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A thesis submitted for the degree of Doctor of Philosophy

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New proposals for the popularization of braid theory

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Contents

Introduction

The request for an informal approach to mathematics comes with increasing regularity from the public. Consequently, there is a wide offer of books, movies and other items to popularize mathematics. In Italy a sign of the renewed interest in this discipline is the growing number of students who enter mathematics courses at university, after some years of inverse trend. However, most students just study the standard topics contained in the school syllabus and have no occasion to approach further themes and methods. In other countries, such as France, few young students are attracted by the study of mathematics. Therefore, teachers are constantly looking for new methods to motivate them.

The aim of this work is to enter in this dynamics and contribute to the popularization of mathematics with new proposals and materials. The main project developed is a cartoon movie: the use of graphical tools is a great opportunity especially in the exposition of topological themes. The chosen subject, braid theory, is indeed topological, and even part of its algebraic formalization can be effectively visualized.

Braids are very rich mathematical objects that can be introduced and studied from several points of view, as we will see in Chapter 1. They have various applications in physics and biology, such as in the study of electromagnetism, of coronal loops in the sun, of the motion of particles, of the DNA structure. Like knot theory, it is not straightforward to explain braid theory in an informal manner: the objects seem to be elementary at a first approach, but the formalization that captures the intuitive ideas is sometimes complicated or subtle.

In this work we provide two possible approaches to braids at a popular level: a movie and a workshop. Before presenting our realizations, we introduce the mathematics of braids, in Chapter 1. We do not aim to give a complete survey on braid theory but focus on the themes that will be developed in the popularization activities. Several references to further reading will be given inside the exposition.

Chapter 2 describes the scientific movie addressed to a non-specialist audience. The drawings were realized using the free ray-tracer POV-Ray. The movie is divided into four parts, each of which has a length of about 15 minutes. The content ranges from the introduction of basic concepts to deep results. The first part of the movie introduces braids as mathematical objects: starting from the topological definition of a braid we arrive to an algebraic formalization of the braid groups. In particular, we give Artin's description of the braid group in terms of a presentation. It is then natural to consider some problems about braid words, one of which is examined in the second part of the movie. Here the attention focuses on algorithms and their complexity. In the third part other topological objects are introduced, that are strongly connected to braids: knots and links. We present some of these connections and one of the major results in knot theory, that was reached through the discovery of a representation of braids in an algebra: the Jones polynomial. The last part of the movie deals with braids as motion of points. In this setting a subgroup of the braid group is discussed that is used in proving results about more connections to knot theory.

The full script of the movie outlined in Chapter 2 is contained in Appendix A.

In Chapter 3 we present a workshop activity based on the action of braids on loops, inviting and leading the audience to a mathematical formalization of the principal concepts involved: braids, curves and group actions. The activity has been developed for high school students and for the general public and has already been tested in two occasions: at a residential laboratory for students in the final year of the high school and at an event for popularizing the scientific research. We analyse the results of the activity in the two settings.

In Appendix B we motivate the choice of offering a workshop in the school and make a concrete proposal for teachers to put it in practice. In particular, we suggest a way to organize the activity and some worksheets that can be used in a class.

Selecting the subjects for the movie and the workshop has not been immediate and has required a wide insight in the matter. The themes we identified are sometimes less topological but can be more comprehensible and closer to the public's experience.

The choices are not only the result of the study of several books and papers, some of which are cited in the bibliography, but also of various discussions with experts in the involved areas and in the communication of mathematics. Indeed, to realize the movie and the workshop I could profit from two visiting grants at the University of Caen and participate in several conferences and seminars. This experience has been very interesting and instructive because of the close contact to many mathematicians who love their research work and believe that popularization is a fundamental part of it.

Chapter 1

Braid theory

In this chapter we introduce the theory of braids. This area of mathematics is quite recent: braids were first defined and studied by Emil Artin in [5]. A second fundamental paper is [6], in which Artin makes a rigorous treatment of the subject, proving the results obtained in the previous paper.

Braids are ubiquitous objects: braids and some of their generalizations are studied from different points of view, such as topology and group theory. This field is also very fertile and applications can be found in knot theory, mathematical physics, robotics, biology, fluid mechanics, solar physics, particle physics.

Our aim is not to give a complete survey on braids: many aspects are not considered here. We focus on the subjects necessary to follow the movie and the workshop described in the next chapters. Some further topics are mentioned in Section 2.2.5. For wider surveys on braid theory we refer to a lot of different books and surveys, such as [41], [50], [54], [51], [9], [8], [33]. These are general references, more specific ones will be pointed out inside the exposition.

The colour figures in this chapter are taken from the movie described in Chapter 2, unless otherwise specified.

1.1 Geometric braids

A braid can be thought as a collection of strands in the space, with ends fixed on two parallel discs, so that the linked part can not "escape" moving beyond the strand ends. Moreover, one requires that the strands "flow" in a fixed direction, without turning back. In other words, any plane between the two discs and parallel to them will intersect each strand exactly once. More formally, we can give the following Definition.

Definition 1.1.1
Fix n points $p_1, p_2, \ldots, p_n$ in the interior of a disc D. A *geometric braid* β

on n strands is a n-tuple of paths $\beta = (b_1, b_2, \dots, b_n)$, $b_i : I \to I \times D$ such that the following conditions hold.

(i) $b_i(t) \in \{t\} \times D$ for all $t \in I$.

(ii) $b_i(0) = (0, p_i)$ for all $1 \leq i \leq n$.

(iii) There exists a permutation $\pi \in S_n$ such that $b_i(1) = (1, p_{\pi(i)})$.

(iv) $b_i(t) \neq b_j(t)$ for all $i \neq j$ and for all $t \in I$.

We will call *geometric braid* not only the n-tuple of paths $(b_1, b_2, \dots, b_n)$ but also the image $(b_1(I), b_2(I), \dots, b_n(I))$ of the paths.
Both the path b_i and its image $b_i(I)$ are called the i-th *strand* (or *string*) of the braid. π is called the *permutation induced by the braid β*.

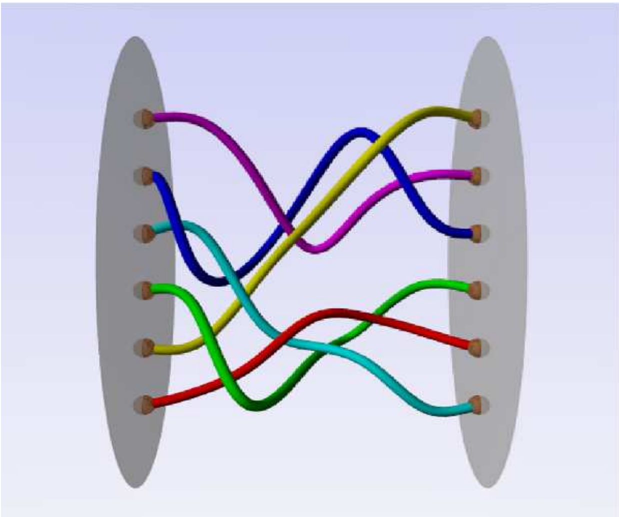


Figure 1.1: A braid is a collection of monotone disjoint paths in $I \times D$ joining some fixed points on the two parallel discs $\{0\} \times D$ and $\{1\} \times D$.

Remark 1.1.2
We will only consider *tame* braids, i.e. we suppose that the strands have pairwise disjoint tubular neighbourhoods. This is equivalent to consider piecewise linear braids and avoids "pathological" situations where the strands are linked together an infinite number of times. The drawings of the movie and of many figures of this chapter are consistent with this restriction: instead of strands we draw tubes, that can be seen as the tubular neighbourhoods of the strands.

To draw braids, we put the left disc in the plane $x = 0$, the right disc in the plane $x = 1$ and the points $p_1, p_2, \dots, p_n$ on the line $z = 0$ such that

$p_{1_y} < p_{2_y} < \ldots < p_{n_y}$. The x direction will be the flowing direction, going from 0 to 1. Hence a braid is contained in the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, y^2 + z^2 \leq 1\}$.

In the literature different conventions are used to draw braids: some authors draw them from bottom to top, some others from top to bottom and others from left to right. We choose this last convention and draw braids in horizontal, from left to right. Our choice allows to better exploit the 4 : 3 format of the movie. Only in the last part of the movie, where braids are characterized as motion of points, we draw braids vertically.

1.2 Braids and isotopies

It is natural to consider two braids the same if their strands are linked in the same manner. In other words, we are not interested in the equations parametrizing the strands (geometry), but in the way the strands are entwined (topology). Then, two braids will be considered the same if we can deform one into the other, keeping the endpoints fixed and without cutting the strands, so that if we stop the deformation at any moment, we get a braid. For example, the four geometric braids in figure 1.2 represent the same braid. More formally, we need to introduce the notion of isotopy.

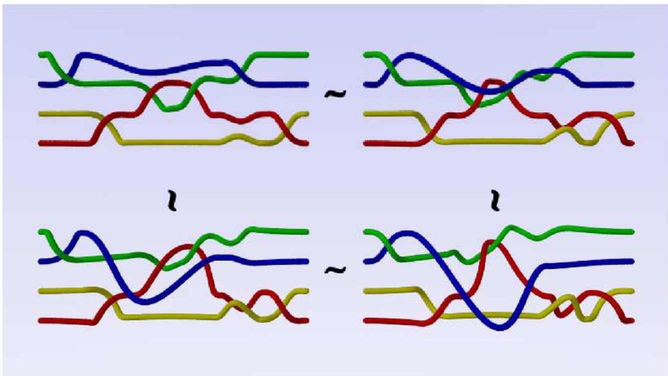


Figure 1.2: We consider these braids as equivalent because one can be deformed into the other, keeping the endpoints fixed and not letting the strands intersect.

Definition 1.2.1
An *isotopy* between geometric braids β and γ on n strings is a n-tuple of continuous maps $F = (F_1, F_2, \ldots, F_n)$, where $F_i : I \times I \to I \times D$, such that the following three conditions are met.

(i) For each $t \in I$, $\{F_i(\{t\} \times I)\}_{i=1,\ldots,n}$ is a geometric braid on n strings.

(ii) $\{F_i(\{0\} \times I)\}_{i=1,\ldots,n} = \beta$.

(iii) $\{F_i(\{1\} \times I)\}_{i=1,\ldots,n} = \gamma$.

It is easy to see that the isotopy between braids is an equivalence relation. The equivalence classes are called *braids*, or *n-braids*, to specify the number of strings.

Note that isotopies between geometric braids do not change the permutation: isotopic braids induce the same permutation. In other words, the permutation induced by a braid is an invariant of the braid. Hence we can speak of the *permutation induced by a braid*.

To draw braids, we will use two dimensional representations like the one in figure 1.3. These are called *braid diagrams* and can formalized as follows.

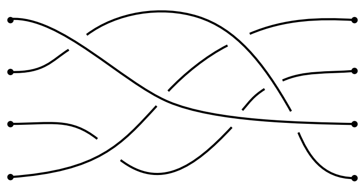


Figure 1.3: A braid diagram.

Definition 1.2.2
A *diagram* of an n-braid is a set of n topological intervals (i.e. spaces homeomorphic to I), called *strands*, in $I \times \mathbb{R}$, such that the projection on I maps each strand homeomorphically onto I, every point of $\{0, 1\} \times \{p_1, p_2, \ldots, p_n\}$ is an endpoint of a strand and every point of $I \times \mathbb{R}$ belongs to at most two strands. The intersections (also called *double points* or *crossings*) are transverse and are a finite number. At each crossing the opposite strands are paired and one of the pairs is labelled: graphically we indicate it removing a little portion of the strand near the crossing.

To obtain a diagram from a braid, take a generic projection of the braid on the plane $z = 0$: a mapping $I \times D \to I \times \mathbb{R}$ such that the image is locally an immersion, except at a finite number of points. If there are singularities other than double points, they can be removed using isotopy of braids. Colour each crossing: "break" the arc that has lower z coordinate at the crossing (the one that passes "beyond" the other).

Vice versa, to obtain a braid from a diagram, consider $I \times \mathbb{R}$ as a subspace of $\mathbb{R}^3$. In a small neighbourhood of each crossing, push the coloured arc such that its z-coordinate becomes positive. This transforms the diagram into a braid. To be more precise we may need to rescale the braid to be contained in the cylinder $I \times D$.

Sometimes, we will not distinguish between braids and diagrams: for example, we will say "the braid in figure 1.3" meaning "the braid represented by the diagram in figure 1.3".

1.3 The Artin braid group

We look for a structure on the set of braids: we will see that this is a group with respect to the operation of composition.

Given two geometric braids, we can compose them, i.e. put one after the other making the endpoints of the first one coincide with the starting points of the second one and rescale the braid to make the new braid fit into the unit interval (as in figure 1.4).

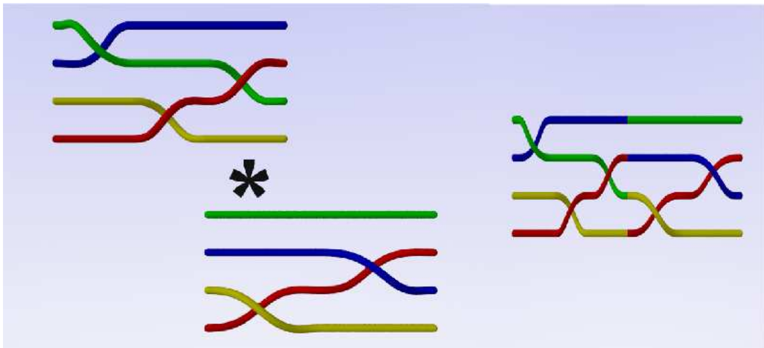


Figure 1.4: Composing two braids means to put them one after the other and rescale to the unit length.

Definition 1.3.1
Let β and γ be geometric n-braids with permutation π and σ respectively. The *composition* $\delta = \beta * \gamma$ (written also $\delta = \beta\gamma$) is defined as the n-tuple of paths $(d_1, d_2, \ldots, d_n)$, where

$$d_i(t) = \begin{cases} b_i(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ c_{\pi(i)}(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Then δ is a geometric braid on n strings with permutation $\sigma\pi$.

Of course, if β is isotopic to β' and γ to γ', then $\beta * \gamma$ is isotopic to $\beta' * \gamma'$ (compose the isotopies). Thus composition is an operation on the set of braids. It is associative: to show this we just need to rescale the intervals. There is a neutral element for the composition: it is the *trivial braid*, also called *identity braid*, i.e. the class of the geometric braid where all the strings are straight. It will be denoted by 1.

From now on we will not distinguish between geometric braids and braids.

Given any braid $\beta = (b_1, b_2, \ldots, b_n)$, we can define a braid β^{-1} by setting $b_i^{-1}(t) = b_i(1-t)$ for all $t \in I$. This corresponds to reverse the x direction, or equivalently, to consider the mirror image of β with respect to the plane $x = 1/2$. It is easy to see that β^{-1} is the inverse of β. So we have found a structure on the set of braids.

Remark and Definition 1.3.2
For each n, the set of braids on n strings forms a group, with respect to composition. It is called the *Artin braid group on* n *strings* or n*-braid group* and denoted by B_n.

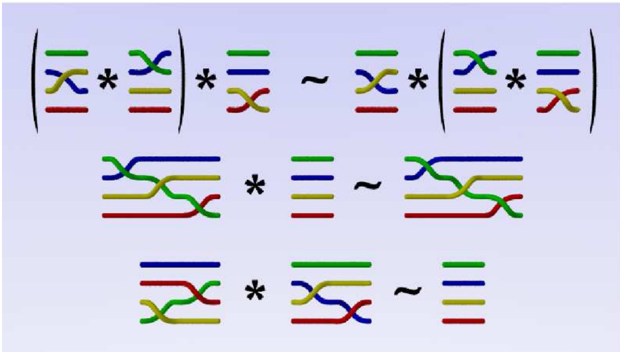


Figure 1.5: The braids with a fixed number of strands form a group: the composition is associative, there is a neutral element and for each braid there is an inverse.

A particular set of braids, the *pure* braids, will be used in many contexts. These are the braids that induce the trivial permutation. Note that the pure n-braids form a subgroup of the n-braid group: the identity braid is pure and composing two pure braids the permutation remains the identity. More formally, the map $B_n \to S_n$ that associates to each braid the permutation induced by it is a homomorphism and we can give the following Definition.

Definition 1.3.3
The *pure braid group* P_n *on* n *strands* is the kernel of the homomorphism $B_n \to S_n$ that associates to each braid the permutation induced by it.

The composition of braids is not commutative. An example is given by the braids in figure 1.6. They are obtained composing two braids with just one crossing, in the two possible ways. Since they induce different permutations (look at the strand colours), they are different.

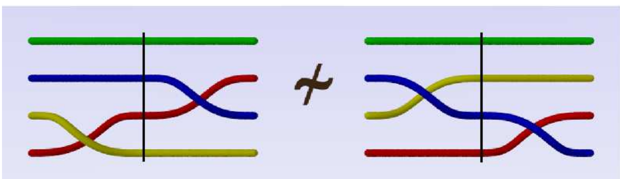


Figure 1.6: An example of non commutativity: the braids are not equivalent because they induce different permutations.

One of the ways to describe a group is to give a presentation.

Definition 1.3.4
Let $\mathcal{S}$ be a set of distinct symbols, and for each $s \in \mathcal{S}$ form the symbol s^{-1}. A *word* w in the symbols $\mathcal{S}$ is a finite sequence $s_1 s_2 \dots s_n$ where each s_i is a symbol s or s^{-1} for some $s \in \mathcal{S}$.
Given a group G and a map $\rho : \mathcal{S} \to G$ that associates to each symbol an element of G, if $\rho(s) = g \in G$ we say that s defines g and s^{-1} defines g^{-1}. If a word w is given by $s_1 s_2 \dots s_n$, then w defines the element of G given by $\rho(w) = \rho(s_1)\rho(s_2)\dots\rho(s_n)$. Let $S = \{\rho(s) \mid s \in \mathcal{S}\}$ be the set of the elements defined by $\mathcal{S}$. If every element of G is defined by a word in the symbols $\mathcal{S}$, then $\mathcal{S}$ is called a *set of generating symbols for* G and S is called a *set of generators* or a *generating set for* G.
A word in the symbols $\mathcal{S}$ that defines the identity of G is called a *relator*. Then $\ker\rho$ is the set of relators. Trivial relators are the empty word and each word ss^{-1} or $s^{-1}s$ for $s \in \mathcal{S}$. Given two words in the symbols $\mathcal{S}$, the equation $w = v$ is called a *relation* if the word wv^{-1} is a relator.
Let $F_{\mathcal{S}}$ be the free group on the generators $\mathcal{S}$. Then $G \cong F_{\mathcal{S}}/\ker\rho$, i.e. two words in the symbols $\mathcal{S}$ define the same element of G if and only if they are related by a finite sequence of relations in $\ker\rho$ and insertion or deletion of sub-words of the form ss^{-1} or $s^{-1}s$, for $s \in \mathcal{S}$.
If $\mathcal{R}$ is a set of generators of $\ker\rho$, we define the set $R = \{\rho(r) \mid r \in \mathcal{R}\}$ and we call R a *complete set of relations for* G. We call the data $\langle S \mid R \rangle$ a *presentation for* G. With abuse of notation we write $G = \langle S \mid R \rangle$ to mean that the group G has a presentation $\langle S \mid R \rangle$.

It is customary not to distinguish $\mathcal{S}$ from S and $\mathcal{R}$ from R when the context is clear. All the groups we consider are finitely presented, i.e. they have a presentation $\langle S \mid R \rangle$ where both S and R are finite.

For example, the free group F_S has a presentation $\langle S \mid \ \rangle$ with no relations. See [48, Chapter 1] for more details about group presentations.

We look now for a set of generators of the braid group. For $1 \le i \le n-1$ we define the *elementary braid* σ_i as the braid where the $(i+1)$-th strand crosses over the i-th strand, while all other strands are straight (figure 1.7).

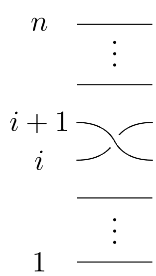


Figure 1.7: The elementary braid σ_i.

Up to isotopy we can assume that any braid consists of arcs such that if we project the braid orthogonally on the plane $z = 0$, all the intersections are transversal and occur at different x-heights. Then, the elementary braids are a set of generators of the braid group: any braid is equivalent to a braid obtained as a composition of the σ_i's and their inverses (see figure 1.8).

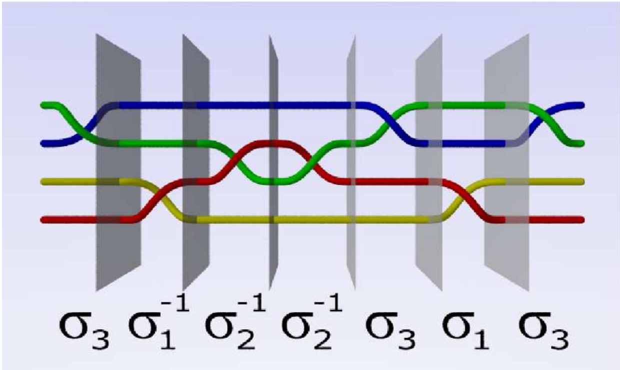


Figure 1.8: Any braid can be described by a word in the σ_i's and their inverses.

Such a decomposition is not unique, but the next theorem exactly says when two words in the σ_i's represent the same braid.

Theorem 1.3.5 (Artin [5], [6], Magnus [47])
The elementary braids $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are a set of generators of the n-braid group. A complete set of relations is given by

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |\, i - j \,| > 1$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad 1 \leq i \leq n-2$$

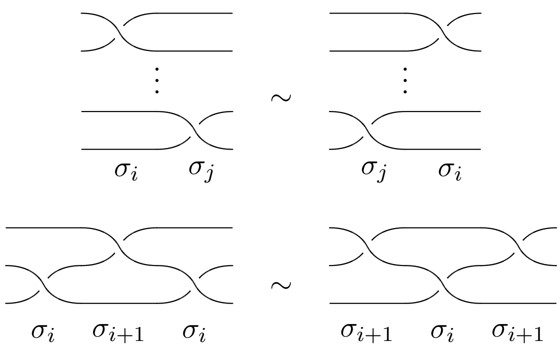


Figure 1.9: The braid relations.

For a proof, see [50, Chapter 2.3].

This theorem is not constructive: if we find a sequence of relations to connect two words, we know that they represent the same braid. But we do not (yet) have a method to find such a sequence.

The next Section 1.4 is devoted to combinatorial problems of this type. In particular, in 1.4.1 we will discuss two algorithms allowing to check whether two words represent the same braid or not.

1.4 Combinatorial group theory

In 1911 Max Dehn [14] formulated the following three fundamental decision problems about group presentations. Let G be a group, defined by means of a presentation.

- *The word problem*: for any word w in the generators, decide whether w defines the identity element of G or not.
- *The conjugacy problem*: for any two words v, w in the generators, decide whether v and w are conjugated or not, i.e. whether there exists a word x in the given generators such that $x^{-1}vx$ defines the same element as w.
- *The isomorphism problem*: given a group G', defined by means of another presentation, decide whether G is isomorphic to G' or not.

The meaning of the word "decide" here is to give an effective procedure that can answer the question in any case. If there is such a procedure, then the problem is said to be *solvable* or *decidable.* The solvability of problems is investigated in computability theory, that is also called recursion theory. More details can be found in [13, Chapter 1].

The three problems presented are investigated in combinatorial group theory: they have been solved (i.e. an algorithm has been found) for some classes of group presentations, but not in general. There are groups in which even the word problem, the easiest of the three, is unsolvable. For more details see [48, Chapters 1 and 6].

Even if a problem is solvable, it can happen that the complexity of an algorithm to solve it is so high that it is infeasible to implement it in practice. The complexity of an algorithm is a measure of how many elementary operations are required to solve the problem, in terms of the input size.

Another problem investigated in combinatorial group theory is the *membership problem* or *generalized word problem*: it asks, given a group G and a subgroup K of G, to decide whether an arbitrary element of G belongs to K. The word problem is then a special case of the membership problem, where the subgroup K is trivial.

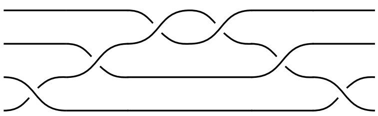
In braid groups, the word and conjugacy problems have been solved, and different algorithms are known, some of which are very efficient. In the following section we present two algorithms to solve the word problem in braid groups.

1.4.1 The word problem in braid groups

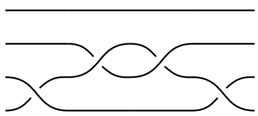
The *Artin braid combing* algorithm solves the word problem. It was discovered by Artin in [5]; a complete proof can be found in [50, Chapter 3].

Let β be a braid in B_n. We have to decide whether $\beta \sim 1$ or not. If β is not pure, then it can not be isotopic to the identity braid. Then, suppose β is pure.

Delete the last strand from β and replace it with a straight one that does not interlace any of the other strands. Call this new braid γ_1.

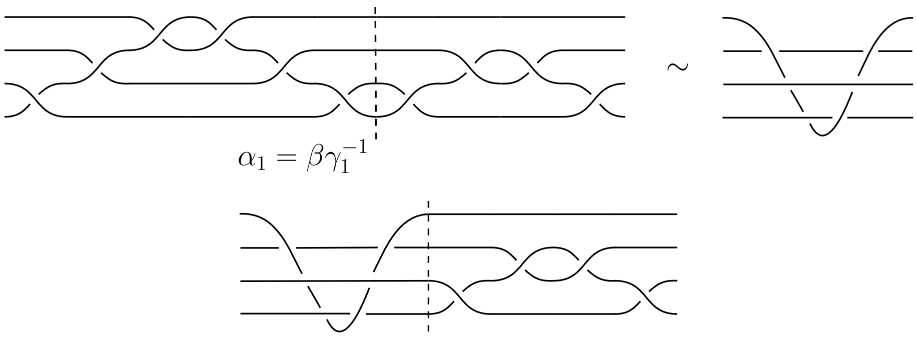


β



γ_1

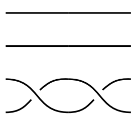
Consider the braid $\alpha_1 = \beta\gamma_1^{-1}$. If we remove the last strand from α_1, we obtain the trivial braid on $n-1$ strands. Hence we can think of α_1 as a braid with $n-1$ straight strands and the n-th strand that can be linked with the strands below. A braid of this form is said to be *combed*.



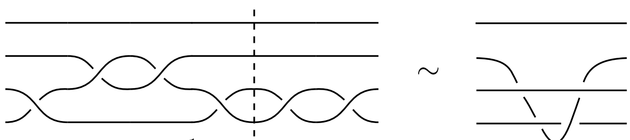
$\alpha_1 = \beta\gamma_1^{-1}$

Figure 1.10: $\beta = \alpha_1\gamma_1$. The left part of β is combed.

Rewriting the definition of α_1, we have $\beta = \alpha_1\gamma_1$. Consider γ_1: it has $n-1$ strands that can be braided, while the last one is straight. Remove the second-to-last strand from γ_1 and replace it with a straight one. Call this new braid γ_2.



γ_2



$\alpha_2 = \gamma_1\gamma_2^{-1}$

The braid $\alpha_2 = \gamma_1\gamma_2^{-1}$ can be put in this form: $n-2$ strands are straight, the $(n-1)$-th strand can be linked with the preceding ones, and the n-th strand is straight. Thus, α_2 is combed and we have $\beta = \alpha_1\alpha_2\gamma_2$.

Iterating the same procedure $n-1$ times we obtain a decomposition $\beta = \alpha_1\alpha_2\cdots\alpha_{n-1}$, where each α_i is a combed braid, i.e. in α_i all the strands are straight, except the $(n+1-i)$-th, which can be linked to the strands $1, 2, \ldots, n-i$.

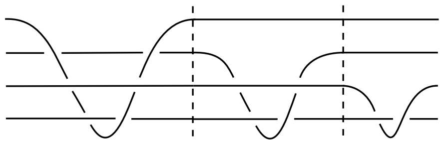


Figure 1.11: The combed braid $\beta = \alpha_1\alpha_2\alpha_3$.

The decomposition of β in this form is unique, in the sense that the braids α_i's are uniquely determined. It is also possible to define a unique form for each α_i, as explained below.

For each $k > 0$, let $\mathcal{A}_k$ be the set of combed k-braids such that the removal of the last strand yields the trivial $(k-1)$-braid. Then, all the α_i's in the above decomposition can be viewed as braids in $\mathcal{A}_k$ for some $k \leq n$: $\alpha_1 \in \mathcal{A}_n$ and $\alpha_i \in \mathcal{A}_{n+1-i}$ if we "forget" the last $i-1$ strands, that are trivial. In other words, each braid group B_k embeds in B_{k+1}, so we can see each $\mathcal{A}_k$ as a subgroup of B_n. The following Proposition ensures that $\mathcal{A}_k$ is a free group and gives a set of generators in terms of the generators of the braid group.

Proposition 1.4.1 (Artin [6])
For every $k > 0$, $\mathcal{A}_k$ is a free group and a set of $k-1$ generators is given by the elements

$$a_i = (\sigma_{k-1}\sigma_{k-2}\cdots\sigma_{i+1})\,\sigma_i^2\,(\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\cdots\sigma_{k-1}^{-1})$$

for $1 \leq i \leq k-1$.

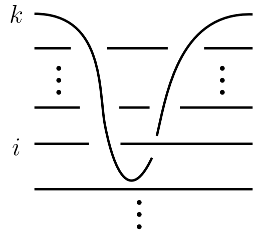


Figure 1.12: The element $a_i \in \mathcal{A}_k$.

The word problem is solvable for free groups. To test if a word is trivial in a free group, it is sufficient to use free reductions, that is to simplify all sub-words of the form ww^{-1}. Thus we can decide whether each α_i is trivial or not.

This result, together with the following, allows to solve the word problem in braid groups.

Theorem 1.4.2 (Artin [6])
The braid $\beta = \alpha_1\alpha_2\cdots\alpha_{n-1}$, with the α_i's obtained as above, is trivial if and only if each α_i is trivial.

The combed form is a *normal form* for the braid: for every braid there is a unique combed braid (with the α_i's written in the a_i's and free reduced) that is equivalent to it. In other words we have a standard way to choose a representative for each braid, i.e. an algorithm to put every braid in its normal form. To compare two braids, it is sufficient to put them in their normal forms and check if these are the same or not.

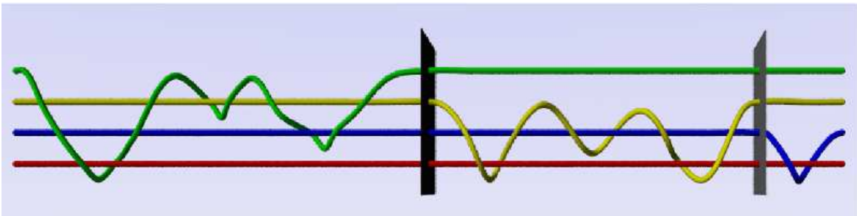


Figure 1.13: A combed braid.

The complexity of the Artin combing algorithm is very high: it is exponential in the length of a word written in the generators. This makes the algorithm unusable in practice.

Other algorithms are known, that are more efficient. In [28], Garside gives a solution to the word problem and to the conjugacy problem. Both Garside's solutions are exponential in the word length and in the number of strands. Garside's ideas have been developed and improved several times since then.

In particular Birman, Ko and Lee in [10] give an algorithm that solves the word problem in B_n in time $(n \cdot m^2)$ for a given word of length m. Their solution relies on another presentation of the braid group, which will be exposed in Section 1.4.3.

Since the other algorithms solve not only the word problem but also the conjugacy problem, we give a list of them in the next section.

Here we outline another approach to solve the word problem in braid groups that was proposed by Dehornoy in [15]. The algorithm discussed is called *handle reduction*. Even if the complexity of handle reduction is only shown to be at most exponential, statistical data of computation times suggest that the algorithm is the fastest found so far. However, it does not generalize to solve the conjugacy problem.

The proofs of the results rely upon an ordering of B_n (see [18, Chapters 1 and 3]) but here we do not go into details and only sum up what a handle is and how (not why) the algorithm works.

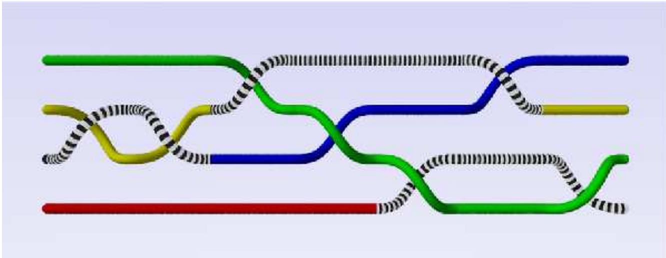


Figure 1.14: In this braid there are three handles.

Definition 1.4.3
A *handle* is a braid (sub)word of the form $\sigma_i^e x \sigma_i^{-e}$, where $e = \pm 1$ and x contains only generators $\sigma_j^{\pm 1}$ with $j < i$.

For each $1 \leq i \leq n-1$ we give a word homomorphism $\varphi_i : B_n \to B_n$ by defining it on the generators as follows. It is easy to check that every φ_i is well defined.

$$\begin{array}{rcl} \sigma_i^{\pm 1} & \mapsto & 1 \\ \sigma_{i-1}^{\pm 1} & \mapsto & \sigma_{i-1}^{-e}\sigma_i^{\pm 1}\sigma_{i-1}^{e} \\ \sigma_j & \mapsto & \sigma_j \quad \text{if } j \notin \{i-1, i\} \end{array}$$

Reducing a i-handle in a word w means to apply the homomorphism φ_i to w. An example is in figure 1.15. Thanks to the following result, the handle reduction algorithm can be written.

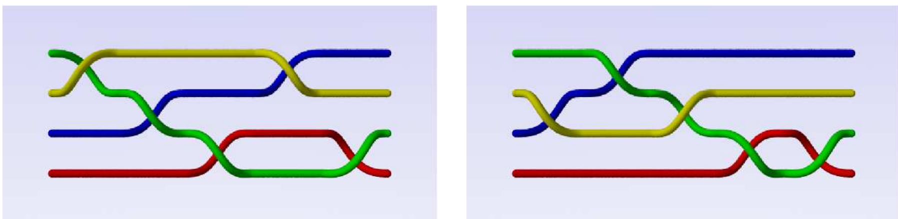


Figure 1.15: Reducing a handle transforms the braid into an equivalent one.

Proposition 1.4.4 (Dehornoy [15])
Handle reduction transforms a word into an equivalent word.
If a braid word w contains no handles, then either it is empty or it is not equivalent to the identity.

A simple algorithm based on handle reduction to check the triviality of a word is the following.

1. If there are no handles, terminate: if the word is trivial, answer "yes", otherwise answer "no".

2. If there are handles, find the handle that ends first and reduce it. Go back to the first step.

Proposition 1.4.5 (Dehornoy [15])
Every sequence of handle reductions from a given word converges, i.e. the algorithm just described always terminates.

1.4.2 The conjugacy problem in braid groups

As already said, the first solution of the conjugacy problem in braid groups is due to Garside [28]. His ideas were later developed by El-Rifai and Morton in [19]. The solution of the conjugacy problem was extended to hold in Garside groups, a generalization of braid groups defined in [16], by Picantin in [52]. Then, the algorithm was improved by Franco and González-Meneses in [26], Gebhardt in [29], Gebhardt and González-Meneses in [30]. For a survey on the improvements see [51, Section 4.4].

Since 1999 and for some time the conjugacy problem in braid groups has been thought to be interesting for possible applications in cryptography, since the braid groups are infinite and non commutative. Extending the conjugacy problem we can define the conjugator search problem, that is, not only decide if two words v, w are conjugated, but also find a conjugating word x such that $xwx^{-1} = v$. This problem was supposed to be "hard enough": there are algorithms to solve it but the complexity was believed to be too high to do it in practice.

Some cryptographic schemes based on braid groups have been proposed but now they are not believed to be secure, since the recent algorithms allow to efficiently solve the conjugacy problem. A wide survey on cryptography in braid groups can be found in [17].

1.4.3 Other presentations of the braid group

We do not deal with the isomorphism problem: it is in general the most difficult of the three problems posed by Dehn. Here we simply give other presentations of the braid groups. We will not present them in the movie, where we just give a hint: in part four, to draw some braids, the points on the disc will be arranged on a circle. This easily yields another presentation of B_n, discussed in the following. More generally, we will see that from any graph with some properties we can obtain a presentation for a braid group.

In the definition of braids we fixed n points inside a disc. Conventionally, the points are usually depicted on a line. In this way it is natural to order the strand from 1 to n and to define the generators of the braid group as the crossings between two neighbouring strands and thus get $n-1$ generators.

But the points can be anywhere in the disc, so if we fix them on a circle, we will obtain a more symmetric presentation, adding a generator σ_n representing the crossing between the first and the last strand. A complete set of relations will then be the following

$$\begin{aligned} \sigma_i\sigma_j = \sigma_j\sigma_i \qquad & |\, i-j \,| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad & 1 \le i \le n \end{aligned}$$

where the indexes are to be considered modulo n.

More generally, take any connected planar graph without loops or intersections, with n vertices, to which we associate the points $p_1, p_2, \ldots, p_n$. To each edge we associate a generator: an edge connecting vertices i and j corresponds to a generator exchanging the points p_i and p_j clockwise (see figure 1.16). In [56] Sergiescu shows how to associate a presentation for B_n to any such graph. We report his result in the following Theorem.

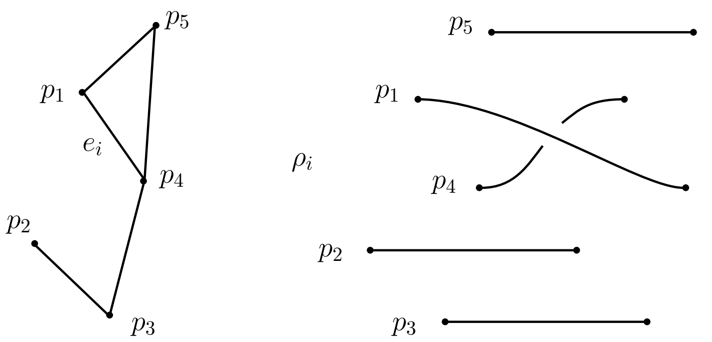


Figure 1.16: To the edge e_i we associate the generator ρ_i.

Theorem 1.4.6 (Sergiescu [56])
Take a connected planar graph with n vertices, without loops or intersections. Label the edges from e_1 to e_k. Then the braid group B_n admits a presentation with generators $\{\rho_i \mid 1 \le i \le k\}$ and the following set of relations.

- $\rho_i\rho_j = \rho_j\rho_i$ *if the edges e_i, e_j are disjoint.*
- $\rho_i\rho_j\rho_i = \rho_j\rho_i\rho_j$ *if the edges e_i, e_j have a common vertex.*
- $\rho_1\rho_2\rho_3\rho_1 = \rho_2\rho_3\rho_1\rho_2 = \rho_3\rho_1\rho_2\rho_3$ *if e_1, e_2, e_3 have a common vertex and are arranged in this order clockwise around it.*
- $\rho_1\rho_2\cdots\rho_{n-1} = \rho_2\rho_3\cdots\rho_n = \rho_3\cdots\rho_n\rho_1 = \ldots = \rho_n\rho_1\cdots\rho_{n-2}$ *if the vertices $e_1, e_2, \ldots, e_n$ form an irreducible pseudo-cycle in clockwise direction.*

To find a solution to the word and conjugacy problem, in [10], Birman, Ko and Lee use another presentation of braid groups, that corresponds to a complete graph, where each vertex is connected to any other through exactly one arc. In the next Proposition we explicitly write the presentation.

For every t, s with $n \geq t > s \geq 1$ let a_{ts} be the element of B_n defined by

$$a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\,\sigma_s\,(\sigma_{s+1}^{-1}\sigma_{s+2}^{-1}\cdots\sigma_{t-1}^{-1})\,.$$

The braid a_{ts} is represented in figure 1.17: the strand t crosses over the strand s, and both strands pass over all the intermediate strands.

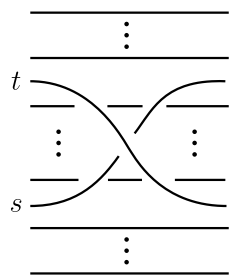


Figure 1.17: The element a_{ts}.

Proposition 1.4.7 (Birman-Ko-Lee [10])
B_n has a presentation with generators

$$a_{ts} \quad n \geq t > s \geq 1$$

and relations

$$a_{ts}a_{rq} = a_{rq}a_{ts} \quad \text{for } (t-r)(t-q)(s-r)(s-q) > 0$$

$$a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} \quad \text{for all } t, s, r \text{ such that } n \geq t > s > r \geq 1.$$

1.5 Other definitions of braids

Braids can be characterized in different ways. In this section we will sketch some other viewpoints on braid groups, all of which are related to the concept of action. This will be the main subject of Chapter 3.

1.5.1 Braids as configuration spaces

The following approach was introduced in [22] and [25].

A braid can be viewed as the n points $\{p_1, p_2, \ldots, p_n\}$ moving in the disc: we can consider each disc $\{t\} \times D$, for $t \in I$, as an instant of the "dance" of the points (see figure 1.18).

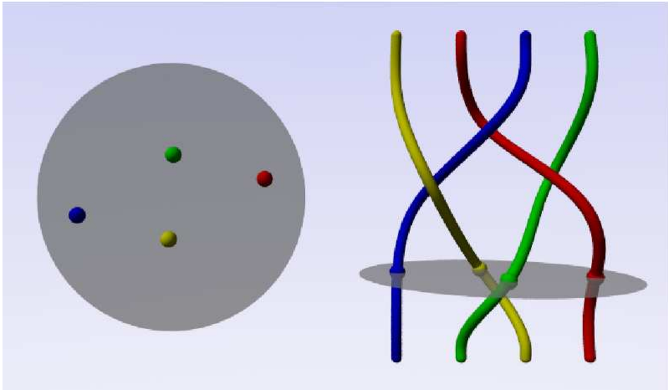


Figure 1.18: An n-braid can be obtained as motion of n points in the disc: to each instant corresponds a configuration of the points.

Consider the n fixed points in the disc as complex numbers, say $p_i = i$, $1 \leq i \leq n$. The big diagonal of $\mathbb{C}^n$,

$$\Delta_n = \{(z_1, z_2, \ldots, z_n) \mid z_i = z_j \text{ for some } i < j\},$$

consists of all the points of $\mathbb{C}^n$ that have at least two coincident coordinates.

We consider the space $N_n = \mathbb{C}^n \setminus \Delta_n$ of ordered configurations of points in $\mathbb{C}$: it is the set of n-tuples $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ such that $z_i \neq z_j$ for $i \neq j$. In other words, each point in this space is a configuration of n distinct points in the plane, an instant of a "dance".

Then, a geometric pure braid can be seen as a loop in N_n, with base point $(p_0, p_1, \ldots, p_n)$. Moving along the loop, the configurations of points change with continuity.

Proposition 1.5.1
Let $P = (p_1, p_2, \ldots, p_n)$ the ordered n-tuple of fixed points. Then the fundamental group of N_n with base point P is isomorphic to the pure braid group on n strands,

$$\pi_1(N_n, P) \cong P_n.$$

See [45, Chapters 13–15] for the definition of fundamental group.

To obtain a similar characterization for B_n, consider the space M_n of not ordered configurations of points in $\mathbb{C}$: this is obtained by permutation of coordinates, $M_n = N_n/S_n$. It is the set of unordered n-tuples $\{z_1, z_2, \ldots, z_n\}$ of pairwise distinct complex numbers. We denote by $[z]$ the element of M_n represented by $z \in N_n$. Then $[P]$ represents the configurations where the points occupy the positions $\{p_1, p_2, \ldots, p_n\}$, but they are possibly permuted. A braid is then a loop in M_n with base point $[P]$.

Proposition 1.5.2
The fundamental group of M_n with base point $[P]$ is isomorphic to the braid

group on n strands,

$$\pi_1(M_n, [P]) \cong B_n.$$

1.5.2 Braids as mapping classes

Here we just sketch the facts needed to understand this characterization of braids. For a wider introduction to mapping class groups, see [23, Chapter 2].

Let $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ be the set of the fixed points in the disc D. We consider the homeomorphisms of the disc fixing the boundary and permuting the punctures. We say that two such homeomorphisms are equivalent if there is an isotopy (a continuous deformation) of one homeomorphism into the other, that keeps the boundary ∂D and the points $\mathcal{P}$ fixed. Each class of homeomorphisms is called a mapping class, and the mapping classes form a group with respect to the composition of maps, called the *mapping class group* of the disc with n punctures. More precisely, we have the following Definition.

Definition 1.5.3
Let $\mathrm{Homeo}^+(D, \mathcal{P})$ denote the group of homeomorphisms $h : D \to D$ that keep the boundary ∂D point-wise fixed and keep $\mathcal{P}$ set-wise fixed. Let $\mathrm{Homeo}_0^+(D, \mathcal{P})$ denote the connected component of the identity homeomorphism in $\mathrm{Homeo}^+(D, \mathcal{P})$, i.e. all the homeomorphisms that are isotopic to the identity. The mapping class group of the pair $(D, \mathcal{P})$ is defined to be

$$\mathrm{mcg}(D, \mathcal{P}) = \pi_0(\mathrm{Homeo}^+(D, \mathcal{P})) = \mathrm{Homeo}^+(D, \mathcal{P})/\mathrm{Homeo}_0^+(D, \mathcal{P}).$$

Theorem 1.5.4 (Artin [6])
The mapping class group of the disc with n punctures $\mathrm{mcg}(D, \mathcal{P})$ *is isomorphic to the n-braid group B_n.*

We now describe how to obtain a braid from a mapping class. Let $\varphi \in \mathrm{Homeo}^+(D, \mathcal{P})$. By a theorem of Alexander we know that any homeomorphism of D in $\mathrm{Homeo}^+(D, \emptyset)$ is isotopic to the identity. This is called Alexander's trick [3], for a proof see for example [33, pages 36–37]. Thus there exists a continuous path $\{\varphi_t\}_{t \in I}$ of homeomorphisms in $\mathrm{Homeo}^+(D, \emptyset)$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$. For each $1 \leq i \leq n$ and for $t \in I$ define the i-th strand as $b_i(t) = (t, \varphi_t(p_i))$. Then the braid corresponding to φ is $\beta = (b_1, b_2, \ldots, b_n)$.

Vice versa, obtaining a mapping class from a braid is more complicated to describe, but the image of the standard generators, called *braid twist*, is easy to visualize: see figure 1.19. To the generator σ_i we associate the class of homeomorphisms of the disc that move only a small disc around the points p_i and p_{i+1}, exchanging them by a clockwise twist.

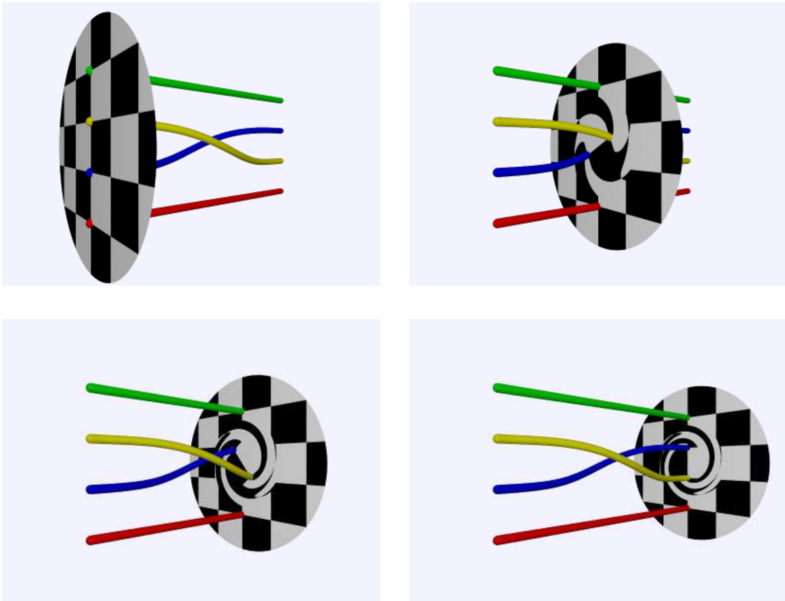


Figure 1.19: The braid twist is the homeomorphism that corresponds to a generator of the braid group. These images are not contained in the movie but are part of further work on braids.

1.5.3 Braids as automorphisms of free groups

We can see the n-braid group as a subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of the free group on n generators. In other words, there is a homomorphism (a representation) of B_n in $\mathrm{Aut}(F_n)$.

Theorem 1.5.5 (Artin representation theorem)
Let F_n be the free group on n generators $x_1, x_2, \ldots, x_n$. Denote by $\mathrm{Aut}(F_n)$ the group of automorphisms of F_n. Then B_n is isomorphic to the subgroup G of $\mathrm{Aut}(F_n)$ consisting of those automorphisms μ such that

- *there is a permutation $\tau \in S_n$ such that for every $1 \leq i \leq n$*

$$x_i\mu = a_i x_{\tau(i)} a_i^{-1}$$

  for some $a_i \in F_n$;

- *μ fixes the loop $x_1 x_2 \cdots x_n$*

$$(x_1 x_2 \cdots x_n)\mu = x_1 x_2 \cdots x_n.$$

The isomorphism $B_n \to G$, $\beta \mapsto \overline{\beta}$, is given as follows. To each elementary braid σ_i, $1 \leq i \leq n$, corresponds the automorphism $\overline{\sigma_i}$ defined by

$$\begin{array}{rcl} x_i\overline{\sigma_i} & = & x_i x_{i+1} x_i^{-1} \\ x_{i+1}\overline{\sigma_i} & = & x_i \\ x_j\overline{\sigma_i} & = & x_j \quad j \notin \{i, i+1\}. \end{array} \tag{1.5.1}$$

Under this isomorphism, the permutation τ corresponding to $\overline{\beta}$ is the permutation induced by β.

A proof of this theorem can be found in [33, Chapter I.5].

Remark 1.5.6
Artin representation theorem provides a faithful (i.e. injective) homomorphism of B_n into $\mathrm{Aut}(F_n)$. It is then immediate to find a solution to the word problem in braid groups. To check whether a given braid β is equivalent to the trivial braid, it is sufficient to verify whether $\overline{\beta}$ fixes each x_i or not.

We now characterize F_n as a fundamental group. The braid action that can be defined in this setting will be the central part of the workshop presented in Chapter 3. We do not treat homotopy and fundamental groups, but refer to introductory texts of algebraic topology, such as [45, Chapters 13–15], or texts on topology, such as [36, Chapter IX.5].

Fix $x \in \partial D$. The fundamental group of the disc with n points removed (also called the *punctured disc*) with base point x is isomorphic to the free group F_n. Free generators for this group are the loops based at x that go round the punctures in anti-clockwise direction (figure 1.20): for $1 \leq i \leq n$ we call x_i the generator that encircles the point p_i.

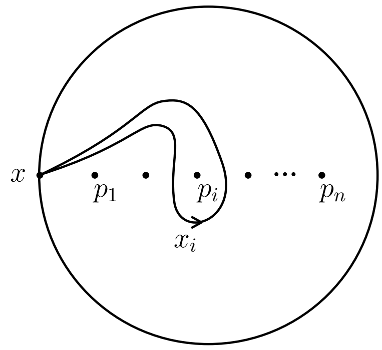


Figure 1.20: The generator x_i of the fundamental group of the punctured disc.

Thus any loop in the disc with base point at x is homotopic to a loop that can be written as a word in the x_i's. This means that each closed curve in the punctured disc based at x can be continuously deformed (by a homotopy) into a curve described as the junction of some curve x_i.

Then, recalling the characterization of braids as homeomorphisms of the punctured disc, we can define the action of a braid β on a loop in the punctured disc: it corresponds to push the loop along the braid in $(I \times D) \setminus \beta$ to the other end of the braid (see figure 1.21).

Then each generator of the braid group σ_i acts on each generator of the free group x_j according to (1.5.1).

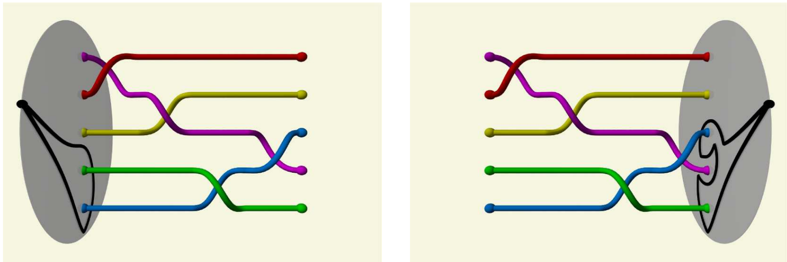


Figure 1.21: The action of a braid on a loop can be visualized moving the disc from one side to the other and deforming the loop such that it never intersects the strands. These images are not contained in the movie but are part of further work on braids.

1.6 Knot theory

One of the first applications of braid theory is in knot theory. Knot theory has been widely narrated in movies, books, articles, both popular and for specialists. A well-known reference for non-specialists is the book by Adams [1]. On the contrary, not much popularizing material has been produced about braid theory.

In this section we introduce some hints about knot theory in order to explain some connections between braids and links.

Knots were first studied by mathematical physicists, especially William Thomson (lord Kelvin), Peter Guthrie Tait and James Clerk Maxwell, in the second half of the nineteenth century. In the paper *On vortex atoms* in 1867 Thomson proposed a theory to describe the microscopic structure of matter, that was strongly connected with the investigation of knots. At the time space was viewed as being filled with the ether, a fluid matter. According to this theory, atoms were tubes of ether that could be knotted and linked. They were stable in the sense that distorting them without cutting did not change the way in which they were knotted. This fact explained the stability of atoms. The existence of different elements depended on the different ways in which the atoms were knotted. This gave different physical and chemical properties to distinct elements.

It was then important to classify knots up to deformations: Tait, influenced by the work of Thomson, compiled some tables of knots. Later he was helped by Kirkman and Little, but they never found a way to classify all the knots. In fact this is a very difficult problem.

Although this physical theory had no evidence and was soon abandoned, there are connections between physical phenomena, such as electromagnetism and the structure of DNA, and topology of closed strings.

For a survey on the history of knot theory see [20] and [21] (in particular Chapters 4 and 5).

Definition 1.6.1
A *knot* is an embedding of the circle S^1 into the Euclidean 3-space. As in the case of braids, we will not distinguish between the embedding and its image and call both knot.
A *link* on n components is a set of n disjoint knots that can be interlaced.

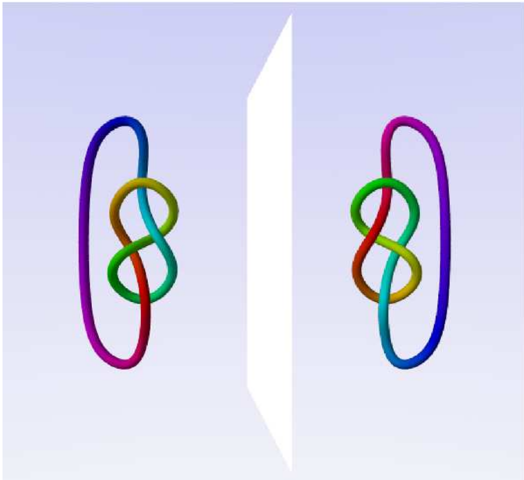


Figure 1.22: The figure eight knot and its mirror image.

Remark 1.6.2
We will only consider *tame* knots and links, i.e. those that are isotopic to piecewise linear knots or links, or equivalently, those that have a tubular non-intersecting neighbourhood. See [46, Chapter 1] for a rigorous definition of tame links.

As in the case of braids, links are usually represented by *diagrams*, drawings as those in figure 1.23.

Definition 1.6.3
A *diagram* of an n-link is a set of n closed curves immersed in $\mathbb{R}^2$, called components, such that every point of $\mathbb{R}^2$ belongs to at most two components. The intersections (called *double points* or *crossings*) are transverse and are a finite number. As for braid diagrams, at each crossing one of the components involved is coloured: a little portion of it near the crossing is removed.

A link diagram can be obtained via a generic projection of the link in a plane, equipped with crossing information. Vice versa, to reconstruct a link from a diagram, it is sufficient to consider the plane as a subspace of $\mathbb{R}^3$ and, in a small neighbourhood of each crossing, push the coloured arc such that its third coordinate becomes positive.

Two links are said to be *isotopic* if there is a continuous deformation of the space (an isotopy) taking one knot into the other. In particular, the strings can neither be cut nor intersect during the isotopy. As in the case of braids, we consider links up to isotopy. In other words, isotopy defines an equivalence relation $\sim$ on the set $\mathcal{L}$ of links. We will denote $\mathcal{K}$ the quotient set $\mathcal{L}/\sim$ of knots and links up to isotopy.

One surprising fact is that some knots are *achiral*, i.e. isotopic to their mirror image, while others are not (and are said to be *chiral*). For example, there are a left trefoil knot and a right one (figure 1.23). An example of an achiral knot is the figure eight knot, depicted in figure 1.22.

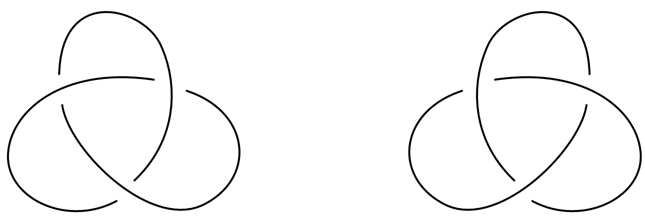


Figure 1.23: The right-hand trefoil and the left-hand trefoil are not isotopic.

One of the first questions that arises is how to understand if two diagrams represent isotopic links or not. It was answered by the following result.

Theorem 1.6.4 (Reidemeister [53], Alexander-Briggs [4])
Two link diagrams represent the same link if and only if they are related by a finite sequence of local moves of the three types depicted in figure 1.24.

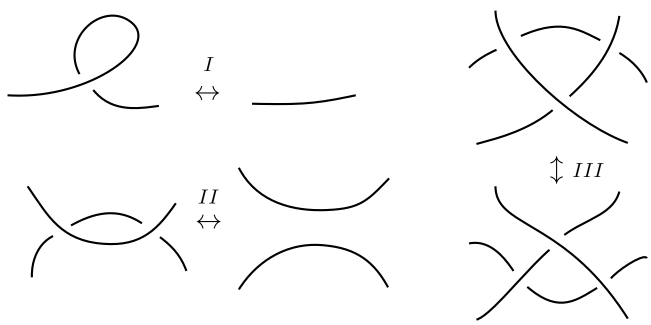


Figure 1.24: The three Reidemeister moves.

Note that move II is analogous to free reduction in braids, while move III can be seen as a braid relation. Move I has not an analogous move for braids, since the braid strands are always monotone.

The application of the Reidemeister moves is not intuitive. For example, there are knot diagrams where it is necessary to initially increase the number of the crossings to obtain the unknot. The simplest example of such a diagram is in figure 1.25.

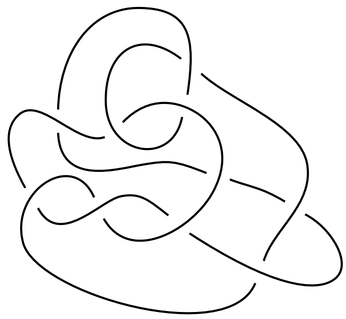


Figure 1.25: An unusual diagram of the unknot.

There exists a "brute-force" algorithm that uses the Reidemeister moves to unknot diagrams: it applies all possible moves, generating a tree of diagrams. If the diagram represents the unknot, this algorithm obtains the trivial diagram in a finite number of moves, which is exponential in the number of crossings. However, the known bound on the number of moves (see [34]) makes it infeasible to use the Reidemeister moves to detect the unknot.

Moreover, this algorithm only allows to trivialize each diagram of the unknot, but does not solve the problem of isotopy of links (i.e. given two link diagrams, decide whether they represent the same link or not). Then, other ways to handle the problem are investigated.

A way to distinguish links is to define *invariants*: we associate a mathematical object to each link, so that equivalent links correspond to the same object. In other words, whenever two links are associated to different objects, they are different. This does not necessary mean that different links correspond to different objects. If this property holds, then the invariant is able to distinguish all the links, and it is said to be *complete*. We summarize this discussion in the following Definition.

Definition 1.6.5
Let $\mathcal{L}$ be the set of links and X a mathematical structure, for example a set, a group or a polynomial ring. An *invariant* of links is a function

$$i : \mathcal{L} \to X$$

that maps equivalent links to the same element of X. In other words the map i induces a map $\tilde{\imath} : \mathcal{K} \to X$ from the set of links up to isotopy to X. If the map $\tilde{\imath}$ is injective, the invariant is said to be *complete*.

This notion can be useful if the way to draw the connection is easily computable and if the objects can be easily distinguished from each other.

For example, to each link we can associate the number of its components: it is elementary to calculate this invariant, but it does not distinguish many links.

Another example is the crossing number: to each link it associates the minimal number of crossings in its diagrams. This invariant is almost intractable, because its computation is very difficult.

Many link invariants are known and some of them are complete, for example the knot complement [32] and the fundamental quandle [39]. However, a complete invariant that is computable has not been found yet. A very strong invariant is the Jones polynomial, that will be described in section 1.6.2.

1.6.1 Links and braids

Given a braid on n strands, it is possible to connect the strand ends to obtain a link. The standard way to connect the strand ends is using n unknotted and unlinked arcs as in figure 1.26. The link obtained is called the *Alexander closure* or simply the *closure* of the braid.

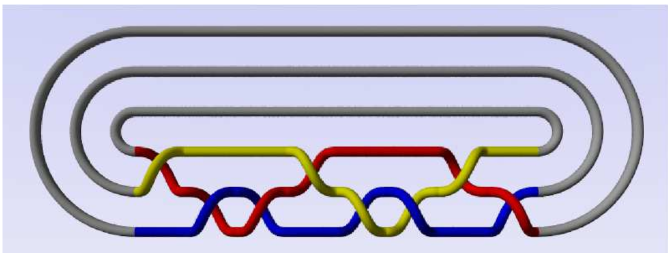


Figure 1.26: The closure of a braid gives a link.

Theorem 1.6.6 (Alexander [2])
Any link can be obtained as the closure of a braid.

A proof of Alexander's theorem can be found in [33, Chapter II.2] and [8, Chapter 2]. [41, Chapter 2] contains a proof and an algorithm based on [59].

In the movie we illustrate the first proof, which uses diagrams of piecewise linear links, though we draw smooth tubes and consider only a knot for simplicity. We report some frames from the movie to sketch the algorithm.

Choose an axis that does not intersect the knot. Our aim is to make the knot into a reel around that axis. Start walking along the knot in the clockwise direction around the axis. At some point, the knot can turn and we will be walking anti-clockwise. Colour all the anti-clockwise arcs. By an isotopy move each coloured arc (or a piece of it) in turn, pulling it to the other side of the axis.

In figure 1.27 this procedure is shown on a knot: the axis is through the purple point. The red pieces wind in the wrong direction around the axis, they become yellow when we move them on the right side.

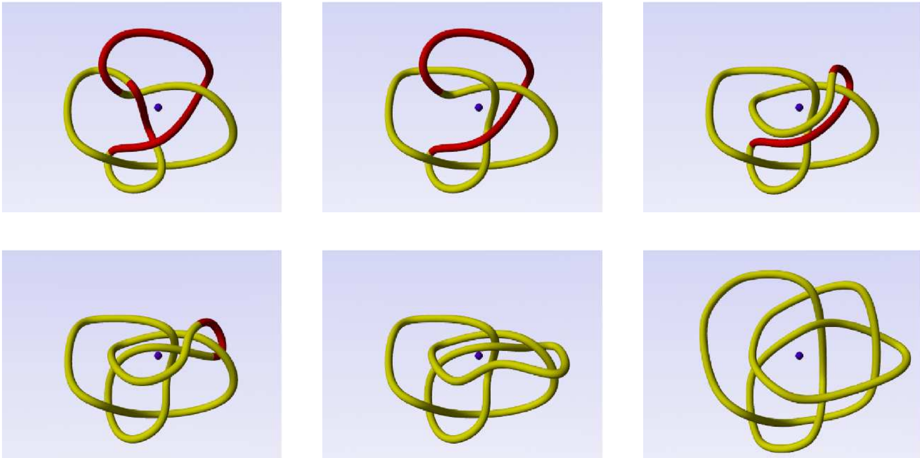


Figure 1.27: Alexander's algorithm: obtaining a reel around an axis.

In this way we get a reel, that is, there is an $n > 0$ such that any half plane with the axis as boundary intersects the knot in exactly n points. Choose one such half plane and "open" the knot to obtain a braid with n strands, as shown in figure 1.28. To emphasize the strands, we draw each of them with a different colour. We pull the half planes (drawn as rectangles) where we cut the knot until they are parallel. The strands are deformed but they never intersect and their endpoints remain fixed on the half planes.

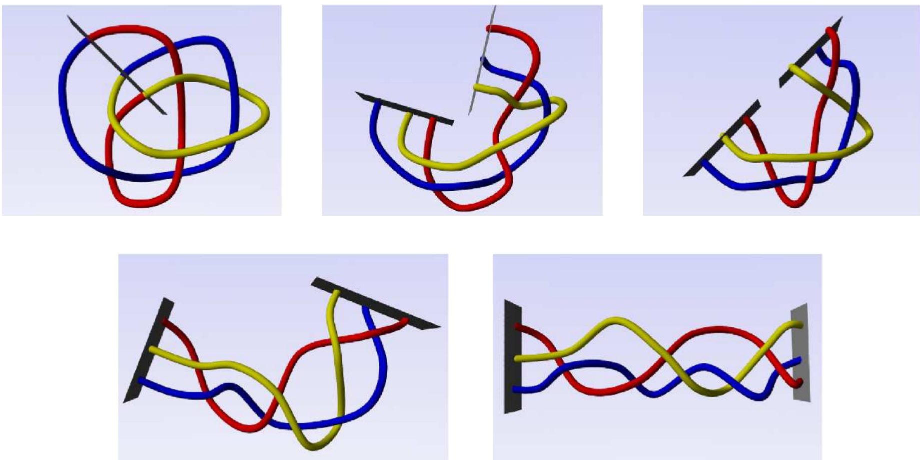


Figure 1.28: Alexander's algorithm: opening the knot to get a braid.

Alexander's Theorem ensures that one can associate a braid to any link, such that the link will be isotopic to the closure of that braid. However, different braids can be associated to a given link, depending on the choices made in the algorithm (which link diagram, which axis, which starting points on the link). The following well-celebrated theorem exactly says when two closed braids give isotopic links.

Theorem 1.6.7 (Markov)
The closure of two braids defines equivalent links if and only if the braids are related by a sequence of braid relations and moves of two kinds:

- *conjugation, that is transform a braid $\beta \in B_n$ into $\alpha\beta\alpha^{-1}$ for some braid $\alpha \in B_n$, and*
- *stabilization, that is transform a braid $\beta \in B_n$ into $\beta\sigma_n^{\pm 1} \in B_{n+1}$, or vice versa.*

The theorem was announced by Markov in [49], but it was probably proved by Ivanovski, one of his students. The first published proof appeared in [8, Chapter 2.2]. Proofs can be found also in [50, Chapter 9] and [41, Chapter 2].

There is another type of closure for braids in B_{2n} with an even number of strands. It is called *plat closure* and consists in putting n arcs on the left and n arcs on the right, such that the i-th arc connects the points p_{2i-1} and p_{2i} on the same x-height. See figure 1.29. The link in this form is sometimes called *plat*.

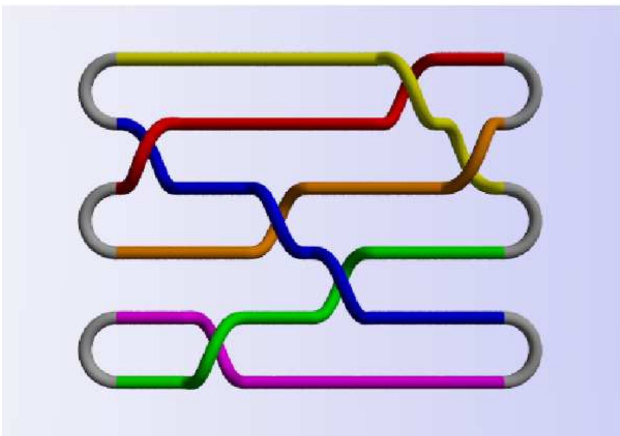


Figure 1.29: The plat closure of a braid gives a link.

Remark 1.6.8
It is possible to pass from a type of closure to the other. To obtain the Alexander closure of an n-braid we add n trivial strands on the top of the braid and connect the strand ends with arcs. By link isotopy we can move the new strands behind the old ones to obtain a link that is put in the form

of the plat closure of a new braid. An example of this operation is shown in the fourth chapter of the movie; some frames can be found in figure 1.30. Vice versa, given a plat, we can use Alexander theorem to put the link in the form of the usual closure of a braid.

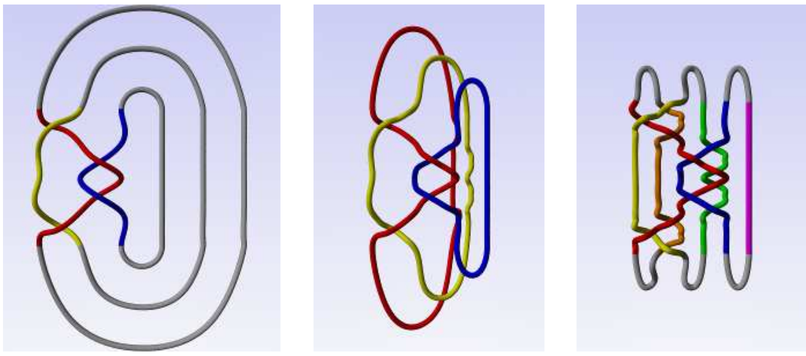


Figure 1.30: Starting from the Alexander closure of a braid, we can isotope the link to obtain a plat.

From this remark it follows that a result analogous to the Alexander theorem holds: every link can be obtained via the plat closure of a braid. Alternatively, this can be shown in the following way: given a link, we can put it in the form of a plat by pulling all local maxima (resp. minima) of the link (with respect to any fixed direction) at the same height.

Also a theorem analogous to that of Markov holds: we present it in Section 1.7 because we need one more definition.

1.6.2 The Jones polynomial

The Jones polynomial is an invariant of links, discovered by Vaughan Jones in 1984 ([37] and [38]). For his work on von Neumann algebras and knot polynomials Jones was awarded a Fields Medal in 1990. In fact, his discovery was unexpected and led to immediate generalizations (see [27]) and developments that permitted to understand connections between different areas of mathematics and physics (knots and links, statistical mechanics, quantum groups, Lie algebras). Interest in knot theory from the mathematical community grew considerably after Jones' discovery.

The original definition of Jones in [37] was given through homomorphisms of the braid group in an algebra. In fact, Jones was studying the Temperley-Lieb algebras that arise in statistical mechanics. In these algebras some relations hold that are similar to the relations in the Artin presentation of the braid group. We sketch Jones' approach.

Definition 1.6.9
Fix $n > 1$ and $\tau \in \mathbb{C}$. The Temperley-Lieb algebra $TL(n, \tau)$ is the algebra

with identity element 1, generated by $e_1, e_2, \ldots, e_n$ with the following three relations.

$$\begin{aligned} e_i^2 &= e_i \\ e_i e_{i+1} e_i &= \tau e_i \\ e_i e_j &= e_j e_i \quad \text{if } |\, i-j \,| > 1 \end{aligned}$$

Proposition 1.6.10 (Ocneanu)
For every (n, τ) there exists a unique linear map $\mathrm{tr} : TL(n, \tau) \to \mathbb{C}$, called Ocneanu's trace, *such that*

(i) $\mathrm{tr}(1) = 1$

(ii) $\mathrm{tr}(ab) = tr(ba)$ *for every* $a, b \in TL(n, \tau)$

(iii) $\mathrm{tr}(xe_{n+1}) = \tau \mathrm{tr}(x)$ *for* $x \in TL(n+1, \tau)$.

Proposition 1.6.11 (Jones)
Let t be such that $\tau^{-1} = 2 + t + t^{-1}$. Then the map $\rho_n : B_n \to TL(n, \tau)$ defined by

$$\rho_n : \sigma_i \mapsto (t+1)e_i - 1$$

is well defined. Moreover, ρ_n is a homomorphism of B_n in $TL(n, \tau)$.

Definition 1.6.12
Let $\beta \in B_n$, let e be the exponent sum of β written in the σ_i's. Denote $\hat{\beta}$ the closure of the braid β. Then the expression

$$V_\beta(t) = \left(-\sqrt{t} - \frac{1}{\sqrt{t}} \right)^{n-1} \sqrt{t}^{-e} \mathrm{tr}\left(\rho_n(\beta)\right)$$

is called the *Jones polynomial* of the braid β.

$V_\beta(t)$ is a Laurent polynomial in the variable $t^{1/2}$ with integer coefficients.

Theorem 1.6.13 (Jones [37])
The Jones polynomial of a braid is invariant under Markov moves. Therefore it is an invariant of the link obtained closing the braid.

In other words, $V_\beta(t)$ depends only on the link $\hat{\beta}$ and not on the braid β. Hence we speak of the *Jones polynomial of the link $\hat{\beta}$*.

The Jones polynomial can be introduced also through the *Kauffman bracket* (see [42] or [43, Chapter I.3]). Let L be an unoriented link. Fix a diagram of L and consider any crossing in it. We can label the regions around the crossing in this way: walking on the over arc towards the crossing, we see an A-region on the right and a B-region on the left (by the way, this colouring of the regions is coherent on the whole diagram).

There are two ways of resolving a crossing in the diagram, that is, of breaking the arcs and connecting them without crossings. We will call A-split the resolution of a crossing where two A-regions are joined and B-split the resolution of a crossing where two B-regions are joined. See the following figure 1.31.

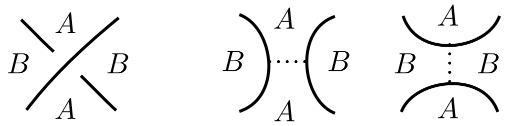


Figure 1.31: Resolving a crossing: A-split and B-split.

The bracket polynomial $\langle D\rangle$ of a link diagram is defined as a state-sum of the resolved diagrams (the definition can be found in [43, Chapter I.3]).

It is possible to show that the following relation between diagrams holds and permits to obtain the same bracket on diagrams differing for Reidemeister moves of type II and III.

$$\langle \times \rangle = A\langle \,)(\, \rangle + A^{-1}\langle \asymp \rangle$$

Here the drawings inside the brackets represent a piece of the diagram, while the rest of the diagram remains unchanged. If we set the "initial condition"

$$\langle \bigcirc \rangle = -A^2 - A^{-2}$$

we can calculate the bracket on every diagram.

The bracket polynomial still depends on the diagram, but it becomes an invariant of the link when we normalize it as follows:

$$\mathcal{L}_D(A) = -A^{-3w(D)}\,\langle\, D\,\rangle$$

where $w(D)$ stands for the *writhe* of the link and is calculated as the sum of the signs of the crossings in an oriented diagram D of L. The writhe is independent of the orientation chosen and is an invariant of links. The sign of the crossings is defined as in figure 1.32.

$\epsilon = +1$ $\qquad\qquad$ $\epsilon = -1$

Figure 1.32: Positive and negative crossings.

Then we define $\mathcal{L}_L(A)$ as $\mathcal{L}_D(A)$ for any diagram D of L.

Theorem 1.6.14 (Kauffman [42])
The Jones polynomial can be calculated as

$$V_L(t) = \mathcal{L}_L(t^{-1/4}).$$

Another relation on link diagrams that can be used to compute the Jones polynomial was found by Jones in [37]. Consider an oriented link diagram and choose a crossing. We call L_+ the positive crossing, L_- the negative crossing and L_0 the resolution of the crossing that preserves the orientations of the arcs (figure 1.33).

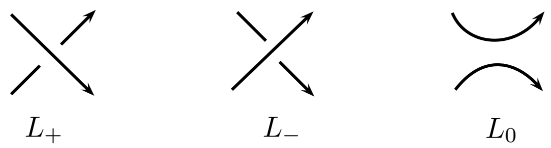


Figure 1.33: A crossing in an oriented diagram can be switched or resolved.

Proposition 1.6.15 (Jones [37])
For each initial condition V_{unknot} there is a unique Laurent polynomial with integer coefficients that is defined by the skein relation

$$t^{-1}V_{L_+} - tV_{L_-} = \left(t^{1/2} - t^{-1/2}\right) V_{L_0}.$$

It is an invariant of links. Setting $V_{unknot} = 1$ we obtain the Jones polynomial.

1.7 The Hilden subgroup

We consider the braids with an even number $2n$ of strands whose plate closure gives a trivial link. Of course a braid with $2n$ strands will give the unlink on n components. An example is in figure 1.34. It is easy to check that this set is a group, called *Hilden group* and denoted H_n. It is a subgroup of the braid group B_{2n}.

To characterize these braids, we can consider motions of disjoint arcs in $D \times I$ with endpoints on the disc. At each instant the configuration of the endpoints gives the intersection of the braid with a disc parallel to D (figure 1.35).

If we see braids as mapping classes, we can characterize these braids as in the following definition.

Definition 1.7.1 (cf. Hilden [35])
In D fix $2n$ points $\mathcal{P} = \{p_1, p_2, \ldots, p_{2n-1}, p_{2n}\}$. Fix n pairwise disjoint

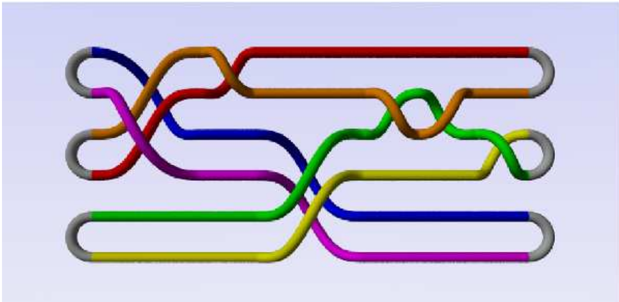


Figure 1.34: This is a Hilden braid since its plat closure gives a trivial link: the arcs that close the braid can be "retracted" by an isotopy to obtain three unknotted unlinked circles.

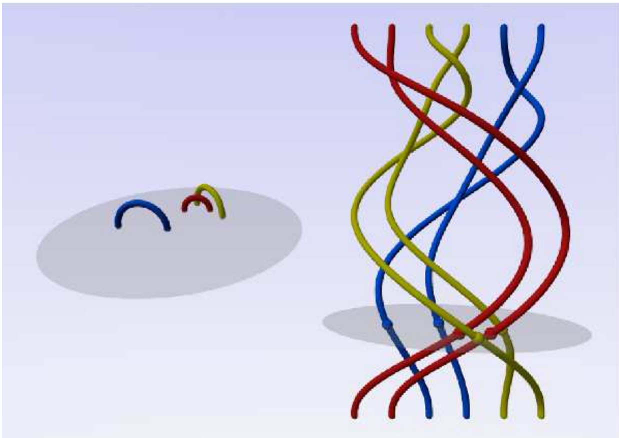


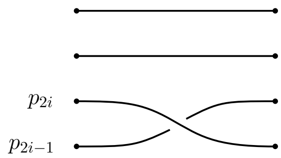
Figure 1.35: A Hilden braid can be seen as a particular "dance" of points connected by pairwise disjoint arcs.

unknotted unlinked arcs $a_* = \{a_1, \ldots, a_n\}$ in $D \times \mathbb{R}^{\geq 0}$ such that a_i has endpoints p_{2i-1}, p_{2i} for $1 \leq i \leq n$. Consider the set of (isotopy classes of) homeomorphisms $D \to D$ that admit an extension to homeomorphisms $D \times \mathbb{R}^{\geq 0} \to D \times \mathbb{R}^{\geq 0}$ fixing a_*. This is a group and is called the *Hilden group* on $2n$ strands. It is denoted by H_n.

The Hilden group is a subgroup of the braid group, since braids can be defined as (isotopy classes of) homeomorphisms $D \to D$ fixing $\mathcal{P} = a_* \cap D$. The arcs a_* can be interpreted as the arcs that close the braid to obtain a plat (see section 1.6.1).

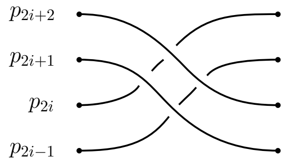
The generators of the Hilden group H_n (already known since Hilden's paper [35]) are of three kinds, that we now describe.

For $1 \leq i \leq n$ the generator t_i corresponds to the arc a_i making a half twist in the clockwise direction. This can be seen as the strand $2i$ that crosses over the strand $2i-1$. In terms of the usual generators of the braid group B_{2n} we can write $t_i = \sigma_{2i-1}$.



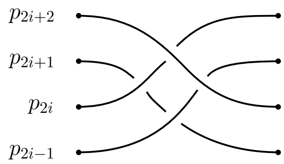
t_i

For $1 \leq i \leq n-1$ we have the generator s_i: the arc a_{i+1} exchanges position with a_i, passing in front of it. This can be seen as the strands $2i+1$ and $2i+2$ crossing over the strands $2i-1$ and $2i$. In terms of the usual generators of B_{2n} we have $s_i = \sigma_{2i+1}\sigma_{2i+2}\sigma_{2i}\sigma_{2i+1}$.



s_i

Finally, for $1 \leq i \leq n-1$ we have the generator r_i: the arc a_i exchanges position with a_{i+1}, passing through it. This can be seen as the strands $2i-1$ and $2i$ passing in front of strand $2i+1$ and behind strand $2i+2$. In terms of the usual generators of B_{2n} we can write $r_i = \sigma_{2i+1}^{-1}\sigma_{2i+2}\sigma_{2i}^{-1}\sigma_{2i+1}$.



r_i

We do not report a complete set of relations to present the Hilden group. A presentation of H_n with the generators we have described and fourteen types of relations was found by Tawn in [57].

Now we can state a theorem analogous to that of Markov to characterize the braids giving the same link after plat closure. A proof can be found in [8, Chapter 5].

Theorem 1.7.2 (Birman, [8])
The plat closure of two braids defines equivalent links if and only if the braids are related by a sequence of braid relations and moves of two kinds:

- *composition with elements of the Hilden subgroup on the left or on the right, and*
- *stabilization, that is transform a braid $\beta \in B_{2n}$ into $\beta\sigma_{2n}^{\pm 1} \in B_{2n+2}$, or vice versa.*

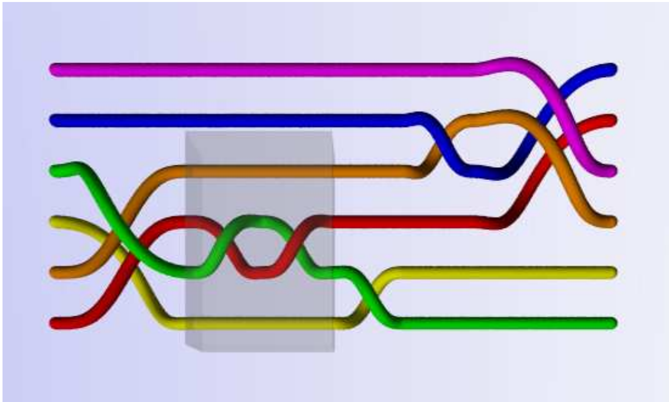


Figure 1.36: The braid in the box gives the same link as the whole braid when they are closed as plats: the long braid is obtained from the short one by composition on the left with r_1, composition on the right with t_1, stabilization and composition on the right with s_2.

The membership problem for the Hilden subgroup inside the braid group is solvable. It asks, given an element β of B_{2n}, to decide whether β belongs to H_n. A proof based on the action of the braid group on the fundamental group of the punctured disc can be found in [58].

Chapter 2

A movie

In this chapter we describe the cartoon realized to present the mathematical theory of braids to a wide public. The subject allows to introduce broader themes in mathematics and can be used as a model to talk about topics such as topology, groups, algorithms and group actions. Together with the workshop described in Chapter 3, the movie can be a tool to make braid theory known to students in the last years of high school and in the first years of university. Another tool based on computer graphics is described in Section 2.6: an interactive applet to illustrate one of the topics presented in the movie.

Some clips from the movie have already been shown to students and to mathematicians. The reactions have been positive: the public was interested and made useful remarks.

2.1 Motivations and aims

A movie allows to approach pure mathematics in an unusual way, which can be pleasant for a general public. Also mathematicians may appreciate to visualize mathematical objects and have an informal tool to discuss mathematics. Braids are ubiquitous objects, there are several ways to look at them, and each one provides new insight into the braid groups. Besides, a lot of different mathematical subjects and results can be narrated starting from braids, involving different fields: topology, group theory, combinatorial problems, algorithms, complexity theory, knot theory. This is a good example to show the interdependence of different fields of mathematics: a question in one field can lead to formalizations, new problems, answers, new insight, in another.

Popularization can help changing the public perception about mathematics: some people think that mathematics is only involved in doing computations or that it is completely abstract and has nothing to do with reality. On the contrary, one of the main aims of mathematics is to observe the world and recognize patterns, formalizing them in mathematical structures, in order to better understand real and abstract objects. Mathematical structures can be found in everyday life.

A further false belief that some people have about mathematics is that it is a dead and static subject, where all the possible theories have already been investigated. Talking of braids, one can stress that research in this area, as in many others, is very active nowadays and that new open questions arise constantly.

One of the main facts, presented in the first chapter of the movie, is that the braids with a fixed number of strings form a group. The groups that are usually introduced at school are numeric groups, such as the integers with respect to the addition and the non-zero reals with respect to the multiplication, or transformation groups, such as planar isometries. However, the first class consists of commutative groups, so that braid groups are more complex and can have more interesting properties. The second class has the disadvantage that their elements are usually visualized as movements, not as static objects. This can be complicated for students in a first approach to groups. On the contrary, the elements of the braid groups are concrete objects, that can be easily visualized, along with the operation of composition.

Another good point is that many concepts in braid theory are suitable to be told in an informal way: many pictures can be drawn and much of the algebra can be "seen". Nevertheless, it is possible to introduce also deep concepts and advanced results without too much effort for the public.

We think that in popularization it is important to have a good and interesting starting point. This can be a practical problem, as in the activity proposed in Chapter 3, or, as in the present case, common objects, known to everyone, that are thought to be simple or suitable as child games. In this way the public is not afraid and gets interested in the formalization of the subject. Everyone has plaited their hair or braided some strings in their life. There are a lot of examples of braids in everyday life: bread and other food, wicker baskets, knitting, ropes, jewellery are some examples. Nevertheless, not many people ever thought that braids are also mathematical objects. The public is often surprised by such an assertion and wants to know more.

Braids can be considered as very naive objects. The public is often convinced that it is easy to handle them. On the contrary, the formalization is sometimes subtle, for example when defining diagrams. Besides, it can be difficult to establish what is intuitive for everyone and can be taken for granted and what needs a definition or an explanation. Furthermore, some concepts that are elementary for a specialist, such as the notion of invariant, are very difficult to be understood by the public.

Even if braids are so common objects, not much popularization has been made around braid theory, contrarily to knot theory, which has been widely narrated in popularization books, movies, articles etc. This is another reason that makes this project worth to be done: we can narrate something new and refer to something known.

Finally, we hope that this tool interests mathematics teachers looking for new ways of teaching and motivating students. The movie can be an occasion for teachers to approach a different field of mathematics and to propose some activities in their classes.

To summarize, through the movie we would like not only to invite the public to approach mathematics without fear and prejudices, giving an idea of a research topic and showing connections among different fields of mathematics, but also to give a tool to teachers. Finally, we hope to give the possibility to see, understand and enjoy braid theory at many different levels, from a young student to a mathematician.

2.2 Summary

The movie is divided into four chapters, so that one can see just one of them or watch each part in a different moment. This can be useful especially if the movie is shown to students: after watching a chapter, the teacher can give some explanations or propose a workshop.

The chapters are thought to be seen sequentially: the first part is the most accessible, while the mathematical content of the following chapters is deeper. To understand the second and the third chapters, the concepts described in the first one are required, while the last chapter contains more advanced topics and requires the contents of the first and the third chapters. The chapters last about 15 minutes, a time after which attention can sink.

Each part is symbolized by an object: the first one is represented by a book. Here braids are introduced as mathematical objects and the structure of the braid group is investigated, to obtain a presentation of the group. A grinder, representing an algorithm, is the object associated to the second chapter, that is devoted to the word problem, algorithms and their complexity. In the third part, symbolized by a trefoil knot, knot theory is introduced and linked to braid theory. The Jones polynomial is presented. Finally, a dancing couple represents the last chapter, that deals with braids as configuration spaces, the plat closure, the Hilden subgroup and the membership problem.

A summary of the contents of the whole movie follows. For a brief exposition of the mathematical theory we refer to Chapter 1, while Appendix A contains the complete script of the movie.

We also spend some words in Section 2.2.5 to point out more themes that could be suitable to be included in the movie.

2.2.1 Chapter 1 - The group structure

We make some examples of braids in the real world and we ask how they can be formalized as mathematical objects. We give a definition of braids, pointing out that we consider braids as equivalent if there is an isotopy transforming them into each other. Some time is spent to informally recall the concept of equivalence class. Then we start to examine the structure of the set of braids: we introduce the composition and remark that it is associative but not commutative. We note that there is an identity element for the composition and that for each braid there is an inverse. Now we can talk of the braid group.

We investigate the group to find the elementary braids, a set of generators. We associate a letter to each generator so that we can describe any braid by a word. We ask when two words represent the same braid and give a complete set of relations. In this way we arrive to Artin's presentation of the braid group and write it in mathematical notation.

2.2.2 Chapter 2 - The word problem

We start from the last question of the first chapter: given two words in the generators, when do they represent the same braid? This is called the word problem.

The group relations are not enough to solve the word problem, because they do not give a constructive answer. Hence we introduce the concept of algorithm. We present two algorithms to solve the word problem in braid groups: Artin braid combing and handle reduction. We motivate the introduction of the latter with complexity issues: the combing is infeasible in practice, while handle reduction seems to be very fast. Besides, this is an occasion to introduce a recent result: the first paper about handle reduction was published in 1997.[1]

2.2.3 Chapter 3 - The world of knots

We introduce knots and links, closed strings in the 3-space. For simplicity we call both objects knots. We present the concept of isotopy of knots, which is similar to that of braids, and show that the figure eight knot is achiral. To distinguish knots, we can define invariants, that is associate to each knot an object in some mathematical structure, such that to equivalent knots corresponds the same object. To introduce the Jones polynomial, a powerful invariant of knots, we relate braids to knots.

It is easy to see that closing a braid we obtain a knot. Two natural questions arise: can we obtain every knot as the closure of a braid? When do two closed braids give the same knot?

[1] In Section 2.6 we also describe an interactive applet about this algorithm.

The answer to the first question is affirmative and is called Alexander theorem. We present Alexander's algorithm to put a knot into the form of a closed braid. To answer the second question, we introduce the operation of conjugation. Together with stabilization, it is a fundamental part of Markov theorem, that characterizes the braids giving the same knot. Unfortunately, this theorem is not constructive.

So, using braids seem not to simplify the problem on knots. But there are more results that relate braids to knots. One of the strongest knot invariant, the Jones polynomial, was found studying representations of braids into an algebra. We present a way to compute it, based on the skein relations, though this is not Jones' original definition.

Finally, we compute the Jones polynomial of the two trefoils, the left-hand trefoil and the right-hand trefoil. Since we obtain different results, we know that the knots are not isotopic and thus the trefoil is chiral.

2.2.4 Chapter 4 - Hilden dances

We characterize braids as motion of points in the disc. This is visualized by dancers performing a dance.

Then we pass to describe particular dances, where dancers are in pairs and hold hands. The formalization of these dances yields a subgroup of the braid group with an even number of strands, called Hilden group. We give a set of generators for this group.

We introduce the plat closure, obtaining it from the usual one. We note that Alexander theorem ensures that every knot can be obtained by plat closure of a braid. Now we state a theorem of Birman, analogous to the Markov theorem but referring to the plat closure. Here the Hilden subgroup comes into play.

Finally we present the membership problem: how is it possible to decide whether a given braid belongs to the Hilden subgroup or not? We do not answer but just note that again we found an algorithmic problem.

2.2.5 What is not in the movie

Having at disposal a big amount of themes and interpretations of braids, it has been difficult to choose the topics to be included in the movie. We tried to make a selection in order to present the subjects in a linear exposition. Each chapter is short and self-contained, but strongly linked to the general narration. However, there are various further themes that could be suitable to be presented in a movie at a popular level.

Here we make a short summary of the topics singled out but not yet developed. We would like to consider them in the future for new parts of the movie or for other popularization activities. We point out some reference for the interested reader.

Surface braids would permit to do beautiful drawings and could be used to fascinate the public. We defined braids taking some points in the disc, but we can allow a general surface instead of the disc. The simplest examples are braids on the sphere and on the torus. To describe surface braid groups other presentations can be written, but the description of the methods to find them would make the narration more technical and difficult to follow. However, these groups can be interesting because they have different properties than the usual braid groups. For example, the braid group on three strands on the sphere is finite, while every n-braid group is infinite for $n > 1$. See [50, Chapter 11] for some results.

A theme that would permit to fascinate the public and show some more connections between different areas of mathematics is the geometric interpretation of the finite subgroups of the braid group on the sphere and on the projective plane. This would also be an occasion for a narration of a wide subject, the classification of surfaces.

In Section 1.5.2 we characterized braids as homeomorphisms. In [23, Chapter 9] braids are presented as a special case of mapping class groups. A further part for the movie could undertake the opposite path: describe braids as mapping classes and then introduce more general results on mapping class groups. However, even the basic theory is not intuitive: it is not immediate to explain the concepts of homeomorphism and isotopy of the disc or a surface. Moreover, we have already inserted a different characterization of braids, as configuration spaces.

Also virtual braids would allow beautiful drawings and the discussion of deep results. They are combinatorial objects that were first defined by Kauffman in [44] via diagrams and moves in the spirit of the Reidemeister moves on link diagrams. However, virtual braids also have a geometric interpretation as motion of two dimensional objects (arcs or circles) in $\mathbb{R}^4$. See [24], [7] and [12].

Similarly, other combinatorial objects have been defined allowing more moves on diagrams. They are called welded braids (see [24]). In [55] Rourke gives a geometric interpretation of these objects. In [40] Kamada proves theorems analogous to those of Alexander (Theorem 1.6.6) and Markov (Theorem 1.6.7) for virtual and welded braids.

Braid cryptography could be another interesting subject for the public, because it gives a concrete application of braid theory. However, we did not insert this theme in the movie, since a lot of time should be used to explain cryptography and cryptographic schemes. This would lead out of the main theme. Moreover, there are probably more effective cryptography schemes that can be considered to popularize the subject, since braid cryptography is not used in practice: all the braid schemes proposed so far are based on the conjugacy problem, which is efficiently solvable in braid groups.

2.3 Some warnings

We describe some difficulties that can be stressed about the movie, especially if it is used as a teaching tool in schools.

Specialists, such as mathematicians, prefer a rigorous written text, where all the concepts are precisely formalized. But we address to a wider public, composed by non specialists, who do not have access to scientific literature or may lack in competences to read it. Hence, we have to reach a compromise between a rigorous and an appealing way of presenting the subject. We can not be too technical and rigorous at some points because the language would soon become cryptic for the audience. On the other hand, we do not have at disposal a long time as in a lesson and we need to pass over some technicalities and not delve into the concepts.

The movie is organized in four short chapters that can be watched by students independently. However, the role of the teacher continues to be crucial to clarify the difficult points, to give more insight to interested students and to give a further formalization if needed. Teachers can find help to achieve this task in Chapter 1, where braid theory is outlined formally, and in Chapter 3, where we propose an activity for students.

It is not so common that teachers propose workshops or complementary activities for students in the last years of high schools. Some teachers are concerned with "completing the syllabus", because of the exam at the end of the high school. Even if it is important to give the students the standard preparation, it can be crucial to give some more insight into mathematics, especially to good students, who have the skill of understanding more advanced concepts. We would like the movie and the workshop to become an occasion to give the students tools to approach advanced themes. However, we keep a popular level to address to all the students, and not only the good ones. Keeping the balance among students with different skills and interests is one of the challenging tasks of the teacher.

2.4 Developing the movie

Building the movie has been a big challenge, in which we had to deal with several matters: we describe the principal ones.

A very big issue has been to identify the target and consequently the level of the movie. We choose to address to a high level public, in particular to students of the first years of scientific faculties and students in the last year of the high school. We think that this is an audience that can comprehend many concepts and appreciate the richness of the subject. Lowering the level, some deep and beautiful results could not be exposed or they would not be appreciated. Vice versa, raising the level further, we would restrict to a very small public, especially to mathematicians, who have the competences to read a technical book or survey on braids.

A critical point is to choose the right speed of the images and the explanations. It is quite difficult to find a balance and not be boring because too slow or incomprehensible because too fast. This also depends on the interest and the knowledge of the audience.

A similar remark applies also to the background music: it should be not too slow and relaxed but also not too invasive. We chose some pieces of Bach, taken from *Das Wohltemperierte Clavier*, the English Suites and the French Suites.

The drawings have not only to be meaningful to describe the mathematics, but also beautiful and convincing. For example, sharp changes of speed in the movements are not pleasant to see.

Another problem is the audio-video synchronization: the rhythms of the speech and of the images might be different: some concepts are explained in a couple of words but need a lot of images to be described. An example is the algorithm proving the Alexander theorem in the third chapter of the movie. Others concepts are more theoretical or deeper, one needs to say a lot to explain them, but the images do not say much or are basic and repetitive. Examples are the conjugation of braids and the Birman theorem. To balance the rhythm we inserted new images or slowed down the drawings in some points and reformulated the text in other ways.

The text has to be clear, at elementary level, but short, precise and direct. We report the movie script in Appendix A.

The major issue consisted in choosing the themes to include in the movie. Of course an introduction to braids was necessary and is presented in the first chapter of the movie. We choose to start from braids in everyday life, to make the subject more appealing and to link mathematics to concrete objects. To present composition of braids, we drew a parallel between this operation and the product of positive real numbers. In this way, we could recall elementary notions that all the audience should possess and we could introduce the structure of group, looking in braid groups for elements and properties corresponding to known notions in numeric groups. Moreover, we could stress a big difference between the operation of product of numbers and that of composition of braids: the commutativity holds in some groups but not in every one.

At the beginning we made a comparison between the composition of braids and the sum of integers: this operation seemed more elementary and we did not have to restrict to positive numbers. However, as some mathematician noted when they saw the first version of the movie, this parallel might cause some confusion because of the difference of notation, since we write composition of braids in the multiplicative manner. Then, we choose to switch and use the multiplicative notation.

Towards the end of the first chapter, we included a picture of Emil Artin and reproduced the first page of his first two papers on braids. We choose to include some historical remarks just to set the birth of braid theory in the time. To give a little colour to the story, also in the following chapters we inserted some pictures portraying the mathematicians who obtained some of the main results: James Waddell Alexander, Andrey Markov, Joan Birman and Vaughan Jones.

We had more freedom in choosing the content of the other chapters. We decided to include a part about combinatorial group theory and algorithms (second chapter), because of different reasons. A concrete problem, such as the word problem in braid groups, can be interesting for the audience: at the beginning we just formalized braids as mathematical objects, but we did not mention what can be "done" with these objects. We convey the idea that the formalization presented in the first chapter is not sterile and that there is much that can still be discovered or improved. Moreover, we can speak of algorithms and their complexity and link braids, and more generally combinatorial problems, to computability theory, giving a connection between different fields of mathematics.

It is possible that part of the public already knows something about knot theory, presented in the third chapter of the movie: this theme has been widely used in popularization of mathematics. The first version of the movie contained the calculation of the Jones polynomial through the Kauffman bracket, but we discarded it because it is less direct than using the skein relation. We choose not to mention Jones' original definition because it is more abstract and involves deeper knowledge, such as the representation of the braid group into an algebra.

In the last chapter we want to show some deeper results. We show that braids can be defined in another way, as motion of points in the disc. We would like to convey the idea that a different insight on the same objects can be useful to better understand them and can lead to generalizations and new theories. We introduce the Hilden subgroup of the braid group, which is an object investigated in present-day research. We think that it is important to give an idea of the fecundity of research in the field. Finally, we go back to a combinatorial problem, the membership problem, to give a stronger link of the last chapter with the others.

2.5 Software

In the production of the movie, we choose to use only free and open source software. In particular, the operative system on the two computers used is Ubuntu Linux (at the beginning version 9.04, then version 10.04 or 11.04).

Here is a list of the software used:

- POV-Ray (`http://www.povray.org`), the Persistence of Vision Raytracer, for 3D image creation. We used the beta version 3.7 because it has the support for multi-processors computers;

- GIMP (http://www.gimp.org), the GNU Image Manipulation Program, as image editor;
- MEncoder (http://www.mplayerhq.hu) to compress frames and produce avi files;
- Kdenlive (http://www.kdenlive.org) as video editor;
- Sound Recorder (a tool of Ubuntu) to record voice and music;
- Audacity (http://audacity.sourceforge.net) as sound editor.

2.6 An applet on handle reduction

The theme of the second chapter of the movie is the word problem in braid groups: there, we present the handle reduction algorithm that solves this problem. To narrate this solution to the word problem, also an interactive applet has been realized. It is now available at http://matematita.science.unitn.it/applet/handlereduction/.

The applet allows to construct braids starting from the generators and to visualize the reduction of the handles. It is supported by a text, organized as a FAQ, that briefly explains in an informal way the concepts needed to understand the algorithm. In particular, we illustrate braids and their representation through diagrams, how to associate a word to a braid (i.e. the generators of the braid group, but without talking of generators nor groups), the braid isotopy problem, the word problem, and finally handles and handle reduction.

The text is a reworking of a text to illustrate an analogous applet[2] by Patrick Dehornoy and Jean Fromentin.

The applet is directed to a general public: it may become a first approach to braids and arouse some interest in mathematics. It can be used in a mathematics exhibition together with other exhibits about braids, such as the one presented in Section 3.4.2. Unlike this exhibit, it does not necessarily need the explanation of a museum guide. However, it is required that the audience has a minimal familiarity with the computer to be able to use and understand the applet.

The applet is written in Java with the JoGL libraries. The software used to realize it is NetBeans (a Java IDE).

2.7 Acknowledgements

I would like to thank Patrick Dehornoy for his request to develop the applet presented in Section 2.6. This was the first occasion I had to approach

[2]available at http://www.math.unicaen.fr/~tressapp/AppletInfo.html

braid theory. I am grateful to my supervisor Paolo Bellingeri, who first introduced me to braid theory, advised me in the choice of the themes and read the scripts many times. He also gave some beautiful ideas to draw variegated figures, not always limited to braids.

For my visiting periods at the university of Caen I was supported by the ANR-Theogar and the GDR Tresses. During my stay I had useful discussions with Jean Fromentin, in particular about the handle reduction algorithm.

I also had the occasion to discuss with Aurélien Alvarez, co-author of the movie *Dimensions* [31], about the movie, and especially the target audience, the software and how to realize the project.

I would like to thank Juan González-Meneses for his helpfulness in answering my questions about braid cryptography and algorithms.

I thank my supervisor Domenico Luminati, who has always supported me. He is particularly interested in knot theory and gave me useful hints about the movie, in particular about the third chapter. He also had the idea to propose the workshop described in Chapter 3.

For many suggestions, especially about the balance between artistic appearance and mathematics, I thank Lilli Fragneto, a mathematician who realized *(é)stran(é)a*, a theatrical performance about fractals. Moreover, she had the idea to link each chapter with an object, to give some concreteness and visually identify the different themes.

I wish to thank Maria Dedò, director of the interuniversity research centre for the communication and informal learning of mathematics *matematita*, who pointed out some critical points of the movie and warned me of some difficulties.

Furthermore, in the development of the movie I was helped by Robin Scott who corrected some mistakes in the script and recorded the first version of the voice in the movie, Damiana Lerose who played the music and Silvia de Toffoli, who gave me comments on the earlier versions of the movie. I am particularly grateful to Roland van der Veen for his clever remarks on the content and the flow of the movie.

Finally, I had further suggestions from Alessia Cattabriga and John Guaschi, who proposed themes that were not included in the movie, although they are surely interesting and suitable for a popular-level narration.

Chapter 3

An activity on braid actions

We propose an activity to introduce the theory of braids in different contexts: we tested it with a group of students and in an event for popularization of science.

The activity is based on the interpretation of a braid as an automorphism of the fundamental group of the punctured disc, presented in Section 1.5.3. It consists of solving a practical problem with concrete objects, such as those depicted in figure 3.1: is it possible to put a closed string on the top of a braid, to obtain a given loop drawn at the bottom?

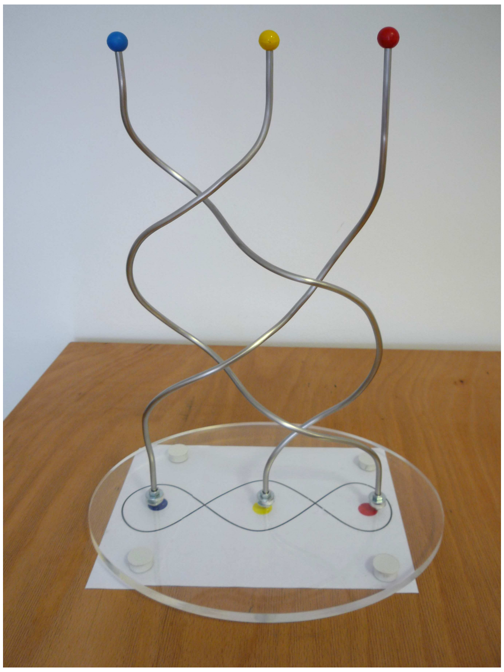


Figure 3.1: Is it possible to put a closed string on the top of a braid, to obtain the loop drawn at the bottom?

In a formal language, the question becomes: is there a loop in the punctured disc that is transformed into the given loop by the action of the given braid on the fundamental group of the puncured disc?

This problem was originally proposed in the exhibition *Matemàtica viva* organized by the *Associação Atractor* from 24/11/2000 to 30/08/2010. A picture of the exhibit and brief instructions to use it can be found at `http://www.atractor.pt/matviva/geral/modulo.html#61`.

3.1 Goals

The main themes of the activity, beyond braids, are the group structure, the fundamental group of a punctured disc and the concept of action of a group on a set. They are introduced in different ways and at different levels depending on the public. We give a description of the aims that can be achieved through a school workshop, bearing in mind that part of them are "high level goals", suitable for good students in the last year of high school or students in the first years of university. The goals have to be balanced on the context, especially in the cases when the same activity is offered to a more general public, or to younger students, or on a different occasion, such as an event for popularization of science.

The activity allows an informal introduction to questions and concepts typical of topology, such as isotopy and equivalence of objects. In this way, the public can approach a new field of mathematics that is usually not treated in schools and they can have an insight into its methods and applications.

Even if the starting problem has a topological nature, its formalization is algebraic and the solution is reached through algebraic computations. Thus, the public can see an example where a problem in a field of mathematics is formalized and solved using tools from another field.

The workshop can be an occasion to look for mathematical models to describe physical phenomena. Students already meet groups when studying planar isometries in Euclidean geometry, but the group structure is usually not completely formalized or even not pointed out in that occasion. Here the group structure arises in totally new contexts (braids, permutations, curves in the punctured disc): it is instructive to recognize similar patterns in different contexts.

The participants need to invent a new language to describe the new objects and work with them. The symbols may be the same used in other contexts (letters of the alphabet) but the meaning is different. Moreover, the groups encountered are not commutative: this forces the participants to make calculations with non numerical objects and non standard rules.

A very interesting aspect of the workshop is that it gives an example of the construction of a scientific model. Starting from observation of physical phenomena and a practical problem, the public develops a formal description using symbols. A theory has to be developed and used to make predictions on the observed phenomena. Finally the exactness of the predictions can be checked through experimental verification.

During the activity it is possible to introduce or recall the notion of algorithm, for example when associating words to braids or curves, or when solving the problem with arbitrary braids and curves. One can also point out the possibility of using computers to do calculations. In a school a further connected activity can consist in implementing a computer program to solve the problem for arbitrary braids and curves.

3.2 A workshop for students

We propose a workshop for students in the final years of high school. It is divided into three parts, focusing respectively on braids, curves and the action of a braid on a curve.

Here we list the objects needed for the workshop and describe the path to follow to solve the problem. Some materials for teachers, including worksheets to guide the activity, can be found in Appendix B.

3.2.1 Materials

This is a list of the objects we used[1] in the workshop described in Section 3.4.1. Some of them are not necessary and can be replaced by drawings. Some more objects can be realized to facilitate the activity, in particular a support to fix strings and contruct braids.

- A big metallic braid as the one in figure 3.1. A braid with three strands and four crossings has a sufficient complexity to allow interesting experiments.
- Some drawings of curves in the plane with three points marked, that correspond to the braid strands. Some curves are elementary, some more complicated and at least one is really complicated.
- Some plastic sheets with three marked points, where it is possible to draw with pencils and cancel.
- Some closed strings, with different lengths.
- A long string (about 3 meters), with open ends. It is opportune to ensure that the string is long enough to reproduce a curve at the top of the braid to obtain the desired curve at the bottom.

[1]We thank the Mechanics Service of the Department of Physics of the University of Trento for the realization of the models.

- A board with rectangular shape and three stripes cut out, so that the braid strands can fit into it. It allows to construct a curve in the horizontal plane at the top of the braid. See figure 3.2 for an example how to use it.

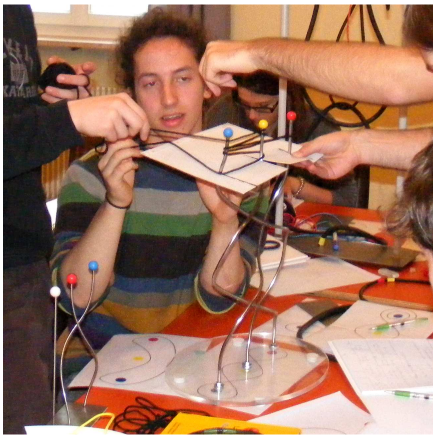


Figure 3.2: Using the objects to reproduce a curve: the board stands for the horizontal plane at the top of the braid.

- The elementary braids with three strands. In fact σ_1 and σ_1^{-1} are sufficient since turning them by a half turn they become σ_2 and σ_2^{-1}, so that all the generators and their inverses can be visualized. These objects are depicted in figure 3.3.

- A board with three long screws to be used with a string. This is useful to study curves in the punctured plane. See figure 3.4.

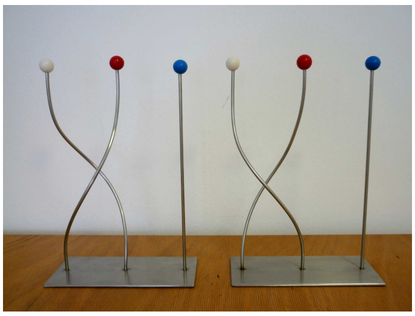


Figure 3.3: The elementary braids: σ_1 and σ_1^{-1}.

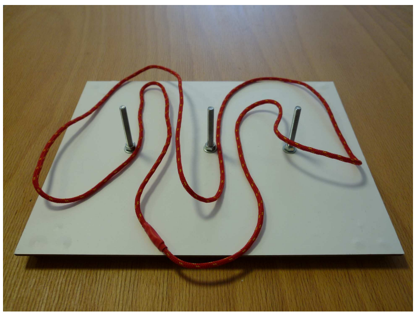


Figure 3.4: A board with three screws.

- Some objects to construct braids and visualize their composition, such as two metallic plates fixed on a support and at least four gum strings (possibly having different colours) with magnets at the ends, to construct braids. To visualize composition of braids, we used a loose metallic plate and some short gum strings. See figure 3.5.

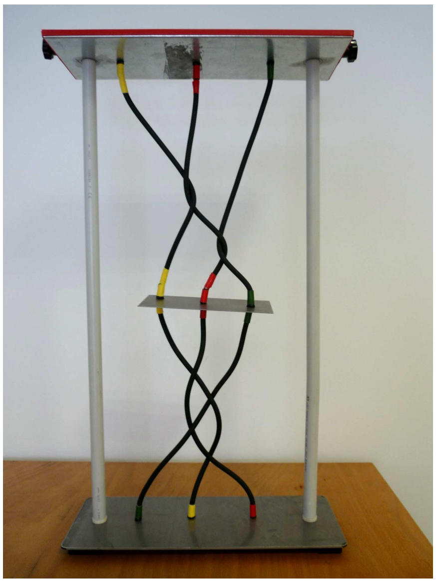


Figure 3.5: The metallic plates to visualize braids and their composition.

3.2.2 The main problem

On the table there are the big braid, the drawing of a curve (figure 3.1) and some closed strings. We begin the activity by submitting the students the main problem:

Is it possible to put a closed string around the braid strands in such a way that, pulling it down, it will overlap with the curve drawn on the bottom?

The students are free to give as many tries as they like, possibly changing the curve on the bottom. In some cases it is immediate to find a solution, in other cases it needs a little work and sometimes it seems to be very difficult: attempts, intuition and imagination are not sufficient any more. It can be surprising to note that in some cases the string has to be very long to reach the bottom, otherwise it will be blocked by the braid strands. Or vice versa, the curve at the bottom can be very simple, but to obtain it, the string at the top has to be very long and complicated. A solution is not too difficult to find if the curve is not too complicated: it is sufficient to consider the pieces of the curve in turn, and place each one around the desired braid strand. However, it is evident that a formalization is needed: in particular a way to describe braids and curves and some tools to calculate how a braid acts on a curve.

3.2.3 Part 1 – How to introduce braids

Now a discussion with the students may be opportune: they are asked to give a definition of braids, looking at the big metallic braid as an example of the object they have to formalize.

Moreover, they have to distinguish different braids, noting that the geometry as rigid objects is not to be considered, but only the way in which the strands are interlaced. In fact it is natural to consider two braids the same if their strands are braided in the same way, i.e. if one braid can be continuously deformed into the other, by keeping the endpoints fixed. This is probably the first approach of the students to topology.

Different physical objects can be used to construct braids. For example, the object in figure 3.5, made of two fixed metallic plates that can support the gum strings with a magnet at the ends. Also a coat hanger or a similar object can be used as support for some strings.

Now the students are ready to handle the second problem:

Is there a way to encode the necessary information to reconstruct a braid?

A more involving way to introduce this problem is the following.

Imagine you have a braid and you are talking at the phone with a friend who wants to reconstruct the same braid. How can you describe it?

The aim of this part is to find the generators of the braid group and to start working with words in the generators. The activity may become an amusing game for the students if they are divided into two groups, each of which has to construct a braid, find a word to encode it, communicate it to the other group and reconstruct the braid chosen by the other group.

It is quite natural to note that the fundamental bricks to construct the braids are the oriented exchanges of positions of two adjacent strings. When the generators have been detected, it is immediate to give a name to each of them and to write a sequence of generators, a word, to describe a braid.

It is now important to highlight the operation of composition: composing two braids means to put one after the other. In the algebraic notation, this means to write a word after the other. This has been implicitly made by the students, who composed the generators to obtain a braid. The next part deals with the properties of the composition.

Is it true that composing a braid β with a braid γ gives the same braid as composing γ with β?

We invite the students to make some attempts with elementary braids such as the generators of the braid group. After a while they will find that composing two generators σ_i and σ_{i+1} in the two ways gives different braids. They will probably say that the braids $\sigma_i\sigma_{i+1}$ and $\sigma_{i+1}\sigma_i$ are evidently different. It is time to require a more formal argument to prove this. A proof involving the permutation induced by a braid is elementary.

Is it possible to find a braid ε such that composing it with any braid β gives β again?

It is quite natural to note that the braid where all the strands are straight does the job. Some perplexity may arise because this is commonly not considered as a braid, since the strands are not braided. It can be necessary to remark that following our definition, this is a braid, even if it is trivial.

Given a braid β, is it possible to find a braid γ such that composing β with γ we obtain the trivial braid ε?

The students can use the short gum strings with magnetic ends and connect them to the metallic plates to create a braid. Using two fixed and one mobile metallic plates it is easy to construct a braid on the top and its inverse on the bottom. The fact that the braids are inverse one to the other can be easily seen by removing the central plate: pulling the strands gently, we can see the isotopy transforming the braid into the trivial one (some attention has to be put on the polarity of the magnets, to be able to connect the string ends).

This experiment leads to note that to construct the inverse of a braid it is sufficient to "look at it in a horizontal mirror". This can be easily translated in the algebraic notation. The students may need to make some practice inverting some words.

The students should note that a left inverse is also a right inverse for a given braid and that the inverse is unique, in the sense that two braids that are inverse to a fixed braid can be deformed into each other.

3.2.4 Part 2 – Curves

Imagine to put a horizontal plane at the top of the braid. Consider the closed curves in the plane that do not jump over the braid strands. Which curves can be considered equivalent?

The students will soon note that they have to study curves in the plane with three points removed.

Observing some curves, it is not difficult to point out some characteristic properties: how many points they encircle, how many self-intersection they have, how many times they wind around a point.

Some experiments can be done with closed strings and the board with three screws bending out from it (figure 3.4). Recalling the isotopies among braids, the students will consider two curves as equivalent if they can be deformed into each other without intersections among themselves and with the three points.

A tricky part is to understand that the curves in the plane do not behave exactly as the strings in the space. With strings, it is not always possible to undo two crossings forming a "bigon", while this "move" is always allowed in the case of planar curves (see figure 3.6).

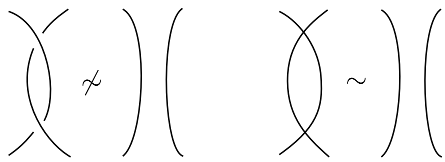


Figure 3.6: On the left, a diagram of curves in the space: the bigon can not be simplified. On the right, curves in the plane: the double points are intersection of the curve and the bigon can be removed.

How can we describe the curves in terms of elementary ones?

This point is maybe the most critical one and requires the intervention of the teacher to help the students. As a matter of fact, it is convenient to consider oriented curves with a fixed base point, that is not obvious in the original problem.

The necessity of considering oriented curves is not self-evident at first: the strings winding around just one point can be described by the same curve with the two possible orientations. Thus oriented curves can seem to be a complication of the model. But a simple example can convince the students of the contrary: the curve winding clockwise around a point and anti-clockwise around another has a self-intersection, while the curve winding around the same two points with a fixed orientation can be represented without self-intersections (see figure 3.7).

Figure 3.7: These two curves are the composition of the same generators, but with different orientations (the curve on the right is to be interpreted as two loops with opposite orientation).

Understanding the necessity of a base point can be even subtler. At first it will probably not be noted and a description of curves without base point will be proposed. The students can then be invited to describe some "simple" curves, those that enclose two points. They will note that the curves enclosing two non-adjacent points can pass "over" or "under" the point in the middle (figure 3.8). However, using a notation that does not take into account a base point, these curves may be described by the same word.

Figure 3.8: These curves are not equivalent. To distinguish them a base point for the loops is needed.

Then, the teacher can ask what "over" and "under" means, and how it is possible to mark one half plane (one of the parts of the plane delimited by the line through the points) to distinguish it from the other one. With some hints, the students will agree that fixing a point in one half plane solves the problem. Now we will require that all the curves pass through that point.

With the help of the board with the screws and a string, the students will understand that each curve can be decomposed into elementary pieces. However, to obtain the generators of the fundamental group (figure 1.20), they may need some hints and help from the teacher.

Finally, the students will fix a symbol for each elementary curve and write any curve as a word in those symbols.

3.2.5 Part 3 – Action of braids on curves

How does a braid act on a curve?

It is important to note that the problem can be decomposed: the students just noticed that each curve can be seen as the juxtaposition of some elementary curves. Acting on a curve with a braid is the same as acting on each elementary curve and then juxtapose the results. Similarly, acting with a braid is the same as acting with a generator after the other. Thus, it is sufficient to calculate how each elementary braid acts on each elementary curve and compose the results.

The students can make experiments using the metallic elementary braids (figure 3.3) and the strings. They will fill in a table reporting the action of each elementary braid on each elementary curve.

Now they can calculate how a curve is transformed by the action of a braid. They can verify the correctness of the computations through a practical test using the metallic braid and a string.

Can we now solve the problem?

The students have to note that they can solve the equation $\beta(\sigma) = x$, where β is a braid, σ is a curve and x is the unknown curve. But the original problem is the inverse: it asks to solve $\beta(x) = \sigma$. They will note that it is sufficient to compose with β^{-1} and make analogous computations as before.

The most satisfactory part is to check practically the result and see that the outcome of the experiment fits together with the predictions derived from the mathematical model and the computations.

Here the objects not yet employed can be useful: a very long string and the board with three cuts can help to construct the curve on the top of the braid.

3.3 A popularization event

The problem of the action of a braid on a curve can also be proposed as a mathematical puzzle in an event for the popularization of science, addressed to the general public.

In this case, the aims of the activity are obviously reduced. They include conveying the idea that mathematics can be hidden inside various objects, that mathematics is not only numbers and computations but deals with recognizing patterns and finding structures. Further, one can give a rough idea of the formalization of braids as mathematical objects and of the concept of action.

The objects needed to propose the activity in this context are the big braid of figure 3.1, some strings and drawings, and the small braids of figure 3.3. The activity can be organized such that the audience can just try the puzzle as a game, or ask for a brief explanation about the mathematics behind it. The explanation of the puzzle is different according to the people trying to solve it.

For children, the exhibit is just a game with some mysterious mathematical background: the ideas that they can receive from such an object is that mathematics can be hidden inside various objects and that the subject can be very interesting and pleasant as a game.

High school students and adults can receive an explanation more similar to the content of the workshop, conveying the concepts of mathematical formalization and of action. It is quite natural for the public to find the elementary braids if they are asked “can you describe this braid?”. Then remarking that one can associate a symbol to each elementary braid and think of composition as an operation, a first formalization of braids can be understood by almost everyone.

A second step is to explain a similar procedure with curves. In this case, the public is not asked to find elementary curves, because it is not so immediate as in the case of braids.

Then, the central idea of an action can be given, pointing out that this gives a new characterization of braids: two braids are the same if and only if they act in the same way on every curve. But in fact, it is sufficient to check how each elementary braid acts on each elementary curve. Finally, it is possible to point out that one can write an equation with a curve as the unknown and solve it dividing the problem into the little pieces that one is able to solve.

Of course, the explanation can be broadened and deepened if the audience is interested. Some of the concepts and the explanations exposed in Section 3.2 can be narrated in a more informal way.

3.4 Testing the activity

The two activities proposed in this chapter have been tested each on one occasion: the workshop was proposed to a group of students on the 21-24 March 2011, while the popularization activity was included in the event *Researcher's night* on the 23rd September 2011. The workshop and the exhibit were proposed in collaboration with Domenico Luminati and the interuniversity research centre for the communication and informal learning of mathematics *matematita*. We comment the activities and the results.

3.4.1 The workshop for students

The activity took place within a residential workshop called *La bottega del matematico*[2]. The basic idea of the event is to reproduce an "atelier", where the students work autonomously but in close contact with a "master". The aims of the workshop consist of giving more education chances to good students and experiment new approaches to the didactics of mathematics and to teachers training. Some more information and reports on the workshops organized in the past are contained in [11].

The 2011 workshop was directed at 26 students, invited by their teachers because of their interest in learning and their good results. The students were divided into three groups, each working on a different theme.

The work began in the late morning of the first day and continued for four days. In the evening of the last day the students gave a talk about the problem they had worked on in the previous days, presenting their results to the other students and teachers. In total, they spent about 30 hours working on the problem and preparing the final presentation.

We report an analysis of the students' work from an educational point of view and some comments about the way they found to develop the mathematical content.

[2] *La bottega del matematico* is organized and financed yearly by the *Intendenza scolastica italiana* of the *Provincia Autonoma di Bolzano – Autonome Provinz Bozen* and is addressed to a group of students attending the final year at the high schools of Italian language of the province and to some teachers of the same schools.

The students did not expect to face a concrete problem. At the very beginning they were not aware of getting into a mathematical problem, which could make them think it is boring or too difficult. At first they took it as a game and were soon curious and interested. In this way they were reassured: they were not expected to know concepts but to give themselves a challenge. All were soon involved, making attempts, comments, questions and conjectures.

The students started to know each other and to work together as a group, making something practical and listening to the ideas of the others. Every one gave at least a try, in turn or together, to manipulate the objects. This enhanced cooperation and gave the basis to work well as a group during all the four days.

Even if the students were able to solve the concrete problem after some tries, they bumped into practical difficulties: in some cases the rope was too short, in other cases the curve was too complicated. They realized that a formalization could be helpful and were curious to discover it.

They began to realize that they would have to work autonomously and would not always receive answers but questions. Finally, the approach to mathematics, completely different from that in school, was perceived as something new and interesting, even if somebody was a little sceptical at the beginning.

Each student could contribute with his own ideas and skills. The cooperation was excellent and the students demonstrated to be very independent, both in the resolution of problems and in the preparation of the final talk. These skills were evident in more than one occasion: students cooperated well, the ones having more intuition gave new ideas, the most rigorous ones asked for more precision from their mates, the most practical ones preferred to handle the objects to make concrete attempts. While preparing the final talk, some of the students prepared posters, some prepared a beamer presentation, while some wrote down and formalized what they had learned.

The teacher who was present during the whole workshop was also very impressed by the students and could learn something about a field that he did not know before. He showed his interest in the workshop, so that the students took the problem seriously and were proud of their work.

We briefly comment the students' activity related to the mathematical content.

They first defined a braid as a collection of strands with endpoint on two horizontal planes, such that each horizontal plane intersects each strand once. Later they noticed that this was the same as to require the strands to be monotone. One of the students was not satisfied with this model and wanted to parametrize the strands. Only after some hours work and some more formalization he became convinced that our problem was topological and not geometrical and that our formalization made sense.

The students used the additive notation to describe both braids and curves in terms of the elementary objects. The generators of the 3-braid group were symbolized by capital Latin letters A, B, while their inverses were preceded by a minus sign: $-A, -B$. It was almost immediate to write an inverse of a given braid, while the notation for the trivial braid was not clear at first.

The description of curves in terms of elementary ones was the most difficult part of the workshop. The students proposed a notation that included symbols for the loop around a point and for an arc passing over, or under, a point. For example, the curve in figure 3.9 would be described as the composition of a loop around the first point, an arc over the second point, an arc over the third point, an arc under the third point and a loop around the second point.

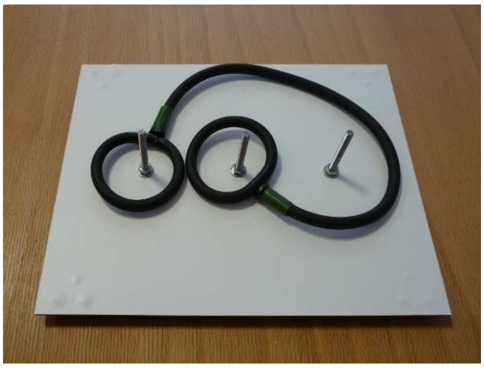


Figure 3.9: The original idea for the decomposition of a curve in terms of elementary ones. It is a possible solution but it makes computations more complicated.

The idea works, if one adds orientations of the loops, but the notation is not simple and would make computations more difficult and long in the next part, when the action of a braid on a curve is considered. We explained that this notation with many symbols is proper, but not very efficient and asked for a simplification. There was a long moment of perplexity because the students did not have new intuitions, until a student, who was handling the board with the screws, started to pull a string with the fingers. With some hints, he realized that the string could be put in a form such that it passed in a base point, encircled a point, went back to the base point, and so on. Then, the students decided to associate Greek letters (eventually preceded by a minus sign) to the loops around a point, the standard generators of the punctured disc.

After this tricky part, the work went on easily. For the students it seemed quite natural to make computations without using the commutative property, even if the notation they had chosen is usually employed for commutative operations.

They filled in a table containing the result of the action of the elementary braids $\pm A$, $\pm B$ on the elementary loops $\pm\alpha$, $\pm\beta$, $\pm\gamma$. A picture of the table is in figure 3.10. Later the students simplified the table, removing the actions on negative loops, because they noted that these could be obtained inverting the curves that are image of the positive loops: for example $A(-\alpha) = -(A(\alpha))$. In fact inverting a loop just means to change its orientation.

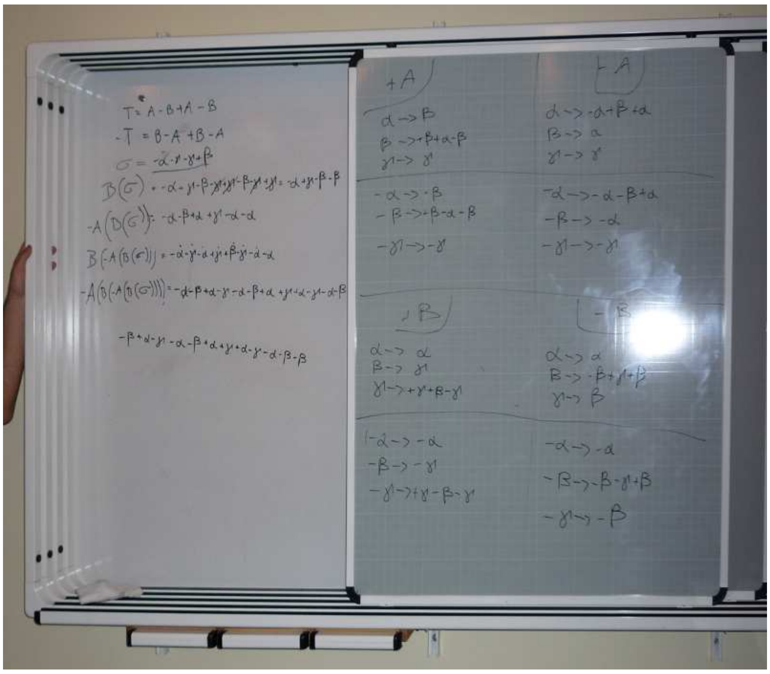


Figure 3.10: The computations performed by the students. On the right side of the board the action of each elementary braid ($\pm A$, $\pm B$) on each elementary loop ($\pm\alpha$, $\pm\beta$, $\pm\gamma$) is reported. On the left side, these actions are used to calculate which curve has as image the curve σ under the action of the braid T.

The students had no difficulty in computing the action of a braid on a curve, using the table. The use a right action caused no problems or mistakes. Neither did the inverse problem of finding a curve x having a given image σ under the action of a braid T. The students wrote it in terms of an equation $Tx = \sigma$, which they soon rewrote as $x = -T\sigma$. This allowed them to find the unknown curve.

In the experimental verification the students were able to see that their model worked. This gave them a great satisfaction. However, they had to do more than one try, because the curve was quite complicated and at first they failed in putting the string in the right way around the strands.

We made some final remarks about the activity, giving some ideas of other problems connected to braids, such as the word problem and the closure of braids. In particular, when we asked the students if they had other ideas for a possible different definition of braids, one of them conjectured that two braids are the same if and only if they act in the same way on all the curves. This is an intuitive notion of representation.

3.4.2 The exhibit for the general public

The materials were used as an exhibit in the popularization event *La notte dei ricercatori* (*The researchers' night*) on the 23rd September 2011. During this event a stand was set up with some mathematical puzzles for the general public, not only involving braids or topology, but different areas of mathematics, to give a wider idea of the subject.

The public in this occasion was very variegated: children, young people and adults.

The use of the braid as a puzzle in this context had a good success: many people tried to solve the practical problem. This was a chance to say a couple of words about braids. Some people were interested and asked questions, while others just wanted to play with the objects.

Appendix A

The movie script

Chapter 1 - The group structure

This movie is about braids and mathematics.
Everyone knows what a braid is.
Braids are everywhere: in hairdressing, in jewellery, in leather belts, in ropes, in cakes, in cheese, in bread and in many other objects. Braiding is one of the oldest ways to decorate objects.
And in mathematics? What is a braid? Why do mathematicians study braids? And how?
Take a disc with n points inside it. Make a parallel copy of it and start drawing strands that connect the marked points.
The strands can not turn back.
The strands can be braided and linked together but two strands can not pass in the same point: intersections are not allowed.
Draw strands until all the points are connected by them. This is a braid.
For simplicity we will not draw the discs and the points any more, but only the strands.
In this way we can construct a lot of braids. Braids with two strands. Very simple ones, and longer ones. Braids with three strands, with four strands, and so on, with any number of strands.
How can we recognize when two braids are combed in the same way? The only thing that matters is the way in which the strands are linked together.
For example, these two braids are the same: we can deform one into the other, keeping the endpoints fixed and not letting the strands cross.
So, we can represent one braid in many different ways. All these braids are obtained deforming the original one keeping the endpoints fixed. All these are equivalent braids.
Still, there are braids that can not be deformed one into the other. So we say that there are different equivalence classes, formed by the braids that can be transformed into each other.

To determine a class we can choose any of its representatives.
When we draw a braid, we do not mean just that particular representative but all the braids it represents.
How can we choose the representatives for each braid?
Further, if we draw two braids, how can we know if they are equal or different? For example, these two braids are the same. We just need to move the blue strand.
But what about these two? Are they the same braid?
And if we have more complex braids?
To answer, we try to find a mathematical structure on the set of braids.
The first thing we note is that we can compose two braids. This just means to put one after the other and connect the strands.
Now, if we have two representatives of the same braid and two of another braid, composing them and making some deformations we see that we get two representatives of the same braid. The result does not depend on which representatives we choose for our braids.
The composition of braids is an operation, like the product of positive numbers.
The product has some beautiful properties: it is associative and commutative. Do these properties hold for the composition of braids?
Here we compose three braids in the two possible ways keeping the order. Of course we get the same braid. So, associativity holds and we can write any number of subsequent compositions without using parentheses.
What about commutativity? Here is an example: we compose two braids in the two possible ways. Look at the red strand, starting from bottom left. It arrives in different positions on the right. So the braids can not be the same. Thus commutativity does not hold.
Thinking of the product again, there is a neutral element, one. Is there such an element for the composition of braids?
Take the trivial braid, where all the strands are parallel. When we compose any braid with it, the result is equivalent to the original braid.
Any number has an inverse with respect to the product. Is there an inverse for a braid?
Look at the braid in a mirror. We can compose these two braids in two ways: first the original braid and then its mirror image, or vice versa, first the mirror image and then the original braid. In both cases, we can simplify them and we get the identity braid. So, for every braid there is an inverse.
So far we have seen that on the set of braids we can define an associative operation. This has a neutral element and for every element in the set there is an inverse. Such a structure is called a group.
To deal with braids, it can be useful to find a way to associate a word, a sequence of symbols, to each braid. Let's begin with the simplest braids.
In the identity braid nothing happens: we associate to it the symbol 1 because this is the neutral element for the composition.

Now consider braids where only two strands are linked together and the others are straight.
When the first two strands are exchanged by a clockwise twist looking from the left, the braid is called σ_1.
In σ_2 the linked strands are the second and the third.
And here is σ_3.
If the strands twist in the other direction, we will have σ_1 inverse. It is called so because it is actually the inverse of σ_1.
This is σ_2 inverse, the inverse of σ_2.
And σ_3 inverse, the inverse of σ_3.
Using these elementary braids as bricks, we can construct many different braids. Here is an example: we put a brick after the other and write the corresponding word.
And vice versa? Can we describe any braid using the elementary braids?
For example, take this braid. It looks quite complicated. But we can deform it. Then, we can cut it into levels. Each level is an elementary braid, with just one crossing. This procedure works on every braid, so to any braid we can associate a word.
Now, there are different words representing the same braid.
For example, an elementary braid and its inverse cancel out when they are side by side. We can replace them by the identity braid.
And we can cancel out the trivial piece.
Vice versa, we can insert in any place an elementary braid followed by its inverse.
And there is more: here the blue strand passes between two strands that form a crossing. Here two distant crossings exchange their positions.
All these movements are easily translated into manipulations of the words.
Surely there are other movements: here the yellow strand passes over a crossing. Here we cancel out an elementary braid and its inverse even if they are distant.
How can we find all the possible manipulations of words? It seems an endless job.
Don't worry, mathematicians have already solved this problem. The first mathematician who wrote a paper about braids was Emil Artin. The paper was written in German in 1928. Artin wrote a second important paper in 1947 in English.
Emil Artin was the first who described the structure of group on the set of braids and noted that the group can be described in this way.
This mathematical notation means that any braid can be described as a word in the σ_i's and their inverses.
In every group we have these reductions. They are local: they change just a little part of the word, keeping the rest fixed.
The only other substitutions permitted in a word are of two kinds. We already saw the corresponding moves: moving the blue strand through a

crossing and exchanging the left and the right crossing.
With these two moves we can generate all the words that describe the same braid.
We have turned braids into mathematical objects.

Chapter 2 - The word problem

We saw in the first chapter a way to describe the braid group: we specified some elementary braids, the generators. Then we found the relations, moves that can transform a word without changing the braid it describes.
But... we still don't know all about this group. For example, the relations don't say how to check if two words are equivalent.
Here we have two words and we construct a sequence of relations that connect them. At each stage we just replace the part of the word inside the yellow box with an equivalent one. We use the two kinds of relations and insertion or deletion of a generator and its inverse. In each level we change the word, but the braid it represents remains the same.
It is not so easy to note immediately, just looking at the two words, that they are equivalent. This becomes evident when we construct the sequence.
What about these two words? We try to construct a sequence as before to connect them.
It can be a hard job to construct such a sequence of words using the relations. We could spend our whole life looking for one, without ever finding it. We will never know whether no sequence exists or we haven't been lucky enough to find it.
That's why we need another way to handle the question. More precisely, we wish to construct a machinery that takes two arbitrary words as input, makes some operations and computations and gives as output the answer "yes", if the two words are equivalent. Following the same procedure, the answer will be "no" if the two words represent different braids. Such a procedure is called an algorithm.
We are lucky: for braids we have such an algorithm. Let's look inside it.
First of all we simplify the problem: two words represent the same braid if and only if one, composed with the inverse of the other, is equivalent to the trivial braid. Therefore, we can reformulate our problem: given an arbitrary word, decide if it is equivalent to the identity. Or in other words: is there a way to transform the given word into "1"?
Look at this braid: the colour of the bubbles is enough to know where each strand ends. If the bubbles at the same level have the same colours, then the braid is called pure.
Of course two equivalent braids will have the bubbles with the same colour arrangement. So a braid that is equivalent to the trivial one has to be pure.
Each pure braid on n strands can be decomposed into n-1 blocks in this

way: in the first block, all the strands but the last are straight, the green strand can link to the others. in the second block, all the strands but the second-to-last are straight. The yellow strand can link only to the strands below. And so on... until the last block, in which the second strand can link to the first one.

We know that for each braid there are lots of different words to represent it. We would like to choose a specific one of them to describe the braid. It will be called the normal form.

Now, we need an algorithm that puts any arbitrary word in its normal form. Artin's braid combing is such an algorithm. Here we show it on a non trivial braid: first copy the braid and delete the last strand, replacing it with a trivial one. Compose this braid with its inverse, getting the identity braid. So, composing the original braid with it, we don't change the braid. On the left side we have a braid that has the desired form: all the strands but the top one are straight. This is the first block.

We deform the strands so that the green strand encircles a strand, passing beyond the other strands, goes back to the initial position, encircles another strand, and so on. This piece is combed.

Do the same procedure on the right side: copy the braid, replace the yellow strand with a trivial one, compose this braid with its inverse, getting a trivial braid, compose this trivial braid with the original one and tidy up the second block.

We have to put it in a form such that the yellow strand links only with the blue one and the red one. We have combed the second block.

Now the last block is easy to rearrange.

We have put the braid in its normal form.

Artin proved that a braid is trivial if and only if, once it is combed, every block is trivial, meaning that it can be represented with no crossings at all. So our braid is not trivial.

And we have an algorithm to check it. Of course the braid has to be pure. Comb it into blocks. If one of the blocks is not trivial, terminate, answering "no". Otherwise continue with the next block. If all the blocks are trivial, we get the trivial braid and the answer is "yes, the input word is equivalent to the identity".

This algorithm works but Artin himself was not satisfied with it. In his paper *Theory of braids* he writes: "Although it has been proved that every braid can be deformed into a similar normal form the writer is convinced that any attempt to carry this out on a living person would only lead to violent protests and discrimination against mathematics. He would therefore discourage such an experiment."

Artin wrote this because combing is an extremely slow method... The complexity of the algorithm is very high: if we draw a graph showing the mean time needed to comb a braid with a fixed number of crossings, the curve is very steep. Increasing the input size by a small amount, the time

grows very fast and may be too long even for a computer. This is why mathematicians look for other, quick, algorithms.
The fastest found so far is probably the one proposed not many years ago, in 1997 by Dehornoy, a French mathematician. It is based on manipulating braids and is called handle reduction.
This is a handle: the red strand passes in front of the strand just above and then in front of the strand just below.
A handle can also pass twice behind the other strands: this is another handle.
This is not a handle because it faces downwards. Neither this is a handle: the blue strand passes once in front of the strand below and once behind.
Reducing handles means to move these pieces of string. In this way we change the word, but the braid it represents remains the same.
A theorem ensures that a word with no handles is either empty or it represents a non-trivial braid.
Let's sketch the handle reduction algorithm: take a pure braid. If there are no handles and the braid is not trivial, terminate, answering "no". Otherwise, if there are handles, find the one that ends first and reduce it. Continue like this until there are no more handles: if the braid is not trivial, the answer is "no". If after reducing all the handles, the braid is trivial, then the answer will be "yes".
But there is a problem: what if the algorithm does not end? It could maybe go into a loop: if it comes back to a word that it has already met, the steps after that one will always be the same and the algorithm will never stop. In fact this is never the case: it can be shown that the algorithm always terminates.
We don't yet know the complexity of this algorithm. But we have some experimental estimates, and it seems much faster than the other known algorithms.
Combing, handles, the formalization of braids in terms of words is really powerful!

Chapter 3 - The world of knots

Let's change topic for a while.
These sailors' knots... are not knotted! The knots can "escape" from the string ends. To capture the knot, we have to glue the string ends together. Now, these are knots.
Closing the ends of the objects on the right we get more complex knots, where two components can be linked together. We call these objects links. If we can untie them, we obtain a very simple link, called unlink.
So, sailor knots are not good for mathematicians and vice versa, mathematical knots are not good for sailors!
Look at this knot. If we take its mirror image, it looks different. But we

can deform the knot on the left into the one on the right without cutting the string. Then, we will consider them the same knot even if they look different at first.
These two knots are the same too. What does it mean? They are tied in the same manner, in some sense. They have different shapes but represent the same knot. One can be deformed into the other, without ever cutting the strings. This knot is called trefoil.
As before, we can ask: is the trefoil the same as its mirror image? In fact, this is a very technical and difficult problem. Actually, the two knots are not the same. They are called left-hand and right-hand trefoil. We will show that they are different using a machinery invented 30 years ago that revolutionized the study of mathematical knot theory.
A way to approach such a problem is to relate the realm of knots, closed strings, to that of braids.
It is easy to see that when we have a braid, we can tie the strand ends together. And we can cross the bridge to the realm of knots.
And vice versa? Can we cross the bridge in the other direction?
A theorem of Alexander ensures that it is possible. Sometimes it is easy: we just need to "open" the knot. But in general we need an algorithm to do it. We describe the one of Alexander even if more efficient ones are known.
We choose an axis and a starting point on the knot. We walk along the knot turning anti-clockwise around the axis. At some time the knot can turn and we will be walking clockwise. We colour all the anti-clockwise pieces red. Now we move each red piece in turn to the other side of the axis. Sometimes we may need to move just a little part of a red piece. In the end we have turned our knot into a reel around the axis. Walking along the knot, we will always be going in the same sense around the axis.
We take a half plane with the axis as border and cut the knot along it. We open the strands keeping the endpoints fixed on the half planes... And here is our braid!
When we close it, we get a knot equivalent to the original one, that is, we can deform one into the other.
Why make life so difficult? The knot on the left seems simpler! But in this way we can exploit the group structure that we know on braids.
Alexander's theorem ensures that we can obtain any knot as the closure of a braid, but two braids can be very different and still give the same knot. For example, they don't even need to have the same number of strands!
So the question now is... given two arbitrary braids, do they give the same knot, once they are closed?
We introduce a new operation, called conjugation: choose a braid. Take another one and its inverse and compose them in this manner: one on the left and the inverse on the right. The new braid is called a conjugate of the first.
Note that the corresponding operation with numbers will not change the

starting number: the product is commutative.
On the contrary, two braids can be different and still be conjugate.
Here is a simpler example: two generators of the braid group are surely different braids. But look: they are conjugate.
In general understanding whether two given braids are conjugate is an intriguing issue.
Let's go back to our problem: when do two braids close to the same knot?
If we conjugate a braid with any other, when we close the new braid, we can shift the lateral pieces so that they cancel out, since one is the inverse of the other. In this way we get the same knot as closing the original braid.
We can modify our braid in another way: add a strand on the top and link the two top strands together. The new braid, when closed, will give the same knot as the old one: we just need to undo the loop. This operation is called stabilization.
Of course stabilization can also link the two strands in the other way and nothing changes.
A Russian mathematician, Markov, noticed that two closed braids give the same knot if and only if they are related by a sequence of moves of the two kinds we have just seen. This is now known as Markov theorem, even if the first proof is probably due to one of his students.
We didn't show the difficult part of this theorem, namely, that the two kinds of moves are enough. We just make an example. we already know that these two braids give the same knot. Now we can prove this, without passing through the realm of knots! We have to find a sequence of conjugations and stabilizations that transforms one braid into the other.
Markov theorem exactly says when two braids give the same knot, but in this form it is of no concrete use: finding a sequence of relations can be very difficult. Approaching knots through braids seems not to simplify things. Then, why did we introduce the closure?
In 1984 Jones, studying braids, proved an outstanding result on knots! It was so important that he won the Fields medal for it, the most important award for mathematicians.
Namely, Jones found a way to associate a formula, a mathematical expression, to each knot. The powerful fact is that this permits us to distinguish many knots: if they have different formulas, then they are different.
Later another algorithm to calculate the so called Jones polynomial was discovered, not involving braids any more. We present it to visualize what happens.
As before, choose a direction to walk along the knot. There are places where we see a crossing. The crossings can be of two types depending on the strand that passes over. Resolving a crossing means to break the arcs and connect them in the other way, respecting their orientation. Introduce a relation between these pieces. The symbol V indicates the Jones polynomial. Now,

associate the polynomial 1 to the unknot.
Using just these two relations, we can calculate the polynomial on every knot. Choose a crossing and apply the first relation to it. simplify... and apply the first relation again to the knot on the right.
In this way we write some equations. Sometimes we can simplify, and using always the same relations, we can calculate the Jones' polynomial of the simplest knots. Then, going back step by step, we can reconstruct the expression for the trefoil.
We didn't check that this machinery is coherent, that is, making different choices always gives the same expression for a fixed knot. This is the difficult part, and the interesting one: the Jones' polynomial is an invariant of knots: calculated on two equivalent knots, it is the same.
If we calculate the Jones polynomial on the mirror image of the trefoil, we obtain an expression that is symmetric to the other, in some sense. But not equal. So the two trefoils can not be equivalent! Even if one is the mirror image of the other!

Chapter 4 - Hilden dances

Look at this dance.
Note that each dancer arrives in a coloured position after each piece of the dance. So, at the end, they occupy the same positions as at the beginning of the dance.
How can we keep track of the movements? and make a static drawing that describes the dance?
Let's formalize the dance. Turn the dancers into points and let them move in the disc.
At each moment the points are distinct. The dance fixes the chosen points as a set: this means that the points go back to their initial positions, but possibly they are permuted.
If we draw the path described by each point on the disc, we see some intersections and we can't reconstruct the dance: we don't know who passed in that intersection first. The lines can make a very complicated drawing.
A solution is to move the disc while the dance is being performed. In other words, we transform time into a spatial dimension, in this case the vertical direction.
Of course in each moment, we have the same number of dancers. This means that at each level a horizontal disc will meet each path exactly once.
We have already seen diagrams like this: this is a braid! Following our convention, we should align the braid horizontally. But just for this chapter we will change notation and draw it vertically.
Vice versa, given any braid, we can turn it into a dance.
In other words, we now have a description of braids as the trace of dancing

points.
Now let's consider some particular dances where dancers are in pairs, always holding hands. As before, all the positions held in the beginning have to be filled at the end of the dance. This time, a pair of dancers becomes an arc. The arcs move on the disc.
They can turn, exchange positions and pass one over the other.
As before, we can translate the dance into a braid.
Of course, the non-dance, where all the points stay still, is of this type. So, the identity braid is a dance of couples.
Performing a dance of this type after another corresponds to composition of braids. We get a new dance of couples. In other words, composing two braids in this set, we stay in the same set.
Then we say that braids describing dancing couples form a subgroup of the braid group. It is called the Hilden subgroup.
There are some simple dances in the subgroup: the two dancers within a single couple can exchange positions, two couples can exchange positions, a couple can pass under the arms of another couple.
Many other dances can be performed, but in fact the three movements just described are sufficient to assemble all Hilden's dances. In other words, these are the generators of the Hilden subgroup. Finding the relations, as in the case of the braid group, is much more complicated.
What is all this good for?
Recall that in chapter 3 we described a way to relate braids to knots, via the closure. When we close a braid, we are actually adding new trivial strands and connecting each old strand with a new one, obtaining a knot.
If we move the new strands behind the old ones, we obtain a different braid, closed in a new way. This closure can be done on each braid with an even number of strands. It is called the plat closure.
Again, closing a braid as a plat we obtain a knot.
Recall the Alexander theorem: any knot can be obtained by closing a braid. We saw how to cut a knot to obtain a braid. From the closed braid we can obtain a plat, thus any knot can be obtained as the closure of a plat.
As before, we would like to understand when two closed plats give the same knot. We can do some simple moves: we can compose our plat with Hilden elements both on the bottom, or on the top. Moreover, we can add two strands and a crossing. And go on like this, using Hilden elements and the new stabilization move.
When we close the plat, the knot type will not change: we can retract the arcs and obtain the initial knot.
Birman showed that these moves are enough: Two plats give the same knot if and only if they can be turned one into the other through a sequence of moves of the two types: stabilization and composition with Hilden elements. So the movements of dancing couples is one of the main ingredients of this theorem.

Now a question arises: can we recognize the elements of the Hilden group?
For example, this is a Hilden braid.
But in general, given a braid with an even number of strands, how can we check if it belongs to the Hilden subgroup? This is called the membership problem and is one of the classic problems in combinatorial group theory.
Again, we have bumped into an algorithmic problem. And again, we have found a connection with knot theory, this time using plat closure!
And all this came from some dancers...

Appendix B

A proposal for teachers

Before offering some materials that can be useful to propose the workshop in a class, we give some grounds why a workshop can be instructive for students.

B.1 Motivations of proposing a workshop

In the last years mathematics teachers note that there are some critical issues in teaching and the workshop methods can be a partial answer to them.

One of the main issues is the lack of motivation to learn: some students are not interested because they think studying mathematics is useless. Another problem is the passivity in learning: some students only take notes but never ask questions or show some interest, while others do not even care to write down what the teacher explains. The bad effect of this way of learning is that students often forget what they have learned just one month or one year before. In addiction to this, students often meet many difficulties to apply the studied methods in different contexts. Moreover, many adults claim that they have never understood or even always hated mathematics and can not remember what they learned at school.

Learning through a workshop can change the students' attitude towards mathematics: this activity can stimulate active participation, cooperation with classmates and intuition. It can also motivate the students, giving an alternative, more practical approach to the subject. The interest is excited through concrete problems and physical objects that can help to have a visualization or a description of the concepts. Manual ability can receive new importance in this context. The main aim is to understand by doing. In this context problems are given and not solutions, to stimulate the active learning.

Working in a workshop does not have the usual fixed scheme of learning at school (teacher's explanation, exercises, test). In this setting, it is natural to proceed by trial and error. Mistakes are accepted as a manner to find the right way to a solution. Then the tasks can be approached in a different way by the students. This approach favours reasoning and not automatisms, that are one of the main causes of the low interest towards mathematics.

One powerful method in teaching is to try and make students associate emotions to mathematics, and unlike one would usually do, positive emotions, such as gratification, interest, a feeling of independence, self-esteem, freedom to try different methods, discovery desire and discovery pleasure, satisfaction in cooperating. During such activities, students usually have "no awareness of making mathematics", thus do not start from a negative approach.

A workshop also helps to get used to apply mathematical skills in different contexts and formalize concrete problems. The step of formalization is fundamental: at the beginning the problem may seem like a game, but the work on it should not stop until it is formalized and a theoretical model is developed.

Finally, cooperation is very important and can help students in difficulty feel as capable as every other student. Moreover, they are encouraged, since the topic is totally new to all of the students and all of them start from the same level.

However, a workshop can be instructive also for good students, having a different attitude towards learning, having more competences than the average and more motivation to learn. We are convinced that an intensive mathematics workshop can be very instructive also for them: it can help them to apply their knowledge and skills in different contexts, give problems to put them to the test, convey the idea that if they are interested in a subject, they can learn more than what is required at school. They can receive much gratification in studying subjects that are usually not included in school syllabi.

Another goal is to stimulate cooperation in order to give an idea of science as based on sharing knowledge and not on competition.

Also the participation of teachers is very important: the workshop can serve as teacher training, to give new ideas and tools that can be used in their classes.

B.2 Workshop materials

We propose some materials for the teachers who want to use the workshop in a high school class. The problem and its solution are suitable for the classes of the last two years of high schools. Even if no prerequisites are necessary, at a lower age students will have more difficulties in formalization, symbolic computation and in understanding the concepts of group and action.

The path to follow is the same as the one described in Chapter 3, but we propose some more steps and some worksheets with notes and questions to guide the students' activity. In fact, an average class would probably have more difficulties in answering the questions and would need more time and some help to formalize and write down the results.

The workshop is divided into four parts, so that it is easier to propose it in schools. The parts are almost the same as those proposed in Section 3.2, the only difference is that the first part there is divided into two parts here. We stress that each part is self-contained and can end with a summary of the learned concepts and results. Of course, the time for the activity can vary a lot, depending on the students' age and their familiarity with autonomous and cooperative work. However, we estimate that the needed time is as follows: from one to two hours for sheet 1, from three to four hours for sheet 2, from two to three hours for sheet 3 and two hours for sheet 4. Times can be reduced if the teacher helps the students in the critical points or if part of the work is left as homework.

If there is not enough time to do all the work, the part on the curves (sheet 3) may be skipped and explained by the teacher, who can introduce the elementary curves and give the students just some minor problems (write the word corresponding to a given curve, or vice versa, draw a curve given by the word, compose two curves, and so on), so that they can practice in handling the new objects.

A very important part, that should not be skipped, is the final practical verification of the theoretical result. This gives the students a deep sense of gratification and reassures of the validity of the model developed to describe the phenomena.

The students should work in small groups; 3 to 5 students each, depending on the class and their familiarity with cooperative work and on the availability of the materials. The groups may have different velocity: this should be modulated depending on the students' comprehension. With the help of the worksheets, the students can work without a constant control of the teacher. However, it is important that the teacher stops the work at some stages to ensure that all the groups have understood the concepts up to that point. It may happen that the groups need a little time to agree on the notation to use (symbols to describe braids, symbols to describe curves, additional or multiplicative notation, right or left action). Moreover, the teacher can give some additional explanations to ensure a deeper understanding.

The worksheets are intended to help the students' work. Some hints are included, but the teacher can decide to remove them and just give them orally at the right moment, so that the students are forced to think without being put soon on the right way. It can happen that the hints are not understood, especially if the students follow a different reasoning from the one proposed in the worksheets. In this case, the teacher has to transfer them into the context and relate them to the work done by the students.

Each worksheet contains some questions and should have a lot of free space so that students can write their observations, conjectures and discoveries. However, experience suggests that the students do not usually write very much. Answers like “yes/no” are not rare. The teacher can choose to force them to write more, or ask them to write a report as homework, containing what they have done and understood. In this case, the worksheets can serve as homework or as an outline for the report. It is then convenient that each student has their own worksheets even if they work in groups and all the group mates will probably write the same.

Writing a report on the work is very important because it forces the students to formalize what they have done and helps them understand and remember the concepts.

Four sheets are proposed. It is convenient to hand out the new sheet only when the work on the previous one has been completed by all the groups and summarized by a student or a group with the help of the teacher.

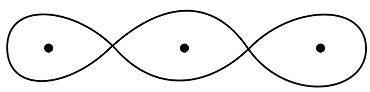
The required materials are the same as the ones used in the workshop and are described in section 3.2. Some of the objects are optional, since students can use drawings instead of them: the board with three screws and the metallic plates with the magnetic strings.

All the objects in the list should be available to every group. Alternatively, two groups or even the whole class can share the same materials, but this can take more time especially at the beginning. Moreover, this does not allow every one to have plenty of time to try the experiments. The objects should not be freely available to the students since the beginning but have to be introduced at the moment when they are needed. Since then, they are always at students' disposal.

We list the questions to guide the activity and give some indications about the right moments to introduce the objects.

Sheet 1

Materials: the big braid, the closed strings, the drawing of the curve in the following figure.[1]



Is it possible to put a closed string around the braid strands in such a way that, pulling it down, it will overlap with the curve drawn at the bottom?

[1]The drawing does not define a unique curve: this depends on the orientation of the loops. However, it is natural to interpret the curve as smooth and hence the neighbouring loops with opposite orientations. The other possible interpretations are not wrong and can be analysed and formalized in Sheet 3.

Make some attempts and compare your ideas with those of your mates.
Did you find a solution?
If the answer is yes, draw it.
Don't worry if you did not, the problem is quite complicated!
What are the difficult points in your opinion?

Materials: drawings of other curves.

What if you change the curve? Make some attempts.
Consider two curves for which you can find a solution. Draw the curves and the solutions.

Which methods have you used to find a solution?

Go back to the initial curve. Do the same methods work? Do you need other procedures?

Hint: take an open string and put it on the bottom overlapping the curve, close it...
Did you find a solution? Draw it. Is it the same as the solution you found before (if you found one)?

Does a solution exist for each curve? And for each braid? Why?

Is the solution unique? Why?

Materials: the board with three cuts.

Now we would like to solve the problem without cheating: the string is closed, we put it on a plane at the top of the braid and pull it down all in a time.

Which mathematical objects do we have to study to solve the problem?

Sheet 2

How can we define a braid?
Hint: Describe the object you used, then try to formalize it to turn the concrete object into a mathematical one.

Materials: the metallic plates with the long gum strings.

How can a braid be described?
Draw (or construct) a braid with four strands and imagine to describe it talking at the phone. The person calling wants to reproduce it.
You can simulate this situation and exchange information with another group if you want.

Did you identify some "simple bricks"?
How many? Draw them.

Give a name to each of them, so that we have a notation to describe braids.

Materials: the loose metallic plate and the short gum strings.

To obtain a braid from the bricks, we put one brick after the other. We call this operation composition.
Is it possible to find a braid such that composing it with any braid β gives β again?

Given a braid β, is it possible to find a braid γ such that composing β with γ we obtain the braid with all the strands parallel?

Then, the braids on four strands form a group. Can you remember what it means?[2]

Can you remember other groups you studied?[3]

Is it true that composing two braids gives the same as composing them in the reverse order? Prove it or find a counterexample.

Sheet 3

Material: the plastic sheets with three marked points and the pencils.

Imagine to cut the top of the braid with a horizontal plane and take away the three intersection points.
Consider these curves (figure B.1) in the plane with the three points removed. Which of them are "the same" curve? In what sense?[4]

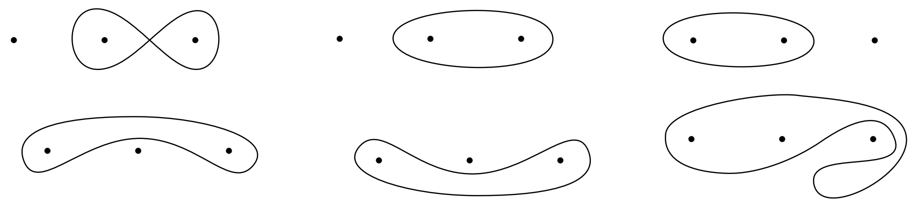


Figure B.1: Which curves are "the same"? Which are different?

When can we consider two curves to be "the same"?

Find some features that permit to distinguish different curves.

[2]If the teacher has never introduced groups, the question can be formulated as: "Rewrite formally the properties we have just seen, about identity and inverse element."

[3]This question can be skipped if the teacher does not want to recall the planar isometries. On the contrary, if the teacher is interested in talking about groups, also the permutations can be introduced, as invariant of braids.

[4]The first curve can be interpreted in two ways, depending on the orientation of the two loops. In the case the two loops are given the same orientation, this curve is equivalent to the second one. However, the loops will be probably considered as oppositely oriented since it is natural to interpret the intersection as transverse.

How can we describe the curves in terms of elementary ones?
Hint: note that to distinguish curves we observe how they go around the three points.
Hint: fix a base point and consider the curves as starting and arriving in that point.
Hint: consider oriented curves, i.e. choose a direction to walk along them.

From now on, consider only oriented curves with a fixed base point.
Draw the elementary curves and assign a symbol to them.

Redraw the original curve in this form and write a sequence of symbols to describe it.[5]

Consider the composition of two curves: this is the curve obtained walking along the first and then along the second. Composition is an operation.
Is there a neutral element (i.e. a curve that composed with any other, does not change the latter)?

Is there an inverse for each curve?

What structure can you recognize on the set of curves?

Sheet 4

Now we have a way to represent algebraically, with symbols, braids and curves in the plane with three points removed.
With the notation you fixed, write an equation that formalizes the initial problem.

Now we can write the unknown as...

Materials: the two braids with one crossing.

Start from the easiest equations that you can write. How does an elementary braid act on an elementary curve? Fill in a table similar to the following, using your notation and the correct number of elementary braids and elementary curves.

		Elementary curves				
		c_1	c_2	c_3	c_4	$\dots$
El. braids	β_1					
	β_2					
	β_3					
	$\dots$					

[5]The following questions on this sheet can be skipped if the teacher does not want to talk (informally) of the fundamental group of the punctured plane.

Note that the braid acts on the curve “at levels”: if we know how each crossing of the braid transforms the curve, we can calculate how the whole braid transforms the curve.
Further, an elementary braid (a brick) acts on a curve in this way: it acts on every elementary curve one after the other.[6]

Can you now describe a way to find how a braid β acts on a curve c?

Hint: start writing the braid and the curve with symbols and use the table.

Can you now describe a way to find which curve we have to put in the plane on the top of a braid if we know what curve we want to obtain at the bottom?

Calculate the solution to the original problem, writing it in symbols. Draw the unknown curve and make a practical test to verify the correctness of the prediction.

Material: the long open string to verify the solution.

[6] It is convenient that the teacher explains this part with more precision.

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