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Dottorato di Ricerca in Matematica

PhD Thesis

Intrinsic Lipschitz graphs in Heisenberg groups and non linear sub-elliptic PDEs

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Ciclo XXIV

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Introduction

In the last few years there has been carried out a large intensive study of the analysis in sub-Riemannian spaces. These objects are the natural generalization of the Riemannian ones (see [3, 12, 27, 75, 91, 92, 114] for an introduction). Precisely, we say that (M, D, g) is a sub-Riemannian space if M is a smooth manifold of dimension n, D is a distribution of $m-$planes $(m \leq n)$ and g is a Riemannian metric on D.
A sub-Riemannian space is also called Carnot-Carathéodory space because, following Gromov ([91]), Carathéodory firstly introduced this type of structure in the mathematical foundation of Carnot's thermodynamic ([30]).

A C-C structure on an open subset $\Omega \subset \mathbb{R}^n$ (or, more generally, a manifold) amounts to a family $X = (X_1, \dots, X_m)$ of vector fields such that every couple of points $x, y \in \Omega$ can be joined by a curve whose derivative belongs to the fiber bundle generated by the family X. Precisely, we require the existence of an absolutely continuous curve $\lambda : [0, T] \longrightarrow \Omega$ and a measurable function $h : [0, T] \longrightarrow \mathbb{R}^m$ such that $\lambda(0) = x$, $\lambda(T) = y$ and

$$\dot{\lambda}(t) = \sum_{j=1}^{m} h_j(t) X_j(\lambda(t)) \quad \text{and} \quad |h(t)| \leq 1 \text{ a.e.} \tag{1}$$

A curve satisfying (1) is said subunit.
Every C-C spaces can be endowed, like in the Riemannian setting, with a canonical distance, named C-C distance, namely:

$$\mathrm{d}_{cc}(x, y) := \inf\{T > 0 \mid \exists\ \lambda : [0, T] \longrightarrow \Omega \text{ subunit}, \ \lambda(0) = x, \lambda(T) = y\}.$$

The metric space $(\Omega, \mathrm{d}_{cc})$ presents new features quite different from the Riemannian ones such as, for istance, the non-uniqueness of geodesics and the gap between the topological and metric dimensions (see, for istance, [3, 12, 91, 114]).

Among the C-C spaces an important role is played by Carnot groups. A Carnot group $\mathbb{G}$ of step k is a connected and simply connected Lie group of dimension n with stratified Lie algebra $\mathcal{G}$ of step k; which means that there are subspaces $V_1, \ldots, V_k$ of $\mathcal{G}$ such that

$$\begin{aligned}&\mathcal{G} \equiv \mathbb{R}^n = V_1 \oplus \ldots \oplus V_k,\\&[V_1, V_j] = V_{j+1} \quad j = 1, \ldots, k-1,\\&[V_1, V_k] = \{0\}.\end{aligned}$$

It is well-known that, by means of the exponential map $\exp : \mathcal{G} \equiv \mathbb{R}^n \longrightarrow \mathbb{G}$, $\mathbb{G}$ can be identified with $\mathbb{R}^n$. There are two main reasons to deal with Carnot groups. First of all they have a rich analytical and geometrical structure. In particular, we point out the presence of a one parameter family of group isomorphism, the so called dilations $\delta_r : \mathbb{G} \longrightarrow \mathbb{G}$, $r > 0$. Moreover, in a Carnot group the distance d_{cc} has a well behaviour with respect to the group law and the dilation family. Precisely, for every $x, y, z \in \mathbb{G}$ and every $r > 0$:

$$\begin{aligned}&\mathrm{d}_{cc}(z \cdot x, z \cdot y) = \mathrm{d}_{cc}(x, y);\\&\mathrm{d}_{cc}(\delta_r(x), \delta_r(y)) = \mathrm{d}_{cc}(x, y).\end{aligned}$$

On the other hand a Carnot group can be considered as a local approximation of a C-C space. Indeed, it can be proved that up to a suitable blow-up procedure (see [109]) a Carnot group is a natural ”tangent” space to a C-C space like an Euclidean space is a tangent space to a Riemannian manifold.

C-C spaces and Carnot groups were applied in several areas of analysis and geometry. Just to mention some of them we recall their role in the understanding of hypoelliptic equations [39, 94, 126], degenerate elliptic and parabolic equations [20, 52, 56, 70, 73, 74, 105, 108, 114, 139], singular integrals [39], potential theory [20, 129], control theory [3] and geometry of Banach spaces [32, 33]. More recently they were fundamental in many applied research areas, such as mathematical finance [49], theoretical computer science and mathematical models in neurosciences [48, 50, 51, 93].

In the PDE's context the role of C-C spaces and Carnot groups has been recognized to be fundamental since the work of Hörmander ([94]) who proved that under some algebraic assumptions on the vector fields $X_0, \ldots, X_k$ (the so called Hörmander's condition) the operator

$$\mathcal{L} = \sum_{i=1}^{k} X_i^2 + X_0$$

is hypoelliptic. In addition, Rothschild and Stein ([126]) proved that for these operators a priori estimates of L^p type for second order derivatives with respect to the family $X_0, \ldots, X_k$ hold. The subsequent literature on these and more general operators in C-C spaces and Carnot groups is huge. We refer the reader to the monograph [20] and the references therein.
Here we want only to specify some typical difficulties which arise when dealing with these kind of problems. To avoid notational complications we restrict ourself to $\mathbb{H}^1$, the first Heisenberg group. Precisely, $\mathbb{H}^1$ is a Carnot group of step 2 with stratification $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, where

$$\mathfrak{h}_1 := \mathrm{span}\{\nabla_1^{\mathbb{H}}, \nabla_2^{\mathbb{H}}\} \quad \text{and} \quad \mathfrak{h}_2 := \mathrm{span}\{\nabla_3^{\mathbb{H}}\}$$

and the only nonvanishing commutator relations are given by $[\nabla_1^{\mathbb{H}}, \nabla_2^{\mathbb{H}}] = 2\nabla_3^{\mathbb{H}}$. Let us consider the following nonlinear equation:

$$\Delta_{\mathbb{H}^1} u := \mathrm{div}_{\mathbb{H}} \left(|\nabla_{\mathbb{H}} u|^{p-2} \nabla_{\mathbb{H}} u \right) = 0 \quad p \geq 2, \tag{2}$$

where $\mathrm{div}_{\mathbb{H}}\, u := \nabla_1^{\mathbb{H}} u + \nabla_2^{\mathbb{H}} u$ and $\nabla_{\mathbb{H}} u := (\nabla_1^{\mathbb{H}} u, \nabla_2^{\mathbb{H}} u)$ (see for istance [15, 61, 62, 63, 105, 108]). Let us note that the vertical derivative $\nabla_3^{\mathbb{H}} u$ does not appear directly in the operator. It rather appears only in an intrinsic way after commutation. Such a lack of ellipticity in the vertical direction is often the basic source of problems in the regularity theory ([108]). Indeed, when attempting to differentiate the equation (2), derivative in the the vertical direction appears and there is no a priori control on the L^p norm of such derivative ([23]).

Another typical problem in PDEs is the study of so-called characteristic points for the boundary of a domain Ω of a Carnot group $\mathbb{G}$. Indeed, in many situations is important to know under which regularity properties on a domain $\Omega \subset \mathbb{G}$ some estimates hold up to the boundary $\partial\Omega$. This problem is already not trivial in the Euclidean setting (see [2, 66]) but in the context of Carnot groups and C-C spaces it is complicated by the presence of the so called characteristic points. Precisely, if $\Omega \subset \mathbb{H}^1$ is an open set with regular boundary, then $x \in \partial\Omega$ is said characteristic if $\nabla_1^{\mathbb{H}}\Phi(x) = 0$ and $\nabla_2^{\mathbb{H}}\Phi(x) = 0$, where $\Phi = 0$ is a local equation for $\partial\Omega$. We refer the reader to the very interesting papers [10, 103, 104, 115, 137] and in particular to the work of Danielli-Garofalo and Nhieu [53] and the references therein for a complete discussion on this and more general problems.

Another very active research line which exponentially grew up in the last few years especially after the work of Pansu [122], is the attempt to develope geometric measure theory in C-C spaces and in particular in Carnot groups, with emphasis on the Heisenberg group $\mathbb{H}^n$. We refer the reader to the monograph [27] for a comprehensive introduction. A very interesting problem in this setting is the possibility of giving good definitions of rectifiability [80, 102, 103, 104]. We point out that the classical Euclidean definition of rectifiability of Federer [69], which use Lipschitz functions, cannot be applied in C-C spaces, which in general are purely unrectifiable [5]. To overcome this problem Franchi-Serapioni and Serra Cassano proposed in their very interesting paper [80] an alternative definition of rectifiability modeled on a different notion of intrinsic regular submanifold. Precisely, they called intrinsic regular hypersurface every $S \subset \mathbb{G}$ (here $\mathbb{G}$ denotes a general Carnot group) which is (locally) the level set of a function $f : \mathbb{G} \longrightarrow \mathbb{R}$ with nonvaninshing continuous horizontal gradient. This notion was also extended, for codimension one, to general Carnot-Carathéodory spaces in [45] and, for general codimensions, firstly in the setting of the Heisenberg group ([82]) and then in general Carnot group ([103]). We point out that an intrinsic hypersurface can be very irregular from an Euclidean point of view and in general these surfaces are not Euclidean C^1 submanifolds, not even locally (see [97]). Nevertheless,

they are invariant with respect to group left-translations or group intrinsic dilations.
The intrinsic regular hypersurfaces share a lot of properties with the Euclidean ones. In particular an implicit function theorem holds ([80] or Theorem 3.1.1). More precisely, given an intrinsic regular hypersurface $S \subset \mathbb{G} \equiv \mathbb{R}^n$ and an open set $\mathcal{U} \subset \mathbb{G}$, there are complementary subgroups $\mathbb{G}_1 \equiv \mathbb{R}^{n-1}$ and $\mathbb{G}_2 \equiv \mathbb{R}^1$(i.e subgroups of $\mathbb{G}$ closed under dilations, such that $\mathbb{G}_1 \cdot \mathbb{G}_2 = \mathbb{G}$ and $\mathbb{G}_1 \cap \mathbb{G}_2 = \{0\}$) and a continuous function $\phi : \omega \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ for which $S \cap \mathcal{U}$ can be written (locally) as the intrinsic graph of ϕ, that is

$$S \cap \mathcal{U} = \{x \cdot \phi(x) \mid x \in \omega\},$$

or, up to a change of coordinates,

$$S \cap \mathcal{U} = \{(\phi(x), x) \mid x \in \omega\}.$$

Let us also recall that the implicit function theorem and the notion of intrinsic graph have been also extended to general Carnot-Carathéodory structures in [45] and later in [102]. Actually, at least in the Heisenberg group, the map $\phi : \omega \subset \mathbb{G}_1 \equiv \mathbb{R}^{2n} \longrightarrow \mathbb{G}_2 \equiv \mathbb{R}$ is not only continuous, indeed in [6] authors proved that ϕ is uniformly $\nabla^\phi-$differentiable. Which means, that there exists an homogeneous homomorphism $L : \mathbb{G}_1 \equiv \mathbb{R}^{2n} \longrightarrow \mathbb{G}_2 \equiv \mathbb{R}$ (i.e. a group homomorphis such that $L(\delta_r(x)) = rL(x)$ for all $x \in \mathbb{G}_1$ and $\mathbb{G}_1$, $\mathbb{G}_2$ are complementary subgroups in $\mathbb{H}^n$) such that

$$\lim_{r \to 0} \mathcal{M}(\phi, z, L, r) = 0$$

where

$$\mathcal{M}(\phi, z, L, r) := \sup_{x,y \in I_r(z), x \neq y} \left\{ \frac{|\phi(x) - \phi(y) - L(\pi_{\mathbb{G}_1}(\Phi(y)^{-1} \cdot \Phi(x)))|}{\mathrm{d}_\phi(x, y)} \right\}.$$

Where $I_r(z) \subset \omega$ is a suitable neighborhood of z, $\Phi(x) := (\phi(x), x)$ and

$$\mathrm{d}_\phi(x, y) := \frac{1}{2} \Big(\|\pi_{\mathbb{G}_1}(\Phi(x)^{-1} \cdot \Phi(y))\| + \|\pi_{\mathbb{G}_1}(\Phi(y)^{-1} \cdot \Phi(x))\| \Big)$$

is a quasidistance on ω. We point out that, as in the Euclidean setting, we can represent the map L using a suitable intrinsic gradient. Precisely, if L

and ϕ are as above then defining the family $\nabla^\phi = (\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi)$ of vector fields (see [6, 45]), namely of first order differential operators, on ω by

$$\begin{aligned} \nabla_i^\phi(x) &= \partial_{x_i} - x_{i+n}\partial_{x_{2n}}, \quad \text{for } n \geq 2 \text{ and } i = 1, \dots, n-1, \\ \nabla_n^\phi(x) &= \partial_{x_n} + 2\phi(x)\partial_{x_{2n}}, \\ \nabla_i^\phi(x) &= \partial_{x_i} + x_{i-n}\partial_{x_{2n}}, \quad \text{for } n \geq 2 \text{ and } i = n+1 \dots, 2n-1, \end{aligned} \tag{3}$$

and by

$$\nabla_1^\phi(x) = \partial_{x_1} + 2\phi(x)\partial_{x_2} \quad \text{if } n = 1.$$

then

$$L(y) = \left\langle \nabla^\phi \phi, \tilde{\pi}(y) \right\rangle \qquad \forall y \in \mathbb{G}_1 \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{2n-1}$ and

$$\tilde{\pi}(x_1, \dots, x_{2n-1}, x_{2n}) := (x_1, \dots, x_{2n-1}) \qquad \forall x \in \mathbb{G}_1.$$

We call the vector $\nabla^\phi \phi$ the $\nabla^\phi-$gradient of ϕ at $x \in \omega$. The intrinsic differentiation has been deeply studiend and generalized to more general spaces in [45].

It is well known that a fundamental object to enstablish a good theory of rectifiable sets is a correct notion of Lipschitz functions. It is easy to see that the classical definition of metric Lipschitz function (i.e the one made using the distance d_{cc})in a Carnot group ([122]) does not fit the geometry. To overcome this problem Franchi-Serapioni and Serra Cassano in [79] proposed to call intrinsic Lipschitz those functions $\phi : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ (where $\mathbb{G}_1$ and $\mathbb{G}_2$ are complementary subgroups of a Carnot group $\mathbb{G}$) such that there exists $\alpha > 0$ for which for every point $q \in \Phi(\mathbb{G}_1)$

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q, 1/\alpha) \cap \Phi(\omega) = \{q\}$$

where

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha) := \{p = (s, x) \in \mathbb{H}^n \mid \|\pi_{\mathbb{G}_1}(q^{-1} \cdot p)\| \leq \alpha \|\pi_{\mathbb{G}_2}(q^{-1} \cdot p)\|\}$$

and $\pi_{\mathbb{G}_1}$, $\pi_{\mathbb{G}_2}$ are the projections on $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively.
First of all notice that this notion is really intrinsic, indeed it is invariant

under left translations of the graph ([79, Propositions 3.11]), i.e. if we left translate an intrinsic Lipschitz graph we obtain an intrinsic Lipschitz graph. We also point out that intrinsic Lipschitz functions are not metric Lipschitz, that is they are not Lipschitz as maps $\phi : (\omega, \mathrm{d}_{cc}) \to (\mathbb{G}_2, \mathrm{d}_{cc})$ where $\omega \subset \mathbb{G}_1$ is an open set (see [79, Remark 3.13]). Nevertheless, the intrinsic Lipschitz functions amount to a thick class of functions. Indeed, it holds that ([79, Propositions 4.8 and 4.11])

$$Lip(\omega) \subsetneq Lip_{\mathbb{G}_1,loc}(\omega) \subsetneq C^{1/2}_{loc}(\omega)\,,$$

where, respectively, $Lip(\omega)$ and $C^{1/2}_{loc}(\omega)$ denote the classes of real valued Euclidean Lipschitz and locally 1/2-Hölder functions on ω. Besides intrinsic Lipschitz functions share a lot of properties with the Euclidean Lipschitz ones as proved in [79]. In particular, if $\phi : \mathbb{G}_1 \equiv \mathbb{R}^{2n} \longrightarrow \mathbb{G}_2 \equiv \mathbb{R}$ is intrinsic Lipschitz where $\mathbb{G}_1$ and $\mathbb{G}_2$ are complementary subgroups of $\mathbb{H}^n$, then ϕ is ∇^ϕ-differentiable for $\mathcal{L}^{2n}$-a.e $x \in \omega$. Moreover, the subgraph

$$E_\phi := \{(s,x) \in \mathbb{H}^n \mid s < \phi(x)\}$$

is a set of locally finite perimeter in $\mathbb{H}^n$ (see [79]). We point out that, in the setting of $\mathbb{H}^n$, it is still open the intriguing question whether a Rademacher type theorem holds for k-codimenisonal intrinsic Lipschitz graphs with $2 \leq k \leq n$.

Using these notions a good rectifiability theory has been established. Nevertheless many interesting questions remain open, see [79, 80, 103].

For example we briefly recall the problem of regularity for the minimal surfaces equation for intrinsic graphs in $\mathbb{H}^n$. Indeed, it can be proved that the analogous of the minimal surface equation in $\mathbb{H}^n$ is:

$$\nabla^\phi\Big(\frac{\nabla^\phi\phi}{\sqrt{1+|\nabla^\phi\phi|^2}}\Big) = 0 \tag{5}$$

where ∇^ϕ is the nonlinear family of vector fields defined in (3). We recall some literature on this equation, which attempt to answer to the problems of existence, unicity and regularity (see [6, 11, 24, 25, 34, 35, 36, 37, 54, 55, 88, 113]). Equation (5) presents some new problems with respect to the formally equivalent Euclidean one. Indeed, it is a priori non trivial, to provide estimates on the missing direction $\nabla^{\phi}_{2n} := [\nabla^{\phi}_{1}, \nabla^{\phi}_{n+1}]$, moreover the structure of the vector fields ∇^{ϕ} and the geometry of $\mathbb{H}^n$ does not allow the use of some classical Euclidean techniques.

In the first part of this thesis we provide some results which improve the theory of intrinsic Lipschitz functions in the Heisenberg group.
In particular we provide an approximation theorem in terms of regular functions for a given intrinsic Lipschitz map $\phi : \omega \subset \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ where $\mathbb{G}_1, \mathbb{G}_2$ are the following complementary subgroups of $\mathbb{H}^n$, $\mathbb{G}_1 := \{(t, x_1, \dots, x_{2n}) \in \mathbb{H}^n \mid t = 0\}$ and $\mathbb{G}_2 := \{(t, x_1, \dots, x_{2n}) \in \mathbb{H}^n \mid x_1 = \dots = x_{2n} = 0\}$ (see Theorem 4.2.7 and [46]). More precisely our first result is the following:

Theorem 1. *Let ϕ be a real valued intrinsic Lipschitz function defined on an open and bounded $\omega \subset \mathbb{G}_1 \equiv \mathbb{R}^{2n}$, then there exists a sequence $\{\phi_i\}_{i\in\mathbb{N}}$ of real valued smooth maps defined on ω such that:*

(i) $\phi_i \to \phi$ *locally uniformly in* ω*;*

(ii) $|\nabla^{\phi_i}\phi_i(x)| \le ||\nabla^{\phi}\phi||_{L^\infty(\omega)}$ $\forall x \in \omega$*;*

(iii) $\nabla^{\phi_i}\phi_i(x) \to \nabla^{\phi}\phi(x)$ $\mathcal{L}^{2n}-a.e$ $x \in \omega$.

The technique used to obtain this result relies on some classical ideas due to Ennio de Giorgi (see [57, 58, 59]) and on some new facts developed in [46]. In particular, we are able to prove an area formula for intrinsic Lipschitz functions (see Theorem 4.2.4 and [46]).We prove the following:

Theorem 2. *If $\phi : \mathbb{G}_1 \equiv \mathbb{R}^{2n} \longrightarrow \mathbb{G}_2 \equiv \mathbb{R}$ is an intrinsic Lipschitz function then there exists a dimensional constant $c_n > 0$ such that the following equality hold:*

$$|\partial E_\phi|_{\mathbb{H}}(\mathbb{R} \times \omega) = c_n \mathcal{S}^{2n+1}(graph(\phi)) = \int_\omega \sqrt{1 + |\nabla^{\phi}\phi|^2}\, \mathrm{d}\mathcal{L}^{2n}.$$

where $\mathcal{S}^{2n+1}$ denotes the spherical Hausdorff measure in $\mathbb{H}^n$ (see Definition 1.1.13) and $|\partial E_\phi|_{\mathbb{H}}$ is the intrinsic perimeter measure (see Definition 1.2.7).

An interesting consequence of our approximation result is an estimate of the Lipschitz constant of a given intrinsic Lipschitz function in terms of the $L^\infty-$norm of its intrinsic gradient (see Proposition 4.2.9). Precisely,

Proposition 1. *Let $\omega \subset \mathbb{G}_1 \equiv \mathbb{R}^{2n}$, $\phi : \omega \longrightarrow \mathbb{G}_2 \equiv \mathbb{R}$ be open and bounded and $n \geq 2$. Then for each $\bar{x} \in \omega$ and each $r > 0$ sufficiently small*

$$Lip(\phi, U_\phi(\bar{x}, r)) \leq c\Big(4\sqrt{\|\nabla^\phi \phi\|_{L^\infty(\omega)}} + 1\Big)\|\nabla^\phi \phi\|_{L^\infty(\omega)}$$

for a suitable geometric positive constant c. Moreover, if $n = 1$ it holds:

$$Lip(\phi, U_\phi(\bar{x}, r)) \leq c\sqrt{1 + \|\nabla^\phi \phi\|^2_{L^\infty(\omega)}} \tag{6}$$

where $Lip(\phi, U_\phi(\bar{x}, r))$ is the intrinsic Lipschitz constant of ϕ on the ball $U_\phi(\bar{x}, r) := \{y \in \omega \mid \mathrm{d}_\phi(\bar{x}, y) < r\}$ (see Definition 3.2.2).

Our second contribution in this framework is a Poincaré inequality for intrinsic Lipschitz functions ([47]), which hopefully could be used in the study of minimal surfaces in $\mathbb{H}^n$. Precisely,

Theorem 3. *Let ω be a bounded and open subset of $\mathbb{G}_1 \equiv \mathbb{R}^{2n}$ with $n \geq 2$. Let $\phi : \omega \longrightarrow \mathbb{R}$ be an intrinsic Lipschitz function. Then there exists a constant C (independent of the Lipschitz constant L of ϕ) such that*

$$\begin{aligned}&\int_{\Omega_{\phi(x)}(x,r)} |\phi(y) - \phi_r(x)| \mathrm{d}\mathcal{L}^{2n}(y) \leq \\ &\leq CrL^{\frac{Q+2}{2}} \int_{\Omega_{\phi(x)}(x,Cr(1+L))} |\nabla^\phi \phi(y)| \mathrm{d}\mathcal{L}^{2n}(y)\end{aligned}$$

for each $x \in \omega$, $r > 0$ such that $\Omega_{\phi(x)}(x, r), \Omega_{\phi(x)}(x, C\,r(1 + L)) \subset \omega$. Where $\phi_r(x)$ is a suitable mean defined in terms of the fundamental solution Γ of a properly defined sub-Laplacian operator (see Definition 4.94), $\Omega_{\phi(x)}(x, r)$ are the super-levels of Γ (see (4.100)) and Q is the homogeneous dimension of $\mathbb{H}^{n-1} \times \mathbb{R}$.

In the second part of the thesis we prove some extensions, to the sub-Riemannian setting, of a couple of PDE's results, well known in the Euclidean context.

Our first contribution ([124]) is a geometric Poincaré type inequality for a stable solution u of the following semilinear equation in the Engel group $\Delta_{\mathbb{E}} u = f(u)$ (see Definition 5.3) where f is smooth. We prove the following,

Theorem 4. *Let u be a stable solution of $\Delta_{\mathbb{E}} u = f(u)$ then for any $\eta \in C_0^\infty(\mathbb{E})$*

$$\int_{\mathbb{E}_0} \mathcal{W} |\nabla_{\mathbb{E}} u|^2 \eta^2 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \eta|^2 |\nabla_{\mathbb{E}} u|^2 \tag{7}$$

where $\mathbb{E}_0 := \{x \in \Omega \mid \nabla_{\mathbb{E}} u(x) \neq 0\}$ and $\mathcal{W}$ is a suitable kernel depending only on u whose explicit expression is contained in Theorem 5.1.1.

Finally we prove a non existence result for solution of $\Delta_{\mathbb{E}} u = f(u)$ using our Poincaré type inequality.

Our second result ([125]) is an extension of the so called Dual Estimate to the obstacle problem for quasilinear elliptic equations in the Heisenberg group. Precisely we prove that for every solution $\bar{u}_\epsilon$ of the following variational problem

$$\inf_{u \in \mathcal{K}} \mathcal{F}_\varepsilon(u; \Omega), \text{ where } \mathcal{F}_\varepsilon(u; \Omega) := \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{p/2}, \tag{8}$$

where $\epsilon > 0$

$$\mathcal{K} := \left\{ u \in W^{1,p}_{\mathbb{H}^n}(\Omega) \text{ s.t. } u \leq \psi, \text{ and } u - u_\star \in W^{1,p}_{\mathbb{H}^n,0}(\Omega) \right\}$$

and for all $p \in \mathcal{P}(\psi, \Omega)$ (see Definition 6.1.1) the following Theorem hold:

Theorem 5. *Let $\bar{u}_\epsilon$, ψ, $\mathcal{K}$ and p be as above then the following Lewy-Stampacchia inequality hold:*

$$\begin{aligned} 0 &\leq \operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \\ &\leq \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+ \end{aligned} \tag{9}$$

in the sense of distributions.

We also prove a similar result for $\epsilon = 0$.

The structure of the thesis is the following:

In Chapter 1 we recall some basic facts about Lie algebras and Lie groups, with particular attention to the class of nilpotent and stratified Lie groups. In particular, in Section 1.1 we provide some basic and well known results on Lie algebras and Lie groups. Then, we recall the definition of the Lie algebra associated to a Lie group and how, starting from a stratified Lie algebra, it is possible to construct a Carnot group. Finally we study in details the main analytical pecurialities of the Carnot group's structure. To this end we introduce the dilations family $\{\delta_\lambda\}$, the C-C distance and the horizontal fiber bundle. In Section 1.2 we outline some basic results of first order calculus and geometric measure theory in Carnot groups. In particular we analyze functions of $\mathbb{G}-$bounded variation and sets of finite $\mathbb{G}-$perimeter and we recall some interesting results about them. Section 1.3 is entirely devoted to the study of three important examples of Carnot groups, namely the Euclidean space $\mathbb{R}^n$, the Heisenberg group $\mathbb{H}^n$ and the Engel group $\mathbb{E}$.

Chapter 2 is a brief introduction to the theory of sub-Laplacian in Carnot groups. We start recalling the main definitions and some easy properties of the sub-Laplacian. Next the introduce the fundamental solution associated to a given sub-Laplacian and we point out some of its computational properties. Particular attention will be given to some representation formulas. With this term we refer to the possibility of represent a given smooth function in terms of some a propri known operators.
In particular we state a result contained in [20] and then we prove a Theorem due to Citti-Lanconelli and Garofalo ([42]) which permits to represent a smooth function defined on a C-C space with vector fields $X_1, \ldots, X_m$ using the fundamental solution Γ associated to the sub-Laplacian $\mathcal{L} = \sum_{j=1}^m X_j^2$ and the super-levels of Γ, which are $\Omega_r(x) := \{y \in \mathbb{R}^m \mid \Gamma(x,y) > \frac{1}{r}\}$.

Chapter 3 is entirely devoted the study of intrinsic hypersurfaces in the Heisenberg group $\mathbb{H}^n$. We start defining the class of intrinsic hypersurfaces in $\mathbb{H}^n$ pointing out some interesting and non-trivial properties. Then we recall the implicit function Theorem for an intrinsic hypersurface S which provides a continuous function ϕ which locally parametrizes S. We continue

analyzing some further differentiability properties of ϕ. In particular we introduce the concept of intrinsic differentiability recallig the original definition of Ambrosio-Serra Cassano and Vittone ([6]) and Citti-Manfredini ([45]) which provides the key tool to characterize all the maps which parametrize intrinsic hypersurfaces.
Finally in subsection 3.2.1 we state some interesting results due to Bigolin and Serra Cassano ([13, 14]) which represent an alternative way to characterize the maps whose graph is an intrinsic hypersurface.

Chapter 4 is devoted to the exposition of the results obtained in [46] and [47] in collaboration with G.Citti, M.Manfredini and F.Serra Cassano. In Section 4.1 we introduce the space of intrinsic Lipschitz functions in the Heisenberg group, $Lip_{\mathbb{W}}(\omega)$, and we point out some of its main properties. Section 4.2 contains the proof of approximation result stated in the Introduction (see also [46]). In subsection 4.2.1 we prove a characterization of the class of intrinsic Lipschitz function in terms of approximating sequences. In other words, we prove that if for a given continuous function ϕ there is a sequence of smooth functions which satisfies $(i), (ii)$ and (iii) in the approximation Theorem then the limit function is locally intrinsic Lipschitz. Moreover we prove the estimate for the Lipschitz constant of a given $\phi \in Lip_{\mathbb{W}}(\omega)$ in terms of the $L^\infty-$norm of its intrinsic gradient mentioned in the Introduction, see Proposition 4.2.9.
The second part of Chapter 4 is dedicated to the work [47]. Subsections 4.3.2 and 4.3.3 are preparatory ones. In particular, we recall the notion of frozen vector fields introduced in [126] and subsequently refined in many works (see [41, 40]) and some useful estimates for the fundamental solution of the sub-Laplacian associated to these frozen vector fields. Then prove the Poincaré inequality stated above.

Chapter 5 contains a work made in collaboration with E.Valdinoci [124]. After a very brief introduction to the theory of semilinear problems in the Engel group we consider the particular equation $\Delta_{\mathbb{E}} u = f(u)$ on $\Omega \subset \mathbb{E}$ and we give the proof of the estimate (7). Finally in Section 5.3 we provide a possible application of our estimate.

Chapter 6 contains another work written in collaboration with E.Valdinoci [125]. In Section 6.2, 6.3 and 6.4 we provide the details of the estimate (9) pointing out some extension in particular when $\epsilon \to 0$. We conclude the exposition with an Appendix which contains the detailed proof of some well known inequality used throughout the chapter.

Acknowledgements: I would like to thank and express my deep gratitude to Francesco Serra Cassano my thesis advisor: he introduced me to this subject and supported me constantly during these three years. He has been a great teacher and it has been nice to work together. Then I would like to thank Giovanna Citti e Maria Manfredini for their constant help, the time they have devoted to me and the invaluable contribution they gave to my mathematical growth. Last but not least, I would like to thank Enrico Valdinoci for his constructive advices and suggentions, it is extremely interesting and stimulating to work with him. I also thank D.Vittone, R.Monti, F.Bigolin and R.Serapioni for useful discussions about my research subjects. Finally, I would like to thank the Department of Mathematics of the University of Trento for its support during my years of PhD study.

Chapter 1

Introduction to Carnot groups

In this chapter we introduce Carnot groups. To this end we start recalling the notion of finite dimensional Lie algebras with particular attention to the nilpotent and stratified ones. Next we briefly recall first order calculus in Carnot groups and we provide the basic tools for the geometric measure theory in this setting, with particular emphasis on the theory of intrinsic finite perimeter sets. At the end of the chapter we examine in detail three important examples of Carnot groups.

1.1 Lie algebras

In this section we recall some well known notions and results on Lie algebras, see [95] for a more detailed treatment.

Definition 1.1.1. *A vector space* $\mathfrak{g}$*, with an operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$*, is called a* Lie algebra *if:*

- $[\cdot,\cdot]$ *is a bilinear map;*
- $[x, y] = -[y, x]\ \forall x, y \in \mathfrak{g}$*;*
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\ \forall x, y, z \in \mathfrak{g}$.

Example. *If* $\mathfrak{g} = \mathbb{R}^n$ *and* $[x, y] = 0$ *for each* $x, y \in \mathfrak{g}$ *then* $\mathfrak{g}$ *is a Lie algebra.*

Example. *The vector space* $\Gamma(TM)$ *of all vector fields on a smooth manifold* M *with the operation defined by*

$$[X,Y] := XY - YX \tag{1.1}$$

is a Lie algebra.

Definition 1.1.2. *A subspace* $\mathfrak{t}$ *of a Lie algebra* $\mathfrak{g}$ *is called a* Lie subalgebra *if it is closed under* $[\cdot,\cdot]$, *i.e if*

$$\forall x, y \in \mathfrak{t} \Rightarrow [x,y] \in \mathfrak{t}.$$

Definition 1.1.3. *Let* U *be a subset of* $\Gamma(TM)$, *where* M *is a smooth manifold. We denote by* $Lie(U)$ *the least sub-algebra of* $\Gamma(TM)$ *containing* U. *Precisely,*

$$Lie(U) := \bigcap \mathfrak{h}$$

where $\mathfrak{h}$ *is a sub-algebra of* $\Gamma(TM)$ *containing* U.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be subalgebras of a Lie algebra $\mathfrak{g}$, we define:

$$[\mathfrak{a},\mathfrak{b}] := \operatorname{span}\{[X,Y] \mid X \in \mathfrak{a},\ Y \in \mathfrak{b}\}.$$

Definition 1.1.4. *A Lie algebra* $\mathfrak{g}$ *is* nilpotent with step equal to k, *if and only if, setting*

$$\begin{cases} \mathfrak{g}^{(1)} := \mathfrak{g} \\ \mathfrak{g}^{(i+1)} := [\mathfrak{g}, \mathfrak{g}^{(i)}] \quad i \geq 1 \end{cases}$$

it holds $\mathfrak{g}^{(k)} \neq \{0\}$ *and* $\mathfrak{g}^{(k+1)} = \{0\}$.

Definition 1.1.5. *A Lie algebra* $\mathfrak{g}$ *is said to be* stratified with step equal to k *if there exists linear subspaces* $V_1, \cdots, V_k$ *of* $\mathfrak{g}$ *such that*

$$\begin{aligned} &\mathfrak{g} = V_1 \oplus \cdots \oplus V_k \\ &V_j = [V_1, V_{j-1}] \quad \text{for } j = 2, \cdots, k \\ &[V_1, V_k] = \{0\}. \end{aligned}$$

Remark 1.1. It is well known that any finite dimensional stratified Lie algebra of step k is in particular nilpotent of the same step, see [103]. We point out that the converse is not true in general. Considering the family of vector fields

$$X = \partial_x - xy\partial_t,\ Y = \partial_y + x\partial_w,\ Z = \partial_z + x\partial_t$$

then the Lie algebra generated by X, Y, Z, that is $Lie(\{X, Y, Z\})$, has dimension 5 and a basis is (X, Y, Z, W, T) where

$$W := [X, Y] = \partial_w + x\partial_t,\ T = [X, Z] = \partial_t.$$

This Lie algebra is nilpotent of step 4 whereas it is not stratified since T is a commutator of both steps 2 and 3, see [19].

Definition 1.1.6. *An homomorphism between Lie algebras* $F : \mathcal{G} \longrightarrow \mathcal{S}$ *is said to be a* Lie homomorphism *if it is linear and*

$$F([X, Y]) = [F(X), F(Y)] \quad \forall X, Y \in \mathcal{G}$$

1.1.1 From a Lie group to its Lie algebra

Definition 1.1.7. *A* Lie group $(\mathbb{G}, \cdot)$ *is a smooth manifold with a group structure* $\cdot$*, such that the maps*

$$\mathbb{G} \times \mathbb{G} \ni (x, y) \mapsto x \cdot y \in \mathbb{G}$$

$$\mathbb{G} \ni x \mapsto x^{-1} \in \mathbb{G}$$

are differentiable. Moreover we say that a Lie group is commutative if $\mathbb{G}$ *is commutative as a group conversely we say that* $\mathbb{G}$ *is non-commutative.*

Example. *If* $(\mathbb{G}, \cdot) = (\mathbb{R}^n, +)$*, where* $+$ *denotes the usual sum operation, is the simplest commutative Lie group. The general linear group* $(\mathbb{G}L(n), \cdot)$ *equipped by the standard matrix multiplication is a noncommutative Lie group.*

Definition 1.1.8. *A smooth vector field* X *on a Lie group* $(\mathbb{G}, \cdot)$ *is* left invariant *if for all* $x, y \in \mathbb{G}$

$$\mathrm{d}_y \tau_x(X(y)) = X(x \cdot y)$$

where for each $x \in \mathbb{G}$ *the map* $\tau_x : \mathbb{G} \longrightarrow \mathbb{G}$ *is defined by* $\tau_x(y) := x \cdot y$ *and* $\mathrm{d}_y \tau_x$ *denotes its differential at the point* $y \in \mathbb{G}$*.*

Let $\mathcal{G}$ be the subset of $\Gamma(T\mathbb{G})$ whose elements are all the left invariant vector fields on a Lie group $\mathbb{G}$, it is easy to see that if $X, Y \in \mathcal{G}$ then

$$[X, Y] := XY - YX \in \mathcal{G}.$$

Hence $\mathcal{G}$ equipped by the bracket defined in (1.1) is a Lie subalgebra of $\Gamma(T\mathbb{G})$ and it is called the *Lie algebra associated to* $\mathbb{G}$. Moreover, let e be the identity element of $\mathbb{G}$, then the map

$$\begin{aligned} \Phi : T_e\mathbb{G} &\longrightarrow \mathcal{G} \\ v &\mapsto \Phi(v) = X \end{aligned}$$

where X is defined by $X(g) = \mathrm{d}_e L_g(v)\ \forall g \in \mathbb{G}$, is an isomorphism of vector spaces; hence the dimension of $\mathcal{G}$ is equal to the topological dimension of $\mathbb{G}$.

Proposition 1.1.1 ([60])**.** *Let* $\mathbb{G}$ *be a Lie group with Lie algebra* $\mathcal{G}$*. For each* $X \in \mathcal{G}$*, there exists a unique solution* $\gamma_X : \mathbb{R} \longrightarrow \mathbb{G}$ *of the system*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\gamma_X(t)_{|t=0} = X(\gamma_X(t)) \\ \\ \gamma_X(0) = e. \end{cases}$$

Remark 1.2. Note that for a Lie group γ_X is defined for all $t \in \mathbb{R}$, while in general this is not true.

Definition 1.1.9. *Let* $\mathbb{G}$ *be a Lie group with Lie algebra* $\mathcal{G}$*. We define the* exponential map $\exp : \mathcal{G} \to \mathbb{G}$ *by*

$$\exp(X) := \gamma_X(1). \tag{1.2}$$

In the following proposition we recall some basic properties of the exponential map. See [60] and [1].

Proposition 1.1.2. *If* $\mathbb{G}$ *is a Lie group and* $\mathcal{G}$ *is its Lie algebra, then*

1. exp *is an analytic function;*

2. exp *is a diffeomorphism in a neighborhood of the origin of* $\mathcal{G}$*;*

3. $\exp(tX) = \gamma_X(t)$ *for all* $X \in \mathcal{G}$, $t \in \mathbb{R}$*;*

4. $\exp((t+s)X) = \exp(tX)\exp(sX)$ *for all* $X \in \mathcal{G}$, $t, s \in \mathbb{R}$.

1.1.2 From a stratified Lie algebra to a Carnot group

Definition 1.1.10. *A Lie group* $\mathbb{G}$ *is said to be* nilpotent with step equal to $k \in \mathbb{N}$ *if the associated Lie algebra* $\mathcal{G}$ *is nilpotent of the same step. Moreover, a finite dimensional, connected and simply connected Lie group with stratified Lie algebra with step equal to* k *is said to be a* Carnot group of step k.

Remark 1.3. By Remark 1.1 it follows that every Carnot group of step k is also nilpotent of step k, but the converse is not true.

The following result is one of the main properties of the exponential map and it generalizes (2) in Proposition 1.1.2, see [20] for a proof.

Theorem 1.1.3. *Let* $\mathbb{G}$ *be a nilpotent, connected and simply connected Lie group with Lie algebra* $\mathcal{G}$ *then* $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ *is a global diffeomorphism.*

Theorem 1.1.3 says that from an analitycal point of view a Lie group is not so far from its Lie algebra. In the remaining part of this section we will prove that a nilpotent Lie group and its Lie algebra are similar in the algebraic sense too. To this end we equip a nilpotent Lie algebra by a group law using the so called Baker-Campbell-Hausdorff formula.

Definition 1.1.11. *Let* $X, Y \in \mathcal{G}$*, where* $\mathcal{G}$ *is a nilpotent Lie algebra of step* k *we define*

$$X \odot Y := \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n} \sum_{1 \le |\alpha|+|\beta| \le k} \frac{(Ad(X))^{\alpha_1}(Ad(Y))^{\beta_1} \ldots (AdX)^{\alpha_n}(AdY)^{\beta_n - 1}(Y)}{\alpha!\beta!|\alpha+\beta|} \tag{1.3}$$

where for any $Z \in \mathcal{G}$ *the map* $AdZ : \mathcal{G} \longrightarrow \mathcal{G}$ *is defined by* $AdZ(W) := [Z, W]$ *and for any* $\alpha \in \mathbb{N}^n$ *we have assumed the convention* $\alpha! := \prod_{s=1}^{n} \alpha_s$ *and* $|\alpha| := \sum_{s=1}^{n} \alpha_s$.

Remark 1.4. Since $\mathcal{G}$ is nilpotent it follows that the sum in (1.3) is finite, moreover it is well known that $\odot$ defines a group law on $\mathcal{G}$, see [20, 95, 103].

Theorem 1.1.4. *Let $(\mathbb{G}, \cdot)$ be a nilpotent connected and simply connected Lie group with Lie algebra $\mathcal{G}$. Then $(\mathcal{G}, \odot)$ is a Lie group and the map*

$$\exp : (\mathcal{G}, \odot) \longrightarrow (\mathbb{G}, \cdot)$$

is a group isomorphism.

The following theorem is due to Lie and is one of the most important and deep result in Lie group theory, see [134].

Theorem 1.1.5. *If $\mathcal{G}$ is a finite dimensional Lie algebra then there exists a connected and simply connected Lie group whose Lie algebra is isomorphic to $\mathcal{G}$.*

Using Theorems 1.1.4 and 1.1.5 we are now in position to prove the following result

Proposition 1.1.6. *Let $\mathcal{G}$ be a stratified Lie algebra of step k, then there exist a natural number m and a group law $\cdot$ on $\mathbb{R}^m$ such that $(\mathbb{R}^m, \cdot)$ is a Carnot group with stratified Lie algebra of step k isomorphic to $\mathcal{G}$.*

Proof. By Remark 1.1 we know that $\mathcal{G}$ is nilpotent of step k. By Theorems 1.1.4 and 1.1.5 there exist a Lie group $\mathbb{G}$ whose Lie algebra is isomorphic to $\mathcal{G}$ and such that the map $\exp : (\mathcal{G}, \odot) \longrightarrow (\mathbb{G}, \cdot_{\mathbb{G}})$ is a group isomorphism. We prove that a coordinate version of $(\mathcal{G}, \odot)$ is a Lie group isomorphic to $\mathbb{G}$ with Lie algebra isomorphic to $\mathcal{G}$. Since $\mathcal{G} = V_1 \oplus \ldots \oplus V_k$ then we can find a basis of $\mathcal{G}$

$$\mathcal{B} := (X_1^{(1)}, \ldots, X_{\dim(V_1)}^{(1)}, \ldots, X_1^{(k)}, \ldots, X_{\dim(V_k)}^{(k)})$$

such that $(X_1^{(i)}, \ldots, X_{\dim(V_i)}^{(i)})$ is a basis of V_i $(i \in \{1, \ldots, k\})$. Let us define

$$m := \sum_{i=1}^{k} \dim(V_k). \tag{1.4}$$

and consider the coordinate map given by

$$\begin{aligned} \pi_{\mathcal{B}} : & \mathcal{G} \longrightarrow \mathbb{R}^m \\ X := & \sum_{i=1}^{k} \sum_{j=1}^{\dim_{V_i}} x_j^i X_j^{(i)} \mapsto (x^{(1)}, \ldots, x^{(r)}) \end{aligned} \tag{1.5}$$

where $x^{(i)} := (x_1^{(i)}, \ldots, x_{\dim V_i}^{(i)}) \in \mathbb{R}^{\dim V_i}$. Next, we set

$$\Psi := \exp \circ (\pi_{\mathcal{B}})^{-1} : \mathbb{R}^m \longrightarrow \mathbb{G}$$

and we equip $\mathbb{R}^m$ with the composition law $\cdot$ defined by

$$x \cdot y := \Psi^{-1}(\Psi(x) \cdot_{\mathbb{G}} \Psi(y))$$

It is easy to see that $(\mathbb{R}^m, \cdot)$ is a Lie group and that Ψ is a Lie group isomorphism between $(\mathbb{R}^m, \cdot)$ and $(\mathbb{G}, \cdot_{\mathbb{G}})$. It remains to prove that the Lie algebra $\mathcal{G}^*$ of $(\mathbb{R}^m, \cdot)$ is isomorphic to $\mathcal{G}$. To this end we consider the map

$$\Phi := \exp^{-1} \circ \Psi^{-1} \circ \exp : \mathcal{G} \longrightarrow \mathcal{G}^*$$

and since all the components of Φ are isomorphism we conclude that Φ is an isomorphism too and hence $\mathcal{G}^*$ is stratified of step k. □

1.1.3 Carnot groups in details

Let $\mathcal{G} = V_1 \oplus \ldots \oplus V_k$ be a stratified Lie algebra of step $k \in \mathbb{N}$, if we denote by $m := \sum_{i=1}^{k} \dim(V_i)$ then by Proposition 1.1.6 there exist a group law $\cdot$ on $\mathbb{R}^m$ such that $(\mathbb{R}^m, \cdot)$ is a Carnot group whose Lie algebra is isomorphic to $\mathcal{G}$.

The presence of a stratification on $\mathcal{G}$ allows us to introduce a group of automorphisms of $\mathcal{G}$, the *dilations*. Indeed, for each $\lambda > 0$ we define $\delta_\lambda : V_1 \longrightarrow V_1$ setting $\delta_\lambda(X) := \lambda X$. This map can be extended to $\mathcal{G}$ by $\delta_\lambda(X) := \lambda^i X$ if $X \in V_i$ and then by linearity. It can be shown that $\forall \lambda, \mu > 0$, $\forall X, Y \in \mathcal{G}$

$$\begin{aligned} &\delta_{\lambda\mu} = \delta_\lambda \circ \delta_\mu; \\ &\delta_\lambda([X, Y]) = [(\delta_\lambda(X), \delta_\lambda(Y)]; \\ &\delta_\lambda(X \odot Y) = \delta_\lambda(X) \odot \delta_\lambda(Y). \end{aligned}$$

By using the map $\pi_{\mathcal{B}}$ defined in Proposition 1.1.6, we can project the dilations on $(\mathbb{R}^m, \cdot)$ as follows

$$\delta_\lambda(x) := \pi_{\mathcal{B}}(\delta_\lambda(\pi_{\mathcal{B}}^{-1}(x))).$$

It can be checked that $\forall \lambda, \mu \geq 0$ and $\forall x, y \in \mathbb{R}^m$

$$\delta_{\lambda\mu} = \delta_\lambda \circ \delta_\mu,$$
$$\delta_\lambda(x \cdot y) = \delta_\lambda(x) \cdot \delta_\lambda(y).$$

and that

$$\delta_\lambda : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$(x_1, \cdots, x_m) \mapsto (\lambda x^{(1)}, \cdots, \lambda^k x^{(k)})$$

where $x^{(i)} := (x_1^{(i)}, \ldots, x_{\dim V_i}^{(i)})$ with $i \in \{1, \ldots, k\}$. From now on we say that every element of $x^{(i)}$ has degree equal to $i \in \mathbb{N}$ e we denote it $deg(x_i)$

Moreover, in a Carnot group $\mathbb{G} \equiv (\mathbb{R}^m, \cdot)$ a natural sub-Riemannian distance between two points $x, y \in \mathbb{G}$ can be introduced as the infimum of all time $T > 0$ for which there exists a subunit curve joining x and y. More precisely, we say that an absolutely continuous curve $\lambda : [0, T] \longrightarrow \mathbb{G}$ is *subunit* if there exists a measurable function $h : [0, T] \longrightarrow \mathbb{R}^{\dim V_1}$ such that

- $\dot{\lambda}(t) = \sum_{i=1}^{\dim_{V_1}} h_i(t) X_i(\lambda(t))$ a.e $t \in [0, T]$
- $||h||_{L^\infty([0,T])} \leq 1$ in $[0, T]$.

where $(X_1, \ldots, X_{\dim V_1})$ denotes a basis of V_1. Therefore we can state the following

Definition 1.1.12. *Let* $\mathbb{G}$ *be a Carnot group, we define the* Carnot-Carathéodory distance *in the following way:*

$$\mathrm{d}_{cc} : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty]$$
$$(x, y) \mapsto \inf\{T > 0 \mid \exists \lambda : [0, T] \longrightarrow \mathbb{G} \ \textit{subunit}, \lambda(0) = x, \lambda(T) = y\}$$

Actually, it is well known that in any Carnot group $\mathbb{G} = (\mathbb{R}^m, \cdot)$ the Hörmander condition is satisfied, i.e.

$$Lie[X_1, \ldots, X_{n_1}](x) = T_x\mathbb{R}^m \quad \forall x \in \mathbb{R}^m, \tag{1.6}$$

hence by the Chow's theorem (see [38, 115]) we have

$$\mathrm{d}_{cc}(x, y) \leq +\infty \quad \forall x, y \in \mathbb{G}.$$

Therefore $(\mathbb{G}, \mathrm{d}_{cc})$ turn out to be a metric space. Using the stratification of $\mathcal{G}$ it can be proved that the distance d_{cc} is translation invariant and homogeneous of degree 1, more precisely

Proposition 1.1.7 ([138]). *For all $x, y, z \in \mathbb{G} \equiv \mathbb{R}^m$ and for all $\lambda \geq 0$*

1. $\mathrm{d}_{cc}(z \cdot x, z \cdot y) = \mathrm{d}_{cc}(x, y)$ *(translation invariance);*
2. $\mathrm{d}_{cc}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda \mathrm{d}_{cc}(x, y)$ *(homogeneity).*

Remark 1.5. For any fixed euclidean compact set $K \subset \mathbb{G}$, there exists a constant $C = C(K) > 0$ such that

$$\frac{1}{C}|x - y| \leq \mathrm{d}_{cc}(x, y) \leq C|x - y|^{\frac{1}{k}} \tag{1.7}$$

for any $x, y \in \mathbb{G}$. Hence inequality (1.7) implies that the topology induced by the Carnot-Carathéodory distance is the same of the one induced by the Euclidean distance.

Remark 1.6. Since $\mathrm{d}_{cc}(x, y)$ is an implicit function, i.e it cannot be directly computed starting from the coordinates of x and y it is often preferible to use an equivalent and explicit distance on $\mathbb{G} \equiv (\mathbb{R}^m, \cdot)$. One possible choice is

$$\mathrm{d}_\infty(x, y) := \|y^{-1} \cdot x\| \tag{1.8}$$

where

$$\|x\| := \|(x_1, \ldots, x_m)\| := \sum_{j=1}^{m} |x_j|^{\frac{1}{deg(x_j)}} \tag{1.9}$$

or

$$\|x\| := \|(x_1, \ldots, x_m)\| := \max_j \{ \ \epsilon_j |x_j|^{\frac{1}{deg(x_j)}} \ \} \tag{1.10}$$

and $\epsilon_1, \dots, \epsilon_m \in (0,1]$ are such that (1.10) defines a norm and $deg(x_j)$ is the degree of x_j. It is easy to prove that the distance d_∞ is such that $\mathrm{d}_\infty(z\cdot x, z\cdot y) = \mathrm{d}_\infty(x,y)$ and $\mathrm{d}_\infty(\delta_\lambda(x), \delta_\lambda(y)) = \lambda \mathrm{d}_\infty(x,y)\ \forall x,y,z \in \mathbb{G}$, $\forall \lambda > 0$. Moreover, for each compact set $K \subset \mathbb{G}$ there exists $C(K) > 0$ such that

$$C^{-1}\mathrm{d}_\infty(x,y) \le \mathrm{d}_{cc}(x,y) \le C\mathrm{d}_\infty(x,y) \quad \forall x,y \in K.$$

Definition 1.1.13. *We shall denote by $\mathcal{H}^m$ the $m-$dimensional Hausdorff measure obtained from the distance d_∞. Analogously, $\mathcal{S}^m$ will denote the corresponding $m-$dimensional spherical Hausdorff measure.*

Definition 1.1.14. *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group with Lie algebra*

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_k.$$

We call the homogeneous dimension *of $\mathbb{G}$ the number*

$$Q := \sum_{i=1}^{k} i \dim(V_i)$$

Remark 1.7. In [109] it is proved that the integer Q is the Hausdorff dimension of $\mathbb{G}$ with respect to the distance d_{cc}.

Proposition 1.1.8 ([115])**.** *If we denote by $\mathcal{L}^m$ the Lebesgue measure on $\mathbb{G} \equiv (\mathbb{R}^m, \cdot)$, then for each measurable $E \subset \mathbb{R}^m$*

$$\mathcal{L}^m(x \cdot E) = \mathcal{L}^m(E \cdot x) \ \ \forall x \in \mathbb{G}.$$

Moreover, for all $x \in \mathbb{G} \equiv (\mathbb{R}^m, \cdot)$ and for all $r \ge 0$ it holds

$$\mathcal{L}^m(B(x,r)) = r^Q \mathcal{L}^m(B(0,1))$$

where $B(x,r) := \{y \in \mathbb{G} \mid \mathrm{d}_{cc}(x,y) < r\}$.

Remark 1.8. From Proposition 1.1.8 it follows that the $m-$dimensional Lebesgue measure is the Haar measure of $\mathbb{G}$.

Definition 1.1.15. *Let $\mathbb{G}$ be a Carnot group with Lie algebra $\mathcal{G} = V_1 \oplus \ldots \oplus V_k$ and let $X_1, \ldots, X_{\dim V_1}$ be a basis of V_1. The* horizontal bundle $H\mathbb{G}$ *is the subbundle of the tangent bundle $T\mathbb{G}$ whose fibers $H_p\mathbb{G}$ are spanned by the* horizontal vectors $X_1(p), \ldots, X_{\dim V_1}(p)$. *We fix a scalar product on $\mathcal{G}$ is such a way that the scalar product induced on each fiber $\langle \cdot, \cdot \rangle_p$ makes the horizontal basis orthonormal.*

As usual, once we have a vector bundle one can introduce its sections, namely a continuous map $F : \mathbb{G} \longrightarrow H\mathbb{G}$ such that $F(p) \in H_p\mathbb{G}$ $\forall p \in \mathbb{G}$. Since we fixed an horizontal basis on each fiber $H_p\mathbb{G}$, there are $F_i : \mathbb{G} \longrightarrow \mathbb{R}$ $(i \in \{1, \ldots, \dim V_1\})$ such that

$$F(p) = \sum_{i=1}^{\dim V_1} F_i(p) X_i(p) \quad \forall p \in \mathbb{G}$$

hence we can identify a section F with its representation in coordinates, that is $F \equiv (F_1, \ldots, F_{\dim V_1})$. Moreover, if $\Omega \subseteq \mathbb{G}$ is an open set, we denote by $C^\infty(\Omega, H\mathbb{G})$ the set of C^∞ section of $H\mathbb{G}$ in Ω where, of course, the C^∞ regularity is understood as regularity between manifolds, similarly we denote by $C_c^\infty(\Omega, H\mathbb{G}) \subset C^\infty(\Omega, H\mathbb{G})$ the set of sections with compact support in Ω.

1.2 Calculus on Carnot Groups

The aim of this section is to outline some basic results of first order calculus and geometric measure theory in a general Carnot group, standard references are [20, 103, 138, 80].

Throughout this section we will denote by $\mathbb{G} = (\mathbb{R}^n, \cdot, \mathrm{d}_{cc,\mathbb{G}})$ and by $\mathbb{S} = (\mathbb{R}^m, \cdot, \mathrm{d}_{cc,\mathbb{S}})$ Carnot groups with Lie algebra $\mathcal{G}$ and $\mathcal{S}$ and homogeneous norms $|| \cdot ||_\mathbb{G}$ and $|| \cdot ||_\mathbb{S}$ respectively. Moreover, we will denote by $n_1 \in \mathbb{N}$ the dimension of the first layer of $\mathcal{G}$, by $(X_1, \ldots, X_{n_1})$ one of its basis and by $U(p, r) := \{q \in \mathbb{G} \mid \mathrm{d}_{cc,\mathbb{G}}(p, q) < r\}$. Finally, $Q \in \mathbb{N}$ will be the homogeneous dimension of $\mathbb{G}$.

1.2.1 First order calculus on Carnot groups

Definition 1.2.1. *A map $L : \mathbb{G} \longrightarrow \mathbb{S}$, is said to be a* homogeneous homomorphism *if L is a group homomorphism and $L \circ \delta_\lambda = \bar{\delta}_\lambda \circ L$, where we have denoted by $\bar{\delta}_\lambda$ the dilations in $\mathbb{S}$.*

Example. *If $\mathbb{G} = \mathbb{R}^N$ and $\mathbb{S} = \mathbb{R}$ then $L : \mathbb{R}^N \longrightarrow \mathbb{R}$ is an homogeneous homomorphism if and only if L is a linear map.*

Definition 1.2.2. *Let $\Omega \subseteq \mathbb{G}$ be an open set. A map $f : \Omega \subset \mathbb{G} \longrightarrow \mathbb{S}$ is* Pansu-differentiable *(or simply* P-differentiable*) at $x \in \Omega$ if there is an homogeneous homomorphism $D_x(f) : \mathbb{G} \longrightarrow \mathbb{S}$ such that*

$$\exists \lim_{y \to x} \frac{\left\| \Big(D_x(f)(x^{-1} \cdot y) \Big)^{-1} \cdot f(x)^{-1} \cdot f(y) \right\|_{\mathbb{S}}}{\|x^{-1} \cdot y\|_{\mathbb{G}}} = 0$$

If a such $D_x(f)$ exist we call it the Pansu-differential of f *(or simply P-differential). If $f : \Omega \subset \mathbb{G} \longrightarrow \mathbb{S}$ is P-differentiable at every $x \in \Omega$ and the P-differential depends continuously on x we say that f is a* continuosly P-differentiable *function and we write $f \in C^1_{\mathbb{G}}(\Omega, \mathbb{S})$.*

Definition 1.2.3. *Let $\Omega \subset \mathbb{G}$ be an open set and $f : \Omega \longrightarrow \mathbb{R}$ be a continuous map, then we define the* horizontal gradient *of f as the function $\nabla_{\mathbb{G}} f : \Omega \longrightarrow \mathbb{R}^{n_1}$*

$$\nabla_{\mathbb{G}} f := (X_1 f, \ldots, X_{n_1} f) \tag{1.11}$$

where $X_i f$ denotes the distributional derivative of f along X_i.

Proposition 1.2.1 ([122])**.** *Let $\Omega \subset \mathbb{G}$ be an open set and let $f : \Omega \longrightarrow \mathbb{R}$ be a continuous function. Then $f \in C^1_{\mathbb{G}}(\Omega, \mathbb{R})$ if and only if the distributional derivative $X_i f \in C^0(\Omega, \mathbb{R})$ with $i \in \{1, \ldots, n_1\}$.*

As in the Euclidean case we have a representation theorem for the P-differential of $f \in C^1_{\mathbb{G}}(\Omega, \mathbb{R})$ in terms of its intrinsic gradient.

Theorem 1.2.2 ([122])**.** *Let $\Omega \subset \mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ be an open set and $f \in C^1_{\mathbb{G}}(\Omega, \mathbb{R})$, then for all $x, y \in \Omega$*

$$D_y(f)(y^{-1} \cdot x) = \left\langle \nabla_{\mathbb{G}} f(y), \pi(y^{-1} \cdot x) \right\rangle_{\mathbb{R}^{n_1}}$$

where $\pi : \mathbb{G} \longrightarrow \mathbb{R}^{n_1}$ *is* $\pi(x) = \pi((x_1, \dots, x_n)) := (x_1, \dots, x_{n_1})$ *and* $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n_1}}$ *denotes the Euclidean scalar product on* $\mathbb{R}^{n_1}$*.*

Proposition 1.2.3 ([115])**.** *Let* $f : \Omega \subseteq \mathbb{G} \longrightarrow \mathbb{S}$ *be P-differentiable at* $x \in \Omega$ *and* $g : f(\Omega) \longrightarrow \mathbb{P}$ *P-differentiable at* $f(x) \in f(\Omega)$*. Then the map* $g \circ f : \Omega \longrightarrow \mathbb{P}$ *is P-differentiable at* x *and*

$$D_x(g \circ f) = D_{f(x)}(g) \circ D_x(f)$$

We conclude this part pointing out the relationship between Euclidean C^1 functions and $C^1_{\mathbb{G}}$ functions.

Proposition 1.2.4 ([103])**.** *Let* $\Omega \subseteq \mathbb{G}$ *be an open set. Then*

$$C^1(\Omega) \subset C^1_{\mathbb{G}}(\Omega).$$

Remark 1.9. We refer to Section 1.3 for an example of function $f \in C^1_{\mathbb{G}}(\Omega) \setminus C^1(\Omega)$.

1.2.2 Lipschitz functions and geometric measure theory

Definition 1.2.4. *We say that a function* $f : \Omega \subset \mathbb{G} \longrightarrow \mathbb{S}$ *is Lipschitz and we write* $f \in Lip_{cc}(\mathbb{G}, \mathbb{S})$*, if there exists a constant* $L > 0$ *such that*

$$\mathrm{d}_{cc,\mathbb{S}}(f(x), f(y)) \leq L \mathrm{d}_{cc,\mathbb{G}}(x, y) \quad \forall x, y \in \Omega. \tag{1.12}$$

We call Lipschitz constant of f*, and we write* $Lip_{cc}(f)$*, the infimum of* $L > 0$ *such that (1.12) hold.*

The following fundamental theorem is due to Pansu and the proof is contained in [122].

Theorem 1.2.5. *If* $f : \mathbb{G} \longrightarrow \mathbb{S}$ *is a Lipschitz function then it is P-differentiable for* $\mathcal{L}^n-$*a.e* $x \in \mathbb{G}$*. Moreover, if* $\mathbb{S} \equiv \mathbb{R}$ *then the derivatives* $X_i f$*,* $i = 1, \dots, n_1$ *exist in distributional sense, are measurable function and*

$$||\nabla_H f||_{L^\infty(\mathbb{G})} \leq Lip_{cc}(f). \tag{1.13}$$

Remark 1.10. We point out that the Carnot group structure is useless in order to prove inequality (1.13), indeed it holds in all Carnot Carathéodory spaces as proved in [115].

Theorem 1.2.5 open the possibility of proving much finer results on Lipschitz functions $f \in Lip_{cc}(\mathbb{G}, \mathbb{S})$ especially when $\mathbb{S} \equiv \mathbb{R}$. To this end we recall some notions of geometric measure theory in Carnot groups, see [75, 80, 115, 103, 138, 87] and the reference therein for the details.

Definition 1.2.5. *Let $\Omega \subset \mathbb{G}$ be an open set and $\varphi \in C^0(\Omega, H\mathbb{G})$. Then $\varphi = (\varphi_1, \dots, \varphi_{n_1})$ and we call* horizontal divergence *of φ the first order operator:*

$$\operatorname{div}_{\mathbb{G}} \varphi := \sum_{j=1}^{n_1} X_j \varphi_j.$$

Remark 1.11. We point out that the previous definition does not depend on the basis $(X_1, \dots, X_{n_1})$, see [103].

Using Definition 1.2.5 and simply rephrasing the classical definition it is possible to introduce the notion of a function of $\mathbb{G}-$bounded variation.

Definition 1.2.6 ([80, 26])**.** *We say that $f : \Omega \longrightarrow \mathbb{R}$ is of bounded $\mathbb{G}$ variation in an open set $\Omega \subset \mathbb{G}$ and we write $f \in BV_{\mathbb{G}}(\Omega)$, if $f \in L^1(\Omega)$ and*

$$|Df|_{\mathbb{G}}(\Omega) := \sup \Big\{ \int_\Omega f \operatorname{div}_{\mathbb{G}} \varphi \, \mathrm{d}\mathcal{L}^n \mid \varphi \in C_c^\infty(\Omega, H\mathbb{G}),\ |\varphi|_{\mathbb{R}^{n_1}} \le 1 \Big\} < +\infty$$

Moreover we say that f is of locally finite $\mathbb{G}-$variation in Ω (in short $f \in BV_{\mathbb{G},loc}(\Omega)$) if $f \in L^1_{loc}(\Omega)$ and $f \in BV_{\mathbb{G}}(\Omega')$ for every $\Omega' \Subset \Omega$.

Definition 1.2.7. *A set $E \subset \mathbb{G}$ is said to be of* finite $\mathbb{G}-$perimeter *in Ω if $\chi_E \in BV_{\mathbb{G}}(\Omega)$, that is*

$$|\partial E|_{\mathbb{G}}(\Omega) := \sup \Big\{ \int_E \operatorname{div}_{\mathbb{G}} \varphi \, \mathrm{d}\mathcal{L}^n \mid \varphi \in C_c^\infty(\Omega, H\mathbb{G}),\ |\varphi|_{\mathbb{R}^{n_1}} \le 1 \Big\} < +\infty$$

Analogously a set $E \subset \mathbb{G}$ is of locally finite $\mathbb{G}-$perimeter in Ω if $\chi_E \in BV_{loc,\mathbb{G}}(\Omega)$.

Remark 1.12. It is well-known that if $E \subseteq \mathbb{G}$ is a set of locally finite perimeter in Ω, then $|\partial E|_{\mathbb{G}}$ is a Radon measure on Ω and its support is such that $\mathrm{spt}(|\partial E|_{\mathbb{G}}) \subseteq (\partial E \cap \Omega)$. Moreover it is not difficult to see that a set of finite Euclidean perimeter has finite $\mathbb{G}-$perimeter too and that this inclusion is strict ([138, Example 3.8]).

Proposition 1.2.6 ([80, 103, 138]). *If E is a Euclidean Lipschitz domain, then*

$$|\partial E|_{\mathbb{G}} = \sqrt{\sum_{i=1}^{n_1} \langle X_i, \nu \rangle}\ \mathcal{H}^{n_1} \llcorner \partial E,$$

where with ν we denote the unit normal to ∂E.

Proposition 1.2.7 ([81, 26]). *Let $f, f_j \in L^1(\Omega)$, $j \in \mathbb{N}$, be such that $f_j \to f$ in $L^1(\Omega)$. Then*

$$|Df|_{\mathbb{G}}(\Omega) \leq \liminf_{j\to\infty} |Df_j|_{\mathbb{G}}(\Omega).$$

In analogy with the Euclidean case, by Riesz's representation Theorem, the following formula holds

Theorem 1.2.8 ([81, 80]). *Let $E \subset \Omega$ be a set with locally finite $\mathbb{G}-$perimeter. Then then there exists a $|\partial E|_{\mathbb{G}}-$ measurable section ν_E of $H\mathbb{G}$ called* generalized inward normal *such that $|\nu_E(p)|_{\mathbb{R}^{n_1}} = 1$ for $|\partial E|_{\mathbb{G}}$ a.e $p \in \Omega$ and for all $\varphi \in C_0^\infty(\mathbb{G}, H\mathbb{G})$ we have*

$$\int_E \mathrm{div}_{\mathbb{G}}\, \varphi \mathrm{d}\mathcal{L}^n = -\int_{\mathbb{G}} \langle \nu_E, \varphi \rangle\, \mathrm{d}|\partial E|_{\mathbb{G}}.$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product defined in Definition 1.2.2.

Definition 1.2.8 ([81]). *(i) Let $E \subset \mathbb{G}$ be a set of locally finite perimeter; we say that $p \in \partial^*_{\mathbb{G}} E$ (the $\mathbb{G}-$reduced boundary of E) if*

1. $|\partial E|_{\mathbb{G}}(U(p,r)) > 0 \quad \forall\ r > 0$;
2. $\exists\ \lim_{r\to 0} ⨍_{U(p,r)} \nu_E\ \mathrm{d}|\partial E|_{\mathbb{G}} =: \nu_E$;
3. $\langle \nu_E(p), \nu_E(p) \rangle = 1$.

(ii) *Let $E \subset \mathbb{G}$ be a measurable set, we say that $p \in \partial_{*,\mathbb{G}} E$, the measure theoretic boundary of E, if*

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(E \cap U(p,r))}{\mathcal{L}^n(U(p,r))} > 0$$

and

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(E^c \cap U(p,r))}{\mathcal{L}^n(U(p,r))} > 0$$

Lemma 1.2.9. *The $\mathbb{G}-$reduced boundary of a set of finite perimeter is invariant under group translations, that is if $q \in \partial^*_{\mathbb{G}} E$ if and only if $p \cdot q \in \partial^*_{\mathbb{G}}(p \cdot E)$, moreover, $\nu_E(q) = \nu_{p \cdot E}(p \cdot q)$.*

Lemma 1.2.10 ([4])**.** *Assume E is a set of locally finite perimeter in $\mathbb{G}$, then*

$$\lim_{r \to 0} \fint_{U(p,r)} \nu_E \, \mathrm{d}|\partial E|_{\mathbb{G}} = \nu_E(p) \quad \textit{for } |\partial E|_{\mathbb{G}} - a.e\ p \in \mathbb{G}$$

Theorem 1.2.11 ([80])**.** *Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group of step two and $E \subseteq \mathbb{G}$ a set with locally finite $\mathbb{G}-$perimeter then there exist $c = c(n) > 0$ such that*

$$|\partial E|_{\mathbb{G}} = c\, \mathcal{S}^n \llcorner \partial^*_{\mathbb{G}} E. \tag{1.14}$$

Remark 1.13. From Definition 1.2.8 and Lemma 1.2.10 we immediately deduce that $|\partial E|_{\mathbb{G}}-$a.e $p \in \mathbb{G}$ belongs to the reduced boundary $\partial^*_{\mathbb{G}} E$.

We end this section with a collection of results that are the Carnot counterpart of the BV function theory in the Euclidean space, see [80], [81] and [27].

Theorem 1.2.12. *For any $f \in BV_{\mathbb{G}}(\Omega)$ the following coarea formula holds*

$$|Df|_{\mathbb{G}}(\Omega) = \int_{\mathbb{R}} |\partial E_t|_{\mathbb{G}}(\Omega) \mathrm{d}t \tag{1.15}$$

where $E_t := \{x \in \Omega \mid f(x) > t\}$.

Now, some observation related to the coarea formula are in order.

Lemma 1.2.13. *If $u \in Lip_{cc}(\Omega, \mathbb{R})$ then, for every $c \in \mathbb{R}$, the set $\{x \in \Omega \mid \nabla_{\mathbb{G}} u(x) \neq 0\} \cap \{x \in \Omega \mid u(x) = c\}$ has zero Lebesgue measure.*

Proof. For every $f \in L^1(\Omega)$ by the coarea formula (1.15) we have

$$\int_\Omega f|\nabla_{\mathbb{G}} u| \mathrm{d}x = \int_{-\infty}^{+\infty} \Big(\int_{\{x\in\Omega \ | \ u(x)=t\}} f \mathrm{d}|\partial E_t|_{\mathbb{G}} \Big) \mathrm{d}t.$$

If we take as f the characteristic function of the set $V \cap \{u = c\}$ where $V \subseteq \Omega$ is a bounded domain, then

$$\int_{\{x\in\Omega \ | \ u(x)=t\}} f \mathrm{d}|\partial E_t|_{\mathbb{G}} = 0 \qquad \forall\ t \neq c$$

hence

$$\int_{V\cap\{u=c\}} |\nabla_{\mathbb{G}} u| \mathrm{d}x = 0$$

that implies the desired result. □

Using Lemma 1.2.13 and (1.7) it easily follows the following

Corollary 1.2.14. *If $u \in Lip_{cc,loc}(\Omega)$ then, for every $c \in \mathbb{R}$, the set $\{x \in \Omega \mid \nabla_{\mathbb{G}} u(x) \neq 0\} \cap \{x \in \Omega \mid u(x) = c\}$ has zero Lebesgue measure.*

Theorem 1.2.15 ([65])**.** *There is a constant $c > 0$ independent of $r > 0$ such that for any set $E \subset \mathbb{G}$ of locally finite $\mathbb{G}-$perimeter, $\forall p \in \mathbb{G}$, $\forall r > 0$*

$$\min\{\mathcal{L}^n(E \cap U(p,r)), \mathcal{L}^n(E^c \cap U(p,r))\}^{\frac{Q-1}{Q}} \leq c|\partial E|_{\mathbb{G}}(U(p,r)) \qquad (1.16)$$

and

$$\min\{\mathcal{L}^n(E), \mathcal{L}^n(E^c)\}^{\frac{Q-1}{Q}} \leq c|\partial E|_{\mathbb{G}}(\mathbb{G}). \qquad (1.17)$$

Definition 1.2.9. *For each $q \in \mathbb{G}$, we define the map $\pi_q : \mathbb{G} \longrightarrow H\mathbb{G}(q)$*

$$\pi_q(p) = \sum_{j=1}^{n_1} x_j X_j(q)$$

where $(x_1, \ldots, x_{n_1}) \in \mathbb{R}^{n_1}$ are the first n_1 coordinates of the point p.

Theorem 1.2.16 ([80])**.** *Let $\mathbb{G}$ be a Carnot group of step two. Then if E is a locally finite $\mathbb{G}-$perimeter set, $p \in \partial^*_{\mathbb{G}} E$ and $\nu_E(p) \in H\mathbb{G}_p$ is the generalized inward normal to E in p, it holds*

$$\lim_{r\to 0} \boldsymbol{1}_{E_{r,p}} = \boldsymbol{1}_{S^+_{\mathbb{G}}(\nu_E(p))} \quad \text{in } L^1_{loc}(\mathbb{G}) \qquad (1.18)$$

where $E_{r,p} := \delta_{1/r}(p^{-1} \cdot E)$. *Moreover, for all* $R > 0$ *it holds*

$$\lim_{r \to 0} |\partial E_{r,p}|_{\mathbb{G}}(U(0,R)) = |S^+_{\mathbb{G}}(\nu_E(p))|_{\mathbb{G}}(U(p,r)), \tag{1.19}$$

where $S^+_{\mathbb{G}}(\nu_E(p)) := \Big\{ q \in \mathbb{G} \mid \langle \pi_p(q), \nu_E(p) \rangle \geq 0 \Big\}$.

1.3 Examples

In this section we describe three important examples of Carnot groups, namely the Euclidean space $\mathbb{R}^n$, the Heisenberg group $\mathbb{H}^n$ and the Engel group $\mathbb{E}$.

1.3.1 Euclidean space

The easiest example of Carnot group is the additive group $(\mathbb{R}^n, +)$. Its dilations are

$$\delta_\lambda(x) = \lambda x \quad (\lambda > 0),$$

and a basis of its Lie algebra is $(\partial_1, \cdots, \partial_n)$, hence $\mathbb{R}^n$ is a Carnot group of step 1 with homogeneous dimension $Q = n$. Moreover, since the subunit curves are straight lines we immediately obtain that

$$\mathrm{d}_{cc}(x,y) = |x - y|_{\mathbb{R}^n}$$

where $|\cdot|_{\mathbb{R}^n}$ denotes the Euclidean norm. We stress that $(\mathbb{R}^n, +)$ is the only Carnot group with step 1 (and n generators).

1.3.2 The Heisenberg group

Our second example is the Heisenberg group $\mathbb{H}^n$ that is the most simple non commutative Carnot group. It is a privileged object of study in analysis and geometry.

Definition 1.3.1. *A Lie algebra* $\mathfrak{h}_n$ *is said to be the Heisenberg algebra if there exists a basis* $(\nabla^{\mathbb{H}}_1, \ldots, \nabla^{\mathbb{H}}_{2n}, \nabla^{\mathbb{H}}_{2n+1})$ *of* $\mathfrak{h}_n$ *such that the only non trivial* *commutation is*

$$[\nabla_i^{\mathbb{H}}, \nabla_{n+i}^{\mathbb{H}}] = 2\nabla_{2n+1}^{\mathbb{H}} \quad \forall i = 1, \dots, n.$$

The Heisenberg group $\mathbb{H}^n$ is the connected, simply connected and nilpotent Lie group associated to $\mathfrak{h}_n$.

Remark 1.14. Defining

$$\begin{aligned} \mathfrak{h}_1 &:= \mathrm{span}\{\nabla_1^{\mathbb{H}}, \dots, \nabla_{2n}^{\mathbb{H}}\} \\ \mathfrak{h}_2 &:= \mathrm{span}\{\nabla_{2n+1}^{\mathbb{H}}\} \end{aligned}$$

$\mathfrak{h}_n$ becomes a stratified Lie algebra of step 2. Hence $\mathbb{H}^n$ is a Carnot group of step 2 with Hausdorff dimension equal to $Q = 2n + 2$ and topological dimension equal to $2n + 1$.

By using Proposition 1.1.6, we identify $\mathbb{H}^n$ with $(\mathbb{R}^{2n+1}, \cdot)$ where, if $p = (s, x) = (s, x_1, \dots, x_{2n})$, $q = (t, y) = (t, y_1, \dots, y_{2n}) \in \mathbb{R} \times \mathbb{R}^{2n}$, then

$$\begin{aligned} q \cdot p = (s + t, x_1 + y_1, \dots \\ \dots, x_{2n} + y_{2n} + (sy_n - tx_n) + \sum_{i=1}^{n} \Big(y_{n+i}x_i - y_i x_{n+i} \Big) \end{aligned} \tag{1.20}$$

and the canonical basis of $\mathfrak{h}_n$ is

$$\begin{cases} \nabla_1^{\mathbb{H}} = \partial_s - x_n \partial_{2n} \\ \nabla_{i+1}^{\mathbb{H}} = \partial_i - x_{i+n} \partial_{2n} \quad \text{if and } i = 1, \dots, n-1 \\ \nabla_{i+1}^{\mathbb{H}} = \partial_i + x_{i-n} \partial_{2n} \quad \text{if and } i = n+1, \dots, 2n-1 \\ \nabla_{2n+1}^{\mathbb{H}} = \partial_{2n} \end{cases} \tag{1.21}$$

In Chapter 4 we will use another coordinate representation of $\mathbb{H}^n$. Namely, instead using the classical exponential map definend in (1.2) we will use the global diffeomorphism $\exp^* : \mathfrak{h}_n \longrightarrow \mathbb{H}^n$ defined by

$$\exp^*(X) = \exp^*(s\nabla_1^{\mathbb{H}} + \sum_{i=1}^{2n} x_i \nabla_i^{\mathbb{H}}) := \exp(s\nabla_1^{\mathbb{H}})\exp(\sum_{i=1}^{2n} x_i \nabla_i^{\mathbb{H}})(0) \tag{1.22}$$

As in the classical case $\exp^*$ defines an isomorphism between $(\mathfrak{h}_n, \odot)$ and $(\mathbb{H}^n, \cdot)$. Hence by Proposition 1.1.6 it follows that $(\mathbb{H}^n, \cdot)$ is isomorphic as a Lie group to $(\mathbb{R}^{2n+1}, \cdot)$ where, if $p = (s, x) = (s, x_1, \dots, x_{2n}), q = (t, y) = (t, y_1, \dots, y_{2n}) \in \mathbb{R} \times \mathbb{R}^{2n}$, then

$$q \cdot p = (s + t, x_1 + y_1, \dots \\ \dots, x_{2n} + y_{2n} + \sum_{i=1}^{n-1}(x_{i+n}y_i - y_{i+n}x_i) + 2(x_n + y_n)t) \tag{1.23}$$

In these new coordinates the canonical basis of $\mathfrak{h}_n$, if $n \geq 2$, is expressed as

$$\begin{cases} \nabla_1^{\mathbb{H}} = \partial_s \\ \nabla_{i+1}^{\mathbb{H}} = \partial_i - x_{i+n}\partial_{2n} \quad \text{if and } i = 1, \dots, n-1 \\ \nabla_{n+1}^{\mathbb{H}} = \partial_n + 2s\partial_{2n} \\ \nabla_{i+1}^{\mathbb{H}} = \partial_i + x_{i-n}\partial_{2n} \quad \text{if and } i = n+1, \dots, 2n-1 \\ \nabla_{2n+1}^{\mathbb{H}} = \partial_{2n} \end{cases} \tag{1.24}$$

The homogeneous dilatations $\delta_\lambda : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1}$ are

$$\delta_\lambda(s, x_1, \dots, x_{2n}) = (\lambda s, \lambda x_1, \dots, \lambda^2 x_{2n}) \quad \lambda > 0$$

and

$$\mathrm{d}_\infty(x, y) = \|y^{-1} \cdot x\| \tag{1.25}$$

where

$$\|(s, x_1, \dots, x_{2n})\| := \max\left\{ \left|(s, x_1, \dots, x_{2n-1})\right|_{\mathbb{R}^{2n}}, \left|x_{2n}\right|^{\frac{1}{2}} \right\} \tag{1.26}$$

or equivalently

$$\|(s, x_1, \dots, x_{2n})\| := \left(\left|(s, x_1, \dots, x_{2n-1})\right|_{\mathbb{R}^{2n}} + \left|x_{2n}\right|^2\right)^{\frac{1}{4}} \tag{1.27}$$

Moreover, we denote by $U(x, r) = \{y \in \mathbb{H}^n \mid \mathrm{d}_\infty(x, y) < r\}$.

Remark 1.15. We are now in position to provide an explicit example of a function $f \in C^1_{\mathbb{H}}(\Omega) \setminus C^1(\Omega)$. Indeed, let us consider

$$f : \mathbb{H}^1 \longrightarrow \mathbb{R}$$
$$(s, x_1, x_2) \mapsto s - \sqrt{s^4 + x_1^4 + x_2^2}$$

then it is clear that f is not C^1 regular at the origin but it is $C^1_{\mathbb{H}}$ regular in a neighbourhood of 0.

1.3.3 The Engel group

Definition 1.3.2. *The Engel algebra is the finite dimensional Lie algebra* $\mathfrak{e}$ *with basis* (X_1, X_2, X_3, X_4) *where the only nonvanishing commutators relationship among the generators are*

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = [X_2, X_3] = X_4. \tag{1.28}$$

The Engel group, denoted by $\mathbb{E}$*, is the connected, simply connected and nilpotent Lie group associated to* $\mathfrak{e}$*.*

Remark 1.16. It is easy to see that the Engel algebra is stratified of step 3. Hence $\mathbb{E}$ is Carnot group of step 3 with homogeneous dimension $Q = 7$ and topological dimension equal to 4.

Since $\mathbb{E}$ is a Carnot group by Prosition 1.1.6 we can represent it by $(\mathbb{R}^4, \cdot)$ where for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{E} \equiv \mathbb{R}^4$

$$\begin{aligned}
&(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) := \\
&= \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1), \\
&x_4 + y_4 + \frac{1}{2}[(x_1 y_3 - x_3 y_1) + (x_2 y_3 - x_3 y_2)] + \\
&+ \frac{1}{12}[(x_1 - y_1 + x_2 - y_2)(x_1 y_2 - x_2 y_1)]. \Big)
\end{aligned} \tag{1.29}$$

The rappresentation of the basis (X_1, X_2, X_3, X_4) in these coordinates gives

$$\begin{cases}
X_1(x_1, x_2, x_3, x_4) = \partial_1 - \dfrac{x_2}{2}\partial_3 - \Big(\dfrac{x_3}{2} + \dfrac{x_2}{12}(x_1 + x_2)\Big)\partial_4 \\
X_2(x_1, x_2, x_3, x_4) = \partial_2 + \dfrac{x_1}{2}\partial_3 - \Big(\dfrac{x_3}{2} - \dfrac{x_1}{12}(x_1 + x_2)\Big)\partial_4 \\
X_3(x_1, x_2, x_3, x_4) = \partial_3 + \dfrac{1}{2}(x_1 + x_2)\partial_4 \\
X_4(x_1, x_2, x_3, x_4) = \partial_4.
\end{cases} \tag{1.30}$$

The homogeneous dilations on $\mathbb{E}$ are

$$\delta_\lambda(x_1, x_2, x_3, x_4) := (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4) \quad \lambda > 0$$

and for every $x = (x_1, x_2, x_3, x_4) \in \mathbb{E}$, we denote by

$$\|x\| := \Big((x_1^2 + x_2^2)^6 + x_3^6 + x_4^4 \Big)^{\frac{1}{12}}$$

and then $\mathrm{d}_\infty(x, y) = \|y^{-1} \cdot x\|$ is a homogeneous distance on $\mathbb{E}$.

Chapter 2

Sub-Laplacian and Fundamental Solution

The goal of this chapter is to present basic aspects of Sub-Laplacian theory on a general Carnot group. In particular we will focus on some representation formulas for smooth functions, which will be fundamental in Chapter 4.

Throughout this chapter we will denote by $\mathbb{G} = (\mathbb{R}^m, \cdot, \delta_\lambda)$ a Carnot group with Lie algebra

$$\mathcal{G} = V_1 \oplus \ldots \oplus V_r.$$

Moreover, $X_1, \ldots, X_{n_1}$ will be a linear basis of V_1 and $Q \in \mathbb{N}$ the homogeneous dimension of $\mathbb{G}$.

2.1 Classical theory

Definition 2.1.1. *We call* sub-Laplacian related to the stratification $(V_1, \ldots, V_r)$ *the second order differential operator defined as*

$$\mathcal{L} := \sum_{j=1}^{n_1} X_j^2. \tag{2.1}$$

Sometimes we will also adopt the notation $\Delta_{\mathbb{G}}$ *in order to emphasize the Carnot group which we use.*

Example. *If* $(\mathbb{G}, \cdot) = (\mathbb{R}^N, +)$ *then, the sub-Laplacian associated to* $\mathbb{G}$ *is the classical Laplacian operator:*

$$\mathcal{L} = \Delta := \sum_{j=1}^{N} \partial_j^2.$$

Example. *If* $\mathbb{G} = \mathbb{H}^1$ *then* $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ *where*

$$\mathfrak{h}_1 := \{\nabla_1^{\mathbb{H}}, \nabla_2^{\mathbb{H}}\}$$
$$\mathfrak{h}_2 := \{\nabla_3^{\mathbb{H}}\}$$

hence $\Delta_{\mathbb{H}^1} = (\nabla_1^{\mathbb{H}})^2 + (\nabla_2^{\mathbb{H}})^2$ *and using the explicit representation of* $\nabla_1^{\mathbb{H}}$ *and* $\nabla_2^{\mathbb{H}}$ *(see (1.21)) we obtain*

$$\Delta_{\mathbb{H}^1} = (\partial_{x,x} + \partial_{y,y}) + 4(x^2 + y^2)\partial_{t,t} + 4(y\partial_x - x\partial_y)\partial_t. \tag{2.2}$$

Example. *If* $\mathbb{G} = \mathbb{E}$ *then the first layer of the stratification is generated by* X_1, X_2 *whose explicit expression is as in (1.30). Then the associated sub-Laplacian is*

$$\begin{aligned}\Delta_{\mathbb{E}} &= \partial_{1,1} + \partial_{2,2} - x_2\partial_{1,3} + x_1\partial_{2,3} - \Big(x_3 + \frac{x_1x_2}{6} + \frac{x_2^2}{6}\Big)\partial_{1,4} + \\ &+ \Big(-x_3 + \frac{x_1x_2}{6} + \frac{x_1^2}{6}\Big)\partial_{2,4} + \frac{1}{6}(x_2 - x_1)\partial_4 + \\ &+ \Big(-\frac{x_1x_3}{2} + \frac{x_1^3}{12} + \frac{x_1^2x_2}{12} + \frac{x_2^3}{12} + \frac{x_2^2x_1}{12} + \frac{x_2x_3}{2}\Big)\partial_{3,4} + \\ &+ \frac{1}{4}(x_1^2 + x_2^2)\partial_{3,3} + \Big[\Big(\frac{x_3}{2} - \frac{x_1}{12}(x_1 + x_2)\Big)^2 + \Big(\frac{x_3}{2} + \frac{x_2}{12}(x_1 + x_2)\Big)^2\Big].\end{aligned}$$

Here $\partial_{i,j} := \partial_{x_i,x_j}$.

Remark 2.1. It is interesting to note that a sub-Laplacian is not necessarly a second order partial differential operator when we write it using Euclidean derivatives.

Lemma 2.1.1 ([20]). *Let* $\mathcal{L}$ *be a sub-Laplacian on* $\mathbb{G}$. *Then*

1. $\mathcal{L}$ *is hypoelliptic, i.e. every distributional solution of* $\mathcal{L}u = f$ *is smooth is* f *is smooth.*

2. *$\mathcal{L}$ is invariant w.r.t. the left traslations on $\mathbb{G}$, i.e. for every fixed $q \in \mathbb{G}$ it holds $\mathcal{L}(u(\tau_q(x))) = \mathcal{L}(u)(\tau_q(x))$ for every $x \in \mathbb{G}$ and every $u \in C^\infty(\mathbb{R}^m)$. Here $\tau_q(p) := q \cdot p$.*

3. *$\mathcal{L}$ is homogeneous of degree 2, i.e for every fixed $\lambda > 0$ it holds*
$$\mathcal{L}(u(\delta_\lambda(x))) = \lambda^2(\mathcal{L}u)(\delta_\lambda(x))$$
for every $x \in \mathbb{G}$ and every $u \in C^\infty(\mathbb{R}^m)$.

4. *Let $A(x)$ be the $m \times m$ matrix obtained as $A(x) := \sigma(x)\sigma(x)^T$ where σ is the $m \times n_1$ matrix whose columns are the coefficients of $X_1, \dots, X_{n_1}$, then*
$$\mathcal{L} = \operatorname{div}(A(x)\nabla^T).$$
Moreover if we define the characteristic form of $\mathcal{L}$ *as*
$$q_{\mathcal{L}}(x, \xi) := \langle A(x)\xi, \xi\rangle$$
it holds $q_{\mathcal{L}}(x,\xi) = \sum_{i=1}^{n_1} \langle A(x)e_i, \xi\rangle^2$ where $\{e_i\}_i$ is the canonical basis of $\mathbb{R}^{n_1}$.

5. *As in the Euclidean case $\mathcal{L}$ is the second order differential operator related to the energy*
$$u \mapsto \int_\Omega |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\mathcal{L}^m.$$
More precisely, it can be proved that u is a critical point of the previous functional if and only if u is a weak solution of $\mathcal{L}u = 0$, that is, $u \in C^\infty(\Omega, \mathbb{R})$ and
$$\int_\Omega \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}}\varphi\rangle \,\mathrm{d}\mathcal{L}^m = 0 \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}).$$

Remark 2.2. Since $q_{\mathcal{L}}(x,\xi) = \sum_{i=1}^{n_1} \langle A(x)e_i, \xi\rangle^2$, then $A(x)$ is positive semidefinite for every $x \in \mathbb{G}$. Moreover, it is easy to prove that if $\mathbb{G}$ has step greater than 2 then $\mathcal{L}$ is not elliptic at any point of $\mathbb{G}$. Indeed, since $q_{\mathcal{L}}(x,\xi) = \sum_{i=1}^{n_1} \langle A(x)e_i, \xi\rangle^2$ we have,
$$\begin{aligned} N(x) : &= \{\xi \in \mathbb{R}^m \mid q_{\mathcal{L}}(x,\xi) = 0\} \\ &= \{\xi \in \mathbb{R}^m \mid \langle A(x)e_i, \xi\rangle, \forall\, i \in \{1, \dots, n_1\}\}. \end{aligned}$$

Since the vectors $A(x)e_i$, $i \in \{1, \dots, n_1\}$, are linearly independent in $\mathbb{R}^m$ for every fixed $x \in \mathbb{G}$, it follows that if $n_1 < m$, that is if the stratification of $\mathbb{G}$ is greater that 2, then for every $x \in \mathbb{G}$ there exists $\xi \in \mathbb{R}^m \setminus \{0\}$ such that $q_{\mathcal{L}}(x, \xi) = 0$. From the previous discussion the non ellipticity of $\mathcal{L}$ follows.

Definition 2.1.2. *Let $\mathbb{G}$ be a Carnot group and let $\mathcal{L}$ be a fixed sub-Laplacian on $\mathbb{G}$. Let $\Omega \subset \mathbb{G}$ be an open set. A smooth function $u : \Omega \longrightarrow \mathbb{R}$ is called $\mathcal{L}-$harmonic on Ω if*

$$\mathcal{L}u = 0 \quad on\ \Omega \tag{2.3}$$

We briefly recall two fundamental results on sub-Laplacian, see [20] for a more detailed treatment.

Theorem 2.1.2 ([20])**.** *(**Weak maximum principle**) Let $\mathcal{L}$ be a sub-Laplacian on a Carnot group $\mathbb{G}$, let $\Omega \subset \mathbb{G}$ be an open bounded set and let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function such that*

$$\begin{aligned} &\mathcal{L}u \geq 0 \quad in\ \Omega \\ &\limsup_{x \to y} u(x) \leq 0 \quad \forall\ y \in \partial\Omega \end{aligned}$$

then

$$u(x) \leq 0 \ \ \forall\ x \in \Omega$$

Corollary 2.1.3 ([20])**.** *Let $\mathcal{L}$ be a sub-Laplacian on a Carnot group $\mathbb{G}$, then the only entire $\mathcal{L}-$harmonic function vanishing at infinity is the null function.*

Theorem 2.1.4 ([20])**.** *(**Strong maximum principle**) Let $\mathcal{L}$ be a sub-Laplacian on a Carnot group $\mathbb{G}$, let $\Omega \subset \mathbb{G}$ be a connected open set and let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function such that*

$$\begin{aligned} u &\leq 0 \quad in\ \Omega \\ \mathcal{L}u &\geq 0 \quad in\ \Omega \end{aligned}$$

If there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$ then $u(x) = 0$ for every $x \in \Omega$.

We are now in position to introduce the concept of fundamental solution of a sub-Laplacian,

Definition 2.1.3. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. A function $\Gamma : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}$ is said to be a* fundamental solution for $\mathcal{L}$ *if*

1. $\Gamma \in C^\infty(\mathbb{R}^m \setminus \{0\})$;
2. $\Gamma \in L^1_{loc}(\mathbb{R}^m)$;
3. $\lim_{|x|\to\infty} \Gamma(x) = 0$;
4. $\int_{\mathbb{R}^m} \Gamma(x)\mathcal{L}\varphi(x)\mathrm{d}x = -\varphi(0) \quad \forall\ \varphi \in C_0^\infty(\mathbb{R}^m)$.

Remark 2.3. Using the hypoellipticity of $\mathcal{L}$ it can be proved that for every sub-Laplacian there exists a fundamental solution, see [20].

The following rappresentation formula holds:

Proposition 2.1.5. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$ and let Γ be a fundamental solution of $\mathcal{L}$, then for every $\varphi \in C_0^\infty(\mathbb{R}^m)$*

$$\int_{\mathbb{R}^n} \Gamma(y^{-1}\cdot x)\mathcal{L}\varphi(x)\mathrm{d}x = \varphi(y) \quad \forall\ y \in \mathbb{R}^m.$$

Proof. If $y^{-1}\cdot x = z$ then

$$\int_{\mathbb{R}^n} \Gamma(y^{-1}\cdot x)\mathcal{L}\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^n} \Gamma(z)\mathcal{L}\varphi(y\cdot z)\mathrm{d}z \tag{2.4}$$

since $\mathcal{L}$ is left invariant on $\mathbb{G}$ we have

$$(\mathcal{L}\varphi)(y\cdot z) = \mathcal{L}(\varphi(y\cdot z))$$

and using (4) in Definition 2.1.3 we get the thesis. $\square$

Now we are going to prove that a fundamental solution of a sub-Laplacian is unique. Before doing so we need the following elementary result, see [20].

Proposition 2.1.6 ([20]). *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. If $u \in L^1_{loc}(\mathbb{R}^m)$ is such that*

$$\int_{\mathbb{R}^m} u(x)\mathcal{L}\varphi(x)\mathrm{d}x = 0 \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^m) \tag{2.5}$$

then for every $\epsilon > 0$ the mollification u_ϵ of u is such that

$$\mathcal{L}u_\epsilon(x) = 0 \qquad \forall x \in \mathbb{G} \tag{2.6}$$

Theorem 2.1.7. *Let $\mathcal{L}$ be a sub-Laplacian on a Carnot group $\mathbb{G}$, then the fundamental solution Γ of $\mathcal{L}$ is unique.*

Proof. If Γ_1 and Γ_2 are fundamental solution of $\mathcal{L}$ then $u := \Gamma_1 - \Gamma_2$ is such that $u \in L^1_{loc}(\mathbb{R}^m)$, $\int_{\mathbb{R}^m} u(x)\mathcal{L}\varphi(x)\mathrm{d}\mathcal{L}^m = 0 \;\; \forall\varphi \in C_0^\infty(\mathbb{R}^m)$ and $u(x) \to 0$ as $|x| \to \infty$. Hence by Proposition 2.1.6 we obtain that $\forall\epsilon > 0$

$$\mathcal{L}u_\epsilon(x) = 0 \;\; \forall x \in \mathbb{R}^m.$$

Then, since $u_\epsilon \to 0$ as $|x| \to \infty$, Corollary 2.1.3 implies $u_\epsilon \equiv 0$. On the other hand $u_\epsilon \to u$ in $L^1_{loc}(\mathbb{R}^m)$, therefore $u(x) = 0$ a.e $x \in \mathbb{R}^m$, so that $\Gamma_1 = \Gamma_2$ in $\mathbb{R}^m \setminus \{0\}$. □

Hence we proved the uniqueness part of the following important theorem, for the existence part we refer to the classical paper of Hörmander [94].

Theorem 2.1.8. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. Then there exists a unique fundamental solution of $\mathcal{L}$.*

Proposition 2.1.9. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$ and let Γ be the fundamental solution of $\mathcal{L}$. Then*

1. Γ is symmetric, *i.e.* $\Gamma(x^{-1}) = \Gamma(x)$ $\forall x \in \mathbb{G} \setminus \{0\}$,
2. Γ is homogeneous of degree $2 - Q$, *i.e.* $\Gamma(\delta_\lambda(x)) = \lambda^{2-Q}\Gamma(x) \;\; \forall\, x \in \mathbb{G} \setminus \{0\}$, $\forall\lambda > 0$,
3. Γ is positive, *i.e.* $\Gamma(x) > 0 \;\; \forall x \in \mathbb{G} \setminus \{0\}$,
4. Γ has a pole at 0, *i.e* $\lim_{x\to 0} \Gamma(x) = \infty$.

Proof. 1. Given $\varphi \in C_0^\infty(\mathbb{R}^m)$, define

$$u(x) := \int_{\mathbb{R}^m} \Gamma(y^{-1}\cdot x)\mathcal{L}\varphi(y)\mathrm{d}\mathcal{L}^m(y), \quad x \in \mathbb{G}.$$

Then, u is smooth and vanishes at infinity. Hence, for every $\psi \in C_0^\infty(\mathbb{R}^m)$,

$$\begin{aligned}\int_{\mathbb{R}^m} \mathcal{L}u(x)\psi(x)\mathrm{d}\mathcal{L}^m &= \int_{\mathbb{R}^m} u(x)\mathcal{L}\psi(x)\mathrm{d}\mathcal{L}^m \\ &= -\int_{\mathbb{R}^m} \mathcal{L}\varphi(y)\Big(\int_{\mathbb{R}^m} \Gamma(y^{-1}\cdot x)\mathcal{L}\psi(x)\mathrm{d}\mathcal{L}^m(x)\Big)\mathrm{d}\mathcal{L}^m \\ &= \int_{\mathbb{R}^m} \mathcal{L}\varphi(x)\psi(x)\mathrm{d}\mathcal{L}^m(x)\end{aligned}$$

so, $\mathcal{L}(u-\varphi) = 0$ in $\mathbb{G}$. Since $u-\varphi$ is an entire harmonic function vanishing at infinity we deduce $u - \varphi \equiv 0$ in $\mathbb{R}^m$. Therefore, for every $\varphi \in C_0^\infty(\mathbb{R}^m)$

$$\varphi(0) = u(0) = -\int_{\mathbb{R}^m} \Gamma(y^{-1})\mathcal{L}\varphi(y)\mathrm{d}\mathcal{L}^m(y)$$

and hence $y \to \Gamma(y^{-1})$ is a fundamental solution of $\mathcal{L}$, the thesis follows by Theorem 2.1.8.

2. For any $\lambda > 0$ let us define $\bar{\Gamma}(x) := \lambda^{Q-2}\Gamma(\delta_{\lambda(x)})$, then obviously $\bar{\Gamma}$ satisfies $(1), (2)$ and (3) in Definition 2.1.3. Moreover for every $\varphi \in C_0^\infty(\mathbb{R}^m)$ we have

$$\begin{aligned}\int_{\mathbb{R}^m} \bar{\Gamma}(x)\mathcal{L}\varphi(x)\mathrm{d}\mathcal{L}^m(x) &= \lambda^{Q-2}\int_{\mathbb{R}^m} \Gamma(\delta_{\lambda(x)}\mathcal{L}\varphi(x)\mathrm{d}\mathcal{L}^m(x) \\ &= \lambda^{-2}\int_{\mathbb{R}^m} \Gamma(y)(\mathcal{L}\varphi)(\delta_{1/\lambda})\mathrm{d}\mathcal{L}^m(y) \\ &= \int_{\mathbb{R}^m} \Gamma\mathcal{L}(\varphi(\delta_{1/\lambda}(y)))\mathrm{d}\mathcal{L}^m(y) = \varphi(0)\end{aligned}$$

and then the thesis follows by the uniqueness of the fundamental solution of $\mathcal{L}$.

3. For every $\varphi \in C_0^\infty(\mathbb{R}^m)$, $\varphi \geq 0$. We define

$$u(y) := \int_{\mathbb{R}^m} \Gamma(y^{-1}\cdot x)\varphi(x)\mathrm{d}\mathcal{L}^m(x), \quad y \in \mathbb{G}$$

then it is easy to see that $u \in C^\infty(\mathbb{R}^m)$ and that u vanishes at infinity. Moreover $\mathcal{L}u = \varphi$, indeed for every $\psi \in C_0^\infty(\mathbb{R}^m)$ one has

$$\begin{aligned}\int_{\mathbb{R}^m} (\mathcal{L}u)(x)\psi(y)\mathrm{d}\mathcal{L}^m(y) &= \int_{\mathbb{R}^m} u(x)\mathcal{L}\psi(y)\mathrm{d}\mathcal{L}^m(y)\\ &= \int_{\mathbb{R}^m} \varphi(x)\Big(\int_{\mathbb{R}^m} \Gamma(y^{-1}\cdot x)\mathcal{L}\psi(y)\mathrm{d}\mathcal{L}^m(y)\Big)\mathrm{d}\mathcal{L}^m\end{aligned}$$

and by symmetry of Γ we conclude

$$\begin{aligned}\int_{\mathbb{R}^m} (\mathcal{L}u)(x)\psi(y)\mathrm{d}\mathcal{L}^m(y) &= \int_{\mathbb{R}^m} \varphi(x)\Big(\int_{\mathbb{R}^m} \Gamma(x^{-1}\cdot y)\mathcal{L}\psi(y)\mathrm{d}\mathcal{L}^m(y)\Big)\mathrm{d}\mathcal{L}^m\\ &- \int_{\mathbb{R}^N} \varphi(x)\psi(x)\mathrm{d}x\end{aligned}$$

and the thesis follows by the fundamental Lemma of calculus of variations. Since $\mathcal{L}u(x) = -\varphi(x) \leq 0\ \forall x \in \mathbb{G}$ and $\lim_{|x|\to\infty} u(x) = 0$ then by the maximum principle $u(x) \geq 0\ \forall x \in \mathbb{G}$. Hence $\Gamma(x) \geq 0\ \forall x \in \mathbb{G}\setminus\{0\}$ and the thesis follows by the strong maximum principle and the fact that a fundamental solution of $\mathcal{L}$ cannot be identically 0.

4. Let d be a fixed homogeneous norm on $\mathbb{G}$. Then

$$k := \min\{\Gamma(x) \mid \mathrm{d}(x) = 1\} > 0$$

therefore

$$\Gamma(x) = \frac{1}{\mathrm{d}(x)}\Gamma(\delta_{1/\mathrm{d}(x)}(x)) \geq \frac{k}{\mathrm{d}(x)^{Q-2}}$$

and the thesis follows.

□

The following fundamental result is proved in [129].

Theorem 2.1.10. *For every open and bounded set $\Omega \subset \mathbb{R}^m$ there exist $C_1, C_2, r_0 > 0$ such that for every $x \in \Omega$ and every $y \in \Omega \setminus \{x\}$ with* $\mathrm{d}_{cc}(x,y) \leq r_0$

1. $C_1\frac{\mathrm{d}_{cc}(x,y)^2}{|B(x,\mathrm{d}_{cc}(x,y))|} \leq \Gamma(x,y) \leq C_2\frac{\mathrm{d}_{cc}(x,y)^2}{|B(x,\mathrm{d}_{cc}(x,y))|}$;

2. $|\nabla_{\mathbb{G}}\Gamma(x,y)| \le C_2 \frac{\mathrm{d}_{cc}(x,y)}{|B(x,\mathrm{d}_{cc}(x,y))|}$

Here $B(x,r)$ is the ball of center x and radious $r>0$ made with respect to the distance d_{cc}.

Definition 2.1.4. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. A norm $\mathrm{d} \in C^{\infty}(\mathbb{R}^m \setminus \{0\})$ is said an $\mathcal{L}-$gauge if it is symmetric (i.e $\mathrm{d}(x^{-1}) = \mathrm{d}(x)$), homogeneous and such that*

$$\mathcal{L}(\mathrm{d}^{2-Q})(x) = 0 \quad \forall\ x \in \mathbb{R}^m \setminus \{0\}.$$

Example. *We know that if $(\mathbb{G},\cdot,\delta_\lambda) = (\mathbb{R}^N,+,\lambda)$ with $N \ge 3$ then the canonical sub-Laplacian is the classical Laplace operator, hence denoting by $\mathrm{d}(x) := |x|$ the Euclidean norm it follows that d is smooth out of the origin, $1-$homogeneous and such that*

$$\Delta(\mathrm{d}^{2-N}) = 0$$

so it is a $\Delta-$gauge on $\mathbb{G}$.

Example. *If $\mathbb{G} = \mathbb{H}^1$ then*

$$\Delta_{\mathbb{H}^1} = (\partial_x^2 + \partial_y^2) + 4(x^2+y^2)\partial_t^2 + 4(y\partial_x - x\partial_y)\partial_t$$

is the canonical sub-Laplacian on $\mathbb{H}^1$. Let us define

$$\mathrm{d}(x,y,t) := (|(x,y)|_{\mathbb{R}^2}^4 + 16|t|^2)^{\frac{1}{4}}$$

then d is smooth out of the origin, $1-$homogeneous and symmetric. Moreover, since $Q=4$ we obtain $2-Q=-2$ and

$$\Delta_{\mathbb{H}^1}(\mathrm{d}^{-2}) = 0 \ \textit{on}\ \ \mathbb{H}^1 \setminus \{0\}.$$

we conclude that d is a $\Delta_{\mathbb{H}^1}-$gauge on $\mathbb{H}^1$.

Remark 2.4. Let us observe that for any Carnot group $\mathbb{G}$ there exists at least one $\mathcal{L}-$gauge on $\mathbb{G}$. Indeed, let Γ be the fundamental solution of $\mathcal{L}$ then, by Proposition 2.1.9, the function

$$\mathrm{d}(x) := \begin{cases} \Gamma(x)^{1/(2-Q)} & \text{if} \quad x \in \mathbb{G} \setminus \{0\} \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is an $\mathcal{L}-$gauge on $\mathbb{G}$.

Now we state, without proof, a representation formula which we will use in Chapter 4, see [20].

Theorem 2.1.11. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$ and let* d *be an $\mathcal{L}-$gauge on $\mathbb{G}$. Let $u \in C^2(\Omega)$ where $\Omega \subset \mathbb{G}$ is open. Then, for every $x \in \Omega$ and $r > 0$ such that $\bar{B}_{\mathrm{d}}(x,r) \subset \Omega$, we have*

$$u(x) = \frac{\beta(Q-2)}{r^{Q-1}} \int_{\partial B_d(x,r)} \frac{|\nabla_{\mathbb{G}}\mathrm{d}|^2(x^{-1}\cdot z)}{|\nabla(\mathrm{d}(x^{-1}\cdot))|(z)} u(z)\mathrm{d}H^{m-1}(z) - \\ - \beta \int_{B_d(x,r)} (\mathrm{d}^{2-Q}(x^{-1}\cdot z) - r^{2-Q})\mathcal{L}u(z)\mathrm{d}\mathcal{L}^m(z) \tag{2.7}$$

where

$$(\beta)^{-1} := (Q-2) \int_{\partial B_d(0,1)} \frac{|\nabla_{\mathbb{G}}\mathrm{d}|^2(z)}{|\nabla(\mathrm{d}(0\circ\cdot))|(z)} \mathrm{d}H^{m-1}(z) \tag{2.8}$$

Corollary 2.1.12. *If $u \in C^2(\Omega)$ is $\mathcal{L}-$harmonic then*

$$u(x) = \frac{\beta(Q-2)}{r^{Q-1}} \int_{\partial B_d(x,r)} \frac{|\nabla_{\mathbb{G}}\mathrm{d}|^2(x^{-1}\cdot z)}{|\nabla(\mathrm{d}(x^{-1}\cdot\cdot))|(z)} u(z)\mathrm{d}H^{m-1}(z)$$

Remark 2.5. If $\mathbb{G} = (\mathbb{R}^N, +, \lambda)$ and $\mathcal{L} = \triangle$ then Corollary 2.1.12 gives the classical Gauss theorem for $\triangle-$harmonic functions. Indeed in this case

$$\frac{|\nabla_{\mathbb{G}}\mathrm{d}|^2(x^{-1}\cdot z)}{|\nabla(\mathrm{d}(x^{-1}\cdot))|(z)} = 1$$

and the second integral is equal to 0.

Using Theorem 2.1.11 we can prove that $\mathcal{L}-$gauges are unique up to multiplicative constants

Proposition 2.1.13. *Let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. If* d *is an $\mathcal{L}-$gauge on $\mathbb{G}$ and β is the constant in (2.8) then*

$$\Gamma := \beta\mathrm{d}^{2-Q} \tag{2.9}$$

is the fundamental solution of $\mathcal{L}$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and let $r > 0$ such that $supp(\varphi) \subset B_d(0,r)$. Appling Theorem 2.1.11 we obtain

$$\varphi(0) = \beta \int_{B_d(0,r)} (\mathrm{d}^{2-Q}(z) - r^{2-Q}) \mathcal{L}\varphi(z) \mathrm{d}\mathcal{L}^m(z) \tag{2.10}$$

Moreover,

$$\int_{B_d(0,r)} \mathcal{L}\varphi(x) \mathrm{d}\mathcal{L}^m(x) = \int_{\partial B_d(0,r)} \langle A \cdot \nabla u, \nu \rangle \, \mathrm{d}H^{m-1} = 0$$

where A is as in Lemma 2.1.1 and ν is the exterior normal to $\partial B_d(0,r)$. Hence, for every $\varphi \in C_0^\infty(\Omega)$

$$\varphi(0) = \beta \int_{B_d(0,r)} \mathrm{d}^{2-Q}(z) \mathcal{L}\varphi(z) \mathrm{d}\mathcal{L}^m(z) \tag{2.11}$$

By the uniqueness of the fundamental solution of $\mathcal{L}$, it remains to prove that the function $\Gamma := \beta \mathrm{d}^{2-Q}$ satisfies (2) and (3) in Definition 2.1.3. To prove (2) we can proceed as follows,

$$\begin{aligned}
\int_{B_d(0,r)} \mathrm{d}^{2-Q}(z) \mathrm{d}\mathcal{L}^m(z) &= \sum_{i=1}^{m} \int_{\{r/2^{i+1} \le \mathrm{d} \le r/2^i\}} \mathrm{d}^{2-Q}(z) \mathrm{d}\mathcal{L}^m(z) \\
&\le \left(\frac{r}{2}\right)^{2-Q} \sum_{i=0}^{m} \frac{1}{2^{i(2-Q)}} \int_{\{r/2^{i+1} \le \mathrm{d} \le r/2^i\}} \mathrm{d}\mathcal{L}^m(z) \\
&= \left(\frac{r}{2}\right)^{2-Q} \sum_{i=0}^{m} \frac{1}{2^{i(2-Q)}} \left(\frac{r}{2^i}\right)^{Q} \int_{\{1/2 \le \mathrm{d} \le 1\}} \mathrm{d}\mathcal{L}^m(z) \\
&= C r^Q \left(\frac{r}{2}\right)^{2-Q} \sum_{i=0}^{m} 2^{-2i}
\end{aligned}$$

and this implies that $\mathrm{d}^{2-Q} \in L^1_{loc}(\mathbb{R}^m)$. Finally, (3) easily follows from the fact that d is a distance and $2 - Q \le 0$. $\square$

We conclude this section providing another representation Theorem due to Citti-Garofalo and Lanconelli and proved in [42].

To this end, for every $x \in \mathbb{G} \equiv \mathbb{R}^m$ and $r > 0$, we define by analogy with the Euclidean case the set

$$\Omega_r(x) := \left\{ y \in \mathbb{R}^m \mid \Gamma(x,y) > \frac{1}{r} \right\} \tag{2.12}$$

and we call it the $\mathcal{L}-$ ball centered at x with radius r.

Lemma 2.1.14. *Let $\Omega \subset \mathbb{R}^m$ be an open set. If $x \in \Omega$ is fixed, there exists $\bar{r} = \bar{r}_x > 0$ such that for every $0 < r < \bar{r}$*

$$\bar{\Omega}_r(x) \subset \Omega.$$

Moreover, $\partial\Omega_r(x)$ is an $(m-1)-$dimensional manifold of class C^∞ for almost all $r \in (0, \bar{r})$.

We are now in position to prove the following representation Theorem,

Theorem 2.1.15 ([42]). *Let A be a symmetric, positive definite matrix such that $\mathcal{L} = \operatorname{div}(A\nabla^T)$ as in Lemma 2.1.14. If $\Omega \subset \mathbb{R}^m$ is a smooth open set then for every $u \in C^\infty(\Omega)$, every $x \in \Omega$ and almost every $r \in (0, \bar{r})$ where $\bar{r} > 0$ is as in Lemma 2.1.14:*

$$u(x) = \int_{\partial\Omega_r(x)} \frac{A\nabla\Gamma \cdot \nabla\Gamma}{|\nabla\Gamma|} u \, \mathrm{d}H^{m-1} + \int_{\Omega_r(x)} A\nabla\Gamma \cdot \nabla u \mathrm{d}\mathcal{L}^m \tag{2.13}$$

where we have let $\Gamma = \Gamma(x, \cdot)$ and ∇ denotes the Euclidean gradient.

Sketch of the Proof: Let us fix $x \in \Omega$ and let $\bar{r} > 0$ as in Lemma 2.1.14. For each $0 < \epsilon < r < \bar{r}$ we denote by U_ϵ the open set defined by

$$U_\epsilon := \Omega_r(x) \setminus \bar{\Omega}_\epsilon(x)$$

and $v := \Gamma(x, \cdot)$. By the divergence theorem and by using the fact that $\mathcal{L}v = 0$ in U_ϵ we obtain

$$\int_{U_\epsilon} A\nabla\Gamma \cdot \nabla u \, \mathrm{d}\mathcal{L}^m = \int_{\partial\Omega_r(x)} (A\nabla\Gamma \cdot \nu) u \, \mathrm{d}H^{m-1} - \int_{\partial\Omega_\epsilon(x)} (A\nabla\Gamma \cdot \nu) u \, \mathrm{d}H^{m-1} \tag{2.14}$$

Now, since $(A\nabla\Gamma \cdot \nabla u)^{1/2}$ is locally integrable (see [42]) we have

$$\lim_{\epsilon \to 0} \int_{U_\epsilon} A\nabla\Gamma \cdot \nabla u \, \mathrm{d}\mathcal{L}^m = \int_{\Omega_r(x)} A\nabla\Gamma \cdot \nabla u \mathrm{d}\mathcal{L}^m.$$

On the other hand since $\partial\Omega_r(x) = \partial\Big\{y \mid \Gamma(x, y) > 1/r\Big\}$, we have

$$\int_{\partial\Omega_r(x)} (A\nabla\Gamma \cdot \nu) u \, \mathrm{d}H^{m-1} = -\int_{\partial\Omega_r(x)} \frac{A\nabla\Gamma \cdot \nabla\Gamma}{|\nabla\Gamma|} u \mathrm{d}H^{m-1}. \tag{2.15}$$

Now, for each $\varphi \in C_c^\infty(\Omega)$ cut off on $\bar{\Omega}_\epsilon(x)$ denoting by $S_\epsilon := \{\Gamma(x,y) \le 1/\epsilon\}$ and $\tilde{S}_\epsilon := \{\Gamma(x,y) = 1/\epsilon\}$ we have

$$u(x) = -\int_{\mathbb{R}^m} \Gamma(x,y)\mathcal{L}(u\varphi)(y)\mathrm{d}\mathcal{L}^m(y) = -\lim_{\epsilon\to 0}\int_{S_\epsilon} \Gamma(x,y)\mathcal{L}(u\varphi)(y)\mathrm{d}\mathcal{L}^m(y) \tag{2.16}$$

$$= \lim_{\epsilon\to 0}\Big(\int_{S_\epsilon} A\nabla\Gamma\cdot\nabla(u\varphi)\mathrm{d}\mathcal{L}^m(y) - \frac{1}{\epsilon}\int_{\tilde{S}_\epsilon} A\nabla(u\varphi)\cdot\nu\mathrm{d}H^{m-1}\Big) \tag{2.17}$$

Integrating by parts and using the divergence theorem we obtain:

$$\begin{aligned} u(x) = &-\lim_{\epsilon\to 0}\int_{S_\epsilon} \mathcal{L}\Gamma u\varphi \mathrm{d}\mathcal{L}^m(y) - \lim_{\epsilon\to 0}\int_{\tilde{S}_\epsilon} (A\nabla\Gamma\cdot\nu)u\ \mathrm{d}H^{m-1} - \\ &-\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\{\Gamma(x,y)\ge 1/\epsilon\}} \mathcal{L}u\ \mathrm{d}\mathcal{L}^m(y) \\ = &-\lim_{\epsilon\to 0}\int_{\tilde{S}_\epsilon} (A\nabla\Gamma\cdot\nu)u\ \mathrm{d}\mathcal{L}^m(y) \end{aligned}$$

where in the last equality we used the fact $\mathcal{L}\Gamma = 0$ on $\{\Gamma(x,y) \le 1/\epsilon\}$ and

$$\frac{1}{\epsilon}\Big|\Big\{\Gamma(x,y) \ge \frac{1}{\epsilon}\Big\}\Big| = \frac{1}{\epsilon}|\Omega_\epsilon(x)| = o(1)$$

as $\epsilon \to 0$ (see [42]). Passing to the limit as $\epsilon \to 0$ in (2.14) we obtain the thesis. □

Chapter 3

Hypersurfaces in the Heisenberg group

The aim of this chapter is to introduce the concept of $\mathbb{H}-$regular hypersurfaces, i.e noncritical level sets of $C^1_{\mathbb{H}}$ functions. These surfaces are the natural Heisenberg counterpart of C^1 Euclidean hypersurfaces, nevetheless they can be very irregular from an Euclidean point of view. These objects were firstly studied in [80] in connection to the theory of rectifiable sets in the Heisenberg group. Successively they were introduced in more general Carnot groups ([81]) and in the context of Carnot-Carathéodory spaces ([45]). One of the main important feature that $\mathbb{H}-$regular hypersurfaces share with C^1 Euclidean hypersurfaces is the presence of an implicit function theorem ([80, 45]). More precisely, for every $\mathbb{H}-$regular hypersurface S, there is a continuous map ϕ which locally parametrizes S in a suitable intrinsic sense, see Theorem 3.1.1. Actually, the map ϕ is not only continuous, indeed in [6, 45] it is proved that ϕ is uniformly $\nabla^\phi-$differentiable, see Definition 3.2.4. As we will see later, it is a difficult task to verify directly the $\nabla^\phi-$differentiability of a given function. Nevertheless this difficulty can be, somehow, bypassed using some interesting results contained in a series of papers ([14, 13, 117]) which we report at the end of the chapter.

Throughout this chapter we will use the notations introduced in section 1.3. More precisely, we refer to $\mathbb{H}^n$ as $\mathbb{R}^{2n+1}$ equipped by the group law defined in (1.23) and we identify points $p \in \mathbb{H}^n$ with $(s, x_1, \ldots, x_{2n}) \in \mathbb{R} \times \mathbb{R}^{2n}$ in such a way that $p = \exp(s\nabla_1^{\mathbb{H}})\exp(\sum_{i=1}^{2n} x_i \nabla_i^{\mathbb{H}})(0)$.

3.1 Hypersurfaces and the implicit function Theorem

Definition 3.1.1. *A set $S \subset \mathbb{H}^n$ is an $\mathbb{H}-$regular hypersurface if for each $p \in S$ there are an open $p \in U \subset \mathbb{H}^n$ and $f \in C^1_{\mathbb{H}}(U, \mathbb{R})$ such that $\nabla_{\mathbb{H}} f \neq 0$ in U and*

$$S \cap U = \{w \in U \mid f(w) = 0\}.$$

Remark 3.1. We point out that the notion of $\mathbb{H}-$regular hypersurface introduced in Definition 3.1.1 is very similar to the classical one. However there are $\mathbb{H}-$regular hypersurfaces that, from an Euclidean point of view, are very irregular. For example, there is an $\mathbb{H}-$regular hypersurface with fractional Euclidean dimension equal to 2.5, see [97]. A more explicit example is

$$S := \{(s, x_1, x_2) \in \mathbb{H}^1 \mid s = \sqrt{s^4 + x_1^4 + x_2^2}\}$$

indeed, by Remark 1.15 we know that S is the zero set of a $C^1_{\mathbb{H}}$ function with non zero intrinsic gradient and hence S is an $\mathbb{H}-$regular hypersurface; on the other hand we also know that this function is not Euclidean regular at the origin.

Remark 3.2. Let $S \subset \mathbb{H}^n$ be an Euclidean C^1 hypersurface then it is an $\mathbb{H}-$regular hypersurface if it hasn't *characteristic points*, i.e if for all $p \in S$

$$H_p S \nsubseteq T_p S$$

where $T_p S$ denotes the Euclidean tangent space to S at p.

We denote by $\nu_S(p)$ the *horizontal normal* to S at $p \in S$, i.e the vector

$$\nu_S(p) := -\frac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|_p} \tag{3.1}$$

and by $T_{\mathbb{H}}S(p)$ the *tangent plane to* S *at* p,

$$T_{\mathbb{H}}S(p) := p \cdot T^g_{\mathbb{H}}S(p)$$

where $T^g_{\mathbb{H}}S(p) := \Big\{ q \in \mathbb{H}^n \mid \langle \nu_{p^{-1}\cdot S}(0), \pi(q) \rangle \Big\}$, where π is as in Definition 1.2.9.

Remark 3.3. A priori it should seem that the horizontal normal depends on the particular choice of f, we will see later that this is not the case, see Remark 3.4

As already said in the introduction of this chapter, one of the main advantage of definition 3.1.1 is the presence of an implicit function theorem. In other words, every $\mathbb{H}-$regular hypersurface is locally the graph of a continuous function in a suitable intrinsic sense. Before stating the precise statement we recall some basic concepts.

The exponential map $\exp^*$ definend in Section 1.3 induces a split in $\mathbb{H}^n$ into homogeneous subgroups, i.e subgroups of $\mathbb{H}^n$ closed under the dilation family. Indeed, denoting:

$$\begin{aligned} \mathbb{W} &:= \{p \in \mathbb{H}^n \mid p = (0, x),\ x \in \mathbb{R}^{2n}\} \\ \mathbb{V} &:= \{p \in \mathbb{H}^n \mid p = (s, 0),\ s \in \mathbb{R}\} \end{aligned} \tag{3.2}$$

we have that $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ and $\mathbb{W} \cap \mathbb{V} = \{0\}$. It is clear that the maps

$$\begin{aligned} \pi_{\mathbb{W}} :& \mathbb{H}^n \longrightarrow \mathbb{W} \\ & (s, x) \mapsto (0, x) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \pi_{\mathbb{V}} :& \mathbb{H}^n \longrightarrow \mathbb{V} \\ & (s, x) \mapsto (s, 0) \end{aligned} \tag{3.4}$$

are continuous and

$$c(\|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\|) \leq \|p\| \leq (\|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\|), \tag{3.5}$$

for some constant $c = c(\mathbb{W}, \mathbb{V}) > 0$ (see also [9] for a generalization of this statement in Carnot groups of any step).
Note that, if $(s, 0) \in \mathbb{V}$, $(0, x) \in \mathbb{W}$, then the sum turn out to be:

$$(0, x) \cdot (s, 0) = (s, x). \tag{3.6}$$

From now on we will denote a point $(0, x) \in \mathbb{W}$ by $x \in \mathbb{R}^{2n}$ and $(s, 0) \in \mathbb{V}$ by $s \in \mathbb{R}$

Definition 3.1.2. *The intrinsic (left) graph of a function $\phi : \omega \subseteq \mathbb{W} \longrightarrow \mathbb{V}$ is the set*

$$\begin{aligned} \mathit{graph}(\phi) : &= \{x \cdot \phi(x) \mid x \in \omega\} \\ &= \{(\phi(x), x) \mid x \in \omega\} \end{aligned} \tag{3.7}$$

the intrinsic subgraph of ϕ is the set

$$E_\phi := \{(s, x) \in \mathbb{R} \times \omega \mid s \leq \phi(x)\} \tag{3.8}$$

and the graph map of ϕ *is the function $\Phi : \omega \longrightarrow \mathbb{H}^n$ defined by*

$$\Phi(x) := x \cdot \phi(x) = (\phi(x), x) \tag{3.9}$$

We are now in position to state the implicit function theorem for $\mathbb{H}-$regular hypersurfaces, see [80, 45].

Theorem 3.1.1. *Let $U \subseteq \mathbb{H}^n$ be open, $0 \in U$ and let $f \in C^1_{\mathbb{H}}(U, \mathbb{R})$ be such that $\nabla^{\mathbb{H}}_1 f(0) > 0$ and $f(0) = 0$. Let*

$$E := \{p \in \Omega \mid f(p) < 0\}$$

and

$$S := \{p \in \Omega \mid f(p) = 0\};$$

Then there exists an open and bounded $\omega \subset \mathbb{W}$ and $h > 0$, such that if we put $U := \omega \times J$ where $J := \{(s, 0) \in \mathbb{R} \times \mathbb{R}^{2n} \mid s \in (-h, h)\}$, then

$$E \text{ has finite } \mathbb{H}-\text{perimeter in } U; \tag{3.10}$$

$$\partial E \cap U = S \cap U; \tag{3.11}$$

$$|\partial E|_{\mathbb{H}} \llcorner U \text{ is concentrated on } S \text{ and } \nu_E = \nu_S \ |\partial E|_{\mathbb{H}} - a.e \text{ on } U \tag{3.12}$$

Furthermore, there is a continuous function $\phi : \omega \longrightarrow (-h, h)$ such that

$$S \cap U = \textit{graph}(\phi).$$

Remark 3.4. By (3.12) it follows that the horizontal normal to an $\mathbb{H}-$regular hypersurface does not depend on the defining function f.

Remark 3.5. Theorem 3.1.1 has been generalized in many directions. In [81] authors proved an analogous result for hypersurfaces defined in general Carnot groups whereas in [45] is given a similar theorem for codimension 1 hypersurfaces in Carnot-Carathéodory spaces and in [102] for general codimension surfaces. Finally, in [9], is proved that an $\mathbb{H}-$regular submanifold (not necessarily of codimension 1) is locally the intrinsic graph of a continuous function.

3.2 Intrinsic differentiability

In this section we provide some necessary conditions for which the image of a continuous function $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V} \equiv \mathbb{R}$ is an $\mathbb{H}-$regular hypersurface, we refer to [6, 45] for the details.

We begin our discussion introducing a suitable function defined on ω.

Definition 3.2.1. *Let $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V}$ be a continuous function. The* graph distance *between $x, y \in \mathbb{W}$ is defined by*

$$\mathrm{d}_\phi(x,y) := \frac{1}{2}\Big(\|\pi_{\mathbb{W}}(\Phi(x)^{-1}\cdot\Phi(y))\| + \|\pi_{\mathbb{W}}(\Phi(y)^{-1}\cdot\Phi(x))\|\Big) \tag{3.13}$$

where Φ is as in (3.9).

Rewriting (3.13) in coordinates we obtain that for each $x = (x_1, \dots, x_{2n})$ $y = (y_1, \dots, y_{2n}) \in \mathbb{W}$

$$\begin{aligned} \mathrm{d}_\phi(x,y) = \frac{1}{2}\max\Big\{ &|(x_1-y_1,\dots,x_{2n-1}-y_{2n-1})|_{\mathbb{R}^{2n-1}}, \\ &|y_{2n}-x_{2n}-2\phi(x)(y_n-x_n)+\sigma(x,y)|^{1/2}\Big\}+ \\ +\frac{1}{2}\max\Big\{ &|(x_1-y_1,\dots,x_{2n-1}-y_{2n-1})|_{\mathbb{R}^{2n-1}}, \\ &|x_{2n}-y_{2n}-2\phi(y)(x_n-y_n)+\sigma(y,x)|^{1/2}\Big\} \end{aligned} \tag{3.14}$$

where

$$\sigma(x,y) := \sum_{i=1}^{n-1}(y_i x_{n+i} - y_{n+i}x_i) \text{ if } n \geq 2$$

and

$$\sigma(x,y) = 0 \text{ if } n = 1.$$

Remark 3.6. By some simple manipulations it is easy to see that d_ϕ is equivalent to the quantity introduced by Ambrosio-Serra Cassano and Vittone in [6], that is:

$$\begin{aligned}\max\{&|(x_1 - y_1, \dots, x_{2n-1} - y_{2n-1})|_{\mathbb{R}^{2n-1}}, \\ &|y_{2n} - x_{2n} - (\phi(x) + \phi(y))(y_n - x_n) + \sigma(x,y)|^{1/2}\},\end{aligned} \tag{3.15}$$

where $\sigma(x,y)$ is as in (3.14).

Definition 3.2.2. *Let $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V} \equiv \mathbb{R}$. We say that ϕ is an (intrinsic) Lipschitz continuous function in ω and we write $\phi \in Lip_{\mathbb{W}}(\omega)$, if there is a constant $L > 0$ such that*

$$|\phi(x) - \phi(y)| \leq L \ \mathrm{d}_\phi(x,y) \qquad \forall x, y \in \omega. \tag{3.16}$$

The Lipschitz constant of ϕ in ω is the infimum of the numbers L such that (3.16) holds and we write $Lip(\phi)$ to denote it. Moreover we say that $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ is a locally (intrinsic) Lipschitz function in ω and we write $\phi \in Lip_{loc,\mathbb{W}}(\omega)$, if $\phi \in Lip_{\mathbb{W}}(\omega')$ for every $\omega' \Subset \omega$, we denote by $Lip(\phi, \omega')$ the Lipschitz constant of $\phi_{|\omega'}$.

Proposition 3.2.1 ([6])**.** *If $\phi \in Lip_{\mathbb{W}}(\omega)$ with Lipschitz constant $L > 0$, then d_ϕ is a quasidistance on ω, that is*

- $\mathrm{d}_\phi(x,y) = 0$ *iff* $x = y$*;*
- $\mathrm{d}_\phi(x,y) = \mathrm{d}_\phi(y,x) \ \ \forall x, y \in \omega$*;*
- *there is* $C = C(L) > 0$ *such that* $\mathrm{d}_\phi(x,y) \leq C\big(\mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(z,y)\big)$ $\forall x, y, z \in \omega$.

Proof. We only prove the third statement, while we refer to [6, 43, 45] for the others ones. It immediately follows from the explicit expression of d_ϕ that:

$$\begin{aligned} &\mathrm{d}_\phi(x,y) \leq \\ &\leq \mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(y,z) + |\phi(x)-\phi(z)|^{1/2}|z_n - y_n|^{1/2} + \\ &+ |\phi(y)-\phi(z)|^{1/2}|z_n - x_n|^{1/2} \end{aligned} \tag{3.17}$$

hence since $\phi \in Lip_{\mathbb{W}}(\omega)$ we have

$$\mathrm{d}_\phi(x,y) \leq (1 + (Lip(\phi))^{1/2})\Big(\mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(y,z)\Big)$$

□

Definition 3.2.3. *An homomorphism $L : \mathbb{W} \longrightarrow \mathbb{V}$ is said an* homogeneous homomorphism *if*

$$L(\delta_r(x)) = rL(x) \quad \forall r > 0, \forall x \in \mathbb{W}$$

Definition 3.2.4 ([6, 45])**.** *Let $\omega \subset \mathbb{W}$ be open and let $\psi, \phi : \omega \longrightarrow \mathbb{R}$ be given continuous functions, then*

1. *We say that ψ is* $\nabla^\phi-$differentiable *at $y \in \omega$ if there is an homogeneous homomorphism $L : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ such that*
$$\lim_{x \to y} \frac{|\psi(x) - \psi(y) - L(\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x)))|}{\mathrm{d}_\phi(x,y)} = 0.$$
2. *We say that ψ is* uniformly $\nabla^\phi-$differentiable *at $z = (z_1, \dots, z_{2n}) \in \omega$ if there is an homogeneous homomorphism $L : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ such that*
$$\lim_{r \to 0} \mathcal{M}(\psi, z, L, r) = 0$$
where
$$\mathcal{M}(\psi, z, L, r) := \sup_{x,y \in I_r(z), x \neq y} \left\{ \frac{\psi(x) - \psi(y) - L(\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x)))}{\mathrm{d}_\phi(x,y)} \right\}.$$

Where we denoted by $I_r(z)$ the set

$I_r(z) :=$
$\{(x_1, \dots, x_{2n}) \in \omega \mid |(x_1, \dots, x_{2n-1}) - (z_1, \dots, z_{2n-1})|_{\mathbb{R}^{2n-1}} < r, |x_{2n} - z_{2n}| < r\}.$

The map L is called the $\nabla^\phi-$differential of ψ at y.

We point out that definition 3.2.4 is well posed. Indeed, it can be proved that if $\psi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ is $\nabla^\phi-$differentiable at $x \in \omega$, then its $\nabla^\phi-$differential is unique, see [6, Lemma 3.4].
Moreover, as in the Euclidean case, the following Proposition holds:

Proposition 3.2.2. *If $\psi, \phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$, then*

(i) If ψ is $\nabla^\phi-$differentiable at $x \in \omega$, then it is continuous at x.

(ii) If ψ is uniformly $\nabla^\phi-$differentiable at $x \in \omega$, then ψ is an intrinsic Lipschitz function in a neighbourhood of x.

Proof. Since ψ is $\nabla^\phi-$differentiable at x, denoting by $L : \mathbb{W} \longrightarrow \mathbb{V}$ its $\nabla^\phi-$differential we can write

$$\begin{aligned}\psi(y) - \psi(x) = &\frac{\psi(y) - \psi(x) - L(\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x)))}{\mathrm{d}_\phi(x, y)} \ \mathrm{d}_\phi(x, y) + \\ &+ L(\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x)))\end{aligned} \tag{3.18}$$

hence (i) follows taking into account that ψ is $\nabla^\phi-$differentiable at x and that $\mathrm{d}_\phi(x, y)$ is bounded near x. In order to prove (ii), we note that, by definition, there exists $C, r > 0$ such that

$$\frac{\psi(y) - \psi(x) - L(\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x)))}{\mathrm{d}_\phi(x, y)} \leq C \tag{3.19}$$

for all $y \in I_r(x)$. Hence by (3.18) it suffices to show that there exists a constant $N = N(\phi, \psi) > 0$ such that for all $y \in I_r(x)$

$$L(\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x))) \leq N \mathrm{d}_\phi(x, y).$$

The previous relation holds since, as we will prove in the next subsection, for every homogeneous homomorphism $L : \mathbb{W} \equiv \mathbb{R}^{2n} \longrightarrow \mathbb{V} \equiv \mathbb{R}$ there is a unique $w_L \in \mathbb{R}^{2n-1}$ such that

$$L(y) = \langle w_L, \tilde{\pi}(y) \rangle \qquad \forall y \in \mathbb{W}.$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean standard scalar product and

$$\tilde{\pi}(x_1, \dots, x_{2n-1}, x_{2n}) := (x_1, \dots, x_{2n-1}) \qquad \forall x \in \mathbb{W}.$$

$\square$

3.2.1 Intrinsic gradient and nonlinear vector fields

In what follow we recall some useful consequences of definition 3.2.4, we refer to [6] and to [45] for a more detailed treatment.

As in the Euclidean case, there is a rappresentation theorem for the $\nabla^\phi-$differential of a map $\psi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ in terms of a properly defined intrinsic gradient of ψ. Precisely, if $\phi : \omega \longrightarrow \mathbb{R}$ is a continuous function defined on an open and bounded set $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$. We introduce the family $\nabla^\phi = (\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi)$ of vector fields (see [6, 45]), namely of first order differential operators, on ω by

$$\begin{aligned} &\nabla_i^\phi(x) = \partial_{x_i} - x_{i+n}\partial_{x_{2n}}, \quad \text{for } n \geq 2 \text{ and } i = 1, \dots, n-1, \\ &\nabla_n^\phi(x) = \partial_{x_n} + 2\phi(x)\partial_{x_{2n}}, \\ &\nabla_i^\phi(x) = \partial_{x_i} + x_{i-n}\partial_{x_{2n}}, \quad \text{for } n \geq 2 \text{ and } i = n+1 \dots, 2n-1, \end{aligned} \tag{3.20}$$

and by

$$\nabla_1^\phi(x) = \partial_{x_1} + 2\phi(x)\partial_{x_2} \quad \text{if } n = 1.$$

Remark 3.7. Let us notice that if $n \geq 2$ then $\forall p \in \mathbb{W} \equiv \mathbb{R}^{2n}$ the family

$$(\nabla_1^\phi(p), \dots, \nabla_{2n-1}^\phi(p))$$

span the horizontal tangent space $H_p\mathbb{W}$ of $\mathbb{W}$, moreover adding, as in the Heisenberg case, a non horizontal vector field

$$\nabla_{2n}^\phi(x) = \partial_{x_{2n}}$$

to the family $\{\nabla_i^\phi\}_i$ we obtain, for every $p \in \mathbb{W}$, a basis of the whole Euclidean tangent plane $T_p\mathbb{W}$ of $\mathbb{W}$,

Proposition 3.2.3 ([6]). *Let $\psi, \phi : \omega \longrightarrow \mathbb{R}$ be such that ψ is ∇^ϕ differentiable at $x \in \omega$ with $\nabla^\phi-$differential equal to L. Then there is a unique vector $w_L \in \mathbb{R}^{2n-1}$ such that*

$$L(y) = \langle w_L, \tilde{\pi}(y) \rangle \qquad \forall y \in \mathbb{W} \tag{3.21}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{2n-1}$ and

$$\tilde{\pi}(x_1, \ldots, x_{2n-1}, x_{2n}) := (x_1, \ldots, x_{2n-1}) \qquad \forall x \in \mathbb{W}.$$

We call the vector w_L the $\nabla^\phi-$gradient of ψ at $x \in \omega$.

Proof. The proof is substantially the Euclidean representation Theorem for linear maps. Indeed For any $x_{2n} \in \mathbb{R}$ we have

$$2L((0, x_{2n})) = L((0, 2x_{2n})) = \sqrt{2}L((0, x_{2n}))$$

hence $L(0, x_{2n}) = 0$. Thus

$$\begin{aligned} L((x_1, \ldots, x_{2n})) &= L((x_1, \ldots, x_{2n-1}, 0)) + L((0, \ldots, 0, x_{2n})) = \\ &= L((x_1, \ldots, x_{2n-1}, 0)). \end{aligned}$$

This implies that an homogeneous homomorphism does not depend on the last coordinate, therefore the map

$$(x_1, \ldots, x_{2n-1}) \to L((x_1, \ldots, x_{2n-1}, 0))$$

is linear and the statement follows from the standard representation theorem for Euclidean linear maps. □

The following two Theorems explain why, in Definition 3.2.3, we called the vector $w_L \in \mathbb{R}^{2n-1}$ the $\nabla^\phi-$gradient of ψ at $x \in \omega$. Precisely, we prove that the family ∇^ϕ plays the role of a gradient at least for regular functions $\phi : \omega \longrightarrow \mathbb{R}$.

Theorem 3.2.4 ([6, 45])**.** *Let $\phi, \psi \in C^0(\omega)$ such that ψ is $\nabla^\phi-$differentiable at $x \in \omega$. For every $j = 1, \ldots, 2n-1$ let $\lambda^j : [-\delta, \delta] \longrightarrow \omega$ be a C^1 integral curve of the vector field ∇^ϕ_j with $\lambda^j(0) = x$ and such that the function*

$$\begin{aligned} F_j :& [-\delta, \delta] \longrightarrow \mathbb{R} \\ & s \mapsto F_j(s) := \phi(\lambda^j(s)) \end{aligned}$$

is of class C^1. Then it holds

$$\lim_{s \to 0} \frac{\psi(\lambda^j(s)) - \psi(\lambda^j(0))}{s} = \nabla^\phi_j \psi(x). \tag{3.22}$$

Proof. We will give the proof only for the case $n \geq 2$, as it can be easily adapted for $n = 1$. If $\lambda^j(t) = (\lambda^j_1(t), \dots, \lambda^j_{2n}(t))$ is as in the statement and $j \neq n$ then by direct computations we have that for every $t \in [\delta, \delta]$,

$$\pi_{\mathbb{W}}\Big(\Phi(\lambda^j(t))^{-1} \cdot \Phi(\lambda^j(0))\Big) = te_j \tag{3.23}$$

$$\mathrm{d}_\phi(\lambda^j(s), \lambda^j(0)) = s \tag{3.24}$$

where e_j denotes the $j-$th element of the canonical basis of $\mathbb{R}^{2n}$. Hence (3.22) follows immediately as a consequence of the $\nabla^\phi-$differentiability of ψ. For $j = n$ we have $\lambda^n_i(t) = x_i$ if $i \neq n, 2n$, whereas

$$\lambda^n_n(t) = x_n + t$$

$$\lambda^n_{2n}(t) = x_{2n} + 2\int_0^t \phi(\lambda^n(\eta))\mathrm{d}\eta.$$

Hence,

$$\pi_{\mathbb{W}}\Big(\Phi(\lambda^n(t))^{-1} \cdot \Phi(\lambda^n(0))\Big) = te_n. \tag{3.25}$$

Moreover, there is a constant $C > 0$ such that

$$\mathrm{d}_\phi(\lambda^n(s), \lambda^n(0)) \leq Cs^2. \tag{3.26}$$

Indeed,

$$\begin{aligned}
\mathrm{d}_\phi(\lambda^n(s), \lambda^n(0)) &= \Big|\Big|\pi_{\mathbb{W}}\Big(\Phi(\lambda^j(t))^{-1} \cdot \Phi(\lambda^j(0))\Big)\Big|\Big| \\
&= |s| + \Big|2s\phi(\lambda^n(s)) - 2\int_0^s \phi(\lambda^n(\eta))\mathrm{d}\eta\Big|^{1/2} \\
&= |s|\Big(1 + \frac{1}{|s|}\Big|2s\phi(\lambda^n(s)) - 2\int_0^s \phi(\lambda^n(\eta))\mathrm{d}\eta\Big|^{1/2}\Big) \\
&:= |s|\Big(1 + \frac{1}{|s|}\Big|\Delta(s)\Big|^{1/2}\Big)
\end{aligned}$$

since

$$\Delta(s) = \Big|2s[\phi(\lambda^n(s)) - \phi(\lambda^n(0))] - 2\int_0^s [\phi(\lambda^n(\eta)) - \phi(\lambda^n(0))]\mathrm{d}\eta\Big|^{1/2}$$

and using the fact that $F_n(s) = \phi(\lambda^n(s))$ is of class C^1 we conclude that $|\Delta(s)| \leq \tilde{C}s^2$ for some positive constant $\tilde{C} > 0$. Hence (3.26) holds with $C := 1 + \sqrt{\tilde{C}}$. By (3.25) and (3.26) we have

$$\frac{|\psi(\lambda^j(s)) - \psi(\lambda^j(0)) - s\nabla_j^\phi\psi(x)|}{|s|}$$
$$\leq (1+\sqrt{\tilde{C}})\frac{\left|\psi(\lambda^j(s)) - \psi(\lambda^j(0)) - L\Big(\pi_{\mathbb{W}}\Big(\Phi(\lambda^n(s))^{-1}\cdot\Phi(\lambda^n(0))\Big)\Big)\right|}{\mathrm{d}_\phi(\lambda^n(s), \lambda^n(0))}$$

and letting $s \to 0$ and using the $\nabla^\phi-$differentiability of ψ the thesis follows. □

Theorem 3.2.5 ([6, 45]). *If $\psi, \phi \in C^1(\omega)$ then ψ is uniformly ∇^ϕ differentiable in ω and*

$$\nabla^\phi\psi(A) = (\nabla_1^\phi\psi(A), \ldots, \nabla_{2n-1}^\phi\psi(A)).$$

We are now in position to recall the important result of Ambrosio-Serra Cassano and Vittone [6], which states that the graph S of a continuous map $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ is an $\mathbb{H}-$regular hypersurface if and only if ϕ is uniformly $\nabla^\phi-$differentiable in ω.

Theorem 3.2.6. *Let $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ be a continuous map and let $S :=$ graph$(\phi) \subset \mathbb{H}^n$ be the intrinsic graph of ϕ, as defined in (3.7). Then the following are equivalent:*

1. *S is an $\mathbb{H}-$regular hypersurface and $\nu_{S,1}(p) < 0$ for all $p \in S$, where $\nu_{S,1}$ denotes the first component of the horizontal normal to S.*

2. *ϕ is uniformly $\nabla^\phi-$differentiable in ω.*

Theorem 3.2.6 is not entirely satisfactory since in general it is hard to prove that a function is uniform $\nabla^\phi-$differentiable. In what follow we recall an interesting result due to Ambrosio-Serra Cassano and Vittone ([6]) and successively refined by Bigolin and Serra-Cassano ([14, 13]) which permit to partially overcome this difficulty.

Definition 3.2.5. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $w = (w_1, \dots, w_{2n-1}) \in C^0(\omega; \mathbb{R}^{2n-1})$. We say that a continuous function $\phi : \omega \longrightarrow \mathbb{R}$ is a* distributional solution *of the system $\nabla^\phi \phi = w$ if for each $\varphi \in C_c^\infty(\omega)$*

$$\int_\omega \phi \, \nabla_i^\phi \varphi \, \mathrm{d}\mathcal{L}^{2n} = -\int_\omega w_i \varphi \, \mathrm{d}\mathcal{L}^{2n} \quad \forall\, i \neq n \tag{3.27}$$

and

$$\int_\omega \left(\phi \frac{\partial \varphi}{\partial x_n} + \frac{1}{2} \phi^2 \frac{\partial \varphi}{\partial x_{2n}} \right) \mathrm{d}\mathcal{L}^{2n} = -\int_\omega w_{n+1} \varphi \, \mathrm{d}\mathcal{L}^{2n}. \tag{3.28}$$

Definition 3.2.6. *Let $\phi : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}$ and $w = (w_2, ..., w_{2n}) : \omega \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ be continuous functions. We call ϕ a broad* solution of the system*

$$\nabla^\phi \phi = w \qquad \text{in } \omega \tag{3.29}$$

if for every $A \in \omega$, $\forall\, j = 1, ..., 2n-1$ there exists a map, which will call exponential map,

$$\exp_A(\cdot \nabla_j^\phi)(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega$$

where $0 < \delta_2 < \delta_1$, such that if $\gamma_j^B(s) = \exp(s\nabla_j^\phi)(B)$,

(E.1) $\gamma_j^B \in C^1([-\delta_2, \delta_2])$

(E.2) $\begin{cases} \dot{\gamma}_j^B = \nabla_j^\phi \circ \gamma_j^B \\ \gamma_j^B(0) = B \end{cases}$

(E.3) $\phi\left(\gamma_j^B(s)\right) - \phi\left(\gamma_j^B(0)\right) = \int_0^s w_j\left(\gamma_j^B(r)\right) dr$

$\forall\, B \in I_{\delta_2}(A)$, $\forall\, j = 2, ..., 2n$.

Remark 3.8. It is important to note that both the uniqueness and the global continuity of the exponential map

$$\exp(\cdot \nabla_j^\phi)(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)}$$

are not guaranteed provided only ϕ, w are continuous as explicitely stated in [138, Remark 3.4].

In the following lemma we provide two sufficient conditions to garantee the existence of a broad* solution of $\nabla^\phi \phi = w$, see [138, 6] for a proof.

Lemma 3.2.7. *Let $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$ be continuous, and suppose that*

(i) *there exists $w \in C^0(\omega)$ such that ϕ is a ditributional solution of the system*
$$\nabla^\phi \phi = w;$$

(ii) *there is a family of functions $\{\phi_\epsilon\}_{\epsilon>0} \subset C^1(\omega)$ such that for each $\omega' \Subset \omega$ we have*
$$\phi_\epsilon \to \phi,\ \nabla^{\phi_\epsilon}\phi_\epsilon \to w \qquad \text{uniformly on } \overline{\omega'}.$$

Then ϕ is a broad solution of the system $\nabla^\phi \phi = w$. Moreover,*
$$w_j(B) = \frac{d}{ds}\phi\left(\exp_A(s\nabla_j^\phi)(B)\right)_{|s=0}$$
for each $B \in I_{\delta_2}(A)$.

Theorem 3.2.8 ([6])**.** *Let $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ be a continuous map. Then the following are equivalent:*

(i) ϕ is uniformly $\nabla^\phi-$differentiable for each $x \in \omega$;

(ii) ϕ is a broad solution of the system $\nabla^\phi \phi = w$ and*
$$\lim_{r\to 0^+} \sup\left\{\frac{|\phi(x)-\phi(y)|}{|x-y|^{1/2}} \mid x, y \in \omega',\ 0 < |x-y| < r\right\} = 0 \qquad (3.30)$$
for each $\omega' \Subset \omega$.

Actually, Theorem 3.2.8 can be refined in the following way.

Theorem 3.2.9 ([14])**.** *Let $\omega \subseteq \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \longrightarrow \mathbb{R}$ and $w = (w_1, ..., w_{2n-1}) : \omega \to \mathbb{R}^{2n-1}$ be a continuous functions. Then the following conditions are equivalent:*

i ϕ is a broad solution of the system $\nabla^\phi \phi = w$ in ω;*

ii $S = \Phi(\omega)$ is an $\mathbb{H}$-regular hypersurface and $\nu_S^{(1)}(P) < 0$ for all $P \in S$.

Another possible way to prove that a continuous function $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ locally parametrizes an $\mathbb{H}-$regular hypersurface in $\mathbb{H}^n$ is contained in the following theorem; see [13] for a proof.

Theorem 3.2.10. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \longrightarrow \mathbb{R}$ be a continuous function. Then following conditions are equivalent:*

(i) The set $S := \Phi(\omega)$ is an $\mathbb{H}$-regular hypersurface and $\nu_S^1(P) < 0$ for all $P \in S$.

(ii) There exists $w = (w_1, \dots, w_{2n-1}) \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a distributional solution of the system $\nabla^\phi \phi = w$.

Remark 3.9. Putting together Theorem 3.2.9 and Theorem 3.2.10 we conclude that a continuous function $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{R}$ is broad* solution of the system $\nabla^\phi \phi = w$ with $w \in C^0(\omega, \mathbb{R}^{2n-1})$ if and only if ϕ is a distributional solution of the same system. Moreover, we also have that $\Phi(\omega)$ is an $\mathbb{H}-$regular hypersurface if and only if the distributional intrinsic gradient of ϕ is a continuous map.

We conclude this chapter recalling an interesting result due to R.Monti and D.Vittone and proved in [117] which provides, in analogy to the Euclidean case ([90, 89]), another way to prove that a set $E \subset \mathbb{H}^n$ with finite $\mathbb{H}-$perimeter is an $\mathbb{H}-$regular hypersurface.

Theorem 3.2.11. *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}-$perimeter in $B_r := B_\infty(0, r)$ with $r > 0$ and let $\mathbb{S}^{2n-1}$ be the unit sphere in $\mathbb{R}^{2n}$. Suppose there exists a continuous mapping $\bar{\nu} : B_r \longrightarrow \mathbb{S}^{2n-1}$ such that $\nu_E(p) = \bar{\nu}(p)$ for $|\partial E|_{\mathbb{H}}-$ a.e $p \in B_r$. Then, possibly modifying E in a set of $\mathcal{L}^{2n+1}$ measure zero, $\partial E \cap B_r$ is an $\mathbb{H}-$regular hypersurface.*

Chapter 4

Intrinsic Lipschitz functions

This chapter is entirely devoted to the study of intrinsic Lipschitz functions, see Definition 3.2.2.

Before starting we spend some words in order to motivate Definition 3.2.2. Indeed, it is not the unique possible. Precisely, if $\mathbb{W}$ and $\mathbb{V}$ are as in (3.2) it is natural to speak of *metric Lipschitz functions*. Restricting the metric d_∞ to $\mathbb{W}$ and $\mathbb{V}$ we say that a function $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V}$ is metric Lipschitz if there is a constant $L > 0$ such that

$$|\phi(x) - \phi(y)| \leq L\mathrm{d}_\infty(x, y) \quad \forall\, x, y \in \omega.$$

We point out that the concepts of intrinsic Lipschitz and metric Lipschitz are different ones. In particular, there are metric Lipschitz functions which are not intrinsic Lipschitz and viceversa (see [79] for some interesting and explicit examples). Nevertheless, there are at least two reasons to adopt Definition 3.2.2 rather than the metric one. First of all, by Proposition 3.2.2 and Theorem 3.2.6 it follows that every $\mathbb{H}-$regular hypersurface is locally the intrinsic graph of an intrinsic Lipschitz map. Moreover, intrinsic Lipschitz graphs are invariant under dilations and left translations (see Proposition 4.1.2) in $\mathbb{H}^n$ whereas this is in general not true for metric Lipschitz functions.

4.1 General properties

In this section we recall an alternative and equivalent characterization of $Lip_{\mathbb{W}}$ introduced in [79]. Moreover, we provide some interesting consequences which follow from this equivalence.

We begin our treatment pointing out what a closed cone is;

Definition 4.1.1. *Let $q \in \mathbb{H}^n$ and $\alpha > 0$. The intrinsic (closed) cone $C_{\mathbb{W},\mathbb{V}}(q,\alpha)$ with base $\mathbb{W}$, axis $\mathbb{V}$, vertex q and opening α is*

$$C_{\mathbb{W},\mathbb{V}}(q,\alpha) := \{p = (s,x) \in \mathbb{H}^n \mid \|\pi_{\mathbb{W}}(q^{-1} \cdot p)\| \leq \alpha \|\pi_{\mathbb{V}}(q^{-1} \cdot p)\|\},$$

where $\pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}$ are defined respectively in (3.3) and (3.4).

The following picture (due to the courtesy of R.Serapioni) is an example of intrinsic cone in $\mathbb{H}^1$:

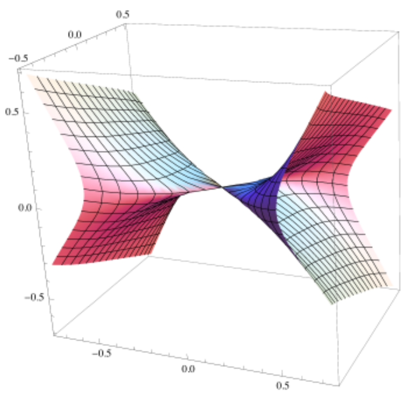

Figure 4.1: Intrinsic Cone in $\mathbb{H}^1$

We are now in position to prove the following:

Proposition 4.1.1. *$\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^{2n} \longrightarrow \mathbb{V} \equiv \mathbb{R}$ is an intrinsic Lipschitz function if and only if there is $L > 0$ such that, for all $q \in graph(\phi)$,*

$$C_{\mathbb{W},\mathbb{V}}(q, 1/L) \cap graph(\phi) = \{q\}. \tag{4.1}$$

Proof. By [79, Proposition 4.6], $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V}$ satisfies (4.1) if and only if there is $L > 0$ such that $\forall p, q \in \omega$

$$|\phi(x) - \phi(y)| \leq L\|\phi(y)^{-1} \cdot y^{-1} \cdot x \cdot \phi(y)\|$$

where $\phi(y) \equiv (\phi(y), 0) \in \mathbb{R}^{2n+1}$. By a direct computation

$$\|\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x))\| = \|\phi(y)^{-1} \cdot y^{-1} \cdot x \cdot \phi(y)\|. \tag{4.2}$$

Moreover,

$$\|\pi_{\mathbb{W}}(\Phi(x)^{-1} \cdot \Phi(y))\| \leq \|\pi_{\mathbb{W}}(\Phi(y)^{-1} \cdot \Phi(x))\| + \sqrt{2}|x_n - y_n|^{\frac{1}{2}}|\phi(x) - \phi(y)|^{\frac{1}{2}}$$

hence

$$2\|\pi_{\mathbb{W}}(\Phi(x)^{-1} \cdot \Phi(y))\| \leq 2\mathrm{d}_{\phi}(x, y) + \sqrt{2}|x_n - y_n|^{\frac{1}{2}}|\phi(x) - \phi(y)|^{\frac{1}{2}} \tag{4.3}$$

since $\phi \in Lip_{\mathbb{W}}(\omega)$

$$\|\pi_{\mathbb{W}}(\Phi(x)^{-1} \cdot \Phi(y))\| \leq \Big(1 + \frac{1}{\sqrt{2}}Lip(\phi)\Big)\mathrm{d}_{\phi}(x, y) \tag{4.4}$$

From (4.2) and (4.4) the thesis follows. $\square$

Remark 4.1. Using cones it is possible to generalize the notion of intrinsic Lipschitz function to more general splitting (see [79]). Precisely, for any couple of homogeneous subspaces $\tilde{\mathbb{W}}$, $\tilde{\mathbb{V}}$ of $\mathbb{H}^n$ such that $\tilde{\mathbb{W}} \cdot \tilde{\mathbb{V}} = \mathbb{H}^n$ and $\tilde{\mathbb{W}} \cap \tilde{\mathbb{V}} = \{0\}$ (hence we don't assume any restriction on the dimension of $\tilde{\mathbb{W}}$ and $\tilde{\mathbb{V}}$) we say that a function $\phi : \tilde{\mathbb{W}} \longrightarrow \tilde{\mathbb{V}}$ is intrinsic Lipschitz if there exists $L > 0$ such that for any point p of its intrinsic graph S, the intrinsic cone with vertex p and opening L (the definition is similar to the one proposed in Definition 4.1.1) intersect S only in p.

In the following proposition we collect some interesting properties of $Lip_{\mathbb{W}}$, see [79].

Proposition 4.1.2. *Let $\phi : \omega \subset \mathbb{W} \longrightarrow \mathbb{V}$, then*

(i) if ϕ is intrinsic Lipschitz and $S = \Phi(\omega) \subset \mathbb{H}^n$ is its intrinsic graph then, for each $\lambda > 0$, the set $\delta_\lambda(S)$ remain a graph. Precisely, denoting by $\phi_\lambda := \delta_\lambda \circ \phi \circ \delta_{1/\lambda}$ it holds:

$$\delta_\lambda(S) = graph(\phi_\lambda)$$

moreover $\phi_\lambda \in Lip_{\mathbb{W}}(\delta_\lambda(\omega))$.

(ii) if ϕ is intrinsic Lipschitz, $S = \Phi(\omega) \subset \mathbb{H}^n$ is its intrinsic graph and $p \in \mathbb{H}^n$. Then $p \cdot S$ is a graph. Precisely, denoting by $\omega_p := \pi_{\mathbb{W}}(p \cdot \omega) \subset \mathbb{W}$ and by $\phi_p : \omega_p \longrightarrow \mathbb{R}$, $\phi_p(y) := \pi_{\mathbb{V}}((p \cdot \pi_{\mathbb{W}}((p \cdot y))^{-1})) \cdot \phi(\pi_{\mathbb{W}}((p \cdot y)^{-1}))$ it holds:

$$p \cdot S = graph(\phi_p).$$

(iii) If ϕ_q and ω_q are as in (ii) then $\phi \in Lip_{\mathbb{W}}(\omega)$ if and only if for each $q \in \mathbb{H}^n$ it holds $\phi_q \in Lip_{\mathbb{W}}(\omega_q)$. Moreover, ϕ is intrinsic Lipschitz if and only if there is $L > 0$ such that, for all $p \in graph(\phi)$ and for all $x \in \omega_{p^{-1}}$ it holds:

$$\|\phi_{p^{-1}}(x)\| \leq L\|x\|. \tag{4.5}$$

(iv) if ϕ is intrinsic Lipschitz function then for every $\omega' \Subset \omega$ there exists a constant $C = C(Lip(\phi), ||\phi||_{L^\infty(\omega)}, \omega') > 0$ such that

$$|\phi(x) - \phi(y)| \leq C|x - y|^{\frac{1}{2}} \quad \forall\ x, y \in \omega'.$$

(v) If ϕ is Euclidean Lipschitz and ω is open and bounded then ϕ is indeed intrinsic Lipschitz in a neighborhood of each point of ω.

Proof. **(i):** Let us observe that $p \in \delta_\lambda(S)$ if and only if there exists $q \in S$ such that $p = \delta_\lambda(q) = \delta_\lambda(\pi_{\mathbb{W}}(q)) \cdot \delta_\lambda(\pi_{\mathbb{V}}(q))$. By definition $q = (\phi(x), x)$ for some $x \in \omega$. Then

$$p = \delta_\lambda((0, x)) \cdot (\lambda\phi(x), 0). \tag{4.6}$$

Hence defining $\omega_\lambda := \{z \in \mathbb{W} \mid \exists\ x \in \omega,\ z = \delta_\lambda(x)\}$ and $\phi_\lambda : \omega_\lambda \longrightarrow \mathbb{V}$ by $\phi_\lambda := \delta_\lambda \circ \phi \circ \delta_{1/\lambda}$ we achive the thesis. The fact that $\phi_\lambda \in Lip_{\mathbb{W}}(\delta_\lambda(\omega))$ is straighforward.

(ii): With an easy algebraic trick (see [79, Proposition 3.6] for the details) we can prove that for each $q \in \mathbb{H}^n$

$$\begin{aligned} \xi_q :&\mathbb{W} \longrightarrow \mathbb{W} \\ &x \to \pi_{\mathbb{W}}(q \cdot (0, x)) \end{aligned} \tag{4.7}$$

is injective. It follows that

$$q \cdot S = \{(\phi_q(x), x) \mid x \in \omega_q\}$$

where $\omega_q := \{\pi_{\mathbb{W}}(q \cdot x) \mid x \in \omega\}$ and, for $y = \pi_{\mathbb{W}}(q \cdot x) \in \omega_q$

$$\phi_q(y) := \pi_{\mathbb{V}}(q \cdot \pi_{\mathbb{W}}(q \cdot y)^{-1}) \cdot \phi(\pi_{\mathbb{W}}(q \cdot y)^{-1}).$$

(iii): By definition, $\text{graph}(\phi_q) = q \cdot \text{graph}(\phi)$. Hence $p \in \text{graph}(\phi_q)$ if and only if $p = q \cdot \bar{p}$ for $\bar{p} \in \text{graph}(\phi)$. Then, if $\phi \in Lip_{\mathbb{W}}(\omega)$,

$$\{p\} = \{q \cdot \bar{p}\} = q \cdot (C_{\mathbb{W},\mathbb{V}}(\bar{p}, \alpha) \cap \text{graph}(\phi)) = C_{\mathbb{W},\mathbb{V}}(p, \alpha) \cap \text{graph}(\phi_q).$$

Hence ϕ_q is intrinsic Lipschitz. For the converse, let us observe that for each $p, q \in \mathbb{H}^n$ it holds

$$(\phi_q)_p = \phi_{p \cdot q}. \tag{4.8}$$

Then, by the first part of the proof, if ϕ_q is intrinsic Lipschitz then $(\phi_q)_{q^{-1}}$ is also intrinsic Lipschitz. Finally, by (4.8), we deduce $(\phi_q)_{q^{-1}} = \phi$ and the thesis follows.

For the second part, let us start pointing out that for each $q \in \text{graph}(\phi)$

$$0 \in \omega_{q^{-1}}, \quad \phi_{q^{-1}} \in Lip_{\mathbb{W}}(\omega_{q^{-1}}) \text{ and } \phi_{q^{-1}}(0) = 0.$$

Hence, by definition, there exists $L > 0$ such that for all $p = (\phi_{q^{-1}}(x), x) \in \text{graph}(\phi_{q^{-1}})$ it holds

$$\|\pi_{\mathbb{V}}(p)\| \leq L\|\pi_{\mathbb{W}}(p)\|$$

therefore

$$|\phi_{q^{-1}}(x)| \leq L\|x\|$$

The reverse inequality is obvious and follows again from (4.8).

(iv): Let $M := \sup_{\bar{\omega}'} |\phi|$ and $\Delta := \sup_{p\in\bar{\omega}'} \|p\|$. Then

$$\begin{aligned} \mathrm{d}_\phi(x,y) &\leq |x-y|_{\mathbb{R}^{2n}} + (1+\sqrt{2M})|x-y|^{\frac{1}{2}}_{\mathbb{R}^{2n}} + |\sigma(x,y)|^{\frac{1}{2}} && (4.9)\\ &\leq (1+2\sqrt{\Delta}+\sqrt{2M})|x-y|^{\frac{1}{2}}_{\mathbb{R}^{2n}} + |x-y|^{\frac{1}{2}}_{\mathbb{R}^{2n}}\|y\|^{\frac{1}{2}} && (4.10)\end{aligned}$$

where in the last inequality we used $|\sigma(x,y)| = |\sigma(x-y,x)| \leq ||x-y||_{\mathbb{R}^{2n}}||y||_{\mathbb{R}^{2n}}$.

(v): Let $M := ||\phi||_{L^\infty(\omega)} < \infty$. To get the thesis it suffices to prove that there exists a constant $C = C(M) > 0$ such for each x, y in a sufficiently small Euclidean neighborhood of each $p \in \omega$ it holds

$$|x-y|_{\mathbb{R}^{2n}} \leq C\mathrm{d}_\phi(x,y).$$

Clearly,

$$|x-y|_{\mathbb{R}^{2n}} \leq |(x_1-y_1,\dots,x_{2n-1}-y_{2n-1})|_{\mathbb{R}^{2n-1}} + |x_{2n}-y_{2n}|,$$

hence we have only to prove that there exists a constant $C > 0$ such that

$$|x_{2n}-y_{2n}| \leq C\mathrm{d}_\phi(x,y).$$

If

$$\frac{1}{2}|x_{2n}-y_{2n}| \geq |-2(\phi(x)+\phi(y))(y_n-x_n)+\sigma(x,y)|$$

then the thesis follows provided x, y are close to p. On the other hand if

$$\frac{1}{2}|x_{2n}-y_{2n}| \leq |-2(\phi(x)+\phi(y))(y_n-x_n)+\sigma(x,y)|$$

then

$$\frac{1}{2}|x_{2n}-y_{2n}| \leq 2||\phi||_{L^\infty(\omega)}|y_n-x_n| + |\sigma(x,y)|$$

and since $|\sigma(x,y)| = |\sigma(x-y,x)| \leq \|x-y\|_{\mathbb{R}^{2n}}\|y\|_{\mathbb{R}^{2n}}$ the conclusion follows. □

As pointed out in [130], $Lip_{\mathbb{W}}(\omega)$ is not a vector space. Nevertheless, it is closed under max and min. Precisely,

Proposition 4.1.3 ([79]). *Let ψ, ϕ be intrinsic Lipschitz functions on $\mathbb{W}$ with the same Lipschitz constant $L > 0$. Then there exists $\bar{L} = \bar{L}(L) \geq L$ such that* $\max\{\phi, \psi\}$ *and* $\min\{\phi, \psi\}$ *are $\bar{L}-$intrinsic Lipschitz.*

Proposition 4.1.4 ([79]). *Let $\phi : \omega \subset \mathbb{W} \to \mathbb{V}$ be an intrinsic Lipschitz function, then there is $\overline{\phi} : \mathbb{W} \longrightarrow \mathbb{V}$ intrinsic Lipschitz such that $Lip(\bar{\phi}) > Lip(\phi)$ and*

$$\overline{\phi}(x) = \phi(x) \quad \forall\ x \in \omega.$$

Theorem 4.1.5 ([117]). *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}-$perimeter in U_r, $r > 0$, ν_E be the measure theoretic inward normal of E, and $\nu \in S^{2n-1}$. Assume there exists $k \in (0, 1]$ such that $\pi_{\mathbb{V}}(\nu_E(p)) \leq -k$ for $|\partial E|_{\mathbb{H}}-a.e$ $p \in U_r$. Then there exists $\alpha > 0$ such that possibly modifying E in a negligible set, for all $p \in \partial E \cap U_r$*

$$\{q \in U_r \mid \|\pi_{\mathbb{W}}(p^{-1} \cdot q)\| < -\alpha\pi_{\mathbb{V}}(p^{-1} \cdot q)\} \subset E$$
$$\{q \in U_r \mid \|\pi_{\mathbb{W}}(p^{-1} \cdot q)\| < \alpha\pi_{\mathbb{V}}(p^{-1} \cdot q)\} \subset \mathbb{H}^n \setminus E$$

If in particular $n = 1$

$$\alpha^2 + 2\alpha \leq \frac{h}{2}, \quad \textit{with } h := \sqrt{\frac{k^2}{2 - k^2}} \tag{4.11}$$

One of the main achivements in the theory of intrinsic Lipschitz functions is the following result.

Theorem 4.1.6 ([79]). *Let $\phi \in Lip_{\mathbb{W}}(\mathbb{W})$ then the subgraph E_ϕ is a set with locally finite $\mathbb{H}-$perimeter.*

Theorem 4.1.6 open the possibility of proving much finer results on intrinsic Lipschitz functions. In particular, by using a blow-up argument it is possible to prove a Rademacher's type Theorem for this class of functions. In other words, as in the Euclidean situation, each intrinsic Lipschitz function is almost everywhere $\nabla^\phi-$differentiable.

Theorem 4.1.7 ([79]). *Let $\omega \subset \mathbb{W}$ be open and $\phi \in Lip_{\mathbb{W}}(\omega)$. Then ϕ is intrinsic $\nabla^\phi-$differentiable $\mathcal{L}^{2n}-a.e$ in ω.*

Sketch of the Proof: First of all, since $\nabla^\phi-$differentiability is a local notion, by Proposition 4.1.4, we may assume that ϕ is intrinsic Lipschitz defined on all $\mathbb{W}$. Hence, by Theorem 4.1.6, we know that E_ϕ is a set of locally finite $\mathbb{H}-$perimeter hence we denote by $\partial^* E_\phi$ the $\mathbb{H}-$reduced boundary of E_ϕ as in Definition 1.2.8 and by ν_{E_ϕ} the inward unit normal to E_ϕ as in Theorem 1.2.8. By Theorem 1.2.16 it follows that:

$$\lim_{r\to 0} \mathbf{1}_{(E_\phi)_{r,p}} = \mathbf{1}_{S^+_{\mathbb{H}}(\nu_{E_\phi}(p))} \ \text{ in } L^1_{loc}(\mathbb{H}^n) \tag{4.12}$$

where $(E_\phi)_{r,p}$ and $S^+_{\mathbb{H}}(\nu_{E_\phi}(p))$ are as in Theorem 1.2.16. As proved in [79, Proposition 4.6], ϕ is differentiable at any point $x \in \omega$ such that $(\phi(x), x) \in \partial^* E_\phi$. Hence to conclude the proof it suffices to prove that

$$\mathcal{L}^{2n}(\pi_{\mathbb{W}}(\text{graph}\phi \setminus \partial^* E_\phi)) = \mathcal{L}^{2n}(\mathbb{W} \setminus \pi_{\mathbb{W}}(\partial^* E_\phi)) = 0. \tag{4.13}$$

We prove here a more general result. Precisely,

$$(\Phi)_*(\mathcal{L}^{2n}) = \nu^1_{E_\phi} |\partial E_\phi|_{\mathbb{H}} \tag{4.14}$$

where $(\Phi)_*(\mathcal{L}^{2n})$ denotes the pushforward measure induced by the graph map of ϕ and $\nu^1_{E_\phi}$ is the first component of ν_{E_ϕ}. Since E_ϕ is a set of locally finite $\mathbb{H}-$perimeter then $\forall g \in C^1_c(\mathbb{H}^n)$

$$\begin{aligned} \int_{E_\phi} \nabla^{\mathbb{H}}_1 g \ \mathrm{d}\mathcal{L}^{2n+1} &= \int_{\mathbb{H}^n} \nu^1_{E_\phi} g \ \mathrm{d}|\partial E_\phi|_{\mathbb{H}} \\ &= \int_{\mathbb{W}} g(\Phi(x)) \ \mathrm{d}\mathcal{L}^{2n}. \end{aligned}$$

By a change of variables (see [107]) we obtain:

$$\begin{aligned} \int_{\mathbb{H}^n} g\nu^1_{E_\phi} \ \mathrm{d}|\partial E_\phi|_{\mathbb{H}} &= \int_{\mathbb{W}} g(\Phi(x)) \ \mathrm{d}\mathcal{L}^{2n} \\ &= \int_{\mathbb{H}^n} g \ \mathrm{d}\Phi_*(\mathcal{L}^{2n}) \end{aligned}$$

which is the thesis. $\square$

Some interesting consequences of Theorems 4.1.6 and 4.1.7 are in order:

Proposition 4.1.8 ([79]). *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be a bounded open set, and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then*

$$\nabla^{\phi}\phi \in (L^{\infty}(\omega))^{2n-1}. \tag{4.15}$$

Proof. Let $S = \Phi(\omega)$. We begin proving that there exists a constant $c = c(\mathbb{W}, \mathbb{V}, \phi) > 0$ such that for all $p \in S$ there is a $r_p = r(\mathbb{W}, \mathbb{V}, p) > 0$ for which

$$cr^{2n+1} \leq \mathcal{L}^{2n}(\pi_{\mathbb{W}}(U(p,r) \cap S)) \quad \forall\, r \in (0, r_p). \tag{4.16}$$

Being S an intrinsic Lipschitz graph there exists $0 < L < \infty$ such that for all $p \in S$

$$C_{\mathbb{W},\mathbb{V}}(p, 1/L) \cap S = \{p\}. \tag{4.17}$$

From (ii) of Proposition 4.1.2, without loss of generality, we can assume that $p = e$. First, let us prove that, if $\alpha := 1/L$, there exists $r_e = r(\mathbb{W}, \mathbb{V}, e) > 0$ such that

$$\pi_{\mathbb{W}}(\overset{\circ}{C}_{\mathbb{W},\mathbb{V}}(e, \alpha) \cap U(e, \frac{c_1}{Lc_2} r)) \subseteq \pi_{\mathbb{W}}(U(e, r) \cap S) \tag{4.18}$$

for each $r \in (0, r_e)$, where c_1 and c_2 are the constants such that

$$c_1(\|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\|) \leq \|p\| \leq c_2(\|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\|) \quad \forall\, p \in \mathbb{H}^n.$$

Because of $e \in S = \Phi(\omega)$, it follows that $e \in \omega$. Thus there exists $r_e > 0$ such that $U(e, r) \subseteq \mathbb{R} \times \omega$ for each $r \in (0, r_e)$. Then, in order to get (4.18) we need to prove that if $p \in \overset{\circ}{C}_{\mathbb{W},\mathbb{V}}(e, \alpha) \cap U(e, (c_1/Lc_2)r)$ then

$$\pi_{\mathbb{W}}(p) \cdot \phi(\pi_{\mathbb{W}}(p)) \subset U(e, r) \quad \forall\, r \in (0, r_e). \tag{4.19}$$

Let $p \in \overset{\circ}{C}_{\mathbb{W},\mathbb{V}}(e, \alpha) \cap U(e, (c_1/Lc_2)r)$ and assume that $p = (s, x)$ with $x \in \omega$. By (4.17) and (iii) in Proposition 4.1.2 it follows that

$$|\phi(x)| < L|s|. \tag{4.20}$$

Since

$$c_1(\|x\| + |s|) \leq \|p\| < \frac{c_1}{Lc_2} r$$

it follows from (4.20) that

$$\|(\phi(x), x)\| \leq c_2(\|x\| + |\phi(x)|) < c_2(\|x\| + |s|) < r.$$

Therefore (4.19) is proved. Moreover, if we define

$$c := (\frac{c_1}{c_2})^{2n+1}\mathcal{L}^{2n}(\pi_{\mathbb{W}}(\overset{\circ}{C}_{\mathbb{W},\mathbb{V}}\ (e,\alpha) \cap U(e,1)))$$

then $c > 0$ because $\pi_{\mathbb{W}}$ is an open map. By group dilations and [79, Lemma 4.3]

$$\mathcal{L}^{2n}(\pi_{\mathbb{W}}(\overset{\circ}{C}_{\mathbb{W},\mathbb{V}}\ (e,\alpha) \cap U(e, \frac{c_1}{c_2}r))) = \mathcal{L}^{2n}(\pi_{\mathbb{W}}(\delta_{\frac{c_1}{c_2}r}(\overset{\circ}{C}_{\mathbb{W},\mathbb{V}}\ (e,\alpha) \cap U(e,1)))) \tag{4.21}$$

$$= cr^{2n+1}$$

By (4.18) and (4.21), the proof is accomplished. We are now in position to prove the corollary.

First, let us prove that

$$-\nu_1 \geq \frac{c}{2}w_{2n-1} \qquad |\partial E_\phi|_{\mathbb{H}} - \text{a.e} \tag{4.22}$$

where ν_1, c, w_{2n-1} denote, respectively, the first component of the intrinsic inward normal to E_ϕ, $\nu = (\nu_1, \dots, \nu_{2n})$, the constant in (4.16) and the $(2n-1)$-dimensional Lebesgue measure of the unit ball of $\mathbb{R}^{2n-1}$. Notice that, by definition, for all $p \in S := \Phi(\omega)$ and for all $r > 0$

$$\Phi^{-1}(U(p,r) \cap S) = \pi_{\mathbb{W}}(U(p,r) \cap S).$$

Then, by (4.16), Theorem 4.1.6 and Corollary 4.14, it follows that, for each $p \in S$ there is a $r_p = r(\mathbb{W}, \mathbb{V}, p) > 0$ such that for each $r \in (0, r_p)$

$$cr^{2n+1} \leq \mathcal{L}^{2n}(\omega \cap \Phi^{-1}(U(p,r) \cap S)) = \int_{U(p,r)} (-\nu_1)\mathrm{d}|\partial E_\phi|_{\mathbb{H}} \tag{4.23}$$

Recall also that

$$\partial^*_{\mathbb{H}} E_\phi \cap (\omega \cdot \mathbb{R}) \subseteq \partial E_\phi \cap (\omega \cdot \mathbb{R}) \subseteq S \tag{4.24}$$

$$\exists \lim_{r\to 0^+} \fint_{U(p,r)} (-\nu_1)\mathrm{d}|\partial E_\phi| = -\nu_1(p) \quad \forall p \in \partial^*_{\mathbb{H}} E_\phi \tag{4.25}$$

$$\exists \lim_{r\to 0} \frac{|\partial E_\phi|(U(p,r))}{r^{2n+1}} = 2w_{2n-1} \quad \forall p \in \partial^*_{\mathbb{H}} E_\phi \tag{4.26}$$

$$|\partial E_\phi|_{\mathbb{H}}(\mathbb{H}^n \setminus \partial^*_{\mathbb{H}} E_\phi) = 0. \tag{4.27}$$

From (4.23),(4.24),(4.25),(4.26) and (4.27), inequality (4.22) follows at once. Indeed, for each $p \in \partial^*_{\mathbb{H}} E_\phi$ we have

$$\begin{aligned}
-\nu_1(p) &\overset{(4.25)}{=} \lim_{r\to 0} \fint_{U(p,r)} (\nu_1) \mathrm{d}|\partial E_\phi|_{\mathbb{H}} \overset{(4.23)}{=} \lim_{r\to 0} \frac{\mathcal{L}^{2n}(\omega \cap \Phi^{-1}(U(p,r)\cap S))}{|\partial E_\phi|_{\mathbb{H}}(U(p,r))} \\
&\overset{(4.23)}{\geq} \lim_{r\to 0} \frac{c r^{2n+1}}{|\partial E_\phi|_{\mathbb{H}}(U,p,r)} \overset{(4.26)}{=} \frac{c}{2w_{2n-1}}.
\end{aligned}$$

Finally, since $|\nu| = 1$ $|\partial E_\phi|_{\mathbb{H}}-$a.e and

$$\nabla^\phi \phi = \frac{(\nu_2, \dots, \nu_{2n})}{-\nu_1} \circ \Phi \quad \mathcal{L}^{2n} - \text{a.e in } \omega$$

by (4.22), the proof is complete. □

Corollary 4.1.9 ([79])**.** *Let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then*

$$\partial_{*,\mathbb{H}} E_\phi \cap (\omega \cdot \mathbb{R}) = \partial E_\phi \cap (\omega \cdot \mathbb{R}) = graph(\phi) \tag{4.28}$$

and

$$\mathcal{S}^{2n+1}(\partial_{*,\mathbb{H}} E_\phi \setminus \partial^*_{\mathbb{H}} E_\phi) = 0. \tag{4.29}$$

Here $\mathcal{S}$ denotes the spherical Hausdorff measure introduced in (1.1.13).

We conclude this section comparing the distance d_ϕ with the distance of points of the graph.
To this end we denote by $U_\phi(x,r) := \{y \in \omega \mid \mathrm{d}_\phi(x,y) < r\}$.

Proposition 4.1.10. *If $\phi \in Lip_{\mathbb{W}}(\omega)$ then there is $C_1 = C_1(Lip(\phi)) > 0$ such that*

$$U_\phi(x, C_1 r) \subseteq \pi_{\mathbb{W}}(U(\Phi(x), r) \cap graph(\phi)) \subseteq U_\phi(x, r/c) \tag{4.30}$$

for all $x \in \omega$ and $r > 0$. Here $U_\phi(x,r) := \{y \in \omega \mid \mathrm{d}_\phi(x,r) < r\}$ and $c > 0$ is defined in (3.5).

Proof. Let $z \in \pi_{\mathbb{W}}(U(\Phi(x),r) \cap \mathrm{graph}(\phi))$ then $\mathrm{d}(\Phi(x),\Phi(z)) < r$. Since the intrinsic projection $\pi_{\mathbb{W}} : \mathbb{H}^n \longrightarrow \mathbb{W}$ is such that $\forall p \in \mathbb{H}^n$

$$\|\pi_{\mathbb{W}}(p)\| \leq \frac{1}{c}\|p\|$$

where $c > 0$ is as in (3.5), we have

$$\begin{aligned} \mathrm{d}_\phi(x,z) &= \frac{1}{2}\Big(\|\pi_{\mathbb{W}}(\Phi(x)^{-1}\cdot\Phi(z))\| + \|\pi_{\mathbb{W}}(\Phi(z)^{-1}\cdot\Phi(x))\|\Big) \\ &\le \frac{1}{\sqrt{2c}}\Big(\|(\Phi(x)^{-1}\cdot\Phi(z))\| + \|(\Phi(z)^{-1}\cdot\Phi(x))\|\Big) \\ &= \frac{1}{c}\mathrm{d}(\Phi(x),\Phi(z)) < \frac{r}{c} \end{aligned}$$

Hence the second inclusion follows.
For the first inclusion, let us note that for all $x, z \in \omega$

$$\mathrm{d}(\Phi(x),\Phi(z)) \le |\phi(z) - \phi(x)| + \mathrm{d}_\phi(x,z)$$

therefore, since $\phi \in Lip_{\mathbb{W}}(\omega)$, for every $z \in U_\phi(x, Cr)$ with $C > 0$ to be determinated we obtain

$$\begin{aligned} \mathrm{d}(\Phi(x),\Phi(z)) &\le Lip(\phi)\mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(x,z) \\ &< C(Lip(\phi)+1)r \end{aligned}$$

and the first inclusion follows choosing $0 < C < 1/(Lip(\phi)+1)$. □

4.1.1 Rectifiable Sets

In this subsection we provide an interesting application of Theorems 4.1.6 and 4.1.7 to the theory of rectifiable sets in $\mathbb{H}^n$. This notion was first introduced in [82] and successively refined, extended and studied by many authors, see for example [81, 97, 80, 44]. The idea is simply to use intrinsic objects to restate the classical notion of rectifiability introduced in [69]. Precisely,

Definition 4.1.2. *We say that $E \subset \mathbb{H}^n$ is* $(2n, \mathbb{H})-$rectifiable *if there exists a sequence of $\mathbb{H}-$regular hypersurfaces $(S_i)_{i\in\mathbb{N}}$ such that, for any bounded $\mathcal{U} \subset \mathbb{H}^n$,*

$$\mathcal{S}^{2n+1}\Big((E\cap\mathcal{U}) \setminus \bigcup_{i\in\mathbb{N}} S_i\Big) = 0. \tag{4.31}$$

We can restate a classical result proved by De Giorgi in [58] and [57], in the context of Heisenberg group.

Theorem 4.1.11 ([80]). *If $E \subset \mathbb{H}^n$ has finite $\mathbb{H}-$perimeter then $\partial^*_{\mathbb{H}} E$ is $(2n, \mathbb{H})$ rectifiable.*

Another natural definition of rectifiability, which in the Euclidean case coincide with the one stated above, is the following one:

Definition 4.1.3. *We say that $E \subset \mathbb{H}^n$ is $(2n, \mathbb{H}_L)-$*rectifiable *if there exists a sequence of Lipschitz graphs $(G_i)_{i\in\mathbb{N}}$ such that, for any bounded $\mathcal{U} \subset \mathbb{H}^n$,*

$$\mathcal{S}^{2n+1}\Big((E \cap \mathcal{U}) \setminus \bigcup_{i\in\mathbb{N}} G_i\Big) = 0. \tag{4.32}$$

Using Theorem 4.1.6 we have the following

Proposition 4.1.12. *$E \subset \mathbb{H}^n$ is $(2n, \mathbb{H})-$rectifiable if and only if E is $(2n, \mathbb{H}_L)$ rectifiable.*

Proof. If E is $(2n, \mathbb{H})-$rectifiable then, by definition, there is a sequence of $\mathbb{H}-$regular hypersurface $(S_i)_{i\in\mathbb{N}}$ for which (4.31) holds. Since each $\mathbb{H}-$regular hypersurface is locally the graph of an intrinsic Lipschitz function (see Proposition 3.2.2 and Theorem 3.2.6) then it clearly follows that E is $(2n, \mathbb{H}_L)$ rectifiable. On the other hand, if E is $(2n, \mathbb{H}_L)-$rectifiable then there exists a sequence of Lipschitz graphs $(G_i)_{i\in\mathbb{N}}$ for which (4.32) holds. Hence, by definition, there are $\phi_i : \omega_i \subset \mathbb{W} \longrightarrow \mathbb{R}$ intrinsic Lipschitz functions such that $G_i = \text{graph}(\phi_i)$. By the extension property (Proposition 4.1.4) we can assume $\omega_i \equiv \mathbb{W}$ for all $i \in \mathbb{N}$. Hence, by Theorem 4.1.6, the subgraph of ϕ_i has locally finite $\mathbb{H}-$perimeter and hence it is $(2n, \mathbb{H})-$rectifiable by Theorem 4.1.11. This proves that all of E is $(2n, \mathbb{H})-$rectifiable. □

We conclude stating an interesting representation result for the intrinsic generalized inward normal of the subgraph of an intrinsic Lipschitz function.

Corollary 4.1.13. *Let $\phi \in Lip_{\mathbb{W}}(\omega)$ then the intrinsic generalized inward normal ν_{E_ϕ} to the subgraph E_ϕ has the following representation*

$$\nu_{E_\phi}(\Phi(x)) = \Big(\frac{-1}{\sqrt{1 + |\nabla^\phi \phi(x)|^2}}, \frac{\nabla^\phi \phi(x)}{\sqrt{1 + |\nabla^\phi \phi(x)|^2}}\Big) \tag{4.33}$$

for $\mathcal{L}^{2n}-a.e\ x \in \omega$.

4.2 C^∞ approximation of intrinsic Lipschitz functions

In this section we face the problem of approximate a given intrinsic Lipschitz map with a more manageable class of functions. Precisely, what we want is an analogous for intrinsic Lipschitz functions of the following well known result for Euclidean Lipschitz functions:

Proposition 4.2.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f : \Omega \longrightarrow \mathbb{R}$ be a Lipschitz function. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of smooth functions $f_n : \Omega \longrightarrow \mathbb{R}$ such that*

1. *f_n converge uniformly to f on the compact sets of Ω;*
2. *$|\nabla f_n(x)| \leq \|\nabla f\|_{L^\infty(\Omega)}$;*
3. *$\nabla f_n(x) \to \nabla f(x)$ a.e in Ω.*

The proof of Proposition 4.2.1 is well known (see [128, 127]) and it is basically done by convolution. On the other hand, for the intrinsic Lipschitz case the approach via convolution fails, essentially because the intrinsic gradient is non linear.

We start proving some auxiliary results:

Proposition 4.2.2. *Let $\omega \subset \mathbb{W}$ be open and bounded and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then for each $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in C^1_c(\mathbb{R}\times\omega, \mathbb{R}^{2n})$*

$$-\int_\Omega \langle \varphi, \nu_E\rangle \, \mathrm{d}|\partial E_\phi|_{\mathbb{H}} = \int_\omega \varphi_1 \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi}\circ\Phi\right\rangle \mathrm{d}\mathcal{L}^{2n} \tag{4.34}$$

where $\hat{\varphi} := (\varphi_2, \dots, \varphi_{2n})$ and $\Phi : \omega \longrightarrow \mathbb{H}^n$ is as in (3.9).

Proof. Let we denote by $E := E_\phi$ the subgraph of ϕ and by $\Omega := \omega \cdot \mathbb{R}e_1 = \mathbb{R}\times\omega$. By Theorem 4.1.6 E is a set of locally finite perimeter in $\mathbb{H}^n$, then there exists a unique $|\partial E|_{\mathbb{H}}$-measurable function $\nu_E : \Omega \longrightarrow \mathbb{R}^{2n}$ such that $|\nu_E|_{\mathbb{R}^{2n}} = 1$ $|\partial E|_{\mathbb{H}}$-a.e in Ω and

$$\int_E \mathrm{div}_{\mathbb{H}}\, \varphi \, \mathrm{d}\mathcal{L}^{2n} = -\int_\Omega \langle \varphi, \nu_E\rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^{2n}),\ |\varphi|_{\mathbb{R}^{2n}} \leq 1.$$

By using Corollary 4.1.13 and 4.14, we have that the first component $\nu_E^{(1)}$ of ν_E is such that $\nu_E^{(1)} < 0$ $|\partial E|_{\mathbb{H}}-$a.e in Ω.

Hence

$$\int_\Omega \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} = \int_\Omega \frac{\langle \varphi, \nu_E \rangle}{\nu_E^{(1)}} \nu_E^{(1)} \, \mathrm{d}|\partial E|_{\mathbb{H}}$$

and by (4.14) we obtain

$$\int_\Omega \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} = -\int_\Omega \frac{\langle \varphi, \nu_E \rangle}{\nu_E^{(1)}} \mathrm{d}\Phi_\#(\mathcal{L}^{2n} \llcorner \mathbb{W})$$

finally by a change of variables

$$\int_\Omega \frac{\langle \varphi, \nu_E \rangle}{\nu_E^{(1)}} \mathrm{d}\Phi_\#(\mathcal{L}^{2n} \llcorner \mathbb{W}) = \int_\omega \frac{\langle \nu_E \circ \Phi, \varphi \circ \Phi \rangle}{\nu_E^{(1)} \circ \Phi} \mathrm{d}\mathcal{L}^{2n}.$$

Now, by the characterization of the inward normal provided in Corollary 4.1.13 we have for every $\varphi \in C_c^1(\Omega, \mathbb{R}^{2n})$ with $|\varphi|_{\mathbb{R}^{2n}} \leq 1$,

$$\begin{aligned}
\int_\Omega \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} &= -\int_\omega \frac{\langle \nu_E \circ \Phi, \varphi \circ \Phi \rangle}{\nu_E^{(1)} \circ \Phi} \mathrm{d}\mathcal{L}^{2n} \\
&= -\int_\omega \varphi_1 \circ \Phi + \sum_{i=2}^{2n} \frac{(\nu_E \circ \Phi)_i (\varphi \circ \Phi)_i}{\nu_E^{(1)} \circ \Phi} \mathrm{d}\mathcal{L}^{2n} \\
&= -\int_\omega \varphi_1 \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi} \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n},
\end{aligned}$$

where $\hat{\varphi} = (\varphi_2, \ldots, \varphi_{2n})$. Hence

$$-\int_\Omega \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} = \int_\omega \varphi_1 \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi} \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n} \tag{4.35}$$

as desidered. □

Now we are going to prove that the gradient $\nabla^\phi \phi$ of a Lipschitz continuous function $\phi \in Lip_{\mathbb{W}}(\omega)$ also agrees with the distributional gradient.

Proposition 4.2.3. *Let $\omega \subset \mathbb{R}^{2n}$ be open and bounded and let $\phi \in Lip_{\mathbb{W}}(\omega)$. Then for each $\psi \in C_c^1(\omega)$*

(i) $\int_\omega \phi \; \nabla_i^\phi \psi \, \mathrm{d}\mathcal{L}^{2n} = -\int_\omega \nabla_i^\phi \phi \; \psi \; , \mathrm{d}\mathcal{L}^{2n} \quad \forall i \neq n$*;*

(ii) $\int_\omega (\phi \partial_{n+1} \psi + \phi^2 \partial_{2n+1} \psi) \, \mathrm{d}\mathcal{L}^{2n} = -\int_\omega \nabla_n^\phi \phi \; \psi \, \mathrm{d}\mathcal{L}^{2n}$*.*

Proof. Let us denote by $M := ||\phi||_{L^\infty(\omega)} < +\infty$. By standard considerations, there is a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\omega)$ converging uniformly to ϕ on every $\omega' \Subset \omega$. We denote by $\Phi_j : \omega \longrightarrow \mathbb{H}^n$ the graph map of ϕ_j and by E_j the subgraph of ϕ_j. Therefore, by Proposition 4.2.2, we obtain that for every $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in C_c^1(\mathbb{R}\times\omega, \mathbb{R}^{2n})$

$$\int_\omega \varphi_1 \circ \Phi - \left\langle \nabla^\phi \phi, \hat{\varphi} \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n} = \int_E \mathrm{div}_{\mathbb{H}}\, \varphi \; \mathrm{d}\mathcal{L}^{2n} =$$

$$= \lim_{j\to\infty} \int_{E_j} \mathrm{div}_{\mathbb{H}}\, \varphi \; \mathrm{d}\mathcal{L}^{2n} = \lim_{j\to\infty} \int_\omega \varphi_1 \circ \Phi_j - \left\langle \nabla^{\phi_j} \phi_j, \hat{\varphi} \circ \Phi_j \right\rangle \mathrm{d}\mathcal{L}^{2n}.$$

where $\hat{\varphi} = (\varphi_2, \dots, \varphi_{2n})$. If $\varphi((s,x)) := \psi(x)\xi(s)$ with $\xi = (\psi_2, \dots, \psi_{2n-1}) \in C_c^1(\omega, \mathbb{R}^{2n-1})$ and $\xi \in C_c^1(\mathbb{R})$ such that $\xi(s) = 1$ for all $s \in \mathbb{R}$ with $M - 1 \le s \le M + 1$, then $\varphi \in C_c^1(\mathbb{R}\times\omega, \mathbb{R}^{2n})$. Hence

$$\int_\omega \left\langle \nabla^\phi \phi(x), \psi(x)\xi(\phi(x)) \right\rangle \mathrm{d}\mathcal{L}^{2n} = \lim_{j\to\infty} \int_\omega \left\langle \nabla^{\phi_j} \phi_j(x), \psi(x)\xi(\phi_j(x)) \right\rangle \mathrm{d}\mathcal{L}^{2n}. \tag{4.36}$$

and since ϕ_j converges uniformly to ϕ, there exist $\bar{j} \in \mathbb{N}$ such that for all $j \ge \bar{j}$ and for all x in the support of ψ,

$$M - 1 \le \phi_j(x) \le M + 1$$

and hence $\xi(\phi_j(x)) = 1$ for all $j \ge \bar{j}$ and for all x in the support of ψ. This implies that

$$\int_\omega \left\langle \nabla^\phi \phi(x), \psi(x) \right\rangle \mathrm{d}\mathcal{L}^{2n} = \lim_{j\to\infty, j\ge\bar{j}} \int_\omega \left\langle \nabla^{\phi_j} \phi_j(x), \psi(x) \right\rangle \mathrm{d}\mathcal{L}^{2n}. \tag{4.37}$$

If $\psi(x) := (0, \dots, \psi_i(x), \dots, 0) \in C_c^1(\omega, \mathbb{R}^{2n-1})$ and $i \ne n$ then by (4.37) we obtain

$$\int_\omega \nabla_i^\phi \phi \; \psi \, \mathrm{d}\mathcal{L}^{2n} = \lim_j \int_\omega \nabla_i^{\phi_j} \phi_j \; \psi \, \mathrm{d}\mathcal{L}^{2n} =$$

$$= - \lim_{j\to\infty, j\ge\bar{j}} \int_\omega \phi_j \nabla_j^\phi \psi \mathrm{d}\mathcal{L}^{2n} = \int_\omega \phi \nabla^\phi \psi \mathrm{d}\mathcal{L}^{2n}$$

where we used the fact that if $j \neq n$ then $\nabla^{\phi_j}\phi_j = \nabla^{\phi}\phi_j$. On the other hand if $j = n$ we obtain

$$\int_\omega \nabla_n^\phi \phi \ \psi \, \mathrm{d}\mathcal{L}^{2n} = \lim_{j\to\infty, j\geq \bar{j}} \int_\omega \nabla_n^{\phi_j}\phi_j \ \psi \, \mathrm{d}\mathcal{L}^{2n} =$$
$$= - \lim_{j\to\infty, j\geq \bar{j}} \int_\omega (\phi_j \partial_n \psi + \phi_j^2 \partial_{2n}\psi) \, \mathrm{d}\mathcal{L}^{2n} = - \int_\omega (\phi \partial_n \psi + \phi^2 \partial_{2n}\psi) \, \mathrm{d}\mathcal{L}^{2n}.$$

□

In the following Theorem we provide a representation formula for the $\mathbb{H}-$perimeter and for the spherical Hausdorff $(2n+1)-$measure of the intrinsic graph of a Lipschitz function ϕ in terms of its $\nabla^\phi-$gradient.

Theorem 4.2.4. *If $\phi \in Lip_{\mathbb{W}}(\omega)$ with $\omega \subset \mathbb{W}$ open and bounded, then there exists a dimensional constant $c_n > 0$ such that the following area formula hold*

$$|\partial E_\phi|_{\mathbb{H}}(\mathbb{R}\times\omega) = c_n \mathcal{S}^{2n+1}(graph(\phi)) = \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2} \ \mathrm{d}\mathcal{L}^{2n}.$$

where $\mathcal{S}^{2n+1}$ denotes the spherical Hausdorff measure in $\mathbb{H}^n$.

Proof. Denoting by E the subgraph of ϕ and by Ω the cylinder $\mathbb{R}\times\omega$, being $|\partial E|_{\mathbb{H}}$ a Radon measure, a classical approximation result ensure the existence of a sequence

$$(\varphi_j)_{j\in\mathbb{N}} = ((\varphi_{j,1}, \ldots, \varphi_{j,2n}))_{j\in\mathbb{N}} \subset C_c^1(\Omega, \mathbb{R}^{2n})$$

with $|\varphi_j|_{\mathbb{R}^{2n}} \leq 1$ such that

$$\varphi_j \to \nu_E \quad |\partial E|_{\mathbb{H}} - \text{a.e in } \Omega$$

moreover by Corollary 4.1.13 it is easy to see that

$$\varphi_j \circ \Phi \to \nu_E \circ \Phi \quad \mathcal{L}^{2n} - \text{a.e in } \omega.$$

Inserting this sequence in (4.35) of Prosposition 4.2.2 we obtain that for all $j \in \mathbb{N}$,

$$- \int_\Omega \langle \varphi_j, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} = \int_\omega \varphi_{j,1}\circ\Phi - \left\langle \nabla^\phi \phi, \hat{\varphi}_j \circ \Phi \right\rangle \mathrm{d}\mathcal{L}^{2n} \tag{4.38}$$

and the first part of the thesis follows taking the limit as $j \to \infty$ in (4.38). The fact that $|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = c_n \mathcal{S}^{2n+1}(grap(\phi))$ for some dimensional constant $c_n > 0$ is a direct consequence of Theorem 1.2.11 and Lemma 4.1.9. $\square$

Before stating the approximation Theorem we need to recall two results which we will be fundamental in the proof.

Theorem 4.2.5 ([136]). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a strictly convex function and let $(g_j)_j$ and g be in $(L^1(\Omega))^n$. If*

1. *$g_j \to g$ weakly in $(L^1(\Omega))^n$;*
2. *$\int_\Omega f \circ g_j \, \mathrm{d}\mathcal{L}^n \to \int_\Omega f \circ g \, \mathrm{d}\mathcal{L}^n$*

then $g_j \to g$ strongly in $(L^1(\Omega))^n$.

Lemma 4.2.6 ([130]). *Suppose that $M > 0$ $c > 0$ and $u \in C^1((-M, M) \times \omega, \mathbb{R}) \cap C^0([-M, M] \times \omega)$ are such that $\nabla_1^{\mathbb{H}} u \leq 0$ and*

$$u(x, M) > c, \quad u(x, -M) \leq 0 \quad \forall x \in \omega.$$

Assume also that $\nabla_1^{\mathbb{H}} u(p) < 0$ on the set $A = \{p \in (-M, M) \times \omega : u(p) = c$. Then there exists $\phi : \omega \longrightarrow (-M, M)$ such that ϕ is uniformly ∇^ϕ-differentiable in ω and

$$\{u > c\} \cap (-M, M) \times \omega = E_\phi \cap (-M, M) \times \omega.$$

We are now in position to state and prove the approximation Theorem. We will strictly follow here the approximation techniques contained in [117] and [130], which are extensions to the Heisenberg setting of the classical De Giorgi's techniques for the Euclidean case [59].

Theorem 4.2.7. *Let $\phi : \mathbb{W} \equiv \mathbb{R}^{2n} \longrightarrow \mathbb{V} \equiv \mathbb{R}$ be a bounded intrinsic Lipschitz function. Then, for each bounded open set $\omega \subset \mathbb{W}$, there exist a a sequence $\{\phi_k\}_{k \in \mathbb{N}}$ of smooth functions on ω such that*

(i) $\phi_k \to \phi$ locally uniformly in ω;

(ii) $|\nabla^{\phi_k} \phi_k(x)| \leq \|\nabla^\phi \phi\|_{L^\infty(\omega)}$ $\forall x \in \omega$;

(iii) $\nabla^{\phi_k}\phi_k(x) \to \nabla^{\phi}\phi(x)$ $\mathcal{L}^{2n}-a.e$ $x \in \omega$;

Proof. Let us assume firstly that $\phi : \mathbb{W} \longrightarrow \mathbb{R}$. Let $M := \|\phi\|_{L^\infty(\mathbb{W})} < +\infty$. For each $\alpha > 0$ we define $u_\alpha : \mathbb{H}^n \longrightarrow \mathbb{R}$ by

$$\begin{aligned} u_\alpha(p) := (\rho_\alpha * \chi_{E_\phi})(p) &= \int_{\mathbb{H}^n} \rho_\alpha(p \cdot q^{-1})\chi_{E_\phi}(q)\, \mathrm{d}\mathcal{L}^{2n+1}(q) \\ &= \int_{\mathbb{H}^n} \rho_\alpha(q)\chi_{E_\phi}(q^{-1} \cdot p)\, \mathrm{d}\mathcal{L}^{2n+1}(q) \end{aligned} \tag{4.39}$$

where $\rho_\alpha(x) := \alpha^{2n+2}\rho(\delta_{1/\alpha}(x))$ and $\rho \in C_c^\infty(U(0,1))$ is a smooth mollifier with $\rho(p^{-1}) = \rho(-p) = \rho(p)$ $\forall p \in \mathbb{H}^n$. Namely let us exploit the classical technique of approximation by convolution in $\mathbb{H}^n$ introduced in [75] of which the main properties are collected in [130, Lemma 2.4].

Claim 0. *Let us first show in detail that u_α is constant far from the graph of ϕ, so that the integral (4.39) is indeed extended only in a neighborhood of the graphs itself.*

To this end, for each $\alpha > 0$ it follows that $u_\alpha \in C_c^\infty(\mathbb{H}^n)$ and

$$spt(u_\alpha) \subset U(0,\alpha) \cdot spt(\chi_{E_\phi})$$

Moreover, observe that for each $\alpha > 0$

$$0 \le u_\alpha(p) \le 1 \ \forall p \in \mathbb{H}^n;$$

and for all sufficiently small $\alpha > 0$

$$u_\alpha(p) = 1 \quad \forall p \in \mathbb{W} \cdot (-\infty, -2M] = (-\infty, -2M] \times \mathbb{W}. \tag{4.40}$$

Notice also that E_ϕ is open in $\mathbb{H}^n$ and

$$\begin{aligned} spt(\chi_{E_\phi}) = \overline{E}_\phi &\subseteq \{(s,x) \mid x \in \mathbb{R}^{2n},\ s \le \phi(x)\} \\ &\subseteq (-\infty, M] \times \mathbb{W}. \end{aligned} \tag{4.41}$$

Hence

$$spt(u_\alpha) \subseteq \bar{U}_\alpha \cdot spt(\chi_{E_\phi}) \subseteq (-\infty, 2M) \times \mathbb{W} \tag{4.42}$$

for $\alpha < M$. In particular, (4.42) implies

$$u_\alpha(p) = 0 \quad \forall\ p \in [2M, +\infty) \times \mathbb{W}. \tag{4.43}$$

Claim 1. *Let us compute explicitly* $\nabla_1^{\mathbb{H}} u_\alpha$.

Let $\varphi \in C_c^\infty((-3M, 3M) \times \omega)$, then

$$\begin{aligned}
&\left\langle \nabla_1^{\mathbb{H}} u_\alpha, \varphi \right\rangle = - \int_{(-3M,3M)\times\omega} u_\alpha(p^{'}) \nabla_1^{\mathbb{H}} \varphi(p^{'}) \mathrm{d}\mathcal{L}^{2n+1}(p^{'}) \qquad (4.44)\\
&= - \int_{\bar{U}_\alpha} \rho_\alpha(p) \mathrm{d}\mathcal{L}^{2n+1}(p) \int_{(-3M,3M)\times\omega} \chi_{E_\phi}(p^{-1} \cdot p^{'}) \nabla_1^{\mathbb{H}} \varphi(p^{'}) \mathrm{d}\mathcal{L}^{2n+1}(p^{'})\\
&= - \int_{\bar{U}_\alpha} \rho_\alpha(p) \mathrm{d}\mathcal{L}^{2n+1}(p) \int_{p^{-1}\cdot((-3M,3M)\times\omega)} \chi_{E_\phi}(q) \nabla_1^{\mathbb{H}} \varphi(p \cdot q) \mathrm{d}\mathcal{L}^{2n+1}(q).
\end{aligned}$$

With the notation $\varphi_p(q) = \varphi(p \cdot q)$ we have $\nabla_1^{\mathbb{H}}(\varphi(p \cdot q)) = \nabla_1^{\mathbb{H}} \varphi_p(q)$, because $\nabla_1^{\mathbb{H}}$ is left-invariant; moreover $\varphi_p \in C_c^\infty(p^{-1} \cdot ((-3M, 3M) \times \omega))$, then

$$\begin{aligned}
&\left\langle \nabla_1^{\mathbb{H}} u_\alpha, \varphi \right\rangle = \qquad (4.45)\\
&= - \int_{\bar{U}_\alpha} \rho_\alpha(p) \mathrm{d}\mathcal{L}^{2n+1}(p) \int_{p^{-1}\cdot((-3M,3M)\times\omega)} \chi_{E_\phi}(q) \nabla_1^{\mathbb{H}} \varphi_p(q) \mathrm{d}\mathcal{L}^{2n+1}(q).
\end{aligned}$$

Put $\mathcal{C}(p, 3M) := p^{-1} \cdot ((-3M, 3M) \times \omega)$ then by an integration by parts, we have

$$\int_{\mathcal{C}(p,3M)} \chi_{E_\phi}(q) \nabla_1^{\mathbb{H}} \varphi_p(q) \mathrm{d}\mathcal{L}^{2n+1}(q) = - \int_{\mathcal{C}(p,3M)} \nu_{E_\phi}^1(q) \varphi_p(q) \mathrm{d}|\partial E_\phi|(q) \tag{4.46}$$

where $\nu_{E_\phi}^1$ is the first component of the horizontal inward normal $\nu_{E_\phi} = (\nu_{E_\phi}^1, \dots, \nu_{E_\phi}^{2n})$ to E_ϕ.

Because $spt(\varphi_p) \Subset \mathcal{C}(p, 3M)$ and $p \in \overline{U}_\alpha$ if α is small enough, we can replace $\mathcal{C}(p, 3M)$ by $\mathcal{C}(0, 3M)$. Thus, by Fubini-Tonelli Theorem and a change of variable, we obtain

$$\left\langle \nabla_1^{\mathbb{H}} u_\alpha, \varphi \right\rangle = \int_{\mathcal{C}(0,3M)} \nu_{E_\phi}^1(q) \mathrm{d}|\partial E_\phi|(q) \Big(\int_{\mathbb{H}^n} \rho_\alpha(q) \varphi(p \cdot q) \mathrm{d}\mathcal{L}^{2n+1}(p) \Big).$$

Then for each $p \in \mathcal{C}(0, 3M) = (-3M, 3M) \times \omega$ and for all small enough $\alpha > 0$

$$\begin{aligned}
\nabla_1^{\mathbb{H}} u_\alpha(p) &= \int_{\mathcal{C}(0,3M)} \rho_\alpha(p \cdot q^{-1}) \nu_1(q) \mathrm{d}|\partial E_\phi|(q) = \qquad (4.47)\\
&= \int_{U^R(p,\alpha)} \rho_\alpha(p \cdot q^{-1}) \nu_1(q) \mathrm{d}|\partial E_\phi|(q)
\end{aligned}$$

where $U^R(p,\alpha) := U(0,\alpha)\cdot p$.

In particular we immediately deduce from (4.47) the following assertion: *For each couple (ω,ω_0) of open and bounded subset of $\mathbb{W}$ with $\omega_0 \Supset \omega$ there exists $\bar\alpha = \bar\alpha(\omega_0) > 0$ such that for all $0<\alpha<\bar\alpha$*

$$\int_{(-2M,2M)\times\omega} |\nabla_{\mathbb{H}} u_\alpha| \mathrm{d}\mathcal{L}^{2n+1} \le |\partial E_\phi|((-2M,2M)\times\omega_0). \tag{4.48}$$

Analogously we also obtain

$$\nabla^{\mathbb{H}}_{n+1}\nabla^{\mathbb{H}}_1 u_\alpha(p) = \int_{\mathcal{C}(0,3M)} \nabla^{\mathbb{H}}_{n+1}\Big(\rho_\alpha(p\cdot q^{-1})\Big)\nu_1(q)\mathrm{d}|\partial E_\phi|(q). \tag{4.49}$$

Claim 2. *For every fixed α and $c\in(0,1)$ the set*

$$A = \{p\in(-2M,2M)\times\omega : u_\alpha(p) = c\}$$

implicitly defines a function $\phi_\alpha : \omega \longrightarrow \mathbb{R}$. This family has a subsequence $\{\phi_k\}_k$ such that $|\nabla^{\phi_k}\phi_k| \le \|\nabla^\phi\phi\|_{L^\infty(\omega)}$ $\forall k\in\mathbb{N}$ on $\omega\subset\mathbb{W}$ and $\{\phi_k\}_k$ converge strongly to ϕ in $L^1(\omega)$.

From Claim 1 we will first deduce that

$$\nabla^{\mathbb{H}}_1 u_\alpha(p) < 0 \quad \forall p\in A. \tag{4.50}$$

Indeed, recalling that (see Corollary 4.1.13)

$$\nu_1\circ\Phi = -\frac{1}{\sqrt{1+|\nabla^\phi\phi|^2}} \quad \text{in } \omega$$

and denoting by

$$I_\alpha(p) := \int_{U^R(p,\alpha)} \rho_\alpha(p\cdot q^{-1})\mathrm{d}|\partial E_\phi|(q) \tag{4.51}$$

we obtain

$$\nabla^{\mathbb{H}}_1 u_\alpha(p) \le -\frac{1}{\sqrt{1+\|\nabla^\phi\phi\|^2_{L^\infty(\omega)}}} I_\alpha(p) \quad \forall p\in(-3M,3M)\times\omega. \tag{4.52}$$

In order to prove (4.50) for every $c \in (0,1)$ let us define

$$E_\alpha = E_{\alpha,c} := \{p \in \omega \cdot \mathbb{R} \mid u_\alpha(p) > c\}$$

and notice that for each $p \in (-2M, 2M) \times \omega$ with $u_\alpha(p) = c$

$$\mathcal{L}^{2n+1}(U^R(p,\alpha) \cap E_\phi) > 0 \qquad \mathcal{L}^{2n+1}(U^R(p,\alpha) \cap E_\phi^c) > 0. \tag{4.53}$$

Otherwise, by contradiction, assume, for instance, that $\mathcal{L}^{2n+1}(U^R(p,\alpha) \cap E_\phi) = 0$. Then, since E_ϕ is open, we can assume $U^R(p,\alpha) \cap E_\phi = \emptyset$. By definition of convolution, it follows that $u_\alpha(p) = 0$ and then a contradiction. Analogously, it follows that $u_\alpha(p) = 1$ if $\mathcal{L}^{2n+1}(U^R(p,\alpha) \cap E_\phi^c) = 0$.

By (4.53) and Theorem 1.2.15, we have $|\partial E_\phi|(U^R(p,\alpha)) > 0$
$\forall p \in (-2M, 2M) \times \omega$ with $u_\alpha(p) = c$, then

$$I_\alpha(p) > 0 \quad\; p \in (-2M, 2M) \times \omega \;\text{ with }\; u_\alpha(p) = c. \tag{4.54}$$

From (4.52) and (4.54), (4.50) follows. Applying the implicit function Lemma 4.2.6 we deduce that there is a function $\phi_\alpha : \omega \longrightarrow [-2M, 2M]$ such that

$$E_\alpha \cap ([-2M, 2M] \times \omega) = E_{\phi_\alpha} \cap ([-2M, 2M] \times \omega), \tag{4.55}$$

From (4.43), (4.40), it follows that

$$\partial E_\alpha \cap (\mathbb{R} \times \omega) = \{p \in [-2M, 2M] \times \omega \mid u_\alpha(p) = c\} = \Phi_\alpha(\omega) \tag{4.56}$$

where $\Phi_\alpha : \omega \longrightarrow \mathbb{H}^n$ is the graph map defined as in (3.9).

We can now estimate from above the gradient of ϕ_α. Letting

$$\hat{\nabla}_{\mathbb{H}} u_\alpha := (\nabla_2^{\mathbb{H}} u_\alpha, \dots, \nabla_{2n}^{\mathbb{H}} u_\alpha),$$

$\hat{\nu}_{E_\phi} = (\nu_{E_\phi}^2, \dots, \nu_{E_\phi}^{2n})$ and arguing as in Claim 1 we get,

$$\begin{aligned} |\nabla^{\phi_\alpha} \phi_\alpha| =& \frac{|\hat{\nabla}_{\mathbb{H}} u_\alpha(p)|}{|\nabla_1^{\mathbb{H}} u_\alpha(p)|} \leq \\ & \frac{1}{|\nabla_1^{\mathbb{H}} u_\alpha(p)|} \int_{U^R(p,\alpha)} |\hat{\nu}_{E_\phi}(q)||\rho_\alpha(p \cdot q^{-1})| \,\mathrm{d}|\partial E_\phi|(q) \\ & \leq I_\alpha(p) \frac{\|\nabla^\phi \phi\|_{L^\infty(U^R(p,\alpha))}}{|\nabla_1^{\mathbb{H}} u_\alpha(p)| \sqrt{1 + \|\nabla^\phi \phi\|^2_{L^\infty(U^R(p,\alpha))}}} \\ & \leq \|\nabla^\phi \phi\|_{L^\infty(U^R(p,\alpha))}, \end{aligned} \tag{4.57}$$

the last inequality being a consequence of (4.52). It follows that for all $\alpha > 0$

$$|\nabla^{\phi_\alpha}\phi_\alpha| \leq \|\nabla^\phi \phi\|_{L^\infty(\omega)} \text{ in } \omega. \tag{4.58}$$

Let us finally prove the convergence of ϕ_k in $L^1(\omega)$. By definition of E_α, it follows that

$$\begin{aligned} &u_\alpha - \chi_{E_\phi} > c \text{ in } E_\alpha \setminus E_\phi \\ &\chi_{E_\phi} - u_\alpha \geq (1-c) \text{ in } E_\phi \setminus E_\alpha \end{aligned}$$

and thus

$$\int_{(-2M,2M)\times\omega} |u_\alpha - \chi_{E_\phi}| \mathrm{d}\mathcal{L}^{2n+1} \geq \bar{c}\mathcal{L}^{2n+1}(E_\alpha \Delta E_\phi)$$

where $\bar{c} := \min\{c, 1-c\}$. Since

$$\lim_{\alpha\to 0} \|u_\alpha - \chi_{E_\phi}\|_{L^1((-2M,2M)\times\omega)} = 0,$$

we also get

$$\lim_{\alpha\to 0} \|\chi_{E_\alpha} - \chi_{E_\phi}\|_{L^1((-2M,2M)\times\omega)} = \lim_{\alpha\to 0} \mathcal{L}^{2n+1}(E_\alpha \Delta E_\phi) = 0. \tag{4.59}$$

A simple application of Fubini theorem shows that

$$\|\phi_\alpha - \phi\|_{L^1(\omega)} = \|\chi_{E_\alpha} - \chi_{E_\phi}\|_{L^1((-2M,2M)\times\omega)}$$

and hence

$$\phi_\alpha \to \phi \text{ in } L^1(\omega). \tag{4.60}$$

Claim 3. *$\{\phi_k\}_k$ converges uniformly to ϕ on the compact subsets of ω.*

Indeed, let (α_h) be a positive sequence converging to 0. Because of (4.58), $|\phi_\alpha| \leq 2M$ $\forall \alpha > 0$ and the fact that the functions ϕ_α are $\frac{1}{2}$−Hölder continuous on every $\omega' \Subset \omega$ with Hölder constant independent from α ([117, Lemma 3.1]), by Ascoli-Arzelá's Lemma there exists a subsequence $(\alpha_{h_k})_k$ and a continuous function $\bar{\phi} \in C^0(\omega)$ such that

$$\phi_{\alpha_{h_k}} \to \bar{\phi} \text{ uniformly on compact subsets of } \omega. \tag{4.61}$$

From (4.60) and (4.61), it follows that $\bar{\phi} \equiv \phi$ and the claim follows.

Claim 4. *There exists a positive sequence* $(\alpha_h)_h$ *such that, if* $\phi_h \equiv \phi_{\alpha_h}$ *then*

$$\nabla^{\phi_h}\phi_h(x) \to \nabla^{\phi}\phi(x) \quad \mathcal{L}^{2n} - a.e\ x \in \omega. \tag{4.62}$$

In order to get (4.62), we need only to prove that there exists a positive sequence $(\alpha_h)_h$ converging to 0 such that there exists

$$\lim_{h\to\infty}\int_\omega \sqrt{1+|\nabla^{\phi_h}\phi_h|^2}\mathrm{d}\mathcal{L}^{2n} = \int_\omega \sqrt{1+|\nabla^{\phi}\phi|^2}\mathrm{d}\mathcal{L}^{2n} \tag{4.63}$$

where $\phi_h \equiv \phi_{\alpha_h}$. Indeed, up to subsequence, by (4.58) and Proposition 4.2.3 we can assume , that the sequence in (4.63) also satisfies

$$\nabla^{\phi_h}\phi_h \to \nabla^{\phi}\phi \ \text{ weakly in } (L^1(\omega))^{2n-1}. \tag{4.64}$$

Then, by Theorem 4.2.5, it follows that

$$\nabla^{\phi_h}\phi_h \to \nabla^{\phi}\phi \ \text{ strongly in } (L^1(\omega))^{2n-1}. \tag{4.65}$$

Therefore, up to a subsequence, (4.62) follows. Let us now prove (4.63). It is sufficient to show that there exists $\bar{c} \in (0,1)$ and $(\alpha_h)_h \subset (0,+\infty)$ converging to 0 such that

$$\exists \lim_{h\to\infty} |\partial E_{\alpha_h,\bar{c}}|_{\mathbb{H}}((-2M,2M)\times\omega) = |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega). \tag{4.66}$$

In fact, by Proposition 4.2.4 and well-known $\mathbb{H}-$perimeter properties

$$\begin{aligned}
&\int_\omega \sqrt{1+|\nabla^{\phi}\phi|^2}\mathrm{d}\mathcal{L}^{2n} = |\partial E_\phi|_{\mathbb{H}}(\mathbb{R}\times\omega) = \qquad (4.67)\\
&= |\partial E_\phi|_{\mathbb{H}}((-\infty,2M]\times\omega) + |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega)+\\
&+ |\partial E_\phi|_{\mathbb{H}}([2M,+\infty)\times\omega) =\\
&= |\partial E_\phi|_{\mathbb{H}}((-\infty,-2M]\times\omega\cap\partial E_\phi) + |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega)+\\
&+ |\partial E_\phi|_{\mathbb{H}}([2M,+\infty)\times\omega\cap\partial E_\phi) =\\
&= |\partial E_\phi|_{\mathbb{H}}((-2M,2M)\times\omega),
\end{aligned}$$

where in the last equality we have used the inequality $|\phi| \leq M$ which implies $(-\infty, -2M] \times \omega \cap \partial E_\phi = [2M, +\infty) \times \omega \cap \partial E_\phi = \emptyset$. Analogously, by (4.55),(4.40) and (4.43)

$$|\partial E_{\alpha_h,c}|_{\mathbb{H}}((-2M, 2M) \times \omega) = |\partial E_{\phi_h}|_{\mathbb{H}}((-2M, 2M) \times \omega) \tag{4.68}$$

and

$$\begin{aligned} \int_\omega \sqrt{1 + |\nabla^{\phi_h}\phi_h|^2} \mathrm{d}\mathcal{L}^{2n} &= |\partial E_{\phi_h}|(\mathbb{R} \times \omega) && (4.69) \\ &= |\partial E_{\phi_h}|_{\mathbb{H}}((-2M, 2M) \times \omega), && (4.70) \end{aligned}$$

where $\phi_h = \phi_{\alpha_h}$. Therefore (4.67),(4.68) and (4.69) imply (4.63). Finally let us prove (4.66). We will follow the technique exploited in [130]. Notice that, by the semicontinuity of $\mathbb{H}-$perimeter measure and (4.59), we have

$$|\partial E_\phi|_{\mathbb{H}}((-2M, 2M) \times \omega) \leq \liminf_{\alpha \to 0^+} |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M, 2M) \times \omega) \tag{4.71}$$

for each $c \in (0, 1)$. On the other hand, by (4.71) and the coarea formula it follows that

$$\begin{aligned} |\partial E_\phi|_{\mathbb{H}}((-2M, 2M) \times \omega) &\leq \int_0^1 \liminf_{\alpha \to 0^+} |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M, 2M) \times \omega) \mathrm{d}c \leq \\ &\leq \liminf_{\alpha \to 0^+} \int_0^1 |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M, 2M) \times \omega) \mathrm{d}c = \\ &= \liminf_{\alpha \to 0^+} \int_{(-2M,2M)\times\omega} |\nabla_{\mathbb{H}} u_\alpha| \mathrm{d}\mathcal{L}^{2n+1} =: I(\omega, c). \end{aligned}$$

Now, for each $\omega_0 \Supset \omega$ open and bounded, by Claim 2, it holds

$$I(\omega, c) \leq |\partial E_\phi|_{\mathbb{H}}((-2M, 2M) \times \omega_0). \tag{4.72}$$

Indeed, by Claim 2, for each $\omega_0 \Supset \omega$ open and bounded there exists a sequence $\{\alpha_h\}_h \subset (0, +\infty)$ which converges to 0 and $\bar{h} = \bar{h}(\omega_0) > 0$ such that for each $h \leq \bar{h}$

$$\int_{(-2M,2M)\times\omega} |\nabla u_{\alpha_h}| \mathrm{d}\mathcal{L}^{2n+1} \leq |\partial E_\phi|_{\mathbb{H}}((-2M, 2M) \times \omega_0). \tag{4.73}$$

Hence

$$I(\omega, c) \leq |\partial E_\phi|_{\mathbb{H}}((-2M, 2M) \times \omega_0) \tag{4.74}$$

for each $c \in (0,1)$ and each $\omega_0 \Supset \omega$ open and bounded. Moreover, since $|\partial E_\phi|_{\mathbb{H}}$ is a Radon measure then by a standard approximation argument we can rewrite (4.72) with ω instead of ω_0. Using again (4.71), we obtain that $\mathcal{L}^1-$a.e $c \in (0,1)$

$$\liminf_{\alpha \to 0} |\partial E_{\alpha,c}|_{\mathbb{H}}((-2M, 2M) \times \omega) = |\partial E_\phi|_{\mathbb{H}}((-2M, 2M) \times \omega).$$

In particular there exists $\bar{c} \in (0,1)$ and a positive sequence $(\alpha_h)_h$ converging to 0 such that (4.66) holds.

We conclude the proof proving that the assumption $\phi : \mathbb{W} \longrightarrow \mathbb{R}$ can be relaxed to $\phi : \omega \longrightarrow \mathbb{R}$ where $\omega \subset \mathbb{W}$ is open and bounded. Indeed, by (iii) of Theorem 4.1.2 ϕ is locally uniformly continuous on ω. Thus ϕ can be extended to a continuous function $\phi : \overline{\omega} \to \mathbb{V} \equiv \mathbb{R}$ and let $M := \sup_\omega |\phi| < +\infty$. By Theorem 4.1.2 (i), there exists a Lipschitz extension $\overline{\phi} : \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{V} \equiv \mathbb{R}$ of ϕ. Define $\phi^* : \mathbb{W} \to \mathbb{V} \equiv \mathbb{R}$ by

$$\phi^*(x) = \max\{\min\{\overline{\phi}(x),\, M\},\, -M\} \quad x \in \mathbb{W}\,.$$

Theorem 4.1.2 (ii) yields that ϕ^* is a bounded Lipschitz function, which still extends ϕ. Applying the previous part of the proof to ϕ^* we get the thesis. $\square$

Remark 4.2. We point out that Theorem 4.1.5 provides another way to prove the uniform convergence of the family $\{\phi_\alpha\}$, see [46].

4.2.1 Some Applications

In this subsection we provide an estimate of the Lipschitz constant of a given $\phi \in Lip_{\mathbb{W}}(\omega)$ in terms of the $L^\infty-$norm of its intrinsic gradient. This result implies a characterization of $Lip_{\mathbb{W}}(\omega)$ in terms of approximating sequences. In other words, if for a given continuous function ϕ there is a sequence of smooth functions which satisfies $(i), (ii)$ and (iii) of Theorem 4.2.7 then $\phi \in Lip_{\mathbb{W},loc}(\omega)$.

The family $\{\nabla_i^\phi\}$ $(i \in \{1, \dots, 2n-1\})$ satisfy the Hörmander condition if $n \geq 2$ (if $n = 1$, it well known that there is no connectivity). Hence we can define another distance in ω, namely the Carnot-Carathéodory distance $\mathrm{d}_{cc,\phi}$ (see [20, 27, 91, 114] for the details). From now on we will denote by $U_{cc,\phi}(x,r) := \{y \in \omega \mid \mathrm{d}_{cc,\phi}(x,y) < r\}$ and we recall that $U_\phi(x,r) := \{y \in \omega \mid \mathrm{d}_\phi(x,y) < r\}$.

Proposition 4.2.8. *Let $\omega \subset \mathbb{W}$ be open and bounded, $n \geq 2$ and $\phi : \omega \longrightarrow \mathbb{R}$ be a Lipschitz function with respect to the distance $\mathrm{d}_{cc,\phi}$ with $Lip_{cc}(\phi)$ its Lipschitz constant and denote*

$$M := \max\Big\{Lip_{cc}(\phi), 2Lip(\phi,\omega)(1 + 9(1 + Lip(\phi,\omega)^{1/2})^4)\Big\}.$$

Then there are positive constants C_1 and C_2 depending only on M in an increasing way such that for each $\bar{x} \in \omega$ and for each $r > 0$ such that $U_{cc,\phi}(\bar{x}, 2r) \subset \omega$ it holds:

$$U_\phi(\bar{x}, r/C_2) \subset U_{cc,\phi}(\bar{x}, r). \tag{4.75}$$

Moreover, for each $\bar{x} \in \omega$ and each $r > 0$ such that $U_\phi(\bar{x}, r) \subset \omega$ it holds:

$$U_{cc,\phi}(\bar{x}, r/C_1) \subset U_\phi(\bar{x}, r). \tag{4.76}$$

Proof. We denote $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{2n})$, and choose $y = (y_1, \dots, y_{2n}) \in U_{cc}(\bar{x}, r)$. In order to establish the inclusion (4.76), we estimate from below $\mathrm{d}_{cc,\phi}(\bar{x}, y)$. By definition there exist $h = (h_1, \dots, h_{2n-1}) \in L^\infty((0,1), \mathbb{R}^{2n-1})$ with $|h| < 2\mathrm{d}_{cc,\phi}(\bar{x}, y)$ and an absolutely continuous curve $\tilde{\gamma}$ such that

$$\dot{\tilde{\gamma}}(t) = \sum_{j=1}^{2n-1} h_j(t) \nabla_j^{\phi_i}(\tilde{\gamma}(t)) \quad a.e \;\; t \in (0,1) \tag{4.77}$$

and

$$\tilde{\gamma}(0) = \bar{x}, \quad \tilde{\gamma}(1) = y.$$

Denoting by $\tilde{\gamma}(t) = (\tilde{\gamma}_1(t), \dots, \tilde{\gamma}_{2n}(t))$ and using the explicit form of the vector fields $\{\nabla^{\phi_i}\}$ we easily obtain that:

$$|y_n - \bar{x}_n| = |\tilde{\gamma}_n(1) - \bar{x}_n| \leq 2\mathrm{d}_{cc,\phi}(\bar{x}, y). \tag{4.78}$$

This implies that the $\mathrm{d}_{cc,\phi}$ distance is equivalent to the exponential distance defined in terms of all the vector fields $(\nabla^\phi)_{i=1\cdots 2n}$, and the equivalence of this last distance and the d_ϕ is already contained in [45] for general vector fields. We repeat here the proof in our special case. Let us consider the curve

$$\gamma : [0,1] \to \mathbb{R}^{2n}, \quad \gamma(s) := \exp(s(y_n - \bar{x}_n)\nabla_n^\phi)(\bar{x})$$

then in coordinates

$$\gamma(s) = (\bar{x}_1, \ldots, \bar{x}_n + s(y_n - \bar{x}_n), \ldots, \bar{x}_{2n-1}, 2(y_n - \bar{x}_n)\int_0^s \phi(\gamma(\tau))\mathrm{d}\tau + \bar{x}_{2n}).$$

Note that the points $\gamma(1)$ and y have the same n-th component, so that while computing their distance, we can discard the vector field ∇_n^ϕ and the family $\{\nabla^\phi\}$ reduces the standard Heisenberg vector fields in $\mathbb{H}^{n-1}$. Hence there exists geometric constants $c_1, c_2 > 0$ (independent of y) such that:

$$c_1 \mathrm{d}_{cc,\phi}(\gamma(1), y) \le \mathrm{d}_\phi(\gamma(1), y) \le c_2\ \mathrm{d}_{cc,\phi}(\gamma(1), y). \tag{4.79}$$

Let us also notice that, by simple calculations:

$$\begin{aligned}\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x}) \le & |y_n - \bar{x}_n| + \sqrt{2}|y_n - \bar{x}_n|^{\frac{1}{2}} M^{\frac{1}{2}} \Big[\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x})\Big]^{\frac{1}{2}} \\ & + 2\sqrt{2}|y_n - x_n|^{\frac{1}{2}} M^{\frac{1}{2}} C(M)^{\frac{1}{2}} \Big[\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x})\Big]^{\frac{1}{2}}\end{aligned} \tag{4.80}$$

where $C(M)$ is as in (3.17). Since $M^{1/2} \le C(M)$ we obtain

$$\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x}) \le |y_n - \bar{x}_n| + 3\sqrt{2}|y_n - \bar{x}_n|^{\frac{1}{2}} C^2(M) \Big[\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x})\Big]^{\frac{1}{2}}$$

hence

$$\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x}) \le |y_n - \bar{x}_n| + 9|y_n - \bar{x}_n| C(M)^4 + \frac{\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x})}{2}$$

and finally:

$$\max_{t\in[0,1]} \mathrm{d}_\phi(\gamma(t), \bar{x}) \le 2(1 + 9C(M)^4)|y_n - \bar{x}_n| \tag{4.81}$$

which implies that $\forall t \in [0,1]$

$$|\phi(\gamma(t)) - \phi(\bar{x})| \le 2M(1 + 9C(M)^4)|y_n - \bar{x}_n|. \tag{4.82}$$

By the triangle inequality stated in (3.17) and using (4.82) with $\tilde{M} := 2M(1+9C(M)^4)$ we infer:

$$\begin{aligned} \mathrm{d}_\phi(\bar{x}, y) &\le \mathrm{d}_\phi(\gamma(1), \bar{x}) + \mathrm{d}_\phi(\gamma(1), y) + |\phi(\gamma(1)) - \phi(\bar{x})|^{1/2}|\bar{x}_n - y_n|^{1/2} \\ &\le \mathrm{d}_\phi(\gamma(1), \bar{x}) + c_2\ \mathrm{d}_{cc,\phi}(\gamma(1), y) + \tilde{M}^{1/2}\mathrm{d}_{cc,\phi}(\gamma(1), \bar{x}) \\ &\le \mathrm{d}_\phi(\gamma(1), \bar{x}) + c_2\ \mathrm{d}_{cc,\phi}(\gamma(1), \bar{x}) + c_2\ \mathrm{d}_{cc,\phi}(y, \bar{x}) + \\ &+ \tilde{M}^{1/2}\mathrm{d}_{cc,\phi}(\gamma(1), \bar{x}). \end{aligned} \tag{4.83}$$

By definition of $\mathrm{d}_{cc,\phi}$ and (4.78):

$$\mathrm{d}_{cc,\phi}(\gamma(1), \bar{x}) \le |y_n - \bar{x}_n| \le 2\mathrm{d}_{cc,\phi}(\bar{x}, y) \tag{4.84}$$

hence by (4.78), (4.83), (4.84) and (4.81) we obtain:

$$\begin{aligned} \mathrm{d}_\phi(\bar{x}, y) &\le \mathrm{d}_\phi(\gamma(1), \bar{x}) + c_2\ \mathrm{d}_{cc,\phi}(\gamma(1), \bar{x}) + c_2\ \mathrm{d}_{cc,\phi}(y, \bar{x}) + \\ &+ \tilde{M}^{1/2}\mathrm{d}_{cc,\phi}(\gamma(1), \bar{x}) \\ &\le (3c_2 + 2\tilde{M}^{1/2})\mathrm{d}_{cc,\phi}(y, \bar{x}) + 4(1 + 9C(M)^4)\mathrm{d}_{cc,\phi}(\bar{x}, y). \end{aligned}$$

The proof of inclusion (4.75) is analogous: by the triangle inequality we have

$$\mathrm{d}_{cc,\phi}(\bar{x}, y) \le \mathrm{d}_{cc,\phi}(\gamma(1), \bar{x}) + \mathrm{d}_{cc,\phi}(\gamma(1), y) \tag{4.85}$$

and calling $c := 1/c_2$ and by simple calculations we obtain:

$$\begin{aligned} \mathrm{d}_{cc,\phi}(\gamma(1), y) &\le c\ \mathrm{d}_\phi(\bar{x}, y) + 2c|y_n - \bar{x}_n|^{1/2} \int_0^1 |\phi(\gamma(\tau)) - \phi(\bar{x})|^{1/2}\mathrm{d}\tau \\ &+ 2c|y_n - \bar{x}_n|^{1/2} \int_0^1 |\phi(\gamma(\tau)) - \phi(y)|^{1/2}\mathrm{d}\tau \\ &\le c\ \mathrm{d}_\phi(\bar{x}, y) + 2cM^{1/2}|y_n - \bar{x}_n|^{1/2} \int_0^1 |\mathrm{d}_{cc,\phi}(\gamma(\tau), \bar{x})|\mathrm{d}\tau \\ &+ 2cM^{1/2}|y_n - \bar{x}_n|^{1/2} \int_0^1 |\mathrm{d}_{cc,\phi}(\gamma(\tau), y)|\mathrm{d}\tau \\ &\le c\ \mathrm{d}_\phi(\bar{x}, y) + 4cM^{1/2}|y_n - \bar{x}_n| \\ &\le c(1 + 4M^{1/2})\mathrm{d}_\phi(\bar{x}, y). \end{aligned}$$

Therefore by (4.85) we have:

$$\mathrm{d}_{cc,\phi}(\bar{x}, y) \le (c(1 + 4M^{1/2}) + 1)\mathrm{d}_\phi(\bar{x}, y), \tag{4.86}$$

and inclusion (4.75) follow.

□

Proposition 4.2.9. *Let $\omega \subset \mathbb{W}$ be open and bounded, $\phi \in Lip_{\mathbb{W}}(\omega)$ and $n \geq 2$. Then for each $\bar{x} \in \omega$ and each $r > 0$ sufficiently small*

$$Lip(\phi, U_\phi(\bar{x}, r)) \leq c\Big(4\sqrt{\|\nabla^\phi \phi\|_{L^\infty(\omega)}} + 1\Big)\|\nabla^\phi \phi\|_{L^\infty(\omega)}$$

for a suitable geometric positive constant c. Moreover, if $n = 1$ it holds:

$$Lip(\phi, U_\phi(\bar{x}, r)) \leq c\sqrt{1 + \|\nabla^\phi \phi\|^2_{L^\infty(\omega)}} \tag{4.87}$$

Proof. Let us fix $\phi \in Lip_{\mathbb{W}}(\omega)$ and $M := ||\nabla^\phi \phi||_{L^\infty(\omega)}$. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be a sequence of smooth functions as in Theorem 4.2.7.
For every $\bar{x} \in \omega$, let us define $\bar{r} := \frac{\mathrm{d}_\phi(\bar{x}, \partial\omega)}{2C_1} > 0$ where C_1 is as in Proposition 4.2.8 and depend only on $||\nabla^\phi \phi||_{L^\infty(\omega)}$.

Then by Proposition 4.2.8, for each $r < \bar{r}$ and for each i sufficiently big it holds:

$$U_{cc,\phi_i}(\bar{x}, r) \Subset \omega, \tag{4.88}$$

hence by [86, Theorem 2.7] we infer that

$$|\phi_i(x) - \phi_i(y)| \leq \|\nabla^{\phi_i}\phi_i\|_{L^\infty(\omega)} \mathrm{d}_{cc,\phi_i}(x, y) \quad \forall x, y \in U_{cc,\phi_i}(\bar{x}, r/2).$$

Hence

$$|\phi_i(x) - \phi_i(y)| \leq \|\nabla^{\phi_i}\phi_i\|_{L^\infty(\omega)} \mathrm{d}_{cc,\phi_i}(x, y) \quad \forall x, y \in U_{\phi_i}(\bar{x}, r/(2C_2)).$$

Moreover, since for each $x, y \in U_{cc,\phi_i}(\bar{x}, r)$, $\mathrm{d}_{cc,\phi_i}(x, y) < r < \bar{r}$, and the local equivalence of the distance $\mathrm{d}_{cc,\phi}$ and d_ϕ we conclude that:

$$|\phi_i(x) - \phi_i(y)| \leq C_2||\nabla^{\phi_i}\phi_i||_{L^\infty(\omega)} \mathrm{d}_{\phi_i}(x, y) \quad \forall x, y \in U_{\phi_i}(\bar{x}, r/(2C_2)). \tag{4.89}$$

By Theorem 4.2.7 we know that for all $i \in \mathbb{N}$ $||\nabla^{\phi_i}\phi_i||_{L^\infty(\omega)} \leq M$, hence taking the limit for $i \to \infty$ in (4.89) we get the thesis.
For $n = 1$, we use the fact, recalled in in Definition 4.1.1, that the cone opening is the inverse of the Lipschitz constant, and the estimate of the cone opening provided in Theorem 4.1.5, with $k = -\frac{1}{\sqrt{1+|\nabla^\phi \phi|^2}}$ □

Using Proposition 4.2.9 we immediately get:

Proposition 4.2.10 (Characterization of locally intrinsic Lipschitz functions). *Let $\omega \Subset \mathbb{W}$ be open and bounded, and let $\phi : \omega \to \mathbb{R}$. Then the following are equivalent:*

(i) $\phi \in Lip_{\mathbb{W},loc}(\omega)$*;*

(ii) there exist $\{\phi_k\}_{k\in\mathbb{N}} \subset C^\infty(\omega)$, $C > 0$ *and* $w \in (L^\infty(\omega))^{2n-1}$ *such that*

- (ii$_1$) $\{\phi_k\}_{k\in\mathbb{N}}$ *uniformly converges to* ϕ *on the compact sets of* ω*;*
- (ii$_2$) $|\nabla^{\phi_k}\phi_k(x)| \le C \quad \mathcal{L}^{2n}$*-a.e.* $x \in \omega$, $k \in \mathbb{N}$*;*
- (ii$_3$) $\nabla^{\phi_k}\phi_k(x) \longrightarrow w(x)$ $\mathcal{L}^{2n}-$*a.e* $x \in \omega$.

Moreover if (ii) holds, then $w \equiv \nabla^\phi \phi$ $\mathcal{L}^{2n}-$*a.e in* ω.

4.3 Poincaré inequality

In this section we prove a Poincaré inequality for intrinsic Lipschitz functions. It is well known that the Poincaré inequality play a crucial role in the study the regularity properties for operators of the form $\sum_{i=1}^{m} X_i^2$. In particular, Poincaré's inequality is fundamental in the so called Moser iteration technique used to obtain Harnack inequalities and Hölder continuity for solutions of various quasilinear degenerate equations. For smooth Hörmander's vector fields, Poincaré's inequality has been proved by Jerison in [96] and improved in [101] and in [78] for different exponents. Some weighted version of Poincaré's inequality are proved in [76], in [78] and in [100]. For the non smooth case: in [77] the authors proved the Poincaré inequality in a low regularity situation for vector fields of diagonal form, i.e. $X_i = \lambda_i(x)\partial_i$, $i = 1, \dots, n$, and the λ_i's were required to satisfy some strong condition,(a strong form of a reverse Hölder inequality involving integral curves of vector fields). Lanconelli-Morbidelli in [98] have developed a general approach to Poincaré's inequality for (possibly nonsmooth) vector fields: they first prove an abstract result, which deduces Poincaré's inequality from a property which they call representability of balls by means of controllable almost exponential maps, introduced in [118]. In the recent paper [112], the Poincaré inequality is proved by developping the method of [98] for not smooth and not diagonal vector fields of step two, assuming Lipschitz condition on the vector fields plus some other structural and regular conditions on the commutators. We also quote the paper [106] where Poincaré inequality for families of Lipschitz continuous vector fields satisfying a Hörmander-type condition of step two, in a low regularity conditions for the commutators. In [21] the author prove Poincaré's inequality for a family of $C^{r-1,1}$ vector fields satisfying Hörmander's rank condition of some step r. These proofs of the Poincaré inequality (also in the non-smooth setting) are based on the Nagel-Stein-Weinger's lemma and doubling condition on the ball of the metric (see [121]).
In our case, the family ∇^ϕ has only Hölder regularity when $\phi \in Lip_{\mathbb{W}}(\omega)$, hence all the previous approaches don't work. Aim of this section is to provide a method to obtain a Poincaré inequality starting from the non smooth

family ∇^ϕ.

4.3.1 Local approximation of the vector fields

If $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is an intrinsic Lipschitz function then, by Proposition 4.1.2, the family $\{\nabla^\phi\}$ has coefficients which are only Hölder continuous. To overcome this lack of regularity on the coefficients we use the approach introduced in [121] and refined in [39]. The idea is to associate to the family $\{\nabla^\phi\}$ a new family of Hörmander vector fields with regular coefficients. Precisely, for each $x_0 \in \omega$ we define

$$\begin{aligned}
\nabla_i^{\phi(x_0)} &:= \nabla_i^\phi \quad \text{if } i \neq n \\
\nabla_n^{\phi(x_0)} &:= \partial_{x_n} + 2\phi(x_0)\partial_{x_{2n}} \\
\nabla_{2n}^{\phi(x_0)} &= \partial_{x_{2n}}.
\end{aligned}$$

Since the point $x_0 \in \omega$ is fixed, then the vector fields $\nabla_i^{\phi(x_0)}$ are C^∞ and nilpotent, whose generated Lie algebra is $\mathcal{G} = \mathfrak{h}_{n-1} \times \mathbb{R}$, moreover we will denote by Q the homogeneous dimension of the Lie group associated to $\mathcal{G}$. Then we can repeat for these vector field the general procedure introduced for the definition of the Heisenberg group.

We use the exponential mapping

$$Exp_{\phi(x_0),x} : \mathcal{G} \longrightarrow \mathbb{W}, \quad Exp_{\phi(x_0),x}(y) = \exp\Big(\sum_{i=1}^{2n} y_i \nabla_i^{\phi(x_0)}\Big)(x).$$

where we have identified the element $y \in \mathcal{G}$ with its coordinates on the basis $\{\nabla^{\phi(x_0)}\}$. Its inverse mapping will be denoted by $Log_{\phi(x_0),x}$ and if $x = (x_1, \dots, x_{2n})$ and $y = (y_1, \dots, y_{2n})$ then:

$$\begin{aligned}
Log_{\phi(x_0),x}(y) = (&y_1 - x_1, \dots, y_{2n-1} - x_{2n-1}, y_{2n} - x_{2n} - 2(y_n - x_n)\phi(x_0) + \\
&+ \sigma(x, y))
\end{aligned}$$

with

$$\sigma(x, y) := \sum_{i=1}^{n-1} (y_i x_{n+i} - y_{n+i} x_i).$$

Moreover, we define the function $\mathrm{d}_{\phi(x_0)} : \mathbb{W} \times \mathbb{W} \longrightarrow \mathbb{R}$ by

$$d_{\phi(x_0)}(x,y) := \|Log_{\phi(x_0),x}(y)\|. \tag{4.90}$$

where $\|(x_1,\dots,x_{2n})\| := \max\Big\{|(x_1,\dots,x_{2n-1}|, |x_{2n}|^{\frac{1}{2}}\Big\}$.

Remark 4.3. We point out that for each $x_0 \in \omega$, $\mathrm{d}_{\phi(x_0)}$ is distance on ω. Moreover, by (3.14), we immediately deduce the following equality:

$$\mathrm{d}_\phi(x,y) = \frac{1}{2}\Big(\mathrm{d}_{\phi(x)}(x,y) + \mathrm{d}_{\phi(y)}(x,y)\Big).$$

In order to study the dependence of the vector fields $\{\nabla^{\phi(x_0)}\}$ on the point $x_0 \in \omega$ we prove the following:

Proposition 4.3.1. *Let $\omega \subset \mathbb{W}$ be open and bounded and let us fix $x_0 \in \omega$. If we denote by $\nabla = (\nabla_1,\dots,\nabla_{2n-1})$ the family of vector fields defined on $\mathbb{R}^{2n}$ whose coordinate representation is*

$$\nabla_i := \nabla_i^{\phi(x_0)} \quad \textit{for } i \neq n,$$
$$\nabla_n := \partial_{x_n},$$

then,

1. *For each $\psi \in C^1(\mathbb{R}^{2n})$ and each $i \in \{1,\dots,2n-1\}$ it holds*

$$\nabla_i^{\phi(x_0)}\psi(x) = \nabla_i\tilde{\psi}(Log_{\phi(x_0),x_0}(x)),$$

where $\tilde{\psi} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is defined by:

$$\tilde{\psi}(\tilde{x}) := \psi(Log^{-1}_{\phi(x_0),x_0}(\tilde{x})).$$

2. *The exponential distance $\tilde{d}$ associated to the family $\{\nabla\}$ is independent of x_0 and of class C^∞. Moreover, the following relation is satisfied*

$$d_{\phi(x_0)}(x,y) = \tilde{d}(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)). \tag{4.91}$$

Proof. Let us first prove (1). If $i \neq n$ the thesis is obvious. On the other hand if $i = n$ then

$$\begin{aligned}(\nabla_n \tilde{\psi})(Log_{\phi(x_0),x_0}(x)) &= \lim_{h\to 0} \frac{\tilde{\psi}(\exp(h\nabla_n)(Log_{\phi(x_0),x_0}(x)) - \tilde{\psi}(Log_{\phi(x_0),x_0}(x))}{h} \\ &= \lim_{h\to 0} \frac{\psi(\exp\big(h\nabla_n^{\phi(x_0)}\big)(x)) - \psi(x)}{h} \\ &= (\nabla_n^{\phi(x_0)}\psi)(x).\end{aligned}$$

Since $\{\nabla\}$ does not depend on x_0 it is clear that $\tilde{\mathrm{d}}$ is independent of x_0. Let us explicitely observe that

$$\mathrm{d}_{\phi(x_0)}(x,y) = \|(z_1,\ldots,z_{2n})\|_\infty \text{ if } y = \exp(\sum_{i=1}^{2n} z_i \nabla_i^{\phi(x_0)})(x) \tag{4.92}$$

and

$$\tilde{\mathrm{d}}(\tilde{x},\tilde{y}) = \|(\tilde{z}_1,\ldots,\tilde{z}_{2n})\|_\infty \text{ if } \tilde{y} = \exp(\sum_{i=1}^{2n} \tilde{z}_i \nabla_i)(\tilde{x}) \tag{4.93}$$

where $\tilde{y} := Log_{\phi(x_0),x_0}(y)$ and $\tilde{x} := Log_{\phi(x_0),x_0}(x)$. Since $\nabla_i = \nabla_i^{\phi(x_0)}$ if $i \neq n$ we have $\tilde{z}_i = z_i$ if $i \neq 2n$. The fact that $\tilde{z}_{2n} = z_{2n}$ follows from a direct computation. □

4.3.2 Sub-Laplacian and fundamental solution

In this subsection we use some ideas already introduced in Chapter 2, in order to study the sub-Laplacian associated to the family $\{\nabla^{\phi(x_0)}\}$.

If

$$\mathcal{L}_{\phi(x_0)} := \sum_{i=1}^{2n-1} \nabla_i^{\phi(x_0)} \nabla_i^{\phi(x_0)} \tag{4.94}$$

is the sub-Laplacian associated to $\{\nabla^{\phi(x_0)}\}$ then, by Theorem 2.1.8, $\mathcal{L}_{\phi(x_0)}$ admits a fundamental solution (see Definition 2.1.3) which we will denote by $\Gamma_{\phi(x_0)}$.

Lemma 4.3.2. *Under the change of variable $Log_{\phi(x_0),x_0}$, the operator $\mathcal{L}_{\phi(x_0)}$ is transformed into the operator*

$$\mathcal{L} := \sum_{i=1}^{2n-1} \nabla_i \nabla_i. \tag{4.95}$$

That is, for each $\psi \in C^1(\mathbb{R}^{2n})$:

$$(\mathcal{L}_{\phi(x_0)}\psi)(x) = (\mathcal{L}\tilde{\psi})(Log_{\phi(x_0),x_0}(x)) \quad \forall\ x \in \mathbb{R}^{2n},$$

where $\tilde{\psi}$ is defined as in Proposition 4.3.1.

Proof. Let $\psi \in C^1(\mathbb{R}^{2n})$, then if $i \neq n$ we have

$$(\nabla_i \nabla_i \tilde{\psi})(Log^{-1}_{\phi(x_0),x_0}(x)).$$

On the other hand, if $i = n$

$$(\nabla_n \nabla_n \tilde{\psi})(Log^{-1}_{\phi(x_0),x_0}(x)) = (\partial_{x_n}\partial_{x_n}\tilde{\psi})(Log^{-1}_{\phi(x_0),x_0}(x)) \tag{4.96}$$

and by a direct calculation:

$$\begin{aligned}(\partial_n \partial_n \tilde{\psi})(Log^{-1}_{\phi(x_0),x_0}(x)) &= \partial^2_{x_2}\psi(x) - 4\phi(x_0)\partial_{x_2 x_4}\psi(x) + 4\phi(x_0)^2\partial^2_{x_4}\psi(x)\\ &= (\nabla_n^{\phi(x_0)}\nabla_n^{\phi(x_0)}\psi)(x)\end{aligned}$$

□

Since the vector fields ∇_i are of class C^∞ and satisfy the Hörmander condition, then the second order differential operator $\mathcal{L}$ is a sub-laplacian operator. Then, by Theorem 2.1.8, it has a fundamental solution Γ of class C^∞ far from the pole $x = y$, which is homogeneous of degree $2 - Q$. This means, by Theorem 2.1.10, that there exist positive constants C_1, C_2 such that for every x and y in $\mathbb{R}^{2n}$, $x \neq y$

$$\begin{aligned}\frac{C_1}{\tilde{d}(x,y)^{Q-2}} \leq \Gamma(x,y) &\leq \frac{C_2}{\tilde{d}(x,y)^{Q-2}};\\ |\nabla_i \Gamma(x,y)| &\leq \frac{C_2}{\tilde{d}(x,y)^{Q-1}};\\ |\nabla_j \nabla_i \Gamma(x,y)| &\leq \frac{C_2}{\tilde{d}(x,y)^{Q}},\end{aligned} \tag{4.97}$$

for every $i, j = 1, \ldots, 2n-1$. We can correlate $\Gamma_{\phi(x_0)}$ and Γ in the following way

Lemma 4.3.3 ([43]). *With the same notations as above, for every $x, y \in \mathbb{R}^{2n}$ with $x \neq y$ and every $i = 1, \ldots, 2n-1$*

$$\nabla_i^{\phi(x_0)} \Gamma_{\phi(x_0)}(x, y) = \nabla_i \Gamma(Log_{\phi(x_0), x_0}(x), Log_{\phi(x_0), x_0}(y)) \tag{4.98}$$

and

$$\Gamma_{\phi(x_0)}(x, y) = \Gamma(Log_{\phi(x_0), x_0}(x), Log_{\phi(x_0), x_0}(y)). \tag{4.99}$$

It follows that inequalities (4.97) are satisfied also for $\Gamma_{\phi(x_0)}(x, y)$ and $d_{\phi(x_0)}(x, y)$ with the same constants. In particular it is clear that these constants are independent of x_0.

Lemma 4.3.4 ([42]). *For every $x, x_0 \in \omega$ and $r > 0$ we define*

$$\Omega_{\phi(x_0)}(x, r) := \left\{ y \in \mathbb{R}^{2n} \mid \Gamma_{\phi(x_0)}(x, y) > r^{2-Q} \right\}, \tag{4.100}$$

then $\Omega_{\phi(x_0)}(x, r)$ is regular and defines spheres locally equivalent to the spheres of the distance $d_{\phi(x_0)}$. Moreover,

$$\Omega_{\phi(x_0)}(x, r) = \left\{ y \in \mathbb{R}^{2n} \mid \Gamma(Log_{\phi(x_0), x_0}(x), Log_{\phi(x_0), x_0}(y)) > r^{2-Q} \right\} \tag{4.101}$$

hence

$$\begin{aligned} \Omega_{\phi(x_0)}(x_0, r) &= \left\{ y \in \mathbb{R}^{2n} \mid \Gamma(0, Log_{\phi(x_0), x_0}(y)) > r^{2-Q} \right\} \\ &= Exp_{\phi(x_0), x_0}\Big(\tilde{\Omega}(0, r) \Big), \end{aligned} \tag{4.102}$$

where

$$\tilde{\Omega}(0, r) := \left\{ \tilde{y} \in \mathbb{R}^{2n} \mid \Gamma(0, \tilde{y}) > r^{2-Q} \right\}. \tag{4.103}$$

From now on we will denote by

$$N(\tilde{y}) := \Gamma^{\frac{-1}{Q-2}}(0, \tilde{y}), \;\; \tilde{y} \in \mathbb{R}^{2n}. \tag{4.104}$$

4.3.3 A representation formula

In this subsection we firstly restate Theorem 2.1.15 using family $\{\nabla^{\phi(x_0)}\}$ and then we improve it in order to obtain a reppresentation formula containing derivatives with respect to the non linear family $\{\nabla^{\phi}\}$.

The following Proposition follows from Theorem 2.1.15 via Coarea Formula:

Proposition 4.3.5 ([42]). *Let $\omega \subset \mathbb{R}^{2n}$ be a bounded open set and let $\phi : \omega \longrightarrow \mathbb{R}$ be of class $C^\infty(\omega)$. Then for every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ we have*

$$\phi(x_0) = \frac{Q}{(Q-2)R^Q} \int_{\Omega_{\phi(x_0)}(x_0,R)} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(Q-1)/(Q-2)}(x_0,y)} \phi(y)\, \mathrm{d}\mathcal{L}^{2n}(y) \quad (4.105)$$

$$+\frac{Q}{R^Q}\int_0^R r^{Q-1} \int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi(x_0)}\phi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.$$

Remark 4.4. We explicitly note that, if we choose $\phi \equiv 1$, then we get from the previous formula

$$1 = \frac{Q}{(Q-2)R^Q} \int_{\Omega_{\phi(x_0)}(x_0,R)} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(Q-1)/(Q-2)}(x_0,y)} \mathrm{d}\mathcal{L}^{2n}(y). \quad (4.106)$$

This remark allows to say that Proposition 4.3.5 represents a function ϕ as the sum of its mean on a suitable level set ball, and its gradient $\nabla^{\phi(x_0)}$. Hence, it is natural to give the following definition

Definition 4.3.1. *Let $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be $L^1_{loc}(\omega)$. For every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ we call mean of ϕ on $\Omega_{\phi(x_0)}(x_0, R)$*

$$\tilde{\phi}_R(x_0) = \frac{Q}{(Q-2)R^Q} \int_{\Omega_{\phi(x_0)}(x_0,R)} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(Q-1)/(Q-2)}(x_0,y)} \phi(y)\, \mathrm{d}\mathcal{L}^{2n}(y).$$

In the sequel we will need an other mean of ϕ on the same ball $\Omega_{\phi(x_0)}(x_0, R)$

$$\phi_R(x_0) = \frac{2}{R}\int_{R/2}^R \tilde{\phi}_r(x_0) dr.$$

We will slightly modify the mean formula in Proposition 4.3.5, which contains derivatives in the direction of the vector fields $\nabla^{\phi(x_0)}$ of ϕ, in order to obtain a mean representation formula which contains derivatives with respect to the vector fields ∇^{ϕ} of ϕ.

Proposition 4.3.6. *Let $\omega \subset \mathbb{R}^{2n}$ be open and bounded and $\phi : \omega \longrightarrow \mathbb{R}$ of class $C^\infty(\omega)$. Then for every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ we have*

$$\begin{aligned}
&\phi(x_0) - \phi_R(x_0) = \\
&= \frac{1}{R}\int_0^R f_1(r)\int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi}\phi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&+ \frac{1}{R}\int_0^R f_1(r)\int_{\Omega_{\phi(x_0)}(x_0,r)} (\phi(y)-\phi(x_0))\left\langle F_2\Big(Log_{\phi(x_0),x_0}(y)\Big), \nabla^{\phi}\phi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&+ \frac{1}{R}\int_{R/2}^R \frac{1}{r^Q}\int_{\Omega_{\phi(x_0)}(x_0,r)} (\phi(y)-\phi(x_0))\left\langle F_3\Big(Log_{\phi(x_0),x_0}(y)\Big), \nabla^{\phi}\phi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r,
\end{aligned}$$

where f_1 is a smooth and bounded function defined on $\mathbb{R}$ and F_2, $F_3 : \mathcal{G} \longrightarrow \mathbb{R}^{2n}$ are defined by;

$$F_2(Log_{\phi(x_0),x_0}(y)) = (\nabla^{\phi(x_0)}_{n+1}\nabla^{\phi(x_0)}_{n}\Gamma_{\phi(x_0)}(x_0,y), 0, -\nabla^{\phi(x_0)}_{1}\nabla^{\phi(x_0)}_{n}\Gamma_{\phi(x_0)}(x_0,y), 0),$$
$$F_3(Log_{\phi(x_0),x_0}(y)) = S(y)\Big(-\nabla^{\phi(x_0)}_{n+1}\Gamma_{\phi(x_0)}(x_0,y), 0, \nabla^{\phi(x_0)}_{1}\Gamma_{\phi(x_0)}(x_0,y), 0\Big).$$

Here

$$S(y) := \frac{Q}{Q-2}\frac{\nabla^{\phi(x_0)}_{n}\Gamma_{\phi(x_0)}(x_0,y)}{\Gamma^{2(Q-1)/(Q-2)}_{\phi(x_0)}(x_0,y)}$$

and the non zero components of F_2 and F_3 are the first and the $(n+1)$th.

Proof. Throughout the proof we will denote by Ω_r the set $\Omega_{\phi(x_0)}(x_0, r)$ and by $\partial\Omega_r$ the boundary of Ω_r where $x_0 \in \omega$ and $0 < r < R$ are as in the statement. By Proposition 4.3.5 we have

$$\begin{aligned}
\phi(x_0) &= \frac{Q}{(Q-2)R^Q}\int_\Omega \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma^{2(Q-1)/(Q-2)}_{\phi(x_0)}(x_0,y)}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y) + \\
&+ \frac{Q}{R^Q}\int_0^R r^{Q-1}\int_\Omega \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi(x_0)}\phi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r
\end{aligned}$$

hence

$$\begin{aligned}\phi(x_0) =&\frac{Q}{(Q-2)R^Q}\int_{\Omega_R}\frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(Q-1)/(Q-2)}(x_0,y)}\,\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)+\\ &+\frac{Q}{R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\left\langle\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y),\nabla^{\phi}\phi(y)\right\rangle\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r+\\ &+\frac{Q}{R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\partial_{2n}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.\end{aligned}$$

Let us compute the last term of the previous equality keeping in mind that

$$\begin{aligned}&\partial_{2n}=\frac{1}{2}\Big(\nabla_1^{\phi(x_0)}\nabla_{n+1}^{\phi}-\nabla_{n+1}^{\phi(x_0)}\nabla_1^{\phi}\Big)\\ &\Omega_{\phi(x_0)}(x_0,r)=\Big\{\Gamma_{\phi(x_0)}^{-1/(Q-2)}(x_0,y)<r\Big\},\end{aligned}$$

hence

$$\frac{Q}{R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\partial_{2n}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r=A-B$$

where

$$A:=\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\nabla_1^{\phi(x_0)}\nabla_{n+1}^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r$$

$$B:=\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\nabla_{n+1}^{\phi(x_0)}\nabla_1^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r$$

we estimate separately the two terms,

$$\begin{aligned}A=&\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\partial\Omega_r}H_1(y)\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\nabla_{n+1}^{\phi}\phi(y)\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r-\\ &-\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_1^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\Big)\nabla_{n+1}^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r=\end{aligned}$$

where

$$H_1(y):=\frac{\nabla_1^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{|\nabla_E\Gamma_{\phi(x_0)}(x_0,y)|},$$

whereas

$$\begin{aligned}B=&\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\partial\Omega_r}\bar{H}_1(y)\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\nabla_1^{\phi}\phi(y)\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r-\\ &-\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_{n+1}^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\Big)\nabla_1^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r=\end{aligned}$$

where

$$\bar{H}_1(y) := \frac{\nabla_{n+1}^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{|\nabla_E\Gamma_{\phi(x_0)}(x_0,y)|}.$$

Now using the fact that

$$\partial\Omega_r = \left\{\Gamma_{\phi(x_0)}^{-1/(Q-2)}(x_0,y) = r\right\},$$

we obtain $r^{Q-1} = \Gamma_{\phi(x_0)}^{-(Q-1)/(Q-2)}(x_0,y)$ on this set. It follows,

$$A = \frac{Q}{2R^Q}\int_0^R\int_{\partial\Omega_r} H_1(y)H_2(y)(\phi(x_0)-\phi(y))\nabla_{n+1}^{\phi}\phi(y)\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r -$$
$$-\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_1^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\Big)\nabla_{n+1}^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r,$$

where

$$H_2(y) := \frac{\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}}$$

and

$$B = \frac{Q}{2R^Q}\int_0^R\int_{\partial\Omega_r} \bar{H}_1(y)H_2(y)(\phi(x_0)-\phi(y))\nabla_1^{\phi}\phi(y)\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r -$$
$$-\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_1^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\Big)\nabla_1^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.$$

By the Coarea formula, see [80] or Chapter 3, we obtain that:

$$A = \frac{Q}{2R^Q(Q-2)}\int_{\Omega_R}\frac{H_2(y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}}(\phi(x_0)-\phi(y))\nabla_{n+1}^{\phi}\phi(y)\nabla_1^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r$$

$$-\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_1^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\Big)\nabla_{n+1}^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r,$$

and

$$B = \frac{Q}{2R^Q(Q-2)}\int_{\Omega_R}\frac{H_2(y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}}(\phi(x_0)-\phi(y))\nabla_1^{\phi}\phi(y)\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r$$

$$-\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_{n+1}^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))\Big)\nabla_1^{\phi}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.$$

Hence, by the reppresentation formula we conclude that:

$$\phi(x_0) = \tilde{\phi}_R(x_0) + \frac{Q}{R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi}\phi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r$$

$$+\frac{Q}{2R^Q(Q-2)}\int_{\Omega_R}\frac{H_2(y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}}(\phi(x_0){-}\phi(y))\nabla^{\phi}_{n+1}\phi(y)\nabla_1^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r$$

$$-\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_1^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0){-}\phi(y))\Big)\nabla^{\phi}_{n+1}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r+$$

$$-\frac{Q}{2R^Q(Q-2)}\int_{\Omega_R}\frac{H_2(y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}}(\phi(x_0){-}\phi(y))\nabla^{\phi}_{1}\phi(y)\nabla_{n+1}^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r-$$

$$+\frac{Q}{2R^Q}\int_0^R r^{Q-1}\int_{\Omega_r}\nabla_{n+1}^{\phi(x_0)}\Big(\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0){-}\phi(y))\Big)\nabla^{\phi}_{1}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.$$

Integrating from $R/2$ to R and denoting by

$$\begin{aligned} D_{1,n} &:= \nabla_1^{\phi(x_0)}\nabla_n^{\phi(x_0)} \\ D_{n+1,n} &:= \nabla_{n+1}^{\phi(x_0)}\nabla_n^{\phi(x_0)}. \end{aligned}$$

we get

$$\begin{aligned}
&\phi(x_0) - \phi_R(x_0) \\
&= \frac{2}{R}\int_{R/2}^R \frac{Q}{\rho^Q}\int_0^\rho r^{Q-1}\int_{\Omega_r}\left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi}\phi(y)\right\rangle\, \mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r\,\mathrm{d}\rho \\
&+ \frac{2}{R}\int_{R/2}^R \frac{Q}{2r^Q(Q-2)}\int_{\Omega_r}\frac{H_2(y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}(x_0,y)}(\phi(y)-\phi(x_0)) \\
&\qquad\qquad\qquad \nabla^{\phi}_{n+1}\phi(y)\nabla_1^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&- \frac{1}{R}\int_{R/2}^R \frac{Q}{\rho^Q}\int_0^\rho r^{Q-1}\int_{\Omega_r} D_{1,n}\Gamma_{\phi(x_0)}(x_0,y)(\phi(y)-\phi(x_0))\,\nabla^{\phi}_{n+1}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&- \frac{1}{R}\int_{R/2}^R \frac{Q}{r^Q(Q-2)}\int_{\Omega_r}\frac{\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{\Gamma_{\phi}^{2(Q-1)/(Q-2)}(x_0)(x_0,y)}(\phi(y)-\phi(x_0)) \\
&\qquad\qquad\qquad \nabla^{\phi}_{1}\phi(y)\nabla_{n+1}^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&+ \frac{1}{R}\int_{R/2}^R \frac{Q}{\rho^Q}\int_0^\rho r^{Q-1}\int_{\Omega_r} D_{n+1,n}\Gamma_{\phi(x_0)}(x_0,y)(\phi(y)-\phi(x_0))\,\nabla^{\phi}_{1}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r.
\end{aligned}$$

Exchanging the order of integration in the first, third, and fifth integral, and we setting

$$f_1(r) = \frac{\left(\frac{r}{R}\right)^{Q-1} - 1}{-Q+1} \text{ if } r \in [R/2, R]$$

and

$$f_1(r) = \left(\frac{r}{R}\right)^{Q-1} \frac{1 - \left(\frac{1}{2}\right)^{-Q+1}}{-Q+1} \text{ if } r \in [0, R]$$

we obtain that the previuos equality is equal to:

$$\begin{aligned}
&\frac{2}{R}\int_0^R f_1(r)\int_{\Omega_r} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi}\phi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&+ \frac{2}{R}\int_{R/2}^R \frac{Q}{2r^Q(Q-2)}\int_{\Omega_r} \frac{H_2(y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}(x_0,y)}(\phi(y)-\phi(x_0)) \\
&\qquad\qquad \nabla^{\phi}_{n+1}\phi(y)\nabla_1^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&- \frac{2}{R}\int_0^R f_1(r)\int_{\Omega_r} D_{1,n}\Gamma_{\phi(x_0)}(x_0,y)(\phi(y)-\phi(x_0))\, \nabla^{\phi}_{n+1}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&- \frac{1}{R}\int_{R/2}^R \frac{Q}{r^Q(Q-2)}\int_{\Omega_r} \frac{H_2(y)}{\Gamma_{\phi(x_0)}^{(Q-1)/(Q-2)}(x_0,y)}(\phi(y)-\phi(x_0)) \\
&\qquad\qquad \nabla^{\phi}_{1}\phi(y)\nabla_{n+1}^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r \\
&+ \frac{2}{R}\int_0^R f_1(r)\int_{\Omega_r} D_{n+1,n}\Gamma_{\phi(x_0)}(x_0,y)(\phi(y)-\phi(x_0))\, \nabla^{\phi}_{1}\phi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\,\mathrm{d}r.
\end{aligned}$$

Now calling F_2 the vector whose components 1 and $n+1$ are in the kernel of the fifth and third integral respectively:

$$F_2(Log_{\phi(x_0),x_0}(y)) := (\nabla^{\phi(x_0)}_{n+1}\nabla^{\phi(x_0)}_{n}\Gamma_{\phi(x_0)}(x_0,y), 0, -\nabla^{\phi(x_0)}_{1}\nabla^{\phi(x_0)}_{n}\Gamma_{\phi(x_0)}(x_0,y), 0),$$

and F_3 the vector whose components 1 and $n+1$ are the kernels of the fourth and second integral respectively:

$$F_3(Log_{\phi(x_0),x_0}(y)) := S(y)\Big(-\nabla^{\phi(x_0)}_{n+1}\Gamma_{\phi(x_0)}(x_0,y), 0, \nabla^{\phi(x_0)}_{1}\Gamma_{\phi(x_0)}(x_0,y), 0\Big).$$

we get the thesis. $\square$

In order to compare two different mean values $\phi_r(x)$, we will first express them as integral on the same sphere:

Lemma 4.3.7. *We have*

$$\tilde{\phi}_r(x_0) = \frac{Q}{(Q-2)}\frac{1}{r^Q}\int_{\tilde{\Omega}(0,r)} \frac{|\nabla\Gamma(0,\tilde{y})|^2}{\Gamma(0,\tilde{y})^{2(Q-1)/(Q-2)}}\phi(Exp_{\phi(x_0),x_0}(\tilde{y}))\ \mathrm{d}\mathcal{L}^{2n}(\tilde{y}),$$

where $\tilde{\Omega}(0,r)$ is defined in (4.103).

Proof. By (4.102) we have that

$$\Omega_{\phi(x_0)}(x_0,r) = Exp_{\phi(x_0),x_0}(\tilde{\Omega}(0,r)).$$

So that, by (4.99)

$$\tilde{\phi}_r(x_0) = \frac{Q}{(Q-2)}\frac{1}{r^Q}\int_{\Omega_{\phi(x_0)}(x_0,r)} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}(x_0,y)^{2(Q-1)/(Q-2)}}\phi(y)\ \mathrm{d}\mathcal{L}^{2n}(y) =$$

$$= \frac{Q}{(Q-2)}\frac{1}{r^Q}\int_{\tilde{\Omega}(0,r)} \frac{|\nabla\Gamma(0,\tilde{y})|^2}{\Gamma(0,\tilde{y})^{2(Q-1)/(Q-2)}}\phi(Exp_{\phi(x_0),x_0}(\tilde{y}))\ \mathrm{d}\mathcal{L}^{2n}(\tilde{y})$$

□

Lemma 4.3.8. *Let $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a C^∞ function.*

For each $x_0, x, \in \omega$, $\tilde{y} \in Log_{\phi(x_0),x_0}(\omega) \cap Log_{\phi(x),x}(\omega)$ we define

$$e = e(x,\tilde{y};x_0) := Log_{\phi(x_0),Exp_{\phi(x_0),x_0}(\tilde{y})}\Big(Exp_{\phi(x),x}(\tilde{y})\Big),$$

and

$$\gamma_{\tilde{y}}(t) = \exp\Big(te\nabla^{\phi(x_0)}\Big)(Exp_{\phi(x_0),x_0}(\tilde{y})).$$

Then,

$$\phi(Exp_{\phi(x),x}(\tilde{y})) - \phi(Exp_{\phi(x_0),x_0}(\tilde{y})) =$$
$$= \int_0^1 \big\langle Log_{\phi(x_0),x_0}(x), \nabla^{\phi(x_0)}\phi(\gamma(t))\big\rangle\, dt + 2(\phi(x)-\phi(x_0))\tilde{y}_n\int_0^1 \partial_{2n}\phi(\gamma(t))\mathrm{d}t$$
$$+ 2\sum_{i=1}^{n-1}\Big((x-x_0)_i\tilde{y}_{n+i} - (x-x_0)_{i+n}\tilde{y}_i\Big)\int_0^1 \partial_{2n}\phi(\gamma(t))\mathrm{d}t.$$

Proof. It is a direct calculation. Indeed the value of e can be explicitly computed and it is:

$$e = Log_{\phi(x_0),x_0}(x)+$$
$$(0, \cdots, 2(\phi(x)-\phi(x_0))\tilde{y}_n + 2\sum_{i=1}^{n-1}((x-x_0)_i\tilde{y}_{n+i} - (x-x_0)_{i+n}\tilde{y}_i)).$$

Let us verify this computation:

$$Exp(\tilde{y}\nabla^{\phi(x)})(x) = \Big(\tilde{y}_1 + x_1, \cdots, \tilde{y}_{2n-1} + x_{2n-1},$$
$$, \sum_{i=1}^{n-1}(x_i\tilde{y}_{n+i} - \tilde{y}_i x_{n+i}) + 2\tilde{y}_n\phi(x) + \tilde{y}_{2n} + x_{2n}\Big).$$

By definition of e

$$e_i = (Exp_{\phi(x),x}(y))_i - (Exp_{\phi(x_0),x_0}(\tilde{y}))_i = (y + x - y - x_0)_i = (x - x_0)_i,$$

for $i = 1, \dots, 2n-1$, and

$$e_{2n} = (Exp_{\phi(x),x}(\tilde{y}))_{2n} - (Exp_{\phi(x_0),x_0}(\tilde{y}))_{2n} -$$
$$-\sum_i (Exp_{\phi(x),x}(\tilde{y}))_{i+n}(Exp_{\phi(x_0),x_0}(\tilde{y}))_i - (Exp_{\phi(x_0),x_0}(\tilde{y}))_{i+n}(Exp_{\phi(x),x}(\tilde{y}))_i -$$
$$-2\Big((Exp_{\phi(x),x}(\tilde{y}))_n - (Exp_{\phi(x_0),x_0}(\tilde{y}))_n\Big)\phi(x_0),$$

that is

$$\begin{aligned} e_{2n} =& \sum_i (x_i\tilde{y}_{n+i} - \tilde{y}_i x_{n+i}) + 2\tilde{y}_n\phi(x) + \tilde{y}_{2n} + x_{2n} \\ &- \Big(\sum_i (x_{0,i}\tilde{y}_{n+i} - \tilde{y}_i x_{0,n+i}) + 2\tilde{y}_n\phi(x_0) + \tilde{y}_{2n} + x_{0,2n}\Big) \\ &- \sum_i \Big((x+\tilde{y})_{i+n}(x_0+\tilde{y})_i - (x+\tilde{y})_i(x_0+\tilde{y})_{i+n}\Big) - 2(x-x_0)\phi(x_0). \end{aligned}$$

Therefore

$$e_{2n} = x_{2n} - x_{0,2n} - \sum_i \Big(x_{i+n}x_{0,i} - x_i x_{0,i+n}\Big) -$$
$$2(x-x_0)\phi(x_0) + 2\tilde{y}_n(\phi(x) - \phi(x_0)) +$$
$$+ 2\sum_{i=1}^{n-1}((x-x_0)_i\tilde{y}_{n+i} - (x-x_0)_{i+n}\tilde{y}_i).$$

Since $\phi \in C^\infty(\omega)$ and γ is a horizontal curve, we obtain

$$\begin{aligned}\phi(Exp_{\phi(x),x}(\tilde y)) - \phi(Exp_{\phi(x_0),x_0}(\tilde y)) &= \int_0^1 (\phi\circ\gamma)'(t)\mathrm{d}t \\ &= \sum_{i=1}^{2n}\int_0^1 e_i \nabla_i^{\phi(x_0)}\phi(\gamma(t))\mathrm{d}t,\end{aligned}$$

so that the thesis immediately follows using the expressions of e_i. □

Let us now prove the following proposition

Proposition 4.3.9. *Let us denote $\gamma_{\tilde y}(t)$ the integral curve introduced in Lemma 4.3.8. Then the function $(t, \gamma_{\tilde y}(t))$ is invertible, and we will denote $(t, \tilde F(z,t))$ its inverse function. We have*

$$\begin{aligned}&\phi_R(x) - \phi_R(x_0) = \\ &= \frac{1}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))\le r} < G_1(z), \nabla^\phi\phi(z) > dzdtdr \\ &+ \sum_{i,j} A_{i,j}(x,x_0)\frac{1}{R}\int_0^1\int_{R/2\le N(\tilde F(z,t))\le R} < G_{2i,j}(z), \nabla^\phi\phi(z) > dzdt \\ &+ \sum_{i,j} B_{i,j}(x,x_0)\frac{1}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))\le r} < G_{3i,j}(z), \nabla^\phi\phi(z) > dzdtdr\end{aligned}$$

for suitable kernels G_1, $G_{2i,j}$, $G_{3i,j}$ (defined in (4.110), (4.111) below), functions $A_{i,j}$, $B_{i,j}$. N is defined in (4.104). The kernel G_1 is homogeneous of order 1, $G_{2i,j}$ is homogeneous of order $1-Q$, $G_{3i,j}$ is homogeneous of order 0, according to (4.112)-(4.114) below. The functions $A_{i,j}$, $B_{i,j}$ satisfies

$$|A_{i,j}(x,x_0)| + |B_{i,j}(x,x_0)| \le d_\phi(x,x_0) + |\phi(x) - \phi(x_0)|.$$

Proof. Calling

$$h(0,\tilde y) = \frac{Q}{(Q-2)}\frac{|\nabla\Gamma(0,\tilde y)|^2}{\Gamma(0,\tilde y)^{2(Q-1)/(Q-2)}},$$

by Lemma 4.3.7 we obtain

$$\begin{aligned}&\tilde\phi_r(x) - \tilde\phi_r(x_0) = \\ &\frac{1}{r^Q}\int_{\tilde\Omega(0,r)} h(0,\tilde y)\Big(\phi(Exp_{\phi(x),x}(\tilde y)) - \phi(Exp_{\phi(x_0),x_0}(\tilde y))\Big)\mathrm{d}\mathcal{L}^{2n}(\tilde y).\end{aligned}$$

Using Lemma 4.3.8

$$\begin{aligned}
&\tilde{\phi}_r(x) - \tilde{\phi}_r(x_0) = \\
&\frac{1}{r^Q}\int_{\tilde{\Omega}(0,r)} h(0,\tilde{y})\Big(\int_0^1 < Log_{\phi(x_0),x_0}(x), \nabla^{\phi(x_0)}\phi(\gamma_{\tilde{y}}(t)) > \mathrm{d}t+ \\
&+ 2(\phi(x) - \phi(x_0))\tilde{y}_n \int_0^1 \partial_{2n}\phi(\gamma_{\tilde{y}}(t))\mathrm{d}t+ \\
&+ 2\sum_{i=1}^{n-1}((x_i - x_{0,i})\tilde{y}_{n+i}\int_0^1 \partial_{2n}\phi(\gamma_{\tilde{y}}(t))\mathrm{d}t- \\
&- 2\sum_{i=1}^{n-1}(x_{i+n} - x_{0,n+i})\tilde{y}_i\int_0^1 \partial_{2n}\phi(\gamma_{\tilde{y}}(t))\mathrm{d}t\Big)\mathrm{d}\mathcal{L}^{2n}(\tilde{y}).
\end{aligned}$$

Let us make the change of variables $(t,z) = (t,\gamma_{\tilde{y}}(t))$. Its inverse will be denoted

$$(t,\tilde{y}) = (t,\tilde{F}(z,t)).$$

$$\tilde{F}_i(z,t) = t(x_{0,i} - x_i) + z_i - x_{0,i} \qquad i = 1,\dots,2n-1 \tag{4.107}$$

In particular, if we consider $\tilde{F}$ as a function of z, its components $\tilde{F}_1$ to $\tilde{F}_{2n-1}$ are homogeneous of order 1 with respect to the vector fields $\nabla^{\phi(x_0)}$. The component $2n$ is homogeneous of order 2. For every fixed t the variable z will belongs to the set

$$\begin{aligned}
D_{t,r} &= \{z \in \mathbb{R}^{2n} : \tilde{F}(t,z) \in \tilde{\Omega}(0,r)\} \\
&= \{z \in \mathbb{R}^{2n} : N(\tilde{F}(t,z)) < r\},
\end{aligned} \tag{4.108}$$

where N is defined in (4.104). Hence

$$\tilde{\phi}_r(x) - \tilde{\phi}_r(x_0) = \tag{4.109}$$

$$\begin{aligned}
&= \frac{1}{r^Q}\int_0^1\int_{D_{t,r}} h(0,\tilde{F}(z,t)) < Log_{\phi(x_0),x_0}(x), \nabla^{\phi(x_0)}\phi(z) > d\mathcal{L}^{2n}(z)dt \\
&+2(\phi(x) - \phi(x_0))\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \tilde{F}_n(z,t)h(0,\tilde{F}(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt \\
&+2\sum_{i=1}^{n-1}((x_i - x_{0,i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \tilde{F}_{n+i}(z,t)h(0,\tilde{F}(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt
\end{aligned}$$

$$-2\sum_{i=1}^{n-1}(x_{i+n}-x_{0,n+i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\tilde F_i(z,t)h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt$$

$$=\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\sum_{i=1}^{2n-1}h(0,\tilde F(z,t))(x-x_0)_i\nabla_i^\phi\phi(z)d\mathcal{L}^{2n}(z)dt+$$

$$-(\phi(x)-\phi(x_0))\frac{(x-x_0)_n}{r^Q}\int_0^1\int_{D_{t,r}}h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt+$$

$$+2(\phi(x)-\phi(x_0))\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\tilde F_n(z,t)h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt+$$

$$+2\sum_{i=1}^{n-1}((x_i-x_{0,i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\tilde F_{n+i}(z,t)h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt$$

$$-2\sum_{i=1}^{n-1}(x_{i+n}-x_{0,n+i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\tilde F_i(z,t)h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt.$$

Let us consider the second term:

$$-(\phi(x)-\phi(x_0))\frac{(x-x_0)_n}{r^Q}\int_0^1\int_{D_{t,r}}h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt$$

(since $\partial_{2n}=[\nabla_1^\phi,\nabla_{n+1}^\phi]$)

$$=-(\phi(x)-\phi(x_0))\frac{(x-x_0)_n}{r^Q}\int_0^1\int_{D_{t,r}}h(0,\tilde F(z,t))[\nabla_1^\phi,\nabla_{n+1}^\phi]\phi(z)d\mathcal{L}^{2n}(z)dt$$

$\Big($integrating by parts and denoting by $Z_i(z,t):=\frac{\nabla_i^\phi N(\tilde F(z,t))}{|\nabla_E N(\tilde F(z,t))|}\Big)$

$$=-(\phi(x)-\phi(x_0))\frac{(x-x_0)_n}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r}h(0,\tilde F(z,t))\nabla_{n+1}^\phi\phi(z)Z_1(z,t)d\mathcal{L}^{2n}(z)dt$$
$$+(\phi(x)-\phi(x_0))\frac{(x-x_0)_n}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r}h(0,\tilde F(z,t))\nabla_1^\phi\phi(z)Z_{n+1}(z,t)d\mathcal{L}^{2n}(z)dt$$
$$+(\phi(x)-\phi(x_0))\frac{(x-x_0)_n}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_1^\phi h(0,\tilde F(z,t))\nabla_{n+1}^\phi\phi(z)d\mathcal{L}^{2n}(z)dt$$
$$-(\phi(x)-\phi(x_0))\frac{(x-x_0)_n}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_{n+1}^\phi h(0,\tilde F(z,t))\nabla_1^\phi\phi(z)d\mathcal{L}^{2n}(z)dt.$$

Let us consider the third term and integrating by parts as before

$$(\phi(x)-\phi(x_0))\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} h(0,\tilde F(z,t))\tilde F_n(z,t)\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt =$$
$$= (\phi(x)-\phi(x_0))\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r} h(0,\tilde F(z,t))\tilde F_n(z,t)\nabla^{\phi}_{n+1}\phi(z)Z_1(z,t)d\mathcal{L}^{2n}(z)dt$$
$$- (\phi(x)-\phi(x_0))\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r} h(0,\tilde F(z,t))\tilde F_n(z,t)\nabla^{\phi}_{1}\phi(z)Z_{n+1}(z,t)d\mathcal{L}^{2n}(z)dt$$
$$- (\phi(x)-\phi(x_0))\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \nabla^{\phi}_{1}\Big(h(0,\tilde F(z,t))\tilde F_n(z,t)\Big)\nabla^{\phi}_{n+1}\phi(z)d\mathcal{L}^{2n}(z)dt$$
$$+ (\phi(x)-\phi(x_0))\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \nabla^{\phi}_{n+1}\Big(h(0,\tilde F(z,t))\tilde F_n(z,t)\Big)\nabla^{\phi}_{1}\phi(z)d\mathcal{L}^{2n}(z)dt.$$

The other two terms can be handled in the same way, and we obtain:

$$(x_i-x_{0,i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \tilde F_{n+i}(z,t)h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt$$
$$= (x_i-x_{0,i})\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r} h(0,\tilde F(z,t))\tilde F_{n+i}(z,t)\nabla^{\phi}_{n+1}\phi(z)Z_1(z,t)d\mathcal{L}^{2n}(z)dt$$
$$- (x_i-x_{0,i})\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r} h(0,\tilde F(z,t))\tilde F_{n+i}(z,t)\nabla^{\phi}_{1}\phi(z)Z_{n+1}(z,t)d\mathcal{L}^{2n}(z)dt$$
$$- (x_i-x_{0,i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \nabla^{\phi}_{1}\Big(h(0,\tilde F(z,t))\tilde F_{n+i}(z,t)\Big)\nabla^{\phi}_{n+1}\phi(z)d\mathcal{L}^{2n}(z)dt$$
$$+ (x_i-x_{0,i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \nabla^{\phi}_{n+1}\Big(h(0,\tilde F(z,t))\tilde F_{n+i}(z,t)\Big)\nabla^{\phi}_{1}\phi(z)d\mathcal{L}^{2n}(z)dt,$$

and

$$- (x_{i+n}-x_{0,n+i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \tilde F_{i}(z,t)h(0,\tilde F(z,t))\partial_{2n}\phi(z)d\mathcal{L}^{2n}(z)dt$$
$$= -(x_{i+n}-x_{0,n+i})\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r} h(0,\tilde F(z,t))\tilde F_{i}(z,t)\nabla^{\phi}_{n+1}\phi(z)Z_1(z,t)d\mathcal{L}^{2n}(z)dt$$
$$+ (x_{i+n}-x_{0,n+i})\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r} h(0,\tilde F(z,t))\tilde F_{i}(z,t)\nabla^{\phi}_{1}\phi(z)Z_{n+1}(z,t)d\mathcal{L}^{2n}(z)dt$$
$$+ (x_{i+n}-x_{0,n+i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \nabla^{\phi}_{1}\Big(h(0,\tilde F(z,t))\tilde F_{i}(z,t)\Big)\nabla^{\phi}_{n+1}\phi(z)d\mathcal{L}^{2n}(z)dt$$
$$- (x_{i+n}-x_{0,n+i})\frac{1}{r^Q}\int_0^1\int_{D_{t,r}} \nabla^{\phi}_{n+1}\Big(h(0,\tilde F(z,t))\tilde F_{i}(z,t)\Big)\nabla^{\phi}_{1}\phi(z)d\mathcal{L}^{2n}(z)dt.$$

Let us integrate (4.109) on the set $[R/2, R]$. Note that

$$\begin{aligned}
&\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{N(\tilde F(z,t))=r} f(z,t)\frac{d\mathcal{L}^{2n}(z)dtdr}{|\nabla_E N(\tilde F(z,t))|}\\
&=\frac{2}{R}\int_{R/2}^{R}\int_0^1\int_{N(\tilde F(z,t))=r}\frac{f(z,t)}{N(\tilde F(z,t))^Q}\frac{d\mathcal{L}^{2n}(z)dtdr}{|\nabla_E N(\tilde F(z,t))|}\\
&=\frac{2}{R}\int_0^1\int_{R/2\le N(\tilde F(z,t))\le R}\frac{f(z,t)}{N(\tilde F(z,t))^Q}d\mathcal{L}^{2n}(z)dt.
\end{aligned}$$

Then, denoting by $g(z,t) := \frac{h(0,\tilde F(z,t))}{N(\tilde F(z,t))^Q}$, $\mathcal{N}_R := \{R/2 \le N(\tilde F(z,t)) \le R\}$, $\phi_{x,x_0} := \phi(x) - \phi(x_0)$ and $\Delta x_i := x_i - x_{0,i}$ we obtain:

$$\begin{aligned}
&\phi_R(x)-\phi_R(x_0)=\\
&=\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\sum_{i=1}^{2n-1}h(0,\tilde F(z,t))(x-x_0)_i,\nabla_i^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&-\phi_{x,x_0}\frac{2}{R}\int_0^1\int_{\mathcal{N}_R}g(z,t)(x-x_0)_n\nabla_{n+1}^{\phi}\phi(z)\nabla_1^{\phi}N(\tilde F(z,t))d\mathcal{L}^{2n}(z)dt\\
&+\phi_{x,x_0}\frac{2}{R}\int_0^1\int_{\mathcal{N}_R}g(z,t)(x-x_0)_n\nabla_1^{\phi}\phi(z)\nabla_{n+1}^{\phi}N(\tilde F(z,t))d\mathcal{L}^{2n}(z)dt\\
&+\phi_{x,x_0}\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_1^{\phi}h(0,\tilde F(z,t))(x-x_0)_n\nabla_{n+1}^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&-\phi_{x,x_0}\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_{n+1}^{\phi}h(0,\tilde F(z,t))(x-x_0)_n\nabla_1^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&+\phi_{x,x_0}\frac{2}{R}\int_0^1\int_{\mathcal{N}_R}g(z,t)\tilde F_n(z,t)\nabla_{n+1}^{\phi}\phi(z)\nabla_1^{\phi}N(\tilde F(z,t))d\mathcal{L}^{2n}(z)dt\\
&-\phi_{x,x_0}\frac{2}{R}\int_0^1\int_{\mathcal{N}_R}g(z,t)\tilde F_n(z,t)\nabla_1^{\phi}\phi(z)\nabla_{n+1}^{\phi}N(\tilde F(z,t))d\mathcal{L}^{2n}(z)dt\\
&-\phi_{x,x_0}\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_1^{\phi}\Big(h(0,\tilde F(z,t))\tilde F_n(z,t)\Big)\nabla_{n+1}^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&+\phi_{x,x_0}\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_{n+1}^{\phi}\Big(h(0,\tilde F(z,t))\tilde F_n(z,t)\Big)\nabla_1^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&+2\sum_{j=1}^{n-1}\Delta x_j\frac{2}{R}\int_0^1\int_{\mathcal{N}_R}g(z,t)\tilde F_{n+j}(z,t)\nabla_{n+1}^{\phi}\phi(z)\nabla_1^{\phi}N(\tilde F(z,t))d\mathcal{L}^{2n}(z)dt
\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{j=1}^{n-1}\Delta x_j\frac{2}{R}\int_0^1\int_{\mathcal{N}_R} g(z,t)\tilde{F}_{n+j}(z,t)\nabla_1^{\phi}\phi(z)\nabla_{n+1}^{\phi}N(\tilde{F}(z,t))d\mathcal{L}^{2n}(z)dt\\
&-2\sum_{j=1}^{n-1}\Delta x_j\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_1^{\phi}\Big(h(0,\tilde{F}(z,t))\tilde{F}_{n+j}(z,t)\Big)\nabla_{n+1}^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&+2\sum_{j=1}^{n-1}\Delta x_j\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_{n+1}^{\phi}\Big(h(0,\tilde{F}(z,t))\tilde{F}_{n+j}(z,t)\Big)\nabla_1^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&-2\sum_{j=1}^{n-1}\Delta x_{j+n}\frac{2}{R}\int_0^1\int_{\mathcal{N}_R} g(z,t)\tilde{F}_{j}(z,t)\nabla_{n+1}^{\phi}\phi(z)\nabla_1^{\phi}N(\tilde{F}(z,t))d\mathcal{L}^{2n}(z)dt\\
&+2\sum_{j=1}^{n-1}\Delta x_{j+n}\frac{2}{R}\int_0^1\int_{\mathcal{N}_R} g(z,t)\tilde{F}_{j}(z,t)\nabla_1^{\phi}\phi(z)\nabla_{n+1}^{\phi}N(\tilde{F}(z,t))d\mathcal{L}^{2n}(z)dt\\
&+2\sum_{j=1}^{n-1}\Delta x_{j+n}\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_1^{\phi}\Big(h(0,\tilde{F}(z,t))\tilde{F}_{j}(z,t)\Big)\nabla_{n+1}^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr\\
&-2\sum_{j=1}^{n-1}\Delta x_{j+n}\frac{2}{R}\int_{R/2}^{R}\frac{1}{r^Q}\int_0^1\int_{D_{t,r}}\nabla_{n+1}^{\phi}\Big(h(0,\tilde{F}(z,t))\tilde{F}_{j}(z,t)\Big)\nabla_1^{\phi}\phi(z)d\mathcal{L}^{2n}(z)dtdr.
\end{aligned}
$$

The kernel in the first term is denoted G_1. The kernel in the terms 2,3,6,7,10,11, 14 and 15 in the right hand side are denoted

$$G_{2i,j} \text{ for } i=1,\dots 8,\ j=1,\dots,n-1 \tag{4.110}$$

with $G_{2i,j}=0$ for $i=1,\dots,4$ and $j=2,\dots,n-1$, and corresponding we will call:

$$A_{1,1}(x,x_0)=A_{2,1}(x,x_0)=A_{3,1}(x,x_0)=A_{4,1}(x,x_0):=\phi(x)-\phi(x_0),$$

$$A_{5,j}(x,x_0)=A_{6,j}(x,x_0):=x_j-x_{0,j},\ \ A_{7,j}(x,x_0)=A_{8,j}(x,x_0):=x_{j+n}-x_{0,n+j}.$$

The kernel in the terms 4,5,8,9,12,13,16 and 17 will be denoted

$$G_{3i,j} \text{ for } i=1,\dots 8,\ j=1,\dots,n-1 \tag{4.111}$$

with $G_{3i,j}=0$ for $i=1,\dots,4$ and $j=2,\dots,n-1$, and corresponding we will call:

$$B_{1,1}=B_{2,1}=B_{3,1}=B_{4,1}:=A_{1,1}\ \ B_{5,j}=B_{6,j}:=A_{5,j},\ \ B_{7,j}=B_{8,j}:=A_{7,j}.$$

Note that the function $|h(0, \tilde{F}(z,t))|$ is bounded by a constant C which depends on C_1 and C_2 in (4.97), then

$$|G_1(z,t)| \leq C d_{\phi(x_0)}(x, x_0). \tag{4.112}$$

In the kernel of type $G_{2i,j}$ the functions $\nabla_k^{\phi}(N(\tilde{F}(z,t)))$, $k \in \{1, n+1\}$ are bounded by a constant C which also depends only on C_1 and C_2.

Then

$$|G_{2i,j}(z,t)| \leq C \frac{d_{\phi(x_0)}(x, x_0)}{(N(\tilde{F}(z,t)))^Q}. \tag{4.113}$$

Analogously,

$$|G_{3i,j}(z,t)| \leq C. \tag{4.114}$$

□

4.4 Main Theorem

The aim of this section is to prove a Poincaré inequality for the vector fields ∇^{ϕ} defined in terms of ϕ. The Poincaré inequality we prove here is partially inspired to the Sobolev type inequality for non regular coefficients contained in [43] and extended to a more general class of vector fields by [111]. The idea is to start with the representation formula proved in Theorem 4.3.9 and deduce, via the approximation theorem, the following result:

Theorem 4.4.1. *Let $\omega \subset \mathbb{R}^{2n}$ be open and bounded with $n \geq 2$ and let $\phi : \omega \longrightarrow \mathbb{R}$ be an intrinsic Lipschitz function. Then there exists a constant C (independent of the Lipschitz constant $L := Lip(\phi)$) such that, for each $x \in \omega$, $r > 0$ such that $\Omega_{\phi(x)}(x, r) \subset \omega$ and $\Omega_{\phi(x)}(x, Cr(1+L)) \subset \omega$*

$$\int_{\Omega_{\phi(x)}(x,r)} |\phi(y) - \phi_r(x)| \mathrm{d}\mathcal{L}^{2n}(y) \leq C L^{\frac{Q+2}{2}} r \int_{\Omega_{\phi(x)}(x, Cr(1+L))} |\nabla^{\phi}\phi(y)| \mathrm{d}\mathcal{L}^{2n}(y), \tag{4.115}$$

We will first establish the representation formula for (intrinsic) Lipschitz continuous functions, which will be carried out by approximation, using the reppresentation formula in Theorem 4.3.9 for C^{∞} functions and the approximation results proved in Theorem 4.2.7. To this end we fix a bounded open set $\omega \subset \mathbb{R}^{2n}$ and an intrinsic Lipschitz function $\phi : \omega \longrightarrow \mathbb{R}$. We also denote $\{\phi_k\} \subset C^\infty(\omega)$ its approximating sequence.

Then for every k we have defined in (4.92) the distance $d_{\phi_k(x_0)}$, in (4.99) the fundamental solution $\Gamma_{\phi_k(x_0)}$ of the operator associated to the function ϕ_k and frozen at the point x_0, and in (4.100) the level set $\Omega_{\phi_k(x_0)}(x_0, r)$ of the fundamental solution $\Gamma_{\phi_k(x_0)}$. Accordingly we will denote $\phi_{k,r}(x_0)$ the mean on the set $\Omega_{\phi_k(x_0)}(x_0, r)$ associated to the function ϕ_k.

Lemma 4.4.2. *Let $\phi : \omega \longrightarrow \mathbb{R}$ be a (intrinsic) Lipschitz continuous function. Let also denote $\{\phi_k\} \subset C^\infty(\omega)$ a sequence such that statements (i), (ii) and (iii) of Theorem 4.2.7 hold. Then for every $r > 0$*

$$\Omega_{\phi_k,x_0}(x_0, r) \to \Omega_{\phi,x_0}(x_0, r) \ \text{as } k \to +\infty,$$

$$\tilde{\phi}_{k,r}(x_0) \to \tilde{\phi}_r(x_0), \ \text{uniformly in } r > 0 \text{ as } k \to +\infty.$$

Proof. We recall that

$$\tilde{\Omega}(0, r) = \left\{\tilde{y} \in \mathbb{R}^{2n} \mid \Gamma(0, \tilde{y}) > r^{2-Q}\right\},$$

then, by (4.102) we have

$$\Omega_{\phi_k(x_0)}(x_0, r) = Exp_{\phi_k(x_0),x_0}(\tilde{\Omega}(0, r)).$$

So that, by definition of $\tilde{\phi}_{k,r}(x_0)$ and (4.99)

$$\tilde{\phi}_{k,r}(x_0) = \frac{Q}{Q-2}\frac{1}{r^Q}\int_{\Omega_{\phi_k(x_0)}(x_0,r)} \frac{|\nabla^{\phi_k(x_0)}\Gamma_{\phi_k(x_0)}(x_0, \tilde{y})|^2}{\Gamma_{\phi_k(x_0)}(x_0, \tilde{y})^{2(Q-1)/(Q-2)}}\phi_k(\tilde{y}) \, \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) =$$

$$= \frac{Q}{Q-2}\frac{1}{r^Q}\int_{\tilde{\Omega}(0,r)} \frac{|\nabla\Gamma(0, \tilde{y})|^2}{\Gamma(0, \tilde{y})^{2(Q-1)/(Q-2)}}\phi_k(Exp_{\phi_k(x_0),x_0}(\tilde{y})) \, \mathrm{d}\mathcal{L}^{2n}(\tilde{y})$$

$$\to \frac{Q}{Q-2}\frac{1}{r^Q}\int_{\tilde{\Omega}(0,r)} \frac{|\nabla\Gamma(0, \tilde{y})|^2}{\Gamma(0, \tilde{y})^{2(Q-1)/(Q-2)}}\phi(Exp_{\phi(x_0),x_0}(\tilde{y})) \, \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) = \tilde{\phi}_r(x_0),$$

uniformly in $r > 0$, as $k \to +\infty$. □

Passing at the limit in the representation formulas in Propositions 4.3.6 and 4.3.9 we obtain the following representation theorem:

Proposition 4.4.3. *Let $\phi : \omega \longrightarrow \mathbb{R}$ be a (intrinsic) Lipschitz continuous function.*

$$
\begin{aligned}
&\phi(x_0) - \phi_R(x) = \\
&= \frac{1}{R}\int_0^R f_1(r)\int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,\tilde{y}), \nabla^{\phi}\phi(\tilde{y})\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \\
&+ \frac{1}{R}\int_0^R f_1(r)\int_{\Omega_{\phi(x_0)}(x_0,r)} (\phi(\tilde{y}) - \phi(x_0)) \left\langle F_2\Big(Log_{\phi(x_0),x_0}(\tilde{y})\Big), \nabla^{\phi}\phi(\tilde{y})\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \\
&+ \frac{1}{R}\int_{R/2}^R \frac{1}{r^Q}\int_{\Omega_{\phi(x_0)}(x_0,r)} (\phi(\tilde{y}) - \phi(x_0)) \left\langle F_3\Big(Log_{\phi(x_0),x_0}(\tilde{y})\Big), \nabla^{\phi}\phi(\tilde{y})\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \\
&- \frac{1}{R}\int_{R/2}^R \frac{1}{r^Q}\int_0^1 \int_{N(\tilde{F}(z,t))\le r} < G_1(z), \nabla^{\phi}\phi(z) > d\mathcal{L}^{2n}(z)dtdr \\
&- \sum_{i,j} A_{i,j}(x,x_0)\frac{1}{R}\int_0^1 \int_{R/2\le N(\tilde{F}(z,t))\le R} < G_{2i,j}(z), \nabla^{\phi}\phi(z) > d\mathcal{L}^{2n}(z)dt + \\
&- \sum_{i,j} B_{i,j}(x,x_0)\frac{1}{R}\int_{R/2}^R \frac{1}{r^Q}\int_0^1 \int_{N(\tilde{F}(z,t))\le r} < G_{3i,j}(z), \nabla^{\phi}\phi(z) > d\mathcal{L}^{2n}(z)dtdr
\end{aligned}
$$

where G_1 is homogeneous of order 1, $G_{2i,j}$ is homogeneous of order $1-Q$, $G_{3i,j}$ is homogeneous of order 0, according (4.112)-(4.114), F_2 and F_3 are homogeneous of order $-Q$ and 0 respectively. Moreover $A_{i,j}$ and $B_{i,j}$ satisfy

$$|A_{i,j}(x,x_0)| + |B_{i,j}(x,x_0)| \le d_\phi(x,x_0) + |\phi(x) - \phi(x_0)|.$$

Proposition 4.4.4. *Let $\phi : \omega \longrightarrow \mathbb{R}$ a (intrinsic) Lipschitz continuous function. Then there exists a constant C such that for every $x_0 \in \omega$ and $R > 0$ such that $x \in \Omega_{\phi(x_0)}(x_0,R) \subset \omega$ we have*

$$|\phi(x_0) - \phi_R(x)| \le CL\int_{\Omega_{\phi(x_0)}(x_0,R)} d^{1-Q}_{\phi(x_0)}(x_0,\tilde{y})|\nabla^{\phi}\phi(\tilde{y})|\mathrm{d}\mathcal{L}^{2n}(\tilde{y}). \qquad (4.116)$$

Proof. Let us estimate the right hand side of the representation formula in Proposition 4.4.3. We have noted that inequalities (4.97) hold for $\Gamma_{\phi(x_0)}$ and $d_{\phi(x_0)}$, with constants independent of L. Hence, we immediately get that

$$\frac{1}{R}\int_0^R f_1(r)\int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,\tilde{y}), \nabla^{\phi}\phi(\tilde{y})\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(\tilde{y})\, \mathrm{d}r$$

$$\leq C \int_{\Omega_{\phi(x_0)}(x_0,R)} d^{1-Q}_{\phi(x_0)}(x_0,\tilde{y})|\nabla^{\phi}\phi(\tilde{y})|\mathrm{d}\mathcal{L}^{2n}(\tilde{y}).$$

Consider the second term in the right hand side of the representation formula in Proposition 4.4.3

$$\frac{1}{R}\int_0^R f_1(r)\int_{\Omega_{\phi(x_0)}(x_0,r)} (\phi(\tilde{y})-\phi(x_0))\left\langle F_2\Big(Log_{\phi(x_0),x_0}(\tilde{y})\Big), \nabla^{\phi}\phi(\tilde{y})\right\rangle \,\mathrm{d}\mathcal{L}^{2n}(\tilde{y})\,\mathrm{d}r$$

(using fact that F_2 is homogeneous of order Q and $|\phi(\tilde{y})-\phi(x_0)| \leq Ld_{\phi(x_0)}(x_0,x)$)

$$\leq CL \int_{\Omega_{\phi(x_0)}(x_0,R)} d^{1-Q}_{\phi(x_0)}(x_0,\tilde{y})|\nabla^{\phi}\phi(\tilde{y})|\mathrm{d}\mathcal{L}^{2n}(\tilde{y}).$$

The third term can be handled in a similar way.

Consider the fourth term in the representation formula in Proposition 4.4.3:

$$\left|\frac{1}{R}\int_{R/2}^R \frac{1}{r^Q}\int_0^1 \int_{N(\tilde{F}(z,t))\leq r} < G_1(z), \nabla^{\phi}\phi(z) > d\mathcal{L}^{2n}(z)dtdr\right|$$

(by (4.112))

$$\begin{aligned}
&\leq C\frac{1}{R}\int_{R/2}^R \frac{d_{\phi(x_0)}(x,x_0)}{r^Q}\int_0^1 \int_{N(\tilde{F}(z,t))\leq r} |\nabla^{\phi}\phi(z)|d\mathcal{L}^{2n}(z)dtdr\\
&\leq C\frac{1}{R}\int_{R/2}^R \int_0^1 \int_{N(\tilde{F}(z,t))\leq r} \frac{1}{(N(\tilde{F}(z,t)))^{Q-1}}|\nabla^{\phi}\phi(z)|d\mathcal{L}^{2n}(z)dtdr\\
&\leq C \int_{\Omega_{\phi(x_0)}(x_0,R)} d^{1-Q}_{\phi(x_0)}(x_0,\tilde{y})|\nabla^{\phi}\phi(\tilde{y})|\mathrm{d}\mathcal{L}^{2n}(\tilde{y}).
\end{aligned}$$

Consider the fifth term:

$$\left|A_{i,j}(x,x_0)\frac{1}{R}\int_0^1 \int_{R/2\leq N(\tilde{F}(z,t))\leq R} < G_{2i,j}(z), \nabla^{\phi}\phi(z) > d\mathcal{L}^{2n}(z)dt\right|$$

(by (4.113))

$$\begin{aligned}
&\leq C\,L\,d_{\phi(x_0)}(x,x_0)\frac{1}{R}\int_0^1 \int_{R/2\leq N(\tilde{F}(z,t))\leq R} \frac{d_{\phi(x_0)}(x,x_0)}{(N(\tilde{F}(z,t)))^{Q}}|\nabla^{\phi}\phi(z)|d\mathcal{L}^{2n}(z)dt\\
&\leq C\,L \int_0^1 \int_{R/2\leq N(\tilde{F}(z,t))\leq R} \frac{1}{(N(\tilde{F}(z,t)))^{Q-1}}|\nabla^{\phi}\phi(z)|d\mathcal{L}^{2n}(z)dt.
\end{aligned}$$

Finally, by (4.114) the sixth term in the representation formula can estimated as the fourth term. □

We are now in position to prove Theorem 4.4.1

Proof. As before we denote by $\Omega_r := \Omega_{\phi(x)}(x,r)$ and $\hat{\Omega}_r := \Omega_{\phi(x)}(x, C(1+L)r)$. Integrating both members on Ω_r in the previous proposition we get

$$\int_{\Omega_r} |\phi(x_0) - \phi_r(x)| \mathrm{d}\mathcal{L}^{2n}(x_0) \leq$$
$$\leq CL \int_{\Omega_r} \Big(\int_{\Omega_{\phi(x_0)}(x_0,r)} d^{1-Q}_{\phi(x_0)}(x_0,\tilde{y}) |\nabla^\phi \phi(\tilde{y})| \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \Big) \mathrm{d}\mathcal{L}^{2n}(x_0).$$

In [45] it is proved that there are constants C_1, C_2 such that

$$\frac{C_1}{1+\sqrt{L}} d_{\phi(\tilde{y})}(x_0,\tilde{y}) \leq d_{\phi(x_0)}(x_0,\tilde{y}) \leq C_2(1+\sqrt{L}) d_{\phi(x)}(x_0,\tilde{y})$$

so that there exists a constant C such that

$$d^{1-Q}_{\phi(x_0)}(x_0,\tilde{y}) \leq C(1+\sqrt{L})^{Q-1} d^{1-Q}_{\phi(\tilde{y})}(x_0,\tilde{y}).$$

Analogously, if $\tilde{y} \in \Omega_{\phi(x_0)}(x_0,r)$, then there exists C such that

$$x_0 \in \Omega_{\phi(\tilde{y})}\big(\tilde{y}, C(1+\sqrt{L})r\big).$$

By the triangular inequality if $d_{\phi(x)}(x,x_0) \leq r$, and $d_{\phi(x_0)}(\tilde{y},x_0) \leq r$ then

$$d_{\phi(x)}(x,\tilde{y}) \leq 2rC(1+L).$$

Hence

$$\int_{\Omega_r} \Big(\int_{\Omega_{\phi(x_0)}(x_0,r)} d^{1-Q}_{\phi(x_0)}(x_0,\tilde{y}) |\nabla^\phi \phi(\tilde{y})| \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \Big) \mathrm{d}\mathcal{L}^{2n}(x_0)$$
$$\leq C(1+\sqrt{L})^{Q-1} \int_{\hat{\Omega}_r} |\nabla^\phi \phi(\tilde{y})| \Big(\int_{\Omega_{\phi(\tilde{y})}(\tilde{y}, r(1+\sqrt{L}))} d^{1-Q}_{\phi(\tilde{y})}(x_0,\tilde{y}) \mathrm{d}\mathcal{L}^{2n}(x_0) \Big) \mathrm{d}\mathcal{L}^{2n}(\tilde{y})$$
$$\leq Cr(1+\sqrt{L})^{Q} \int_{\hat{\Omega}_r} |\nabla^\phi \phi(\tilde{y})| \mathrm{d}\mathcal{L}^{2n}(\tilde{y}).$$

□

Chapter 5

Stable solutions in Engel groups

In this chapter we investigate the stable solution of a semilinear elliptic problem set in the Engel group $\mathbb{E}$.

5.1 The problem: basic tools

While we refer to Section 1.3 for the standard definitions and properties of the Engel group, we now introduce the problem we study.

Given a domain $\Omega \subseteq \mathbb{E}$ and $f \in C^1(\mathbb{R})$, we consider $u \in C^2(\Omega)$ to be a (weak) solution of

$$\Delta_{\mathbb{E}} u = f(u), \tag{5.1}$$

that is we suppose that

$$-\int_{\mathbb{E}} \langle \nabla_{\mathbb{E}} u, \nabla_{\mathbb{E}} \eta \rangle_{\mathbb{E}} = \int_{\mathbb{E}} f(u)\eta \tag{5.2}$$

for every $\eta \in C_0^\infty(\Omega)$.

We assume that u is stable, that is

$$0 \leq \int_{\mathbb{E}} \langle \nabla_{\mathbb{E}} \eta, \nabla_{\mathbb{E}} \eta \rangle_{\mathbb{E}} + \int_{\mathbb{E}} \dot{f}(u)\eta^2 \tag{5.3}$$

for every $\eta \in C_0^\infty(\Omega)$.

The stability condition in (5.3) has been widely studied in the calculus of variation setting: indeed, it states that the second variation of the energy functional associated to (5.1) is nonnegative at the critical point u – hence, for instance, minimal solutions are always stable, but, in principle, stability is a weaker condition than minimality.

Equation (5.1) is called semilinear, since the only nonlinearity depends on the solution u (not on the space, neither on the derivatives of u): such kind of equations have been studied in detail in the Euclidean framework, and in the subRiemannian one as well (see, e.g. [85, 16, 17, 20]), and they possess the remarkable geometric property that the operator is constant along the level sets of the solution.

At any point of

$$\mathbb{E}_0 := \{x \in \Omega \mid \nabla_{\mathbb{E}} u \neq 0\}$$

we denote by ν the opposite of the intrinsic unit normal to the level set of u as defined in (3.1), that is

$$\nu = \frac{\nabla_{\mathbb{E}} u}{|\nabla_{\mathbb{E}} u|}.$$

We shall also consider the intrinsic tangent direction to the level set of u

$$v := \frac{X_2 u}{|\nabla_{\mathbb{E}} u|} X_1 - \frac{X_1 u}{|\nabla_{\mathbb{E}} u|} X_2$$

Let us observe that $\forall p \in \mathbb{E}_0$

$$\langle \nu(p), v(p) \rangle_{p,\mathbb{E}} = 0$$

where $\langle \cdot, \cdot \rangle_{p,\mathbb{E}}$ is the standard scalar product defined in (1.1.15). We denote by Hu the intrinsic Hessian matrix, i.e.

$$Hu := \begin{pmatrix} X_1 X_1 u & X_2 X_1 u \\ X_1 X_2 u & X_2 X_2 u \end{pmatrix}$$

As usual, we define

$$(Hu)^2 := (Hu)(Hu)^T$$

and

$$|Hu| := \sqrt{|\nabla_{\mathbb{E}} X_1 u|^2 + |\nabla_{\mathbb{E}} X_2 u|^2}$$

Also, in $\mathbb{E}_0$, following an analogy in the Heisenberg group (see [116, 123, 7, 8]), we define the horizontal mean curvature

$$h := \operatorname{div}_{\mathbb{E}} \nu \tag{5.4}$$

and the imaginary curvature [7, 8]:

$$p := -\frac{X_3 u}{|\nabla_{\mathbb{E}} u|} \tag{5.5}$$

5.1.1 The inequality

Denoting by

$$\mathcal{J} := 2(X_3 X_2 u X_1 u - X_3 X_1 u X_2 u) + (X_4 u)(X_1 u - X_2 u)$$

we have:

Theorem 5.1.1.

$$\int_{\mathbb{E}_0} \left[|\nabla_{\mathbb{E}} u|^2 \Big\{ \Big(p + \frac{\big\langle (Hu)^T \nu, v \big\rangle}{|\nabla_{\mathbb{E}} u|} \Big)^2 + h^2 \Big\} - \mathcal{J} \right] \eta^2 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \eta|^2 |\nabla_{\mathbb{E}} u|^2$$

for any $\eta \in C_0^\infty(\mathbb{E})$.

Before giving the proof some comments are in order. Theorem 5.1.1 is a sort of geometric weighted Poincaré inequality, in the sense that the weighted L^2-norm of any test function is bounded by a weighted L^2-norm of its gradient, and the weights are built with geometric objects.

In the Euclidean case, the analogue of Theorem 5.1.1 was established in [131, 132], and recently many extensions have been performed (see, in particular, [67, 68]). As far as we know, the first applications in the sub-Riemannian setting, were performed in [71, 15] for the Heisenberg group and in [72] for the Grushin plane. In several cases, these type of geometric weighted inequalities lead to rigidity results (such as classification, symmetry, or non existence, of solutions). Differently from the Euclidean case, the weight on the left hand side of the inequality does not need to be positive in general, due to the presence of $\mathcal{J}$. Thus, the presence of noncommutating vector fields, complicates the geometry of the level sets via the sign of $\mathcal{J}$. Indeed, if $\mathcal{J} \leq 0$, when the right hand side of the inequality in Theorem 5.1.1 vanishes, one obtains that the level sets of u satisfy the geometric equations, see Corollary 5.3.2,

$$\begin{cases} p + \dfrac{\big\langle (Hu)^T\nu, v\big\rangle}{|\nabla_{\mathbb{E}} u|} = 0 \\ \\ h = 0 \end{cases} \tag{5.6}$$

The higher the step of the group, the more complicated are the combinatorics occurring in the inequality, and the more difficult is the geometric interpretation of the quantities involved. Nevertheless, the Engel group still mantains a reasonable level of geometric insight and provides a challenging source of problems for this approach.

Remark 5.1. If, in Theorem 5.1.1, u does not depend on x_4, then the situation boils down to the one in the Heisenberg group (note indeed that $X_3X_1u = X_1X_3u$, so Theorem 5.1.1 reduces to Theorem 2.3 in [71]).

5.2 Proof of the estimate

The proof of our first result needs some preliminary, technical computations, by which we obtain some useful identities.

Lemma 5.2.1. *Let $j \in \{1, 2\}$. If $u \in C^2(\Omega)$ then in $\mathbb{E}_0$ we have*

$$X_j|\nabla_{\mathbb{E}} u| = \langle X_j(\nabla_{\mathbb{E}} u), \nu\rangle_{\mathbb{E}}. \tag{5.7}$$

Moreover, for each $\eta \in C_0^\infty(\Omega)$,

$$\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}} u|\eta) = \frac{\eta}{|\nabla_{\mathbb{E}} u|}(Hu)^T\nabla_{\mathbb{E}} u + |\nabla_{\mathbb{E}} u|\nabla_{\mathbb{E}}\eta \tag{5.8}$$

and

$$|\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}} u|\eta)|^2 = \frac{\eta^2}{|\nabla_{\mathbb{E}} u|^2}|(Hu)^T\nabla_{\mathbb{E}} u|^2 + 2\eta\,\langle \nabla_{\mathbb{E}} u, (Hu)\nabla_{\mathbb{E}}\eta\rangle_{\mathbb{E}} + |\nabla_{\mathbb{E}}\eta|^2|\nabla_{\mathbb{E}} u|^2 \tag{5.9}$$

Proof. Equation (5.7) is straightforward. Also, the proof of (5.8) follows from the following simple calculation:

$$\begin{aligned} \nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}} u|\eta) &= \eta\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}} u|) + |\nabla_{\mathbb{E}} u|\nabla_{\mathbb{E}}\eta = \\ &= \frac{\eta}{|\nabla_{\mathbb{E}} u|}(Hu)^T\nabla_{\mathbb{E}} u + |\nabla_{\mathbb{E}} u|\nabla_{\mathbb{E}}\eta \end{aligned} \tag{5.10}$$

Furthermore

$$\begin{aligned}|\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)|^2 &= \langle \nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta), \nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)\rangle_{\mathbb{E}} = \\ &= \left\langle \frac{\eta}{|\nabla_{\mathbb{E}}u|}(Hu)^T\nabla_{\mathbb{E}}u, \frac{\eta}{|\nabla_{\mathbb{E}}u|}(Hu)^T\nabla_{\mathbb{E}}u \right\rangle_{\mathbb{E}} + \\ &+ 2\left\langle \frac{\eta}{|\nabla_{\mathbb{E}}u|}(Hu)^T\nabla_{\mathbb{E}}u, |\nabla_{\mathbb{E}}u|\nabla_{\mathbb{E}}\eta \right\rangle_{\mathbb{E}} + \\ &+ \langle |\nabla_{\mathbb{E}}u|\nabla_{\mathbb{E}}\eta, |\nabla_{\mathbb{E}}u|\nabla_{\mathbb{E}}\eta\rangle_{\mathbb{E}}\,.\end{aligned}$$

Hence

$$\begin{aligned}|\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)|^2 &= \Big(\frac{\eta}{|\nabla_{\mathbb{E}}u|}\Big)^2|(Hu)^T\nabla_{\mathbb{E}}u|^2 + 2\eta\left\langle (Hu)^T\nabla_{\mathbb{E}}u, \nabla_{\mathbb{E}}\eta\right\rangle_{\mathbb{E}} + \\ &+ |\nabla_{\mathbb{E}}\eta|^2|\nabla_{\mathbb{E}}u|^2 \\ &= \Big(\frac{\eta}{|\nabla_{\mathbb{E}}u|}\Big)^2|(Hu)^T\nabla_{\mathbb{E}}u|^2 + 2\eta\left\langle \nabla_{\mathbb{E}}u, (Hu)\nabla_{\mathbb{E}}\eta\right\rangle_{\mathbb{E}} + \\ &+ |\nabla_{\mathbb{E}}\eta|^2|\nabla_{\mathbb{E}}u|^2\end{aligned}$$

and this proves (5.9). □

Lemma 5.2.2. *Let $u \in C^2(\Omega)$ then*

$$|Hu|^2 - \left\langle (Hu)^2\nu, \nu\right\rangle_{\mathbb{E}} = |(Hu)^T v|^2. \tag{5.11}$$

Moreover, in $\mathbb{E}_0$

$$|(Hu)^T v|^2 = |\nabla_{\mathbb{E}}u|^2\Big\{\Big(p + \frac{\left\langle (Hu)^T\nu, v\right\rangle}{|\nabla_{\mathbb{E}}u|}\Big)^2 + h^2\Big\}. \tag{5.12}$$

Proof. We note that for each $p \in \mathbb{E}$ $(\nu(p), v(p))$ is an orthonormal basis of $H_p\mathbb{E}$. Then (5.11) follows, for instance, from Lemma 3 in [15].

In order to prove (5.12), we begin observing that

$$\begin{aligned}(Hu)^T &= \begin{pmatrix} X_1X_1u & X_1X_2u \\ X_2X_1u & X_2X_2u\end{pmatrix} = \\ &= \begin{pmatrix} X_1X_1u & X_2X_1u \\ X_1X_2u & X_2X_2u\end{pmatrix} + \begin{pmatrix} 0 & X_1X_2u - X_2X_1u \\ X_2X_1u - X_1X_2u & 0\end{pmatrix} = \\ &= Hu + \begin{pmatrix} 0 & X_3u \\ -X_3u & 0\end{pmatrix}.\end{aligned} \tag{5.13}$$

Now we define

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let also $\mathcal{Z}$ and $H_\nu \in \operatorname{Mat}(\mathbb{R}, 2\times 2)$ be defined as

$$\mathcal{Z}_{ij} := \nu_i\big((Hu)^T\nu\big)_j \quad \text{and} \quad (H_\nu)_{ij} := X_j(\nu_i)$$

for i, $j \in \{1,2\}$. So, we use (5.7) to obtain that

$$\begin{aligned} & \mathcal{Z}_{ij} + |\nabla_{\mathbb{E}} u|(H_\nu)_{ij} = \\ = \; & \nu_i\langle X_j(\nabla_{\mathbb{E}} u), \nu\rangle_{\mathbb{E}} + |\nabla_{\mathbb{E}} u| X_j\Big(\frac{X_i u}{|\nabla_{\mathbb{E}} u|}\Big) = \\ = \; & \frac{X_i u}{|\nabla_{\mathbb{E}} u|}\langle X_j(\nabla_{\mathbb{E}} u), \nu\rangle_{\mathbb{E}} + X_j X_i u - \frac{X_i u}{|\nabla_{\mathbb{E}} u|} X_j |\nabla_{\mathbb{E}} u| = \\ = \; & X_j X_i u \end{aligned}$$

that is

$$\mathcal{Z} + |\nabla_{\mathbb{E}} u|\, H_\nu = Hu$$

Hence, we can rewrite (5.13) in the following way

$$(Hu)^T = (X_3 u)J + \mathcal{Z} + |\nabla_{\mathbb{E}} u| H_\nu \tag{5.14}$$

Furthermore

$$Jv = -\nu \tag{5.15}$$

and

$$\begin{aligned} (\mathcal{Z} v)_i &= \sum_{j=1}^{2} \mathcal{Z}_{ij} v_j = \\ &= \sum_{j=1}^{2} \nu_i\big((Hu)^T\nu\big)_j v_j = \nu_i\,\langle (Hu)^T\nu, v\rangle_{\mathbb{E}} \end{aligned}$$

that is

$$\mathcal{Z} v = \big\langle (Hu)^T\nu, v\big\rangle_{\mathbb{E}}\, \nu \tag{5.16}$$

By plugging (5.15) and (5.16) into (5.14), we conclude that

$$(Hu)^T v = \Big(- X_3 u + \big\langle (Hu)^T\nu, v\big\rangle_{\mathbb{E}}\Big)\nu + |\nabla_{\mathbb{E}} u| H_\nu v \tag{5.17}$$

and so

$$\begin{aligned}
|(Hu)^T v|^2 &= \big\langle (Hu)^T v, (Hu)^T v \big\rangle_{\mathbb{E}} = \\
&= \Big(-X_3 u + \big\langle (Hu)^T \nu, v \big\rangle_{\mathbb{E}} \Big)^2 + |\nabla_{\mathbb{E}} u|^2 |H_\nu v|^2 + \\
&\quad + 2\Big(-X_3 u + \big\langle (Hu)^T \nu, v \big\rangle_{\mathbb{E}} \Big) |\nabla_{\mathbb{E}} u| \, \langle H_\nu v, \nu \rangle_{\mathbb{E}}
\end{aligned}$$

From this and the definitions in (5.4) and (5.5), we obtain that the proof of (5.12) is completed if we prove that

$$\langle H_\nu v, \nu \rangle_{\mathbb{E}} = 0 \tag{5.18}$$

and that

$$|H_\nu v| = |\operatorname{div}_{\mathbb{E}} \nu| \tag{5.19}$$

To this end, let us observe that, by (5.17),

$$|\nabla_{\mathbb{E}} u| \, \langle H_\nu v, \nu \rangle_{\mathbb{E}} = \big\langle (Hu)^T v, \nu \big\rangle_{\mathbb{E}} - \langle (Hu) v, \nu \rangle_{\mathbb{E}} + X_3 u \tag{5.20}$$

Now, by (5.13),

$$(Hu)^T v - (Hu) v = \begin{pmatrix} -\dfrac{X_1 u}{|\nabla_{\mathbb{E}} u|} X_3 u \\ \\ -\dfrac{X_2 u}{|\nabla_{\mathbb{E}} u|} X_3 u \end{pmatrix} = -(X_3 u)\nu$$

hence

$$\big\langle (Hu)^T v - (Hu) v, \nu \big\rangle_{\mathbb{E}} = -X_3 u$$

By plugging this into (5.20), we obtain (5.18).

To obtain (5.19), we argue as follows. By (5.18), we know that $H_\nu v$ is parallel (or antiparallel) to v, therefore

$$H_\nu v = \pm |H_\nu v| v$$

Hence, by (5.17),

$$\begin{aligned}
\pm \, |\nabla_{\mathbb{E}} u| \, |H_\nu v| &= \langle |\nabla_{\mathbb{E}} u| H_\nu v, v \rangle_{\mathbb{E}} = \\
&= \langle (Hu)^T v, v \rangle_{\mathbb{E}} = \sum_{i,j=1}^{2} (X_i X_j u) v_i v_j
\end{aligned} \tag{5.21}$$

Now, we remark that

$$\nu_i^2 = 1 - v_i^2 \tag{5.22}$$

To prove this, we take $i = 1$ (the case $i = 2$ being analogous), and we observe that

$$\nu_1^2 = v_2^2 = 1 - v_1^2$$

which establishes (5.22).

On the other hand, if $i \neq j$,

$$\nu_i \nu_j = \nu_1 \nu_2 = (-v_2)(v_1) = -v_i v_j \tag{5.23}$$

So, by (5.7), (5.22) and (5.23), we obtain

$$\begin{aligned}
|\nabla_{\mathbb{E}} u| \, |\operatorname{div}_{\mathbb{E}} \nu| &= |\nabla_{\mathbb{E}} u| \sum_{i=1}^{2} X_i \Big(\frac{X_i u}{|\nabla_{\mathbb{E}} u|} \Big) = \\
&= \sum_{i=1}^{2} X_i X_i u - \frac{X_i u}{|\nabla_{\mathbb{E}} u|} \langle X_i (\nabla_{\mathbb{E}} u), \nu \rangle_{\mathbb{E}} = \\
&= \sum_{i=1}^{2} X_i X_i u - \sum_{i,j=1}^{2} (X_i X_j u) \nu_i \nu_j = \\
&= \sum_{i=1}^{2} X_i X_i u - \sum_{i=1}^{2} (X_i X_i u) \nu_i^2 - \sum_{i \neq j=1}^{2} (X_i X_j u) \nu_i \nu_j = \\
&= \sum_{i=1}^{2} X_i X_i u - \sum_{i=1}^{2} (X_i X_i u)(1 - v_i^2) + \sum_{i \neq j=1}^{2} (X_i X_j u) v_i v_j = \\
&= \sum_{i=1}^{2} (X_i X_i u) v_i^2 + \sum_{i \neq j=1}^{2} (X_i X_j u) v_i v_j = \\
&\sum_{i,j=1}^{2} (X_i X_j u) v_i v_j
\end{aligned} \tag{5.24}$$

By comparing (5.21) and (5.24), we see that

$$\pm |\nabla_{\mathbb{E}} u| \, |H_\nu v| = |\nabla_{\mathbb{E}} u| \, |\operatorname{div}_{\mathbb{E}} \nu|$$

which implies (5.19), as desired. □

Lemma 5.2.3. *For each $u \in C^2(\Omega)$ it holds that*

$$X_1\Delta_{\mathbb{E}}u = \Delta_{\mathbb{E}}X_1u + 2X_3X_2u + X_4u; \tag{5.25}$$

$$X_2\Delta_{\mathbb{E}}u = \Delta_{\mathbb{E}}X_2u - 2X_1X_3u + X_4u; \tag{5.26}$$

$$X_3\Delta_{\mathbb{E}}u = \Delta_{\mathbb{E}}X_3u - 2X_4X_1u - 2X_4X_2u; \tag{5.27}$$

$$\text{and } X_4\Delta_{\mathbb{E}}u = \Delta_{\mathbb{E}}X_4u. \tag{5.28}$$

Proof. For the first equality

$$\begin{aligned}
X_1\Delta_{\mathbb{E}}u &= X_1(X_1X_1u) + X_1(X_2X_2u)\\
&= \Delta_{\mathbb{E}}X_1u + X_1X_2X_2u - X_2X_2X_1u\\
&= \Delta_{\mathbb{E}}X_1u + X_3X_2u + X_2X_1X_2u - X_2X_2X_1u\\
&= \Delta_{\mathbb{E}}X_1u + 2X_3X_2u + X_2X_3u - X_3X_2u\\
&= \Delta_{\mathbb{E}}X_1u + 2X_3X_2u + X_4u.
\end{aligned}$$

The second and the third equality follow in a similar way, indeed

$$\begin{aligned}
X_2\Delta_{\mathbb{E}}u &= X_2(X_1X_1u) + X_2(X_2X_2u)\\
&= \Delta_{\mathbb{E}}X_2u - X_1X_1X_2u + X_2X_1X_1u\\
&= \Delta_{\mathbb{E}}X_2u - X_3X_1u + X_1X_2X_1u - X_1X_1X_2u\\
&= \Delta_{\mathbb{E}}X_2u - X_3X_1u - X_1X_3u\\
&= \Delta_{\mathbb{E}}X_2u - X_3X_1u + X_1X_3u - 2X_1X_3u\\
&= \Delta_{\mathbb{E}}X_2u - 2X_1X_3u + X_4u
\end{aligned}$$

and

$$\begin{aligned}
X_3\Delta_{\mathbb{E}}u &= X_3(X_1X_1u) + X_3(X_2X_2u)\\
&= X_1X_3X_1u + X_2X_3X_2u - X_4X_1u - X_4X_2u\\
&= \Delta_{\mathbb{E}}X_3u - 2X_4X_1u - 2X_4X_2u.
\end{aligned}$$

The last is a direct consequence of $X_1X_4u = X_4X_1u$ and $X_2X_4u = X_4X_2u$.
□

Using Lemma 5.2.3, we obtain

Corollary 5.2.4. *Let $u \in C^2(\Omega)$ be a weak solution of (5.2) then*

$$\Delta_{\mathbb{E}} X_1 u + 2X_3X_2u + X_4u = \dot{f}(u)X_1u$$
$$\text{and} \quad \Delta_{\mathbb{E}} X_2 u - 2X_1X_3u + X_4u = \dot{f}(u)X_2u$$

We are now in the position of proving the following geometric inequality:

Proposition 5.2.5. *Let $u \in C^2(\Omega)$ be a stable weak solution of (5.1). Then, for each $\eta \in C_0^\infty(\Omega)$,*

$$\int_{\mathbb{E}_0} [|Hu|^2 - \langle (Hu)^2\nu, \nu\rangle_{\mathbb{E}}]\eta^2 - 2\int_{\mathbb{E}_0} (X_3X_2uX_1u - X_3X_1uX_2u)\eta^2 - \\ - \int_{\mathbb{E}_0} (X_4u)(X_1u - X_2u)\eta^2 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}}\eta|^2 |\nabla_{\mathbb{E}} u|^2$$

Proof. Multiplying by $(X_1u)\eta^2$ equation (5.25) in Corollary 5.2.4 and by $(X_2u)\eta^2$ equation (5.26) and then integrating by parts we obtain

$$-\int_{\mathbb{E}} \left\langle \nabla_{\mathbb{E}} X_1 u, \nabla_{\mathbb{E}}(X_1u\eta^2)\right\rangle_{\mathbb{E}} + 2\int_{\mathbb{E}} X_3X_2u(X_1u)\eta^2 + \int_{\mathbb{E}} X_4u(X_1u)\eta^2 = \\ = \int_{\mathbb{E}} \dot{f}(u)(X_1u)^2\eta^2$$

$$-\int_{\mathbb{E}} \left\langle \nabla_{\mathbb{E}} X_2 u, \nabla_{\mathbb{E}}(X_2u\eta^2)\right\rangle_{\mathbb{E}} - 2\int_{\mathbb{E}} X_1X_3u(X_2u)\eta^2 + \int_{\mathbb{E}} X_4u(X_2u)\eta^2 = \\ = \int_{\mathbb{E}} \dot{f}(u)(X_2u)^2\eta^2$$

Consequently, by summing term by term, we get

$$-\int_{\mathbb{E}} \Big(|\nabla_{\mathbb{E}} X_1u|^2 + |\nabla_{\mathbb{E}} X_2u|^2\Big)\eta^2 - \\ - \int_{\mathbb{E}} \left\langle \nabla_{\mathbb{E}} X_1u, \nabla_{\mathbb{E}}\eta^2\right\rangle_{\mathbb{E}} X_1u - \int_{\mathbb{E}} \left\langle \nabla_{\mathbb{E}} X_2u, \nabla_{\mathbb{E}}\eta^2\right\rangle_{\mathbb{E}} X_2u + \tag{5.29}$$
$$+ 2\int_{\mathbb{E}} (X_3X_2uX_1u - X_1X_3uX_2u)\eta^2 + \int_{\mathbb{E}} (X_4u)(X_1u + X_2u)\eta^2 = \\ = \int_{\mathbb{E}} \dot{f}(u)|\nabla_{\mathbb{E}} u|^2\eta^2 \tag{5.30}$$

On the other hand, since u is stable, by choosing $|\nabla_{\mathbb{E}} u|\eta$ as a test function in (5.3) we obtain

$$0 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}} u|\eta)|^2 + \int_{\mathbb{E}} \dot{f}(u)|\nabla_{\mathbb{E}} u|^2\eta^2 \tag{5.31}$$

By Corollary 1.2.14 we have that $\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)=0$ almost everywhere outside $\mathbb{E}_0$; hence making use of (5.9) we obtain from (5.31) that

$$0\le\int_{\mathbb{E}_0}\Big(\frac{\eta^2}{|\nabla_{\mathbb{E}}u|^2}|(Hu)^T\nabla_{\mathbb{E}}u|^2+2\eta\left\langle(Hu)^T\nabla_{\mathbb{E}}u,\nabla_{\mathbb{E}}\eta\right\rangle_{\mathbb{E}}+|\nabla_{\mathbb{E}}\eta|^2|\nabla_{\mathbb{E}}u|^2\Big)+ \\ +\int_{\mathbb{E}}\dot f(u)|\nabla_{\mathbb{E}}u|^2\eta^2$$

So, noticing that $2\eta\nabla_{\mathbb{E}}\eta=\nabla_{\mathbb{E}}\eta^2$, and using (5.29), after a simplification we obtain that

$$\int_{\mathbb{E}}|Hu|^2\eta^2-\int_{\mathbb{E}_0}\frac{\eta^2}{|\nabla_{\mathbb{E}}u|^2}|(Hu)^T\nabla_{\mathbb{E}}u|^2-2\int_{\mathbb{E}}(X_3X_2uX_1u-X_1X_3uX_2u)\eta^2- \\ -\int_{\mathbb{E}}(X_4u)(X_1u+X_2u)\eta^2\le\int_{\mathbb{E}_0}|\nabla_{\mathbb{E}}\eta|^2|\nabla_{\mathbb{E}}u|^2.$$

Recalling that

$$X_1X_3u=X_3X_1u+X_4u$$

we get the thesis. □

Then, from Proposition 5.2.5 and Lemma 5.2.2 we immediately obtain Theorem 5.1.1.

We end this section by giving some more geometric insight on the quantity $\mathcal{J}$, in relation with the intrinsic normal and tangent vectors:

Lemma 5.2.6. *For every $u\in C^1(\Omega)$ and every $x\in\mathbb{E}_0$ it holds*

$$\mathcal{J}(x)=-|\nabla_{\mathbb{E}}u|(x)\left\langle\nabla_{\mathbb{E}}X_3u(x),v(x)\right\rangle_{\mathbb{E}}-|\nabla_{\mathbb{E}}u|(x)^2\left\langle X_3\nu(x),v(x)\right\rangle_{\mathbb{E}}\tag{5.32}$$

Proof. By definition in $\mathbb{E}_0$

$$\langle\nabla_{\mathbb{E}}X_3u,v\rangle_{\mathbb{E}}=\frac{1}{|\nabla_{\mathbb{E}}u|}(X_1X_3uX_2u-X_2X_3uX_1u)\tag{5.33}$$

and using (1.28) we obtain

$$\langle\nabla_{\mathbb{E}}X_3u,v\rangle_{\mathbb{E}}=\frac{1}{|\nabla_{\mathbb{E}}u|}\Big[(X_3X_1uX_2u-X_3X_2uX_1u)+X_4u(X_2u-X_1u)\Big].\tag{5.34}$$

Moreover, in $\mathbb{E}_0$,

$$\langle X_3\nu,v\rangle_{\mathbb{E}}=\frac{1}{|\nabla_{\mathbb{E}}u|^2}\Big(X_3X_1uX_2u-X_3X_2uX_1u\Big)\tag{5.35}$$

hence adding (5.34) and (5.35) we get the thesis. □

Using Theorem 5.1.1 and Lemma 5.2.6 it immediately follows that

Corollary 5.2.7. *Let $u \in C^2(\Omega)$ be a stable weak solution of (5.1). Then, for each $\eta \in C_0^\infty(\Omega)$,*

$$\int_{\mathbb{E}_0} \Big(|\nabla_{\mathbb{E}} u|^2 \Big\{ \Big(p + \frac{\big\langle (Hu)^T \nu, v \big\rangle_{\mathbb{E}}}{|\nabla_{\mathbb{E}} u|} \Big)^2 + h^2 \Big\} + |\nabla_{\mathbb{E}} u| \left\langle \nabla_{\mathbb{E}} X_3 u, v \right\rangle_{\mathbb{E}} + \tag{5.36}$$
$$+ |\nabla_{\mathbb{E}} u|^2 \left\langle X_3 \nu, v \right\rangle_{\mathbb{E}} \Big) \eta^2 \le \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \eta|^2 |\nabla_{\mathbb{E}} u|^2$$

5.3 Some applications to entire stable solutions: geometric equations and non existence results

It is interesting to investigate whether or not rigidity results and geometric properties of stable solutions may be obtained from inequalties of the type proved in the previous section (or by other methods as well).
In this spirit we prove a first non-existence result for semilinear equations in the Engel group, see Theorem 5.3.3.

From now on, we will denote by

$$B(0,R) := \{x \in \mathbb{E} \mid \|x\| < R\}$$

the gauge open ball centered at 0 of radius R, where $\|\cdot\|$ is as in Section 1.3.

The following Lemma is proved in [71].

Lemma 5.3.1. *Let $g \in L^\infty_{loc}(\mathbb{R}^n, [0, +\infty))$ and let $q > 0$. Let also, for any $\tau > 0$,*

$$\eta(\tau) := \int_{B(0,\tau)} g(x) \mathrm{d}x \tag{5.37}$$

Then, for every $0 < r < R$,

$$\int_{B(0,R) \setminus B(0,r)} \frac{g(x)}{|x|^q} \, \mathrm{d}x \le q \int_r^R \frac{\eta(\tau)}{\tau^{q+1}} \, \mathrm{d}\tau + \frac{1}{R^q} \eta(R)$$

Corollary 5.3.2. *Let u be a stable solution of $\Delta_{\mathbb{E}} u = f(u)$ in the whole of $\mathbb{E}$ with*

$$\mathcal{J} \leq 0 \quad \textit{in } \mathbb{E}_0 \tag{5.38}$$

For any $\tau > 0$ and any $x = (x_1, x_2, x_3, x_4) \in \mathbb{E}$, let us define

$$\eta(\tau) := \int_{B(0,\tau)} |\nabla_{\mathbb{E}} u(x)|^2 \mathrm{d}x \tag{5.39}$$

If

$$\liminf_{R\to\infty} \frac{\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^3}\mathrm{d}\tau + \frac{\eta(R)}{R^2}}{(\log R)^2} = 0 \tag{5.40}$$

then, the level set of u in the proximity of noncharacteristic points are such that

$$\mathrm{div}_{\mathbb{E}}\, \nu = 0 \tag{5.41}$$

and on such sets the following equation holds

$$p = -\frac{1}{|\nabla_{\mathbb{E}} u|} \langle Huv, \nu \rangle_{\mathbb{E}} \tag{5.42}$$

Proof. This is a modification of the proof of Corollary 3.2 of [71], where we take into account the more complicated algebraic calculations of the Engel group. Given $R > 1$, we define

$$\phi_R(x) := \begin{cases} 1 & \text{if } x \in B(0, \sqrt{R}) \\ 2(\log R)^{-1} \log(R/|x|) & \text{if } x \in B(0, R) \setminus B(0, \sqrt{R}) \\ 0 & \text{if } x \in \mathbb{E} \setminus B(0, R) \end{cases}$$

We observe that

$$\begin{aligned} & X_1|x|^{12} = 12(x_1^2 + x_2^2)^5 x_1 - Ax_2 - 2x_3x_4^3 \\ \text{and} \qquad & X_2|x|^{12} = 12(x_1^2 + x_2^2)^5 x_2 + Ax_1 - 2x_3x_4^3 \end{aligned}$$

with $A := 3x_3^5 + (1/3)(x_1 + x_2)x_4^3$. Since $|x_1| \leq |x|$, $|x_2| \leq |x|$, $|x_3| \leq |x|^2$ and $|x_4| \leq |x|^3$, we conclude that $|A| \leq C_1|x|^{10}$ and so

$$|\nabla_{\mathbb{E}}|x|^{12}| \leq C_2|x|^{11}$$

for some for some C_1, $C_2 > 0$.

Notice also that, in $B(0,R) \setminus B(0,\sqrt{R})$,

$$\phi_R(\xi) = C(R) - (1/6)(\log R)^{-1} \log |\xi|^{12}$$

for some $C(R) \in \mathbb{R}$, thus

$$|\nabla_{\mathbb{E}} \phi_R(\xi)| = (1/6)(\log R)^{-1} |\xi|^{-12} |\nabla_{\mathbb{E}} |\xi|^{12}| \le C_3 (\log R)^{-1} |\xi|^{-1}$$

in $B(0,R) \setminus B(0,\sqrt{R})$, for some $C_3 > 0$. Therefore, by (5.38) and Theorem 5.1.1,

$$\begin{aligned} &\int_{\mathbb{E}_0} \left[|\nabla_{\mathbb{E}} u|^2 \left\{ \left(p + \frac{\langle (Hu)^T \nu, v \rangle}{|\nabla_{\mathbb{E}} u|} \right)^2 + h^2 \right\} \right] \phi_R^2 \\ &\le \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \phi_R|^2 |\nabla_{\mathbb{E}} u|^2 \le C_4 (\log R)^{-2} \int_{B(0,R)\setminus B(0,\sqrt{R})} \frac{|\nabla_{\mathbb{E}} u|^2}{|\xi|^2} \end{aligned}$$

for some $C_4 > 0$. On the other hand, by Lemma 5.3.1,

$$\int_{B(0,R)\setminus B(0,\sqrt{R})} \frac{|\nabla_{\mathbb{E}} u|^2}{|\xi|^2} \le 2 \int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3}\, \mathrm{d}\tau + \frac{1}{R^2} \eta(R)$$

All in all,

$$\begin{aligned} &\int_{\mathbb{E}_0} \left[|\nabla_{\mathbb{E}} u|^2 \left\{ \left(p + \frac{\langle (Hu)^T \nu, v \rangle}{|\nabla_{\mathbb{E}} u|} \right)^2 + h^2 \right\} \right] \phi_R^2 \\ &\qquad \le 2 C_4 (\log R)^{-2} \left[\int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3}\, \mathrm{d}\tau + \frac{1}{R^2} \eta(R) \right] \end{aligned}$$

Then the claim follows by sending $R \to \infty$, thanks to (5.40). □

Remark 5.2. Recalling Lemma 5.2.6, we observe that (5.38) is implied by the following monotonicity conditions:

$$\langle X_3 \nu, v \rangle_{\mathbb{E}} \ge 0 \quad \text{and} \quad \langle \nabla_{\mathbb{E}} X_3 u, v \rangle_{\mathbb{E}} \ge 0$$

Remark 5.3. Condition (5.40) may be seen as a bound on the energy growth: for instance, it is satisfied if $\eta(R)/R^2$ stays bounded for large R, i.e. if the energy in $B(0,R)$ does not grow more than R^2. Of course, this is quite a strong assumption on the decay of $\nabla_{\mathbb{E}} u$ in the variables (ξ_3, ξ_4) and it would be desirable to investigate in which way such condition may be weakened.

Remark 5.4. We stress that equations (5.41) and (5.42) may be seen as geometric equations along the level sets of the solution u. In particular, (5.41) may be stated as saying that the level set is a minimal surface for the Engel framework (in analogy with the Euclidean setting and in the terminology of [116]). Also, (5.42) is a prescription on the imaginary curvature p, in relation with the Hessian, the normal, and the tangent vectors.

Remark 5.5. Let us observe that if u is solution of (5.1.1) that do not depend on ξ_3 and ξ_4 then u satisfies $\Delta u = f(u)$, where Δ is the classical Euclidean Laplacian. Moreover, by [71, Remark 3.4], every bounded stable solution of (5.1.1) that do not depend on the last two coordinates and satisties (5.40) has to be constant.

Theorem 5.3.3. *There exists no $u \in C^3(\mathbb{E})$ stable solution of $\Delta_{\mathbb{E}} u = f(u)$ satisfying*

i. $f \in C^2(\mathbb{E})$ *and the zeros of* $\ddot{f}$ *(if any) are isolated;*

ii. $\{\xi \in \mathbb{E} \mid \nabla_{\mathbb{E}} u(\xi) = 0\} = \emptyset$*;*

iii. $u \in L^\infty(\mathbb{E})$*;*

iv. $\langle X_3 \nu, v \rangle_{\mathbb{E}} \geq 0$ *in* $\mathbb{E}$*;*

v. $\langle \nabla_{\mathbb{E}} X_3 u, v \rangle_{\mathbb{E}} \geq 0$ *in* $\mathbb{E}$*;*

vi. the set $\{(X_1 u + X_2 u) = 0\}$ *has zero Lebesgue measure;*

vii. $\liminf_{R \to \infty} \dfrac{\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^3} d\tau + \frac{\eta(R)}{R^2}}{(\log R)^2} = 0$*;*

where η is as in Corollary 5.3.2.

Proof. By contradiction: let $u \in C^3(\mathbb{E})$ be a stable solution of (5.1) and satisfying (*i*), (*ii*), (*iii*), (*iv*), (*v*), (*vi*) and (*vii*). By (*iii*) and [29, Th. 2.10] we have

$$|\nabla_{\mathbb{E}} u| \in L^\infty(\mathbb{E}) \tag{5.43}$$

We claim that

$$X_3 u = 0 \quad \text{in } \mathbb{E} \tag{5.44}$$

To this end, we argue by contradiction, supposing that there exists $Q \in \mathbb{E}$ such that

$$X_3 u(Q) \neq 0. \tag{5.45}$$

Thus we consider the following Cauchy problem

$$\begin{cases} \phi'(s) = v(\phi(s)) \\ \phi(0) = Q \end{cases}$$

where v is the intrinsic tangent direction. By (ii) and the fact that v is bounded it follows that the solution exists and it is defined for any $s \in \mathbb{R}$. Moreover,

$$u(\phi(s))' = \left\langle \nabla_{\mathbb{E}} u(\phi(s)), \phi'(s) \right\rangle_{\mathbb{E}} = |\nabla_{\mathbb{E}}(\phi(s))| \left\langle \nu(\phi(s)), v(\phi(s)) \right\rangle_{\mathbb{E}} = 0 \;\; \forall s \in \mathbb{R}$$

that is, ϕ lies on the level set of u, namely

$$\phi(s) \in \{\xi \in \mathbb{E} \mid u(\xi) = u(Q)\} \qquad \forall s \in \mathbb{R}.$$

Furthermore,

$$|\nabla_{\mathbb{E}} u(\phi(s))|' = \left\langle \nabla_{\mathbb{E}} |\nabla_{\mathbb{E}} u|(\phi(s)), \phi(s)' \right\rangle_{\mathbb{E}} \quad \forall s \in \mathbb{R}$$

and by (5.10) (applied here with $\eta \equiv 1$) and Corollary 5.3.2 (recall also Remark 5.2) we get

$$\begin{aligned} |\nabla_{\mathbb{E}} u(\phi(s))|' &= \frac{1}{|\nabla_{\mathbb{E}} u(\phi(s))|} \left\langle (Hu)^T \nabla_{\mathbb{E}} u(\phi(s)),\, v(\phi(s)) \right\rangle_{\mathbb{E}} = \\ &= \langle \nu(\phi(s)),\, (Hu) v(\phi(s)) \rangle_{\mathbb{E}} = -|\nabla_{\mathbb{E}} u(\phi(s))|\, p(\phi(s) = \\ &= X_3 u(\phi(s)) \;\; \forall s \in \mathbb{R} \end{aligned} \tag{5.46}$$

which, via (5.45), implies

$$|\nabla_{\mathbb{E}} u(\phi(s))|'_{|s=0} \neq 0 \;\; \forall s \in \mathbb{R} \tag{5.47}$$

From (5.46) we deduce

$$|\nabla_{\mathbb{E}} u(\phi(s))|^{''} = (X_3 u(\phi(s)))^{'} = \left\langle \nabla_{\mathbb{E}} X_3 u(\phi(s)), \phi^{'}(s) \right\rangle_{\mathbb{E}} = \\ = \langle \nabla_{\mathbb{E}} X_3 u(\phi(s)), v(\phi(s)) \rangle_{\mathbb{E}} \quad \forall s \in \mathbb{R} \tag{5.48}$$

and by (iv) we deduce also that

$$|\nabla_{\mathbb{E}} u(\phi(s))|^{''} \geq 0 \quad \forall s \in \mathbb{R} \tag{5.49}$$

Therefore, defining $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\Phi(s) := |\nabla_{\mathbb{E}} u(\phi(s))| - |\nabla_{\mathbb{E}} u(Q)|$$

we have that $\Phi \in C^2(\mathbb{R})$, $\Phi(0) = 0$, $\Phi^{'}(s) \neq 0$ $\forall s \in \mathbb{R}$ and $\Phi(s)^{''} \geq 0$ $\forall s \in \mathbb{R}$, thanks to (5.47) and (5.49). It follows that

$$\sup_{\mathbb{R}} \Phi = +\infty$$

but this is in contradiction with (5.43), hence (5.44) is established.
Now we claim that

$$X_4 u = 0 \tag{5.50}$$

By Lemma 5.2.3, we obtain

$$\Delta_{\mathbb{E}} X_3 u - 2X_4 X_1 u - 2X_4 X_2 u = X_3 \Delta_{\mathbb{E}} u = \\ = X_3(f(u)) = \dot{f}(u) X_3 u$$

and so by (5.44) it follows that

$$X_4(X_1 u + X_2 u) = 0 \tag{5.51}$$

Moreover, by Corollary 5.2.4 and (5.44),

$$\Delta_{\mathbb{E}} X_1 u - X_4 u = \Delta_{\mathbb{E}} X_1 u - 2X_4 u + X_4 u = \\ = \Delta_{\mathbb{E}} X_1 u - 2(X_2 X_3 - X_3 X_2) u + X_4 u = \Delta_{\mathbb{E}} X_1 u + 2X_3 X_2 u + X_4 u = \\ = \dot{f}(u) X_1 u \tag{5.52}$$

and

$$\begin{aligned}\Delta_{\mathbb{E}} X_2 u + X_4 u &= \Delta_{\mathbb{E}} X_2 u - 2X_1 X_3 u + X_4 u = \\ &= \dot{f}(u) X_2 u\end{aligned} \tag{5.53}$$

By adding (5.52) and (5.53) we obtain

$$\Delta_{\mathbb{E}}(X_1 u + X_2 u) = \dot{f}(u)(X_1 u + X_2 u) \tag{5.54}$$

and so, by Lemma 5.2.3,

$$\begin{aligned}\Delta_{\mathbb{E}} X_4(X_1 u + X_2 u) &= X_4 \Delta_{\mathbb{E}}(X_1 u + X_2 u) = \\ &= X_4\big(\dot{f}(u)(X_1 u + X_2 u)\big) = \\ &= \ddot{f}(u) X_4 u (X_1 u + X_2 u) + \dot{f}(u) X_4 (X_1 u + X_2 u)\end{aligned} \tag{5.55}$$

Accordingly, using (5.55) and (5.51), we conclude that

$$\ddot{f}(u) X_4 u (X_1 u + X_2 u) = 0 \ \text{ in } \ \mathbb{E}$$

Hence, by (vi)

$$\ddot{f}(u) X_4 u = 0 \ \text{ almost everywhere in } \ \mathbb{E}$$

and so, by continuity,

$$\ddot{f}(u) X_4 u = 0 \ \text{ everywhere in } \ \mathbb{E}$$

This implies that (5.50) holds at any point of the open set $G := \{\xi \in \mathbb{E} \mid \ddot{f}(u(\xi)) \neq 0\}$. So, by continuity, (5.50) holds at any point of its closure $\overline{G}$.

We show that (5.50) also holds at points of $\mathbb{E} \setminus \overline{G}$ (if any). For this, let us take $\xi_o \in \mathbb{E} \setminus \overline{G}$. Since the latter is an open set, there exists an open neighborhood V such that

$$\xi_o \in V \subseteq (\mathbb{E} \setminus \overline{G}) \subseteq \mathbb{E} \setminus G = \{\xi \in \mathbb{E} \mid \ddot{f}(u(\xi)) = 0\}.$$

In particular, $\ddot{f}(u(\xi)) = 0$ for any $\xi \in V$. Thus, by (i), $u(\xi)$ must be constant for any $\xi \in V$. Therefore, $X_4 u(\xi) = 0$ for any $\xi \in V$, and, in particular, $X_4 u(\xi_o) = 0$.

This shows that (5.50) holds at points of $\mathbb{E} \setminus \overline{G}$ too, and so the proof of (5.50) is completed.

Now, by (5.44) and (5.50), we conclude that u does not depend on ξ_3 and ξ_4 and by Remark 5.5 we conclude that u is constant but this is impossible by (ii), which proves Theorem 5.3.3. $\square$

Remark 5.6. Of course, we do not believe that our Theorem 5.3.3 is optimal: we just consider it a first attempt towards the understanding of semilinear equations in the Engel framework and, as far as we know, this is the first non-existence result in this setting. We think it would be interesting to develop a stronger theory and possibly to drop some structural assumptions in Theorem 5.3.3.

Chapter 6

A Lewy-Stampacchia Estimate for quasilinear variational inequalities in the Heisenberg group

In this chapter, we extend the so called Dual Estimate of [99] to the obstacle problem for quasilinear elliptic equations in the Heisenberg group.

6.1 An introduction to the problem and some basic tools

For the notations and the definitions we refer to Chapter 1. Throughout this chapter we denote by $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ a point in $\mathbb{H}^n$. We are interested in studying the obstacle problem in this framework. For this, we consider a smooth function $\psi : \mathbb{H}^n \to \mathbb{R}$, which will be our obstacle (more precisely, ψ is supposed to have continuous derivatives of second order in X and Y).

Fixed a bounded open set Ω with smooth boundary, and $p \in (1, +\infty)$, we consider the space $W^{1,p}_{\mathbb{H}^n}(\Omega)$ to be the set of all functions u in $L^p(\Omega)$ whose distributional horizontal derivatives $X_j u$ and $Y_j u$ belong to $L^p(\Omega)$, for $j = 1, \dots, n$.

Such space is naturally endowed with the norm

$$\|u\|_{W^{1,p}_{\mathbb{H}^n}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{j=1}^{n} \Big(\|X_j u\|_{L^p(\Omega)} + \|Y_j u\|_{L^p(\Omega)} \Big).$$

We call $W^{1,p}_{\mathbb{H}^n,0}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to this norm.

We fix a smooth domain $\Omega_\star \Supset \Omega$, $u_\star \in W^{1,p}_{\mathbb{H}^n}(\Omega_\star) \cap L^\infty(\Omega_\star)$ and we introduce the space

$$\mathcal{K} := \Big\{ u \in W^{1,p}_{\mathbb{H}^n}(\Omega) \text{ s.t. } u \le \psi, \text{ and } u - u_\star \in W^{1,p}_{\mathbb{H}^n,0}(\Omega) \Big\}.$$

Loosely speaking, $\mathcal{K}$ is the space of all the functions having prescribed Dirichlet boundary datum equal to $u_\star$ along $\partial\Omega$ and that stay below the obstacle ψ.

Now we consider a parameter $\varepsilon \ge 0$ and we deal with the variational problem

$$\inf_{u \in \mathcal{K}} \mathcal{F}_\varepsilon(u;\Omega), \text{ where } \mathcal{F}_\varepsilon(u;\Omega) := \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{p/2}. \tag{6.1}$$

By direct methods, it is seen that such infimum is attained (see, e.g., the compactness result in [135, 52] or references therein) and so we consider a minimizer $\bar{u}_\varepsilon$.

Then, $\bar{u}_\varepsilon$ is a solution of the variational inequality[1]

$$\int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n}(v - \bar{u}_\varepsilon) \ \ge\ 0, \tag{6.2}$$

[1]Formula (6.2) may be easily obtained this way. Fixed $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$ with $v \le \psi$, and $v - \bar{u}_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, for any $t \ge 0$, let $u^{(t)} := \bar{u}_\varepsilon + t(v - \bar{u}_\varepsilon)$. Notice that

$$u^{(t)} := (1-t)\bar{u}_\varepsilon + tv \le (1-t)\psi + t\psi \le \psi,$$

hence $u^{(t)} \in \mathcal{K}$. So, by the minimality of $\bar{u}_\varepsilon$, we have $\mathcal{F}_\varepsilon(u^{(0)};\Omega) = \mathcal{F}_\varepsilon(\bar{u}_\varepsilon;\Omega) \le \mathcal{F}_\varepsilon(u^{(t)};\Omega)$ for any $t \ge 0$. Consequently,

$$\begin{aligned} 0 \ &\le\ \lim_{t \searrow 0} \frac{\mathcal{F}_\varepsilon(u^{(t)};\Omega) - \mathcal{F}_\varepsilon(u^{(0)};\Omega)}{t} \\ &=\ \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n}(v - \bar{u}_\varepsilon), \end{aligned}$$

that is (6.2).

for any $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$ with $v \le \psi$, and $v - \bar{u}_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$.

These kind of variational inequalities has now receiving a considerable attention (see, e.g., [56] and references therein), even when $p = 2$ (notice that in such a case ε does not play any role). We observe that, when $p \neq 2$, the operator driving the variational inequality in (6.2) is not linear anymore (in fact, it may be seen as the Heisenberg group version of the p-Laplace operator): for these kind of operators even the regularity theory is more problematic than expected at a first glance, and many basic fundamental questions are still open (see, e.g., [62], [105], [108] and [139]): this is a crucial difference with respect to the Euclidean case, so we think it is worth dealing with the problem in such a generality.

Now, we introduce the set of p's for which our main result holds. The definition we give is slightly technical, but, roughly speaking, consists in taking the set of all the p's for which a pointwise bound for the operator of a sequence of minimal solutions is stable under uniform limit. The further difficulty of taking this assumption is due to the lack of a thoroughgoing regularity theory for the quasilinear Heisenberg equation (as remarked in Lemma 6.5.7 at the end of this chapter, this technicality may be skipped when $p = 2$).

Definition 6.1.1. *Let $p \in (1, +\infty)$. We say that $p \in \mathcal{P}(\psi, \Omega)$ if the following property holds true:*

For any $\varepsilon > 0$, any $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, any $M > 0$, any sequence $F_k = F_k(r, \xi) \in C([-M, M] \times \Omega)$, with $F_k(\cdot, \xi) \in C^1([-M, M])$ and

$$0 \le \partial_r F_k \le \left(\mathrm{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}\psi\right)\right)^+, \tag{6.3}$$

if $u_k : \Omega \to [-M, M]$ is a sequence of minimizers of the functional

$$\int_\Omega \frac{1}{p}(\varepsilon + |\nabla_{\mathbb{H}^n} u(\xi)|^2)^{p/2} + F_k(u(\xi), \xi)\, d\xi \tag{6.4}$$

over the functions $u \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, $u - v \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, with the property that u_k converges to some u_∞ uniformly in Ω, we have that

$$\begin{aligned} 0 &\le \mathrm{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} u_\infty|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\infty\right) \\ &\le \left(\mathrm{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}\psi\right)\right)^+ \end{aligned} \tag{6.5}$$

in the sense of distributions.

As remarked[2] in Lemma 6.5.7 at the end of this chapter, we always have that

$$2 \in \mathcal{P}(\psi, \Omega).$$

In particular, our main result (i.e., the forthcoming Theorem 6.1.1) always holds for $p = 2$ without any further restriction. We think that it is an interesting open problem to decide whether or not other values of p belong to $\mathcal{P}(\psi, \Omega)$, in general, or at least when the right hand side of (6.5) is close to zero (e.g., when the obstacle is almost flat). For instance, the property in Definition 6.1.1 would be satisfied in presence of a Hölder regularity theory for the horizontal gradient for solutions of quasilinear equations in the Heisenberg group – namely, if one knew that bounded solutions of

$$\mathrm{div}_{\mathbb{H}^n}\left((\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} u\right) = f,$$

with f bounded, have Hölder continuous horizontal gradient, with interior estimates (this would be the Heisenberg counterpart of classical regularity results for the Euclidean case, see, e.g., Theorem 1 in [133]); notice also that this would be a regularity theory for the equation, not for the obstacle problem. As far as we know, such a theory has not been developed yet, not even for minimal solutions (see, however, [28, 108, 139] for the case of homogeneous equations). Nevertheless, we think it is worth stating our result in the more abstract setting of $\mathcal{P}(\psi, \Omega)$, because, once the regularity theory becomes available, our result would be valid in general – and also because the setting we use is somewhat more general and weaker than the regularity theory itself.

The result we prove is:

Theorem 6.1.1. *If* $p \in \mathcal{P}(\psi, \Omega)$ *then*

$$\begin{aligned} 0 &\le \mathrm{div}_{\mathbb{H}^n}\left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon\right) \\ &\le \left(\mathrm{div}_{\mathbb{H}^n}\left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} \psi\right)\right)^+ \end{aligned} \tag{6.6}$$

[2] As usual, the superscript "+" denotes the positive part of a function, i.e. $f^+(x) := \max\{f(x), 0\}$.

in the sense of distributions.

The result in Theorem 6.1.1 is quite intuitive: when $\bar{u}_\varepsilon$ does not touch the obstacle, it is free to make the operator vanish. When it touches and sticks to it, the operator is driven by the one of the obstacle – and on these touching points the obstacle has to bend in a somewhat convex fashion, which justifies the first inequality in (6.6) and superscript "+" in the right hand side of (6.6).

Figure 1, in which the thick curve represents the touching between $\bar{u}_\varepsilon$ and the obstacle, tries to describe this phenomena. On the other hand, the set in which $\bar{u}_\varepsilon$ touches the obstacle may be very wild, so the actual proof of Theorem 6.1.1 needs to me more technical than this.

In fact, the first inequality of (6.6) is quite obvious since it follows, for instance, by taking $v := \bar{u}_\varepsilon - \varphi$ in (6.2), with an arbitrary $\varphi \in C_0^\infty(\Omega, [0, +\infty)))$, so the core of (6.6) lies on the second inequality: nevertheless, we think it is useful to write (6.6) in this way to emphasize a control from both the sides of the operator applied to the solution.

We remark that the right hand side of (6.6) is always finite when $\varepsilon > 0$, and when $\varepsilon = 0$ and $p \geq 2$. In this case, (6.6) is an L^∞-bound and may be seen as a regularity result for the solution of the obstacle problem. It is worth noticing that such regularity result holds for $\varepsilon = 0$ as well, only assuming that $p \in \mathcal{P}(\psi, \Omega)$, which is a requirement on the problem when $\varepsilon > 0$.

On the other hand, if $\varepsilon = 0$ and $p \in (1, 2)$, the right hand side of (6.6) may become infinite (in this case (6.6) is true, but maybe meaningless, stating that something is less than or equal to an infinite quantity).

In the Euclidean setting, the analogue of (6.6) was first obtained in [99] for the Laplacian case, and it is therefore often referred to with the name of Lewy-Stampacchia Estimate. It is also called Dual Estimate, for it is, in a sense, obtained by the duality expressed by the variational inequality (6.2). Other Authors refer to it with the name of Penalization Method, for the role played by ε.

After [99], estimates of these type became very popular and underwent many important extensions and strengthenings: see, among the others, [119,

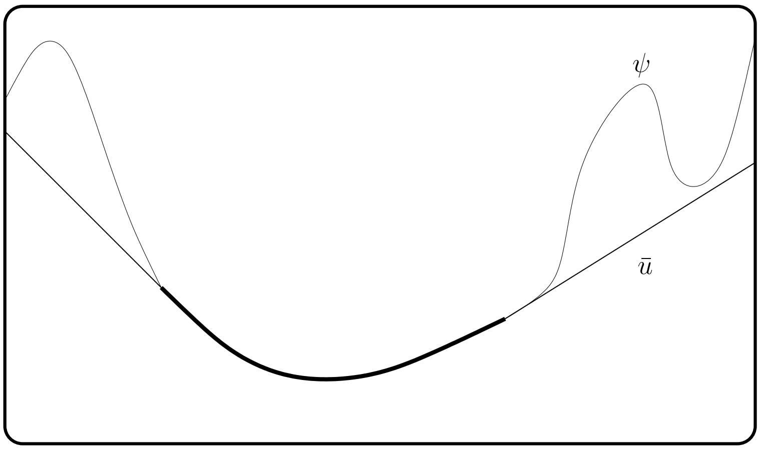

Figure 6.1: *Touching the obstacle*

83, 64, 18, 110]. As far as we know, the estimate we prove is new in the Heisenberg group setting, even for $p = 2$.

Hereafter, we deal with the proof of Theorem 6.1.1. First, in § 6.2, we prove Theorem 6.1.1 when $\varepsilon > 0$.

The proof when $\varepsilon = 0$ is contained in § 6.4 and it is based on a limit argument, i.e., we consider the problem with $\varepsilon > 0$, we use Theorem 6.1.1 in such a case, and then we pass $\varepsilon \searrow 0$. This procedure is quite delicate though, because, as far as we know, it is not clear whether or not the Heisenberg group setting allows a complete Hölder regularity theory for first derivatives (see [62]). To get around this point, in § 6.3, we study the L^p-convergence of the solution $\bar{u}_\varepsilon$ of the ε-problem to the solution $\bar{u}_0$ of the problem with $\varepsilon = 0$, which, we believe, is of independent interest (see, in particular Propositions 6.3.1 and 6.3.2).
We point out that the same arguments hold verbatim for nilpotent stratified Lie groups of any steps and we work in $\mathbb{H}^n$ only for the sake of notational simplicity.

6.2 Lewy-Stampacchia estimate when $\varepsilon > 0$

We prove (6.6) in the simpler case $\varepsilon > 0$ (the case $\varepsilon = 0$ will be dealt with in § 6.4). The technique used in this proof is a variation of a classical penalized test function method (see, e.g., [119, 83, 64, 18, 110] and references therein), and several steps of this proof are inspired by some estimates obtained by [31] in the Euclidean case.

First of all, we set

$$\mu := -1 + \min\left\{ \inf_{\overline{\Omega}} \psi,\ \inf_{\overline{\Omega}} u_\star \right\} \in \mathbb{R}$$

and we observe that

$$\bar{u}_\varepsilon \geq \mu \tag{6.7}$$

a.e. in Ω. Indeed, let $w := \max\{\bar{u}_\varepsilon, \mu\}$. Since ψ and $u_\star$ are below μ in Ω, we have that $w \in \mathcal{K}$, thus

$$0 \leq \mathcal{F}_\varepsilon(w;\Omega) - \mathcal{F}_\varepsilon(\bar{u}_\varepsilon;\Omega) = -\int_{\Omega\cap\{\bar{u}_\varepsilon<\mu\}} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2} \leq 0,$$

and, from this, (6.7) plainly follows.

Now, let $\eta \in (0,1)$, to be taken arbitrarily small in the sequel. Let also

$$h := \left(\operatorname{div}_{\mathbb{H}^n}\left((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\right)\right)^+. \tag{6.8}$$

Notice that

$$\|h\|_{L^\infty(\Omega)} < +\infty, \tag{6.9}$$

because $\varepsilon > 0$. For any $t \in \mathbb{R}$, we consider the truncation function

$$H_\eta(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t/\eta & \text{if } 0 < t < \eta, \\ 1 & \text{if } t \geq \eta. \end{cases}$$

Now, we take u_η to be a weak solution of

$$\begin{cases} \operatorname{div}_{\mathbb{H}^n}\left((\varepsilon + |\nabla_{\mathbb{H}^n}u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}u_\eta\right) = h \cdot \left(1 - H_\eta(\psi - u_\eta)\right) & \text{in } \Omega, \\ u_\eta = \bar{u}_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{6.10}$$

where, as usual, the boundary datum is attained in the trace sense: such a function u_η may be obtained by the direct method in the calculus of variations, by minimizing the functional

$$\int_\Omega \frac{1}{p}(\varepsilon + |\nabla_{\mathbb{H}^n} u(\xi)|^2)^{p/2} + F_\eta(u(\xi),\xi)\, d\xi$$

over $u \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, $u - \bar{u}_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, where

$$F_\eta(r,\xi) := \int_0^r h(\xi)\cdot\big(1 - H_\eta(\psi(\xi)-\sigma)\big)\, d\sigma.$$

Now, we claim that

$$u_\eta \le \psi \text{ a.e. in } \Omega. \tag{6.11}$$

To establish this, we use the test function $(u_\eta-\psi)^+$ in (6.10). Since, on $\partial\Omega$, we have $(u_\eta-\psi)^+ = (\bar{u}_\varepsilon-\psi)^+ = 0$, we obtain that

$$\begin{aligned} & -\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} u_\eta\Big)\cdot \nabla_{\mathbb{H}^n}(u_\eta-\psi)^+ \\ = & \int_\Omega h\cdot\big(1 - H_\eta(\psi - u_\eta)\big)(u_\eta-\psi)^+ = \int_\Omega h\cdot(u_\eta-\psi)^+. \end{aligned}$$

Consequently, by (6.8),

$$\begin{aligned} & \int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} u_\eta\Big) \\ & \qquad -\Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} \psi\Big)\Big]\cdot \nabla_{\mathbb{H}^n}(u_\eta-\psi)^+ \\ = & \int_\Omega \Big[\mathrm{div}_{\mathbb{H}^n}\ \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} \psi\Big) - h\Big]\cdot(u_\eta-\psi)^+ \\ \le & \ 0. \end{aligned}$$

By the strict monotonicity of the operator (i.e., by the strict convexity of the function $\mathbb{R}^{2n} \ni \zeta \mapsto (\varepsilon + |\zeta|^2)^{p/2}$), it follows that $(u_\eta-\psi)^+$ vanishes almost everywhere in Ω, proving (6.11).

Now, we claim that

$$\bar{u}_\varepsilon \ge u_\eta \text{ a.e. in } \Omega. \tag{6.12}$$

To verify this, we consider the test function

$$\tau := \bar{u}_\varepsilon + (u_\eta - \bar{u}_\varepsilon)^+.$$

We notice that

$$\tau = \begin{cases} \bar{u}_\varepsilon & \text{in } \{u_\eta \le \bar{u}_\varepsilon\}, \\ u_\eta & \text{in } \{u_\eta > \bar{u}_\varepsilon\}, \end{cases}$$

hence $\tau \le \psi$, due to (6.11). Furthermore, on $\partial\Omega$, we have that $\tau = \bar{u}_\varepsilon$, due to the boundary datum in (6.10). Therefore the obstacle problem variational inequality (6.2) gives that

$$\begin{aligned} 0 &\le \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\tau - \bar{u}_\varepsilon) \\ &= \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar{u}_\varepsilon)^+. \end{aligned} \tag{6.13}$$

On the other hand, from (6.10),

$$\begin{aligned} &\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar{u}_\varepsilon)^+ \\ &\qquad = - \int_\Omega h \cdot \big(1 - H_\eta(\psi - u_\eta)\big) \cdot (u_\eta - \bar{u}_\varepsilon)^+ \le 0. \end{aligned} \tag{6.14}$$

By (6.13) and (6.14), we obtain that

$$\begin{aligned} &\int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \\ &\qquad - \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \Big] \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar{u}_\varepsilon)^+ \le 0. \end{aligned}$$

This and the strict monotonicity of the operator implies that $(u_\eta - \bar{u}_\varepsilon)^+$ vanishes almost everywhere in Ω, hence proving (6.12).

Now, we claim that

$$\bar{u}_\varepsilon \le u_\eta + \eta \text{ in } \Omega. \tag{6.15}$$

To do this, we set

$$\theta := \bar{u}_\varepsilon - (\bar{u}_\varepsilon - u_\eta - \eta)^+.$$

Notice that $\theta \le \bar{u}_\varepsilon \le \psi$, and also that, on $\partial\Omega$, $\theta = \bar{u}_\varepsilon$. As a consequence, (6.2) gives that

$$\begin{aligned} 0 &\le \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\theta - \bar{u}_\varepsilon) \\ &= - \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\bar{u}_\varepsilon - u_\eta - \eta)^+. \end{aligned} \tag{6.16}$$

On the other hand, $(\bar{u}_\varepsilon - u_\eta - \eta)^+ = 0$ on $\partial\Omega$, and

$$\begin{aligned}\{\bar{u}_\varepsilon - u_\eta - \eta > 0\} &\subseteq \{\psi - u_\eta > \eta\}\\ &\subseteq \big\{1 - H_\eta(\psi - u_\eta) = 0\big\},\end{aligned}$$

and therefore, by (6.10),

$$\begin{aligned}&\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n}(u_\eta + \eta)|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}(u_\eta + \eta)\Big)\cdot \nabla_{\mathbb{H}^n}(\bar{u}_\varepsilon - u_\eta - \eta)^+\\ &= \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n}u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}u_\eta\Big)\cdot \nabla_{\mathbb{H}^n}(\bar{u}_\varepsilon - u_\eta - \eta)^+ \\ &= -\int_\Omega h\cdot \big(1 - H_\eta(\psi - u_\eta)\big)\cdot(\bar{u}_\varepsilon - u_\eta - \eta)^+ = 0.\end{aligned} \tag{6.17}$$

Then, (6.16) and (6.17) yield that

$$\begin{aligned}&\int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\Big)\\ &\qquad - \Big((\varepsilon + |\nabla_{\mathbb{H}^n}(u_\eta + \eta)|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}(u_\eta + \eta)\Big)\Big]\cdot \nabla_{\mathbb{H}^n}(\bar{u}_\varepsilon - u_\eta - \eta)^+\\ &\qquad \le 0.\end{aligned}$$

Thus, in this case, the strict monotonicity of the operator implies that $(\bar{u}_\varepsilon - u_\eta - \eta)^+$ vanishes almost everywhere in Ω, and so (6.15) is established.

In particular, by (6.11), (6.15) and (6.7),

$$\|u_\eta\|_{L^\infty(\Omega)} \le 2 + \|\psi\|_{L^\infty(\Omega)} + \|u_\star\|_{L^\infty(\Omega)}. \tag{6.18}$$

Moreover, by (6.12) and (6.15), we have that

$$u_\eta \text{ converges uniformly in } \Omega \text{ to } \bar{u}_\varepsilon \tag{6.19}$$

as $\eta \searrow 0$.

Furthermore

$$0 \le \partial_r F_\eta(r,\xi) \le h(\xi) = \Big(\mathrm{div}_{\mathbb{H}^n}\ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)\Big)^+$$

hence (6.6) follows[3] from (6.19) and the fact that $p \in \mathcal{P}(\psi,\Omega)$ (recall (6.5) in Definition 6.1.1). □

[3]It is worth pointing out that this is the only place in the chapter where we use the condition that $p \in \mathcal{P}(\psi,\Omega)$. In particular, all the estimates in § 6.3 are valid for all $p \in (1,+\infty)$.

6.3 Estimating the L^p-distance between $\nabla_{\mathbb{H}^n}\bar{u}_0$ and $\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon$

The purpose of this section is to consider the solution $\bar{u}_\varepsilon$ of the ε-problem and the solution $\bar{u}_0$ of the problem with $\varepsilon = 0$, and to bound the L^p-norm of $|\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|$. Such estimate is quite technical and it is different according to the cases $p \in (1, 2]$ and $p \in [2, +\infty)$: see the forthcoming Propositions 6.3.1 and 6.3.2.

As a matter of fact, we think that the estimates proved in Propositions 6.3.1 and 6.3.2 are of independent interest, since they also allow to get around the more difficult (and in general not available in the Heisenberg group) Hölder-type estimates.

For all $g \in L^1(U_r)$, we define the average of g in $\mathcal{B}_r$ as

$$(g)_r := \frac{1}{\mathcal{L}(U_r)} \int_{U_r} g,$$

where B_r is the ball centered at $0 \in \mathbb{H}^n$ with radius $r > 0$ with respect to the norm definend in 1.27. In what follows, we focus on L^p-estimates around a fixed point, say $\xi_\star$, of Ω. Without loss of generality, we take $\xi_\star$ to be the origin, and we fix $R \in (0, 1)$ so small that $U_R \Subset \Omega$.

Then, we denote by $\bar{u}_0 : \Omega \to \mathbb{R}$ the minimizer of problem (6.1) with $\varepsilon = 0$. Then, for a fixed $\varepsilon > 0$, we take $\bar{u}_\varepsilon : U_R \to \mathbb{R}$ to be the minimizer of $\mathcal{F}_\varepsilon(u; U_R)$ among all the functions $u \in W^{1,p}_{\mathbb{H}^n}(U_R)$, $u \leq \psi$, and $u - \bar{u}_0 \in W^{1,p}_{\mathbb{H}^n,0}(U_R)$. We can then extend $\bar{u}_\varepsilon$ also on $\Omega \setminus U_R$ by setting it equal to $\bar{u}_0$ in such a set. By construction

$$\begin{aligned} \int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0|^p &= \mathcal{F}_0(\bar{u}_0; \Omega) - \int_{\Omega\setminus U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0|^p \\ &\leq \mathcal{F}_0(\bar{u}_\varepsilon; \Omega) - \int_{\Omega\setminus U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0|^p = \int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2} &= \mathcal{F}_\varepsilon(\bar{u}_\varepsilon; U_R) \\ &\leq \mathcal{F}_\varepsilon(\bar{u}_0; U_R) = \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{p/2}. \end{aligned} \tag{6.21}$$

Proposition 6.3.1. *Assume that*

$$p \in (1, 2]. \tag{6.22}$$

Then, there exists $C > 0$, only depending on n and p, such that

$$\int_{U_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \leq C\big(1 + (|\nabla_{\mathbb{H}^n} \bar{u}_0|^p)_R\big)^{1-(p/2)} \varepsilon^{(p/2)^2} R^Q. \tag{6.23}$$

Proof. The technique for this proof is inspired by the one of Lemma 2.3 of [120], where a similar result was obtained in the quasilinear Euclidean case (however, our proof is self-contained). We have

$$\begin{aligned} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 &\leq \big(|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon| + |\nabla_{\mathbb{H}^n} \bar{u}_0|\big)^2 \\ &\leq C\big(|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2\big). \end{aligned} \tag{6.24}$$

Here, C is a positive constant, which is free to be different from line to line. By (6.22), (6.21) and (6.24), we obtain

$$\begin{aligned} &\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 \\ &\leq C \int_{U_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \\ &= C \left(\int_{U_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \right. \\ &\qquad \left. + \int_{U_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \right) \\ &\leq C \left(\int_{U_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} + \int_{U_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{1-(p/2)}} \right) \\ &\leq C \left(\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} + \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2} \right) \\ &\leq C \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2}. \end{aligned} \tag{6.25}$$

Thus, (6.25) and Lemma 6.5.4, applied here with $a := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $b := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$, yield that

$$\begin{aligned}
&\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2}\\
&\qquad\le C\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon-\nabla_{\mathbb{H}^n}\bar{u}_0|^2+\\
&\qquad\qquad+C\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)}\\
&\qquad\le C\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)}.
\end{aligned}\tag{6.26}$$

Now, from (6.20),

$$\begin{aligned}
&\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)}-\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)}\\
&\qquad\le\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)}-\int_{U_R}|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p\\
&\qquad\le\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)}-\int_{U_R}|\nabla_{\mathbb{H}^n}\bar{u}_0|^p.
\end{aligned}\tag{6.27}$$

Moreover, using (6.22) and some elementary calculus, we see that

$$|(1+\tau)^{p/2}-\tau^{p/2}|\le C$$

for any $\tau\ge 0$. Therefore, taking $\tau:=\theta/\varepsilon$, we obtain that

$$|(\varepsilon+\theta)^{p/2}-\theta^{p/2}|\le C\varepsilon^{p/2}\tag{6.28}$$

for any $\theta\ge 0$. Thus, using (6.27) and (6.28) with $\theta:=|\nabla_{\mathbb{H}^n}\bar{u}_0|^2$, we conclude that

$$\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_0|^2)^{(p/2)}-\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)}\le C\varepsilon^{p/2}R^Q.\tag{6.29}$$

Now, we estimate the left hand side of (6.29) from below. For this scope, we define

$$\begin{aligned}
&h:=t\nabla_{\mathbb{H}^n}\bar{u}_0+(1-t)\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon,\\
&J:=p\int_{U_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0-\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon)\\
\text{and}\quad&\tilde{J}:=p\int_{U_R}\Big[\int_0^1(1-t)\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\cdot(\nabla_{\mathbb{H}^n}\bar{u}_0-\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon)\Big)\,dt\Big].
\end{aligned}$$

We observe that the variational inequality in (6.2) for $\bar u_\varepsilon$ gives that

$$J \geq 0. \tag{6.30}$$

Also, using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}
&\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_0|^2)^{(p/2)} - \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)} \\
&\quad = \int_{U_R} \Big[\int_0^1 \frac{d}{dt} (\varepsilon + |t\nabla_{\mathbb{H}^n} \bar u_0 + (1-t)\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)}\, dt \Big] \\
&\quad = p \int_{U_R} \Big[\int_0^1 (\varepsilon + |t\nabla_{\mathbb{H}^n} \bar u_0 + (1-t)\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \\
&\qquad\quad \times (t\nabla_{\mathbb{H}^n} \bar u_0 + (1-t)\nabla_{\mathbb{H}^n} \bar u_\varepsilon) \cdot (\nabla_{\mathbb{H}^n} \bar u_0 - \nabla_{\mathbb{H}^n} \bar u_\varepsilon)\, dt \Big] \\
&\quad = p \int_{U_R} \Big[\int_0^1 (\varepsilon + |h|^2)^{(p/2)-1} h \cdot (\nabla_{\mathbb{H}^n} \bar u_0 - \nabla_{\mathbb{H}^n} \bar u_\varepsilon)\, dt \Big].
\end{aligned}$$

Integrating by parts the latter integral in t (by writing $dt = \frac{d}{dt}(t-1)\,dt$), and exploiting (6.30), we obtain

$$\begin{aligned}
&\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_0|^2)^{(p/2)} - \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)} \\
&\quad = J + \tilde J \geq \tilde J.
\end{aligned} \tag{6.31}$$

Making use of Lemma 6.5.3 – applied here with $a := \nabla_{\mathbb{H}^n} \bar u_0$ and $b := \nabla_{\mathbb{H}^n} \bar u_\varepsilon$ – we have that

$$\tilde J \geq \frac{1}{C} \int_{U_R} \Big[\int_0^1 (1-t)(\varepsilon + |t\nabla_{\mathbb{H}^n} \bar u_0 + (1-t)\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar u_0 - \nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2\, dt \Big].$$

From this and Lemma 6.5.5 – applied here with $\kappa := 1$ and $\Psi(x) := x^{1-(p/2)}$, which is nondecreasing, thanks to (6.22) – we deduce that

$$\tilde J \geq \frac{1}{C} \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_0|^2 + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar u_0 - \nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2. \tag{6.32}$$

In view of (6.29), (6.31) and (6.32), we conclude that

$$\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_0|^2 + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar u_0 - \nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2 \leq C\varepsilon^{p/2} R^Q. \tag{6.33}$$

Then, (6.23) follows from (6.26), (6.33) and Lemma 6.5.6, applied here with $f := \nabla_{\mathbb{H}^n} \bar u_0$ and $g := \nabla_{\mathbb{H}^n} \bar u_\varepsilon$. □

In the degenerate case $p \in [2, +\infty)$ the estimate obtained in Proposition 6.3.1 for the singular case $p \in (1, 2]$ needs to be modified according to the following result:

Proposition 6.3.2. *Suppose that*

$$p \in [2, +\infty). \tag{6.34}$$

Then, there exists $C > 0$, only depending on n and p, such that

$$\int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \le C\big(1 + (|\nabla_{\mathbb{H}^n}\bar{u}_0|^p)_R\big)^{1-(1/p)}\varepsilon R^Q.$$

Proof. The variational inequalities (6.2) for $\bar{u}_0$ and $\bar{u}_\varepsilon$ imply that

$$\int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_0 \cdot (\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon - \nabla_{\mathbb{H}^n}\bar{u}_0) \ge 0$$

$$\text{and} \qquad \int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon \cdot (\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon) \ge 0.$$

Consequently,

$$\int_{U_R} \Big(|\nabla_{\mathbb{H}^n}\bar{u}_0|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_0 - (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\Big) \cdot (\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon) \le 0.$$

Using this and (6.40) of Lemma 6.5.1, applied here with $A := \nabla_{\mathbb{H}^n}\bar{u}_0$ and $B := \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon$, we obtain

$$\begin{aligned}
&\int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
&\le C\int_{U_R} \Big(|\nabla_{\mathbb{H}^n}\bar{u}_0|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_0 - |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\Big) \cdot (\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon) \\
&\le C\int_{U_R} \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon - |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-2}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon\Big) \cdot (\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon).
\end{aligned}$$

This and Corollary 6.5.2, applied here with $a := \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon$, give

$$\begin{aligned}
&\int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
&\le C\int_{U_R} \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1} - |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-2}\Big)\,|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|\,|\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon| \\
&\le C\varepsilon\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p-2)/2}\big(|\nabla_{\mathbb{H}^n}\bar{u}_0| + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|\big).
\end{aligned}$$

Therefore, recalling (6.34), noticing that

$$\frac{p-2}{p}+\frac{1}{p}+\frac{1}{p}=1$$

and using the Generalized Hölder Inequality with the three exponents $p/(p-2)$, p and p, we obtain

$$\int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p$$
$$\leq C\varepsilon \left(\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2}\right)^{(p-2)/p} \left(\int_{U_R} \left(|\nabla_{\mathbb{H}^n}\bar{u}_0|^p + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p\right)\right)^{1/p} R^{Q/p}.$$

Then, by the minimal property of $\bar{u}_0$ in (6.20),

$$\begin{aligned}
&\int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0 - \nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p \\
&\leq C\varepsilon \left(\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2}\right)^{(p-2)/p} \left(\int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p\right)^{1/p} R^{Q/p} \\
&\leq C\varepsilon \left(\int_{U_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2}\right)^{(p-1)/p} R^{Q/p} \\
&\leq C\varepsilon \left(R^Q + \int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^p\right)^{(p-1)/p} R^{Q/p} \\
&\leq C\varepsilon \left(R^Q + \int_{U_R} |\nabla_{\mathbb{H}^n}\bar{u}_0|^p\right)^{(p-1)/p} R^{Q/p},
\end{aligned}$$

from which the desired result follows. □

Corollary 6.3.3. *For all $p \in (1, +\infty)$, we have that*

$$\lim_{\varepsilon \searrow 0} \|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon - \nabla_{\mathbb{H}^n}\bar{u}_0\|_{L^p(U_R)} = 0. \tag{6.35}$$

Also, there exist a subsequence of ε's and a function $G \in L^p(U_R)$ such that

$$|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon(x)| \leq G(x) \tag{6.36}$$

for almost every $x \in U_R$.

Furthermore, if we set

$$\Gamma_\varepsilon := (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon, \tag{6.37}$$

then there exist a subsequence of ε's and a function $G_\star \in L^1(U_R)$ such that

$$|\Gamma_\varepsilon(x)| \le G_\star(x) \tag{6.38}$$

for almost every $x \in U_R$.

Proof. We obtain (6.35) from Propositions 6.3.1 and 6.3.2, according to whether $p \in (1,2)$ or $p \in [2,+\infty)$.

From (6.35), one deduces (6.36) (see, e.g., Theorem 4.9(b) in [22]).

Now, we define $G_\star := 2^{(p/2)}(G + G^{p-1})$. We observe that $G_\star \in L^1(U_R)$, since $G \in L^p(U_R) \subseteq L^1(U_R)$ and $G^{p-1} \in L^{p/(p-1)}(U_R) \subseteq L^1(U_R)$. So, in order to obtain the desired result, we have only to show that the inequality in (6.38) holds true.

For this, we notice that, if $p \in (1,2)$,

$$\begin{aligned}
|\Gamma_\varepsilon| &= \frac{|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{1-(p/2)}} = \frac{|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-1}|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{2-p}}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{1-(p/2)}} \\
&\le \frac{|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-1}(\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{(2-p)/2}}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{1-(p/2)}} = |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-1} \le G^{p-1},
\end{aligned}$$

which implies (6.38) in this case.

On the other hand, if $p \in [2,+\infty)$,

$$\begin{aligned}
|\Gamma_\varepsilon| &\le 2^{(p/2)-1}\left(\varepsilon^{(p/2)-1} + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^{p-2}\right)|\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon| \\
&\le 2^{(p/2)-1}(1 + G^{p-2})G,
\end{aligned}$$

which implies (6.38) in this case too. □

6.4 Lewy-Stampacchia estimate when $\varepsilon = 0$

By Theorem 6.1.1 (for $\varepsilon > 0$, which has been proved in § 6.2), we know that, for a sequence $\varepsilon \searrow 0$,

$$0 \le \int_{U_R} \Gamma_\varepsilon \cdot \nabla\varphi \le \int_{U_R} \left(\mathrm{div}_{\mathbb{H}^n}\left((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\right)\right)^+ \varphi, \tag{6.39}$$

for any $\varphi \in C_0^\infty(U_R, [0,+\infty))$, as long as $U_R \subset \Omega$, where Γ_ε is as in (6.37).

By possibly taking subsequences, in the light of (6.35) and (6.38), we have that

$$\lim_{\varepsilon \searrow 0} \Gamma_\varepsilon = |\nabla_{\mathbb{H}^n} \bar{u}_0|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_0$$

almost everywhere in U_R, and that Γ_ε is equidominated in $L^1(U_R)$. Consequently, we can pass to the limit in (6.39) via the Dominated Convergence Theorem and obtain (6.6) for $\bar{u}_0$. This completes the proof of Theorem 6.1.1 also when $\varepsilon = 0$. □

6.5 Appendix

In this appendix, we collect some technical and well known estimates of general interest that will be used in the proofs of the main results.

We start with some classical estimates (see, e.g. Lemma 3 in [84] and references therein), which turns out to be quite useful to deal with nonlinear operators of degenerate type:

Lemma 6.5.1. *Let $M \in \mathbb{N}$, $M \geq 1$, and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on M and p, such that, for any A, $B \in \mathbb{R}^M$,*

$$|A - B|^p \leq C\Big(|A|^{p-2}A - |B|^{p-2}B\Big) \cdot (A - B) \tag{6.40}$$

and

$$\Big||A|^{p-2}A - |B|^{p-2}B\Big| \leq C|A - B| \left(|A|^{p-2} + |B|^{p-2}\right). \tag{6.41}$$

Corollary 6.5.2. *Let $N \in \mathbb{N}$ and and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on N and p, such that for any $\varepsilon > 0$ and any $a \in \mathbb{R}^N$*

$$\left((\varepsilon + |a|^2)^{(p/2)-1} - |a|^{p-2}\right)|a| \leq C\varepsilon(\varepsilon + |a|^2)^{(p-2)/2}.$$

Proof. We let $A := (a, \varepsilon)$ and $B := (a, 0) \in \mathbb{R}^{N+1}$ and we exploit (6.41). We obtain

$$\begin{aligned}
&2C\varepsilon(\varepsilon+|a|^2)^{(p-2)/2}\\
&\qquad\geq C\varepsilon\Big((\varepsilon+|a|^2)^{(p-2)/2}+|a|^{p-2}\Big)\\
&\qquad= C|A-B|\left(|A|^{p-2}+|B|^{p-2}\right)\\
&\qquad\geq \Big||A|^{p-2}A-|B|^{p-2}B\Big|\\
&\qquad= \Big|(\varepsilon+|a|^2)^{(p-2)/2}(a,\varepsilon)-|a|^{p-2}(a,0)\Big|\\
&\qquad= \Big|\Big(\big((\varepsilon+|a|^2)^{(p-2)/2}-|a|^{p-2}\big)a,\,(\varepsilon+|a|^2)^{(p-2)/2}\varepsilon\Big)\Big|\\
&\qquad\geq \left((\varepsilon+|a|^2)^{(p-2)/2}-|a|^{p-2}\right)|a|,
\end{aligned}$$

as desired. $\square$

In the subsequent Lemmata 6.5.3 and 6.5.4, we collect some simple, but interesting, estimates that are used in Proposition 6.3.1:

Lemma 6.5.3. *Let $N\in\mathbb{N}$, $N\geq 1$, $t\in[0,1]$, $\varepsilon>0$, and a, $b\in\mathbb{R}^N$. Let $h(t):=ta+(1-t)b$. Then, there exists $C>1$, only depending on N and p, such that*

$$\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\cdot(a-b)\Big)\geq\frac{1}{C}(\varepsilon+|ta+(1-t)b|^2)^{(p/2)-1}|a-b|^2.$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\cdot(a-b)\Big)&=\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\Big)\cdot(a-b)\\
&=(\varepsilon+|h|^2)^{(p/2)-2}\big(\varepsilon+(p-1)|h|^2\big)\frac{dh}{dt}\cdot(a-b)\\
&\geq\frac{1}{C}(\varepsilon+|h|^2)^{(p/2)-1}|a-b|^2\\
&=\frac{1}{C}(\varepsilon+|ta+(1-t)b|^2)^{(p/2)-1}|a-b|^2,
\end{aligned}$$

as desired. $\square$

Lemma 6.5.4. *Let*

$$p\in(1,2]. \tag{6.42}$$

Let $N \in \mathbb{N}$, $N \geq 1$, $\varepsilon > 0$, and a, $b \in \mathbb{R}^N$. Then, there exists $C > 1$, only depending on N and p, such that

$$(\varepsilon + |a|^2 + |b|^2)^{p/2} \leq C\Big[(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1}|b-a|^2 + (\varepsilon + |a|^2)^{(p/2)}\Big].$$

Proof. We have

$$|b|^2 = |b-a+a|^2 \leq \big(|b-a| + |a|\big)^2 \leq C\big(|b-a|^2 + |a|^2\big)$$

and so

$$\begin{aligned} & (\varepsilon + |a|^2 + |b|^2)^{p/2} \\ = \;& (\varepsilon + |a|^2 + |b|^2)^{(p/2)-1}(\varepsilon + |a|^2 + |b|^2) \\ \leq \;& C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1}(\varepsilon + |a|^2 + |b-a|^2) \\ = \;& C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1}|b-a|^2 + C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1}(\varepsilon + |a|^2). \end{aligned}$$

Therefore, by (6.42),

$$\begin{aligned} & (\varepsilon + |a|^2 + |b|^2)^{p/2} \\ & \qquad \leq C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1}|b-a|^2 + C(\varepsilon + |a|^2)^{(p/2)}, \end{aligned}$$

that is the desired claim. □

The following result deals with some technical estimates on monotone integrands.

Lemma 6.5.5. *Let $N \in \mathbb{N}$, $N \geq 1$. Let $\kappa \in \{0,1\}$. Let ε, $\varepsilon' > 0$. Let a, $b \in \mathbb{R}^N$. Let $\Psi : [\varepsilon, +\infty) \to [\varepsilon', +\infty)$ be a measurable and nondecreasing function. Then*

$$\int_0^1 \frac{(1-t)^\kappa}{\Psi(\varepsilon + |ta + (1-t)b|^2)}\, dt \geq \frac{1}{2\Psi(\varepsilon + |a|^2 + |b|^2)}. \tag{6.43}$$

Proof. If $|a| \leq |b|$, for any $t \in [0,1]$,

$$\begin{aligned} |ta + (1-t)b|^2 &\leq t^2|a|^2 + (1-t)^2|b|^2 + 2t(1-t)|a||b| \\ &\leq t^2|b|^2 + (1+t^2-2t)|b|^2 + 2t(1-t)|b|^2 = |b|^2. \end{aligned}$$

On the other hand, if $|a| \geq |b|$, for any $t \in [0,1]$,

$$
\begin{aligned}
|ta + (1-t)b|^2 &\leq t^2|a|^2 + (1-t)^2|b|^2 + 2t(1-t)|a||b| \\
&\leq t^2|a|^2 + (1+t^2-2t)|a|^2 + 2t(1-t)|a|^2 = |a|^2.
\end{aligned}
$$

In any case,

$$
\varepsilon + |ta + (1-t)b|^2 \leq \varepsilon + |a|^2 + |b|^2
$$

and the claim follows from the monotonicity of Ψ. □

The next is a useful Hölder/L^p type estimate, that is exploited in Proposition 6.3.1.

Lemma 6.5.6. *Let $N \in \mathbb{N}$, $N \geq 1$. Let f, $g \in L^p(\mathcal{B}_R, \mathbb{R}^N)$. Suppose that*

$$
p \in (1, 2]. \tag{6.44}
$$

Then

$$
\begin{aligned}
&\int_{\mathcal{B}_R} |f-g|^p \\
&\qquad \leq \left(\int_{\mathcal{B}_R} (\varepsilon + |f|^2 + |g|^2)^{(p/2)-1} |f-g|^2 \right)^{p/2} \\
&\qquad\qquad \times \left(\int_{\mathcal{B}_R} (\varepsilon + |f|^2 + |g|^2)^{p/2} \right)^{(2-p)/2}.
\end{aligned}
$$

Proof. We observe that

$$
\begin{aligned}
&|f-g|^p \\
= \ &\left[(\varepsilon + |f|^2 + |g|^2)^{(p/2)-1} |f-g|^2 \right]^{p/2} \left[(\varepsilon + |f|^2 + |g|^2)^{p/2} \right]^{(2-p)/2},
\end{aligned}
$$

and so the desired result follows from the Hölder Inequality with exponents $2/p$ and $2/(2-p)$, which can be used here due to (6.44). □

To end this chapter, we remark that Definition 6.1.1 is always nonvoid (independently of ψ and Ω), in the sense that

Lemma 6.5.7. $2 \in \mathcal{P}(\psi, \Omega)$.

Proof. The functional in (6.4) when $p = 2$ boils down to

$$\int_\Omega \frac{1}{2}|\nabla_{\mathbb{H}^n} u(\xi)|^2 + F_k(u(\xi), \xi)\, d\xi, \tag{6.45}$$

up to an additive constant that does not play any role in the minimization. Hence, if u_k minimizes this functional, we have that

$$-\int_\Omega \nabla_{\mathbb{H}^n} u_k(\xi) \cdot \nabla_{\mathbb{H}^n} \varphi(\xi)\, d\xi = \int_\Omega \partial_r F_k(u_k(\xi), \xi)\, \varphi(\xi)\, d\xi$$

for any $\varphi \in C_0^\infty(\Omega)$.

Accordingly, if also u_k approaches some u_∞ uniformly in Ω, it follows that

$$\begin{aligned} \int_\Omega u_\infty \Delta_{\mathbb{H}^n} \varphi &= \lim_{k\to+\infty} \int_\Omega u_k \Delta_{\mathbb{H}^n} \varphi \\ &= \lim_{k\to+\infty} -\int_\Omega \nabla_{\mathbb{H}^n} u_k \cdot \nabla_{\mathbb{H}^n} \varphi = \lim_{k\to+\infty} \int_\Omega \partial_r F_k(u_k, \xi)\, \varphi \end{aligned} \tag{6.46}$$

for any $\varphi \in C_0^\infty(\Omega)$.

Also, from (6.3),

$$0 \le \partial_r F_k \le (\Delta_{\mathbb{H}^n} \psi)^+$$

and so (6.46) gives that

$$0 \le \int_\Omega u_\infty \Delta_{\mathbb{H}^n} \varphi \le \int_\Omega (\Delta_{\mathbb{H}^n} \psi)^+\, \varphi \tag{6.47}$$

for any $\varphi \in C_0^\infty(\Omega, [0, +\infty))$.

On the other hand, since u_k is a minimizer for (6.45), we have that

$$\sup_{k\in\mathbb{N}} \|\nabla_{\mathbb{H}^n} u_k\|_{L^2(\Omega)} < +\infty$$

and so, up to a subsequence, we may suppose that $\nabla_{\mathbb{H}^n} u_k$ converges to some $\nu \in L^2(\Omega)$ weakly in $L^2(\Omega)$. It follows from the uniform convergence of u_k that

$$\begin{aligned} -\int_\Omega \nu \cdot \nabla_{\mathbb{H}^n} \varphi &= -\lim_{k\to+\infty} \int_\Omega \nabla_{\mathbb{H}^n} u_k \cdot \nabla_{\mathbb{H}^n} \varphi \\ &= \lim_{k\to+\infty} \int_\Omega u_k\, \Delta_{\mathbb{H}^n} \varphi = \int_\Omega u_\infty\, \Delta_{\mathbb{H}^n} \varphi \end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$. That is, $\nabla_{\mathbb{H}^n} u_\infty = \nu$ in the sense of distributions, and so as a function. In particular, $\nabla_{\mathbb{H}^n} u_\infty \in L^2(\Omega)$, and therefore (6.47) yields that

$$0 \leq \int_\Omega \nabla_{\mathbb{H}^n} u_\infty \cdot \nabla_{\mathbb{H}^n} \varphi \leq \int_\Omega (\Delta_{\mathbb{H}^n} \psi)^+ \varphi,$$

for any $\varphi \in C_0^\infty(\Omega, [0, +\infty))$. This shows that u_∞ satisfies (6.5) in the distributional sense. $\square$

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