \\ \text{and zero otherwise,} \quad (8.91)$$

$$P_2^{(ab)(cd)} = 1 \quad \text{if } (ab) \neq (cd) \text{ and zero otherwise.} \quad (8.92)$$

Their meaning is simple: a couple (ab) identifies a link of the graph Γ_5 ; two links can be either coincident, or touching at a node, or disjoint. The matrices $P_k^{(ab)(cd)}$ correspond to these three different cases. Using the basis $P_k^{(ab)(cd)}$ we can write the matrix $\alpha^{(ab)(cd)}$ as

$$\alpha^{(ab)(cd)} = \sum_{k=0}^2 \alpha_k P_k^{(ab)(cd)}. \quad (8.93)$$

As a result our ansatz for a semiclassical boundary state $|\Psi_0\rangle$ is labelled by a (large) half-integer l_0 and has only three complex free parameters, the numbers α_k .

8.5 The new spin foam dynamics

The dynamics is implemented in terms of a spin foam functional $\langle W|$. Here we are interested in its components on the Hilbert space spanned by spin networks with graph Γ_5 . The sum over two-complexes can be implemented in terms of a formal perturbative expansion in the parameter λ of a Group Field Theory [84]:

$$\langle W|\Gamma_5, l_{ab}, i_a\rangle = \sum_{\sigma} \lambda^{N_{\sigma}} W(\sigma) \quad (8.94)$$

The sum is over spinfoams (colored 2-complexes) whose boundary is the spin network (Γ_5, l_{ab}, i_a) , $W(\sigma)$ is the spinfoam amplitude

$$W(\sigma) = \prod_{f \subset \sigma} W_f \prod_{v \subset \sigma} \mathcal{A}_{EP R}^v \quad (8.95)$$

where $\mathcal{A}_{EP R}^v$ and W_f are the vertex and face amplitude respectively. The quantity N_{σ} in (8.94) is the number of vertices in the spin foam σ , therefore the formal expansion in λ is in fact a *vertex expansion*.

The vertex amplitude is given by

$$\mathcal{A}_{EP R}^v(l_{ab}, i_a) = \sum_{i_a^+ i_a^-} \{15l\}(j_{ab}^+, i_a^+) \{15j\}(j_{ab}^-, i_a^-) \prod_a f_{i_a^+ i_a^-}^{i_a}(l_{ab}) \quad (8.96)$$

where the unbalanced spins j^+, j^- are

$$j_{ab}^{\pm} = \beta^{\pm} l_{ab}, \quad \beta^{\pm} = \frac{1 \pm \beta}{2}. \quad (8.97)$$

This relation puts restrictions⁸ on the value of β and of l_{ab} . The fusion coefficients $f_{i_a^+ i_a^-}^{i_a}(l_{ab})$ are defined in [57] (see also [85]) and built out of the intertwiner $v_i^{m_1 \dots m_4}$ in $\mathcal{H}_{l_1} \otimes \dots \otimes \mathcal{H}_{l_4}$ and the intertwiners $v_{i_{\pm}}^{m_1^{\pm} \dots m_4^{\pm}}$ in $\mathcal{H}_{j_1^{\pm}} \otimes \dots \otimes \mathcal{H}_{j_4^{\pm}}$. Defining a map $Y : \mathcal{H}_l \rightarrow \mathcal{H}_{j^+} \otimes \mathcal{H}_{j^-}$ with matrix elements $Y_{m^+ m^-}^m = \langle j^+, m^+; j^-, m^- | Y | l, m \rangle$ given by Clebsh-Gordan coefficients, we have that the fusion coefficients $f_{i_a^+ i_a^-}^{i_a}$ are given by

$$f_{i_a^+ i_a^-}^{i_a} = Y_{m_1 m_1^+ m_1^-} \dots Y_{m_4 m_4^+ m_4^-} v_i^{m_1 \dots m_4} v_{i_+}^{m_1^+ \dots m_4^+} v_{i_-}^{m_1^- \dots m_4^-}. \quad (8.98)$$

Indices are raised and lowered with the Wigner metric.

⁸Formula (8.96) is well-defined only for j^{\pm} half-integer. As a result, for a fixed value of β , there are restrictions on the boundary spin l . For instance, if we choose $\beta = 1/n$ with n integer, then we have that l has to be integer and $l \geq n$, i.e. $l \in \{n, n+1, n+2, \dots\}$.

Throughout this paper we will restrict attention to the lowest order in the vertex expansion. To this order, the boundary amplitude of a spin network state with graph Γ_5 is given by

$$\langle W | \Gamma_5, l_{ab}, i_a \rangle = \mu(l_{ab}) \mathcal{A}_{EP R}^v(l_{ab}, i_a), \quad (8.99)$$

i.e. it involves a single spin foam vertex.

The function μ is defined as $\mu(l) = \prod_{ab} W_{fab}(l)$. A natural choice for the face amplitude is $W_f(j^+, j^-) = (2j^+ + 1)(2j^- + 1) = (1 - \beta^2)l^2 + 2l + 1$. Other choices can be considered. We assume that $\mu(\lambda l_{ab})$ scales as λ^p for some p for large λ . We will show in the following that, at the leading order in large l_0 , the LQG propagator (8.3) is in fact independent from the choice of face amplitude, namely from the function $\mu(l)$.

8.6 LQG propagator: integral formula

In this section we define the LQG propagator and then provide an integral formula for it. The dynamical expectation value of an operator \mathcal{O} on the state $|\Psi_0\rangle$ is defined via the following expression

$$\langle \mathcal{O} \rangle = \frac{\langle W | \mathcal{O} | \Psi_0 \rangle}{\langle W | \Psi_0 \rangle}. \quad (8.100)$$

The geometric operator we are interested in is the metric operator $E_n^a \cdot E_n^b$ discussed in section 8.4. We focus on the *connected* two-point correlation function G_{nm}^{abcd} on a semiclassical boundary state $|\Psi_0\rangle$. It is defined as

$$G_{nm}^{abcd} = \langle E_n^a \cdot E_n^b E_m^c \cdot E_m^d \rangle - \langle E_n^a \cdot E_n^b \rangle \langle E_m^c \cdot E_m^d \rangle. \quad (8.101)$$

We are interested in computing this quantity using the boundary state $|\Psi_0\rangle$ introduced in section 8.4 and the spin foam dynamics (8.96). This is what we call the *LQG propagator*. As the boundary state is a superposition of coherent spin networks, the LQG propagator involves terms of the form $\langle W | \mathcal{O} | l, \Phi(\vec{n}) \rangle$. Its explicit formula is

$$\begin{aligned} G_{nm}^{abcd} &= \frac{\sum_l \psi(l) \langle W | E_n^a \cdot E_n^b E_m^c \cdot E_m^d | l, \Phi(\vec{n}) \rangle}{\sum_l \psi(l) \langle W | l, \Phi(\vec{n}) \rangle} - \\ &\quad - \frac{\sum_l \psi(l) \langle W | E_n^a \cdot E_n^b | l, \Phi(\vec{n}) \rangle}{\sum_l \psi(l) \langle W | l, \Phi(\vec{n}) \rangle} \frac{\sum_l \psi(l) \langle W | E_m^c \cdot E_m^d | l, \Phi(\vec{n}) \rangle}{\sum_l \psi(l) \langle W | l, \Phi(\vec{n}) \rangle} \end{aligned} \quad (8.102)$$

In the following two subsections we recall the integral formula for the amplitude of a coherent spin network $\langle W | l, \Phi(\vec{n}) \rangle$ [70, 72],[73] and derive analogous integral expressions for the amplitude with metric operator insertions $\langle W | E_n^a \cdot E_n^b | l, \Phi(\vec{n}) \rangle$ and $\langle W | E_n^a \cdot E_n^b E_m^c \cdot E_m^d | l, \Phi(\vec{n}) \rangle$.

8.6.1 Integral formula for the amplitude of a coherent spin network

The boundary amplitude of a coherent spin network $|l, \Phi(\vec{n})\rangle$ admits an integral representation [70, 72],[73]: see section 7.3 in chapter 7. Here we go through its derivation

as we will use a similar technique in next section.

The boundary amplitude $\langle W|l, \Phi(\vec{n})\rangle$ can be written as an integral over five copies of $SU(2) \times SU(2)$ (with respect to the Haar measure)⁹:

$$\begin{aligned} \langle W|l_{ab}, \Phi_a(\vec{n})\rangle &= \mu(l) \mathcal{A}_{EP R}^v(l_{ab}, i_a) \\ &= \sum_{i_a} \left(\prod_a \Phi_{i_a}(\vec{n}) \right) \langle W|l_{ab}, i_a \rangle \\ &= \mu(l) \int \prod_{a=1}^5 dg_a^+ dg_a^- \prod_{ab} P^{ab}(g^+, g^-). \end{aligned} \quad (8.103)$$

The function $P^{ab}(g^+, g^-)$ is given by

$$P^{ab}(g^+, g^-) = \langle l_{ab}, -\vec{n}_{ba} | Y^\dagger D^{(j_{ab}^\pm)}((g_a^+)^{-1} g_b^+) \otimes D^{(j_{ab}^-)}((g_a^-)^{-1} g_b^-) Y | l_{ab}, \vec{n}_{ab} \rangle. \quad (8.104)$$

where the map Y is defined in section 8.5. Using the factorization property of spin coherent states,

$$Y|l, \vec{n}\rangle = |j^+, \vec{n}\rangle \otimes |j^-, \vec{n}\rangle, \quad (8.105)$$

we have that the function $P^{ab}(g^+, g^-)$ factorizes as

$$P^{ab}(g^+, g^-) = P^{ab+}(g^+) P^{ab-}(g^-) \quad (8.106)$$

with

$$P^{ab\pm} = \langle l_{ab}, -\vec{n}_{ba} | D^{(j_{ab}^\pm)}((g_a^\pm)^{-1} g_b^\pm) | l_{ab}, \vec{n}_{ab} \rangle = \left(\langle \frac{1}{2}, -\vec{n}_{ba} | (g_a^\pm)^{-1} g_b^\pm | \frac{1}{2}, \vec{n}_{ab} \rangle \right)^{2j_{ab}^\pm}. \quad (8.107)$$

In the last equality we have used (again) the factorization property of spin coherent states to exponentiate the spin j_{ab}^\pm . In the following we will drop the $1/2$ in $|\frac{1}{2}, \vec{n}_{ab}\rangle$ and write always $|\vec{n}_{ab}\rangle$ for the coherent state in the fundamental representation.

The final expression we get is

$$\langle W|l, \Phi(\vec{n})\rangle = \mu(l) \int \prod_{a=1}^5 dg_a^+ dg_a^- e^S \quad (8.108)$$

where the “action” S is given by the sum $S = S^+ + S^-$, with

$$S^\pm = \sum_{ab} 2j_{ab}^\pm \log \langle -\vec{n}_{ab} | (g_a^\pm)^{-1} g_b^\pm | \vec{n}_{ba} \rangle. \quad (8.109)$$

8.6.2 LQG operators as group integral insertions

In this section we use a similar technique to derive integral expressions for the expectation value of metric operators. In particular we show that

$$\langle W|E_n^a \cdot E_n^b | l_{ab}, \Phi_a(\vec{n})\rangle = \mu(l) \int \prod_{a=1}^5 dg_a^+ dg_a^- q_n^{ab}(g^+, g^-) e^S \quad (8.110)$$

⁹See section 7.3 in chapter 7.

and that

$$\langle W | E_n^a \cdot E_n^b \cdot E_m^c \cdot E_m^d | l_{ab}, \Phi_a(\vec{n}) \rangle = \mu(l) \int \prod_{a=1}^5 dg_a^+ dg_a^- q_n^{ab}(g^+, g^-) q_m^{cd}(g^+, g^-) e^S \quad (8.111)$$

where we assume¹⁰ $n \neq m$ and $a, b, c, d \neq n, m$. The expression for the insertions $q_n^{ab}(g^+, g^-)$ in the integral is derived below.

We start focusing on $\langle E_n^a \cdot E_n^b \rangle$ in the case $a \neq b$. The metric field $(E_n^b)_i$ acts on a state $|l_{ab}, m_{ab}\rangle$ as β times the generator J_i of $SU(2)$. As a result we can introduce a quantity Q_i^{ab} defined as

$$Q_i^{ab}(g^+, g^-) = \langle l_{ab}, -\vec{n}_{ba} | Y^\dagger D^{(j_{ab}^+)}((g_a^+)^{-1} g_b^+) \otimes D^{(j_{ab}^-)}((g_a^-)^{-1} g_b^-) Y(E_n^b)_i | l_{ab}, \vec{n}_{ab} \rangle, \quad (8.112)$$

so that

$$\langle W | E^{na} \cdot E^{nb} | l, \Phi(\vec{n}) \rangle = \int \prod_{a=1}^5 dg_a^+ dg_a^- \delta^{ij} Q_i^{na} Q_j^{nb} \prod'_{cd} P^{cd}(g^+, g^-). \quad (8.113)$$

The product \prod' is over couples (cd) different from $(na), (nb)$. Thanks to the invariance properties of the map Y , we have that

$$Y J_i^{ab} | l_{ab}, m_{ab} \rangle = (J_i^{ab+} + J_i^{ab-}) Y | l_{ab}, m_{ab} \rangle. \quad (8.114)$$

Thus Q_i^{ab} can be written as

$$Q_i^{ab} = Q_i^{ab+} P^{ab-} + P^{ab+} Q_i^{ab-} \quad (8.115)$$

with

$$Q_i^{ab\pm} = \beta \langle j_{ab}^\pm, -\vec{n}_{ba} | D^{(j_{ab}^\pm)}((g_a^\pm)^{-1} g_b^\pm) J_i^{ab\pm} | j_{ab}^\pm, \vec{n}_{ab} \rangle. \quad (8.116)$$

Now we show that $Q_i^{ab\pm}$ is given by a function $A_i^{ab\pm}$ linear in the spin j_{ab}^\pm , times the quantity $P^{ab\pm}$ defined in (8.107),

$$Q_i^{ab\pm} = A_i^{ab\pm} P^{ab\pm}. \quad (8.117)$$

The function $A_i^{ab\pm}$ is determined as follows. The generator $J_i^{ab\pm}$ of $SU(2)$ in representation j_{ab}^\pm can be obtained as the derivative

$$i \frac{\partial}{\partial \alpha^i} D^{(j_{ab}^\pm)}(h(\alpha)) \Big|_{\alpha^i=0} = J_i^{ab\pm} \quad (8.118)$$

where the group element $h(\alpha)$ is defined via the canonical parametrization $h(\alpha) = \exp(-i\alpha^i \frac{\sigma_i}{2})$. Therefore, we can write $Q_i^{ab\pm}$ as

$$\begin{aligned} Q_i^{ab\pm} &= i\beta \frac{\partial}{\partial \alpha^i} \left(\langle j_{ab}^\pm, -\vec{n}_{ba} | D^{(j_{ab}^\pm)}((g_a^\pm)^{-1} g_b^\pm) D^{(j_{ab}^\pm)}(h(\alpha)) | j_{ab}^\pm, \vec{n}_{ab} \rangle \right) \Big|_{\alpha^i=0} \\ &= i\beta \frac{\partial}{\partial \alpha^i} \left(\beta \langle -\vec{n}_{ba} | (g_a^\pm)^{-1} g_b^\pm h(\alpha) | \vec{n}_{ab} \rangle \right) \Big|_{\alpha^i=0}^{2j_{ab}^\pm} \\ &= \beta j_{ab}^\pm \langle -\vec{n}_{ba} | (g_a^\pm)^{-1} g_b^\pm \sigma^i | \vec{n}_{ab} \rangle \langle -\vec{n}_{ba} | (g_a^\pm)^{-1} g_b^\pm | \vec{n}_{ab} \rangle^{2j_{ab}^\pm - 1}. \end{aligned} \quad (8.119)$$

¹⁰Similar formulae can be found also in the remaining cases but are not needed for the calculation of the LQG propagator.

Comparing expression (8.119) with (8.117) and (8.107), we find that $A_i^{na\pm}$ is given by

$$A_i^{na\pm} = \beta j_{na}^{\pm} \frac{\langle -\vec{n}_{an} | (g_a^{\pm})^{-1} g_n^{\pm} \sigma^i | \vec{n}_{na} \rangle}{\langle -\vec{n}_{an} | (g_a^{\pm})^{-1} g_n^{\pm} | \vec{n}_{na} \rangle}. \quad (8.120)$$

A vectorial expression for $A_i^{na\pm}$ can be given, introducing the rotation $R_a^{\pm} = D^{(1)}(g_a^{\pm})$,

$$A_i^{na\pm} = \beta j_{na}^{\pm} (R_n^{\pm})^{-1} \frac{R_n^{\pm} n_{na} - R_a^{\pm} n_{an} - i(R_n^{\pm} n_{na} \times R_a^{\pm} n_{an})}{1 - (R_a^{\pm} n_{an}) \cdot (R_n^{\pm} n_{na})}. \quad (8.121)$$

Thanks to (8.115) and (8.117), we have that the expression for Q_i^{ab} simplifies to

$$Q_i^{ab} = A_i^{ab} P^{ab} \quad (8.122)$$

with

$$A_i^{ab} = A_i^{ab+} + A_i^{ab-}. \quad (8.123)$$

As a result, equation (8.113) reduces to

$$\langle W | E_n^a \cdot E_n^b | l, \Phi(\vec{n}) \rangle = \int \prod_{a=1}^5 dg_a^+ dg_a^- \delta^{ij} A_i^{na} A_j^{nb} \prod_{cd} P^{cd}(g^+, g^-), \quad (8.124)$$

which is of the form (8.108) with the insertion $A^{na} \cdot A^{nb}$. Therefore, comparing with equation (8.110), we have that

$$q_n^{ab}(g^+, g^-) = A^{na} \cdot A^{nb} \quad (8.125)$$

for $a \neq b$. The case with $a = b$ can be computed using a similar technique but the result is rather simple and expected, thus we just state it

$$q_n^{aa}(g^+, g^-) = \beta^2 l_{na} (l_{na} + 1). \quad (8.126)$$

As far as $\langle E_n^a \cdot E_n^b E_m^c \cdot E_m^d \rangle$ a similar result can be found. In particular, for $n \neq m$ and $a, b, c, d \neq n, m$ the result is stated at the beginning of this section, equation (8.111), with the same expression for the insertion $q_n^{ab}(g^+, g^-)$ as in equation (8.125) and equation (8.126).

Substituting (8.110)-(8.111) in (8.102) we obtain a new expression for the propagator in terms of group integrals:

$$\begin{aligned} G_{nm}^{abcd} &= \frac{\sum_l \mu(l) \psi(l) \int dg^{\pm} q_n^{ab} q_m^{cd} e^S}{\sum_l \mu(l) \psi(l) \int dg^{\pm} e^S} - \\ &\quad - \frac{\sum_l \mu(l) \psi(l) \int dg^{\pm} q_n^{ab} e^S \sum_l \mu(l) \psi(l) \int dg^{\pm} q_m^{cd} e^S}{\sum_l \mu(l) \psi(l) \int dg^{\pm} e^S \sum_l \mu(l) \psi(l) \int dg^{\pm} e^S}. \end{aligned} \quad (8.127)$$

This expression with metric operators written as insertions in an integral is the starting point for the large l_0 asymptotic analysis of next section.

8.7 Stationary phase approximation

The correlation function (8.127) depends on the scale j_0 fixed by the boundary state. We are interested in computing its asymptotic expansion for large j_0 . The technique we use is an (extended) stationary phase approximation of a multiple integral over both spins and group elements. In 8.7.1 we put expression (8.127) in a form to which this approximation can be applied. Then in 8.7.2 we recall a standard result in asymptotic analysis regarding connected two-point functions and in 8.7.3-8.7.4 we apply it to our problem.

8.7.1 The total action and the extended integral

We introduce the “total action” defined as $S_{tot} = \log \psi + S$ or more explicitly as

$$S_{tot}(j_{ab}, g_a^+, g_a^-) = -\frac{1}{2} \sum_{ab, cd} \alpha^{(ab)(cd)} \frac{j_{ab} - j_{0ab}}{\sqrt{j_{0ab}}} \frac{j_{cd} - j_{0cd}}{\sqrt{j_{0cd}}} - i \sum_{ab} \phi_0^{ab} (j_{ab} - j_{0ab}) + S^+(j_{ab}, g_a^+) + S^-(j_{ab}, g_a^-). \quad (8.128)$$

Notice that the action $S^+ + S^-$ is a homogeneous function of the spins j_{ab} therefore, rescaling the spins j_{0ab} and j_{ab} by an interger λ so that $j_{0ab} \rightarrow \lambda j_{0ab}$ and $j_{ab} \rightarrow \lambda j_{ab}$, we have that the total action goes to $S_{tot} \rightarrow \lambda S_{tot}$. We recall also that $q_n^{ab} \rightarrow \lambda^2 q_n^{ab}$. In the large λ limit, the sums over spins in expression (8.127) can be approximated with integrals over continuous spin variables¹¹:

$$\sum_j \mu \int d^5 g^\pm q_n^{ab} e^{\lambda S_{tot}} = \int d^{10} j d^5 g^\pm \mu q_n^{ab} e^{\lambda S_{tot}} + O(\lambda^{-N}) \quad \forall N > 0. \quad (8.129)$$

Moreover, notice that the action, the measure and the insertions in (8.127) are invariant under a $SO(4)$ symmetry that makes an integration $dg^+ dg^-$ redundant. We can factor out one $SO(4)$ volume, e.g. putting $g_1^+ = g_1^- = 1$, so that we end up with an integral over $d^4 g^\pm = \prod_{a=2}^5 dg_a^+ dg_a^-$.

As a result we can re-write expression (8.127) in the following integral form

$$G_{nm}^{abcd} = \lambda^4 \left(\frac{\int d^{10} j d^4 g^\pm \mu q_n^{ab} q_m^{cd} e^{\lambda S_{tot}}}{\int d^{10} j d^4 g^\pm \mu e^{\lambda S_{tot}}} - \frac{\int d^{10} j d^4 g^\pm \mu q_n^{ab} e^{\lambda S_{tot}}}{\int d^{10} j d^4 g^\pm \mu e^{\lambda S_{tot}}} \frac{\int d^{10} j d^4 g^\pm \mu q_m^{cd} e^{\lambda S_{tot}}}{\int d^{10} j d^4 g^\pm \mu e^{\lambda S_{tot}}} \right). \quad (8.130)$$

To this expression we can apply the standard result stated in the following section.

8.7.2 Asymptotic formula for connected two-point functions

Consider the integral

$$F(\lambda) = \int dx f(x) e^{\lambda S(x)} \quad (8.131)$$

¹¹The remainder, i.e. the difference between the sum and the integral, can be estimated via Euler-Maclaurin summation formula. This approximation does not affect any finite order in the computation of the LQG propagator.

over a region of \mathbf{R}^d , with $S(x)$ and $f(x)$ smooth complex-valued functions such that the real part of S is negative or vanishing, $Re S \leq 0$. Assume also that the stationary points x_0 of S are isolated so that the Hessian at a stationary point $H = S''(x_0)$ is non-singular, $\det H \neq 0$. Under these hypothesis an asymptotic expansion of the integral F for large λ is available: it is an extension of the standard stationary phase approximation that takes into account the fact that the action S is complex [86]. A key role is played by *critical points*, i.e. stationary points x_0 for which the real part of the action vanishes, $Re S(x_0) = 0$. Here we assume that there is a unique critical point. Then the asymptotic expansion of $F(\lambda)$ for large λ is given by

$$F(\lambda) = \left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}} \frac{e^{i \text{Ind} H} e^{\lambda S(x_0)}}{\sqrt{|\det H|}} \left(f(x_0) + \frac{1}{\lambda} \left(\frac{1}{2} f''_{ij}(x_0) (H^{-1})^{ij} + D \right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \right) \quad (8.132)$$

with $f''_{ij} = \partial^2 f / \partial x^i \partial x^j$ and $\text{Ind} H$ is the index¹² of the Hessian. The term D does not contain second derivatives of f , it contains only¹³ $f(x_0)$ and $f'_i(x_0)$. Now we consider three smooth complex-valued functions g , h and μ . A connected 2-point function relative to the insertions g and h and w.r.t. the measure μ is defined as

$$G = \frac{\int dx \mu(x) g(x) h(x) e^{\lambda S(x)}}{\int dx \mu(x) e^{\lambda S(x)}} - \frac{\int dx \mu(x) g(x) e^{\lambda S(x)}}{\int dx \mu(x) e^{\lambda S(x)}} \frac{\int dx \mu(x) h(x) e^{\lambda S(x)}}{\int dx \mu(x) e^{\lambda S(x)}}. \quad (8.134)$$

Using (8.132) it is straightforward to show that the (leading order) asymptotic formula for the connected 2-point function is simply

$$G = \frac{1}{\lambda} (H^{-1})^{ij} g'_i(x_0) h'_j(x_0) + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \quad (8.135)$$

Notice that both the measure function μ and the disconnected term D do not appear in the leading term of the connected 2-point function; nevertheless they are present in the higher orders (loop contributions). The reason we are considering the quantity G , built from integrals of the type (8.131), is that the LQG propagator has exactly this form. Specifically, in sections 8.7.3 we determine the critical points of the total action, in 8.7.4 we compute the Hessian of the total action and the derivative of the insertions evaluated at the critical points, and in 8.9 we state our result.

8.7.3 Critical points of the total action

The real part of the total action is given by

$$Re S_{tot} = - \sum_{ab,cd} (Re \alpha)^{(ab)(cd)} \frac{l_{ab} - l_{0ab}}{\sqrt{l_{0ab}}} \frac{l_{cd} - l_{0cd}}{\sqrt{l_{0cd}}} +$$

¹²The index is defined in terms of the eigenvalues of h_k of the Hessian as $\text{Ind} H = \frac{1}{2} \sum_k \arg(h_k)$ with $-\frac{\pi}{2} \leq \arg(h_k) \leq +\frac{\pi}{2}$.

¹³More explicitly, the term D is given by

$$D = f'_i(x_0) R''_{jkl}(x_0) (H^{-1})^{ij} (H^{-1})^{kl} + \frac{5}{2} f(x_0) R'''_{ijk}(x_0) R'''_{mnl}(x_0) (H^{-1})^{im} (H^{-1})^{jn} (H^{-1})^{kl} \quad (8.133)$$

with $R(x) = S(x) - S(x_0) - \frac{1}{2} H_{ij}(x_0) (x - x_0)^i (x - x_0)^j$.

$$+ \sum_{ab} j_{ab}^- \log \frac{1 - (R_a^- n_{ab}) \cdot (R_b^- n_{ba})}{2} + \sum_{ab} j_{ab}^+ \log \frac{1 - (R_a^+ n_{ab}) \cdot (R_b^+ n_{ba})}{2} . \quad (8.136)$$

Therefore, having assumed that the matrix α in the boundary state has positive definite real part, we have that the real part of the total action is negative or vanishing, $Re S_{tot} \leq 0$. In particular the total action vanishes for the configuration of spins j_{ab} and group elements g_a^\pm satisfying

$$l_{ab} = l_{0ab} , \quad (8.137)$$

$$g_a^\pm \text{ such that } R_a^\pm n_{ab}(l) = -R_b^\pm n_{ba}(l) . \quad (8.138)$$

Now we study the stationary points of the total action and show that there is a unique stationary point for which $Re S_{tot}$ vanishes.

The analysis of stationary points of the action $S^+ + S^-$ with respect to variations of the group variables g_a^\pm has been performed in full detail by Barrett et al. in [73]. Here we briefly summarize their result as they apply unchanged to the total action. We invite the reader to look at the original reference for a detailed derivation and a geometrical interpretation of the result.

The requirement that the variation of the total action with respect to the group variables g_a^\pm vanishes, $\delta_g S_{tot} = 0$, leads to the two sets of equations (respectively for the real and the imaginary part of the variation):

$$\sum_{b \neq a} j_{ab}^\pm \frac{R_a^\pm n_{ab} - R_b^\pm n_{ba}}{1 - (R_a^\pm n_{ab}) \cdot (R_b^\pm n_{ba})} = 0 \quad , \quad \sum_{b \neq a} j_{ab}^\pm \frac{(R_a^\pm n_{ab}) \times (R_b^\pm n_{ba})}{1 - (R_a^\pm n_{ab}) \cdot (R_b^\pm n_{ba})} = 0 . \quad (8.139)$$

When evaluated at the maximum point (8.138), these two sets of equations are trivially satisfied. In fact the normals \vec{n}_{ab} in the boundary state are chosen to satisfy the closure condition (8.78) at each node. Therefore the critical points in the group variables are given by all the solutions of equation (8.138).

For normals \vec{n}_{ab} which define non-degenerate tetrahedra and satisfy the continuity condition (8.82), the equation $R_a n_{ab} = -R_b n_{ba}$ admits two distinct sets of solutions, up to global rotations. These two sets are related by parity. The two sets can be lifted to $SU(2)$. We call them \bar{g}_a^+ and \bar{g}_a^- . Out of them, four classes of solutions for the couple (g_a^+, g_a^-) can be found. They are given by

$$(\bar{g}_a^+, \bar{g}_a^-), (\bar{g}_a^-, \bar{g}_a^+), (\bar{g}_a^+, \bar{g}_a^+), (\bar{g}_a^-, \bar{g}_a^-) . \quad (8.140)$$

The geometrical interpretation is the following. The couples $(l_{ab} \vec{n}_{ab}, l_{ab} \vec{n}_{ab})$ are interpreted as the selfdual and anti-selfdual parts (with respect to some ‘‘time’’ direction, e.g. $(0, 0, 0, 1)$) of area bivectors associated to triangles in 4-dimensions; since these bivectors are diagonal, they live in the 3-dimensional subspace of \mathbf{R}^4 orthogonal to the chosen ‘‘time’’ direction. Because of the closure condition (8.78), for a fixed n the four bivectors $(l_{na} \vec{n}_{na}, l_{na} \vec{n}_{na})$ define an embedding of a tetrahedron in \mathbf{R}^4 . The two group elements g_a^+ and g_a^- of the action (8.109) define an $SO(4)$ element which rotates the ‘‘initial’’ tetrahedron. The system (8.138) is a gluing condition between tetrahedra.

The first two classes of solutions in (8.140) glue five tetrahedra into two Euclidean non-degenerate 4-simplices related by a reflection, while the second two classes correspond to degenerate configurations with the 4-simplex living in the three-dimensional plane orthogonal to the chosen “time” direction.

The evaluation of the action $S(l_{ab}, g_a^+, g_a^-) = S^+(l_{ab}, g_a^+) + S^-(l_{ab}, g_a^-)$ on the four classes of critical points gives

$$S(l_{ab}, \bar{g}_a^+, \bar{g}_a^-) = +S_{Regge}(l_{ab}), \quad (8.141)$$

$$S(l_{ab}, \bar{g}_a^-, \bar{g}_a^+) = -S_{Regge}(l_{ab}), \quad (8.142)$$

$$S(l_{ab}, \bar{g}_a^+, \bar{g}_a^+) = +\beta^{-1}S_{Regge}(l_{ab}), \quad (8.143)$$

$$S(l_{ab}, \bar{g}_a^-, \bar{g}_a^-) = -\beta^{-1}S_{Regge}(l_{ab}), \quad (8.144)$$

where $S_{Regge}(l_{ab})$ is Regge action for a single 4-simplex with triangle areas $A_{ab} = \beta l_{ab}$ and dihedral angles $\phi_{ab}(l)$ written in terms of the areas

$$S_{Regge}(l_{ab}) = \sum_{ab} \beta l_{ab} \phi_{ab}(l). \quad (8.145)$$

Now we focus on stationarity of the total action with respect to variations of the spin labels l_{ab} . We fix the group elements (g_a^+, g_a^-) to belong to one of the four classes (8.140). For the first class we find

$$0 = \left. \frac{\partial S_{tot}}{\partial l_{ab}} \right|_{(\bar{g}_a^+, \bar{g}_a^-)} = - \sum_{cd} \frac{\alpha^{(ab)(cd)}(l_{cd} - l_{0cd})}{\sqrt{l_{0ab}}\sqrt{l_{0cd}}} - i\phi_0^{ab} + i \frac{\partial S_{Regge}}{\partial l_{ab}}. \quad (8.146)$$

The quantity $\partial S_{Regge}/\partial l_{ab}$ is β times the extrinsic curvature at the triangle f_{ab} of the boundary of a 4-simplex with triangle areas $A_{ab} = \beta l_{ab}$. As the phase ϕ_0^{ab} in the boundary state is chosen to be exactly β times the extrinsic curvature, we have that equation (8.146) vanishes for $l_{ab} = l_{0ab}$. Notice that, besides being a stationary point, this is also a critical point of the total action as stated in (8.137).

On the other hand, if equation (8.146) is evaluated on group elements belonging to the classes $(\bar{g}_a^-, \bar{g}_a^+)$, $(\bar{g}_a^-, \bar{g}_a^-)$, $(\bar{g}_a^+, \bar{g}_a^+)$, we have that there is no cancellation of phases and therefore no stationary point with respect to variations of spins. This is the feature of the phase of the boundary state: it selects a classical contribution to the asymptotics of a spin foam model, a fact first noticed by Rovelli for the Barrett-Crane model in [45].

8.7.4 Hessian of the total action and derivatives of the insertions

Here we compute the Hessian matrix of the total action S_{tot} at the critical point $l_{ab} = l_{0ab}$, $(g_a^+, g_a^-) = (\bar{g}_a^+, \bar{g}_a^-)$. We introduce a local chart of coordinates $(\bar{p}_a^+, \bar{p}_a^-)$ in a neighborhood of the point $(\bar{g}_a^+, \bar{g}_a^-)$ on $SU(2) \times SU(2)$. The parametrization is defined as follows: we introduce

$$g_a^\pm(p_a^\pm) = h(p_a^\pm) \bar{g}_a^\pm \quad (8.147)$$

with $h(p_a^\pm) = \sqrt{1 - |\vec{p}_a^\pm|^2} + i\vec{p}_a^\pm \cdot \vec{\sigma}$. The vector \vec{p}_a^\pm is assumed to be in a neighborhood of the origin, which corresponds to the critical point \vec{g}_a^\pm . We introduce also the notation n_a^\pm

$$n_a^\pm = \bar{R}_a^\pm n_a \quad (8.148)$$

where \bar{R}_a^\pm is the rotation associated to the $SU(2)$ group element \vec{g}_a^\pm . The bivectors $(l_{ab}n_{ab}^+, l_{ab}n_{ab}^-)$ have the geometrical interpretation of area bivectors associated to the triangles of a 4-simplex with faces of area proportional to l_{ab} .

The Hessian matrix is obtained computing second derivatives of the total action with respect to l_{ab} , p_a^+ and p_a^- , and evaluating it at the point $l_{ab} = l_{0ab}$ and $p_a^\pm = 0$. With this definitions we have that the (gauge-fixed) Hessian matrix is a $(10 + 12 + 12) \times (10 + 12 + 12)$ matrix (as it does not contain derivatives w.r.t. g_1^\pm) and has the following structure:

$$S''_{tot} = \begin{pmatrix} \frac{\partial^2 S_{tot}}{\partial l \partial l} & 0_{10 \times 12} & 0_{10 \times 12} \\ 0_{12 \times 10} & \frac{\partial^2 S_{tot}}{\partial p^+ \partial p^+} & 0_{12 \times 12} \\ 0_{12 \times 10} & 0_{12 \times 12} & \frac{\partial^2 S_{tot}}{\partial p^- \partial p^-} \end{pmatrix} \quad (8.149)$$

as

$$\left. \frac{\partial^2 S_{tot}}{\partial p_a^{i\pm} \partial p_b^{j\mp}} \right|_{\vec{p}=0} = 0, \quad \left. \frac{\partial^2 S_{tot}}{\partial l_{ab} \partial p_c^{j\mp}} \right|_{\vec{p}=0} = 0. \quad (8.150)$$

For the non-vanishing entries we find

$$Q_{(ab)(cd)} = \left. \frac{\partial^2 S_{tot}}{\partial l_{ab} \partial l_{cd}} \right|_{\vec{p}=0} = -\frac{\alpha^{(ab)(cd)}}{\sqrt{l_{0ab}} \sqrt{l_{0cd}}} + (S''_{Regge})_{(ab)(cd)}, \quad (8.151)$$

$$H_{(ai)(bj)}^\pm = \left. \frac{\partial^2 S_{tot}}{\partial p_a^{i\pm} \partial p_b^{j\pm}} \right|_{\vec{p}=0} = 2i\beta^\pm l_{0ab} (\delta^{ij} - n_{ab}^{i\pm} n_{ab}^{j\pm} + i\epsilon^{ijk} n_{ab}^{k\pm}), \quad (8.152)$$

$$H_{(ai)(aj)}^\pm = \left. \frac{\partial^2 S_{tot}}{\partial p_a^{i\pm} \partial p_a^{j\pm}} \right|_{\vec{p}=0} = -2i\beta^\pm \sum_{b \neq a} l_{0ab} (\delta^{ij} - n_{ab}^{i\pm} n_{ab}^{j\pm}), \quad (8.153)$$

where we have defined the 10×10 matrix of second derivatives of the Regge action

$$(S''_{Regge})_{(ab)(cd)} = \left. \frac{\partial^2 S_{Regge}}{\partial l_{ab} \partial l_{cd}} \right|_{l_{0ab}}. \quad (8.154)$$

We report also the first derivatives of the insertion $q_n^{ab}(g^+, g^-)$ evaluated at the critical point:

$$\left. \frac{\partial q_n^{ab}}{\partial p_a^\pm} \right|_{\vec{p}=0} = i\beta^2 \beta^\pm l_{0na} l_{0nb} (\vec{n}_{nb}^\pm - \vec{n}_{na} \cdot \vec{n}_{nb} \vec{n}_{na}^\pm + i \vec{n}_{na}^\pm \times \vec{n}_{nb}^\pm), \quad (8.155)$$

$$\left. \frac{\partial q_n^{ab}}{\partial p_n^\pm} \right|_{\vec{p}=0} = -i\beta^2 \beta^\pm l_{0na} l_{0nb} (\vec{n}_{na}^\pm + \vec{n}_{nb}^\pm) (1 - \vec{n}_{na} \cdot \vec{n}_{nb}), \quad (8.156)$$

$$\left. \frac{\partial q_n^{ab}}{\partial l_{cd}} \right|_{\vec{p}=0} = \beta^2 \frac{\partial (l_{na} \vec{n}_{na} \cdot l_{nb} \vec{n}_{nb})}{\partial l_{cd}} \Big|_{l_{0ab}}. \quad (8.157)$$

We recall that in all these expressions the normals \vec{n}_{ab} are functions of l_{ab} as explained in section 8.4. These expressions will be used in section 8.9 to compute the leading order of the LQG propagator.

8.8 Expectation value of metric operators

Before focusing on the LQG propagator, i.e. on the two-point function, here we briefly discuss the one-point function $\langle E_n^a \cdot E_n^b \rangle$. Its meaning is the dynamical expectation value of the metric operator. The fact that it is non-vanishing provides the background for the propagator. Using the technique developed in the previous sections we can compute it at the leading order in the large spin expansion. We use the integral formula for the metric operator (8.110)-(8.111) and the stationary phase analysis of section 8.7 and find that the expectation value of the metric operator is simply given by the evaluation of the insertion $q_n^{ab}(g^+, g^-)$ at the critical point

$$\langle E_n^a \cdot E_n^b \rangle = q_n^{ab}(g^+, g^-) \Big|_{l_{0ab}, \bar{g}^+, \bar{g}^-} + \mathcal{O}(l_0). \quad (8.158)$$

For the diagonal components $a = b$ we have that the insertion is simply given by $q_n^{aa} = (\beta l_{na})^2$ so that its evaluation at the critical point gives the area square of the triangle f_{na} . For the off-diagonal components we have that $q_n^{ab} = A_n^a \cdot A_n^b$ where A_n^{ai} is given in equation (8.121). Its evaluation at the critical point can be easily found using equation (8.138) in expression (8.121). We find

$$\begin{aligned} \vec{A}^{na} \Big|_{l_{0ab}, \bar{g}^+, \bar{g}^-} &= \vec{A}^{na+} \Big|_{l_{0ab}, \bar{g}^+} + \vec{A}^{na-} \Big|_{l_{0ab}, \bar{g}^-} \\ &= \beta l_{0na}^+ \vec{n}_{na}(l_0) + \beta l_{0na}^- \vec{n}_{na}(l_0) = \beta l_{0na} \vec{n}_{na}(l_0) \end{aligned} \quad (8.159)$$

so that \vec{A}^{na} at the critical point evaluates to the classical value $\vec{E}_{ncl}^a = \beta l_{0na} \vec{n}_{na}(l_0)$, the normal to the face a of the tetrahedron n (normalized to the area of the face). It is the classical counterpart of the operator $(E_n^a)^i$. Therefore we have that at the leading order the expectation value of the off-diagonal components is given by the dihedral angle between two faces of a tetrahedron

$$\begin{aligned} \langle E_n^a \cdot E_n^b \rangle &= \vec{E}_{ncl}^a \cdot \vec{E}_{ncl}^b + \mathcal{O}(l_0) \\ &= \beta^2 l_{0na} l_{0nb} \vec{n}_{na}(l_0) \cdot \vec{n}_{nb}(l_0) + \mathcal{O}(l_0). \end{aligned} \quad (8.160)$$

They have the expected geometrical meaning. We observe that the same quantities computed with the Barrett-Crane spinfoam dynamics do not show the right behavior when the off-diagonal components of the metric operator are considered.

Using the same technique we can evaluate the leading order of the two-point function. We have that

$$\langle E_n^a \cdot E_n^b E_m^c \cdot E_m^d \rangle = E_{ncl}^a \cdot E_{ncl}^b E_{mcl}^c \cdot E_{mcl}^d + \mathcal{O}(l_0^3). \quad (8.161)$$

The quantity we are specifically interested in in this paper is the *connected* two-point function. It is of order $\mathcal{O}(l_0^3)$, therefore it requires the next-to-leading orders in equations (8.160) and (8.161). Such orders depend on the measure $\mu(l)$. However in the

computation of the connected part, these contributions cancel. The technique we use in next section for the calculation of the connected two-point function is the one introduced in section (8.7.2) and captures directly the leading order.

8.9 LQG propagator: the leading order

We have defined the LQG propagator as the connected two-point function $G_{nm}^{abcd} = \langle E_n^a \cdot E_n^b E_m^c \cdot E_m^d \rangle - \langle E_n^a \cdot E_n^b \rangle \langle E_m^c \cdot E_m^d \rangle$. Using the integral formula (8.110)-(8.111) and the result (8.135) for the asymptotics of connected two-point functions, we can compute the LQG propagator in terms of (the inverse of) the Hessian of the total action and of the derivative of the metric operator insertions at the critical point. These two ingredients are computed in section 8.7.4. Using them, we find that the LQG propagator is given by

$$\begin{aligned} G_{nm}^{abcd}(\alpha) &= \sum_{p,q,r,s} Q_{(pq)(rs)}^{-1} \frac{\partial q_n^{ab}}{\partial l_{pq}} \frac{\partial q_m^{cd}}{\partial l_{rs}} + \\ &+ \sum_{r,s=2}^5 \sum_{i,k=1}^3 \left((H^+)_{(ri)(sk)}^{-1} \frac{\partial q_n^{ab}}{\partial p_r^{i+}} \frac{\partial q_m^{cd}}{\partial p_s^{k+}} + (H^-)_{(ri)(sk)}^{-1} \frac{\partial q_n^{ab}}{\partial p_r^{i-}} \frac{\partial q_m^{cd}}{\partial p_s^{k-}} \right) \\ &+ \mathcal{O}(l_0^2) \end{aligned} \quad (8.162)$$

where all the terms appearing in this expression are defined in section 8.7.4. From this expression we can extract the dependence on the boundary spin l_0 and on the Immirzi parameter β . We notice that the combinations

$$R_{nm}^{abcd} = \frac{1}{\beta^3 l_0^3} \sum_{p<q,r<s} Q_{(pq)(rs)}^{-1} \frac{\partial q_n^{ab}}{\partial l_{pq}} \frac{\partial q_m^{cd}}{\partial l_{rs}}, \quad (8.163)$$

$$X_{nm}^{abcd} = \frac{1}{2\beta^4 l_0^3} \sum_{r,s=2}^5 \sum_{i,k=1}^3 \left(\frac{1}{\beta^+} (H^+)_{(ri)(sk)}^{-1} \frac{\partial q_n^{ab}}{\partial p_r^{i+}} \frac{\partial q_m^{cd}}{\partial p_s^{k+}} + \frac{1}{\beta^-} (H^-)_{(ri)(sk)}^{-1} \frac{\partial q_n^{ab}}{\partial p_r^{i-}} \frac{\partial q_m^{cd}}{\partial p_s^{k-}} \right), \quad (8.164)$$

$$Y_{nm}^{abcd} = \frac{1}{2\beta^4 l_0^3} \sum_{r,s=2}^5 \sum_{i,k=1}^3 \left(\frac{1}{\beta^+} (H^+)_{(ri)(sk)}^{-1} \frac{\partial q_n^{ab}}{\partial p_r^{i+}} \frac{\partial q_m^{cd}}{\partial p_s^{k+}} - \frac{1}{\beta^-} (H^-)_{(ri)(sk)}^{-1} \frac{\partial q_n^{ab}}{\partial p_r^{i-}} \frac{\partial q_m^{cd}}{\partial p_s^{k-}} \right), \quad (8.165)$$

are in fact independent from l_0 and from β . In terms of these quantities we have that the LQG propagator has the following structure

$$G_{nm}^{abcd}(\alpha) = (\beta l_0)^3 (R_{nm}^{abcd}(\alpha) + \beta X_{nm}^{abcd} + \beta^2 Y_{nm}^{abcd}) + \mathcal{O}(l_0^2) \quad (8.166)$$

where the dependence on β and on l_0 has been made explicit now. The matrices R_{nm}^{abcd} , X_{nm}^{abcd} and Y_{nm}^{abcd} can be evaluated algebraically. Only the matrix R_{nm}^{abcd} depends on

the three parameters $\alpha_0, \alpha_1, \alpha_2$ appearing on the boundary state. We find that

$$R_{nm}^{abcd} = \begin{pmatrix} \begin{pmatrix} c_1 & c_3 & c_3 \\ c_3 & c_2 & c_4 \\ c_3 & c_4 & c_2 \end{pmatrix} & \begin{pmatrix} c_3 & c_5 & c_6 \\ c_5 & c_3 & c_6 \\ c_6 & c_6 & c_4 \end{pmatrix} & \begin{pmatrix} c_3 & c_6 & c_5 \\ c_6 & c_4 & c_6 \\ c_5 & c_6 & c_3 \end{pmatrix} \\ \begin{pmatrix} c_3 & c_5 & c_6 \\ c_5 & c_3 & c_6 \\ c_6 & c_6 & c_4 \end{pmatrix} & \begin{pmatrix} c_2 & c_3 & c_4 \\ c_3 & c_1 & c_3 \\ c_4 & c_3 & c_2 \end{pmatrix} & \begin{pmatrix} c_4 & c_6 & c_6 \\ c_6 & c_3 & c_5 \\ c_6 & c_5 & c_3 \end{pmatrix} \\ \begin{pmatrix} c_3 & c_6 & c_5 \\ c_6 & c_4 & c_6 \\ c_5 & c_6 & c_3 \end{pmatrix} & \begin{pmatrix} c_4 & c_6 & c_6 \\ c_6 & c_3 & c_5 \\ c_6 & c_5 & c_3 \end{pmatrix} & \begin{pmatrix} c_2 & c_4 & c_3 \\ c_4 & c_2 & c_3 \\ c_3 & c_3 & c_1 \end{pmatrix} \end{pmatrix} \quad (8.167)$$

where

$$c_1 = 4\gamma_1, \quad c_2 = 4\gamma_2, \quad c_3 = -\frac{2}{3}(2\gamma_0 - 3\gamma_1 + 3\gamma_2), \quad (8.168)$$

$$c_4 = \frac{1}{3}(8\gamma_0 - 12\gamma_1), \quad c_5 = \frac{1}{9}(49\gamma_0 - 93\gamma_1 + 48\gamma_2), \quad c_6 = -\frac{1}{9}(23\gamma_0 - 42\gamma_1 + 15\gamma_2), \quad (8.169)$$

and¹⁴

$$\gamma_0 = \frac{1}{10} \left(-\frac{1}{\alpha_0 + 6\alpha_1 + 3\alpha_2} + \frac{32}{-8\alpha_0 - 8\alpha_1 + 16\alpha_2 + i\sqrt{15}} - \frac{5}{\alpha_0 - 2\alpha_1 + \alpha_2 + i\sqrt{15}} \right), \quad (8.172)$$

$$\gamma_1 = \frac{1}{30} \left(-\frac{3}{\alpha_0 + 6\alpha_1 + 3\alpha_2} + \frac{16}{-8\alpha_0 - 8\alpha_1 + 16\alpha_2 + i\sqrt{15}} + \frac{5}{\alpha_0 - 2\alpha_1 + \alpha_2 + i\sqrt{15}} \right), \quad (8.173)$$

$$\gamma_2 = \frac{1}{30} \left(-\frac{3}{\alpha_0 + 6\alpha_1 + 3\alpha_2} - \frac{64}{-8\alpha_0 - 8\alpha_1 + 16\alpha_2 + i\sqrt{15}} - \frac{5}{\alpha_0 - 2\alpha_1 + \alpha_2 + i\sqrt{15}} \right). \quad (8.174)$$

The matrices X_{nm}^{abcd} and Y_{nm}^{abcd} turn out to be proportional

$$X_{nm}^{abcd} = \frac{7}{36} Z_{nm}^{abcd}, \quad Y_{nm}^{abcd} = -i \frac{\sqrt{15}}{36} Z_{nm}^{abcd}, \quad (8.175)$$

¹⁴We notice that the inverse of the matrix $Q_{(ab)(cd)}$ can be written in terms of the parameters γ_k using the formalism (8.93) introduced in section 8.4,

$$(Q^{-1})^{(ab)(cd)} = \sum_{k=0}^2 l_0 \gamma_k P_k^{(ab)(cd)}. \quad (8.170)$$

The matrix $Q_{(ab)(cd)}$ is defined in equation (8.151) and is given by

$$Q^{(ab)(cd)} = \frac{1}{l_0} \sum_{k=0}^2 (ih_k - \alpha_k) P_k^{(ab)(cd)}, \quad (8.171)$$

with $h_0 = -\frac{9}{4}\sqrt{\frac{3}{5}}$, $h_1 = \frac{7}{8}\sqrt{\frac{3}{5}}$, $h_2 = -\sqrt{\frac{3}{5}}$.

with the matrix Z_{nm}^{abcd} given by

$$Z_{nm}^{abcd} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 & e^{i\frac{\pi}{3}} \\ -1 & 0 & e^{-i\frac{\pi}{3}} \\ e^{i\frac{\pi}{3}} & e^{-i\frac{\pi}{3}} & 0 \end{pmatrix} & \begin{pmatrix} 0 & e^{-i\frac{\pi}{3}} & -1 \\ e^{-i\frac{\pi}{3}} & 0 & e^{i\frac{\pi}{3}} \\ -1 & e^{i\frac{\pi}{3}} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 & e^{i\frac{\pi}{3}} \\ -1 & 0 & e^{-i\frac{\pi}{3}} \\ e^{i\frac{\pi}{3}} & e^{-i\frac{\pi}{3}} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & e^{i\frac{\pi}{3}} & e^{-i\frac{\pi}{3}} \\ e^{i\frac{\pi}{3}} & 0 & -1 \\ e^{-i\frac{\pi}{3}} & -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & e^{-i\frac{\pi}{3}} & -1 \\ e^{-i\frac{\pi}{3}} & 0 & e^{i\frac{\pi}{3}} \\ -1 & e^{i\frac{\pi}{3}} & 0 \end{pmatrix} & \begin{pmatrix} 0 & e^{i\frac{\pi}{3}} & e^{-i\frac{\pi}{3}} \\ e^{i\frac{\pi}{3}} & 0 & -1 \\ e^{-i\frac{\pi}{3}} & -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}. \quad (8.176)$$

This is the main result of the paper, the scaling and the tensorial structure of metric correlations in LQG. In the following we collect some remarks on this result:

- The LQG propagator scales as l_0^3 , as expected for correlations of objects with dimensions of area square, $E_n^a \cdot E_n^b \sim (\beta l_0)^2$.
- The off-diagonal components are not suppressed as happened for the Barrett-Crane model [50, 51] and have the same scaling as the diagonal ones.
- The contribution R_{nm}^{abcd} in (8.166) matches exactly with the matrix of correlations of areas and angles computed in perturbative quantum Regge calculus with a boundary state as done in [48].
- On the other hand, the ‘ β -terms’ in (8.166), $\beta X_{nm}^{abcd} + \beta^2 Y_{nm}^{abcd}$, are new and proper of the spin foam model. They come from $SU(2) \times SU(2)$ ‘group’ fluctuations. They don’t contribute to area-area correlations, nor to area-angle correlations. On the other hand, their contribution to angle-angle correlations is non-trivial.
- In the limit $\beta \rightarrow 0$ and $l_0 \rightarrow \infty$ with $\beta l_0 = \text{const} = A_0$, only the Regge contribution survives. It is interesting to notice that the same limit was considered in [87] in the context of loop quantum cosmology.
- The ‘ β -terms’ have an interesting feature that we now describe. Let us focus on the tensorial components $G_4 = G_{12}^{(34)(45)}$ and $G_5 = G_{12}^{(35)(45)}$. They are related by a permutation of the vertices 4 and 5, keeping the other three vertices fixed. The ‘Regge-term’ is invariant under this permutation, $R_{12}^{(34)(45)} = R_{12}^{(35)(45)}$. On the other hand the ‘ β -terms’ are not. In particular we have that

$$\beta X_{12}^{(35)(45)} + \beta^2 Y_{12}^{(35)(45)} = e^{i\frac{2\pi}{3}} \left(\beta X_{12}^{(34)(45)} + \beta^2 Y_{12}^{(34)(45)} \right). \quad (8.177)$$

It would be interesting to identify the origin of the phase $\frac{2\pi}{3}$. We notice that the permutation of the vertex 4 with the vertex 5 of the boundary spin network corresponds to a parity transformation of the four-simplex. In this sense the ‘ β -terms’ are parity violating.

In next section we investigate the relation of the result found with the graviton propagator computed in perturbative quantum field theory.

8.10 Comparison with perturbative quantum gravity

The motivation for studying the LQG propagator comes from the fact that it probes a regime of the theory where predictions can be compared to the ones obtained perturbatively in a quantum field theory of gravitons on flat space [42, 43, 44]. Therefore it is interesting to investigate this relation already at the preliminary level of a single spin foam vertex studied in this paper. In this section we investigate this relation within the setting discussed in [50, 51, 52].

In perturbative quantum gravity¹⁵, the graviton propagator in the harmonic gauge is given by

$$\langle h_{\mu\nu}(x)h_{\rho\sigma}(y) \rangle = \frac{-1}{2|x-y|^2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma}). \quad (8.178)$$

Correlations of geometrical quantities can be computed perturbatively in terms of the graviton propagator. For instance the angle at a point x_n between two intersecting surfaces f_{na} and f_{nb} is given by¹⁶

$$q_n^{ab} = g_{\mu\nu}(x_n)g_{\rho\sigma}(x_n)B_{na}^{\mu\rho}(x_n)B_{nb}^{\nu\sigma}(x_n) \quad (8.179)$$

where $B^{\mu\nu}$ is the bivector associated to the surface¹⁷. As a result the angle fluctuation can be written in terms of the graviton field

$$\delta q_n^{ab} = h_{\mu\nu}(x_n)(T_n^{ab})^{\mu\nu}, \quad (8.181)$$

where we have defined the tensor $(T_n^{ab})^{\mu\nu} = 2\delta_{\rho\sigma}B_{na}^{\mu\rho}(x_n)B_{nb}^{\nu\sigma}(x_n)$. The angle correlation $(G_{nm}^{abcd})_{qft}$ is simply given by

$$(G_{nm}^{abcd})_{qft} = \langle h_{\mu\nu}(x_n)h_{\rho\sigma}(x_m) \rangle (T_n^{ab})^{\mu\nu}(T_m^{cd})^{\rho\sigma}. \quad (8.182)$$

In particular, this quantity can be computed for couples of surfaces identified by triangles of area A_0 living on the boundary of a regular Euclidean 4-simplex. This quantity has been computed in [51] and we report it here for reference,

$$(G_{nm}^{abcd})_{qft} = \frac{-A_0^3}{18\sqrt{3} \times 512} \times$$

¹⁵Here we consider the Euclidean case.

¹⁶We thank E. Alesci for a discussion on this point.

¹⁷To be more specific, we consider local coordinates (σ^1, σ^2) for a surface t and call $t^\mu(\sigma)$ its embedding in the $4d$ manifold. The bivector $B_t^{\mu\nu}(x)$ is defined as

$$B_t^{\mu\nu}(x) = \frac{\partial t^\mu}{\partial \sigma^\alpha} \frac{\partial t^\nu}{\partial \sigma^\beta} \varepsilon^{\alpha\beta}. \quad (8.180)$$

$$\times \left(\begin{array}{c} \left(\begin{array}{ccc} -16 & 6 & 6 \\ 6 & -28 & 16 \\ 6 & 16 & -28 \end{array} \right) \\ \left(\begin{array}{ccc} 6 & 4 & -7 \\ 4 & 6 & -7 \\ -7 & -7 & 16 \end{array} \right) \\ \left(\begin{array}{ccc} 6 & -7 & 4 \\ -7 & 16 & -7 \\ 4 & -7 & 6 \end{array} \right) \\ \left(\begin{array}{ccc} 6 & 4 & -7 \\ -28 & 6 & 16 \\ 6 & -16 & 6 \\ 16 & 6 & -28 \end{array} \right) \\ \left(\begin{array}{ccc} 6 & -7 & 4 \\ 16 & -7 & -7 \\ -7 & 6 & 4 \\ -7 & 4 & 6 \end{array} \right) \\ \left(\begin{array}{ccc} 6 & -7 & 4 \\ -28 & 16 & 6 \\ 16 & -28 & 6 \\ 6 & 6 & -16 \end{array} \right) \end{array} \right) \quad (8.183)$$

The question we want to answer here is if the quantity $(G_{nm}^{abcd})_{qft}$ and the leading order of the LQG propagator given by equation (8.166) can match. As we can identify βl_0 with the area A_0 , we have that the two have the same scaling. The non-trivial part of the matching is the tensorial structure. Despite the fact that we have 9×9 tensorial components, only six of them are independent as the others are related by symmetries of the configuration we are considering. On the other hand the semiclassical boundary state $|\Psi_0\rangle$ we used in the LQG calculation has only three free parameters, $\alpha_0, \alpha_1, \alpha_2$. Therefore we can ask if there is a choice of these 3 parameters such that we can satisfy the 6 independent equations given by the matching condition

$$(G_{nm}^{abcd}(\alpha))_{lqg} = (G_{nm}^{abcd})_{qft}. \quad (8.184)$$

We find that a solution in terms of the parameters α_k can be found only in the limit of vanishing Immirzi parameter, keeping constant the product $\beta l_0 = A_0$. In this limit we find a unique solution for α_k given by

$$\alpha_0 = \frac{1}{100}(495616\sqrt{3} - 45\sqrt{15}i), \quad (8.185)$$

$$\alpha_1 = \frac{1}{200}(-299008\sqrt{3} + 35\sqrt{15}i), \quad (8.186)$$

$$\alpha_2 = \frac{1}{25}(31744\sqrt{3} - 5\sqrt{15}i). \quad (8.187)$$

Therefore the matching condition (8.184) can be satisfied, at least in the specific limit considered. Having found a non-trivial solution, it is interesting to study the real part of the matrix $\alpha^{(ab)(cd)}$ in order to determine if it is positive definite. Its eigenvalues (with the associated degeneracy) are

$$\lambda_5 = 9216\sqrt{3}, \quad deg = 5, \quad (8.188)$$

$$\lambda_4 = \frac{4608\sqrt{3}}{5}, \quad deg = 4, \quad (8.189)$$

$$\lambda_1 = -\frac{1024\sqrt{3}}{5}, \quad deg = 1. \quad (8.190)$$

We notice that all the eigenvalues are positive except one, λ_1 . The corresponding eigenvector represents conformal rescalings of the boundary state, $l_{0ab} \rightarrow \lambda l_{0ab}$. It

would be interesting to determine its origin and to understand how the result depends on the choice of gauge made for the graviton propagator (8.178).

8.11 Conclusions

We have studied correlation functions of metric operators in loop quantum gravity. The analysis presented involves two distinct ingredients:

- The first is a setting for defining correlation functions. The setting is the boundary amplitude formalism. It involves a boundary semiclassical state $|\Psi_0\rangle$ which identifies the regime of interest, loop quantum gravity operators $E_n^a \cdot E_n^b$ which probe the quantum geometry on the boundary, a spin foam model $\langle W |$ which implements the dynamics. The formalism allows to define semiclassical correlation functions in a background-independent context.
- The second ingredient consists in an approximation scheme applied to the quantity defined above. It involves a vertex expansion and a large spin expansion. It allows to estimate the correlation functions explicitly. The explicit result can then be compared to the graviton propagator of perturbative quantum gravity. In this chapter we focused on the lowest order in the vertex expansion and the leading order in the large spin expansion.

The results found in the chapter can be summarized as follows:

1. In section 8.4 we have introduced a semiclassical state $|\Psi_0\rangle$ peaked on the intrinsic and the extrinsic geometry of the boundary of a regular Euclidean 4-simplex. The technique used to build this state is the following: (i) we use the coherent intertwiners introduced in [65, 72] to define coherent spin networks as in [73]; (ii) we choose the normals labelling intertwiners so that they are compatible with a simplicial 3-geometry (8.82). This addresses the issue of discontinuous lengths identified in [80]; (iii) then we take a gaussian superposition over coherent spin networks in order to peak on extrinsic curvature as in [45, 46]. This state is an improvement of the ansatz used in [50, 51, 52], as it depends only on the three free parameters $\alpha_0, \alpha_1, \alpha_2$.
2. In section 8.6 we have defined expectation values of geometric observables on a semiclassical state. The LQG propagator is defined in equation (8.101) as a connected correlation function for the product of two metric operators

$$G_{nm}^{abcd} = \langle E_n^a \cdot E_n^b E_m^c \cdot E_m^d \rangle - \langle E_n^a \cdot E_n^b \rangle \langle E_m^c \cdot E_m^d \rangle. \quad (8.191)$$

This is the object that in principle can be compared to the graviton propagator on flat space: the background is coded in the expectation value of the geometric operators and the propagator measures correlations of fluctuations over this background.

3. In section 8.6.2 we have introduced a technique which allows to write LQG metric operators as insertions in a $SO(4)$ group integral. It can be interpreted as the

covariant version of the LQG operators. The formalism works for arbitrary fixed triangulation. Having an integral formula for expectation values and correlations of metric operators allows to formulate the large spin expansion as a stationary phase approximation. The problem is studied in detail restricting attention to the lowest order in the vertex expansion, i.e. at the single-vertex level.

4. The analysis of the large spin asymptotics is performed in section 8.7. The technique used is the one introduced by Barrett et al in [73]. There, the large spin asymptotics of the boundary amplitude of a coherent spin network is studied and four distinct critical points are found to contribute to the asymptotics. Two of them are related to different orientations of a 4-simplex. The other two come from selfdual configurations. Here, our boundary state is peaked also on extrinsic curvature. The feature of this boundary state is that it selects only one of the critical points, extracting $\exp iS_{Regge}$ from the asymptotics of the EPRL spin foam vertex. This is a realization of the mechanism first identified by Rovelli in [45] for the Barrett-Crane model.
6. In 8.8 we compute expectation values of LQG metric operators at leading order and find that they reproduce the intrinsic geometry of the boundary of a regular 4-simplex.
7. Computing correlations of geometric operators requires going beyond the leading order in the large spin expansion. In section 8.7.2 we derive a formula for computing directly the *connected* two-point correlation function to the lowest non-trivial order in the large spin expansion. The formula is used in section 8.7.4.
8. The result of the calculation, the LQG propagator, is presented in section 8.9. We find that the result is the sum of two terms: a “Regge term” and a “ β -term”. The Regge term coincides with the correlations of areas and angles computed in Regge calculus with a boundary state [48]. It comes from correlations of fluctuations of the spin variables and depends on the parameters α_k of the boundary state. The “ β -term” comes from fluctuations of the $SO(4)$ group variables. An explicit algebraic calculation of the tensorial components of the LQG propagator is presented.
9. The LQG propagator can be compared to the graviton propagator. This is done in section 8.10. We find that the LQG propagator has the correct scaling behaviour. The three parameters α_k appearing in the semiclassical boundary state can be chosen so that the tensorial structure of the LQG propagator matches with the one of the graviton propagator. The matching is obtained in the limit $\beta \rightarrow 0$ with βl_0 fixed.

Now we would like to put these results in perspective with respect to the problem of extracting the low energy regime of loop quantum gravity and spin foams (see in particular [88]).

Deriving the LQG propagator at the level of a single spin foam vertex is certainly only a first step. Within the setting of a vertex expansion, an analysis of the LQG

propagator for a finite number of spinfoam vertices is needed. Some of the techniques developed in this chapter generalize to this more general case. In particular superpositions of coherent spin networks can be used to build semiclassical states peaked on the intrinsic and the extrinsic curvature of an arbitrary boundary Regge geometry. Moreover, the expression of the LQG metric operator in terms of $SO(4)$ group integrals presented in this chapter works for an arbitrary number of spin foam vertices and allows to derive an integral representation of the LQG propagator in the general case, analogous to the one of [70] but with non-trivial insertions. This representation is the appropriate one for the analysis of the large spin asymptotics along the lines discussed for Regge calculus in [89]. The non-trivial question which needs to be answered then is if the semiclassical boundary state is able to enforce semiclassicality in the bulk. Another feature identified in this chapter which appears to be general is that, besides the expected Regge contribution, correlations of LQG metric operators have a non-Regge contribution which is proper of the spin foam model. It would be interesting to investigate if this contribution propagates when more than a single spin foam vertex is considered.

From the Hamiltonian formalism to Spin-Foams

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Starting from a BF-type formulation of General Relativity in the canonical formalism, we construct a physical scalar product with no restrictions for the boundary states that, for particular cases, reproduce the E.P.R. amplitude.

I. INTRODUCTION

Loop quantum gravity has so far achieved splendid results in the description of the kinematic of pure gravity states in terms of colored graphs, whose nodes represent quanta of volume connected by links representing quanta of area [1]. However, it is still incomplete as far as the dynamics is concerned. A hamiltonian operator has been described by T. Thiemann [2] for the formulation of General Relativity via Ashtekar variables but, in spite of the formal definition, we have not yet been able to decipher the precise action on the states because of its complexity. In alternative, a quantum dynamics has been constructed by describing gravity as a topological theory (BF-theory) with constraints on the states [3]; in particular, here we will refer to the work by Engle Pereira and Rovelli (E.P.R.) in [4]. The amplitude obtained, sometimes referred to as E.P.R. has the correct low energy limit [11]. But so far the results are limited to a formulation via path integral over space-times of Regge type. Here we take some steps towards the derivation of the E.P.R. amplitude from the hamiltonian theory.

We start in section (II) with a standard introduction on the spin-network as functions on a compact group G and we define the approximation of a spin-network on a lattice. We perform a series of integrals on G over products of holonomies along the squares of the lattice. This allows us to calculate the expectation value of an operator that will be revealed in section (III) as the projector on the physical states. For doing this we extend to the general case the approach adopted by Karim Noui and Alejandro Perez in [6] for the three dimensional case. In section (III) we consider BF-theory and the classical constraints that give General Relativity starting from this theory. In the quantum theory, these constraints have to be imposed on the physical states. We choose to impose them already on the kinematical states. We extract some indication for doing this from the work of Jonathan Engle, Etera Livine, Roberto Pereira, Carlo Rovelli [4] where is largely explored the case of 4-valent spin-networks. We get a projector on the physical states and we calculate a physical scalar product between the two simpler 4-valent spin-networks, obtaining the same amplitude obtained in [4] via path integral over space-times of Regge type, called E.P.R. amplitude. For another attempt to relate the spin-foam vertex with a hamiltonian operator see [10].

II. SPIN-NETWORKS

We use greek letters for spacetime indices and latin letters for space indices and follow the approach of [8]. We consider S an oriented smooth m -dimensional manifold representing space, parameterized with coordinates x^a , $a = 1, \dots, m$. A finite collection γ of real-analytic paths γ_i such that

$$\begin{aligned} \gamma_i &: [0, 1] \rightarrow S \\ \tau &\rightarrow \gamma_i(\tau), \end{aligned} \tag{1}$$

with $\gamma_i(\tau)$ points of S , form a graph in S if they intersect, if at all, only in their endpoints. We then call them links and call their endpoints nodes. Given a node v , we say a link γ_i is outgoing from v if $\gamma_i(0) = v$, and we say γ_i is incoming to v if $\gamma_i(1) = v$. We take a connection A of a compact group G . We can think of the holonomies along these paths as elements of G . The functionals of the form

$$\psi(A) = f(Te^{\int_{\gamma_1} A}, \dots, Te^{\int_{\gamma_n} A})$$

form an algebra, that we call $F^n(A)$, that is isomorphic to the algebra of all continuous complex-valued functions on G^n . Given two function in this algebra, we can thus define their inner product by

$$\langle \psi, \phi \rangle = \int_{G^n} \bar{\psi} \phi \quad (2)$$

where the integral is done using normalized Haar measure on G^n . Obviously if $\psi \in F^n(A)$ we have also $\psi \in F^k(A)$ if $k > n$. This is because we can see $F^n(A)$ as a $F^k(A)$, constant with respect to the group elements $n+1, \dots, k$. In this way it is possible to extend the definition of the scalar product to arbitrary couple of function, respectively in $F^n(A)$ and $F^m(A)$, by considering both as element of $F^k(A)$ with $k \geq m, n$. It is easy to see that this is independent from the choice of k .

A **spin-network** in S consist of:

- a graph γ in S ;
- for each link e of γ , an irreducible representation ρ_e ;
- for each node v of γ , an intertwiner, that is an operator which maps

$$i_v : \rho_{e_1} \otimes \dots \otimes \rho_{e_p} \rightarrow \rho_{e'_1} \otimes \dots \otimes \rho_{e'_q}$$

where $e_1 \dots e_p$ are the links incoming to v and e'_1, \dots, e'_q are the links outgoing from v .

A way to get a function in $F^n(A)$ from a spin-network in S is to take the holonomy along each link of γ , think of it as a group element, and write it in the representation labeling the link. Picking a basis for this representation we think of the result as a matrix with one superscript (representing the beginning of the link) and one subscript (representing the ending of the link). In addition we write the intertwiner for every node as a tensor and contract all indices.

$$F(A) = [R^\rho]^a_b \otimes \dots \otimes [R^\sigma]^c_d \quad (i_1)^b_{\dots} \otimes \dots \otimes (i_l)^{\dots}_a \otimes \dots \otimes (i_n)^{\dots}_c \otimes \dots \otimes (i_q)^d_{\dots}$$

It's easy to see that a normalized state (respect to (2)) can be obtained by multiplying

$$F(A)_{NORMALIZED} = F(A) \cdot \sqrt{\dim \rho \dots \dim \sigma}. \quad (3)$$

This holds because

$$\int dU [R^\rho(U)]^I_J \dots [R^\sigma(U)]^K_L = \sum_i i^{I\dots K} i^*_{J\dots L} \quad (4)$$

with i a normalized intertwiner between the representations $\rho \dots \sigma$. Hence the normalized intertwiner between only two representation is the “delta” divided by the square root of the dimensions of the representation

$$\int dU [R^\rho(U)]^I_J [R^\sigma(U)]^K_L = \frac{\delta^{IK}}{\sqrt{\dim \rho}} \frac{\delta_{JL}}{\sqrt{\dim \sigma}} \delta^{\rho\sigma} = \frac{1}{\dim \rho} \delta^{\rho\sigma} \delta^{IK} \delta_{JL}. \quad (5)$$

These functions are gauge invariant. In [7] it is shown that these functions span $L^2(A/G)$, with A/G the space of the connections in S modulo gauge transformations. So, we have a one to one correspondence from spin networks to functions in $L^2(A/G)$ and we can speak about a spin-network meaning the relative function in $L^2(A/G)$.

We consider the image in R^m of a part of the graph γ contained in a open set of an atlas covered S . It is given by the $x^a(\gamma)$. We say that two images made with x and y coordinates are equivalent if exists a continuous, piecewise differentiable, map from R^m to R^m , $y : x \rightarrow y(x)$ such that $y^a(\gamma_i) = y^a[x(\gamma_i)]$ for every $\gamma_i \in \gamma$. This take into account for the different possible choices of coordinates. A change in the coordinates (or a diffeomorphism), deform the image of the graph in R^m without change its corresponding abstract graph, that is the combinatorial relation between links and nodes. The “physical scalar product” that we construct in this work depend only on abstract graphs and therefore it is diffeomorphism-invariant.

Taken an m -dimensional lattice with step ϵ , we can always approximate such an image with links of the lattice. In the limit $\epsilon \rightarrow 0$ we recover the smooth image. In this framework we can consider a minimal link of length ϵ as an element $(a_1, \dots, a_m, z+)$ where $(a_1, \dots, a_m) \in N^m$ labels the starting point, $z = 1, \dots, m$ indicates the direction and \pm the orientation, with $(a_1, \dots, a_z, \dots, a_m, z+) = (a_1, \dots, a_z + 1, \dots, a_m, z-)$. We indicate with $U_{(a, \dots, d, e, \dots, m, d+)}$ the holonomy of the link $(a, \dots, d, e, \dots, m, d+)$. We consider the holonomy in the representation σ along one of this minimal link, we say $R(U)_{(a, \dots, d, e, \dots, m, d+)}^\sigma$. Consider the square path

$$\begin{aligned} (a, \dots, d, e, \dots, m, d+) &\rightarrow (a, \dots, d+1, e, \dots, m, e+) \\ &\rightarrow (a, \dots, d+1, e+1, f, g, \dots, m, d-) \\ &\rightarrow (a, \dots, d, e+1, \dots, m, e-), \end{aligned} \quad (6)$$

We indicate the product of the holonomies associated to the side of the square as $U_{(a, \dots, d, e, \dots, m)}^{d+e+}$. We have the simple result

$$\begin{aligned} \int dU_{(a, \dots, d, e, \dots, m, d+)} \delta(U_{(a, \dots, d, e, \dots, m)}^{d+e+}) R(U)_{(a, \dots, d, e, \dots, m, d+)}^\sigma &= \\ = R(U)_{(a, \dots, d, e, \dots, m, e-)}^\sigma R(U)_{(a, \dots, d, e+1, \dots, m, d+)}^\sigma R(U)_{(a, \dots, d+1, e+1, \dots, m, e+)}^\sigma, \end{aligned} \quad (7)$$

graphically in figure (1), where we have used

$$\delta(U_{(a, \dots, d, e, \dots, m)}^{d+e+}) = \sum_{\rho} (\dim \rho) R(U)_{(a, \dots, d, e, \dots, m)}^{\rho, d+e+}.$$

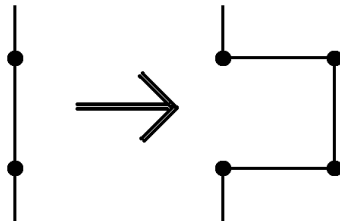


FIG. 1:

With little work we can see that any 2 graphs related by a diffeomorphism can be related using a sequence of such integrations. Similarly, the product of two deltas along two squares with a common link, integrated in the common link, results in a delta along the rectangle (U^{REC}) containing the two squares

$$\int dU_{(a, \dots, d, e, \dots, m, d+)} \delta(U_{(a, \dots, d, e, \dots, m)}^{d+e+}) \delta(U_{(a, \dots, d, e, \dots, m)}^{d+e-}) = \delta(U^{REC}). \quad (8)$$

So, if we consider a product of delta functions associates to all the squares of a 2-dimensional region on the lattice, and we integrate on the internal links, we obtain a delta over the holonomy along the boundary of the region. This is illustrated in figure (2)

Consider now the following scalar product between two spin-networks

$$\langle \gamma_1, \rho_1, i_{v1} | \prod_{\substack{INDEP. \\ SQUARES}} \delta(U^{SQ}) | \gamma_2, \rho_2, i_{v2} \rangle \quad (9)$$

The sum is over “independent” squares. By this we mean the following. We write the deltas one square at a time and, when we should consider a square of which all sides have been already drawn from other squares occupied by deltas, we don’t write it. We continue until we touch all the possible squares of the lattice. This procedure is largely arbitrary, depending on the sequence that we follow in considering the squares, but we are satisfied if we can calculated (9) at least for one sequence. This procedure avoid situation like this: you takes a cube and the six deltas on its faces

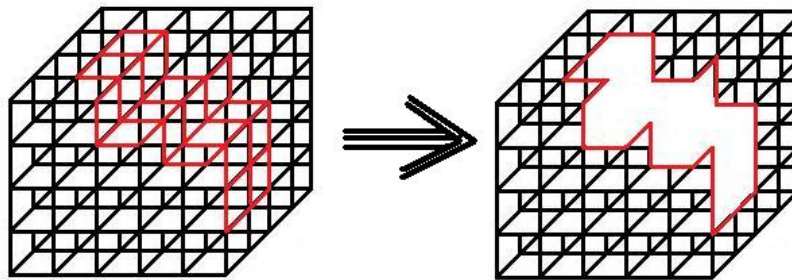


FIG. 2:

and integrate on the twelve group elements of the links. The result is an infinitive constant. This is because, if we integrate the common links of the lower and lateral faces, we obtain a delta on the upper face. But we have already a delta on this face! In this way we have an integral of the type

$$\int dU_1 dU_2 dU_3 dU_4 \delta(U_1 U_2 U_3 U_4) \delta(U_1 U_2 U_3 U_4) = \delta(1) = \infty.$$

The formula (9) contains an integration for every link of the lattice and a delta for every independent square of the lattice. Now we note that some link appears only in the deltas and not in the initial and final states, we call them “free links”. Integrating on this links and using repeatedly the result (8) we obtain

$$\langle \gamma_1, \rho_1, i_{v1} | \prod_{\substack{INDIP. \\ SQUARES}} \delta(U^{SQ}) | \gamma_2, \rho_2, i_{v2} \rangle = W \langle \gamma_1, \rho_1, i_{v1} | \prod_{\substack{INDIP. \\ FACES}} \delta(U^{FC}) | \gamma_2, \rho_2, i_{v2} \rangle \quad (10)$$

where for faces we intend the closed figures formed by the links of the graph $\gamma_1 \cup \gamma_2$. Indeed if, by absurd, after this integration we would obtain deltas over closed paths whose links don't overlap the links of the graph, we would have some links of these paths that isn't in the final or initial state. But this is impossible because we have already integrated over this free links. The arbitrarily in the choice of the sequence of squares reflect an arbitrarily in the sequence of faces, following the same idea that a face doesn't appear if its contour is drawn by the other faces. Obviously, in the calculation we obtain some “infinity”: for example the integration of a delta over a path which goes to infinitive, over a free link at the infinite, gives 1. But, after this integration, remain so much integrations over now empty links at the infinitive of that canceled path, which generate an “infinite” constant, that we call “W”¹.

Let us now restrict the dimensions of the space to 3 and to the case where the spin-network are constructed on the dual 1-skeleton of a triangulated manifold or a Regge space. As before, we start with a 3-dimensional real-analytic manifold S representing space. Given any triangulation of S we can choose a graph in S called the “dual 1-skeleton”, having one node at the center of each 3-simplex and one link intersecting each $(3 - 1)$ -simplex. In a 3-dimensional space we have a node for every tetrahedron and an link for every face. Given two nodes, the link which connects them correspond at the common face of the two tetrahedron. Note that all the vertices are quadrivalent. In this case the simplest transition from a normalized (truncated) initial state (the left) to a normalized final (truncated) state (the right) is represented in the figure (3) and gives, for the scalar product

$$W^{-1} \langle \gamma_2, \rho_2, i_{v2} | \prod_{\substack{INDIP. \\ SQUARES}} \delta(U^{SQ}) | \gamma_1, \rho_1, i_{v1} \rangle = \langle \gamma_2, \rho_2, i_{v2} | \prod_{\substack{INDIP. \\ FACES}} \delta(U^{FC}) | \gamma_1, \rho_1, i_{v1} \rangle, \quad (11)$$

¹ For a more rigorous treatment, we have to analyze (the images of) the parts of the graph $\gamma_1 \cup \gamma_2$ contained in the open sets of the atlas, once at a time. We can construct a partial “face”, joining by integrations the squares, until we arrive near the frontier of (the image of) the open set, named O_A . Then we pass to another open set O_B , with $O_B \cap O_A \neq \emptyset$, containing the links of the partial face near the frontier of O_A . The diffeomorphism from the cart A to the cart B can deform this links but not their mutual relations, so it is not a problem. Now we can proceed with the integration of the squares in the image of O_B and so on, obtaining the complete “face”.



FIG. 3:

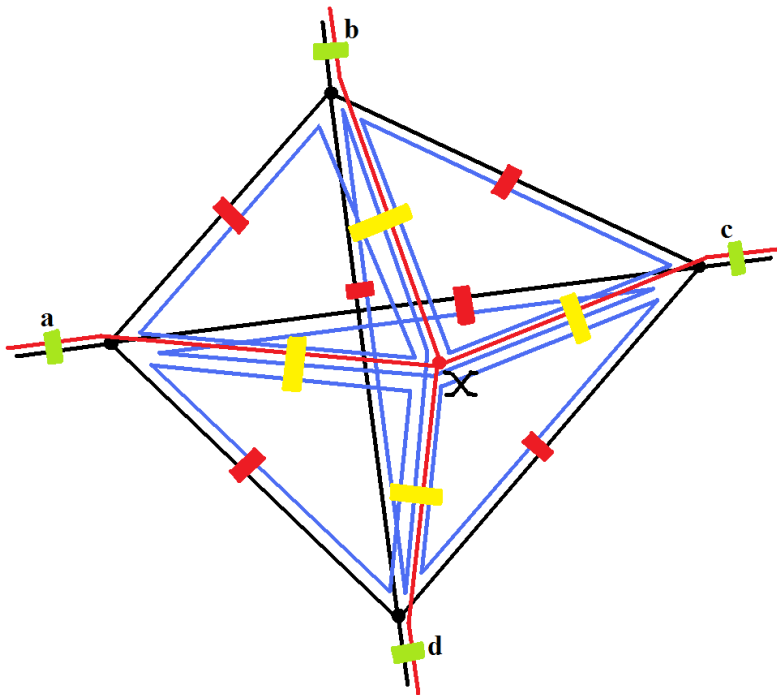


FIG. 4: In red the initial state, in black the final state, in blue the paths of the deltas. The colored rectangles represent the integrations.

$$\langle \gamma_2, \rho_2, \iota_{v2} | \prod_{\substack{INDIP. \\ FACE}} \delta(U^{FC}) | \gamma_1, \rho_1, \iota_{v1} \rangle = 15j \cdot \prod_{i=1}^6 \sqrt{\dim \rho_i}, \quad (12)$$

In figure (4) the paths of the deltas are represented in blue and the integrations on the group are represented as colored rectangles. The yellow rectangles generate, through the relation (4), five four-valent intertwiners. They result contracted with the pattern in figure (5) that defines the so called 15j-symbol

$$15j = i_{\rho}^{abcd} i_{\sigma}^{aefg} i_{\alpha}^{behf} i_{\beta}^{cfhm} i_{\gamma}^{dglm},$$

where $\rho, \sigma, \alpha, \beta, \gamma$ are the representations of the spins of the intertwiners when we consider them as couples of trivalent ones linked by links of spin $\rho, \sigma, \alpha, \beta, \gamma$. The red integrations select from the decomposition of the deltas (the blue ones) the representations equal to the final state representations and cancel the factor $\dim \rho$ in the decomposition. The green integrations cancel the factor $\sqrt{\dim \rho} \cdot \sqrt{\dim \rho} = \dim \rho$ coming from the normalizations and impose equivalence at the truncated parts. The other contractions fix the intertwiners coming from the yellow integrations to be equal to the intertwiners of the final state.

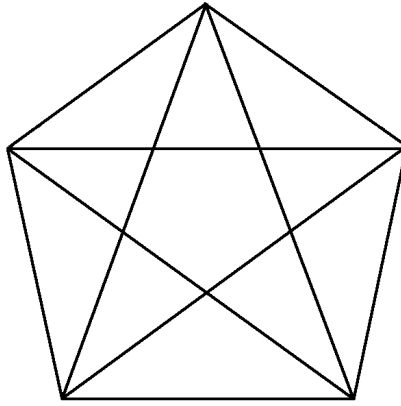


FIG. 5: Pattern of contraction of the five 4-valent intertwiners resulting from the yellow integrations.

We can now describe the amplitude (9) computed on these states using a spin-foam language. Let us start joining the initial and the final states by their external lines (the truncated parts). Returning at the triangulated manifold, this mean to identify the free face of each tetrahedron in the final states with one face of the initial tetrahedron, obtaining a 4 dimensional figure called 4-simplex, part of a Regge-spacetime. From a 4-simplex we pass at the 2-complex or spin-foam, that is its dual. For construct the dual we draw a vertex in the center of the 4-simplex and, for every tetrahedron, we draw an edge which intersect its and is incident to the vertex. These edges form the boundaries of the faces of the symplex. In this case we can see the faces as marked by a couple of edges, see (6). We note that every face marked by a couple of edges corresponding to final tetrahedrons (the black upper ones) includes a link in the final spin-network. So we associate the spins of these links to these faces. In a similar way every face marked by a mixed couple includes a link in the initial state and one in the final state with the same spin. We associate these spins to these faces. Furthermore every node of the spin-networks bounds an edge of the spin-foam and so we associate the spins of the corresponding nodes to the edges.

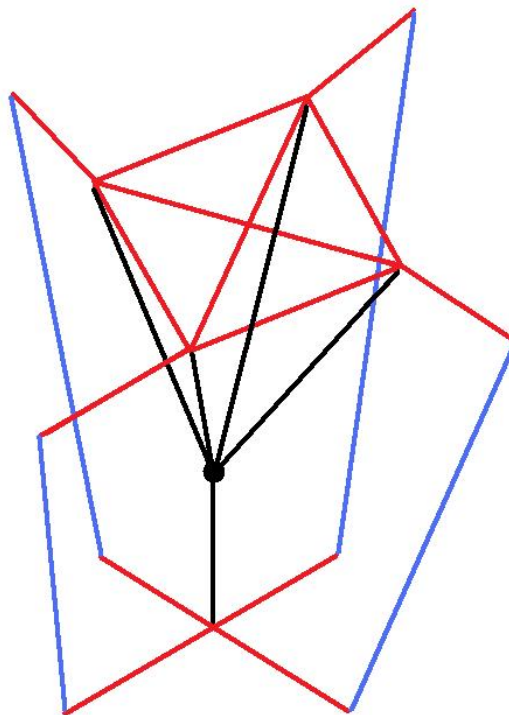


FIG. 6: The 2-complex, in black, with the initial and final states, in red.

In the spin-foam formalism we associate an amplitude to the 2-complex, understood as a product of a faces amplitude, an edges amplitude and a vertex amplitude. The vertex amplitude is the part of this that is unfactorizable in terms of faces and edges. The unfactorizable part in (12) is the $15j$. The spins which enter in the symbol are the six spins from the final states, the four spins from the initial state and the five spins of the spin-networks nodes. As we have seen, these 15 spins can be associated to the ten faces of the 4-simplex and to its five edges. We remain with the six factors $\sqrt{\dim \rho}$ from the normalization of the non-truncated part. We can understand them as the faces amplitude in the spin-foam.

In the spin-foam framework we have a condition on the amplitude for preserving the unitarity: we image of separating an arbitrary 2-complex in two parts cutting it with a plane. The total amplitude has to be described as the product of the amplitudes of the two parts. This imposes that the face amplitude for the faces border by a link included in one of the two boundary states has to return a factor $(\dim \rho)^{n/2}$ if an internal face gives $(\dim \rho)^n$. This is because, dividing a 2-complex in two parts, we can divide an internal face in two parts. Every parts will be bordered by a link in a boundary state. The amplitude of the original *internal* face has to be the product of the amplitudes of the resulting faces *bounded by a link in a boundary state*. The relation $(\dim \rho)^n = (\dim \rho)^{n/2} \cdot (\dim \rho)^{n/2}$ ensures this. For the same reason, a face bounded by a link in the initial state and a link in the final state has to return an amplitude equal to 1. If it is not so, we can divide the 2-complex with an infinite number of planes, generating an infinite number of faces with the same character. In this way the amplitude would explode. The amplitude (12) respects this conditions with $n = 1$. Notice that the four faces of the final tetrahedron do not have associated deltas, because of our choice of associating deltas only to independent faces.

So far we have shown that the scalar product (9) between spin-network states can be expressed in terms of a spin-foam with vertex amplitude given be a $15j$ -symbol. Let us can see how this is related to quantum general relativity.

III. BF-THEORY AND GENERAL RELATIVITY

We assume that the Lie Algebra \mathfrak{g} of the group G is equipped with an invariant nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. The action for BF-theory is

$$S = \int_{M=S \otimes R} F \wedge B d^{m+1}x \quad (13)$$

where $F(A)$ is the curvature of A , that is a \mathfrak{g} -valued two-form, and B is a \mathfrak{g} -valued $(m-1)$ form. In the 4-dimensional- $SO(4)$ case we have

$$S = \frac{1}{4!} \int_{M=S \otimes R} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} B_{\rho\sigma} d^4x \quad (14)$$

$$S = \frac{1}{2} \int_{M=S \otimes R} [\epsilon^{abc} F_{0a} B_{bc} + \epsilon^{abc} F_{ab} B_{c0}] d^4x \quad (15)$$

$$S = \int_{M=S \otimes R} [F_{0a} E^a + \epsilon^{abc} F_{ab} B_c] d^4x \quad (16)$$

$$S = \int_{M=S \otimes R} [\dot{A}_a E^a + A_0 D_a E^a + \epsilon^{abc} F_{ab} B_c] d^4x \quad (17)$$

where we have set $\epsilon^{abc} B_{bc} = \frac{1}{2} E^a$ and $B_{0c} = \frac{1}{2} B_c$. We have two first class constraints

$$D_a E^a = 0$$

which imposes gauge invariance respect to transformations of G , and

$$F_{ab} = 0 \quad (18)$$

which codes the dynamic and diffeomorphisms-invariance. Riemaniann general relativity can be obtained by imposing that B has the form

$$B_{\mu\nu}^{IJ} = \left(\frac{1}{\beta} + 1 \cdot * \right) (e_\mu^I e_\nu^J - e_\nu^I e_\mu^J)$$

where IJ are indices of the $so(4)$ algebra, β is the Immirzi parameter and $*$ is an Hodge dual which acts in the Lie-algebra indices ($*B_{\mu\nu}^{IJ} = \epsilon^{IJ}_{KL} B_{\mu\nu}^{KL}$). This decomposition is possible if and only if

$$*B_{\mu\nu} \cdot B_{\mu\nu} \left(1 + \frac{1}{\beta^2} \right) - \frac{2}{\beta} B_{\mu\nu} \cdot B_{\mu\nu} = 0 \quad (19)$$

$$n_I \left((*B)_{\mu\nu}^{IJ} - \frac{1}{\beta} B_{\mu\nu}^{IJ} \right) = 0 \quad (20)$$

for some direction n_I . No sum on repeated indices is assumed. This can be rewritten as

$$*E^a \cdot E^a \left(1 + \frac{1}{\beta^2} \right) - \frac{2}{\beta} E^a \cdot E^a = 0 \quad (21)$$

$$n_I \left((*E^a)^{IJ} - \frac{1}{\beta} E^{aIJ} \right) = 0 \quad (22)$$

We see that

$$\{A_a(x), E^b(x')\} = \delta_a^b \delta^3(x - x').$$

In the quantum theory we can represent E^a as

$$E^a(x) = -i \frac{\delta}{\delta A_a(x)}.$$

Acting on the holonomy state $e^{\int_{\gamma_1} A}$ and choosing $n_I = \delta_I^0$ we see that the (21) and (22) can rewrite as

$$C_2 \left(1 + \frac{1}{\beta^2} \right) - \frac{2}{\beta} C_1 = 0 \quad (23)$$

$$L^I - \frac{1}{\beta} K^I = 0 \quad (24)$$

where $L^I = \frac{1}{2} \epsilon_{KL}^I J^{KL}$, $K^I = J^{0I}$ are respectively the generators of the $SO(3)$ subgroup that leaves n_I invariant and the generators of the corresponding boosts. C_1 and C_2 are the Casimir and pseudo-Casimir operators of g

$$C_1 = J \cdot J = 2(L^2 + K^2) \quad (25)$$

$$C_2 = *J \cdot J = 4L \cdot K. \quad (26)$$

The constraint (24) are of second class, but we can substitute them with the first class master-constraint

$$\sum_I \left(L^I - \frac{1}{\beta} K^I \right)^2 = L^2 \left(1 - \frac{1}{\beta} \right) + \frac{1}{2\beta^2} C_1 - \frac{1}{2\beta} C_2 = 0 \quad (27)$$

Substituted the (23) we have

$$C_2 = 4\beta L^2 \quad (28)$$

The (23) is solved, up to \hbar corrections by

$$(j^+)^2 = \left(\frac{\beta + 1}{\beta - 1} \right)^2 (j^-)^2 \quad (29)$$

where (j^+, j^-) label the unitary representation of $SO(4)$. So the (23) poses only a restriction on the representations in the spin-network states. For $\beta > 0$ the (28) imposes

$$l^2 = \left(\frac{2j^-}{1 - \beta} \right)^2 = \left(\frac{2j^+}{1 + \beta} \right)^2 \quad (30)$$

that for $0 < \beta < 1$ corresponds to

$$l = j^+ + j^-. \quad (31)$$

That is, the constraint selects the highest irreducible in the decomposition of the representation space, that we call $H_{(j^+,j^-)}$, when viewed as the carrying space of a reducible representation of the $SO(3)$ subgroup

$$H_{(j^+,j^-)} \rightarrow H_{j^+} \otimes H_{j^-} = H_{|j^+-j^-|} \oplus \dots \oplus H_{j^++j^-}.$$

This can be implemented posing a particular form on the intertwiners k -valent in the spin-networks

$$i^{(q^+q^-)_1 \dots (q^+q^-)_k} = i^{m_1 \dots m_k} \bigotimes_{i=1}^k \int_{SO(4)} dU i_{m_i}^{q_i^+ q_i^-} [R^{\frac{(1+\beta)l_i}{2} \frac{|1-\beta|l_i}{2}}(U)]^{q_i^+ q_i^-} \quad (32)$$

with $i^{m_1 \dots m_k}$, $i_{m_i}^{q_i^+ q_i^-}$ intertwiners of $SO(3)$. For $k = 4$ this node reduces to the *E.P.R.*-node. Once we have imposed these constraints we can implement the dynamics. Equation (18) becomes

$$F_{ab}(x)|physical\ state \rangle = 0.$$

The approximation of this condition on a lattice is

$$U^{SQ} - 1|physical\ state \rangle = 0 \quad \forall\ INDIP.\ SQUARE.$$

It gives (remember that the delta on the group has support on the unity) the physical scalar product; that is

$$\langle final\ state|N \prod_{\substack{INDIP. \\ SQUARES}} \delta(U^{SQ})|initial\ state \rangle.$$

It simplifies, as we have seen in the discussion after equation (8), to

$$NW \langle final\ state| \prod_{\substack{INDIP. \\ FACES}} \delta(U^{FC})|initial\ state \rangle, \quad (33)$$

where *FACES* are the closed surfaces bounded by the union of the links of the graphs in the initial and final states. The constant N is fixed by the condition

$$\langle state|N \prod_{\substack{INDIP. \\ SQUARES}} \delta(U^{SQ})|state \rangle = 1$$

and absorbs the divergence of W . The operator

$$P = N \prod_{\substack{INDIP. \\ SQUARES}} \delta(U^{SQ})$$

is called a projector on the physical states. At this point we can forget the lattice. If the states are the 4-valent basic graphs that we have seen in the previous section we obtain now the *E.P.R.* amplitude with the $15j$ -symbol of $SO(4)$ which splits into two $15j$ -symbols of $SO(3)$ contracted in a way given by the *E.P.R.* node, that is

$$\begin{aligned} A_{EPR} &= \prod_f \sqrt{(|1-\beta|j_f+1)(|1+\beta|j_f+1)} \cdot \\ &\cdot \sum_{i_a^+ i_a^-} 15j\left(\frac{(1+\beta)j_{ab}}{2}; i_a^+\right) 15j\left(\frac{|1-\beta|j_{ab}}{2}; i_a^-\right) \bigotimes_a f_{i_a^+ i_a^-}^{i_a}(j_{ab}) \end{aligned} \quad (34)$$

where $a, b = 1, \dots, 5$ are abstract labels to identify the a -th intertwiner or the contraction from the a -th and the b -th. The label f indicates the faces of the spin-foam bounded by the initial and final states. Moreover

$$f_{i^+ i^-}^i = i^{m^1 \dots m^4} i_{(q_1^+ q_1^-) \dots (q_4^+ q_4^-)} \bigotimes_{i=1 \dots 4} i_{m_i}^{q_i^+ q_i^-}.$$

In this way we have recovered the E.P.R. spin-foam transition amplitude of reference [4]. In this framework we apparently have not need to insert a lot of complete sets between initial and final state: the only difficult for doing the calculation (33) seems to be finding the closed surfaces in the union of the graphs. But we consider

$$\langle s_1 | P | s_2 \rangle = \langle s_1 | P P | s_2 \rangle = \sum_i \langle s_1 | P | s_i \rangle \langle s_i | P | s_2 \rangle. \quad (35)$$

The sum is on all the spin-networks s_i , not only on those which satisfied the restrictions on the representations, and so the scalar product is incorrect. The solution can be found by inserting infinite projectors and complete sets, restricting all the sums to the states that satisfied the restrictions.

$$\langle s_1 | P | s_2 \rangle_{Phys} = \sum_{i_1, \dots, i_n} \langle s_1 | P | s_{i_1} \rangle \langle s_{i_1} | P | s_{i_2} \rangle \dots \langle s_{i_{n-1}} | P | s_{i_n} \rangle \langle s_{i_n} | P | s_2 \rangle \quad (36)$$

Some “internal” scalar product will be trivially equal to one. Discarding these we obtain a sum on all the non-trivial sequence of spin-networks which connect the initial and final states. In this way we recover, calculating the single physical scalar products, the ordinary sum on spin-foams connecting the external spin-networks.

IV. CONCLUSIONS

We have shown how to reconstruct the E.P.R. spin-foam amplitude of reference [4] starting from a formulation of the canonical theory. We can add a few remarks:

- The final amplitude is expressed in terms of invariant quantities of the group theory for G . Therefore it can be generalized to non compact groups. We can pass freely, for example, from $SO(4)$ to $SO(1,3)$, substituting the relative invariants for the ones with those of the second. The unitary representation for the last are labeled by (n, ρ) , with n positive integer and ρ real. In this case the condition (23) becomes:

$$n\rho\left(\beta - \frac{1}{\beta}\right) = \rho^2 - n^2 \quad (37)$$

and fixes $\rho = \beta n$ or $\rho = -n/\beta$. This makes the area spectrum discretized also in the lorentzian theory [4].

- The form of the scalar product (33) doesn't depend to the dimension of the space S . So we can in principle calculate the probability for a state to pass from a graph embedded in a N -dimensional space to a graph embedded in a $N + 1$ dimensional space. In this way we can study if a 3-dimensional space is statistically favored.
- The theory can be extended to gauge interactions. We can move from a 4-dimensional manifold M^4 to an 11-dimensional one with topology of $M^4 \times S^1 \times S^2 \times CP^2$. The form B becomes

$$B_{\mu_1, \dots, \mu_9}^{IJ} = \frac{*T}{\beta} (b^I \wedge b^J)_{\mu_1, \dots, \mu_9} + *L (b_{\mu_1} \dots b_{\mu_9})^{IJ}$$

where the dual $*T$ acts on tangent space indices and $*L$ on Lie algebra indices. The theory classically admits a solution of the form

$$g_{\alpha\beta} = \begin{pmatrix} g_{ab} + Q_a^r K_l^r Q_b^s K^{sl} & Q_a^r K_j^r \\ Q_b^r K_i^r & g_{ij} \end{pmatrix}$$

$$g_{\alpha\beta} = b_{\alpha}^{\Sigma} b_{\beta}^{\Sigma} \quad g_{ij} = \hat{e}_i^I \hat{e}_j^I \quad g_{ab} = e_a^A e_b^A$$

$$b = \begin{pmatrix} e & Q^r K^{rl} \hat{e}_l \\ 0 & \hat{e} \end{pmatrix}$$

$$\alpha, \beta, \Sigma = 1, \dots, 11 \quad a, b, A = 1, \dots, 4 \quad i, j, l, I = 5, \dots, 11$$

Capital letters indicate Lie $SO(11)$ algebra indices. Indices r, s move on the $1 + 3 + 8 = 12$ generators of $U(1) \times SU(2) \times SU(3)$ with gauge fields Q^r and Killing vectors K^r . We suppose that the extra dimensions have size $\sim L_{PLANCK}$ and, at the leading order (L_{PLANCK}^0), we recover the standard model.

The dual spin-networks will be of valence 11 with every link intersecting a 9-simplex. Only 4 links for nodes intersect (in a single point) a triangle in M^4 and a 7-simplex in $S^1 \times S^2 \times CP^2$. The others 7 intersects (in a single point) a tetrahedron in M^4 and a 6-simplex in $S^1 \times S^2 \times CP^2$. We recover the semiclassical limit putting $j \rightarrow \infty$ for the first four and $j \sim L_{PLANCK}^2$ for the last seven. We conjecture that, in this limit, the dominant contribution comes from intertwiners of the form

$$i_{m_1, \dots, m_{11}} = h_{m_1, \dots, m_4} \times k_{m_5, \dots, m_{11}},$$

where intertwiners h glue themselves by links $j \rightarrow \infty$ only with intertwiners h , and intertwiners k gluing themselves by links $j \sim L_{PL}^2$ only with intertwiners k . Moreover, all the intertwiners h are also intertwiners of $SO(4)$, because $SO(4) \subset SO(11)$. It will be interesting to discover if this subset of $SO(4)$ intertwiners corresponds or not (after solving the simplicity constraints) to the set of E.P.R. intertwiners.

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Some considerations on the physical projector

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Several efforts have been made to construct the projection operator on physical states in LQG. The complexity of the Hamiltonian operator defined by Thiemann [1] has led many researchers to follow a different path. The most promising one defines general relativity as a topological BF-theory supported by constraints. Attempts to define the projection operator into this way were made by Emanuele Alesci, Karim Noui and Francesco Sardelli [2]. A precise definition has been given by Karim Noui and Alejandro Perez in the three-dimensional case [3]. Here we simply understand what should be the behavior of this operator, to have guidelines to follow in its precise definition.

I. INTRODUCTION

Look at the scheme in figure 1: it gives a good description of what we mean with physical state. We restrict us to the closed universe case, where N is finite. We define the projector on physical states as

$$P|k\rangle = \frac{1}{N} \sum_{j=0}^{N-1} M^j |k\rangle \quad (1)$$

where $|k\rangle = |1\rangle, |2\rangle, \dots, |N\rangle$ is some Cauchy spin-network and

$$M|k\rangle = |k+1\rangle \Rightarrow M^j |k\rangle = |k+j\rangle.$$

If $k+j > N$ we consider

$$M|N\rangle = |1\rangle \Rightarrow M^j |k\rangle = |k+j \bmod N\rangle$$

in the general case. It is easy to see that

$$P|k\rangle = P|k+m\rangle, \quad \forall m \in \mathbb{Z} \quad (2)$$

$$P^2|k\rangle = P|k\rangle \quad (3)$$

$$MP|k\rangle = PM|k\rangle = P|k\rangle. \quad (4)$$

It seems impossible to impose

$$\langle k+m|P|k\rangle = 1 \quad \text{if} \quad \langle k|k\rangle = \langle k+m|k+m\rangle = 1, \quad (5)$$

so that the projector can be a not unitary operator. Consider now the operator $H = M - 1$. From the (4) we know that it is a null operator on the physical states

$$HP|k\rangle = (M-1)P|k\rangle = (MP-P)|k\rangle = (P-P)|k\rangle = 0. \quad (6)$$

So, we can credibly suppose that the hamiltonian \mathcal{H} (with $P = \delta(H)$) could be expanded in a series

$$\mathcal{H} = \sum_{n=1}^{\infty} a_n H^n \Rightarrow \mathcal{H}[H=0] = 0$$

for some succession $\{a_n\}$. The action of the hamiltonian is to modify a state $|k\rangle$, adding new loops, transforming it in a weighted sum of states $|k+j\rangle$.

We emphasize that the Hamiltonian **doesn't** realize simply a step in a foliation along a time parameter t , because the states $|k\rangle$ don't belong all to the same foliation.

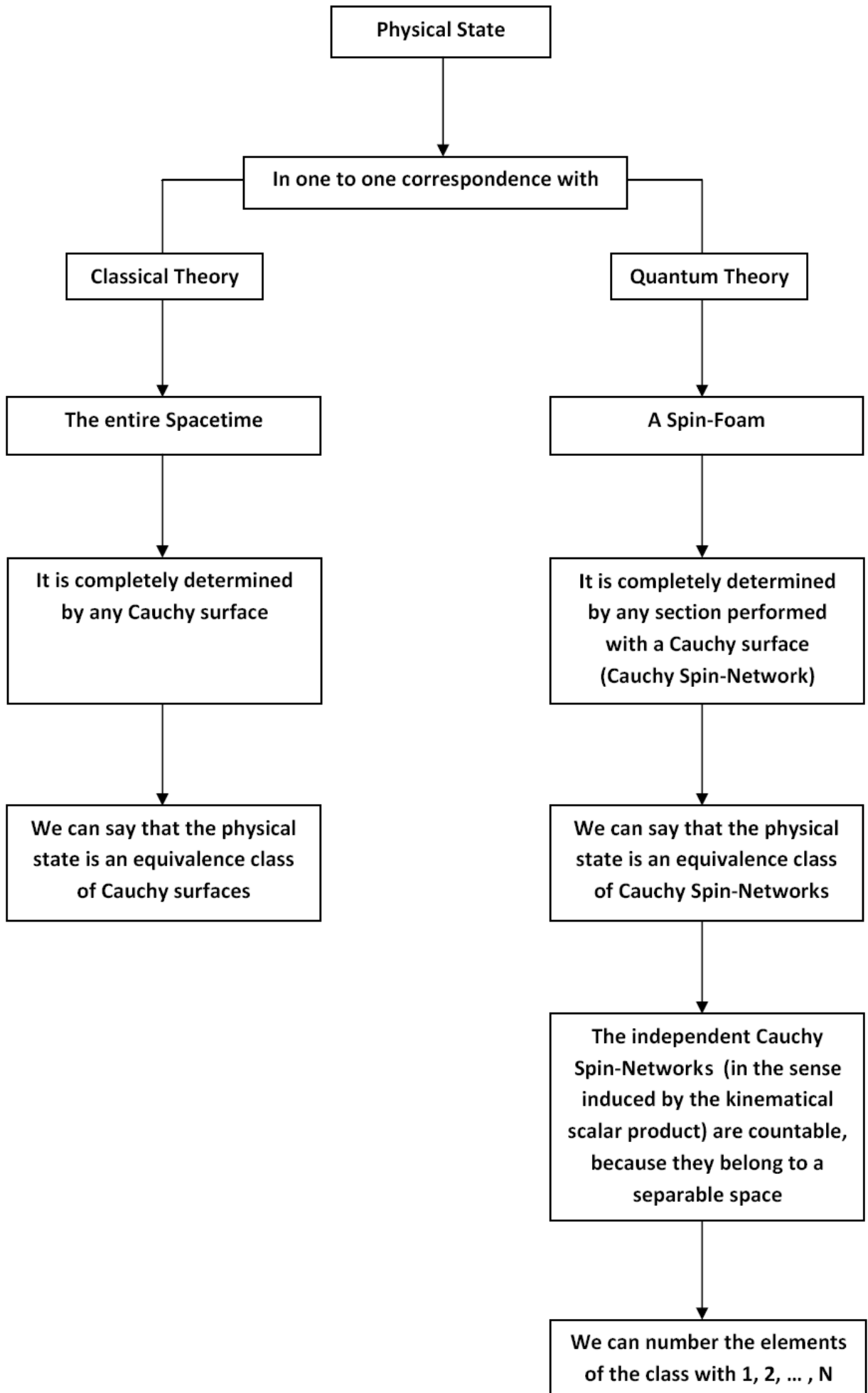


FIG. 1:

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