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SEMI-CLASSICAL ASPECTS
OF
BLACK HOLE PHYSICS

Candidato: Roberto Di Criscienzo

Relatore: Prof. Luciano Vanzo

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*... Ojalá por lo menos que me lleve la muerte
para no ver tanto Anna para no verla siempre
en todos los segundos, en todas las visiones...*

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1. Introduction

Since the discovery of quantum black hole thermal radiance by Hawking [1], it became pretty clear that something remarkable concerning the interface of gravity, quantum theory and thermodynamics was at work. In the usual picture, a radiating black hole loses energy and therefore shrinks, evaporating away to a fate which is still debated. Many new ideas came out from the recognition that quantum field theory implied a *thermal spectrum*: the most impressive probably being 't Hooft's idea of a *dimensional reduction* in quantum gravity and the associated holographic description (indicating a drastic reduction in the counting of degrees of freedom in finite regions, which scale with area rather than volume) [2,3]; and the *principle of black hole complementarity* aimed to reconcile the apparent loss of unitarity implied by the Hawking process with the rest of physics as seen by external observers [4]. But there were also other, more practical, issues regarding these matters, some of which bewildered scientists since the very beginning and that have been only partly resolved. A key issue is that the original derivation of Hawking radiation applied only to stationary black holes, while the picture above uses quasi-stationary arguments. Actually, an evaporating black hole is non-stationary. Nevertheless, a surprising aspect of the semi-classical result is that the radiation caused by the changing metric of the collapsing star approaches a steady outgoing flux at large times, implying a drastic violation of energy conservation. This certainly means that one cannot neglect the back-reaction problem which, however, has not been solved yet in a satisfactory way. A sample of other key issues have to deal with the final state of the evaporation process, the thermal nature and the related information loss paradox, the Bekenstein–Hawking entropy and the associated micro-states counting, the trans-planckian problem, and so on. It was to address some of these questions that some alternative derivations and descriptions of the Hawking's emission process began to appear over the years [5–13], one of these being the so called *tunnelling method*, the *leitmotiv* of my doctoral studies.

Before introducing it in full mathematical detail, we shall spend the next few paragraphs to spell out some general features of this method and how it compares with the general properties of quantum fields and states in the presence of a black hole.

The tunnelling method has to do with the particle interpretation of the emission process, which would be the most natural way to explain the loss of energy suffered by the radiating black hole, but also poses a problem. It is that while the original derivation of Hawking radiation used the theory of quantum free fields propagating in a fixed gravitational background, for the case at hand a particle interpretation in the near horizon region of a black hole space-time was found to be far from unique and therefore inherently ambiguous. This fact is deeply rooted into the physical nature of the problem, which is that the general covariance of the physical laws allows many inequivalent choices of time, as was explained several times (see, *e.g.* [14,15]). It was partly to overcome such difficulties that DeWitt launched in the Seventies the “stress-energy-tensor program” aimed to compute the expectation values of the stress tensor and other observables to

better describe quantum effects in a curved space-time [16]. Thus, it was found that it is not in general possible to divide $\langle T_{\mu\nu} \rangle_{\text{ren}}$ into a real particle contribution and a vacuum polarisation[‡] part in an unambiguous way. However, it will always be possible to choose any coordinate system, as long as it is regular across the horizon, and use it to define an observer dependent vacuum relative to which a particle definition is feasible. As we will see, with the tunnelling method we are only concerned with the *probability* that such an observer dependent notion of particle be emitted from the horizon of the black hole. If this probability is a coordinate scalar it will not depend strongly on what particle concept one employs. In fact, we can call it *a click event in a particle detector* if we like, without committing ourselves with the particle concept; concept which remains of great heuristic value however, and will find its realisation in the choice of the particle action used to compute the tunnelling probability. Such an observer dependent notion of particles has been advocated also by Gibbons and Hawking in their treatment of de Sitter radiation [17], on the ground that an observer independent definition of particles is not relevant to what a given observer would measure with a particle detector.

The second aspect we would like to mention is the fact that, as it will be seen, the tunnelling method uses only the classical action of a single massless, spin-less particle and therefore appears to be state independent. It does what it does by relating the particle emission to an imaginary contribution to the classical action localised at the horizon, which only comes from the local geometry. This can be traced back to the work of Damour and Ruffini, with contributions of Padmanabhan *et al* , Massar and Parentani [18] and Kraus and Wilczek, Parikh and Wilczek [19–21]. To what extent this is so was recently studied in [22] where, for a large class of quantum states with Hadamard short distance singularity, it was shown that the two-point correlation function, which is directly related to the tunnelling probability, has a universal thermal appearance whose temperature is, needless to say, the Hawking temperature.

Now all these important findings have the drawback that they have been proved, and one can easily understand why, only for stationary black holes. In that case the tunnelling picture has not much more to say than was already known, at least for the simplest solutions, besides helping to understand black hole radiation from a different viewpoint. Where exact calculations can be done, it typically gives a less detailed picture of the radiation process since it is mainly related to a semi-classical emission rate. The cases where the method is more powerful have to do either with more intricate stationary solutions or else with truly dynamical black holes. We can list here some of its achievements and properties:

- (i) The estimation of the leading correction to the semi-classical emission rate arising from back-reaction to the background geometry, whose introduction by Kraus and

[‡] In quantum theory, negative energy density may arise in a region of space because an external field applied to the vacuum may either increase or reduce the local energy density of zero point fluctuations. This phenomenon, known as *vacuum polarisation*, is expected to play an important role in the proximity of a black hole.

Wilczek, Parikh and Wilczek [19–21] motivated the tunnelling method in a form that will be discussed soon;

- (ii) The original tunnelling method can be generalised to a Hamilton–Jacobi variant, originated with the work of Padmanabhan and collaborators and systematically applied either to stationary or dynamical black holes (see *e.g.* [7, 23–28] for a sample of papers). For dynamical black holes this was particularly important, since even approximate quantum calculations are notoriously hard. Moreover, in this variant it manifestly preserves general covariance;
- (iii) Supplemented with a more precise and more general mathematical definition of a local horizon, the Hamilton–Jacobi variant can be applied to any sort of horizon within this class, and in particular to cosmological and weakly isolated horizons [29]. It can also be applied to past horizons and white holes, in which case a clear notion of temperature emerges in complete analogy to black holes;
- (iv) The tunnelling method strengthens the connection of the semi-classical rate with the surface gravity of the horizon even for dynamical black holes, provided that opportune definitions of such quantities are employed;
- (v) The tunnelling picture can promptly give indications on what occurs in other situations. One example is the WKB approach to Unruh’s radiation, reviewed in [30]. It can also be applied to see whether naked singularities are going to emit radiation, or to study the decay of unstable particles that in the absence of gravity would be stable by the action of some conservation law. This is valuable given the great efforts that are normally necessary to analyse quantum effects in the presence of gravity.

It remains to explain how the tunnelling picture works and to what extent it gives a consistent picture, at least from a logical point of view. This will cover the rest of the thesis that, we anticipate, will essentially follow the material just indicated in listed form above.

In Section 2, the methodological core of the topic is outlined and applied to various stationary solutions. In Section 2.1 we present the original method introduced by Parikh and Wilczek to study the black hole radiance — namely the *null geodesic method* — which is subsequently applied to verify the thermal properties of a static, spherically symmetric Schwarzschild black hole in the scalar sector. In Section 2.2 we comment results pertaining to statistical correlations of successive emission events, as displayed by the tunnelling method. In Section 2.3 and Section 2.4, after pointing out some flaws of this approach, we extend it to the more recently conceived *Hamilton–Jacobi tunnelling method*: this approach is examined in great detail in order to stress those points that — given the vastness and heterogeneity of opinions in the literature — might lead to confusion.

In Section 3, we study the mathematical foundation of the two tunnelling methods. The Hamilton–Jacobi method, preferable in view of its manifest covariance, is examined in Section 3.1. It is found to be equivalent to the null geodesic method, at least

in its generalised form, in Section 3.2. The aforementioned variety of opinions also comprehends a number of alternative proposals to the two methods, some of which are summarised in Section 3.3.

In Section 4, we present (the only) two solutions of static (spherically symmetric) dirty black holes arising in modified theories of gravity and describe their thermodynamic properties. In particular, we shall see that being these solutions not asymptotically flat, the Killing vector is ambiguously defined. This implies a discrepancy between the Killing temperature associated to the conical singularity of the Euclidean sections and the Kodama–Hayward temperature selected by the tunnelling method. This discrepancy does not deserve to bother us too much by itself since intimately related to the notion of energy we implement. By the end of the day, in fact, the ratio “energy/temperature” turns out to be a scalar invariant. If the entropy of these black holes does not present conceptual difficulties thanks to Wald, the same thing cannot be said for the definition of their mass. Recently, however, it has been pointed out by [31] how the first law of thermodynamics can help to identify the mass of these dirty black holes but only at the condition that the entropy is the Wald entropy and the temperature is the Killing temperature.

In Section 5, we approach the dynamical case but restricting ourselves to spherically symmetric non vacuum black holes. Though it may appear strange that stationary black holes, being a particular case of dynamical ones, are discussed first, there is a reason: in the stationary case there is much more control on the mathematics of the tunnelling picture. In particular, it is possible to use both the null geodesic method as well as its Hamilton–Jacobi variant to deduce the exact semi-classical thermal spectrum of the Hawking radiation, a fact which greatly enhances the understanding of the tunnelling picture in a way which is not available to dynamical situations. At the same time it provides the foundation for the dynamical treatment and inspires extensions to more general cases. The Section is further divided into several sub-sections. In Section 5.1 we try to characterise a black hole horizon and to give an opportune definition of its surface gravity. Again few dynamically inequivalent definitions will be reviewed, putting particular emphasis on the one which the tunnelling method privileges. In Section 5.2 we review material pertaining to spherically symmetric space-times and the form that surface gravities and horizons take in this highly symmetric case. In Section 5.3 we finally come to grasp the tunnelling mechanism. After a careful scrutiny of the escaping paths we derive the basic results; the computation of the imaginary part of the action, its relation with the surface gravity and the ensuing quasi-thermal interpretation. We end in Section 5.4 by presenting the extension of the Hamilton–Jacobi tunnelling method to the fermionic sector of some dynamical spherically symmetric solution.

In Section 6, we discuss how the tunnelling picture generalises to cosmological horizons, naked singularities and particle decays. We shall see that the first two cases are governed by a simple pole singularity while decays correspond to branch cut singularities of the classical action. In Section 6.1 we review few facts regarding Friedmann–Robertson–Walker space-times (abbr. FRW) with emphasis on the existence

of horizons, their structure and surface gravity. In Section 6.2 we apply the tunnelling method to cosmological horizons and find their emission rate. In Section 6.3 and 6.4 we analyse its application to particle decays and particle creation by black hole singularities. The former was extensively studied in de Sitter space-time, where exact results can be obtained, but the tunnelling method is not restricted to particular space-time choices. In Section 6.5 we review the possibility that naked singularities themselves radiate away their (suitably defined) mass. Indications of this phenomenon are supported by the tunnelling method.

In cosmology, the physical interpretation of Hayward’s dynamical surface gravity as a temperature is not as clear as in de Sitter or black hole case, but it is strongly favoured by the tunnelling picture. In particular, an Unruh–DeWitt detector moving in de Sitter or black hole space-time always thermalises at the corresponding Gibbons–Hawking temperature even if it is not on a geodesic path, while for cosmology there is not an analogous result. One may conjecture that the cosmological temperature given by Equation (6.4) is the one that a point-like detector registers when moving on a Kodama trajectory, whose four-velocity is the Kodama vector field. In general, such a trajectory is accelerated relative to geodesic observers so one expects a mixed contribution from the horizon and the acceleration together. These expectations were actually confirmed years ago in de Sitter and anti-de Sitter space and resulted in beautiful formulas [32, 33]. It would be very interesting to apply tunnelling methods also to Unruh’s like detectors (but see [30]). A first order of approximation along this direction is undertaken in Section 7. Section 7.1 contains a brief introduction on point-like detectors from a quantum field theoretic point of view. In Section 7.2 we focus on the Unruh–DeWitt detector which is introduced in order to confirm through quantum field theoretic calculations the results of previous tunnelling considerations. In Section 7.3 we show how the new formalism applies to a generic static black hole, and finite time effects are investigated. Section 7.4 is devoted to FRW space-times, and to a discussion of the realistic model of a universe filled with matter and cosmological constant $(\Omega_m, \Omega_\Lambda)$, where the main conclusions are presented as consequence of analytic results supported by numerical computations.

We end with Section 8 where conclusions are summarised.

We shall use the following conventions throughout: the metric signature is $(-, +, \dots, +)$; the constants c , G , k_B and \hbar set equal to unity unless explicitly stated; we use an index notation for tensor components. Greek indices are used to denote space-time components of a tensor; mid-Latin indices as i, j are typically used to denote the 0, 1 component; late-Latin indices as m, n for purely spatial tensors part. If not otherwise stated, an overdot will represent a time derivative, and a prime a spatial derivative in the (case by case) relevant spatial coordinate.

The research work presented in this thesis is largely based on the following papers:

- Di Criscienzo R, Nadalini M, Vanzo L, Zerbini S and Zoccatelli G 2007 *Phys. Lett. B* **657** 107–11
- Di Criscienzo R and Vanzo L 2008 *Europhys. Lett.* **82** 60001
- Hayward S A, Di Criscienzo R, Nadalini M, Vanzo L and Zerbini S 2009 *Class. Quantum Grav.* **26** 062001
- Hayward S A, Di Criscienzo R, Nadalini M, Vanzo L and Zerbini S 2009 Comment on “On the tunneling through the black hole horizon” *Preprint* gr-qc/0911.4417; Hayward S A, Di Criscienzo R, Nadalini M, Vanzo L and Zerbini S 2009 Reply to Comments on “Invariance of the tunneling method for dynamical black holes” *Preprint* gr-qc/0909.2956
- Di Criscienzo R and Zerbini S 2009 *J. Math. Phys.* 103517 [Erratum-Ibid. 2010 **51** 059901]
- Di Criscienzo R, Hayward S, Nadalini M, Vanzo L and Zerbini S 2010 *Class. Quantum Grav.* **27** 015006
- Bellini E, Di Criscienzo R, Sebastiani L and Zerbini S 2010 *Entropy* **12** 2186–98
- Di Criscienzo R, Vanzo L and Zerbini S 2010 *JHEP* **1005** 092
- Casadio R, Chiodini S, Orlandi A, Acquaviva G, Di Criscienzo R and Vanzo L 2011 *Mod. Phys. Lett. A* **26** 2149–58
- Acquaviva G, Di Criscienzo R, Vanzo L and Zerbini S 2011 Unruh–DeWitt detectors in spherically symmetric dynamical space-times *Preprint* arXiv: 1101.5254
- Vanzo L, Acquaviva G and Di Criscienzo R 2011 *Class. Quantum Grav.* **28** 183001

2. Quantum Tunnelling: Static Black Holes

According to quantum field theory, the physical vacuum – the state with no real particles – is in general a complex entity populated by virtual particles which constantly are created, interact among themselves and are then annihilated in this vacuum. In the absence of external fields, the vacuum is usually stable in the sense that virtual particles are not able to survive long enough to become real. However, it is now well proved that external fields are able to convert short-living particles into real ones just supplying enough energy to the process.

Considering particle creation by a static gravitational field endowed with a Killing vector ξ^α , the energy of a particle is $\omega = -p_\alpha \xi^\alpha$, where p^α is the particle's four-momentum. For massive (massless) particles, the momentum p^α is a future-directed time-like (null) vector. Thus, the energy ω of a particle is always positive in the region where the Killing vector is also future-directed and time-like. It follows that particle creation in a space-time region with ξ^α future-directed and time-like is forbidden by energy conservation. Such considerations provide a heuristic argument [34] to conclude that a static gravitational field can create particles only if the space-time contains a black hole. In fact, when a virtual particle pair is created just inside the horizon, the positive energy virtual particle can tunnel outside — no classical escape route existing — where it materialises as a real particle; in alternative, for a pair created just outside the horizon, the negative energy virtual particle, forbidden outside the horizon, can tunnel inwards [35]. In either cases, the negative energy particle is absorbed by the black hole, resulting in a black hole mass decrease, while the positive energy particle escapes to infinity, visible to distant observers (Hawking radiation).

In 1974 Hawking [1] proved that indeed black holes are classically stable objects that, due to quantum fluctuations, become unstable with respect to particle emission. Nowadays, several methods are known for deriving Hawking radiation and calculating its temperature, but none of them has been able to catch the intuitive picture above as the tunnelling method proposed a decade ago by Parikh and Wilczek [21]: it were not for the inclusion of back reaction, their method could be considered a semi-classical version of Hartle–Hawking's path integral derivation of black hole radiance [36].

2.1. The null geodesic method

The so called *null geodesic method* — as named by Kerner and Mann [25] — derives black hole radiance from few reasonable assumptions, namely: energy conservation, detailed balance and the use of non-singular coordinates across the horizon.

Energy conservation requires fixing the total energy of the space-time before and after particle emission. Since black hole mass and volume are linked together, a mass reduction due to the emission of a particle translates into a size contraction; so one might worry how to deal with quantum fluctuations of the metric originating from such contraction. However, this is not a problem as far as we consider transitions between zero-spin geometries. In this case in fact, no graviton quantisation is involved or, said

in other words, passing from different spherically symmetric configurations does not produce gravitational waves. As a consequence, the only degree of freedom remained in the problem is the position of the emitted particle (actually, a thin shell). Thus, to keep things as simple as possible, we can restrict to consider uncharged, static, spherically symmetric black holes emitting neutral matter, referring to next sections treatment of the most general case.

Because of Birkhoff's theorem [37], we are dealing with Schwarzschild geometry which, written in the Schwarzschild frame, reads

$$ds^2 = - \left(1 - \frac{r_g}{r}\right) dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.1)$$

where the *gravitational radius* is $r_g = 2M$. Computing the free fall acceleration of a body initially at rest in the Schwarzschild frame, we get

$$a := \sqrt{a_n a^n} = \frac{M}{r^2 \sqrt{1 - r_g/r}}. \quad (2.2)$$

The geometry (2.1) becomes singular at the gravitational radius and the acceleration (2.2) becomes infinite in the same limit. Nonetheless, near r_g the space-time is regular (*i.e.* neither curvature singularities or geodesic incompleteness): an indication of the fact that the Schwarzschild frame ceases to be defined at $r = r_g$ (For $r \leq r_g$ systems must be non-rigid in the sense that the metric components $g_{\alpha\beta}$ must be functions of time, cf. [34]).

It is well-known that the gravitational radius r_g represents the Killing event horizon of Schwarzschild black hole [37]; thus, in order to deal with tunnelling of a particle across it, we need to cover the region $r \sim r_g$ with regular coordinates. A possible choice, not the only one, is represented by Painlevé–Gullstrand coordinates [38]. In order to construct this new regular-across-the-horizon frame, we consider the geodesic equations of a family of observers, freely falling ($dr/d\tau < 0$) in the geometry (2.1), along a radial trajectory (θ, ϕ constant):

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - \frac{r_g}{r}}, \quad \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_g}{r}\right) = \tilde{E}^2 \quad (2.3)$$

τ representing observer's proper time and $\tilde{E} = E/m$ is the observer's conserved energy per unit mass. The energy parameter is related to observer's physical three-velocity $v^2 = g_{mn}dx^m dx^n / (-g_{00})dt^2$: in fact, given the Killing vector $\xi_{(t)}^\mu$ — generating the time-symmetry transformations of (2.1), the energy reads

$$\tilde{E} = -u_\mu \xi_{(t)}^\mu = -u_0 = -g_{00} u^0 = \frac{\sqrt{-g_{00}}}{\sqrt{1 - v^2}}. \quad (2.4)$$

We specialise to the case of observers starting at infinity with null initial velocity so that $\tilde{E} = 1$. For this family of observers, there exists a function t_p such that

$$\nabla_\mu t_p = -u_\mu. \quad (2.5)$$

Taking (2.3) into consideration, we get the Painlevé time t_p in terms of Schwarzschild coordinates

$$t_p = t + \int^r dr' \sqrt{\frac{r_g}{r'}} \left(1 - \frac{r_g}{r'}\right)^{-1} = t + 2\sqrt{r_g r} + r_g \ln \left(\frac{\sqrt{r} - \sqrt{r_g}}{\sqrt{r} + \sqrt{r_g}} \right). \quad (2.6)$$

In terms of (t_p, r, θ, ϕ) , the Schwarzschild geometry is written as

$$ds^2 = - \left(1 - \frac{r_g}{r}\right) dt_p^2 + 2 \sqrt{\frac{r_g}{r}} dt_p dr + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.7)$$

At fixed time, the spatial geometry described by (2.7) is Euclidean; while at any fixed radius, the boundary geometry is the same as that in (2.1). Most importantly, the metric is no more singular at the gravitational radius r_g ; the space-time is stationary, but no more static. The t_p coordinate — being the time perceived by infalling observers who reach the curvature singularity $r = 0$ in finite time — remains a valid coordinate as far as it remains finite.

The geodesic for a null s-wave is given by

$$\dot{r} \equiv \frac{dr}{dt_p} = \pm 1 - \sqrt{\frac{r_g}{r}}, \quad (2.8)$$

with upper (lower) sign corresponding to outgoing (ingoing) geodesic. The basic idea behind the approach by Parikh and Wilczek [21] is to study the emission of such a null-radial thin shell from the black hole through quantum tunnelling. Imposing energy conservation means that the total space-time energy is fixed and one allows the black hole mass to fluctuate. As showed in [19], the motion of a shell of energy ω emitted from a black hole of initial mass M is described by (2.8) in the space-time geometry of (2.7) with M replaced by $M - \omega$.

Differently by tunnelling in quantum mechanics, where two configurations (classical turning points) separated by a potential barrier are connected by a trajectory in imaginary time, here we have the perception that nothing similar happens. In fact, the problem with black hole radiation is that, as soon as a particle is displaced outside the horizon, it can escape along a classical trajectory. It is worth to mention how the crucial ingredient of the story here is played by energy conservation. As the black hole of mass M and area $16\pi M^2$ emits a particle of energy ω , its mass reduces to $M - \omega$ and its horizon recedes to $2(M - \omega) = r_g - 2\omega$. This amount of contraction, evidently, depends upon the mass of the emitted particle, so that quoting Parikh “it is the tunnelling particle itself that secretly defines the tunnelling barrier” [35].

Before evaluating the black hole emission rate, we can ask to what extent is justified the point-particle approximation. Given that a distant observer detects a wave with frequency Ω_{ob} , this has been emitted with frequency $\Omega_{\text{em}} \propto (1 - r_g/r)^{-1/2} \Omega_{\text{ob}}$. In the limit of $r \rightarrow r_g$, where the emission process occurs, the wavelength vanishes, making the point-particle (WKB) approximation fully reliable.

In the WKB approximation, the tunnelling probability rate Γ_{em} turns out to be equal to

$$\Gamma_{\text{em}} \sim \exp(-2 \text{Im } I) \quad (2.9)$$

where I is the action for the tunnelling process evaluated to the appropriate order. We shall derive later, in Section 3.1, the exact correspondence between Γ_{em} and the exp-function. Eventually, one would expect to be able to read off the black hole radiation temperature from comparison of the the probability rate (2.9) with the Boltzmann factor $e^{-\omega/T}$. The imaginary part of the action for a null s-wave outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} can be expressed as

$$\text{Im } I = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} dr p_r = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_0^{p_r} dp'_r \quad (2.10)$$

Changing the integration from momentum to energy thanks to Hamilton's equation $\dot{r} = dH/dp_r$ and noting that $H = M - \omega'$ with constant M and $0 \leq \omega' \leq \omega$, we have

$$\text{Im } I = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_M^{M-\omega} \frac{dH}{\dot{r}} = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_0^\omega \frac{(-d\omega')}{\dot{r}}. \quad (2.11)$$

Using (2.8) and switching the order of integration,

$$\text{Im } I = \text{Im} \int_0^\omega (-d\omega') \int_{r_{\text{in}}}^{r_{\text{out}}} dr \frac{1}{1 - \sqrt{\frac{2(M-\omega')}{r}}}. \quad (2.12)$$

With the understanding that the particle starts at $r_{\text{in}} = 2M$ and materialises at $r_{\text{out}} = 2(M-\omega)$, $r_{\text{in}} > r_{\text{out}}$, the integral over r is divergent and needs to be appropriately regularised. A tunnelling process corresponds to an energy which is not in the spectrum of the Hamiltonian. We need to continue the energy ω to complex values, that is $\omega' \rightarrow \omega' + i\epsilon$, in order to ensure that positive energy solutions decay in time:

$$\begin{aligned} \text{Im } I &= \text{Im} \int_0^\omega (-d\omega') \int_{r_{\text{in}}}^{r_{\text{out}}} dr \frac{\sqrt{r}}{\sqrt{r} - \sqrt{2(M-\omega')} + i\epsilon} \\ &= -\text{Im } i\pi \int_0^\omega (-d\omega') 4(M-\omega') = +4\pi\omega \left(M - \frac{\omega}{2}\right). \end{aligned} \quad (2.13)$$

Of course, Hawking radiation can also be regarded as pair creation outside the horizon, with the negative energy particle tunnelling into the black hole. The calculation proceeds as above with slight change of signs due to the fact that anti-particles travel back in time (cf. [21] for details). That both particle and anti-particle channels contribute to black hole emission is something which only affects the amplitude of the process, that is something which enters the proportionality factor of (2.9). In conclusion, the emission rate obtained is

$$\Gamma \sim \exp(-2 \text{Im } I) = e^{-8\pi M\omega \left(1 - \frac{\omega}{2M}\right)}. \quad (2.14)$$

The expected Boltzmann factor $e^{-\omega/T_{\text{H}}}$ is recovered only to order $O(\omega)$: in this order of approximation, the black hole temperature perfectly coincides with standard Hawking's result, $T_{\text{H}} = (8\pi M)^{-1}$. The $O(\omega^2)$ correction arising from the physics of energy conservation, makes higher the effective temperature of the hole as it radiates in agreement with the well-known negative thermal capacity of Schwarzschild black hole.

Neglecting the $\omega(2M)^{-1}$ term in (2.14) and invoking detailed balance[§], the Planck distribution is recovered.

A note – It appears that by neglecting the back-reaction no member of a pair of particles created just inside the event horizon can escape it, by reason of causality. In the terminology to be used in Section 5 this correspond to a type-II tunnelling path, the particles created just outside forming instead a type-I path. The inclusion of the back-reaction is truly fundamental for the existence of both possibilities. Since in the dynamical case we will mainly ignore the back-reaction effect, only type-I path will be accessible to the tunnelling probability.

2.2. Entropy and correlations

The exact expression for the emission rate (2.14) can be intriguingly re-written as the change in the hole’s Bekenstein–Hawking entropy [40],

$$\Gamma \sim e^{-8\pi M\omega(1-\frac{\omega}{2M})} = e^{\Delta S_{\text{BH}}} . \quad (2.15)$$

This is an interesting form since it agrees with Einstein’s formula for the probability of fluctuations about statistical equilibrium [41]. In this case, it refers to the probability rate with which a static Schwarzschild black hole gets out of equilibrium by emission of a scalar quantum. This formula is derived to order $O(\omega)$ from the fact that [42]

$$\Gamma_{\text{em}} \sim e^{-2\text{Im}I} \equiv e^{-\beta\omega} = \exp\left(-\frac{\partial S}{\partial M}\omega\right) \simeq \exp-[S(M) - S(M - \omega) + O(\omega^2)] , \quad (2.16)$$

where the standard relation involving the inverse temperature at equilibrium, $\beta = \partial S/\partial M$, has been used. It is remarkable that Equation (2.14) puts forward this result to order $O(\omega^2)$. Since the exp-function in Equations (2.15) and (2.16) can be re-written as $\exp \Delta S_{\text{BH}} = \exp S_{\text{final}}/\exp S_{\text{initial}}$, we can re-interpret it as a factor counting the internal

[§] Consider an ensemble of many identical copies of the same quantum system. Let the energy and the number of accessible states of the system be fixed. The probability that a system randomly chosen out of the ensemble will be in state i is denoted by p_i , with $\sum_i p_i = 1$. The *transition probability* $\lambda_{i \rightarrow j}$ denotes the conditional probability per unit time of a system going from state i to state j , *i.e.* $p_i \lambda_{i \rightarrow j} dt$ is the probability of a system originally being in state i and going to state j within the time interval dt [39]. From this definition, $\sum_j p_i \lambda_{i \rightarrow j} dt$ represents the probability of transition from state i to any other possible state of the system in time dt ; and $\sum_j p_j \lambda_{j \rightarrow i} dt$ represents the probability of transition into state i from any other possible state of the system in time dt . It follows that, $\dot{p}_i = \sum_j (p_j \lambda_{j \rightarrow i} - p_i \lambda_{i \rightarrow j})$ is the net rate of change of p_i . The *detailed balance condition* requires that,

$$\text{at equilibrium,} \quad p_j \lambda_{j \rightarrow i} = p_i \lambda_{i \rightarrow j} , \quad \forall i, j . \quad (*)$$

In the case at hand, the system is made of “black hole + radiation”. The initial state i represents, for example, the black hole with $N - 1$ particles (*e.g.* photons). Through the physical mechanism described before, the black hole emits one more particle, so that the state j becomes “black hole + N particles”. Neglecting back-reaction, the internal degrees of freedom of the black hole do not change and by (*), we have that $\lambda_{i \rightarrow j}/\lambda_{j \rightarrow i} \equiv \Gamma_{\text{em}}/\Gamma_{\text{ab}} = p_N/p_{N-1} = e^{-\omega/T}$. This result combined with the classical constraint, $\Gamma_{\text{ab}} \mp \Gamma_{\text{em}} = |T_l(\omega)|^2$, reproduces Planck (minus sign) and Fermi–Dirac (plus sign) distributions. Here, $T_l(\omega)$ represents the transmission coefficient of the black hole barrier which in general can depend on the energy ω and the angular momentum l of the particle.

degrees of freedom of the black hole. It is a long debated question how the thermal nature of Hawking radiation can be reconciled with unitarity (*information loss puzzle*). From one side, unitarity is a milestone of classical and quantum physics; from the other, Hawking radiation is the most reliable result of quantum gravity derived with semi-classical techniques. This is not the place to enter the details of such a wide subject: it is enough to say that the discovery of non-thermal corrections to the black hole radiation spectrum due to the inclusion of back-reaction effects showed by Parikh and Wilczek, and the possibility of writing the emission rate in the form (2.15) gave hope for a possible solution of the problem. For example, let us take into consideration the emission of a quantum of energy $\omega = \omega_1 + \omega_2$ and, separately, the emission of a quantum of energy ω_1 followed by the emission of a quantum of energy ω_2 . The function

$$\chi(\omega_1, \omega_2) = \log \left| \frac{\Gamma(\omega)}{\Gamma(\omega_1)\Gamma(\omega_2)} \right| \quad (2.17)$$

can be considered as a measure of the statistical correlation between the two events which is expected to be zero whenever the events are independent. This is indeed the case for thermal spectrum, but also for the emission rate (2.14) it was argued in [43] to be the case. Even the inclusion in Equations (2.15), (2.16) and (2.17) of logarithmic corrections to the black hole entropy, as suggested by direct count of black hole microstates both in string theory and loop quantum gravity, seemed not to be able to show the appearance of correlations in the spectrum of black hole radiation [44].

However, a more careful statistical treatment due to Zhang *et al* in two significant papers [45] showed that, on the contrary, the statistical correlation (2.17) is actually non zero and equal to $8\pi\omega_1\omega_2$, a result later confirmed by Israel and Yun [46]. To make clear the point, we display the black hole mass into $\Gamma(\omega)$ and, following Israel and Yun, we write the probability that a black hole of mass M emits a quantum of energy ω as

$$\Gamma(M, \omega) = \exp[-4\pi M^2 + 4\pi(M - \omega)^2] \quad (2.18)$$

which is the tunnelling result. Then this yields the non trivial correlation

$$\chi(\omega_1, \omega_2) = \log \left| \frac{\Gamma(M, \omega_1 + \omega_2)}{\Gamma(M, \omega_1)\Gamma(M, \omega_2)} \right| = 8\pi\omega_1\omega_2 \quad (2.19)$$

which is the result of Zhang *et al*. In general, one may define a measure of the correlation between events x and y as

$$\chi(x, y) = \log \frac{P(x, y)}{P(x)P(y)} = \log \frac{P(y|x)}{P(y)} \quad (2.20)$$

where

$$P(y|x) = \frac{P(x, y)}{P(x)} \quad (2.21)$$

is the conditional probability of y given x and $P(x, y)$ is the probability of both x and y . Clearly, if we substitute $P(y|x)$ in place of $P(y)$ into $\chi(x, y)$, we get zero correlation for any events x and y , because in this way one absorbs the correlation itself into the test for its existence. Thus, by the same token, if we put $\Gamma(M - \omega_1, \omega_2)$ into $\chi(\omega_1, \omega_2)$

in place of $\Gamma(M, \omega_2)$, as it might be thought in order to take into account the mass loss accompanying the first emission, we find no correlation at all, as can be directly verified.

We see that the tunnelling method leads to a very important conclusion, that the radiation does indeed contain correlations that possibly can carry off the information content of the hole.

2.3. The Hamilton–Jacobi method

Despite the merits of the seminal work by Parikh and Wilczek, we cannot omit to point out a couple of unpleasant features of their null geodesic method, as the fact that: (i) it strongly relies on a very specific choice of (regular-across-the-horizon) coordinates; and (ii) it turns upside down the relationship between Hawking radiation and back-reaction. As regard the former point, it should be clear how irrelevant is, in the spirit of general relativity, the choice of coordinates: being physical observables invariant with respect to the group of diffeomorphisms (the hole temperature is such an observable), there is no reason why Painlevé–Gullstrand coordinates should be favourable with respect to other (equally well-behaved) coordinates. With respect to the latter, we notice that, in the null geodesic description, apparently there cannot be Hawking radiation without back-reaction: watching carefully, however, it is the discovery of Hawking radiation that justifies back-reaction and makes commendable the treatment of Hawking radiation’s self-gravity. The so-called *Hamilton–Jacobi method* can cope with both issues. The intent is to give a particle description of Hawking radiation, ignoring its self-gravitation, under the assumption that the emitted (scalar) particle’s action does satisfy the Lorentzian Hamilton–Jacobi equation. Later we will show that the null geodesic method can do the same job using instead the reduced action.

As it will become clear later, this method applies to any well-behaved coordinate system across the horizon; it generalises beyond the assumption of spherical symmetry; it makes possible to include the study of tunnelling by fermionic particles. Some sceptics about Hawking radiation contest the fact that as soon as the black hole starts radiating, the same assumptions on space-time stationarity drops down, invalidating the whole derivation. Given that the departure from perfect stationarity is ridiculously small, nonetheless, as we shall show in Section 5, the Hamilton–Jacobi method can prove black hole evaporation even for slowly varying, time dependent, space-times.

Finally, we wish to notice that since the methods of tunnelling are intimately related to the physics of (some type of) horizons, we may apply them even to space-times with multiple horizons. In standard computations of Hawking radiation, this is typically a hard task, if not impossible. In fact, it is well known that, as an example, Reissner–Nordström–de Sitter space does not possess regular Euclidean section for general values of mass, electric charge and cosmological constant [47]. This means that for arbitrary values of the parameters, it is not possible to compute the Hawking temperature of event or cosmological horizons by Euclidean continuation, simply because it could be that no Euclidean section of the given Lorentzian space-time exists.

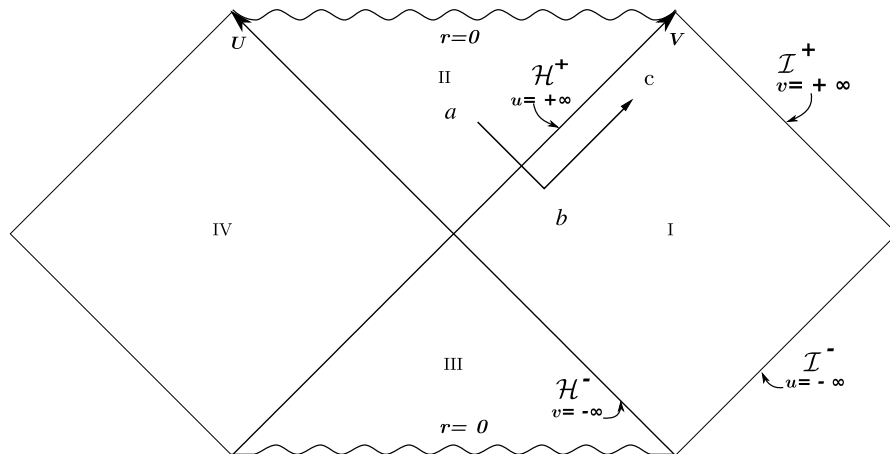


Figure 1. Carter–Penrose diagram of the eternal Schwarzschild black hole. \mathcal{H}^\pm denote future and past horizon; \mathcal{I}^\pm future and past null infinity; $r = 0$ the curvature singularity; v, u represent advanced and retarded null coordinates, respectively. \vec{abc} is a null piecewise continuous path from inside to outside the hole, with \vec{ab} running backward in time (classically forbidden trajectory).

In its minimal formulation, the Hamilton–Jacobi method works according to the following strategy [48]: (a) assume that the tunnelling particle’s action satisfies the relativistic Hamilton–Jacobi equation,

$$g^{\mu\nu} \partial_\mu I \partial_\nu I + m^2 = 0 \quad (2.22)$$

where m^2 represents the invariant mass; (b) reconstruct the whole action I , starting from the symmetries of the problem and the partial derivatives $\partial_\mu I$, by means of

$$I = \int dx^\mu \partial_\mu I, \quad (2.23)$$

where the integration is carried along an oriented, null, curve (like for example \vec{abc} in Figure 1), to which we refer as the *tunnelling path*, with at least one point on the horizon; (c) split the integration along the null path in two pieces, one along the segment crossing the horizon (\vec{ab} in Figure 1), the remaining contribution living in the outer domain of the space-time (\vec{bc} in Figure 1); (d) perform a near-horizon approximation in the first integral and regularise its divergence according to Feynman’s $i\epsilon$ -prescription.

Once the procedure is done, the imaginary part of the classical action, relevant for the tunnelling rate (2.9), reads

$$\text{Im } I = \frac{\pi\omega}{\kappa}, \quad (2.24)$$

where κ and ω are the surface gravity and particle’s energy, respectively.

As we have already stressed, Hawking radiation is a semi-classical result of quantum field theory in curved space-time. In the spirit of tunnelling, we allow particles to travel along classically forbidden trajectories by starting just behind the horizon onward to infinity. Thus, particles must travel back in time, since the horizon is locally to the future

of the static external region. For this reason, we must implement coordinates well defined through regions I and II , a requirement which automatically discards Schwarzschild-like or isotropic-like coordinates (it is easy to show that, in Schwarzschild space-time, isotropic coordinates, (t, ρ, θ, ϕ) , defined through $r(\rho) := (1 + M/2\rho)^2\rho$, cover regions I and IV of Figure 1). The classical one-particle action becomes complex as an indication of the classical impossibility of the motion and gives, in this way, an imaginary part to the emission rate of the hole. In this sense, we can say that, of the actual path travelled by the tunnelling particle (*e.g.* \overrightarrow{abc}), only the infinitesimal region across the horizon plays a crucial role in the whole calculation. Even if null classically forbidden curves like \overrightarrow{ab} do not have anything special with respect to other forbidden paths, their choice is preferable to computational purposes. Also, it will become clearer later the reason why we specifically regularise divergent integrals by Feynman’s prescription rather than others. In consideration of our choices, the mass term in Equation (2.22) is irrelevant to the physics of the horizon (Hawking radiation) but, as we shall show in Section 6, under quite general conditions particles masses play an important role in relation to bulk particle creation phenomena.

To show in practice how the Hamilton–Jacobi method works, we are going to derive the emission rate (2.9) and prove the identity (2.24) in the case of a scalar, uncharged, particle tunnelling from a Schwarzschild black hole, in complete analogy with what we did in the previous subsection. This will give us the opportunity to unveil other debated points of the method. Later on we shall free the derivation from any dependence by special coordinate systems.

2.3.1. Painlevé–Gullstrand coordinates In this case the space-time geometry is described by Equation (2.7). Because of Equation (2.6), expressing the relation between Painlevé and Schwarzschild times, the particle energy associated to a Killing observer is expressed by $\omega = -\partial_{t_p} I$ (according to (2.4), the particle energy is $\omega = -p \cdot \xi$ where $p_\mu = -\partial_\mu I$ is the four-momentum). Since we are dealing with a spherically symmetric space-time, we may neglect without fault the angular coordinates. We consider a null direction $(\Delta t_p, \Delta r)$, for which

$$-\left(1 - \frac{r_g}{r}\right) \Delta t_p^2 + 2\sqrt{\frac{r_g}{r}} \Delta t_p \cdot \Delta r + \Delta r^2 = 0. \quad (2.25)$$

In a neighbourhood of the event horizon (*e.g.* \overrightarrow{ab} in Figure 1), the relation between Δt_p and Δr is uniquely fixed to be $\Delta t_p = -\Delta r/2$ since, in this limit, the alternative solution $\Delta t_p = -(1 - \sqrt{r_g/r})^{-1} \Delta r$ diverges in contrast to the physical meaning of the coordinate t_p . It simply means that the segment \overrightarrow{ab} is part of null direction across the horizon. The reduced Hamilton–Jacobi equation

$$-(\partial_{t_p} I)^2 + 2\sqrt{\frac{r_g}{r}} \partial_{t_p} I \partial_r I + \left(1 - \frac{r_g}{r}\right) (\partial_r I)^2 = 0 \quad (2.26)$$

can be written as

$$-\omega^2 - 2\omega \sqrt{\frac{r_g}{r}} \partial_r I + \left(1 - \frac{r_g}{r}\right) (\partial_r I)^2 = 0. \quad (2.27)$$

The imaginary part of the action is

$$\text{Im } I = \text{Im} \int_{a \rightarrow b \rightarrow c} (\partial_r I dr + \partial_{t_p} I dt_p) \quad (2.28)$$

$$= \text{Im} \int_{a \rightarrow b} dr \left(\partial_r I + \frac{1}{2} \omega \right) \quad (2.29)$$

$$= \text{Im} \int_{\curvearrowright} dr \partial_r I . \quad (2.30)$$

Passing from (2.28) to (2.29), we took into consideration that: (i) only the classically forbidden part \overrightarrow{ab} of the whole null path \overrightarrow{abc} contributes to complexifying the action of the tunnelling particle; (ii) as far as we are concerned with the infinitesimal region of intersection between the horizon and the path \overrightarrow{ab} , dt_p and dr are related to each other in the way specified above; (iii) the particle’s energy goes as $-\partial_{t_p} I$. Being ω a real positive constant, the second term in the integrand of (2.29) does not enter the imaginary part of the action so it drops down in passing to (2.30). From now on, integration along the classically forbidden path (e.g. \overrightarrow{ab} or, by extension, \overrightarrow{abc}) will be denoted simply by \int_{\curvearrowright} . It is remarkable that also the integration over the “time” coordinate enters, in general, the reconstruction process of the particle’s action. In this specific case, we see that this “temporal” contribution — being manifestly real — does not affect the result. The next subsection will show a coordinate system where instead the “temporal” contribution vanishes exactly. It should be clear that names such as “temporal” or “spatial” contributions are coordinate dependent terms in no way fundamentally related to the physics of the process, which must be covariant. The general covariance of the tunnelling method will be continuously emphasised as one of the main features.

Inserting the non-manifestly real solution of Equation (2.27) into (2.30), we get

$$\text{Im } I = 2 \text{Im} \int_{\curvearrowright} dr \frac{\omega}{1 - \frac{r_g}{r}} = 2 \text{Im} \int_{\curvearrowright} dr \frac{\omega r}{r - r_g} , \quad (2.31)$$

and regularising the divergent integral according to Feynman’s $i\epsilon$ -prescription, the imaginary part of the action becomes

$$\text{Im } I = 2 \text{Im} \int_{\curvearrowright} dr \frac{\omega r}{r - r_g - i\epsilon} = 4M\pi\omega . \quad (2.32)$$

Strictly speaking ω should be computed on the horizon, because this is how the method of residues works; however this is the same thing as the energy measured at infinity since on the outgoing trajectory ω is conserved. That said, after the known identification $\kappa = 1/4M$, the identity (2.24) is fully recovered. Whether we had used the opposite prescription, $r \rightarrow r + i\epsilon$, a corresponding change of sign of the imaginary part would have resulted. Its meaning will be explained in Section 3.

2.3.2. Eddington–Finkelstein coordinates Let us introduce another reference frame without singularities on r_g originally constructed by Eddington (1924) and Finkelstein (1958), [49]. This frame is fixed to radially moving photons. Since no observer can move together with photons, this new frame is not, strictly speaking, a reference frame.

Nevertheless, this system of test photons proves to be very convenient, [34]. According to (2.1), the equation of motion of a radial incoming photon is $dr/dt = -(1 - r_g/r)$. From the viewpoint of a distant observer, the photon, starting from r_1 at time t_1 , arrives in r ($r_g < r < r_1$) at time

$$t = r_1 - r - r_g \ln \left(\frac{r - r_g}{r_1 - r_g} \right) + t_1. \quad (2.33)$$

This expression can be opportunely re-written as $v = v_1$, where

$$v := t + r_*, \quad r_* := r + r_g \ln \left| \frac{r}{r_g} - 1 \right| \quad (2.34)$$

r_* is the so-called *tortoise coordinate* with v_1 a constant characterising the initial data of the photon at (t_1, r_1) . Because of the logarithm in (2.33), r_* is defined for any $r > 0$. Chosen a set of photons at fixed t , we may label each photon of the set through a number v , which will identify uniquely that photon during its whole motion: v rises to the role of a new null coordinate, usually called *advanced time*. After differentiation of (2.34) and substitution in (2.1), the line element takes the so called Eddington–Finkelstein form,

$$ds^2 = - \left(1 - \frac{r_g}{r} \right) dv^2 + 2 dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.35)$$

Because of (2.34), the Killing vector field $\xi^\mu = \partial_t$ transforms to $\xi'^\mu = \partial_v$ and particle's energy as measured by such (natural) observer is simply $\omega = -\partial_v I$. Favoured by the spherical symmetry of the problem, we may consider only radial motions. Expanding the trajectory along a null direction in a neighbourhood of the horizon (*e.g.* \vec{ab} in Figure 1), gives

$$0 = 2 \Delta v \cdot \Delta r. \quad (2.36)$$

However, along the null path \vec{ab} , the v coordinate is constant, so that

$$\Delta v|_{\searrow} = 0 \quad (2.37)$$

is the right solution. The Hamilton–Jacobi equation,

$$2 \partial_v I \partial_r I + \left(1 - \frac{r_g}{r} \right) (\partial_r I)^2 = 0 \quad (2.38)$$

can be re-written as

$$\left(1 - \frac{r_g}{r} \right) \partial_r I = 2\omega. \quad (2.39)$$

According to the reconstruction assumption,

$$\text{Im } I = \text{Im} \int_{a \rightarrow b \rightarrow c} (\partial_r I dr + \partial_v I dv), \quad (2.40)$$

and thanks to (2.37) and (2.39),

$$\text{Im } I = 2 \text{Im} \int_{\searrow} dr \frac{\omega}{\left(1 - \frac{r_g}{r} \right)} + \int_{b \rightarrow c} (\dots). \quad (2.41)$$

Neglecting the real contribution coming from the classically allowed path \vec{bc} and following the same procedure as in (2.31), we end with the same result as in Equations (2.24) and (2.32).

Among the coordinate systems of a certain importance covering the space-time region across the gravitational radius there is the so-called Lemaître frame. In this frame the Schwarzschild space-time appears as truly dynamical. We shall see how the Hamilton–Jacobi method deals with it in Section 5.2.1.

We end this section pointing out the role played by the local observer. As discussed in [50], the particle’s energy $\omega = -\partial_t I$ given in the two previous examples is obviously the conserved energy as measured by an observer living at infinity. It follows that the Hawking temperature of a Schwarzschild black hole, $T = 1/8\pi M$, is actually the one measured by a Killing observer at infinity. The particle’s energy as detected by a Killing observer at position \mathbf{x}_{ob} is given instead by $\omega_{\text{ob}} = \omega/\sqrt{-g_{00}(\mathbf{x}_{\text{ob}})}$. Comparing the tunnelling probability derived from (2.24) with the thermal distribution as measured by the observer $\beta_{\text{ob}} \omega_{\text{ob}} = 2\pi\omega/\kappa$, we obtain

$$T_{\text{ob}}\sqrt{-g_{00}(\mathbf{x}_{\text{ob}})} = T_{\infty} = \text{constant}, \quad (2.42)$$

which expresses the expected result given by Tolman years ago [51].

2.4. More general static solutions

Up to now we have only considered the Schwarzschild solution in order to keep the discussion as simple as possible and to illustrate the principles involved without unnecessary complications. However more general static solutions are of interest for a variety of reasons. In order to include a broader class, the metric can be written in a diagonal gauge as

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + C(r)^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.43)$$

which describes what Visser [52] termed *dirty black holes*. Black hole solutions are defined by functions $V(r)$ and $W(r)$ having simple and positive zeroes. This is only a necessary condition to have a black hole; we must also require that the domain of outer communication be “outside of the black hole” *i.e.*, it should correspond to values of the radial coordinate larger than the horizon and extending up to spatial infinity. Interesting black holes described by such metrics can be obtained in the Einstein–Maxwell–dilaton coupled system. An example illustrating the feature is the following two-parameter family of solutions

$$ds^2 = -\left(1 - \frac{r_+}{r}\right) dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.44)$$

The dilaton is $\exp(2\varphi) = (1 - r_-/r)^{-1/2}$; the hole has magnetic charge $q_m = 3r_+r_-/16$, horizon radius $r_+ = 2M$ (which defines M) and can be extended to a non singular, geodesically complete solution with horizons and asymptotically flat infinities.

As long as $V \neq W$ in Equation (2.43), it is possible to set $C(r) = r$ by an opportune radius redefinition. But for $V = W$, this is not always the case as can be seen from the following example

$$ds^2 = -\frac{1 - 2M/r}{\sqrt{\Delta}} dt^2 + \frac{\sqrt{\Delta}}{1 - 2M/r} dr^2 + \sqrt{\Delta} r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.45)$$

Here $C(r) := \Delta^{1/2}r$ with $\Delta = 1 + 2Mr^{-1} \sinh^2 \gamma$ and γ a real constant. The metric is asymptotically flat with an event horizon at $r = 2M$ and an electric field due to a charge $Q = M \sinh(2\gamma)/2$. This solution can be obtained from dimensional reduction of a boosted Schwarzschild solution in Kaluza–Klein theory.

The metric (2.43) admits also a Painlevé–Gullstrand representation associated with a redefinition of the time coordinate,

$$t_p = t \pm \int dr \sqrt{\frac{1 - W(r)}{V(r)W(r)}}. \quad (2.46)$$

A simple computation leads to

$$ds^2 = -V(r)dt_p^2 \pm 2\sqrt{\frac{1 - W(r)}{W(r)}} \cdot V(r) dr dt_p + dr^2 + C(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.47)$$

Turning our attention back to (2.43) with $C(r) \equiv r$, by computing the Einstein's tensor one sees that the stress-energy tensor must have the following form

$$T_{\nu}^{\mu} = \text{diag}(-\varrho(r), P(r), S(r), S(r)). \quad (2.48)$$

Thus, in general, the metric (2.43) has continuously distributed sources and is not asymptotically flat. If the space-time we have in mind is such that

$$\begin{aligned} V(r) &= \frac{(r - r_1) \cdots (r - r_s)}{r^s}, \\ W(r) &= \frac{(r - r_1) \cdots (r - r_q)}{r^q}, \quad s \leq q, \end{aligned} \quad (2.49)$$

where $r_1 \leq r_2 \leq \cdots \leq r_s \leq r_q$, then

$$V(r) \cdot W(r) > 0 \begin{cases} \forall r > 0, & s = q, \\ \forall r > r_q, & s < q, \end{cases} \quad (2.50)$$

which means that outside the outermost event horizon $r = r_q$, $V(r)$ and $W(r)$ are both positive functions. In this case from Einstein's equations it is possible to prove that [53]

$$V(r) = W(r) \iff \varrho + P = 0 \quad \text{through space-time} \quad (2.51)$$

which is satisfied only in vacuum space-times with cosmological constant. This condition is completely equivalent to the fact that [54] *in a static space-time*,

$$g_{tt}g_{rr} = -1 \iff T_{\mu\nu}n^{\mu}n^{\nu} = 0, \quad (2.52)$$

for all radial null vectors n^{μ} . Another important fact regarding such dirty black holes is that as a consequence of the dominant energy condition and the Einstein's equations, if $V(r)$ has a simple zero at some $r = r_H$ then $W(r)$ also has a simple zero at the same point (for a proof of this and other properties see [53]).

The metric (2.43) is of interest also because it admits two inequivalent definitions of conserved energy: one is the Killing energy, $\omega = -p_{\mu}\xi^{\mu} = -\partial_t I$; the other is the Kodama energy which uses the vector field $K = \sqrt{W/V} \partial_t$. This vector field comes about because, in spherical symmetry (as spelled out extensively in Section 5), it has

the amazing property that both K^μ as well as $J_\nu = -K^\mu T_{\mu\nu}$ are conserved: $\nabla_\mu K^\mu = 0$, $\nabla_\mu J^\mu = 0$. As a consequence it is possible to define two different notions of temperature, depending on which energy one is using.

As we shall prove in detail in Section 3, even if the metric (2.43) is singular on the horizon, the tunnelling method works. Using for example the Hamilton–Jacobi version, the massless Hamilton–Jacobi equation for radial motion reads

$$-\frac{1}{V(r)}(\partial_t I)^2 + W(r)(\partial_r I)^2 = 0. \quad (2.53)$$

The classical action is given by (plus/minus sign corresponding to outgoing/ingoing particles, respectively)

$$I_\pm = -\omega t \pm \int \omega \frac{dr}{\sqrt{V(r)W(r)}}, \quad (2.54)$$

where $\omega = -\partial_t I$ represents the Killing energy. Assuming the near horizon expansion (as noted above, the occurrence of the same zero in $V(r)$ and $W(r)$ is a theorem, given Einstein’s equations and some energy condition)

$$V(r) = V'(r_H)(r - r_H) + \dots, \quad W(r) = W'(r_H)(r - r_H) + \dots, \quad (2.55)$$

and Feynman’s prescription, we readily obtain

$$\text{Im} \int dI_+ - \text{Im} \int dI_- = \frac{\pi\omega}{\kappa} \quad (2.56)$$

where

$$\kappa = \frac{1}{2} \sqrt{V'(r_H)W'(r_H)} \quad (2.57)$$

is the Killing surface gravity. One may use the Kodama energy, $\omega_K = \sqrt{W/V} \omega$, which on the horizon takes the form $\omega_K = \sqrt{W'(r_H)/V'(r_H)} \omega$ via de l’Hôpital rule; in this case one obtains the so called Hayward’s surface gravity

$$\kappa_H = \frac{1}{2} W'(r_H) \quad (2.58)$$

and the corresponding temperature $T_H = \kappa_H/2\pi$. As we shall see in Section 5, this is the one that can be generalised naturally to dynamical situations, where there are no more Killing vectors in general. Notice however that $\omega/\kappa = \omega_K/\kappa_H$, so that the tunnelling probability is invariant. In Section 4, we shall show two explicit examples of static, spherically symmetric solutions arising in modified theories of gravity where Killing and Hayward temperatures naturally cohabit according to the notion of energy we choose to implement.

With this calculation, we basically end the whole story about tunnelling of neutral scalar particles in static, spherically symmetric black hole space-times. We have seen how the tunnelling picture arose and developed in a variety of different proposals; in Section 3 we shall confront with more technical aspects as the equivalence between so called null geodesic and Hamilton–Jacobi methods and their mathematical foundations. We refer to [55] for a complete review on how the tunnelling method works in more exotic scenarios.

3. Analytic Continuation Arguments

In the previous sections, we have described the Hamilton–Jacobi strategy in a list of four steps, from (a) to (d). It is clear, however, that, at least at first sight, not all of them stay on equal footing: besides some irremissible (*e.g.* postulation of Hamilton–Jacobi equation) or very natural requirements (*e.g.* trajectory splitting as \overrightarrow{abc} into $\overrightarrow{ab} + \overrightarrow{bc}$), we find other less tolerable points. Why, in fact, should it be that particles traveling along classically forbidden trajectories from inside the black hole to outside must follow null paths? And which fundamental principle suggests us to regularise divergent integrals according to one prescription rather than others? By the end of the day, we shall show that — in contrast to the common sense — even the Hamilton–Jacobi equation is an accessory requirement. A certain experience in the field tells us how these points can result in some sense cryptic to the same experts. With the purpose of clarifying some of the points mentioned above, we are going to outline the foundations of the Hamilton–Jacobi method in order to point out what is fundamental and what is only an additional assumption; and comparing, if possible, alternative solutions attempted in literature with present proposals.

3.1. Foundation of Hamilton–Jacobi method

Let us consider the motion of a scalar particle from region *II* to region *I* in the eternal version of Schwarzschild black hole (\overrightarrow{ab} in Figure 2). This motion is classically forbidden since the particle should travel back in time to follow it. Notice that, in general, nothing we can say about the causal nature of the forbidden path. However, if the coordinates of the starting point *a* are displaced to complex values, then such an allowed path exists. Thus, the reader must think to Figure 2 as a section of the complexified Schwarzschild

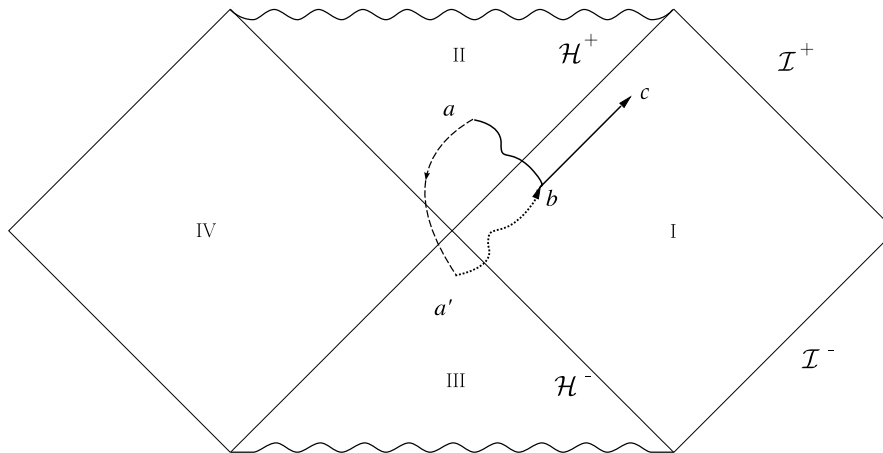


Figure 2. A section of the complexified U, V plane of the eternal Schwarzschild black hole. θ, ϕ coordinates are suppressed. \overrightarrow{abc} is a path from inside to outside the hole, with \overrightarrow{ab} a classically forbidden trajectory and $\overrightarrow{a'b}$ representing its reflected trajectory.

plane with coordinates (θ, ϕ) constant and suppressed. By analytically continuing the

point a in the complexified Schwarzschild space, the amplitude to propagate to c from a real point a in region *II* can be related to the amplitude to propagate to c from a reflected point a' in region *III* [36]. This latter process is just the time-reversed of absorption of a particle by the black hole. In this way, the emission probability for a black hole is related to the probability for it to absorb. All that we need in order to prove the case, is to take as a fundamental assumption the analyticity of the particle action in the complex (U, V) plane as a function of the space-time coordinates of point a [56].

In region *II* (*III*), the Kruskal–Szekeres coordinates U, V are both positive (negative) definite. So, let us rotate U and V from positive to negative values by posing

$$\tilde{U} = e^{i\vartheta} U, \quad \tilde{V} = e^{-i\vartheta} V, \quad \vartheta \in [0, \pi], \quad (3.1)$$

a choice which clearly preserves the invariant product $UV = (1 - r/r_g)e^{r/r_g}$. The integral of dI over the tunnelling path $a \rightarrow b \rightarrow c$ will now be replaced by the integral over the path (say γ) consisting of the semi-circle $a \rightarrow a'$ with ϑ ranging from 0 to π (over which $|U|, |V|$ are constants) plus the integral over the path $a' \rightarrow b \rightarrow c$ which crosses the past horizon (over which ϑ is constant). In formulas

$$\begin{aligned} I &\equiv \int_{\nearrow} dI = \int_{\gamma} (\partial_{\tilde{U}} I d\tilde{U} + \partial_{\tilde{V}} I d\tilde{V}) \\ &= \int_{\nearrow} (\partial_U I dU + \partial_V I dV) - i \int_0^\pi d\vartheta (\tilde{V} \partial_{\tilde{V}} - \tilde{U} \partial_{\tilde{U}}) I, \end{aligned} \quad (3.2)$$

where, this time, the upward arrow denotes integration along the classically allowed trajectory $\overrightarrow{a'bc}$. Since in region *I*, $U = -e^{-\kappa u}$ and $V = e^{+\kappa v}$, where u, v are retarded and advanced time coordinates and κ is the horizon surface gravity, $\kappa = 1/4M$, we have that $\partial_t = \kappa(V\partial_V - U\partial_U)$. Hence we obtain

$$\int_{\nearrow} dI = \int_{\nearrow} dI - \frac{i\pi}{\kappa} \partial_t I = \int_{\nearrow} dI + \frac{i\pi\omega}{\kappa}, \quad (3.3)$$

with $\omega = -p \cdot \xi = -\partial_t I$ the conserved Killing energy of the emitted particle. Taking into consideration only the imaginary part of the action, which is the relevant part for tunnelling purposes, we get

$$-2 \operatorname{Im} \left(\int_{\nearrow} dI - \int_{\nwarrow} dI \right) = -\frac{2\pi\omega}{\kappa}. \quad (3.4)$$

Since the geometry is static, it is left invariant by time inversion \hat{T} :

$$\operatorname{Im} \int_{\nwarrow} dI = \hat{T} \operatorname{Im} \int_{\nearrow} dI = \operatorname{Im} \int_{b \rightarrow a} dI \quad (3.5)$$

where $b \rightarrow a$ is the time inverse of the path $a' \rightarrow b$ (Figure 2). With a justified abuse of terminology, \int_{\nwarrow} will denote also the integration along the time-reversed path. From a physical point of view, this is the path followed by an incoming particle absorbed by the black hole, so that exponentiating (3.4), we can write

$$\Gamma_{\text{em}} = \Gamma_{\text{abs}} e^{-\frac{2\pi\omega}{\kappa}}, \quad (3.6)$$

in agreement with the result of Hartle and Hawking [36] derived by path–integral methods.

The rotation of the (U, V) coordinates in the complex plane, Equation (3.1), has been chosen in the form given because it corresponds to a Wick-like rotation of Schwarzschild time in the lower half complex plane, namely $t \rightarrow t - i\vartheta/\kappa$, which is the analyticity region of positive energy solutions of field equations. In fact:

$$e^{-2\kappa\tilde{t}} =: \frac{\tilde{U}}{\tilde{V}} \stackrel{(3.1)}{=} e^{2i\vartheta} \frac{U}{V} = e^{2i\vartheta} \cdot e^{-2\kappa t} = \exp \left[-2\kappa \left(t - \frac{i\vartheta}{\kappa} \right) \right]. \quad (3.7)$$

One might legitimately ask for the meaning of an anti-Wick rotation, $t \rightarrow t + i\vartheta/\kappa$. In this case, one rotates U (V) clockwise (counter-clockwise) with consequent change of signs in Equation (3.1). In turn, Equation (3.4) changes in favour of

$$-2 \operatorname{Im} \left(\int_{\rightarrow} dI - \int_{\leftarrow} dI \right) = \frac{2\pi\omega}{\kappa}. \quad (3.8)$$

The plus sign at the right hand side of this equation, tells us that the object we are talking about is not a black hole, but rather its time reverse (*white hole*), for which absorption is exponentially suppressed with respect to emission, $\Gamma_{\text{abs}} = \Gamma_{\text{em}} e^{-\frac{2\pi\omega}{\kappa}}$.

The Reissner–Nordström solution It is easy to generalise the present discussion to include charged, spherically symmetric black holes. In this case, Einstein–Maxwell equations can be solved in terms of the Reissner–Nordström metric,

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q}{r^2} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q}{r^2} \right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (3.9)$$

and the Maxwell one-form,

$$A = \frac{Q}{r} dt \quad (3.10)$$

where M and Q represent the black hole mass and electric charge, respectively. Without entering the details of the solution, for which we refer the reader to standard texts (*e.g.* [37,57]), it is enough to say that Reissner–Nordström black hole possesses an outer (inner) event horizon r_+ (r_-) in correspondence of which we can still define a tortoise coordinate r_* ; null retarded and advanced coordinates u, v ; Kruskal like coordinates U, V ; and an outer (inner) surface gravity κ_+ (κ_-). This time, the action to be integrated on the tunnelling path is $dI = dI_0 + qA$, where I_0 is the free action and q is the electric charge of the tunnelling particle. The form A is ill-defined at the horizon, for this reason one usually makes a gauge transformation to a form ${}^\Lambda A = A + d\Lambda$ which is regular there. In our case the analytic continuation takes A away from the horizon so this will actually be unnecessary. Using as above complex (\tilde{U}, \tilde{V}) coordinates, we easily obtain

$$\operatorname{Im} \int_{\rightarrow} dI = \operatorname{Im} \int_{\leftarrow} dI + \frac{\pi}{\kappa_+} (\omega - q\Phi) \quad (3.11)$$

where $\omega = -\partial_t I_0$ is the mechanical energy and $\Phi = Q/r_+$. The quantity $\omega - q\Phi$ is gauge invariant and conserved along the outgoing path from the past horizon to

infinity. This is because for any Killing vector field ξ such that the Lie derivative $(\mathcal{L}_\xi A)_\mu \equiv \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu = 0$, the action I is invariant, to first order in α , under transformation $x \rightarrow x + \alpha \xi$. The Nöther charge associated to this symmetry transformation is represented by $-\xi^\mu (p_\mu - qA_\mu)$, given that p_μ is the (mechanical) momentum of the particle. All these considerations are valid, in particular, for the special choice $\xi = \xi_{(t)} = \partial_t$. Finally

$$2 \operatorname{Im} \left[\int_{\mathcal{X}} (dI_0 + qA) - \int_{\mathcal{X}} (dI_0 + qA) \right] = \frac{2\pi}{\kappa_+} (\omega - q\Phi), \quad (3.12)$$

which reproduces the standard result

$$\Gamma_{\text{em}} = \Gamma_{\text{abs}} e^{-\frac{2\pi}{\kappa_+} (\omega - q\Phi)}. \quad (3.13)$$

Note that this derivation applies equally well to particles with non vanishing angular momentum, since adding a term $\partial_\phi I d\phi$ to the differential dI would not change the imaginary part. Hence equations such as (3.6) and (3.13) are actually true for any angular momentum, though of course the implicit grey body factors do depend on it.

The Kerr solution The observation that $\int \partial_\phi I d\phi$ is real will not be true for rotating black holes. We can extend the calculation to the Kerr solution by noticing that throughout the complex manifold the azimuthal angle must also be rotated in order to keep the metric regular. More precisely, one has to shift $\phi \rightarrow \phi - i\Omega^{\text{H}}\vartheta/\kappa$, where $\Omega^{\text{H}} = a/(r_+^2 + a^2)$ indicates the angular velocity of the horizon and $a = J/M$ (cf. Section 3.2 for more details on the solution). Then, adding the term $\partial_\phi I d\phi$ to the differential dI would produce an imaginary term after a π -rotation, equal to $-i\pi\Omega^{\text{H}}\partial_\phi I/\kappa$. The outgoing trajectory from the past horizon is a classical solution so $\partial_\phi I = J$, the conserved angular momentum. We thus obtain the result

$$\operatorname{Im} \int_{\mathcal{X}} dI = \operatorname{Im} \int_{\mathcal{X}} dI + \frac{\pi}{\kappa} (\omega - \Omega^{\text{H}} J) \quad (3.14)$$

or

$$\Gamma_{\text{em}} = \Gamma_{\text{abs}} e^{-\frac{2\pi}{\kappa} (\omega - \Omega^{\text{H}} J)}. \quad (3.15)$$

Of course in quantum theory the angular momentum is quantised. As is well known the emission and absorption probabilities for particles with energy ω and angular momentum j are related to the Bogoliubov β -coefficients, whose computation is a classical problem involving only the relevant field equations. Unitarity in the space of classical solutions relates them to the transmission coefficient $T_{\omega lm}$ through the potential barrier around the horizon

$$\Gamma_{\text{abs}} \pm \Gamma_{\text{em}} = |T_{\omega lm}|^2 \quad (3.16)$$

where the plus (minus) sign is for fermions (bosons). Together with (3.15) it gives the Fermi–Dirac (Bose–Einstein) spectrum.

Equation (3.6), or its cousins for the charged and Kerr black holes, turns out to be perfectly consistent with equations (2.13) and (2.32) previously obtained, but it has

been derived under the unique assumption of the analyticity of the particle action in the complexified Schwarzschild/Reissner–Nordström/Kerr geometry, and nothing more. The simplicity of this result seems to us particularly important and probably deserves further discussion. In first place, it can be considered as the analogue of the well known “path integral derivation of black hole radiance” due to Hartle and Hawking [36], derived uniquely from the classical action I . In second place, it helps us to understand better how the Hamilton–Jacobi method works. At this regard we immediately see that reference to the Hamilton–Jacobi equation and insistence on considering only null trajectories (on which performing near-horizon expansions) are additional ingredients of the method which have nothing to do with the physical phenomenon of tunnelling. Even if they are not absolutely necessary from a conceptual point of view, their role turns out to be fundamental for computational purposes. As already said, most of the times the analytic continuation does not even exist, since only for special manifolds there is a complexified extension; while considering arbitrary forbidden trajectories as \overrightarrow{ab} in Figure 2 would rule out any exact computation.

As mentioned above, the regularisation scheme typically implemented in the tunnelling method is the Feynman’s prescription, *e.g.* (2.13) and (2.32). At the beginning, we introduced it with no justification, either physical or mathematical, but now it is clear that Feynman’s prescription

$$\int_{\mathcal{Q}} dx \frac{f(x)}{x} \rightarrow \int_{\mathcal{Q}} dx \frac{f(x)}{x - i\epsilon} = i\pi f(0) + \dots$$

where \mathcal{Q} is any interval such that $0 \in \mathcal{Q}$, has to be the preferred since it is the only one consistent with analytic continuation results, when available. From now on, Feynman’s prescription will be adopted systematically and its justification given for granted even in those cases where there is no analytic proof of its validity.

As a last remark, in the previous sections we repeatedly insisted on the importance of using well-behaved coordinates across the horizon. However, even this restriction seems to be less compelling in the light of analytic continuation. Equation (3.4), which is the best achievement of this section, expresses the remarkable fact that it is the difference of $\text{Im} \int_{\leftarrow} dI$ and $\text{Im} \int_{\rightarrow} dI$ which provides the correct ratio between the emission and absorption amplitudes. As far as we are concerned with regular coordinates, only the integration over \overrightarrow{abc} trajectories carries a non-vanishing imaginary part, being the absorption amplitude completely real. Insisting on singular gauges as Schwarzschild coordinates, however, nothing prevents the entering trajectory from picking up an imaginary contribution. It is exemplary, in this sense, the (Sg for) Schwarzschild gauge where both the escaping

$$I_{\text{out}} = \int_{\leftarrow}^{\rightarrow} dI = -\omega t + \int_{\text{Sg}}^{\rightarrow} \frac{\omega r}{r - r_g - i\epsilon} dr = -\omega t + i\pi r_g \omega \quad (3.17)$$

and the classically allowed ingoing trajectory

$$I_{\text{in}} = \int_{\rightarrow}^{\leftarrow} dI = -\omega t - \int_{\text{Sg}}^{\leftarrow} \frac{\omega r}{r - r_g - i\epsilon} dr = -\omega t - i\pi r_g \omega, \quad (3.18)$$

pick up an imaginary contribution providing in this way the correct result. This statement, if true for the Schwarzschild gauge, cannot be taken as a general rule valid for any singular gauge, as the reader may easily convince himself by taking into consideration isotropic coordinates.

3.2. Equivalence of null geodesic and Hamilton–Jacobi methods

At this point of the discussion, it seems natural to ask what relation exists between the aforementioned null geodesic and Hamilton–Jacobi methods. In literature, (see for instance [25]) we find many specific examples suggesting the essential equivalence between the two methods, but nothing resembling a mathematical proof. To this aim, notice that in the null geodesic approach one starts with

$$\text{Im} \int_{\gamma} p_r dr \quad (3.19)$$

which looks non covariant. However we can use the full Liouville one-form $\varpi = p_\mu dx^\mu$ and write the more general expression,

$$\text{Im} \int_{\gamma} \varpi \quad (3.20)$$

which, without the “Im”, is nothing but the reduced action. We can perform the analytic continuation of this integral just as we did for the complete action, first by writing $\varpi = p_U dU + p_V dV$, then rotating (U, V) from zero to π and finally integrating along the rotated curve. In this way, the imaginary part will be $i\pi(U p_V - V p_U)$: but this is $-i\pi\omega/\kappa$, where $\omega = -p_t$ is the Killing energy as measured at infinity. In all we get

$$\text{Im} \int_{\gamma} \varpi = \text{Im} \int_{\gamma} \varpi + \frac{\pi\omega}{\kappa} . \quad (3.21)$$

As another example we consider the line element of the Kerr–Newman solution,

$$\begin{aligned} ds_{\text{KN}}^2 = & \frac{G^2 a^2 \sin^2 \theta}{4\kappa_+^2 \Sigma} \cdot \frac{(r - r_-)(r + r_+)}{(r^2 + a^2)(r_+^2 + a^2)} \left[\frac{\Sigma}{r^2 + a^2} + \frac{\Sigma_+}{r_+^2 + a^2} \right] (U^2 dV^2 + V^2 dU^2) + \\ & + \frac{G(r - r_-)}{2\kappa_+^2 \Sigma} \left[\frac{\Sigma^2}{(r^2 + a^2)^2} + \frac{\Sigma_+^2}{(r_+^2 + a^2)^2} \right] dU dV + \\ & + \frac{G^2 a^2 \sin^2 \theta}{4\kappa_+^2 \Sigma} \cdot \frac{(r + r_+)^2}{(r_+^2 + a^2)^2} (U dV - V dU)^2 + \\ & + \frac{Ga \sin^2 \theta}{\kappa_+ \Sigma (r_+^2 + a^2)} [\Sigma_+(r - r_-) + (r^2 + a^2)(r + r_+)] (U dV - V dU) d\phi_+ \\ & + \Sigma d\theta^2 + \frac{[(r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta] \sin^2 \theta}{\Sigma} d\phi_+ , \end{aligned} \quad (3.22)$$

where U, V are Kruskal-like null coordinates, defined as,

$$|U| = \exp \kappa(r_* - t) , \quad |V| = \exp \kappa(r_* + t) , \quad (3.23)$$

$dr_* = (r^2 + a^2)dr/\Delta$ and the surface gravity κ can refer to κ_+ or κ_- according to the space-time region of interest. The co-rotating angular coordinate ϕ_+ is defined as

$$\phi_+ := \phi - \Omega^H t \quad (3.24)$$

with Ω^H the horizon angular velocity, $G := (r - r_+)/UV$ and as usual $\Sigma := r^2 + a^2 \cos^2 \theta$ and $\Delta := r^2 + a^2 - 2Mr$. This metric is particularly important since in the appropriate limits it comprises the maximal analytic extension of Kerr–Newman, Kerr, Reissner–Nordström and Schwarzschild black holes [34].

The analytic continuation of the coordinates in the Euclidean section requires

$$t \rightarrow t - i\vartheta/\kappa \quad \text{and} \quad \phi \rightarrow \phi - i\Omega^H\vartheta/\kappa, \quad \vartheta \in [0, \pi] \quad (3.25)$$

where from now on, we consider $\kappa \equiv \kappa_+$. As seen before, this is the same as requiring (3.1) and sending $\phi_+ \rightarrow \phi_+$:

$$\text{Im} \int_{\mathcal{A}} \varpi = \text{Im} \int^{\mathcal{A}} \varpi + \pi(U_{p_U} - V_{p_V}). \quad (3.26)$$

But $U_{p_U} - V_{p_V} = -\kappa^{-1}p_t = \kappa^{-1}\hat{\omega}$, where $\hat{\omega} = -\partial_t I$ represents the energy measured by a co-rotating observer. From its same definition, $d\phi = d\phi_+ + \Omega^H dt$, so to get $\hat{\omega} = \omega - (\partial\phi/\partial t)\partial_\phi I = \omega - \Omega^H J$, with J the black hole angular momentum. We end with our main formula, Equation (3.4),

$$-2 \text{Im} \left(\int_{\mathcal{A}} \varpi - \int^{\mathcal{A}} \varpi \right) = -\frac{2\pi}{\kappa} (\omega - \Omega^H J) \quad (3.27)$$

which proves the equivalence of a *generalized null geodesic method* with Hamilton–Jacobi ansatz for a very general class of stationary, axisymmetric, charged black holes. In retrospect, this could not be otherwise because the Hawking effect is an energy conserving process, so that the reduced action is all one needs in a stationary geometry.

3.3. *Alternative proposals*

The tunnelling method has been explored by many authors and counts a considerable literature which is all but monolithic in presenting the subject. The large number of viewpoints is probably the best indication of the non-trivial nature of the topic. Conversely, it has been also the origin for a certain confusion [58] which made difficult to export its ideas outside the community of experts in the field.

In this sense, it is paradigmatic to consider the so called “factor-two problem” [59], actually a fake problem consisting in the fact that a naive application of the tunnelling method may end with a black hole temperature which is twice the expected result. Roughly speaking, the troubles typically arise when one tries to adopt singular coordinates (like Schwarzschild) and uncritically assumes that the absorption amplitude is real. Then, as it is clear from Equation (3.17), one ends with a temperature equal to $1/4\pi M$. As we have seen in Equation (3.18), this is simply due to the fact that we are missing an equal and opposite contribution coming from the absorbed particle which, entered the master Equation (3.4), provides indeed the correct temperature.

Mitra, for example, circumvents this bug in the following way [60]. The Hamilton–Jacobi equation is a differential equation for the action of both the ingoing and outgoing particle. Because of the staticity and spherical symmetry, we can solve the action in terms of the standard *ansatz* for the separation of variables to get, $I_{\text{out,in}} = -\omega t \pm i\pi\omega r_g + \dots$. It is worth to stress that in general, from reconstruction of the action can arise some constant of integration \mathcal{C} . On the other hand, it was argued, the classical theory of black holes tells us that an incoming particle is absorbed with probability equal to one. In order to match these apparently contradictory facts, we use the freedom of choosing the integration constant \mathcal{C} to impose the classical constraint on the absorption probability, that is $\mathcal{C} = \pi\omega r_g$, so that $\text{Im } I_{\text{out}} = 2\pi r_g \omega$ and $\Gamma_{\text{em}} \propto \exp(-8\pi M\omega)$, as expected.

However, if we try to repeat the same argument for the Schwarzschild geometry in isotropic coordinates,

$$ds^2 = - \left(\frac{1 - M/2\rho}{1 + M/2\rho} \right)^2 dt^2 + \left(1 + \frac{M}{2\rho} \right)^4 [d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)] , \quad (3.28)$$

where this time the horizon is located at the minimal two-sphere $\rho_0 = M/2$, the in-out particle action derived from the Hamilton–Jacobi equation reads

$$\begin{aligned} I_{\text{in}} &= (-\omega t + \mathcal{C}) - i4\pi M\omega \\ I_{\text{out}} &= (-\omega t + \mathcal{C}) + i4\pi M\omega , \end{aligned} \quad (3.29)$$

and the classical constraint $\text{Im } I_{\text{in}} = 0$ fixes $\text{Im } \mathcal{C} = 4\pi M\omega$ implying a Hawking temperature of $1/16\pi M$ which, this time, is a half of the correct, well-known, result.

A relevant achievement in the discussion of the “factor-two problem” has been pursued by Angheben *et al* [23] and Stotyn *et al* [50], who showed that correctly using the theory of distributions in curved static space-time it is possible to recover the correct result for the Hawking temperature even working in singular gauges. In fact, the non locally integrable function $1/r$ does not lead to a covariant distribution $1/(r \pm i\epsilon)$; where instead, Feynman’s regularisation of an invariantly defined distance makes unnecessary to work with the complex action of ingoing particles. The natural choice in substitution of $1/r$ is represented by the proper spatial distance, as defined by the spatial metric, $d\sigma^2 \equiv g_{mn}dx^m dx^n$ ($m, n = 1, 2, 3$), a quantity that is invariant under re-definitions of the spatial coordinate system and, more generally, with respect to the subgroup of the gauge group consisting of time re-calibrations and spatial diffeomorphisms, defined by

$$\begin{cases} t' = t'(t, x^i) \\ x'^j = x'^j(x^i) . \end{cases} \quad (3.30)$$

We consider the previously discussed spherically symmetric metric in diagonal gauge,

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \quad (3.31)$$

where we suppose $V(r)$ and $W(r)$ to have both a simple pole in correspondence of the horizon r_{H} ,

$$V(r) = V'(r)(r - r_{\text{H}}) + \dots , \quad W(r) = W'(r)(r - r_{\text{H}}) + \dots . \quad (3.32)$$

The proper distance reads,

$$\sigma = \int \frac{dr}{\sqrt{W(r)}} = \frac{2}{\sqrt{W(r_{\text{H}})}} \sqrt{r - r_{\text{H}}}, \quad (3.33)$$

so the classical action of the outgoing massless particle is

$$I_{\text{out}} = -\omega t + \int_{\rightarrow} dr \frac{\omega}{\sqrt{V(r)W(r)}} = -\omega t + \frac{2}{\sqrt{V'(r_{\text{H}})W'(r_{\text{H}})}} \int_{\rightarrow} \omega \frac{d\sigma}{\sigma}. \quad (3.34)$$

The integral is still divergent as soon as $\sigma \rightarrow 0$, namely the horizon is reached, but now the prescription corresponding to the Feynman's propagator $1/\sigma \rightarrow 1/(\sigma - i\epsilon)$ selects the correct imaginary part for the classical action,

$$I_{\text{out}} = \frac{2\pi\omega i}{\sqrt{V'(r_{\text{H}})W'(r_{\text{H}})}} + (\text{real contribution}). \quad (3.35)$$

From comparison of (3.35) with (2.24), we soon read off the horizon surface gravity κ for the dirty black hole (3.31),

$$\kappa = \frac{\sqrt{V'(r_{\text{H}})W'(r_{\text{H}})}}{2}, \quad (3.36)$$

which is amazingly in agreement with the result obtained through the conical singularity method [61], cf. Section 4. From a qualitative point of view, this coincidence can be ascribed to the fact that both the conical singularity and the proper distance methods refer the energy ω to the same Killing observer. However, we notice that, without any further assumption, a dirty black hole space-time is not asymptotically flat, so it is not completely clear how to normalise in general the Killing vector $\xi_{(t)}^{\mu}$.

Can we say that the factor-two problem has been solved? We have seen that, at least confining to the evaluation of the Hawking temperature of a Schwarzschild black hole in Schwarzschild coordinates, there are two satisfactory ways to circumvent the factor-two problem arisen because of the use of singular charts: the former, essentially based on the classical requirement that the absorption probability be unity; the latter, related to an appropriate way of dealing with distributions in curved space-times. However, the combination of these techniques as made in [50] has the deleterious drawback of making the temperature again a half of the correct result. According to these authors, the point is that to apply the Feynman prescription, the proper distance should be extended to negative values, which seems geometrically impossible. The authors then note that to recover the standard result $T_{\text{H}} = 1/8\pi M$, one ought to use an alternative contour of integration when dealing with proper distance formulation, $\int d\sigma/\sigma$, namely the quarter-contour of Figure 3 (left) rather than the standard semi-circular contour of Figure 3 (right).

Another point of view is the one advocated in [62] where the authors are among the first to point out the possible role of a ‘‘temporal contribution’’ to solve the factor-two problem, at least in the context of singular gauges. In fact, temporal and spatial contributions are on equal footing in the Hamilton–Jacobi method as formulated in Section 2.2, since it is covariant. For example, the master equation (3.4) does not

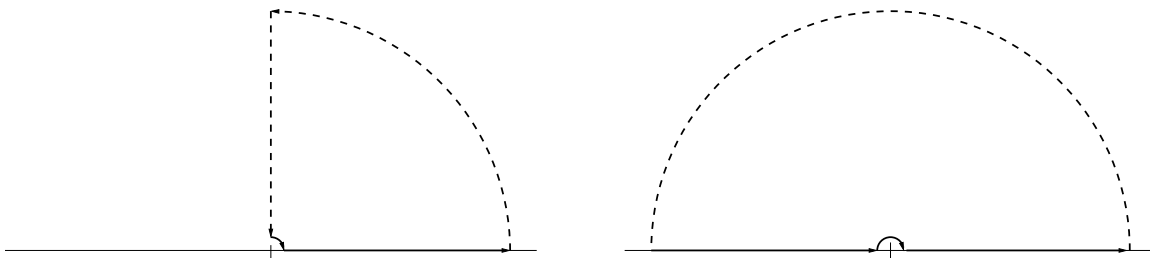


Figure 3. Standard semi-circular contour of integration in the complex plane of $\int d\sigma/\sigma$ and the quarter-contour proposed in [50].

distinguish at all between “time” or “space” coordinates, something which makes perfectly sense according to the principles of general relativity. On the other hand, the null geodesic method as originally formulated does not involve a time integration, but appears non covariant and also non canonically invariant. Along the line of thought initially formulated by [63], Singleton *et al* make use of closed integration paths, necessary to ensure invariance of the emission rate under canonical transformations. To all the extents, the integration over the closed circuit $\oint p_r dr$ corresponds to taking into consideration the imaginary contribution related to the action of both the incoming and outgoing particle, exactly as in (3.18): $\oint \rightarrow \int_{\leftarrow} - \int_{\rightarrow}$. The null geodesic method as formulated in Section 3.2, evidently, is covariant as well as invariant under canonical transformations in phase space. Hence, even acknowledging a certain loss of importance of canonical transformations in the passage from classical to quantum theories, this is certainly an interesting point. On the other hand, since not all the canonical transformations correspond to unitary transformations in the quantum theory, perhaps a more urgent issue is not the canonical invariance of the decay rate, but the covariance of the result – as demanded by general relativity – and the unitarity of the process – as demanded by quantum mechanics – something that nobody knows how to solve.

As stressed above, the Hamilton–Jacobi method was born also, surely not only, by the desire of disentangling Hawking radiation from back-reaction, in the sense that it is the existence of Hawking radiation that founds the problem of back-reaction, not vice versa. Given that the Hamilton–Jacobi method can prove black hole radiance without taking into account the back-reaction, it is natural to ask how the back-reaction can be incorporated in this method. As shown by Medved and Vagenas [64, 65], a tunnelled particle of instantaneous energy ω effectively “sees” a space-time metric of the form (3.31) in which the horizon radius $r_H = r_H(M - \omega)$, M being the original Misner–Sharp black hole mass. Because of quantum uncertainty principle, it seems too crude the approximation of a discontinuous jump from M to $M - \omega$ for the black hole mass. Rather, the particle action corrected by energy conservation, turns out to be given by

$$I'_{\text{out}} = \int_0^\omega d\omega' I_{\text{out}} \Big|_{M \rightarrow M - \omega'} \stackrel{(3.35)}{=} \int_0^\omega d\omega' \frac{2\pi i}{\sqrt{V'_H(M - \omega') W'_H(M - \omega')}} \quad (3.37)$$

where the subscript H is a reminder that we are evaluating this quantity at the black

hole horizon. The quantum corrected tunnelling probability deduced from (3.37) turns out to be that expected from statistical mechanics and in perfect agreement with the null geodesic method of Section 2.1, namely $\Gamma_{\text{em}} \sim \exp \Delta S_{\text{BH}}$. A somewhat different treatment of this problem with equivalent results, within the Hamilton-Jacobi method, can also be found in [66].

As our last point we report on a case which has been put forward in [67, 68] and further extended in [69–71], regarding corrections of higher order in \hbar to the Hawking temperature. The argument starts from the semi-classical expansion

$$I = I_0 + \sum_{n \geq 1} \hbar^n I_n \quad (3.38)$$

and the assumption that the field obeying the Klein–Gordon equation in the black hole background is as usual $\varphi = \exp(iI/\hbar)$. It follows from this that all I_n obey the same partial differential equation, giving rationale to the further assumption that $I_n \propto I_0$ for every n . Therefore, writing $I_n = c_n I_0$, the action takes the form

$$I = \left(1 + \sum_{n \geq 1} c_n \hbar^n \right) I_0 \quad (3.39)$$

where I_0 , to fix ideas, obeys the Hamilton–Jacobi Equation (2.53) and $\{c_n\}_n$ remain unpredictable quantities. It is clear that the tunnelling probability $\Gamma = \exp(-2\text{Im}I)$ will predict a modified temperature

$$T = \frac{T_0}{\left(1 + \sum_{n \geq 1} c_n \hbar^n \right)} \quad (3.40)$$

where

$$T_0 = \frac{1}{4\pi} \sqrt{V'(r_{\text{H}})W'(r_{\text{H}})} = \frac{\kappa}{2\pi} \quad (3.41)$$

is the uncorrected black hole temperature associated to the Killing surface gravity. Obviously this is not changed by simply passing to Kodama–Hayward surface gravity and energy. The procedure looked very suspicious to several authors [72], who did not find any higher order correction neither from scalar emission nor for bosons or fermions. The contradiction was settled by Yale [73], who noted that the result (3.40) assumes an incorrect definition of energy, namely $\omega_0 = -\partial_t I_0$, while Hamilton–Jacobi theory would require $\omega = -\partial_t I$ instead. Then, from (3.39)

$$\omega = \left(1 + \sum_{n \geq 1} c_n \hbar^n \right) \omega_0 \quad (3.42)$$

so that

$$T = \frac{\omega}{2\text{Im}I} = \frac{\omega_0}{2\text{Im}I_0} = T_0 \quad (3.43)$$

and all supposed corrections disappear. Of course, one should also allow complex solutions of the array of semi-classical equations, so even the assumption $I_n \propto I_0$ for all n is really unjustified.

In this section we have taken into consideration alternative proposals to the null geodesic and Hamilton–Jacobi methods outlined in Section 2.1 and 2.2. Without claim for completeness, we have tried to point out their salient features, referring to original papers for details. We cannot deny that all of them contain small pieces of the complete picture, but at the same time the persistent lack of manifest general covariance is to be considered a defect, demanding a suitable reformulation. Exactly this was our desire in writing this section.

4. Dirty Black Holes in Higher Derivative Theories of Gravity

Since the discovery by Riess and Perlmutter and respective collaborators (cf. [61] and references therein) that the universe is in an accelerating epoch, the dark energy issue has become probably the most ambitious field of research in fundamental physics. That the dark energy fluid has an equation of state index w very close to minus one, represents an important point in favour of those who propose to explain dark energy in terms of a cosmological constant, Λ . Still, a non-vanishing cosmological constant does not exhaust the range of models that have been proposed so far in order to solve the aforementioned issue. This is justified, in part, by the whole sort of well-known problems raised by the existence of a strictly positive cosmological constant. On the other hand, it is well accepted the idea according to which general relativity is not the ultimate theory of gravity, but an extremely good approximation valid in the present day range of detection. It basically comes from this viewpoint the input to so-called modified theories of gravity which nowadays enjoy great popularity. Without any claim for unification, such models propose to change the Einstein–Hilbert Lagrangian to a more general form able to reproduce the same general relativity tests on solar distance scales and further justify both inflationary and current acceleration of the universe.

The arena of feasible models is in principle infinite while departures from Einstein’s theory are most of the times all but minimal. Of crucial interest is, of course, the existence and, if this is the case, the properties of black holes in modified gravities. It is quite easy to find the conditions allowing the existence of de Sitter–Schwarzschild black holes. Here, we are interested in non-trivial and static black holes solutions. However, the number of exact non-trivial static black hole solutions so far known in modified theories of gravity is extremely small: just two, both spherically symmetric. They have been obtained by Barrow and Clifton [74] in a modified theory of the type $f(R) = R^{1+\delta}$ with δ a small real parameter; and by Deser, Sarioglu and Tekin [75] by adding to Einstein–Hilbert Lagrangian a non-polynomial contribution of the type $\sqrt{C^2}$, with $C_{\alpha\beta\gamma\delta}$ being the Weyl tensor. These black hole solutions are not expected to share the same laws of their Einsteinian counterparts: for this reason we refer to them as dirty black holes. Some of the physical quantities one would like to address to dirty black holes are their mass, the horizon entropy, their temperature and so on. Thanks to the large amount of work carried over in the last decade, we can firmly say that the issue of entropy and temperature of dirty black holes represents a well posed problem [76]. However, with regard to the mass issue, all considerations still lay on a much less precise ground.

In Section 4.1, we are going to derive the Deser *et al* solution: emphasis will be put on temperature and entropy for this class of black holes. The same will be done in Section 4.2 for the Clifton–Barrow model. Finally, in Section 4.3 the issue of mass for these classes of black holes is taken on and discussed with reference to recent literature.

4.1. The Deser–Sarioglu–Tekin solution

The Deser–Sarioglu–Tekin solution [75] derives from the action

$$I_{\text{DST}} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R + \sqrt{3}\sigma \sqrt{C^2} \right) + \text{Boundary Term} \quad (4.1)$$

where σ is a real parameter and $C^2 := C_{\alpha\beta}{}^{\gamma\delta} C_{\gamma\delta}{}^{\alpha\beta}$ is the trace of the Weyl tensor squared. Looking for static, spherically symmetric solutions of the type,

$$ds^2 = -a(r)b(r)^2 dt^2 + \frac{dr^2}{a(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (4.2)$$

the action (4.1) becomes

$$I_{\text{DST}}[a(r), b(r)] = \frac{1}{2} \int dt \int_0^\infty dr [(1-\sigma)(ra(r)b'(r) + b(r)) + 3\sigma a(r)b(r)]. \quad (4.3)$$

Imposing the stationarity condition $\delta I[a(r), b(r)] = 0$ gives the equations of motion for the unknown functions $a(r)$ and $b(r)$.

$$\begin{aligned} (1-\sigma)rb'(r) + 3\sigma b(r) &= 0 \\ (1-\sigma)ra'(r) + (1-4\sigma)a(r) &= 1-\sigma. \end{aligned} \quad (4.4)$$

According to σ , the space of solutions of (4.4) can be different, in particular:

- $\sigma = 0$ corresponds to Einstein–Hilbert action. In fact, $a(r) = 1 - \frac{\hat{c}}{r}$ and $b(r) = \hat{k}$ and for \hat{c}, \hat{k} positive constants, the Schwarzschild solution of general relativity is recovered;
- $\sigma = 1$: only the trivial, physically unacceptable, solution $a(r) = 0 = b(r)$ exists;
- $\sigma = \frac{1}{4}$: then, for some positive constants \tilde{k} and r_0 :

$$a(r) = \ln\left(\frac{r_0}{r}\right) \quad \text{and} \quad b(r) = \frac{\tilde{k}}{r}; \quad (4.5)$$

- In all other cases, the general solution to (4.4) turns out to be

$$a(r) = \frac{1-\sigma}{1-4\sigma} - cr^{-\frac{1-4\sigma}{1-\sigma}} \quad \text{and} \quad b(r) = \left(\frac{r}{k}\right)^{\frac{3\sigma}{\sigma-1}}, \quad (4.6)$$

for some positive constants c, k .

The constants \hat{k}, \tilde{k} and k appearing in $b(r)$ are removable by time re-scaling. Notice also that, in (4.5), g_{00} and g_{11} go to zero as $r \rightarrow \infty$ so that the model is unphysical. For this reason, we shall mainly concentrate on the solution (4.6) parametrized by all the $\sigma \notin \{0, 1, 1/4\}$.

In order to treat (4.6), let us introduce the parameter $p(\sigma) := \frac{1-\sigma}{1-4\sigma}$ so that the metric becomes

$$ds^2 = -(p - cr^{-\frac{1}{p}}) \left(\frac{r}{k}\right)^{2\left(\frac{1-p}{p}\right)} dt^2 + \frac{dr^2}{(p - cr^{-\frac{1}{p}})} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.7)$$

For $p < 0$, or $\frac{1}{4} < \sigma < 1$, $a(r) = -(|p| + cr^{\frac{1}{|p|}}) < 0$ for all r , that is, the parameter region $\frac{1}{4} < \sigma < 1$ needs to be excluded to preserve the metric signature. As regard the asymptotic behaviour of (4.7), we see that:

- for $p > 1$ or $0 < \sigma < \frac{1}{4}$, we have that $g_{00} \rightarrow 0$ and $g_{11} \rightarrow \frac{1}{p}$ as $r \rightarrow \infty$;
- for $0 < p < 1$ or $\sigma \in (-\infty, 0) \cup (1, +\infty)$, we have that $g_{00} \rightarrow \infty$ and $g_{11} \rightarrow \frac{1}{p}$ as $r \rightarrow \infty$.

As noted by Deser *et al.* the fact that the asymptotics of g_{00} and g_{11} differ means that the equivalence principle is violated: something which is intimately related with the difficulty of defining a “mass” in this theory [75].

Looking at the solution (4.7), we see that the hypersurface $r = r_H := (c/p)^p$ defined by the condition $a(r_H) = 0$ behaves as a Killing horizon with respect to the time-like Killing vector field ξ^α . To prove this, let us define a complex null tetrad $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ (an overbar denoting complex conjugation) for the metric (4.7) according to the following rules [34]:

- (i) l^μ is s.t. on the horizon $l_H^\mu \equiv \xi^\mu$;
- (ii) The normalization conditions hold $l \cdot n = -1$ and $m \cdot \bar{m} = 1$;
- (iii) All the other scalar products vanish.

Since the metric (4.7) is not asymptotically flat, it is not clear at all what is the right normalization for ξ^μ . Assuming $\xi^\mu = \lambda \partial_t^\mu$, $\lambda \in \mathbb{R}^+$, it's not difficult to see that

$$\begin{aligned} l^\mu &= (\lambda, \lambda a(r)b(r), 0, 0) , \\ n^\mu &= \left(\frac{1}{2\lambda a(r)b(r)^2}, -\frac{1}{2\lambda b(r)}, 0, 0 \right) , \\ m^\mu &= \left(0, 0, \frac{i}{\sqrt{2}r}, \frac{1}{\sqrt{2}r \sin \theta} \right) , \end{aligned} \quad (4.8)$$

satisfy the list of conditions to form a complex null tetrad. As a consequence, for example, the metric can be re-written as $g_{\mu\nu} = -2l_{(\mu}n_{\nu)} + 2m_{(\mu}\bar{m}_{\nu)}$. The null expansions are, by definition,

$$\begin{aligned} \theta_- &:= \nabla_\mu n^\mu + n^\mu l_\nu \nabla_\mu n^\nu + n_\nu l^\mu \nabla_\mu n^\nu = -\frac{1}{\lambda r b(r)} , \\ \theta_+ &:= \nabla_\mu l^\mu + l^\mu n_\nu \nabla_\mu l^\nu + l_\nu n^\mu \nabla_\mu l^\nu = \frac{2\lambda}{r} a(r)b(r) . \end{aligned} \quad (4.9)$$

Thus, in-going light rays always converge ($\theta_- < 0$ for all $r > 0$); out-going light rays, instead, focus inside the horizon ($\theta_+ < 0$ as $r < r_H$), diverge outside it ($\theta_+ > 0$ as $r > r_H$) and run paralely at the horizon ($\theta_+|_H = 0$). When they are slightly perturbed in the in-direction (that is, along n), the out-going null ray is absorbed inside the horizon r_H as it is confirmed by the fact that the in-going Lie derivative $\mathcal{L}_n \theta_+|_H = -1/r_H^2 < 0$ is everywhere negative. Computing the convergence ($\varrho := -m^\mu \bar{m}^\nu \nabla_\nu l_\mu$) and the shear ($\varsigma := -m^\mu m^\nu \nabla_\nu l_\mu$) of the null congruences at the horizon we can immediately check they vanish, as expected for any Killing horizon. The Killing surface gravity

$$\kappa_H := -l^\mu n^\nu \nabla_\mu l_\nu|_H = \lambda a'(r)_H b(r)_H , \quad (4.10)$$

turns out to depend by the normalization of the Killing vector ξ^μ . In order to fix λ , we may implement the conical singularity method. To this aim, let us start by the Euclidean metric

$$ds_E^2 = +\frac{dr^2}{W(r)} + V(r)d\tau^2 + r^2d\Omega^2, \quad (4.11)$$

where we suppose that both $V(r)$ and $W(r)$ have a structure like

$$V(r) = (r - \tilde{r})v(r) \quad \text{and} \quad W(r) = (r - \tilde{r})w(r), \quad (4.12)$$

with $v(r), w(r)$ regular for $r > \tilde{r}$. \tilde{r} may be identified with some type of horizon close to which we are interested in the behaviour of the metric.

$$r - \tilde{r} \equiv \zeta x^2, \quad (4.13)$$

with ζ a constant we are going to fix very soon.

$$\begin{aligned} ds_E^2 &= \frac{1}{w(r)} \left[\frac{dr^2}{r - \tilde{r}} + (r - \tilde{r})v(r)w(r)d\tau^2 \right] + (\tilde{r} + \zeta x^2)^2 d\Omega^2 \\ &\stackrel{x \ll 1}{\approx} \left(\frac{4\zeta}{w(\tilde{r})} dx^2 + \zeta v(\tilde{r})x^2 d\tau^2 \right) + \tilde{r}^2 d\Omega^2. \end{aligned} \quad (4.14)$$

Let us choose $\zeta = w(\tilde{r})/4$, the Euclidean metric takes the form

$$ds_E^2 \approx dx^2 + x^2 d \left(\frac{\sqrt{v(\tilde{r})w(\tilde{r})}}{2} \tau \right)^2 + \tilde{r}^2 d\Omega^2, \quad x \ll 1. \quad (4.15)$$

(4.15) shows that close to the horizon ($r \approx \tilde{r}$ or $x \ll 1$) the metric factorizes into $\mathcal{K}_2 \times \mathbb{S}_{\tilde{r}}^2$: \mathcal{K}_2 being the metric of flat two-dimensional manifold on behalf of identifying x with the polar distance and τ with the angular coordinate. However, \mathcal{K}_2 is regular if and only if

$$\frac{\sqrt{v(\tilde{r})w(\tilde{r})}}{2} \tau \sim \frac{\sqrt{v(\tilde{r})w(\tilde{r})}}{2} \tau + 2\pi \quad (4.16)$$

or, in other words,

$$\tau \sim \tau + \frac{4\pi}{\sqrt{v(\tilde{r})w(\tilde{r})}} \equiv \tau + \beta. \quad (4.17)$$

β representing the (unique) τ -period which allows to impose a smooth flat metric on \mathbb{R}^2 . In quantum field theory, the KMS propagator exhibits a periodicity in time when the system is at finite temperature. The period of the compactified time, β , is directly related to the temperature of the system in Lorentzian signature, through ($k_B = 1$)

$$T = \frac{1}{\beta}. \quad (4.18)$$

If we assume the standard Hawking temperature formula, $T = \kappa_H/2\pi$, the period β in (4.17) can be re-written according to

$$\kappa_H = \frac{\sqrt{V'(\tilde{r})W'(\tilde{r})}}{2}, \quad (4.19)$$

which for the metric (4.2) reads $\kappa_{\text{H}} = a'(r)_{\text{H}} b(r)_{\text{H}}/2$. Comparison between the latter and (4.10) fixes the normalization of the Killing vector ξ^μ to be $\lambda = 1/2$. What is most important to us is that $\kappa_{\text{H}} \neq 0$, so that we may conclude that the Killing horizon is of the bifurcate type. As we already know (and further discussion will follow in the next Section), this is not the unique surface gravity which can be defined for a generic spherically symmetric static black hole, but it is the one compatible with Wald's entropy formula.

Given these preliminary remarks, we are now in the position to apply Wald's argument [76] to derive the black hole entropy associated to the Killing horizon of the solution (4.7). The explicit calculation of the black hole entropy S_{W} of the horizon $r = r_{\text{H}} = (c/p)^p$ is provided by the formula

$$S_{\text{W}} = -2\pi \oint_{\substack{r = r_{\text{H}} \\ t = \text{const}}} \left(\frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\gamma\delta}} \right)^{(0)} \hat{\epsilon}_{\alpha\beta} \hat{\epsilon}_{\gamma\delta} \sqrt{h_{(2)}} d\theta d\phi, \quad (4.20)$$

where $\mathcal{L} = \mathcal{L}(R_{\alpha\beta\gamma\delta}, g_{\alpha\beta}, \nabla_\alpha R_{\beta\gamma\delta\epsilon}, \dots)$ is the Lagrangian density of any general theory of gravity, in the specific case,

$$\mathcal{L}(R_{\alpha\beta\gamma\delta}, g_{\alpha\beta}, \nabla_\alpha R_{\beta\gamma\delta\epsilon}, \dots) = \frac{1}{16\pi} (R + \sqrt{3}\sigma\sqrt{C^2}). \quad (4.21)$$

The hatted variable, $\hat{\epsilon}_{\alpha\beta}$, is the binormal vector to the (bifurcate) horizon: it is antisymmetric under the exchange of $\alpha \leftrightarrow \beta$ and normalized so that $\hat{\epsilon}_{\alpha\beta}\hat{\epsilon}^{\alpha\beta} = -2$. For the metric (4.2), the binormal turns out to be

$$\hat{\epsilon}_{\alpha\beta} = b(r)(\delta_\alpha^0 \delta_\beta^1 - \delta_\alpha^1 \delta_\beta^0). \quad (4.22)$$

The induced volume form on the bifurcate surface $r = r_{\text{H}}$, $t = \text{constant}$ is represented by $\sqrt{h_{(2)}} d\theta d\phi$, where, for any spherically symmetric metric, $\sqrt{h_{(2)}} = r^2 \sin\theta$ and the angular variables θ, ϕ run over the intervals $[0, \pi]$, $[0, 2\pi)$, respectively. Finally, the superscript (0) indicates that the partial derivative $\delta\mathcal{L}/\delta R_{\alpha\beta\gamma\delta}$ is evaluated on shell. The variation of the Lagrangian density with respect to $R_{\alpha\beta\gamma\delta}$ is performed as if $R_{\alpha\beta\gamma\delta}$ and the metric $g_{\alpha\beta}$ are independent. In the specific case, equation (4.20) becomes

$$S_{\text{W}} = -8\pi \mathcal{A}_{\text{H}} b^2(r_{\text{H}}) \left(\frac{\delta \mathcal{L}}{\delta R_{0101}} \right)^{(0)}, \quad (4.23)$$

with \mathcal{A}_{H} the area of the black hole horizon. Let us compute the Lagrangian variation,

$$\begin{aligned} 16\pi (\delta \mathcal{L}) &= \delta R + \sqrt{3}\sigma \delta(\sqrt{C^2}) \\ &= \frac{1}{2}(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \delta R_{\alpha\beta\gamma\delta} + \frac{\sqrt{3}\sigma}{2} (C^2)^{-\frac{1}{2}} \delta(C^2). \end{aligned} \quad (4.24)$$

Using the fact that $C^2 = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{3}R^2$, we get,

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta R_{\alpha\beta\gamma\delta}} &= \frac{1}{16\pi} \left\{ \frac{1}{2}(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) + \frac{\sqrt{3}\sigma}{2} (C^2)^{-\frac{1}{2}} \right. \\ &\quad \cdot \left. \left[2R^{\alpha\beta\gamma\delta} - (g^{\alpha\gamma} R^{\beta\delta} + g^{\beta\delta} R^{\alpha\gamma} - g^{\alpha\delta} R^{\beta\gamma} - g^{\beta\gamma} R^{\alpha\delta}) + \frac{1}{3}(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma})R \right] \right\} \end{aligned} \quad (4.25)$$

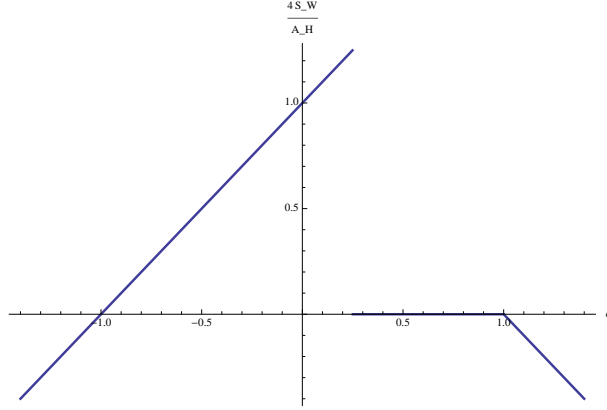


Figure 4. Wald's entropy in units of $\mathcal{A}_H/4$ versus σ parameter for the Deser *et al.* black hole.

In the specific,

$$\left(\frac{\delta\mathcal{L}}{\delta R_{0101}}\right)^{(0)} = \frac{1}{32\pi} \left[g^{00}g^{11} + \frac{\sqrt{3}\sigma}{\sqrt{C^2}} \left(2R^{0101} - g^{00}R^{11} - g^{11}R^{00} + \frac{1}{3}g^{00}g^{11}R \right) \right] \Big|_H. \quad (4.26)$$

Since in general, $\text{Tr}(C^n) = (-\frac{1}{3})^n [2 + (-2)^{2-n}]X^n$, for $n > 0$ and

$$X(r) = \frac{1}{r^2} [r^2 a'' + 2(a-1) - 2ra'] + \frac{1}{rb} [3ra'b' - 2a(b' - rb'')] \quad (4.27)$$

for the metric (4.2), we may write

$$\sqrt{C^2} \Big|_H = \frac{1}{\sqrt{3}} \left[\frac{1}{r^2} [r^2 a'' + 2(a-1) - 2ra'] + \frac{1}{rb} [3ra'b' - 2a(b' - rb'')] \right] \Big|_H. \quad (4.28)$$

Taking together (4.23), (4.26) and (4.28), for both the solutions (4.5) and (4.7), we finally have that the horizon entropy for the Deser *et al.* black hole is

$$S_W = \frac{\mathcal{A}_H}{4} (1 + \varepsilon\sigma), \quad \text{where} \quad \varepsilon := \begin{cases} +1, & \sigma \leq \frac{1}{4} \\ -1, & \sigma > \frac{1}{4} \end{cases}. \quad (4.29)$$

According to (4.29) the entropy predicted by Wald's formula restricts considerably the space of the σ parameter with respect to our previous considerations. In fact, as shown by Figure 4, the entropy of the black hole is positive only as far as $\sigma \in (-1, \frac{1}{4}]$. For $\sigma = -1$, the entropy vanishes suggesting (but we leave this to the level of a speculation) that, for this value of σ , the number of microscopic configurations realizing the black hole is only one. For $\sigma \in (-1, 0)$, the entropy of Deser's black hole is always smaller than its value in general relativity. Notice also that for $\sigma = \frac{1}{4}$, the entropy function is continuous even if the black hole metric changes. However, as pointed out above, such solution is not physical because of its pathological asymptotic behaviour.

En passant, we notice how Wald's entropy could be computed equally well by introducing a new radial co-ordinate ρ such that

$$\rho(r) := \frac{k^{-\frac{1-p}{p}}}{p} r^{\frac{1}{p}} \quad (4.30)$$

the metric (4.7) transforms to

$$ds^2 = -h(\rho)dt^2 + \frac{d\rho^2}{h(\rho)} + q(\rho)d\Omega^2 \quad (4.31)$$

with

$$h(\rho) = \left(\frac{p\rho}{k}\right)^{2(1-p)} \left(p - \frac{c}{pk^{\frac{1-p}{p}}\rho}\right), \quad q(\rho) = (pk^{\frac{1-p}{p}}\rho)^{2p}. \quad (4.32)$$

This time, Wald's entropy (4.29) would follow from

$$S_W = -8\pi \oint_{\substack{r= \\ t= \text{const}}}^{r_H} \left(\frac{\delta\mathcal{L}}{\delta R_{\rho t \rho t}}\right)^{(0)} q(\rho) d\Omega^2. \quad (4.33)$$

4.2. The Clifton–Barrow solution

The Clifton–Barrow solution starts from the following modified-gravity action (evaluated in the vacuum space):

$$I_{CB} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left(\frac{R^{1+\delta}}{\chi}\right). \quad (4.34)$$

Here, δ is a constant and χ is a dimensional parameter. We can choose $\chi = 16\pi G^{1+\delta}$. When $\delta = 0$, we recover the Hilbert–Einstein action of general relativity.

Taking the variation of the action with respect to the metric $g_{\mu\nu}$, we obtain:

$$R_{\mu\nu} = \delta \left(\frac{\partial^\sigma \partial^\tau R}{R} - (1-\delta) \frac{\partial^\sigma R \partial^\tau R}{R^2}\right) \left(g_{\mu\sigma} g_{\nu\tau} + \frac{1+2\delta}{2(1-\delta)} g_{\mu\nu} g_{\sigma\tau}\right). \quad (4.35)$$

Looking for static, spherically symmetric metric of the type,

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + r^2 d\Omega^2, \quad (4.36)$$

we find the Clifton–Barrow solution of Equation (4.35):

$$V(r) = \left(\frac{r}{r_0}\right)^{2\delta(1+2\delta)/(1-\delta)} \left(1 + \frac{c}{r^{(1-2\delta+4\delta^2)/(1-\delta)}}\right), \quad (4.37)$$

$$W(r) = \frac{(1-\delta)^2}{(1-2\delta+4\delta^2)(1-2\delta-2\delta^2)} \left(1 + \frac{c}{r^{(1-2\delta+4\delta^2)/(1-\delta)}}\right), \quad (4.38)$$

c and r_0 being dimensional constants.

In a similar way with respect to the previous class of black holes, we can see that the hypersurface $r = r_H := (-c)^{(1-\delta)/(1-2\delta+4\delta^2)}$, for which $W(r_H) = 0$ and $\partial_r W(r_H) \neq 0$, determines an event horizon, and, since $c < 0$, the Clifton–Barrow metric is a black hole solution.

According to Equation (4.19), we recover the Killing horizon surface gravity

$$\kappa_H = \frac{1}{2} \sqrt{\frac{(1-2\delta+4\delta^2) r_H^{(2\delta+2\delta^2-1)/(1-\delta)}}{(1-2\delta-2\delta^2) r_0^{\delta(1+2\delta)/(1-\delta)}}}, \quad (4.39)$$

which can be used to find the Hawking temperature $T = \kappa_H/2\pi$.

As a last remark, we are able to derive the black hole entropy associated to the event horizon of the Clifton–Barrow solution. For modified gravity $f(R)$ -theories (where the gravity Lagrangian is a function $f(R)$ of the Ricci scalar only), it is easy to see that Wald’s entropy formula (4.20) simplifies to

$$S_W = 4\pi\mathcal{A}_H \left. \frac{df(R)}{dR} \right|_{r_H}. \quad (4.40)$$

In our case, $f(R) = R^{1+\delta}/\chi$, so we find:

$$S_W = \frac{4\pi\mathcal{A}_H}{\chi}(1+\delta) \left[\frac{6\delta(1+\delta)}{(2\delta^2+2\delta-1)r_H^2} \right]^\delta, \quad (4.41)$$

due to the fact that on the Clifton–Barrow solution $R = 6\delta(1+\delta)/(2\delta^2+2\delta-1)r^2$.

In order to have positive entropy, we require $\delta > (\sqrt{3}-1)/2$. The solutions with $0 < \delta < (\sqrt{3}-1)/2$ are unphysical, whereas for $\delta = 0$ we find the result of general relativity.

4.3. The mass issue

We have shown that the solutions we considered in the previous two subsections possess a Killing horizon with an associated Killing vector $\xi^\mu \sim \partial_t^\mu$ which cannot be defined unambiguously due to the fact that the space-times are not asymptotically flat. Of course, this is not the only temperature we can define for such horizons. As shown in previous sections and further discussed in Section 5, in static, asymptotically flat space-times, both the Killing and Kodama vectors coincide, so that they give rise to the same concepts of energy and temperature. In static, non-asymptotically flat spaces, they are both ambiguous and can differ by normalisations but nonetheless, the ratio “energy/surface gravity” remains fixed. This means that as far as the Killing temperature associated with the black holes mentioned here is non vanishing, also their Kodama–Hayward temperature will be so. On the other hand, that the horizons we are concerned are of the bifurcate type means we are in Wald’s hypothesis to compute their entropy. In this sense, Equations (4.29) and (4.41) and Figure 4 represent the main results of this section.

To complete the picture of thermodynamic properties of dirty black holes in modified theories of gravity, it would be necessary to formulate a consistent definition of their mass. As it is well known, in modified theories of gravity the first law of thermodynamics generally requires a work term even in vacuum solutions something which makes the first law useless in the situations at hand. Quite recently some attempts have been put forward in order to answer the question, but only for asymptotically flat space-times [77]. In principle, a powerful tool to evaluate the black hole mass in a theory of the type $\mathcal{L} = R+(\dots)$ is represented by the so-called Brown–York quasi-local mass [78]. In static, spherically symmetric space-times where the metric can be put in the form (4.2) the Brown–York mass reads

$$M_{\text{BY}} = ra(r)b(r) \left[\sqrt{\frac{a^{(0)}(r)}{a(r)}} - 1 \right], \quad (4.42)$$

with $a^{(0)}(r)$ is an arbitrary function which determines the zero of the energy for a background space-time and r is the radius of the space-like hypersurface boundary. When the space-time is asymptotically flat, the ADM mass M is the M_{BY} determined in (4.42) in the limit $r \rightarrow \infty$. If no cosmological horizon is present, the large r limit of (4.42) is used to determine the mass. However, this approach is known to fail whenever the matter action (*i.e.* what we have represented with (...) few lines above) contains derivatives of the metric as it is the case of the Deser *et al.* action, (4.1).

Another quasi-local energy definition well known in general relativity and fully employed in spherical symmetry is the so-called Misner–Sharp energy (cf. Section 5). In the last few years, different authors have tried to generalize the Misner–Sharp energy definition to wider classes of gravity theories [79]. But even if Cai *et al.* provide a general formula for the generalised Misner–Sharp energy in $f(R)$ gravity, this does not produce any explicit, useful, result for the Clifton–Barrow black hole.

We finally mention the recent attempt by Cognola *et al.* [31] to identify the mass of dirty black holes as the integration constant $|c|$ through the differential relation $d|c| = \kappa_{\text{H}} dS_{\text{W}}/8\pi$, κ_{H} being the Killing surface gravity of the horizon. This result seems to favour the Killing temperature with respect to the Kodama–Hayward temperature with a selection made, this time, by Wald’s entropy formula. This does not seem to us particularly surprising in consideration of the fact that Wald’s entropy law applies specifically to Killing horizons of bifurcating type. It would be interesting to see whether a first law as the one above can apply to the Kodama–Hayward temperature allowing for departures from Wald’s entropy.

5. Quantum Tunnelling: Dynamical Black Holes

In the previous sections we saw that further insight into the Hawking's effect can be obtained by the use of the tunnelling methods. We showed that in some circumstances it is even possible to recover exact results by analytically continuing the integral of the classical action from a tunnelling path to a classical path, and how this possibility provided a foundation to the tunnelling method. But of course everything is strictly true only for stationary black holes.

In this section we shall extend this work to what we shall aptly name *spherically symmetric dynamical black holes*, epitomised by the Vaidya solution [80] and its generalisations by Bardeen [81] and York [82]. As a matter of fact, it has long been felt that the usual semi-classical treatment of stationary black holes had to be extended to cover at least slowly changing black holes. By this expression, we mean black holes that can be still described in terms of few multipole moments such as mass, angular momentum and the charges associated to local gauge symmetries, except that the parameters and the causal structure are allowed to change with time. Although a technical definition of a “slowly varying black hole” can be given in some cases, an example being the Booth–Fairhurst slowly evolving horizon [83], in general this depends on the actual physical processes involved. For example, in the case of Hawking's evaporation, conditions for slowness in the presence of a near-horizon viscous fluid have been given by Brevik [84]. In general, it is understood that the black hole temperature has to be much smaller than the Planck mass, while in order to study the effects of the expansion, the Hubble rate H^{-1} should be comparable with the black hole emission/absorption rates.

Now, even for the case of a slow evolution, it was pointed out by Fredenhagen and Haag long ago [85] that by letting the mass of the black hole to change with time, the radiation would originate from the surface of the black hole at all times after its formation. This poses the question: *what and where is the surface of a dynamical black hole?* The issue baffled scientists since the beginning of black hole physics and produced several reactions during the Nineties, which eventually culminated with the notion of *outer trapping horizon* by Hayward [86, 87] and the isolated and dynamical horizons of Ashtekar and co-workers [88, 89] (an extensive review is in [90]).

One is concerned to study at first the dynamical version of the event horizon and to provide a mathematical definition which is able to capture a useful local notion of it, encompassing the time-lasting textbook definition given in Hawking–Ellis renowned book [37]. Armed with a precise notion of horizon, we shall proceed to study the instabilities occurring near the horizon of the changing black hole. This question looks non trivial since a changing horizon is typically embedded in a dynamical space-time and it is not even expected to be a null hyper-surface, though it is still one of infinite red-shift. Without entering the details (cf. [55]), we can anticipate that the arrival point is the very important local notion of *future trapping horizons* and their associated *surface gravity*, as defined by Hayward.

5.1. Horizons and surface gravity

Horizons Not long after the classical definition of the event (EH) and the apparent horizons (AH) (boundaries of trapped 3-dimensional space-like regions within partial Cauchy surfaces), several quasi-local notions of dynamical horizons were proposed in the literature, perhaps starting with the notion of perfect horizon due to Hájíček [91]. But this only applied to equilibrium black holes while the apparent horizon, being tied to a partial Cauchy surface, only represents a localisation in time. Moreover, it has proven not possible to formulate thermodynamic laws for AH similar to those holding good for the event horizons.

The first successful attempt to go beyond the limitations imposed either by the instantaneous character of apparent horizons or by the global, teleological nature of event horizons is due to Hayward. His concept of a future outer trapping horizon (to be abbreviated as FOTH) then evolved either into less constrained definitions, like the Ashtekar–Krishnan dynamical horizons, or more specialised ones, like the Booth–Fairhurst slowly evolving FOTH. The horizon as defined by Hayward will be central for our purposes: for sake of completeness, however, we shall give an updated list of locally or quasi-locally defined horizons which appeared over the years, each playing some role in the problem of understanding dynamical black hole. At least four types of horizon have been defined over the years:

- (i) non expanding and perfect horizons (Hájíček [91]);
- (ii) trapping horizons (Hayward [86, 87]);
- (iii) isolated and weakly isolated horizons (Ashtekar *et al* [88]);
- (iv) dynamical horizons (Ashtekar and Krishnan [89]);
- (v) slowly evolving horizons (Booth and Fairhurst [83]).

Most of these newly defined horizons have very desirable properties: they do not require a space-like hypersurface, no notion of interior and exterior and no conditions referring to infinity, like asymptotic flatness for example (all are non local conditions). Moreover, they are not endowed with teleological features (they do not anticipate the future, so to speak) and, given a solution of Einstein equations, one can find whether they exist or do not exist by purely local computations. Finally, unlike EH they are related to regions endowed by strong gravitational fields and are typically absent in weak field regions.

We recall that the expansion θ of a bundle of null rays is the rate of change of area transverse to the bundle

$$\frac{dA}{dv} = \int \theta d^2S$$

where v is a parameter along the rays. All quasi-local horizons rely on the local concept of *trapped or marginally trapped surface*: this is a space-like closed two-manifold \mathbf{S} such that $\theta_+\theta_- \geq 0$; if ℓ_\pm are the future-directed null normals to \mathbf{S} , normalised to $g_{\mu\nu}\ell_+^\mu\ell_-^\nu \equiv \ell_+ \cdot \ell_- = -1$, then θ_+ , θ_- are the respective expansions or optical scalars, that is the expansions of the two bundles of null rays orthogonal to \mathbf{S} . It is further

assumed that ℓ_+ is associated to an outgoing null geodesic beam, so that in a region of not too strong gravity $\theta_+ > 0$ and the beam is expanding, as for example within the exterior of a black hole at a safe distance. It follows then that ℓ_- is associated to an ingoing null geodesic beam, with $\theta_- < 0$ and the beam contracting along the way. It is always possible to choose double null coordinates x^\pm such that

$$\theta_\pm = \frac{2}{r} \partial_\pm r \quad (5.1)$$

where r is the areal radius, defined so that a metric sphere has area $A = 4\pi r^2$. To cover black holes rather than white holes it is further assumed that both expansions are negative (or non positive) on a trapped (marginally trapped) surface. If $\theta_+\theta_- < 0$ the surface is untrapped and marginal if $\theta_+\theta_- = 0$. In spherical symmetry with radial coordinate r this means the co-vector dr is temporal, spatial or null respectively. A further subdivision may be made: a trapped surface is *future* if $\theta_\pm < 0$ and *past* if $\theta_\pm > 0$. A marginal \mathcal{S} with $\theta_+ = 0$ is *future* if $\theta_- < 0$; *past* if $\theta_- > 0$; *bifurcating* if $\theta_- = 0$; *outer* if $\partial_- \theta_+ < 0$; *inner* if $\partial_- \theta_+ > 0$ and *degenerate* if $\partial_- \theta_+ = 0$ [86, 87, 92, 93]. Equivalently, the vector field $g^{r\mu} \partial_\mu$ is future causal (it means non-space-like) or past causal, respectively. For a better understanding of these terms we may perhaps note that

$$\partial_r = \partial_+ r \partial_+ + \partial_- r \partial_- = \frac{r}{2} (\theta_+ \partial_+ + \theta_- \partial_-).$$

So, for example, on a future trapped surface with $\theta_\pm < 0$ the radial increasing vector is past directed, confirming the intuition that one cannot escape the trapped surface without moving in the past.

So much far for black holes, using the mental picture that we, the observers, are outside it at some radius larger than the radius of the trapped region. For white holes one encounters also marginally trapped surfaces with $\theta_- = 0$ and $\theta_+ > 0$. In cosmology one may encounter again both possibilities, marginally trapped surfaces with either $\theta_- = 0$ or $\theta_+ = 0$, but we defer their description to Section 6.

The abstract definitions will become much more simple and intuitive when referred to spherically symmetric space-times in Section 5.2, where only few of them will be actually used. Here we take the opportunity for few more definitions. The optical scalars can also be defined as follows: the induced metric on each \mathcal{S} is

$$q_{\mu\nu} = g_{\mu\nu} + \ell_{+\mu} \ell_{-\nu} + \ell_{-\mu} \ell_{+\nu} \quad (5.2)$$

for in fact $q_{\mu\nu} \ell_\pm^\nu = 0$. Let $q^{\mu\nu} = g^{\mu\nu} + \ell_+^\mu \ell_-^\nu + \ell_-^\mu \ell_+^\nu$, not the inverse of $q_{\mu\nu}$. Then q^μ_ν is the projection tensor to $T_*(\mathcal{S})$, the tangent space to \mathcal{S} . Associated to the null vector fields ℓ_\pm are the projected tensor fields $(\theta_\pm)_{\mu\nu} = q_\mu^\alpha q_\nu^\beta \nabla_\alpha \ell_{\pm\beta}$ and their decomposition into symmetric, anti-symmetric and trace part. They are tensors on \mathcal{S} because $(\theta_\pm)_{\mu\nu} \ell_\pm^\nu = (\theta_\pm)_{\mu\nu} \ell_\mp^\nu = 0$. The twists (anti-symmetric parts) vanish since the geodesic beams are normal to \mathcal{S} (we have to assume that \mathcal{S} is part of a continuous family of surfaces \mathcal{S}_t). The expansions are then given by the traces

$$\theta_+ = q_{\mu\nu} \nabla^\mu \ell_+^\nu, \quad \theta_- = q_{\mu\nu} \nabla^\mu \ell_-^\nu. \quad (5.3)$$

Finally, the shear is the trace-free symmetric part (note that $g^{\mu\nu}q_{\mu\nu} = q^{\mu\nu}q_{\mu\nu} = 2$)

$$\sigma_{\mu\nu}^{(\pm)} = \theta_{\pm\mu\nu} - \frac{1}{2}q_{\mu\nu}\theta_{\pm}. \quad (5.4)$$

As before, we indicate the Lie-derivative along a vector field X by the calligraphic symbol \mathcal{L}_X and $\mathcal{L}_{\pm} = \mathcal{L}_{\ell_{\pm}}$. Thus, for example, $\mathcal{L}_-\theta_{\mu\nu} = \ell_-^{\alpha}\partial_{\alpha}\theta_{\mu\nu} + \partial_{\mu}\ell_-^{\alpha}\theta_{\alpha\nu} + \partial_{\nu}\ell_-^{\alpha}\theta_{\mu\alpha}$, while on scalars it acts as an ordinary partial derivative.

Future outer trapping horizons — A future outer trapping horizon (FOTH) is a smooth three-dimensional sub-manifold H of space-time which is foliated by closed space-like two-manifolds \mathbf{S}_t , $t \in \mathbb{R}$, with future-directed null normals ℓ_{\pm} such that: (i) the expansion θ_+ of the null normal ℓ_+ vanishes; (ii) the expansion θ_- of ℓ_- is negative; (iii) $\mathcal{L}_-\theta_+ < 0$. A future inner trapping horizon (FITH) is as above except that on it $\mathcal{L}_+\theta_- > 0$. ▼

This proved to be the most important definition since forms the basis for almost all other definitions. Condition (i) requires strong fields since certainly $\theta_+ > 0$ in weak fields. The condition (ii) is related to the idea that H is of the future type (e. g. a black hole rather than a white hole); (iii) says that H is of the outer type^{||}, since a motion of \mathbf{S}_t along ℓ_- makes it trapped. It also distinguishes black hole horizons from cosmological ones, which for an expanding universe are FITH. Note that no reference is made to any space-like hypersurface, nor to infinity. Trapping horizons are locally defined and have physical properties such as mass, angular momentum and surface gravity, satisfying conservation laws [89, 94]. They are a geometrically natural generalisation of Killing horizons, which are stationary trapping horizons. A non-stationary trapping horizon is not null, but still has infinite red-shift. Unlike event and apparent horizon they do not require asymptotic flatness.

One can always find a scalar field C on H so that the vector fields

$$V^{\mu} = \ell_+^{\mu} - C\ell_-^{\mu} \quad \text{and} \quad N^{\mu} = \ell_+^{\mu} + C\ell_-^{\mu}, \quad (5.5)$$

are respectively tangent and normal to the trapping horizon. Note that $V \cdot V = -N \cdot N = 2C$. Hayward [86, 87] showed that if the *null energy condition* (abbr. NEC) holds then $C \geq 0$ on a FOTH. Thus, the horizon must be either space-like or null, and it is null if and only if the shear $\sigma_{\mu\nu}^{(+)}$ and $T_{\mu\nu}\ell_+^{\mu}\ell_+^{\nu}$ both vanish across H . Intuitively, H is space-like in the dynamical regime where gravitational radiation and matter are pouring into it and is null when it reaches equilibrium. Conversely, on a FITH V^{μ} is either null or time-like.

It is worth mentioning that *the second law of the mechanics of trapping horizons* follows quite easily from this apparatus. Taking the Lie-derivative of \sqrt{q} , which is the area density corresponding to the metric $q_{\mu\nu}$ on the cross-sections of H , we get

$$\mathcal{L}_V\sqrt{q} = -C\sqrt{q}\theta_-. \quad (5.6)$$

By definition θ_- is negative on H and we have just seen that, barring violations of the null energy condition, C also is non-negative. Since V is future directed we obtain

^{||} For example the Cauchy horizon in the Reissner–Nordström solution is of inner type.

the local form of the second law: *If the null energy condition holds, then the area element \sqrt{q} on a FOTH is non-decreasing along future directions.*

Integrating over \mathbf{S}_t the same law applies to the total area of the trapped sections. As long as the null energy condition is maintained it will be non-decreasing, reaching a constant value if and only if the horizon becomes a null hypersurface.

The main difference between an apparent horizon as defined in Hawking–Ellis and the trapping horizon of Hayward is that the AH represents the instantaneous surface of a black hole, *i.e.* it needs a (partial) Cauchy surface Σ and it is very sensitive to the choice of Σ . To compute the AH one needs only its metric and the second fundamental form of Σ , namely the initial data for Einstein’s equations. Hayward’s horizon instead is a null hypersurface H which is insensitive to a choice of Σ and does not refer to spatial infinity. The trapping horizon is a foliation of H .

The causal character as well as the area law required the validity of the NEC; an evaporating black hole violates NEC, therefore the area law will also be violated and the horizon will be time-like. How it could be that tunnelling along a classically forbidden path is still possible will be seen soon and represents the real possibility of the radiation process even in the temporary absence of a global event horizon.

Surface gravity The surface gravity associated to an event horizon is a well known concept in black hole physics whose importance can be hardly overestimated. Surprisingly, a number of inequivalent definitions beyond the standard one appeared over the last 15 years or so in the field, with various underlying motivations. We have collected the following (we rely on the review of Nielsen and Yoon [95]):

- (i) the standard Killing surface gravity (Bardeen *et al* [96], textbooks);
- (ii) a first definition given by Hayward in [86];
- (iii) the effective surface gravity appearing in Ashtekar–Krishnan [90];
- (iv) the Fodor *et al* definition for dynamical spherically symmetric space-times [97];
- (v) the Visser [7] and Nielsen–Visser [28] surface gravity;
- (vi) still one more definition by Hayward [87], using Kodama theory of spherically symmetric space-times [98].

In addition there are some more technical definitions due to Mukohyama and Hayward [99] and to Booth and Fairhurst, the latter related to their notion of evolving horizons [83]. Except for the last item, which is what the tunnelling method leads to, the remaining definitions will be less relevant to us, so we reserve to them only some brief considerations.

The Killing surface gravity is related to the fact that the integral curves of a Killing vector are not affinely parametrized geodesics on the Killing horizon H , where the norm $\xi^2 = 0$. Hence

$$\xi^\mu \nabla_\mu \xi_\nu \cong \kappa \xi_\nu$$

defines the Killing surface gravity κ on H , where \cong means evaluation on the horizon. The Killing field is supposed to be normalised at infinity by $\xi^2 = -1$. The definition can be extended to EHs that are not Killing horizons or to Killing horizons in non asymptotically flat space-times, by replacing ξ with the null generator of the horizon. However there is no preferred normalisation in this case (cf. Section 4), and this is one reason of the debating question regarding the value of the surface gravity in dynamical situations.

Hayward's first definition was motivated by the desire to get a proof of the first law for THs. It is defined for a future trapping horizon without appeal to un-affinity of null geodesics, as

$$\kappa \cong \frac{1}{2} \sqrt{-n^\mu \nabla_\mu \theta_\ell}.$$

This quantity is independent on the parametrisation of the integral curves of the vector field ℓ^μ , since the evaluation is on a marginal outer surface where $n \cdot \ell = -1$ and $\theta_\ell = 0$.

Given a weakly isolated horizon H , Ashtekar and Krishnan showed that for any vector field t^a along H with respect to which energy fluxes across H are defined, there is an area balance law that takes the form

$$\delta E^t = \frac{\bar{\kappa}}{8\pi G} \delta A_S + \text{work terms}$$

with an effective surface gravity given by

$$\bar{\kappa} = \frac{1}{2R} \frac{dr}{dR}.$$

R is the areal radius of the marginally trapped surfaces, *i.e.* $A_S = 4\pi R^2$, the function r is related to a choice of a lapse function and finally E^t is the energy associated with the evolution vector field t^a . For a spherically symmetric dynamical horizon a natural choice would be $r = R$ so $\bar{\kappa} = 1/2R$, just the result for a Schwarzschild black hole. To illustrate the naturalness of this definition, consider a slowly changing spherically symmetric black hole with mass $M(v)$, where v is a time coordinate. Defining the horizon radius at each time by $R = 2M(v)$ and $A_S = 4\pi R^2$, we can differentiate M

$$\dot{M} = \frac{\dot{R}}{2} = \frac{1}{2R} \frac{\dot{A}_S}{8\pi}$$

so to obtain $\delta M = \bar{\kappa} \delta A_S / 8\pi$. One recognises the usual area law in differential form with surface gravity $\bar{\kappa} = 1/2R = 1/4M$. Consider, however, the more general case where the horizon is the solution of the implicit equation $R = 2M(v, R)$, as it happens for example in the Bardeen–Vaidya metric. The same computation leads to $\dot{M} = \dot{A}_S (1 - 2M') / (16\pi R)$, leading to

$$\kappa \cong \frac{1}{4M} (1 - 2M') \tag{5.7}$$

a prime denoting the radial derivative. The surface gravity here deduced does not conform to Ashtekar *et al* definitions, suggesting that its value depends on the definition of the black hole mass one is adopting.

The definition of Fodor *et al* looks like the Killing form of the surface gravity in that $\kappa \ell^\nu = \ell^\mu \nabla_\mu \ell^\nu$, where now ℓ^μ is an outgoing null vector orthogonal to a trapped or marginally trapped surface. This is because, as a rule, such null vectors are not affinely parametrized, although they can always be parametrized so that $\kappa = 0$. Fodor *et al* choose to fix the parametrization so that

$$\kappa = -n^\mu \ell^\nu \nabla_\nu \ell_\mu$$

with n^μ affinely parametrized and normalised to $n \cdot t = -1$ at space-like infinity, where t^a is the asymptotic Killing field. Note that this definition is non local but looks like a natural generalisation of the Killing surface gravity.

As far as we know, the Visser and Visser–Nielsen surface gravity is only defined in dynamical Painlevé–Gullstrand coordinates, so there is no guarantee that it is a geometrical invariant. Indeed, we shall see that for this class of metrics it is different from the invariant surface gravity computed within the general Hayward formalism.

The last item is a local geometrical definition of the surface gravity for the trapping horizon of a spherically symmetric black hole [87], so we postpone a full discussion to Section 5.2.4. Basically, one introduces local null coordinates x^\pm in a tubular neighbourhood of a FOTH and for any sphere of radius r one defines the quantity

$$\kappa = \frac{r}{2} \left(g^{+-} \partial_- \theta_+ + \frac{1}{2} g^{+-} \theta_+ \theta_- \right). \quad (5.8)$$

Evaluated on a trapping horizon, $\theta_+ = 0$, it will be positive precisely when the horizon is of outer type ($\partial_- \theta_+ < 0$, recall that with our conventions $g^{+-} < 0$) and $\kappa = 0$ if degenerate. The definition may look somewhat artificial, but in fact it can be put in a form that strongly resembles the Killing surface gravity of stationary black holes. To see this, we anticipate a result of the next section according to which, following Kodama [98], any spherically symmetric metric admits a unique (up to normalisation) vector field K^μ such that $\nabla^\nu (K^\mu G_{\mu\nu}) = 0$, where $G_{\mu\nu}$ is the Einstein tensor; for instance, using the double-null form, one finds

$$K = -g^{+-} (\partial_+ r \partial_- - \partial_- r \partial_+). \quad (5.9)$$

The defining property of K shows that it represents a natural generalisation of the time translation Killing field of a static black hole. Now consider the expression $K_\mu \nabla_{[\nu} K_{\mu]}$: it is not hard to see that on H it is proportional to K_ν . So one defines the dynamical surface gravity as $K^\mu \nabla_{[\nu} K_{\mu]} \cong -\kappa K_\nu$. For a Killing vector field $\nabla_\nu K_\mu$ is anti-symmetric so the definition reduces to the usual one.

5.2. Spherically symmetric fields

In this section we discuss the general time-dependent spherically symmetric metric that will be the arena of our dynamical tunnelling computations. We have a two-fold intent in doing this. If Hawking radiation proceeds by emission of discrete energy quanta, a continuous description of the changing metric would only be possible when their number is so large so as to simulate a continuous streaming of energy. In such a case, a

dynamical excreting black hole may be modelled by a continuously differentiable solution of Einstein's equations for most of its history. Given that, it would be quite ironic if there were not evidence of a continuously operating quantum emission process. In other words, if the tunnelling method only worked for the event horizons of stationary black holes then there would be a problem with the use of such dynamical metrics as models of black hole evaporation.

The second aspect has to do with the region where the radiation originates. The tunnelling calculation suggests the the outgoing radiation is emitted from the trapping horizon, not the global event horizon. And it further suggests that the semi-classical probability is related to the horizon surface gravity as defined in Equation (5.8) [87]. Thus we have first of all to learn how to compute these things in a spherically symmetric, time dependent metrics.

5.2.1. Forms of the metric The paradigm of a spherically symmetric metric is of course the Schwarzschild vacuum solution of Equation (2.1). To introduce dynamics one could think to make the replacement $M \rightarrow M(t)$; however the resulting metric has a curvature singularity at $r = 2M(t)$; for instance, the scalar curvature is

$$\mathcal{R} = \frac{2r(r - 2M)\ddot{M} + 4\dot{M}^2}{(r - 2M)^3}$$

so only for special values of the mass function is $\mathcal{R} = 0$; other invariants will diverge though. A more general ‘‘Schwarzschild gauge’’ can be written down

$$ds^2 = -e^{2\Phi(r,t)} \left(1 - \frac{2M(r,t)}{r}\right) dt^2 + \left(1 - \frac{2M(r,t)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi) \quad (5.10)$$

which was studied in depth by Visser in his classic ‘‘Dirty Black Holes’’ [52]. We shall not make use of his metric in the following.

It was discovered by Vaidya [80] that by taking the Schwarzschild metric in advanced Eddington–Finkelstein coordinates (v, r, θ, ϕ) (or retarded, (u, r, θ, ϕ) for other purposes)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi) \quad (5.11)$$

and by making the replacement $M \rightarrow M(v)$, one obtains a regular solution of Einstein's equations except at the origin $r = 0$: this is the shining star solution of Vaidya. The metric is still not the most general one allowed by the symmetry: in fact, a spherically symmetric metric can depend at most on two arbitrary functions of the coordinates (v, r) . One form is due to Bardeen [81]

$$ds^2 = -e^{2\Phi} \left(1 - \frac{2M}{r}\right) dv^2 + 2e^\Phi dvdr + r^2(d\theta^2 + \sin^2\theta d\phi) \quad (5.12)$$

where Φ and M are functions of v, r . Another important form is obtained by passing from Schwarzschild time to the proper time of a radially infalling observer

$$t_p = t + 2\sqrt{2Mr} + 4M \log \left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}} \right). \quad (5.13)$$

The metric is the Painlevé–Gullstrand metric we encountered in Section 2

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt_p^2 + 2\sqrt{\frac{2M}{r}} dt_p dr + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi). \quad (5.14)$$

Taking now $M = M(t_p)$ we get a dynamical black hole long studied by Lindesay and Brown [100, 101] and others [102]. There is a more general form whereby M can also depend on r and a second function $\Phi(\tilde{t}, r)$ is introduced, say

$$ds^2 = -e^{2\Phi(\tilde{t}, r)} \left(1 - \frac{2M(\tilde{t}, r)}{r} \right) d\tilde{t}^2 + 2e^{\Phi(\tilde{t}, r)} \sqrt{\frac{2M(\tilde{t}, r)}{r}} d\tilde{t} dr + dr^2 + r^2 d\Omega^2, \quad (5.15)$$

where $d\Omega^2 := d\theta^2 + \sin^2 \theta d\phi$. This metric is locally diffeomorphic to the metric (5.10) under a change of time $t \rightarrow \tilde{t}(t, r)$, with the mass treated as a scalar field, *i.e.* $M(\tilde{t}, r) = M(t(\tilde{t}, r), r)$. It was thoroughly studied by Visser and Nielsen in [28]. In this rather impressive work the dynamics and decay of evolving horizons were investigated mainly using a dynamical version of Painlevé–Gullstrand coordinates. The metric (5.15) is also locally diffeomorphic to the metric (5.12), under a change $v \rightarrow \tilde{t}(v, r)$, where now $M(\tilde{t}, r) = M(v(\tilde{t}, r), r)$.

All these spherically symmetric metrics (though not only these) are special cases of metrics that can locally be expressed in the warped form

$$ds^2 = \gamma_{ij}(x^i) dx^i dx^j + R^2(x^i) d\Omega^2, \quad (5.16)$$

where the two-dimensional metric

$$d\gamma^2 = \gamma_{ij}(x^i) dx^i dx^j \quad (5.17)$$

is referred to as the normal metric (in the space normal to the sphere of symmetry), x^i are associated coordinates and $R(x^i)$ is the areal radius, considered as a scalar field in the normal two-dimensional space. The two-dimensional metric γ_{ij} has only one degree of freedom and the function $R(x^i)$ is the second one; however the two functions on which the metric depends can be moved both to $d\gamma^2$ using the coordinates freedom. That is we can rename $x^1 = r$ and put $R(x^i) = r$. The coordinate r is known as the areal radius, and is a geometrical invariant being the quotient of the area of space-like spheres by 4π . This will be referred to as the *r-gauge*. Sometimes the normal metric can be further written, somewhat redundantly, in the expanded form

$$d\gamma^2 = -E(r, t) dt^2 + 2F(r, t) dt dr + G(r, t) dr^2, \quad \gamma = -(EG + F^2) \quad (5.18)$$

especially if one wants to discuss certain limits, but we will always try to get metrics regular across the trapping horizons. If we wish, we can also locally write the line element in a double-null form which is conformally flat

$$ds^2 = -2e^f dx^+ dx^- + r^2(x^+, x^-) d\Omega^2 \quad (5.19)$$

with $f = f(x^+, x^-)$. For instance, starting with Equation (5.12) we can introduce null coordinates via

$$e^f dx^- = \frac{1}{2} \left(1 - \frac{2m(r, v)}{r} \right) e^{2\Phi} dv - e^\Phi dr, \quad dx^+ = dv$$

where f is an integrating factor. The normal section of (5.12) then takes the form (5.19). One may use one spatial and one temporal direction on putting $dx^+dx^- = dt^2 - dr^2$, but there is no unique choice of such directions. The remaining coordinate freedom consists of conformal diffeomorphisms (in the language of two-dimensional metrics)

$$x^\pm \rightarrow \tilde{x}^\pm(x^\pm).$$

The double-null form will be also referred to as the *conformal gauge*.

Another form we would like to describe is the metric in the so called *synchronous gauge*. Let us consider the Schwarzschild space-time in coordinates (T, r, θ, ϕ) such that the line element can be expressed as

$$ds^2 = -dT^2 + \frac{dr^2}{B} + (r_g B)^2 d\Omega^2, \quad (5.20)$$

where $r_g = 2M$ is the usual gravitational radius, and

$$B(T, r) := \left[\frac{3}{2r_g}(r - T) \right]^{\frac{2}{3}}. \quad (5.21)$$

We shall refer to these coordinates as the Lemaître–Rylov gauge. This is indeed an interesting (time-dependent) gauge since, contrary for example to isotropic coordinates, (T, r) extend beyond the gravitational radius, $r < r_g$. Now considering $B(T, r)$ as an arbitrary function and replacing $r_g B$ with a general function $R(r, T)$ we obtain the metric in synchronous gauge

$$ds^2 = -dT^2 + \frac{dr^2}{B(T, r)} + R^2(T, r) d\Omega^2 \quad (5.22)$$

in which the metric is diagonal but the areal radius is a function of r and T .

The last form we would like to mention is the *isotropic gauge* in which the spatial part of the metric is conformally flat

$$ds^2 = -A(t, \rho) dt^2 + B(\rho, t) (d\rho^2 + \rho^2 d\Omega^2). \quad (5.23)$$

An important example of this latter form is the McVittie solution describing in author's mind a point mass in a FRW flat cosmology [103]. It reads as in (5.23) with

$$A(\rho, t) = \left(1 + \frac{M}{2a(t)\rho} \right)^{-2} \left(1 - \frac{M}{2a(t)\rho} \right)^2 \quad (5.24)$$

$$B(\rho, t) = a(t)^2 \left(1 + \frac{M}{2a(t)\rho} \right)^4. \quad (5.25)$$

When $M = 0$ it reduces to a spatially flat FRW metric with scale factor $a(t)$; when $a(t) = 1$ it reduces to the Schwarzschild metric with mass M in isotropic coordinates. This solution had a strong impact on the general problem of matching the Schwarzschild solution with cosmology, a problem faced also by Einstein and Dirac. Besides McVittie, it has been extensively studied by Nolan in a series of papers [104]. To put the metric in the general r -gauge form, the coordinate transformation

$$r = a(t)\rho \left(1 + \frac{M}{2a(t)\rho} \right)^2$$

transforms the metric in the so called Nolan gauge, in which it reads

$$ds^2 = - (A_s - H^2(t)r^2) dt^2 - \frac{H(t)r}{2\sqrt{A_s}} dr dt + \frac{dr^2}{A_s} + r^2 d\Omega^2 \quad (5.26)$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter and, for example, in the charged four-dimensional case, $A_s = 1 - 2M/r + Q^2/r^2$. Unlike the Schwarzschild case, $A_s = 0$, or $r = 2M$ in the neutral case or $\rho = M/2a(t)$, is a curvature singularity rather than a global event horizon. In fact, it represents a Big Bang singularity very similar to $r = 0$ in FRW models. When $H = 0$ one recovers the Schwarzschild solution. For constant H , it reduces to the Schwarzschild–de Sitter solution in Painlevé coordinates. As we shall see, the McVittie solution possesses in general black hole and cosmological trapping horizons, and the space-time is dynamical.

Other coordinate systems which can be used to include dynamics in the Schwarzschild metric are discussed in [105]. We may note that Equation (5.22) contains as a special case the FRW family of metrics. We stress that all forms discussed above are completely equivalent ways to describe the space-time structure of a spherically symmetric field, the use of any particular form being dictated only by computational convenience. We also note that in the warped form (5.16) the sphere of symmetry can be replaced by anyone among the two-dimensional manifolds with constant curvature which appear in the static topological black hole solutions embedded in Anti–de Sitter space. Only, the rotational isometry group must be replaced by the appropriate isometry group of the surface, which in the static case are torii or compact Riemann surfaces with higher genus.

5.2.2. Trapping horizons To compute the trapping horizons we shall start from the Bardeen–Vaidya form (5.12). We can take the two null, future directed congruences normal to spheres of constant radius as

$$\ell_+ = \left[1, \frac{1}{2}e^\Phi \left(1 - \frac{2M}{r} \right), 0, 0 \right], \quad \ell_- = [0, -e^{-\Phi}, 0, 0] \quad (5.27)$$

where as always $\ell_+ \cdot \ell_- = -1$. A simple computation gives the optical scalars

$$\theta_+ = \frac{1}{r} \left(1 - \frac{2M}{r} \right) e^\Phi, \quad \theta_- = -\frac{2e^{-\Phi}}{r}. \quad (5.28)$$

Trapped or marginally trapped spheres have $\theta_+ \theta_- \geq 0$, therefore there are no such surfaces in the region $r > 2M(r, v)$. We also see that $\theta_- < 0$ along the surface defined by $\theta_+ = 0$, which is therefore a trapping horizon of the future type. Its defining equation is $r = 2M(r, v)$, which defines a line $r = r(v)$ in normal space and therefore a hypersurface with topology $\mathbb{R}^1 \times \mathbb{S}^2$ in space-time.

Consider now $\partial_- \theta_+$, where the symbol ∂_- denotes the directional derivative along ℓ_- . One obtains, using $M' = \partial_r M$,

$$\partial_- \theta_+ = -\frac{1}{r^2} \left(\frac{2M}{r} - 2M' \right).$$

On the trapping horizon $\partial_- \theta_+ \cong -(1 - 2M')/4M^2$ will be negative if and only if $M' < 1/2$. As explained in Section 5.2.4, this is the condition which ensures the positivity of the surface gravity. Thus in this case $r = 2M$ is a FOTH. The areal radius of the horizon will be denoted by r_{H} from now on. The signs of θ_{\pm} are geometrical invariants, but their actual values are not because the null directions are defined up to an overall scale. An invariant combination is $\chi = 2g^{\pm}\theta_+\theta_-$ or, using (5.16),

$$\chi = \gamma^{ij} \partial_i R \partial_j R \equiv g^{\mu\nu} \partial_\mu R \partial_\nu R. \quad (5.29)$$

$\chi = 0$ is then the condition for a trapping horizon in Hayward's sense. For example in any coordinate system where one coordinate is the areal radius $R = r$, the condition is $g^{rr} = 0$. The trapping horizon of McVittie solution can also be computed using (5.29): using Nolan form it is a solution of the equation $\sqrt{A_s} = Hr_{\text{H}}$, which in turn implies $1 - 2M/r_{\text{H}} = H(t)^2 r_{\text{H}}^2$. As M is a constant this is a cubic algebraic equation with a priori more than one real root and in any case at most two positive roots. The situation is similar to de Sitter space, except that the horizon radius is here a function of time, $r_{\text{H}}(t)$. For positive H both horizons are of the future type, but as a rule one is outer (meaning that there is a black hole) while the larger root corresponds to an inner horizon, hence to a FITH, if and only if

$$\frac{M}{r_{\text{H}}^2} - H^2 r_{\text{H}} - \frac{\dot{H}}{2H} > 0. \quad (5.30)$$

For $M = 0$, *i.e.* for homogeneous cosmology, only the inner cosmological horizon survives. This is the case, for example, in de Sitter space-time. We will make use of these results in Section 6.

5.2.3. Misner–Sharp–Kodama energy One special feature of spherically symmetric space-times is the absence of gravitational radiation. This feature makes it possible the existence of a special, privileged notion of energy, the Misner–Sharp mass, which for spheres with areal radius r is the same as the Hawking mass [106] (a general reference for energy in general relativity is the review [107]). The energy may be defined by [93] (we recall that r is the areal radius)

$$E = \frac{r}{2} - r g^{+-} \partial_+ r \partial_- r = \frac{r}{2} - \frac{r^3}{4} g^{+-} \theta_+ \theta_- \quad (5.31)$$

and interpreted as the energy inside a sphere of radius r ; so, by definition, a metric sphere is trapped if and only if $E > r/2$, marginal if and only if $E = r/2$ and untrapped if and only if $E < r/2$. Note that (5.31) is a special case of the Hawking mass

$$E(S) = \sqrt{\frac{\text{Area}(S)}{16\pi}} \left(1 - \frac{1}{8\pi} \oint g^{+-} \theta_+ \theta_- d^2 S \right) \quad (5.32)$$

and is a geometrical invariant. Two very important properties of E were proved by Hayward [93]: *in an asymptotically flat space-time, E coincides with the Bondi–Sachs*

scalar energy at null infinity, and with the Arnowitt–Deser–Misner (ADM) mass at spatial infinity. Using (5.12) the energy takes the implicit form

$$g^{\mu\nu}\partial_\mu r\partial_\nu r = g^{rr} = 1 - \frac{2E}{r} \quad (5.33)$$

which shows the relation with the Schwarzschild mass. Then $E = M$ and the FOTH is at $r = 2E$. Many important properties of E can be displayed using also the Einstein gravitational field equations. We report two of them which seem remarkable, leaving the interested reader to the literature. Using double-null coordinates the variation of E as determined by Einstein's equations is

$$\partial_\pm E = 2\pi e^{-f} r^3 (T_{+-}\theta_\pm - T_{\pm\pm}\theta_\mp) \quad (5.34)$$

where we remember that $e^{-f} = -g^{+-}$. These field equations can also be written in the Bardeen–Vaidya form, using Equation (5.12), where it is seen that $E = M$ and

$$\partial_v M = 4\pi r^2 T_v^r, \quad \partial_r M = -4\pi r^2 T_r^v. \quad (5.35)$$

Thus in vacuo E is a constant. It can then be shown that the solution is locally isometric to a Schwarzschild solution with energy E . This is an improvement of Birkhoff's theorem.

The second result is essentially the area law: *if the NEC holds on a FOTH then $E = r_H/2$ is non-decreasing along the horizon.*

The question arises naturally whether E is the charge associated to a conserved current. It was discovered by Kodama [98] that in spherical symmetry there is a vector field K^μ such that $\nabla^\mu(G_{\mu\nu}K^\nu) = 0$; by Einstein equations it follows also the conservation equation

$$\nabla_\mu (T^{\mu\nu} K_\nu) = 0 \quad (5.36)$$

and a corresponding charge. If we define the two-dimensional Levi–Civita skew tensor

$$\epsilon_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} \tau_1^\alpha \tau_2^\beta$$

where τ_1, τ_2 are tangent vectors to constant radius spheres, then the Kodama vector may be defined by

$$K^\mu = \epsilon^{\mu\nu} \partial_\nu r. \quad (5.37)$$

It can easily be seen that

$$\nabla_\mu K^\mu = 0. \quad (5.38)$$

Let us give few examples:

- (a) for static, non-dirty, black holes it is the Killing field;
- (b) for dirty black holes, (2.43) or (3.31), $K = \sqrt{W/V} \partial_t$;
- (c) in Bardeen–Vaidya gauge $K = e^{-\Phi} \partial_v$;
- (d) for the metrics (5.14) and (5.15), $K = \partial_{t_p}$ or $K = e^{-\Phi} \partial_{\tilde{t}}$, respectively;
- (e) for the metric (5.18) $K = (EF + G^2)^{-1/2} \partial_t$;

- (f) in conformal gauge $K = -g^{+-}(\partial_+ r \partial_- - \partial_- r \partial_+)$;
- (g) in synchronous gauge $K = \sqrt{B}(R' \partial_\tau - \dot{R} \partial_r)$;
- (h) for FRW flat cosmology $K = \partial_t - Hr \partial_r$.

In every case it follows that

$$K^2 = \frac{2E}{r} - 1$$

so that K is time-like, space-like or null if and only if $r > 2E$, $r < 2E$ or $r = 2E$, respectively. Let us also define the current

$$j^\mu = -T^{\mu\nu} K_\nu. \quad (5.39)$$

We have just seen that $\nabla_\mu j^\mu = 0$, $\nabla_\mu K^\mu = 0$, and therefore there exist two conserved charges

$$Q_j = - \int_\Sigma j^\mu n_\mu d^3V \quad (5.40)$$

$$Q_K = - \int_\Sigma K^\mu n_\mu d^3V \quad (5.41)$$

where Σ is a space-like three-dimensional surface with fixed boundary at some constant r and future pointing time-like normal n_μ . Here “conserved” means independent on the choice of Σ . The charges as defined will be positive in regions where j^μ , K^μ are both time-like. Using (5.34) one can easily see that

$$j = \frac{1}{4\pi r^2} (\partial_+ E \partial_- - \partial_- E \partial_+). \quad (5.42)$$

To compute the charges we pass to a synchronous gauge by choosing coordinates (τ, ζ) adapted to Σ , with ∂_τ normal and ∂_ζ tangent to Σ . In these coordinates we can always write the metric in the form

$$ds^2 = -d\tau^2 + e^\lambda d\zeta^2 + r^2(\tau, \zeta) d\Omega^2. \quad (5.43)$$

From (5.37) and (5.39) we obtain

$$K^\mu = e^{-\lambda/2}(r', -\dot{r}, 0, 0), \quad j^\mu = \frac{e^{-\lambda/2}}{4\pi r^2}(E', -\dot{E}, 0, 0) \quad (5.44)$$

where $\dot{r} = dr/d\tau$, $r' = dr/d\zeta$. Following Hayward, we shall say that the point $r = 0$ is a *regular centre* if it is a boundary point of the normal space and $E/r \rightarrow 0$ as the centre is approached. Otherwise it is a *central singularity*. We now assume that Σ extends from a regular centre to some $r > 0$: from (5.44) and the metric we have the normal $n^\mu = \delta_0^\mu$ and

$$-K^\mu n_\mu = e^{-\lambda/2} r', \quad -j^\mu n_\mu = \frac{e^{-\lambda/2}}{4\pi r^2} E'.$$

Therefore integrating over Σ with the invariant measure $dV = e^{\lambda/2} r^2 \sin \theta d\zeta d\theta d\phi$, we finally obtain

$$Q_K = \frac{4\pi}{3} r^3, \quad Q_j = E. \quad (5.45)$$

We conclude that E is indeed the charge associated to a conserved current. The charge Q_j is the definition of energy of Kodama. For static solutions the Kodama vector coincides with the Killing field, the generator of the time translation symmetry.

5.2.4. *The surface gravity* The properties of the Kodama vector field discussed so far prompt for a natural definition of the surface gravity of a trapping horizon. We have seen that K becomes null precisely on a trapping horizon and space-like within. The time-like integral curves of K are in general not contained within the horizon. Consider the quantity $K^\mu \nabla_{[\nu} K_{\mu]}$: it can be seen that on a trapping horizon it is proportional to K_ν

$$K^\mu \nabla_{[\nu} K_{\mu]} \cong -\kappa K_\nu. \quad (5.46)$$

The function κ is, by definition, the horizon surface gravity of Hayward. For static black holes K is the Killing field so $\nabla_\mu K_\nu$ is anti-symmetric and the definition reduces to the usual one. A formula to compute κ efficiently was found by Hayward [87]. Working in double-null coordinates we have $K_+ = -\partial_+ r$, $K_- = \partial_- r$ (see point (f) of the examples list) therefore

$$\partial_+ K_- = -\partial_- K_+ = \partial_- \partial_+ r = \frac{g^{+-}}{2} \square_\gamma r \quad (5.47)$$

where $\square_\gamma = 2g^{+-} \nabla_+ \nabla_-$ is the two-dimensional Klein–Gordon operator acting on scalars. Computing the left hand side of (5.46) then gives the wanted formula

$$\kappa = \frac{1}{2} \square_\gamma r = \frac{1}{2} r \left(g^{+-} \partial_- \theta_+ + \frac{1}{2} g^{+-} \theta_+ \theta_- \right), \quad (5.48)$$

where the last form is obtained by using $\theta_\pm = 2r^{-1} \partial_\pm r$. That is, κ is the “Box” of r ; when evaluated on the trapping horizon, where $\theta_+ = 0$, it is the surface gravity and is positive if and only if $\partial_- \theta_+ < 0$, that is if the horizon is of outer type. Its invariant character is manifest. Let us give few examples with comparison to other definitions. For the Bardeen–Vaidya metric one obtains

$$\kappa = \frac{1}{4M} (1 - 2M') \quad (5.49)$$

which is also the Visser dynamical surface gravity as defined in [7] in a Painlevé–Gullstrand frame. The first Hayward’s definition (see the list in Section 5.1), invoked without appeal to un-affinity, would have given instead

$$\tilde{\kappa} = \frac{1}{4M} \sqrt{1 - 2M'}$$

which is not even correct for the static Reissner–Nordström solution ¶. The definition of Fodor *et al* gives

$$\hat{\kappa} = \frac{e^\Phi}{4M} (1 - 2M') + \dot{\Phi}.$$

For the dynamic Painlevé–Gullstrand metric (5.14) with $M = M(t_p, r)$, we obtain

$$\kappa = \frac{1}{4M} (1 - 2M' + 2\dot{M}) \quad (5.50)$$

while the Visser and Visser–Nielsen surface gravity for this kind of metrics would give the same formula without the time derivative term. Incidentally, this shows that

¶ For which $M' = q^2/2(2M^2 - q^2 + 2M\sqrt{M^2 - q^2})$.

the dynamical Bardeen–Vaidya metric is not diffeomorphic to a dynamical Painlevé–Gullstrand metric (5.14) with the same mass function, as it would happen in the stationary case, and therefore represent physically different gravitational fields. For the dynamic Painlevé–Gullstrand metric (5.15), with mass function $M(\tilde{t}, r)$, we obtain instead

$$\kappa = \frac{1}{4M}(1 - 2M' + 2\dot{M}e^{-\Phi}) \quad (5.51)$$

again different from the Visser–Nielsen surface gravity for the same mass function, which has no time derivative terms.

5.3. Tunnelling from trapping horizons

We now come to see what the tunnelling method has to say about dynamical, spherically symmetric black holes. We shall start by identifying the dynamical version of the tunnelling path which was displayed and discussed in Section 2, see Figure 1. This will be accomplished by using a specific and convenient form of the metric, which we start to review a little more than already done. In absence of analytical techniques we shall make explicit use of the Hamilton–Jacobi equation, the null geodesic method being quite inconvenient to treat truly dynamical metrics (but see Clifton [102]). The covariance of the method will be stressed throughout.

5.3.1. Metrics to be used We shall use for the time being the Bardeen–Vaidya (BV) metric (5.12), which we recall here

$$ds^2 = -e^{2\Phi(r,v)} \left(1 - \frac{2M(r,v)}{r} \right) dv^2 + 2e^{\Phi(r,v)} dvdr + r^2 d\Omega^2. \quad (5.52)$$

The sphere of symmetry will not play any role here, though. To illustrate the covariance of the results we shall occasionally make use of the metric in Painlevé–Gullstrand (PG) form, either (5.14) or (5.15). Few things about metric (5.52) will help with the understanding. The field equations read

$$\frac{\partial M}{\partial v} = 4\pi r^2 T_v^r, \quad \frac{\partial M}{\partial r} = -4\pi r^2 T_v^v, \quad \frac{\partial \Phi}{\partial r} = 4\pi r e^{\Phi} T_r^v, \quad (5.53)$$

and the stress tensor can be written as

$$T_{\mu\nu} = \frac{\dot{M}}{4\pi r^2} \nabla_\mu v \nabla_\nu v - \frac{M'}{2\pi r^2} \nabla_{(\mu} r \nabla_{\nu)} v. \quad (5.54)$$

If M only depends on v , it describes a null fluid obeying the dominant energy condition for $\dot{M} > 0$. For the excreting black hole $\dot{M} < 0$, so the null energy condition will also be violated. We already know that $r = 2M(r, v)$ is a trapping horizon, that is a FOTH, if and only if $2M' < 1$, which we shall assume from now on. Putting $r = 2M$ into the metric gives

$$ds^2 = \frac{4e^{\Phi} \dot{M}}{1 - 2M'} \Big|_{\text{H}} dv^2$$

so we conclude that the horizon is certainly time-like if $\dot{M} < 0$; we know from general results that it will be space-like or null if $\dot{M} \geq 0$. The Hayward surface gravity is

$$\kappa = \frac{1}{4M}(1 - 2M') \quad (5.55)$$

and is positive under our assumptions. As we extensively explained, the Misner–Sharp mass, or energy for short, is the value $E(v)$ taken by $M(v, r)$ on the trapping horizon, *i.e.*

$$E(v) = M(v, r_{\text{H}}(v)) = \frac{r_{\text{H}}(v)}{2}. \quad (5.56)$$

Using (5.53) one can show that an observer at rest at $r \gg r_{\text{H}}$ sees a quasi-static geometry with a luminosity $L = -dE/dv$. We shall not assume, initially, that L has the Hawking form $L = \hbar \mathcal{N} E^{-2}$, with \mathcal{N} a constant proportional to the number of massless species radiated from the black hole.

For sake of completeness, we note that a cosmological constant can be introduced via

$$ds^2 = -e^{2\Phi(r,v)} \left(1 - \frac{2M(r,v)}{r} - \frac{\Lambda r^2}{3} \right) dv^2 + 2e^{\Phi(r,v)} dvdr + r^2 d\Omega^2. \quad (5.57)$$

We shall admit that $\Lambda > 0$; if M is constant and $\Phi = 0$ one recognises the Schwarzschild–de Sitter metric in advanced coordinates. Let us call C the expression in round brackets above; the null normals and their expansions are

$$\ell_+ = \left[1, \frac{1}{2} e^{\Phi} C, 0, 0 \right], \quad \theta_+ = \frac{1}{r} e^{\Phi} C \quad (5.58)$$

$$\ell_- = [0, -e^{-\Phi}, 0, 0], \quad \theta_- = -\frac{2e^{-\Phi}}{r}. \quad (5.59)$$

Hence the horizons are located in correspondence of the roots of the equation $C = 0$; the tractable case is $M = M(v)$. Then this becomes a cubic equation which for $0 < 9\Lambda M^2 < 1$, as is well known, admits precisely two real positive roots r_c, r_b , with $r_c > r_b$ by definition. We see that θ_+ vanishes at both roots and $\theta_- < 0$, therefore the horizon spheres $r = r_b, r = r_c$, are marginally trapped surfaces of the future type which foliate a black hole and a cosmological trapping horizon, respectively. One can easily show that

$$\partial_- \theta_+ = -\frac{1}{r} \partial_r C.$$

Computing the radial derivatives at both horizons we see that this is negative at r_b and positive at r_c . The cosmological horizon is therefore an example of a trapping horizon of inner type, the black hole horizon at $r = r_b$ remaining of the outer type.

5.3.2. Rays tracing The most important features of a dynamical black hole of the kind discussed here are: (i) the existence of the irremovable space-like singularity at the origin $r = 0$ of the coordinate system; (ii) the possible existence of a global event horizon (\mathcal{H} in Figure 5); and (iii) the time-like future trapping horizon (TH in Figure 5).

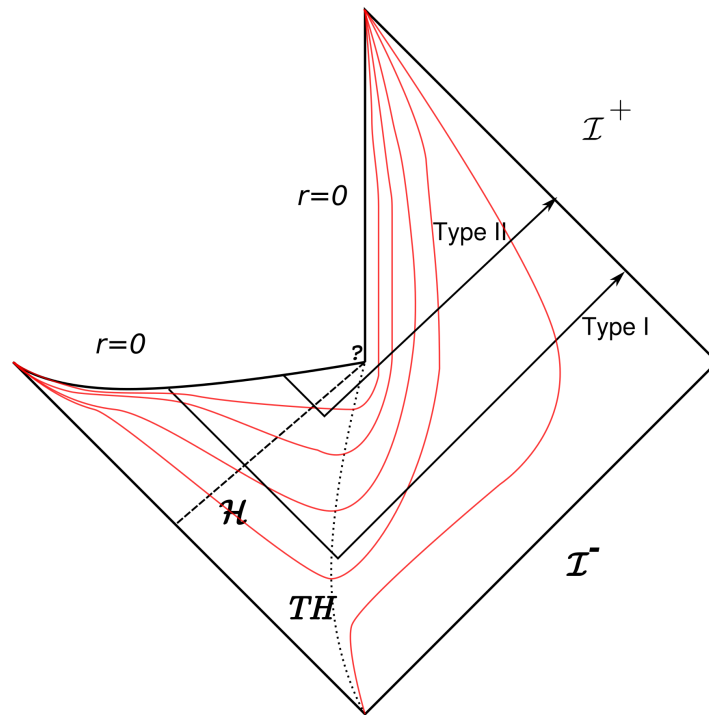


Figure 5. Evaporating black hole: the trapping horizon is at points where the radial outgoing rays have zero speed (dotted line). The event horizon \mathcal{H} is represented by a dashed line. Red lines represent curves of constant r .

We have seen that the BV form and the PG form are locally diffeomorphic to each other with the mass function transforming as a scalar field, so the causal structure of both solutions must be the same. A visual picture of the trapping horizon for an evaporating black hole is displayed in Figure 5. The horizontal line represents the space-like singularity $r = 0$. Consider the radial outgoing null rays: in BV form they obey the differential equation

$$\dot{r} \equiv \frac{dr}{dv} = \frac{1}{2} e^{\Phi(v,r)} \left(1 - \frac{2M(v,r)}{r} \right) \quad (5.60)$$

while ingoing rays simply are $v = v_0$, for some constant v_0 . For comparison, in the PG form (5.14) the same equation takes the form

$$\dot{r} \equiv \frac{dr}{d\tilde{t}} = \left(\pm 1 - \sqrt{\frac{2M(\tilde{t},r)}{r}} \right) e^{\Phi(\tilde{t},r)} \quad (5.61)$$

where $M(\tilde{t},r) = M(v(\tilde{t},r),r)$ and the plus (minus) sign referring to outgoing (ingoing) rays. Returning to the BV form, the event horizon is an outgoing null surface,

$$\dot{r}_{\text{EH}} = \frac{1}{2} \left(1 - \frac{2M}{r_{\text{EH}}} \right) e^{\Phi} < 0 \quad (5.62)$$

so that $r_{\text{EH}} < r_{\text{H}} = 2M$. The acceleration close to the trapping horizon is

$$\ddot{r} = -e^{\Phi} \frac{\dot{M}}{2M} + \frac{e^{\Phi}}{4M} (1 - 2M') \dot{r} \simeq \frac{e^{\Phi_{\text{H}}}}{r_{\text{H}}} L + \frac{e^{\Phi_{\text{H}}}}{4M} (1 - 2M') \dot{r} > 0. \quad (5.63)$$

We see that in both cases at the TH, which we recall is the line $r = 2M(v, r)$ for BV or $r = 2\tilde{M}(\tilde{t}, r)$ for PG, photons are only momentarily at rest, subsequently escaping on a dynamical time scale κ^{-1} , where κ is the surface gravity (5.55). Therefore outgoing photons (massless particles) have $\dot{r} < 0$ before reaching the trapping horizon and r must decrease on going to the horizon. This can be seen from the lines of constant r in Figure 5. Implicit in the Figure is also the idea that the Misner–Sharp mass has a non zero limit as $v \rightarrow -\infty$. More complicated dynamics are possible in general depending on the mass function, including the possibility of horizon formation at finite times (some of these are discussed in Kodama’s original paper [98]).

The fact that $\dot{r} = 0$ at the trapping horizon and only there is one of the most important facts at the root of the quantum tunnelling phenomenon. Now we would like to consider the case where $M(r, v)$ goes to a constant at very large negative advanced time and the trapping horizon extends to past infinity, since this is the region when the black hole does not radiate yet. In this case all radial null geodesics emerging from the trapping horizon seem to do so at $v = -\infty$, because the speed vanishes at $r = 2M$ (or $2\tilde{M}$). For particles with non zero angular momentum the trapping horizon is not a surface of momentarily zero speed. For instance, in the equatorial plane the velocity is given by

$$\dot{r} = \frac{1}{2} \left(1 - \frac{2M}{r} \right) e^{\Phi} - \frac{J^2}{2r^2} \quad (5.64)$$

where J is the conserved angular momentum per unit mass. The essential point is that $\dot{r} < 0$ at $r = 2M$, so that for a rotating particle it will even be harder to escape quantum mechanically than to a non rotating one. We will see in the next section that the presence of angular momentum will not affect the main conclusion that the radiation originates close to the trapping horizon, not the global event horizon. Actually, the global event horizon cannot even be computed without knowledge of the dynamics near the evaporation end point. The event horizon is a null hypersurface defined by the outermost locus traced by outgoing photons that can “never” reach arbitrarily large distances r . A practical way to locate the event horizon was given by Bardeen and York [81, 82]: one looks for photons that can only reach large r in a time comparable to the evaporation time $\sim E/L \gg E$. This implies that the event horizon is located by the unaccelerated photons with $\ddot{r} = 0$. From (5.62) and (5.63) it follows then that

$$\dot{r}_{\text{EH}} \simeq -\frac{L}{r_{\text{H}}\kappa}, \quad r_{\text{EH}} \simeq 2E \left(1 - \frac{2L}{r_{\text{H}}\kappa} \right) \simeq 2E(1 - 4L) \quad (5.65)$$

should be a solution to Equation (5.62) to first order in the black hole luminosity L (which is very small for large black holes) and ignoring contributions due to Φ and M' which are known to be negligible to first order in L .

5.3.3. *Tunnelling paths* We propose now to compute the probability amplitude for a quantum particle to cross a trapping horizon. We do this by making use of the Hamilton–Jacobi method. Recall that, in the spherically symmetric dynamical case, it is possible to introduce the Kodama vector field K such that $(K^\alpha G_{\alpha\beta})^{;\beta} = 0$, actually its defining property. Given the metric (5.16), the Kodama vector components are

$$K^i(x) = \frac{1}{\sqrt{-\Upsilon}} e^{ij} \partial_j R, \quad K^\theta = 0 = K^\phi. \quad (5.66)$$

where e^{ij} is the numerical skew tensor with $e^{01} = 1$. The Kodama vector gives a preferred flow of time and in this sense it generalises the flow of time given by the Killing vector in the static case (see [108] for a detailed study of the space-time foliation it determines). The conserved charges associated to K , Equation (5.40) and (5.41), are respectively the volume and the Misner–Sharp mass of space-time. We may also use the Kodama flow to define the invariant energy associated with a particle by means of the scalar quantity on the normal space

$$\omega = -K^i \partial_i I, \quad (5.67)$$

where I denotes the classical action of the massless particle, which we assume to satisfy the reduced Hamilton–Jacobi equation

$$\Upsilon^{ij} \partial_i I \partial_j I = 0. \quad (5.68)$$

Thus, for example, $\omega = -e^{-\Phi} \partial_v I$ will be the particle energy on an extremal in a BV gauge; we may note that this gauge is only fixed up to diffeomorphisms $v \rightarrow v' = v'(v)$, under which the field $\Phi(v, r)$ transforms as a conformal field, $\Phi(v, r) \rightarrow \Phi'(v', r) = \Phi(v, r) + \log |dv'/dv|$. Therefore ω will not depend on the choice of advanced time which respect to the BV form of the metric. Similarly, in Painlevé–Gullstrand gauge the invariant energy is $\omega = e^{-\Phi} \partial_{\tilde{t}} I$, and is gauge invariant under time re-parametrisation $\tilde{t} \rightarrow \bar{t} = \bar{t}(\tilde{t})$.

To illustrate the method we shall work in s-wave and omit the angular dependence of I . We stress the importance to have at disposal an invariant definition of energy. Equation (5.67) certainly satisfies this requirement if the action is a scalar. In the following our aim will be to show that there is a precise invariant prescription to deal with the imaginary part of the action, in case there is one, which is valid for all solutions in all coordinate systems which are regular across the horizon. The task is to compute

$$\text{Im} \int_{\gamma} dI \quad (5.69)$$

along a tunnelling path, say γ . A priori there are two types of such tunnelling paths, which we shall call type-I and type-II, because a pair may form on either sides of the trapping horizon. A type-I path corresponds to a pair forming outside and therefore will be a backward null ray coming out from the future singularity at $r = 0$, crossing the trapping horizon and subsequently escaping to infinity, as illustrated in Figure 5. The figure is the dynamical analogue of the corresponding figure in Schwarzschild space-time, except that it is smooth at $\dot{r} = 0$, namely at the trapping horizon. A type-II

path corresponds to a pair forming in the trapped region, so we may take a backward null ray crossing the event horizon from the singularity till some interior point, which subsequently escapes to infinity along a null ray crossing the trapping horizon at some other event. In some cases such a path (namely an outgoing light ray emerging from the trapping horizon) seems to emerge from the trapping horizon at $v = -\infty$ in advanced time, therefore in this case, type-II paths are those with exactly one point, the starting point, lying on the horizon. Type-II paths are absent in static geometries, since there is no region in between the two horizons. The segment of type-II paths crossing the horizon outward seems nevertheless an allowed classical path so one may wonder whether they can contribute to an emission amplitude: does the pole disappear? We will see indeed that on such paths the radial momentum vanishes, basically for the reason that the speed is zero on the trapping horizon — and only there — so in fact they do not contribute semi-classically.

We shall illustrate these results in some of the gauges we discussed above for spherically symmetric space-times, having two main purposes: to illustrate the intrinsic covariance of the result and to display how contributions to the tunnelling amplitude may or may not come from all terms in the differential form $dI = \partial_i I dx^i$. We also have the opportunity to test the formalism and to see how different can be the prosaic description of the same physical effect from the perspective of other coordinate systems, a fact well known in special and general relativity.

The BV gauge

Here the metric is

$$ds^2 = -e^{2\Phi} C dv^2 + 2e^\Phi dv dr + r^2 d\Omega^2 \quad (5.70)$$

where $M = (1 - C)r/2$ is the Misner–Sharp mass discussed in Section 5.2.3. The Kodama vector and the invariant particle energy assume the simple expressions given by $K = (e^{-\Phi}, \mathbf{0})$ and $\omega = -e^{-\Phi} \partial_v I$, while the invariant surface gravity is just given by $\kappa_H = \partial_r C_H/2$. As noted above Φ transforms as an ordinary Liouville field, *i.e.* $\Phi \rightarrow \Phi + \log |\partial \tilde{v} / \partial v|$, under $v \rightarrow \tilde{v}(v)$, making ω invariant under re-parametrisation of the advanced time coordinate.

Type-II tunnelling paths are easily dispensed for: they cross the horizon along a null direction which is outward with zero speed at the horizon, so $\dot{r} = 0$ at and in a neighbourhood of the horizon, or $dr = 0$ along the null direction; therefore on a small segment crossing the trapping horizon we have

$$\int \partial_r I dr = 0.$$

On a real trajectory the energy is real and finite everywhere, which shows that the temporal v -contribution does not play any role, too, in what concerns us at the moment (the evaluation of $\text{Im } I$). Thus $\text{Im } I = 0$ on type-II paths.

For a type-I path the ray crosses the horizon along a null ingoing ray but in the backward direction; such a ray has $v = v_0$ (some constant) and $\delta v = 0$ exactly, therefore

again the $\partial_v I dv$ term will vanish at the horizon and $\delta r > 0$. From the Hamilton–Jacobi equation we see that, for outgoing modes of the kind we consider,

$$C(\partial_r I) = 2\omega . \quad (5.71)$$

Thus one has

$$\begin{aligned} \text{Im } I &= \text{Im} \int_{\gamma} (\partial_r I dr + \partial_v I dv) = \text{Im} \int_{\gamma} dr \frac{2\omega}{C} \\ &= 2 \text{Im} \int_{\gamma} dr \frac{\omega}{\partial_r C|_{\text{H}}(r - r_{\text{H}} - i\epsilon)} = \frac{\pi\omega_{\text{H}}}{\kappa_{\text{H}}} , \end{aligned} \quad (5.72)$$

where ω_{H} is ω evaluated on the horizon, the quantity C has been expanded around the horizon pole, that is

$$C(v, r) = \partial_r C|_{\text{H}} \Delta r + \dots \quad (5.73)$$

and Feynman’s $i\epsilon$ -prescription has been implemented in order to deal with the simple pole. $\kappa_{\text{H}} = \partial_r C|_{\text{H}}/2$ is the horizon surface gravity and coincides with our geometrical expectations. Unlike the stationary black holes, where ω is a constant of motion, here we get the local energy and temperature as measured near the horizon, which are connected to the quantities at infinity by the corresponding red-shift factors. We see that, in a BV coordinate system, the temporal integration does not give any contribution to the imaginary part of the action of particles tunnelling through the trapping horizon.

It will be very important to discuss the physical meaning of this result but we prefer to postpone this discussion to the end of the section, after we will have shown that the tunnelling amplitude is not an artefact due to a special coordinate system but holds as good as in any other.

The conformal two-dimensional gauge

A coordinate system where the temporal contribution to the action plays an essential role is the general diagonal form of a spherically symmetric metric, which reads

$$ds^2 = e^{\psi(t,r)} (-dt^2 + dr^2) + R^2(t, r) d\Omega^2 . \quad (5.74)$$

In this form, the normal metric is conformally related to the two-dimensional Minkowski space-time. The χ function simply reads

$$\chi = e^{-\psi} [-(\partial_t R)^2 + (\partial_r R)^2] , \quad (5.75)$$

If $R(t, r)$ is a monotonically increasing function of r (the normal case) then this leads to the future trapped horizon condition

$$(\partial_t R)_{\text{H}} + (\partial_r R)_{\text{H}} = 0 . \quad (5.76)$$

If instead $R(r, t)$ is decreasing as a function of r , then we have to take the opposite relative sign, because we expect that an evaporating horizon should have a negative time derivative, $(\partial_t R)_{\text{H}} < 0$. A computation of the expansion scalars will confirm this fact. The Kodama vector and the associated invariant energy are

$$K = e^{-\psi} (\partial_r R, -\partial_t R, 0, 0) , \quad (5.77)$$

$$\omega = e^{-\psi} (-\partial_r R \partial_t I + \partial_t R \partial_r I) . \quad (5.78)$$

The dynamical surface gravity reads

$$\kappa_{\text{H}} = \frac{1}{2} e^{-\psi_{\text{H}}} (-\partial_t^2 R + \partial_r^2 R) \Big|_{\text{H}} . \quad (5.79)$$

Due to conformal invariance, the Hamilton–Jacobi equation is the same as in two-dimensional Minkowski space-time, namely using double null coordinates $\partial_+ I \partial_- I = 0$, and for the outgoing particle we may take

$$\partial_+ I = \partial_t I + \partial_r I = 0 , \quad (5.80)$$

since the radial momentum $p_r = \partial_r I > 0$ and $\partial_t I < 0$ for a real outgoing particle. Given that for a type-II path the null expansion condition leads to $\delta x^- = \delta t - \delta r = 0$, we get $dI = \partial_+ I dx^+ + \partial_- I dx^- = 0$ on account of the Hamilton–Jacobi equation, and there is no imaginary part. On reflection, this result seems to violate our basic tenet that there is no amplitude because the photon speed vanishes at the horizon. In fact, the coordinate speed $\dot{r} = \pm 1$ everywhere in conformal gauge and it never vanishes. However, the speed of the wave front is

$$\dot{R} = \partial_r R \dot{r} + \partial_t R$$

and this vanishes on H for an outgoing photon with $\dot{r} = 1$, on account of the horizon equation (5.76). Thus, what really matters is that the speed of the areal radius should vanish at the horizon.

But there are also the type-I tunnelling paths, for which the null expansion gives $\delta x^+ = \delta t + \delta r = 0$. Therefore in this case

$$I = \int_{\gamma} (dr \partial_r I + dt \partial_t I) = 2 \int_{\gamma} dr \partial_r I . \quad (5.81)$$

Furthermore, due to (5.78) and (5.80), one has

$$\partial_r I = \frac{\omega}{e^{-\psi} (\partial_r R + \partial_t R)} . \quad (5.82)$$

and we have a pole at the horizon. Making use of near horizon approximation along the null direction, from (5.76) and (5.79), one has $(\partial_r R)_{\text{H}} + (\partial_t R)_{\text{H}} = 0$, $\delta t + \delta r = 0$, thus with the understanding that we shall put $\delta r = r - r_{\text{H}}$, we get

$$\begin{aligned} e^{-\psi} [\partial_r R + \partial_t R] &= e^{-\psi_{\text{H}}} (\partial_{rr}^2 R - \partial_{rt}^2 R - \partial_{tt}^2 R + \partial_{tr}^2 R) \Big|_{\text{H}} (r - r_{\text{H}}) + \dots \\ &= 2\kappa_{\text{H}} (r - r_{\text{H}}) + \dots \end{aligned} \quad (5.83)$$

and the imaginary part follows. If $R(t, r)$ is a decreasing function of r then we shall have to take $p_r = \partial_r I < 0$ but again $\partial_t I < 0$ so the Hamilton–Jacobi equation implies now $\partial_- I = 0$ along the tunnelling path. Moreover since r is decreasing as R increases (*i.e.* toward infinity), on an outgoing (ingoing) null direction we have $\delta t + \delta r = 0$ ($\delta t - \delta r = 0$) and the horizon condition becomes

$$(\partial_t R - \partial_r R) \Big|_{\text{H}} = 0$$

instead of Equation (5.76). It is now simple to see that a tunnelling type-II path gives no amplitude at all, but that a type-I does produce an imaginary part with the right magnitude and sign. In fact in this case we obtain first

$$\partial_r I = -\frac{\omega}{e^{-\psi}(\partial_r R - \partial_t R)},$$

then, recalling $\partial_- I = 0$,

$$dI = \partial_+ I dx^+ = 2\partial_r I dr = -2\frac{\omega}{e^{-\psi}(\partial_r R - \partial_t R)} dr \cong -2\frac{\omega}{\kappa_H(r - r_H)} dr.$$

The integral of dI has to be done on the segment of the path crossing the horizon along decreasing r , say from r_2 to r_1 with $r_1 < r_2$, and this is of course minus the usual integral in increasing order; reversing the orientation gives back the usual result

$$\text{Im} \int_{\curvearrowright} dI = \frac{\pi\omega}{\kappa_H}.$$

The synchronous gauge

Another coordinate system where the coordinate speed does not vanish at the trapping horizon is described by the line element

$$ds^2 = -dt^2 + \frac{1}{B(r,t)} dr^2 + R^2(r,t) d\Omega^2 = d\gamma^2 + R^2(r,t) d\Omega^2, \quad (5.84)$$

in which the metric is diagonal, but R is a function of r and t . In this case, one has

$$\chi = -(\partial_t R)^2 + B(\partial_r R)^2, \quad (5.85)$$

If $R(r,t)$ is an increasing function of r , the horizon $\chi_H = 0$ should be defined by

$$(\partial_t R)_H = -\sqrt{B_H}(\partial_r R)_H, \quad (5.86)$$

in which we are assuming again a regular coordinate system on the horizon, namely that B_H and the partial derivatives are non-vanishing. For R decreasing with r one should change the sign to the right of Equation (5.86). The Kodama vector reads

$$K = (\sqrt{B}\partial_r R, -\sqrt{B}\partial_t R, 0, 0), \quad (5.87)$$

and the invariant energy

$$\omega = \sqrt{B}(\partial_r R \partial_t I - \partial_t R \partial_r I). \quad (5.88)$$

The dynamical surface gravity is evaluated to be

$$\kappa_H = \frac{1}{4} \left(-2\partial_t^2 R_H + 2B_H \partial_r^2 R_H + \frac{1}{B_H} \partial_t R_H \partial_t B_H + \partial_r R_H \partial_r B_H \right). \quad (5.89)$$

Making use of the horizon condition, we may rewrite it

$$\kappa_H = \frac{1}{4} \left(-2\partial_t^2 R_H + 2B_H \partial_r^2 R_H - \frac{1}{\sqrt{B_H}} \partial_r R_H \partial_t B_H + \partial_r R_H \partial_r B_H \right). \quad (5.90)$$

In this case, the Hamilton–Jacobi equation reads simply

$$-(\partial_t I)^2 + B(\partial_r I)^2 = 0 \quad (5.91)$$

and for an outgoing particle we have to choose the root with $\partial_r I > 0$. Therefore, in a type-II path crossing the trapping horizon we have $\sqrt{B_H} \delta t = \delta r$, in a neighbourhood of TH, because $\dot{r} = \sqrt{B}$. As a consequence the outgoing temporal contribution will cancel the radial one. Type-II paths will then give no tunnelling amplitude. On reflection, this happens because although the coordinate speed is non vanishing at the trapping horizon, the areal velocity indicating the speed of the wave front

$$\dot{R} = \partial_r R \dot{r} + \partial_t R$$

does vanish on TH for an outgoing photon, which has $\dot{r} = \sqrt{B}$, on account of Equation (5.86). On a type-I path instead $\sqrt{B_H} \Delta t = -\Delta r$, and the two contributions add so that

$$I = 2 \int_{\gamma} dr \partial_r I. \quad (5.92)$$

The Hamilton–Jacobi equation and the expression for the invariant energy lead to

$$\partial_r I = \frac{\omega}{B \partial_r R + \sqrt{B} \partial_t R}. \quad (5.93)$$

Making the expansion along the null curve, for which $\sqrt{B_H} \Delta t + \Delta r = 0$, in the near-horizon approximation, one gets

$$\text{Im } I = 2 \cdot \text{Im} \int_{\gamma} dr \frac{\omega}{2\kappa_H(r - r_H - i\epsilon)} = \frac{\pi\omega_H}{\kappa_H}, \quad (5.94)$$

leading to the desired amplitude. But notice that in this gauge, the temporal contribution is essential in order to obtain the correct result.

The r -gauge

The fact that type-II paths never contributed to the tunnelling amplitude suggest that it is a general fact. Our last example will use a general metric in the r -gauge to confirm this. The normal metric here is non-diagonal, but as in BV gauge $R = r$. We have

$$ds^2 = d\gamma^2 + r^2 d\Omega^2, \quad (5.95)$$

where the reduced normal metric is now taken in the (redundant) form

$$d\gamma^2 = -E(r, t) dt^2 + 2F(r, t) dt dr + G(r, t) dr^2, \quad F \neq 0. \quad (5.96)$$

The horizon is located at the zeroes of

$$\chi(t, r) = \gamma^{ij} \partial_i R \partial_j R = \gamma^{rr}(t, r) = \frac{E}{EG + F^2} = 0 \quad (5.97)$$

i.e. at $E_H = E(t, r_H) = 0$, provided $F_H \neq 0$. The Misner–Sharp mass inside a sphere of radius r is

$$M = \frac{r}{2} \left(1 - \frac{E}{EG + F^2} \right) \quad (5.98)$$

and its value on the horizon is the black hole mass by definition, $M = r_H/2$. The other ingredient, the Kodama vector, reads

$$K = \left(\frac{1}{\sqrt{F^2 + EG}}, 0, 0, 0 \right), \quad (5.99)$$

and the invariant particle energy follows

$$\omega = -\frac{\partial_t I}{\sqrt{F^2 + EG}}. \quad (5.100)$$

The dynamical surface gravity is computed to be

$$\kappa_{\text{H}} = \frac{1}{2} \square_{\gamma} r \Big|_{\text{H}} = \left[\frac{1}{2F^3} \left(E'F - \frac{1}{2} \dot{E}G \right) \right] \Big|_{\text{H}}, \quad (5.101)$$

where an overdot and a prime denote differentiation with respect to t and r , respectively. Now the horizon will be future and of outer type provided $F > 0$ and $\kappa_{\text{H}} > 0$. We expect the mass within the sphere to be a decreasing function of r ; also, for an excreting black hole the time derivative \dot{M} should be negative as well. From (5.98) this will be true if $E' > 0$ and $\dot{E} > 0$, although these are only sufficient conditions. It follows that for $F < 0$, $\kappa_{\text{H}} > 0$ always but the horizon is foliated by past marginally trapped surfaces ($G > 0$ of course), while for $F > 0$ it is future and outer if the inequality

$$E' \Big|_{\text{H}} > \left(\frac{\dot{E}G}{2F} \right) \Big|_{\text{H}} \quad (5.102)$$

is true, which we shall assume from now on. The outgoing null rays at the horizon satisfy either $\dot{r} = 0$ or, if $F < 0$, $\dot{r} = -2F/G$. Only the first type will concern us, since photons from the past horizon should always be allowed classically. In fact, from the Kodama energy expression (5.100) and equation (5.97) as well, the Hamilton–Jacobi equation reads

$$\chi(\partial_r I)^2 - 2 \frac{\omega F}{\sqrt{EG + F^2}} \partial_r I - \omega^2 G = 0, \quad (5.103)$$

Solving this, one sees that $\partial_r I$ has no pole at H (which is past for $F < 0$) and therefore no imaginary part is found. Returning to our case with $F > 0$, on a type-II path $\dot{r} = 0$ at the horizon, therefore, as before, the radial momentum vanishes and no imaginary term can come from $\int \partial_r I dr$.

It remains to consider type-I paths. From the metric the null radial expansion across the segment crossing the horizon outward gives $\delta t = -(G/2F)|_{\text{H}} \delta r$, now with $F > 0$, so we can expand the χ function along this ingoing null direction to first order in $\delta r = r - r_{\text{H}}$. We obtain easily

$$\chi \cong \dot{\chi} \delta t + \chi' \delta r = \left(\chi' - \frac{G}{2F} \dot{\chi} \right) \Big|_{\text{H}} (r - r_{\text{H}}) + \dots = 2\kappa_{\text{H}}(r - r_{\text{H}}) + \text{O}((r - r_{\text{H}})^2) \quad (5.104)$$

where \cong means evaluation on the horizon and Equation (5.101) has been used. Also, $\partial_t I = -F_{\text{H}} \omega$ from definition (5.100) and the horizon condition $E = 0$. Hence we end up with I given by the sum of a real term and a possibly imaginary part coming from the horizon:

$$I = \int_{\gamma} (dr \partial_r I + dt \partial_t I) = \int_{\gamma} dr \left[\partial_r I + \frac{1}{2} G_{\text{H}} \omega_{\text{H}} \right]. \quad (5.105)$$

What is remarkable is that in this gauge the temporal part is present but, being regular, it does not contribute to the imaginary part of the action. From the Hamilton–Jacobi equation (5.103) we get, choosing the solution with positive radial momentum,

$$\partial_r I = \frac{\omega F}{\sqrt{EG + F^2} \chi} (2 + \mathcal{O}(\chi)) . \quad (5.106)$$

and we see that this has a pole at $\chi = 0$. Making use of this equation, Feynman’s prescription and (5.104), one has once more

$$\text{Im } I = \text{Im} \int_{\gamma} dr \partial_r I = \text{Im} \int_{\gamma} dr \frac{\omega F}{\sqrt{F^2 + EG}} \cdot \frac{1 + \sqrt{1 + \mathcal{O}(\chi)}}{2 \kappa_H (r - r_H - i\epsilon)} = \frac{\pi \omega_H}{\kappa_H} . \quad (5.107)$$

We have shown in general that type-II paths will not contribute a tunnelling amplitude to future trapping horizons ($F > 0$), because $\delta r = 0$ to first order in χ and we are assuming the energy to be real and finite. As to type-I paths the situation also is clear, they give a amplitude fully controlled by the value of the geometrical surface gravity κ_H . This example also shows the possibility of a quantum tunnelling through past trapping horizons of outer type, as for $F < 0$ there is an amplitude for a photon to cross with negative radial momentum, *i.e.* for a photon to be absorbed (we are using the mental picture of a past horizon as one from which it is impossible to enter but very easy to escape). The fact that there is no imaginary part on type-II paths either because the coordinate speed of a photon vanishes or because the temporal contribution cancels the radial one was the source of much confusion in the past literature. For instance, Belinski ventured to say that there is no Hawking radiation from black holes just from this fact [109], since he apparently forgot about the existence of type-I paths.

We end this section by noticing once more the special role of the trapping horizon. The imaginary part being produced on crossing it, it should be natural to think that this is the place where most of radiation forms. This has been confirmed in [102], where it is shown that asymptotic observers register a radiation flux that starts increasing at a time they see the collapsing shell crosses the surface $r = 2M$ of the model used (a special case of Painlevé–Gullstrand model with a time dependent mass, see Equation (5.14)). The stress-tensor for Vaidya space-time has been considered in [110] and should be consistent with this result.

A note — In the previous computations various choices of signs have been applied in such a way that it may seem they were chosen somewhat *ad hoc* in order to get the wanted result. This is not so. Once the future sheet of the trapping horizon has been chosen, and the sign of the Kodama vector determined so that it is future directed, no other sign uncertainties will occur for either outgoing or ingoing particles if one uses consistently the Hamilton–Jacobi equation for outgoing particles on every segment of the tunnelling path; furthermore, questions such as the right form of the equations determining the horizon, the orientation of the space-time coordinates and the sign of the metric coefficients are carefully established once for all. On the other hand, if there exists a past sheet in the trapping horizon then using the tunnelling picture we may as well compute the action along an inward directed curve at the horizon (the ambiguity

inherent in this and analogous terms is easily resolved if the manifold is asymptotically flat). Then there will be again a non-vanishing imaginary part, but we can interpret it as a small absorption probability, as we did for the static case in relation to white holes.

5.4. *Tunnelling of fermions*

At this point of the discussion, one might ask what particles are to be found in the Hawking radiation spectrum of a black hole. Since a black hole has a well defined temperature, in principle it should radiate all the standard model particles like a black body at that temperature (ignoring grey body factors). The emission spectrum, therefore, is expected to contain particles of all spins, in particular fermions as well. The basic reference here is the seminal paper by Kerner and Mann [111], to which others followed for both stationary and dynamical black holes [72, 112–115]. The fact that massless fermions are emitted with the same temperature as massless bosons is not a trivial result, given the fact that fermionic and bosonic vacua are in general distinct. It is important to mention that these expectations have been recently extended to include spin-1 bosons and gravitinos as well; and that the Hawking temperature does not receive higher order corrections in \hbar beyond the semi-classical one [72, 73] as originally proposed in [67] (see also [68–71] for further extensions).

Now, what do we take as the action of fermionic particles? Undoubtedly the most convenient choice would be to take the phase of the spinor wave function which, as is well known, satisfies the Hamilton–Jacobi equation as for spin-less particles. However, one may also consider the action from another point of view. Virtually, all known variational formulations of the motion of spinning particles have an action of the form [116]

$$I_f = I_0 + (\text{spin corrections})$$

where I_0 is the kinetic term equal to the classical action of scalar particles as considered in previous sections, and the correction terms contain the coupling of the spin degrees of freedom with the spin connection of the manifold. These can either be derived from covariance considerations or more directly by the semi-classical treatment of the Dirac equation itself. Some of these terms give additive corrections to the particle four-momentum but in no case the remaining terms contain horizon singularities, as they are only responsible for spin precession effects. In the following course, therefore, we shall ignore them. Next, we neglect any change of angular momentum of the black hole due to the spin of the emitted particle. For zero-angular momentum black holes with mass much larger than the Planck mass ($m_{\text{Pl}} \sim 10^{-5}$ gr) this is a good approximation. Statistically, as many particles with spin in one direction will be emitted as particles with spin in the opposite direction, producing no net change in the black hole angular momentum (although second-order statistical fluctuations will be present in general).

Here, we are going to follow [112] and describe the tunnelling of spin-1/2 particles in dynamical, spherically symmetric, geometries in BV or McVittie forms, equation (5.70) and (5.23)-(5.26), respectively. The static cases can always be recovered as an opportune

limit of the dynamical metrics above. We refer to the original paper [111] for analogous discussions in generalised Painlevé–Gullstrand and Kruskal–Szekeres coordinates.

Temporarily re-introducing \hbar , the Dirac equation reads

$$\left(\gamma^\mu D_\mu + \frac{m}{\hbar}\right) \Psi(v, r, \theta, \phi) = 0, \quad (5.108)$$

where

$$D_\mu = \partial_\mu + \frac{i}{2} \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta}, \quad (5.109)$$

$$\Sigma_{\alpha\beta} = \frac{i}{4} [\gamma_\alpha, \gamma_\beta]_-. \quad (5.110)$$

The γ^μ -matrices satisfy the Clifford algebra,

$$[\gamma_\alpha, \gamma_\beta]_+ = 2g_{\alpha\beta} \mathbb{I}, \quad (5.111)$$

where \mathbb{I} is the (4×4) -identity matrix. In order to get the Dirac- γ matrices for the BV metric (5.70) at hand, we define firstly a tetrad of orthogonal vectors E_μ^a such that

$$\eta_{ab} E_\mu^a E_\nu^b = g_{\mu\nu}. \quad (5.112)$$

The convention is that (first) Latin indices are Minkovskian so they run over $(0, 1, 2, 3)$; Greek indices are coordinate indices so they run over (v, r, θ, ϕ) . Of course there are many different tetrads, but the simplest choice is the following:

$$E_v^a = \left(e^\Phi \sqrt{C}, 0, 0, 0 \right), \quad (5.113)$$

$$E_r^a = \left(-\frac{1}{\sqrt{C}}, \frac{1}{\sqrt{C}}, 0, 0 \right), \quad (5.114)$$

$$E_\theta^a = (0, 0, r, 0), \quad (5.115)$$

$$E_\phi^a = (0, 0, 0, r \sin \theta). \quad (5.116)$$

The γ_μ -matrices are expressed in terms of the tetrad in the following way:

$$\gamma_\mu = \gamma_a E_\mu^a. \quad (5.117)$$

With our choice (5.113) - (5.116), it turns out that

$$\gamma_v = e^\Phi \sqrt{C} \gamma_0, \quad \gamma_r = \frac{1}{\sqrt{C}} (\gamma_1 - \gamma_0), \quad \gamma_\theta = r \gamma_2, \quad \gamma_\phi = r \sin \theta \gamma_3 \quad (5.118)$$

and

$$\gamma^v = \frac{e^{-\Phi}}{\sqrt{C}} (\gamma_1 - \gamma_0), \quad \gamma^r = \sqrt{C} \gamma_1, \quad \gamma^\theta = \frac{1}{r} \gamma_2, \quad \gamma^\phi = \frac{1}{r \sin \theta} \gamma_3. \quad (5.119)$$

We can also express the

$$\gamma^5 \stackrel{def}{=} i \gamma^v \gamma^r \gamma^\theta \gamma^\phi = \frac{i e^{-\Phi}}{r^2 \sin \theta} (\mathbb{I} - \gamma_0 \gamma_1) \gamma_2 \gamma_3. \quad (5.120)$$

All what we need now is a convenient representation of the Dirac- γ_a matrices satisfying $[\gamma_a, \gamma_b]_+ = 2\eta_{ab}$ with $\eta = \text{diag}(-1, 1, 1, 1)$.

$$\gamma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}; \quad (5.121)$$

$$\gamma_2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \quad (5.122)$$

The σ -matrices are the Pauli matrices satisfying the usual relations:

$$\sigma_i \sigma_j = \mathbb{I}^{(2 \times 2)} \delta_{ij} + i \varepsilon_{ijk} \sigma_k, \quad i, j, k = 1, 2, 3. \quad (5.123)$$

Calculating the $\Gamma_\mu^{\alpha \beta}$ symbols and the $\Sigma_{\alpha\beta}$ defined in (5.110), we have

$$\not{\partial} \Psi + i \left[\frac{1}{2} \gamma^\mu \Gamma_\mu^{\alpha \beta} \Sigma_{\alpha\beta} \Psi \right] + \frac{m}{\hbar} \Psi = 0. \quad (5.124)$$

Let us employ the following ansatz for the spin-up Dirac field (We shall perform a detailed analysis only for the spin-up case, being confident that *mutatis mutandis* everything applies in the same way to the spin-down case):

$$\Psi(v, r, \theta, \phi)_\uparrow = \begin{pmatrix} H(v, r, \theta, \phi) \\ 0 \\ Y(v, r, \theta, \phi) \\ 0 \end{pmatrix} \exp \left[\frac{i}{\hbar} I_\uparrow(v, r, \theta, \phi) \right]. \quad (5.125)$$

Plugging the ansatz (5.125) into the Dirac equation (5.124), it turns out that the term in square brackets is of order $O(\hbar)$. Thus, we do not need to work out its precise form, since in the $\hbar \rightarrow 0$ limit it vanishes. To leading order in \hbar , equation (5.124) becomes $0 = (\hbar \not{\partial} + m) \Psi_\uparrow + O(\hbar)$, or

$$0 = \exp \left[\frac{i}{\hbar} I_\uparrow(v, r, \theta, \phi) \right] \cdot \left\{ \frac{e^{-\Phi}}{\sqrt{C}} \begin{pmatrix} iH\partial_v I_\uparrow - Y\partial_v I_\uparrow \\ 0 \\ -H\partial_v I_\uparrow - iY\partial_v I_\uparrow \\ 0 \end{pmatrix} + \sqrt{C} \begin{pmatrix} -Y\partial_r I_\uparrow \\ 0 \\ -H\partial_r I_\uparrow \\ 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 0 \\ -Y\partial_\theta I_\uparrow \\ 0 \\ -H\partial_\theta I_\uparrow \end{pmatrix} + \frac{1}{r \sin \theta} \begin{pmatrix} 0 \\ -iY\partial_\phi I_\uparrow \\ 0 \\ -iH\partial_\phi I_\uparrow \end{pmatrix} + im \begin{pmatrix} H \\ 0 \\ Y \\ 0 \end{pmatrix} \right\}.$$

Thus, we get the following equations:

$$v : \quad \frac{iH}{\sqrt{C}} (e^{-\Phi} \partial_v) I_\uparrow - \frac{Y}{\sqrt{C}} (e^{-\Phi} \partial_v) I_\uparrow - \sqrt{C} Y \partial_r I_\uparrow + iHm = 0 \quad (5.126)$$

$$r : \quad -\frac{Y}{r} \left(\partial_\theta I_\uparrow + \frac{i}{\sin \theta} \partial_\phi I_\uparrow \right) = 0 \quad (5.127)$$

$$\theta : \quad -\frac{H}{\sqrt{C}} (e^{-\Phi} \partial_v) I_\uparrow - \frac{iY}{\sqrt{C}} (e^{-\Phi} \partial_v) I_\uparrow - \sqrt{C} H \partial_r I_\uparrow + iYm = 0 \quad (5.128)$$

$$\phi : \quad -\frac{H}{r} \left(\partial_\theta I_\uparrow + \frac{i}{\sin \theta} \partial_\phi I_\uparrow \right) = 0 \quad (5.129)$$

and the Kodama vector $K = e^{-\Phi} \partial_v$ has been put in evidence throughout. K plays the role of the Killing vector ∂_t for dynamical black holes. Therefore, it makes sense the following *ansatz* for the action:

$$I_\uparrow = - \int dv e^{\Phi(r,v)} \omega + \mathcal{F}(r) + J(\theta, \phi), \quad (5.130)$$

which inserted into (5.126) - (5.129) gives:

$$v : \quad \frac{1}{\sqrt{C}}(iH - Y)\omega - \sqrt{CY}\mathcal{F}'(r) + iHm = 0 \quad (5.131)$$

$$r : \quad -\frac{Y}{r} \left(J_\theta(\theta, \phi) + \frac{i}{\sin\theta} J_\phi(\theta, \phi) \right) = 0 \quad (5.132)$$

$$\theta : \quad -\frac{1}{\sqrt{C}}(H + iY)\omega - \sqrt{CH}\mathcal{F}'(r) + iYm = 0 \quad (5.133)$$

$$\phi : \quad -\frac{H}{r} \left(J_\theta(\theta, \phi) + \frac{i}{\sin\theta} J_\phi(\theta, \phi) \right) = 0 . \quad (5.134)$$

Equations (5.132) and (5.134) imply that $J(\theta, \phi)$ is a complex function. The same solution for J is obtained for the spin-down case, then its contribution to the rate emission Γ cancels out and we can forget about it. As regard the remaining equations, we have, both in the massless and massive cases, that

- (i) $H = -iY$, so that $\mathcal{F}'(r) = 0$, corresponding to the case of the incoming particle absorbed in the classical limit with probability equal to 1, since

$$\gamma^5 \Psi = \begin{pmatrix} iH - Y \\ 0 \\ H + iY \\ 0 \end{pmatrix} . \quad (5.135)$$

- (ii) $H = iY$ so that $\mathcal{F}'(r) = -2\omega/C(r, v)$, describing the emission process, indeed.

This implies that

$$\text{Im } \mathcal{F}(r) = -\text{Im} \int dr \frac{\omega}{C(r, v)/2} = \frac{2\pi\omega}{C'(r_H(v), v)} , \quad (5.136)$$

confirming the predictions of the Kodama–Hayward theory. We conclude that our dynamical black hole is unstable against the emission of spinor particles. It is essential, for this result to hold, that the black hole be slowly evolving on the timescale of the wave, since otherwise no meaningful notion of a frequency is available.

For the McVittie's solution, we see that previous calculations showed that the derivatives of the action were strongly mixed by the matrix structure of the Dirac equation. Nevertheless things combined so that only the radial derivative of the action was really important. Hence, we do not study the full Dirac equation in the following, but use instead a shortcut. Writing as before

$$\Psi = U \exp \left[\frac{i}{\hbar} I(t, r, \theta, \phi) \right] \quad (5.137)$$

where U is a slowly varying spinor amplitude, from the Dirac equation we get

$$\not{D}U + \hbar^{-1} (i\not{\partial}I + m) U = 0 . \quad (5.138)$$

In the semi-classical limit the second term dominates, so $(i\not{\partial}I + m) U = 0$; thus the matrix $(i\not{\partial}I + m)$ must be singular, or equivalently,

$$g^{\mu\nu} \partial_\mu I \partial_\nu I + m^2 = 0 . \quad (5.139)$$

The action will have a simple pole at the location of the trapping horizon affecting to outgoing modes, so for these we also neglect the mass term. Then for an outgoing particle we find

$$\partial_r I = -\frac{\partial_t I}{F(r, t)}, \quad (5.140)$$

where

$$F(r, t) = \sqrt{A_s(r)}(\sqrt{A_s(r)} - rH(t)). \quad (5.141)$$

We pick the imaginary part by expanding this function at the horizon along a future null direction, using the fact that for two neighbouring events on a null direction in the metric (5.26), one has $t - t_0 = (2H_0^2 r_0^2)^{-1}(r - r_0)$, where $H_0 = H(t_0)$. We find the result

$$F(r, t) = \left(\frac{1}{2} A'_s(r_0) - r_0 H_0^2 - \frac{\dot{H}_0}{2H_0} \right) (r - r_0) = \kappa_0 (r - r_0) \quad (5.142)$$

where this time $r_0 = r_H(t_0)$. From this equation we see that $\partial_r I$ has a simple pole at the trapping horizon; hence, making use again of Feynman $i\epsilon$ -prescription, one finds

$$\text{Im} I = \pi \kappa(t_0)^{-1} \omega(t_0), \quad (5.143)$$

where $\omega(t) = -\partial_t I$ is the energy at time t , in complete agreement with the geometric evaluation of the previous sections. However, we stress that a full justification of the given shortcut really requires the full spinor amplitudes, as we showed above while discussing the BV black hole.

6. Cosmological Horizons, Decays and Naked Singularities

In this section we review the formalism of horizon tunnelling as applied to cosmological horizons, decay of unstable particles that in the absence of gravity would be otherwise stable and radiation from naked singularities. Especially the subject of horizon tunnelling in cosmology attracted much interest recently, leading to a related stream of papers. A sample of these articles which seemed relevant to us can be found in [117,118], although we rely mainly on the papers [53,119] and the references cited therein. We shall follow mainly the Hamilton–Jacobi version of the tunnelling picture, although the null geodesic method can be still applied. Results in this direction appeared recently in [29], where the Parikh–Wilczek and Hamilton–Jacobi methods are compared and showed to agree. An early study of the evolution of evaporating black holes in inflationary cosmology is in [120].

6.1. The FRW space-time

Consider a generic FRW space-time, namely one with constant curvature spatial sections. Its line element can be written as

$$ds^2 = -dt^2 + \frac{a^2(t)}{1 - kr^2} dr^2 + [a(t)r]^2 d\Omega^2 \quad (6.1)$$

where r is measured in units of the curvature radius and, as usual, $k = 0, -1, +1$ labels flat, open and closed three-geometries, respectively. In this gauge, the normal reduced metric is diagonal and the horizon is implicitly given by the equation $\chi = 0$, where

$$\chi(t, r) = 1 - [a(t)r]^2 \left[H^2(t) + \frac{k}{a^2(t)} \right], \quad (6.2)$$

namely

$$R_{\text{H}} := a(t)r_{\text{H}} = \frac{1}{\sqrt{H^2(t) + \frac{k}{a^2(t)}}}, \quad \text{with} \quad H(t) = \frac{\dot{a}(t)}{a(t)}, \quad (6.3)$$

provided the space-time energy density $\varrho(t)$ is positive. As always it corresponds to the vanishing of the wave front velocity whose radius is $R = a(t)r$. The surface $R_{\text{H}}(t)$ coincides also with the Hubble radius as defined by astronomers for vanishing curvature, but we shall call it Hubble radius in any case. The important case $k = 0$ deserves special attention. The horizon is the surface $r\dot{a}(t) = 1$; substituting into the metric reveals the causal character of the horizon: it will be null if and only if either $p = \varrho/3$ (radiation) or $\varrho = -p$ (vacuum energy), ϱ and p being the energy density and pressure, respectively. It will be time-like or space-like according to whether $|(\varrho + 3p)/2\varrho|$ is less or greater than one. The former case $-\varrho < p < \varrho/3$ covers almost the totality of cosmological models, while the space-like character can only be achieved if $p < -\varrho$ (dubbed phantom energy) or $p > \varrho/3$, which includes stiff matter. These results actually hold for non zero curvature too. The horizon is a surface with $\theta_- = 0$, $\theta_+ > 0$ that for $-\varrho < p < \varrho/3$ satisfies also $\partial_+\theta_- > 0$. In the terminology of Section 5.1 it can be classified as a past

inner trapping horizon (abbr. PITH), because all metric spheres at larger radii are trapped with $\theta_{\pm} > 0$. It means that given a comoving observer, a spherical light beam approaching him from cosmic distance will have increasing area due to cosmological expansion. One has to be careful here that the areal radius is $R = a(t)r$, not r , so the area of the wave front is $4\pi R^2$ and the horizon can be written as the condition $HR_{\text{H}} = 1$. Examples of a FITH are provided by the de Sitter horizon and the collapsing Vaidya solution in de Sitter space, Equation (5.57).

The dynamical surface gravity reads

$$\kappa_{\text{H}} = \frac{1}{2}\square_{\gamma}[a(t)r] = -\left(H^2(t) + \frac{1}{2}\dot{H}(t) + \frac{k}{2a^2(t)}\right)R_{\text{H}}(t) < 0, \quad (6.4)$$

and the minus sign refers to the fact the Hubble horizon is of the inner type. Similarly, the Kodama vector is

$$K = \sqrt{1 - kr^2}(\partial_t - rH(t)\partial_r) \quad (6.5)$$

so that the invariant Kodama energy of a particle is equal to

$$\omega = \sqrt{1 - kr^2}(-\partial_t I + rH(t)\partial_r I) \equiv \sqrt{1 - kr^2}\tilde{\omega}. \quad (6.6)$$

Notice that K is space-like for $ra > (H^2 + k/a^2)^{-1/2}$, *i.e.* beyond the horizon, so that particles will eventually tunnel from outside to the inner region, $r < r_{\text{H}}$.

The next ingredient will be the reduced Hamilton–Jacobi equation for a relativistic particle with mass parameter m , which reads

$$-(\partial_t I)^2 + \frac{(1 - kr^2)}{a^2(t)}(\partial_r I)^2 + m^2 = 0. \quad (6.7)$$

Solving Equation (6.6) and (6.7) for $\partial_r I$ and $\partial_t I$ we obtain

$$\partial_t I = -\tilde{\omega} + rH\partial_r I \quad (6.8)$$

$$\partial_r I = -\frac{aH\tilde{\omega}(ar) \pm a\sqrt{\omega^2 - m^2 + m^2\left(H^2 + \frac{k}{a^2}\right)(ar)^2}}{1 - \left(H^2 + \frac{k}{a^2}\right)(ar)^2}, \quad (6.9)$$

with the sign chosen according to which direction we think the particle is propagating. The effective mass here defines two important and complementary energy scales: if one is interested in the horizon tunnelling then only the pole matters (since the denominator vanishes), and we may neglect to all the extents the mass parameter setting $m = 0$ (since its coefficient vanishes on the horizon). On the opposite, in investigating other effects in the bulk away from the horizon, such as the decay rate of composite particles, the role of the effective mass becomes relevant as the energy of the particle can be smaller than the energy scale settled by m , and the square root can possibly acquire a branch cut singularity. Equation (6.9) will be the starting point of our considerations since it embraces all semi-classical quantum effects we are interested in.

6.2. Horizon tunnelling

Using Equation (6.9) we may derive, following [53], the cosmic horizon tunnelling rate. To this aim, as we have anticipated, the energy scale is such that near the horizon, we may neglect the particle's mass, and note that radially moving massless particles follow a null direction. The horizon region being a region with strong gravity we expect indeed some "particle creation". To a pair created near the horizon in the inner region $r < r_H$ corresponds a type-I path, one segment crossing the horizon backward in cosmological time. To a pair created outside the horizon corresponds a type-II path, the segment crossing the horizon inward being classically allowed. As in Section 5.3.3 only type-I paths will contribute to horizon tunnelling.

Take then a null radial direction crossing the horizon to the inner region: we have

$$\delta t = \frac{a(t)}{\sqrt{1 - kr^2}} \delta r. \quad (6.10)$$

The action for particles coming out of the horizon towards the inner region is to be integrated on a type-I path: then we must choose the solution with negative radial momentum, $\partial_r I < 0$ and we obtain

$$I = \int dt \partial_t I + \int dr \partial_r I = 2 \int dr \partial_r I, \quad (6.11)$$

upon solving the Hamilton–Jacobi equation (6.7) with zero mass and using (6.10). Note that the time derivative of the action contributes to the total imaginary part which, as we have seen, is a general feature of the tunnelling method. For $\partial_r I$ we use now Equation (6.9), which exhibits a pole at the vanishing of the function $F(r, t) := 1 - (a^2 H^2 + \hat{k})r^2$, defining the horizon position. Expanding $F(r, t)$ again along the null direction (6.10) we obtain

$$F(r, t) \approx +4\kappa_H a(t)(r - r_H) + \dots, \quad (6.12)$$

where κ_H given in Equation (6.4) represents the dynamical surface gravity associated with the horizon. In order to deal with the simple pole in the integrand, we implement Feynman's $i\epsilon$ – prescription. In the final result, beside a real (irrelevant) contribution, we obtain the following imaginary part

$$\text{Im } I = -\frac{\pi\omega_H}{\kappa_H}. \quad (6.13)$$

This imaginary part is usually interpreted as arising because of a non-vanishing tunnelling probability rate of massless particles across the cosmological horizon,

$$\Gamma \sim \exp(-2\text{Im } I) \sim \exp\left(-\frac{2\pi\omega_H}{-\kappa_H}\right). \quad (6.14)$$

Notice that, since $\kappa_H < 0$ and $\omega_H > 0$ for physical particles, (6.13) is positive definite. Due to the invariant character of the quantities involved, we may interpret the scalar $T = -\kappa_H/2\pi$ as the dynamical temperature parameter associated to FRW space-times. In particular, this gives naturally a positive temperature for de Sitter space-time, a long debated question years ago, usually resolved by changing the sign of the horizon's

energy. It should be noted that in literature, the dynamical temperature is usually given in the form $T = H/2\pi$ (exceptions are the papers [121]). Of course this is the expected result for de Sitter space in inflationary coordinates, but it ceases to be correct in any other coordinate system. In this regard, the \dot{H} and k terms are crucial in order to get an invariant temperature. The horizon's temperature and the ensuing heating of matter was foreseen several years ago in the interesting paper [122].

6.3. Decay rate of unstable particles

We consider the decay rate of a composite particle in a regime where the energy of the decaying particle is lower than the proper mass m of the decayed products. A crucial point is to identify the energy of the particle before the decay with the Kodama energy. We also denote m the effective mass parameter of one of the decay products (recall it may contain a curvature terms). The relevant contribution to the action comes from the radial momentum given by equation (6.9). If we introduce the instantaneous radius r_0 by

$$[a(t)r_0]^2 = R_0^2 := \left(1 - \frac{\omega^2}{m^2}\right) R_H^2, \quad (6.15)$$

where R_H is the horizon radius given by Equation (6.3), then the classically forbidden region is $0 < r < r_0$. From (6.9), we see that for the unstable particle sitting at rest at the origin of the comoving coordinates, one has an imaginary part of the action as soon as the decay product is tunnelling into this region to escape beyond r_0 ,

$$\text{Im } I = mR_H \int_0^{R_0} dR \frac{\sqrt{R_0^2 - R^2}}{R_H^2 - R^2}. \quad (6.16)$$

The integral can be computed exactly at fixed t , and the result is

$$\text{Im } I = \frac{\pi}{2} R_H (m - \omega) > 0, \quad (6.17)$$

leading to a rate which, assuming a two-particle decay, takes the form

$$\Gamma = \Gamma_0 e^{-2\pi R_H (m - \omega)}, \quad (6.18)$$

where Γ_0 is an unknown pre-factor depending on the coupling constant of the interaction responsible of the decay (for instance, for a $\lambda\phi^3$ interaction one should have $\Gamma_0 \sim \lambda^2$). Of course, each newly produced particle will itself decay, leading possibly to the instability mechanism first discussed by Myhrvold [123] in de Sitter space. Since the tunnelling process locally conserves energy one should put $\omega = m/2$, so that the tunnelled particle will emerge in the classical region at $r = r_0$ with vanishing momentum. Furthermore, the result is again invariant against coordinate changes, since both ω and R_H are invariantly defined quantities.

A particularly interesting case is represented by de Sitter space. Taking for example the line element in the static patch

$$ds^2 = -(1 - H_0^2 r^2) dt^2 + \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 d\Omega^2, \quad (6.19)$$

for the imaginary part (6.18) we obtain

$$\text{Im } I = \frac{\pi}{2H_0}(m - \omega) \quad (6.20)$$

a result actually independent of the coordinate system in use. Putting $\omega = m/2$, the above result has been obtained by Volovik [124] using the so-called “fluid” static form of de Sitter space

$$ds^2 = -dt^2 + (dR - H_0 R dt)^2 + R^2 d\Omega^2. \quad (6.21)$$

in agreement with the asymptotic approximation of the exact result due to [125].

6.4. Particle creation by black holes singularities

One may also use the tunnelling formalism to investigate whether particle creation in the bulk of space-time is possible due to the presence of space-time singularities, for example due to static black holes. With regard to this, we consider the exterior region of a spherically symmetric static black hole space-time and repeat the same argument. Quite generally, we can write the line element as

$$ds^2 = -e^{2\Phi(r)} C(r) dt^2 + C^{-1}(r) dr^2 + r^2 d\Omega^2. \quad (6.22)$$

From the Hamilton–Jacobi equation, the radial momentum turns out to be

$$\int dr \partial_r I = \int dr \frac{\sqrt{\omega^2 - m^2 C(r) e^{2\Phi(r)}}}{C(r) e^{\Phi(r)}}. \quad (6.23)$$

If we are interested in particle creation we should set $\omega = 0$: in fact, according to the interpretation of the Kodama energy we gave before, this approximation simulates the vacuum condition. Then

$$\int dr \partial_r I = m \int_{r_1}^{r_2} dr \frac{1}{\sqrt{-C(r)}}, \quad (6.24)$$

where the integration is performed in every interval (r_1, r_2) in which $C(r) > 0$. Equation (6.24) shows that, under very general conditions, in static black hole space-times there could be a production rate whenever a region where $C(r) > 0$ exists.

As a first example, let us analyse the Schwarzschild black hole. For the exterior (static) solution, one has $C(r) = 1 - 2M/r > 0$ and $\Phi(r) = 0$, thus the imaginary part diverges since the integral has an infinite range. We conclude that the space-like singularity does not create particles in the semi-classical regime. In the interior the Kodama vector is space-like, thus no energy can be introduced. A similar conclusion has been obtained also for the Big Bang cosmic singularity, the only scale factor leading to particle emission being $a(t) \sim t^{-1}$. This is like a big rip in the past.

The situation is different when a naked singularity is present. Consider a neutral particle in the Reissner–Nordström solution with mass M and charge $Q > 0$ (for definiteness) given by the spherically symmetric line element

$$ds^2 = -\frac{(r - r_-)(r - r_+)}{r^2} dt^2 + \frac{r^2}{(r - r_-)(r - r_+)} dr^2 + r^2 d\Omega^2. \quad (6.25)$$

Here $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ are the horizon radii, connected to the black hole mass and charge by the relations

$$M = \frac{r_+ + r_-}{2}, \quad Q = \sqrt{r_+ r_-}. \quad (6.26)$$

The Kodama energy coincides with the usual Killing energy and

$$C(r) = \frac{(r - r_-)(r - r_+)}{r^2}. \quad (6.27)$$

The metric function $C(r)$ is negative in between the two horizons, where the Kodama vector is space-like, so there the action is real. On the other hand, it is positive within the outer communication domain, $r > r_+$, and also within the region contained in the inner Cauchy horizon, that is $0 < r < r_-$. Thus, because of (6.24) and assuming the particles come created in pairs, we obtain

$$\text{Im } I = -m \int_0^{r_-} \frac{r}{\sqrt{(r_- - r)(r_+ - r)}} dr = mq - \frac{mM}{2} \ln \left(\frac{M + Q}{M - Q} \right). \quad (6.28)$$

Modulo the pre-factor over which we have nothing to say, with the usual interpretation there is a probability

$$\Gamma \sim \exp(-2\text{Im } I) = \left(\frac{M - Q}{M + Q} \right)^{mM} e^{-2mQ}. \quad (6.29)$$

Pleasantly, (6.29) vanishes in the extremal limit $M = Q$. Being computed for particles with zero energy, we can interpret this as an effect of particle creation by the strong gravitational field near the singularity. Since the electric field is of order Q/r^2 near $r = 0$, there should be also a strong Schwinger's effect. In that case one should write the Hamilton–Jacobi equation for charged particles.

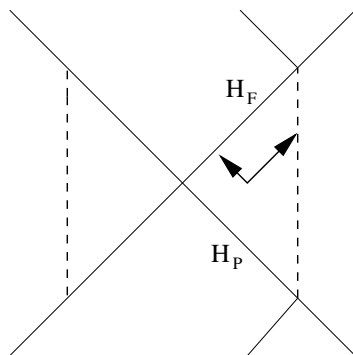


Figure 6. A pair of particles is created in the compact region inside the inner horizon.

The processes just discussed should bear a bit on the question of the stability of the Cauchy horizons. Due to infinite blue-shift of perturbations coming in from the asymptotically flat exterior regions both sheets of the Cauchy horizon (H_F and H_P in Figure 6) are believed to be classically unstable. Of course, if the naked singularity is formed from collapse of charged matter, one asymptotically flat region (say, the left one) disappears. Taking into account particle creation, it can be easily seen that escaping

particles will reach the future portion H_F of the inner horizon with infinite blue-shift, or infinite Kodama energy, as measured by an observer on a Kodama trajectory. Hence, the future sheet will probably become unstable, this time by quantum effects. On the other hand, the particle reaching the singularity will do so with infinite red-shift, that is with zero energy, giving a negligible back-reaction and substantially not changing the nature of the singularity (its time-like character, for example).

Thus, the only process potentially capable of modifying the singularity would be its conjectured screening by absorption of charged particles. Of course the mass inflation phenomenon is an issue here but, in the context of perfect spherical symmetry, exact solutions of the Einstein–Maxwell system exist having both event and Cauchy horizons and describing collapse of spherical shells. For these what we said should apply. The present formalism also predicts the absence of particle creation in the region in between the two horizons, despite the metric there is dynamical. The possibility that the naked singularity itself radiates away its mass will be taken up in the next section.

6.5. Naked singularities

A general reference on the physics of naked singularities is [126]. Sticking to two-dimensional models for simplicity, consider the following metric [127]

$$ds^2 = \sigma^{-1} dx^+ dx^-, \quad \sigma = \lambda^2 x^+ x^- - a(x^+ - x_0^+) \Theta(x^+ - x_0^+) \quad (6.30)$$

where λ is related to the cosmological constant by $\Lambda = -4\lambda^2$. This metric arises as a solution of two-dimensional dilaton gravity coupled to a bosonic field with stress tensor $T_{++} = 2a\delta(x^+ - x_0^+)$, describing a shock wave.

A look at Figure 7 reveals that $\sigma = 0$ is a naked singularity partly to the future of a flat space region, usually named the linear dilaton vacuum. The heavy arrow represents the history of the shock wave responsible for the existence of the time-like singularity. The Hamilton–Jacobi equation implies either $\partial_+ I = 0$ or $\partial_- I = 0$, I being the action.

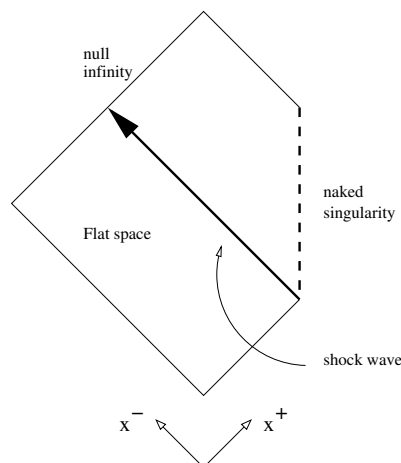


Figure 7. The naked singularity formed by the shock wave.

To find the ingoing flux, we integrate along x^+ till we encounter the naked singularity, using $\partial_- I = 0$, so that

$$I = \int dx^+ \partial_+ I = \int \omega \frac{dx^+}{2\sigma} = \int \frac{\omega dx^+}{2(\lambda^2 x^- - a)(x^+ + ax_0^+/C - i\epsilon)} \quad (6.31)$$

where $C = C(x^-) := (\lambda^2 x^- - a)$ and $\omega = 2\sigma \partial_+ I$ is the familiar Kodama's energy. The imaginary part immediately follows on using $(x - i\epsilon)^{-1} = P_x^{\frac{1}{2}} + i\pi\delta(x)$, giving the absorption probability as a function of retarded time

$$\Gamma(\omega) = \Gamma_0 e^{-2\text{Im} I} = \Gamma_0 e^{-\pi\omega/C(x^-)}, \quad (6.32)$$

Γ_0 being some pre-factor of order one. The flux is computed by integrating the probability over the coordinate frequency (that is, the variable conjugated to the coordinate time) $\hat{\omega} = \omega/\sigma$, with the density of states measure $d\hat{\omega}/2\pi$, which gives

$$T_{++} = \frac{\Gamma_0}{2\pi} \int \Gamma(\sigma\hat{\omega}) \hat{\omega} d\hat{\omega} = \Gamma_0 \frac{(\lambda^2 x^- - a)^2}{2\pi^3 \sigma^2}. \quad (6.33)$$

Similarly, in order to find the outgoing flux we integrate along x^- starting from the naked singularity, this time using $\partial_+ I = 0$. A similar calculation gives

$$\text{Im} I = \frac{\pi\omega}{2\lambda^2 x^+}, \quad (6.34)$$

then, integrating the probability over the coordinate frequency, we obtain

$$T_{--} = \Gamma_0 \frac{\lambda^4 (x^+)^2}{2\pi^3 \sigma^2}. \quad (6.35)$$

The outgoing flux is $2(T_{++} - T_{--})$. The conservation equations

$$\sigma \partial_+ T_{--} + \partial_- (\sigma T_{+-}) = 0, \quad \sigma \partial_- T_{++} + \partial_+ (\sigma T_{+-}) = 0 \quad (6.36)$$

determine the components only up to arbitrary functions $B(x^-)$ and $A(x^+)$, something which corresponds to the freedom in the choice of a vacuum. For instance, requiring the fluxes to vanish in the linear dilaton vacuum fixes them uniquely. As it is well known, T_{+-} is given by the conformal anomaly: $T = 4\sigma T_{+-} = R/24\pi$ (for one bosonic degree of freedom). Matching to the anomaly gives the pre-factor $\Gamma_0 = \pi^2/24$, of order one indeed. These results agree with the one-loop calculation to be found in [127]. Note that the stress tensor diverges while approaching the singularity, indicating that its resolution will not be possible within classical gravity but requires quantum gravity instead [128].

We return now to the Reissner–Nordström solution. Could it be that the naked singularity emits particles? In the four-dimensional case one easily sees that the action has no imaginary part along null trajectories either ending or beginning at the singularity. Formally this is because the Kodama energy coincides with the Killing energy in such a static manifold and there is no infinite red-shift from the singularity to infinity. Even considering the metric as a genuinely two-dimensional solution, this would lead to an integral for I

$$I = \int \omega \frac{(r - r_+)(r - r_-)}{r^2} dx^+ \quad (6.37)$$

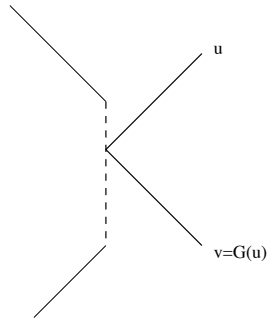


Figure 8. Part of the Penrose diagram of the Reissner–Nordström region close to the singularity.

where $x^\pm = t \pm r_*$, with

$$r_* = r + \frac{r_+^2}{r_+ - r_-} \ln \left(\frac{r_+ - r}{r_+} \right) - \frac{r_+^2}{r_+ - r_-} \ln \left(\frac{r_- - r}{r_-} \right) = \frac{x^+ - x^-}{2}. \quad (6.38)$$

But close to the singularity

$$r^2 = \left[\frac{3r_+r_-}{2}(x^+ - x^-) \right]^{2/3} + \dots \quad (6.39)$$

not a simple pole, rather, an integrable singularity. It is fair to say that the Reissner–Nordström naked singularity will not emit particles in this approximation. This seems to be coherent with quantum field theoretical results. With the customary $u = x^-$ and $v = x^+$, there is a map $u \rightarrow v = G(u)$ which gives the family of ingoing null geodesics, characterised by constant values of v , which after reflection in the origin emerge as the family of outgoing null geodesics with constant u . According to [129], the radiated s-wave power of a minimally coupled scalar field is given in terms of the map $G(u)$ by the Schwarzian derivative

$$\mathcal{W} = \frac{1}{24\pi} \left[\frac{3}{2} \left(\frac{G''}{G'} \right)^2 - \frac{G'''}{G'} \right]. \quad (6.40)$$

The (u, v) section of the Reissner–Nordström metric is conformally flat, hence the above map is trivial (*i.e.* linear) and $\mathcal{W} = 0$.

7. Unruh–DeWitt Detector vs Horizon Tunnelling

As we have repeatedly emphasised, one of the most beautiful achievements of the semi-classical theory of black holes is represented by the interpretation of the surface gravity κ associated to the (event, FOTH) horizon as the Hawking temperature associated to the radiation emitted from that horizon: a quantum effect as explicitly shown by the famous formula $T_H = \hbar c^3 \kappa / 2\pi k_B$. As proved in Section 6.1, the tunnelling method provides a non-vanishing emission rate even in the vicinity of the Hubble sphere in most of the cosmological models described by FRW metrics simply by choosing Kodama observers as privileged observers. This apparently seems to be interpretable as an indication that the correspondence between surface gravity and Hawking temperature can be pushed even forward than traditionally expected. If it is true that semi-classical methods such as the tunnelling method or the Hamilton–Jacobi method have proven so far to be reliable in all the testable conditions; still a comparison between such methods and standard quantum field theoretic calculations in dynamical spaces remains something deserving to be done. With the purpose of partially filling this gap, we try to understand how far the correspondence between surface gravity and temperature in dynamical spherically symmetric spaces can be established. We shall do this by searching whether a point-like detector will register a quasi-thermal excitation rate of the form

$$\dot{F} \sim E \exp\left(-\frac{2\pi E}{\kappa_H(t)}\right)$$

where $\kappa_H(t)$ is the dynamical horizon surface gravity (6.4) and E is the energy of the detector’s quantum jump. We shall take this feature as a hint that the detector feels the vacuum as a mixed quasi-thermal state with an effective temperature parameter $T_H(t) = \kappa_H(t)/2\pi$. The analysis will be done for conformally coupled scalar fields, the best approximation we know to massless radiation, but we think the conclusions to be drawn will have general validity.

7.1. Point-like detectors

A key concept of physics — no matter if classical or quantum — is the concept of *particle*. The question “What is a particle?” is so important in our atomistic viewpoint that even in quantum field theory we feel necessary to find an answer.

As far as gravitation is not considered (an approximation which turns out to be often natural since gravity is the weakest known interaction), the answer runs straight without many accidents: in this case, in fact, the space-time where quantum fields propagate is Minkowskian. It is well known [14] that Minkowski space-time possesses a rectangular coordinate system (t, x, y, z) naturally associated with the Poincaré group, whose action leaves the Minkowski line element unchanged. The vector field ∂_t is a Killing vector of Minkowski space-time orthogonal to the space-like hypersurfaces $t = \text{constant}$ and the wave modes $u_{\mathbf{k}} \propto e^{-i(\omega_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})}$ are eigenfunctions of this Killing vector with eigenvalues $-i\omega$. Thus, a particle mode solution in Minkowski space is one which

is positive frequency ($\omega > 0$) with respect to the time coordinate t . Under Poincaré transformations, positive frequency solutions transform to positive frequency solutions and the concept of particle is the same for every inertial observer, in the sense that all inertial observers agree on the number of particles present. Further, the Minkowski vacuum state, defined as the state with no particles present, is invariant under the Poincaré group.

Problems arise as one turns gravity on. In curved space-time, in fact, the Poincaré group is no longer a symmetry group. In general, there are no Killing vectors with which to define positive frequency modes and no coordinate choice is available to make the field decomposition in some modes more natural than others. This, of course, is not just an accident but is rooted in the same guiding principle of general relativity: that coordinate systems are physically irrelevant. As a possible way out, DeWitt and others suggested an operational definition of a particle: “a particle is something that can be detected by a particle detector”. (That this definition may look as a tautology will not concern us further.) The particle detector proposed by Unruh [130] and later, in simplified version, by DeWitt [131] can be described as a quantum mechanical particle with many energy levels linearly coupled to a massless scalar field via a monopole moment operator. This is the construction we will employ in the following.

Let us consider a massless scalar field $\hat{\phi}$ with Hamiltonian \hat{H}_ϕ obeying the massless Klein–Gordon equation. The free field operator is expanded in terms of a complete orthonormal set of solutions to the field equation as

$$\hat{\phi}(t, \mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(\hat{a}_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}_{\mathbf{k}}^\dagger e^{+i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} \right), \quad (7.1)$$

where, in the massless case, $\omega_{\mathbf{k}} = |\mathbf{k}|$. Field quantization is realized by imposing the usual commutation relations on the creation and annihilation operators,

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}'), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0 = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger]. \quad (7.2)$$

The Minkowski vacuum is the state $|0\rangle$ annihilated by $\hat{a}_{\mathbf{k}}$, for all \mathbf{k} .

The detector is a quantum mechanical system with a set of energy eigenstates $\{|0\rangle_{\text{d}}, |E_i\rangle\}$ which moves along a prescribed classical trajectory $t = t(\tau)$, $\mathbf{x} = \mathbf{x}(\tau)$, where τ is the detector’s proper time. The detector is coupled to the scalar field ϕ via the interaction Hamiltonian

$$\hat{H}_{\text{int}} = \lambda \hat{M}(\tau) \hat{\phi}(\tau). \quad (7.3)$$

λ is treated here as a small parameter while $\hat{M}(\tau)$ is the detector’s monopole moment operator whose evolution is provided by

$$\hat{M}(\tau) = e^{i\hat{H}_{\text{d}}\tau} \hat{M}(0) e^{-i\hat{H}_{\text{d}}\tau}, \quad (7.4)$$

\hat{H}_{d} being the detector’s Hamiltonian. This model is also known as the *point-like detector* since the interaction takes place at a point along the given trajectory at any given time.

Suppose that at time τ_0 the detector and the field are in the product state $|0, E_0\rangle = |0\rangle |E_0\rangle$, where $|E_0\rangle$ is a detector state with energy E_0 [132]. We want

to know the probability that at a later time $\tau_1 > \tau_0$ the detector is found in the state $|E_1\rangle$ with energy $E_1 \geq E_0$, no matter what is the final state of the field ϕ . The answer is provided by the so-called interaction picture where we let both operators and states evolve in time according to the following understandings: operators evolution is governed by free Hamiltonians; states evolution is governed by the Schrödinger equation depending on the interaction Hamiltonian, that is

$$i \frac{d}{d\tau} |\varphi(\tau)\rangle = \hat{H}_{\text{int}} |\varphi(\tau)\rangle. \quad (7.5)$$

The amplitude for the transition from the state $|0, E_0\rangle$ at time $\tau = \tau_0$ to the state $|\varphi, E_1\rangle$ at time $\tau = \tau_1$ is then provided by

$$\langle \varphi, E_1 | 0, E_0 \rangle = \langle \varphi, E_1 | \hat{\mathcal{T}} \exp \left(-i \int_{\tau_0}^{\tau_1} d\tau \hat{H}_{\text{int}}(\tau) \right) | 0, E_0 \rangle, \quad (7.6)$$

where $\hat{\mathcal{T}}$ is the time-ordering operator. To first order in perturbation theory, we get

$$\begin{aligned} \langle \varphi, E_1 | 0, E_0 \rangle &= \langle \varphi, E_1 | \hat{\mathbb{I}} | 0, E_0 \rangle - i \lambda \langle \varphi, E_1 | \int_{\tau_0}^{\tau_1} d\tau e^{i\hat{H}_d \tau} \hat{M}(0) e^{-i\hat{H}_d \tau} \hat{\phi}(\tau) | 0, E_0 \rangle + \dots \\ &= -i \lambda \langle E_1 | \hat{M}(0) | E_0 \rangle \int_{\tau_0}^{\tau_1} d\tau e^{i\tau(E_1 - E_0)} \langle \varphi | \hat{\phi}(\tau) | 0 \rangle + \dots \end{aligned} \quad (7.7)$$

The transition probability to all possible finale states of the field $\hat{\phi}$ is given by squaring (7.7) and summing over the complete set $\{|\varphi\rangle\}$ of final unobserved field states,

$$\begin{aligned} \sum_{\varphi} |\langle \varphi, E_1 | 0, E_0 \rangle|^2 &= \lambda^2 |\langle E_1 | \hat{M}(0) | E_0 \rangle|^2 \times \\ &\times \int_{\tau_0}^{\tau_1} d\tau' \int_{\tau_0}^{\tau_1} d\tau e^{-i(E_1 - E_0)(\tau' - \tau)} \langle 0 | \hat{\phi}(\tau') \hat{\phi}(\tau) | 0 \rangle. \end{aligned} \quad (7.8)$$

This expression has two parts: the pre-factor $\lambda^2 |\langle E_1 | \hat{M}(0) | E_0 \rangle|^2$ which depends only on the peculiar details of the detector and the *response function*

$$F_{\tau_0, \tau_1}(\Delta E) = \int_{\tau_0}^{\tau_1} d\tau \int_{\tau_0}^{\tau_1} d\tau' e^{-i\Delta E(\tau - \tau')} \langle 0 | \hat{\phi}(\tau) \hat{\phi}(\tau') | 0 \rangle, \quad (7.9)$$

which is insensitive to the internal structure of the detector and is thus the same for all possible detectors. Here, we have set the energy gap $\Delta E \equiv E_1 - E_0 \geq 0$ for excitations or decay, respectively. From now on, we will only consider the model-independent response function.

Introducing new coordinates $u := \tau$, $s := \tau - \tau'$ for $\tau > \tau'$ and $u := \tau'$, $s := \tau' - \tau$ for $\tau' > \tau$, the response function can be re-written as

$$F_{\tau_0, \tau_1}(\Delta E) = 2 \int_{\tau_0}^{\tau_1} du \int_0^{u - \tau_0} ds \operatorname{Re} \left(e^{-i\Delta E s} \langle 0 | \hat{\phi}(u) \hat{\phi}(u - s) | 0 \rangle \right) \quad (7.10)$$

having used $\langle 0 | \hat{\phi}(\tau') \hat{\phi}(\tau) | 0 \rangle = \langle 0 | \hat{\phi}(\tau) \hat{\phi}(\tau') | 0 \rangle^*$, since $\hat{\phi}$ is a self-adjoint operator. Equation (7.10) can be differentiated with respect to τ_1 in order to obtain the *transition rate*

$$\dot{F}_{\tau_0, \tau}(\Delta E) = 2 \int_0^{\tau - \tau_0} ds \operatorname{Re} \left(e^{-i\Delta E s} \langle 0 | \hat{\phi}(\tau) \hat{\phi}(\tau - s) | 0 \rangle \right), \quad (7.11)$$

where we set $\tau_1 \equiv \tau$. If the correlation function $\langle 0 | \hat{\phi}(\tau) \hat{\phi}(\tau - s) | 0 \rangle$ is invariant under τ -translations, (7.11) can be further simplified,

$$\dot{F}_{\tau_0, \tau}(\Delta E) = \int_{-(\tau-\tau_0)}^{\tau-\tau_0} ds e^{-i\Delta E s} \langle 0 | \hat{\phi}(s) \hat{\phi}(0) | 0 \rangle . \quad (7.12)$$

The correlation function $\langle 0 | \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle$ which appears in these expressions is the positive frequency Wightman function that can be obtained from (7.1),

$$\langle 0 | \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} e^{i\mathbf{k} \cdot (x-x')} \quad (7.13)$$

The integral in $|\mathbf{k}|$ contains ultra-violet divergences and can be regularized [14] by introducing the exponential cut-off $e^{-\epsilon|\mathbf{k}|}$, with $\epsilon > 0$ and small, in the high frequency modes. The resulting expression is

$$\langle 0 | \hat{\phi}(x(\tau)) \hat{\phi}(x(\tau')) | 0 \rangle = \frac{1/4\pi^2}{|\mathbf{x}(\tau) - \mathbf{x}(\tau')|^2 - [t(\tau) - t(\tau') - i\epsilon]^2} \equiv W_\epsilon(x(\tau), x(\tau')) \quad (7.14)$$

so that, we finally have

$$\dot{F}_{\tau_0, \tau}(\Delta E) = \lim_{\epsilon \rightarrow 0^+} \int_{-(\tau-\tau_0)}^{\tau-\tau_0} ds e^{-i\Delta E s} W_\epsilon(x(s), x(0)) . \quad (7.15)$$

Let us apply the above construction to Kodama detectors in de Sitter space-time. We first recall that de Sitter metric in the cosmological global system reads

$$ds^2 = d\tau^2 - \frac{1}{H^2} \cosh^2(H\tau) d\Omega_3^2 . \quad (7.16)$$

An easy calculation then gives the equivalent definitions of the Kodama trajectory

$$r = K e^{-Ht} \iff \bar{r} = K \iff \sin \chi = \frac{KH}{\cosh(H\tau)} , \quad (7.17)$$

K being a constant. As we mentioned before, the relevant equality is the second one: it means that the Kodama observers are just the stationary de Sitter observers at constant distance from their cosmological horizon. We know that these observers will perceive a thermal bath at de Sitter temperature $T = H/2\pi$, so this must be true in the inflationary patch as well.

We can confirm this expectation by rewriting Equation (7.14) with the following relations for de Sitter space

$$\eta = -H^{-1} e^{-Ht}, \quad r = K e^{-Ht} = -KH\eta, \quad \tau = t\sqrt{1 - K^2 H^2} . \quad (7.18)$$

Then, provided $1 - K^2 H^2 > 0$, the Wightman function (7.14) becomes

$$W_\epsilon(x, x') = -\frac{1}{4\pi^2} \frac{H^2}{e^{H(t+t')}} \frac{1}{(1 - K^2 H^2)(e^{-Ht} - e^{-Ht'} - i\epsilon)^2} . \quad (7.19)$$

and, in the limit of $\tau_0 \rightarrow -\infty$, the detector transition rate (7.15) becomes

$$\dot{F}(\Delta E) = -\frac{H^2}{4\pi^2} \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} ds \frac{\exp(-i\Delta E \sqrt{1 - K^2 H^2} s)}{(e^{Hs/2} - e^{-Hs/2} - i\epsilon)^2} . \quad (7.20)$$

As long as $H > 0$ (expanding universe) we can write the denominator as

$$(e^{Hs/2} - e^{-Hs/2} - i\epsilon)^2 = 4 \sinh^2 [H(s - i\epsilon)/2] . \quad (7.21)$$

This function has infinitely many double poles in the complex s -plane, namely for $s = s_j$ with

$$s_j = \frac{2\pi i}{H} j , \quad j \in \mathbb{Z} . \quad (7.22)$$

Since we are interested in the case of $\Delta E > 0$, we can close the contour of integration in the lower half plane, summing over the residues of all the double poles in the lower complex s -plane, with the exception of the $s = 0$ pole which has been slightly displaced above the integration path by the $i\epsilon$ -prescription. The well known result turns out to be confirmed, namely

$$\dot{F}(\Delta E) = \frac{1}{2\pi} \cdot \frac{\Delta E \sqrt{1 - K^2 H^2}}{\exp\left(\frac{2\pi \Delta E \sqrt{1 - K^2 H^2}}{H}\right) - 1} , \quad (7.23)$$

showing the presence of a cosmological Unruh effect with de Sitter temperature

$$T_H = \frac{H}{2\pi} , \quad (7.24)$$

red-shifted by the Tolman factor $\sqrt{1 - K^2 H^2}$, here appearing as a Doppler shift due to the proper motion of the detector. In fact, as remarked after Equation (7.17), K is also the value of the static coordinate \bar{r} of the detector relative to the static patch.

If we recall that the value of the four-acceleration along the Kodama trajectories (7.17) is

$$A^2 \equiv a^\mu a_\mu = -\frac{H^4 K^2}{1 - H^2 K^2} , \quad (7.25)$$

one easily sees that

$$\frac{T_H}{\sqrt{1 - H^2 K^2}} = \frac{\sqrt{A^2 + H^2}}{2\pi} . \quad (7.26)$$

One can interpret this formula by saying that the temperature is actually due to a mixing of a pure Unruh effect (the acceleration term) plus a cosmological expansion term (the H term), and is the de Sitter version [32] of a formula discovered by Deser and Levine for detectors in anti-de Sitter space [32, 33].

Alternatively, we can understand this effect as the transition from cosmological energy ΔE , conjugated to cosmic de Sitter time t , to the energy $\Delta E \sqrt{1 - K^2 H^2}$ as measured locally by Kodama's observers (with proper time τ).

7.2. Unruh–DeWitt detector revisited

More realistic models of particle detectors have been proposed in the last few years (cf. [132] for an exhaustive analysis). These models deal, for example, with the fact that a realistic detector should have a finite size in space; its motion could be along non stationary worldlines (that is, worldlines which are not the orbits of some time-like

Killing vector); the time for which it is switched on is only finite.

Motivated by the observation that calculations of the Unruh effect based on idealised particle detectors are usually made in a way that involves integrations along the entire detector trajectory up to the infinitely remote future, Takagi and Schlicht [133] have independently derived an expression which allows time-dependence of the detector response in the case of a non-stationary trajectory and explicitly conforms to the principle of causality: namely, that the response at a given instant of time depends only on the detectors past movements. But, on trying to reproduce the thermal Unruh spectrum, they are led to an unphysical result the authors trace down to the use of the standard regularisation $t \rightarrow t - i\epsilon$ of the correlation function (7.14), as shown in detail at the end of the previous sub-section. By consistently employing a rigid detector of finite extension, they are led to a different regularisation which works fine with the causal response function. This is the regularisation scheme we shall employ to interpret the tunnelling emission rate from a FRW horizon in terms of a particle detector. In this case, we consider detectors following Kodama trajectories in a generic spherically symmetric space-time. The problem of back-reaction on these Kodama trajectories has been investigated in [134]. As we shall see soon, the Unruh–DeWitt thermometer gives a clean answer only in the stationary case, while for FRW space-time the situation is not so simple, since horizon effects are entangled with highly non trivial kinematic effects.

Introducing the conformal time η

$$\eta(t) = \int^t \frac{d\tilde{t}}{a(\tilde{t})}, \quad (7.27)$$

the flat FRW space-time becomes conformally flat,

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2), \quad x = (\eta, \mathbf{x}). \quad (7.28)$$

Thus, we can adapt Schlicht’s proposal to the case we are concerned here, that is

$$W_\epsilon^{(S)}(x, x') = \lim_{\epsilon \rightarrow 0^+} \int \frac{d\mathbf{k}}{2k} \frac{e^{-ik(x-x') - i\epsilon(\dot{x} + \dot{x}')}}{a(\eta)a(\eta')} \quad (7.29)$$

where an over dot stands for derivative with respect to proper time and (S) stands for Schlicht. After an integration over \mathbf{k} , one arrives at

$$W^{(S)}(x, x') = \lim_{\epsilon \rightarrow 0^+} \frac{1}{a(\eta)a(\eta')} \frac{1/4\pi^2}{[(x - x') + i\epsilon(\dot{x} + \dot{x}')]^2}. \quad (7.30)$$

In the flat case, this result has been generalised by Milgrom and Obadia, who made use of an analytical proper-time regularisation [135].

The transition probability per unit proper time of the detector depends on the response function per unit proper time which, for radial trajectories, may be written as

$$\frac{dF}{d\tau} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi^2} \text{Re} \int_0^{\tau - \tau_0} ds \frac{e^{-iEs}}{a(\tau)a(\tau - s)[x(\tau) - x(\tau - s) + i\epsilon(\dot{x}(\tau) + \dot{x}(\tau - s))]^2} \quad (7.31)$$

where τ_0 is the detector’s proper time at which we turn on the detector, and $E \equiv \Delta E$ is the energy associated with the excited detector state (we are considering $E > 0$). The limit $\tau_0 \rightarrow -\infty$ can be taken whenever there is not a Big Bang singularity.

Although the covariant $i\epsilon$ -prescription is necessary in order to deal with the second order pole at $s = 0$, one may try to avoid the awkward limit $\epsilon \rightarrow 0^+$ by omitting the ϵ -terms and subtracting the leading pole at $s = 0$ (see [136] for details). In the calculation, the normalization condition

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \equiv [a(\tau)\dot{x}(\tau)]^2 = -1, \quad (7.32)$$

characteristic of time-like four-velocities, has to be imposed. As a consequence, for $\Delta\tau = \tau - \tau_0 > 0$, one can re-write the detector transition probability per unit time in the form⁺

$$\frac{dF}{d\tau} = \frac{1}{2\pi^2} \int_0^\infty ds \cos(Es) \left(\frac{1}{\sigma^2(\tau, s)} + \frac{1}{s^2} \right) + J \quad (7.33)$$

where

$$J(\Delta\tau) := -\frac{1}{2\pi^2} \int_{\Delta\tau}^\infty ds \frac{\cos(Es)}{\sigma^2(\tau, s)}, \quad (7.34)$$

and

$$\sigma^2(\tau, s) \equiv a(\tau)a(\tau - s)[x(\tau) - x(\tau - s)]^2. \quad (7.35)$$

Due to (7.32), for small s , one has

$$\sigma^2(\tau, s) = -s^2[1 + O(s^2)] \quad (7.36)$$

so that

$$\sigma^2(\tau, s) = -s^2[1 + s^2d(\tau, s)]. \quad (7.37)$$

We introduce the even $\sigma_e^2(\tau, s)$ and odd parts $\sigma_o^2(\tau, s)$ in s of the quantity $\sigma^2(\tau, s)$ to get the expression

$$\frac{dF}{d\tau} = \frac{dF_e}{d\tau} + \frac{dF_o}{d\tau} + J \quad (7.38)$$

where the two contributions of definite parity are

$$\frac{dF_e}{d\tau} = \frac{1}{4\pi^2} \int_{-\infty}^\infty ds \cos(Es) \left(\frac{1}{\sigma_e^2(\tau, s)} + \frac{1}{s^2} \right) \quad (7.39)$$

and

$$\frac{dF_o}{d\tau} = \frac{1}{2\pi^2} \int_0^\infty ds \frac{\cos(Es)}{\sigma_o^2(\tau, s)}. \quad (7.40)$$

This is the main formula which we shall use in the following. Equations (7.33) and (7.38) are much more convenient to deal with than the original expression (7.31) containing the ϵ -terms, since in the latter the limit in distributional sense must be taken at the end of any computation.

In the important stationary case in which $\sigma(\tau, s)^2 = \sigma^2(s) = \sigma^2(-s)$, the odd part drops out and Equation (7.33) simply becomes

$$\frac{dF}{d\tau} = \frac{1}{4\pi^2} \int_{-\infty}^\infty ds e^{-iEs} \left(\frac{1}{\sigma^2(s)} + \frac{1}{s^2} \right) + J(\Delta\tau). \quad (7.41)$$

⁺ This is not quite the original expression found in [136] but can be obtained from it by simple manipulations.

In this case – examples are the static black hole and the FRW de Sitter space – taking the limit $\Delta\tau \rightarrow \infty$, the fluctuating tail drops down as well.

As a result we have a manageable expression, Equation (7.38), which neatly separates the relevant contributions to the response function: (i) an even term in the variable s , always present; (ii) an odd part which participates to non-stationary effects; and (iii) the fluctuating tail J which incorporates the finite-time effects and, as we shall see in the case of asymptotically stationary situations, controls how fast the thermal equilibrium is reached.

7.3. Quantum thermometers in static spaces

As an application of the formalism previously developed, we are going to consider the case of the generic static black hole which, eventually, applies with minor changes to de Sitter space as well in order to reproduce Equation (7.23).

The general metric for a static black hole reads

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + r^2d\Omega^2, \quad (7.42)$$

where, for sake of simplicity, we shall assume $W(r) = V(r)$, with $V(r)$ having just simple poles in order to describe what we might call a *nice black hole* (as suggested by S. A. Hayward). Let r_H be the (greatest) solution of $V(r) = 0$, the general formalism tells us that the horizon is located at $r = r_H$; the Kodama vector coincides with the usual Killing vector $(1, \mathbf{0})$; and the Hayward surface gravity is the Killing surface gravity, namely $\kappa_H = \kappa = V'_H/2$. We now introduce the Kruskal-like gauge associated with this static black hole solution. The first step consists in introducing the tortoise coordinate

$$r^*(r) = \int^r \frac{d\tilde{r}}{V(\tilde{r})}. \quad (7.43)$$

One has $-\infty < r^* < \infty$ and $ds^2 = V(r^*)[-dt^2 + (dr^*)^2] + r^2(r^*)d\Omega_{(2)}^2$. The Kruskal-like coordinates are

$$R = \frac{1}{\kappa} e^{\kappa r^*} \cosh(\kappa t), \quad T = \frac{1}{\kappa} e^{\kappa r^*} \sinh(\kappa t), \quad (7.44)$$

so that

$$-T^2 + R^2 = \frac{e^{2\kappa r^*}}{\kappa^2}, \quad (7.45)$$

and the line element becomes

$$\begin{aligned} ds^2 &= e^{-2\kappa r^*} V(r^*)[-dT^2 + dR^2] + r^2(T, R)d\Omega^2 \\ &\equiv e^{\Psi(r^*)}(-dT^2 + dR^2) + r^2(T, R)d\Omega^2 \end{aligned} \quad (7.46)$$

where now the coordinates are T and R , $r^* = r^*(T, R)$, $e^{\Psi(r^*)} := V(r^*)e^{-2\kappa r^*}$ and the normal metric turns out to be conformally flat. The key point to recall here is that in the Kruskal gauge (7.46) the normal metric – the important one for radial trajectories – is conformally related to two dimensional Minkowski space-time. The second observation is that Kodama observers are defined by the integral curves associated with the Kodama

vector, thus the areal radius $r(T, R)$ and r^* are *constant*. As a consequence, the proper time along Kodama trajectories reads

$$d\tau^2 = V(r^*)dt^2 = e^{\Psi(r^*)}(dT^2 - dR^2) = a^2(r^*)(dT^2 - dR^2) \quad (7.47)$$

so that $t = \tau/\sqrt{V(r^*)}$ and

$$R(\tau) = \frac{1}{\kappa} e^{\kappa r^*} \cosh\left(\kappa \frac{\tau}{\sqrt{V(r^*)}}\right), \quad T(\tau) = \frac{1}{\kappa} e^{\kappa r^*} \sinh\left(\kappa \frac{\tau}{\sqrt{V(r^*)}}\right). \quad (7.48)$$

The geodesic distance reads

$$\sigma^2(\tau, s) = e^{\Psi(r^*)} [-(T(\tau) - T(\tau - s))^2 + (R(\tau) - R(\tau - s))^2], \quad (7.49)$$

and one gets, using (7.48),

$$\sigma^2(\tau, s) = -\frac{4V(r^*)}{\kappa^2} \sinh^2\left(\frac{\kappa s}{2\sqrt{V(r^*)}}\right). \quad (7.50)$$

Since $\sigma^2(\tau, s) = \sigma^2(s) = \sigma^2(-s)$, we can use equation (7.41) in the limit when $\Delta\tau$ goes to infinity:

$$\frac{dF}{d\tau} = \frac{\kappa}{8\pi^2\sqrt{V^*}} \int_{-\infty}^{\infty} dx \exp\left(-\frac{2i\sqrt{V^*}E x}{\kappa}\right) \left[-\frac{1}{\sinh^2 x} + \frac{1}{x^2}\right]. \quad (7.51)$$

The integral can be evaluated by the theorem of residues and the final result is

$$\frac{dF}{d\tau} = \frac{1}{2\pi} \frac{E}{\exp\left(\frac{2\pi\sqrt{V^*}E}{\kappa}\right) - 1}. \quad (7.52)$$

Since the transition rate exhibits the characteristic Planck distribution, it means that the Unruh–DeWitt thermometer in the generic spherically symmetric black hole space-time detects a quantum system in thermal equilibrium at the local temperature

$$T = \frac{\kappa}{2\pi\sqrt{V^*}}. \quad (7.53)$$

With regard to the factor $\sqrt{V^*} = \sqrt{-g_{00}}$, recall the Tolman’s theorem which states that, for a gravitational system at thermal equilibrium, $T\sqrt{-g_{00}} = \text{constant}$. For asymptotically flat black hole space-times, one obtains the “intrinsic” constant temperature of the Hawking effect, *i.e.*

$$T_{\text{H}} = \frac{\kappa}{2\pi} = \frac{V'_{\text{H}}}{4\pi}. \quad (7.54)$$

We would like to point out that this is a quite general result, valid for a large class of nice black holes, as for example Reissner–Nordström and Schwarzschild–AdS black holes. On the other hand, the Schwarzschild–dS black hole cannot be included, due to the presence of two horizons. However, as an important particular case, we may consider the static patch of de Sitter space, with a metric defined by

$$V(r) = 1 - H_0^2 r^2, \quad H_0^2 = \frac{\Lambda}{3}. \quad (7.55)$$

The unique horizon is located at $r_{\text{H}} = H_0^{-1}$ and the Gibbons–Hawking temperature is [17] $T_{\text{GH}} = H_0/2\pi$.

A point firstly stressed in de Sitter space by Garbrecht and Prokopec [137] regards how the thermal distribution of the response function is reached in the limit of very large times. To answer this, we consider the finite time contribution due to the fluctuating tail – the J term in Equation (7.41) – for de Sitter or black hole cases compared to the thermal value given by the even, time-independent part. A direct calculation of the tail (7.34) using Equation (7.50) for black holes (in particular Equation (7.55) for de Sitter) and the fact that

$$\text{csch}^2(x) = 4 \sum_{n=1}^{\infty} n e^{-2nx} \quad (7.56)$$

gives

$$\begin{aligned} J_{\text{dS}}(\Delta\tau) &= \frac{\kappa_l^2}{8\pi^2} \int_{\Delta\tau}^{\infty} ds \frac{\cos(Es)}{\sinh^2\left(\frac{\kappa_l s}{2}\right)} \\ &= \frac{E}{2\pi^2} \sum_{n=1}^{\infty} \frac{n e^{-n\kappa\Delta\tau}}{n^2 + E^2/\kappa^2} \left(\frac{\kappa}{E} n \cos(E\Delta\tau) - \sin(E\Delta\tau) \right) \end{aligned} \quad (7.57)$$

where κ_l is the rescaled surface gravity and $T_{\text{H}} = \kappa_l/2\pi \equiv \kappa/2\pi\sqrt{V}$ the local Hawking temperature. We recall that $\kappa = H_0$ for de Sitter space and $\kappa = V'(r_{\text{H}})/2$ for the black hole: these quantities in fact determine the characteristic time-scales the thermalization time has to be compared to.

We consider as before the peculiar Kodama observer for which $V = 1$, so that $T_{\text{H}} = \kappa/2\pi$. As a general feature, the fluctuating tail term drops out exponentially for large $\Delta\tau$, that is for long proper time intervals in which the detector stays on. In order to analyze the approaching to an equilibrium condition of the response function, we consider the ratio between the finite-time expression – the sum of the even part \dot{F}_e and the tail J – and the even part alone, with the agreement that equilibrium is attained whenever

$$R_{\text{eq}} = \frac{\dot{F}_e + J}{\dot{F}_e} \sim \text{O}(1).$$

Looking at (7.57) one can say that irrespective of the absolute value of κ , a detector that is switched on for a short time $\Delta\tau \ll \kappa^{-1}$ detects a thermal bath only with particles of energy $E \ll \kappa$. To see entering the thermal bath particles of energy $E \gg \kappa$, we need to keep the detector turned on for a much longer time $\Delta\tau \gg \kappa^{-1}$. The Hubble scale corresponds to an extremely small energy scale of order 10^{-42}GeV , therefore $E \gg \kappa$ corresponds to the physical region of interest.

7.4. The FRW and asymptotically de Sitter space-times

To apply the Unruh–DeWitt detector formalism to cosmology we consider a generic FRW spatially flat space-time. Recall that here the areal radius is $R = ra(t)$, so for the

Kodama observer, one has

$$r(t) = \frac{K}{a(t)}, \quad (7.58)$$

with constant K . For a radial trajectory, the proper time in FRW is

$$d\tau^2 = a(\eta)(d\eta^2 - dr^2). \quad (7.59)$$

As a function of the proper time, the conformal time along a Kodama trajectory is

$$\eta(\tau) = - \int d\tau \frac{1}{a(\eta)\sqrt{1 - K^2 H^2(\tau)}} \quad (7.60)$$

$H(\tau)$ being the Hubble parameter as a function of proper time. In general, we may use Equation (7.38) in which, for radial Kodama observer, one has

$$x(\tau) = (\eta(\tau), r(\tau), 0, 0) = \left(- \int \frac{1}{a(\tau)\sqrt{V(\tau)}} d\tau, \frac{K}{a(\tau)}, 0, 0 \right). \quad (7.61)$$

Let us come to consider the more realistic scenario of a truly dynamical space-time of cosmological interest. From previous considerations, our basic formulas for the transition rate of the detector (7.33) are manageable – in the sense that we can extract quantitative information – only in the few highly symmetrical circumstances mentioned in the previous subsection. As it will be clear at the end of this section, any departure from those models is responsible for significant computational complications. For instance, let us take on the case of homogeneous, spatially flat, universe dominated by cold matter and cosmological constant. The scale factor is (e.g. [138])

$$a(t) = a_0 \sinh^{2/3} \left(\frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right) \quad (7.62)$$

where $a_0 = (\Omega_m/\Omega_\Lambda)^{1/3}$ and $\Omega_m + \Omega_\Lambda = 1$; $H_0 = \sqrt{8\pi\rho_{cr}/3}$ and Ω_Λ represents the relative density of matter (if $\Lambda = m$) or cosmological constant (if $\Lambda = \Lambda$). Set $h \equiv \sqrt{\Omega_\Lambda} H_0$ for simplicity, its current value is of order $h \approx 2 \times 10^{-18} \text{sec}^{-1}$. Upon integration the conformal time becomes

$$\eta(t) = \frac{1}{a_0 h} \left\{ \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{4}{3}\right)}{\sqrt{\pi}} - \text{sech}^{2/3}(ht) {}_2F_1\left(\frac{5}{6}, \frac{1}{3}, \frac{4}{3}; \text{sech}^2\left(\frac{3}{2}ht\right)\right) \right\} \quad (7.63)$$

where ${}_2F_1(a, b, c; z)$ is a hypergeometric function and the constant has been opportunely chosen so that at the Big Bang $\eta(t=0) = 0$. The detector's proper time is related to the cosmic time through a manageable expression only if we limit ourselves to consider co-moving detectors: $\tau(t) - \tau_0 = \int dt \sqrt{1 - H^2(t)K^2}$ so that for $K = 0$, $\Delta\tau(t) = t$, $\Delta\tau$ being the proper time interval during which the detector is turned on. Unlike the stationary cases analyzed previously, this model presents a Big Bang singularity at the origin of the time coordinate, so that the detector must be switched on at some $\tau_0 > 0$. In particular, the Big Bang prevents taking the limit as $\tau_0 \rightarrow -\infty$. By the same reason, the scale factor (7.62) is defined only for positive values of the argument: a new feature with respect to what we have seen above. As a consequence, $a(t-s)$ or $a(t+s)$ are defined as in (7.62) only for $t-s > 0$ or $t+s > 0$, respectively; and trivially continued

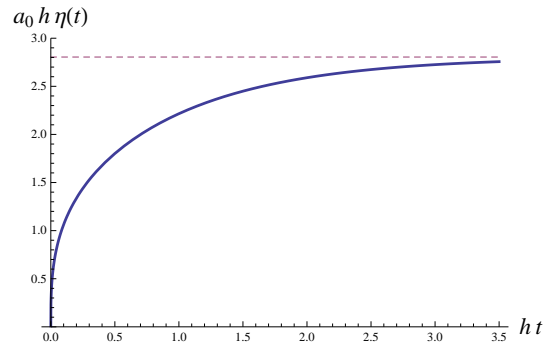


Figure 9. The conformal time $\eta(t)$ as a function of cosmological time and its asymptotic value $\Gamma(1/6)\Gamma(4/3)/\sqrt{\pi}$.

outside these intervals to make well defined the transition rate (7.33) or (7.38). Leaving the technical details to the original paper [139], we obtain the following response function

$$\begin{aligned} \dot{F}_\tau = \dot{F}_{\text{dS}} + J_{\text{dS}}(\Delta\tau) - \frac{h^2}{2\pi^2} \sum_{n=1}^{\infty} \sum_{k=1}^{3n-1} g(n, k) e^{-3nh\Delta\tau} \times \\ \times \frac{e^{hk\Delta\tau} (hk \cos(E\Delta\tau) + E \sin(E\Delta\tau)) - hk}{h^2 k^2 + E^2} \end{aligned} \quad (7.64)$$

in which \dot{F}_{dS} is the de Sitter τ independent contribution given by Equation (7.23) with $K = 0$ and effective Hubble constant $h = \sqrt{\Omega_\Lambda} H_0$ and J_{dS} is the related tail given by Equation (7.57). The numerical coefficients $g(n, k)$ can in principle be computed but enter in a tail which decays exponentially fast in the switching time and which also contain oscillating terms. We may take the limit $\Delta\tau \rightarrow \infty$ and observe that every τ dependent term of this expression goes to zero. This is the main result of the section.

To summarise, we may say that the detector clicks close to a de Sitter response and reaches thermalization (possibly, through decaying oscillations) as $\Delta\tau$ is sufficiently large. In fact, as far as the regime $h\Delta\tau \gg 1$ is concerned, de Sitter space-time is recovered. We may think of this as describing a de Sitter thermal noise continuously perturbed by the expansion (or contraction) of the universe. In particular, insofar as we can speak of temperature, in this large-time regime the detector registers the de Sitter temperature $h/2\pi$, equal to the large-time limit of the horizon temperature parameter given by the surface gravity which, in the present case, has the exact but slow long-time evolution

$$T_{\text{H}} = \frac{h}{2\pi} \left[\coth\left(\frac{3ht}{2}\right) - \frac{3}{4} \operatorname{sech}\left(\frac{3ht}{2}\right) \operatorname{csch}\left(\frac{3ht}{2}\right) \right].$$

It is worth noting that, while in the stationary phase $h\Delta\tau \gg 1$ the limiting result is consistent with the limiting value of the surface gravity, in the non-stationary regime it seems less trivial to compare the results of the two methods, because it has not been possible to extract a temperature parameter from the transition rate of the detector, but asymptotically.

In conclusion, we have seen that for black holes and pure de Sitter space semi-classical and quantum field theoretic analysis are mutually consistent and even predict the dependence of the temperature on position or acceleration. Moreover, the analysis of the oscillating tail can be extended to stationary black holes as well.

For cosmology and away from de Sitter space the thermal interpretation, strictly speaking, is lost but the detector still gets excited by the expansion of the universe. By accepting the surface gravity versus temperature paradigm, we would expect a quasi-thermal excitation rate of the form

$$\dot{F} \sim E \exp(-E/T_{\text{H}}(t))$$

$T_{\text{H}}(t)$ being given by our last expression above. That is, although in a generic FRW space-time the thermal interpretation breaks down in most of the cases because of the time-dependence of the background, still the time dependence of the transition rate could be expected to mainly reside in an effective temperature parameter, as written. But, using a co-moving detector, this is not what we have found. For instance, in the Einstein–de Sitter regime there seems to be excitations of non-thermal type; and, as we have just shown, the scale factor of the flat Λ CDM-cosmology has no other temperatures in action than the de Sitter one.

It remains to see whether there is any non trivial quasi-thermal effect on accelerated, or more general, Kodama trajectories. In the affirmative case, that would mean that the horizon surface gravity and temperature should be associated more likely to vacuum correlations than to particle creation forcing, in our view, a different interpretation of the tunnelling picture. One possibility is that the horizon surface gravity could represent an intrinsic property of the horizon itself, leading to some kind of holographic description, while the detector in the bulk simply clicks because it is embedded in a changing geometry. In fact, we would expect the clicks in almost any changing geometry, even for those lacking a trapping horizon.

8. Conclusions

We have completed our journey into the realm of quantum gravitational effects that can be understood as a result of tunnelling processes. In this journey, we visited black hole radiation, de Sitter and cosmological radiation, decays and naked singularities, but we have not included tunnelling between different gravitational field configurations, nor topology changing processes, nor sonic black holes or other analogue objects. Thus we may generically qualify the subject as covering semi-classical effects in a given background field. From this point of view it seemed remarkable that it could be extended to cover back-reaction effects to first order in the ratio E/M , where E is the energy scale of the tunnelling process and M the black hole mass or some generalisation thereof.

The most robust results in the subject were obtained in Section 2 and Section 3 and comprise:

- (i) The ability of the method to include first order back-reaction effects in order to comply with energy conservation and the predicted correlations among successive emission events is one of the key results.
- (ii) In the stationary case, it is possible to provide a foundation of tunnelling methods using analytic continuation of the classical action alone, avoiding in this way the seemingly impossible task of crossing the horizon. The analytic continuation could be extended to cover white holes and to associate with them a well defined geometric temperature, a point not always appreciated in literature.
- (iii) In the same vein, one can prove the complete equivalence of the null geodesic method with the Hamilton–Jacobi method for stationary fields. We also gave a “covariantised” version of the null geodesic method, which allows to avoid questions such as covariance or canonical invariance.

Section 4 is again devoted to the treatment of two dirty black hole solutions in modified theories of gravity, with emphasis on their thermodynamic properties. We have shown that they possess in general Killing horizons with an associated Killing surface gravity. This is typically different from Hayward’s surface gravity due to the ambiguity in defining the normalization of Killing vectors in non asymptotically flat spaces and, as a consequence, of energy. We have seen how this is not a big problem to the fact that the ratio temperature/energy remains an invariant scalar whatever the choice for the temperature or energy. The entropy *ala* Wald of these black holes is in general well defined; the definition of a sensible mass parameter is, instead, a still debated question.

In Section 5, using a local notion of horizon and dynamical surface gravity for time varying spherically symmetric black holes, the main results comprise:

- (i) The application of the tunnelling method to non stationary, spherically symmetric black holes and the fact that it naturally selects a trapping horizon of the future non degenerate type, either inner or outer. It can also be applied to past horizons under the same condition and by interchanging emission with absorption. In the

limiting situation where the surface gravity vanishes it is consistent with the idea that extremal black holes have zero temperature.

- (ii) One of the principal consequences of the tunnelling method is the observation that the radiation seems to originate near the local trapping horizon, not the global event horizon.
- (iii) Using the Kodama–Hayward theory of spherically symmetric space-times, and noting that one can possibly associate an observable temperature to a black hole only if it is a coordinate scalar, we showed that the tunnelling probability depends on the ratio, $2\pi E/\kappa$, of the Kodama invariant charge taken as energy and the geometrical surface gravity as defined by Hayward.
- (iv) One can include non interacting fermions in the formalism, as demonstrated in several papers. Also noticeable is the fact that any mass term gets strongly suppressed by the horizon pole, which dominates the rate well over mass contributions.
- (v) The time-like nature of the trapping horizon of an excreting black hole is consistent with the physical interpretation of the formalism. It is true that it can be crossed either ways, but it remains the fact that there is a class of paths for which an imaginary part exists, and these paths precisely correspond in Feynman diagram language to particle creation.

In Section 6 we addressed other themes – cosmology, decays and naked singularities. Since FRW spaces are spherically symmetric, the general theory outlined in Section 5.2 applies. We showed that, with obvious modifications, the tunnelling picture works equally well. In particular the type-II paths never contribute, while type-I paths do. We may consider these findings for cosmological horizons as a generalisation of the well known facts discovered for de Sitter space-time, in complete analogy with the parallel extension of the theory from stationary to dynamical black holes. We also considered the decay of particles which in the absence of gravity would not occur, and find that the Hamilton–Jacobi equation can manage them through the presence of a branch point singularity in the radial momentum, that would not be present in absence of an external field. From this point of view the tunnelling method has universally valid features. Again, its value is to be found where it provides approximate results when exact calculations are in general impossible, at the same time being consistent with the exact calculations when available (such is the case of de Sitter space). We showed also that the presence of strong fields in a certain space-time region, even in the absence of horizons, can result in particle production by means of a tunnelling process “from nothing”. It can be considered as the gravitational analogue of the Schwinger effect in a strong external electric field. Finally, it is of interest that the tunnelling method can also be applied to naked singularities, although in this case there is nothing to tunnel through, the singularity being the boundary of space-time. We showed nevertheless that the emerging picture is quite in agreement with quantum field theory in two-dimensional models, making one confident of its general validity.

In Section 7, Unruh–DeWitt detectors have been used in order to investigate the issue of temperature associated with spherically symmetric dynamical space-times. We have shown that for the generic static black hole case and the FRW de Sitter case, making use of peculiar Kodama trajectories, semiclassical and quantum field theoretic techniques give the same standard and well known thermal interpretation, with an associated temperature, corrected by appropriate Tolman factors. For a FRW space-time interpolating de Sitter space with the Einstein–de Sitter universe (that is a more realistic situation in the frame of Λ CDM cosmologies), we have seen that the detector response splits into a de Sitter contribution plus a fluctuating term containing no trace of Boltzmann-like factors, but rather describing the way thermal equilibrium is reached in the late time limit. As a consequence, and unlike the case of black holes, the identification of the dynamical surface gravity of a cosmological trapping horizon as an effective temperature parameter seems lost, at least for our co-moving simplified detectors. The possibility remains that a detector performing a proper motion along a Kodama trajectory may register something more, in which case the horizon surface gravity would be associated more likely to vacuum correlations than to particle creation.

We conclude with few general remarks on the tunnelling method extensively used insofar. One can probably complain that the tunnelling method gives only approximate results. But for black holes with really intricate metrics exact calculations are illusory even in stationarity, so we feel that this is not a serious drawback. Even if there is little doubt that the method is correct, still it remains of a hypothetical nature. However, we think that also in case of a failure due to some internal inconsistency, that should be nonetheless an important message. In fact, tunnelling processes are predictions of quantum theory, so a failure of them in gravity theory would certainly signal something interesting. Despite all the desirable properties discussed above, the tunnelling method is and remains an essentially semi-classical procedure carrying along with it all the limitations inherent to its nature. In particular, it applies only to free particles while in principle quantum field theory methods certainly have a wider scope.

The tunnelling method is not a closed subject. For instance, in striking contrast with the stationary case, an important missing point of the dynamical case is the absence of an extension of Kodama–Hayward’s theory to dynamical axis-symmetric black holes, to be used as a tool for studying tunnelling. We mention here the papers [140] where a tentative theory is developed. We feel that this is perhaps the most important missing point. The first back-reaction corrections are also of great interest, as they are not so well developed in dynamical situations. Processes where small black holes are emitted by large ones should also be within the range of problems where the tunnelling picture could work, and that also is an important “to do”.

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9. References

- [1] Hawking S W 1974 *Nature* **248** 30; Hawking S W 1975 *Commun. Math. Phys.* **43** 199–220
- [2] 't Hooft G 1993 Dimensional reduction in quantum gravity *Preprint* gr-qc/9310026; 't Hooft G 1995 *Nucl. Phys. Proc. Suppl.* **43** 1–11
- [3] Susskind L 1995 *J. Math. Phys.* **36** 6377–96
- [4] Susskind L, Thorlacius L and Uglum J 1993 *Phys. Rev. D* **48** 3743–61
- [5] Jacobson T 1993 *Phys. Rev. D* **48** 728–41
- [6] Visser M 1998 *Phys. Rev. Lett.* **80** 3436–9
- [7] Visser M 2003 *Int. J. Mod. Phys. D* **12** 649
- [8] Barcelo C, Liberati S, Sonogo S and Visser M 2006 *Phys. Rev. Lett.* **97** 171301
- [9] Schutzhold R and Unruh W G 2008 *Phys. Rev. D* **78** 041504
- [10] Kiefer C 2001 *Class. Quantum Grav.* **18** L151
- [11] Vaz C, Kiefer C, Singh T P and Witten L 2003 *Phys. Rev. D* **67** 024014
- [12] Unruh W G and Schutzhold R 2005 *Phys. Rev. D* **71** 024028
- [13] Kiefer C, Muller–Hill J, Singh T P and Vaz C 2007 *Phys. Rev. D* **75** 124010
- [14] Birrell N D and Davies P C W 1982 *Quantum fields in curved space* (Cambridge: Cambridge University Press)
- [15] Schutzhold R 2001 *Phys. Rev. D* **63** 024014
- [16] DeWitt B S 1975 *Phys. Rept.* **19** 295–357
- [17] Gibbons G W and Hawking S W 1977 *Phys. Rev. D* **15** 2738–51
- [18] Damour T and Ruffini R 1976 *Phys. Rev. D* **14** 332–4; Srinivasan K and Padmanabhan T 1999 *Phys. Rev. D* **60** 24007; Shankaranarayanan S, Srinivasan K and Padmanabhan T 2001 *Mod. Phys. Lett. A* **16** 571–8; Shankaranarayanan S, Padmanabhan T and Srinivasan K 2002 *Class. Quantum Grav.* **19** 2671–88; Padmanabhan T 2004 *Mod. Phys. Letts. A* **19** 2637–43; Massar S and Parentani R 2000 *Nucl. Phys. B* **575** 333–56
- [19] Kraus P and Wilczek F 1995 *Nucl. Phys. B* **433** 403–20
- [20] Kraus P and Wilczek F 1995 *Nucl. Phys. B* **437** 231–42
- [21] Parikh M K and Wilczek F 2000 *Phys. Rev. Lett.* **85** 5042–5
- [22] Moretti V and Pinamonti N 2010 State independence for tunnelling processes through black hole horizons and Hawking radiation *Preprint* gr-qc/1011.2994
- [23] Angheben M, Nadalini M, Vanzo L and Zerbini S 2005 *JHEP* **0505** 014; Nadalini M, Vanzo L and Zerbini S 2006 *J. Phys. A* **39** 6601
- [24] Di Criscienzo R, Nadalini M, Vanzo L, Zerbini S and Zoccatelli G 2007 *Phys. Lett. B* **657** 107–11
- [25] Kerner R and Mann R B 2006 *Phys. Rev. D* **73** 104010
- [26] Chatterjee B, Ghosh A and Mitra P 2008 *Phys. Lett. B* **661** 307–11
- [27] Kerner R and Mann R B 2007 *Phys. Rev. D* **75** 084022
- [28] Nielsen A B and Visser M 2006 *Class. Quantum Grav.* **23** 4637
- [29] Wu X-n and Gao S 2007 *Phys. Rev. D* **75** 044027
- [30] de Gill A, Singleton D, Akhmedova V and Pilling T 2010 *Am. J. Phys.* **78** 685–91
- [31] Cognola G, Gorbunova O, Sebastiani L and Zerbini S 2011 *Phys. Rev. D* **84** 023515
- [32] Narnhofer H, Peter I and Thirring W E 1996 *Int. J. Mod. Phys. B* **10** 1507–20

- [33] Deser S and Levin O 1997 *Class. Quantum Grav.* **14** L163–8; Deser S and Levin O 1998 *Class. Quantum Grav.* **15** L85–7; Deser S and Levin O 1999 *Phys. Rev. D* **59** 064004
- [34] Frolov V P and Novikov I D 1998 *Black Hole Physics. Basic Concepts and New Developments* (Dordrecht: Kluwer Academic Publishers)
- [35] Parikh M K 2004 A Secret Tunnel Through The Horizon *Preprint* hep-th/0405160
- [36] Hartle J B and Hawking S W 1976 *Phys. Rev. D* **13** 2188
- [37] Hawking S W and Ellis G F R 1973 *The large scale structure of space-time* (Cambridge: Cambridge University Press)
- [38] Painlevé P 1921 *Compt. Rend. Acad. Sci.* **173** 677; Gullstrand A 1922 *Arkiv. Mat. Astron. Fys.* **16** 1
- [39] Thomsen J S 1953 *Phys. Rev.* **91** 1263–6
- [40] Hawking S W 1971 *Phys. Rev. Lett.* **26** 1344; Bekenstein J D 1972 *Lett. Nuov. Cimento* **4** 737; Bekenstein J D 1973 *Phys. Rev. D* **7** 2333–46
- [41] Landau L D and Lifshitz E M 1980 *Course of Theoretical Physics. Vol. 5. Statistical Physics* (Oxford: Pergamon Press)
- [42] Gibbons G W and Hawking S W 1977 *Phys. Rev. D* **15** 2752–6
- [43] Parikh M K 2004 Energy conservation and Hawking radiation *Preprint* hep-th/0402166
- [44] Arzano M, Medved A J M and Vagenas E C 2005 *JHEP* **0509** 037; Medved A J M and Vagenas E C 2005 *Mod. Phys. Lett. A* **20** 1723–8
- [45] Zhang B, Cai Q, You L and Zhan M 2009 *Phys. Lett. B* **675** 9–101; Zhang B, Cai Q, Zhan M and You L 2011 *Ann. Phys.* 326350–63
- [46] Israel W and Yun Z 2010 *Phys. Rev. D* **82** 124036
- [47] Mann R B and Ross S 1995 *Phys. Rev. D* **52** 2254; Bousso R and Hawking S W 1996 *Phys. Rev. D* **54** 6312
- [48] Di Criscienzo R, Hayward S, Nadalini M, Vanzo L and Zerbini S 2010 *Class. Quantum Grav.* **27** 015006
- [49] Eddington A S 1924 *Nature* **113** 192; Finkelstein D 1958 *Phys. Rev.* **110** 965
- [50] Stotyn S, Schleich K and Witt D 2009 *Class. Quantum Grav.* **26** 065010
- [51] Tolman R C 1934 *Relativity, Thermodynamics and Cosmology* (Oxford: Clarendon Press)
- [52] Visser M 1992 *Phys. Rev. D* **46** 2445–51
- [53] Hayward S A, Di Criscienzo R, Nadalini M, Vanzo L and Zerbini S 2009 *Class. Quantum Grav.* **26** 062001
- [54] Jacobson T 2007 *Class. Quantum Grav.* **24** 5717–9
- [55] Vanzo L, Acquaviva G and Di Criscienzo R 2011 *Class. Quantum Grav.* **28** 183001
- [56] Vanzo L 2011 On tunneling across horizons *Preprint* gr-qc/1104.1569, in press on Europhysics Lett.
- [57] Wald R M 1984 *General Relativity* (Chicago: University of Chicago Press)
- [58] Pizzi M 2009 Second comment to “Invariance of the tunneling method” *Preprint* gr-qc/0909.3800; Hayward S A, Di Criscienzo R, Nadalini M, Vanzo L and Zerbini S 2009 Comment on “On the tunneling through the black hole horizon” *Preprint* gr-qc/0911.4417; Hayward S A, Di Criscienzo R, Nadalini M, Vanzo L and Zerbini S 2009 Reply to Comments on “Invariance of the tunneling method for dynamical black holes” *Preprint* gr-qc/0909.2956
- [59] Akhmedov E T, Akhmedova V and Singleton D 2006 *Phys. Lett. B* **642** 124; Nakamura T K 2007 Factor Two Discrepancy of Hawking Radiation Temperature *Preprint* 0706.2916
- [60] Mitra P 2007 *Phys. Lett. B* **648** 240–2
- [61] Bellini E, Di Criscienzo R, Sebastiani L and Zerbini S 2010 *Entropy* **12** 2186–98
- [62] Akhmedov E T, Pilling T and Singleton D 2008 *Int. J. Mod. Phys. D* **17** 2453–8
- [63] Chowdhury B D 2008 *Pramana* **70** 3–26
- [64] Medved A J M and Vagenas E C 2005 *Mod. Phys. Lett. A* **20** 2449–54
- [65] Vagenas E C 2003 *Phys. Lett. B* **559** 65–73; Jiang Q Q, Li H L, Yang S Z and Chen D Y 2007 *Mod. Phys. Lett. A* **22** 891

- [66] Yang S and Chen D 2007 *Int. J. Theor. Phys.* **46** 1747–52
- [67] Banerjee R and Majhi B R 2008 *JHEP* **0806**
- [68] Majhi B R 2009 *Phys. Rev. D* **79** 044005
- [69] Majhi P R and Samanta S 2010 *Annals Phys.* **325** 2410–24
- [70] Banerjee R, Kiefer C and Majhi B R 2010 *Phys. Rev. D* **82** 044013
- [71] Banerjee R and Majhi B R 2009 *Phys. Lett. B* **675** 243–5
- [72] Yale A 2011 *Phys. Lett. B* **697** 398–403; Chatterjee B and Mitra P 2009 *Phys. Lett. B* **675** 240–2; Wang W, Ding C, Chen S and Jing J 2010 *Gen. Rel. Grav.* **42** 347–57
- [73] Yale A 2011 *Eur. Phys. J. C* **71** 1622
- [74] Clifton T and Barrow J D 2005 *Phys. Rev. D* **72** 103005
- [75] Deser S, Sarioglu O and Tekin B 2008 *Gen. Rel. Grav.* **40** 1
- [76] Wald R M 1993 *Phys. Rev. D* **48** 3427
- [77] Deser S and Tekin B 2007 *Phys. Rev. D* **75** 084032; Abreu G and Visser M 2010 *Phys. Rev. Lett.* **105** 041302
- [78] Brown J D and York J W 1993 *Phys. Rev. D* **47** 1407; Brown J D, Creighton J and Mann R B 1994 *Phys. Rev. D* **50** 6394; Chan K C K, Horne J H and Mann R B 1995 *Nucl. Phys. B* **447** 441
- [79] Maeda H and Nozawa M 2008 *Phys. Rev. D* **77** 06403; Cai R G, Cao L M, Hu Y P and Ohta N 2009 *Phys. Rev. D* **80** 104016
- [80] Vaidya P C 1951 *Proc. Indian Acad. Sci. A* **33** 264; Vaidya P C 1953 *Nature* **171** 260; Narlikar V V and Vaidya P C 1947 *Nature* **159** 642
- [81] Bardeen J M 1981 *Phys. Rev. Lett.* **46** 382
- [82] York J W 1983 *Phys. Rev. D* **28** 2929
- [83] Booth I and Fairhurst S 2004 *Phys. Rev. Lett.* **92** 011102
- [84] Brevik I 2000 *Phys. Rev. D* **61** 124017
- [85] Fredenhagen K and Haag R 1990 *Commun. Math. Phys.* **127** 273
- [86] Hayward S A 1994 *Phys. Rev. D* **49** 6467; Hayward S A 1994 *Class. Quantum Grav.* **11** 3025
- [87] Hayward S A 1998 *Class. Quantum Grav.* **15** 3147–62
- [88] Ashtekar A, Beetle C and Fairhurst S 1999 *Class. Quantum Grav.* **16** L1; Ashtekar A, Corichi A and Krasnov K 2000 *Adv. Theor. Math. Phys.* **3** 419; Ashtekar A, Fairhurst S and Krishnan B 2000 *Phys. Rev. D* **62** 104025; Ashtekar A, Beetle C, Dreyer O, Fairhurst S, Krishnan B, Lewandowski J and Wisniewski J 2000 *Phys. Rev. Lett.* **85** 3564
- [89] Ashtekar A and Krishnan B 2002 *Phys. Rev. Lett.* **89** 261101; Ashtekar A and Krishnan B 2003 *Phys. Rev. D* **68** 104030
- [90] Ashtekar A and Krishnan B 2004 *Living Rev. Rel.* **7** 10
- [91] Hájiček P 1973 *Comm. Math. Phys.* **34** 37; Hájiček P 1977 *Lett. Nuovo Cim.* **18** 251–4
- [92] Hayward S A 1994 *Class. Quantum Grav.* **11** 3037–48
- [93] Hayward S A 1996 *Phys. Rev. D* **53** 1938–49
- [94] Hayward S A 2004 *Phys. Rev. Lett.* **93** 251101; Hayward S A 2006 *Phys. Rev. D* **74** 104013
- [95] Nielsen A B and Yoon J H 2008 *Class. Quantum Grav.* **25** 085010
- [96] Bardeen J M, Carter B and Hawking S W 1973 *Commun. Math. Phys.* **31** 161
- [97] Fodor G, Nakamura K, Oshiro Y and Tomimatsu A 1996 *Phys. Rev. D* **54** 3882
- [98] Kodama H 1980 *Prog. Theor. Phys.* **63** 1217
- [99] Mukohyama S and Hayward S A 2000 *Class. Quantum Grav.* **17** 2153
- [100] Brown B A and Lindesay J 2010 *AIP Conf. Proc.* **1280** 3–16; Lindesay J 2009 *Class. Quantum Grav.* **26** 125014
- [101] Lindesay J and Sheldon P 2010 *Class. Quantum Grav.* **27** 215015
- [102] Clifton T 2008 *Class. Quantum Grav.* **25** 175022
- [103] McVittie G C 1933 *Mon. Not. R. Astronomic Soc.* **93** 325
- [104] Nolan B C 1998 *Phys. Rev. D* **58** 064006; Nolan B C 1999 *Class. Quantum Grav.* **16** 1227; Nolan B C 1999 *Class. Quantum Grav.* **16** 3183

-
- [105] Francis M R and Kosowsky A 2004 *Am. J. Phys.* **72** 1204–9
- [106] Misner C W and Sharp D H 1964 *Phys. Rev.* **136** B571; Hawking S W 1968 *J. Math. Phys.* **9** 598
- [107] Szabados L B 2004 *Living Rev. Relativity* **7** 4
- [108] Abreu G and Visser M 2010 *Phys. Rev. D* **82** 044027
- [109] Belinski V A 2009 On the Tunnelling Through the Black Hole Horizon *Preprint* gr-qc/0910.3934
- [110] Hiscock W A 1981 *Phys. Rev. D* **23** 2813; Balbinot R 1986 *Phys. Rev. D* **33** 1611
- [111] Kerner R and Mann R B 2008 *Class. Quantum Grav.* **25** 09501
- [112] Di Criscienzo R and Vanzo L 2008 *Europhys. Lett.* **82** 60001
- [113] Li H L, Yang S Z, Zhou T J and Lin R 2008 *Europhys. Lett.* **84** 20003
- [114] Kerner R and Mann R B 2008 *Phys. Lett. B* **665** 4
- [115] Yale A and Mann R B 2009 *Phys. Lett. B* **673** 168–72
- [116] Barducci A, Casalbuoni R and Lusanna L 1976 *Nuovo Cimento A* **35** 377; Berezin F A and Marinov M S 1977 *Annals Phys.* **104** 336; Cognola G, Vanzo L, Zerbini S and Soldati R 1981 *Phys. Lett. B* **104** 67–9
- [117] Arzano M 2006 *Mod. Phys. Lett. A* **21** 41–8; Cai R G, Cao L M and Hu Y P 2009 *Class. Quantum Grav.* **26** 155018; Cai R G, Cao L M and Hu Y P 2008 *JHEP* **0808** 090; Cai R G and Cao L M 2007 *Nucl. Phys. B* **785** 135–48; Cai R G and Cao L M 2007 *Phys. Rev. D* **75** 064008
- [118] Volovik G E 2008 On de Sitter radiation via quantum tunneling *Preprint* gr-qc/0803.3367
- [119] Di Criscienzo R, Vanzo L and Zerbini S 2010 *JHEP* **1005** 092
- [120] Mallet R L 1986 *Phys. Rev. D* **33** 2201
- [121] Wu S F, Wang B, Yang G H and Zhang P M 2008 *Class. Quantum Grav.* **25** 235018
- [122] Brout R, Horwitz G and Weil D 1987 *Phys. Lett. B* **192** 318
- [123] Myhrvold N P 1983 *Phys. Rev. D* **28** 2439
- [124] Volovik G E 2009 *JETP Lett.* **90** 1
- [125] Bros J, Epstein H and Moschella U 2008 *JCAP* **0802** 003; Bros J, Epstein H and Moschella U 2010 *Annales Henri Poincaré* **11** 611–58; Bros J, Epstein H, Gaudin M, Moschella U and Pasquier V 2010 *Commun. Math. Phys.* **295** 261–88
- [126] Harada T, Iguchi H and Nakao K I 2002 *Prog. Theor. Phys.* **107** 449–524
- [127] Vaz C and Witten L 1994 *Phys. Lett. B* **325** 27
- [128] Harada T, Iguchi H, Nakao K I, Singh T P, Tanaka T and Vaz C 2001 *Phys. Rev. D* **64** 041501; Iguchi H and Harada T 2001 *Class. Quantum Grav.* **18** 3681
- [129] Ford L H and Parker L 1978 *Phys. Rev. D* **17** 1485
- [130] Unruh W G 1976 *Phys. Rev. D* **14** 870
- [131] DeWitt B S 1979 “Quantum gravity: the new synthesis” in *General Relativity an Einstein centenary survey* (Cambridge: Cambridge University Press)
- [132] Langlois P 2005 Imprints of spacetime topology in the Hawking–Unruh effect *Preprint* gr-qc/0510127
- [133] Takagi S 2004 *Prog. Theor. Phys. Supp. Grav.* **88** 1; Schlicht S 2004 *Class. Quantum Grav.* **21** 4647
- [134] Casadio R, Chiodini S, Orlandi A, Acquaviva G, Di Criscienzo R and Vanzo L 2011 *Mod. Phys. Lett. A* **26** 2149–58
- [135] Obadia N and Milgrom M 2007 *Phys. Rev. D* **75** 065006
- [136] Louko J and Satz A 2006 *Class. Quantum Grav.* **23** 6321; Louko J and Satz A 2008 *Class. Quantum Grav.* **25** 055012
- [137] Garbrecht B and Prokopec T 2004 *Class. Quantum Grav.* **21** 4993
- [138] Mukhanov V 2005 *Physical Foundations of Cosmology* (Cambridge: Cambridge University Press)
- [139] Acquaviva G, Di Criscienzo R, Vanzo L and Zerbini S 2011 Unruh–DeWitt detectors in spherically symmetric dynamical space-times *Preprint* arXiv: 1101.5254
- [140] Jing J and Wang Y 1996 *Int. J. Theor. Phys.* **35** 1481–91; Yang X J, He H and Zhao Z 2003 *Gen. Rel. Grav.* **35** 579–94; Chen D and Yang S 2007 *Gen. Rel. Grav.* **39** 1503–15