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Tesi di Dottorato

General aspects of modified theories of gravity

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Est igitur natura loci spatiumque profundi,
quod neque clara suo percurrere fulmina cursu
perpetuo possint aevi labentia tractu,
nec prorsum facere ut restet minus ire meando.

Lucretius, De Rerum Natura, Liber I

Introduction

Recent observational data imply -against any previous belief- that the current expansion of the universe is accelerating [1, 2, 3]. Since this discovery, the so called Dark Energy issue has become the "Mystery of the Millennium" [4]. Today, dark energy is probably the most ambitious and tantalizing field of research because of its implications in fundamental physics. There exist several descriptions of the acceleration of the universe. Among them, the simplest one is the introduction of small positive Cosmological Constant in the framework of General Relativity, the so called ΛCDM model, where the dark energy, whose energy density is given by Cosmological Constant, drives the accelerated expansion of the universe. Alternatively, accelerating Friedmann-Robertson-Walker universe may be described by quintessence/phantom-fluid or other kind of inhomogenous fluid, satisfying suitable Equation of State. That the dark fluid has an Equation of State index ω very close to minus one represents an important point in favour of a Cosmological Constant-like representation of the dark energy, but in principle quintessence/phantom-fluid is not excluded. However, the estimated extremely small value of Cosmological Constant leads to several well-know problems. The first one, is the so called 'cosmological constant problem'. In quantum field theory, the Cosmological Constant appears as the vacuum energy density, which has to be included in gravity theory, as the vacuum effect [5] may suggest. On the other hand, the expected value of vacuum energy density results to be of 122 orders (!) of magnitude larger than the observed value. Supersymmetry and strings theories aim to solve this problem by different ways, but up to now a successful answer seems to be far away (with regard to this issue, see the recent paper of Maggiore *et al.* in Ref. [6]).

Other questions arise from standard cosmology (the so-called 'coincidence problem', linked with the same order of magnitude of matter and dark energy density in the universe today, the origin of dark matter, the absence of a consistent quantum theory of gravity and so on), and, despite the successful results obtained by General Relativity in describing the universe and the Solar System, it is well accepted the idea according to which General Relativity plus Cosmological Constant is not the ultimate theory of gravity, but an extremely good approximation valid in the present day range of detection.

The existence of an early accelerated epoch in our universe, namely the 'hot universe' scenario or inflation, adds a new problem to the standard cosmology, and various proposals to construct acceptable inflationary model exist (scalar, spinor, (non-)abelian vector theory and so on). Otherwise, the scenarios to describe the early-time and the late-time accelerations are usually very similar and is quite natural to expect that same theory lies behind both they. Since General Relativity with matter and radiation correctly describes the intermediate (decelerated) expansion of the universe, it is reasonable to expect that a different gravitational theory dictates the (Friedman-Robertson-Walker) background evolution at high and small energy (curvatures) without the introduction of any other dark components.

The modified theories of gravity represent a generalization of Einstein's gravity, where some combination of curvature invariants (The Riemann tensor, the Weyl tensor, the Ricci tensor and so on) replaces or is added into the classical Hilbert-Einstein action formed by the Ricci scalar term R. Thus, in this framework, the early-time and the late-time acceleration may be caused by the fact that some (sub)-dominant terms of gravitational action become essential at high or small curvatures. Moreover, some other related problem of Cosmological Constant could be solved in this way. Of course, the complete understanding of gravity and the fundamental theory remains to be an open problem of modern physics.

The original idea of introducing a correction to the Hilbert-Einstein action in the form of $R + R^2$ was proposed long time ago by Starobinsky [7] in order to solve many of the problems left open by the inflation, so that the Starobinsky model can be considered as the first modified gravity inflationary model. The interest in models of modified gravity grew up in cosmology. In Refs. [8, 9, 10] is possible to find some examples. Here, the first candidate proposed to explain the current acceleration was the model $R - \mu_0^4/R$, with μ_0 on the same order of Hubble parameter today, but this theory is subject to cosmological instabilities. The first work of a viable unification of the early- and late-time acceleration was proposed by Nojiri & Odintsov in Ref. [11], and in Ref. [12] Capozziello *et al.* suggested that both, dark matter and dark energy, are curvature effects of some modification to standard gravity.

The mathematical structure of modified theories of gravity and their physical properties are an exciting field of research. Furthermore, despite the arena of modified gravity-models is in principle infinite, the very accurate data arisen from observation of our universe, restrict the field of viable models.

The aim of this work is to investigate the both, some mathematical and physical general aspect of modified gravity, and, more specifically, the proprieties of viable, realistic models of modified gravity which can be used to reproduce the inflation and the dark energy epoch of universe today.

The work is organized as the following. In Chapter **1**, the formalism of $\mathcal{F}(R, G)$-modified gravity is presented. In this kind of theories, the modification to the Hilbert-Einstein action is given by the function $\mathcal{F}(R, G)$ of the Ricci scalar R and the Gauss-Bonnet invariant G only. A remark is in order. As a rule, modification of gravity may contain a huge list of invariants. Otherwise, we will often work with the above specific class of modified gravity. The popularity of modifications through some function of the Ricci scalar, $F(R)$, is clearly motivated by the easier formalism and by the prospect to find a final theory of gravity in the simplest form of R plus a suitable function $f(R)$ of R itself. Furthermore, the Gauss-Bonnet modified gravity is a string-inspired theory. In Chapter **2**, we will discuss the black hole solutions in modified gravity, with particular attention to $F(R)$-gravity. These black hole solutions are not expected to share the same laws of their Einsteinian counterparts. Some of the physical quantities one would like to address to modified gravity black holes are their mass, the horizon entropy, their temperature and so on. Here, we propose and identify the mass with a quantity proportional to the constant of integration, which appears in the explicit solutions, making use of derivation of the First Law of black hole thermodynamics from the equations of motion in $F(R)$-gravity, and evaluating independently the entropy via Wald method and the Hawking temperature via quantum mechanical methods in curved space-times. The results are extended to general classes of modified gravity theories. In Chapter **3** we consider the (Friedman-Robertson-Walker) cosmological context of $\mathcal{F}(R, G)$-gravity and we study the finite-time future singularities which often occur in dark energy models (the most popular case is the Big Rip one of phantom fluid). It is explicitly demonstrated that Gauss-Bonnet modified gravity as the $F(R)$-gravity may show singularities during cosmological evolution. However, the introduction of specific form of modified gravity may naturally solve the problem of singularities in general theories of gravity. In Chapter **4**, as a prosecution of Chapter **3**, we study inhomogeneous viscous fluids, especially relating with modified gravity and singularities. In Chapters **5** and **6**, we restrict our analysis to realistic models of $F(R)$-modified gravity producing (unstable) de Sitter inflation and (stable) de Sitter of dark energy epoch (the so called 'one step' and 'two steps'-models). It is quite interesting to note how, despite this models mimic with high precision the ΛCDM Model, the dynamical behaviour of Equation of State and the introduction of new degree of freedom in the equations of motion, involve a very accurate analysis in order to reach the feasibility of the models and in order to fit the all the most recent and accurate observational data. In particular, in Chapter **6**, we consider the perturbation theory and we find that fluctuations of effective dark energy-modified gravity at high red-shift may present a strong divergence in the frequency. As a consequence, a divergence in high derivatives of Hubble parameter may appear. Since a dynamical correction of the Einstein equations in the small curvature region seems to introduce in the theory singularities in the high curvature region, we conclude that a modification of gravity to reproduce the current acceleration requires a modification at inflationary scale. In other words, the dark energy scenario is strictly related with the inflation and an unified description is suggested. We conclude with Chapter **7**, where black hole and de Sitter solutions are considered in a covariant-renormalizable field theory of gravity. The popularity of this kind of theories is related with the possibility to reach a quantum theory of gravity.

The present thesis is based on the following papers published in the referred journal, pre-prints and conference proceedings (Refs.[13]-[24]):

- G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, “A class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion”, Phys. Rev. D **77**, 046009 (2008) [arXiv:0712.4017 [hep-th]];
- K. Bamba, S. D. Odintsov, L. Sebastiani and S. Zerbini, “Finite-time future singularities in modified Gauss-Bonnet and $\mathcal{F}(R,G)$ gravity and singularity avoidance”, Eur. Phys. J. C **67**, 295 (2010) [arXiv:0911.4390 [hep-th]];
- O. Gorbunova and L. Sebastiani, “Viscous Fluids and Gauss-Bonnet Modified Gravity”, Gen. Rel. Grav. **42**, 2873 (2010) [arXiv:1004.1505 [gr-qc]];
- L. Sebastiani, “Dark Viscous Fluid coupled with Dark Matter and future singularity”, Eur. Phys. J. C **69**, 547 (2010) [arXiv:1006.1610 [gr-qc]];
- E. Bellini, R. Di Criscienzo, L. Sebastiani and S. Zerbini, “Black Hole entropy for two higher derivative theories of gravity”, Entropy **12**, 2186 (2010) [arXiv:1009.4816 [gr-qc]];
- G. Cognola, E. Elizalde, L. Sebastiani and S. Zerbini, “Black hole and de Sitter solutions in a covariant renormalizable field theory of gravity”, Phys. Rev. D **83**, 063003 (2011) [arXiv:1007.4676 [hep-th]];
- E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, “Non-singular exponential gravity: a simple theory for early- and late-time accelerated expansion”, Phys. Rev. D **83**, 086006 (2011) [arXiv:1012.2280 [hep-th]];
- L. Sebastiani and S. Zerbini, “Static Spherically Symmetric Solutions in F(R) Gravity”, Eur. Phys. J. C **71**, 1591 (2011) [arXiv:1012.5230 [gr-qc]];
- G. Cognola, O. Gorbunova, L. Sebastiani and S. Zerbini, “On the Energy Issue for a Class of Modified Higher Order Gravity Black Hole Solutions”, Phys. Rev. D **84**, 023515 (2011) [arXiv:1104.2814 [gr-qc]];
- E. Elizalde, S. D. Odintsov, L. Sebastiani and S. Zerbini, “Oscillations of the F(R) dark energy in the accelerating universe”, to appear in Eur. Phys. J. C, arXiv:1108.6184 [gr-qc];
- G. Cognola, L. Sebastiani and S. Zerbini, “Stability in Generalized Modified Gravity”, To appear in the proceedings of 12th Marcel Grossmann Meeting on General Relativity (MG 12), Paris, France, 12-18 Jul 2009, arXiv:1006.1586 [gr-qc];
- L. Sebastiani, “Finite-time singularities in modified $\mathcal{F}(R,G)$-gravity and singularity avoidance”, A workshop with a celebration of Emilio Elizalde’s sixtieth birthday, ICE/CSIC, Universitat Autònoma de Barcelona, 8-10 March 2010, arXiv:1008.3041 [gr-qc].

Units: We use units of $k_{\rm B} = c = \hbar = 1$ and denote the gravitational constant G_N by $\kappa^2 \equiv 8\pi G_N$, such that $G_N^{-1/2} = M_{\rm Pl}$, being $M_{\rm Pl} = 1.2 \times 10^{19}$GeV the Planck mass.

Contents

Chapter 1

The formalism of $\mathcal{F}(R, G)$ theories of gravity.

In modified theories of gravity the Hilbert-Einstein term of General Relativity (GR), that is the Ricci scalar R, is substituted by a more general combination of curvature invariants (the Riemann tensor, the Weyl tensor, the Ricci tensor, the Ricci scalar and tensors formed from these by the operations of taking duals, contractions or covariant differentiations...). The simplest class of modified gravitational theories is the $F(R)$-gravity, where the modification is given by a function $F(R)$ of the Ricci scalar R only (for a general reviews, see Refs. [25, 26, 27]). Among them, the most reasonable choice is writing $F(R)$ as the Ricci scalar R plus an arbitrary function $f(R)$ of R. In this way, one can reproduce the accelerating Friedman Robertson Walker (FRW) universe by including modified gravity into an effective energy density and pressure of the universe. An other interesting class of modified gravity which may easily produce the late-time acceleration epoch is string-inspired modified Gauss-Bonnet gravity, so-called $F(G)$-gravity [28], where $F(G)$ is an arbitrary function of the Gauss-Bonnet four dimensional topological invariant G. In this Chapter, we explore the formalism of $\mathcal{F}(R, G)$-gravity models, where the modification to GR is given by a combination of both, R and G, and we briefly derive the gravitational field equations.

1.1 The action and FRW equations of motion

The action of $\mathcal{F}(R, G)$-gravity in four dimension space-time is given by

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{\mathcal{F}(R,G)}{2\kappa^2} + \mathcal{L}^{(\text{matter})}\right], \tag{1.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{L}^{(\text{matter})}$ is the matter Lagrangian[1] and $\mathcal{M}$ is the space-time manifold. $\mathcal{F}(R, G)$ is a generic function of the Ricci scalar R and the Gauss Bonnet four dimensional topological invariant[2] G:

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}\,. \tag{1.2}$$

The Gauss Bonnet invariant is a combination of the Riemann Tensor $R_{\mu\nu\xi\sigma}$, the Ricci Tensor $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$ and its trace $R = g^{\alpha\beta}R_{\alpha\beta}$.

[1] The Lagrangian of matter perfect fluid-like is

$$\mathcal{L}^{(\text{matter})} = -\frac{1}{2}g^{00}(p_{\text{m}} + \rho_{\text{m}}) - \frac{(\rho_{\text{m}} - p_{\text{m}})}{2}\,.$$

This relation can be derived by starting from the Lagrangian $\mathcal{L}$ of free scalar field ϕ, namely, $\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\phi^2}{2}$, where the energy density ρ_{m} is $\rho_{\text{m}} = \partial_t\phi^2/2 + \phi^2/2$ and the pressure p_{m} is $p_{\text{m}} = \partial_t\phi^2/2 - \phi^2/2$. We are assuming, the fluid depends on the time coordinate only.

[2] One has $\int_{\mathcal{M}} d^4x\sqrt{-g}[G] = 0$.

From the action in Eq. (1.1), the gravitational field equation is derived as

$$\begin{aligned}
\mathcal{F}'_R\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right) &= \kappa^2 T^{(\mathrm{matter})}_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\left(\mathcal{F}-\mathcal{F}'_R R\right)+\nabla_\mu\nabla_\nu\mathcal{F}'_R-g_{\mu\nu}\Box\mathcal{F}'_R\\
&+\left(-2RR_{\mu\nu}+4R_{\mu\rho}R_\nu{}^\rho-2R_\mu{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}+4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\rho\sigma}\right)\mathcal{F}'_G\\
&+2\left(\nabla_\mu\nabla_\nu\mathcal{F}'_G\right)R-2g_{\mu\nu}\left(\Box\mathcal{F}'_G\right)R+4\left(\Box\mathcal{F}'_G\right)R_{\mu\nu}-4\left(\nabla_\rho\nabla_\mu\mathcal{F}'_G\right)R_\nu{}^\rho\\
&-4\left(\nabla_\rho\nabla_\nu\mathcal{F}'_G\right)R_\mu{}^\rho+4g_{\mu\nu}\left(\nabla_\rho\nabla_\sigma\mathcal{F}'_G\right)R^{\rho\sigma}-4\left(\nabla_\rho\nabla_\sigma\mathcal{F}'_G\right)g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}\,. \qquad (1.3)
\end{aligned}$$

Now, $\mathcal{F}(R,G)$ has been replaced with $\mathcal{F}$ and we have used the following expressions:

$$\mathcal{F}'_R\equiv\frac{\partial\mathcal{F}}{\partial R}\,,\quad \mathcal{F}'_G\equiv\frac{\partial\mathcal{F}}{\partial G}\,. \qquad (1.4)$$

Here, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box\phi\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$ is the covariant d'Alembertian for a scalar field ϕ, and $T^{(\mathrm{matter})}_{\mu\nu}=\mathrm{diag}\left(\rho_\mathrm{m},p_\mathrm{m},p_\mathrm{m},p_\mathrm{m}\right)$ is the contribution to the stress energy-momentum tensor from all ordinary matters[3], with ρ_m and p_m being, respectively, the matter energy-density and pressure.
If we put $\mathcal{F}(R,G)=R$, we recover the Einstein's Equation.

The most general flat FRW space-time is described by the metric

$$ds^2=-N^2(t)dt^2+a(t)^2d\mathbf{x}^2\,, \qquad (1.5)$$

where $a(t)$ is the scale factor of the Universe and $N(t)$ is an arbitrary function of the cosmic time t. In what follows, we take the gauge $N(t)=1$.

In the FRW background, from $(\mu,\nu)=(0,0)$ and the trace part of $(\mu,\nu)=(i,j)$ (with $i,j=1,\cdots,3$) components in Eq. (1.3), we obtain the equations of motion (EOM):

$$\begin{aligned}
\left(\frac{3}{\kappa^2}H^2\right)\mathcal{F}'_R &= \rho_\mathrm{m}+\frac{1}{2\kappa^2}\left[\left(\mathcal{F}'_R R+G\mathcal{F}'_G-\mathcal{F}\right)-6H\dot{\mathcal{F}}'_R-24H^3\dot{\mathcal{F}}'_G\right]\,, \qquad (1.6)\\
-\frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right)\mathcal{F}'_R &= p_\mathrm{m}+\frac{1}{2\kappa^2}\Big[-\left(\mathcal{F}'_R R+G\mathcal{F}'_G-\mathcal{F}\right)+4H\dot{\mathcal{F}}'_R+2\ddot{\mathcal{F}}'_R\\
&+16H\left(\dot{H}+H^2\right)\dot{\mathcal{F}}'_G+8H^2\ddot{\mathcal{F}}'_G\Big]\,. \qquad (1.7)
\end{aligned}$$

Here, $H=\dot{a}(t)/a(t)$ is the Hubble parameter and the dot denotes the time derivative of ∂_t. Moreover, we have

$$\begin{aligned}
R &= 6\left(2H^2+\dot{H}\right)\,, \qquad (1.8)\\
G &= 24H^2\left(H^2+\dot{H}\right)\,. \qquad (1.9)
\end{aligned}$$

In a large class of modified gravity models which reproduce the cosmology of Standard Model ($\mathcal{F}(R,G)=R$) plus the suitable correction terms of inflation and/or current acceleration, one has

$$\mathcal{F}(R,G)=R+f(R,G)\,. \qquad (1.10)$$

Thus, the modification to gravity is encoded in the function $f(R,G)$ of R and G, which is added to the classical term R of the Einstein-Hilbert action of General Relativity. In what follows we will often discuss modified gravity in this form, by explicitly separating the contribution of GR from its modification. In this case, it is reasonable to write Eq. (1.3) as

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\kappa^2\left(T^{\mathrm{MG}}_{\mu\nu}+\tilde{T}^{(\mathrm{matter})}_{\mu\nu}\right)\,. \qquad (1.11)$$

[3]In general, it includes matter and radiation.

Here, the part of modified gravity is formally included into the 'modified gravity' stress-energy tensor $T^{\mathrm{MG}}_{\mu\nu}$, given by

$$
\begin{aligned}
T^{\mathrm{MG}}_{\mu\nu} \equiv & \frac{1}{\kappa^2 \mathcal{F}'(R)}\left\{\frac{1}{2}g_{\mu\nu}\left(\mathcal{F}-\mathcal{F}'_R R\right)+\nabla_\mu\nabla_\nu\mathcal{F}'_R - g_{\mu\nu}\Box\mathcal{F}'_R\right. \\
& +\left(-2RR_{\mu\nu}+4R_{\mu\rho}R_\nu{}^\rho - 2R_\mu{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}+4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\rho\sigma}\right)\mathcal{F}'_G \\
& +2\left(\nabla_\mu\nabla_\nu\mathcal{F}'_G\right)R - 2g_{\mu\nu}\left(\Box\mathcal{F}'_G\right)R+4\left(\Box\mathcal{F}'_G\right)R_{\mu\nu}-4\left(\nabla_\rho\nabla_\mu\mathcal{F}'_G\right)R_\nu{}^\rho \\
& \left.-4\left(\nabla_\rho\nabla_\nu\mathcal{F}'_G\right)R_\mu{}^\rho+4g_{\mu\nu}\left(\nabla_\rho\nabla_\sigma\mathcal{F}'_G\right)R^{\rho\sigma}-4\left(\nabla_\rho\nabla_\sigma\mathcal{F}'_G\right)g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}\right\}. \qquad (1.12)
\end{aligned}
$$

Hence, one must not forget that gravitational terms enter in both sides of the Eq. (1.11). Furthermore, $\tilde{T}^{(\mathrm{matter})}_{\mu\nu}$ is given by the non-minimal coupling of the ordinary matter stress-energy tensor $T^{(\mathrm{matter})}_{\mu\nu}$ with geometry, namely,

$$
\tilde{T}^{(\mathrm{matter})}_{\mu\nu} = \frac{1}{\mathcal{F}'_R}T^{(\mathrm{matter})}_{\mu\nu}\,. \qquad (1.13)
$$

It should be noted that only $T^{(\mathrm{matter})}_{\mu\nu}$ is covariant conserved, and formally $\kappa^2/\mathcal{F}'_R$ may be interpreted as an effective gravitational constant.
Eqs. (1.6)-(1.7) read

$$
\rho_{\mathrm{eff}} = \frac{3}{\kappa^2}H^2\,, \qquad (1.14)
$$

$$
p_{\mathrm{eff}} = -\frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right)\,, \qquad (1.15)
$$

where ρ_{eff} and p_{eff} are the effective energy density and pressure of the universe, respectively, and these are defined as

$$
\rho_{\mathrm{eff}} \equiv \frac{1}{\mathcal{F}'_R}\left\{\rho_{\mathrm{m}}+\frac{1}{2\kappa^2}\left[\left(\mathcal{F}'_R R+G\mathcal{F}'_G-\mathcal{F}\right)-6H\dot{\mathcal{F}}'_R-24H^3\dot{\mathcal{F}}'_G\right]\right\}\,, \qquad (1.16)
$$

$$
\begin{aligned}
p_{\mathrm{eff}} \equiv & \frac{1}{\mathcal{F}'_R}\left\{p_{\mathrm{m}}+\frac{1}{2\kappa^2}\left[-\left(\mathcal{F}'_R R+G\mathcal{F}'_G-\mathcal{F}\right)+4H\dot{\mathcal{F}}'_R+2\ddot{\mathcal{F}}'_R+16H\left(\dot{H}+H^2\right)\dot{\mathcal{F}}'_G\right.\right. \\
& \left.\left.+8H^2\ddot{\mathcal{F}}'_G\right]\right\}. \qquad (1.17)
\end{aligned}
$$

On shell, one has

$$
\rho_{\mathrm{eff}} \equiv \rho_{\mathrm{m}}+\frac{1}{2\kappa^2}\left[\left(\mathcal{F}'_R R+G\mathcal{F}'_G-\mathcal{F}\right)-6H^2(\mathcal{F}'_R-1)-6H\dot{\mathcal{F}}'_R-24H^3\dot{\mathcal{F}}'_G\right]\,, \qquad (1.18)
$$

$$
\begin{aligned}
p_{\mathrm{eff}} \equiv & p_{\mathrm{m}}+\frac{1}{2\kappa^2}\left[-\left(\mathcal{F}'_R R+G\mathcal{F}'_G-\mathcal{F}\right)+(4\dot{H}+6H^2)(\mathcal{F}'_R-1)+4H\dot{\mathcal{F}}'_R+2\ddot{\mathcal{F}}'_R\right. \\
& \left.+16H\left(\dot{H}+H^2\right)\dot{\mathcal{F}}'_G+8H^2\ddot{\mathcal{F}}'_G\right]. \qquad (1.19)
\end{aligned}
$$

For General Relativity with $\mathcal{F}(R,G)=R$, $\rho_{\mathrm{eff}}=\rho_{\mathrm{m}}$ and $p_{\mathrm{eff}}=p_{\mathrm{m}}$ and therefore Eqs. (1.14)-(1.15) are the Friedman equations.

The following matter conservation law results

$$
\dot{\rho}_{\mathrm{m}}+3H(\rho_{\mathrm{m}}+p_{\mathrm{m}})=0\,. \qquad (1.20)
$$

For a perfect fluid, it gives the equation of State (EoS)

$$p_{\rm m} = \omega \rho_{\rm m} \,, \tag{1.21}$$

ω being the thermodynamical EoS-parameter for matter. For standard matter, $\omega = 0$ and $\rho_{\rm m} = \rho_{\rm m(0)} a(t)^{-3}$ while, for radiation, $\omega = 1/3$ and $\rho_{\rm r} = \rho_{\rm r(0)} a(t)^{-4}$, $\rho_{\rm r}$ being the radiation density and $\rho_{\rm m(0)}$, $\rho_{\rm r(0)}$ generic constants.

We also can introduce the effective EoS by using the corresponding parameter $\omega_{\rm eff}$,

$$\omega_{\rm eff} \equiv \frac{p_{\rm eff}}{\rho_{\rm eff}} \,, \tag{1.22}$$

and get

$$\omega_{\rm eff} = -1 - \frac{2\dot{H}}{3H^2} \,. \tag{1.23}$$

If the strong energy condition (SEC) is violated ($\omega_{\rm eff} < -1/3$), the universe expands in an accelerating way, and vice-versa.

1.2 F(R)-gravity: critical points and stability of cosmological perturbations

This Section is devoted to the specific study of $F(R)$-gravity, while in the next Section we will generalize the results of the De sitter space to a more general class of modified gravity.

The action of modified $F(R)$-theories is:

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{F(R)}{2\kappa^2} + \mathcal{L}^{(\rm matter)} \right] . \tag{1.24}$$

Now, $F(R)$ is a generic function of the Ricci scalar R only. Eq. (1.3) simply reads

$$F'(R)\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) = \kappa^2 T^{(\rm matter)}_{\mu\nu} + \left\{ \frac{1}{2} g_{\mu\nu}[F(R) - RF'(R)] + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)F'(R) \right\} . \tag{1.25}$$

The prime denotes derivative with respect to the curvature R. The starting point is the trace of this equation, which is trivial in Einstein gravity, $R = -\kappa^2 \mathrm{T}^{(\rm matter)}$, with $\mathrm{T}^{(\rm matter)}$ the trace of the matter stress energy-momentum tensor, but, for $F(R)$-gravity, it reads

$$3\Box F'(R) + RF'(R) - 2F(R) = \kappa^2 \mathrm{T}^{(\rm matter)} . \tag{1.26}$$

We can rewrite this equation as

$$\Box F'(R) = \frac{\partial V_{\rm eff}}{\partial F'(R)} \,, \tag{1.27}$$

where

$$\frac{\partial V_{\rm eff}}{\partial F'(R)} = \frac{1}{3}\left[2F(R) - RF'(R) + \kappa^2 \mathrm{T}^{(\rm matter)}\right] , \tag{1.28}$$

$F'(R)$ being the so-called 'scalaron' or the effective scalar degree of freedom. On the critical points of the theory, the effective potential $V_{\rm eff}$ has a maximum (or minimum), so that

$$\Box F'(R) = 0 \,, \tag{1.29}$$

and

$$2F(R) - RF'(R) = -\kappa^2 \mathrm{T}^{(\rm matter)} . \tag{1.30}$$

For example, in absence of matter, i.e. $\mathrm{T}^{(\mathrm{matter})} = 0$, one has the de Sitter (dS) critical point associated with a constant scalar curvature R_{dS}, such that

$$2F(R_{\mathrm{dS}}) - R_{\mathrm{dS}}F'(R_{\mathrm{dS}}) = 0\,. \tag{1.31}$$

We have derived the de Sitter condition without using specific metric. It is valid in FRW space-time, as in the static spherically symmetric one, like for the case of Scwarzshild-de Sitter solution.

Performing the variation of Eq. (1.26) with respect to $R = R^{(0)} + \delta R$, by evaluating $\Box F'(R)$ as $\Box F'(R) = F''(R)\Box R + F'''\nabla^\mu R \nabla_\nu R$, we find, to first order in δR,

$$\begin{aligned}
&\Box R^{(0)} + \frac{F'''(R^{(0)})}{F''(R^{(0)})} g^{\mu\nu}\nabla_\mu R^{(0)}\nabla_\nu R^{(0)} - \frac{1}{3F''(R^{(0)})}\left[2F(R^{(0)}) - R^{(0)}F'(R^{(0)}) + \kappa^2\mathrm{T}^{\mathrm{matter}}\right] \\
&+\Box\delta R + \left\{\left[\frac{F''''(R^{(0)})}{F''(R^{(0)})} - \left(\frac{F'''(R^{(0)})}{F''(R^{(0)})}\right)^2\right] g^{\mu\nu}\nabla_\mu R^{(0)}\nabla_\nu R^{(0)} + \frac{R^{(0)}}{3} - \frac{F'(R^{(0)})}{3F''(R^{(0)})}\right. \\
&\left.+\frac{F'''(R^{(0)})}{3(F''(R^{(0)}))^2}\left[2F(R^{(0)}) - R^{(0)}F'(R^{(0)}) + \kappa^2\mathrm{T}^{(\mathrm{matter})}\right]\right\}\delta R \\
&+2\frac{F'''(R^{(0)})}{F''(R^{(0)})} g^{\mu\nu}\nabla_\mu R^{(0)}\nabla_\nu \delta R + \mathcal{O}(\delta R^2) - \frac{\kappa^2}{3F''(R^{(0)})}\delta\mathrm{T}^{(\mathrm{matter})} \simeq 0\,.
\end{aligned} \tag{1.32}$$

Here, $\delta\mathrm{T}^{(\mathrm{matter})}$ is the variation of the trace of stress energy tensor. In fact, stress energy tensor is given by Eq. (1.20) of matter. The above equation can be used to study perturbations around critical points. The simplest case is the de Sitter one. We put $R^{(0)} = R_{dS}$ (which is a constant) and neglect the contribute of matter. By using Eq. (1.31) one has

$$\left(\Box - m^2\right)\delta R = \mathcal{O}(\delta R^2)\,, \tag{1.33}$$

where

$$m^2 = \frac{1}{3}\left(\frac{F'(R_{dS})}{F''(R_{dS})} - R_{dS}\right)\,. \tag{1.34}$$

Note that

$$m^2 = \frac{\partial^2 V_{\mathrm{eff}}}{\partial F'(R_{dS})^2}\,. \tag{1.35}$$

The second derivative of the effective potential represents the effective mass of the scalaron. Thus, if $m^2 > 0$ (in the sense of the quantum theory, the scalaron, which is a new scalar degree of freedom, is not a tachyon), one gets a stable solution. In the case of the de Sitter solution, Eq. (1.33) is written as

$$-\left(\ddot{\delta R} + 3H_{dS}\dot{\delta R} + m^2\delta R\right) = \mathcal{O}(\delta R^2)\,, \tag{1.36}$$

where $H_{dS} = \sqrt{R_{dS}/12}$. Perturbation δR decreases or oscillates with time if m^2 is positive and exponentially diverges if m^2 is negative. Therefore, the de Sitter stability condition reads

$$\frac{F'(R_{\mathrm{dS}})}{R_{\mathrm{dS}}F''(R_{\mathrm{dS}})} > 1\,. \tag{1.37}$$

As an example, let us consider modified gravity in the form

$$F(R) = R + \alpha R^n\,, \tag{1.38}$$

where α is a constant dimensional parameter and n is a positive number. In vacuum, this model leads to the de Sitter solution

$$R_{dS} = \left(\frac{1}{\alpha(n-2)}\right)^{\frac{1}{n-1}} \quad , \quad n \neq 2\,, \tag{1.39}$$

as a consequence of Eq. (1.31). We assume $\alpha > 0$ if $n > 2$ and $\alpha < 0$ if $0 < n < 2$. The stability condition (1.37) reads

$$\frac{1}{n} > 1\,. \tag{1.40}$$

It means, that if $0 < n < 1$, the de Sitter point is stable and vice versa. Note that the term R^2 is trivial on the the Sitter solution, since in this case Eq. (1.31) is zero for any value of R_{dS}.

1.3 De Sitter solution and stability in $\mathcal{F}(R,P,Q)$-modified gravity

In this Section, we deal with modified generalized models described by the Lagragian density $\mathcal{F}(R,P,Q)$ [23],

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{\mathcal{F}(R,P,Q)}{2\kappa^2} + \mathcal{L}^{(\mathrm{matter})}\right]\,, \tag{1.41}$$

where $\mathcal{F}(R,P,Q)$ is a function of the Ricci scalar and the quadratic curvature invariants P and Q such that,

$$P = R_{\mu\nu}R^{\mu\nu}\,, \quad Q = R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}\,. \tag{1.42}$$

The Gauss-Bonnet can be written as $G = R^2 - 4P + Q$, according with Eq. (1.2).

The field equation for such class of models reads[29]:

$$\begin{aligned}
\mathcal{F}'_R\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) &+ \frac{1}{2}g_{\mu\nu}(\mathcal{F}'_R R - \mathcal{F}) + 2\mathcal{F}'_P\, R^{\alpha}{}_{\mu}\, R_{\alpha\nu} + 2\mathcal{F}'_Q\, R_{\alpha\beta\gamma\mu}\, R^{\alpha\beta\gamma}{}_{\nu} \\
&+ g_{\mu\nu}\,\Box\mathcal{F}_R - \nabla_\mu\nabla_\nu\mathcal{F}'_R - 2\nabla_\alpha\nabla_\beta[\mathcal{F}'_P\, R^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}] + \Box(\mathcal{F}'_P\, R_{\mu\nu}) \\
&+ g_{\mu\nu}\,\nabla_\alpha\nabla_\beta(\mathcal{F}'_P\, R^{\alpha\beta}) - 4\nabla_\alpha\nabla_\beta[\mathcal{F}'_Q\, R^{\alpha}{}_{\mu\nu}{}^{\beta}] = 8\pi G\, T_{\mu\nu}\,.
\end{aligned} \tag{1.43}$$

Here, we are using $\mathcal{F}(R,P,Q) \equiv \mathcal{F}$ and

$$\mathcal{F}'_R \equiv \frac{\partial\mathcal{F}}{\partial R}\,, \quad \mathcal{F}'_P \equiv \frac{\partial\mathcal{F}}{\partial P}\,, \quad \mathcal{F}'_Q \equiv \frac{\partial\mathcal{F}}{\partial Q}\,. \tag{1.44}$$

The trace of Eq. (1.43) is

$$\nabla^2\left(3\mathcal{F}'_R + R\mathcal{F}'_P\right) + 2\nabla_\mu\nabla_\nu\left[\left(\mathcal{F}'_P + 2\mathcal{F}'_Q\right)R^{\mu\nu}\right] - 2\mathcal{F} + R\mathcal{F}'_R + 2\left(\mathcal{F}'_P + \mathcal{F}'_Q\right) = \kappa^2 \mathrm{T}^{(\mathrm{matter})}\,. \tag{1.45}$$

Requiring $R = R_0$, $P = P_0$, and $Q = Q_0$, where R_0, P_0, Q_0 are constants, one has the de Sitter existence condition in vacuum

$$2\mathcal{F}_{(0)} - R_0\mathcal{F}'_{R(0)} - 2P_0\mathcal{F}'_{P(0)} - 2Q_0\mathcal{F}'_{Q(0)} = 0\,. \tag{1.46}$$

The adding subscript '0' indicates that the function is evaluated on R_0, P_0, Q_0 (ex: $\mathcal{F}_{(0)} = \mathcal{F}(R_0, P_0, Q_0)$). Perturbing around de Sitter-space, namely $R = R_0 + \delta R$, $P = P_0 + \delta P$ and $Q = Q_0 + \delta Q$, observing that[4] $P_0 = R_0^2/4$ and $Q_0 = R_0^2/6$, and $\delta P = (R_0/2)\delta R$ and $\delta Q = (R_0/3)\delta R$, one arrives at the perturbation equation

$$\left(\Box - M^2\right)\delta R = \mathcal{O}(\delta R^2)\,, \tag{1.47}$$

in which the scalaron effective mass reads

$$M^2 = \frac{R_0}{3}\left(\frac{\mathcal{F}'_{R(0)} + \frac{2R_0}{3}\left(\mathcal{F}'_{P(0)} + \mathcal{F}'_{Q(0)}\right)}{R_0\left[A_{R(0)} + A_{P(0)} + A_{Q(0)} + \frac{2}{3}\left(\mathcal{F}'_{P(0)} + \mathcal{F}'_{Q(0)}\right)\right]} - 1\right)\,, \tag{1.48}$$

where

$$A_{R(0)} = \left(\mathcal{F}''_{RR} + \frac{R_0}{2}\mathcal{F}''_{RP} + \frac{R_0}{3}\mathcal{F}''_{RQ}\right)\Big|_{R_0,P_0,Q_0}\,, \tag{1.49}$$

$$A_{P(0)} = \frac{R_0}{3}\left(\mathcal{F}''_{RQ} + \frac{R_0}{2}\mathcal{F}''_{QP} + \frac{R_0}{3}\mathcal{F}''_{QQ}\right)\Big|_{R_0,P_0,Q_0}\,, \tag{1.50}$$

$$A_{Q(0)} = \frac{R_0}{2}\left(\mathcal{F}''_{RP} + \frac{R_0}{2}\mathcal{F}''_{PP} + \frac{R_0}{3}\mathcal{F}''_{PQ}\right)\Big|_{R_0,P_0,Q_0}\,. \tag{1.51}$$

Thus, if $M^2 > 0$, one has stability of the de Sitter solution. In the particular case $\mathcal{F}(R,P,Q) = \mathcal{F}(R,G)$, one has [30, 31]

$$\frac{9\mathcal{F}'_R}{R_0[9\mathcal{F}''_{RR} + 6R_0\mathcal{F}''_{RG} + R_0^2\mathcal{F}''_{GG}]}\Big|_{R_0,G_0} > 1\,. \tag{1.52}$$

We note, $G_0 = R_0^2/6$. In the case of a $F(R)$ model, one finds the condition (1.37).

1.4 Lagrangian derivation and static spherically symmetric metric

In this Section we write the EOM for static spherically symmetric metric (SSS, like the Schwarzschild one) in $\mathcal{F}(R,G)$-gravity. A convenient Lagrangian derivation is shown.

We shall look for SSS solutions of the type,

$$ds^2 = -\mathrm{e}^{2\alpha(r)}B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2\,, \tag{1.53}$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$ and $\alpha(r)$ and $B(r)$ are functions of the sphere radius r. With this

[4]One has:

$$\delta P = 2R_{\mu\nu}\delta R^{\mu\nu} = \frac{2}{4}R_{\mu\nu}g^{\mu\nu}\delta R = \frac{R_0}{2}\delta R\,,$$

$$\delta Q = 2R_{\mu\nu\xi\sigma}\delta R^{\mu\nu\xi\sigma} = \frac{2}{6}R_{\mu\nu\xi\sigma}\frac{(g^{\mu\nu}g^{\xi\sigma} - g^{\nu\sigma}g^{\xi\mu})}{2}\delta R = \frac{R_0}{3}\delta R\,.$$

Ansatz, the scalar curvature R and the Gauss Bonnet invariant G read

$$\begin{aligned} R &= -\frac{1}{r^2}\left[3r^2\left(\frac{dB(r)}{dr}\right)\left(\frac{d\alpha(r)}{dr}\right)+2r^2\,B\left(r\right)\left(\frac{d\alpha(r)}{dr}\right)^2+r^2\left(\frac{d^2B(r)}{dr^2}\right)\right. \\ &\left.+2r^2\,B\left(r\right)\left(\frac{d^2\alpha(r)}{dr^2}\right)+4r\left(\frac{dB(r)}{dr}\right)+4rB(r)\left(\frac{d\alpha(r)}{dr}\right)+2B(r)-2\right], \end{aligned} \tag{1.54}$$

$$\begin{aligned} G &= \frac{4}{r^2}\left[\left(\frac{d\alpha(r)}{dr}\right)\left(\frac{dB(r)}{dr}\right)(5B(r)-3)+\left(\frac{dB(r)}{dr}\right)^2+\left(\frac{d^2B(r)}{dr^2}\right)(B(r)-1)\right. \\ &\left.+2(B(r)-1)B(r)\left(\left(\frac{d\alpha(r)}{dr}\right)^2+\frac{d^2\alpha(r)}{dr^2}\right)\right]. \end{aligned} \tag{1.55}$$

By plugging this expression into the action (1.1), one obtains a higher derivative Lagrangian theory. In order to work with a first derivatives Lagrangian system, we may use the method of Lagrangian multipliers, used for the FRW space-time in Refs. [30, 32, 33]. In the static case we are dealing with, the method permits to consider as independent Lagrangian coordinates the scalar curvature R, the Gauss Bonnet invariant G and the quantities $\alpha(r)$ and $B(r)$, appearing in the spherically static symmetric Ansatz. As a consequence, we will obtain two equations of motion where the quantities $\alpha(r)$ and $B(r)$ appear in a very simple way, as we will see for the special case of $F(R)$-gravity in Chapter **2**. The main difference with respect to the other general approaches is that we do not directly make use of field equation (1.3).

By introducing the Lagrangian multipliers λ and μ and making use of Eq. (1.54)-(1.55), the action (1.1) may be written as

$$\begin{aligned} I \equiv& \frac{1}{2\kappa^2}\int dt\int dr\left(\mathrm{e}^{\alpha(r)}r^2\right)\left\{\mathcal{F}(R,G)-\lambda\left[R+3\left(\frac{d}{dr}B\left(r\right)\right)\frac{d}{dr}\alpha\left(r\right)\right.\right. \\ &+2\,B\left(r\right)\left(\frac{d}{dr}\alpha\left(r\right)\right)^2+\frac{d^2}{dr^2}B\left(r\right)+2\,B\left(r\right)\frac{d^2}{dr^2}\alpha\left(r\right)+\frac{4}{r}\frac{d}{dr}B(r) \\ &\left.+4\,\frac{B\left(r\right)}{r}\frac{d}{dr}\alpha\left(r\right)+2\,\frac{B\left(r\right)}{r^2}-\frac{2}{r^2}\right] \\ -\mu&\left[G-\frac{4}{r^2}\left[\left(\frac{d\alpha(r)}{dr}\right)\left(\frac{dB(r)}{dr}\right)(5B(r)-3)+\left(\frac{dB(r)}{dr}\right)^2+\left(\frac{d^2B(r)}{dr^2}\right)(B(r)-1)\right.\right. \\ +2(B(r)&\left.\left.\left.-1)B(r)\left(\left(\frac{d\alpha(r)}{dr}\right)^2+\frac{d^2\alpha(r)}{dr^2}\right)\right]\right]+\frac{\kappa^2}{\mathrm{e}^{2\alpha(r)}B(r)}(p_\mathrm{m}+\rho_\mathrm{m})-\kappa^2(\rho_\mathrm{m}-p_\mathrm{m})\right\}. \end{aligned} \tag{1.56}$$

Here, we have explicitly written the contribute of the matter Lagrangian $\mathcal{L}^{(\mathrm{matter})}$ (see Note 1). Making the variation with respect to R and G, one gets

$$\lambda=\mathcal{F}'_R(R,G)\,, \tag{1.57}$$

$$\mu=\mathcal{F}'_G(R,G)\,. \tag{1.58}$$

Thus, by substituting this values and by making an integration by part, the total Lagrangian $\mathscr{L}$ of the system takes the form

$$
\begin{aligned}
\mathscr{L}(\alpha, d\alpha/dr, B, dB/dr, R, dR/dr, G, dG/dr) = \mathrm{e}^{\alpha(r)} \Big\{ & r^2 \left(\mathcal{F} - \mathcal{F}'_R R - \mathcal{F}'_G G\right) \\
& + 2\mathcal{F}'_R \left(1 - r\frac{dB(r)}{dr} - B(r)\right) + \mathcal{F}''_{RR}\frac{dR}{dr} r^2 \left(\frac{dB(r)}{dr} + 2B(r)\frac{d\alpha(r)}{dr}\right) \\
& - \mathcal{F}''_{GG}\frac{dG}{dr}\left(4\frac{dB(r)}{dr} + 8B(r)\frac{d\alpha(r)}{de}\right)(B(r) - 1) \\
& + \frac{\kappa^2 r^2}{\mathrm{e}^{2\alpha(r)} B(r)}(p_{\mathrm{m}} + \rho_{\mathrm{m}}) - \kappa^2 r^2 (\rho_{\mathrm{m}} - p_{\mathrm{m}}) \Big\} \,.
\end{aligned} \tag{1.59}
$$

It is easy to see that, if $\mathcal{F}'_G = \text{const}$, i.e. the Gauss Bonnet simply is an additive term, the contribute of Gauss-Bonnet vanishes. Making the variation with respect to $\alpha(r)$ and with respect to $B(r)$, one finally get the EOM:

$$
\begin{aligned}
& \mathrm{e}^{\alpha(r)} \Big\{ r^2(\mathcal{F} - \mathcal{F}'_R R - \mathcal{F}'_G G) + 2\mathcal{F}'_R \left[1 - r\left(\frac{dB(r)}{dr}\right) - B(r)\right] - \frac{d\mathcal{F}'_R}{dr}\left[r^2\left(\frac{dB(r)}{dr}\right) + 4rB(r)\right] \\
& -2r^2 B(r)\frac{d^2\mathcal{F}'_R}{dr^2} + 4(3B(r) - 1)\left(\frac{dB(r)}{dr}\right)\frac{d\mathcal{F}'_G}{dr} + 8B(r)(B(r) - 1)\frac{d^2\mathcal{F}'_G}{dr^2}\Big\} = \\
& \kappa^2 r^2 \mathrm{e}^{\alpha(r)} \left(\frac{(p_{\mathrm{m}} + \rho_{\mathrm{m}})}{\mathrm{e}^{2\alpha(r)} B(r)} - (p_{\mathrm{m}} - \rho_{\mathrm{m}})\right) \,,
\end{aligned} \tag{1.60}
$$

$$
\begin{aligned}
& \mathrm{e}^{\alpha(r)} \Big\{ \frac{d\alpha(r)}{dr}\left(2r\mathcal{F}'_R + r^2\frac{d\mathcal{F}'_R}{dr} - 4(3B(r) - 1)\frac{d\mathcal{F}'_G}{dr}\right) - r^2\frac{d^2\mathcal{F}'_R}{dr^2} + 4(B(r) - 1)\frac{d^2\mathcal{F}'_G}{dr^2}\Big\} = \\
& \kappa^2 r^2 \mathrm{e}^{\alpha(r)} \left(\frac{(p_{\mathrm{m}} + \rho_{\mathrm{m}})}{\mathrm{e}^{2\alpha(r)} B(r)^2}\right) \,.
\end{aligned} \tag{1.61}
$$

The above equations with Eqs. (1.54)-(1.55) form a system of four ordinary differential equations in the four unknown quantities $\alpha(r)$, $B(r)$, $R = R(r)$ and $G = G(r)$. By explicitly written R and G in Eqs. (1.60)-(1.61) as functions of $B(r)$ and $\alpha(r)$, we reduce the system to two differential equations.

When $\mathcal{F}(R, G) = R$ and $\rho_{\mathrm{m}} = p_{\mathrm{m}} = 0$ (vacuum space), the equations lead to the Schwarzschild solution, namely

$$
\alpha(r) = \text{const} \,, \tag{1.62}
$$

$$
B(r) = \left(1 - \frac{2MG_N}{r}\right) \,. \tag{1.63}
$$

Here, we have introduced the Newton constant G_N, so that M is a mass constant. The Ricci scalar is $R = 0$.
Another well known vacuum solution when $\mathcal{F}(R, G)$ is a function of R only ($\mathcal{F}(R, G) = F(R)$) is the one associated with R constant. As a result, with $\alpha = \text{const}$, Eq. (1.61) is trivially satisfied, and Eq. (1.54) with Eq. (1.60) lead to the Schwarschild-de Sitter solution

$$
B(r) = \left(1 - \frac{2MG_N}{r} - \frac{\Lambda r^2}{3}\right) \,, \tag{1.64}
$$

when the de Sitter condition (1.31) is verified. Here, M is an integration constant again and $\Lambda = \text{const}$ so that $R = 4\Lambda$.

It is easy to see that we can obtain the Schwarzshild-de Sitter solution also by considering the GR case, $\mathcal{F}(R,G)=R$, in the presence of a dark energy fluid with $\rho_{\rm DE}=\Lambda/\kappa^2$ such that $p_{\rm DE}=-\rho_{\rm DE}$, $\rho_{\rm DE}$ and $p_{\rm DE}$ being the energy density and pressure of dark energy fluid, respectively.

In Gauss Bonnet gravity $\mathcal{F}(R,G)=R+f(G)$, where $f(G)$ is a function of the Gauss Bonnet invariant only, the vacuum pure de Sitter solution,

$$B(r)=\left(1-\frac{\Lambda r^2}{3}\right), \tag{1.65}$$

with $\alpha=$ const, always exists. In this case, $R=4\Lambda$ and $G=8\Lambda^2/3$ are constants, and Λ, in principle, is the solution of Eq. (1.60), namely $f(G)-G(df(G)/dG)+2\Lambda=0$.

1.5 Conformal transformations in $F(R)$-gravity

In (non-minimally) scalar-tensor theories of gravity, a scalar field strongly coupled to the metric field through the Ricci scalar in the action is used. The first model of scalar-tensor theory was proposed by Brans & Dicke in 1961 [34], trying to incorporate Mach's principle into the theory of gravity. In Brans-Dicke theory a scalar field ϕ, whose kinetic term is proportional to $1/\phi$, is coupled with the Ricci scalar. Furthermore, in scalar tensor theories, a potential $V(\phi)$ of scalar field may appear. The success of this kind of theories principally is related with the possibility to reproduce the primordial acceleration of the universe, namely the inflation.

A modified gravity theory may be rewritten in scalar-tensor or Einstein frame form. We analyze the case of $F(R)$-gravity. One can rewrite the Jordan frame action of Eq. (1.24) by introducing a scalar field which couples to the curvature. Of course, this is not exactly physically-equivalent formulation, as it is explained in Ref. [35]. However, Einstein frame formulation may be used for getting some of intermediate results in simpler form (especially, when the matter is not accounted for).

Let us introduce the field A into Eq. (1.24):

$$I_{JF}=\frac{1}{2\kappa^2}\int_{\mathcal{M}}\sqrt{-g}\,[F'(A)\,(R-A)+F(A)]\,d^4x\,. \tag{1.66}$$

Here 'JF' means 'Jordan frame' and we neglect the contribute of matter. By making the variation of the action with respect to A, we have $A=R$. The scalar field σ is defined as

$$\sigma=-\ln[F'(A)]\,. \tag{1.67}$$

Consider now the following conformal transformation of the metric,

$$\tilde{g}_{\mu\nu}=\mathrm{e}^{-\sigma}g_{\mu\nu}\,, \tag{1.68}$$

for which we get the 'Einstein frame' (EF) action of the scalar field σ:

$$\begin{aligned} I_{EF} &= \frac{1}{2\kappa^2}\int_{\mathcal{M}}d^4x\sqrt{-\tilde{g}}\left\{\tilde{R}-\frac{3}{2}\left(\frac{F''(A)}{F'(A)}\right)^2\tilde{g}^{\mu\nu}\partial_\mu A\partial_\nu A-\frac{A}{F'(A)}+\frac{F(A)}{F'(A)^2}\right\} \\ &= \frac{1}{2\kappa^2}\int_{\mathcal{M}}d^4x\sqrt{-\tilde{g}}\left(\tilde{R}-\frac{3}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma-V(\sigma)\right)\,, \end{aligned} \tag{1.69}$$

where

$$V(\sigma)\equiv\frac{A}{F'(A)}-\frac{F(A)}{F'(A)^2}=\mathrm{e}^{\sigma}R(\mathrm{e}^{-\sigma})-\mathrm{e}^{2\sigma}F[R(\mathrm{e}^{-\sigma})]\,. \tag{1.70}$$

Here, $R(\mathrm{e}^{-\sigma})$ is the solution of Eq. (1.67) with $A=R$, becoming R a function of $\mathrm{e}^{-\sigma}$, and $\tilde{R}$ denotes the Ricci scalar evaluated with respect to the conformal metric $\tilde{g}_{\mu\nu}$. Furthermore, $\tilde{g}$ denotes the determinant of conformal metric, $\tilde{g}=\mathrm{e}^{-4\sigma}g$.

Let us consider a matter Lagrangian. After the scale transformation $g_{\mu\nu} \to \mathrm{e}^{-\sigma} g_{\mu\nu}$ is done, there appears a coupling of the scalar field σ with matter. For example, if matter is a scalar field Φ, with mass M_Φ, whose action is given by

$$I_{JF(\Phi)} = \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - M_\Phi^2\Phi^2\right), \tag{1.71}$$

then there appears a coupling with σ (in this Einstein frame):

$$I_{EF(\Phi)} = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\left(-\mathrm{e}^{\sigma}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - M_\Phi^2\mathrm{e}^{2\sigma}\Phi^2\right). \tag{1.72}$$

The strength of the coupling is of the same order as that of the gravitational coupling, κ^2 in Eq. (1.69). Unless the mass corresponding to σ, which is defined by m_σ^2 so that

$$m_\sigma^2 \equiv \frac{3}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{3}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{(F'(A))^2} + \frac{1}{F''(A)}\right\}, \tag{1.73}$$

is big, the system is unstable. If on the de Sitter solution the scalaron mass defined by Eq. (1.34) is exactly equal to one, it is necessary to check stability with a more detailed investigation on the mass of σ in conformal transformation.

Conformal FRW metric

By using a Lagrangian derivation in a similar way of § 1.4, we complete this Section by giving the FRW and the SSS-conformal equations of motion.

Let us consider the conformal transformation (1.68) of FRW metric (1.5), namely

$$d\tilde{s}^2 = -N(t)^2\mathrm{e}^{-\sigma(t)}dt^2 + a(t)^2\mathrm{e}^{-\sigma(t)}d\mathbf{x}^2. \tag{1.74}$$

Here, $\sigma(t)$ is a function of t. The scalar curvature $\tilde{R}$ reads

$$\begin{aligned}\tilde{R} &= 6e^{\sigma(t)}\left(\frac{\ddot{a}(t)}{a(t)N(t)^2} + \frac{\dot{a}(t)^2}{a(t)^2N(t)^2} - \frac{\dot{a}(t)\dot{N}(t)}{a(t)N(t)^3}\right)\\ &+3e^{\sigma(t)}\left(\frac{\dot{\sigma}(t)^2}{2N(t)^2} + \frac{\dot{N}(t)\dot{\sigma}(t)}{N(t)^3} - \frac{3\dot{\sigma}(t)\dot{a}(t)}{a(t)N(t)^2} - \frac{\ddot{\sigma}(t)}{N(t)^2}\right).\end{aligned} \tag{1.75}$$

If we put $\sigma(t) = 0$ and $N(t) = 1$, we obtain Eq. (1.8). By plugging this expression into the Einstein frame action (1.69) with conformal metric (1.74) and by making an integration by part, one arrives at the Lagrangian,

$$\begin{aligned}&\mathscr{L}\left(a(t), \dot{a}(t), N(t), \dot{N}(t), \sigma(t), \dot{\sigma(t)}\right) =\\ &\frac{6e^{-\sigma(t)}}{N(t)}\left[\dot{a}(t)a(t)^2\dot{\sigma}(t) - \dot{a}(t)^2a(t)\right] - V(\sigma(t))a(t)^3N(t)e^{-2\sigma(t)},\end{aligned} \tag{1.76}$$

and we deal with a first a first derivatives Lagrangian system. The Hamilton-Jacobi equations of

$N(t)$, $a(t)$ and $\sigma(t)$, give the following EOM for the gauge $N(t)=1$:

$$\begin{aligned} 6H\left(H-\dot{\sigma}(t)\right) &= V(\sigma(t))e^{-\sigma(t)}\,, &(1.77)\\ 2\left(3H^2+2\dot{H}\right) &= V(\sigma(t))e^{-\sigma(t)}\,, &(1.78)\\ 3\left(2H^2+\dot{H}\right) &= e^{-\sigma(t)}\left(V(\sigma(t))-\frac{1}{2}\frac{dV(\sigma(t))}{d\sigma}\right)\,. &(1.79)\end{aligned}$$

Note that, due to the presence of scalar field $\sigma(r)$, in conformal theories we work with an additional equation of motion. If $V(\sigma(t))=0$, the latter equation is redundant.

Conformal SSS metric

Let us consider the conformal transformation (1.68) of SSS metric (1.53), namely

$$d\tilde{s}^2=-B(r)\mathrm{e}^{2\alpha(r)-\sigma(r)}dt^2+\frac{dr^2}{B(r)\mathrm{e}^{\sigma(r)}}+r^2\mathrm{e}^{-\sigma(r)}d\Omega\,. \tag{1.80}$$

Here, $\sigma(r)$ is a function of r. The scalar curvature $\tilde{R}$ reads

$$\begin{aligned}\tilde{R} &= -2\mathrm{e}^{\sigma(r)}B(r)\left(\frac{d\alpha(r)}{dr}\right)^2-\frac{4\mathrm{e}^{\sigma(r)}B(r)}{r}\left(\frac{d\alpha(r)}{dr}\right)-3\mathrm{e}^{\sigma(r)}B_r(r)\left(\frac{d\alpha(r)}{dr}\right)\\ &\quad -2\mathrm{e}^{\sigma(r)}B(r)\left(\frac{d^2\alpha(r)}{dr^2}\right)+3\mathrm{e}^{\sigma(r)}B(r)\left(\frac{d\alpha(r)}{dr}\right)\left(\frac{d\sigma(r)}{dr}\right)+\frac{6\mathrm{e}^{\sigma(r)}B(r)}{r}\left(\frac{d\sigma(r)}{dr}\right)\\ &\quad +3\mathrm{e}^{\sigma(r)}\left(\frac{dB(r)}{dr}\right)\left(\frac{d\sigma(r)}{dr}\right)-\frac{3}{2}\mathrm{e}^{\sigma(r)}B(r)\left(\frac{d\sigma(r)}{dr}\right)^2+3\mathrm{e}^{\sigma(r)}B(r)\left(\frac{d^2\sigma(r)}{dr^2}\right)\\ &\quad -\frac{4\mathrm{e}^{\sigma(r)}}{r}\left(\frac{dB(r)}{dr}\right)-\mathrm{e}^{\sigma(r)}\left(\frac{d^2B(r)}{dr^2}\right)-\frac{2\mathrm{e}^{\sigma(r)}B(r)}{r^2}+\frac{2\mathrm{e}^{\sigma(r)}}{r^2}\,.\end{aligned} \tag{1.81}$$

If we put $\sigma(r)=0$, we obtain Eq. (1.54). By plugging this expression into the Einstein frame action (1.69) with conformal metric (1.80), and by making an integration by part, one arrives at the Lagrangian,

$$\begin{aligned}&\mathscr{L}(\alpha(r),d\alpha(r)/dr,B(r),dB(r)/dr,\sigma(r),d\sigma(r)/dr) =\\ &e^{\alpha(r)-\sigma(r)}\left(2-2B(r)-2\frac{B(r)}{dr}r-\frac{\sigma(r)}{dr}\frac{B(r)}{dr}r^2-2\frac{\alpha(r)}{dr}\frac{\sigma(r)}{dr}B(r)r^2-e^{-\sigma(r)}V(r)r^2\right)\,.\end{aligned} \tag{1.82}$$

The Hamilton-Jacobi equations of $\alpha(r)$, $B(r)$ and $\sigma(r)$, give the following EOM:

$$\begin{aligned}&2-2\left(\frac{dB(r)}{dr}\right)r-2B(r)+\left(\frac{dB(r)}{dr}\right)\left(\frac{d\sigma(r)}{dr}\right)r^2+2B(r)\left(\frac{d^2\sigma(r)}{dr^2}\right)r^2\\ &\quad+4r\left(\frac{d\sigma(r)}{dr}\right)B(r)-2r^2\left(\frac{d\sigma}{dr}\right)^2B(r)=r^2\mathrm{e}^{-\sigma(r)}V(\sigma)\,,\end{aligned} \tag{1.83}$$

$$2r\left(\frac{d\alpha(r)}{dr}\right)-\left(\frac{d\alpha(r)}{dr}\right)\left(\frac{d\sigma(r)}{dr}\right)r^2+\left(\frac{d^2\sigma(r)}{dr^2}\right)r^2-\left(\frac{d\sigma(r)}{dr}\right)^2 r^2=0\,, \tag{1.84}$$

$$\begin{aligned}&-\left(\frac{d^2B(r)}{dr^2}\right)r^2-4\left(\frac{dB(r)}{dr}\right)r-3\left(\frac{dB(r)}{dr}\right)\left(\frac{d\alpha(r)}{dr}\right)r^2-2\left(\frac{d\alpha(r)}{dr}\right)^2 B(r)r^2\\&-2\left(\frac{d^2\alpha(r)}{dr^2}\right)B(r)r^2-4\left(\frac{d\alpha(r)}{dr}\right)B(r)r-2B(r)+2=r^2\mathrm{e}^{-\sigma(r)}\left(2V(\sigma)-\frac{dV(\sigma)}{d\sigma}\right)\,.\end{aligned} \tag{1.85}$$

The scalar field formulation may be used also in Gauss-Bonnet modified gravity. In Ref. [36] a non-local model of modified gravity, which depends on Gauss Bonnet and other higher-derivative invariants (like $\Box^{-1}G$), is presented. By introducing a scalar field coupled with the metric through the Gauss-Bonnet invariant, it is shown that a local form can be obtained, and the analysis of the model (i.e., the derivation of FRW or SSS equations of motion) results considerably simplified.

Chapter 2

The black hole solutions and the energy issue in modified gravity

In modified gravity, it is of crucial interest the existence and the properties of black holes (BH). Static, spherically symmetric solutions have been investigated in several papers. Typically, modified models admit the de Sitter space as a solution, but the issue to find exact SSS metrics different from the Schwarzschild-dS one appears a formidable task, since also for a reasonable model, the equations of motion are much more complicated with respect to the ones in vacuum of GR. Furthermore, within the class of higher order gravitational models, the issue associated with the energy (mass) of black hole solutions is problematic (in Ref. [37], Visser refers to the BH in theories of modified gravity as "dirty black holes"). Several attempts in order to find a satisfactory answer to the mass problem have been investigated (see for example Refs. [38, 39, 40] and references therein). In this Chapter we will exhibit some non trivial SSS-solutions of $F(R)$-gravity and an expression for the associated BH-energy is proposed and identified with a quantity proportional to the constant of integration, which appears in the explicit solutions. The identification is achieved making use of derivation of the First Law of black hole thermodynamics from the equations of motion of $F(R)$-gravity, evaluating independently the entropy via Wald method [41] and the Hawking temperature [42] via quantum mechanical methods in curved space-times. The results are extended to general class of modified gravity theories and several non trivial examples are discussed. This work has been developed in Refs. [17], [20], [21].

2.1 $F(R)$-static spherically symmetric solutions

In § **1.4** we have written the equations of $F(R,G)$-gravity in the case of static spherically symmetric-metric of Eq. (1.53). In this Section, we will consider the simple case of vacuum solutions in $F(R)$-gravity.
For the specific choice $R^{1+\delta}$, with δ a real parameter, a class of exact SSS solutions has been presented by Barrow & Clifton (2005) in Ref. [43]. In higher order gravitational models, namely by adding to Einstein-Hilbert Lagrangian a non-polynomial contribution of the type $\sqrt{C^2}$, with $C_{\mu\nu\xi\sigma}$ being the Weyl tensor, a SSS solution has been proposed by Deser, Sarioglu & Tekin (2008) in Ref. [44].
A general discussion on SSS solutions has been presented in Refs. [45, 46, 47, 48], where one can find further references.

Here, we would like to present the exact SSS solutions shown in Ref. [20]. They are found in a simple way by starting from the EOM (1.60)-(1.61). In particular, we are able to present the most general form of the important class of SSS metric with $\alpha(r) = \text{const}$ that can be realized in $F(R)$-gravity. In absence of matter, by putting $\mathcal{F}(R,G) = F(R)$, Eqs. (1.60)-(1.61) read:

$$
\begin{aligned}
&\mathrm{e}^{\alpha(r)}\left\{RF'(R)-F(R)-2\left(\frac{1-B(r)-r(dB(r)/dr)}{r^2}\right)F'(R)\right. &(2.1)\\
&\left.+2B(r)\left[\frac{d^2R}{dr^2}+\left(\frac{2}{r}+\frac{(dB(r)/dr)}{2B(r)}\right)\frac{dR}{dr}+\frac{F'''(R)}{F''(R)}\left(\frac{dR}{dr}\right)^2\right]F''(R)\right\}=0\,,\\
&\mathrm{e}^{\alpha(r)}\left[\frac{d\alpha(r)}{dr}\left(\frac{2}{r}+\frac{F''(R)}{F'(R)}\frac{dR}{dr}\right)-\frac{F''(R)}{F'(R)}\frac{d^2R}{dr^2}-\frac{F'''(R)}{F'(R)}\left(\frac{dR}{dr}\right)^2\right]=0\,. &(2.2)
\end{aligned}
$$

Once $F(R)$ is given, together with equation (1.54), the above equations form a system of three differential equations in the three unknown quantities $\alpha(r), B(r)$ and $R=R(r)$. We note that one advantage of this system (in the case of $F(R)$-gravity) is that the $B(r)$ variable does not explicitly appear in Eq. (2.2) and vice versa for $\alpha(r)$.

2.1.1 Solutions with constant $\alpha(r)$

Now, let us consider the case of $\alpha(r)=$ const. We can directly put $\alpha(r)=0$ without loss of generality. In general, the Ricci scalar R is not constant. From Eq. (2.2) one has:

$$
F'''\left(\frac{dR}{dr}\right)^2+F''\left(\frac{d^2R}{dr^2}\right)=\frac{d^2}{dr^2}F'(R)=0\,. \tag{2.3}
$$

Thus,

$$
F'(R)=ar+b\,. \tag{2.4}
$$

Here, a and b are two integration constants, b is adimensional. If we give the explicit form of $R=R(r)$, we may find r as a function of Ricci scalar and reconstruct $F'(R)$ realizing such solution. The equation (1.54) leads to:

$$
R=-\frac{d^2B(r)}{dr^2}-\frac{4}{r}\frac{dB\,(r)}{dr}-2\,\frac{B\,(r)}{r^2}+\frac{2}{r^2}\,. \tag{2.5}
$$

Since $(F''(R))(dR/dr)\equiv dF'(R)/dr=a$ and $dF(R)/dr\equiv(F'(R))dR/dr$, by deriving Eq. (2.1) with respect to r, one has

$$
-\frac{d^2B(r)}{dr^2}\left(a+\frac{b}{r}\right)+\frac{2a}{r^2}\left(2B(r)-1\right)+\frac{2b}{r^3}(B(r)-1)-\frac{a}{r}\frac{dB(r)}{dr}=0\,. \tag{2.6}
$$

If $b=0$, the general solution is

$$
B(r)=\frac{1}{2}\left(1-\frac{C_1}{r^2}+C_2r^2\right)\,, \tag{2.7}
$$

where C_1 and C_2 are constants. Eq. (2.5) leads to

$$
R=\frac{1}{r^2}-6C_2\,, \tag{2.8}
$$

so that

$$
F'(R)=a\sqrt{\frac{1}{R+6C_2}}\,. \tag{2.9}
$$

We then can reconstruct $F(R)$ and we get the model

$$F(R) = \frac{a}{2}\sqrt{R + 6C_2}\,, \tag{2.10}$$

for which Eq. (2.7) is a spherically symmetric static solution with $\alpha(r) = 0$. If we put $C_2 = 0$, we recover a special case of Clifton-Barrow solution [43]. If $C_1 = 0$, we recover a special case reported in Ref. [45].

The most general solution of Eq. (2.6) for $b \neq 0$ is

$$\begin{aligned} B(r) \;=\; & \Big[-\frac{r^2 C_2 \log(r)\, a^3}{b^4} + \frac{r^2 C_2 \log(b+ar)\, a^3}{b^4} + \frac{3r^2 a^2}{2b^2} - \frac{r C_2 a^2}{b^3} - \frac{r^2 \log(r)\, a^2}{b^2} \\ & + \frac{r^2 \log(b+ar)\, a^2}{b^2} - \frac{ra}{b} + \frac{C_2 a}{2b^2} + r^2 C_1 - \frac{C_2}{3br} + 1\Big]\,, \end{aligned} \tag{2.11}$$

where C_1 and C_2 are generic constants again. As a check, it is easy to see that, if $a = 0$, the solution of this equation is the Schwarzschild-dS one (1.64), which corresponds to $F'(R) = b$, and one possibility is GR plus a cosmological constant F_0, $F(R) = bR + F_0$, with a constrain on F_0. Eq. (2.11) is the starting point of the reconstruction method and it is compatible with the result obtained in Ref. [48]. Implicitly $F(R)$ is determined by Eqs. (2.4), (2.5) and (2.11). If a $F(R)$-model realizes the metric

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2\,, \tag{2.12}$$

the coefficient $B(r)$ assumes the generic form of Eq. (2.11).

On the other hand, we have to note that, since Eq. (2.6) has been obtained trough a derivation, the left side of Eq. (2.1) evaluated on the solution (2.11) could be different to zero, so that some other constraints on free parameters could be necessary. As an example, let us consider the simple case where

$$C_2 = -\frac{b^2}{a}\,. \tag{2.13}$$

Eq. (2.11) becomes

$$B(r) = \frac{1}{2} + \frac{b}{3ar} + \frac{3a^2 r^2}{2b^2} + C_1 r^2\,, \tag{2.14}$$

and the Ricci scalar reads

$$R = \frac{1}{r^2} - 12C_1 - \frac{18a^2}{b^2}\,. \tag{2.15}$$

By using Eq. (2.4) one has

$$F(R) = b\left(R + 2\left(\frac{a}{b}\right)\sqrt{R + \frac{18a^2}{b^2} + 12C_1}\right)\,. \tag{2.16}$$

Now, we find that the left side of Eq. (2.1) is $(9(a^2/b^2) + 6C_1)b$, so that we have to require $C_1 = -(3/2)(a^2/b^2)$. As a result, the reconstruction gives

$$F(R) = R + 2a\sqrt{R}\,. \tag{2.17}$$

Since b is a global adimensional parameter, we are assuming $b = 1$. This model admits a spherically symmetric solution of the type (2.12) with

$$B(r) = \frac{1}{2}\left(1 + \frac{2}{3ar}\right)\,, \tag{2.18}$$

and Ricci curvature $R = 1/r^2$.

2.1.2 Solutions with non constant $\alpha(r)$

Suppose to have a non constant $\alpha(r)$. Thus, in order to find some solutions, we make the Ansatz

$$\alpha(r) = \frac{1}{2}\log\left(\frac{r}{r_0}\right)^q , \tag{2.19}$$

$$R = \lambda \cdot r^s , \tag{2.20}$$

where r_0, q, λ and s are constants. From Eq. (2.2), it follows that the modified gravity model which realizes this kind of solution is

$$F(R) = k(R^\gamma + \mu) , \tag{2.21}$$

where k and μ are dimensional constants and γ is given by

$$\gamma = \frac{(2+q+4s)s \pm s\sqrt{q^2+20q+4}}{4s^2} . \tag{2.22}$$

If we put $s = -2$, one has $q = 2(\gamma-1)(2\gamma-1)/(2-\gamma)$. In this case, Eq. (2.1) has a simple solution by choosing $\mu = 0$ and $\lambda = 6\gamma(\gamma-1)/(2\gamma^2-2\gamma-1)$. Thus, by rescaling γ as $\gamma = \delta+1$, one recovers the Clifton-Barrow solution [43] for the following model:

$$F(R) = R^{\delta+1}(\kappa^2)^\delta . \tag{2.23}$$

Motivated by dimensional reasons, we have written k in terms of the Newton constant encoded in κ^2. When $\delta = 0$ the Hilbert-Einstein term of GR is recovered. Note that in this model the modification with respect GR is not additive. Clifton-Barrow solution reads

$$\alpha(r) = \log\left[\left(\frac{r}{r_0}\right)^{\delta(1+2\delta)/(1-\delta)}\left(\frac{(1-2\delta+4\delta^2)(1-2\delta-2\delta^2)}{(1-\delta)^2}\right)^{1/2}\right] , \tag{2.24}$$

$$B(r) = \frac{(1-\delta)^2}{(1-2\delta+4\delta^2)(1-2\delta-2\delta^2)}\left(1-\frac{C}{r^{(1-2\delta+4\delta^2)/(1-\delta)}}\right) . \tag{2.25}$$

Here, C is an integration constant. This is consistent with Eq. (1.54), compatibly with assumption (2.20) leading to

$$R = \frac{6\delta(1+\delta)}{(2\delta^2+2\delta-1)}\left(\frac{1}{r^2}\right) . \tag{2.26}$$

However, there exists also another solution with $q = 1$ and $s = -1$. It means, $\gamma = -1$ or $\gamma = 3/2$. Eq. (1.54) is solved by

$$B(r) = \frac{4}{7} - \frac{2}{3\sqrt{-6\mu}}r + \frac{C_1}{r^{7/2}} + \frac{C_2}{r} . \tag{2.27}$$

Here, C_1 and C_2 are arbitrary constants and μ is related with λ. On the other hand, it is easy to see that Eq. (2.1) is inconsistent if $\gamma = 3/2$, but admits a solution for $C_2 = 0$ if $\gamma = -1$ and $\lambda = \sqrt{-6/\mu}$, with μ negative. As a consequence, we may put $\mu = -h^2/6$, $h > 0$, and we have found that the model

$$F(R) = k\left(\frac{1}{R} - \frac{h^2}{6}\right) , \tag{2.28}$$

admits the following SSS metric

$$ds^2 = -\left(\frac{r}{r_0}\right)B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega , \tag{2.29}$$

with

$$B(r) = \frac{4}{7}\left(1 - \frac{7}{6h}r - \frac{C}{r^{7/2}}\right), \tag{2.30}$$

where C is a constant. The Ricci scalar becomes $R = 6/(hr)$.

We note that the functional freedom in the choice of $F'(R)$ gives the possibility of its cosmological reconstruction. In fact, following the construction developed above, we can generate more SSS metrics in $F(R)$-gravity via its reconstruction.

2.2 $F(R)$-black holes

The static solutions describe a black hole as soon as there exists a real positive solution $r = r_H$, such that

$$B(r_H) = 0, \tag{2.31}$$

$$\left.\frac{dB(r)}{dr}\right|_{r_H} \neq 0. \tag{2.32}$$

If this happens, there exists an event horizon[1].

For example, for the model $F(R) = k(\sqrt{R + 6C_2})$ of Eq. (2.10), where k is a dimensional constant, since the positive solutions of $B(r_H) = 0$ read

$$r_{H(1,2)} = \sqrt{\frac{-1 \pm \sqrt{1 + 4C_1C_2}}{2C_2}}, \tag{2.33}$$

one has to require $C_1 > 0$ and $C_2 > 0$ (for the plus sign) or $C_2 < 0$ and $C_1 \lesssim -1/(4C_2)$ (for the minus sign). The first choice leads to the positive sign of $dB(r)/dr|_{r_H}$, and the metric signature changes when $r < r_H$, like in the Schwarzschild BH.

For the other model in Eq. (2.17), we can choose $a = -g$, $g > 0$, thus $F(R) = R - 2g\sqrt{R}$ and we have

$$r_H = \frac{2}{3g}. \tag{2.34}$$

However, we note that any integration constant appears in the solution, since g is a fixed (mass) parameter of the model. This is probably a particular solution of a wider one.

Let us consider the Clifton-Barrow solution (2.25) of the class of models $F(R) = R^{\delta+1}(\kappa^2)^{\delta}$. One has that

$$r_H = C^{(1-\delta)/(1-2\delta+4\delta^2)}, \tag{2.35}$$

determines an event horizon, and, since $C > 0$, this is a BH metric.

For the solution (2.30) corresponding to the model $F(R) = k\left(\frac{1}{R} - \frac{h^2}{6}\right)$, if $C > 0$, with $C > \sqrt{(2/9)^9(3h)^7}$ (we remember, h is positive), we have two positive roots, r_+ and r_{++} such that $r_{++} > r_+$, of

$$1 - \frac{7}{6h}r_H - \frac{C}{r_H^{7/2}} = 0, \tag{2.36}$$

since $B(r \to 0^+, +\infty) \to -\infty$, and $B(r) > 0$ when $dB(r)/dr = 0$. Thus, as in the Schwarzshild-de Sitter space-time, the solutions denote an event horizon (corresponding to the one of the BH) in $r = r_+$, and a cosmological horizon in $r = r_{++}$, due to the fact that $B'(r_+) > 0$ and $B'(r_{++}) < 0$, making the changing of metric signature out of the region $\{r_+, r_{++}\}$. On the other hand, if $C < 0$, one has only a positive root of Eq. (2.36), since $B(r \to 0^+) \to +\infty$ and $B(r \to +\infty) \to -\infty$, and $dB(r)/dr < 0$, namely is non vanishing. In this case we have a cosmological horizon.

[1] We will consider non extremal BH, so that $dB(r)/dr|_{r_H} \neq 0$.

2.3 The First Law of BH-thermodynamics

In order to study the issue associated with the energy of black hole solutions in modified gravity, let us remind the case of GR, in which several notions of quasi-local energies may be introduced. In particular we mention the so called Misner-Sharp mass, which has the important property to be defined for dynamical, spherically symmetric space-time [49], where the use of invariant quantities play a crucial role [50, 51]. For the sake of completeness, we recall that in four dimensions, any spherically symmetric metric can locally be expressed in the form

$$ds^2 = \gamma_{ij}(x^i)dx^i dx^j + \mathcal{R}^2(x^i)d\Omega_2^2\,, \qquad i,j \in \{0,1\}\,, \tag{2.37}$$

where $d\Omega_2^2$ here is the usual metric on the two sphere S^2, but it could be the metric of a generic two-dimensional maximally symmetric space. Of course, in such cases the black hole will have a different topology. The two-dimensional metric

$$d\gamma^2 = \gamma_{ij}(x^i)dx^i dx^j \tag{2.38}$$

is referred to as the normal one. The related coordinates are $\{x^i\}$, while $\mathcal{R}(x^i)$ is the areal radius, considered as a scalar field in the two dimensional normal space. A relevant scalar quantity in the reduced normal space is

$$\chi(x^i) = \gamma^{ij}(x^i)\partial_i\mathcal{R}(x^i)\partial_j\mathcal{R}(x^i)\,, \tag{2.39}$$

since the dynamical trapping horizon, if it exists, is located in correspondence of

$$\chi(x^i)\Big|_H = 0\,, \tag{2.40}$$

provided that $\partial_i\chi(x^i)|_H \neq 0$ (we use the suffix 'H' for all quantities evaluated on the horizon). In the static case, this is equivalent to Eqs. (2.31)-(2.32). The quasi-local Misner-Sharp gravitational energy is defined by [52]

$$E_{MS}(x^i) := \frac{1}{2G_N}\mathcal{R}(x^i)\left[1-\chi(x^i)\right]\,. \tag{2.41}$$

This is an invariant quantity on the normal space. Note also that, on the horizon, $E_{MS}|_H = (1/2)\mathcal{R}(x^i)|_H \equiv E$, where E is the energy of black hole.

Recall that in a non dynamical space-time (static or stationary), one can introduce the Killing vector field $\xi_\mu(x^\nu)$ such that

$$\nabla_\mu\xi^\nu(x^\nu) + \nabla^\nu\xi_\mu(x^\nu) = 0\,, \tag{2.42}$$

with the Killing surface gravity κ_K given by the relation

$$\kappa_K\xi^\mu(x^\nu) = \xi^\nu\nabla_\nu\xi^\mu(x^\nu)\,. \tag{2.43}$$

In the spherical symmetric, dynamical case, the real geometric object which generalizes the Killing vector field is the Kodama vector field $\mathcal{K}(x^i)$ [53]. Given the metric (2.37), it is defined by

$$\mathcal{K}^i(x^i) := \frac{1}{\sqrt{-\gamma}}\varepsilon^{ij}\partial_j\mathcal{R}(x^i)\,,\, i=0,1 \qquad \mathcal{K}^i := 0\,,\, i\neq 0,1 \tag{2.44}$$

ε^{ij} being the completely antisymmetric Levi-Civita tensor on the normal space and γ the determinant associated with γ metric. The Hayward surface gravity associated with dynamical horizon is given by the normal-space scalar

$$\kappa_H := \frac{1}{2}\Box_\gamma\mathcal{R}(x^i)\Big|_H\,, \tag{2.45}$$

where $\Box_\gamma$ is the Laplacian corresponding to the γ metric.

Assuming Einstein equations of GR, in a generic four-dimensional spherically symmetric space-time, a geometric dynamical identity holds true in general. This can be derived as follows. Let us introduce the normal space invariant

$$\mathrm{T}^{(2)}(x^i) = \gamma^{ij} T_{ij}^{(\mathrm{matter})}(x^i) \,, \tag{2.46}$$

which is the reduced trace of the matter stress energy tensor $T_{\mu\nu}^{(\mathrm{matter})} \equiv T_{\mu\nu}^{(\mathrm{matter})}(x^i)$. Then, making use of Einstein equations, it is possible to show that, on the dynamical horizon [49],

$$\kappa_H = \frac{1}{2\mathcal{R}_H G_N} + 2\pi\mathcal{R}_H \frac{\mathrm{T}_H^{(2)}}{G_N} \,, \tag{2.47}$$

where $\mathcal{R}_H = \mathcal{R}(x^i)|_H$ and $\mathrm{T}_H^{(2)} = \mathrm{T}^{(2)}(x^i)|_H$. Introducing the horizon area $\mathcal{A}_H$ and the (formal) three-volume V_H enclosed by the horizon, with their respective 'thermodynamical' differentials $d\mathcal{A}_H = 8\pi\mathcal{R}_H d\mathcal{R}_H$ and $dV_H = 4\pi\mathcal{R}_H^2 d\mathcal{R}_H$ (we are assuming a horizon with the topology of a sphere), we get

$$\frac{\kappa_H}{8\pi G_N} d\mathcal{A}_H = d\left(\frac{\mathcal{R}_H}{2G_N}\right) + \frac{\mathrm{T}_H^{(2)}}{2G_N} dV_H \,. \tag{2.48}$$

This equation can be recast in the form of a geometrical identity, once the Misner-Sharp energy at the horizon (which defines the BH energy E in GR) has been introduced. It reads

$$dE = \frac{\kappa_H}{2\pi} d\left(\frac{\mathcal{A}_H}{4G_N}\right) - \frac{\mathrm{T}_H^{(2)}}{2G_N} dV_H \,. \tag{2.49}$$

Let us restrict the discussion to static case in the absence of matter where the metric in Eq. (2.37) can be written in the simpler form of SSS metric of Eq. (1.53). Of course the general formalism is also valid in the static case, and leads to the horizon conditions of Eqs. (2.31)-(2.32).

The Kodama vector reduces to

$$\mathcal{K}^\mu = \left(\mathrm{e}^{-\alpha(r)}, \vec{0}\right) \,. \tag{2.50}$$

When $\alpha(r) = 0$, which corresponds to the case of GR in vacuum, the static Kodama vector coincides with the usual Killing vector K^μ,

$$K^\mu = (1, \vec{0}) \,. \tag{2.51}$$

It is easy to verify that also the Killing surface gravity defined by Eq. (2.43) coincides with the Hayward's one of Eq. (2.45).

The Killing/Hawking temperature of the related black hole reads

$$T_K^{(\mathrm{GR})} := \frac{\kappa_K}{2\pi} = \frac{1}{4\pi} \frac{dB(r)}{dr}\Big|_{r_H} \,. \tag{2.52}$$

This is a well known result, and it can be justified in several ways, for example making use of standard derivations of Hawking radiation [54], or by eliminating the conical singularity in the corresponding Euclidean metric, or making use of the tunneling method, recently introduced in Refs. [55, 56], and discussed in details in several papers.

In this case, Eq. (2.49) can be written as

$$dE = T_K^{(\mathrm{GR})} d\left(\frac{\mathcal{A}_H}{4G_N}\right) \,. \tag{2.53}$$

However, as we have seen in explicit examples, within modified gravity it happens to deal with black hole solutions with $\alpha(r) \neq 0$. In this case, the Kodama vector (2.50) does not coincide with the Killing vector (2.51). Then Killing surface gravity in Eq. (2.43) reads

$$\kappa_K := \frac{1}{2}\sqrt{\frac{d(\mathrm{e}^{2\alpha(r)}B(r))}{dr}\frac{dB(r)}{dr}}\Big|_{r_H} = \frac{1}{2}\mathrm{e}^{\alpha(r_H)}\frac{dB(r)}{dr}\Big|_{r_H} . \tag{2.54}$$

The Killing/Hawking temperature is

$$T_K := \frac{\kappa_K}{2\pi} = \frac{1}{4\pi}\mathrm{e}^{\alpha(r_H)}\frac{dB(r)}{dr}\Big|_{r_H} . \tag{2.55}$$

However, making use of Eq. (2.45), one has for the Hayward surface gravity

$$\kappa_H := \frac{1}{2}\sqrt{\frac{dB(r)}{dr}\frac{dB(r)}{dr}}\Big|_{r_H} = \frac{1}{2}\frac{dB(r)}{dr}\Big|_{r_H} . \tag{2.56}$$

The Kodama/Hayward temperature reads

$$T_H := \frac{\kappa_H}{2\pi} = \frac{1}{4\pi}\frac{dB(r)}{dr}\Big|_{r_H} , \tag{2.57}$$

which is trivially related to the previous one by $T_K = \mathrm{e}^{\alpha(r_H)}T_H$. If $\alpha(r) = 0$ we recover Eq. (2.52), namely $T_K = T_H$. A detailed discussion about this issue can be found in Refs. [51], in which also the dynamical case is discussed.

In the static case, all derivations of Hawking radiation (see Appendix A for a brief review) leads to a semi-classical expression for the black hole radiation rate Γ,

$$\Gamma \equiv \mathrm{e}^{-\frac{\Delta E_K}{T_K}} , \tag{2.58}$$

in terms of the change ΔE_K of the Killing energy E_K [56], but if one uses the Kodama energy E_H for the emitted particle, one has

$$\Gamma \equiv \mathrm{e}^{-\frac{\Delta E_H}{T_H}} . \tag{2.59}$$

This fact derives by the relationship $\Delta E_H = \mathrm{e}^{-\alpha(r)}\Delta E_K$. From the Eqs. (2.58)-(2.59), one arrives at the identity

$$\frac{\Delta E_H}{T_H} = \frac{\Delta E_K}{T_K} , \tag{2.60}$$

which may interpreted as the First Law of black hole thermodynamics as soon as

$$\Gamma \equiv \mathrm{e}^{-\Delta S} , \tag{2.61}$$

with ΔS the change of the entropy S of the irradiating black hole itself.

With regard to entropy of the black hole, it is well known that in GR the so called Area Law is satisfied, and we have

$$S = \frac{\mathcal{A}_H}{4G_N} . \tag{2.62}$$

In GR and in the static case, the First Law of black hole thermodynamics in vacuum reduces to

$$dE = T_K dS , \tag{2.63}$$

where we remember E is the Misner-Sharp energy evaluated on the horizon. This equation is equivalent to Eq. (2.49) by using Eq. (2.62).

Now we come to the key point of our proposal. For a generic modified gravity theory, it seams very difficult to define in a reasonable way the analogue of the local Misner-Sharp mass (a conserved current can not be found whit high order differential field equations). As we will see, an exception is the higher-dimensional Lovelock gravity.

For this reason, an attempt is made for obtaining an expression of energy associated with black holes solutions in modified theories of gravity. The proposal of Ref. [21] consists in the identification of the black hole energy with a quantity proportional to the constant of integration, which appears in the explicit solutions, and positive defined, in accordance with the sign of the integration constant necessary to define an event horizon. The identification is achieved making use of derivation of the First Law of black hole thermodynamics from the equations of motion, evaluating in an independent way the related black hole entropy via Wald method ($S = S_W$) and the Killing/Hawking temperature via the quantum mechanics in curved space-time, for example the tunneling method [55] or other standard equivalent methods, as in the case of General Relativity. In our proposal, the Killing energy seems to be preferable with respect to the Hayward energy in general theories of gravity.

This approach is also supported by the results obtained in Refs. [57, 58], where, on quite general grounds, generalizing the Jacoboson results on GR (see the seminal paper of Ref. [59]), the equations of a modified gravitational theories are shown to be equivalent to the First Law of black hole thermodynamics. As it is well known, this issue may be of high relevance in substantiating the idea that gravitation might be a manifestation of thermodynamics of quantum vacuum [60].

2.4 Lovelock Black Hole Solutions

In this Section, as warm up, we review Lovelock theory [61] with the related static and spherically symmetric black hole solutions. This theory is a very interesting higher dimensional generalization of Einstein gravity introduced by Lovelock in 1971. It is the most general theory of gravity which conserves second order equations of motion in arbitrary dimensions. In general, by making use of higher order geometrical invariants in the action, in the metric formalism for the field equations one obtains fourth order partial differential equations. However, as Lovelock had shown, one can obtain second order differential equations by making use of higher dimensional extended Euler densities, the so called m-th order Lovelock terms $\mathcal{L}_m$ defined by

$$\mathcal{L}_m = \frac{1}{2^m}\delta^{\lambda_1\sigma_1\cdots\lambda_m\sigma_m}_{\rho_1\kappa_1\cdots\rho_m\kappa_m} R_{\lambda_1\sigma_1}{}^{\rho_1\kappa_1}\cdots R_{\lambda_m\sigma_m}{}^{\rho_m\kappa_m}\ , \qquad m=1,2,3,\dots . \tag{2.64}$$

Here, $R_{\lambda\sigma}{}^{\rho\kappa}$ is the Riemann tensor in arbitrary D-dimensions and $\delta^{\lambda_1\sigma_1\cdots\lambda_m\sigma_m}_{\rho_1\kappa_1\cdots\rho_m\kappa_m}$ is the generalized totally antisymmetric Kronecker delta defined by

$$\delta^{\mu_1\mu_2\cdots\mu_p}_{\nu_1\nu_2\cdots\nu_p} = \det\begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \cdots & \delta^{\mu_1}_{\nu_p} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \cdots & \delta^{\mu_2}_{\nu_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{\mu_p}_{\nu_1} & \delta^{\mu_p}_{\nu_2} & \cdots & \delta^{\mu_p}_{\nu_p} \end{pmatrix} .$$

The action for Lovelock gravitational theory reads

$$I = \int_{\mathcal{M}} d^D x\sqrt{-g}\left[-2\Lambda + \sum_{m=1}^{k}\left\{\frac{a_m}{m}\mathcal{L}_m\right\}\right] , \tag{2.65}$$

where we defined the maximum order $k \equiv [(D-1)/2]$ and a_m are arbitrary constants. Here, $[z]$ represents the maximum integer satisfying $[z] \leq z$. Hereafter, we set $a_1 = 1$.

For such a kind of theory, the equations of motion in vacuum are second order quasi-linear partial differential equations in the metric tensor and read

$$\mathcal{G}_\mu{}^\nu = 0, \tag{2.66}$$

the Lovelock tensor $\mathcal{G}_\mu{}^\nu$ being given by

$$\mathcal{G}_\mu{}^\nu = \Lambda\delta_\mu^\nu - \sum_{m=1}^{k} \frac{1}{2^{m+1}} \frac{a_m}{m} \delta^{\nu\lambda_1\sigma_1\cdots\lambda_m\sigma_m}_{\mu\rho_1\kappa_1\cdots\rho_m\kappa_m} R_{\lambda_1\sigma_1}{}^{\rho_1\kappa_1} \cdots R_{\lambda_m\sigma_m}{}^{\rho_m\kappa_m} \,. \tag{2.67}$$

As we said in previous Section, we shall focus our attention on static, spherically symmetric solutions, thus we look for metric of the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_n^2, \tag{2.68}$$

where $d\Omega_n^2$ is the metric of a n-dimensional sphere S^n (one has $n = D-2$). Such kind of theories become quite interesting for $D > 4$, the four-dimensional case being equivalent to Schwarzschild-de Sitter, since $\mathcal{L}_1 = R$ and $\mathcal{L}_2$ is equal to the Gauss-Bonnet quadratic term, which in four-dimensions is a topological invariant.

A direct evaluation of field equations gives [62]

$$\mathcal{G}_t^t = \mathcal{G}_r^r = -\frac{n}{2r^n} \frac{d\left[r^{n+1}W(r)\right]}{dr} \,, \tag{2.69}$$

$$\mathcal{G}_i^j = -\frac{1}{2r^{n-1}} \frac{d^2\left[r^{n+1}W(r)\right]}{d^2r} \,, \tag{2.70}$$

where $W(r)$ is given by

$$W(r) = \sum_{m=2}^{k} \frac{\alpha_m}{m} [1-B(r)]^m r^{-2m} + [1-B(r)]r^{-2} - \frac{2\Lambda}{n(n+1)} \,, \tag{2.71}$$

with $\alpha_m = a_m \prod_{p=1}^{2m-2}(n-p)$.

For example, for $D = 4$, $k = 1$, and so one has the Schwarzschild-de Sitter solution, while for $D = 5$, $k = 2$, there is one Lovelock non trivial term (the Gauss-Bonnet, which in five-dimensions is not a topological invariant) and one has the Boulware-Deser solution [63]. For higher dimensions one has an algebraic equation of increasing complexity, but, as we shall see in the following, for our purposes it will be not necessary to know explicitly the expression for the solution $B(r)$.

For the static metric in Eq. (2.68) one has the Killing vector $K^\mu = (1, \vec{0})$ and since

$$\nabla_\nu \mathcal{G}_\mu^\nu = 0 \,, \qquad \mathcal{G}_{\mu\nu} = \mathcal{G}_{\nu\mu} \,, \tag{2.72}$$

the vector $J_\mu = \mathcal{G}_{\mu\nu}K^\nu$ is covariantly conserved and gives rise to a Killing conserved charge. This corresponds to the quasi-local generalized Misner-Sharp mass which reads

$$E_{MS}(r) \equiv -\frac{1}{\kappa^2}\int_\Sigma J^\mu d\Sigma_\mu = \frac{nV(\Omega_n)}{2\kappa^2}\int_0^r d\rho \frac{d(\rho^{n+1}W)}{d\rho} = \frac{nV(\Omega_n)}{2\kappa^2} r^{n+1} W(r) \,, \tag{2.73}$$

where Σ is a spatial volume at fixed time, $d\Sigma_\mu = (d\Sigma, \vec{0})$, and assuming spherical horizons, $V(\Omega_n) = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$, with $\Gamma(z)$ the Euler-Gamma function.

In the absence of matter Eq. (2.69) can be integrated and one has

$$r^{n+1}W(r) = C \,, \tag{2.74}$$

C being a constant of integration which we will show to be related to the mass of the black hole. In particular, on shell, that is at the horizon $r = r_H$ such that $B(r_H) = 0$, Eqs. (2.73)-(2.74) lead to $E_{MS}(r_H) = E_K$,

$$E_K = \frac{nV(\Omega_n)}{2\kappa^2} C \,. \tag{2.75}$$

Now, let us show that a First Law of black hole thermodynamics holds true, with the "energy" of the black hole solution, namely the Killing charge obtained below, proportional to constant of integration C. In the case of Lovelock gravity the validity of the First Law of black hole thermodynamics has been investigated in many places (see for example Refs. [64, 65, 66, 67]). For the static case we present a direct and simple proof.

First of all we introduce the horizon defined by the existence of the largest positive root r_H of $B(r)$ which satisfies Eqs. (2.31)-(2.32). Then, from Eq. (2.71) and Eq. (2.74), we have the identity

$$C = r_H^{n+1} W(r_H) = \sum_{m=2}^{k} \frac{\alpha_m}{m} r_H^{n+1-2m} + r_H^{n-1} - \frac{2\Lambda r_H^{n+1}}{n(n+1)} \,. \tag{2.76}$$

On the other hand, taking the derivative with respect to r of Eq. (2.74) and putting $r = r_H$, and making use again of Eq. (2.74), we obtain

$$\sum_{m=2}^{k} \frac{\alpha_m(n+1-2m)}{m} r_H^{n+1-2m} + (n-1) r_H^{n-1} - \frac{2\Lambda r_H^{n+1}}{n} = \frac{dB(r)}{dr}\Big|_{r_H} \left(\sum_{m=2}^{k} \alpha_m r_H^{n+2-2m} + r_H^{n} \right) . \tag{2.77}$$

Now, let us compute the 'thermodynamical' change of C with respect to a small change of r_H. From Eq. (2.76) one has

$$dC = \left(\sum_{m=2}^{k} \frac{\alpha_m(n+1-2m)}{m} r_H^{n-2m} + (n-1) r_H^{n-2} - \frac{2\Lambda r_H^{n}}{n} \right) dr_H \,. \tag{2.78}$$

Making use of Eq. (2.77) this expression may be rewritten in the form

$$dC = \frac{dB(r)}{dr}\Big|_{r_H} \left(\sum_{m=2}^{k} \alpha_m r_H^{n+1-2m} + r_H^{n-1} \right) dr_H \,. \tag{2.79}$$

Let us interpret the right side of the latter identity. Here we are dealing with a static, spherically symmetric metric admitting a Killing vector. If there is an event horizon located at r_H, then the Hawking temperature of the related black hole is given by Eq. (2.55).

Now, all thermodynamical quantities associated with these black holes solutions can be computed by standard methods. In particular, the entropy S_W can be calculated by the Wald method [37, 41, 68] or other methods if you like, and one has (see for example Refs. [65, 66, 69]):

$$S_W = \frac{2\pi V(\Omega_n)}{\kappa^2} r_H^n \left(1 + n \sum_{m=2}^{k} \frac{\alpha_m}{n+2-2m} r_H^{2-2m} \right) . \tag{2.80}$$

As a result, from Eqs. (2.55), (2.75), (2.78) and (2.80), one has the First Law of black hole thermodynamics for Lovelock gravity, that is

$$T_K \, dS_W = dE_K \,. \tag{2.81}$$

We have shown that for a generic Lovelock gravity, the First Law of black hole thermodynamics holds and one can identify the energy of a static, spherically symmetric black hole with the constant of integration and Killing conserved charge.

The generalization to topological Lovelock black holes has been investigated in Ref. [70], and again the First Law of black hole thermodynamics has been shown to hold.

2.5 $F(R)$ four-dimensional modified gravity

In this Section we will come back to black hole solutions in $F(R)$-four dimensional modified gravity.

The entropy associated to these black holes solutions can be calculated by the Wald method. Following Refs. [37, 41, 68], the explicit calculation of the black hole entropy S_W is provided by the formula

$$S_W = -2\pi \int_\Sigma \left(\frac{\delta \mathscr{L}}{\delta R_{\mu\nu\alpha\beta}} \right) \Big|_H e_{\mu\nu} e_{\alpha\beta} d\Sigma \,, \tag{2.82}$$

where $\mathscr{L} = \mathscr{L}(R_{\mu\nu\alpha\beta}, R_{\mu\nu}, R, g_{\mu\nu}...)$ is the Lagrangian density of any general theory of gravity and $e_{\alpha\beta} = -e_{\beta\alpha}$ is the binormal vector to the (bifurcate) horizon. It is normalized so that $e_{\alpha\beta}\mathrm{e}^{\alpha\beta} = -2$. For the SSS metric (1.53), the binormal turns out to be

$$e_{\alpha\beta} = \mathrm{e}^{\alpha(r)} (\delta^0_\alpha \, \delta^1_\beta - \delta^1_\alpha \, \delta^0_\beta) \,, \tag{2.83}$$

δ^α_β being the Kronecker delta. The induced area form, on the bifurcate surface $\{r = r_H, t = \text{const}\}$, is represented by $d\Sigma$. Finally, the subscript 'H' indicates, as usually, that the partial derivative is evaluated on the horizon, and the variation of the Lagrangian density with respect to $R_{\mu\nu\alpha\beta}$ is performed as if $R_{\mu\nu\alpha\beta}$ and the metric $g_{\alpha\beta}$ are independent. Since

$$\frac{\delta R}{\delta R_{\mu\nu\alpha\beta}} = \frac{1}{2} \left(g^{\alpha\mu} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta} \right) \,, \tag{2.84}$$

for the modified gravity models of the $F(R)$-class, such that $\mathscr{L} = F(R)/(2\kappa^2)$, one obtains

$$S_W = \frac{\mathcal{A}_H \, F'(R_H)}{4G_N} \,. \tag{2.85}$$

For $F(R) = R$, we recover the formula (2.62) of GR. For simplicity we will consider only spherical horizons, thus the area is $\mathcal{A}_H = 4\pi r_H^2$ and the volume $V_H = 4\pi r_H^3/3$.

By evaluating the first equation of motion (2.1) on the event horizon, and multiplying both sides of equation by dr_H, we directly obtain

$$T_K dS_W = \mathrm{e}^{\alpha(r_H)} \left(\frac{F'(R_H)}{2G_N} - \frac{R_H F'(R_H) - F(R_H)}{4G_N} r_H^2 \right) dr_H \,. \tag{2.86}$$

Here, $R_H = R(r_H)$. T_K is the Killing temperature given by Eq. (2.55), whose validity, derived in GR by quantum mechanics, is now extended to $F(R)$-gravity. Thus, we have derived for a generic $F(R)$ gravitational model the First Law of black hole thermodynamics as soon as the identifications

$$dE_K := T_K dS_W \,, \tag{2.87}$$

where dE_K is the variation of Killing energy, and

$$E_K := \int \mathrm{e}^{\alpha(r_H)} \left(\frac{F'(R_H)}{2G_N} - \frac{R_H F'(R_H) - F(R_H)}{4G_N} r_H^2 \right) dr_H \,, \tag{2.88}$$

can be made. In what follows, by making use of exact solutions, we will provide a support for this identification. In particular, we will justify the use of Killing temperature instead the one of Hayward by showing how it permits to identify the mass of the black hole with the integration constant of the SSS solutions.

Our proposal, expressed by Eq. (2.88), should be compared with a similar proposal contained in Ref. [71]. In Ref. [40] an attempt to define a local Misner-Sharp mass has been presented. There, however, the proposed formula is not really satisfactory, because the quasi-local form is only present in some particular cases, one of which will be discussed.

The simplest but important example is the class of static solutions with constant curvature R. In § 1.4 we have just shown that it leads to Schwarzshild-de Sitter solution (1.64) with $R = R_H = 4\Lambda$. We can take $\alpha = 0$ without losing of generalization. If we use the de Sitter condition (1.31), we obtain

$$E_K = \frac{F'(R_H)}{2G_N}\left(r_H - \frac{\Lambda}{3}r_H^3\right) . \tag{2.89}$$

Since $B(r_H) = 0$, one finally gets

$$E_K = (F'(R_H))\, M \, , \tag{2.90}$$

which is our identification of mass-energy expression for this class of black hole, in agreement with Ref. [40]. Note that $F'(R_H)$ simply is a constant which do not depend on r_H.

2.5.1 BH solutions with $\alpha(r) = 0$

For this kind of models, $T_K = T_H$ and the Fist Law can be written by using T_K or T_H, indifferently.

Let us consider the model $F(R) = k\sqrt{R + 6C_2}$ of Eq. (2.10). Since $R_H = 1/r_H^2 - 6C_2$, one has

$$E_K = \frac{3k}{16G_N}\left(r_H^2 + C_2 r_H^4\right) . \tag{2.91}$$

By using the fact $B(r_H) = 0$, it reads

$$E_K = \frac{3k}{16G_N}C_1 \, , \tag{2.92}$$

C_1 being the integration constant of SSS metric. Note that $k > 0$ in order to have positive entropy.

For the model $F(R) = R - 2g\sqrt{R}$ of Eq. (2.17), where $a = -g$, $g > 0$, since $R_H = 1/r_H^2$, one gets from Eqs. (2.86)-(2.87):

$$dE_K \equiv T_K dS_W = \frac{1}{2G_N}\left(1 - \frac{3g}{2}r_H\right) dr_H \, . \tag{2.93}$$

Since $r_H = 2/(3g)$, this equation leads to

$$dE_K = 0 \, . \tag{2.94}$$

We remind that g is a fixed parameter of the model, being the solution (2.18) without integration constants, and being null the variation on the horizon (i.e. $dr_H = 0$). Since $T_K \neq 0$, it is simple to see that $dS_W = 0$ and our BH is stable (it does not irradiate).

2.5.2 The Clifton-Barrow solution

Let us apply the same procedure for the highly non-trivial Clifton-Barrow solution of Eqs. (2.24)-(2.25) of the model $F(R) = R^{\delta+1}(\kappa^2)^\delta$, for which $\alpha(r)$ is not a constant.

According to Equation (2.55) the Killing temperature reads,

$$T_K = \frac{1}{4\pi}\sqrt{\frac{(1-2\delta+4\delta^2)}{(1-2\delta-2\delta^2)}\frac{r_H^{(2\delta+2\delta^2-1)/(1-\delta)}}{r_0^{\delta(1+2\delta)/(1-\delta)}}} \,. \tag{2.95}$$

With regard to the black hole entropy associated with the event horizon of the Clifton-Barrow solution, from the Wald formula in Eq. (2.85) we find:

$$S_W = \frac{\mathcal{A}_H}{4G_N^{1-\delta}}(1+\delta)\left[\frac{6\delta(1+\delta)}{(2\delta^2+2\delta-1)r_H^2}\right]^\delta \,. \tag{2.96}$$

In order to have the positive sign of entropy, we must require $\delta > (\sqrt{3}-1)/2$ or $-1 < \delta < 0$. The solutions with $0 < \delta < (\sqrt{3}-1)/2$ or $\delta < -1$ are unphysical, whereas for $\delta = 0$ we find the result of General Relativity. On the other hand, only the solutions of $-1 < \delta < 0$ give a real value for the temperature. If $\delta > (\sqrt{3}-1)/2$ the Killing/Hawking Temperature becomes imaginary.

Making use of Eqs. (2.86)-(2.87), one has

$$dE_K = A_\delta(r_H)\left(r_H^{(4\delta^2-\delta)/(1-\delta)}\right)dr_H \,, \tag{2.97}$$

where $A_\delta(r_H)$ is a function of r_H,

$$A_\delta(r_H) = \frac{1}{4}\,\mathrm{e}^{\alpha(r_H)}\left(G^{\delta-1}(c_\delta)^\delta\right)[2(1+\delta) - c_\delta\,\delta] \,. \tag{2.98}$$

Here, c_δ has been chosen so that $R = c_\delta/r^2$,

$$c_\delta = \frac{6\delta(1+\delta)}{(2\delta^2+2\delta-1)} \,. \tag{2.99}$$

As a result, the energy turns out to be

$$E_K = A_\delta(r_H)\frac{(1-\delta)}{(1+4\delta^2-\delta)}\,r_H^{(1+4\delta^2-\delta)/(1-\delta)} \,. \tag{2.100}$$

Finally, since $r_H = C^{(1-\delta)/(1-2\delta+4\delta^2)}$, one gets again that the energy is proportional to the constant of integration C of the BH solution, since

$$E_K = \Psi_\delta\frac{G_N^{\delta-1}}{r_0^{\delta(1+2\delta)/(1-\delta)}}\,C \,, \tag{2.101}$$

where we have introduced the dimensionless constant Ψ_δ depending on δ,

$$\Psi_\delta = \left(\frac{2^{\delta-1}3^\delta\delta^\delta(\delta-1)^2(\delta+1)^{\delta+1}}{\sqrt{1-2\delta-2\delta^2}\sqrt{1-2\delta+4\delta^2}}\frac{1}{(2\delta^2+2\delta-1)^\delta}\right) \,. \tag{2.102}$$

We conclude with some remarks. In the above expression, the range of parameter δ has to be restricted to the ranges already discussed in order to have a positive temperature and entropy. As a check, it is easy to show that in the limit $\delta \to 0$, one gets the GR value $C = 2E_K G_N$, E_K being the BH energy/mass of GR. Furthermore, the Killing energy E_K and the Killing temperature depend on the dimensional constant r_0, and we may take it proportional to Planck length $G_N^{1/2}$.

2.5.3 BH solution of $1/R$ model

As the last non trivial example, let us consider the solutions of Eqs. (2.29)-(2.30) for the model $F(R) = k\left(\frac{1}{R} - \frac{h^2}{6}\right)$. Eq. (2.85) gives for the related entropy,

$$S_W = k\frac{\pi h^2 r_H^4}{36 G_N}\,, \tag{2.103}$$

the entropy being positive, since $k > 0$. The Killing temperature associates with the horizon reads

$$T_K = \frac{1}{\pi}\left(\frac{r_H}{r_0}\right)^{1/2}\left(\frac{C}{2r_H^{9/2}} - \frac{1}{6h}\right)\,, \tag{2.104}$$

which is positive on the event horizon (see §2.2). C is the integration constant of the solution. By computing the Killing energy we have

$$E_K = k\frac{h}{54 G_N}\left(\frac{1}{r_0}\right)^{1/2}\left(r_H^{9/2} - \frac{6}{7h}r_H^{7/2}\right)\,. \tag{2.105}$$

Thus, making use of $B(r_H) = 0$, one arrives at

$$E_K = k\frac{h^2}{63 G_N}\left(\frac{1}{r_0}\right)^{1/2} C\,. \tag{2.106}$$

Also in this case we can identify the integration constant of the model as a quantity proportional to the black hole Killing energy, r_0 being a suitable dimensional parameter introduced in the solution because of dimensional reasons.

2.6 The Deser-Sarioglu-Tekin topological black hole solutions

In this Section, first we generalize the modified gravity black hole solution of Deser et al. [44], and then we shall show that also for these solutions the First Law of black hole thermodynamics is valid and the constant of integration is proportional to the Killing energy.

For the sake of simplicity we shall restrict ourselves to the four-dimensional case, but, since we are interested in black hole with generalized topological horizon, we have to include a non vanishing 'cosmological constant' Λ (see for example the GR case in Refs. [72, 73, 74]). The D-dimensional case as well as the inclusion of Electromagnetism presents no difficulties.

To begin with, we write down the action of the model

$$I = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\left(R - 2\Lambda + \sqrt{3}\sigma\sqrt{F}\right)\,, \tag{2.107}$$

where Λ is constant, σ is a real dimensionless parameter and $F = C_{\mu\nu\xi\sigma}C^{\mu\nu\xi\sigma}$ is the square of the Weyl tensor, which is an important measure of the curvature of space-time,

$$F = \frac{1}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}\,. \tag{2.108}$$

For $\sigma = 0$ the Weyl contribution turns off and GR result is recovered. This model is a very interesting additive modification of GR with cosmological constant.

For more generality we look for static, (pseudo)-spherically symmetric solutions with various topology and so we write the metric in the form

$$ds^2 = -a^2(r)b(r)dt^2 + \frac{dr^2}{b(r)} + r^2\left(\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\phi^2\right)\,, \tag{2.109}$$

where $a(r)$ and $b(r)$ are functions of r and the horizon manifold will be a sphere S_2, a torus T_2 or a compact hyperbolic manifold Y_2, according to whether $k = 1, 0, -1$.

A direct computation shows that the noteworthy properties of the Weyl scalar F discussed in Ref. [44] for $k = 1$, are still valid for $k = 0, -1$. Thus the unknown functions $a(r)$ and $b(r)$ can be obtained by imposing the stationary condition $\delta\hat{I} = 0$, where, $\hat{I}$ is the original action evaluated on the metric (2.109), up to integration by parts and on the 'topological' variables $\{\rho, \phi\}$. It reads

$$\hat{I} = \frac{1}{\kappa^2}\int dr\,\left\{(1-\sigma)\left[r\left(\frac{da(r)}{dr}\right)b(r) + ka(r)\right] + 3\sigma a(r)b(r) - \Lambda r^2 a(r)\right\}\,. \tag{2.110}$$

From this equation follow the EOM

$$(1-\sigma)r\left(\frac{da(r)}{dr}\right) + 3\sigma a(r) = 0\,, \tag{2.111}$$

$$r\left(\frac{db(r)}{dr}\right) + \frac{(1-4\sigma)}{1-\sigma}\,b(r) = k - \Lambda\,\frac{r^2}{1-\sigma}\,. \tag{2.112}$$

The general solutions of Eqs. (2.111)-(2.112) are

$$a(r) = \left(\frac{r}{r_0}\right)^{\frac{3\sigma}{\sigma-1}}\,, \tag{2.113}$$

$$b(r) = k\,\frac{(1-\sigma)}{(1-4\sigma)} - Cr^{-\frac{1-4\sigma}{1-\sigma}} - \Lambda\,\frac{r^2}{3(1-2\sigma)}\,, \tag{2.114}$$

where $C > 0$ is an integration constant and $r_0 > 0$ has to be introduced for dimensional reasons.

We assume $\sigma \neq 1, \frac{1}{4}$. For $\sigma = 1$ only the trivially, physically unacceptable solution $a(r) = 0$ exists. For $\sigma = 1/4$ and $\Lambda = 0$ a simple solution can be found on the sphere S_2 ($k = 1$). It is $a(r) = \tilde{k}/r$ and $b(r) = \log(r/r_0)$, where $\tilde{k}, r_0$ are dimensional constants.

One can see that in GR ($\sigma = 0$) black hole solutions exist only for negative cosmological constant, except for the case $k = 1$, where Λ can assume any arbitrary value. One also must pay attention to the metric signature. For example, in the case of $k = 1$ and $\Lambda = 0$, the region $1/4 < \sigma < 1$, for which $b(r) < 0$, needs to be excluded to preserve the metric signature.

As usual, the horizon is given by the positive root r_H of $b(r) = 0$ with $db(r)/dr|_{r_H} \neq 0$. The algebraic equation can be easily solved and gives

$$r_H = \left(k\,\frac{1-\sigma}{1-4\sigma} - \Lambda\,\frac{r_H^2}{3(1-2\sigma)}\right)^{\frac{\sigma-1}{1-4\sigma}} C^{\frac{1-\sigma}{1-4\sigma}}\,. \tag{2.115}$$

Thus, since the Killing-Hawking temperature is $T_K = (a(r_H)/4\pi)(db(r)/dr)|r_H$, we get

$$T_K = \frac{1}{4\pi r_H}\left(k - \Lambda\,\frac{r_H^2}{(1-\sigma)}\right)\left(\frac{r_H}{r_0}\right)^{\frac{3\sigma}{\sigma-1}}\,. \tag{2.116}$$

A direct computation of the entropy via Wald method has been done in Ref. [17]. Since for the metric (2.109), the binormal vector $\epsilon_{\mu\nu}$ turns out to be

$$\epsilon_{\mu\nu} = b(r)(\delta^0_\mu\delta^1_\nu - \delta^1_\mu\delta^0_\nu)\,, \tag{2.117}$$

δ^i_j being the Kronecker delta, formula (2.82) becomes

$$S_W = -8\pi \mathcal{A}_H \, b^2(r_H) \left(\frac{\delta \mathscr{L}}{\delta R_{0101}} \right) \Big|_H . \tag{2.118}$$

Above, $\mathcal{A}_H = V_k r_H^2$, in which $V_1 = 4\pi$ (the sphere), $V_0 = |\Im \tau|$, with τ the Teichmueller parameter for the torus, and finally $V_{-1} = 4\pi g$, $g > 2$, for the compact hyperbolic manifold with genus g [72]. Let us compute the Lagrangian variation, where the constant Λ vanishes,

$$\begin{aligned} \delta \mathscr{L} &= \frac{1}{2\kappa^2} \left[\delta R + \sqrt{3}\, \sigma \, \delta(\sqrt{F}) \right] \\ &= \frac{1}{2\kappa^2} \left[\frac{1}{2} (g^{\mu\xi} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\xi}) \delta R_{\mu\nu\xi\sigma} + \frac{\sqrt{3}\sigma}{2} (F)^{-\frac{1}{2}} \right] \delta(F) \, . \end{aligned} \tag{2.119}$$

Using Eq. (2.108), we get

$$\begin{aligned} \frac{\delta \mathscr{L}}{\delta R_{\mu\nu\xi\sigma}} &= \frac{1}{2\kappa^2} \left\{ \frac{1}{2} (g^{\mu\xi} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\xi}) + \frac{\sqrt{3}\sigma}{2} (F)^{-\frac{1}{2}} \times \right. \\ & \left. \left[2R^{\mu\nu\xi\sigma} - (g^{\mu\xi} R^{\nu\sigma} + g^{bd} R^{\mu\xi} - g^{\mu\sigma} R^{\nu\xi} - g^{\nu\xi} R^{\mu\sigma}) + \frac{1}{3} (g^{\mu\xi} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\xi}) R \right] \right\} . \end{aligned} \tag{2.120}$$

In the specific,

$$\left(\frac{\delta \mathscr{L}}{\delta R_{0101}} \right) \Big|_H = \frac{1}{4\kappa^2} \left[g^{00} g^{11} + \frac{\sqrt{3}\sigma}{\sqrt{F}} \left(2R^{0101} - g^{00} R^{11} - g^{11} R^{00} + \frac{1}{3} g^{00} g^{11} R \right) \right] \Big|_H . \tag{2.121}$$

For the metric (2.109), we may write[2]

$$\begin{aligned} \sqrt{F}|_H &= \frac{1}{\sqrt{3}} \Big| \frac{1}{r^2} \left[r^2 \left(\frac{d^2 b(r)}{dr^2} \right) + 2\,(b(r) - k) - 2r \left(\frac{db(r)}{dr} \right) \right] \\ & + \frac{1}{r a(r)} \left[3r \left(\frac{db(r)}{dr} \right) \left(\frac{da(r)}{dr} \right) - 2b(r) \left(\frac{da(r)}{dr} - r \frac{d^2 a(r)}{dr^2} \right) \right] \Big|_{r_H} . \end{aligned} \tag{2.122}$$

Taking together Eq. (2.118), Eq. (2.121) and Eq. (2.122), for both the solutions (2.113) and (2.114),

[2]The trace of the Weyl tensor to the n, with $n > 0$, is

$$\mathrm{tr} F^{\frac{n}{2}} = \left(-\frac{1}{3} \right)^n [2 + (-2)^{2-n}] X(r)^n \, ,$$

where

$$X(r) = \frac{1}{r^2} \left[r^2 \frac{d^2 b(r)}{dr^2} + 2(b(r) - k) - 2r \frac{db(r)}{dr} \right] + \frac{1}{r a(r)} \left[3r \frac{db(r)}{dr} \frac{da(r)}{dr} - 2b(r) \left(\frac{da(r)}{dr} - r \frac{d^2 a(r)}{dr^2} \right) \right] .$$

In our case, $n = 2$.

we finally have that the horizon entropy for the Deser *et al.* black hole is

$$S_W = \frac{\mathcal{A}_H}{4G_N}\left(1+\varepsilon\sigma\right)\,, \qquad \text{where} \qquad \varepsilon := \begin{cases} +1, & \sigma \leq \frac{1}{4} \\ -1, & \sigma > 1/4\,, \sigma \neq 1 \end{cases} \,. \tag{2.123}$$

Here, we have extended the result to the case $\sigma = 1/4$, for which it is easy to see it is still valid and the entropy function is continuous even if the black hole metric changes.

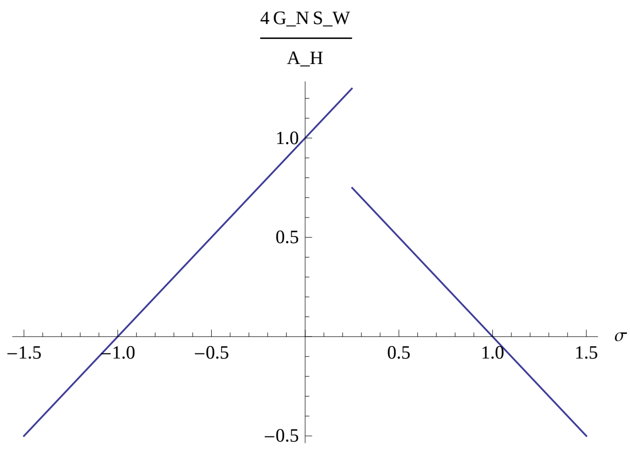

Figure 2.1: Wald's entropy in units of $\mathcal{A}_H/4G_N$ versus σ parameter for the Deser *et al.* BH.

In fact, as shown by Fig. 2.1, the entropy of the black hole is positive only as far as $\sigma \in (-1,1)$. For $\sigma = -1$, the entropy vanishes suggesting that, for this value of σ, the number of microscopic configurations realizing the black hole is only one. For $\sigma \in (-1,0) \cup (1/4,1)$, the entropy of Deser's black hole is always smaller than its value in General Relativity.

As a final result we obtain

$$T_K dS_W = \frac{V_k\left(1+\sigma\right)}{8\pi G_N}\left(k - \Lambda\,\frac{r_H^2}{(1-\sigma)}\right)\left(\frac{r_H}{r_0}\right)^{\frac{3\sigma}{\sigma-1}} dr_H\,. \tag{2.124}$$

Furthermore, by using Eq. (2.115), as a consequence of the First Law, we get

$$E_K = \frac{V_k\left(1+\sigma\right)}{8\pi G_N} C\,. \tag{2.125}$$

In this class of modified gravitational models the energy of black hole is particularly simple, since the modification is described by the dimensionless parameter σ.

2.7 Topological conformal Weyl gravity solutions

In this Section, first we revisit the higher gravity black hole solution of Riegert *et al.* [75, 76], and its topological version [77]. To begin with, we write down the action of the model in the form

$$I = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left[\gamma(R-2\Lambda) + 3wF\right]\,, \tag{2.126}$$

where γ is an arbitrary parameter, which may be proportional to the square of Plank mass, w is a dimensionless parameter, Λ is a 'cosmological constant' and $F = C_{\mu\nu\xi\sigma}C^{\mu\nu\xi\sigma}$ is the square of the Weyl tensor. The pure conformal invariant model $\gamma = 0$ is very interesting and its phenomenology has been investigated in Ref. [78].

As in previous Section, also here we shall consider various topology and this means that the metric will have the form (2.109), and the arbitrary functions $a(r), b(r)$ will be obtained from the reduced action $\hat{I}$

$$\hat{I} = \int dr \left[\gamma\left(rb(r)\left(\frac{da(r)}{dr}\right) + ka(r) - 2\Lambda\, r^2 a(r)\right) + w\frac{A(r)^2}{r^2 a(r)}\right], \tag{2.127}$$

where we have put

$$\begin{aligned} A(r) \quad = \quad & r^2 a(r)\left(\frac{d^2 b(r)}{dr^2}\right) + 3r^2\left(\frac{da(r)}{dr}\right)\left(\frac{db(r)}{dr}\right) - 2ra(r)\left(\frac{db(r)}{dr}\right) + 2r^2\left(\frac{d^2 a(r)}{dr^2}\right)b(r) \\ & -2r\left(\frac{da(r)}{dr}\right)b(r) + 2a(r)b(r) - 2ka(r)\,. \end{aligned} \tag{2.128}$$

We are dealing with a higher order Lagrangian system, the Lagrangian depending on the first and second derivative of the unknown functions $a(r)$ and $b(r)$.

The equations of motion read

$$\begin{aligned} & 4\left[\frac{d^2}{dr^2}\left(\frac{A(r)b(r)}{a(r)}\right)\right] - 2\frac{d}{dr}\left\{\frac{A(r)}{ra(r)}\left[3r\left(\frac{db(r)}{dr}\right) - 2b(r)\right]\right\} \\ & +2\frac{A(r)}{r^2 a(r)}\left[r^2\left(\frac{d^2 b(r)}{dr^2}\right) - 2r\left(\frac{db(r)}{dr}\right) + 2b(r) - 2k\right] \\ & -\frac{A(r)^2}{r^2 a^2(r)} + \frac{\gamma}{w}\left[k - b(r) - r\left(\frac{db(r)}{dr}\right) - 2\Lambda r^2\right] = 0\,, \end{aligned} \tag{2.129}$$

$$\begin{aligned} & \frac{d^2 A(r)}{dr^2} - \frac{d}{dr}\left[\frac{A(r)}{ra(r)}\left(3r\frac{da(r)}{dr} - 2a(r)\right)\right] \\ & +\frac{A(r)}{r^2 a(r)}\left[2r^2\frac{d^2 a(r)}{dr^2} - 2r\left(\frac{da(r)}{dr}\right) + 2a(r)\right] + \frac{\gamma r}{2w}\left(\frac{da(r)}{dr}\right) = 0\,. \end{aligned} \tag{2.130}$$

For simplicity let us look for exact solutions with $a(r) = 1$. With this Ansatz, Eq. (2.130) can be integrated and one obtains

$$b(r) = \frac{b_1}{r} + c_0 + c_1 r + c_2 r^2\,, \tag{2.131}$$

b_1 and c_k, $k = 0, 1, 2$ being integration constants. In order to satisfy Eq. (2.129) we have to distinguish the two cases $\gamma \neq 0$ (a modified Einstein gravity) and $\gamma = 0$ (pure conformal gravity), since they provide completely different solutions.

In the case $\gamma \neq 0$, Eq. (2.129) is satisfied only if

$$c_0 = k\,, \quad c_1 = 0 \quad c_2 = -\frac{1}{3}\Lambda\,, \tag{2.132}$$

while b_1 remains a free parameter. We see that this is a topological Schwarzschild-de Sitter (AdS) black hole like solution, since

$$b(r) = k - \frac{C}{r} - \frac{1}{3}\Lambda r^2\,, \tag{2.133}$$

where here b_1 has been replaced by $-C$. It has to be noted that this is the solution which one would have obtained from the Hilbert-Einstein action[3] with cosmological constant, that is with $w = 0$.

[3]It means, the Weyl tensor is vanishing on the Schwarzschild-de Sitter solution.

As we already said, if $\gamma = 0$, the solution is completely different and in fact, in such a case Eq. (2.129) is satisfied only if

$$c_1 = \frac{c_0^2 - k^2}{3b_1} \,. \tag{2.134}$$

Now the solution depends on the three arbitrary parameters c_0, c_2 and b_1. By a redefinition of them by $c_0 \to k + 3c_0$, $c_2 \to \lambda$, $b_1 \to -C$, we write it in the form

$$b(r) = k + 3c_0 - \frac{c_0}{C}(2k + 3c_0)\, r + \lambda r^2 - \frac{C}{r} \,, \tag{2.135}$$

in agreement with the topological black hole solution already found by Klemm in Ref. [77].

The event horizon exists as soon as there is positive solution r_H of $b(r_H) = 0$. For example, if $C > 0$ and $\lambda > 0$, it is easy to show that there exists always a positive root independently on the values of c_0 and of λ, while, in the opposite case $\lambda < 0$, a positive root of $b(r_H) = 0$ exists only if $c_0 \geq 0$ and the value of $|\lambda|$ is sufficiently small. The special $\lambda = 0$ case will be discussed at the end of this Section.

With regard to the computation of entropy, assuming that there exists an event horizon $b(r_H) = 0$, with $r_H > 0$ and $db(r)/dr|_{r_H} \neq 0$, for the pure Weyl gravity case the Wald method, in a very similar way of the previous Section, gives

$$S_W = 2wV_K \left(\frac{C}{r_H} - c_0 \right) = 2wV_K \left(x - c_0 \right) \,, \tag{2.136}$$

and

$$dS_W = 2wV_K \, dx \,. \tag{2.137}$$

For convenience we have introduced the variable $x = C/r_H$. $\mathcal{A}_H = V_k r_H^2$ $(k = 1, 0, -1)$ as in the previous Section.

The integration constant C in Eq. (2.137) can be seen as a function of r_H obtained by solving the equation $b(r_H) = 0$, which, as it follows from (2.135), it is a second-order algebraic equation in C. Of course, in order to have a positive entropy we have to choose $c_0 < C/r_H$ and moreover C has to be positive being proportional to the energy.

Now we restrict ourselves to the case $\lambda > 0$ such that

$$\lambda = 1/L^2 \,. \tag{2.138}$$

In this way, by solving $b(r_H) = 0$ of Eq. (2.135) with respect to C we get:

$$2x \equiv \frac{2C}{r_H} = \frac{r_H^2}{L^2} + k + 3c_0 + \sqrt{W} \,, \tag{2.139}$$

where

$$W = \left(\frac{r_H^2}{L^2} + k + 3c_0 \right)^2 - 4c_0(2k + 3c_0) > 0 \,. \tag{2.140}$$

From the latter equation it follows

$$dx = \frac{r_H}{L^2} \left(1 + \frac{(r_H^2/L^2) + k + 3c_0}{\sqrt{W}} \right) dr_H \,. \tag{2.141}$$

On the other hand the Killing temperature can be written in the convenient form

$$T_K = \frac{1}{4\pi\, r_H} \left(\frac{2r_H^2}{L^2} + \sqrt{W} \right) \,, \tag{2.142}$$

and using Eq. (2.137) and Eq. (2.141) we obtain

$$\begin{aligned} T_K dS_W &= \frac{\omega V_k}{2\pi L^2}\left(\frac{3r_H^2}{L^2} + k + 3c_0 + \sqrt{W} + \left(\frac{2r_H^2}{L^2}\right)\frac{(r_H^2/L^2) + k + 3c_0}{\sqrt{W}}\right) dr_H \\ &= \frac{\omega V_k}{\pi L^2}(r_H dx + x\, dr_H)\,. \end{aligned} \quad (2.143)$$

Since $dC = (r_H dx + x\, dr_H)$, we finally see that the First Law of black hole thermodynamics reads

$$T_K dS_W = \frac{wV_k}{\pi L^2} dC\,. \quad (2.144)$$

As a result, we may again identify the energy as

$$E_K = \frac{wV_k}{\pi L^2} C\,. \quad (2.145)$$

We conclude this Section with some remarks. The pure Weyl conformal gravity does not contain dimensional parameters. Thus, one could think that there exists a trivial entropy and a vanishing energy, but, as we have shown above, the solution gives rise to a length scale L related to the integration constant λ. In such a case the First Law of black hole thermodynamics holds and the energy of black hole solution is proportional to the other mass-constant of integration C.

The situation is different when $\lambda = 0$, since in such a case the scale does not emerge and for the horizon one gets

$$r_H = \left(\frac{k + 3c_0 + \sqrt{(c_0 + k)(k - 3c_0)}}{2c_0(2k + 3c_0)}\right) C\,. \quad (2.146)$$

The latter equation gives a positive r_H for $k \neq 0$ and a suitable value for c_0. In any case we see that $x = C/r_H$ is a pure number, and so $dx = 0$ and the entropy is trivially constant. The First Law of black hole thermodynamics is trivially valid. This is the particular case discussed in Ref. [77].

We have seen, that in all explicit and known examples, the First Law of black hole thermodynamics (Clausius relation), that emerges from equations of motion, gives a reasonable value for the energy, which results proportional to the integration constant of the SSS solutions.

Chapter 3

The finite-time future singularities in $\mathcal{F}(R, G)$-modified gravity

Many of $\mathcal{F}(R, G)$-modified gravity models suffer from the fact that they bring the future universe evolution to finite-time singularities. It means, there is a finite time, for which some physical quantity (scale factor, effective energy density/pressure of the universe or, more simplicity, some derivatives of Hubble parameter and therefore the components of Riemann Tensor) becomes singular rendering unphysical the solution. Some of these singularities are softer than other and not all physical quantities necessarily diverge in rip time. The study of singularities in $F(R)$-gravity has been done at the Ref. [79]. Since singular solutions correspond to accelerated universe, they may appear as the final attractor in realistic models which mimic the de Sitter universe where we live, leading to various instabilities in the universe and destroying the feasibility of the models. Thus, before analyzing in the following Chapters the viable conditions of modified gravity, it is of some interest to explore in detail the $\mathcal{F}(R, G)$-gravity realizing future time singularities. In principle, Dark Energy could be described by scalar field theories, fluids, and so on. Any of such DE models (including modified gravity) may be represented as the effective fluid with corresponding characteristics. Otherwise, we will see that, unlike to convenient DE-fluids which may be singular or not, modified gravity suggests an universal scenario to cure the finite-time future singularities. This Chapter is based on Refs. [14], [24].

3.1 Four types of the finite-time future singularities

In general, in FRW Universe described by the metric (1.5), singularities appear during cosmological evolution when the Hubble parameter is expressed as

$$H = \frac{h_0}{(t_0 - t)^\beta} + H_0 \,, \tag{3.1}$$

where h_0, t_0 and H_0 are positive constants, β is a constant, and $t < t_0$ because it has to be for expanding Universe. We can see that if $\beta > 0$, H becomes singular in the limit $t \to t_0$. Hence, t_0 is the time when a singularity appears. On the other hand, if $\beta < 0$, even for non-integer values of β some derivative of H and therefore the curvature or some combination of curvature invariants, becomes singular. We assume $\beta \neq 0$ because $\beta = 0$ corresponds to de Sitter space, which has no singularity. Since H_0 is not a dynamic term, in the next Sections we will often put it equal to zero.

The finite-time future singularities can be classified in the following way [80]:

- Type I (Big Rip [81]): for $t \to t_0$, $a(t) \to \infty$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$. It corresponds to $\beta = 1$ and $\beta > 1$.

- Type II (sudden [82]): for $t \to t_0$, $a(t) \to a_0$, $\rho_{\text{eff}} \to \rho_0$ and $|p_{\text{eff}}| \to \infty$. It corresponds to $-1 < \beta < 0$.
- Type III: for $t \to t_0$, $a(t) \to a_0$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$. It corresponds to $0 < \beta < 1$.
- Type IV: for $t \to t_0$, $a(t) \to a_0$, $\rho_{\text{eff}} \to \rho_0$, $|p_{\text{eff}}| \to p_0$ and higher derivatives of H diverge. It corresponds to $\beta < -1$ but β is not any integer number.

Here, $a_0(\neq 0)$ and ρ_0, p_0 are constants. We call singularities for $\beta = 1$ and those for $\beta > 1$ as the 'Big Rip' singularities and the 'Type I' singularities, respectively.

The Type I, II or III singularity appears when the Ricci scalar in Eq. (1.8) diverges and becomes singular. The cosmological expansion of modified gravity models could tend towards such asymptotic solution (if exists), that may be the final attractor of the system[1]. It is interesting to note that, since singular solutions often are energetically accessible for the system (see also Refs. [83, 84] where the Starobinsky model [85] has been considered), they can appear and destabilize the models also in the presence of other stable solutions. As a qualitative example, we can consider the case of a realistic $F(R)$-model, namely the Hu-Sawiki Model [86], able to reproduce the de Sitter phase of our universe,

$$F(R) = R - \frac{\tilde{m}^2 c_1 (R/\tilde{m}^2)^n}{c_2 (R/\tilde{m}^2)^n + 1} = R - \frac{\tilde{m}^2 c_1}{c_2} + \frac{\tilde{m}^2 c_1/c_2}{c_2 (R/\tilde{m}^2)^n + 1} \,. \tag{3.2}$$

Here, $\tilde{m}^2$ is a mass scale, c_1 and c_2 are positive parameters and n is a natural positive number. The model is very carefully constructed, such that $c_1 \tilde{m}^2/c_2 \simeq 2\Lambda$, where Λ is the Cosmological Constant, and in the high curvature region the physics of ΛCDM Model can be found. We note that the scalaron $F'(R)$,

$$F'(R) = 1 - \frac{\tilde{m}^2 c_1/c_2}{\left(c_2 (R/\tilde{m}^2)^n + 1\right)^2} (n) \left(\frac{c_2}{\tilde{m}^2}\right) \left(\frac{R}{\tilde{m}^2}\right)^{n-1} \,, \tag{3.3}$$

tends to a constant when $R \to \pm\infty$. Furthermore, by writing $\partial V_{\text{eff}}/\partial R$ as $F''(R)(\partial V_{\text{eff}}/\partial F'(R))$, in principle one can evaluate the potential V_{eff} of Eq. (1.28) through an integration. By neglecting the contribute of matter, when $R \to \pm\infty$ one easily finds

$$V_{\text{eff}}(R \to \pm\infty) \simeq -\frac{\tilde{m}^2 c_1/c_2}{3 c_2 (R/\tilde{m}^2)^n} \left(\frac{c_2}{\tilde{m}^2}\right) (n+1) \,. \tag{3.4}$$

Up to now, we are not able to say if some singular solution appears in this model. Furthermore, the Hu-Sawiki Model exhibits a stable de Sitter solution in vacuum, that may be the final attractor of the system. However, we observe that, if a singular solution with R diverging exists, it is at a finite value of V_{eff} (in particular, it tends to zero) and the scalaron $F'(R)$ can crossover the potential in some point of cosmological evolution and arise the value $F'(R) = 0$ for which catastrophic curvature singularity emerges. In the Appendix B the energy conditions related with occurrence of singularities are discussed. In general, it is possible to see that singularities violate the strong energy condition (SEC) describing acceleration. This is the reason for which realistic models of modified gravity describing the current acceleration of our universe could become unstable and fall into a singularity. We will better analyze the singularities in Hu-Sawiki Model in § 3.4.1 and we will see that for some choice of parameters the model exhibits singularities in expanding universe.

Finally, the Type IV singularity appears for finite values of R. Since in this case only higher derivatives of Hubble rate diverge, then some combination of curvature invariants also diverges and leads to singularity. As a consequence, the solution becomes unphysical or may cause serious

[1] As regards this point, it is well know that phantom dark energy ($\omega_{\text{eff}} < -1$) reproduces the acceleration of the universe ending in the Big Rip. We will briefly analyze in §4.1.1 of the next Chapter a quintessence/phantom inhomogeneous fluid with de Sitter solution and final attractor in the Big Rip.

problems in the black holes or stellar astrophysics [87].

The study of the singularities is fundamental in order to achieve a correct description of the universe. In the next Sections we will reconstruct the typical forms of modified gravity producing finite-time future singularities and the curing terms which protect the theory against singularities. Since near the singularities the Hubble parameter or its derivative diverge, we often analyze the problem in the asymptotic limit, when t is close to t_0. We will also reasonably assume that the contribute of matter in expanding universe is too small with respect to the one of modified gravity and we will neglect it.

3.2 Effective parameters and singular solutions

It could be useful to introduce the effective parameters $\rho_{\rm eff}$ and $p_{\rm eff}$ to verify the presence of singularities in specific $f(R,G)$-models, when $\mathcal{F}(R,G) = R + f(R,G)$, as in Eq. (1.10). In this case, we can treat modified gravity like an effective dark energy fluid. By using the metric (1.5) with $N(t) = 1$, we can write the Eqs. (1.18)-(1.19) as

$$\rho_{\rm eff} = \frac{1}{2\kappa^2}\left[(Rf'_R + Gf'_G - f) - 6H\dot{f}'_R - 24H^3\dot{f}'_G - 6H^2 f'_R\right] + \rho_{\rm m}\,, \tag{3.5}$$

and

$$\begin{aligned} p_{\rm eff} &= \frac{1}{2\kappa^2}\Big[(f - Rf'_R - Gf'_G) + 4H\dot{f}'_R + 2\ddot{f}'_R \\ &\quad +16H(\dot{H}+H^2)\dot{f}'_G + 8H^2\ddot{f}'_G + (4\dot{H}+6H^2)f'_R\Big] + p_{\rm m}\,, \end{aligned} \tag{3.6}$$

which solve Eqs.(1.14)-(1.15). Here, $f(R,G)$ has been replaced by f and the subscript 'R' denotes the derivative with respect the Ricci scalar and the subscript 'G' denotes the derivative with respect the Gauss-Bonnet. The point denotes, as usually, the time derivative.

The matter has a constant EoS-parameter $\omega = p_{\rm m}/\rho_{\rm m}$. By combining the two equations in Eqs. (1.14)-(1.15), we obtain

$$\mathcal{G}(H,\dot{H}...) = -\frac{1}{\kappa^2}\left[2\dot{H} + 3(1+w)H^2\right]\,, \tag{3.7}$$

where

$$\mathcal{G}(H,\dot{H}...) = p_{\rm eff} - \omega\rho_{\rm eff}\,. \tag{3.8}$$

When a cosmology is given by Hubble parameter H as a function of the cosmic time t, such that $H = H(t)$, the right-hand side of Eq. (3.7) is described by a function of t. If the function $\mathcal{G}(H,\dot{H}...)$ in Eq. (3.8), which is the combination of H, $\dot{H}$, $\ddot{H}$ and the higher derivatives of H, reproduces the above function of t, this cosmology could be realized[2]. Hence, the function $\mathcal{G}(H,\dot{H}...)$ can be used to judge whether the particular cosmology could be realized or not [79]. The form of $\mathcal{G}(H,\dot{H}...)$ is determined by the gravitational theory which one considers. In the case of $f(R,G)$-gravity, by substituting Eqs. (3.5) and (3.6) into Eq. (3.8), we find

$$\begin{aligned} \mathcal{G}(H,\dot{H}...) &= \frac{1}{2\kappa^2}\Big\{(1+\omega)(f - Rf'_R - Gf'_G) + f'_R\left[6H^2(1+\omega) + 4\dot{H}\right] \\ &\quad +H\dot{f}'_R(4+6\omega) + 8H\dot{f}'_G\left[2\dot{H} + H^2(2+3\omega)\right] + 2\ddot{f}'_R + 8H^2\ddot{f}'_G\Big\}\,. \end{aligned} \tag{3.9}$$

[2]When matter is relevant, we need two EOM, so that the using of the above equation only is not sufficient. However, we will consider singular solutions without matter.

If any singularity occurs, Eq. (3.7) behaves as

$$\mathcal{G}(H,\dot{H}...) \simeq \begin{cases} -\frac{3(1+\omega)h_0^2+2\beta h_0}{\kappa^2}(t_0-t)^{-2} & (\beta=1) \text{ Big Rip} \\ -\frac{3(1+\omega)h_0^2}{\kappa^2}(t_0-t)^{-2\beta} & \beta>1 \text{ (Type I)} \\ -\frac{2\beta h_0}{\kappa^2}(t_0-t)^{-\beta-1} & \beta<1 \text{ (Types II, III, IV)}\,. \end{cases} \quad (3.10)$$

Here, we have used the singular form of H in Eq. (3.1) with $H_0 = 0$ and we have considered the limit $t \to t_0$.

If a specific model of $f(R,G)$-gravity is given, it could be useful verify the consistence of Eq. (3.9) with Eq. (3.10) in order to check the presence of singularities. The behavior of Eq. (3.9) takes two different asymptotic forms which depend on the parameter β as follows:

- Case of $\beta \geq 1$: In the limit $t \to t_0$, we find

$$\begin{aligned}\mathcal{G}(H,\dot{H}...) \quad &\sim \quad \alpha\left[f+\frac{f'_R}{(t_0-t)^{2\beta}}+\frac{f'_G}{(t_0-t)^{4\beta}}\right]+\gamma\frac{\dot{f}'_R}{(t_0-t)^{\beta}} \\ &\quad +\delta\frac{\dot{f}'_G}{(t_0-t)^{3\beta}}+\epsilon\ddot{f}'_R+\zeta\frac{\ddot{f}'_G}{(t_0-t)^{2\beta}}\,, \end{aligned} \quad (3.11)$$

  where α, γ, δ, ϵ and ζ are constants. To realize a I Type singularity, we must verify the consistence with the first cases of Eq. (3.10). Hence, if for $G \sim 1/(t_0-t)^{4\beta}$ and $R \sim 1/(t_0-t)^{2\beta}$ with $\beta \geq 1$, the highest term of Eq. (3.11) is proportional to $1/(t_0-t)^{2\beta}$, it is possible to have a Type I singularity[3]. This condition is necessary and not sufficient. Another very important condition that must be satisfied is the concordance of the signs in Eq. (3.10), which depends on the parameters of the model.

- Case of $\beta < 1$: In the limit $t \to t_0$, we obtain

$$\begin{aligned}\mathcal{G}(H,\dot{H}...) \quad &\sim \quad \alpha\left[f+\frac{f'_R}{(t_0-t)^{\beta+1}}+\frac{f'_G}{(t_0-t)^{3\beta+1}}\right]+\gamma\frac{\dot{f}'_R}{(t_0-t)^{\beta}} \\ &\quad +\delta\frac{\dot{f}'_G}{(t_0-t)^{2\beta+1}}+\epsilon\ddot{f}'_R+\zeta\frac{\ddot{f}'_G}{(t_0-t)^{2\beta}}\,. \end{aligned} \quad (3.12)$$

  To realize this kind of singularities, the last case of Eq.(3.10) has to be verified. Thus, if for $G \sim 1/(t_0-t)^{3\beta+1}$ and $R \sim 1/(t_0-t)^{\beta+1}$ with $\beta < 1$, the highest term of Eq. (3.12) is proportional to $1/(t_0-t)^{\beta+1}$, it is possible to have a Type II, III or IV singularity. Also this condition is necessary and not sufficient.

In the next Sections, we will reconstruct the typical terms of R and/or G which could produce singularities.

3.3 The reconstruction of singular $f(G)$-gravity

In this Section, as an explicit example of $\mathcal{F}(R,G)$-gravity, we reconstruct the $f(G)$-gravity models where finite-time future singularities may occur. The action is given by Eq. (1.1) with $\mathcal{F}(R,G) = R + f(G)$, it means, the modification to GR is represented by a function of the Gauss-Bonnet invariant only. The form of H is taken by Eq. (3.1). To find such $f(G)$-gravity models, we use the reconstruction method of modified gravity [79, 88]. By using proper functions $P(t)$ and $Q(t)$ of a scalar field t which we identify with the cosmic time, we can write the action (in vacuum) as

$$I = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R+P(t)G+Q(t)\right]\,. \quad (3.13)$$

[3] Here, we are considering the dynamic behaviour of R and G on the singular solution of H.

The variation with respect to t yields

$$\dot{P}(t)G + \dot{Q}(t) = 0\,, \tag{3.14}$$

from which, in principle, we can find t as a function of G, $t = t(G)$. By substituting $t = t(G)$ into Eq. (3.13), we find the action in terms of $f(G)$,

$$f(G) = P(t(G))G + Q(t(G))\,. \tag{3.15}$$

We describe the scale factor as

$$a(t) = a_0 \exp\left(g(t)\right)\,, \tag{3.16}$$

where a_0 is a constant and $g(t)$ is a proper function of t. By using the explicit form of the EOM (1.6)-(1.7), and by writing $f(G)$ as in Eq. (3.15) and the scale factor into the Hubble parameter ($H = \dot{g}(t)$), and by using the matter conservation law (1.20) and then neglecting the contribution from matter, we get the differential equation

$$2\frac{d}{dt}\left(\dot{g}^2(t)\dot{P}(t)\right) - 2\dot{g}^3(t)\dot{P}(t) + \ddot{g}(t) = 0\,. \tag{3.17}$$

By using the first EOM (1.6), $Q(t)$ is given by

$$Q(t) = -24\dot{g}^3(t)\dot{P}(t) - 6\dot{g}^2(t)\,. \tag{3.18}$$

Big Rip singularity

First, we examine the Big Rip singularity. If $\beta = 1$ in Eq. (3.1) with $H_0 = 0$, H and G are given by

$$H = \frac{h_0}{(t_0 - t)}\,, \tag{3.19}$$

$$G = \frac{24h_0^3}{(t_0 - t)^4}(1 + h_0)\,. \tag{3.20}$$

The scale factor results

$$a(t) = \frac{a_0}{(t_0 - t)^h}\,. \tag{3.21}$$

The most general solution of Eq. (3.17) when $h_0 \neq 1$ is given by

$$P(t) = \frac{1}{4h_0(h_0 - 1)}(2t_0 - t)t + c_1\frac{(t_0 - t)^{3-h_0}}{(3 - h_0)} + c_2\,, \tag{3.22}$$

where c_1 and c_2 are constants. We can take $c_1 = 0$ if $h_0 = 3$. From Eq. (3.18), we get

$$Q(t) = -\frac{6h_0^2}{(t_0 - t)^2} - \frac{24h_0^3\left[(t_0 - t)/(2h_0(h_0 - 1)) - c_1(t_0 - t)^{2-h_0}\right]}{(t_0 - t)^3}\,. \tag{3.23}$$

Furthermore, from Eq. (3.14) we obtain

$$t = \left[\frac{24(h_0^3 + h_0^4)}{G}\right]^{1/4} + t_0\,, \tag{3.24}$$

which is consistent with Eq. (1.9). By solving Eq. (3.15), we find the most general form of $f(G)$ which realizes the Big Rip singularity,

$$f(G) = \frac{\sqrt{6h_0^3(1 + h_0)}}{h_0(1 - h_0)}\sqrt{G} + c_1 G^{\frac{h_0+1}{4}} + c_2 G\,. \tag{3.25}$$

This is an exact solution of Eq. (3.7) in the case of Eq. (3.19). In the model $R+\alpha G^{1/2}$, where $\alpha(\neq 0)$ is a constant, the Big Rip singularity for $G\to+\infty$ could appear realizing any value of $h_0\neq 1$. Note that $G^{(1+h_0)/4}$ is an invariant with respect to the Big Rip solution.

In the case of $h_0=1$, it is possible to find another exact solution of $P(t)$,

$$P(t)=\alpha(t_0-t)^q\ln\left[\gamma(t_0-t)^z\right]\,, \tag{3.26}$$

where γ is a positive constant and q and z are constants. The equation (3.17) is satisfied for the case of Eq. (3.19) if $q=3-h_0=2$ (and therefore $h_0=1$) and $z\alpha=-1/4$. From Eq. (3.18), we have

$$Q(t)=-\frac{12}{(t_0-t)^2}\ln\left[\gamma(t_0-t)\right]\,. \tag{3.27}$$

The form of $f(G)$ is given by

$$f(G)=\frac{\sqrt{3}}{2}\sqrt{G}\ln(\gamma G)\,. \tag{3.28}$$

This is another exact solution of Eq. (3.7) for $H=1/(t_0-t)$. In general, in the model $R+\alpha\sqrt{G}\ln(\gamma G)$ with $\alpha>0$ and $\gamma>0$, the Big Rip singularity could appear.

Other types of singularities

Next, we investigate the other types of singularities. If $\beta\neq 1$, Eq. (3.1) implies that the scale factor $a(t)$ behaves as

$$a(t)=(a_0)\exp\left[\frac{h_0(t_0-t)^{1-\beta}}{\beta-1}\right]\,. \tag{3.29}$$

We restrict our investigation to the case of $H_0=0$.

We consider the case in which H and G are given by

$$H = \frac{h_0}{(t_0-t)^\beta}\,,\quad \beta>1\,, \tag{3.30}$$

$$G \simeq \frac{24h_0^4}{(t_0-t)^{4\beta}}\,. \tag{3.31}$$

A solution of Eq. (3.17) in the limit $t\to t_0$ is given by

$$P(t)\simeq\frac{\alpha}{(t_0-t)^z}\,, \tag{3.32}$$

with $z=-2\beta$ and $\alpha=-1/4h_0^2$. The form of $f(G)$ is expressed as

$$f(G)=-12\sqrt{\frac{G}{24}}\,. \tag{3.33}$$

Hence, if $f(G)\simeq-\alpha\sqrt{G}$ with $\alpha>0$, a Type I singularity for $G\to+\infty$ could appear.

When $\beta<1$, the forms of H and G are given by

$$H = \frac{h_0}{(t_0-t)^\beta}\,,\quad \beta<1\,, \tag{3.34}$$

$$G \simeq \frac{24h_0^3\beta}{(t_0-t)^{3\beta+1}}\,. \tag{3.35}$$

An asymptotic solution of Eq. (3.17) in the limit $t\to t_0$ is given by

$$P(t)\simeq\frac{\alpha}{(t_0-t)^z}\,, \tag{3.36}$$

with $z = -(1+\beta)$ and $\alpha = 1/(2h_0(1+\beta))$. The form of $f(G)$ becomes

$$f(G) = \frac{6h_0^2}{(\beta+1)}(3\beta+1)\left(\frac{|G|}{24h_0^3|\beta|}\right)^{2\beta/(3\beta+1)} . \tag{3.37}$$

Hence, if $f(G)$ behaves as

$$f(G) \simeq \alpha|G|^\gamma , \quad \gamma = \frac{2\beta}{3\beta+1} , \tag{3.38}$$

with $\alpha > 0$ and $0 < \gamma < 1/2$, we find $0 < \beta < 1$ and a Type III singularity for $G \to +\infty$ could emerge.

If $\alpha > 0$ and $-\infty < \gamma < 0$, we find $-1/3 < \beta < 0$ and a Type II (sudden) singularity for $G \to -\infty$ could appear. Moreover, if $\alpha < 0$ and $1 < \gamma < \infty$, we obtain $-1 < \beta < -1/3$ and a Type II singularity with $G \to 0^-$ could occur.

Finally, if $\alpha > 0$ and $2/3 < \gamma < 1$, we obtain $-\infty < \beta < -1$ and a Type IV singularity for $G \to 0^-$ could appear. We also require that $\gamma \neq 2n/(3n-1)$, where n is a natural number.

We can generate all the possible Type II singularities as shown above except for the case $\beta = -1/3$, that is, $H = h_0/(t_0-t)^{1/3}$. In this case, we have the following form of G:

$$G = -8h_0^3 + 24h_0^4(t_0-t)^{4/3} . \tag{3.39}$$

To find t in terms of G, we must consider the whole expression of G by taking into account also the low dynamic term of (t_0-t). We obtain

$$f(G) \simeq \frac{1}{4\sqrt{6}h_0^3}G|G+8h_0^3|^{1/2} + \frac{2}{\sqrt{6}}|G+8h_0^3|^{1/2} , \tag{3.40}$$

As a consequence, the specific model $R + \sigma_1 G|G+c_3|^{1/2} + \sigma_2|G+c_3|^{1/2}$, where σ_1, σ_2 and c_3 are positive constants, can generate the Type II singularity where G tends to the negative constant $-c_3$ as in Eq. (3.39).

All the asymptotic solutions we have found satisfy Eq. (3.10) in the corresponding cases.

3.3.1 Example of realistic $f(G)$-models generating singularities

Here, we study the presence of singularities in the following realistic models of $f(G)$-gravity which reproduce the current acceleration, namely [89]

$$f_1(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2} , \tag{3.41}$$

$$f_2(G) = \frac{a_1 G^{n+N} + b_1}{a_2 G^n + b_2} , \tag{3.42}$$

$$f_3(G) = a_3 G^n (1 + b_3 G^m) , \tag{3.43}$$

$$f_4(G) = (G^m)\frac{a_1 G^n + b_1}{a_2 G^n + b_2} . \tag{3.44}$$

Here, a_1, a_2, b_1, b_2, a_3, b_3, n, m and N are constants. For the model (3.42), with $n > 0$, Types I, II and III singularities may be present. In fact, for $N = 1/2$, one could have Big Rip singularities, since in this case, in the Big Rip limit large G, Eq. (3.42) gives $\alpha G^{1/2}$. Thus, as discussed above, one has a Big Rip singularity. Moreover, again with $N = 1/2$, if $a_1/a_2 < 0$, Eq. (3.42) for large value of G, leads to $-\alpha G^{1/2}$ with $\alpha > 0$ and thus Type I singularity could appear. If

n and N are integers and $n+N>0$, for large and negative value of G, $f_2(G)\sim a_1/(a_2G^{-N})$. As a result, a Type II singularity could appear, when $-n<N<0$, N even and $a_1/a_2>0$ or N odd and $a_1/a_2<0$ (see Eq. (3.38) and the related discussion). If $0<N<1/2$ and $a_1/a_2>0$, we have the Type III singularity (see Eq. (3.38)) for large and positive values of G, such that $f_2(G)\sim(a_1/a_2)G^N$. When $G\to 0^-$, we do not recover any example of singularity of the preceding Subsection.

If there exists any singularity solution, the consistence of Eq. (3.9) and Eq. (3.10) has to be verified, as we have already discussed in Section **3.2**.

We see that our model in Eq. (3.42) with $n>0$ and $N>0$ could also produce Type II singularity for $0<\beta<-1/3$, or Type IV singularity for $\beta<-1$, when $G\to 0^-$. We get

$$f_2(G)\sim\frac{b_1}{b_2}\,,\quad \frac{df_2(G)}{dG}\sim -n\frac{b_1a_2}{b_2^2}G^{n-1}\,.$$

We are assuming b_1/b_2 very small and neglecting in Eq. (3.9), otherwise we have to consider $H_0\neq 0$ in Eq. (3.1) and a different analysis of Eq. (3.9) has to be done. It can be shown that, under the requirement $n>2/3$ (the relation between n and β is $n=2\beta/(3\beta+1)$), the asymptotic behavior of Eq. (3.12) when $G\simeq 24h_0^3\beta/(t_0-t)^{3\beta+1}$ is proportional to $1/(t_0-t)^{\beta+1}$ and therefore it is possible to realize the Type II or IV singularity. Here, we include some examples:

- For $N=1$ and $n=2$, $\mathcal{G}(H,\dot{H}...)\simeq-[(24h_0^5)b_1a_2/(\kappa^2b_2^2)](t_0-t)^{-1/2}$ when $\beta=-1/2$. Hence, if $b_1a_2<0$, the model can become singular when $G\to 0^-$ (Type II singularity).
- For $N=1$ and $n=3$, $\mathcal{G}(H,\dot{H}...)\simeq[b_1a_2/(\kappa^2b_2^2)](t_0-t)^{-4/7}$ when $\beta=-3/7$. Thus, if $b_1a_2>0$, the model can become singular when $G\to 0^-$ (Type II singularity).
- For $N=1$ and $n=8/9$, $\mathcal{G}(H,\dot{H}...)\simeq[2(8/9)^2(32^{-1/9})h_0^{5/3}b_1a_2/(\kappa^2b_2^2)](t_0-t)^{1/3}$ when $\beta=-4/3$. Hence, if $b_1a_2>0$, the model can become singular when $G\to 0^-$ (Type IV singularity).

In a certain sense, the model $f_1(G)$ in Eq. (3.41) is a particular case of the one in Eq. (3.42). For large values of G, it is easy to see that $\mathcal{G}(H,\dot{H}...)$ in Eq. (3.11) and in Eq. (3.12) tends to a constant, so that it is impossible to find singularities. Nevertheless, similarly to the above, Type II or III singularities can occur when $G\to 0^-$ for $n>2/3$. For example, if $n=2$, and therefore $\beta=-1/2$, one finds $\mathcal{G}(H,\dot{H})\simeq[(24h_0^5/(\kappa^2b_2^2))(a_1b_2-a_2b_1)](t_0-t)^{-1/2}$. If $(a_1b_2-a_2b_1)>0$, the model can become singular when $G\to 0^-$ (Type II singularity).

With regard to $f_3(G)$ in Eq. (3.43), it is interesting to find the conditions on m, n, a_3 and b_3 for which we do not have any type of singularities. When $G\to\pm\infty$ or $G\to 0^-$, it is possible to write this model in the form $f(G)\simeq\alpha G^\gamma$, α and γ being constants, which we have investigated on in the first part of this Section. We do not consider the trivial case $n=m$. The no-singularity conditions follow directly from the preceding results as complementary conditions to the singularity ones:

- Case (1): $n>0$, $m>0$, $n\neq 1$ and $m\neq 1$. We avoid any singularity if $0<n+m<1/2$ and $a_3b_3<0$; $n+m>1/2$, $n>1$ and $a_3>0$; $n+m>1/2$, $2/3<n<1$ and $a_3<0$; $n+m>1/2$, $0<n\leq 2/3$ and if $n=1/2$, $a_3>0$.
- Case (2): $n>0$, $m<0$ and $n\neq 1$. We avoid any singularity if $0<n<1/2$ and $a_3<0$; $n>1/2$, $n+m>1$ and $a_3b_3>0$; $n>1/2$, $2/3<n+m<1$ and $a_3b_3<0$; $n>1/2$, $n+m\leq 2/3$ and if $n+m=1/2$, $a_3b_3>0$.
- Case (3): $n<0$, $m>0$ and $m\neq 1$. We avoid any singularity if $m+n>1/2$; $m+n<1/2$ and $a_3b_3<0$.
- Case (4): $n<0$ and $m<0$. We avoid any singularity if $a_3<0$.

We end this Subsection considering the last realistic model of Eq. (3.44), again for $n > 0$. Since for large G, one has $f_4(G) \simeq (a_1/a_2)G^m$ and for small G, one has $f_4(G) \simeq (b_1/b_2)G^m$, the preceding analysis leads to the absence of any type of singularities for

$$\frac{1}{2} < m \leq \frac{2}{3}\,. \tag{3.45}$$

In fact, for this range of values, the asymptotic behavior of the right-hand side of Eq. (3.7) is different from the asymptotic behavior of its left-hand side on the singularity solutions. Thus, Eq. (3.44) provides an example of realistic model free of all possible singularities when Eq. (3.45) is satisfied, independently on the coefficients. Moreover, this model suggests the universal scenario to cure finite-time future singularities. In § 3.5 we will see that adding αG^m, α being a constant and $1/2 < m \leq 2/3$, to any singular Dark Energy, results in combined non-singular model.

3.4 The reconstruction of singular $\mathcal{F}(R,G)$-gravity

In this Subsection, we reconstruct the generic $\mathcal{F}(R,G)$-gravity models producing finite-time future singularities.

In a similar way of the previous Section, we rewrite the action (1.1) in vacuum by using proper functions $Z(t)$, $P(t)$ and $Q(t)$ of a scalar field which is identified with the time t,

$$I = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[Z(t)R + P(t)G + Q(t)\right]\,. \tag{3.46}$$

By the variation with respect to t, we obtain

$$\dot{Z}(t)R + \dot{P}(t)G + \dot{Q}(t) = 0\,, \tag{3.47}$$

from which in principle it is possible to find t as a function of R and G, $t = t(R,G)$. By substituting $t = t(R,G)$ into Eq. (3.46), we find the action in terms of $\mathcal{F}(R,G)$,

$$\mathcal{F}(R,G) = Z(t(R,G))R + P(t(R,G))G + Q(t(R,G))\,. \tag{3.48}$$

By using the conservation law and the first equation of motion (1.6), and then neglecting the contribution from matter, we get the differential equation

$$\ddot{Z}(t) + 4\dot{g}^2(t)\ddot{P}(t) - \dot{g}(t)\dot{Z}(t) + [8\dot{g}(t)\ddot{g}(t) - 4\dot{g}^3(t)]\dot{P}(t) + 2\ddot{g}(t)Z(t) = 0\,, \tag{3.49}$$

where we have used the expression of the scale factor in Eq. (3.16) and the Hubble parameter $H = \dot{g}(t)$. By using the first EOM again, $Q(t)$ becomes

$$Q(t) = -24\dot{g}^3(t)\dot{P}(t) - 6\dot{g}^2(t)Z(t) - 6\dot{g}(t)\dot{Z}(t)\,. \tag{3.50}$$

Big Rip singularity

First, we investigate the Big Rip singularity. If $\beta = 1$ in Eq. (3.1) with $H_0 = 0$, we have

$$\begin{aligned} H &= \frac{h_0}{(t_0 - t)}\,, \\ R &= \frac{6h_0}{(t_0 - t)^2}(2h_0 + 1)\,, \\ G &= \frac{24h_0^3}{(t_0 - t)^4}(1 + h_0)\,. \end{aligned} \tag{3.51}$$

A simple (trivial) solution of Eq. (3.49) is given by

$$Z(t) = \alpha(t_0-t)^z\,, \tag{3.52}$$
$$P(t) = \delta(t_0-t)^x\,, \tag{3.53}$$

with α and δ being constants, where

$$x = 3-h_0\,, \tag{3.54}$$

and $z = z_+, z_-$ is given by

$$z_\pm = \frac{1-h_0 \pm \sqrt{h_0^2-10h_0+1}}{2}\,. \tag{3.55}$$

Thus, the most general solution of $Z(t)$ is expressed as

$$Z(t) = \alpha_1(t_0-t)^{z_+} + \alpha_2(t_0-t)^{z_-}\,, \tag{3.56}$$

where α_1 and α_2 are constants. From Eq. (3.50), we have

$$Q(t) = \frac{24h_0^3\delta(3-h_0)}{(t_0-t)^{h_0+1}} + \frac{6h_0\alpha_1(z_+-h_0)}{(t_0-t)^{2-z_+}} + \frac{6h_0\alpha_2(z_--h_0)}{(t_0-t)^{2-z_-}}\,. \tag{3.57}$$

Under the condition $0 < h_0 < 5-2\sqrt{6}$ or $h_0 > 2+\sqrt{6}$, the solution of $\mathcal{F}(R,G)$, by absorbing some factor into the constants, results

$$\mathcal{F}(R,G) = \alpha_1 R^{1-\frac{z_+}{2}} + \alpha_2 R^{1-\frac{z_-}{2}} + \delta\cdot G^{\frac{h_0+1}{4}}\,. \tag{3.58}$$

If $\delta = 0$, we find a $F(R)$-model realizing Big Rip according with Ref. [79]. $G^{\frac{h_0+1}{4}}$, combined with R, is an invariant of the Big Rip solution in $f(G)$-gravity and produces the Big Rip in a general $\mathcal{F}(R,G)$-gravity theory. Note that $1-(z_\pm/2) \neq 1$ (pure Einstein gravity is free of singularities).

Another exact solution of Eq. (3.49) is given by

$$Z(t) = \frac{\alpha}{(t_0-t)^z}\,, \tag{3.59}$$
$$P(t) = \frac{\delta}{(t_0-t)^x}\,, \tag{3.60}$$

where δ and x are constants, and

$$z = x+2\,, \tag{3.61}$$

$$\alpha = \frac{4h_0^2x\delta(h_0-x-3)}{x^2+(5-h_0)x+6}\,. \tag{3.62}$$

From Eq. (3.50), we find

$$Q(t) = -\frac{6h_0}{(t_0-t)^{x+4}}\left[4h_0^2x\delta + \alpha(x+2+h_0)\right]\,. \tag{3.63}$$

The solution of Eq. (3.47) is given by

$$(t_0-t) = g(R,G)$$
$$= \left\{\frac{-\alpha(x+2)R \pm \sqrt{\alpha^2(x+2)^2R^2 + 24h_0\left[4h_0^2x\delta + \alpha(x+2+h_0)\right](x+4)(x\delta)G}}{2(x\delta)G}\right\}^{1/2}\,, \tag{3.64}$$

with $x \neq 0$ and $\delta \neq 0$.

To have real solutions, we must require that the arguments of the roots in Eq. (3.64) are positive. Since $h_0 > 0$, the principal cases are as follows:

- Case (1): $x > 0$, $\delta > 0$, $1 + x \leq h_0 < x + 5 + \frac{6}{x}$. We must use the sign $+$ in Eq. (3.64).
- Case (2): $-\frac{3}{2} \leq x < 0$, $\delta < 0$, $h_0 \geq x + 1$. We must use the sign $+$.
- Case (3): $-4 < x < -\frac{3}{2}$, $\delta < 0$, $h_0 > x + 5 + \frac{6}{x}$. We must use the sign $+$.
- Case (4): $x > 0$, $\delta < 0$, $x + 5 + \frac{6}{x} > h_0 \geq 1 + x$. We must use the sign $-$.
- Case (5): $-\frac{3}{2} \leq x < 0$, $\delta > 0$, $h_0 \geq x + 1$. We must use the sign $-$.
- Case (6): $-4 < x < -\frac{3}{2}$, $\delta > 0$, $h_0 > x + 5 + \frac{6}{x}$. We must use the sign $-$.
- Case (7): $x = -4$, $\delta > 0$. We must use the sign $-$.
- Case (8): $x = -4$, $\delta < 0$. We must use the sign $+$.

The solution of $\mathcal{F}(R,G)$ reads

$$\mathcal{F}(R,G) = \frac{\alpha}{(g(R,G))^{x+2}} R + \frac{\delta}{(g(R,G))^{x}} G - \frac{6h_0}{(g(R,G))^{x+4}} \left[4h_0^2 x\delta + \alpha(x + 2 + h_0)\right] , \qquad (3.65)$$

where $g(R,G)$ is given by Eq. (3.64). This is an exact solution of the EOM for the Big Rip case.

We show several examples. In the case $\alpha = 1$ and $x = -2$, we find

$$\mathcal{F}(R,G) = R + \frac{\sqrt{6}\sqrt{h_0(1+h_0)}}{(1-h_0)} \sqrt{G} , \quad h_0 \neq 1 , \qquad (3.66)$$

which is in agreement with the result of the previous Section.

If $\alpha = 0$ and $x = h_0 - 3$ (this case corresponds to the cases (1)–(6) presented above), we find

$$\mathcal{F}(R,G) = \delta \cdot G^{\frac{h_0+1}{4}} , \quad \delta \neq 0 , \qquad (3.67)$$

which is equivalent to Eq. (3.58) with $\alpha_1 = \alpha_2 = 0$.

If $x = -4$, the result is given by

$$\mathcal{F}(R,G) = \frac{16h_0^4\delta}{(1+2h_0^2)^2} \left[(9 + 21h_0 + 6h_0^2) - (1+h_0)^2 \frac{R^2}{G}\right] , \quad \delta \neq 0 . \qquad (3.68)$$

Hence, if $\mathcal{F}(R,G) = \pm\alpha \mp \delta \cdot (R^2/G)$ with $\alpha > 0$ and $\delta > 0$, the Big Rip singularity could appear for large values of R and G.

If $x = h_0 - 1$, by absorbing some constant, the solution becomes

$$\mathcal{F}(R,G) = \delta \cdot G \left(\frac{R}{G}\right)^{\frac{1-h_0}{2}} , \quad \delta \neq 0 , \quad h_0 \neq 1 . \qquad (3.69)$$

Thus, if $\mathcal{F}(R,G) = \delta \cdot G^{\gamma}/R^{\gamma-1}$ with $\delta \neq 0$ and $1/2 < \gamma < 1$ or $1 < \gamma < +\infty$, the Big Rip singularity could appear for large values of R and G.

Furthermore, it is possible to directly verify that the model:

$$\mathcal{F}(R,G) = \delta \cdot \left(\frac{G^m}{R^n}\right) , \qquad (3.70)$$

with δ being a generic constant, is a solution of the EOM (1.6)-(1.7) in the case of the Big Rip singularity ($\beta = 1$) for some value of h_0. In general, it is possible to obtain solutions for $h_0 > 0$ if $m > 0$, $n > 0$ and $m > n$. For example, the case $n = 2$ and $m = 3$ realizes the singularity in $h_0 = 5$; the case $n = 1$ and $m = 3$ realizes the singularity in $h_0 = 4 + \sqrt{19}$ and so forth. This is a generalization of Eq. (3.69). Note that we do not recover a physical solution for $m = -1$ and $n = -2$ because in this case $h_0 = -3$. For a similar kind of model, where $\mathcal{F}(R,G)$ is a function of R^2/G (i.e., $\mathcal{F}(R,G) = \mathcal{F}(R^2/G)$), which produces the Big Rip singularity, see Eq. (3.68). For $m = 0$ or $n = 0$, we recover Eq. (3.58).

Other types of singularities

Next, we study the other types of singularities. We consider the case in which H is given by

$$H = \frac{h_0}{(t_0 - t)^\beta}\,, \quad \beta \neq 1\,. \tag{3.71}$$

An exact solution of Eq. (3.49) is

$$Z(t) = -\lambda(4h_0^2)(t_0 - t)\,, \tag{3.72}$$

$$P(t) = \lambda(t_0 - t)^{2\beta+1}\,, \tag{3.73}$$

where λ is a generic constant. The form of $Q(t)$ is given by

$$Q(t) = \frac{24h_0^4\lambda}{(t_0 - t)^{2\beta-1}} + \frac{48h_0^3\beta}{(t_0 - t)^\beta}\,. \tag{3.74}$$

For $\beta = 1$, we find a special case of Eq. (3.65). For $\beta > 1$, we obtain the asymptotic real solution of Eq. (3.47):

$$(t_0 - t) = g(R,G) = 2^{1/2\beta}\left[\frac{h_0^2 R + \sqrt{h_0^4 R^2 + 6h_0^4(4\beta^2 - 1)G}}{(1 + 2\beta)G}\right]^{1/2\beta}\,. \tag{3.75}$$

The form of $\mathcal{F}(R,G)$ is expressed as

$$\mathcal{F}(R,G) = -4h_0^2\lambda(g(R,G))R + \lambda(g(R,G)^{1+2\beta})G + 24h_0^4\lambda(g(R,G)^{1-2\beta})\,, \quad \beta > 1\,. \tag{3.76}$$

In the case $\beta \gg 1$, the form of $\mathcal{F}(R,G)$ can be written as

$$\mathcal{F}(R,G) \simeq R - \frac{\alpha G}{R + \sqrt{R^2 + \gamma G}}\,, \quad \alpha > 0\,, \quad \gamma > 0\,. \tag{3.77}$$

This is the behavior of a $\mathcal{F}(R,G)$ model in which a "strong" Type I singularity ($\beta \gg 1$) could appear for $R, G \to +\infty$ (asymptotically solve Eq. (3.7)).

To find other models, we can consider the results of § 3.3. The Type I singularities ($\beta > 1$) correspond to the asymptotic limits for R and G

$$\begin{aligned} R &\simeq \frac{12h_0^2}{(t_0 - t)^{2\beta}}\,, \\ G &\simeq 24\frac{h_0^4}{(t_0 - t)^{4\beta}}\,. \end{aligned} \tag{3.78}$$

These are two functions of the Hubble parameter only, so that

$$\lim_{t\to t_0} 24\left(\frac{R}{12}\right)^2 = \lim_{t\to t_0} G\,. \tag{3.79}$$

If we substitute G for R in Eq. (3.33) by taking into account Eq. (3.79), we obtain a zero function (this is because Eq. (3.33) is zero on the singularity solution). If we substitute G for G/R, however, we obtain the following model:

$$\mathcal{F}(R,G) = R - \frac{6G}{R}\,. \tag{3.80}$$

This is an asymptotic solution of Eq. (3.7). Thus, there appears Type I singularity with $R, G \to +\infty$ for the model $\mathcal{F}(R,G) = R - \alpha(G/R)$ with $\alpha > 0$.

In the case of $H = h_0/(t_0 - t)^\beta$ with $\beta < 1$, it is not possible to write G and R like functions of the same variable (H or the same combination of H and $\dot{H}$). Nevertheless, if we examine the asymptotic behavior of G and R, we have

$$\begin{aligned} R &\simeq \frac{6h_0\beta}{(t_0-t)^{\beta+1}}\,, \\ G &\simeq \frac{24h_0^3\beta}{(t_0-t)^{3\beta+1}}\,, \end{aligned} \tag{3.81}$$

and

$$\frac{G}{R} \sim G^{\frac{2\beta}{3\beta+1}}\,. \tag{3.82}$$

If we use G/R for G in Eq. (3.37) as in Eq. (3.82), we see that Eq. (3.7) is asymptotically verified for $\beta < 1$. Under this consideration, it is possible to derive a $\mathcal{F}(R,G)$-gravity theory (by setting some parameters) from Eq. (3.37) as

$$\mathcal{F}(R,G) = R + \frac{3}{2}\frac{G}{R}\,, \tag{3.83}$$

in which the other types of singularities appear. Thus, in the model $\mathcal{F}(R,G) = R + \alpha(G/R)$ with $\alpha > 0$, the Type II, III and IV singularities could appear. Then, by substituting G for R we get

$$\mathcal{F}(R,G) \simeq R - \delta\frac{(1+\beta)}{(\beta-1)}|R|^{\frac{2\beta}{1+\beta}}\,, \quad \delta > 0\,. \tag{3.84}$$

This is the result of Ref [79]. In the model $F(R) = R + \alpha R^\gamma$, with $0 < \gamma < 1$ and $\alpha > 0$, a Type III singularity could appear for $R \to +\infty$. In the model $F(R) = R + \alpha|R|^\gamma$, with $-\infty < \gamma < 0$ and $\alpha > 0$, a Type II singularity could appear for $R \to -\infty$. In the model $F(R) = R + \alpha|R|^\gamma$, with $2 < \gamma < +\infty$ ($\gamma \neq 2n/(n-1)$, where n is a natural number) and $\alpha < 0$, a Type IV singularity could appear for $R \to 0^-$ [4].

In the next Subsection we will analyze an example of realistic $F(R)$-gravity generating singularity.

[4] Note that in the Big Rip case, we have found exact solutions. This kind of reasoning is therefore inapplicable.

3.4.1 Example of realistic singular-$F(R)$-model: the Hu-Sawicki Model

Let us return to Hu-Sawicki Model of Eq. (3.2). The Hu-Sawiki Model could become singular when R diverges. In particular, it shows a Type II singularity when H behaves as:

$$H = \frac{h_0}{(t_0 - t)^{\beta}} + H_0\,, \quad -1 < \beta < 0\,, \tag{3.85}$$

where we have reintroduced the positive constant H_0. As usually, the constant h_0 has to be positive.

In the asymptotic limit, Eq. (3.7) with Eq. (3.9) are verified by putting:

$$\beta = -\frac{n}{n+2}\,, \tag{3.86}$$

$$h_0 = \left[\frac{6n^2(n+1)}{(n+2)^2}\left(\frac{2+n}{-6n}\right)^{n+2}\left(\frac{c_1}{c_2^2}\ \left(\tilde{m}^2\right)^{(n+1)}\right)\right]^{n+2}\,, \tag{3.87}$$

$$H_0 = \sqrt{\frac{c_1\tilde{m}^2}{6c_2}}\,. \tag{3.88}$$

Here, h_0 is positive if n is an even number and the model may show the Type II singularity in expanding universe (if n is an odd number, this kind of singularity could appear for contracting universe, as the Big Crunch). Note that H_0 is the constant Hubble parameter H_{dS} of the de Sitter universe, $H_0 = H_{dS}$. We have just discussed in § 3.1 the problems generated by the possibility to have singular solutions in the cosmological scenario described by Hu-Sawiki Model. Let us have a look for the strategy to use in order to cure singularity occurrence.

3.5 Curing the finite-time future singularities

In this last Section, we discuss a possible way to cure the finite-time future singularities in $\mathcal{F}(R,G)$-gravity. We will see some simple curing term, i.e., some power function of R or G, to add into the theory in order to prevent the singularities. In the last Subsection, the quantum effects in the range of high curvature are also discussed.

3.5.1 Power terms of R and G

First, we consider $f(G)$-modified gravity. If any singularity occurs, $\mathcal{G}(H,\dot{H}...)$ evaluated on the singular form of H of Eq. (3.1) with $H_0 = 0$, behaves as in Eq. (3.10).

The singularities appear in two cases: (a) $G \to \pm\infty$ (Big Rip, Type I and Type III singularities and Type II singularities with $-1/3 < \beta < 0$); (b) $G \to 0^-$ (Type IV singularities and Type II singularities with $-1 < \beta < -1/3$)[5].

- Case of $G \to \pm\infty$
  Let us consider the $f^*(G)$ curing term

  $$f^*(G) = \gamma G^m\,, \quad m \neq 1\,, \tag{3.89}$$

  with $\gamma \neq 0$ and m being a constant. One way to prevent a singularity appearing could be that the function $\mathcal{G}(H,\dot{H}...)$ becomes inconsistent with the behavior of Eq. (3.10). In general, $\mathcal{G}(H,\dot{H}...)$ must tend to infinity faster than Eq. (3.10). For $H = h_0/(t_0 - t)$, this is the Big Rip, the (additive) contribute of $f^*(G)$ to $\mathcal{G}(H,\dot{H}...)$ is $\mathcal{G}^*(H,\dot{H}...)$,

  $$\mathcal{G}^*(H,\dot{H}...) \sim \frac{\alpha}{(t_0 - t)^{4m}}\,. \tag{3.90}$$

[5] Note that if H tends to a non avoidable constant H_0, the Gauss Bonnet diverges for any value of $-1 < \beta < 0$, i.e. for any kind of Type II singularity, as the Ricci scalar R.

Here, α is a generic constant. Hence, if $m > 1/2$, we avoid the singularity. Nevertheless, there is one specific case in which the Big Rip singularity could still occur. If $m = (1+h_0)/4$, $\mathcal{G}^*(H, \dot{H}...)$ is exactly equal to zero, so that (for example) the following specific model with $m > 1/2$, admits the Big Rip singularity:

$$R + f(G) = R + \frac{\sqrt{24m(4m-1)^3}}{2h_0(1-2m)} G^{1/2} + \gamma G^m \,. \tag{3.91}$$

This is because the power function G^m is an invariant with respect to the Big Rip singularity generated by $G^{1/2}$-term. If we have the model $R + \alpha\sqrt{G}$, we can eliminate the Big Rip singularity with a power function γG^m ($m > 1/2$) only if $\alpha > 0$ (such that the configuration of Eq. (3.91) can not be realized).

For $H = h_0/(t_0 - t)^\beta$ with $\beta > 1$, this is the Type I singularity, the curing term in Eq. (3.89) leads to

$$\mathcal{G}^*(H, \dot{H}...) \sim \frac{\alpha}{(t_0 - t)^{4\beta m}} \,. \tag{3.92}$$

Also in this case, if $m > 1/2$, we avoid the singularity.
For example, the model $R + \alpha\sqrt{G} + \gamma G^2$ with $\alpha > 0$ is free of Type I singularities, while if $\alpha < 0$, the Big Rip singularity could appear.

For $H = h_0/(t_0 - t)^\beta$ with $0 < \beta < 1$, this is the Type III singularity, the curing term in Eq. (3.89) leads to

$$\mathcal{G}^*(H, \dot{H}...) \sim \frac{\alpha}{(t_0 - t)^{m(3\beta+1)+(1-\beta)}} \,. \tag{3.93}$$

If $m > 2\beta/(3\beta + 1)$, i.e. $m > 1/2$, we avoid the singularity.

Also for $H = h_0/(t_0 - t)^\beta$ with $-1/3 < \beta < 0$, this is the case of Type II singularity when $G \to -\infty$, we have to require the same condition.
For example, $R + \alpha|G|^m + \gamma G^2$ with $m < 1/2$, is free of Type I, II (with $-1/3 < \beta < 0$) and III singularities.

- Case of $G \to 0^-$
  For $H = h_0/(t_0 - t)^\beta$ with $\beta < -1/3$ (Type II and IV singularities), the curing term in Eq. (3.89) leads to Eq. (3.93) again, which diverges and hence becomes inconsistent with Eq. (3.10) if $m \leq 2/3$.
  For example, $R + \alpha|G|^\zeta + \gamma G^{-1}$ with $\zeta > 2/3$ is free of Types IV singularities.

As a result, the term γG^m with $m > 1/2$ and $m \neq 1$ cures the singularities occurring when $G \to \pm\infty$. Moreover, the term γG^m with $m \leq 2/3$ cures the singularities occurring when $G \to 0^-$.

In $f(R)$-gravity, by using the term γR^m, the same consequences are found. The term γR^m with $m > 1$ cures the Type I, II and III singularities occurring when $R \to \pm\infty$. On the other hand, the term γR^m with $m \leq 2$ cures the Type IV singularity occurring when $R \to 0^-$.

Note that γG^m or γR^m are invariants with respect to the Big Rip solution (see Eq. (3.58)), so it is necessary to pay attention to the whole form of the theory.

A general important result is the following: the terms like γR^n or γG^m with $1 < n \leq 2$ or $1/2 < m \leq 2/3$ respectively, avoid any types of singularities in $f(R, G)$-gravity or in the presence of dark energy fluid producing singularities (with regard to R^2 curing term see Refs. [79, 90, 91]).

3.5.2 Combinations of R and G

Within the framework of $f(R, G)$-gravity, i.e. $\mathcal{F}(R, G) = R + f(R, G)$, we can use the terms $f^*(R, G)$ such as

$$f^*(R, G) = \gamma \frac{G^m}{R^n} \,, \tag{3.94}$$

with $\gamma \neq 0$ and m, n constants, to cure the singularities.

The singularities appear in the following three cases: (a) $R \to \pm\infty$, $G \to \pm\infty$ (Types I and Type III singularities and Type II singularities for $-1/3 < \beta < 0$); (b) $R \to -\infty$, $G \to 0^-$ (Type II singularities for $-1 < \beta < -1/3$), and (c) $R \to 0^-$, $G \to 0^-$ (Type IV singularities).

We investigate general possibilities.

In the case of the Big Rip singularity, the contribute of $f^*(R,G)$ to $\mathcal{G}(H,\dot{H}...)$ in Eq. (3.9) diverges as

$$\mathcal{G}^*(H,\dot{H}...) \sim \frac{1}{(t_0-t)^{4m-2n}}\,. \tag{3.95}$$

Thus, if $m > (n+1)/2$, we avoid the singularity. Nevertheless, there is the possibility that $\mathcal{G}^*(H,\dot{H}...)$ is exactly equal to zero and the curing term does not protect the theory against the Big Rip (see Eq. (3.69) and Eq. (3.70) in the case of $m = n+1$, where such combination of R and G alone produces the Big Rip, and therefore is trivial in $R + f(R,G)$ models). Hence, the whole form of $\mathcal{F}(R,G)$ as well as its form in the asymptotic limit must be examined.

In the case of Type I singularities, $\mathcal{G}^*(H,\dot{H}...)$ diverges as

$$\mathcal{G}^*(H,\dot{H}...) \sim \frac{1}{(t_0-t)^{4\beta m-2\beta n}}\,. \tag{3.96}$$

Also in this case, if $m > (n+1)/2$, we avoid the singularity.

When $\beta < 1$, $\mathcal{G}^*(H,\dot{H}...)$ behaves as

$$\mathcal{G}^*(H,\dot{H}...) \sim \frac{\alpha}{(t_0-t)^{(3\beta+1)m-(\beta+1)n+(1-\beta)}}\,. \tag{3.97}$$

As a result, $\mathcal{G}^*(H,\dot{H}...)$ diverges faster than $(t_0-t)^{-\beta-1}$ and therefore the Type III singularity ($0 < \beta < 1$) is avoided if $m\,, n > 0$ such that $m > (1+n)/2$ (for example, one can choose $n = 1$ and $m = 2$). The Type II singularity for $-1/3 < \beta < 0$ is avoided if $m > 0$ and $n < 0$. The Type II singularity for $-1 < \beta < -1/3$ is avoided if $m < 0$ and $n < 0$. Finally, the Type IV singularity ($\beta < -1$) is avoided if $n > 0$ and $m < 0$.

3.5.3 Quantum effects

In the high curvature limit, quantum effects could become relevant and they have to be taken into account. Consider next the quantum contribution to the conformal anomaly. The complete energy density $\rho_{\rm tot}$ and pressure $p_{\rm tot}$ of matter are:

$$\rho_{\rm tot} = \rho_{\rm m} + \rho_{\rm A}\,, \tag{3.98}$$

$$p_{\rm tot} = p_{\rm m} + p_{\rm A}\,, \tag{3.99}$$

where $\rho_{\rm m}$ and $p_{\rm m}$ are, as usually, the standard contributes of matter and $\rho_{\rm A}$ and $p_{\rm A}$ are given by quantum effects. Taking the trace T_A of the conformal anomaly energy-momentum tensor,

$$\mathrm{T}_A = -\rho_{\rm A} + 3p_{\rm A}\,, \tag{3.100}$$

plus observing the energy conservation law,

$$\dot{\rho_{\rm A}} + 3H(\rho_{\rm A} + p_{\rm A}) = 0\,, \tag{3.101}$$

we find that:

$$p_{\rm A} = -\rho_{\rm A} - \frac{\dot{\rho_{\rm A}}}{3H}\,. \tag{3.102}$$

Thus we obtain for the conformal anomaly energy density [92]:

$$
\begin{aligned}
\rho_{\mathrm{A}} &= -\frac{1}{a(t)^4}\int a(t)^4 H\,(\mathrm{T}_A)\,dt \\
&= -\frac{1}{a(t)^4}\int a(t)^4 H\Big\{-12b\dot{H}^2 + 24b_1(-\dot{H}^2 + H^2\dot{H} + H^4) - \\
&\qquad (4b_0 + 6b_2)(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H})\Big\}dt\,.
\end{aligned}
\tag{3.103}
$$

Here, b_0, b_1 and b_2 are constants, occuring in the expression for the conformal trace anomaly:

$$
\mathrm{T}_{\mathrm{A}} = b_0(F + \frac{2}{3}\Box R) + b_1 G + b_2 R\,. \tag{3.104}
$$

As usually, F is the square of the Weyl Tensor and G the Gauss-Bonnet invariant. Explicitly, if there are N scalars, $N_{1/2}$ spinors, N_2 gravitons and N_{HD} higher derivative conformal scalars, one has for b_0 and b_1 the following expressions:

$$
b_0 = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{HD}}{120(4\pi)^2}\,, \tag{3.105}
$$

$$
b_1 = \frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{HD}}{360(4\pi)^2}\,, \tag{3.106}
$$

whereas b_2 is an arbitrary constant whose value depends on the regularization.

The quantum corrected EOM (1.14) is:

$$
\rho_{\mathrm{eff}} + \rho_{\mathrm{A}} = \frac{3}{\kappa^2}H^2\,. \tag{3.107}
$$

Quantum effects become relevant for large values of curvature R and when the effective energy density of the universe is not too much large. In particular, this is the case of Type II singularities, when $H = h_0/(t_0 - t)^\beta$ with $-1 < \beta < 0$. Eq. (3.103) gives:

$$
\rho_{\mathrm{A}} \simeq \frac{\alpha}{(t_0 - t)^{\beta+2}}\,. \tag{3.108}
$$

Here, α is a number. In some scenario, quantum effects have to be taken into account. In this case, ρ_{A} diverges in Eq.(3.107) faster than H^2, so that the Type II singularity is not realized.

Chapter 4

Viscous fluids and singularities

Here, we examine the results obtained in Refs. [15, 16]. We will study some features of inhomogeneous viscous fluids, especially relating to singularities. Fluids in general have been considered as candidate to dark energy into the context of GR, since the evolution of cosmological parameters is not defined with precise accuracy, except for the current values with 3-5% error at least, and the observations do not exclude the possibility to have dark energy with a dynamical Equation of State, and not static as in the case of Cosmological Constant of ΛCDM Model. We stress that modified gravity also has an equivalent description as effective (viscous) fluid. In this Chapter, as a prosecution of the previous one, we analyze the behaviour of dark energy fluids in singular theories of modified gravity, investigating how the singularities may change or disappear, due to the contribution of these fluids. After that, a Section is devoted to the study of viscous dark energy-fluids (DE-fluids) coupled with dark matter (DM).

4.1 Viscous fluids and modified gravity

The most general form of inhomogeneous viscous fluid in FRW background is given by the Equation of State [93]:

$$p_{\mathrm{F}} = \omega(\rho_{\mathrm{F}})\rho_{\mathrm{F}} + B(\rho_{\mathrm{F}}, a(t), H, \dot{H}...)\,, \tag{4.1}$$

where p_{F} and ρ_{F} are the pressure and energy density of fluid, respectively, and the thermodynamical variable $\omega(\rho_{\mathrm{F}})$ is an arbitrary function of the density ρ_{F}. The bulk viscosity $B(\rho_{\mathrm{F}}, a(t), H, \dot{H}...)$ is a function of the density ρ_{F}, the scale factor $a(t)$, and the Hubble parameter H and its derivatives. The motivation to consider this general form of time-dependent bulk viscosity comes from the modification of gravity, which can always treat like a fluid in this form. For example, in $\mathcal{F}(R, G)$-gravity, if we define the effective energy density and pressure ρ_{eff} and p_{eff} as in Eqs. (1.16)-(1.17), we can take $B(\rho_{\mathrm{F}}, a(t), H, \dot{H}...) = 0$ and $\omega(\rho_{\mathrm{F}}) = \omega_{\mathrm{eff}}$ defined by Eq. (1.22). In this way, the EoS parameter depends on H and its derivatives. Otherwise, in § 3.2, we have used an other fluid representation for $f(R, G)$-gravity with $\omega(\rho_{\mathrm{F}})$ constant, such that $\omega(\rho_{\mathrm{F}}) = \omega$, where ω is the EoS parameter of matter. In this case, we can find Eq. (4.1), by identifying $B(\rho_{\mathrm{F}}, a(t), H, \dot{H}...)$ with $\mathcal{G}(H, \dot{H}...)$ of Eq. (3.9).

In general, we identify as quintessence, fluids with $-1 < \omega(\rho_{\mathrm{F}}) < -1/3$, and as phantom, fluids with $\omega(\rho_{\mathrm{F}}) < -1$.

4.1.1 Example of realistic fluid model generating the Big Rip

In principle DE-fluids -as modified gravity- may bring the future universe evolution to become singular. Let us analyze in some detail an interesting inhomogeneous non viscous fluid introduced in Ref. [94], whose EoS is

$$p_{\mathrm{F}} = -\rho_{\mathrm{F}} + f(\rho_{\mathrm{F}})\,, \tag{4.2}$$

where

$$f(\rho_{\mathrm{F}}) = +\frac{2\rho_{\mathrm{F}}}{3n}\left(1-\frac{4n}{\delta}\left(\frac{3\tilde{m}^2}{\kappa^2\rho_{\mathrm{F}}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \quad t \leq t_0\,, \tag{4.3}$$

$$f(\rho_{\mathrm{F}}) = -\frac{2\rho_{\mathrm{F}}}{3n}\left(1-\frac{4n}{\delta}\left(\frac{3\tilde{m}^2}{\kappa^2\rho_{\mathrm{F}}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \quad t > t_0\,. \tag{4.4}$$

Here, $n \geq 1$ and δ are constant positive parameters, $\tilde{m}^2$ is a mass scale and t_0 is the fixed time for which $f(\rho_{\mathrm{F}})$ assumes the smallest value and it is equal to zero. If we define the EoS parameter $\omega(\rho_{\mathrm{F}})$ as $\omega(\rho_{\mathrm{F}}) = p_{\mathrm{F}}/\rho_{\mathrm{F}}$, one has

$$\omega(\rho_{\mathrm{F}}) = -1 + \sigma(t)\frac{2}{3n}\left(1-\frac{4n}{\delta}\left(\frac{3\tilde{m}^2}{\kappa^2\rho_{\mathrm{F}}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{4.5}$$

where $\sigma(t) = 1$ when $t \leq t_0$ and $\sigma(t) = -1$ when $t > t_0$. We note that $t = t_0$, such that $f(\rho_{\mathrm{F}}) = 0$, corresponds to the transition point between quintessence $(-1 < \omega(\rho_{\mathrm{F}}) < -1/3)$ and phantom $(\omega_{\mathrm{F}} < -1)$ region, for which $\omega(\rho_{\mathrm{F}}) = -1$. When $t < t_0$, $-1 < \omega(\rho_{\mathrm{F}}) < -1 + 2/(3n) \leq -1/3$, and when $t > t_0$, $-5/3 \leq -1 - 2/(3n) < \omega(\rho_{\mathrm{F}}) < -1$.

This model may be used to correctly reproduce the matter era and the present accelerated epoch at the time $t = t_0$ (it is well know that ω_{eff} is very close to -1 today). The fluid energy conservation law reads

$$\dot{\rho}_{\mathrm{F}} + 3Hf(\rho_{\mathrm{F}}) = 0\,, \tag{4.6}$$

which leads

$$\rho_{\mathrm{F}} = \frac{3\tilde{m}^2\left(\frac{a(t)}{n}\right)^{\frac{2}{n}}\left(4n + C^{-\left(\frac{1}{2}\right)}\left(\frac{a(t)}{n}\right)^{-\frac{1}{n}}\right)^4 C}{16\delta^2\kappa^2}\,. \tag{4.7}$$

Here, $a(t)$ is the scale factor of the universe and $C > 0$ is an integration constant and it has been defined positive. We can put $a(t_0) = 1$. The fluid energy density at the present time t_0 is defined as $\rho_{\mathrm{F}(0)}$.

If the mass scale $\tilde{m}^2$ corresponds to the energy density of matter at the present time $\rho_{\mathrm{m}(0)}$, i.e. $\rho_{\mathrm{m}(0)} = 3\tilde{m}^2/\kappa^2$, by imposing $\rho_{F(0)}/\rho_{\mathrm{m}(0)} = \Lambda/(3\tilde{m}^2)$, such that Λ/κ^2 is the observed dark energy density in our universe, Λ being the Cosmological Constant, and $\dot{\rho}_{\mathrm{F}(0)} = 0$ (this is the condition to have $\omega(\rho_{\mathrm{F}(0)}) = -1$), one finds:

$$C = \frac{1}{16}\left(n^{1-\frac{1}{n}}\right)^{-2}, \tag{4.8}$$

$$\frac{16n^2}{\delta^2} = \frac{\Lambda}{3\tilde{m}^2}\,. \tag{4.9}$$

It is easy to see that, for $t \ll t_0$, since matter evolves as $\rho_m \sim a(t)^{-3}$, its energy density grows up in the past faster than the one of fluid and we have the matter era, but since for $t = t_0$, $\rho_{\mathrm{F}(0)} > \rho_{\mathrm{m}(0)}$, there is a point in the past when the energy density of fluid overtakes the energy density of matter and an accelerated epoch driven in a first step by quintessence fluid (for $t < t_0$) and therefore by phantom fluid (for $t > t_0$) takes place. The solution of equation of motion $\rho_{\mathrm{F}} = 3H^2/\kappa^2$ is

$$H = \frac{n\left(\frac{\delta}{\sqrt{\tilde{m}^2}}\right)}{(t_s - t)\left(t - t_s + \frac{\delta}{\sqrt{\tilde{m}^2}}\right)}, \tag{4.10}$$

where $t_s > 0$ is a fixed time parameter. Here, we have used condition (4.8). The cosmic time t has to be $t < t_s$. The Hubble parameter diverges at finite-future time, when $t \to t_s$ and, since Eq. (4.10) behaves as Eq. (3.1) for $\beta = 1$, the Big Rip singularity appears. Therefore, t_s corresponds to the life time of the universe. We observe that, in order to have an expanding universe ($H > 0$), $\delta/\sqrt{\tilde{m}^2}$ has to be larger than t_s. The present de Sitter solution (for which we have putted $\dot{\rho}_{\mathrm{F}(0)}$ and therefore $\dot{H}$ equal to zero) corresponds to $t_0 = t_s - (\delta/2\sqrt{\tilde{m}^2})$.

In conclusion, we have seen that fluid exits from de Sitter-phase involving in a phantom region. The the De Sitter solution is not a final attractor of the system, which becomes singular. In order to cure such singularity, it is possible to use some power functions of R or some power functions of G into a wider $f(R, G)$-modified gravity framework, via scenario suggested in Chapter 3. Now, we will see how (vice versa) inhomogeneous fluids can cure singularities in $f(R, G)$-gravity.

4.2 Viscous fluids in singular universe

In this Section [15] we take a simple theory of modified gravity where $\mathcal{F}(R, G) = R + f(R, G)$ as in Eq. (1.10). Moreover, we consider the presence of a viscous fluid, whose Equation of State is a simple formulation of Eq. (4.1) and it is given by

$$p_{\mathrm{F}} = \omega(\rho_{\mathrm{F}})\rho_{\mathrm{F}} - 3H\zeta(H)\,, \tag{4.11}$$

where $\zeta(H)$ is the bulk viscosity and it depends on the Hubble parameter H only. On thermodynamical grounds, in order to have the positive sign of the entropy change in an irreversible process, $\zeta(H)$ has to be a positive quantity, so we assume $\zeta(H) > 0$ [95, 96]. For the stress-energy tensor of fluid $T_{\mu\nu}^{(\mathrm{fluid})}$, one has :

$$T_{\mu\nu}^{(\mathrm{fluid})} = \rho_{\mathrm{F}} u_\mu u_\nu + (\omega(\rho_{\mathrm{F}})\rho - 3H\zeta(H))\,(g_{\mu\nu} + u_\mu u_\nu)\,, \tag{4.12}$$

where $u_\mu = (1, 0, 0, 0)$ is the four velocity vector. Into the effective parameters of Eqs. (1.18)-(1.19), we have to take into account the contribute of fluid also, so that we redefine ρ_{eff} of Eq. (1.18) as ρ_{MG} and p_{eff} of Eq. (1.19) as p_{MG}, where the suffix 'MG' indicates the 'modified gravity' contribute. We also neglect the contribute of ordinary matter and radiation. For $f(R, G)$-gravity, in analogy with Eqs.(3.5)-(3.6), one has

$$\rho_{\mathrm{MG}} = \frac{1}{2\kappa^2}\left[(Rf'_R + Gf'_G - f) - 6H\dot{f}'_R - 24H^3\dot{f}'_G - 6H^2 f'_R\right], \tag{4.13}$$

$$\begin{aligned} p_{\mathrm{MG}} = \frac{1}{2\kappa^2}\Big[&(f - Rf'_R - Gf'_G) + 4H\dot{f}'_R + 2\ddot{f}'_R \\ &+16H(\dot{H} + H^2)\dot{f}'_G + 8H^2\ddot{f}'_G + (4\dot{H} + 6H^2)f'_R\Big]. \end{aligned} \tag{4.14}$$

The new effective energy density and pressure describing universe become

$$\rho_{\mathrm{eff}} = \rho_{\mathrm{MG}} + \rho_{\mathrm{F}}\,, \tag{4.15}$$

$$p_{\mathrm{eff}} = p_{\mathrm{MG}} + p_{\mathrm{F}}\,. \tag{4.16}$$

The equations of motion are:

$$\rho_{\mathrm{MG}} + \rho_{\mathrm{F}} = \frac{3}{\kappa^2}H^2\,, \tag{4.17}$$

$$p_{\mathrm{MG}} + p_{\mathrm{F}} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right). \tag{4.18}$$

The fluid energy conservation law is a consequence of the EOM (4.17)-(4.18):

$$\dot{\rho_{\mathrm{F}}} + 3H\rho_{\mathrm{F}}(1 + \omega(\rho_{\mathrm{F}})) = 9H^2\zeta(H)\,. \tag{4.19}$$

In what follows, we will concentrate again on the singular form of Hubble parameter as in Eq. (3.1), namely $H = h_0/(t_0 - t)^\beta + H_0$, such that the scale factor behaves as

$$
\begin{aligned}
a(t) &= \frac{a_0}{(t_0-t)^h}\,, \quad \beta = 1 \quad (\text{Big Rip})\,,\\
a(t) &= a_0 \exp\left[\frac{h_0(t_0-t)^{1-\beta}}{\beta-1}\right]\,, \quad \beta(\neq 1) > 0 \quad (\text{Type I, III singularities})\,,\\
a(t) &= a_0 \exp\left[\frac{h_0(t_0-t)^{1-\beta}}{\beta-1} + H_0\right]\,, \quad \beta < 0 \quad (\text{Type II, IV singularities})\,. \qquad (4.20)
\end{aligned}
$$

As usually, a_0 and h_0 are positive constants and t_0 is the finite time for which singularity appears. Here, the positive constant $H_0 \neq 0$ has been considered in the significant cases of Type II and IV singularities only.

By using Eq. (4.19), we will check the solution of the fluid energy density when H is singular. We will see how changes the total effective energy density (and, as a consequence, the total effective pressure) of the universe due to the fluid contribute in the case of singular theories of $f(R, G)$-modified gravity, and if the singularities are still realized. In particular, we are interested in the quintessence ($-1 < \omega(\rho_\mathrm{F})$) and phantom ($\omega(\rho_\mathrm{F}) < -1$) region.

We investigate the cases of $\omega(\rho_\mathrm{F})$ constant and $\omega(\rho_\mathrm{F})$ dependent on energy density.

4.2.1 $\omega(\rho_\mathrm{F})$ constant

Let us start considering the simple case when $\omega(\rho_\mathrm{F})$ is a constant, such that $\omega(\rho_\mathrm{F}) = \omega_\mathrm{F}$, where ω_F is the constant EoS parameter of fluid. We take different choices of bulk viscosity $\zeta(H)$.

Non-viscous case

In the non-viscous case $\zeta(H) = 0$ (perfect fluid), the solution of Eq. (4.19) assumes the classical form:

$$
\rho_\mathrm{F} = \rho_0 a(t)^{-3(1+\omega_\mathrm{F})}\,, \qquad (4.21)
$$

where ρ_0 is a positive constant and $a(t)$ is the scale factor. As a consequence, on the sigular forms of $a(t)$ in Eq. (4.20), ρ_F behaves as

$$
\rho_\mathrm{F} = \rho_0 (t_0-t)^{3h_0(1+\omega_\mathrm{F})}\,, \quad \beta = 1\,, \qquad (4.22)
$$

$$
\rho_\mathrm{F} = \rho_0 \mathrm{e}^{\frac{3h_0(1+\omega_\mathrm{F})(t_0-t)^{1-\beta}}{1-\beta}}\,, \quad \beta(\neq 1) > 0\,, \qquad (4.23)
$$

$$
\rho_\mathrm{F} = \rho_0 \mathrm{e}^{3(1+\omega_\mathrm{F})(t_0-t)\left(H_0 - \frac{h_0(t_0-t)^{-\beta}}{\beta-1}\right)}\,, \quad \beta < 0\,. \qquad (4.24)
$$

For $\beta = 1$ (Big Rip) and $\beta > 1$ (Type I singularity), ρ_F grows up and becomes relevant when t is close to t_0 only if $\omega_\mathrm{F} < -1$. It means that phantom fluids increases the effective density and pressure of the universe in the case of Big Rip and Type I singularities, whereas quintessence fluid ($\omega_\mathrm{F} > -1$) becomes negligible and do not influence the asymptotic behaviour of $f(R, G)$ models that realize this kind of singularities. As a consequence, for modified gravity which produces Type I singularities, we will examine the case of phantom fluid only.

In Einstein's gravity ($f(R, G) = 0$), Eq. (4.17) and Eq. (4.22) admit the solution:

$$
H = -\frac{2}{3(1+\omega_\mathrm{F})}\frac{1}{(t_0-t)}\,, \qquad (4.25)
$$

and we can see that the phantom fluid produces the Big Rip for $H = h_0/(t_0 - t)$, where $h_0 = -2/3(1 + \omega_{\rm F})$.

In general, in $f(R, G)$-modified gravity, in the presence of phantom fluid, the asymptotically Big Rip singularity could appear if $\rho_{\rm MG}$ diverges less than H^2 ($\sim (t_0 - t)^{-2}$) on the singular solution of Eq. (4.25), namely the modified gravity becomes negligible with respect to the fluid contribute in Eq. (4.17). On the other hand, if a $f(R, G)$ model realizes the Big Rip for a certain value of h_0, the fluid energy density $\rho_{\rm F}$ of Eq. (4.22) becomes negligible on this singular solution if $\omega_{\rm F} > -(1 + 2/(3h_0))$, because in this case it diverges less than H^2.

When $\beta > 1$, the energy density $\rho_{\rm F}$ of phantom fluid exponentially diverges in Eq. (4.23), so that the EOM (4.17)-(4.18) become inconsistent and the Type I singularity is never realized.

When $0 < \beta < 1$, $\rho_{\rm F}$ tends to ρ_0 with time in Eq. (4.23), and it is asymptotically negligible with respect to H^2 ($\sim (t_0 - t)^{-2\beta}$). In this case, a $f(R, G)$-model realizing Type III singularity, is not influenced by perfect fluids on this kind of singularity.

For Type II and IV singular models ($\beta < 0$), the presence of quintessence or phantom fluids can make the singularities more difficult to realize. Note that H^2 of Type II and IV singularities tends to the constant $3H_0^2/\kappa^2$ like $\sim (t_0 - t)^{-\beta}$, while $\rho_{\rm F}$ in Eq. (4.24), after the developing of the exponential function in power series, tends to ρ_0 like $\sim (1 + \omega_{\rm F})(t_0 - t)$.

In the case of $-1 < \beta < 0$, the presence of fluid may change the numerical value of H_0 for which the singularity appears in $f(R, G)$-gravity, but does not necessarily avoid the singularity.

In the case of $\beta < -1$, since $\rho_{\rm F}$ behaves as $(t_0 - t)$ and it is asymptotically larger than the time-dependent part of H^2 ($\sim (t_0 - t)^{-\beta}$), Eq. (4.17) could become inconsistent. In particular, the softest Type IV singularities with $|\beta| \gg 1$ are very difficult to realize in the presence of phantom or quintessence perfect fluids.

Examples:

- In the model $R - \alpha\sqrt{G}$, where α is a positive constant, the Type I singularity or the Big Rip for some values of $h_0 > 1$ could occur (see Eq. (3.25) and Eq. (3.33) together). If we add a phantom fluid ($\omega_{\rm F} < -1$), the Type I singularity is avoided, while the Big Rip could still appear.
  If $\omega_{\rm F} < -5/3$ (namely, $\omega_{\rm F} < -(1+2/(3h_0))$ for any value of $h_0 > 1$), the fluid energy density of Eq. (4.22) grows up faster than H^2 in the case of the Big Rip, and the Big Rip with $h_0 > 1$ is not realized. On the other hand, the phantom fluid could produce the Big Rip for some value of $0 < h_0 < 1$, when $h_0 = -2/3(1 + \omega_{\rm F})$ like in Eq. (4.25). However, it is possible to verify, by using Eq. (4.13), that $\rho_{\rm MG}$ of this model, when $0 < h_0 < 1$, diverges still like H^2, but is negative. If the effective energy density of the universe becomes negative, the Big Rip is not a physical (real) solution.

- The model $R + \alpha R^\gamma$, where α is a constant, could realize the Type II singularity when $\gamma < 0$ or the Type IV singularity when $2 < \gamma$ (see Eq. (3.84)). In both cases we assume H_0 negligible in Eq. (3.1).
  The presence of quintessence or phantom fluids does not avoid the Type II singularity, because the numerical value of H_0 changes on the singular solution ($H_0 = \sqrt{\kappa^2\rho_0/3}$), but the dynamical behaviour of the modified function $f(R)$ keeps the same, due to the fact that R tends to infinity, and is not influenced by the constant H_0. Moreover, if we use a phantom fluid, there is the possibility that the Type II singularity is changed into the Big Rip in the form of Eq. (4.25), because, when $H \sim (t_0 - t)^{-1}$, it is easy to verify that $\rho_{\rm MG}$ of the model tends to zero, so that the fluid is dominant and makes the future singularity stronger.
  The Type IV singularity could be avoided by phantom or quintessence fluids, especially if γ parameter is very close to two (it means, $|\beta| \gg 1$). As a consequence, other future scenarios for the universe are possible. For example, if $\gamma = 3$, the model admits an unstable de Sitter solution with $R_{dS} = \sqrt{1/\alpha}$ (see Eq. (1.39)), or the phantom fluid may produce an accelerating phase.

- The model $R - \alpha G^\gamma$, where $\alpha > 0$ and $\gamma > 1$, shows the Type II singularity with $H_0 = 0$

and $-1 < \beta < -1/3$ (see Eq. (3.38) and the following discussion). Now, the presence of phantom or quintessence fluids with suitable boundary conditions on ρ_0, avoids the Type II singularity. Unlike the preceding example, the value of H_0 and the dynamical behaviour of $f(G)$ change together, because in the case of $H_0 = 0$, when $-1 < \beta < -1/3$, G tends to zero, but if $H_0 \neq 0$, G diverges to infinitive (is easy to see it by using Eq. (1.9)) and Eq. (4.17) for this kind of model becomes inconsistent on the Type II singularity.

Constant viscosity

Now, we introduce bulk viscosity in cosmic fluid. Note that viscous fluids belong to more general inhomogeneous EoS fluids introduced in Ref. [94, 97].

Suppose to have the bulk viscosity equal to a constant ζ_0, i.e. $\zeta(H) = \zeta_0$. Eq. (4.19) yields:

$$\rho_{\mathrm{F}} = \rho_0 a^{-3(1+\omega_{\mathrm{F}})} + 9\zeta_0 a^{-3(1+\omega_{\mathrm{F}})} \int^t a(t')^{1+3\omega_{\mathrm{F}}} \dot{a}(t')^2 dt' \,. \tag{4.26}$$

For the Big Rip ($\beta = 1$), ρ_{F} behaves as

$$\rho_{\mathrm{F}} = \rho_0 (t_0 - t)^{3h_0(1+\omega_{\mathrm{F}})} + \frac{9h_0^2\zeta_0}{(t_0 - t)(1 + 3h_0 + 3h_0\omega_{\mathrm{F}})} \,. \tag{4.27}$$

In this case, in Einstein's framework ($f(R, G) = 0$), the solution of Eq. (4.17) becomes (see also Ref. [95]):

$$H = \frac{\sqrt{3\kappa^2\rho_0}\,\mathrm{e}^{(3\kappa^2\zeta_0/2)t}}{3 + \left[\frac{3}{\zeta_0}(1+\omega_{\mathrm{F}})\sqrt{\frac{\rho_0}{3\kappa^2}}(\mathrm{e}^{(3\kappa^2\zeta_0/2)t} - 1)\right]} \,. \tag{4.28}$$

H shows a finite-time future singularity when t tends to t_0, where

$$t_0 = \frac{2}{3\kappa^2\zeta_0} \ln\left[1 - \sqrt{\frac{3\kappa^2}{\rho_0}} \frac{\zeta_0}{(1+\omega_{\mathrm{F}})}\right] \,. \tag{4.29}$$

If we expand the exponential functions around t_0, we obtain:

$$H \simeq -\frac{2}{3(1+\omega_{\mathrm{F}})} \frac{1}{(t_0 - t)} + \frac{\kappa^2}{1+\omega_{\mathrm{F}}}\zeta_0 + \mathcal{O}(t_0 - t) \,, \tag{4.30}$$

that corresponds to Eq. (3.1) with $\beta = 1$ (Big Rip), $h_0 = -2/(3 + 3\omega_{\mathrm{F}})$, where $\omega_{\mathrm{F}} < -1$, and $H_0 = \kappa^2\zeta_0/(1 + \omega_{\mathrm{F}})$. The viscosity ζ_0 is not relevant in the asymptotic singular limit of H (here, H_0 is negative, but the first positive term of H is much larger), and we recover Eq. (4.25), that is valid for phantom perfect fluids, and the linked discussion already done is still valid.

In order to study the effects of the viscosity on Type I, II, III and IV singular models, it is worth considering the asymptotic behaviour of the conservation law in Eq. (4.19). We require that the left part diverges like the right part on the singular solutions:

$$\dot{\rho_{\mathrm{F}}} + 3\rho_{\mathrm{F}}(1+\omega_{\mathrm{F}})\left(\frac{h_0}{(t_0 - t)^\beta} + H_0\right) \simeq \frac{9h_0^2\zeta_0}{(t_0 - t)^{2\beta}} + \frac{18h_0 H_0\zeta_0}{(t_0 - t)^\beta} + 9H_0^2\zeta_0 \,, \tag{4.31}$$

where we take $H_0 = 0$ if $\beta > 0$. In what follows, we neglect the homogeneous solutions, already discussed above.

The following asymptotic solutions of Eq. (4.31) are found:

$$\rho_{\rm F} \simeq \frac{3h_0\zeta_0}{(1+\omega_{\rm F})(t_0-t)^\beta}\,, \quad \beta>1\,, \tag{4.32}$$

$$\rho_{\rm F} \simeq \frac{9\zeta_0 h_0^2}{(2\beta-1)(t_0-t)^{2\beta-1}}\,, \quad 1>\beta>0\,, \tag{4.33}$$

$$\rho_{\rm F} \simeq \frac{9h_0H_0\zeta_0}{(\beta-1)(t_0-t)^{\beta-1}}+\frac{3H_0\zeta_0}{1+\omega_{\rm F}}\,, \quad \beta<0\,, H_0\neq 0\,. \tag{4.34}$$

In the first case ($\beta > 1$), it is possible to see that fluid energy density diverges more slowly than H^2 in Eq. (4.17), so that viscous fluid does not influence the asymptotically behaviour of Type I $f(R,G)$-singular models, due to the constant viscosity.

Also in the second case ($0<\beta<1$), viscous fluid is asymptotically avoidable in the case of Type III $f(R,G)$-singular models, since fluid energy density diverges less than H^2.

In the end, we consider fluid which tends to a constant when $\beta < 0$. Large bulk viscosity ζ_0 becomes relevant in the EOM and, if $\omega_{\rm F} < -1$, the effective energy density, due to the fluid contribute, could become negative avoiding the Type II and IV singularities.

Example:

- We have seen in Chapter 3 that the Hu-Sawicki Model in some cases produces the Type II singularity for a certain positive value of $H_0 = H_{dS}$ as in Eq. (3.88).
  A fluid with $\omega_{\rm F} > -1$ and constant viscosity ζ_0 large with respect to H_{dS}, may change the value of H_0 for which singularity appears, but does not avoid it.
  On the other hand, a fluid with $\omega_{\rm F} < -1$ and $\zeta_0 \gg H_{dS}$, makes the singularity unphysical, since the solution of Eq. (4.17) and Eq. (4.34) leads to H_0 imaginary and the singularity does not appear.

Viscosity proportional to H

This is the case $\zeta(H) = 3H\tau$. As ζ is assumed to be positive, the constant τ has to be positive. Eq. (4.19) yields:

$$\rho_{\rm F} = \rho_0 a^{-3(1+\omega_{\rm F})} + 27\tau a^{-3(1+\omega_{\rm F})}\int^t dt' a(t')^{3\omega_{\rm F}}\dot{a}(t')^3\,. \tag{4.35}$$

For the Big Rip ($\beta = 0$), $\rho_{\rm F}$ behaves as:

$$\rho_{\rm F} = \frac{27h_0^3\tau}{(t_0-t)^2(2+3h_0+3h_0\omega_{\rm F})}\,, \tag{4.36}$$

In Einstein's gravity ($f(R,G)=0$), the Eq. (4.17) with the Eq. (4.36) admit the solution:

$$H = \frac{2}{(9\kappa^2\tau-3(1+\omega_{\rm F}))}\frac{1}{(t_0-t)}\,, \tag{4.37}$$

and realize the Big Rip for $H = h_0/(t_0-t)$, where $h_0 = 2/(9\kappa^2\tau - 3(1+\omega_{\rm F}))$. h_0 is positive if [95]:

$$(1+\omega_{\rm F}) - 3\kappa^2\tau < 0\,. \tag{4.38}$$

It means that phantom fluid or fluid in the quintessence region with sufficiently large bulk viscosity could produce the Big Rip. On the other hand, if $(1+\omega_{\rm F}) - 3\kappa^2\tau > 0$, the fluid does not realize the Big Rip for expanding universe.

The other asymptotic solutions of Eq. (4.19) are:

$$\rho_{\mathrm{F}} \simeq \frac{9h_0^2\tau}{(1+\omega_{\mathrm{F}})(t_0-t)^{2\beta}}, \qquad \beta > 1, \tag{4.39}$$

$$\rho_{\mathrm{F}} \simeq \frac{27\tau h_0^3}{(3\beta-1)(t_0-t)^{3\beta-1}}, \qquad 0<\beta<1, \tag{4.40}$$

$$\rho_{\mathrm{F}} \simeq \frac{27hH_0^2\tau}{(\beta-1)(t_0-t)^{\beta-1}} + \frac{9H_0^2\tau}{1+\omega_{\mathrm{F}}}, \qquad \beta<0\,, H_0\neq 0\,. \tag{4.41}$$

For $\beta > 1$, ρ_{F} diverges like H^2 if $\omega_{\mathrm{F}} > -1$. Thus, the fluid could asymptotically produce the Type I singularity and generally does not influence the $f(R,G)$-gravity producing such kind of singularity. On the other hand, if $\omega_{\mathrm{F}} < -1$, for large values of viscosity τ, the theory is protected against Type I singularity, since the effective energy density of the universe may become negative.

When $0 < \beta < 1$, since ρ_{F} diverges less than H^2, the fluid does not influence the $f(R,G)$-singular models on Type III singularity and can be neglected on singular solutions.

When $\beta < 0$, the fluid can influence the $f(R,G)$-models producing Type II and IV singularity with $H_0 \neq 0$, if the viscosity τ is large. In particular, if $\omega_{\mathrm{F}} < -1$, the fluid energy density becomes negative and may avoid Type II and IV singularities.

Examples:

- The model $R - \alpha(G/R)$, where α is a positive constant, shows the Type I singularity (see Eq. (3.80)).
  A fluid with $\omega_{\mathrm{F}} > -1$ and energy density in the form of Eq. (4.39), may influence some feature of singularity, but the Type I singularity is still realized. In addition, if τ is sufficiently large, Eq. (4.38) is satisfied and an other possible scenario is the Big Rip solution.
  If $\omega_{\mathrm{F}} < -1$, large values of τ make negative the effective energy density of the universe on the Type I singularity, which could be changed into the Big Rip.

- In the model $R+\alpha(G/R)$, where α is a positive constant, the Type II, III and IV singularities could appear (see Eq. (3.83)). The presence of fluids with $\omega_{\mathrm{F}} < -1$ and energy density in the form of Eq. (4.39) and Eq. (4.40), does not influence the Type III singularity, but could change the Types II and IV into the Big Rip, like in the previous example.

4.2.2 $\omega(\rho_{\mathrm{F}})$ not a constant

In this general case, the fluid EoS parameter $\omega(\rho_{\mathrm{F}})$ explicitly depends on the fluid energy density ρ_{F}. We are interested in some simple case. We consider viscous fluid, whose thermodynamical parameter $\omega(\rho_{\mathrm{F}})$ is given by:

$$\omega(\rho_{\mathrm{F}}) = A_0\rho_{\mathrm{F}}^{\alpha-1} - 1\,, \tag{4.42}$$

where $A_0(\neq 0)$ and α are constants. When $\alpha = 1$, we find the case when $\omega(\rho_{\mathrm{F}})$ is a constant. Let us suppose the following form of bulk viscosity $\zeta(H)$:

$$\zeta(H) = (3H)^n\tau\,. \tag{4.43}$$

Here, $\tau > 0$ and n are constants.

The energy conservation law (4.19) leads:

$$\dot{\rho_{\mathrm{F}}} + 3HA_0\rho_{\mathrm{F}}^{\alpha} = 9H^2(3H)^n\tau\,, \tag{4.44}$$

from which we may get the (asymptotic) solutions of the fluid energy density when H is singular.

In what follows, we consider several examples in the cases of $\tau \neq 0$ (viscous case) and $\tau = 0$ (non viscous case).

Viscous case

Let us take τ positive constant different to zero. For the Big Rip singularity ($\beta = 1$), some simple (asymptotic) solutions of Eq. (4.44) are given by:

$$\rho_{\mathrm{F}} = \frac{3^{n+2} h_0^{n+2} \tau}{(n+1+3h_0 A_0)(t_0-t)^{n+1}}\,, \quad \alpha = 1\,, \tag{4.45}$$

$$\rho_{\mathrm{F}} \simeq \left(\frac{3^{n+1} h_0^{n+1} \tau}{A_0 (t_0-t)^{n+1}}\right)^{\frac{1}{\alpha}}\,, \quad \alpha > 1\,. \tag{4.46}$$

Eq. (4.45) corresponds to the cases when $\omega(\rho_{\mathrm{F}})$ is a constant. For $n = 0, 1$, we find Eq. (4.27) and Eq. (4.36). When $\alpha = 1$ and $n > 1$, the fluid energy density diverges faster than H^2 ($\sim (t_0 - t)^{-2}$) and the EOM (4.17)-(4.18) become inconsistent on the Big Rip. We can say that fluids with $\omega(\rho_{\mathrm{F}})$ constant and bulk viscosity proportional to H^n, where $n > 1$, avoid the Big Rip. The same happens in the presence of this kind of viscous fluids with $n + 1 > 2\alpha$, where $\alpha > 1$, as in Eq. (4.46).

For Type I singularities ($\beta > 1$), an asymptotic, simple solution of Eq. (4.44) is:

$$\rho_{\mathrm{F}} \simeq \left(\frac{3^{n+1} h_0^{n+1} \tau}{A_0 (t_0-t)^{(n+1)\beta}}\right)^{\frac{1}{\alpha}}\,, \quad \alpha \geqslant 1\,. \tag{4.47}$$

The cases $\alpha = 1$ and $n = 0, 1$ correspond to Eq. (4.32) and Eq. (4.39). The fluid avoid the Type I singularities if $2\alpha < n + 1$ when $\alpha \geqslant 1$, so that its energy density diverges faster than H^2 in Eq. (4.17). It means that, if the viscosity behaves as a power function of H larger than one, the fluid with $\omega(\rho_{\mathrm{F}})$ constant is able to protect the theory against the Big Rip and Type I singularities together.

Note that the viscosity is introduced in the EOM by the fluid pressure of Eq. (4.11). On the Big Rip and Type I singularities, the curvature R behaves as H^2. Motivated by fact that the correction term γR^m, with γ constant and $m > 1$, cures Big Rip and Type I singularities in $f(R, G)$ gravity (see Chapter 3), we may directly conclude that the term $-3H\zeta(H)$ proportional to H^{1+n}, with $n > 1$, shows the same effect, like we have just seen.

For Type III singularities ($0 < \beta < 1$), an asymptotic solution of Eq. (4.44) is:

$$\rho_{\mathrm{F}} \simeq \frac{3^{n+2} h_0^{n+2} \tau}{(2\beta + n\beta - 1)(t_0-t)^{2\beta+n\beta-1}}\,, \quad 1/2 < \alpha \leqslant 1\,. \tag{4.48}$$

The cases $\alpha = 1$ and $n = 0, 1$ correspond to Eq. (4.33) and Eq. (4.40). The fluid energy density diverges faster than H^2 when $n > 1/\beta$. In principle, if a $f(R, G)$-theory shows the Type III singularity for a certain value of β, the presence of a fluid with viscosity proportional to H^n, where $n > 1/\beta$ (and, as a consequence, always $n > 1$), can make inconsistent the EOM and avoid this kind of singularity. Otherwise, it could appear a new Type III singularity realized by fluid for $H = h_0/(t_0 - t)^{1/n}$, so that $\rho_{\mathrm{F}} \sim H^2$, solving in some cases Eq. (4.17). We will see a nice example in the end of the Section.

For Type II and IV singularities ($\beta < 0$), if $H_0 \neq 0$, an asymptotic solution of Eq. (4.44) is given by:

$$\rho_{\mathrm{F}} \simeq \frac{3^{n+2} H_0^{n+1} h_0 \tau}{(\beta-1)(t_0-t)^{\beta-1}} + \left(\frac{3^{n+1} H_0^{n+1} \tau}{A_0}\right)^{\alpha}\,, \quad \alpha \geqslant 1\,. \tag{4.49}$$

The cases $\alpha = 1$ and $n = 0, 1$ correspond to Eq. (4.34) and Eq. (4.41). In general, this kind of fluid influences the feature of Type II and IV singularities in $f(R, G)$-gravity, but not necessarily avoid they.

Non viscous case

If the viscosity is equal to zero, i.e. $\tau = 0$, Eq. (4.44) yields:

$$\rho_{\mathrm{F}} = \left[(\alpha - 1)\left(3A_0 \ln \frac{a(t)}{a_0}\right)\right]^{\frac{1}{1-\alpha}}, \tag{4.50}$$

where $a(t)$ is, as usual, the scale factor, a_0 is a positive parameter and $\alpha \neq 1$ (non perfect fluids). We may take $A_0(\alpha - 1)$ positive, so that, in general, ρ_{F} is positive.

In addition, we set

$$[3A_0(\alpha - 1)]^{\frac{1}{1-\alpha}} = \frac{H_0^2}{\kappa^2}, \tag{4.51}$$

where in this case H_0 is a positive parameter. As a consequence, one has:

$$\rho_{\mathrm{F}} = \frac{H_0^2}{\kappa^2}\left[\ln \frac{a(t)}{a_0}\right]^{\frac{1}{1-\alpha}}. \tag{4.52}$$

In Einstein's gravity ($f(R, G) = 0$), the first EOM (4.17) reads:

$$a(t) = (a_0)\,\mathrm{Exp}\left\{6^{\frac{2-2\alpha}{2\alpha-1}}\left[\pm\frac{(2\alpha - 1)(\sqrt{3}H_0 t)}{\alpha - 1}\right]^{2(\alpha-1)/(2\alpha-1)}\right\}. \tag{4.53}$$

Note that for large values of α, the fluid energy density tends to H_0^2/κ^2, and Eq. (4.42), by using Eq. (4.51), leads to $\omega(\rho_{\mathrm{F}}) \simeq -1$, and $a(t) \simeq a_0 \mathrm{e}^{H_0 t/3}$ (de-Sitter universe).

Moreover, one can see that Eq. (4.53) produces the following form of H,

$$H = \frac{h_0}{(t_0 - t)^{\frac{1}{(2\alpha-1)}}}. \tag{4.54}$$

Here, the constants of Eq. (4.53) have been encoded into parameters h_0 and t_0.

In principle, this inhomogeneous non viscous fluid can generate any Type of singularity in the form of Eq. (3.1) with $\beta = 1/(2\alpha - 1)\,, \alpha \neq 1$ (except the Big Rip case, for which one has to consider the perfect fluids).

We conclude with a special case of non-viscous fluid with $\omega(\rho_{\mathrm{F}})$ non-constant, namely the Chaplygin gas [98], which also has been considered as a candidate to dark energy and whose Equation of State is:

$$p_{\mathrm{F}} = -\frac{A_0}{\rho_{\mathrm{F}}}, \tag{4.55}$$

where A_0 is a positive constant. Eq. (4.19) leads to:

$$\rho_{\mathrm{F}} = \sqrt{A_0 + \frac{1}{a(t)^6}}. \tag{4.56}$$

Since $a(t)$ diverges or tends to a constant for Big Rip, Type I and Type III singularities, it is easy to see that the Chaplygin gas does not influence in the EOM (4.17)-(4.18) the asymptotic behaviour of $f(R, G)$-models in wich such kind of singularities appear. At least, it could influence

$f(R, G)$-models in which Type II and IV singularities are realized, but not necessarily prevents the singularities.

Example:

- In the model $R + \alpha R^{1/2}$, with α positive constant, the Type III singularity for $\beta = 1/3$ may appear (see Eq. (3.84)). A fluid with $\omega(\rho_{\mathrm{F}})$ constant and energy density in ther form of Eq. (4.48), where $n > 3$, avoids this kind of singularity. It is interesting to see that in this case, since if $\beta = 1/n$, ρ_{MG} of Eq. (4.13) diverges faster than H^2 in Eq. (4.17), the fluid does not produce a new Type III singularity, due to the contribute of modified gravity. Moreover, the model is free of any type of singularity, being the theory protected against singularities of $f(R)$-gravity by fluid and against fluid singularities by modified gravity itself.

4.3 Viscous fluids coupled with Dark Matter

In this Section [16] we consider viscous fluid coupled with dark matter. Their energy conservation laws are given by:

$$\dot{\rho}_{\mathrm{F}} + 3H(\rho_{\mathrm{F}} + p_{\mathrm{F}}) = -Q_0 \rho_{\mathrm{F}} \,, \tag{4.57}$$

$$\dot{\rho}_{\mathrm{DM}} + 3H\rho_{\mathrm{DM}} = Q_0 \rho_{\mathrm{F}} \,. \tag{4.58}$$

Here, Q_0 is the coupling constant, ρ_{DM} is the energy density of dark matter (the corresponding pressure is equal to zero), whereas ρ_{F} and p_{F} are, as usually, the energy density and pressure of viscous fluid. The fluid pressure p_{F} is written as in Eq. (4.11).

The equations of motion simply read

$$\rho_{\mathrm{F}} + \rho_{\mathrm{DM}} = \frac{3}{\kappa^2} H^2 \,, \tag{4.59}$$

$$p_{\mathrm{F}} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) \,. \tag{4.60}$$

We will motivate this study by showing how this coupling may solve the coincidence problem and remove singular solutions of DE-fluids.

4.3.1 $\omega(\rho_{\mathrm{F}})$ constant

Suppose to have $\omega(\rho_{\mathrm{F}}) = \omega_{\mathrm{F}}$ constant for the fluid and bulk viscosity in the form of Eq. (4.43), namely $\zeta(H) = \tau(3H)^n$, $\tau > 0$ and n being constants. In this case, the general solution of Eq. (4.57) is

$$\rho_{\mathrm{F}} = \rho_{F(0)} \frac{\mathrm{e}^{-Q_0 t - 3\omega_{\mathrm{F}} \log a(t)}}{a(t)^3} + \frac{\tau 3^{2+n} \mathrm{e}^{-Q_0 t - 3\omega_{\mathrm{F}} \log a(t)}}{a(t)^3} \int^t \mathrm{e}^{Q_0 t' + 3\omega_{\mathrm{F}} \log a(t')} a(t') \dot{a}(t')^2 \left(\frac{\dot{a}(t')}{a(t')}\right)^n dt' \,, \tag{4.61}$$

where $\rho_{F(0)}$ is a positive constant of integration.

One possible solution is the de Sitter space, where $H = \dot{a}(t)/a(t) = H_{dS}$ is a constant. One may identify the Hubble parameter H_{dS} with the present value of accelerated universe. In this case, Eq. (4.61) can be solved as

$$\rho_{\mathrm{F}} = \rho_{F(0)} \mathrm{e}^{-t(Q_0 + 3H_{dS}(1+\omega_{\mathrm{F}}))} + \frac{(3H_{dS})^{n+2}\tau}{(Q_0 + 3H_{dS}(1+\omega_{\mathrm{F}}))} \,. \tag{4.62}$$

It follows the solution of Eq. (4.58) for dark matter

$$\rho_{\mathrm{DM}} = \rho_{DM(0)} \mathrm{e}^{-3H_{dS} t} - \rho_{F(0)} \frac{Q_0}{Q_0 + 3H_{dS}\omega_{\mathrm{F}}} \mathrm{e}^{-t(Q_0 + 3H_{dS}(1+\omega_{\mathrm{F}}))} + \frac{(3H_{dS})^{n+1} Q_0 \tau}{(Q_0 + 3H_{dS}(1+\omega_{\mathrm{F}}))} \,, \tag{4.63}$$

where $\rho_{DM(0)}$ is a positive constant. It is easy to see that, if $\tau \neq 0$, the EOM (4.59)-(4.60) are satisfied only if $\rho_{F(0)} = \rho_{DM(0)} = 0$. Therefore, we note that, if the de Sitter solution is an attractor and it is able to describe our universe today, we can require

$$\frac{\rho_{\mathrm{DM}}}{\rho_{\mathrm{F}}} = \frac{Q_0}{3H_{dS}} = \frac{1}{3}\,, \tag{4.64}$$

and the coincidence problem is solved by setting

$$Q_0 = H_{dS}\,. \tag{4.65}$$

The ratio of DM and fluid is approximately $1/3$, almost independent from initial conditions. By evaluating Eq. (4.60) on the de Sitter solution, one has the relation between ω_{F} and τ, namely

$$\omega_{\mathrm{F}} = -\frac{4}{3} + 4\kappa^2 (3H_{dS})^{n-1}\tau\,. \tag{4.66}$$

Here, Eq. (4.65) has been used. Note that ρ_{F} of Eq. (4.62) results positive. For example, a DE-fluid with $\omega_{\mathrm{F}} = -1$ admits the de Sitter solution for $H = H_{dS}$ if its bulk viscosity is

$$\zeta(H) = \frac{(3H)^n}{12\kappa^2 (3H_{dS})^{n-1}}\,,$$

and the coupling constant with DM is $Q_0 = H_{dS}$.

This is a generalization of the result achieved in Ref. [99] for coupled non viscous DE-fluid with DM. If $\tau = 0$, it is easy to see that Eqs. (4.62)-(4.63) are solutions of the EOM in the de Sitter case $H = H_{dS}$, if $Q_0 = -3(1+\omega_{\mathrm{F}})H_{dS}$ and $\rho_{DM(0)} = 0$, so that the coincidence problem is solved by putting

$$\frac{\rho_{\mathrm{DM}}}{\rho_{\mathrm{F}}} = -(1+\omega_{\mathrm{F}}) \sim \frac{1}{3}\,, \tag{4.67}$$

which leads to the condition of phantom fluid

$$\omega_{\mathrm{F}} = -\frac{4}{3}\,. \tag{4.68}$$

Let us return to the case of $\tau \neq 0$. In order to investigate if the de Sitter solution is an attractor or not, we consider the perturbation as

$$H(t) = H_{dS} + \Delta(t)\,. \tag{4.69}$$

Here, $\Delta(t)$ is a function of the cosmic time t and it is assumed to be small. The second EOM (4.60) gives

$$2\dot{\Delta}(t) + 6H_{dS}\Delta(t) \simeq 3H_{dS}(n+1)\Delta(t)\,, \tag{4.70}$$

where we have used Eq. (4.62) and Eq. (4.66). By assuming $\Delta(t) = \mathrm{e}^{\lambda t}$, we find

$$\lambda + 3H_{dS} - \frac{3}{2}H_{dS}(n+1) \simeq 0\,, \tag{4.71}$$

that is

$$\lambda \simeq \frac{3}{2}H_{dS}(n-1)\,. \tag{4.72}$$

Then, if $n < 1$, the de Sitter solution is stable and the coupling of viscous fluid and dark matter at last generates a stable accelerated universe with a constant rate of DM and DE-fluid. If $n > 1$, the de Sitter solution is not stable and other future scenarios are possible.

We have seen in §4.2 that phantom (viscous) fluid ($\omega_{\mathrm{F}} < -1$) can generate the Big Rip singularity. On the other hand, the coupling with DM seems to avoid such problem, being constant the value of fluid energy density in stable de Sitter universe.

4.3.2 $\omega(\rho_{\rm F})$ not a constant

To complete this Section, let us consider a more general case, when the thermodynamical parameter $\omega(\rho_{\rm F})$ of viscous fluid is not a constant. A simple example is given by Eq. (4.42), namely $\omega(\rho_{\rm F}) = A_0\rho_{\rm F}^{\alpha-1} - 1$, A_0 and α being constant parameters. The energy conservation law (4.57) of viscous fluid becomes

$$\dot{\rho}_{\rm F} + 3HA_0\rho_{\rm F}^{\alpha} + Q_0\rho_{\rm F} = 9H^2(3H)^n\tau\,. \tag{4.73}$$

Here, we suppose the bulk viscosity proportional to H^n, namely $\zeta(H) = \tau(3H)^n$ as in Eq. (4.43), $\tau > 0$ and n being constants. If we assume $\alpha \gg 1$, on the de Sitter solution $H = H_{dS}$, we obtain

$$\rho_{\rm F} \simeq \left(\frac{\tau(3H_{dS})^{n+1}}{A_0}\right)^{\frac{1}{\alpha}}\,. \tag{4.74}$$

By using Eq. (4.58), the energy density of dark matter reads

$$\rho_{\rm DM} \simeq \frac{Q_0}{3H_{dS}}\rho_{\rm F}\,, \tag{4.75}$$

and in order to solve the coincidence problem we have to require $Q_0 = H_{dS}$.

From the EOM (4.59)-(4.60), by assuming that the fluid drives the accelerated expansion of the universe, it follows

$$A_0 \simeq \tau(3H_{dS})^{n+1}\left(\frac{\kappa^2}{3H_{dS}^2}\right)^{\alpha}\,, \tag{4.76}$$

and for $\omega(\rho_{\rm F})$ in the de Sitter space one has

$$\omega(\rho_{\rm F}) \simeq -1 + 3(3H_{dS})^{n-1}\kappa^2\tau\,, \tag{4.77}$$

being $\rho_{\rm F}$ constant.

In order to investigate if the de Sitter solution is an attractor or not, we consider the perturbation as in Eq. (4.69). The second EOM (4.60) gives

$$2\dot{\Delta}(t) + 6H_{dS}\Delta(t) \simeq H_{dS}\left(\frac{n+1}{\alpha}\right)\Delta(t)\,, \tag{4.78}$$

where we have used Eq. (4.74) and Eq. (4.76). By assuming $\Delta(t) = {\rm e}^{\lambda t}$, we find

$$\lambda + 3H_{dS} - \frac{1}{2}H_{dS}\left(\frac{n+1}{\alpha}\right) \simeq 0\,, \tag{4.79}$$

that is

$$\lambda \simeq H_{dS}\left(\frac{1}{2}\left(\frac{n+1}{\alpha}\right) - 3\right)\,. \tag{4.80}$$

Then, if $(n+1)/\alpha < 6$, the de Sitter solution is stable.

Chapter 5

Realistic $F(R)$-gravity

Here, we review viable conditions of realistic $F(R)$-gravity able to reproduce the universe where we live. The simplest class of $F(R)$-modified gravity models is given by $f(R)$-gravity, i.e. $F(R) = R + f(R)$, with the aim to mimic the cosmology of ΛCDM Model. We discuss a class of viable exponential models presented in Refs. [13], [19].

5.1 Viability conditions in $F(R)$-gravity

In this and in the next Chapter we will concentrate on $F(R)$-modified gravity, whose action is given by Eq. (1.24). We remember, that the suffix ($'$) will denote the derivative with respect to R.

The viability conditions [100] follow from the fact that the theory has to be consistent with the results of General Relativity and with the important goals arisen with ΛCDM Model (which corresponds to $F(R) = R - 2\Lambda$, Λ being the Cosmological Constant) in the description of the universe and our Solar System.

If $R = 0$ it is reasonable to have the consistence with Special Relativity, so in general we require $F(0) = 0$ in order to obtain the Minkowski solution of flat space.

Recall that, in order to avoid anti-gravity effects, it is required that $F'(R) > 0$, namely the positivity of the effective gravitational coupling G_{eff}, where $G_{\text{eff}} = G_N/F'(R)$, at least when R assumes the curvature values of present and past universe (in general, when $R \geq 4\Lambda$).

5.1.1 Existence of a matter era and stability of cosmological perturbations

On the critical points of the theory, one has $\dot{F}'(R) = 0$ (see Eq. (1.29)). In particular, during matter era, modified gravity has to vanish, so that $\rho_{\text{eff}} = \rho_{\text{m}}$ and $p_{\text{eff}} = p_{\text{m}} \equiv 0$ in Eqs. (1.14)-(1.15) and Ricci scalar of Eq. (1.8) results $R = 3H^2$. As a consequence, from Eqs. (1.18)-(1.19) we obtain the conditions on critical point of matter era, namely (the critical points in $F(R)$-modified gravity have been carefully investigated in Ref. [101])

$$\frac{RF'(R)}{F(R)} = 1\,, \tag{5.1}$$

and

$$F'(R) = 1\,. \tag{5.2}$$

In order to reproduce the results of the Standard Model, where $R = \kappa^2\rho_{\text{m}}$ when matter drives the cosmological expansion, a $F(R)$-theory is acceptable if the modified gravity contribution vanishes during this era and $F'(R) \simeq 1$. However, another condition is required on the second derivative of $F(R)$: it has to be positive [102]. This last condition arises from the stability of the cosmological perturbations. If we consider a small region of space-time in the weak-field regime, so that the curvature is approximated by $R = R^{(0)} + \delta R$, were $R^{(0)} = -\kappa^2 T^{(\text{matter})}$ is the matter solution, we get Eq. (1.32). By using Eq. (1.30), it is easy to see that, since $F'(R^{(0)}) > 0$, the solution is stable when

$$F''(R) > 0 \tag{5.3}$$

during matter era. However, a more detailed evaluation on local perturbations in matter era will be shown in Section **6**.

5.1.2 Existence and stability of a late-time de Sitter point

A reasonable theory of modified gravity which reproduces the current acceleration of the universe needs to show an accelerating solution for $R_{dS} = 4\Lambda$, Λ being the cosmological constant and typically $\Lambda \simeq 10^{-66}\text{eV}^2$. In principle, it is sufficient to require that the EoS parameter ω_{eff} of Eq. (1.23) is smaller than $-1/3$, but note that all available cosmological data confirm that its value is actually very close to -1. The possibility of the effective quintessence/phantom dark energy and different future scenarios of the universe evolution, such as the so-called 'Little Rip cosmology' [103, 104], are not exluded, but the most realistic solution for our current universe is a (asymptotically) stable de Sitter solution given by Eq. (1.31) under condition (1.37).

5.1.3 Local tests and the stability on a planet's surface

The results of GR were first confirmed by local tests at the level of the Solar System. A theory of modified gravity has to admit a static spherically-symmetric solution of the type of Schwarzshild solution (1.63) or, more in general, the Schwarzshild-de Sitter solution (1.64) with Λ very small. The typical value of the curvature in the Solar System far from sources is $R = R^*$, where $R^* \simeq 10^{-61}\text{eV}^2$ (it corresponds to one hydrogen atom per cubic centimeter). If a Scwarzshild-de Sitter solution exists, it will be stable provided by Eq. (1.37) evaluated on R^*. The stability of the solution is necessary in order to find the post-Newtonian parameters in GR [105].

Concerning the matter instability [106], this might also occur when the curvature is rather large, as on a planet ($R \simeq 10^{-38}\text{eV}^2$), as compared with the average curvature of the universe today ($R \simeq 10^{-66}\text{eV}^2$). In order to arrive to a stability condition, we can start from Eq. (1.32), where $R^{(0)} = R_b$ is the curvature of the planet surface and δR is a perturbation due to the curvature difference between the internal and the external solution. The curvature $R_b = -\kappa^2 T^{(\text{matter})}$ depends on the radial coordinate r. By assuming δR depending on time only, one has

$$-\partial_t^2(\delta R) \sim U(R_b)\delta R\,, \tag{5.4}$$

where

$$\begin{aligned} U(R_b) \quad = \quad & \left[\left(\frac{F'''(R_b)}{F''(R_b)}\right)^2 - \frac{F''''(R_b)}{F''(R_b)}\right] g^{rr}\nabla_r R_b \nabla_r R_b - \frac{R_b}{3} + \frac{F'(R_b)}{3F''(R_b)} \\ & \frac{F'''(R_b)}{3(F''(R_b))^2}(2F(R_b) - R_b F'(R_b) - R_b)\,. \end{aligned} \tag{5.5}$$

Here, $g_{\mu\nu}$ is the diagonal metric describing the planet. If $U(R_b)$ is negative, then the perturbation δR becomes exponentially large and the whole system becomes unstable. Thus, the planet stability condition is

$$U(R_b) > 0\,. \tag{5.6}$$

This expression has to be evaluated for typical values $R_b \simeq 10^{-38}\text{eV}^2$.

5.1.4 Existence of an early-time acceleration and the future singularity problem

In order to reproduce the early-time acceleration of our universe, namely the inflation epoch, the modified gravity models have to admit a solution for ω_{eff} in Eq. (1.23) smaller than $-1/3$. An important point is that this solution should be unstable.

If the model reproduces the de Sitter solution when $R_{\text{dS}} \simeq 10^{20-38}\text{GeV}^2$ (this is the typical curvature value at inflation), we have to require that Eq. (1.37) is violated. Thus, the characteristic time of the instability t_i is given by the inverse of the mass of the scalaron in Eq. (1.34):

$$t_i \simeq \left|\frac{1}{m}\right| = \left|\sqrt{\frac{F'(R_{\text{dS}})}{F'(R_{\text{dS}}) - R_{\text{dS}}F''(R_{\text{dS}})}}\right| . \tag{5.7}$$

In principle, other scenarios for the very early universe are possible instead the standard cosmic inflation, such as the ekpyrotic one [107], which also accommodates the Big-Bang physics.

Furthermore, we have seen in Chapter **3** and in Chapter **4** that many DE-models, including modified gravity, bring the future universe evolution to a finite-time singularity. The presence of a finite-time future singularity may cause serious problems to the cosmological evolution or to the corresponding black hole or stellar astrophysics. Thus, it is always necessary to avoid such scenario in realistic models of modified gravity. It is remarkable that modified gravity actually provides a very natural way to cure such singularities by adding, for instance, higher-power term of R (see § 3.5.1). Simultaneously with the removal of any possible future singularity, the addition of this terms supports the early-time inflation caused by modified gravity (it may be the case of R^2-term, which protects the theory against singularities and could produce inflation [7]). Remarkably, even in the case inflation were not an element of the alternative gravity dark energy model considered, it eventually occurs after adding such higher-power term. Hence, the removal of future singularities is a natural prescription for the unified description of inflation and current acceleration.

5.2 The $f(R)$-'one step' and 'two steps' models

In Refs. [85, 86, 108] several versions of viable modified $f(R)$-gravity have been proposed, namely so-called 'one-step' models, which reproduce the current acceleration of the universe in as simple way. This models show a correction to the Hilbert-Einstein action as $F(R) = R + f(R)$, being the modification a function $f(R)$ of the Ricci scalar R. They incorporate a vanishing (or fast decreasing) cosmological constant in the flat ($R \to 0$) limit, and exhibit a suitable, constant asymptotic behavior for large values of R. These models can be collected in the following class of $f(R)$-gravity toy models [13]:

$$F(R) = R + f(R) ,$$

$$f(R) = -2\Lambda\, \theta(R - R_0) . \tag{5.8}$$

Here, $\theta(R - R_0)$ is Heaviside's step distribution and Λ is the Cosmological Constant. Models in this class are characterized by the existence of one transition scalar curvature R_0. For $R = 0$, $f(0) = 0$ and we recover the limit of Special Relativity. When $R \gg R_0$, $f(R) \simeq -2\Lambda$ and we mimic the ΛCDM Model.

These models contain a sort of 'switching on' of the Cosmological Constant as a function of the scalar curvature R. The simplest version of this kind reads

$$f(R) = -2\Lambda(1 - \mathrm{e}^{-\frac{R}{R_0}}) . \tag{5.9}$$

Here the transition is smooth around R_0.

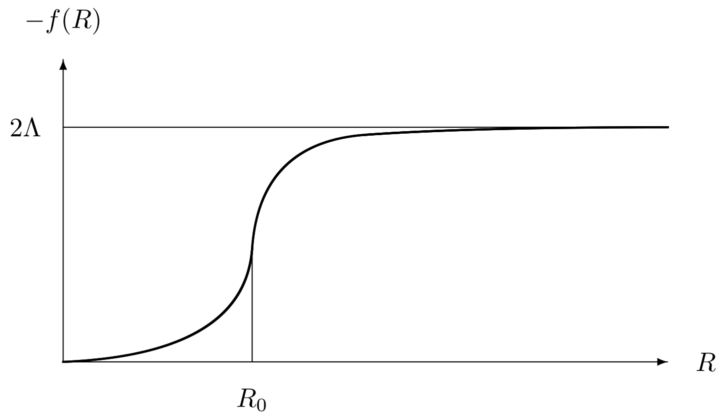

Figure 5.1: Typical behavior of $f(R)$ in the 'one-step' models.

Models in the form of Eq. (5.8) may be combined in a natural way, if one is also interested in the phenomenological description of the inflationary epoch. For example, a 'two-steps' model may be the smooth version of

$$F(R) = R + f(R)\,,$$

$$f(R) = -2\Lambda\,\theta(R - R_0) - 2\Lambda_{\rm i}\,\theta(R - R_i)\,. \tag{5.10}$$

Here, R_i is the transition scalar curvature at inflationary scale, and $\Lambda_{\rm i}$ is a suitable Cosmological Constant producing the acceleration of inflation, when $R \gg R_i$. The effective Cosmological Constant $\Lambda_{\rm eff}$ at inflation results

$$\Lambda_{\rm eff} = \Lambda + \Lambda_{\rm i}\,. \tag{5.11}$$

The typical behavior of $f(R)$ associated with the 'one'- and 'two-step(s)' models is given, in Fig. 5.1 and Fig. 5.2, respectively. The main problem associated with these sharp models is the appearance of possible antigravity regime in a region around the transition point between the inflation and the universe of ΛCDM Model and antigravity in a past epoch[1], what is not phenomenologically acceptable. On the other hand, an analytical study of these models can be easily carried out.

The exponential gravity seems to give a viable possibility to unify early and late-time acceleration as in Eq. (5.10). A natural possibility derived by Eq. (5.9) is

$$f(R) = -2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - 2\Lambda_{\rm i}\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right)\,, \tag{5.12}$$

where n is a natural number larger than one. In this way, effects of inflation are neglected when $R \ll R_i$. In what follows we will study in detail this kind of model, and we will see how it could become acceptable in order to reproduce the whole history of our universe. Some adding parameters will be necessary.

[1]The derivative of $-\theta(R - R_i)$ is the Dirac-delta distribution $-\delta(R_i)$.

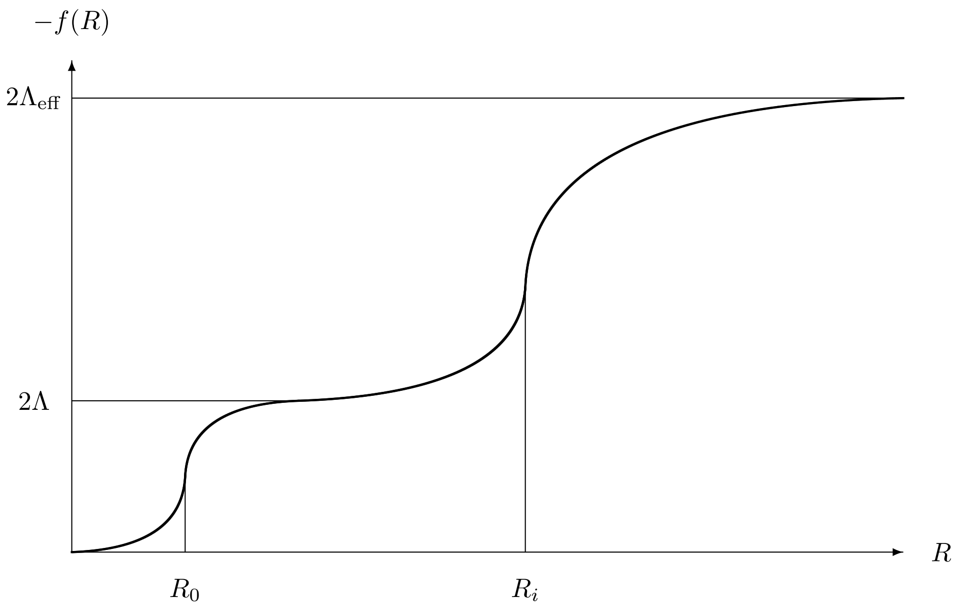

Figure 5.2: Typical behavior of $f(R)$ in the 'two-steps' models.

We conclude this Section by showing other two (more complicate) exponential models which can be used to reproduce the Cosmological Constant in high curvature regime, namely

$$f_1(R) = 2\Lambda\left(\frac{1+\mathrm{e}^{-\beta R_0}}{1+\mathrm{e}^{-\beta(R_0-R)}}-1\right), \tag{5.13}$$

$$\begin{aligned} f_2(R) &= -2\Lambda\left[\tanh\left(\frac{\beta}{2}(R-R_0)\right)+\tanh\left(\frac{\beta}{2}R_0\right)\right] \\ &= = -2\Lambda\left(\frac{\mathrm{e}^{\beta(R-R_0)}-1}{\mathrm{e}^{\beta(R-R_0)}+1}+\frac{\mathrm{e}^{\beta R_0}-1}{\mathrm{e}^{\beta R_0}+1}\right). \end{aligned} \tag{5.14}$$

Here, β is a positive parameter which regulates the amplitude of the transition between the region $R < R_0$ and the region $R > R_0$. The advantage of this models is the analytical possibility to pass to the scalar tensor theory (we have an exact solution of Eq. (1.70)), as it has been shown in Ref. [13].

5.3 Realistic exponential gravity

Let us analyze the exponential model of Eq. (5.9), namely

$$F(R) = R - 2\Lambda\left(1-\mathrm{e}^{-R/R_0}\right), \tag{5.15}$$

where the curvature parameter R_0 is on the same order of the Cosmological Constant, $R_0 \sim \Lambda$. In flat space $F(0) = 0$ and one recovers the Minkowski solution. For $R \gg R_0$, $F(R) \simeq R - 2\Lambda$, and the theory mimics the ΛCDM model. Note that late-time cosmology of such exponential gravity was also considered in Ref. [109]. We have:

$$F'(R) = 1 - 2\frac{\Lambda}{R_0}\mathrm{e}^{-R/R_0}\,, \tag{5.16}$$

$$F''(R) = 2\frac{\Lambda}{R_0^2}\mathrm{e}^{-R/R_0}\,. \tag{5.17}$$

It is remarkable that the function in Eq. (5.15) corresponds to a polynomial modification of gravity without a true cosmological constant. One can write

$$F(R) = R + 2\Lambda\sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\left(\frac{R}{R_0}\right)^k\,, \tag{5.18}$$

and modified gravity can be viewed as a correction to the Einstein's gravity given by a sum of power terms which become relevant at different scales of energy.

Since $|F'(R \gg R_0) - 1| \ll 1$, the model is protected against anti-gravity during the cosmological evolution until the de Sitter solution ($R_{dS} = 4\Lambda$) of today's universe is reached.

For large values of the curvature, $F(R \gg R_0) \simeq R$ and we can reconstruct the matter-dominated era as in GR. In particular, $F''(R) > 0$, and we do not have any instability problems related to the matter epoch, obtaining matter stability on a planet's surface ($U(R_b) \sim 1/(3F''(R_b))$ in Eq. (5.5) and at the Solar System scale.

In order to study the de Sitter era, it is convenient to introduce the following function, $G(R)$,

$$G(R) = 2F(R) - RF'(R)\,. \tag{5.19}$$

On the zeros of $G(R)$ we recover the condition in Eq. (1.31) and we have the de Sitter solution which describes the accelerated expansion of the universe. In our case, since $G(0) = 0$, one has a trivial de Sitter solution for $R = 0$. Consider now

$$G'(R) = F'(R) - RF''(R)\,. \tag{5.20}$$

If $G'(0) > 0$, the function $G(R)$ becomes positive and it is quite simple to see that any non-trivial zero (i.e. de Sitter solution) exists. In order to obtain the de Sitter solution of universe today, we have to require

$$R_0 < 2\Lambda\,. \tag{5.21}$$

In this case, since $G'(0) < 0$, the function $G(R)$ becomes negative and starts to increase after $R = R_0$. For $R = 4\Lambda$, $F(R) \simeq -2\Lambda$, $F'(R) \simeq 1$ and $F''(R) \simeq 0^+$. It means that $G(4\Lambda) \simeq 0$ and we find the de Sitter solution of the dark energy phase which is able to describe the current acceleration of our universe. After this stage, $G(R > 4\Lambda) \simeq R$ is positive and we do not find other de Sitter solutions. Note that the de Sitter solution for $R_{dS} = 4\Lambda$ is stable, since stability condition (1.37) leads to

$$\left(\frac{R_0}{\Lambda}\right)^2\frac{e^{\left(\frac{4\Lambda}{R_0}\right)}}{2} - \left(\frac{R_0}{\Lambda}\right) > 4\,. \tag{5.22}$$

This condition always is satisfied and $G'(4\Lambda) > 0$. On the other hand, the Minkowski space solution is unstable, proved by condition (5.21) which leads to $G'(0) < 0$. In Fig. 5.3 the graphic of $G(R/\Lambda)$ for the case $R_0 = 0.6\Lambda$ is shown. Summing up, we have two FRW-vacuum solutions, which correspond to the trivial de Sitter point for $R = 0$ and to the stable de Sitter point of current acceleration, for $R = 4\Lambda$.

Finally, we have to consider the existence of spherically-symmetric solution. In $R = 0$ we find the Schwarzschild solution, which is unstable. On the other hand, the physical Schwarzschild-de Sitter solutions are obtained for $R \gg R_0$. For example, in the Solar System, $R^* \simeq 10^{-61}\mathrm{eV}^2$, and $F(R^*) \simeq R^* - 2\Lambda$, since $R^* \gg R_0(\sim \Lambda)$. In this case we find the Schwarzschild-de Sitter solution as in Eq.(1.64), which can be approximated with the Schwarzschild solution of Eqs. (1.62)-(1.63),

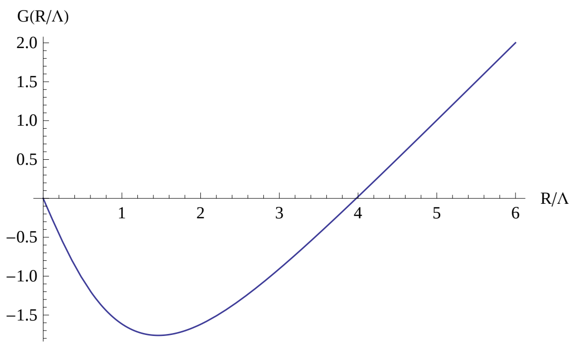

Figure 5.3: Plot of $G(R/\Lambda)$ of exponential model for $R_0 = 0.6\Lambda$. The zeros correspond to the de Sitter solutions.

owing to the fact that Λ is very small, as in ΛCDM Model. Otherwise, stability condition (1.37) has to be verified for $R = R^*$, and the solution results stable.

The description of the cosmological evolution in exponential gravity has been carefully studied in Refs. [109, 110] where it has been explicitly demonstrated that the late-time cosmic acceleration following the matter-dominated stage, as final attractor of the universe, can indeed be realized. By carefully fitting the value of R_0, the correct rate between matter and dark energy of the current universe follows. We will analyze the dynamic of exponential 'one-step' model in Section **6**.

As our next step, we want to generalize the model in order to describe inflation.

5.4 Exponential gravity describing inflation

A simple modification of the 'one-step' model which incorporates the inflationary era is given by a combination of the function discussed above with another 'one-step' function reproducing the cosmological constant during inflation, as in Eq.(5.12). A natural possibility is [19]

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - 2\Lambda_{\mathrm{i}}\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^{\alpha}\,. \tag{5.23}$$

For simplicity, we call

$$f_i = -2\Lambda_{\mathrm{i}}\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right)\,, \tag{5.24}$$

where R_i and Λ_{i} assume the typical values of the curvature and expected cosmological constant during inflation, namely R_i, $\Lambda_{\mathrm{i}} \sim 10^{20-38}\mathrm{eV}^2$ (note that, since $\Lambda \ll \Lambda_{\mathrm{i}}$, Eq. (5.11) gives $\Lambda_{\mathrm{eff}} \simeq \Lambda_{\mathrm{i}}$), while n is a natural number larger than one. The presence of this additional parameter is motivated by the necessity to avoid the effects of inflation during the matter era, when $R \ll R_i$, so that, for $n > 1$, one gets

$$R \gg |f_i(R)| \simeq \frac{2R^n}{R_i^{n-1}}\,, \quad R \gg R_i\,. \tag{5.25}$$

We have introduced the last term γR^{α}, where γ is a positive dimensional constant and α a real number, in order to obtain the exit from inflation. If $\gamma \sim 1/R_i^{\alpha-1}$ and $\alpha > 1$, also the effects of this term vanish in the small curvature regime, when $R \ll R_i$ and

$$R \gg \frac{R^{\alpha}}{R_i^{\alpha-1}}\,, \quad R \gg R_i\,. \tag{5.26}$$

Note that $f_i(0)=0$ and $f_i(R\gg R_i)\simeq -2\Lambda_{\rm i}$. We also obtain

$$f_i'(R)=-\frac{2\Lambda_{\rm i}nR^{n-1}}{R_i^n}\mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\,, \tag{5.27}$$

$$f_i''(R)=-\frac{2\Lambda_{\rm i}n(n-1)R^{n-2}}{R_i^n}\mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}+2\Lambda_{\rm i}\left(\frac{nR^{n-1}}{R_i^n}\right)^2\mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\,. \tag{5.28}$$

The first derivative $f_i'(R)$ has a minimum at $R=\tilde{R}$, where $f_i''(\tilde{R})=0$. One gets

$$\tilde{R}=R_i\left(\frac{n-1}{n}\right)^{\frac{1}{n}}\,. \tag{5.29}$$

Thus, in order to avoid the anti-gravity effects $(1+f_i'(R)>0)$, it is sufficient to require $|f_i'(\tilde{R})|<1$. This leads to

$$R_i>2\Lambda_{\rm i}\,(n)\left(\frac{n-1}{n}\right)^{\frac{n-1}{n}}\mathrm{e}^{-\frac{n-1}{n}}\,. \tag{5.30}$$

For example, one can choose $n=4$. In this case Eq. (5.30) is satisfied for $R_i>3.046\,\Lambda_{\rm i}$. A reasonable choice is $R_i=4\Lambda_{\rm i}$. The last power-term of Eq. (5.23) does not give any problems with anti-gravity, because its first derivative is positive, proved by $\alpha>1$.

It is necessary that the modification of gravity describing inflation does not have any influence on the stability of the matter era in the small curvature range. When $R\ll R_i$, the second derivative of such modification, namely

$$f_i''(R)+\alpha(\alpha-1)\gamma R^{\alpha-2}\simeq\frac{2}{R}\left[-n(n-1)\left(\frac{R}{R_i}\right)^{n-1}+\frac{\alpha(\alpha-1)}{2}\left(\frac{R}{R_i}\right)^{\alpha-1}\right]\,, \tag{5.31}$$

must be positive, that is

$$n>\alpha\,. \tag{5.32}$$

We require the existence of the de Sitter critical point $R_{\rm dS}$ which describes inflation in the high-curvature regime of $f_i(R)$, so that $f_i(R_{\rm dS}\gg R_i)\simeq-2\Lambda_{\rm i}$ and $f_i'(R_{\rm dS}\gg R_i)\simeq 0^+$. In this region, the role of the first term of Eq. (5.23) is negligible, while the term γR^α needs to be taken into account. For simplicity, we shall assume that

$$\gamma=\frac{1}{R_{\rm dS}^{\alpha-1}}\,. \tag{5.33}$$

The function $G(R)$ in Eq. (5.19),

$$G(R)=R+2f_i-Rf_i'+\frac{(2-\alpha)}{R_{\rm dS}^{\alpha-1}}R^\alpha\,, \tag{5.34}$$

has to be zero on the de Sitter solution. We get

$$R_{\rm dS}=\frac{4\Lambda_{\rm i}}{3-\alpha}\,,\quad R_{\rm dS}\gg R_i\,. \tag{5.35}$$

Let us consider the scalaron mass of Eq. (1.34) on the de Sitter solution:

$$m^2\simeq\frac{R_{\rm dS}}{3}\left(\frac{1+2\alpha-\alpha^2}{\alpha(\alpha-1)}\right)\,. \tag{5.36}$$

It is negative if $\alpha > 2.414$. In this case inflation is strongly unstable. Using Eq. (5.7) we derive the characteristic time of the instability as

$$t_i \sim \frac{1}{\sqrt{R_{\rm dS}}} \sim 10^{-10} - 10^{-19}\,\text{sec}\,, \tag{5.37}$$

in accordance with the expected value. If $\Lambda_{\rm i} = 4R_i$, in order to satisfy the two condition (5.35) simultaneously (which lead to $2 < \alpha < 3$) and in order to have unstable inflation, we can require

$$5/2 \leq \alpha < 3\,, \quad \Lambda_{\rm i} = 4R_i\,. \tag{5.38}$$

Now, we will try to reconstruct the evolution of the function $G(R)$ in Eq. (5.34) as we did in the previous Section for current acceleration. When $R = 0$, we find a trivial de Sitter point and $G(0) = 0$. For the first derivative of $G(R)$, one has

$$G'(R) = 1 + f_i'(R) - Rf_i''(R) + \alpha(2-\alpha)\frac{R^{\alpha-1}}{R_{\rm dS}^{\alpha-1}}\,. \tag{5.39}$$

$G'(0) > 0$ and $G(R)$ increases. Since $f_i''(R)$ starts being positive for $R > \tilde{R}$ (where $\tilde{R}$ is expressed as in Eq. (5.29)) and $2 - \alpha < 0$, it is easy to see that $G(R)$ begins to decrease at around $R = R_i$ and that it is zero when $R = R_{\rm dS}$. After this point, $0 < G'(R > R_{\rm dS})$ and we do not have other de Sitter solutions. On the other hand, it is possible to have a fluctuation of $G(R)$ along the R-axis just before the de Sitter point describing inflation takes over. In order to avoid other de Sitter solutions (i.e., possible final attractors for the system), we need to verify the fulfillment of the following condition:

$$G(R) > 0\,, \quad 0 < R < R_{\rm dS}\,. \tag{5.40}$$

Precise analysis of this condition leads to a transcendental equation. In the next Subsection we will limit ourselves to a graphical evaluation. In general, it will be sufficient to choose n sufficiently large in order to avoid such effects.

5.4.1 Construction of a model for inflation

By taking into account all the conditions met above, the simplest choice of parameters to introduce in the function of Eq. (5.23) is:

$$n = 4\,, \quad \alpha = \frac{5}{2}\,, \tag{5.41}$$

while the curvature R_i is set as

$$R_i = 4\Lambda_{\rm i}\,, \tag{5.42}$$

and condition (5.38) is satisfied. In this way, R_i satisfies Eq. (5.30) and we have no anti-gravity effects, and $n > \alpha$ as in Eq. (5.32) avoiding undesirable instability effects in the small-curvature regime.

From Eq. (5.35) one recovers the unstable de Sitter solution describing inflation as

$$R_{\rm dS} = 8\Lambda_{\rm i}\,. \tag{5.43}$$

In Fig. 5.4 a plot of $G(R)$ is shown. The zeros of $G(R)$ correspond to de Sitter solutions. One can see that the only non-trivial zero is the de Sitter point of Eq. (5.43), and here the function crosses the R-axis up-down, according to the instability of such solution (since $F''(R > \tilde{R}) > 0$, we get $G'(R) \sim m^2{<}0$). This means that the inflationary de Sitter point corresponds to a maximum of the theory (without matter/radiation). The system gives rise to the de Sitter solution where the universe expands in an accelerating way but, suddenly, it exits from inflation and tends towards the minimal attractor at $R = 0$, unless the theory develops a singularity solution for $R \to \infty$. In such case, the model could exit from inflation and move in the wrong direction, where the curvature would grow up and diverge, and a singularity would appear. In the next Section singularities will be considered in the context of exponential gravity. We will see that the theory is free of future-time singularities.

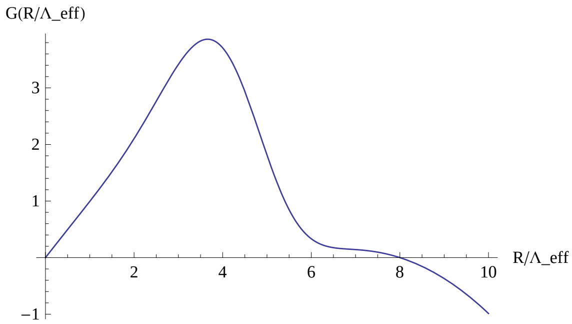

Figure 5.4: Plot of $G(R/\Lambda_{\rm i})$ of exponential model for inflation. The zeros correspond to the de Sitter solutions.

5.5 Singularities in exponential gravity

In § 3.5.1 we have seen that the terms of the type γR^α with $\alpha > 1$ protect the theory against singularities occurring when $R \to \pm\infty$, it means Type I, II, III singularities, and the Big Rip. As a consequence, our model in Eq. (5.23) is free of catastrophic divergences in the curvature, due to the contribution of the last power term necessary to exit from inflation. A remark is in order. Since in 'one-step' models for large values of curvature the high derivatives of $F(R)$ tend to zero, it is easy to see that effective energy density never diverges, and the Big Rip and the Type I and III singularities are not realized. Nevertheless, the Type II singularity may affect this kind of models. For example, in §3.4.1, we explicit found the possibility to realize the Type II singularity in the Hu-Sawiki Model. This fact suggests the presence of curvature high power terms which prevent such singularity and induce also inflationary effects.

As regard the Type IV singularities ($H \sim 1/(t_0 - t)^\beta\,, \beta < -1$), a short analysis has to be done. When $R \to 0^-$, $F(R \to 0^-) \simeq R - R/R_0$, and it is easy to see that the effective energy density defined by Eq. (3.5) behaves as $\sim 1/(t_0 - t)^{\beta+1}$, and it is larger than $H^2(\sim 1/(t_0 - 1)^{2\beta})$. For this reason, the EOM become inconsistent with Type IV singularities. The argument is valid also if we consider the more general case where H tends to a positive constant in the asymptotic singular limit: also in this situation $F(R)$ approaches a constant like $\sim 1/(t_0 - 1)^{\beta+1}$, while the time dependent part of H^2 behaves as $\sim 1/(t_0 - t)^\beta$. However, in Chapter **6** we will analyze better such kind of singularity into the context of DE-oscillation in exponential gravity.

In the next Section we will see that in the very asymptotic limit $R \gg R_i$ the model $F(R) = \gamma R^\alpha$ exhibits a (disconnected) singularity. Since the de Sitter solution of inflation is unstable, we can ask if the model could move toward such extreme limit. We will provide an argument to exclude this possibility.

Thus, we have found that our theory is free from singularities. In particular, when we are in the region of inflation, Types I, II or III singularities do not occur. When inflation ends, it is reasonable to suppose that the model moves to the attractor at $R \to 0$. In this way, the small curvature regime arises, the first term of Eq. (5.23) becomes dominant and the physics of the ΛCDM model is reproduced.

5.6 Asymptotic behavior

As last issue, we will analyze the solutions of our model when R is very large in comparison with the curvature R_i of inflation. This means that Eq. (5.23) can be approximated by

$$F(R \gg R_i) \simeq \gamma R^\alpha\,, \tag{5.44}$$

which is proved by the fact that $\alpha > 2$ and, by setting $\gamma = R_{\rm dS}^{\alpha-1}$, one has $\gamma R^\alpha \gg R$. By using Eq. (1.6) and Eq. (1.8) for $F(R)$-gravity in vacuum, one can deal with a first order differential autonomous system in R and H as

$$\dot{R} = -\frac{1}{F''(R)}\left(HF'(R) + \frac{F(R) - RF'(R)}{6H}\right), \tag{5.45}$$

$$\dot{H} = \frac{R}{6} - 2H^2. \tag{5.46}$$

For the model of Eq. (5.44), at the limit $t \to 0^+$, we can find the following solutions:

$$H \simeq \frac{H_0}{t^\beta}, \tag{5.47}$$

$$R \simeq 12\frac{H_0^2}{t^{2\beta}}. \tag{5.48}$$

Here, H_0 is a large positive constant and β a positive parameter so that $\beta = 1$ or $\beta > 1$.

This result shows that in the limit $R \to +\infty$ the model exhibits a past singularity, which could be identified with the Big Bang one. However, with a change $t \to (t_0 - t)$, we may obtain a future-time singularity solution. It is important to stress that this kind of solution is disconnected from the de Sitter phase of inflation, where the term R is of the same order of γR^α and is therefore not negligible as in Eq. (5.44). In the very asymptotic limit the scalaron $F'(R)$ results

$$F'(R) = \gamma\alpha R^{\alpha-1}. \tag{5.49}$$

We can also evaluate the potential V_{eff} of Eq. (1.28), through integration of $F''(R)(\partial V_{\text{eff}}/\partial F'(R)) = \partial V_{\text{eff}}/\partial R$. By neglecting the contribute of matter, one easily finds

$$V_{\text{eff}}(R \gg R_i) \simeq \frac{\gamma^2\alpha(\alpha-1)(2-\alpha)}{3(2\alpha-1)}R^{(2\alpha-1)}. \tag{5.50}$$

We observe that, in order to reach the singularity, the scalaron has to crossover an infinite potential barrier ($V_{\text{eff}}(R \to \infty) \to \infty$) and go to infinity ($F'(R \to \infty) \to \infty$), but clearly this dynamical behavior is forbidden.

We may safely assume that, just after the Big Bang, a Planck epoch takes over where physics is not described by GR and where quantum gravity effects are dominant. When the universe exits from the Planck epoch, its curvature is bound to be the characteristic curvature of inflation and the unstable de Sitter solution takes over.

Chapter 6

Oscillations of dark energy in realistic $F(R)$-gravity

Here, we use the so-called fluid representation of $F(R)$-gravity. The equations of motion are presented in the standard gravity FRW form but with the addition of DE-fluid derived by the modification of gravity, which is separated from matter fluid and for which we define a corresponding EoS parameter $\omega_{\rm DE}$. Specifically, we will study the oscillations that are generated in $\omega_{\rm DE}$ around the phantom divide crossing ($\omega_{\rm DE} = -1$) in realistic models of $F(R)$-modified gravity, during the matter dominance era and in the de Sitter epoch. The evolution of the $F(R)$-dark energy in these regimes will be carefully investigated. As an example, we will complete the analysis of exponential gravity of Chapter **5** via numerical evaluation of the corresponding DE-fluid, by setting the parameters in order to reproduce the last very accurate observational data. Some problems appearing in relation with the big values generated for the dark energy fluctuations, for large values of the red shift, will be discussed, by considering the relation between the oscillations obtained and possible (soft) singularities. The results shown in this Chapter have been presented in Ref. [22].

6.1 DE-oscillations in the de Sitter universe

We will be interested in the cosmological behavior of realistic $F(R)$-models of modified gravity describing the de Sitter epoch of the universe today. The tag 'realistic' has been defined in Chapter **5** and has to do with the feasibility of the models in view the all the most recent and accurate observational data. In particular, we will still consider modified gravity in the form $F(R) = R + f(R)$, by explicitly separating the contribution of GR from its modification. Up to now, with this class of models, it is possible to turn out the most realistic reproduction of our universe, as in the case of exponential modified gravity.

Let us consider the effects of modified gravity and matter together as the ones of an effective fluid with energy density and pressure given by Eqs. (1.18)-(1.19). Now, for $F(R)$-gravity, we define the dark energy density $\rho_{\rm DE}$ and the dark energy pressure $p_{\rm DE}$ as

$$\rho_{\rm DE} = \rho_{\rm eff} - \rho_{\rm m}\,, \tag{6.1}$$

$$p_{\rm DE} = p_{\rm eff} - p_{\rm m}\,, \tag{6.2}$$

by explicit separate the contribute of matter. In this way, we obtain a fluid representation of $F(R)$ gravity. We introduce the variable [110]

$$y_H(z) \equiv \frac{\rho_{\rm DE}}{\rho_{\rm m(0)}} = \frac{H^2}{\tilde{m}^2} - (z+1)^3 - \chi(z+1)^4\,. \tag{6.3}$$

Here, $\rho_{\mathrm{m}(0)}$ is the energy density of matter at present time, $\tilde{m}^2$ is the mass scale

$$\tilde{m}^2 \equiv \frac{\kappa^2 \rho_{\mathrm{m}(0)}}{3} \simeq 1.5 \times 10^{-67} \mathrm{eV}^2 \,, \tag{6.4}$$

and χ is defined as[1]

$$\chi \equiv \frac{\rho_{\mathrm{r}(0)}}{\rho_{\mathrm{m}(0)}} \simeq 3.1 \times 10^{-4} \,, \tag{6.5}$$

where $\rho_{\mathrm{r}(0)}$ is the energy density of radiation at present, z the redshift parameter[2], $z = 1/a(t) - 1$, and $y_H(z)$ is written as a function of z.

The EoS-parameter for dark energy, ω_{DE}, is

$$\omega_{\mathrm{DE}} \equiv \frac{p_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 + \frac{1}{3}(z+1)\frac{1}{y_H(z)}\frac{dy_H(z)}{d(z)} \,. \tag{6.6}$$

By combining Eq. (1.14) with Eq. (1.8) and using Eq. (6.3), one gets

$$\frac{d^2 y_H(z)}{dz^2} + J_1 \frac{dy_H(z)}{dz} + J_2 \left(y_H(z)\right) + J_3 = 0 \,, \tag{6.7}$$

where

$$J_1 = \frac{1}{(z+1)}\left(-3 - \frac{1}{y_H + (z+1)^3 + \chi(z+1)^4}\frac{1 - F'(R)}{6\tilde{m}^2 F''(R)}\right) \,, \tag{6.8}$$

$$J_2 = \frac{1}{(z+1)^2}\left(\frac{1}{y_H + (z+1)^3 + \chi(z+1)^4}\frac{2 - F'(R)}{3\tilde{m}^2 F''(R)}\right) \,, \tag{6.9}$$

$$J_3 = -3(z+1) - \frac{(1 - F'(R))((z+1)^3 + 2\chi(z+1)^4) + (R - F(R))/(3\tilde{m}^2)}{(z+1)^2(y_H + (z+1)^3 + \chi(z+1)^4)}\frac{1}{6\tilde{m}^2 F''(R)} \,. \tag{6.10}$$

Thus, we have

$$R = 3\tilde{m}^2 \left(4y_H(z) - (z+1)\frac{dy_H(z)}{dz} + (z+1)^3\right) \,. \tag{6.11}$$

Here, we have used $-(z+1)(H(z))d/dz = (H(t))d/d(\ln a(t)) = d/dt$, where H could be an explicit function of the red shift, $H(z)$, or an explicit function of the time, $H(t)$.

In general, Eq. (6.7) can be solved in a numerical way, once we write the explicit form of the $F(R)$-model.

Let us now study perturbations around the de Sitter solution R_{dS} given by Eq. (1.31), to see that we are able to recover the stability condition (1.37). Here we restrict our analysis to homogeneous perturbations. The behavior of general, linear, inhomogeneous perturbations has been discussed in Ref. [111], where the equivalence between the two approaches has been shown explicitly (see also the independent proof contained in Ref. [112]).

The starting point will be

$$y_H(z) \simeq y_0 + y_1(z) \,, \tag{6.12}$$

where $y_0 = R_{\mathrm{dS}}/12\tilde{m}^2$ is the dark energy constant of the dS-universe and $|y_1(z)| << 1$. Eq. (6.11) leads to

$$R = 3\tilde{m}^2 \left(4y_0 + 4y_1(z) - (z+1)\frac{dy_1(z)}{dz} + (z+1)^3\right) \,. \tag{6.13}$$

[1] Here, we have used the data of Ref. [3].
[2] At the present, when $z = 0$, $a(t) = 1$.

In this case, by neglecting the contribution of radiation and assuming the matter one to be much smaller than y_0, Eq. (6.7) becomes at first order in $y_1(z)$

$$\frac{d^2 y_1(z)}{dz^2} + \frac{\alpha}{(z+1)}\frac{dy_1(z)}{dz} + \frac{\beta}{(z+1)^2} y_1(z) = 4\zeta(z+1)\,, \tag{6.14}$$

where

$$\alpha = -2\,, \tag{6.15}$$

$$\beta = -4 + \frac{4F'(R_{\rm dS})}{RF''(R_{\rm dS})}\,, \tag{6.16}$$

$$\zeta = 1 + \frac{1 - F'(R_{\rm dS})}{R_{\rm dS}F''(R_{\rm dS})}\,. \tag{6.17}$$

Here, we have used the de Sitter condition (1.31). The solution of Eq. (6.14) is

$$y_1(z) = C_0 (z+1)^{\frac{1}{2}\left(1-\alpha\pm\sqrt{(1-\alpha)^2-4\beta}\right)} + \frac{4\zeta}{\beta}(z+1)^3\,, \tag{6.18}$$

where C_0 is a constant. It is easy to see that $|y_1(z)| \ll 1$ when $z \to -1^+$, and, therefore, the de Sitter solution is stable, provided by Eq. (1.37), i.e. $F'(R_{\rm dS})/R_{\rm dS}F''(R_{\rm dS}) > 1$.

We have two possible behaviors for the dark energy density in viable models of modified gravity, for a stable de Sitter universe [113]. If

$$\frac{25}{16} > \frac{F'(R_{\rm dS})}{R_{\rm dS}F''(R_{\rm dS})} > 1\,, \tag{6.19}$$

the solution approaches the de Sitter point as a power function of $(z+1)$, that is $y_1(z) \sim (z+1)^\gamma$, $\gamma > 0$. Otherwise, if

$$\frac{F'(R_{\rm dS})}{R_{\rm dS}F''(R_{\rm dS})} > \frac{25}{16}\,, \tag{6.20}$$

the discriminant in the square root of Eq. (6.18) is negative and the dark energy density shows an oscillatory behavior whose amplitude decreases as $(z+1)^{3/2}$, when $z \to -1^+$. As a consequence, we can write $y_H(z)$ as

$$y_H(z) = \frac{R_{\rm dS}}{12\tilde{m}^2} + \left(\frac{1}{F'(R_{\rm dS}) - R_{\rm dS}F''(R_{\rm dS})} - 1\right)(z+1)^3 + (z+1)^{\frac{3}{2}} \times \tag{6.21}$$

$$\left[A_0 \cos\left(\sqrt{\left(\frac{4F'(R_{\rm dS})}{R_{\rm dS}F''(R_{\rm dS})} - \frac{25}{4}\right)}\log(z+1)\right) + B_0 \sin\left(\sqrt{\left(\frac{4F'(R_{\rm dS})}{R_{\rm dS}F''(R_{\rm dS})} - \frac{25}{4}\right)}\log(z+1)\right)\right]\,,$$

A_0 and B_0 being constants which depend on the boundary conditions.

Using Eq. (6.6), we can evaluate the $\omega_{\rm DE}$ parameter

$$\omega_{\rm DE} = -1 + \frac{4\zeta}{\beta}\frac{(z+1)^3}{y_0} + \frac{1}{3}\gamma\frac{(z+1)^\gamma}{y_0}\,, \tag{6.22}$$

where

$$\gamma = \frac{1}{2}\left(1 - \alpha \pm \sqrt{(1-\alpha)^2 - 4\beta}\right)\,. \tag{6.23}$$

In the case of oscillating models which satisfy Eq. (6.20), one has

$$\omega_{\mathrm{DE}} = -1 + \frac{12\tilde{m}^2}{R_{\mathrm{dS}}}\left(\frac{1}{F'(R_{\mathrm{dS}}) - R_{\mathrm{dS}}F''(R_{\mathrm{dS}})} - 1\right)(z+1)^3 + 4\tilde{m}^2\frac{(z+1)^{\frac{3}{2}}}{R_{\mathrm{dS}}} \times \tag{6.24}$$

$$\left[A_0'\cos\left(\sqrt{\left(\frac{4F'(R_{\mathrm{dS}})}{R_{\mathrm{dS}}F''(R_{\mathrm{dS}})} - \frac{25}{4}\right)}\log(z+1)\right) + B_0'\sin\left(\sqrt{\left(\frac{4F'(R_{\mathrm{dS}})}{R_{\mathrm{dS}}F''(R_{\mathrm{dS}})} - \frac{25}{4}\right)}\log(z+1)\right)\right],$$

with

$$A_0' = \frac{3}{2}A_0 + \sqrt{\left(\frac{4F'(R_{\mathrm{dS}})}{R_{\mathrm{dS}}F''(R_{\mathrm{dS}})} - \frac{25}{4}\right)}B_0\,, \tag{6.25}$$

and

$$B_0' = \frac{3}{2}B_0 - \sqrt{\left(\frac{4F'(R_{\mathrm{dS}})}{R_{\mathrm{dS}}F''(R_{\mathrm{dS}})} - \frac{25}{4}\right)}A_0\,. \tag{6.26}$$

We observe that ω_{DE} exhibits the same oscillation period of $y_H(z)$ and that its amplitude is amplified by its frequency, written in the coefficients A_0' and B_0'.

By writing $F(R) = R + f(R)$, the oscillatory condition leads to

$$\frac{1 + f'(R_{\mathrm{dS}})}{R_{\mathrm{dS}}f''(R_{\mathrm{dS}})} > \frac{25}{16}\,. \tag{6.27}$$

The large class of the 'one-step' models which mimic the cosmological constant in the high-curvature regime (where $f(R_{\mathrm{dS}}) \simeq -2\Lambda$, Λ being the Cosmological Constant) satisfies this condition ($0 < f''(R_{\mathrm{dS}}) \ll 1$), and we expect that the corresponding ω_{DE} parameter oscillates around the value -1.

6.1.1 Time evolution

Let us now consider the stable de Sitter solution of Eq. (6.12) and Eq. (6.18) in the case of γ being a positive real number (what means that the condition (6.19) is satisfied):

$$y_H(z) = y_0 + C_0(z+1)^\gamma + \frac{4\zeta}{\beta}(z+1)^3\,. \tag{6.28}$$

Here β, ζ and γ have been given by Eqs. (6.16)-(6.17) and Eq. (6.23), $y_0 = H_{\mathrm{dS}}^2/\tilde{m}^2 = R_{\mathrm{dS}}/(12\tilde{m}^2)$, such that $R_{\mathrm{dS}} = 12H_{\mathrm{dS}}^2$, and C_0 is a constant. We will assume $C_0 > 0$. The first EOM (1.14) leads to

$$\frac{H^2}{\tilde{m}^2} = y_H(z) + (z+1)^3 = y_0 + C_0(z+1)^\gamma + \left(\frac{1}{F'(R_{\mathrm{dS}}) - R_{\mathrm{dS}}F''(R_{\mathrm{dS}})}\right)(z+1)^3\,. \tag{6.29}$$

We will explicitly solve H as a function of the cosmic time t. By writing $z+1$ as $1/a(t)$, one gets

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H_{\mathrm{dS}}^2 + (C_0\tilde{m}^2)\left(\frac{1}{a(t)}\right)^\gamma\,. \tag{6.30}$$

We have here omitted the matter contribution. By considering $t > 0$, the general solution for the expanding universe is

$$a(t) = \left(\frac{C_0\tilde{m}^2}{H_{dS}^2}\right)^{\frac{1}{\gamma}}\left[\sinh\left(\frac{H_{\mathrm{dS}}}{2}\gamma t + \phi\right)\right]^{\frac{2}{\gamma}}, \tag{6.31}$$

being ϕ a positive constant. It is then easy to obtain

$$a(t) = a_0 e^{H_{\rm dS} t} \left[1 - e^{-\left(\frac{H_{\rm dS}}{2}\gamma t + \phi\right)}\right]^{\frac{2}{\gamma}}, \qquad (6.32)$$

where a_0 is a constant which depends on ϕ. As $\gamma > 0$, we get $a(t) \simeq a_0 e^{H_{\rm dS} t}$. For the Hubble parameter, one has

$$H = H_{\rm dS} \coth\left(\frac{1}{2} H_{\rm dS}\gamma t + \phi\right), \qquad (6.33)$$

and, in general, for $t > 0$, it is $H \simeq H_{\rm dS}$.

Consider now the case of an oscillatory behavior followed by the condition (6.20). Then

$$H = \left(\frac{\dot{a}(t)}{a(t)}\right) = \sqrt{H_{\rm dS}^2 + \tilde{m}^2 a(t)^{-\frac{3}{2}} \left[A \cos\left[\nu \log(a(t)^{-1})\right] + B \sin\left[\nu \log(a(t)^{-1})\right]\right]}, \qquad (6.34)$$

where we have used Eq. (6.21), omitting matter contributions, and the frequency ν is

$$\nu = \sqrt{\frac{4F'(R_{\rm dS})}{R_{\rm dS} F''(R_{\rm dS})} - \frac{25}{4}}. \qquad (6.35)$$

Assuming $a(t) \simeq \exp(H_0 t)$, Eq. (6.34) yields

$$H \simeq \sqrt{H_{\rm dS}^2 + \tilde{m}^2 \left(\frac{1}{e^{H_0 t}}\right)^{\frac{3}{2}} \left[-A\cos(\nu H_0 t) + B \sin(\nu H_0 t)\right]}. \qquad (6.36)$$

Also, in this case, $H \simeq H_{\rm dS}$. Finally, we should stress that, as

$$\frac{d^n}{dt^n} H(t) = \left(-H(z)(z+1)\frac{d}{dz}\right)^n H(z), \qquad (6.37)$$

it is quite simple to see that any derivative of $H(t)$, and, as a consequence, any time derivative of $y_H(z)$, becomes singular around the zeros of the sinus and cosinus functions or in the limit $z \to -1$.

6.2 DE-oscillations in the matter era

The critical points associated with matter dominated era for $F(R)$-gravity have been briefly discussed in § 5.1.1. An important viable condition is the positivity of the second derivative of the cosmological function, $F''(R) > 0$, as in Eq. (5.3), during the matter era. When $F''(R) < 0$ perturbations grow up and, as a consequence, the theory becomes strongly unstable. Now, a detailed analysis of such perturbations will be carried out, by considering the dynamical behavior of dark energy. Since in the late-time matter era it is not stable (raugly speaking, $\dot{F}(R) \neq 0$), it will be necessary to introduce some physical assumptions in order to find correct results.

At first, we assume that $y_H(z) \ll (1+z)^3$ and neglect the contribute of radiation. Eq. (6.7) reads, to first order in $y_H(z)/(z+1)^3$,

$$\begin{aligned} y_H''(z) - \frac{y_H'(z)}{(z+1)}(3) + \frac{y_H(z)}{(z+1)^2}\left(\frac{4F'(R)-3}{RF''(R)}\right) &= \\ (z+1)\left[3 + \frac{1}{2F''(R)R}\left((1-F'(R)) + \frac{(R-F(R))}{R}\right)\right], \end{aligned} \qquad (6.38)$$

where R is written in full form, as in Eq. (6.11), and we have used the condition (5.1). In what follows we will consider

$$R - F(R) \simeq 6\tilde{m}^2 y_H(z)\,, \tag{6.39}$$

$$F'(R) \simeq 1 + \mathcal{O}(y_H(z))\,, \tag{6.40}$$

as a consequence of the matter era conditions (5.1)-(5.2).

In the standard cosmological scenario, the effects of dark energy are completely neglected when $z \gg 3$ and Eq. (6.38) can be expanded at first order in $y_H(z)$ as

$$\begin{aligned} y_H''(z) + y_H'(z)\frac{1}{(z+1)}\left(-\frac{7}{2} - \frac{(1-F'(R))F'''(R)}{2F''(R)^2}\right) &+ \\ y_H(z)\frac{1}{(z+1)^2}\left(2 + \frac{1}{RF''(R)} + \frac{2(1-F'(R))F'''(R)}{F''(R)^2}\right) &= \left(3 + \frac{2-F'(R)-F(R)/R}{2RF''(R)}\right)(z+1)\,. \end{aligned} \tag{6.41}$$

In this case the Ricci scalar simply reads

$$R = 3\tilde{m}^2(z+1)^3\,. \tag{6.42}$$

In order to solve Eq. (6.41) we can set $z = z_0 + (z - z_0)$, where $|z - z_0| \ll z_0$, and perform a variation with respect to z. To first order in $(z - z_0)$, we find

$$\begin{aligned} & y_H''(z) + y_H'(z)\frac{1}{(z_0+1)}\left(-\frac{7}{2} - \frac{(1-F'(R_0))F'''(R_0)}{2F''(R_0)^2}\right) + \\ & \qquad y_H(z)\frac{1}{(z_0+1)^2}\left(2 + \frac{1}{R_0F''(R_0)} + \frac{2(1-F'(R_0))F'''(R_0)}{F''(R_0)^2}\right) = \\ & \left(3 + \frac{2-F'(R_0)-F(R_0)/R_0}{2R_0F''(R_0)}\right)(z_0+1) + \\ & \quad 3\left(\frac{1}{2} + \frac{5F(R_0)/R_0 - F'(R_0) - 4}{6R_0F''(R_0)} - \frac{(2-F'(R_0)-F(R_0)/R_0F'''(R_0))}{2F''(R_0)^2}\right)(z-z_0)\,, \end{aligned} \tag{6.43}$$

where

$$R_0 = 3\tilde{m}^2(z_0+1)^3\,. \tag{6.44}$$

The solution of this equation is

$$y_H(z) = a + b\cdot(z-z_0) + C_0\cdot e^{\frac{1}{2(z_0+1)}\left(\alpha\pm\sqrt{\alpha^2-4\beta}\right)(z-z_0)}\,, \tag{6.45}$$

where C_0 is constant and

$$\begin{aligned} a \;=\; & \left(\frac{1}{6\tilde{m}^2}\right)\frac{6R_0^2F''(R_0) + (2-F'(R_0))R_0 - F(R_0)}{1 + 2R_0F''(R_0) + 2(2-F'(R_0)-F(R_0)/R_0)R_0F'''(R_0)/F''(R_0)} + \\ & \left(\frac{R_0^2}{4\tilde{m}^2}\right)\frac{7F''(R_0)^2 + (2-F'(R_0)-F(R_0)/R_0)F'''(R_0)}{[2R_0F''(R_0)^2 + F''(R_0) + 2R_0(2-F'(R_0)-F(R_0)/R_0)F'''(R_0)]^2}\times \\ & [RF''(R_0)^2 + (5F(R_0)/R_0 - F'(R_0) - 4)F''(R_0)/3 - R_0(2-F'(R_0)-F(R_0)/R_0)F'''(R_0)]\,, \end{aligned} \tag{6.46}$$

$$
\begin{aligned}
b &= \frac{R_0}{2\tilde{m}^2(z_0+1)} \\
&\quad \frac{R_0 F''(R_0)^2 + (5F(R_0)/R_0 - F'(R_0) - 4)F''(R_0)/3 - (2 - F'(R_0) - F(R_0))R_0 F'''(R_0))}{2R_0 F''(R_0)^2 + F''(R_0) + 2(2 - F'(R_0) - F(R_0)/R_0)R_0 F'''(R_0)}\,,
\end{aligned} \tag{6.47}
$$

$$
\alpha = \frac{7}{2} + \frac{(1 - F'(R_0))F'''(R_0)}{2F''(R_0)^2}\,, \tag{6.48}
$$

$$
\beta = 2 + \frac{1}{R_0 F''(R_0)} + \frac{2(1 - F'(R_0))F'''(R_0)}{F''(R_0)^2}\,. \tag{6.49}
$$

Let us now analyze this result. Since in the expanding universe $(z - z_0) < 0$, it turns out that the matter solution is stable around R_0 if $\alpha > 0$ and $\beta > 0$. This means that

$$
\frac{(1 - F'(R))F'''(R_0)}{2F''(R_0)^2} > -\frac{7}{2}\,, \tag{6.50}
$$

$$
\frac{1}{R_0 F''(R_0)} > 12\,. \tag{6.51}
$$

We can thus have an oscillatory behavior of the dark energy if the discriminant of the square root of Eq. (6.45) is negative.

6.2.1 The late-time matter era in realistic $F(R)$-gravity

In realistic models of modified gravity, the de Sitter universe follows the matter era. The effects of dark energy could be relevant at a late-time matter era, near the transition between the matter and de Sitter epochs ($1 \lesssim z \lesssim 3$). In this case, we can not do an expansion of the $F(R)$-functions in terms of $y_H(z)$, as we did before. On the other hand, in realistic models of modified gravity, $y_H(z)$ tends to a constant value, as in Eq. (6.12), $y_H(z) = y_0 + y_1(z)$, where $y_0 \simeq R_{dS}/12\tilde{m}^2$ is related to the de Sitter solution and $|y_1(z)| \ll y_0$ (in this way, we can reproduce the correct dynamical evolution of the universe, as in the ΛCDM model). As a consequence, we can actually perform the variation of Eq. (6.38) with respect to $y_1(z)$, to obtain

$$
\begin{aligned}
y_1''(z) + y_1'(z)\frac{1}{(z+1)}\left[-\frac{7}{2} - \frac{(1 - F'(R))F'''(R)}{2F''(R)^2}\right] + \frac{y_0 + y_1(z)}{(z+1)^2}\left(\frac{4F'(R) - 3}{RF''(R)}\right) &= \\
(z+1)\left[3 + \frac{1}{2F''(R)R}\left((1 - F'(R)) + \frac{(R - F(R))}{R}\right)\right]&\,,
\end{aligned} \tag{6.52}
$$

where

$$
R = 3\tilde{m}^2\left[(z+1)^3 + 4y_0\right]\,. \tag{6.53}
$$

Also in this case, we can take $z = z_0 + (z - z_0)$, where $|z - z_0| \ll z_0$, and doing the variation with respect to z, we find, up to first order in $(z - z_0)$,

$$
\begin{aligned}
&y_1''(z) + y_1'(z)\frac{1}{(z_0+1)}\left[-\frac{7}{2} - \frac{(1 - F'(R_0))F'''(R_0)}{2F''(R_0)^2}\right] + \frac{y_0 + y_1(z)}{(z_0+1)^2}\left(\frac{1}{R_0 F''(R_0)}\right) = \\
&(z_0+1)\left[3 + \frac{1}{2F''(R_0)R_0}\left(1 - F'(R_0) + \frac{R_0 - F(R_0)}{R_0}\right)\right] + \\
&\qquad 3\left[\frac{1}{2} - \frac{1 - F'(R_0)}{2F''(R_0)^2}F'''(R_0) + \frac{1 - F'(R_0)}{6F''(R_0)R_0}\right](z - z_0)\,,
\end{aligned} \tag{6.54}
$$

where

$$R_0 = 3\tilde{m}^2((z_0+1)^3 + 4y_0)\,. \tag{6.55}$$

In this expression we have used the conditions (6.39)-(6.40). Owing to the fact that $(y_0\tilde{m}^2) \ll R_0$, we have considered terms at least of first order in $(y_0\tilde{m}^2)/R_0$. The solution of this equation is

$$\begin{aligned} y_0 &= a\,, \\ y_1(z) &= b\cdot(z-z_0) + C_0 \cdot e^{\frac{1}{2(z_0+1)}\left(\alpha\pm\sqrt{\alpha^2-4\beta}\right)(z-z_0)}\,, \end{aligned} \tag{6.56}$$

where C_0 is a constant, and

$$\begin{aligned} a &\simeq \frac{R_0}{6\tilde{m}^2(4F'(R_0)-3)}(6F''(R_0)R_0 + 2 - F'(R_0) - F(R_0)/R_0) + \\ &\quad \frac{R_0^2}{4\tilde{m}^2}(7F''(R_0)^2 + 2 - F'(R_0) - F(R_0)/R_0)F'''(R_0)) \times \\ &\quad (R_0F''(R_0)^2 - (1-F'(R_0))R_0F'''(R_0) + (1-F'(R_0))F''(R_0)/3)\,, \end{aligned} \tag{6.57}$$

$$b = \frac{3(z_0+1)^2(R_0F''(R_0)^2 - (1-F'(R_0))R_0F'''(R_0) + (1-F'(R_0))F''(R_0)/3)}{2\,F''(R_0)}\,, \tag{6.58}$$

$$\alpha = \frac{7}{2} + \frac{(1-F'(R_0))F'''(R_0)}{2F''(R_0)^2}\,, \tag{6.59}$$

$$\beta = \frac{1}{R_0F''(R_0)}\,. \tag{6.60}$$

The solution is stable around R_0 if $\alpha > 0$ and $\beta > 0$. This means that

$$\frac{(1-F'(R_0))F'''(R_0)}{2F''(R_0)^2} > -\frac{7}{2}\,, \tag{6.61}$$

$$\frac{1}{R_0F''(R_0)} > 0\,. \tag{6.62}$$

This conditions are in perfect agreement with Eq. (5.3). The oscillatory behavior of the dark energy occurs when the discriminant of the square root of Eq. (6.56) is negative.

We observe that the expression (6.56) is more accurate than (6.45). In general, if the conditions (6.61)-(6.62) are satisfied for $R_0 = 3\tilde{m}^2(z_0+1)^3 + 12\tilde{m}^2y_0$, the conditions (6.50)-(6.51) will be also satisfied, provided it is possible to use the approximation $R_0 = 3\tilde{m}^2(z_0+1)^3$. In particular, since $F'(R) \simeq 1$, the condition

$$\frac{1}{F''(R)} \gg 1\,, \tag{6.63}$$

is sufficient to obtain stability during the matter era and an oscillating behavior of the dark energy. The large class of the 'one-step' models satisfies this condition.

We can use the solutions (6.45) or (6.56) and the expression (6.6) to evaluate the parameter ω_{DE} around $z = z_0$ during the matter era. We get

$$\omega_{\mathrm{DE}} \simeq -1 + \frac{1}{3a}\left[b(z_0+1) + C_0\frac{\alpha\pm\sqrt{\alpha^2-4\beta}}{2}e^{\frac{1}{2(z_0+1)}\left(\alpha\pm\sqrt{\alpha^2-4\beta}\right)(z-z_0)}\right]\,. \tag{6.64}$$

In the case of the oscillating models, for which $\alpha^2 - 4\beta < 0$, this equation reads

$$\omega_{\mathrm{DE}} \simeq -1 + \frac{b}{3a}(z_0+1) + \frac{e^{\frac{\alpha}{2(z_0+1)}}}{6a}\left[A_0 \cos\frac{\sqrt{4\beta-\alpha^2}}{2(z_0+1)}(z-z_0) + B_0 \sin\frac{\sqrt{4\beta-\alpha^2}}{2(z_0+1)}(z-z_0)\right], \quad (6.65)$$

where A_0 and B_0 are constant, proportional to the period $\sqrt{4\beta-\alpha^2}/(2(z_0+1))$.

6.3 DE-oscillations in exponential gravity

Let us return to exponential gravity. In Section **5** we have shown how with this kind of model we may reproduce the universe where we live. Now, we are interested in the cosmological evolution of the corresponding dark energy during the de Sitter phase. In particular, we will set the parameters in order to recover the last results of the $WMAP$, BAO and SN surveys [3].

We will consider the 'one step' model in Eq. (5.15) by setting the parameters as follows:

$$\Lambda = (7.93)\tilde{m}^2\,,$$

$$R_0 = 0.6\Lambda\,, \quad 0.8\Lambda\,, \quad \Lambda\,.$$

Note that R_0 has been chosen according with condition (5.21).

Eq. (6.7) has been solved in a numerical way[3], in the range of $R_0 \ll R \ll R_i$ (matter era/current acceleration), taking the following initial conditions at $z = z_i$,

$$\left.\frac{dy_H(z)}{d(z)}\right|_{z_i} = 0\,,$$

$$\left.y_H(z)\right|_{z_i} = \frac{\Lambda}{3\tilde{m}^2}\,,$$

which correspond to the ones of the ΛCDM model. This choice obeys to the fact that in the high red shift regime the exponential model is very close to the ΛCDM Model. The values of z_i have been chosen so that $RF''(z=z_i) \sim 10^{-7}$, assuming $R = 3\tilde{m}^2(z+1)^3 + 4\Lambda$. We have $z_i = 1.8$, 2.2, 2.6 for $R_0 = 0.6\Lambda$, 0.8Λ, Λ, respectively.

Using Eq. (6.6), one derives ω_{DE} from y_H. In Figs. 6.1, 6.2 and 6.3, we plot ω_{DE} as a function of the redshift z for $R_0 = 0.6\Lambda$, 0.8Λ, Λ, respectively. Note that ω_{DE} is very close to minus one. In the present universe ($z=0$), one has $\omega_{\mathrm{DE}} = -0.994$, -0.975, -0.950 for $R_0 = 0.6\Lambda$, 0.8Λ, Λ. The smaller R_0 is, our model becomes more indistinguishable from the ΛCDM model, where $\omega_{\mathrm{DE}} = -1$.

We can also extrapolate the behavior of the density parameter of dark energy Ω_{DE},

$$\Omega_{\mathrm{DE}} \equiv \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{eff}}} = \frac{y_H}{y_H + (z+1)^3 + \chi\,(z+1)^4}\,. \quad (6.66)$$

Plots of Ω_{DE} as a function of the redshift z for $R_0 = 0.6\Lambda$, 0.8Λ, Λ, are shown in Figs. 6.4, 6.5 and 6.6. For the present universe ($z=0$), one has $\Omega_{\mathrm{DE}} = 0.726$, 0.728, 0.732 for $R_0 = 0.6\Lambda$, 0.8Λ, Λ, respectively. The data are in accordance with the last and very accurate observations of our universe today, where:

$$\omega_{\mathrm{DE}} = -0.972^{+0.061}_{-0.060}\,,$$

$$\Omega_{\mathrm{DE}} = 0.721 \pm 0.015\,.$$

[3] Mathematica 7 ©.

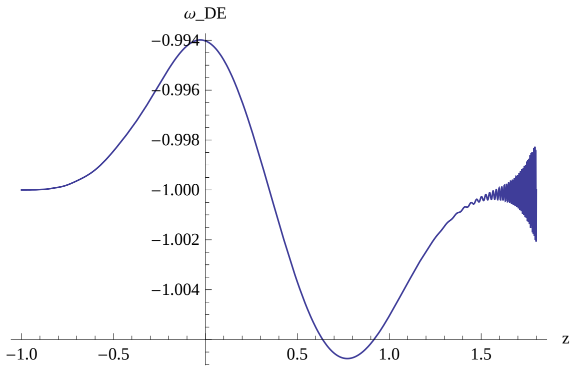

Figure 6.1: Plot of ω_{DE} for $R_0 = 0.6\Lambda$. At redshift $z = 0$, $\omega_{\mathrm{DE}} = -0.994$.

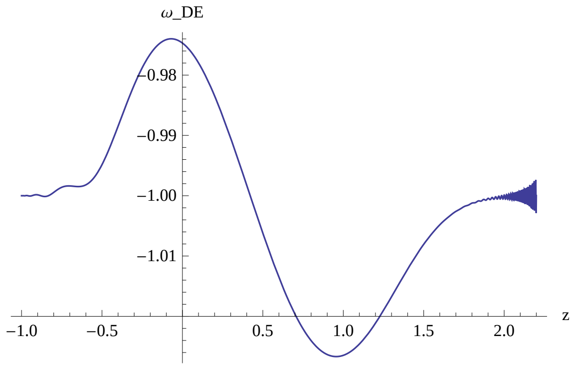

Figure 6.2: Plot of ω_{DE} for $R_0 = 0.8\Lambda$. At redshift $z = 0$, $\omega_{\mathrm{DE}} = -0.975$.

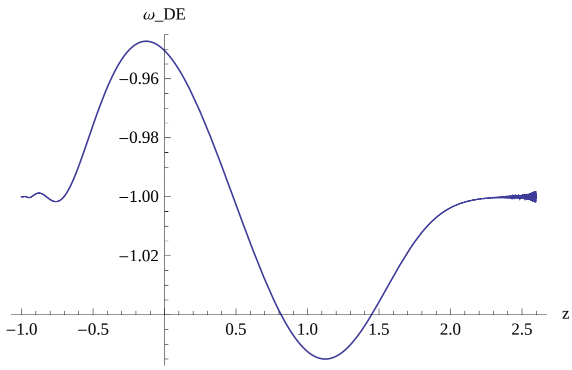

Figure 6.3: Plot of ω_{DE} for $R_0 = \Lambda$. At redshift $z = 0$, $\omega_{\mathrm{DE}} = -0.950$.

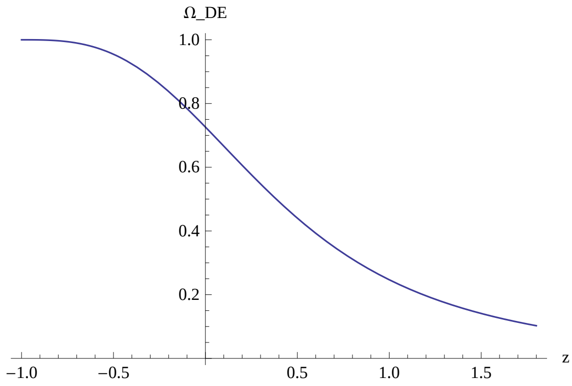

Figure 6.4: Plot of Ω_{DE} for $R_0 = 0.6\Lambda$. At redshift $z = 0$, $\Omega_{\mathrm{DE}} = 0.726$.

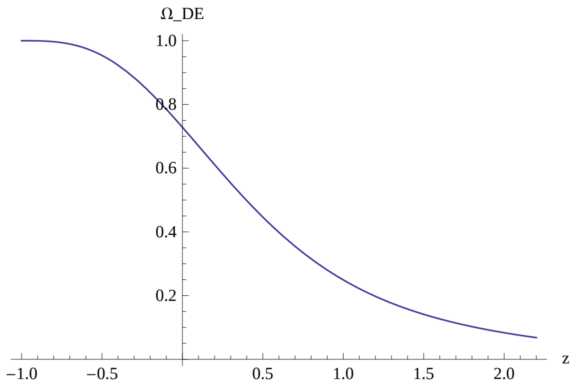

Figure 6.5: Plot of Ω_{DE} for $R_0 = 0.8\Lambda$. At redshift $z = 0$, $\Omega_{\mathrm{DE}} = 0.728$.

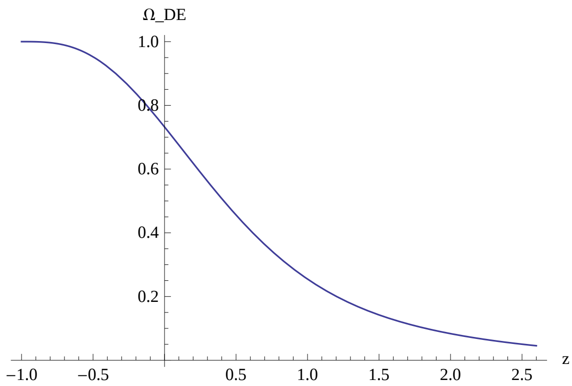

Figure 6.6: Plot of Ω_{DE} for $R_0 = \Lambda$. At redshift $z = 0$, $\Omega_{\mathrm{DE}} = 0.732$.

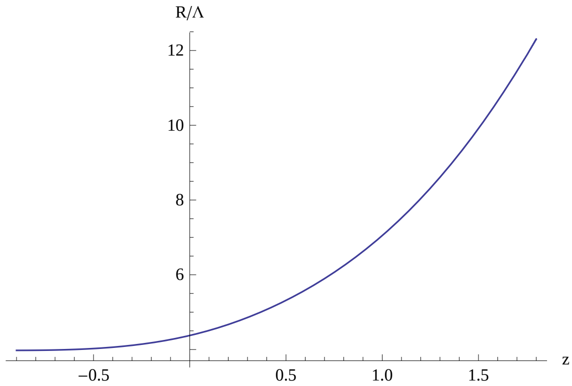

Figure 6.7: Plot of R/Λ for $R_0 = 0.6\Lambda$. At redshift $z = 0$, $R/\Lambda = 4.376$ and at $z = -0.9$, $R/\Lambda = 3.978$.

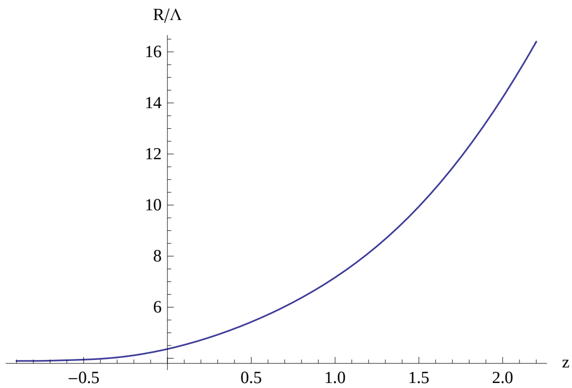

Figure 6.8: Plot of R/Λ for $R_0 = 0.8\Lambda$. At redshift $z = 0$, $R/\Lambda = 4.362$ and at $z = -0.9$, $R/\Lambda = 3.895$.

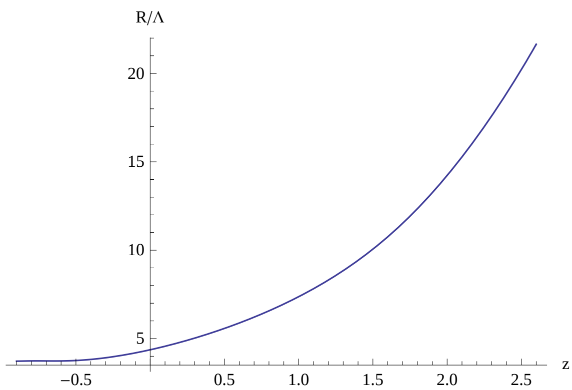

Figure 6.9: Plot of R/Λ for $R_0 = \Lambda$. At redshift $z = 0$, $R/\Lambda = 4.365$ and at $z = -0.9$, $R/\Lambda = 3.725$.

As last point, we want to analyze the behavior of the Ricci scalar in Eq. (6.11) for $R_0 = 0.6\Lambda$, 0.8Λ, Λ. Results are shown in Figs. 6.7, 6.8 and 6.9. We clearly see that the transition crossing the phantom divide does not cause any serious problem to the accuracy of the cosmological evolution arising from our model. In particular, $R(z \to -1^+)$ tends to $12\tilde{m}^2 y_H(z \to -1^+)$, which is an effective cosmological constant (note that R_0 is small and we are close to the value of the ΛCDM model, where $12\tilde{m}^2 y_H = 4\Lambda$). As a consequence, the de Sitter solution is a final attractor of our system and describes an eternal accelerating expansion.

6.3.1 De Sitter oscillations

In general, in 'one step' models, oscillating conditions (6.27) and (6.63) are satisfied. In what follows, we will verify the results of § **6.1**-**6.2** by considering exponential gravity with

$$R_0 = \Lambda\,.$$

In Fig. 6.10 the corresponding plot of $y_H(z)$ is shown. In Fig. 6.11, the plot of $\Omega_{\rm DE}(z)$ of Fig. 6.6 is overlapped with the one for $(y_H(z) - \Lambda/(3\tilde{m}^2))$.

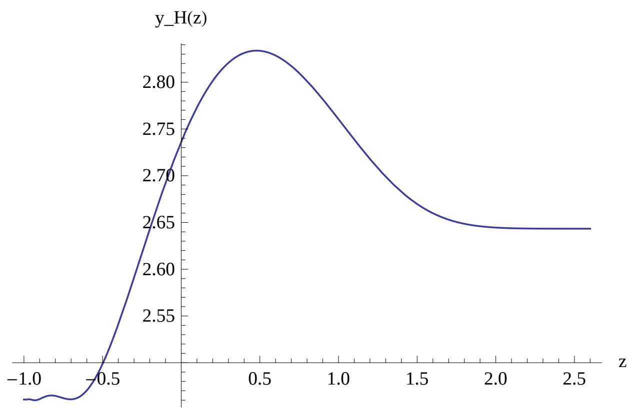

Figure 6.10: Plot of $y_H(z)$.

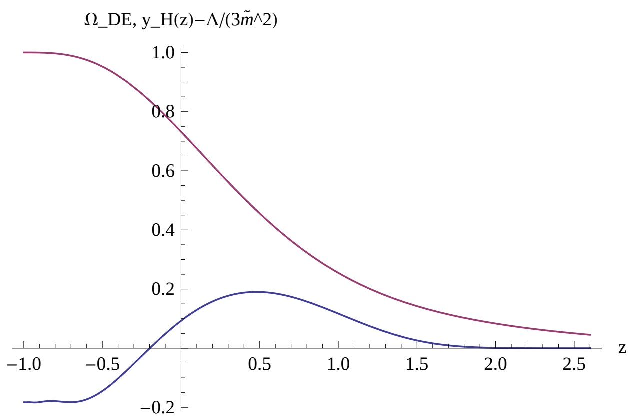

Figure 6.11: Plot of $\Omega_{\rm DE}(z)$ overlapped with $(y_H(z) - \Lambda/(3\tilde{m}^2))$.

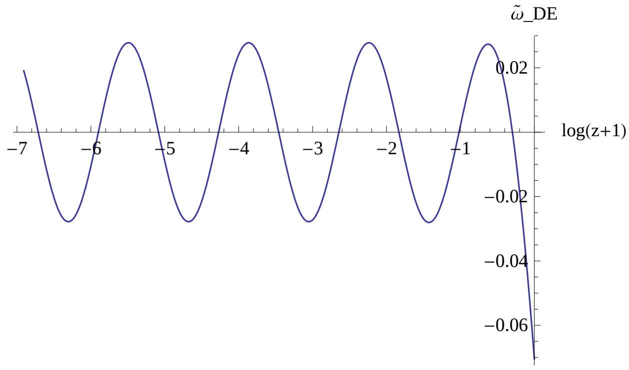

Figure 6.12: Plot of $\tilde{\omega}_{DE}$ for $\log[-0.999+1] < \log[z+1] < \log[0+1]$.

The computational evaluation (extrapolated from $y_H(z=0.9)$) gives $R_{dS} \simeq 3.725\Lambda$ (Fig. 6.9) so that we are remarkably close to the corresponding value for the ΛCDM model, namely $R_{dS} = 4\Lambda$. Further, as the condition (6.20) is satisfied, we can predict an oscillatory behavior of $y_H(z)$, which is shown in fact in Fig. 6.10. The value of ω_{DE} oscillates infinitely often around the line of phantom divide at $\omega_{DE} = -1$, as in Eq. (6.24), as it has been depicted in Fig. 6.3. In order to appreciate the constant frequency of ω_{DE} with respect to the logarithmic scale during the de Sitter phase, in Fig. 6.12 we plot the values of

$$\tilde{\omega_{DE}}(\log(z+1)) = \left[\omega_{DE} + 1 - \frac{12\tilde{m}^2}{R_{dS}}\left(\frac{1}{F'(R_{dS}) - R_{dS}F''(R_{dS})} - 1\right)(z+1)^3\right](z+1)^{-\frac{3}{2}}\,, \quad (6.67)$$

as a function of $\log(z+1)$, for $-0.999 < z < 0$ (here, $R_{dS} = 3.725\Lambda$). In this way, we stress the oscillating part of Eq. (6.24), whose frequency is proportional to $2\pi/\sqrt{4/(R_{dS}f''(R_{dS})) - 25/4} \simeq 1.570$. As last point, we should remark that the amplitude of ω_{DE} is amplified with respect to $y_H(z)$ by its frequency, and it decreases as $(z+1)^{3/2}$.

6.3.2 Matter oscillations

In the high curvature region matter is dominant ($z > 1.5$ with $\Omega_{DE}(z) \sim 0.1$) and $y_H \simeq \Lambda/3\tilde{m}^2$, as it is clear in Fig. 6.11. Eq. (6.56) yields an estimation of the dark energy density value around z_0. Owing to the fact that, in this region, $F''(R)$ is very close to 0^+, dark energy oscillates as

$$y_H(z) \simeq a + e^{\frac{\alpha(z-z_0)}{2(z_0+1)}}\left[A_0 \sin\left(\frac{\sqrt{\beta}}{(z_0+1)}(z-z_0)\right) + B_0 \cos\left(\frac{\sqrt{\beta}}{(z_0+1)}(z-z_0)\right)\right], \quad (6.68)$$

where A_0 and B_0 are constant, and

$$a \simeq \Lambda/3\tilde{m}^2\,, \quad (6.69)$$

$$\alpha = 3\,, \quad (6.70)$$

$$\beta = \frac{\Lambda}{6\tilde{m}^2[(z_0+1)^3 + 4\Lambda/3\tilde{m}^2]} e^{\frac{3\tilde{m}^2((z_0+1)^3+4\Lambda/3\tilde{m}^2)}{\Lambda}}\,. \quad (6.71)$$

Here we have used Eqs. (6.57)-(6.60). We must stress that, by using our formula, it turns out that $b \sim \exp(-R/\Lambda) \ll (y_0/R_0)^2$, which we have not been able to evaluate, its value being extremely close to zero. The frequency $\nu(z_0)$ of dark energy oscillations is

$$\nu(z_0) = \sqrt{\frac{\Lambda}{6\tilde{m}^2[(z_0+1)^3 + 4\Lambda/3\tilde{m}^2]} e^{\frac{3\tilde{m}^2((z_0+1)^3+4\Lambda/3\tilde{m}^2)}{\Lambda}}}\,\frac{1}{(z_0+1)}\,, \quad (6.72)$$

while the amplitude decreases as $\exp\left[3(z-z_0)/(2(z_0+1))\right]$.

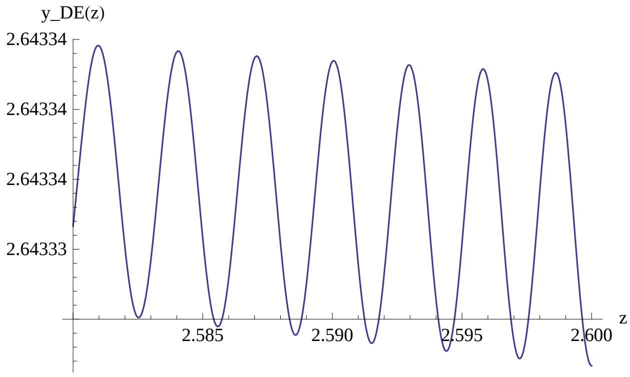

Figure 6.13: Plot of $y_H(z)$ in the vicinity of $z_0 = 2.60$.

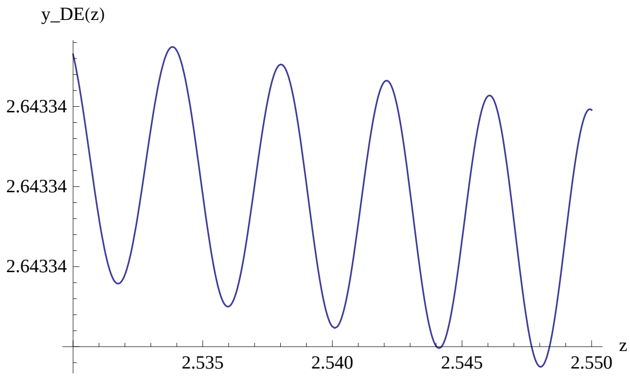

Figure 6.14: Plot of $y_H(z)$ in the vicinity of $z_0 = 2.55$.

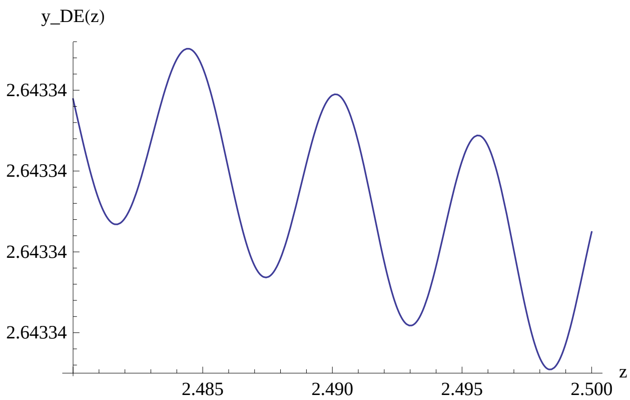

Figure 6.15: Plot of $y_H(z)$ in the vicinity of $z_0 = 2.50$.

We can verify the validity of our formula by analyzing in detail the graphics of $y_{DE}(z)$ in the vicinity of $z_0 = 2.60$, 2.55 and 2.50. Such graphics are shown in Figs. 6.13, 6.14 and 6.15. We have chosen an interval of $|z - z_0| = 0.02$. The period $T(z_0)$ of dark energy oscillations has to be $T(z_0) = 2\pi/\nu(z_0)$ and the number of crests in our interval is $n_{|z-z_0|}(z_0) = 0.02/T(z_0)$. The predicted values are

$$T(2.60) \simeq 0.003\,, \qquad n_{0.02}(2.60) \simeq 6.667\,;$$
$$T(2.55) \simeq 0.004\,, \qquad n_{0.02}(2.55) \simeq 5.000\,;$$
$$T(2.50) \simeq 0.006\,, \qquad n_{0.02}(2.50) \simeq 3.333\,.$$

These values are in good accordance with the numerical computation. The dark energy density is very close to $\Lambda/(3\tilde{m}^2) \simeq 2.64333$ and one can check that the amplitude of oscillation decreases with red shift z.

Since the dependence of the amplitude $\nu(z_0)$ on z_0 is weaker when $z_0 < 2.5$, we can study the oscillations of the dark energy density around $z_0 = 2.45$ and $z_0 = 2.40$ by using a longer interval $|z - z_0| = 0.05$. We find

$$T(2.45) \simeq 0.008\,, \qquad n_{0.05}(2.45) \simeq 6.250\,;$$
$$T(2.40) \simeq 0.010\,, \qquad n_{0.05}(2.40) \simeq 5.000\,.$$

These results can be compared with Figs. 6.16-6.17.

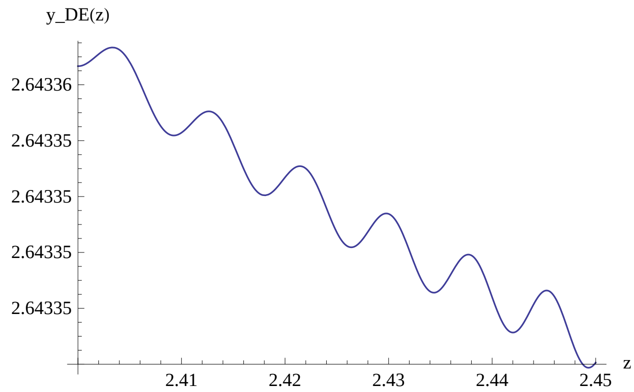

Figure 6.16: Plot of $y_H(z)$ in the vicinity of $z_0 = 2.45$.

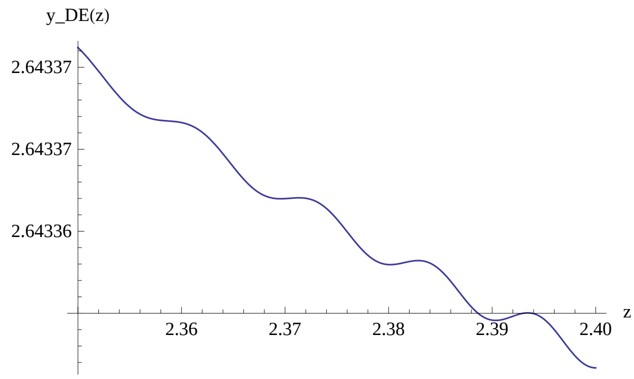

Figure 6.17: Plot of $y_H(z)$ in the vicinity of $z_0 = 2.40$.

When $z < 2.40$ the periods become too large with respect to the change of $\nu(z_0)$ and we can no more distinguish oscillations in the dark energy until the beginning of the de Sitter epoch, when the dark energy behavior is governed by Eq. (6.21). On the other hand, the effects of such oscillations are amplified in the expression (6.65) for ω_{DE}, where the amplitude of oscillations is proportional to the period. Fig. 6.18 shows the behavior of ω_{DE} inside the region $2.3 < z < 2.6$. Since the frequency of ω_{DE} is the same as for $y_{DE}(z)$, we can observe there how it decreases with red shift z.

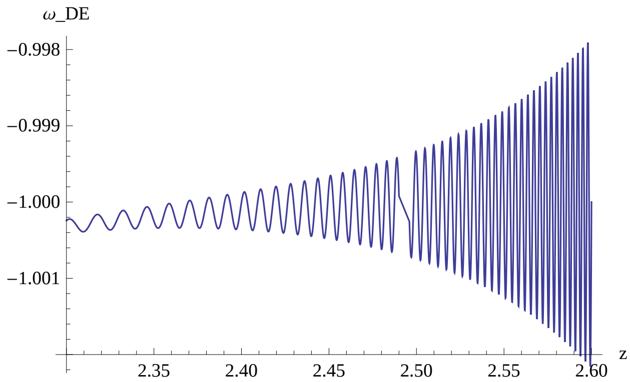

Figure 6.18: Plot of ω_{DE} as a function of $2.3 < z < 2.6$.

6.4 Oscillations and effective singularities

We have seen how exponential model could become indistinguishable with respect to the ΛCDM model. It mimics, in fact, the cosmological constant to high precision and ω_{DE} can be made extremely close to the value of -1. The transition crossing the phantom divide does not cause any serious problem to the accuracy of the cosmological evolution arising from such kind of models. We can see in Figs. 6.7-6.9 how Ricci scalar decreases with the red-shift, as one would expect, and the amplitude oscillations of the dark energy density are so small that one cannot observe them.

On the other hand, we have seen how, in a model of this kind, $F''(R \to \infty) \to 0^+$. As a consequence, the frequency of the dark energy oscillations increases with the redshift, as in Eq. (6.72). Furthermore, when $z \gg 3$, Eq. (6.45) predicts a strong divergence of the frequency.

Since the method we have used to analytically evaluate such oscillations yields a linear expansion of the EOM, we cannot observe any singularity in the time derivative of $y_H(z)$ and, correspondingly, in the Hubble parameter, but we have that

$$\left| \frac{d^n}{dt^n} H(t)|_{t_0} \right| \propto (\nu(z_0))^n \ , \tag{6.73}$$

where t_0 is the time at redshift z_0 and $\nu(z_o)$ is the frequency of dark energy at the same redshift. For extremely large values of the redshift, the high time derivatives of the Hubble parameter become infinitely large and approach an effective (IV Type) future-time singularity, showing a different behavior of the exponential model with respect to the ΛCDM one. This fact suggests that a correction of the Einstein's equations in the small curvature region must be related with a modification of gravity at high curvature, produced by the leading terms of inflation.

For example, if we consider the 'two step' exponential model of Eq. (5.23), it is easy to see that the frequency $\nu_0(z_0) \simeq \sqrt{1/F''(R_0)}/(z_0+1)$ reads

$$\nu(z_0) \simeq \sqrt{\frac{1}{\frac{2R_0}{\Lambda} e^{-\frac{R_0}{\Lambda}} + \alpha(\alpha-1)\left(\frac{R_0}{R_i}\right)^{\alpha-1}}} \, \frac{1}{(z_0+1)} \ , \tag{6.74}$$

where R_0 is the curvature at redshift z_0. Since $\alpha > 2$, while the exponential term becomes small on approaching the effective singularity at $R_0 \to \infty$, the power-like term becomes larger, and the frequency of dark energy oscillations does not diverge.

Chapter 7

BH and dS-solutions in a covariant renormalizable field theory of gravity

We conclude this work with an example of coviariant renormalizable gravity-model.

Attempts to quantize gravity have been mainly carried out by considering the perturbations of a flat, Lorentz invariant background and using the principles of General Relativity. By doing this, unavoidable, non-renormalizable divergences coming from the ultraviolet region in momentum space show up. To escape this severe difficulty, higher derivative theories have been invoked but there a new problem, namely the unitarity issue, appears (see for example Ref. [114]).

Recently, Hořava had the idea to directly modify the ultraviolet behavior of the graviton propagator in a Lorentz non-invariant way [115], as $1/|\mathbf{k}|^{2z}$, with $\mathbf{k}$ the spatial momenta and $z = 2, 3$ or higher. This exponent comes from the different (anisotropic) scaling properties of the space-time coordinates $(t, \mathbf{x})$, as $t \to b^z t$, $\mathbf{x} \to b\mathbf{x}$, b being a rescaling parameter. When $z = 3$, the theory appears to be UV power-counting renormalizable (which leads to conjecture renormalizability). To exhibit the Lorentz non-invariance, terms explicitly breaking Lorentz invariance (or more precisely, the full diffeomorphism invariance) are written down, by treating the temporal and the spatial coordinates differently. The Horawa model has diffeomorphism invariance with respect to the time coordinate t only, while for the spatial coordinates one has for the variations $\delta x^i = \zeta^i(t, \mathbf{x})$, $\delta t = f(t)$, with $\zeta^i(t, \mathbf{x})$ and $f(t)$ arbitrary functions of t and $\mathbf{x}$ and of t only, respectively.

In Ref. [116] a Hořava-like gravity model with full diffeomorphism invariance was proposed. There, when considering perturbations from a flat, Lorentz invariant background, the Lorentz invariance of the propagator was dynamically broken by a non-standard coupling with a perfect fluid. The propagator behaved as $1/|\mathbf{k}|^{2z}$ with $z = 2, 3$ or higher in the ultraviolet region and the model could be perturbatively power counting (super-)renormalizable, if $z \geq 3$. The price to pay for such covariant renormalizability was the presence of an unknown fluid, which might have a stringy origin but cannot correspond to a usual fluid, like radiation, baryons, dust, or the like. The model could be consistently constructed when the EoS parameter of fluid ω_{F} is $\omega_{\mathrm{F}} \neq -1\,, 1/3$. For usual particles in the high energy region, the corresponding fluid is relativistic radiation, for which $\omega_{\mathrm{F}} \to 1/3$. Actually, the non-relativistic fluid was needed even in the high energy region. Later, a dust fluid with $\omega_{\mathrm{F}} = 0$ was constructed for the scalar theory by introducing a Lagrange-multiplier field, which gives a constraint on the first scalar field [117, 118].

More recently, in Ref. [119], a fluid with arbitrary constant ω_{F} from a scalar field which satisfies a constraint has been constructed. Owing to the constraint, the scalar field is not dynamical and, even in the high energy region, a non-relativistic fluid could be obtained. Through coupling with the fluid, a full diffeomorphism invariant Lagrangian results (in fact a class of such gravitational Lagrangians), which is given completely in terms of field's variables. It has been demonstrated that such theory has all the good properties of the Lorentz non-invariant gravities, as in the previously mentioned theories (like its conjectured renormalizability [120]), while having the advantage of being at the same time a covariant theory. It was also conjectured there, that the spatiallyflat FRW cosmology for such covariant field gravity might exhibit accelerating solutions. In the present Chapter we will show that this is the case. We will consider the covariant renormalizable theory and we will demonstrate that Schwarzschild black hole and de Sitter solutions exist as exact solutions. This work has been developed in Ref. [18].

7.1 Black hole solutions in covariant (power-counting) renormalizable gravity.

To start, let us briefly review the covariant (power-counting) renormalizable gravity of Ref. [116]. It is described by the action

$$I = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left\{R - 2\Lambda - \alpha\left[\left(R^{ij} - \frac{\beta}{2}\,Rg^{ij}\right)\nabla_i\phi\nabla_j\phi\right]^n - \lambda\left(\frac{1}{2}\,g^{ij}\nabla_i\phi\nabla_j\phi + U_0\right)\right\}\,, \tag{7.1}$$

where ϕ is a cosmological scalar field, λ a Lagrangian multiplier, $\alpha, \beta, \Lambda, U_0$ are arbitrary constants and, finally, $n \geq 1$ is an arbitrary number.

Variation of the action with respect to λ gives the constraint

$$g^{ij}\nabla_i\phi\nabla_j\phi = -2U_0\,, \tag{7.2}$$

while the field equations for the scalar field read

$$\begin{aligned} 0 &= \nabla_i\left\{\left[2n\alpha\,F^{n-1}\left(R^{ij} - \frac{\beta}{2}\,Rg^{ij}\right) + \lambda\,g^{ij}\right]\nabla_j\phi\right\} \\ &= \frac{1}{\sqrt{-g}}\,\partial_i\left\{\left[2n\alpha\,F^{n-1}\left(R^{ij} - \frac{\beta}{2}\,Rg^{ij}\right) + \lambda\,g^{ij}\right]\sqrt{-g}\,\partial_j\phi\right\}\,, \end{aligned} \tag{7.3}$$

where, for convenience, we have putted

$$F = T_{ij}R^{ij} - \frac{\beta}{2}\,RT\,, \qquad T_{ij} = \nabla_i\phi\nabla_j\phi\,, \qquad T = g^{ij}T_{ij} = -2U_0\,. \tag{7.4}$$

The field equations related to the gravitational field have the form

$$\begin{aligned} G_{ij} + \Lambda g_{ij} + \frac{\alpha}{2}\,F^n\,g_{ij} &= n\alpha F^{n-1}\left[R_i^k T_{kj} + R_j^k T_{ki} - \frac{\beta}{2}\,(TR_{ij} + RT_{ij})\right] + \frac{\lambda}{2}\,T_{ij} \\ &\quad + n\alpha\left[D_{rsij}(T^{rs}F^{n-1}) - \frac{\beta}{2}\,D_{ij}(TF^{n-1})\right] + \Omega^{rs}\,\frac{\delta T_{rs}}{\delta g^{ij}}\,, \end{aligned} \tag{7.5}$$

where G_{ij} is the usual Einstein's tensor ($G_{ij} = R_{ij} - Rg_{ij}/2$), Ω_{rs} is a tensor which will play no role in the following, and we have introduced the differential operators

$$D_{ij} = g_{ij}\Box - \frac{1}{2}\,(\nabla_i\nabla_j + \nabla_j\nabla_i)\,, \tag{7.6}$$

$$\begin{aligned} D_{rsij} &= \frac{1}{4}\,[(g_{ir}g_{js} + g_{jr}g_{is})\Box + g_{ij}(\nabla_r\nabla_s + \nabla_s\nabla_r) \\ &\quad -(g_{ir}\nabla_s\nabla_j + g_{jr}\nabla_s\nabla_i + g_{is}\nabla_r\nabla_j + g_{js}\nabla_r\nabla_i)]\,. \end{aligned} \tag{7.7}$$

Note that the field equations (7.5) are valid for an arbitrary, symmetric 'energy-momentum' tensor T_{ij}, but in our particular case such a tensor does not depend on the metric and so the last term in Eq. (7.5), depending on Ω_{rs}, drops out. Now, we look for interesting physical solutions of the field equations above.

Schwarzschild solution

This is the simplest one and can be easily obtained for $\Lambda = 0$ and $n > 1$. In fact, in all such cases $R_{ij} = 0, \lambda = 0$ satisfy all field equations and solution in Eqs. (1.62)-(1.63) can be recovered. The scalar field ϕ has to fulfill the constraint (7.2) only.

Einstein-space solutions

These are generalizations of the previous solution. They have the form

$$R_{ij} = \frac{1}{4} R_0 \, g_{ij} \, . \tag{7.8}$$

Here, $R = R_0$ is a constant Ricci scalar. In such a case,

$$F = \left(\beta - \frac{1}{2}\right) R_0 U_0 \equiv F_0 \, , \tag{7.9}$$

where F_0 is a constant and, from Eq. (7.3) and Eq. (7.5), we get

$$g^{ij} \nabla_i \left[n\alpha \left(\frac{1}{2} - \beta \right) R_0 F_0^{n-1} + \lambda \right] \nabla_j \phi = 0 \, , \tag{7.10}$$

$$\left[\Lambda - \frac{R_0}{4} + \frac{\alpha}{2} \left(1 + \frac{n\beta}{1-2\beta} \right) F_0^n \right] g_{ij} = \frac{\lambda}{2} T_{ij} + n\alpha \, F^{n-1} \left(D_{rsij} \, T^{rs} + \frac{1-\beta}{2} \, R_0 T_{ij} \right) \, . \tag{7.11}$$

We see that non-trivial solutions effectively exist. For example, if λ and ϕ satisfy the equations

$$\lambda = n\alpha \left(\beta - \frac{1}{2} \right) R_0 F_0^{n-1} \, , \tag{7.12}$$

$$D_{rsij} \, T^{rs} + \frac{1}{4} \, R_0 T_{ij} = \Sigma g_{ij} \, , \tag{7.13}$$

Σ being a constant, the curvature can be derived from the algebraic equation

$$\frac{R_0}{4} - \Lambda + \alpha \left\{ n\Sigma + \frac{R_0 U_0}{4} \, [1 - (n+2)\beta] \right\} \left[\left(\beta - \frac{1}{2} \right) R_0 U_0 \right]^{n-1} = 0 \, . \tag{7.14}$$

Of course this is a solution if the Eqs. (7.12)-(7.13) are compatible with the constraint (7.2). In principle, more general solutions with non-constant Λ may exist too.

7.2 Cosmological applications

We shall now look for cosmological solutions and thus we start with a FRW metric of the type (1.5) and for the gage $N(t) = 1$, and a scalar field which depends on time only. Then, $\phi = \phi(t)$ is completely determined by the constraint (7.2) and, as a consequence, the tensor T_{ij} has only one non-vanishing component, namely

$$T_{00} \equiv \dot{\phi}^2 = 2U_0 \, , \tag{7.15}$$

where we have used the constraint (7.2).

Since all quantities depend on time only, Eq. (7.3) gives

$$\lambda - n\alpha\left[6\left((\beta-1)\dot{H}+(2\beta-1)H^2\right)\right]^n U_0^{n-1} = \frac{C_0}{a^3}\,, \tag{7.16}$$

H being, as usually, the Hubble parameter and C_0 an arbitrary integration constant. Moreover, due to the symmetry of the metric in field equations (7.5), only two equations are independent. It is clear that, by choosing $\beta = 1$, one has a simplification, namely

$$0 = \Lambda - 3H^2 + \frac{1}{2}\alpha\,(1-4n)(6U_0H^2)^n + U_0\lambda\,, \tag{7.17}$$

$$0 = \Lambda - 3H^2 - 2\dot{H} + \frac{1}{2}\alpha(1-2n)(6U_0H^2)^n + \frac{1}{3}\alpha n(1-2n)\dot{H}(6U_0)^n H^{2n-1}\,. \tag{7.18}$$

Now, in the latter equations, λ can be eliminated by means of Eq. (7.16), getting in this way the generalized Friedmann equations for the pure gravitational field. We have

$$0 = \Lambda - 3H^2 + \frac{1}{2}\alpha\,(1-2n)(6U_0H^2)^n - \frac{C_0}{a^3}\,, \tag{7.19}$$

$$0 = \Lambda - 3H^2 - 2\dot{H} + \frac{1}{2}\alpha(1-2n)(6U_0H^2)^n + \frac{1}{3}\alpha n(1-2n)\dot{H}(6U_0)^n H^{2n-1}\,. \tag{7.20}$$

One easily sees that, in order to get de Sitter solutions, one has to choose a vanishing integration constant, that is $C_0 = 0$. In this way the previous equations become equivalent and one obtains a constant Hubble parameter H_0, namely $H = H_0$, by solving

$$\frac{1}{2}\alpha\,(2n-1)(6U_0H_0^2)^n + 3H_0^2 - \Lambda = 0\,. \tag{7.21}$$

On the contrary, choosing $C_0 \neq 0$ one gets a second-order differential equation in the scale factor $a(t)$. A simple way to get such equation is to make use of the well known minisuperspace approach, which we have briefly described in § 1.4 referring to SSS solutions of $\mathcal{F}(R,G)$-gravity.

Recall we are dealing with the FRW space-time (1.5) with non constant $N(t)$ function, which describes the reparametrization invariance of the model. As a result, for β generic, one has

$$F = K^{ij}\partial_i\phi\partial_j\phi = \left(R^{ij} - \frac{\beta}{2}Rg^{ij}\right)\partial_i\phi\partial_j\phi = \frac{3(\dot{\phi})^2}{N^4}\left[(\dot{a})^2a^{-2} + (\beta-1)\left(\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{N}}{aN}\right)\right]. \tag{7.22}$$

Here, $N = N(t)$, $a = a(t)$ and $\phi = \phi(t)$ are functions of time t only. One can see the particular role played by the dimensionless parameter β. If one makes the choice $\beta = 1$, namely $K_{ij} = G_{ij}$, where G_{ij} is the Einstein's tensor, the dependence on the acceleration $\ddot{a}$ and $\dot{N}$ drops out. In fact, due precisely to the diffeomorphism invariance of the model, G_{00} is the Hamiltoniam constraint of GR and the modified gravitational fluid model becomes very simple, so that one has the following simplified minisuperspace action

$$I = \frac{V}{2\kappa^2}\int_{\mathcal{M}} dt\left[-6a(\dot{a})^2N^{-1} - 2\Lambda a^3N - \alpha 3^n N^{(1-4n)}(\dot{a})^{2n}a^{-2n+3}(\dot{\phi})^{2n} - \lambda a^3N\left(U_0 - \frac{(\dot{\phi})^2}{2N^2}\right)\right], \tag{7.23}$$

where V is the spatial-volume. In this case, one has two Lagrange multipliers λ and N, the first one implements the constraint

$$U_0 = \frac{(\dot{\phi})^2}{2N^2}\,, \tag{7.24}$$

while the second gives the Hamilonian constraint of our covariant model. After the variation, one has to take the gauge $N = 1$. The other two Lagrangian coordinates are ϕ and a, and one has the corresponding equations of motion. Let us continue with the equation of motion associated with N. On shell, one has

$$6H^2 - \alpha(1-4n)(6U_0)^n H^{2n} - 2\Lambda = 2\lambda U_0 \,. \tag{7.25}$$

On the other hand, since the Lagrangian does not depend on ϕ, the associated equation of motion reads

$$C_0 = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \,, \tag{7.26}$$

where $\mathscr{L}$ is the Lagrangian and C_0 is a constant of integration. On shell,

$$-2n\alpha(6U_0)^n H^{2n} + 2\lambda U_0 = \frac{C_0\sqrt{2U_0}}{a^3} \,. \tag{7.27}$$

Making use of the two last equations, we arrive at

$$6H^2 - \alpha(1-2n)(6U_0)^n H^{2n} - 2\Lambda = \frac{C_0\sqrt{2U_0}}{a^3} \,. \tag{7.28}$$

Finally, the last equation of motion is the one associated with a. It reads

$$(6H^2 - \alpha(1-2n)(6U_0)^n H^{2n} - 2\Lambda) = -\left(4 + \alpha\frac{2n}{3}(2n-1)(6U_0)^n H^{2n-2}\right)\dot{H} \,. \tag{7.29}$$

Making use of above equations, we also have

$$\frac{C_0\sqrt{2U_0}}{a^3} = -\left(\alpha\frac{2n}{3}(2n-1)(6U_0)^n H^{2n-2} + 4\right)\dot{H} \,. \tag{7.30}$$

Some remarks are here in order. The equations we have obtained are identical to the ones coming directly from the equations of motion. In particular, as in General Relativity, the equation of motion associated with a is not an independent one, since it can be obtained by taking the derivative with respect to t of the other equations and de Sitter solution, for which $\dot{H} = 0$ and $H = H_0$, where H_0 is a constant, corresponds to the choice $C_0 = 0$. In this case, Eq. (7.30) is satisfied, and we find Eq. (7.21).

With regard to the dS-solution of Eq. (7.21), one needs to look for positive H_0^2 solutions with $\alpha > 0$, a necessary condition in order to have a correct non linear graviton dispersion relation [116]. With regard to this issue, let us consider the simplest non trivial case, namely $n = 2$. One has as a solution

$$H_0^2 = \frac{-1 + \sqrt{1 + 24\alpha U_0^2 \Lambda}}{36\alpha U_0^2} \,. \tag{7.31}$$

Note that, for $\Lambda = 0$, the de Sitter solution exists only for $\alpha < 0$, which would correspond to an unusual dispersion relation for the graviton.

The stability of all de Sitter solutions is not difficult to study. In fact taking the first variation of Eq. (7.29) around $H = H_0$, one obtains

$$\delta\dot{H} = -3H_0\delta H \,. \tag{7.32}$$

As a consequence, all the de Sitter solutions are stable.

Let us investigate the case when C_0 is non-vanishing. In this case a de Sitter solution does not exist. Then, we may take $\Lambda = 0$. First, let us study the model with $n = 2$. In this case, with $\alpha > 0$, one has the differential equation from Eq. (7.30):

$$\frac{dH}{dt} = -\frac{3}{2}\frac{H^2 + 18\alpha U_0^2 H^4}{1 + 36\alpha U_0^2 H^2} \,. \tag{7.33}$$

Separating variables, one gets

$$\frac{1}{H} - 6U_0\sqrt{\frac{\alpha}{2}}\arctan\left(6U_0\sqrt{\frac{\alpha}{2}}H\right) = \frac{3}{2}t + B_0\,, \tag{7.34}$$

where B_0 is an integration constant. The solution is given in an implicit way only. However, even then it is easy to show that the model is protected against future-time singularities. In fact, let us look for solutions in which the Hubble parameter is as in Eq. (3.1), $H = h_0/(t_0 - t)^\beta$. When $\beta > 0$, $1/H$ tends to zero and the arctangent tends to a constant and the sign of the first leading term on the left hand side of Eq. (7.34) is inconsistent with the sign of the right hand side. Moreover, when $\beta < 0$, the left side of Eq. (7.34) diverges. As a consequence, no singular future solution can exist.

In the general case, we can investigate the possible presence of acceleration. In fact, with $\Lambda = 0$, one has

$$\frac{\dot{H}}{H^2} + 1 = \frac{1}{(2 + \alpha\frac{n}{3}(2n-1)(6U_0)^n H^{2n-2}}\left[-1 + \alpha(2n-1)(2n-3)6^{n-1}U_0^n H^{2n-2}\right]\,. \tag{7.35}$$

As a result, one may have acceleration as long as

$$H^{2n-2} > \frac{1}{\alpha(2n-1)(2n-3)6^{n-1}U_0^n}\,. \tag{7.36}$$

In particular, for $n = 2$ this condition becomes

$$H^2 > \frac{1}{18\alpha U_0^2}\,. \tag{7.37}$$

Coming back to the general model, it turns out that for $\beta \neq 1$ calculations are much more involved, since $\ddot{a}$ is present in the Lagrangian, and the model becomes a higher-derivative system in the sense of Ostrogradsky. However, we may carry out a direct calculation, which shows that a dS solution is not possible there.

7.3 Entropy calculation

It is of interest to evaluate the black hole entropy associated with the different solutions we have discussed. Since we are dealing with a covariant theory, we can make use of the Noether charge Wald methods, as in Chapter **2**. A direct evaluation of formula (2.82) yields (cf. with Ref. [112])

$$S_W = -2\pi\int_{\mathcal{M}}\frac{\partial\mathscr{L}}{\partial R_{ijrs}}\Big|_H\varepsilon_{ij}\varepsilon_{rs}d\Sigma = -\frac{1}{8G_N}\int_{\mathcal{M}}\left[\varepsilon_{ij}\varepsilon^{ij} - n\alpha F^{n-1}\frac{\partial F}{\partial R_{ijrs}}\varepsilon_{ij}\varepsilon_{rs}\right]\Big|_H d\Sigma\,. \tag{7.38}$$

The first term is the GR contribution, while the other one is due to the modification of GR in the considered model. However, in the case of the Schwarzschild solution one has $F = 0$ and Eq. (2.62) is found. As a consequence, in this modified gravity model, the entropy of the Schwarzschild black hole satisfies the usual Area Law ($S_W = \mathcal{A}_H/(4G_N)$).

Let us now consider the dS solution we have found for $\beta = 1$ and $n = 2$ in Eq. (7.31). The simplest way to perform the calculation is to make use of the static gauge, namely

$$ds^2 = -V(\rho)dt_s^2 + \frac{d\rho^2}{V(\rho)} + \rho^2 d\Omega^2\,, \tag{7.39}$$

being $V(\rho) = 1 - H_0^2\rho^2$ and $d\Omega = (d\theta^2 + \sin^2\theta d\phi^2)$. This static form of the dS metric can be obtained from the FRW by the coordinate transformation

$$\rho = re^{H_0 t}\,, \quad t_s = t - \frac{1}{2H_0}\ln V(\rho)\,. \tag{7.40}$$

The solution of Eq. (7.15) corresponding to the scalar fluid reads

$$\phi(t_s,\rho)=\sqrt{2U_0}\left[t_s+\frac{1}{2H_0}\ln V(\rho)\right]. \tag{7.41}$$

The relevant scalar quantity to be evaluated is

$$\frac{\partial F}{\partial R_{ijrs}}\varepsilon_{ij}\varepsilon_{rs}=-2U_0+\varepsilon_{ij}\varepsilon_{rs}\partial^i\phi\partial^r\phi g^{js}\,, \tag{7.42}$$

In general, the binormal tensor is given by $\varepsilon_{ij}=v_iu_j-v_ju_i$ and, in a static gauge, it is easy to show that one may choose $v_i=(\sqrt{V},0,0,0)$ and $u_i=\left(0,\frac{1}{\sqrt{V}},0,0\right)$. A direct calculation yields

$$\varepsilon_{ij}\varepsilon_{rs}\partial^i\phi\partial^r\phi g^{js}=2U_0\,. \tag{7.43}$$

Thus, the Area Law is also satisfied for the de Sitter solution we have found, confirming that, for $\beta=1$, we are dealing with a minimal modification of GR.

Appendix A

The Tunneling method

In this Appendix, we present a short review of the tunneling method in its Hamilton-Jacobi variant. The method is based on the computation of the classical action I along a trajectory starting slightly behind the trapping horizon but ending in the bulk, and the associated WKB approximation ($c = 1$)

$$\text{Amplitude} \propto e^{i\frac{I}{\hbar}}\,.$$

The related semi-classical emission rate Γ reads

$$\Gamma \propto |\text{Amplitude}|^2 \propto e^{-2\frac{\Im I}{\hbar}}\,.$$

The imaginary part of the classical action is due to deformation of the integration path according to the Feynman prescription, in order to avoid the divergence present on the horizon. As a result, one asymptotically gets a Boltzmann factor β, and an energy ω_K appears, i.e.,

$$\Gamma \propto e^{-\frac{\beta}{\hbar}\omega_K}\,.$$

The Hawking temperature T_K is identified as

$$T_K = \frac{1}{\beta}\,.$$

To evaluate the action I, let us start with a generic static, spherically symmetric solution in D-dimension, written in Eddington-Finkelstein gauge, which, as it is well known, is regular gauge on the horizon

$$ds^2 = -B(r)e^{2\alpha(r)}dv^2 + 2e^{\alpha(r)}dr\,dv + r^2 d\Omega^2_{D-2} = \gamma_{ij}(x^i)dx^i dx^j + r^2 d\Omega^2_{D-2}\,.$$

Here $x^i = (v, r)$, where v is the advanced time. Since we are dealing with static, spherically symmetric solution space-times, one may restrict to radial trajectories, and only the two-dimensional normal metric is relevant, and the Hamilton-Jacobi equation for a (massless) particle is

$$\gamma^{ij}\partial_i I \partial_j I = +2e^{\alpha(r)}\partial_v I \partial_r I + e^{2\alpha(r)}B(r)(\partial_r I)^2 = 0\,.$$

Thus

$$\partial_r I = \frac{2\omega_K}{e^{\alpha(r)}B(r)}\,,$$

in which $\omega_K = -\partial_v I$ is the Killing energy of the emitted particle. In the near horizon approximation, $B(r) \simeq B'(r_H)(r - r_H)$. As a consequence, making use of Feynman prescription for the simple pole in $r - r_H$, one has

$$I = \int dr \partial_r I = \int dr \frac{2\omega_K}{e^{\alpha(r)}B'(r_H)(r - r_H - i\varepsilon)}\,,$$

where the range of integration over r contains the location of the horizon r_H. Thus

$$\Im I = \frac{2\pi\omega_K}{e^{\alpha(r_H)}B'(r_H)}\,,$$

and the Hawking-Killing temperature is

$$T_K = \frac{e^{\alpha(r_H)}B'(r_H)}{4\pi}\,.$$

If one had introduced the Kodama energy $\omega_H = e^{-\alpha_H}\omega_K$, one would have obtained the Hayward temperature

$$T_H = \frac{B'_H}{4\pi}\,.$$

Appendix B

Energy conditions near the singularities

We briefly discuss the energy conditions maybe related with occurrence of singularities. We have four types of energy conditions:

- Weak energy condition (WEC): $\rho_{\text{eff}} \geqslant 0$ and $\rho_{\text{eff}} + p_{\text{eff}} \geqslant 0$;
- Strong energy condition (SEC): $\rho_{\text{eff}} + p_{\text{eff}} \geqslant 0$ and $\rho_{\text{eff}} + 3p_{\text{eff}} \geqslant 0$;
- Null energy condition (NEC): $\rho_{\text{eff}} + p_{\text{eff}} \geqslant 0$;
- Dominant energy condition (DEC): $\rho_{\text{eff}} \geqslant |p_{\text{eff}}|$.

On the singular solution $H = h/(t_0 - t)^\beta + H_0$, we have

$$\rho_{\text{eff}} + p_{\text{eff}} = -\frac{2}{\kappa^2}\frac{h\beta}{(t_0 - t)^{\beta+1}}\,,$$

where ρ_{eff} and p_{eff} are the effective energy density and pressure of the universe (deriving from modified gravity, fluids, scalar fields...).

The effective DE related with Type I and III singularities ($\beta > 0\,, \beta \neq 1$) violate the SEC and the NEC also, whereas DE related with Types II and III satisfy the NEC.

Note that

$$\rho_{\text{eff}} + 3p_{\text{eff}} = -\frac{6}{\kappa^2}\left(H_0^2 + 2\frac{hH_0}{(t_0 - t)^\beta} + \frac{h^2}{(t_0 - t)^{2\beta}} + \frac{h\beta}{(t_0 - t)^{\beta+1}}\right)\,.$$

The effective DE related with Type II singularities ($-1 < \beta < 0$) violate the SEC for small value of t. Only when t is close to t_0, the last term of this equation is dominant and the SEC is satisfied on the singular solution.

In the case of Type IV singularities ($\beta < -1$), when t is really close to t_0, the term H_0^2 could be dominant and the SEC is violated, expecially if $|\beta| \ll 1$.

At last, it is easy to see that, on the singular solutions, when t is near to t_0, the DEC is always violated except for large value of H_0 in the case of Type IV singularities, but also in this case the behaviour of universe approaching the singular solution violate the DEC. As a consequence, since ρ_{eff} has to be positive, the WEC always is satisfied on singular solutions.

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Bibliography

Introduction

[1] A. G. Riess *et al.* [SNST Collaboration], Astron. J. **116**, 1009 (1998) [arXiv:astro-ph/9805201]; S. Perlmutter *et al.* [SNCP Collaboration], Astrophys. J. **517**, 565 (1999) [arXiv:astro-ph/9812133]; P. Astier *et al.* [SNLS Collaboration], Astron. Astrophys. **447**, 31 (2006) [arXiv:astro-ph/0510447]; A. G. Riess *et al.*, Astrophys. J. **659**, 98 (2007), [arXiv:astro-ph/0611572].

[2] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003) [arXiv:astro-ph/0302209]; H. V. Peiris *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 213 (2003) [arXiv:astro-ph/0302225]; D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **170**, 377 (2007) [arXiv:astro-ph/0603449].

[3] E. Komatsu *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **180**, 330 (2009) [arXiv:0803.0547 [astro-ph]].

[4] T. Padmanabhan, AIP Conf. Proc. **861**, 179 (2006) [arXiv:astro-ph/0603114].

[5] Ya. B. Zeldovich, Soviet Physics Uspekhi **11**, 381- 393, (1968).

[6] L. Hollenstein, M. Jaccard, M. Maggiore and E. Mitsou, “Zero-point quantum fluctuations in cosmology”, arXiv:1111.5575 [astro-ph.CO].

[7] A.A.Starobinsky, Phys. Lett. B **91**, 99 (1980).

[8] S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002) [gr-qc/0201033]. S. Capozziello, S. Carloni and A. Troisi, Recent Res. Dev. Astron. Astrophys. **1**, 625 (2003) [astro-ph/0303041]; S. Capozziello, V. F. Cardone and A. Troisi, Phys. Rev. D **71**, 043503 (2005) [astro-ph/0501426].

[9] R. Dick, Gen. Rel. Grav. **36**, 217 (2004) [gr-qc/0307052].

[10] S.M.Carroll, V.Duvvuri, M.Trodden and M.S.Turner, Phys. Rev. D **70**, 043528 (2004) [astro-ph/0306438].

[11] S.Nojiri and S. D. Odintsov, Phys. Rev. **D68**, 123512 (2003) [hep-th/0307288].

[12] S. Capozziello, V. F. Cardone and A. Troisi, JCAP **0608**, 001 (2006) [astro-ph/0602349].

[13] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **77**, 046009 (2008) [arXiv:0712.4017 [hep-th]].

[14] K. Bamba, S. D. Odintsov, L. Sebastiani and S. Zerbini, Eur. Phys. J. C **67**, 295 (2010) [arXiv:0911.4390 [hep-th]].

[15] O. Gorbunova and L. Sebastiani, Gen. Rel. Grav. **42**, 2873 (2010) [arXiv:1004.1505 [gr-qc]].

[16] L. Sebastiani, Eur. Phys. J. C **69**, 547 (2010) [arXiv:1006.1610 [gr-qc]].

[17] E. Bellini, R. Di Criscienzo, L. Sebastiani and S. Zerbini, Entropy **12**, 2186 (2010) [arXiv:1009.4816 [gr-qc]].

[18] G. Cognola, E. Elizalde, L. Sebastiani and S. Zerbini, Phys. Rev. D **83**, 063003 (2011) [arXiv:1007.4676 [hep-th]].

[19] E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **83**, 086006 (2011) [arXiv:1012.2280 [hep-th]].

[20] L. Sebastiani and S. Zerbini, Eur. Phys. J. C **71**, 1591 (2011) [arXiv:1012.5230 [gr-qc]].

[21] G. Cognola, O. Gorbunova, L. Sebastiani and S. Zerbini, Phys. Rev. D **84**, 023515 (2011) [arXiv:1104.2814 [gr-qc]].

[22] E. Elizalde, S. D. Odintsov, L. Sebastiani and S. Zerbini, "Oscillations of the F(R) dark energy in the accelerating universe", to appear in Eur. Phys. J. C, arXiv:1108.6184 [gr-qc].

[23] G. Cognola, L. Sebastiani and S. Zerbini, "Stability in Generalized Modified Gravity", To appear in the proceedings of 12th Marcel Grossmann Meeting on General Relativity (MG 12), Paris, France, 12-18 Jul 2009, arXiv:1006.1586 [gr-qc].

[24] L. Sebastiani, "Finite-time singularities in modified $\mathcal{F}(R,G)$-gravity and singularity avoidance", A workshop with a celebration of Emilio Elizalde's sixtieth birthday, ICE/CSIC, Universitat Autònoma de Barcelona, 8-10 March 2010, arXiv:1008.3041 [gr-qc].

Chapter 1

[25] S.Nojiri and S. D. Odintsov, eConfC **0602061**, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)] [hep-th/0601213]; S. Nojiri and S. D. Odintsov, Phys. Rept. **505** 59 , (2011), [arXiv:1011.0544 [gr-qc]].

[26] S. Capozziello and M. Francaviglia, Gen. Rel. Grav. **40**, 357 (2008) [arXiv:0706.1146 [astro-ph]].

[27] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010) [arXiv:0805.1726 [gr-qc]].

[28] S. Nojiri and S. D. Odintsov, Phys. Lett. B **631**, 1 (2005) [arXiv:hep-th/0508049]. S. Nojiri, S. D. Odintsov and O. G. Gorbunova, J. Phys. A **39**, 6627 (2006) [arXiv:hep-th/0510183]; G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. D **73**, 084007 (2006) [arXiv:hep-th/0601008]; S. Nojiri, S. D. Odintsov and M. Sami, Phys. Rev. D **74**, 046004 (2006) [arXiv:hep-th/0605039]; S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. **66**, 012005 (2007) [arXiv:hep-th/0611071]; G. Cognola, E. Elizalde, S. Nojiri, S. Odintsov and S. Zerbini, Phys. Rev. D **75**, 086002 (2007) [arXiv:hep-th/0611198]; B. Li, J. D. Barrow and D. F. Mota, Phys. Rev. D **76**, 044027 (2007) [arXiv:0705.3795 [gr-qc]]; M. Gurses, Gen. Rel. Grav. **40**, 1825 (2008) A. De Felice and S. Tsujikawa, Phys. Lett. B **675**, 1 (2009) [arXiv:0810.5712 [hep-th]]; M. Alimohammadi and A. Ghalee, Phys. Rev. D **79**, 063006 (2009) [arXiv:0811.1286 [gr-qc]]; C. G. Boehmer and F. S. N. Lobo, Phys. Rev. D **79**, 067504 (2009) [arXiv:0902.2982 [gr-qc]]; K. Uddin, J. E. Lidsey and R. Tavakol, Gen. Rel. Grav. **41**, 2725 (2009) [arXiv:0903.0270 [gr-qc]]; S. Y. Zhou, E. J. Copeland and P. M. Saffin, JCAP **0907**, 009 (2009) [arXiv:0903.4610 [gr-qc]]. A. De Felice and T. Suyama, JCAP **0906**, 034 (2009) [arXiv:0904.2092 [astro-ph.CO]]; N. Goheer, R. Goswami, P. K. S. Dunsby and K. Ananda, Phys. Rev. D **79**, 121301 (2009) [arXiv:0904.2559 [gr-qc]]; J. Sadeghi, M. R. Setare and A. Banijamali, Phys. Lett. B **679**, 302 (2009) [arXiv:0905.1468 [hep-th]]; JSadeghi, M RSetare and ABanijamali, Eur. Phys. J. C **64**, 433 (2009) [arXiv:0906.0713 [hep-th]]; A. De Felice and S. Tsujikawa, Phys. Rev. D **80**, 063516 (2009) [arXiv:0907.1830 [hep-th]]; M. Alimohammadi and A. Ghalee, Phys. Rev. D **80**, 043006 (2009) [arXiv:0908.1150

[gr-qc]]; M. Mohseni, Phys. Lett. B **682**, 89 (2009) [arXiv:0911.2754 [hep-th]]; K. Bamba, C. - Q. Geng, S. Nojiri and S. D. Odintsov, Europhys. Lett. **89**, 50003 (2010) [arXiv:0909.4397 [hep-th]].

[29] S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden and M. S. Turner, Phys. Rev. D **71**, 063513 (2005) [astro-ph/0410031].

[30] G. Cognola, M. Gastaldi and S. Zerbini, Int. J. Theor. Phys. **47**, 898 (2008) [arXiv:gr-qc/0701138].

[31] G. Cognola and S. Zerbini, Int. J. Theor. Phys. **47**, 3186 (2008) [arXiv:0802.3967 [hep-th]].

[32] A. Vilenkin, Phys. Rev. **D 32** 2511 (1985).

[33] S. Capozziello, Int. J. Mod. Phys. **D 11** 4483 (2002).

[34] C. Brans and R. H. Dicke, Phys. Rev. 124:925 (1961).

[35] S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, Phys. Lett. B **639**, 135 (2006) [arXiv:astro-ph/0604431].

[36] S. Capozziello, E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Lett. B **671**, 193 (2009) [arXiv:0809.1535 [hep-th]].

Chapter 2

[37] M. Visser, Phys. Rev. D **48**, 5697 (1993).

[38] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003) [hep-th/0212292]; Phys. Rev. D **75**, 084032 (2007) [arXiv:gr-qc/0701140].

[39] G. Abreu and M. Visser,*Phys. Rev. Lett.* **105**, 041302 (2010).

[40] R. G. Cai, L. M. Cao, Y. P. Hu and N. Ohta, Phys. Rev. D **80**, 104016 (2009) [arXiv:0910.2387 [hep-th]].

[41] R.M. Wald, Phys. Rev. D **48**, 3427 (1993).

[42] S. W. Hawking, Nature **248** 30 (1974); Commun. Math. Phys. **43** 199-220 (1975).

[43] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 103005 (2005) [gr-qc/0509059].

[44] S. Deser, O. Sarioglu and B. Tekin, Gen. Rel. Grav. **40**, 1 (2008) [arXiv:0705.1669 [gr-qc]].

[45] T. Multamaki and I. Vilja, Phys. Rev. D **74**, 064022 (2006) [astro-ph/0606373]; Phys. Rev. D **76**, 064021 (2007) [astro-ph/0612775].

[46] S. Capozziello, A. Stabile and A. Troisi, Class. Quant. Grav. **25**, 085004 (2008) [arXiv:0709.0891 [gr-qc]].

[47] T. R. P. Carames, E. R. Bezerra de Mello, Eur. Phys. J. **C64**, 113-121 (2009) [arXiv:0901.0814 [gr-qc]].

[48] R. Saffari and S. Rahvar, Phys. Rev. D **77**, 104028 (2008) [arXiv:0708.1482 [astro-ph]]; Mod. Phys. Lett. A **24** 305 (2009) [arXiv:0710.5635 [astro-ph]].

[49] S. A. Hayward, Class. Quant. Grav. **15**, 3147 (1998).

[50] R. Di Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini and G. Zoccatelli, Phys. Lett. B **657** (2007) 107 [arXiv:0707.4425 [hep-th]]; R. Di Criscienzo and L. Vanzo, Europhys. Lett. **82**, 60001 (2008) [arXiv:0803.0435 [hep-th]].

[51] S. A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini and S. Zerbini, Class. Quant. Grav. **26**, 062001 (2009) [arXiv:0806.0014 [gr-qc]]; Class. Quant. Grav. **27** 015006 (2010).

[52] C. W. Misner and D. H. Sharp Phys. Rev. **136** B571 (1964).

[53] H. Kodama, Prog. Theor. Phys. **63**, 1217 (1980).

[54] M. Visser, Int. J. Mod. Phys. D **12**, 649 (2003) [hep-th/0106111].

[55] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).

[56] M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini, JHEP **0505**, 014 (2005) [arXiv:hep-th/0503081]. M. Nadalini, L. Vanzo and S. Zerbini, J. Phys. A **39**, 6601 (2006) [arXiv:hep-th/0511250].

[57] R. Brustein, M. Hadad, Phys. Rev. Lett. **103**, 101301 (2009) [arXiv:0903.0823 [hep-th]].

[58] E. Elizalde, P. J. Silva, Phys. Rev. D **78** , 061501 (2008) [arXiv:0804.3721 [hep-th]].

[59] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995) [gr-qc/9504004].

[60] T. Padmanabhan, Class. Quant. Grav. **19** , 5387 (2002) [gr-qc/0204019]; Phys. Rept. **406** , 49 (2005) [gr-qc/0311036]; Rept. Prog. Phys. **73** , 046901 (2010) [arXiv:0911.5004 [gr-qc]].

[61] D. Lovelock, J. Math. Phys. **12** , 498-501 (1971).

[62] J. T. Wheeler, Nucl. Phys. B **268**, 737 (1986).

[63] D. G. Boulware, S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).

[64] R. C. Myers, J. Z. Simon, Phys. Rev. D **38**, 2434 (1988).

[65] A. Paranjape, S. Sarkar and T. Padmanabhan, Phys. Rev. D **74**, 104015 (2006).

[66] H. Maeda, M. Nozawa, Phys. Rev. D **77**, 064031 (2008) [arXiv:0709.1199 [hep-th]]; H. Maeda, S. Willison and S. Ray, Class. Quant. Grav. **28**, 165005 (2011) [arXiv:1103.4184 [gr-qc]].

[67] R. G. Cai, Phys. Lett. **B582** , 237 (2004) [hep-th/0311240].

[68] V. Faraoni, Entropy **12**, 1246 (2010) [arXiv:1005.2327 [gr-qc]].

[69] G. Kofinas, R. Olea, JHEP **0711**, 069 (2007) [arXiv:0708.0782 [hep-th]].

[70] R. G. Cai, K. S. Soh, Phys. Rev. D **59** , 044013 (1999) [gr-qc/9808067]; R. G. Cai, Phys. Rev. D **65** , 084014 (2002) [hep-th/0109133].

[71] M. Akbar, R. -G. Cai, Phys. Lett. B **648**, 243 (2007) [gr-qc/0612089].

[72] L. Vanzo, Phys. Rev. D **56**, 6475 (1997) [gr-qc/9705004].

[73] D. R. Brill, J. Louko, P. Peldan, Phys. Rev. D **56**, 3600 (1997) [gr-qc/9705012].

[74] R. B. Mann, Class. Quant. Grav. **14**, 2927 (1997) [gr-qc/9705007].

[75] R. J. Riegert, Phys. Rev. Lett. **53**, 315-318 (1984).

[76] P. D. Mannheim and D. Kazanas, Astrophys. J. **342**, 635-638 (1989).

[77] D. Klemm, Class. Quant. Grav. **15**, 3195-3201 (1998) [gr-qc/9808051].

[78] P. D. Mannheim, Astrophys. J. **479**, 659 (1997) [astro-ph/9605085]; Prog. Part. Nucl. Phys. **56**, 340-445 (2006) [astro-ph/0505266].

Chapter 3

[79] K. Bamba, S. Nojiri and S. D. Odintsov, JCAP **0810**, 045 (2008) [arXiv:0807.2575 [hep-th]].

[80] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005) [arXiv:hep-th/0501025].

[81] B. McInnes, JHEP **0208** 029 (2002) [arXiv:hep-th/0112066]; V. Faraoni, Int. J. Mod. Phys. D **11**, 471 (2002) [arXiv:astro-ph/0110067]; R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003) [arXiv:astro-ph/0302506]; S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003) [arXiv:hep-th/0303117]; E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004) [arXiv:hep-th/0405034]; P. F. Gonzalez-Diaz, Phys. Lett. B **586**, 1 (2004) [arXiv:astro-ph/0312579]; S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70**, 123529 (2004) [arXiv:astro-ph/0410309]; M. Sami and A. Toporensky, Mod. Phys. Lett. A **19**, 1509 (2004) [arXiv:gr-qc/0312009]; H. Stefancic, Phys. Lett. B **586**, 5 (2004) [arXiv:astro-ph/0310904]; L. P. Chimento and R. Lazkoz, Mod. Phys. Lett. A **19**, 2479 (2004) [arXiv:gr-qc/0405020]; C. Csaki, N. Kaloper and J. Terning, Annals Phys. **317**, 410 (2005) [arXiv:astro-ph/0409596]; P. X. Wu and H. W. Yu, Nucl. Phys. B **727**, 355 (2005) [arXiv:astro-ph/0407424]; E. Babichev, V. Dokuchaev and Yu. Eroshenko, Class. Quant. Grav. **22**, 143 (2005) [arXiv:astro-ph/0407190]; I. Y. Aref'eva, A. S. Koshelev and S. Y. Vernov, Phys. Rev. D **72**, 064017 (2005) [arXiv:astro-ph/0507067]; X. F. Zhang, H. Li, Y. S. Piao and X. M. Zhang, Mod. Phys. Lett. A **21**, 231 (2006) [arXiv:astro-ph/0501652]; M. P. Dabrowski and T. Stachowiak, Annals Phys. **321**, 771 (2006) [arXiv:hep-th/0411199]; E. M. Barboza and N. A. Lemos, Gen. Rel. Grav. **38**, 1609 (2006) [arXiv:gr-qc/0606084].

[82] J. D. Barrow, Phys. Lett. B **235**, 40 (1990);

[83] A. V. Frolov, Phys. Rev. Lett. **101**, 061103 (2008) [arXiv:0803.2500 [astro-ph]].

[84] E. V. Arbuzova and A. D. Dolgov, Phys. Lett. B **700** 289 (2011) [arXiv:1012.1963 [astro-ph.CO]].

[85] A. A. Starobinsky, JETP Lett. **86**, 157 (2007) [arXiv:0706.2041 [astro-ph]].

[86] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007) [arXiv:0705.1158 [astro-ph]].

[87] T.Kobayashi and K. Maeda, Phys.Rev. D **78**, 064019 (2008) [arXiv:0807.2503v2 [astro-ph]].

[88] S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, Phys. Lett. B **639**, 135 (2006) [arXiv:astro-ph/0604431]; S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006) [arXiv:hep-th/0608008]; S. Nojiri and S. D. Odintsov, J. Phys. A **40**, 6725 (2007) [arXiv:hep-th/0610164]; S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. **66**, 012005 (2007) [arXiv:hep-th/0611071].

[89] S. Nojiri, S. D. Odintsov and P. V. Tretyakov, Prog. Theor. Phys. Suppl. **172**, 81 (2008). [arXiv:0710.5232 [hep-th]].

[90] M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **22**, L35 (2005) [arXiv:hep-th/0409177].

[91] S. Nojiri and S. D. Odintsov, Phys. Rev. D **78**, 046006 (2008) [arXiv:0804.3519 [hep-th]].

[92] S.Noijri and S.D.Odintsov, Phys.Lett.**B** 595 (2004) [arXiv:0405078v3 [hep-th]]; I. H. Brevik and O. Gorbunova, Eur. Phys. J. C **56** 425 (2008) [arXiv:0806.1399 [gr-qc]].

Chapter 4

[93] S. Nojiri and S. D. Odintsov, ECONF C0602061:06 (2006)[arXiv:hep-th/0601213]; S. Capozziello, V. F. Cardone, E.Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **73**:043512 (2006) [astro-ph/0508350].

[94] S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005) [hep-th/0505215];

[95] I. H. Brevik, O. Gorbunova, Gen.Rel.Grav. **37** 2039-2045 (2005) [arXiv: 0508038v1 [gr-qc]].

[96] I. H. Brevik, O. Gorbunova and Y. A. Shaido, Int. J. Mod. Phys. D **14** 1899 (2005) [arXiv:gr-qc/0508038].

[97] V. F. Cardone, C. Tortora, A. Troisi and S. Capozziello, Phys. Rev. D **73**, 043508 (2006) [astro-ph/0511528].

[98] A. Yu. Kamenshchik, U.Moschella, V. Pasquier, Phys.Lett. **B** 511:265-268 (2001) [arXiv:0103004 [gr-qc]].

[99] S.Nojiri and S.D.Odintsov, Phys.Lett.B **686** 44-48 (2010) [arXiv:0911.2781 [hep-th]].

Chapter 5

[100] S. Capozziello and V. Faraoni, “Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics”, Springer, Berlin (2010).

[101] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Phys. Rev. D **75**, 083504 (2007) [gr-qc/0612180].

[102] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573**, 1 (2003) [arXiv:astro-ph/0307285]; V. Faraoni, Phys. Rev. D **74** (2006) 104017 [arXiv:astro-ph/0610734].

[103] P. H. Frampton, K. J. Ludwick and R. J. Scherrer, Phys. Rev. D **84**, 063003 (2011) [arXiv:1106.4996 [astro-ph.CO]].

[104] I. Brevik, E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **84**, 103508 (2011) [arXiv:1107.4642 [hep-th]].

[105] T. Chiba, T. L. Smith and A. L. Erickcek, Phys. Rev. D **75**, 124014 (2007) [arXiv:astro-ph/0611867].

[106] S. Nojiri and S. D. Odintsov, Phys. Lett. B **652**, 343 (2007) [arXiv:0706.1378 [hep-th]].

[107] J. Khoury, B. Ovrut, P. Steinhardt and N. Turok, Phys. Rev. **D64**, 123522 (2001); Phys. Rev. **D66**, 046005 (2002).

[108] S. A. Appleby and R. A. Battye, Phys. Lett. B **654**, 7 (2007) [arXiv:0705.3199 [astro-ph]].

[109] E. V. Linder, Phys. Rev. D **80**, 123528 (2009) [arXiv:0905.2962 [astro-ph.CO]].

[110] K. Bamba, C. Q. Geng and C. C. Lee, JCAP **1008**, 021 (2010) [arXiv:1005.4574 [astro-ph.CO]].

Chapter 6

[111] V. Faraoni, Phys. Rev. D **72**, 124005 (2005), arXiv:gr-qc/0511094.

[112] G. Cognola, E. Elizalde, S.Nojiri, S. D. Odintsov and S. Zerbini, JCAP **0502**, 010 (2005), hep-th/0501096.

[113] H. Motohashi, A. A. Starobinsky and J. Yokoyama, JCAP **1106**, 006 (2011) [arXiv:1101.0744 [astro-ph.CO]].

Chapter 7

[114] I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, IOP Publishing (1992).

[115] P. Horava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].

[116] S. Nojiri and S. D. Odintsov, Phys. Rev. D **81**, 043001 (2010) [arXiv:0905.4213 [hep-th]].

[117] E. A. Lim, I. Sawicki and A. Vikman, JCAP **1005**, 012 (2010) [arXiv:1003.5751 [astro-ph.CO]].

[118] S. Capozziello, J. Matsumoto, S. Nojiri and S. D. Odintsov, Phys. Lett. B **693**, 198 (2010) [arXiv:1004.3691 [hep-th]].

[119] S. Nojiri and S. D. Odintsov, Phys. Lett. B **691**, 60 (2010) [arXiv:1004.3613 [hep-th]].

[120] S. Carloni, M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu, Phys. Rev. D **82**, 065020 (2010) [arXiv:1003.3925 [hep-th]].