UNIVERSITÀ DEGLI STUDI DI TRENTO

Dipartimento di Matematica

DOTTORATO DI RICERCA IN MATEMATICA

XXIV CICLO

A thesis submitted for the degree of Doctor of Philosophy

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Mixed quasi-étale surfaces and new surfaces of general type

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Introduction

In this thesis we define and study the *mixed quasi-étale surfaces*. In particular we classify all the mixed quasi-étale surfaces whose minimal resolution of the singularities is a regular surface with $p_g = 0$ and $K^2 > 0$.

It is a well known fact that each Riemann surface with $p_g = 0$ is isomorphic to $\mathbb{P}^1$. At the end of XIX century M. Noether conjectured that an analogous statement holds for the surfaces: in modern words, he conjectured that every smooth projective surface with $p_g = q = 0$ be rational.
The first counterexample to this conjecture is due to F. Enriques (1869). He constructed the so called Enriques surfaces (see [Enr96]).

The Enriques-Kodaira classification divides compact complex surfaces in four main classes according to their Kodaira dimension κ: $-\infty$, 0, 1, 2. A surface is said to be of *general type* if $\kappa = 2$. Nowadays this class is much less understood than the other three. The Enriques surfaces have $\kappa = 0$.

The first examples of surfaces of general type with $p_g = 0$ have been constructed in the 30's by L. Campedelli e L. Godeaux.

The idea of Godeaux to construct surfaces was to consider the quotient of simpler surfaces by the free action of a finite group. In this spirit, Beauville (see [Bea96, Page 118]) proposed a simple construction of surfaces of general type, considering the quotient of a product of two curves C_1 and C_2 by the free action of a finite group G. Moreover he gave an explicit example with $p_g = q = 0$ considering the quotient of two Fermat curves of degree 5 in $\mathbb{P}^2$.

There is no hope at the moment to achieve a classification of the whole class of the surfaces of general type. Since for a surface in this class the Euler characteristic of the structure sheaf χ is strictly positive, one could hope that a classification of the boundary case $\chi = 1$ is more affordable.

Some progresses in this direction have been done in the last years through the work of many authors, but this (a priori small) case has proved to be very challenging, and we are still very far from a classification of it. At the same time, this class of surfaces, and in particular the subclass of the surfaces with $p_g = 0$ contains some of the most interesting surfaces of general type, see [BCP11] for more details.

If S is a surface of general type with $\chi = 1$, which means $p_g = q$, then by Beauville ([Bea82]), $p_g = q \leq 4$, and if $p_g = q = 4$, then S is birational to the product of curves of genus 2. The case $p_g = q = 3$ has been studied in

[CCML98], [Pir02] and [HP02] and the surfaces in this class are completely classified. The cases $p_g = q \leq 2$ are still far from being classified.

Generalizing the Beauville example, Catanese considers the quotient $(C_1 \times C_2)/G$, where the C_i are Riemann surfaces of genus at least two, and G is a finite group. Following [Cat00], there are two cases: the *mixed case* where the action of G exchanges the two factors (and then $C_1 \cong C_2$); and the *unmixed case* where G acts diagonally.

After [Cat00] many authors studied the surfaces birational to a quotient of a product of two curves, mainly in the case of surfaces of general type with $\chi = 1$. We refer to [BC04], [BCG08], [BCGP08] and [BP10] for the case $p_g = q = 0$, to [CP09], [Pol08],[Pol09] and [MP10] for the case $p_g = q = 1$ and to [Pen11] for the case $p_g = q = 2$. In all these works the authors work either in the unmixed case or in the mixed case under the assumption that the group acts freely.

The main purpose of this thesis is to extend the results and the strategies of the above mentioned papers in the non free mixed case. Let C be a Riemann surface of genus $g \geq 2$, let G be a finite group that acts on $C \times C$ with a mixed action, i.e. there exists an element in G that exchanges the two factors. Let $G^0 \triangleleft G$ be the index two subgroup of the elements that do not exchange the factors. We say that $X = (C \times C)/G$ is a *mixed quasi-étale surface* if the *quotient map* $C \times C \to (C \times C)/G$ has finite branch locus.

We present an algorithm to construct regular surfaces as the minimal resolution of the singularities of mixed quasi-étale surfaces. We give a complete classification of the regular surfaces with $p_g = 0$ and $K^2 > 0$ that arise in this way. Moreover we show a way to compute the fundamental group of these surfaces and we apply it to the surfaces we construct; we follow the idea in [BCGP08] (see also [DP10]) for the unmixed case, and we adapt it to the mixed case.

The main theorem of the thesis is the following:

Theorem. *Let S be the minimal resolution of the singularities of a mixed quasi-étale surface X with $p_g(S) = q(S) = 0$ and $K_S^2 > 0$, then*

1. *S is minimal and of general type.*
2. *S belongs to one of the 17 families collected in Table 1.*

In the first column of Table 1 we report the value K_S^2 of the self-intersection of the canonical class of the surface, $\mathrm{Sing}(X)$ represents the singularities of X (see Definition 5.1.12 for the notation we use). The column Type gives the type of the set of spherical generators of G^0 (see Section 2.3) in a compacted way, e.g. $2^3, 4 = (2, 2, 2, 4)$. The columns G and G^0 give the group and its index two subgroup. The groups denoted by $G(a, b)$ are groups of order a, while b is the MAGMA identifier of the group. The column $b_2(X)$, $H_1(S, \mathbb{Z})$, and $\pi_1(S)$ give respectively the second Betti number of X, the first homology group and the fundamental group of S.

K_S^2	Sing(X)	Type	G^0	G	$b_2(X)$	$H_1(S,\mathbb{Z})$	$\pi_1(S)$	Label
1	$2\,C_{2,1} + 2\,D_{2,1}$	$2^3\,,4$	$D_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$	7.3.1
2	$6\,C_{2,1}$	2^5	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	2	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	7.3.2
2	$6\,C_{2,1}$	4^3	$(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	G(64, 82)	2	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	7.3.3
2	$C_{2,1} + 2\,D_{2,1}$	$2^3\,,4$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$	7.3.4
2	$C_{2,1} + 2\,D_{2,1}$	$2^2\,,3^2$	$\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$	7.3.5
2	$2\,C_{4,1} + 3\,C_{2,1}$	$2^3\,,4$	G(64, 73)	G(128, 1535)	3	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	7.3.6
2	$2\,C_{3,1} + 2\,C_{3,2}$	$3^2\,,4$	G(384, 4)	G(768, 1083540)	2	$\mathbb{Z}_4$	$\mathbb{Z}_4$	7.3.7
2	$2\,C_{3,1} + 2\,C_{3,2}$	$3^2\,,4$	G(384, 4)	G(768, 1083541)	2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	7.3.8
3	$C_{8,3} + C_{8,5}$	$2^3\,,8$	G(32, 39)	G(64, 42)	2	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	7.3.9
4	$4\,C_{2,1}$	2^5	$D_4 \times \mathbb{Z}_2$	$D_{2,8,5} \rtimes \mathbb{Z}_2$	2	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_8$	7.3.10
4	$4\,C_{2,1}$	2^5	$\mathbb{Z}_2^4$	$(\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$	2	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	K-N	7.3.11
4	$4\,C_{2,1}$	4^3	G(64, 23)	G(128, 836)	2	$\mathbb{Z}_2^3$	$\mathbb{Z}_4^2 \rtimes \mathbb{Z}_2$	7.3.12
8	$\emptyset$	2^5	$D_4 \times \mathbb{Z}_2^2$	$(D_{2,8,5} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$	2	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$	$1 \to \Pi_{17} \times \Pi_{17} \to \pi_1 \to G \to 1$	7.3.13
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	$\mathbb{Z}_4^3$	$1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$	7.3.14
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	$1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$	7.3.15
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	$1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$	7.3.16
8	$\emptyset$	4^3	G(128, 36)	G(256, 3679)	2	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	$1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$	7.3.17

Table 1: The surfaces and their fundamental group. See Section 7.3 for a detailed description.

The last column gives a label, referring to a subsection of Section 7.3, where we give more details on each construction.

Some of our construction are more interesting than others. We would like to point out the surfaces 7.3.4 and 7.3.7: these are numerical Campedelli surfaces ($K_S^2 = 2$) with topological fundamental group (and therefore algebraic fundamental group) $\mathbb{Z}_4$, we discuss the role of these surfaces in the classification of the numerical Campedelli surfaces in Section 3.6.1. Moreover, according to [BCP11], two of our constructions realize surfaces whose topological type was not present in the literature before. These surfaces are tagged by 7.3.10 and 7.3.12. We also note that the surfaces 7.3.1, 7.3.4 and 7.3.5 are $\mathbb{Q}$-homology projective planes in sense of [HK11].

The thesis is divided in seven chapters. The new results are contained in the last three chapters, whereas the first four chapters collect known results from the literature, which we used. More precisely we have organized the thesis as follows.

- In Chapter 1, we recall some standard definitions and properties about covering spaces and lifts, in particular we recall how the fundamental group $\pi_1(X, x)$ acts on the fibre $p^{-1}(x)$ of a covering space $p\colon \tilde{X} \to X$.

  We give and prove the theorem of existence of covering spaces. Finally we discuss the monodromy of a covering space.

- In Chapter 2, we recall the basic properties of branched and Galois coverings; here we give the definition of quasi-étale morphism. We recall some classical results about Riemann surfaces as the Hurwitz's formula and the Riemann existence theorem.

  In Section 2.3 we explain how to associate an algebraic datum, an *appropriate orbifold homomorphism*, to any Galois covering $c\colon C \to C/H$. In Section 2.4 we give the inverse construction, obtaining a Galois covering $c\colon C \to C/H$ from any appropriate orbifold homomorphism. Theorem 2.4.3 shifts our geometric classification problem into an algebraic problem.

  Finally, in Section 2.5, we consider a Riemann surface C (and a finite subgroup H of $\mathrm{Aut}(C)$). We extend the action of $\pi_1(C)$ on the universal cover of C to the action of a bigger group, an *orbifold surface group*. We will use it later for computing the fundamental group of the surfaces that we construct.

- In Chapter 3 we recall some standard definitions and classical properties of smooth complex surfaces.

  In the Sections 3.5 and 3.6 we explain the Enriques-Kodaira classification of compact complex surfaces and we focus on the surfaces of general type. In particular, in the last part of the chapter we present the actual knowledge about the classification of the surfaces of general type with $\chi = 1$.

- In the fourth chapter we consider group actions on product of curves. Following [Cat00] the action can be of two types: *mixed* or *unmixed*.

  In Section 4.2, we give the definition of cyclic quotient singularity (type $C_{n,a}$) and we give their resolution graphs (in terms of the continued fraction of $\frac{n}{a}$).

  In Section 4.3 we give the definition of *product quotient surfaces*, i.e. the surfaces S that are minimal resolution of the singularities of a surface $X := (C_1 \times C_2)/G$ where G acts with an unmixed action. We recall the properties of these surfaces (in particular the formulae for their numerical invariant).

  In section 4.4 we introduce the *mixed surfaces*, and the *mixed quasi-étale surfaces.*

  In the last section of this chapter we summarize the actual knowledge about the classification of the surfaces with $\chi = 1$ that are birational to a quotient of product of curves.

- Chapter 5 is dedicated to investigate the mixed quasi-étale surfaces, their singularities and the numerical invariants of the minimal resolution of their singularities.

  Let $X = (C \times C)/G$ be a mixed surface, let G^0 be the index two subgroup of the elements that do not exchange the factors. We denote by Y the surface $(C \times C)/G^0$ and by π the natural map $Y \to X$.
  We start translating the quasi-étale condition in algebraic terms, by showing (Theorem 5.0.12) that a mixed surface is mixed quasi-étale if and only if the exact sequence

  $$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

  does not split.
  We show that for a mixed q.e. surface it holds $\mathrm{Sing}(X) = \pi(\mathrm{Sing}(Y))$.

  Then the singular points of X are naturally divided in two subsets, according if they are branch points of π or not, and the second set of points is a set of cyclic quotient singularities. In Section 5.1.1 we investigate the singular points of X that are also branch points of π, introducing what we call singularities of *type* $D_{n,a}$.

  Let S be the minimal resolution of the singularities of a mixed qe surface $X = (C \times C)/G$. Following the ideas of the unmixed case, in Section 5.2 we relate the numerical invariants e and K^2 of S with the genus of C, the order of G and $\mathrm{Sing}(X)$. In Section 5.2.1 we prove some inequalities relating the invariants of S with the possible *signatures* of the orbifold surface groups which are domain of the appropriate orbifold homomorphisms involved.

- In Chapter 6 we develop an algorithm to classify all the smooth regular surfaces with fixed values of the invariants K^2 and p_g which are minimal resolution of the singularities of a mixed quasi-étale surface. As byproduct we get the second part of the main theorem of the thesis (see Theorem 6.1.1).

  In Section 6.1 we provide the theoretical background of the algorithm, in particular giving explicit bounds for the algebraic data depending on the invariants of the surface (necessary for the finiteness of the algorithm) and explaining how to read the singularities of the mixed quasi-étale surfaces from the algebraic data.

  In Section 6.2 we explain the strategy of the algorithm, that we have implemented in MAGMA. Running the script in the case $p_g = 0$ and $K^2 > 0$ we get the surfaces in Table 1. The algorithm needs to "skip" few cases: in Section 6.3 we prove the second part of the main theorem excluding these cases.

  Finally, in Section 6.4 we report the MAGMA script.

- In the last chapter we show a method to compute the fundamental group of a smooth regular surface birational to a mixed q.e. surface and we apply it to the surfaces we construct.

  In Section 7.2 we determine the minimal model of the constructed surfaces, proving that they are all minimal, so completing the proof of the main theorem.

  In the last section we report a detailed description of all the regular surfaces in Table 1.

Contents

Chapter 1

Covering spaces

In this first chapter, we recall some definitions and properties related to covering spaces. In particular we give the basic definitions and the lifting properties, for further details we refer to [Hat02, Section 1.3] and [Mas02, Chapter 5]. We will give and prove the theorem of existence of covering spaces. Finally we will discuss the monodromy of a covering space.

If not different stated, we shall assume that all spaces are path-connected and locally path-connected.

1.1 Generalities on covering spaces

Definition 1.1.1. Let X be a topological space. A *covering space* (or *étale-covering*) of X is a pair consisting of a topological space $\tilde{X}$ and a continuous map $p\colon \tilde{X} \to X$ such that the following condition holds: each point $x \in X$ has a path-connected open neighborhood U such that each component of $p^{-1}(U)$ is mapped homeomorphically onto U by p. Any open neighborhood U that satisfies this condition is called an *elementary neighborhood.*

Remark 1.1.2. For every $x \in X$ the topology induced by the topology of $\tilde{X}$ on the fiber $p^{-1}(x)$ is the discrete topology.

Definition 1.1.3. Let X be a topological space and let G be a group that acts on X. If for all $g \in G$ the map $\theta_g\colon x \mapsto g \cdot x$ is continuous then X is called *G-space.*

Remark 1.1.4. If X is a G-space, then θ_g is an homeomorphism for each $g \in G$.

Definition 1.1.5. Let X be a G-space. The action of G on X is *discontinuous* if:

(i) the stabilizer of each point is finite;

(ii) each point x of X has a neighborhood U such that any element of G not in the stabilizer of x maps U outside itself.

Moreover, the action of G on X is *properly discontinuous* if the stabilizer of each point is trivial.

Proposition 1.1.6. *Let X be an Hausdorff G-space, with G finite and that acts freely on X, then the action of G on X is properly discontinuous.*

Proof. Since the action is free, the stabilizer of each point is trivial.

Let $1_G, g_1, \ldots, g_k$ be the elements of G and let $x \in X$. The points $y_i = g_i \cdot x$ are all distinct since the action is free. X is an Hausdorff space and so there exist a neighborhood U_0 of x and neighborhoods V_i of y_i such that $U_0 \cap V_i = \emptyset$ for $i = 1, \ldots, k$. Let $U_i = \theta_{g_i}^{-1}(V_i)$ for $i = 1, \ldots, k$. The U_i are open neighborhoods of x and so also $U = \bigcap_i U_i$ is an open neighborhood of x. We claim that U has the required property: $U \cap \theta_{g_j}(U) = \emptyset$ for each j. Since $U \subseteq U_j = \theta_{g_j}^{-1}(V_j)$ we get $\theta_{g_j}(U) \subseteq V_j$, while $U \subseteq U_0$. We conclude remembering that $U_0 \cap V_j = \emptyset$. □

Proposition 1.1.7. *Let X be a G-space; if the action of G is properly discontinuous, then $p\colon X \to X/G$ is a covering space.*

Proof. We start showing that the map $p\colon X \to X/G$ is open. Let V an open subset of X, then

$$\begin{aligned} p^{-1}(p(V)) &= \{x \in X \mid p(x) \in p(V)\} = \{x \in X \mid p(x) = p(y),\ y \in V\} \\ &= \{x \in X \mid x = \theta_g(y),\ y \in V\} = \{x \in X \mid x \in \theta_g(V)\} \\ &= \bigcup_{g \in G} \theta_g(V) \end{aligned}$$

hence $p^{-1}(p(V))$ is open in X, and by definition of quotient topology, $p(V)$ is open in X/G.

Let U be an open neighborhood of a point x that satisfies the condition (ii) of Definition 1.1.5 (the stabilizer is trivial), hence $p^{-1}(p(U)) = \cup \theta_g(U)$ is a disjoint union of open subsets. The restriction of p on one of these open subsets is continuous, open and bijective and so it is an homeomorphism. □

If $p\colon \tilde{X} \to X$ is a covering space, then the cardinality of the fiber $p^{-1}(x)$ is locally constant over X. Since we are assuming X connected this cardinality is constant over X, it is called the *number of sheets* or *degree* of the covering. If the number of sheets is finite, we say that the covering is *finite*.

Definition 1.1.8. Let $p\colon \tilde{X} \to X$ be a covering space, a *lift* of a map $f\colon Y \to X$ is a map $\tilde{f}\colon Y \to \tilde{X}$ such that $p\tilde{f} = f$.

We now collect some results concerning uniqueness and existence of lifts.

Lemma 1.1.9 (Uniqueness of the lift)**.** *Let $p\colon \tilde{X} \to X$ be a covering space and let Y be a connected space. Given any two continuous maps $\tilde{f}_0, \tilde{f}_1\colon Y \to \tilde{X}$ such that $p\tilde{f}_0 = p\tilde{f}_1$ the set $W = \{y \in Y : \tilde{f}_0(y) = \tilde{f}_1(y)\}$ is either empty or all of Y.*

Proof. Since Y is connected it suffices to show that W is both open and closed. Let $y \in Y$, and let U be an elementary neighborhood of $x = p\tilde{f}_0(y) = p\tilde{f}_1(y)$. By definition $p^{-1}(U) = \sqcup V_j$, assume V_0 and V_1 are the components of $p^{-1}(U)$ which contain $\tilde{f}_0(y)$ and $\tilde{f}_1(y)$ respectively. By continuity there exists a neighborhood Z of y such that $\tilde{f}_i(Z) \subseteq V_i$, $i = 0, 1$.

If $y \notin W$, then $V_0 \cap V_1 = \emptyset$ and Z is a neighborhood of y in W^c and so W is closed. If $y \in W$, then $V_0 = V_1$; since $p\tilde{f}_0(y) = p\tilde{f}_1(y)$ and that p is an homeomorphism on V_0, hence injective, we get that $\tilde{f}_0 = \tilde{f}_1$ on Z, and so W is open. □

Definition 1.1.10. A *path* in X is a continuous map f from $I := [0,1]$ to X.

If α and β are two paths in X such that $\alpha(1) = \beta(0)$, we can define the composition path as follows:

$$(\alpha\beta)(t) := \begin{cases} \alpha(2t) & \text{if } t \in [0, 1/2] \\ \beta(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

A path $\alpha\colon I \to X$ is called *loop* if $\alpha(0) = \alpha(1)$.
The *inverse path* of α is the path $\overline{\alpha}\colon I \to X$ defined by $\overline{\alpha}(t) := \alpha(1-t)$.

Lemma 1.1.11 (Lifting paths)**.** *Let $p\colon \tilde{X} \to X$ be a covering space. Let $\gamma\colon I \to X$ be a path with starting point x_0, for any $\tilde{x}_0 \in p^{-1}(x_0)$ there exists a unique lift $\tilde{\gamma}\colon I \to \tilde{X}$ with starting point $\tilde{x}_0$.*

Proof. Let $\{U_j\}_{j\in J}$ be a open cover of X by elementary neighborhoods; then $\{\gamma^{-1}(U_j)\}$ is an open cover of the compact space I, so it is possible to find a finite sequence of points $0 = t_0, t_1, \ldots, t_k = 1$ such that for each k there exists $j_k \in J$ such that $\gamma([t_k, t_{k+1}]) \subset U_{j_k}$.

We construct the lift by induction on $[0, t_k]$. For $k = 0$ we set $\tilde{\gamma}(0) = \tilde{x}_0$. Now suppose to have defined $\tilde{\gamma}_k\colon [0, t_k] \to \tilde{X}$ with $\tilde{\gamma}_k(0) = \tilde{x}_0$ and that this lift is unique. By construction $\gamma([t_k, t_{k+1}]) \subset U_{i_k}$ and $p^{-1}(U_{i_k})$ is the disjoint union of some open subsets $W_j \subset \tilde{X}$ homeomorphic to U_{i_k} via p. Among these open subsets, let W be the one that contains $\tilde{\gamma}_k(t_k)$; we define $\tilde{\gamma}_{k+1}$ as follows:

$$(\tilde{\gamma}_{k+1})(t) := \begin{cases} \tilde{\gamma}_{k+1}(t) & \text{if } t \in [0, t_k] \\ (p_{|W})^{-1}(\gamma(t)) & \text{if } t \in [t_k, t_{k+1}] \end{cases}$$

It follows immediately that $\tilde{\gamma}_{k+1}$ is continuous, the uniqueness follows by Lemma 1.1.9.

□

Using the same strategy of Lemma 1.1.11 it is possible to prove the following statement:

Lemma 1.1.12 ([Mas02, Lemma V.3.3]). *Let $p\colon \tilde{X} \to X$ be a covering space and let $\gamma_0, \gamma_1 \colon I \to \tilde{X}$ be paths in $\tilde{X}$ which have the same starting point. If $p\gamma_0$ and $p\gamma_1$ are homotopic, then γ_0 and γ_1 are homotopic; in particular, γ_0 and γ_1 have the same end point.*

As corollary of Lemma 1.1.12, we have the following theorem:

Theorem 1.1.13. *Let $p\colon \tilde{X} \to X$ be a covering space, let $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0)$. Then, the induced homomorphism*

$$\begin{array}{rcl} p_*\colon \pi_1(\tilde{X}, x_0) & \longrightarrow & \pi_1(X, x_0) \\ p_*[\gamma] & = & [p\gamma] \end{array}$$

is a monomorphism.

Proof. It is obvious that p_* is a homomorphism. Let $[\gamma] \in \pi_1(\tilde{X}, x_0)$ such that $p_*[\gamma] = [c]$, with c the constant path of base point x_0, so $p \circ \gamma$ and c are homotopic. γ is the lift of $p \circ \gamma$ of base point $\tilde{x}_0$ and the constant path $\tilde{c}$ with base $\tilde{x}_0$ is the unique lift of c of base point $\tilde{x}_0$, hence they are homotopic by Lemma 1.1.12. Hence $[\gamma] = [\tilde{c}]$ and so p_* is injective. □

Proposition 1.1.14. *Let $p\colon \tilde{X} \to X$ be a covering space, let $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0)$. The number of sheets of the covering equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.*

Proof. For a loop g in X based at x_0, let $\tilde{g}$ be its unique lift based at $\tilde{x}_0$. A product $h \cdot g$ with $[h] \in H := p_*\pi_1(\tilde{X}, \tilde{x}_0)$ lifts to $\widetilde{(h \cdot g)} = \tilde{h} \cdot \tilde{g}$ ending at the same point as $\tilde{g}$ since $\tilde{h}$ is a loop based at $\tilde{x}_0$. Thus we may define a function ν from the cosets $H[g]$ to $p^{-1}(x_0)$ by sending $H[g]$ to $\tilde{g}(1)$. It is well defined and the path-connectedness of $\tilde{X}$ implies that ν is surjective, since $\tilde{x}_0$ can be jointed to any point in $p^{-1}(x_0)$ by a path $\tilde{g}$ projecting to a loop g based at x_0. To see that ν is injective, we observe that $\nu(H[g_1]) = \nu(H[g_2])$ implies that $g_1\overline{g_2}$ lifts to a loop in $\tilde{X}$ based at $\tilde{x}_0$ so $[g_1][g_2]^{-1} \in H$ an hence $H[g_1] = H[g_2]$. □

Theorem 1.1.15 ([Mas02, Lemma V.4.2]). *Let $p\colon \tilde{X} \to X$ be a covering space and let $x_0 \in X$. Then, the subgroups $p_*\pi_1(\tilde{X}, \tilde{x})$ for $\tilde{x} \in p^{-1}(x_0)$ are exactly a conjugacy class of subgroups of $\pi_1(X, x_0)$.*

Theorem 1.1.16 (Existence of lifts, [Hat02, Proposition 1.33]).
Let Y be a connected and locally path-connected space. Let $p\colon \tilde{X} \to X$ be a covering space and let $f\colon Y \to X$ be a continuous map. Let $y_0 \in Y$, $x_0 = f(y_0)$ and $\tilde{x}_0 \in p^{-1}(x_0)$. There exists a unique lift $\tilde{f}$ of f such that $\tilde{f}(y_0) = \tilde{x}_0$ if and only if

$$f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)\,.$$

Definition 1.1.17. An *isomorphism* between covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ is a homeomorphism $\phi : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2\phi$. In particular, the isomorphisms from the covering $p : \tilde{X} \to X$ to itself are said *deck transformations* or *automorphisms* of covering spaces, they form a group that is denoted by $A(\tilde{X}, p)$.

Obviously $A(\tilde{X}, p)$ acts on the left on $\tilde{X}$. We have that this action has no fixed points, indeed:

Lemma 1.1.18. *Let $\varphi \in A(\tilde{X}, p)$. If $\varphi \neq 1$ then $\varphi(q) \neq q$ for each $q \in \tilde{X}$.*

Proof. By contradiction, assume that $\varphi(q) = q$ for some $q \in \tilde{X}$. Applying Theorem 1.1.9, we have that the unique lift of p with $\varphi(q) = q$ is the identity and so $\varphi = 1$. □

Using Lemma 1.1.9 and Theorem 1.1.16, we have immediately

Proposition 1.1.19. *Two covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic via an isomorphism $\phi : \tilde{X}_1 \to \tilde{X}_2$ taking $\tilde{x}_1 \in p_1^{-1}(x_0)$ to $\tilde{x}_2 \in p_2^{-1}(x_0)$ if and only if $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.*

A consequence of Theorem 1.1.16 is that a simply-connected covering space of a space X is also a covering space of every other covering space of X. A simply connected covering space of X is called a *universal cover*. By Proposition 1.1.19 it is unique up to isomorphism, so we can call it `the` universal cover.

1.1.1 The action of the group $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$

We now define an action of the group $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$ for any $x_0 \in X$; i.e., we make $\pi_1(X, x_0)$ operating on the left on the set $p^{-1}(x_0)$.

Let $p : \tilde{X} \to X$ be a covering space and let γ be a path in X. By Lemma 1.1.11, there exists a unique lift $\tilde{\gamma}$ of $\overline{\gamma}$, the inverse path of γ, starting at a given point of $p^{-1}(\gamma(1))$. In this way we get a well-defined map

$$L_\gamma : p^{-1}(\gamma(1)) \longrightarrow p^{-1}(\gamma(0)) \tag{1.1}$$

by sending the starting point $\tilde{\gamma}(0)$ of each lift $\tilde{\gamma}$ to its ending point $\tilde{\gamma}(1)$.

Remark 1.1.20. The reason for taking a lift of $\overline{\gamma}$ and not of γ is that in this way we have that $L_{\gamma\eta} = L_\gamma L_\eta$, otherwise we have $L_{\gamma\eta} = L_\eta L_\gamma$.

By Lemma 1.1.12, L_γ depends only on the homotopy class of γ, this means that if we restrict to loops base at $x_0 \in X$, then the association $\gamma \mapsto L_\gamma$ gives a homomorphism from $\pi_1(X, x_0)$ to the group of permutation of $p^{-1}(x_0)$. By Remark 1.1.20, we get a left action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

Lemma 1.1.21. *The action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ is transitive.*

Proof. Let $\tilde{x}_1$ and $\tilde{x}_2$ be points in $p^{-1}(x_0)$, since $\tilde{X}$ is path-connected, there exists a path $\gamma\colon I \to \tilde{X}$ such that $\gamma(0) = \tilde{x}_1$ and $\gamma(1) = \tilde{x}_2$. The path $\eta = \overline{p \circ \gamma}\colon I \to X$ is a loop based at x_0. Since $\overline{\gamma}$ is the unique lift of η with starting point $\tilde{x}_2$, we have

$$L_\eta(\tilde{x}_1) = \tilde{x}_2\,.$$

□

Lemma 1.1.22. *The stabilizer of $\tilde{x} \in p^{-1}(x_0)$ for the $\pi_1(X, x_0)$ action is the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}))$.*

Proof. The stabilizer of $\tilde{x}$ is the subgroup of $\pi_1(X, x_0)$ given by the classes $[\alpha]$ such that $L_\alpha(\tilde{x}) = \tilde{x}$, in other words, the classes whose lift is a loop based at $\tilde{x}$. So, if $[\alpha]$ belongs to the stabilizer, then it is the image of a loop based at $\tilde{x}$ and so $[\alpha] \in p_*\pi_1(\tilde{X}, \tilde{x})$.

Conversely, let $[\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}))$, then $[\gamma] = p_*[\tilde{\gamma}]$ with $\tilde{\gamma}$ loop of base point $\tilde{x}$, hence

$$L_\gamma(\tilde{x}) = \tilde{x}$$

hence the stabilizer of $\tilde{x}$ is $p_*(\pi_1(\tilde{X}, \tilde{x}))$. □

The following statement shows the connection between the group $A(\tilde{X}, p)$ of automorphism of a covering space and the action of $\pi_1(X, x_0)$ on $p^{-1}(x)$.

Proposition 1.1.23. *For any $\varphi \in A(\tilde{X}, p)$, any $\gamma \in \pi_1(X, x)$ and any $\tilde{x} \in p^{-1}(x)$, it holds:*

$$\varphi(L_\gamma(\tilde{x})) = L_\gamma(\varphi(\tilde{x}))\,.$$

Proof. Let α be the unique lift of $\overline{\gamma}$ in $\tilde{X}$ with base point $\tilde{x}$, then $L_\gamma(\tilde{x})$ is the end point of α. Let consider the path $\varphi_*(\alpha)$ in $\tilde{X}$; its starting point is $\varphi(\tilde{x})$ and its end point is $\varphi(L_\gamma(\tilde{x}))$. We observe that

$$p_*(\varphi_*(\alpha)) = (p\varphi)_*(\alpha) = p_*(\alpha) = \overline{\gamma}\,,$$

that is $\varphi_*(\alpha)$ is the lift of $\overline{\gamma}$ with base point $\varphi(\tilde{x})$, hence $L_\gamma(\varphi(\tilde{x}))$ is the end point of $\varphi_*(\alpha)$ that is $\varphi(L_\gamma(\tilde{x}))$. □

1.1.2 Regular covering spaces and quotient spaces

Definition 1.1.24. Let $p\colon \tilde{X} \to X$ be a covering space and let $\tilde{x} \in p^{-1}(x)$. If $p_*\pi_1(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$, the covering is called *regular*.

For a regular covering space $p\colon \tilde{X} \to X$, it holds the following nice description of $A(\tilde{X}, p)$:

Lemma 1.1.25 ([Hat02, Proposition 1.39])**.** *Let $p\colon \tilde{X} \to X$ be a regular covering, then $A(\tilde{X}, p)$ is isomorphic to the quotient group $\pi_1(X, x)/p_*(\pi_1(\tilde{X}, \tilde{x}))$ for any $x \in X$ and any $\tilde{x} \in p^{-1}(x)$.*

By Theorem 1.1.15 and Proposition 1.1.19 we get

Lemma 1.1.26. *Let $p\colon \tilde{X} \to X$ be a covering space. The automorphism group $A(\tilde{X}, p)$ operates transitively on $p^{-1}(x)$, $x \in X$, if and only if the covering is regular.*

As consequence we have the following:

Proposition 1.1.27 (see [Mas02, Section 5.8])**.** *Let $p\colon \tilde{X} \to X$ be a regular covering space, then X is homeomorphic to $\tilde{X}/A(\tilde{X}, p)$.*

Conversely

Theorem 1.1.28 ([Hat02, Proposition 1.40])**.** *Let Y be a connected, locally path-connected and let G be a group of homeomorphisms that acts properly discontinuous.*

Then $p\colon Y \to Y/G$ is a regular covering and $A(Y, p) \cong G$.

Corollary 1.1.29. *In the same assumptions of Theorem 1.1.28, we have the following short exact sequence:*

$$1 \longrightarrow \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(Y/G, p(y_0)) \longrightarrow G \longrightarrow 1\,.$$

Proof. By Theorem 1.1.13 p_* is injective, while by Lemma 1.1.25 and Theorem 1.1.28 we have $\pi_1(X, x)/p_*(\pi_1(\tilde{X}, \tilde{x})) \cong A(\tilde{X}, p) \cong G$. □

1.2 Existence Theorem of covering spaces

Every covering space $p\colon \tilde{X} \to X$ induces a subgroup $p_*\pi_1(\tilde{X}, \tilde{x})$ of $\pi_1(X, p(\tilde{x}))$ for any point $\tilde{x} \in \tilde{X}$.

In this section we want to investigate the "inverse" problem, that is: given a subgroup $K \subseteq \pi_1(X, x_0)$, is there a covering space $p\colon X_K \to X$ such that $p_*\pi_1(X_K, \tilde{x}) = K$ for a suitable choice of the base point $\tilde{x} \in X_K$?

Definition 1.2.1. A topological space X is *semilocally simply connected* if any point $x \in X$ has a neighborhood U_x such that every loop in U_x is homotopic in X to the constant path.

The following statement gives a positive answer to our question.

Theorem 1.2.2 ([Hat02, Proposition 1.36])**.** *Let X be a topological space which is path-connected, locally path-connected, and semilocally simply connected. Then, for every subgroup of $K \subseteq \pi_1(X, x_0)$, there exists a covering space $p\colon X_K \to X$ such that $p_*(\pi_1(X_K, \tilde{x})) \cong K$ for a suitable choice of the base point $\tilde{x} \in p^{-1}(x_0)$.*

Proof. The proof is divided in two steps: the first step is to show how construct an universal cover $\tilde{X}$ of X; the second step explains how to construct X_K from $\tilde{X}$.
Step 1: we start defining $\tilde{X}$ as set:

$$\tilde{X} := \{[\gamma] \mid \gamma \text{ is a path in } X \text{ s.t. } \gamma(0) = x_0\},$$

where, as usual, $[\gamma]$ denotes the homotopy class of γ.
The function $p\colon \tilde{X} \to X$ sending $[\gamma]$ to $\gamma(1)$ is well defined and surjective since X is path-connected.

In order to give a covering space, we have to define a topology on $\tilde{X}$. We make a few preliminary observations. Let $\mathcal{U}$ be the collection of path-connected open sets $U \subseteq X$ such that $\pi_1(U) \hookrightarrow \pi_1(X)$ is trivial. Note that if the map $\pi_1(U) \hookrightarrow \pi_1(X)$ is trivial for one choice of base point in U, it is trivial for all choices of base point since U is path-connected. A path-connected open subset $V \subseteq U \subseteq \mathcal{U}$ is also in $\mathcal{U}$ since the composition $\pi_1(V) \hookrightarrow \pi_1(U) \hookrightarrow \pi_1(X)$ will also be trivial. It follows that $\mathcal{U}$ is a basis for the topology on X if X is locally path-connected and semilocally simply-connected.

Given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U let

$$U_{[\gamma]} := \{[\gamma\eta] \mid \eta \text{ is a path in } U \text{ s.t. } \eta(0) = \gamma(1)\}.$$

We note that $U_{[\gamma]}$ depends only on the homotopy class $[\gamma]$. We also observe that the restriction of p to $U_{[\gamma]}$ is surjective since U is path-connected and injective since different choices of η joining $\gamma(1)$ to a fixed $u \in U$ are all homotopic in X.

If $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$, indeed if $\gamma' = \gamma\eta$ then elements of $U_{[\gamma']}$ have the form $[\gamma\eta\nu]$ and hence lie in $U_{[\gamma]}$, while elements in $U_{[\gamma]}$ have the form $[\gamma\nu] = [\gamma\eta\overline{\eta}\nu] = [\gamma'\overline{\eta}\nu]$ and hence lie in $U_{[\gamma']}$.

This property can be used to show that the sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$. Let $U_{[\gamma]}$ and $V_{[\gamma']}$ be two sets and let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, we have $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. So if $W \in \mathcal{U}$ is contained in $U \cap V$ and contains $\gamma''(1)$ then $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$.

The bijection $U_{[\gamma]} \to U$ given by the restriction of p is a homeomorphism since it gives a bijection between the subsets $V_{[\gamma']} \subseteq U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ contained in U. Namely, in one direction we have $p(V_{[\gamma']}) = V$ and in the other direction we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for any $[\gamma'] \in U_{[\gamma]}$ with end point in V since $V_{[\gamma']} \subseteq U_{[\gamma']} = U_{[\gamma]}$ and $V_{[\gamma']}$ maps onto V.

The previous paragraph implies that $p\colon \tilde{X} \to X$ is continuous. We can also deduce that this is a covering space since for fixed $U \in \mathcal{U}$, the sets $U_{[\gamma]}$ for varying $[\gamma]$ partition $p^{-1}(U)$, because if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ then $U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$.

It remains only to show that $\tilde{X}$ is simply-connected. For a point $[\gamma] \in \tilde{X}$ let γ_t be the path in X equals γ on $[0, t]$ and is stationary at $\gamma(t)$ on $[t, 1]$.

Then the function $t \mapsto \gamma_t$ is a path in $\tilde{X}$ that starts at $[x_0]$, the homotopy class of the constant path at x_0, and ends at $[\gamma]$. Since $[\gamma]$ was an arbitrary point in $\tilde{X}$, this shows that $\tilde{X}$ is path-connected. To show that $\pi_1(\tilde{X}, [x_0])$ is trivial, it suffices to show that the image of this group under p_* is trivial since p_* is injective. Elements in the image of p_* are represented by loops γ at x_0 that lift to loops in $\tilde{X}$ based at $[x_0]$. We have observed that the path $t \mapsto [\gamma_t]$ lifts γ starting at $[x_0]$, and for this lifted path to be a loop means that $[\gamma_1] = [x_0]$. Since $\gamma_1 = \gamma$, this says that $[\gamma] = [x_0]$, so γ is nullhomotopic and the image of p_* is trivial.

This completes the construction of a universal cover space $\tilde{X} \to X$.

Step 2: For points $[\gamma]$ and $[\gamma']$ in the simply-connected covering space $\tilde{X}$ we define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma\overline{\gamma'}] \in K$. This is an equivalence relation since K is a subgroup: it is reflexive since K contains the identity element, symmetric since K is closed under inverses, and transitive since K is closed under multiplication. Let X_K be the quotient space of $\tilde{X}$ obtained by identifying $[\gamma]$ with $[\gamma']$ if $[\gamma] \sim [\gamma']$, with the quotient topology. Note that if $\gamma(1) = \gamma'(1)$, then $[\gamma] \sim [\gamma']$ if and only if $[\gamma\eta] \sim [\gamma'\eta]$. This means that if any two points in basic neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified in X_K then the whole neighborhoods are identified. Hence the natural projection $X_K \to X$ induced by $[\gamma] \mapsto \gamma(1)$ is a covering space.

If we choose for the base point $\tilde{x}_0 \in X_K$ the equivalence class of the constant path c at x_0, then the image of $p_*\colon \pi_1(X_K, \tilde{x}_0) \to \pi_1(X, x_0)$ is exactly K. This is because for a loop γ in X based at x_0, its lift to $\tilde{X}$ starting at $[c]$ ends at $[\gamma]$, so the image of this lifted path in X_K is a loop if and only if $[\gamma] \sim [c]$, or equivalently $[\gamma] \in K$. □

Remark 1.2.3. If the subgroup K in Theorem 1.2.2 is normal, then X_K is a regular covering.

Remark 1.2.4. If K is normal in $\pi_1(X, x_0)$, then $\pi_1(X, x_0)$ acts on the left on X_K in the following way: let $t \in \pi_1(X, x_0)$ and let $[\gamma] \in X$, then $t \cdot [\gamma] := [t\gamma]$. First of all we observe that this is equivalent to take the final point of the unique lift of $t\gamma$ of base point $[c] \in X_K$, the class of the constant path based at x_0.
The action is well-defined, indeed if $\gamma \sim \gamma'$ then $t\gamma\overline{t\gamma'} = t(\gamma\overline{\gamma'})\bar{t}$. By assumption $\gamma\overline{\gamma'} \in K$ that is normal in $\pi_1(X, x_0)$ and so $t(\gamma\overline{\gamma'})\bar{t} \in K$.

$$s \cdot (t \cdot [\gamma]) = s \cdot [t\gamma] = [st\gamma] = (st) \cdot [\gamma]$$

proves that it is a left action. It is clear that $\gamma(1) = t\gamma(1)$, thus $p([\gamma]) = p([t\gamma]) = \gamma(1)$.

1.3 The monodromy of a covering space

Let $p\colon \tilde{X} \to X$ be a covering space of degree d, so that all points have exactly d preimages. Let $x \in X$, by Proposition 1.1.14 we have that d is exactly the index of $p_*(\pi_1(\tilde{X}, \tilde{x}))$ in $\pi_1(X, x)$ for $\tilde{x} \in p^{-1}(x)$.

Let us consider the fiber $p^{-1}(x) = \{y_1, \ldots, y_d\}$ over x. To every loop γ in X based at x we can associate a map L_γ as in (1.1). Next consider the images $L_\gamma(y_i)$, these also lie over x, and indeed they form the entire fibre $p^{-1}(x)$. Hence the map L_γ permutes the indexes $\{1, \ldots, d\}$ and it is obvious that it depends only on the homotopy class of γ, so we have a group homomorphism

$$\psi\colon \pi_1(X, x) \longrightarrow \mathfrak{S}_d$$

where $\mathfrak{S}_d$ denotes the symmetric group of all permutations on d elements. This map is indeed a group homomorphism by Remark 1.1.20.

Definition 1.3.1. The *monodromy representation* of a covering $p\colon \tilde{X} \to X$ of finite degree d is the group homomorphism $\psi\colon \pi_1(X, x) \longrightarrow \mathfrak{S}_d$ defined above.

Proposition 1.3.2. *Let $p\colon \tilde{X} \to X$ be a regular covering. Then the image of the monodromy representation of p is $A(\tilde{X}, p)$.*

Proof. Since the covering is regular we have that $X \cong \tilde{X}/G$, with $G := A(\tilde{X}, p)$.
Let $y \in p^{-1}(x)$ and let $\gamma \in \pi_1(X, x)$. By Lemma 1.1.11 there exists an unique lift η of $\overline{\gamma}$ with base point y. By construction there exists an element $h \in G$ such that $h(y) = L_\gamma(y)$, the uniqueness follows by Lemma 1.1.18. So we have a map:

$$\psi'\colon \pi_1(X, x) \longrightarrow G\,.$$

The action of G on $p^{-1}(y)$ is transitive so for each $y' \in p^{-1}(x)$ there exists $h \in G$ and let $y' = h(y)$. Since C is path-connected, there exists a path η from y' to y. Let $\gamma := p_*(\eta) \in \pi_1(X, x)$, then $y' = L_\gamma(y) = h(y)$ and so $\psi'(\gamma) = h$. $\square$

Chapter 2

Branched coverings of Riemann surfaces

From now on we work over the field of complex numbers: $\mathbb{C}$.

We refer to [Har77, Chapter II] for the basic definition and properties concerning algebraic varieties (irreducible, normal, dimension) and morphism (proper, finite) between algebraic varieties.

2.1 Branched, Galois and quasi-étale coverings

In this section we assume that all the varieties are algebraic, irreducible and normal.

Definition 2.1.1. Let $f\colon X \to Y$ be a finite proper morphism between varieties of the same dimension. Then the inverse image of every point is a finite set of points. We call such a map a *branched covering.*

Definition 2.1.2. Let X be a variety and let G be a finite subgroup of $\mathrm{Aut}(X)$. We say that $f\colon X \to Y = X/G$ is a *Galois covering.*

Definition 2.1.3. Let X and Y be varieties of the same dimension and $f\colon X \to Y$ be a regular map such that $f(X) \subseteq Y$ is dense. The degree of the field extension $f^*(\mathbb{C}(Y)) \subseteq \mathbb{C}(X)$, which is finite, is called the *degree* of f:

$$\deg(f) := [\mathbb{C}(X) : f^*(\mathbb{C}(Y))]\,.$$

Proposition 2.1.4 ([Sha77, Theorem 6.3.3])**.** *Let $f\colon X \to Y$ be a finite map between varieties of the same dimension. Then for all $y \in Y$ it holds $|f^{-1}(y)| \leq \deg(f)$.*

Definition 2.1.5. Let $f\colon X \to Y$ be a branched covering, let $x \in X$ and $y = f(x)$. If the number of preimages of y is strictly less than $\deg(f)$, then we say that y is a *branch point* and that x is a *ramification point.* The set of all branch points is called *branch locus* (or branch set).

Definition 2.1.6. Let $f\colon X \to Y$ be a branched covering, let $x \in X$ and $y = f(x)$. Let V be a neighborhood of y such that the connected component U of $f^{-1}(V)$ that contains x, does not contain other preimages of y. The *ramification index* of x, denoted by r_x, is the number of preimages in U for a general point other than y in V.

Remark 2.1.7. For any unramified point, its ramification index is $r_x = 1$.

Proposition 2.1.8 ([Sha77, Theorem 6.3.4])**.** *Let $f\colon X \to Y$ be a branched covering. Then the set of unramified points in Y is an open set in the Zariski topology.*

Remark 2.1.9. If $f\colon X \to Y$ is a branched covering without branch points then $f\colon X \to Y$ is a covering space of Y and in this case we say that f is étale.

Quasi-étale covering are special cases of branched coverings, and they have been firstly introduced in [Cat07].

Definition 2.1.10 (cf. [Cat07, Definition 1.1])**.** Let $f\colon Y \to X$ be a surjective morphism between varieties of the same dimension. We say that f is a *quasi-étale morphisms* if it is étale in codimension 1, i.e. there exists $Z \subset Y$ of codimension ≥ 2 such that $f_{|(Y\setminus Z)}\colon Y \setminus Z \to f(Y \setminus Z)$ is étale.

Lemma 2.1.11 ([Cat07, Remark 3.1])**.** *Let $f\colon Y \to X$ be a quasi-étale morphism. If Y is smooth and X is normal, then f is étale.*

2.2 Some facts on Riemann surfaces

In this section we recall some facts on Riemann surfaces, we refer to [Mir90] for further details.

By proposition 2.1.8, in the compact Riemann surfaces case the branch locus is finite.

Lemma 2.2.1 ([Mir90, pages 48-49])**.** *Let $f\colon X \to Y$ be a non constant holomorphic map between compact Riemann surfaces. Then f is a branched covering.*

For compact Riemann surfaces it holds the well known "Hurwitz's formula":

Theorem 2.2.2 (Hurwitz's formula, see [Mir90, Theorem II.4.16])**.**
Let $f\colon X \to Y$ be a non constant holomorphic map between compact Riemann surfaces. Then

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{x \in X} (r_x - 1)\,.$$

Let C be a compact Riemann surface, we want to investigate C/G where G is a finite group acting holomorphically and faithfully on C.

Remark 2.2.3. We first observe that we can always assume that G acts faithfully. Indeed, if $K \triangleleft G$ is the normal subgroup of the elements that act trivially, then we can replace G by $G' := G/K$ and obviously we have $C/G \cong C/G'$.

From now on we always assume that G acts faithfully and holomorphically, so that G embeds in $\mathrm{Aut}(C)$.

Proposition 2.2.4 ([Mir90, Proposition III.3.1])**.** *Let C be a Riemann surface, let $G \triangleleft \mathrm{Aut}(C)$ and let $p \in C$. Suppose that the stabilizer subgroup* $\mathrm{Stab}(p)$ *is finite. Then* $\mathrm{Stab}(p)$ *is cyclic.*

Proof. Fix a local coordinate z centered at p. For any $g \in \mathrm{Stab}(p)$, write $g(z) = \sum_{n=1}^{\infty} a_n(g) z^n$; this power series has no constant term since $g(p) = p$ and $a_1(g) \neq 0$ since g is an automorphism of X an hence it has multiplicity one at every point.

Consider the function $a_1 \colon \mathrm{Stab}(p) \to \mathbb{C}^*$. Note that it is a homomorphism of groups: $a_1(gh)$ is calculated by computing the power series for $g(h(z))$, so that $a_1(gh) = a_1(g)a_1(h)$.

To finish the proof it suffices to prove that this map is injective, since the only finite subgroups of $\mathbb{C}^*$ are cyclic. Let $g \in \ker(a_1)$, i.e. $g(z) = z +$ (higher order terms); we have to show that in fact $g(z) = z$. Suppose that this is not the case and let $m \geq 2$ be the exponent of the first non zero higher order term of g, therefore $g(z) = z + az^m \mod z^{m+1}$ with $a \neq 0$. It is not difficult to prove by induction that $g^k = z + kaz^m \mod z^{m+1}$. But since the stabilizer is finite, this element must have finite order; hence for some k, $g^k(z) = z$. It follows that for some k, $ka = 0$ hence $a = 0$ and so g is the identity. ☐

Proposition 2.2.5 ([Mir90, Proposition III.3.2])**.** *Let C be a Riemann surface, let G be a finite group acting faithfully and holomorphically. Then the points of C with non trivial stabilizer are discrete.*

Proof. Suppose that there exists a sequence $\{p_k\}$ converging to p such that each p_i has a nontrivial element g_i fixing it. Since G is finite, we may pass to a subsequence and assume that each p_i is fixed by the same nontrivial element g that is continuous and so it fixes the limit point p too. Since g and the identity 1_G agree on set $S \subset C$ with an accumulation point, they must be equal (see [Mir90, Identity Theorem, Theorem II.1.35]). ☐

Remark 2.2.6. In the same assumptions of the previous proposition, if C is compact, then only finitely many points have non trivial stabilizer.

Proposition 2.2.7 ([Mir90, Proposition III.3.3]). *Let C be a Riemann surface and let $G \lhd \mathrm{Aut}(C)$ finite. Fix a point $x \in C$. Then there is an open neighborhood U of x such that:*

- *U is invariant under the action of $\mathrm{Stab}_G(x)$: $g(u) \in U$ for every $g \in G$ and $u \in U$;*
- *$U \cap g(U) = \emptyset$ for every $g \notin \mathrm{Stab}_G(x)$;*
- *the natural map $\alpha\colon U/\mathrm{Stab}_G(x) \to C/G$, induced by sending a point in U to its orbit, is a homeomorphism onto an open subset of C/G;*
- *no point of U except x is fixed by any element of $\mathrm{Stab}_G(x)$.*

Using the previous statement, it is possible to define a complex structure on C/G. We get the following:

Theorem 2.2.8 ([Mir90, Theorem III.3.4]). *Let C be a Riemann surface and let $G \lhd \mathrm{Aut}(C)$ finite. Then C/G is a Riemann surface, the quotient map $f\colon C \to C/G$ is holomorphic of degree $|G|$ and $r_p(f) = |\mathrm{Stab}_G(p)|$ for any $p \in C$.*

2.2.1 The Riemann Existence Theorem

Let C be a Riemann surface and let $G \lhd \mathrm{Aut}(C)$ finite. By Theorem 2.2.8 we can define a structure of Riemann surface on $C' := C/G$. Let

$$f\colon C \longrightarrow C'$$

be the quotient map; it is a Galois covering. Let $B := \{p_1, \dots, p_r\}$ be the branch locus of f. Let $X := C' \setminus B$ and $C_0 := f^{-1}(X)$ thus the restriction

$$f_0\colon C_0 \to X$$

of f to C_0 is a covering space.

The aim of this section is to reverse this construction. We start from a Riemann surface C', r points $x_1, \dots, x_r$ of C' and an étale covering

$$F\colon \overline{C} \to C' \setminus \{x_1, \dots, x_r\}\,.$$

We will show that F can be extended to a Galois covering $f\colon C \to C'$, and that the Riemann surface C is unique up to isomorphism.

Proposition 2.2.9. *Let $f'\colon X \setminus A \to X'$ be a holomorphic map between Riemann surfaces, where $A \subset X$ is finite. If there exists a continuous map $f\colon X \to X'$ that extends f' then f is holomorphic.*

Proof. Let $x \in A$ and let $\varphi\colon U \to \mathbb{C}$ and $\psi\colon V \to \mathbb{C}$ local charts in X and X' respectively, such that $x \in U$ and $f(x) \in V$. The map

$$\psi \circ f \circ \varphi^{-1}\colon \varphi(U \cap f^{-1}(V)) \to \mathbb{C}$$

is holomorphic in $\varphi(U \cap f^{-1}(V)) \setminus \varphi(x)$ and it is bounded in a neighborhood of $\varphi(x)$. Using the Riemann extension theorem we conclude that the map is holomorphic also in $\varphi(x)$, thus f is holomorphic in x. $\square$

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the unitary open disc and let $D^* := D \setminus \{0\}$ be the punctured disc. In order to prove the Riemann existence theorem we need the following:

Theorem 2.2.10 ([For81, Theorem 5.10]). *Let X be a Riemann surface and let $s\colon X \to D^*$ be a connected covering space of degree $m < +\infty$. Then there exists a biholomorphic map $\psi\colon X \to D^*$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "s"'] & & D^* \arrow[dl, "p_m"] \\ & D^* & \end{tikzcd}$$

where $p_m(z) = z^m$.

Theorem 2.2.11 (Riemann existence theorem). *Let $\overline{C}$ and C' be Riemann surfaces and let $A \subset C'$ be a finite subset. Let*

$$f\colon \overline{C} \longrightarrow C' \setminus A$$

be a proper étale covering.

Then f can be extended to a branched covering of C', that is there exist a Riemann surface C, a proper holomorphic map

$$F\colon C \longrightarrow C'$$

and a biholomorphic map

$$\varphi\colon C \setminus F^{-1}(A) \longrightarrow \overline{C}$$

such that the following diagram commutes:

$$\begin{tikzcd} \overline{C} \arrow[r, leftrightarrow, "\varphi"] \arrow[d, "f"'] & C \setminus F^{-1}(A) \arrow[r, hook] & C \arrow[d, "F"] \\ C' \setminus A \arrow[rr, hook] & & C' \end{tikzcd}$$

Moreover C is unique up to isomorphisms.

Proof. For each $x \in A$ let (U_x, ψ_x) a chart centered in x, i.e. $\psi_x(x) = 0$; moreover we can assume that $\psi_x(U_x) \cong D$ and that $U_{x_1} \cap U_{x_2} = \emptyset$ if $x_1 \neq x_2$. Let $U_x^* := U_x \setminus \{x\}$, since f is proper we have that $f^{-1}(U_x^*)$ has a finite number of connected components $V_{x,i}^*$:

$$f^{-1}(U_x^*) = V_{x,1}^* \sqcup \ldots \sqcup V_{x,N}^* ,$$

where each $V_{x,i}^* \to U_x^*$ is a connected covering of finite degree m_i. By Theorem 2.2.10, for each $i = 1, \ldots, N$ there exists a biholomorphism $h_i \colon V_{x,i}^* \to D^*$ such that the following diagram of holomorphic maps commutes:

$$\begin{tikzcd} V_{x,i}^* \arrow[r, "h_i"] \arrow[d, "f"'] & D^* \arrow[d, "p_{m_i}"] \\ U_x^* \arrow[r, "\psi_x"] & D^* \end{tikzcd}$$

with $p_{m_i}(z) = z^{m_i}$.

Adding a point $y_{x,i}$ to each $V_{x,i}^*$ we get sets $V_{x,i} := V_{x,i}^* \cup \{y_{x,i}\}$ on which we consider the natural topology that makes the natural extension of h_i to a map $V_{x,i} \to D$ (sending $y_{x,i}$ into 0) an homeomorphism. We define

$$C := \overline{C} \cup \{y_{x,i}, i = 1, \ldots, N\}_{x \in A} .$$

On C there exists an unique topology such that the inclusion $\overline{C} \hookrightarrow C$ is continuous and for any W neighborhood of x then

$$\{y_{x,i}\} \cup (f{-}1(W) \cap V_{x,i}^*)$$

is a neighborhood of $y_{x,i}$. This topology is Hausdorff.

We define $F \colon C \to C'$ with $F(z) = f(z)$ for each $z \in \overline{C}$ and $F(y_{x,i}) = x$. It is easy to prove that F is proper. The charts $(V_{x,i}, h_i)$ defined above are compatible with the charts of $\overline{C}$ and so they define a complex structure on C. The covering

$$f \colon \overline{C} \longrightarrow C' \setminus A$$

extends to a continuous map

$$F \colon C \longrightarrow C'$$

that is holomorphic because of Proposition 2.2.9. Since $C \setminus F^{-1}(A) = \overline{C}$, we can choose as $\varphi \colon \overline{C} \setminus F^{-1}(A) \longrightarrow \overline{C}$ the identity. This prove the existence.

We construct C in such a way that for each point $x \in A$, $F^{-1}(x)$ has cardinality equal to the number of connected components of $F^{-1}(U_x^*)$. Let $F_1 \colon C_1 \to C'$ be a map satisfying the conditions of the statement. Then

$F^{-1}(U_x^*) = F_1^{-1}(U_x^*)$ and so, since F_1 is proper, $F_1^{-1}(x)$ contains at least a point for each connected component of $F^{-1}(U_x^*)$.

$F_1^{-1}(x)$ does not contains other points, because if it contains an other point z it must be isolated and so C_1 is not a Riemann surface in a neighborhood of z. So we can extend the identity map $Id\colon \overline{C} \to \overline{C}$ to a bijective continuous map $\alpha\colon C \to C_1$ sending each point $y_{x,i}$ in the unique accumulation point for $V_{x,i}^*$ in $F_1^{-1}(U_x^*)$. By Proposition 2.2.9, this map is holomorphic and hence an isomorphism. $\square$

2.2.2 Finiteness of $\mathrm{Aut}(C)$

Lemma 2.2.12 (Linearization of the action, [Mir90, Corollary III.3.5])**.** *Let C be a Riemann surface and let $G \triangleleft \mathrm{Aut}(C)$ finite. Fix a point $p \in C$ with non trivial stabilizer of order m. Let $g \in \mathrm{Stab}(p)$ be a generator of the stabilizer subgroup. Then there is a local coordinate z on C centered at p such that $g(z) = \lambda z$, where $\lambda = \exp(\frac{2\pi i}{m})$.*

Theorem 2.2.13 ([Mir90, Lemma III.3.6])**.** *Let C be a compact Riemann surface and let $G \triangleleft \mathrm{Aut}(C)$ finite. Let $f\colon C \to Y = C/G$. Then for every branch point $y \in Y$ there is an integer $r \geq 2$ such that $f^{-1}(y)$ consists of exactly $|G|/r$ points of C, and each of these preimages f has multiplicity r.*

Proof. Suppose that $y \in Y$ is a branch point of the map f. Let $f^{-1}(y) = \{x_1, \ldots, x_s\}$; they form a single orbit for the action of G on C. Moreover their stabilizers subgroups are conjugates and in particular they have the same order, say r. The number s of points in this orbit is the index of the stabilizer, and so $s = |G|/r$. $\square$

Applying Theorem 2.2.2 to the previous statement, we get the following:

Corollary 2.2.14. *Let C be a compact Riemann surface and let G be a finite subgroup of $\mathrm{Aut}(C)$. Let $f\colon C \to Y = C/G$. Suppose that there are k branch points $y_1, \ldots, y_k$ in Y, with f having multiplicity r_i at the $|G|/r_i$ points above y_i. Then*

$$\begin{aligned} 2g(C) - 2 &= |G|(2g(C/G) - 2) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1) \\ &= |G|\left(2g(C/G) - 2 + \sum_{i=1}^{k}\left(\frac{r_i - 1}{r_i}\right)\right) \end{aligned}$$

In next chapters we will consider only Riemann surfaces of genus $g \geq 2$, hence the assumption of finiteness of $G \triangleleft \mathrm{Aut}(C)$ is automatic; indeed, studying the Weierstrass points of Riemann surfaces, Schwartz in 1890 proved

Theorem 2.2.15 (Schwartz, see [Sch90])**.** *Any compact Riemann surface of genus $g \geq 2$ has a finite number of automorphisms.*

Moreover, for Riemann surfaces of genus at least 2, Corollary 2.2.14 leads to a bound on the order of the groups G which act holomorphically and effectively.

Theorem 2.2.16 (Hurwitz's Theorem, [Mir90, Theorem III.3.9])**.** *Let C be a compact Riemann surface of genus $g \geq 2$ and let $G \triangleleft \mathrm{Aut}(C)$. Then*

$$|G| \leq 84(g-1)\,.$$

Proof. Since G is finite, by Corollary 2.2.14:

$$2g-2 = |G|(2g'-2+R)\,,$$

where g' is the genus of C/G and $R = \sum_{i=1}^{k}(1-1/r_i)$, $r_i \geq 2$.

Suppose first that $g' \geq 1$. If $R = 0$, so there is no ramification to the quotient map, then $g' \geq 2$, which implies that $|G| \leq g-1$. If $R \neq 0$, this force $R \geq 1/2$, then $2g'-2+R \geq 1/2$ and so $|G| \leq 4(g-1)$.

Let us assume that $g' = 0$, hence $2g-2 = |G|(R-2)$ which forces $R > 2$. In this case $k \geq 3$; we now assume that $r_1 \leq r_2 \leq \ldots \leq r_k$.

Let $k = 3$, then only r_1 can be equal to 2; in this case if $r_2 = 3$ then $r_3 \geq 7$ and $R \geq 2 + 1/42$. If $r_2 = 4$ then $r_3 \geq 5$ and $R \geq 2 + 1/20$; if $r_2 \geq 5$ then for any r_3 we get $R \geq 2 + 1/10$.
If $r_1 = 3$, we only exclude the case $r_1 = r_2 = r_3 = 3$: in this case $R = 2$, otherwise $R \geq 2 + 1/12$ (see [Mir90, Lemma III.3.8]).
If $r_1 = 4$, then $R \geq 2 + 1/4$.

Let $k = 4$ and $r_1 = r_2 = r_3 = r_4 = 2$ then $R = 2$; if $r_i \geq 3$ for at least one i then $R \geq 2 + 1/6$.

Finally, if $k \geq 5$ then $R \geq 2 + 1/2$.

So the minimal value for R is obtained with $r_1 = 2$, $r_2 = 3$, $r_3 = 7$: we get that $R - 2 \geq 1/42$. Therefore $|G| \leq 84(g-1)$. □

If we make stronger assumptions on G we get the following results:

Proposition 2.2.17 (Nakajima's Theorem, see [Nak87])**.** *Let C be a compact Riemann surface of genus $g \geq 2$ and let G be an abelian subgroup of* $\mathrm{Aut}(C)$*. Then*

$$|G| \leq 4g + 4\,.$$

Proposition 2.2.18 (Wiman's Theorem, see [Wim95])**.** *Let C be a compact Riemann surface of genus $g \geq 2$ and let G be a cyclic subgroup of* $\mathrm{Aut}(C)$*. Then*

$$|G| \leq 4g + 2\,.$$

2.3 The appropriate orbifold homomorphism of a Galois covering

We start this section with some definitions of group theory.

Given integers $g \geq 0$ and $m_1, \ldots, m_r > 1$ the *orbifold surface group of signature (or type)* $(g; m_1, \ldots, m_r)$ is defined as:

$$\mathbb{T}(g; m_1, \ldots, m_r) := \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \mid c_1^{m_1}, \ldots, c_r^{m_r}, \textstyle\prod_{i=1}^{g}[a_i, b_i] \cdot c_1 \cdots c_r \rangle .$$

For $r = 0$ we have the *surface group* of genus g

$$\Pi_g := \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i] \rangle$$

that is the fundamental group of a Riemann surface of genus g.

For $g = 0$ we get the *polygonal group*

$$\mathbb{T}(m_1, \ldots, m_r) := \langle c_1, \ldots, c_r \mid c_1^{m_1}, \ldots, c_r^{m_r}, c_1 \cdots c_r \rangle . \tag{2.1}$$

Let H be a finite group, we say that an homomorphism

$$\psi \colon \mathbb{T}(g; m_1, \ldots, m_r) \longrightarrow H$$

is an *appropriate orbifold homomorphism* if it is surjective and $\psi(c_i)$ has order m_i.

Definition 2.3.1. Let H be a finite group and let $g, m_1, \ldots, m_r$ as above. A *generating vector* for H of type $(g; m_1, \ldots, m_r)$ is a $(2g+r)$-tuple of elements of H:

$$V := (d_1, e_1, \ldots, d_g, e_g, h_1, \ldots, h_r)$$

such that V generates H, $\prod_{i=1}^{g}[d_i, e_i] \cdot h_1 \cdot h_2 \cdots h_r = 1$ and there exists a permutation $\sigma \in \mathfrak{S}_r$ such that $\mathrm{ord}(h_i) = m_{\sigma(i)}$ for $i = 1, \ldots, r$. If such a V exists, then H is said to be $(g; m_1, \ldots, m_r)$*-generated*.

In the particular case $g = 0$, we have the following:

Definition 2.3.2. Let H be a finite group. A *spherical system of generators* of H of type (or signature) $(m_1, \ldots, m_r)$ is a set of generators $\{h_1, \ldots, h_r\}$ of H such that $h_1 \cdots h_r = 1$ and there exists a permutation $\sigma \in \mathfrak{S}_r$ such that $\mathrm{ord}(h_i) = m_{\sigma(i)}$ for $i = 1, \ldots, r$.

Remark 2.3.3. To give a generator vector of signature $(g; m_1, \ldots, m_r)$ for a finite group H is equivalent to give an appropriate orbifold homomorphism $\psi \colon \mathbb{T}(g; m_1, \ldots, m_r) \to H$.

Let C be a compact Riemann surface of genus $g(C) \geq 2$ and let $G \triangleleft \mathrm{Aut}(C)$; we denote by C' the compact Riemann surface C/G, and by $g(C')$ its genus. Let $B := \{p_1, \dots, p_r\}$ be the branch locus of $f \colon C \to C'$. Let $X := C' \setminus B$ and $C_0 := f^{-1}(X)$ thus the restriction $f_0 \colon C_0 \to X$ of f to C_0 is a covering space. We observe that $G \hookrightarrow \mathrm{Aut}(C_0)$ considering the restriction of each automorphism to C_0 and that $C_0/G \cong X$. The action of G on C_0 is properly discontinuous, hence $f_0 \colon C_0 \to X$ is a regular covering by Theorem 1.1.28.

Let us fix a point of B, say p_1, and let $\{q_1, \dots, q_t\}$ be its fiber $f^{-1}(p_1)$. By Proposition 2.2.4 we have that $H := \mathrm{Stab}(q_1) \cong \mathbb{Z}_n$ for some integer $n \geq 2$. By construction we have that there exist $g_2, \dots, g_t \in G$ such that $g_i q_1 = q_i$.

Lemma 2.3.4. *There is a G-equivariant bijection*

$$f^{-1}(p_1) \longleftrightarrow \{gH\}$$

where $\{gH\}$ is the set of the left cosets of H.

Proof. Two elements g, $g' \in G$ are in the same coset if and only if there exists $h \in H$ such that $gh = g'$, that is $g'(q_1) = g(hq_1) = g(q_1)$. Hence

$$q_j \longmapsto \{g \in G \mid gq_1 = q_j\}$$

gives a bijection. □

Lemma 2.3.5. $g_i H g_i^{-1} \cong \mathrm{Stab}_G(q_i)$.

Proof. $\mathrm{Stab}(q_i) \supseteq g_i H g_i^{-1}$, since $(g_i H g_i^{-1}) g_i H = g_i H H = g_i H$.
For the other inclusion we note that if $g \in \mathrm{Stab}(q_i)$, then $g g_i H = g_i H$ and so there exists $h \in H$ such that $g g_i = g_i h$. Hence $g = g_i h g_i^{-1} \in g_i H g_i^{-1}$. □

Hence the stabilizers of the q_i are isomorphic in particular they have all the same cardinality $n = \frac{|G|}{t}$.

Let $p \in X$ and $g' := g(C')$. We have

$$\pi_1(C', p) = \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} \mid \prod_{i=1}^{g'} [\alpha_i, \beta_i] \rangle .$$

Removing the points of B, we have that we cannot contract loops around the p_i and so we have that the fundamental group changes as follows: for each i let γ_i be a loop based at p going once around p_i. Up to relabel the points in B, we have that

$$\pi_1(X, p) = \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \gamma_1, \dots, \gamma_r \mid \prod_{i=1}^{g'} [\alpha_i, \beta_i] \cdot \gamma_1 \cdots \gamma_r \rangle .$$

Since $f_0\colon C_0 \to X$ is regular, by Proposition 1.3.2 we get a surjective map

$$\theta\colon \pi_1(X,p) \longrightarrow G\,.$$

For $i = 1,\dots,r$ let $h_i := \theta(\gamma_i)$ and let m_i be the cardinality of the stabilizers of the points in $f^{-1}(p_i)$. For $j = 1,\dots,g'$ let $a_j := \theta(\alpha_j)$ and $b_j := \theta(\beta_j)$. We get

Lemma 2.3.6. $\{a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r\}$ *is a generating vector of type* $(g'; m_1, \dots, m_r)$.

Proof. It is obvious that $\prod_{i=1}^{g'}[a_i,b_i]\cdot h_1\cdots h_r = 1$ since θ is a homomorphism and $\prod_{i=1}^{g'}[\alpha_i,\beta_i]\cdot\gamma_1\cdots\gamma_r = 1$.

The non trivial part of the statement is that $\mathrm{ord}(h_i) = m_i$. It is equivalent to the fact that $\overline{\gamma_i}^d$ lifts to a closed path for $d = m_i$ and does not for $0 < d < m_i$.

Let V' be an open neighborhood of p_i such that $V := V' \setminus \{p_i\}$ is a elementary neighborhood for f_0. Let $v \in V$, let τ a path in X from p to v and let δ a loop in V around p_i such that $\overline{\gamma_1} = \tau\delta\tau^{-1}$. It is clear that if we prove that δ^d lifts to a closed path for $d = m_i$ and does not for $0 < d < m_i$, we are done.

Let $q_j \in f^{-1}(p_i)$, by Proposition 2.2.7, there exists an open neighborhood U invariant under the action of $\mathrm{Stab}(q_j) = \langle h\rangle \cong \mathbb{Z}_{m_i}$. So, up to shrinking V', we can assume that each component $U_j \ni q_j$ of $f^{-1}(V')$ is invariant under the action of $\mathrm{Stab}(q_j)$. By Lemma 2.2.12, we can choose local coordinate w centered in q such that $h(w) = \lambda w$ with $\lambda = \exp(\frac{2\pi i}{m_i})$, and so for each point $q \in f^{-1}(p_i)$ we can choose appropriate local coordinate w centered in q in such a way that $f(w) = w^{m_i}$. We note that we can assume that $v = (\frac{1}{2})^{m_i}$ and that δ is the loop $\delta(t) = \frac{\exp(2\pi i t)}{2^{m_i}}$. We can also assume $j = 1$; let $z = \exp(\frac{2\pi i}{m_i})$ be a primitive n^{th}-root of the unity, hence the preimages of v are the points $f^{-1}(v) \cap U_1 = \{\frac{z^k}{2}\}_k$. We have that δ lifts to paths $\tilde{\delta}_k(t) = z^k\,\dfrac{\exp(\frac{2\pi i t}{m_i})}{2}$ from $\frac{z^k}{2}$ to $\frac{z^{k+1}}{2}$. Hence δ^{m_i} lifts to a closed path and it does not happen for any integer in $\{1,\dots,m_i-1\}$. □

We get that every Galois covering $f\colon C \to C/G \cong C'$ with C and G as above, induces an appropriate orbifold homomorphism

$$\psi\colon \mathbb{T}(g(C'); m_1, \dots, m_r) \longrightarrow G$$

or equivalently, a generating vector of type $(g(C'); m_1, \dots, m_r)$ for G.

2.4 From the appropriate orbifold homomorphism to the Galois covering

In this section we show how to invert the construction of the previous section. In other words, given the compact Riemann surface C', the finite group G

and the appropriate orbifold homomorphism

$$\psi\colon \mathbb{T}(g(C'); m_1, \dots, m_r) \longrightarrow G$$

we construct a compact Riemann surface C such that $C' = C/G$.

Let $g' := g(C')$ be the genus of C' and

$$\pi_1(C', p) = \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \mid \prod_{i=1}^{g'} [\alpha_i, \beta_i] \rangle \,.$$

Let $(a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r)$ be a generating vector of type $(g'; m_1, \dots, m_r)$ for G. Fix $B := \{p_1, \dots, p_r\} \subset C'$ and choose $p \in X := C' \setminus B$. For each j, let γ_j be a geometric loop around p_j such that $\prod_{i=1}^{g'} [\alpha_i, \beta_i] \cdot \prod \gamma_j = 1$, so

$$\pi_1(X, p) = \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \gamma_1, \dots, \gamma_r \mid \prod_{i=1}^{g'} [\alpha_i, \beta_i] \cdot \prod \gamma_j = 1 \rangle \,.$$

The vector $(a_1, b_1, \dots, a_{g'}, b_{g'}, h_1, \dots, h_r)$ induces an epimorphism:

$$\begin{array}{rcl} \theta\colon \pi_1(X, p) & \longrightarrow & G \\ \gamma_i & \longmapsto & h_i \\ \alpha_j & \longmapsto & a_j \\ \beta_j & \longmapsto & b_j \end{array}$$

let K be its kernel:

$$1 \longrightarrow K \longrightarrow \pi_1(X, p) \xrightarrow{\theta} G \longrightarrow 1 \,.$$

By Theorem 1.2.2, we can associate to the normal subgroup $K \triangleleft \pi_1(X, p)$ a Galois covering space $f\colon X_K \longrightarrow X$ such that $\pi_1(X_K, y) \cong K$.

By Remark 1.2.4 we have that $\pi_1(X, p)$ acts on the left on X_K: let $t \in \pi_1(X, p)$ and let $[\gamma] \in X_K$, then $t \cdot [\gamma] := [t\gamma]$. Since for $\eta \in K$ we have $[\gamma] = \eta[\gamma]$, we have a left G-action on X_K: let $h \in G$ then $h \cdot [\gamma] := [\theta^{-1}(h)\gamma]$. This action is well defined, indeed if $\theta^{-1}(h)$ and $\theta^{-1}(h')$ are two different preimages of h, then they differ for some $k \in K$; hence $[\theta^{-1}(h)\gamma] = [k\theta^{-1}(h)\gamma]$, indeed $\theta^{-1}(h)\gamma(\overline{k\theta^{-1}(h)\gamma}) = \overline{k} \in K$.

G acts faithfully:

$$\begin{aligned} [\gamma] = h[\gamma] = [\theta^{-1}(h)\gamma] & \iff \gamma\overline{\gamma}\theta^{-1}(h^{-1}) \in K \\ & \iff \theta^{-1}(h^{-1}) \in K \\ & \iff (h^{-1}) = 1_K \end{aligned}$$

Using Theorem 2.2.11, we extend the étale covering $f\colon X_K \to X = C' \setminus B$ to a Galois covering $F\colon C \to C'$ of Riemann surfaces.

Now fix a point in B, say p_1, and let W be a small open neighborhood of p_1 in C', so that $W \setminus \{p_1\}$ is isomorphic to a punctured disc. Let $D_1^* \sqcup \cdots \sqcup D_s^* = f^{-1}(W \setminus \{p_1\})$, where the D_i^* are pairwise disjoint.
Let τ be a loop going around p_1 once in W of base point p' and let $\gamma\colon I \to X$ be a path from p to p', such that γ_1 is homotopic to $\gamma\tau\overline{\gamma}$.

Claim 1. *$f^{-1}(W \setminus \{p_1\})$ has $|G : \langle h_1\rangle|$ connected components.*

Proof. By construction of X_K $[\gamma] \in f^{-1}(p')$, let $[\gamma']$ be another point in $f^{-1}(p')$. We have that $[\gamma]$ and $[\gamma']$ belong to the same component D_i^* if and only if there exists a path $\delta\colon I \to X_K$ from $[\gamma]$ to $[\gamma']$ such that $\eta := f \circ \delta$ is contained in W. In other words, η is a loop in W with base point p', so $\eta = \tau^k$ and $[\gamma'] = [\gamma\eta]$. Now we have $\theta(\gamma\eta\overline{\gamma}) = h_1^k$ and so to each point of $f^{-1}(p') \cap D_i^*$ corresponds an unique element of $S := \langle h_1\rangle$. Conversely, to each power of γ_1 is associated a point in D_i^*: $[\gamma_1^k\gamma]$. These points are exactly $m_1 = \operatorname{ord}(h_1)$, since we have that

$$\begin{aligned} [\gamma_1^a\gamma] = [\gamma_1^b\gamma] \iff \gamma_1^{a-b} \in K &\iff 1_K = \theta(\gamma_1^{a-b}) = h_1^{a-b} \\ &\iff a \cong b \mod m_1 \end{aligned}$$

Hence there are m_1 elements in each component, hence there are $|G : \langle h_1\rangle|$ connected components. □

From this proof it follows also that $[\gamma]$ is in the same D_i^* of $[\gamma\tau^k]$ for each $k \in \mathbb{Z}$.

Let $S := \langle h_1\rangle$ be the cyclic subgroup of G generated by h_1, a straightforward computation shows that $\{[\gamma\tau^k]\}_k = S \cdot [\gamma]$

Claim 2. *The correspondence $h \mapsto h \cdot [\gamma]$ is a bijection between G and $f^{-1}(p')$.*

Proof. We start proving the surjectivity, if $[\eta] \in f^{-1}(p)$ then let $h = \theta(\eta\overline{\gamma})$, hence $h \cdot [\gamma] = [\theta^{-1}(h)\gamma] = [\eta]$.

For the injectivity we consider h and h' such that $h[\gamma] = h'[\gamma]$ that is $\theta^{-1}\gamma\overline{\gamma}\theta^{-1}(h'^{-1}) = \theta^{-1}(hh'^{-1}) \in K$, hence $hh'^{-1} = 1_K$. □

Claim 3. *To be in the same D_i^* corresponds to be in the same left coset gS.*

Proof. To each $h \in G$ is associated an unique point in $f^{-1}(p')$: $[\theta^{-1}(h)\gamma]$. Let h and h' be two elements of G, we have that $hS = h'S$ if and only if there exists k such that $h\theta(\gamma\tau^k\gamma^{-1}) = h'$ that is $h\theta(\gamma\tau^k\gamma^{-1})[\gamma] = h'[\gamma]$. In other words, $[\theta^{-1}(h')\gamma] = [\theta^{-1}(h)\gamma\tau^k\overline{\gamma}\gamma] = [\theta^{-1}(h)\gamma\tau^k]$, that is equivalent to be in the same connected component, by the argument of Claim 1. □

We have just proved

Lemma 2.4.1. *There is a bijection*

$$\begin{array}{rcl} F^{-1}(p_i) & \longleftrightarrow & \{kS\} \\ y_j & \longleftrightarrow & k_j S \end{array}$$

where $S := \langle h_i \rangle$.

Moreover

Lemma 2.4.2. $\mathrm{Stab}_G(y_j) = k_j S k_j^{-1}$.

Proof. The proof is exactly the same as Lemma 2.3.5. □

Summarizing we get the following statement:

Theorem 2.4.3 (cf. [BCP11, Theorem 4.2])**.** *A finite group G acts as a group of automorphisms on a compact Riemann surface C of genus g if and only if there are natural numbers $g', m_1, \dots, m_r$ and an appropriate orbifold homomorphism*

$$\theta \colon \mathbb{T}(g'; m_1, \dots, m_r) \longrightarrow G$$

such that the Riemann-Hurwitz relation holds:

$$2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).$$

If this is the case, then g' is the genus of $C' = C/G$ and the Galois covering $f \colon C \to C'$ is branched in r points $p_1, \dots, p_r$ with branch indexes $m_1, \dots, m_r$, respectively.

Remark 2.4.4. The appropriate orbifold homomorphism θ is induced by the monodromy of the Galois étale G-covering $f_0 \colon C_0 \to C'_0$ given by f, where C'_0 is the Riemann surface obtained from C' by removing the branch points of f, and $C_0 := f^{-1}(C'_0)$. In particular, $\theta(\gamma_i)$ generates the stabilizer of a point in $f^{-1}(p_i)$.

If we denote by $h_i \in G$ the image of γ_i under θ, then

$$\Sigma(h_1, \dots, h_r) := \cup_{a \in G} \cup_{i \in \mathbb{Z}} \{a h_1^i a^{-1}, \dots, a h_r^i a^{-1}\},$$

is the set of stabilizers for the action of G on C.

2.5 Lifting automorphisms to the universal cover

Let C' be a Riemann surface of genus g', let $\{p_1, \dots, p_r\} \subset C'$, let $p \in X := C' \setminus \{p_1, \dots, p_r\}$ and let

$$\theta \colon \mathbb{T}(g'; m_1, \dots, m_r) \longrightarrow G$$

be an appropriate orbifold homomorphism. Let $f\colon C \to C'$ be the Galois covering of C' obtained by these data, as seen in the previous section.

Next consider $u\colon \Delta \to C$, the universal cover of C; as seen in the proof of Theorem 1.2.2, the elements of Δ are the homotopy classes of paths in C with base point $y = [p]$, the constant path in X of base point p.

By Remark 1.2.4, we have that $\pi_1(C, y)$ acts on Δ as follows: let $\eta \in \pi_1(C, y)$ and $[\xi] \in \Delta$, $\eta \cdot [\xi] = [\eta\xi]$, that is equivalent to take the final point of the unique lift of $\eta\xi$ (that is a path in C) with base point $[x_0]$, the class of the constant path in C with base point y.

Let $C_0 := f^{-1}(X)$ so $f_0\colon C_0 \to X$ is an étale covering. Let g be the genus of C, we recall that

$$\pi_1(C_0, y) \cong \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r \mid \prod_{i=1}^{g}[a_i, b_i] \cdot c_1 \cdots c_r\rangle\,.$$

Plugging the holes, the fundamental group changes: by Van Kampen's Theorem we have to quotient by the normal subgroup generated by the c_i, each c_i is a simple loop around a hole; they are nullhomotopic in C:

$$\pi_1(C, y) \cong \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i]\rangle = \Pi_g\,.$$

Hence we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \langle\langle c_i\rangle\rangle \arrow[r, "c_i\mapsto\gamma_i^{m_i}"] \arrow[d] & \langle\langle \gamma_i^{m_i}\rangle\rangle \arrow[r] \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \pi_1(C_0, y) \arrow[r, "(f_0)_*"] \arrow[d] & \pi_1(X, p) \arrow[r, "\theta"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(C, y) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

where $\langle\langle A\rangle\rangle$ denotes the normal subgroup generated by A. By construction, it follows that

$$\begin{aligned}
F &= \pi_1(X, p)/\langle\langle \gamma_i^{m_i}\rangle\rangle \\
&= \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}\gamma_1, \dots, \gamma_r \mid \prod[\alpha_i, \beta_i] \cdot \gamma_1 \cdots \gamma_r, \gamma_j^{m_j}\rangle =: \mathbb{T}
\end{aligned}$$

thus we have proved:

Lemma 2.5.1. *The sequence:*

$$1 \to \pi_1(C, y) \to \mathbb{T} \to G \to 1\,.$$

is exact.

Lemma 2.5.2. *The action of $\pi_1(C, y)$ on Δ extends to an action of $\mathbb{T}$ on Δ.*

Proof. Let $[\xi] \in \Delta$, $w := u([\xi]) \in C$, and $z := f(w) \in C'$; let $t \in \mathbb{T}$. Suppose that $w = \widehat{\gamma}$ is the class of the path γ (path in C' based at p).
We have defined $t \cdot w = t \cdot \widehat{\gamma} = \widehat{t\gamma} = w'$, which is the final point of the unique lift η of $t\gamma$ with base point $y = \widehat{p}$: the constant path based at p. Now we lift η to the unique lift with base point $[x_0]$, the constant path in C with base point y; we define $t \cdot [\xi]$ as the final point of this lift. Using the uniqueness of the lift, it is easy to see that this is a well-defined left action and that on $\pi_1(C, y)$ it coincides with the action defined before. □

Remark 2.5.3. We observe that we already know how $\mathbb{T}$ acts on C and $u(t \cdot [\xi]) = t \cdot u([\xi])$.

We will use this construction in Section 7.1 to compute the fundamental group of the surfaces that we construct.

The next step is to understand which points of Δ have non-trivial stabilizer for the action just described:

Lemma 2.5.4. *Let $[\xi] \in \Delta$ then*

$$\mathrm{Stab}_{\mathbb{T}}([\xi]) = \begin{cases} \{1\} & \textit{if } f(u([\xi])) \notin \{p_1, \ldots, p_r\} \\ \alpha\langle\gamma_i\rangle\alpha^{-1} & \textit{if } f(u([\xi])) = p_i, \textit{ for some } \alpha \in \mathbb{T} \end{cases}$$

Proof. Let $[\xi] \in \Delta$, $w := u[\xi] = \widehat{\gamma}$ and $z := f(w)$; let $t \in \mathbb{T}$. If $t \cdot [\xi] = [\xi]$, then we have also that $t \cdot \widehat{\gamma} = \widehat{\gamma}$.
Now there are two cases: either $z \notin \{p_1, \ldots, p_r\}$ or $z = p_i$ for some i.

If $z \notin \{p_1, \ldots, p_r\}$ then w is not a ramification point for f, so the $\mathbb{T}$ acts as $\pi_1(X, p)$ that acts freely on C_0 and so $\mathrm{Stab}(w) = \{1\}$ and so $t = 1$.

If $z = p_i$ for some i, we have that w is a ramification point for f, by Lemma 2.4.2, we get that $\mathrm{Stab}_G(w) = kSk^{-1}$ where $S = \langle h_i\rangle$ and $k \in G$, but we recall that G acts as follows: $g[\gamma] = [\theta^{-1}(g)\gamma] = (\alpha\gamma_i^d\alpha)[\gamma]$ for some $\alpha \in \mathbb{T}$ and $d \in \{1, \ldots, m_i - 1\}$, and so $\mathrm{Stab}_{\mathbb{T}}(w) = \alpha\langle\gamma_i\rangle\alpha^{-1}$. □

Chapter 3

Generalities on surfaces

In this chapter we recall some definitions and properties about divisors, intersection theory on surfaces and birational transformations. Some of them are taken from [Bea96], but we refer also to [Har77] and [GH78] for further details and discussions.

We also recall the Enriques-Kodaira classification and we give some properties of surfaces of general type (see also [BHPV04, Chapter VI]).

3.1 Intersection theory on surfaces

Let S be a smooth projective complex variety of dimension n. We recall that the Picard group ([Har77, page 143]) of S is the group of isomorphism classes of invertible sheaves (or line bundle) on S, and it is denoted by $\mathrm{Pic}(S)$. To every divisor D on S there corresponds an invertible sheaf $\mathcal{O}_S(D)$ and a meromorphic global section s unique up to scalar multiplication such that $div(s) = D$. The map $D \mapsto \mathcal{O}_S(D)$ identifies $\mathrm{Pic}(S)$ with the group of linear equivalence classes of divisors on S, see [Har77, Section II.6] for further details.

Let Ω^p_S be the sheaf of the holomorphic p-forms; let $\omega_S = \Omega^n_S$ be the line bundle of the holomorphic n-forms on S. A *canonical divisor* is any divisor K_S such that $\mathcal{O}_S(K_S) = \omega_S$.

Let X be another smooth variety and let $f\colon S \to X$ be a morphism. We can define the inverse image with respect to f of an invertible sheaf (see [Har77, Section II.5]), to get a homomorphism

$$f^*\colon \mathrm{Pic}X \longrightarrow \mathrm{Pic}S\,.$$

If f is a morphism of surfaces which is generically finite of degree d, then we define the direct image f_*C of an irreducible curve C by setting

$$f_*C = \begin{cases} 0 & \text{if } C \text{ is contracted to a point by } f \\ \\ rf(C) & \text{if } f(C) \text{ is a curve on } S\text{, and where } r \text{ is the degree} \\ & \text{of } C \to \pi(C) \text{ induced by } f \end{cases}$$

We define f_*D for all divisors D on S by linearity. It follows by definition that

$$f_*f^*D = dD \qquad \text{for all divisors } D \text{ on } S.$$

Definition 3.1.1. Let C, C' be two irreducible distinct curves on a surface S, let $x \in C \cap C'$, and let $\mathcal{O}_x$ be the local ring (see [Har77, page 16]) of S at x. If f and g are equation of C and C' in $\mathcal{O}_x$, the *intersection multiplicity* of C and C' at x is defined to be

$$m_x(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_x / \langle f, g \rangle \,.$$

By Nullstellensatz the ring $\mathcal{O}_x/\langle f, g\rangle$ is a finite-dimensional vector space over $\mathbb{C}$. We note that $m_x(C \cap C') = 1$ if and only if f and g generate the maximal ideal, i.e. form a system of local coordinates in a neighborhood of x: in this case C and C' are said to be *transverse* at x.

Definition 3.1.2. If C and C' are two distinct irreducible curves on a surface S, the *intersection number* $C.C'$ is defined by:

$$C.C' = \sum_{x \in C \cap C'} m_x(C \cap C') \,.$$

We define the intersection number on divisors extending by linearity the previous one and we get the following:

Proposition 3.1.3 ([Bea96, Theorem I.4])**.** *For L, $L' \in \operatorname{Pic}(S)$, define*

$$L.L' := \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}) \,.$$

Then . is a symmetric bilinear form on $\operatorname{Pic}(S)$, such that if C and C' are two distinct irreducible curves on S then

$$\mathcal{O}_S(C).\mathcal{O}_S(C') = C.C' \,.$$

Remark 3.1.4. If D, D' are divisors on S, we write $D.D'$ for $\mathcal{O}_S(D).\mathcal{O}_S(D')$. By the previous statement, we can calculate this product by replacing D or D' by a linear equivalent divisor.

Lemma 3.1.5 ([Bea96, Lemma I.6])**.** *Let C be a non-singular irreducible curve on S. For all $L \in \operatorname{Pic}(S)$, we have*

$$\mathcal{O}_S(C).L = \deg(L_{|C}) \,. \tag{3.1}$$

Definition 3.1.6. If D is any divisor on the surface S, we say that $D.D$, usually denoted by D^2, is the *self-intersection* of D.

In order to compute C^2 it would be useful the following remark:

Remark 3.1.7. Let C be a non singular irreducible curve on a surface S. Then $C^2 = \deg_C(\mathcal{N}_{C,S})$, where $\mathcal{N}_{C,S}$ is the normal bundle to C in S.

Lemma 3.1.8 ([Bea96, Proposition I.8])**.** *Let C be a smooth curve and let $f\colon S \to C$ be a surjective morphism. Let F be a fibre of f. Then $F^2 = 0$.*

Proposition 3.1.9 (Projection formula, [Bea96, Proposition I.8])**.** *Let S' be a surface, let $g\colon S \to S'$ be a generically finite morphism of degree d, let D and D' divisors on S'. Then*

$$g^*D.g^*D' = d(D.D')\,.$$

3.2 Riemann-Roch Theorem

We start this section recalling the Serre duality theorem:

Theorem 3.2.1 (Serre duality theorem, [Har77, Section II.7])**.** *Let M be a compact complex manifold of dimension n, and let L be a line bundle on M. Then for each $0 \le j \le n$ the vector spaces*

$$H^j(M, L) \quad \textit{and} \quad H^{n-j}(M, \omega_M \otimes L^{-1})$$

are dual. In particular,

$$\chi(L) = (-1)^n \chi(\omega_M \otimes L^{-1})\,.$$

Using the previous theorem we can prove the Riemann-Roch theorem.

Theorem 3.2.2 (Riemann-Roch)**.** *Let S be a smooth surface, let L a line bundle on S, it holds:*

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - L.\omega_S)\,.$$

Proof. Let us compute $L^{-1}.L \otimes \omega_S^{-1}$. By definition of the intersection product we get

$$L^{-1}.L \otimes \omega_S^{-1} = \chi(\mathcal{O}_S) - \chi(L) - \chi(\omega_S \otimes L^{-1}) + \chi(\omega_S)\,.$$

By Serre duality, we have $\chi(\mathcal{O}_S) = \chi(\omega_S)$ and $\chi(L) = \chi(\omega_S \otimes L^{-1})$, therefore we get:

$$L^{-1}.L \otimes \omega_S^{-1} = 2(\chi(\mathcal{O}_S) - \chi(L))\,.$$

Using the bilinearity of the intersection form we get

$$L^{-1}.L \otimes \omega_S^{-1} = L^{-1}.L + L^{-1}.\omega_S^{-1} = -L^2 + L.\omega_S$$

and this concludes the proof. ∎

This theorem is usually written in terms of divisors:

$$h^0(D) + h^0(K_S - D) - h^1(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D.K_S)\,.$$

As consequence of the Riemann-Roch theorem we have the following:

Lemma 3.2.3 (Genus formula, [Bea96, Lemma I.15])**.** *Let C be an irreducible curve on a smooth surface S. The geometric genus of C ($= h^1(C, \mathcal{O}_C)$) is given by $g(C) = 1 + \frac{1}{2}(C^2 + C.K_S)$.*

The genus formula can be written

$$2g(C) - 2 = (C + K_S).C = \deg(K_S + C)_{|C}\,.$$

This formula can also be deduced by (3.1) using the adjunction formula:

Proposition 3.2.4 (Adjunction formula, [GH78, page 147])**.** *Let M be a compact complex manifold, let $V \subset M$ be a smooth analytic hypersurface. Then*

$$K_V = (K_M + V)_{|V}\,.$$

3.3 Birational transformation and minimality

Let S be a smooth surface and let $p \in S$. Take a neighborhood U of p such that there exist local coordinates x, y at p (i.e. curves $x = 0$ and $y = 0$ which meet transversely at p). Up to shrink U, we can assume that p is the only point of U in the intersection of the two curves. We define $\hat{U}$ as the subvariety of $U \times \mathbb{P}^1$ given by the equation $xY - Xy = 0$, where X, Y are the homogeneous coordinates of $\mathbb{P}^1$.

It is obvious that the projection $\epsilon\colon \hat{U} \to U$ is an isomorphism over the point of U where at most one coordinate vanishes, while $\epsilon^{-1}(p) = \{p\} \times \mathbb{P}^1$. Let $\hat{S}$ be the surface obtained by passing $\hat{U}$ and $S \setminus \{p\}$ along $\hat{U} \setminus \epsilon^{-1}(p) \cong U \setminus \{p\}$.

Definition 3.3.1. We call $\epsilon\colon \hat{S} \to S$ the *blow-up of S in p*. $E := \epsilon^{-1}(p) \cong \mathbb{P}^1$ is the *exceptional curve* of the blow-up.

Remark 3.3.2. The restriction of ϵ to $\epsilon^{-1}(S \setminus \{p\})$ is an isomorphism onto $S \setminus \{p\}$.

Let $\epsilon\colon \hat{S} \to S$ be the blow-up in $p \in S$, and consider an irreducible curve C on S passing through p with multiplicity m. The closure of $\epsilon^{-1}(C \setminus \{p\})$ in $\hat{S}$ is an irreducible curve $\hat{C}$ which is called the *strict transform* of C.

Lemma 3.3.3 ([Bea96, Lemma II.2])**.** *Let $\epsilon\colon \hat{S} \to S$ be the blow-up of S in p. Let C be an irreducible curve on S passing through p with multiplicity m, then*

$$\epsilon^* C = \hat{C} + mE\,.$$

Proposition 3.3.4 ([Bea96, Proposition II.3]). *Let $\epsilon\colon \hat{S} \to S$ be the blow-up of a point $p \in S$. Let E be the exceptional curve, then*

- *there exists an isomorphism* $\mathrm{Pic}(S) \oplus \mathbb{Z} \to \mathrm{Pic}(\hat{S})$ *defined by* $(D, n) \mapsto \epsilon^* D + nE$.
- *Let D be a divisor on S. Then $\epsilon^* D.E = 0$ and $E^2 = -1$.*

Lemma 3.3.5. *Let $\epsilon\colon \hat{S} \to S$ be the blow-up of a point $p \in S$. The canonical divisor of $\hat{S}$ is given by $\epsilon^* K_S + E$ and $K_{\hat{S}}^2 = K_S^2 - 1$.*

Proof. Since the canonical sheaf on $\hat{S} \setminus E$ and $S \setminus \{p\}$ is the same, we have $K_{\hat{S}} = \epsilon^* K_S + nE$, for some integer n. Using the adjuction formula we get

$$-2 = 2g(E) - 2 = (K_{\hat{S}} + E).E \quad \Longrightarrow \quad K_{\hat{S}}.E = -1\,.$$

It follows that $-1 = K_{\hat{S}}.E = \epsilon^* K_S.E + nE^2 = 0 - n$ and so $n = 1$.

The formula for K^2 follows immediately using Proposition 3.3.4 and Proposition 3.1.9. □

We now recall some statements taken from [Bea96] that relate blow-ups and rational maps.

Theorem 3.3.6 (elimination of indeterminacy, [Bea96, Theorem II.7]). *Let $\varphi\colon S \dashrightarrow X$ be a rational map from a surface to a projective variety.*

Then there exists a surface S' and a morphism $\eta\colon S' \to S$ which is the composition of a finite number of blow-ups, and a morphism $f\colon S' \to X$ such that the diagram

$$\begin{tikzcd} & S' \arrow[dl, "\eta"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "\varphi"'] & & X \end{tikzcd}$$

commutes.

Theorem 3.3.7 (universal property of blowing-up, [Bea96, Proposition II.8]). *Let $f\colon S \to X$ be a birational morphism of surfaces, and suppose that the rational map f^{-1} is not defined at the point p of X.*

Then f factorizes as

$$f\colon S \xrightarrow{g} \hat{X} \xrightarrow{\epsilon} X$$

where g is a birational morphism and ϵ is the blow-up at p.

Theorem 3.3.8 ([Bea96, Theorem III.11]). *Let $f\colon S \to S_0$ be a birational morphism of surfaces.*

Then there is a finite sequence of blow-ups $\epsilon_k\colon S_k \to S_{k-1}$ $(k = 1, \ldots, n)$ and an isomorphism $u\colon S \to S_n$ such that $f = \epsilon_1 \circ \ldots \circ \epsilon_n \circ u$.

Corollary 3.3.9 ([Bea96, Corollary II.12]). *Let $\varphi\colon S' \dashrightarrow S$ be a birational map of surfaces.*

Then there are a surface $\hat{S}$ and morphisms $f\colon \hat{S} \to S'$ and $g\colon \hat{S} \to S$ which are the composition of a finite number of blow-ups and isomorphisms such that the diagram

$$\begin{tikzcd} & \hat{S} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "\varphi"'] & & X \end{tikzcd}$$

commutes.

Definition 3.3.10. Let S_1 and S_2 be two surfaces, we say that S_1 *birationally dominates* S_2 if there exists a birational morphism $S_1 \to S_2$.

A smooth surface S is *minimal* if every birational morphism $S \to S'$ is an isomorphism.

Proposition 3.3.11 ([Bea96, Proposition II.16]). *Every smooth surface birationally dominates a minimal surface.*

Definition 3.3.12. Let $S' \to S$ be a birational morphism between smooth surfaces. If S is minimal, we say that S is the *minimal model* of S'.

Remark 3.3.13. By Theorem 3.3.8 we have that a surface is minimal if and only if it contains no exceptional curve.

Let E be an exceptional curve, by definition $E \cong \mathbb{P}^1$ and by Proposition 3.3.4 $E^2 = -1$. The next important statement gives the converse:

Theorem 3.3.14 (Castelnuovo's contractibility criterion, [Bea96, Theorem II.17]). *Let S be a surface and let $E \subset S$ be a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then E is an exceptional curve on S.*

Proposition 3.3.15 ([BHPV04, Proposition III.2.2]). *An irreducible curve $C \subset S$ is an exceptional curve if and only if*

$$C^2 < 0 \quad \textit{and} \quad K_S.C < 0\,.$$

3.4 Numerical invariants

To every smooth projective surface S we can associate some birational invariants (see [Bea96, Proposition III.20]):

$$\begin{aligned} q(S) &= h^1(S, \mathcal{O}_S) \\ p_g(S) &= h^0(S, \mathcal{O}_S(K_S)) = h^2(S, \mathcal{O}_S) \text{ (by Serre duality)} \\ P_n(S) &= h^0(S, \mathcal{O}_S(nK_S)) \text{ for } n \geq 1 \end{aligned}$$

$q(S)$ is called the *irregularity* of S, $p_g = P_1$ is the *geometric genus*, and the P_n are called the *plurigenera* of S. We have $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$.

We denote by $e(S)$ the topological Euler-Poincaré characteristic of S: $e(S) = \sum(-1)^i b_i$ where $b_i = \dim_{\mathbb{C}} H^i(S, \mathbb{C})$ are the Betti's numbers. By Poincaré duality, we get that

$$b_0 = b_4 = 1 \qquad \text{and} \qquad b_1 = b_3\,. \tag{3.2}$$

$e(S)$ is not a birational invariant, indeed if $\epsilon\colon S' \to S$ is the blow-up of S in p, then $e(S') = e(S) + 1$, since we replace a point ($e(p) = 1$) with a rational curve E ($e(E) = 2$).

The invariants $q(S)$ and $b_1(S)$ are related by the following equation ([Bea96, Fact III.19]):

$$q(S) = h^0(S, \Omega^1_S) = \frac{1}{2} b_1(S)\,, \tag{3.3}$$

in particular q is a topological invariant.

The self-intersection of the canonical divisor K_S^2 is a topological (but not birational, see Lemma 3.3.5) invariant, indeed by Topological index theorem (see [BHPV04, Theorem I.3.1]),

$$K_S^2 = 3\tau(S) + 2e(S)\,,$$

where $\tau(S)$ is the index of S (see [BHPV04, page 22]) that is a topological invariant.

Theorem 3.4.1 (Noether's formula, [GH78, page 438])**.** *Let S be a smooth surface:*

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + e(S))\,.$$

It follows that $\chi(\mathcal{O}_S)$ (and so $p_g(S)$) is a topological invariant.

Lemma 3.4.2 ([Bea96, Lemma VI.3])**.** *Let $p\colon S' \to S$ be an étale map of degree d between surfaces. Then $K_{S'}^2 = d \cdot K_S^2$, $e(S)' = d \cdot e(S)$ and $\chi(S') = d \cdot \chi(S)$.*

Proof. The last equation follows from the first two using Noether's formula. The first follows immediately from projection formula, since $K_{S'} = p^* K_S$.

To prove the second equation we start choosing a triangulation of S, then $e(S) = \sum(-1)^i f_i(S)$, where $f_i(S)$ is the number of faces of dimension i. Since the faces are simply connected, their inverse images in S' triangulate it. Clearly $f_i(S') = d \cdot f_i(S)$ and we are done. □

3.5 The Enriques-Kodaira classification

Let X be any compact complex manifold, let ω_X be its canonical bundle. To X one associates its *canonical ring*:

$$R(X) = \bigoplus_{m \geq 0} H^0(\omega_X^{\otimes m}) .$$

This ring is commutative; let $(tr(R(X))$ be its degree of trascendency over $\mathbb{C}$.

Definition 3.5.1. Let X be a compact complex manifold. We define the *Kodaira dimension* $\kappa(X)$ as follows:

$$\kappa(X) = \begin{cases} -\infty & \text{if } R(X) \cong \mathbb{C} \\ tr(R(X)) - 1 & \text{otherwise} \end{cases}$$

The Kodaira dimension is a birational invariant, and for a compact complex manifold X, $\kappa(X)$ can assume the values: $-\infty$, 0, $\dots$, $n = \dim X$.

Remark 3.5.2 ([Har77, page 421]). Let X be a smooth compact complex variety, let K be a canonical divisor of X, let ϕ_{mK} be the rational map from X to the projective space associated with the linear system $|mK|$. The Kodaira dimension of X is equal to the maximal dimension of the images $\phi_{mK}(X)$, for $n \geq 1$.

Definition 3.5.3. A variety X is said to be *of general type* if its Kodaira dimension is maximal: $\kappa(X) = \dim X$.

Theorem 3.5.4 ([BHPV04, Theorem I.7.2] or [Uen75, Theorem 8.1])**.** *Let X be a smooth compact complex variety. Then*

- $\kappa(X) = -\infty$ *if and only if* $P_m(X) = 0$ *for all* $m \geq 1$*.*
- $\kappa(X) = 0$ *if and only if* $P_m(X) = 0$ *or* 1 *for* $m \geq 1$*, but not always 0.*
- $\kappa(X) = k$*, for* $1 \leq k \leq \dim X$ *if and only if there are real constants* $\alpha > 0$ *and* $\beta > 0$ *such that* $\alpha m^k < P_m(X) < \beta m^k$ *for* m *large enough.*

Corollary 3.5.5. *Let X be a smooth compact complex variety of dimension k. X is of general type if and only if*

$$\lim_{m \to \infty} \frac{P_m(X)}{m^k} > 0$$

Remark 3.5.6 ([Bea96, Example VII.2]). For a curve it is easy to give the Kodaira dimension explicitly. Let C be a smooth curve of genus g. Then

$$\begin{aligned} \kappa(C) = -\infty &\iff g = 0 \\ \kappa(C) = 0 &\iff g = 1 \\ \kappa(C) = 1 &\iff g \geq 2 \end{aligned}$$

Proposition 3.5.7 ([Bea96, Proposition VII.4]). *Let C and D be two smooth curves, let $S = C \times D$. Then*

- *if C or D is rational, then S is ruled: $\kappa(S) = -\infty$.*
- *if C and D are elliptic, then $\kappa(S) = 0$.*
- *if C is elliptic and $g(D) \geq 2$ then $\kappa(S) = 1$.*
- *if $g(C) \geq 2$ and $g(D) \geq 2$then $\kappa(S) = 2$.*

Proof. If p_1 and p_2 are the projection of S to C and D respectively, we have $\omega_S \cong p_1^*\omega_C \otimes p_2^*\omega_D$ ([Har77, Section II.8]) and $H^0(S, \mathcal{O}_S(nK_S)) \cong H^0(C, \omega_C^{\otimes n}) \otimes H^0(D, \omega_D^{\otimes n})$ ([Bea96, Fact III.22]) , so that the rational map $\phi_{nK} \colon S \dashrightarrow \mathbb{P}^N$ factorizes as

$$\phi_{nK} \colon C \times D \overset{(\phi_{nK_C}, \phi_{nK_D})}{\dashrightarrow} \mathbb{P}^{N'} \times \mathbb{P}^{N''} \overset{s}{\hookrightarrow} \mathbb{P}^N$$

where s is the Segre embedding. The proposition follows from Remark 3.5.6. □

The previous proposition is a particular case of a more general theorem:

Theorem 3.5.8 ([Uen75, page 69]). *If X_1 and X_2 are connected compact complex manifolds, then $\kappa(X_1 \times X_2) = \kappa(X_1) + \kappa(X_2)$.*

Theorem 3.5.9. *Let A be a compact complex manifold, and let G be a finite group acting on A, let $S \to X$ be the minimal resolution of the singularities of $X := A/G$. Then $\kappa(A) \geq \kappa(S)$.*

Proof. Let us consider the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"'] & A \arrow[d] \\ S \arrow[r] & X \end{tikzcd}$$

where Y is the fibred product of A and S over X.
We note that π is a branched covering and so $K_Y = \pi^*K_S + D$, with D effective divisor on Y; hence for each integer $m \geq 1$ we have $H^0(mK_S) \hookrightarrow H^0(mK_Y)$, and so $h^0(mK_Y) \geq h^0(mK_S)$. Let $k := \kappa(S)$; if $k \leq 0$ it follows immediately by Theorem 3.5.4 that $\kappa(Y) \geq \kappa(S)$; otherwise it is enough to note that

$$\lim_{m\to\infty} \frac{P_m(Y)}{m^k} \geq \lim_{m\to\infty} \frac{P_m(S)}{m^k}$$

in order to conclude that $\kappa(Y) \geq \kappa(S)$. Since f is a birational map we get $\kappa(A) = \kappa(Y) \geq \kappa(S)$ and we are done. □

As already noted, the Kodaira dimension of a n-dimensional compact complex manifold can assume the values $-\infty, 0, 1, \ldots, n$. In the case $n = 2$, the surfaces in the classes $\kappa = -\infty$ and $\kappa = 0$, and to a lesser extent those with $\kappa = 1$ can be classified in detail. This classification is called the "Enriques-Kodaira classification" and is collected in the following result.

Theorem 3.5.10 ([BHPV04, Theorem VI.1.1])**.** *Every compact complex surface has a minimal model in exactly one of the ten classes of Table 3.1. This model is unique (up to isomorphism) except for the surfaces with minimal model in classes 1) and 3).*

$\kappa(X)$	Class of X	K_X^2	$e(X)$
$-\infty$	1) minimal rational surfaces	8 or 9	4 or 3
	2) minimal surfaces of class VII	≤ 0	≥ 0
	3) ruled surfaces of genus $g \geq 1$	$8(1-g)$	$4(1-g)$
0	4) Enriques surfaces	0	12
	5) bi-elliptic surfaces	0	0
	6) Kodaira surfaces	0	0
	7) K3 surfaces	0	24
	8) tori	0	0
1	9) minimal properly elliptic surfaces	0	≥ 0
2	10) minimal surfaces of general type	> 0	> 0

Table 3.1:

A *rational surface* is a surface birational to $\mathbb{P}^2$. The only minimal surfaces of this type are $\mathbb{P}^2$ and the Hirzebruch surfaces $\Sigma_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, with $n = 0, 2, 3, \ldots$ ($\mathbb{P}^1 \times \mathbb{P}^1 = \Sigma_0$).

Theorem 3.5.11 (Castelnuovo's Rationality Criterion, [Bea96, Theorem V.1])**.** *Let S be a surface with $q = P_2 = 0$. Then S is rational.*

Remark 3.5.12. The condition $P_2 = 0$ implies $p_g = 0$. In analogy with the case of the curves, it seems more natural to replace the hypothesis of the statement with the weaker assumption $q = p_g = 0$, but in 1896 Enriques constructed a surface with $q = p_g = 0$ and $\pi_1 = \mathbb{Z}_2$ and so not rational.

A *surface of class VII* is a surface X with $\kappa(X) = -\infty$ and $b_1(X) = 1$, moreover $q = 1$. These surfaces are neither algebraic nor Kähler. Examples of this type of surfaces are Hopf surfaces ([Hop48]) and Inoue surfaces ([Ino74]).

Ruled surfaces of genus g have a smooth morphism to a curve of genus g whose fibres are lines $\mathbb{P}^1$.

Theorem 3.5.13 (Enriques, [Bea96, Corollary VI.18])**.** *Let S be a smooth projective complex surface, the following are equivalent:*

- *S is ruled;*
- *$P_n = 0$ for all n;*
- *$P_{12} = 0$.*

An *Enriques surface* X is a surface with $q(X) = 0$ and non-trivial canonical K_X, but with $2K_X$ trivial.

A *bi-elliptic surface* (or hyperelliptic surface) is a surface X with $b_1(X) = 2$ and an elliptic fibration over an elliptic curve. Any such surface is the quotient of a product of two elliptic curves by a finite abelian group.

Kodaira surfaces are usually divided into two subtypes: the primary Kodaira surfaces are surfaces with $b_1 = 3$ and an elliptic fibration over an elliptic curve; the secondary Kodaira surfaces are surfaces which admits a primary Kodaira surface as unramified covering of degree ≥ 2. These surfaces are not algebraic.

A *K3 surface* X is a surface with $q = 0$ and trivial canonical bundle. They are all Kähler varieties.

A *torus* is a surface isomorphic to the quotient of $\mathbb{C}^2$ by a lattice of real rank 4. A torus is diffeomorphic to $S^1 \times S^1 \times S^1 \times S^1$ so its fundamental group is $\mathbb{Z}^4$.

A *properly elliptic surface* is a surface admitting an elliptic fibration with $\kappa(X) = 1$. A very simple example is provided by the product of two curve, one elliptic and the other of genus ≥ 2.

3.6 Surfaces of general type

Following the Enriques-Kodaira classification we can divide compact complex surfaces in four main classes according to their Kodaira dimension: $-\infty$, 0, 1, 2. Nowadays the first three classes are much better understood than the last one.

Definition 3.6.1. A surface X is said to be *of general type* if $\kappa(X) = 2$.

Remark 3.6.2 (BHPV, Corollary 6.5). Every smooth surface of general type is projective.

Theorem 3.6.3 ([BHPV04, Theorem VII.2.2])**.** *If X is a minimal surface of general type, then $K_X^2 > 0$.*

Theorem 3.6.4 ([Bea96, Theorem X.4])**.** *If X is any surface of general type, then $e(X) \geq 0$ and $\chi(\mathcal{O}_S) \geq 1$.*

By Noether's formula, the condition $e(S) \geq 0$ is equivalent to $K_S^2 \leq 12\chi(\mathcal{O}_S)$. For a surface of general type Bogomolov and Miyaoka, and independently Yau proved the stronger

Theorem 3.6.5 ([BHPV04, Theorem VII.4.1])**.** *Let S be a smooth surface of general type. Then*

$$K_S^2 \leq 9\chi(\mathcal{O}_S)\,. \tag{BMY}$$

In literature are well-known other inequalities that involve the invariants of minimal surfaces of general type:

Theorem 3.6.6 ([BHPV04, Theorem VII.3.1])**.** *Let S be a smooth surface of general type. Then*

$$K_S^2 \geq 2p_g(S) - 4 \tag{N}$$

$$\textit{if } q > 0 \Longrightarrow K_S^2 \geq 2p_g(S) \tag{D}$$

The inequality (N) is due to Noether, while (D) is due to Debarre.

In the following picture there are drawn the limit lines of the inequalities in the (χ, K^2) plane.

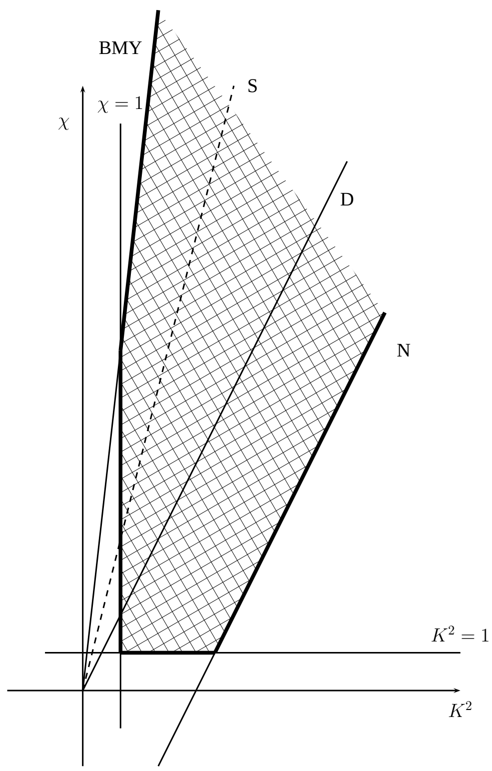


Figure 3.1:

The above listed inequalities show that the pair $(\chi(\mathcal{O}_S), K_S^2)$ for a surface S of general type gives a point with integral coordinates in the convex region limited by the "bold" piecewise linear curves. Moreover if $q > 0$ this point cannot be at the "right" of the line D. The line labeled by S is the Severi line $K^2 = 4\chi$.

In order to explain the meaning of this line we have to introduce the Albanese variety and the Albanese morphism.

Let X be a connected compact Kähler manifold. To X is associated a complex torus of dimension $g = h^{1,0}(X)$: the *Albanese variety* $\mathrm{Alb}(X)$ as follows. Let $w_1, \ldots, w_g$ be a basis for $H^0(X, \Omega^1_X)$, so that $w_1, \ldots, w_g, \overline{w_1}, \ldots, \overline{w_g}$ form a basis of $H^1(X, \mathbb{C})$. Furthermore, let $h_1, \ldots, h_{2g}$ be a basis for $H_1(X, \mathbb{Z})$ modulo torsion. We consider the vectors

$$v_j = \left(\int_{h_j} w_1, \ldots, \int_{h_j} w_g \right) \in \mathbb{C}^g , \qquad j = 1, \ldots, 2g .$$

It can be proved that they are linearly independent over $\mathbb{R}$ (see [BHPV04, page 46]).

The vectors $v_1, \ldots, v_{2g}$ span an integral lattice Γ in $\mathbb{C}^g$ and thus determine a complex torus $\mathbb{C}^g/\Gamma$. Replacing the h_j's or the w_k's by another basis, we obtain the same torus, up to isomorphism. This torus is $\mathrm{Alb}(X)$.

Fixing a point $x_0 \in X$ define the holomorphic map $\alpha\colon X \to \mathrm{Alb}(X)$ by $\alpha(x) = \left(\int_{x_0}^x w_1, \ldots, \int_{x_0}^x w_g\right)$. Changing x_0 amounts to change α by a translation of $\mathrm{Alb}(X)$. This map is called the *Albanese morphism.*

The Albanese morphism is a very useful tool for studying irregular surfaces, in particular:

Definition 3.6.7. A variety X is called of *maximal Albanese dimension* if the image of the Albanese morphism has dimension $\dim X$.

This is the general case for a surface, since otherwise the Albanese morphism is a fibration onto a smooth curve of genus $q(X)$.

We can now explain the Severi line:

Theorem 3.6.8 ([Par05]). *If S is a smooth complex minimal surface of maximal Albanese dimension then $K_S^2 \geq 4\chi$.*

3.6.1 Surfaces of general type with $\chi = 1$

There is no hope at the moment to achieve a classification of the whole class of the surfaces of general type. Since for a surface in this class the Euler characteristic of the structure sheaf χ is strictly positive, one could hope that a classification of the boundary case $\chi = 1$ is more affordable. We report here some progresses in this direction, we refer to [BCP06] for more details.

$p_g = q \geq 4$

Theorem 3.6.9 (Beauville, [Bea82]). *If S is a minimal surface of general type, then $p_g \geq 2q - 4$. Moreover, if $p_g = 2q - 4$, then S is a product of a curve of genus 2 with a curve of genus $q - 2$.*

Corollary 3.6.10. *If $p_g = q$, (i.e. $\chi(\mathcal{O}_S) = 1$), then $p_g = q \leq 4$. Moreover, minimal surfaces of general type with $p_g = q = 4$ are exactly the products of two genus 2 curves.*

Hence this case is clear, we just mention that $K_S^2 = 8$.

$p_g = q = 3$

These surfaces have been studied in [CCML98], [Pir02] and [HP02] and they are completely classified:

Theorem 3.6.11. *A minimal surface of general type with $p_g = q = 3$ has $K^2 = 6$ or $K^2 = 8$ and, more precisely,*

- *if $K^2 = 6$, S is the symmetric square of a genus 3 curve;*
- *otherwise $S = (C_2 \times C_3)/\tau$, where C_g denotes a curve of genus g and τ is an involution of product type acting on C_2 as an elliptic involution (i.e. , with elliptic quotient), and on C_3 as a fixed point free involution*

$p_g = q = 2$

This case is still far from being classified, but Ciliberto and Mendes Lopes in [CML02] classify the surfaces in this class with non-birational bicanonical map (not presenting the standard case).

Definition 3.6.12. A surface S of general type *presents the standard case* (for the non birationality of the bicanonical map), if there exists a dominant rational map onto a curve $f\colon S \dashrightarrow B$ whose general fibre is irreducible of genus 2.

Theorem 3.6.13. *If S is a minimal surface of general type with $p_g = q = 2$ and non birational bicanonical map not presenting the standard case, then S is a double covering of a principally polarized abelian surface (A, Θ), with Θ irreducible. The double covering $S \to A$ is branched along a divisor $B \in |2\Theta|$, having at most double points. In particular $K_S^2 = 4$.*

Other results on the classification of minimal surfaces of general type with $p_g = q = 2$ are due to Zucconi and Penegini (see [Zuc03] and [Pen11]). They produced a complete classification of surfaces with $p_g = q = 2$ and $K^2 = 8$ which are isogenous to a product of curves (see Definition 4.1.2); as a by-product, they obtained the classification of all surfaces with these invariant such that the image of the Albanese morphism is a curve (see Section 4.5).

$p_g = q = 1$

In this case the classical inequalities give $2 \leq K^2 \leq 9$, and the Albanese morphism is a map onto an elliptic curve, in particular all these surfaces have a fibration with base a curve of genus 1.

We denote by $\mathcal{M}_{K_S^2, p_g, q}$ the projective moduli scheme of surfaces of general type with fixed K_S^2, p_g, q.

Theorem 3.6.14. *It holds:*

- $\mathcal{M}_{2,1,1}$ *is irreducible and unirational and it has dimension 7, and the Albanese map of all these surfaces is a genus 2 fibration.*
- $\mathcal{M}_{3,1,1}$ *has 4 connected components, all unirational of dimension 5. The Albanese map is a genus 3 fibration for the surfaces in one of those components, while it is a genus 2 fibration for the others.*
- $\mathcal{M}_{j,1,1}$ *is non empty for* $j = 4, \ldots, 8$.

The cases $K^2 = 2, 3$ are completely classified. In [Cat81b] the author proves that all the surfaces with $K^2 = 2$ are double covers of the symmetric square of their Albanese curve.
In [CC91] the authors study the case $K^2 = 3$. They show that the Albanese map could be either a genus 2 or 3 fibration. The case $g = 3$ was classified in [CC93], while in [CP06] was classified the case $g = 2$.

Some examples of surfaces of with $p_g = q = 1$ and $K^2 = 4, 5$ are due to Catanese ([Cat99]), and these examples are constructed as bidouble covers.

Rito ([Rit07]) and Polizzi ([Pol08]) constructed some examples of surfaces of general type with $p_g = q = 1$ and $K^2 = 6$. Also the first example with $K^2 = 7$ is due to Rito ([Rit10b]).

The case $K^2 = 8$ was studied by Polizzi ([Pol06]) who consider the case of surfaces having bicanonical map of degree 2. He could prove that all these surfaces are isogenous to a product (see Definition 4.1.2) and they form three components of the moduli space, one of dimension 5 and two of dimension 4.

It remains unsettled the existence of surfaces of general type with $p_g = q = 1$ and $K^2 = 9$.

Other results towards the classification of minimal surfaces of general type with $p_g = q = 1$ are due to Carnovale, Mistretta and Polizzi; we comment these results in Section 4.5.

$p_g = q = 0$

This class of surfaces is one of the most complicated and intriguing classes of surfaces of general type. By the standard inequalities we have $1 \leq K^2 \leq 9$.

The first examples of surfaces in this class are due to Campedelli ([Cam32]) and Godeaux ([God34b]) in the 30's, and in their honor minimal surfaces of general type with $K^2 = 1$ are called *numerical Godeaux surfaces*, and those with $K^2 = 2$ are called *numerical Campedelli surfaces*.

Concerning the classification of minimal surfaces of general type with $p_g = q = 0$, there have been many recent progresses. Nowadays there are examples for each value of $1 \leq K^2 \leq 9$.

If $K_S^2 = 9$, then S is a quotient of the unit ball in $\mathbb{C}^2$ by a discrete group acting freely ([Yau77],[Yau78]). This surfaces are called *fake projective planes*: they have the same Betti numbers of $\mathbb{P}^2$, but they are not birational to it. Thanks to the new works of Prasad and Yeung and of Steger and Cartright ([PY07], [PY10], [CS10]) asserting that the moduli space consists exactly of 100 points, corresponding to 50 pairs of complex conjugate surfaces (cf. [KK02]), this case is completely classified.

Let $K_S^2 = 8$. In this case, is the bidisk in $\mathbb{C}^2$ the universal cover of S? If this is the case, then a complete classification should be possible. The classification has already been accomplished in [BCG08] for the reducible case where there is a finite étale covering which is isomorphic to a product of curves, see Section 4.5 for further details.

There are many examples, due to Kuga and Shavel ([Kug75], [Sha78]) for the irreducible case, which yield rigid surfaces, but a complete classification of this second case is still missing.

Let $K_S^2 = 1$. In this case it is known that the algebraic fundamental group is finite:

Theorem 3.6.15 (cf. [Rei78]). *Let S be a numerical Godeaux surface, then $\hat{\pi}_1(S) \cong \mathbb{Z}_m$ for $1 \leq m \leq 5$.*

The first example of a numerical Godeaux surfaces with $\hat{\pi}_1 \cong \mathbb{Z}_5$ is due to Godeaux: see [God34b]. M. Reid in [Rei78] constructs the first examples of numerical Godeaux surfaces with $\hat{\pi}_1 \cong \mathbb{Z}_m$ for $m = 3, 4$. The first examples with $\hat{\pi}_1 \cong \mathbb{Z}_2$ or $\hat{\pi}_1$ trivial, are due to R. Barlow, see [Bar84] and [Bar85] respectively.

Moreover there is the following conjecture:

Conjecture (M. Reid). The moduli space of the canonical models of minimal surfaces of general type with $\chi = 1$ and $K^2 = 1$, has exactly five irreducible components corresponding to each choice $\pi_1 = \mathbb{Z}_m$ for $1 \leq m \leq 5$.

By [Rei78], it is known that the conjecture holds for $m \geq 3$.

Let $K_S^2 = 2$. Also in this case the algebraic fundamental group $\hat{\pi}_1(S)$ is finite:

Theorem 3.6.16 (cf. [Rei], [Xia85]). *Let S be a numerical Campedelli surface, then $|\hat{\pi}_1(S)| \leq 9$.*

The question whether all these groups can occur has been open for a while. By the works of many authors the answer is affirmative:

Theorem 3.6.17. *Let S be a numerical Campedelli surface, then $\hat{\pi}_1(S)$ is either the quaternion group or an abelian group of order at most 9.*
All these cases are possible.

By the works of Mendes Lopes, Pardini and Reid ([Rei], [MLP08], [MLPR09]), the cases of order 8 and 9 are classified. In particular, they show that the topological fundamental group equals the algebraic fundamental group and that cannot be the dihedral group D_4 of order 8. In [Nai99] the author proves that the symmetric group $\mathfrak{S}_3$ of order 6 cannot occur as the fundamental group of a numerical Campedelli surface.

The last open case, $\mathbb{Z}_4$, is realized by our examples (see Section 7.1) and by a completely different construction found independently by [PPS10a]. We note that the topological fundamental group of [PPS10a] is not known.

In [BCP11], two question about the topological fundamental group has been posed:

Question 1. Let S be a numerical Campedelli surface.

- Is $\pi_1(S)$ finite? In particular, $|\pi_1(S)| \leq 9$?
- Does every group of order ≤ 9 except $\mathfrak{S}_3$ and D_4 occur as topological fundamental group (not only as algebraic fundamental group)?

We mention (cf. [BCP11]) that after our constructions, the only open case left for the latter question is $\mathbb{Z}_6$.

The constructions of minimal surfaces of general type with $p_g = 0$ and $K^2 \leq 7$ available in literature are listed in Table 3.2 and Table 3.3 (cf. [BCP11, Table 1, 2, 3]). We remark that we have included in the tables also the surfaces constructed in [BCGP08] and [BP10]; in Section 4.5, we will comment with more details these results.

Table 3.2: Minimal surfaces of general type with $p_g = 0$ and $K^2 \leq 3$ available in the literature

K^2	π_1	π_1^{alg}	H_1	References
1	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[God34a][Rei78][Miy76]
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[Rei78][OP81][Bar84][Nai94] [BP10]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[Rei78]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Bar84][Ino94][KLP10] [BP10]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[Wer94][Wer97]
	$\{1\}$	$\{1\}$	$\{0\}$	[Bar85][LP07]
	?	$\{1\}$	$\{0\}$	[CG94][DW99]
2	$\mathbb{Z}_9$	$\mathbb{Z}_9$	$\mathbb{Z}_9$	[MLP08]
	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^2$	[Xia85][MLP08]
	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	[Cam32][Rei][Pet76][Ino94] [Nai94][BCGP08]
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[Rei][Nai94][Keu88] [BCGP08]
	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	[Rei] [BP10]
	Q_8	Q_8	$\mathbb{Z}_2^2$	[Rei] [Bea99][BP10]
	$\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_7$	[Rei91]
	?	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[NP09]
	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	[Cat81a][Sup98][BCGP08][BP10]
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	[Ino94][Keu88] [BCGP08][BP10]
	?	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[PPS10a]
	?	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[LP09]
	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[BCGP08][BP10]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[KLP10]
	?	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[LP09]
	$\{1\}$	$\{1\}$	$\{0\}$	[LP07]
3	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	[Nai94] [Keu88] [MLP04a]
	$Q_8 \times \mathbb{Z}_2$	$Q_8 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$	[Bur96][Pet77] [Ino94][NP11][BC10]
	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}$	[CS10]
	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}$	[CS10]
	Q_8	Q_8	$\mathbb{Z}_2^2$	[CS10]
	D_4	D_4	$\mathbb{Z}_2^2$	[CS10]
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[CS10][BP10]
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	[BP10]
	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	[BP10]
	$\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_7$	[CS10]
	$\mathfrak{S}_3$	$\mathfrak{S}_3$	$\mathbb{Z}_2$	[CS10]
	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[CS10][BP10]
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	[CS10]
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	[CS10]
	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	[CS10]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[KLP10][CS10]
	?	?	$\mathbb{Z}_2$	[PPS10b]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS09a][CS10]

Table 3.3: Minimal surfaces of general type with $p_g = 0$ and $4 \leq K^2 \leq 7$ available in the literature

K^2	π_1	π_1^{alg}	H_1	References
4	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	[Nai94][Keu88][BCGP08]
	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$	[BCGP08]
	$Q_8 \times \mathbb{Z}_2^2$	$Q_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	[Bur96][Pet77][Ino94]
	$(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	$(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	$\mathbb{Z}_4^2$	[BCGP08]
	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	[BCGP08]
	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	[BCGP08]
	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	[BCGP08]
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	[BP10]
	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$	$\mathbb{Z}_3^2$	[BCGP08]
	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2^3$	[BCGP08]
	$\mathbb{Z}_8$	$\mathbb{Z}_8$	$\mathbb{Z}_8$	[BP10]
	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	[BP10]
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	[KLP10]
	$\{1\}$	$\{1\}$	$\{0\}$	[PPS09b]
5	$Q_8 \times \mathbb{Z}_2^3$	$Q_8 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	[Bur96][Pet77][Ino94]
	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$	[BP10]
	$\mathbb{Z}_5 \times Q_8$	$\mathbb{Z}_5 \times Q_8$	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	[BP10]
	$1 \to \mathbb{Z}^2 \to \pi_1 \to D_{2,8,3} \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	[BP10]
	$1 \to \mathbb{Z}^2 \to \pi_1 \to \mathbb{Z}_8 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	[BP10]
	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	[BP10]
	$D_{8,4,3}$	$D_{8,4,3}$	$\mathbb{Z}_2 \times \mathbb{Z}_8$	[BP10]
	$D_{8,5,-1}$	$D_{8,5,-1}$	$\mathbb{Z}_8$	[BP10]
	?	?	?	[Ino94]
6	$\mathbb{Z}^2 \rtimes \mathbb{Z}_{15}$	$\mathbb{Z}^2 \rtimes \mathbb{Z}_{15}$	$\mathbb{Z}_3 \times \mathbb{Z}_{15}$	[BCGP08]
	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^6$	[Bur96][Pet77][Ino94]
	$1 \to \mathbb{Z}^2 \times \Pi_2 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	[BCGP08]
	$1 \to \Pi_2 \to \pi_1 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	[BCGP08]
	$\mathbb{Z}_7 \times \mathfrak{A}_4$	$\mathbb{Z}_7 \times \mathfrak{A}_4$	$\mathbb{Z}_{21}$	[BCGP08]
	$\mathbb{Z}_5 \times \mathfrak{A}_4$	$\mathbb{Z}_5 \times \mathfrak{A}_4$	$\mathbb{Z}_{15}$	[BCGP08]
	$1 \to \mathbb{Z}^6 \to \pi_1 \to \mathbb{Z}_3^3 \to 1$	$\hat{\pi}_1$	$\mathbb{Z}_3^3 \subset H_1$	[Kul04]
	$\mathfrak{S}_3 \times D_{4,5,-1}$	$\mathfrak{S}_3 \times D_{4,5,-1}$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	[BCGP08]
	?	?	?	[Ino94][MLP04b]
7	$1 \to \Pi_3 \times \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^3 \to 1$	$\hat{\pi}_1$	?	[Ino94][MLP01] [BCC]

Chapter 4

Group action on product of curves

The first examples of surfaces of general type with $p_g = 0$ have been constructed in the 30's by L. Campedelli and L. Godeaux.

The idea of Godeaux was to consider the quotient of simpler surfaces by the free action of a finite group. In this spirit, Beauville (see [Bea96, page 118]) proposed a simple construction of surfaces of general type, considering the quotient of a product of two curves C_1 and C_2 by the free action of a finite group G. Moreover he gave an explicit example considering the quotient of two Fermat curves of degree 5 in $\mathbb{P}^2$.

After [Cat00] many authors started studying the surfaces that appear as quotient of a product of curves.

4.1 Group action on product of curves

In this chapter $C_1, \ldots, C_n$ will be smooth projective curves of respective genus $g_i := g(C_i)$ and G will be a finite group acting on $C_1 \times \ldots \times C_n$. Following [Cat00] the action can be of two types:

- *Unmixed*: G acts independently on each factor $G \hookrightarrow \mathrm{Aut}(C_i)$, and the action of G on $C_1 \times \ldots \times C_n$ is the diagonal action:

$$g(x_1, \ldots, x_n) = (g(x_1), \ldots, g(x_n))\,;$$

  in this case $G \hookrightarrow \mathrm{Aut}(C_1 \times \ldots \times C_n) \subseteq \mathrm{Aut}(C_1) \times \ldots \times \mathrm{Aut}(C_n)$. The latter inclusion is an equality if and only if the curves are not pairwise isomorphic.

- *Mixed*: there are elements of G that permute some factors $C_1, \ldots, C_n$, in this case these factors are isomorphic;
  for example in the case $C_1 \cong \ldots \cong C_n$, we have $G \hookrightarrow \mathrm{Aut}(C \times \ldots \times C) = \mathrm{Aut}(C)^n \rtimes \mathcal{S}_n$.

From now on, we always consider $g_i \geq 2$ for each i. The reason of this choice is given by the following:

Lemma 4.1.1. *Let G be a finite group acting on $C_1 \times \ldots \times C_n$, where the C_i are Riemann surfaces. Let $S \to X$ be the minimal desingularization of $X := (C_1 \times \ldots \times C_n)$. If S is of general type then $g(C_i) \geq 2$ for each i.*

Proof. By Lemma 3.5.9, we have $\kappa(C_1 \times \ldots \times C_n) \geq \kappa(S) = n$, S is of general type. By Theorem 3.5.8, we get $\kappa(C_1 \times \ldots \times C_n) = \kappa(C_1) + \ldots + \kappa(C_n) = n$, hence $\kappa(C_i) = 1$ for each i, by Remark 3.5.6 it is equivalent to $g(C_i) \geq 2$. □

Since we want to construct surfaces S of general type as minimal desingularization of surfaces of the form $(C_1 \times C_2)/G$, we shall consider only curves of genus at least 2.

Definition 4.1.2. A variety X is said to be *isogenous to a higher product* if it is a quotient $(C_1 \times \ldots \times C_n)/G$ where the C_i are curves of genus at least two, and G is a finite group acting freely on $C_1 \times \ldots \times C_n$.

The adjective *higher* emphasizes that the curves have genus at least two. From now on we will drop this adjective and we will simple say "isogenous to a product".

Proposition 4.1.3 ([Cat00, Proposition 3.11])**.** *A surface S is isogenous to a higher product if and only if S admits a finite unramified covering which is isomorphic to a product of curves of genera at least two.*

In the isogenous case we have a very nice description of the fundamental group of the variety.

Proposition 4.1.4 (cf. [Cat00])**.** *Let $S := (C_1 \times \ldots \times C_n)/G$ be isogenous to a product. Then the fundamental group of S sits in an exact sequence*

$$1 \to \Pi_{g_1} \times \ldots \times \Pi_{g_n} \to \pi_1(S) \to G \to 1$$

where $\Pi_{g_i} := \pi_1(C_i)$.

Proof. Since the action of G is free, it is properly discontinuous (Lemma 1.1.6). By Corollary 1.1.29 we have the following short exact sequence:

$$1 \longrightarrow \pi_1(C_1 \times \ldots \times C_n) \longrightarrow \pi_1(S) \longrightarrow G \longrightarrow 1\,,$$

but $\pi_1(C_1 \times \ldots \times C_n) = \pi_1(C_1) \times \ldots \times \pi_1(C_n) = \Pi_{g_1} \times \ldots \times \Pi_{g_n}$. □

Now we focus on the surfaces case, i.e. let G be a finite group acting on the product $C_1 \times C_2$, where the C_i are Riemann surfaces of genus at least two. There are two cases: the *unmixed case* where G acts diagonally; and the *mixed case* where the action of G exchanges the two factors (and then $C_1 \cong C_2$).

Lemma 4.1.5 ([Cat00, Lemma 3.8]). *Let $f\colon C_1 \times C_2 \to B_1 \times B_2$ be a surjective holomorphic map between product of curves. Assume both B_1, B_2 have genus at least two. Then, after possibly exchanging B_1 with B_2, there are holomorphic maps $f_i\colon C_i \to B_i$ such that $f(x, y) = (f_1(x), f_2(y))$.*

Lemma 4.1.6 ([Cat00, Corollary 3.9]). *Assume that both C_1, C_2 are curves of genus ≥ 2. Then the inclusion $\mathrm{Aut}(C_1 \times C_2) \supseteq \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$ is an equality if C_1 is not isomorphic to C_2, whereas $\mathrm{Aut}(C \times C)$ is the semidirect product of $\mathrm{Aut}(C)^2$ with the $\mathbb{Z}_2$ generated by the involution Φ exchanging the two factors.*

Definition 4.1.7. Let C_1, C_2 be Riemann surfaces of genus at least two, let $G \subset \mathrm{Aut}(C_1 \times C_2)$ be a finite group and let $G^0 := G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2))$. Then G^0 acts on each factor and diagonally on the product. If G^0 acts faithfully on both curves, we say that the action is *minimal*, and we refer to $X := (C_1 \times C_2)/G$ as a *minimal realization* of X.

Proposition 4.1.8 ([Cat00, Proposition 3.13]). *If S is a surface isogenous to a higher product, then a minimal realization is unique.*

A particular class of surfaces isogenous to a higher product is the following:

Definition 4.1.9 ([Cat00, Proposition 3.15]). A surface isogenous to a product $S := (C_1 \times C_2)/G$ is said to be of *generalized hyperelliptic type* if

- the Galois covering $p\colon C_1 \to C_1/G$ is unramified;
- the quotient curve C_2/G is isomorphic to $\mathbb{P}^1$.

The invariants of surfaces isogenous to a product may be computed using the following result:

Proposition 4.1.10. *Let $S := (C_1 \times C_2)/G$ be a surface isogenous to a higher product of curves, then*

$$e(S) = \frac{4(g(C_1) - 1)(g(C_2) - 1)}{|G|}$$

$$K_S^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G|}$$

$$\chi(S) = \frac{(g(C_1) - 1)(g(C_2) - 1)}{|G|}$$

Proof. Let $p\colon (C_1 \times C_2) \to S$ be the projection on the quotient; p is an étale covering of degree $|G|$.

The topological Euler-Poincaré characteristic is multiplicative: $e(C_1 \times C_2) = e(C_1) \times e(C_2) = (2 - 2g(C_1))(2 - 2g(C_2))$. By Lemma 3.4.2, we have $e(C_1 \times C_2) = |G| \cdot e(S)$ which implies the first equation.

By [Bea96, Fact III.22], it follows

$$H^1(C_1 \times C_2, \mathcal{O}_{C_1 \times C_2}) = H^0(C_1 \times C_2, \Omega^1_{C_1 \times C_2}) = H^0(C_1, \Omega^1_{C_1}) \oplus H^0(C_2, \Omega^1_{C_2})$$

and so:

$$q(C_1 \times C_2) = g(C_1) + g(C_2)\,.$$

Since $\Omega^2_{C_1 \times C_2} \cong p_1^*\Omega_{C_1} \otimes p_2^*\Omega_{C_2}$, where p_1 and p_2 are the projection of $C_1 \times C_2$ to C_1 and C_2 respectively, we have

$$H^0(C_1 \times C_2, \Omega^2_{C_1 \times C_2}) = H^0(C_1, \Omega^1_{C_1}) \otimes H^0(C_2, \Omega^1_{C_2})$$

hence

$$p_g(C_1 \times C_2) = g(C_1) \cdot g(C_2)\,.$$

We get

$$\begin{aligned} \chi(C_1 \times C_2) &= 1 + p_g(C_1 \times C_2) - q(C_1 \times C_2) \\ &= 1 + g(C_1) \cdot g(C_2) - g(C_1) - g(C_2) \\ &= (g(C_1) - 1)(g(C_2) - 1)\,. \end{aligned}$$

By Lemma 3.4.2, we have $\chi(C_1 \times C_2) = |G| \cdot \chi(S)$ which implies the last equation.

Using Noether's formula it is easy to prove the second equation too. □

Theorem 4.1.11 (cf. [Fre71, Hilfsatz 3 and Satz 1]). *Let V be a smooth algebraic variety and let G be a finite group acting on V. Let $X := V/G$, and assume* $\operatorname{codim}\operatorname{Sing}(X) > 1$. *Let S be the minimal resolution of the singularities of X, then*

$$H^0(S, \Omega^1_S) \cong H^0(V, \Omega^1_V)^G\,.$$

Corollary 4.1.12 ([MP10, Proposition 3.5]). *Let V be a smooth algebraic surface and let G be a finite group acting on V with only isolated fixed points. Let S be the minimal desingularization of $X := V/G$, then*

$$H^0(S, \Omega^1_S) \cong H^0(V, \Omega^1_V)^G\,.$$

4.2 Cyclic quotient singularities

In this section we introduce the cyclic quotient singularities and we discuss their minimal resolution. This class of singularities will be crucial in the next chapters; we will see that a quotient surface of unmixed type (see Definition 4.3.1) has only singularities of this type (see Proposition 4.3.6).

Definition 4.2.1. A variety Z has a *quotient singularity* in $z \in Z$ if there exists a neighborhood U of z such that $U \cong \mathbb{C}^m/H$ with H finite subgroup of $\operatorname{Aut}(\mathbb{C}^m, 0)$, the group of the holomorphic automorphism of $\mathbb{C}^m$ fixing 0.

Lemma 4.2.2 (Cartan, cf. [Bri68, Lemma 2.2])**.** *If H is a finite subgroup of* $\mathrm{Aut}(\mathbb{C}^m, 0)$*, then there exists a system of coordinates such that the action of H can be linearized.*

Thanks to the previous lemma, we can assume that $H \subset \mathrm{GL}(m, \mathbb{C})$.

Definition 4.2.3. A variety Z has a *cyclic quotient singularity* in $z \in Z$ if there exists a neighborhood U of z such that $U \cong \mathbb{C}^m/H$ with H cyclic finite subgroup of $\mathrm{GL}(m, \mathbb{C})$.

We are interested on singularities on surface, so now we consider the case H finite cyclic subgroup of $\mathrm{GL}(2, \mathbb{C})$. In this case we have that H has the following form

$$H = \left\langle \begin{pmatrix} e^{\frac{2\pi i p}{r}} & 0 \\ 0 & e^{\frac{2\pi i q}{r}} \end{pmatrix} \right\rangle .$$

for some p, q, $r \in \mathbb{Z}$, and we say that $\frac{1}{r}(p, q)$ is the *type* of the cyclic quotient singularity $\mathbb{C}^2/H$.

Lemma 4.2.4 ([BHPV04, pages 104-105])**.** *Each cyclic quotient singularity of type $\frac{1}{r}(p, q)$ is isomorphic to a cyclic quotient singularity of type $\frac{1}{n}(1, a)$ with $1 \le a \le n$ and* $\gcd(a, n) = 1$.

Definition 4.2.5. Let $1 \le a \le n$ and $\gcd(a, n) = 1$. We denote a cyclic quotient singular point of type $\frac{1}{n}(1, a)$ by $C_{n,a}$.

Remark 4.2.6. Let a and n as above, we denote by a' the unique integer in $\{1, \dots, n-1\}$ such that $a \cdot a' \cong 1 \mod n$.

Lemma 4.2.7. *$C_{n,a}$ and $C_{n,a'}$ are locally analytically isomorphic.*

Proof. Let x, y be the coordinates of $\mathbb{C}^2$ and assume that $H = \langle h \rangle$ acts in this way: $h(x, y) = (\varepsilon x, \varepsilon^a y)$, with $\varepsilon = e^{\frac{2\pi i}{n}}$. We define new coordinates: $(x', y') := g(x, y) = (y, x)$. We now note that $H = \langle h^{a'} \rangle$ since $\gcd(a', n) = 1$, in these new coordinates $h' = h^{a'}$ acts as follows:

$$\begin{aligned} h'(x', y') &= g(h^{a'}(g^{-1}(x', y'))) = g(\varepsilon^{a'} x, \varepsilon^{aa'} y) \\ &= (y, \varepsilon^{a'} x) = (x', \varepsilon^{a'} y') \end{aligned}$$

since $a \cdot a' \cong 1 \mod n$.

□

Definition 4.2.8. Let n and a be coprime integers with $n > a > 0$. The *continued fraction* of $\frac{n}{a}$ is the finite expression

$$\frac{n}{a} = b_1 - \cfrac{1}{b_2 - \frac{1}{b_3 - \dots}} = [b_1, \dots, b_l] .$$

The resolution of a cyclic quotient singularity of type $\frac{1}{n}(1,a)$ is well known, see [BHPV04, Section III.5] or [Rei03]. The exceptional divisor E of the minimal resolution of a cyclic quotient singularities is a so called *Hirzebruch-Jung string of type* (n,a) (for short HJ-string), that is $E = \sum_{i=1}^{l} E_i$, where the E_i are smooth rational curves with $E_i^2 = -b_i$, $E_i.E_{i+1} = 1$ for $i = 1,\dots,l-1$ and $E_i.E_j = 0$ for $|i-j| \geq 1$. The b_i are given by the continued fraction $\frac{n}{a}$, and the dual graph is:

Lemma 4.2.9. *Let n and a be coprime integers with $n > a > 0$ and let $\frac{n}{a} = [b_1,\dots,b_l]$. Let $A_{1,l}$ be the intersection matrix determined by the Hirzebruch-Jung string of a singularity of type $C_{n,a}$, i.e.*

$$A_{1,l} = \begin{pmatrix} -b_1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -b_2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & \dots & 0 & 1 & -b_{l-1} & 1 \\ 0 & \dots & 0 & 0 & 1 & -b_l \end{pmatrix}.$$

Then $\det A_{1,l} = (-1)^l n$.

Proof. We prove the statement by induction on l.

If $l = 1$ then $n/a = [b_1]$ and $n = b_1$; $\det A_{1,1} = -b_1 = (-1)^1 b_1$.
If $l = 2$ then $n/a = [b_1, b_2] = b_1 - \frac{1}{b_2} = \frac{b_1 b_2 - 1}{b_2}$ and $n = b_1 b_2 - 1$;
$\det A_{1,2} = b_1 b_2 - 1 = (-1)^2 n$.

Now, we assume that the statement holds for $1 \leq i < l$ and we prove it for l. We note that

$$\det A_{1,l} = -b_1 \det A_{2,l} - \det A_{3,l}\,,$$

where

$$A_{i,l} = \begin{pmatrix} -b_i & 1 & & & \\ 1 & -b_{i+1} & 1 & & \\ & \dots & \dots & & \\ & 1 & -b_{l-1} & 1 \\ & & 1 & -b_l \end{pmatrix}.$$

From the other side we have:

$$\frac{n}{q} = b_1 - \frac{1}{[b_2,\dots,b_l]} \implies [b_2,\dots,b_l] = \frac{q}{b_1 q - n}\,.$$

In the same way we get that $[b_3, \dots, b_l] = \dfrac{b_1 q - n}{k}$, for some $k \in \mathbb{N}_{>0}$. By inductive hypothesis,

$$\det A_{2,l} = (-1)^{l-1} q \qquad \det A_{3,l} = (-1)^{l-2}(b_1 q - n)$$

and so

$$\begin{aligned} \det A_{1,l} &= -b_1(-1)^{l-1} q - (-1)^{l-2}(b_1 q - n) \\ &= (-1)^{l-2}(b_1 q - b_1 q + n) \\ &= (-1)^l n\,. \end{aligned}$$

□

Let $\rho\colon S \to X$ be a minimal resolution of singularities, we have that (in a neighborhood of a singular point $x \in X$):

$$K_S = \rho^* K_X + \sum_{i=1}^{l} r_i E_i\,.$$

Since E_i is exceptional, $\rho^* K_X . E_i = 0$. Moreover, by the genus formula we get $(K_S^2 + E_i).E_i = 2g(E_i) - 2 = -2$. So

$$(K_S + E_k).E_k = -2, \qquad \left(K_S - \sum_{i=1}^{l} r_i E_i\right).E_k = 0, \quad \forall k = 1, \dots, l\,. \tag{4.1}$$

These equations determines the r_i as follows:

Lemma 4.2.10 ([Bar99], [Hir53])**.** *Let $\mu_0 = n$, $\mu_1 = a$ and $\mu_{i+1} = b_i \mu_i - \mu_{i-1}$ for $i = 2, \dots, l$. Let $\lambda_0 = 0$, $\lambda_1 = 1$ and $\lambda_{i+1} = \lambda_i b_i - \lambda_{i-1}$ for $i = 2, \dots, l$.*

Then in a neighborhood of a singular point $z \in X$ of type $C_{n,a}$, we have that

$$nK_S = \rho^* n K_X + \sum_{i=1}^{l} a_i E_i\,,$$

where $a_i = \lambda_i + \mu_i - n$ for $i = 1, \dots, l$.

Proof. By (4.1):

$$nK_S.E_k = n(K_S + E_k).E_k + nb_k = n(b_k - 2) \tag{4.2}$$

and the a_i satisfy the following equation:

$$nK_S.E_k = n\left(\sum_{i=1}^{l} r_i E_i\right).E_k = a_{k-1} - b_k a_k + a_{k+1} \tag{4.3}$$

where $a_0 = a_{l+1} = 0$ and $a_j = nr_j$ for $j = 1, \dots, l$. The thesis follows if we prove that the a_i defined in the statement give the unique solution for the linear system

$$a_{k-1} - b_k a_k + a_{k+1} = n(b_k - 2) \qquad k = 1, \dots, l\,.$$

The coefficients matrix is the matrix $A_{1,l}$ of Lemma 4.2.9 that has non zero determinant, hence the linear system has an unique solution. We claim that a_i as above are the unique solution. Noting that $\mu_i \equiv \lambda_i a \mod n$, it is easy to show that $\mu_{l-1} = b_l$, $\lambda_{l-1} = a' b_l - n$, $\mu_l = 1$, $\lambda_l = a'$, $\mu_{l+1} = 0$, $\lambda_{l+1} = n$. Hence

$$-b_1 a_1 + a_2 = -b_1(a + 1 - n) + (b_1 + b_1 a - n - n) = n(b_1 - 2)\,,$$

$$a_{l-1} - b_l a_l = b_l(a' + 1) - 2n - b_l(a' + 1) + b_l n = n(b_l - 2)\,.$$

For each $k \in \{2, \dots, l - 1\}$

$$\begin{aligned} a_{k-1} + a_{k+1} &= (\lambda_{k-1} + \mu_{k-1} - n) + (\lambda_k + \mu_k - n) \\ &= \lambda_{k-1} + \lambda_{k+1} + \mu_{k-1} + \mu_{k+1} - 2n \\ &= \lambda_k b_k + \mu_k b_k - 2n \\ &= b_k(\lambda_k + \mu_k - n) + n(b_k - 2) \qquad = b_k a_k + n(b_k - 2) \end{aligned}$$

□

Lemma 4.2.11. *For a singular point x of type $C_{n,a}$, we have that in a neighborhood of x*

$$K_S^2 = K_X^2 - \left(\frac{a + a' + 2}{n} - 2 + \sum_{i=1}^{l} (b_i - 2) \right).$$

Proof. We have

$$nK_S = \rho^* nK_X + \sum_{i=1}^{l} a_i E_i$$

Since $\rho \colon S \to X$ is a biratonal morphism we get

$$\begin{aligned} K_S^2 &= (\rho^* K_X)^2 + 2\left(\frac{1}{n} \sum a_i E_i\right).(\rho^* K_X) + \frac{1}{n^2}\left(\sum a_i E_i\right)^2 \\ &= K_X^2 + \frac{1}{n^2}\left(\sum a_i E_i\right)^2. \end{aligned}$$

Moreover

$$\begin{aligned} \left(\sum_{i=1} a_i E_i\right)^2 &= \left(\sum_{i=1} a_i E_i\right).(nK_S - \rho^* nK_X) \\ &= n \sum a_i (E_i.K_S) \\ &= n \sum a_i (b_i - 2) \end{aligned}$$

where the last equality follows by the first equation of (4.1), and so

$$\begin{aligned}\left(\sum_{i=1} a_iE_i\right)^2 &= n\sum_i(\lambda_i+\mu_i-n)(b_i-2)\\ &= n[\sum(\lambda_i+\mu_i)(b_i-2)-n\sum(b_i-2)] \qquad (4.4)\end{aligned}$$

where the λ_i and the μ_i are as in Lemma 4.2.10. Extending the first sum in (4.4) we have:

$$\begin{aligned}\sum(\lambda_i+\mu_i)(b_i-2) &= \sum(\lambda_ib_i-2\lambda_i)+\sum(\mu_ib_i-2\mu_i)\\ &= \sum(\lambda_{i+1}+\lambda_{i-1}-2\lambda_i)+\sum(\mu_{i+1}+\mu_{i-1}-2\mu_i)\end{aligned}$$

and these two last sums are telescopic sums, thus:

$$\begin{aligned}\sum(\lambda_i+\mu_i)(b_i-2) &= \lambda_0-\lambda_1+\lambda_{l+1}-\lambda_l+\mu_0-\mu_1+\mu_{l+1}-\mu_l\\ &= -1+n-a'+n-a-1=2n-2-a-a'\end{aligned}$$

Hence equation (4.4) becomes:

$$\left(\sum_{i=1} a_iE_i\right)^2 = n[2n-(a+a'+2+n\sum(b_i-2))]$$

and it follows:

$$\begin{aligned}K_X^2 &= K_S^2-\frac{1}{n^2}\left(\sum a_iE_i\right)^2\\ &= K_S^2-\frac{[2n-(a+a'+2+n\sum(b_i-2))]}{n}\\ &= K_S^2-2+\frac{a+a'+2}{n}+\sum(b_i-2)\,.\end{aligned}$$

□

Definition 4.2.12. A singular point p of a normal surface X is a *Rational Double Point* (R.D.P.) or Du Val singularity if X has a minimal resolution of singularities $f\colon S\to X$ such that every irreducible component E_i of the exceptional divisor E over p satisfies $K_S.E_i=0$, or equivalently, $E_i^2=-2$.

Definition 4.2.13 ([Rei87, Definition 1.1])**.** A normal variety X of dimension n has *canonical singularities* if

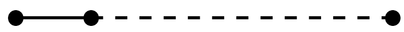
1. for some $r\geq 1$, the Weil divisor rK_X is Cartier;
2. if $f\colon Y\to X$ is a resolution of the singularities of X and $\{E_i\}$ is the family of all exceptional irreducible divisors of f, then

$$rK_Y=f^*(rK_X)+\sum a_iE_i\,,\qquad \text{with } a_i\geq 0\,.$$

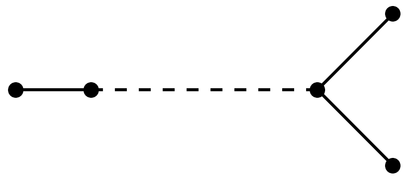
Theorem 4.2.14 (cf. [Mat67, Theorem 4.6.7]). *Let p be a singular point on a normal surface X. Then p is a canonical singularity if and only if p is a Rational Double Point.*

Moreover the dual graph of the exceptional curves of the minimal resolution is one of the following 5 types:

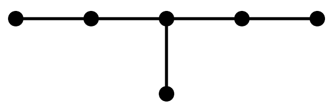
A_n



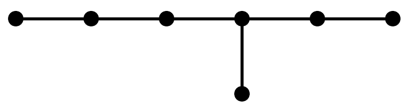
D_n



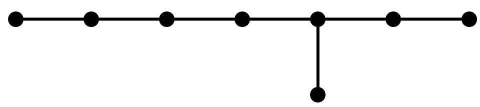
E_6



E_7



E_8



Remark 4.2.15. Let X be a surface with at most canonical singularities. Let $\rho\colon S \to X$ be the minimal resolution of the singularities. By [Mat67, Theorem 4.6.2], we have that

$$K_S = \rho^* K_X \,.$$

Remark 4.2.16. In the particular case $a = n-1$: $C_{n,n-1}$, we have that all the curves of the minimal resolution are (-2)-curves and $H \subset \mathrm{SL}(2, \mathbb{C})$. This class of singularities is the class of the R.D.P. singularities of type A_n.

4.3 Surfaces: the unmixed case

Definition 4.3.1. Let C_1, C_2 be two Riemann surface of respective genus g_1, $g_2 \geq 2$, and let G be a finite group that acts diagonally on $C_1 \times C_2$. An *umixed action* of G on $C_1 \times C_2$ is a monomorphism $G \hookrightarrow \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$. We say that $X := (C_1 \times C_2)/G$ is a *quotient surface of unmixed type.*

Remark 4.3.2 (cf. [Cat00, Remark 3.10]). Every quotient surface X of unmixed type may be obtained by a minimal (see Definition 4.1.7) unmixed action. Let $(C_1 \times C_2)/G$ be a realization of X. If G does not embed in $\mathrm{Aut}(C_i)$, then the kernel K_i acts trivially on C_i for $i = 1, 2$. Thus we replace $C_1 \times C_2$ by $Z := C_1/K_2 \times C_2/K_1$ and G by $G' := G/\langle\langle K_1, K_2\rangle\rangle$; we get that $X = Z/G'$ and this is a minimal realization.

Definition 4.3.3. An *unmixed surface* $X = (C_1 \times C_2)/G$ is the minimal realization of a quotient surface of unmixed type. The minimal resolution S of X is called a *product quotient* surface or a *standard isotrivial fibration.* X is also called the *quotient model* of the product quotient surface.

The name "standard isotrivial fibration" comes from [Ser96]:

Definition 4.3.4. A *fibration* is a morphism $\phi\colon S \to C$ with connected fibres from an algebraic smooth surface onto a smooth projective curve.
A fibration is said to be *isotrivial* if all smooth fibres are isomorphic to each other.

Remark 4.3.5. Let $X = (C_1 \times C_2)/G$ be an unmixed surface, the natural maps $\alpha_i\colon X \to C_i/G$ $(i = 1, 2)$ are two isotrivial fibration: the general fibre of α_1 is isomorphic to C_2 and the general fibre of α_2 is isomorphic to C_1.

Proposition 4.3.6. *Let $X = (C_1 \times C_2)/G$ be a quotient surface of unmixed type. Then X has finitely many singular points that are cyclic quotient singularities.*

Proof. By Remark 2.2.6 we have that on both C_i there are finitely many points with non trivial stabilizer, which is cyclic by Theorem 2.2.4. Since G acts diagonally we have that $\mathrm{Stab}(x, y) = \mathrm{Stab}(x) \cap \mathrm{Stab}(y)$ that is cyclic. $\square$

Remark 4.3.7. The map $C_1 \times C_2 \to X$ is quasi-étale, indeed the singular locus of X: $\mathrm{Sing}(X)$ is also the branch locus of the quotient map.

Theorem 4.3.8 (cf. [Ser96, Theorem 2.1])**.** *Let $f\colon S \to X = (C_1 \times C_2)/G$ be a standard isotrivial fibration and let us consider the natural projection $f_1\colon S \to C_1/G$. Take any point over $\overline{y} \in C_1/G$ and let $F = f_1^*(\overline{y})$. Then*

(i) The reduced structure of F is the union of an irreducible curve Y, called the central component of F, and either none or at least two mutually disjoint HJ-strings, each meeting Y at one point, and each being contracted by λ to a singular point of X.

(ii) The intersection of a string with Y is transversal, and it takes place at only one of the end components of the string.

An analogous statement holds if one considers the projection $f_2\colon S \to C_2/G$.

Proposition 4.3.9. *Let $\rho\colon S \to X = (C_1 \times C_2)/G$ be a standard isotrivial fibration and let $\alpha_i\colon X \to C_i/G$ be the natural fibrations. Let $y \in X$ be a singular point of type $C_{n,a}$, and let $x_i := \alpha_i(y) \in C_i/G$. Consider the two fibres $Y_1 := \alpha_1^*(x_1)$ and $Y_2 := \alpha_2^*(x_2)$ taken with the reduced structure. Let $\tilde{Y}_i := \rho_*^{-1}(Y_i)$ be the strict transforms of Y_i ($i = 1, 2$) and let E be the exceptional divisor over y.*

Then $\tilde{Y}_1$ intersects one of the extremal curves of E, while $\tilde{Y}_2$ intersects the other extremal curve.

Proof. Let $f\colon C_1 \times C_2 \to X$ be the projection to the quotient and let $p \in f^{-1}(y)$. By assumption, there exists $g \in \mathrm{Aut}(C_1 \times C_2)$ with $g(p) = p$ and $H := \langle g \rangle = \mathrm{Stab}(p)$. By Lemma 4.2.2 there exist coordinates in a neighborhood U of p, with $p = (0,0)$ and $g = M \in \mathrm{GL}(2, \mathbb{C})$.
Since $\mathrm{ord}(g)$ is finite, then there exist coordinates (x, y) with $M(x, y) = (\epsilon x, \epsilon^a y)$ with $\epsilon = e^{\frac{2\pi i}{n}}$.

Let $\tilde{Y}_i$ be the connected component of $f^{-1}(Y_i)$ passing through p. We note that $T_0\tilde{Y}_i$ is an eigenspace for M since $\tilde{Y}_i$ is a connected component of a fibre of the natural map $C_1 \times C_2 \to C_i/G$, that is invariant for the action of M. We also note that $T_0\tilde{Y}_1 \cap T_0\tilde{Y}_2 = \{0\}$.

If $a \neq 1$ then $T_0\tilde{Y}_1$ and $T_0\tilde{Y}_2$ are the coordinate axes; while if $a = 1$ then $M = \epsilon \cdot Id_2$ and, up to a linear change of coordinate (that does not change M), $T_0\tilde{Y}_1$ and $T_0\tilde{Y}_2$ become the coordinate axes.

Since $T_0\tilde{Y}_1 = \{x + f_1(x, y) = 0\}$ and $T_0\tilde{Y}_2 = \{y + f_2(x, y) = 0\}$ with $\mathrm{mult}_0 f_i \geq 2$, we define new coordinates $w := x + f_1(x, y)$ and $t = y + f_2(x, y)$; in these coordinates, $g(w) = \epsilon w$ and $g(t) = \epsilon^a t$.

So we have found coordinates such that $g(w, t) = (\epsilon w, \epsilon^a t)$ and such that $f(\{w = 0\})$ and $f(\{t = 0\})$ are Y_1 and Y_2.

By [Rei03, Proposition-Definition 1.1], U is the spectrum of the ring of invariant monomials: $\mathbb{C}[w, t]^H$. This ring is generated by monomials (see [Rei03, Corollary 2.5])

$$u_0 = w^n\,,\; u_1 = w^{n-a}t\,, \ldots, u_{k+1} = t^n$$

that satisfy

$$u_{i-1}u_{i+1} = u_i^{d_i} \qquad \text{for} \qquad i = 1, \ldots, k\,,$$

where the exponents d_i are given by $\dfrac{n}{n-a} =: [d_1, \ldots, d_k]$. In other words, $U \cong \mathrm{Spec}\,\mathbb{C}[u_0, u_1, \ldots, u_{k+1}]/J$ where J is the ideal of the relations between the u_i. Hence in U the two fibres Y_1 and Y_2 are, set-theoretically, $\{u_0 = 0\}$ and $\{u_{k+1} = 0\}$.

Let L be the overlattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1, a)$ of $\mathbb{Z}^2$ (see [Rei03, Proposition 2.2]) and let $M = \{(\alpha, \beta) \mid \alpha + a\beta \equiv 0 \mod n\} \subset \mathbb{Z}^2$ be the dual lattice of invariant monomials. Let

$$e_0 = (0,1),\ e_1 = \frac{1}{r}(1,a), \dots, e_{l+1} = (1,0)$$

be the lattice points of the boundary of $N(L)$ in $[0,1]^2$, where $N(L)$ is the convex hull in $\mathbb{R}^2$ of all nonzero lattice points in the first quadrant. Let $[b_1, \dots, b_l] := \frac{n}{a}$, by [Rei03, Proposition 2.2], the points e_i are related as follows:

$$e_{i+1} + e_{i-1} = b_i e_i\,.$$

Let ζ_i, η_i be monomials forming the dual basis to e_i, e_{i+1}, that is:

$$e_i(\zeta_i) = 0,\ e_i(\eta_i) = 1,\ e_{i+1}(\zeta_i) = 1,\ e_{i+1}(\eta_i) = 0\,.$$

By [Rei03, Theorem 3.2], the resolution of singularities $Z \to U$ is constructed as follows:

$$Z = Z_0 \cup Z_1 \cup \dots \cup Z_l\,,$$

where $Z_i \cong \mathbb{C}^2$ with coordinates ζ_i, η_i, for $i = 1, \dots, l$. The gluing $Z_i \cup Z_{i+1}$ and the morphism $f\colon Z \to U$ are determined by the isomorphism:

$$Z_i \setminus \{\eta_i = 0\} \overset{\cong}{\to} Z_{i+1} \setminus \{\zeta_{i+1} = 0\}$$

defined by

$$\zeta_{i+1} = \eta_i^{-1}\,, \qquad \eta_{i+1} = \zeta_i \eta_i^{b_{i+1}}$$

where $[b_1, \dots, b_l] := \frac{n}{a}$.

The preimage $f^{-1}(\{u_0 = 0\}) = f^{-1}(Y_1)$ is the complex line $\mathbb{C}_{\eta_l}$, that is contained in the Z_l-chart. $f^{-1}(Y_1)$ intersects only one extremal exceptional curve, namely the one obtained by the gluing $\{\zeta_{l-1} = 0\} \cup \{\eta_l = 0\}$.
Analogously, the preimage $f^{-1}(\{u_{k+1} = 0\}) = f^{-1}(Y_2)$ is the complex line $\mathbb{C}_{\zeta_0}$ contained in the Z_0-chart. $f^{-1}(Y_2)$ intersects only one extremal exceptional curve, namely the one obtained by the gluing $\{\zeta_0 = 0\} \cup \{\eta_1 = 0\}$. $\square$

Definition 4.3.10. Let $\rho\colon S \to X = (C_1 \times C_2)/G$ be a standard isotrivial fibration and let $\alpha\colon X \to C_i/G$ be one of the two natural fibrations. Let $y \in \mathrm{Sing}(X)$ be a point of type $C_{n,a}$ with $\frac{n}{a} = [b_1, \dots, b_l]$. Let $E := \sum_{i=1}^{l} \gamma_i E_i$ be the exceptional divisor over y. E is a tree of rational curves: $E_i^2 = -b_i$, with $[b_1, \dots, b_l]$, $E_i.E_{i+1} = 1$ while $E_i.E_j = 0$ if $|i-j| \geq 2$. Let $\tilde{Y}$ be the strict transform in S of Y: the fibre $\alpha^*(\alpha(y))$ taken with the reduced structure.
We say that y is *of type $C_{n,a}$ with respect to α* if $\tilde{Y}$ intersects E_1.

Remark 4.3.11. If y is of type $C_{n,a}$ with respect to α_1 then y is of type $C_{n,a'}$ with respect to α_2.

Definition 4.3.12 (cf. [Pol10, Definition 2.7])**.** Let $S \to X = (C_1 \times C_2)/G$ be a standard isotrivial fibration. We say that a reducible fibre F of $\alpha\colon X \to C_i/G$ for $i = 1$ or 2 is *of type* $\left(\frac{a_1}{n_1}, \ldots, \frac{a_r}{n_r}\right)$ if F contains exactly r singular points $y_1, \ldots, y_r$, where each y_i is of type C_{n_i,a_i} with respect to α.

Proposition 4.3.13 (cf. [Pol10, Proposition 2.8])**.** *Let $S \to X = (C_1 \times C_2)/G$ be a standard isotrivial fibration. Let F be a fibre of $\alpha\colon X \to C_i/G$ for $i = 1$ or 2, and let $\tilde{F}$ its strict transform in S.*
If F of type $\left(\frac{a_1}{n_1}, \ldots, \frac{a_r}{n_r}\right)$ with respect to α, then

$$\sum_{i=1}^{r} \frac{a_i}{n_i} = -\tilde{F}^2 \,.$$

As corollary of Proposition 4.3.13 we get

Lemma 4.3.14. *Let $S \to X = (C_1 \times C_2)/G$ be a standard isotrivial fibration. Let $\alpha\colon X \to C_1/G$ be the natural fibration. If* $\operatorname{Sing}(X) = \{y_1, \ldots, y_r\}$ *where each y_i is of type C_{n_i,a_i} with respect to α, then*

$$\sum_{i=1}^{r} \frac{a_i}{n_i} \in \mathbb{Z} \,.$$

Lemma 4.3.15 ([Ser96, Proposition 2.2])**.** *If S is a smooth surface birational to the quotient surface of unmixed type $X := (C_1 \times C_2)/G$ then*

$$q(S) = g(C_1/G) + g(C_2/G) \,.$$

Proof. Let p_1 and p_2 be the projections of $C_1 \times C_2$ onto its factors, we have $\Omega^1_{C_1 \times C_2} = p_1^*(\Omega^1_{C_1}) \oplus p_2^*(\Omega^1_{C_2})$ ([Bea96, Fact III.22]), hence

$$\begin{aligned} q(S) &= \dim H^0(\Omega^1_{C_1 \times C_2})^G = \dim H^0(\Omega^1_{C_1})^G + \dim H^0(\Omega^1_{C_2})^G \\ &= g(C_1/G) + g(C_2/G) \end{aligned}$$

where the first and last equalities are given by Corollary 4.1.12. □

Therefore, for a product quotient surface $S \to (C_1 \times C_2)/G$ it holds $q(S) = 0$ if and only if $g(C_1/G) = g(C_2/G) = 0$. This implies that a product quotient surface S of general type with quotient model $(C_1 \times C_2)/G$ has $p_g(S) = 0$ if and only if $\chi(\mathcal{O}_S) = 1$ and $C_1/G \cong C_2/G \cong \mathbb{P}^1$.

4.4 Surfaces: the mixed case

Definition 4.4.1. Let C be a Riemann surface of genus $g(C) \geq 2$, and let G be a finite group. A *mixed action* of G on $C \times C$ is a monomorphism

$G \hookrightarrow \mathrm{Aut}(C \times C)$ whose image is not contained in $\mathrm{Aut}(C)^2$. Given a mixed action we will denote by $G^0 \triangleleft G$ the index two subgroup $G \cap \mathrm{Aut}(C)^2$. A *quotient surface of mixed type* is a surface which arises as quotient $X := (C \times C)/G$ by a mixed action of G on $C \times C$.

Remark 4.4.2 (cf. [Cat00, Remark 3.10]). Every quotient surface X of mixed type may be obtained by a minimal mixed action.

Let $(C \times C)/G$ be a realization of X and let $\tau' \in G$ be a transformation not in G^0: $\tau'(x, y') = (\tau_2 y', \tau_1 x)$. We choose $y = \tau_2 y'$ as a new coordinate on the second factor, and then $\tau'(x, y) = (y, \tau x)$, where $\tau = \tau_2\tau_1$.
Let $K_2 \times \mathrm{Id} := G^0 \cap (\mathrm{Aut}(C) \times \mathrm{Id})$ and $\mathrm{Id} \times K_1 := G^0 \cap (\mathrm{Id} \times \mathrm{Aut}(C))$, then $K_1 \cong K_2$ as subgroups of $\mathrm{Aut}(C)$; indeed if $\psi \in K_1$ then $(\mathrm{Id}, \psi) \in G^0$, conjugating it by τ', we get $(\psi, \mathrm{Id}) \in G^0$.
We obtain that $K_1 \times K_1$ is a normal subgroup of G, and $G/(K_1 \times K_1)$ acts mixed and minimally on $(C/K_1) \times (C/K_1)$.

Definition 4.4.3. Let X be a quotient surface of mixed type. By the previous remark we may obtain X as $C \times C/G$ by a minimal mixed action; we will call the map $C \times C \to X$ *the quotient map* of X.

Theorem 4.4.4 (cf. [Cat00, Proposition 3.16])**.** *Let $G \hookrightarrow \mathrm{Aut}(C \times C)$ be a minimal mixed action. Fix $\tau' \in G \setminus G^0$; it determines an element $\tau := \tau'^2 \in G^0$ and an element $\varphi \in \mathrm{Aut}(G^0)$ defined by $\varphi(h) := \tau' h \tau'^{-1}$. Then, up to a coordinate change, G acts as follows:*

$$\begin{array}{rcl} g(x,y) & = & (gx, (\varphi\, g)y) \\ \tau' g(x,y) & = & (\varphi(g)y, \tau g\, x) \end{array} \qquad for\ g \in G^0 \tag{4.5}$$

Conversely, for every $G^0 \subseteq \mathrm{Aut}(C)$ and G extension of degree 2 of G^0, fixed $\tau' \in G \setminus G^0$ and defined τ and φ as above, (4.5) defines a minimal mixed action on $C \times C$.

Proof. The argument in Remark 4.4.2 shows that, if the action is minimal and mixed, then there are coordinates such that G acts as in (4.5).

Observing that $\tau' g \tau' h = \varphi(g)\tau h$, $\varphi(\tau' g \tau' h) = \tau g \varphi(h)$ and that $\varphi(\tau) = \tau$ it is easy to prove that (4.5) defines a mixed G-action on $C \times C$. Moreover, the action is minimal by definition of G^0. □

Definition 4.4.5. A *mixed surface* $X = (C \times C)/G$ is a quotient surface of mixed type provided by the corresponding minimal mixed action, as described in Theorem 4.4.4. If the quotient map is quasi-étale (see Definition 2.1.10) we say that X is a *mixed quasi-étale surface* (for short "mixed q.e. surface"). Let S be the minimal resolution of the singularities of a mixed surface X, if S is regular ($q(S) = 0$), then we say that X is a *regular mixed surface.*

Remark 4.4.6. Note that when we use Theorem 4.4.4 to define a mixed action on $C \times C$, we choose an element $\tau' \in G \setminus G^0$, but the mixed surface $(C \times C)/G$ obtained does not depend on this choice.

Remark 4.4.7. Let $X = (C \times C)/G$ be a mixed surface, and let G^0 be the index two subgroup of G of the elements that do not exchange the factors: $G^0 = G \cap \mathrm{Aut}(C)^2$. Then $Y = (C \times C)/G^0$ is an unmixed surface.

4.5 Surfaces quotient of product of curves with $\chi = 1$: the classification so far

In this section we collect the main results of classification of the surfaces S (of general type) that appear as minimal resolution of the singularities of $X = (C_1 \times C_2)/G$ where C_1 and C_2 are Riemann surfaces and $G \subseteq \mathrm{Aut}(C_1 \times C_2)$ is a finite group.

We have already seen (Section 3.6.1) that the minimal surfaces of general type with $p_g = q \geq 3$ are completely classified and they are isogenous to a product of curves.

The $p_g = q = 0$ case

We start noting that if S is a surface of general type with $p_g(S) = 0$, we automatically have that $q = 0$, since $\chi(\mathcal{O}_S) = 1 + p_g(S) - q(S) \geq 1$ (see Theorem 3.6.4).

In [BC04] Bauer and Catanese study the above situation under the assumption that the action of G is free and of unmixed type and $p_g(S) = 0$. They completely solve this case under the further assumption that G is abelian and give some examples in the non abelian case.

In [BCG08] all the surfaces of general type with $p_g = 0$ and isogenous to a product of curves are classified, in particular they prove the following:

Theorem 4.5.1 ([BCG08])**.** *There are exactly 18 families of minimal surfaces of general type with $p_g = 0$ isogenous to a product of curve.*
13 of these families are of unmixed type, while 5 are of mixed type.

Remark 4.5.2. We observe that in [BCG08] the authors claim that there are 4 families of mixed type. They missed a family, that we have tagged by 7.3.13 in Table 6.1.

In [BCGP08] Bauer, Catanese, Grunewald and Pignatelli start to study the case of non free action assuming that the quotient surface $(C_1 \times C_2)/G$ has at most R.D.P. as singularities. They prove that indeed only nodes (Du Val singularities of type A_1) can occur as singularities and they state:

Theorem 4.5.3 ([BCGP08]). *Surfaces of general type with $p_g(S) = 0$, whose canonical model is a singular quotient surface $X = (C_1 \times C_2)/G$ by an unmixed action of G form 27 irreducible families.*

We note that here automatically $K_S^2 > 0$, since $K_X^2 > 0$ and $K_S = \rho^* K_X$, where $\rho \colon S \to X$ is the minimal desingularization of X.

Finally in [BP10] the authors remove all the assumption on the singularities and they give a complete classification of the product-quotient surfaces S with $K_S^2 > 0$ and $p_g = 0$:

Theorem 4.5.4 ([BP10]). *If S is a product-quotient surface with $p_g(S) = 0$ and $K_S^2 > 0$, then one of the following is true:*

1. *S is minimal and of general type.*
2. *S is the "fake Godeaux surface" which has $K_S^2 = 1$, $\pi_1(S) = \mathbb{Z}_6$ and its minimal model has $K^2 = 3$.*

Moreover, their classification yields 32 irreducible families of minimal surfaces with $p_g = 0$ which are the minimal resolution of the singularities of $X = (C_1 \times C_2)/G$ where the G-action is of unmixed type and X does not have canonical singularities.

Dropping the assumption that G acts freely, Proposition 4.1.4 does not hold. In [BCGP08] (see also [DP10]) the authors proved that the fundamental group still has a very similar description:

Theorem 4.5.5 (see [BCGP08, Theorem 0.10], [DP10]). *Let $C_1, \dots, C_n$ be compact complex curves of respective genus $g_i \geq 2$ and let G be a finite group acting faithfully on each C_i (unmixed action). Let $X := (C_1 \times \ldots \times C_n)/G$, and let S the minimal resolution of the singularities of X.*

Then the fundamental group $\pi_1(X) \cong \pi_1(S)$ has a normal subgroup $\mathcal{N}$ of finite index which is isomorphic to the product of surface groups (see Section 2.3).

The $p_g = q = 1$ case

In [Pol08], Polizzi investigates the surfaces $S = (C_1 \times C_2)/G$ with $p_g(S) = q(S) = 1$, such that the action of G is of unmixed type and free. He classifies this case under the assumption that G is abelian and gives some examples in the non abelian case.

In [CP09] all the surfaces of general type with $p_g = q = 1$ and isogenous to a product of curves are classified, in particular Carnovale and Polizzi prove the following:

Theorem 4.5.6 ([CP09]). *The surfaces $S = (C \times F)/G$ with $p_g = q = 1$ isogenous to a product of curves are minimal of general type and form 49 families. In particular, 44 families are of unmixed type, while 5 are of mixed type.*

In [Pol09] the author starts to study the singular case admitting that the quotient surface $(C \times F)/G$ has at most R.D.P. as singularities. He proves that indeed only nodes can occur as singular point and he shows

Theorem 4.5.7 ([Pol09])**.** *Let $\lambda\colon S \to X = (C \times F)/G$ be a standard isotrivial fibration of general type $p_g = q = 1$ not isogenous to a product of curves. Assume that X contains only R.D.P.'s. Then S is a minimal surface, K_S^2 is even and the singularities of X are exactly $8 - K_S^2$ nodes.*

Moreover the occurrences for K_S^2, $g(F)$, $g(C)$ and G are precisely described and there are 28 possibilities.

Finally in [MP10], Mistretta and Polizzi remove all the assumption on the singularities and they prove:

Theorem 4.5.8 ([Pol09])**.** *Let $\lambda\colon S \to X = (C \times F)/G$ be a standard isotrivial fibration of general type $p_g = q = 1$ and assume that X contains at least one singularity which is not a R.D.P. and that S is a minimal model. Then there are 15 possible 4-tuples $(K_S^2, g(F), g(C), G)$.*

Moreover they describe the basket of singularities.

The $p_g = q = 2$ case

In [Zuc03] the author proves the following:

Theorem 4.5.9. *There are two classes of minimal surfaces S of general type with $p_g = q = 2$ whose Albanese image is a surface and having an irrational pencil, and they are both isogenous to a higher product.*

More precisely, $S = (C_1 \times C_2)/\mathbb{Z}_2$ where, either $g(C_1) = g(C_2) = 2$ or $g(C_1) = g(C_2) = 3$.

Zucconi manages also to remove the hypothesis on the Albanese map using the generalized hyperelliptic surfaces (see Definition 4.1.9); he proves:

Theorem 4.5.10 ([Zuc03, Proposition 4.2])**.** *Let S be a surface of general type with $p_g = q = 2$ and not of Albanese general type. Then S is a generalized hyperelliptic surface.*

In [Pen11], Penegini deals the case $p_g = q = 2$. He investigates both the isogenous case both the singular case, in particular he proves:

Theorem 4.5.11. *Let S be a minimal surface of general type with $p_g = q = 2$ such that it is either a surface isogenous to a product of curves of mixed type or it admits an isotrivial fibration. Let $\alpha\colon S \to Alb(S)$ be the Albanese map. Then we have the following possibilities:*

1. *If $\dim(\alpha(S)) = 1$, then $S = (C \times F)/G$ and it is generalized hyperelliptic. There are exactly 24 families of these surfaces.*

2. *If* $\dim(\alpha(S)) = 2$*, then there are three cases:*
   - S *is isogenous to product of unmixed type* $(C \times F)/G$*, and there are 3 families of such surfaces.*
   - S *is isogenous to a product of mixed type* $(C \times C)/G$*, there is only one family of these surfaces.*
   - $S \to T := (C \times F)/G$ *is an isotrivial standard fibration, and there are 5 families of these surfaces.*

Penegini also gives a detailed description for the basket of singularities and for the possible 4-tuples $(K_S^2, g(F), g(C), G)$.

Chapter 5

Mixed quasi-étale surfaces

In this chapter we study the mixed quasi-étale surfaces; our aim is to produce an algorithm to construct and classify all surfaces S with given values of the invariants that appear as minimal resolution of a mixed quasi-étale surface.

In this chapter C will denote a Riemann surface of genus $g(C) \geq 2$, $G \subseteq \operatorname{Aut}(C \times C)$ a finite group with a mixed action on $C \times C$ and $G^0 := G \cap \operatorname{Aut}(C)^2 \triangleleft G$ the index two subgroup of elements that do not exchange the factors.

Let $X := (C \times C)/G$ be a mixed surface. We note that the quotient map factors as follows

$$C \times C \xrightarrow{\sigma} Y := (C \times C)/G^0 \xrightarrow{\pi} X\,.$$

We are in the following situation:

$$\begin{tikzcd}
 & C \times C \arrow[dl, "p_1"'] \arrow[d, "\sigma"] \arrow[dr, "p_2"] & \\
C \arrow[d, "c"'] & Y = (C \times C)/G^0 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] \arrow[dd, "\pi"] & C \arrow[d, "c"] \\
C/G^0 & & C/G^0 \\
S \arrow[r, "\rho"] & X = (C \times C)/G &
\end{tikzcd} \tag{5.1}$$

where p_1, $p_2 \colon C \times C \to C$ are the projections to the first and the second factor. By definition, $G^0 \hookrightarrow \operatorname{Aut}(C)$. Let $c\colon C \to C/G^0$ be the projection to the quotient. Let α_1, $\alpha_2 \colon Y \to C/G^0$ be the morphisms defined by

$$\alpha_1(\sigma(u,v)) = c(u)\,, \qquad \alpha_2(\sigma(u,v)) = c(v)\,. \tag{5.2}$$

Note that they are well defined! Let $\rho\colon S \to X$ be the minimal resolution of X. Moreover we denote by $Q\colon Y \to C/G^0 \times C/G^0$ the map

$$Q(\sigma(u,v)) = (c(u), c(v))\,. \tag{5.3}$$

Theorem 5.0.12 (cf. [Cat00, Proposition 3.16])**.** *Let $X = (C \times C)/G$ be a mixed surface. Then the quotient map $C \times C \to X$ is quasi-étale if and only if the exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{5.4}$$

does not split.

Proof. $(\Rightarrow)$ We have to prove that the extension (5.4) does not split.

Let τ' and φ as in Theorem 4.4.4. If there exists $h \in G^0$ such that $(\tau' h)^2 = 1$, i.e. $\varphi(h)\tau h = 1$, then we get

$$\tau' h(x, \tau h x) = (\varphi(h)\tau h x, \tau h x) = (x, \tau h x) \quad \forall x \in C\,,$$

hence the quotient map $C \times C \to X$ is ramified along the curve $y = (\tau h)x$, contradicting our assumptions.

$(\Leftarrow)$ We factor the quotient map of $X := (C \times C)/G$ as

$$C \times C \xrightarrow{\sigma} Y := (C \times C)/G^0 \xrightarrow{\pi} X\,.$$

From the minimality of Y (G^0 acts faithfully on both factors), we have that σ is branched only in a finite number of points $r_1, \dots, r_t$, therefore our claim follows if we prove that the branch locus of the double cover π is finite.

Aiming for a contradiction, we assume that there exists a curve $D \subseteq X$ such that $|\pi^{-1}(q)| = 1$ for all $q \in D$.

Let $q \in D$ be such that $\pi^{-1}(q) = p' \notin \{r_1, \dots, r_t\}$. Since σ is a $|G^0| =: n$ to 1 map, we have $\sigma^{-1}(p') = \{p_1, \dots, p_n\}$. Since $|(\pi \circ \sigma)^{-1}(q)| = n$, we get that $|\mathrm{Stab}(p_1)| = 2$, hence $\mathrm{Stab}(p_1) \cong \mathbb{Z}_2$ is generated by an element not in G^0. Then (5.4) splits, a contradiction.

□

Theorem 5.0.13. *Let $X = (C \times C)/G$ be a mixed quasi-étale surface. Then* $\mathrm{Sing}(X) = \pi(\mathrm{Sing}(Y))$.

Proof. Let $\{r_1, \dots, r_t\}$ be the singular locus of Y. If $q \in \mathrm{Sing}(X) \setminus \pi(\mathrm{Sing}(Y))$ then $\pi^{-1}(q) = p' \notin \{r_1, \dots, r_t\}$ and we can argue as in the proof of Theorem 5.0.12 to get a contradiction; therefore $\mathrm{Sing}(X) \subseteq \pi(\mathrm{Sing}(Y))$.

Let $y \in \mathrm{Sing}(Y)$. If y is not a ramification point for π, then it is obvious that $\pi(y) \in \mathrm{Sing}(X)$. Let $Z := Y \setminus \pi^{-1}(\mathrm{Sing}(X))$. Then

$$\pi_{|Z} \colon Z \to X \setminus \mathrm{Sing}(X)$$

is a quasi-étale morphism with Z normal and $X \setminus \mathrm{Sing}(X)$ smooth hence, by Lemma 2.1.11, $\pi_{|Z}$ is étale. So the branch points for π are contained in $\mathrm{Sing}(X)$. It follows that if $y \in \mathrm{Sing}(Y)$ and it is a ramification point for π, then $\pi(y) \in \mathrm{Sing}(X)$. □

Remark 5.0.14. From the previous theorem it follows immediately that if $X = (C \times C)/G$ is a mixed q.e. surface then the map

$$\pi \colon Y := (C \times C)/G^0 \longrightarrow X$$

is quasi-étale, since its branch locus is contained in $\mathrm{Sing}(X)$.

Lemma 5.0.15. *Let S be the minimal resolution of the mixed quasi-étale surface $X = (C \times C)/G$.*

Then $q(S) = g(C/G^0)$.

Proof. From Corollary 4.1.12 it follows

$$H^0(\Omega^1_S) = (H^0(\Omega^1_{C\times C}))^G .$$

Arguing as in [Cat00, Proposition 3.15], we get

$$\begin{array}{rcl} H^0(\Omega^1_S) & = & (H^0(\Omega^1_{C\times C}))^G \\ & = & (H^0(\Omega^1_C) \oplus H^0(\Omega^1_C))^G \\ & = & (H^0(\Omega^1_C)^{G^0} \oplus H^0(\Omega^1_C)^{G^0})^{G/G^0} \\ & = & (H^0(\Omega^1_{C'}) \oplus H^0(\Omega^1_{C'}))^{G/G^0}. \end{array}$$

Since X is a quotient surface of mixed type, the quotient $\mathbb{Z}_2 = G/G^0$ exchange the last summands, hence

$$H^0(\Omega^1_S) \cong H^0(\Omega^1_{C'}) .$$

We get $q(S) = h^0(\Omega^1_S) = h^0(\Omega^1_{C'}) = g(C')$. □

5.1 On the singularities

By construction $Y = (C \times C)/G^0$ is an unmixed surface and so its singularities are all cyclic quotient singularities. In this section we want to understand which kind of singularities a mixed q.e. surface can have. In particular we study their resolution graph.

We start with the following observation:

Remark 5.1.1. The group G induces an involution ι on $Y = (C \times C)/G^0$ in the following way: let $\sigma(u,v)$ be a point in Y, $\iota(\sigma(u,v)) = \sigma(g(u,v))$ for some $g \in G \setminus G^0$. It is easy to prove that it is well defined.

Let $\lambda\colon T \to Y$ be the minimal resolution of the singularities of Y.

Lemma 5.1.2. *The involution ι on Y lifts to an involution μ on T.*

Proof. Let $\mu\colon T \dashrightarrow T$ be the birational map defined by $\mu := \lambda^{-1} \circ \iota \circ \lambda$. Let $\Gamma \subset T \times T$ be the graph of μ; let f_1 be the projection on the first factor and let f_2 be the projection on the second factor.

If μ is not defined in a point $p \in T$, then Γ contains a (-1)-curve C contracted to p by f_1. f_2 maps C to a curve $D \subset T$ contracted to $\iota(\lambda(p))$ by λ. But $D^2 \geq -1$ and all the exceptional curves have self-intersection ≤ -2, a contradiction. $\square$

Remark 5.1.3. If μ fixes $p \in T$ then $P \in D$, the exceptional divisor of λ.

Let $y = \sigma(u,v)$ be a singular point in Y of type $C_{n,a}$. Consider the morphisms α_1, $\alpha_2\colon Y \to C/G^0$ defined as in (5.2):

$$\alpha_1(\sigma(u,v)) = c(u)\,, \qquad \alpha_2(\sigma(u,v)) = c(v)\,.$$

Proposition 5.1.4. *If y is a point of type $C_{n,a}$ with respect to α_1 (see Definition 4.3.10), then $\iota(y)$ is a point of type $C_{n,a'}$ with respect to α_1, where $aa' \cong 1 \mod n$.*

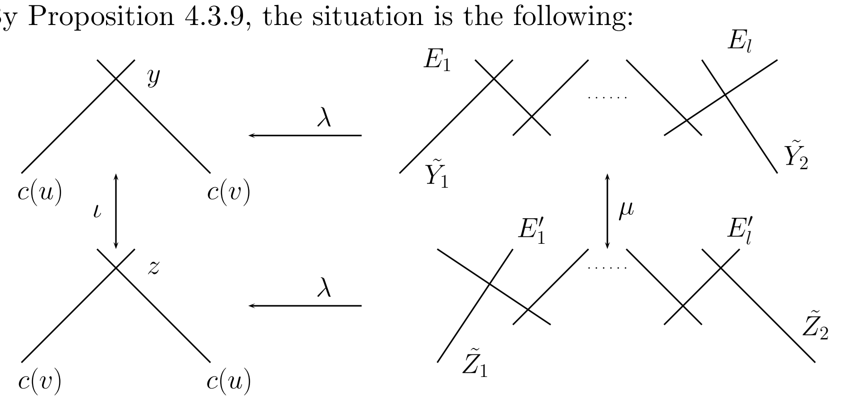
Proof. Let $y = \sigma(u,v)$ and let $z := \iota(y) = \iota(\sigma(u,v)) = \sigma(\tau'(u,v)) = \sigma(v,\tau u)$; $Q(z) = (c(v), c(\tau u)) = (c(v), c(u))$. Consider the following fibres:

$$Y_1 := \alpha_1^*(c(u))\,, \qquad Y_2 := \alpha_2^*(c(v))\,,$$

$$Z_1 := \alpha_1^*(c(v)) \quad \text{and} \quad Z_2 := \alpha_2^*(c(u))\,,$$

all of them taken with the reduced structure.

Let $\tilde{Y}_i := \lambda_*^{-1}(Y_i)$ and $\tilde{Z}_i := \lambda_*^{-1}(Z_i)$ $(i = 1,2)$ be their strict transforms in T. By Proposition 4.3.9, the situation is the following:



Then there exists $\nu_1, \dots, \nu_4, \gamma_1, \dots, \gamma_l, \delta_1, \dots, \delta_l \in \mathbb{N}$ such that:

$$F_1 := (\lambda \circ \alpha_1)^*(c(u)) = \nu_1 \tilde{Y_1} + \sum_{i=1}^{l} \gamma_i E_i + \Gamma_1$$

$$F_2 := (\lambda \circ \alpha_2)^*(c(v)) = \nu_2 \tilde{Y_2} + \sum_{i=1}^{l} \gamma_i E_i + \Gamma_2$$

$$F_3 := (\lambda \circ \alpha_1)^*(c(v)) = \nu_3 \tilde{Z}_1 + \sum_{j=1}^{l} \delta_j E'_j + \Gamma_3$$

$$F_4 := (\lambda \circ \alpha_2)^*(c(u)) = \nu_4 \tilde{Z}_2 + \sum_{j=1}^{l} \delta_j E'_j + \Gamma_4$$

Here the E_i are the irreducible components of the exceptional divisor lying over y and the E'_j are the irreducible components of the exceptional divisor lying over z. Since μ is an isomorphism that exchanges these two divisors, we have that they have the same number of irreducible components. Here the Γ_i are unions of HJ-strings disjointed from the E_i and E'_j; they are the exceptional divisors lying over the other singular points of Y_i and Z_i. We note that by assumptions,

$$E_i \cap \tilde{Y_1} = \begin{cases} \{pt.\} & \text{if } i = 1 \\ \emptyset & \text{if } i \neq 1 \end{cases} \qquad E_i \cap \tilde{Y_2} = \begin{cases} \{pt.\} & \text{if } i = l \\ \emptyset & \text{if } i \neq l \end{cases} \tag{5.5}$$

$$E'_i \cap \tilde{Z}_1 = \begin{cases} \{pt.\} & \text{if } i = 1 \\ \emptyset & \text{if } i \neq 1 \end{cases} \qquad E'_i \cap \tilde{Z}_2 = \begin{cases} \{pt.\} & \text{if } i = l \\ \emptyset & \text{if } i \neq l \end{cases} \tag{5.6}$$

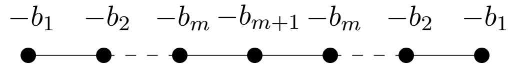
By (5.5), $\{pt.\} = \mu(E_1 \cap \tilde{Y_1}) = \mu(E_1) \cap \mu(\tilde{Y_1})$, but $\mu(\tilde{Y_1}) = \tilde{Z}_2$ and the unique curve of $\lambda^{-1}(z)$ that intersect $\tilde{Z}_2$ is E'_l, hence $\mu(E_1) = E'_l$. We get $-b_1 = E_1^2 = E'^2_l = -b'_l$; analogously $\mu(E_l) = E'_1$ and $-E_l^2 = E'^2_1 = -b_l$. Arguing in this way, it is easy to prove, by induction, that $\mu(E_i) = E'_{l+1-i}$ and so $b_i = b_{l+1-i}$. In particular, we get that $\tilde{Z}_1$ intersects the extremal curve with self-intersection $-b_l$, hence $z = \iota(y)$ is of type $C_{n,a'}$ with respect to α_1. □

5.1.1 Singularities of type $D_{n,a}$

Proposition 5.1.5. *Let $X = (C \times C)/G^0$ be a mixed q.e. surface and let $y \in \operatorname{Sing}(Y)$ be a point of type $C_{n,a}$ with $\frac{n}{a} = [b_1, \dots, b_l]$. Let $\lambda\colon T \to Y$ be the minimal resolution of Y. If y is a ramification point of $\pi\colon Y \to X$ then*

(i) n is even;

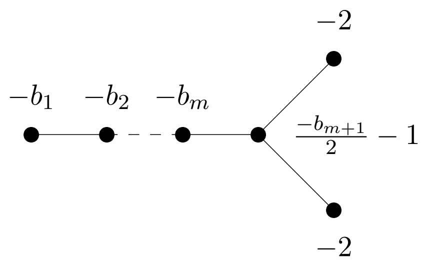
(ii) $b_i = b_{l+1-i}$ for all $i = 1, \dots, l$ and l is odd: $l = 2m + 1$ for some $m \in \mathbb{N}$, in particular the resolution graph of y is:



(iii) the minimal resolution of the singular point $\pi(p)$ *is the connected union* E *of* $m+3$ *rational curves* $E_1, \dots, E_{m+1}, F_1, F_2$ *with the following intersection numbers:*

$$\begin{cases} E_i^2 = -b_i \text{ for } i = 1, \dots, m \quad , \\ E_{m+1}^2 = -1 - \frac{b_{m+1}}{2}\,, \quad \text{ in particular } b_{m+1} \text{ is even}\,, \\ E_i.E_j = 0 \text{ if } |i-j| \geq 2\,, 1 \leq i, j \leq m\,, \\ F_1^2 = F_2^2 = -2\,, \\ E_{m+1}.F_1 = E_{m+1}.F_2 = E_i.E_{m+1} = 1 \text{ for all } 1 \leq i \leq m\,, \\ F_1.E_j = F_2.E_j = 0 \text{ if } j \neq m+1\,. \end{cases}$$

The resolution graph is:



Proof. (i) Let $y = \sigma(u, v)$, $|\mathrm{Stab}_G(u, v)| = 2n$ and $|\mathrm{Stab}_{G^0}(u, v)| = n$. If n is odd, then there exists an element g of order 2 in $\mathrm{Stab}_G(u, v) \setminus \mathrm{Stab}_{G^0}(u, v)$, by Sylow's theorem. In particular $g \in G \setminus G^0$, a contradiction.

(ii) Let $D := \lambda^{-1}(y)$, since y is of type $C_{n,a}$ we have that D is a tree of l rational curves $D_1, \dots, D_l$ with $D_i^2 = -b_i$, $D_i.D_{i+1} = 1$ and $D_i.D_j = 0$ if $|i-j| \geq 2$. Arguing as in the proof of Proposition 5.1.4, we get that $\mu(D_i) = D_{l+1-i}$ and $-b_i = D_i^2 = D_{l+1-i}^2 = -b_{l+1-i}$.

Assume now that $l = 2m$ be even, the involution μ exchanges D_i with D_{l+1-i}, hence $p = D_m \cap D_{m+1}$ is the unique point fixed by μ .

Let us consider local coordinates in a neighborhood U of p centered in p, in these coordinates the involution is $\varsigma \colon \mathbb{C}^2 \to \mathbb{C}^2$ with a unique fixed point: $(0, 0)$. Up to a change of coordinates (Lemma 4.2.2), we can assume that ς is linear. The Jordan form of ς is one of the following:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

We note that

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}^2 = \begin{pmatrix} a^2 & 2a \\ 0 & a^2 \end{pmatrix} \neq \mathrm{Id} \qquad \forall a \in \mathbb{C}$$

therefore ς is of the form

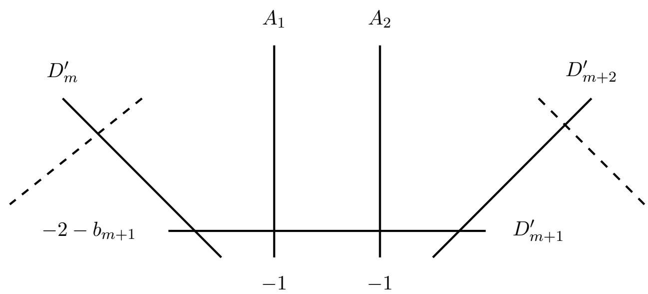
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{with } a^2 = b^2 = 1\,.$$

We have, up to a linear coordinate change, three cases:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

ς cannot be neither the identity, which fixes every point, nor the second matrix which fixes a line. The third matrix fixes only the point $(0,0)$ but it sends every line that passes through the origin into itself, a contradiction since we have that μ exchanges two lines through p. Hence l is odd.

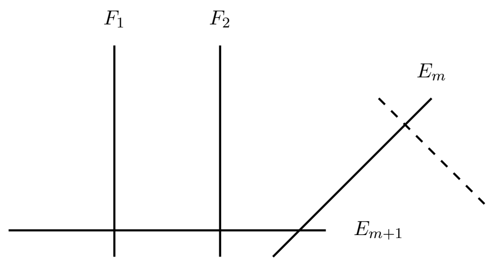
(iii) By point (ii), $l = 2m+1$ and $\mu(D_{m+1}) = D_{m+1}$. The restriction of μ to D_{m+1} is an involution $\varsigma\colon \mathbb{P}^1 \to \mathbb{P}^1$, by Hurwitz's formula we get that ς fixes exactly two points p_1 and p_2 that cannot be the points of intersection of D_{m+1} with D_m or D_{m+2}. Let $\epsilon\colon T' \to T$ be the blow-up of T in p_1 and p_2, we denote by D'_i the strict transform of D_i and by A_1 and A_2 the two (-1)-exceptional curves. We have that ${D'}_i^2 = {D'}_{l+1-i}^2 = D_i^2 = D_{l+1-i}^2 = -b_i$ for $i = 1\dots, m$, ${D'}_{m+1}^2 = -2 - b_{m+1}$ and $A_1^2 = A_2^2 = -1$.



Let us consider local coordinates (x,y) in a neighborhood $U \cong \mathbb{C}^2$ of p_1 centered in p_1; arguing as before, we can assume that in these coordinates, the involution is $\varsigma\colon (x,y) \mapsto (-x,-y)$; moreover the blow-up of U in p_1 is given by $(x_1,y_1) \mapsto (x_1, x_1y_1)$ on a chart (say U_1) and by $(x_2,y_2) \mapsto (x_2y_2, y_2)$ on the other chart (say U_2). The gluing $U_1 \cup U_2$ is given by $(x_1,y_1) \mapsto (x_1^{-1}, x_1y_1)$ and the exceptional curve E is $\{x_1 = 0\} \cup \{y_2 = 0\}$. The involution ς lifts to an involution on the blow-up:

$$(x_1,y_1) \longmapsto (-x_1,y_1) \qquad (x_2,y_2) \longmapsto (x_2,-y_2)\,.$$

So the set of points fixed by the lift of ς is $\{x_1 = 0\} \cup \{y_2 = 0\} = E$. Arguing in the same way for p_2 we lift the involution μ to an involution μ' on T'. Let V be an open set of T containing $D' := \bigcup_i D'_i \cup A_1 \cup A_2$ and let $p\colon V \to V/\mu'_{|V}$ be the projection to the quotient. Up to shrinking V, the restriction of μ' to V is an isomorphism of V that fixes only A_1 and A_2 and so the quotient $V/\mu'_{|V}$ is smooth and it has the form:



where $F_1 = p(A)$, $F_2 = p(B)$, $E_i = p(D'_i) = p(D'_{l+1-i})$ for $i = 1, \dots, m+1$. All these curves are rational, indeed the restriction of p to each of these curves is an isomorphism onto its image, except for one case: D'_{m+1}. In this case the map $p_{|D'_{m+1}}\colon D'_{m+1} \to E_{m+1}$ has degree 2 and it is the quotient of $D'_{m+1} \cong \mathbb{P}^1$ by an involution that fixes two points, hence by Hurwitz's formula we get:

$$-2 = 2\left(2g(E_{m+1}) - 2 + \frac{1}{2} + \frac{1}{2}\right).$$

It follows that $g(E_{m+1}) = 0$ and so E_{m+1} is a rational curve. Using the projection formula, we can compute the self-intersection of the curves $E_1, \dots, E_{m+1}, F_1, F_2$:

$$\begin{aligned} E_i^2 = \frac{1}{2}(p^*(E_i).p^*(E_i)) &= \frac{1}{2}((D'_i + D'_{l+1-i})^2) \\ &= \frac{1}{2}((D'_i)^2 + (D'_{l+1-i})^2) = -b_i \quad \forall i = 1, \dots, m \end{aligned}$$

$$E_{m+1}^2 = \frac{1}{2}(p^*(E_{m+1}).p^*(E_{m+1})) = \frac{1}{2}(D'_{m+1}.D'_{m+1}) = -1 - \frac{b_{m+1}}{2}$$

$$F_i^2 = \frac{1}{2}(p^*(F_i).p^*(F_i)) = \frac{1}{2}(2A_i.2A_i) = -2 \quad \text{for } i = 1, 2$$

□

Corollary 5.1.6. *Let $y \in Y$ as in Proposition 5.1.5, then $a = a'$, i.e. $a^2 = 1$ mod (n).*

Proof. This follows directly by Proposition 5.1.5 (ii). □

Lemma 5.1.7. *Let $X = (C \times C)/G^0$ be a mixed q.e. surface and let $y \in \operatorname{Sing}(Y)$ be a point of type $C_{n,a}$ with $\frac{n}{a} = [b_1, \dots, b_m, b_{m+1}, b_m, \dots, b_1]$. Let*

$$\frac{p}{q} := [b_1, \dots, b_m], \qquad \beta := \frac{b_{m+1}}{2} + 1 \quad \text{and} \quad \alpha := (\beta - 1)p - q.$$

Then $x := \pi(y)$ is a quotient singularity isomorphic to $\mathbb{C}^2/H$ with:

- *if* $\alpha = 0$ *(i.e.* $p = 0$*), then*

$$H = \left\langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{n+1} \end{pmatrix} \right\rangle, \qquad \textit{with } \epsilon = e^{\frac{2\pi i}{2n}},$$

- *if* $\alpha \neq 0$ *and odd, then*

$$H = \left\langle \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \ \textit{with } \eta = e^{\frac{2\pi i}{2\alpha}}, \omega = e^{\frac{2\pi i}{2p}},$$

- *if* $\alpha \neq 0$ *and even, then*

$$H = \left\langle \begin{pmatrix} 0 & \zeta \\ -\zeta & 0 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \right\rangle, \ \textit{with } \zeta = e^{\frac{2\pi i}{4\alpha}} \textit{ and } \omega = e^{\frac{2\pi i}{2p}}.$$

Proof. The statement follows immediately from the classification of finite subgroups of $\mathrm{GL}(2, \mathbb{C})$ without quasi-reflection (i.e. with only one fixed point), see [Bri68, Satz 2.11] or [Mat67, Theorem 4.6.20]. □

Definition 5.1.8. We say that a singular point x as in Lemma 5.1.7 is a singular point of type $D_{n,a}$.

Remark 5.1.9.

1. A singular point of type $D_{n,n-1}$ is a Rational Double Point. It is a Du Val singularity of type D_m, where $2m - 4 = n$.
2. We note that $a = 1$ if and only if $p/q = 0$. In this case we have a point of type $D_{n,1}$ which is isomorphic to a cyclic quotient singularity type $C_{2n,n+1}$.

Remark 5.1.10. Let $X = (C \times C)/G$ be a mixed quasi-étale surface and let $S \xrightarrow{\rho} X$ be its minimal resolution of the singularities. Let $T \xrightarrow{\lambda} Y$ be the minimal resolution of $Y = (C \times C)/G^0$. By the proof of Proposition 5.1.5, it follows that the involution μ on T fixes $2d$ points, where d is the number of branch points for π. The involution μ lifts to an involution μ' of T' that fixes the exceptional divisor and so T'/μ' is smooth and it is isomorphic to S. Moreover $\tilde{\pi} : T' \to S$ is a double cover ramified along the $2d$ exceptional curves.

In the following the term *multiset* will be used in the sense of MAGMA, that is a set with some of its members repeated.

Definition 5.1.11 (cf. [BP10, Definition 1.2])**.** Let Y be an unmixed surface. Then we define the *basket of singularities of* Y to be the multiset

$$\mathcal{B}(Y) := \left\{ \lambda \times C_{n,a} \, : \, Y \text{ has exactly } \lambda \text{ singularities of type } C_{n,a} \right\}.$$

Let $X = (C \times C)/G$ be a mixed q.e. surface. We recall that $\mathrm{Sing}(X) = \pi(\mathrm{Sing}(Y))$. We define the following two multisets:

$$\mathcal{B}_C := \big\{\eta \times C_{n,a} \,:\, X \text{ has exactly } \eta \text{ singularities of type } C_{n,a} \text{ not in the branch locus } \big\}.$$

$$\mathcal{B}_D := \big\{\zeta \times D_{m,b} \,:\, X \text{ has exactly } \zeta \text{ singularities of type } D_{m,b} \text{ in the branch locus } \big\}.$$

Definition 5.1.12. The *basket of singularities of* X is the multiset

$$\mathcal{B}(X) = \mathcal{B}_C \cup \mathcal{B}_D\,.$$

Remark 5.1.13. As noted in [BP10, Remark 1.3], in the above definitions there is some ambiguity: a point of type $C_{n,a}$ is also a point of type $C_{n,a'}$ with $a' = a^{-1}$ in $\mathbb{Z}_n$. We consider these different representations as equal and usually we do not distinguish between them.

Lemma 5.1.14. *Let $X = (C \times C)/G$ be a mixed q.e. surface. Let $\mathcal{B}(X) = \mathcal{B}_C \cup \mathcal{B}_D$ be the basket of singularities of X with $\mathcal{B}_C := \{\eta_i \times C_{n_i,a_i}\}_i$ and $\mathcal{B}_D := \{\zeta_j \times D_{m_j,b_j}\}_j$. Then*

$$\sum_i \eta_i \frac{a_i + a_i'}{n_i} + \sum_j \zeta_j \frac{b_j}{m_j} \in \mathbb{Z}\,.$$

Proof. If $x \in X$ is a singular point of type $D_{m,b}$, then $\pi^{-1}(x) = y$ (with $y = \iota(y)$) is a singular point of type $C_{m,b}$. If $x \in X$ is a singular point of type $C_{n,a}$, then $\pi^{-1}(x) = \{y, z\}$ ($z = \iota(y)$) are two singular points of type $C_{n,a}$, hence

$$\mathcal{B}(Y) = \{2\eta_i \times C_{n_i,a_i}, \zeta_j \times C_{m_j,b_j}\}_{i,j}\,. \tag{5.7}$$

Let $\alpha\colon Y \to C/G^0$ the fibration given by $\alpha_1(\sigma(u,v)) = c(u)$. By Proposition 5.1.4, if $y \in \mathrm{Sing}(Y)$ is a point of type $C_{n,a}$ with respect to α, then $z = \iota(y)$ is a point of type $C_{n,a'}$ with respect to α. So to each element $D_{m,b}$ in $\mathcal{B}_D$ corresponds a singular point of type $C_{m,b}$ with respect to α, while to each element $C_{n,a}$ in $\mathcal{B}_C$ corresponds a pair of singular points: one of type $C_{n,a}$ and one of type $C_{n,a'}$ with respect to α. By Lemma 4.3.14, we get

$$\sum_i \eta_i \frac{a_i + a_i'}{n_i} + \sum_j \zeta_j \frac{b_j}{m_j} \in \mathbb{Z}\,.$$

□

5.2 On the invariants

Definition 5.2.1 (see [BP10, Definition 1.4])**.** Let x be a singular point of type $C_{n,a}$ and let $\frac{n}{a} := [b_1, \ldots, b_l]$.
We define the following correction terms:

i) $k_x = k(C_{n,a}) := -2 + \dfrac{2+a+a'}{n} + \sum_{i=1}^{l}(b_i - 2) \geq 0$;

ii) $e_x = e(C_{n,a}) := l + 1 - \dfrac{1}{n} \geq 0$;

iii) $B_x = B(C_{n,a}) := 2e_x + k_x$.

Definition 5.2.2. Let x be a singular point of type $D_{n,a}$ and let $\frac{n}{a} := [b_1, \ldots, b_m, b_{m+1}, b_m, \ldots, b_1]$.
We define the following correction terms:

i) $k_x = k(D_{n,a}) := -2 + \frac{a+1}{n} + \sum_{i=1}^{m}(b_i - 2) + \frac{b_{m+1}}{2} \geq 0$;

ii) $e_x = e(D_{n,a}) := m + 4 - \dfrac{1}{2n}$;

iii) $B_x = B(D_{n,a}) := 2e_x + k_x$.

Remark 5.2.3. From the definition it follows that

$$k(D_{n,a}) = \frac{k(C_{n,a})}{2} \qquad e(D_{n,a}) = \frac{e(C_{n,a})}{2} + 3\,, \qquad B(D_{n,a}) = \frac{B(C_{n,a})}{2} + 6\,.$$

Let $\mathcal{B}$ be the basket of singularities of X. We use the following notation:

$$k(\mathcal{B}) = \sum_{x \in \mathcal{B}} k_x, \quad e(\mathcal{B}) = \sum_{x \in \mathcal{B}} e_x, \quad B(\mathcal{B}) = \sum_{x \in \mathcal{B}} B_x\,.$$

Proposition 5.2.4. *Let $\rho \colon S \to X$ be the minimal resolution of singularities of $X = (C \times C)/G$, and let $\mathcal{B}$ be the basket of singularities of X. Then*

$$K_S^2 = \frac{8(g-1)^2}{|G|} - k(\mathcal{B})\,; \tag{5.8}$$

$$e(S) = \frac{4(g-1)^2}{|G|} + e(\mathcal{B})\,. \tag{5.9}$$

Proof. Arguing as in Proposition 4.1.10, we get $K_{C\times C}^2 = 8(g-1)^2$ and $e(C \times C) = 4(g-1)^2$, where $g := g(C)$. By construction $\sigma \colon C \times C \to Y$ has finite branch locus, then $K_{C\times C} = \sigma^* K_Y$. We get $K_{C\times C}^2 = \deg \sigma \cdot K_Y^2 = |G^0| \cdot K_Y^2$, so

$$K_Y^2 = \frac{8(g-1)^2}{|G^0|}\,.$$

Since $\pi\colon Y \to X$ has finite branch locus, then $K_Y = \pi^* K_X$. We get $K_Y^2 = \deg \pi \cdot K_X^2 = 2 \cdot K_X^2$, so

$$K_X^2 = \frac{K_Y^2}{2} = \frac{8(g-1)^2}{|G|} .$$

Let $\mathcal{B} = \mathcal{B}_C \cup \mathcal{B}_D = \{\eta_i \times C_{n_i,a_i}\}_i \cup \{\zeta_j \times D_{n_j,a_j}\}_j$, then the basket of singularities of $Y := (C \times C)/G^0$ is

$$\mathcal{B}(Y) = \{\eta_i \times C_{n_i,a_i}, \eta_i \times C_{n_i,a_i'}\}_i \cup \{\zeta_j \times C_{n_j,a_j}\}_j ,$$

hence by definition

$$k(\mathcal{B}(Y)) = 2k(\mathcal{B}) .$$

Let $\lambda\colon T \to Y$ be the minimal resolution of singularities of Y, it is a product-quotient surface (see [BP10, Definition 0.1]) and so by Lemma 4.2.11 we have

$$K_T^2 = \frac{8(g-1)^2}{|G^0|} - k(\mathcal{B}(Y)) .$$

The involution μ on T has an even number of fixed points: $2d$ with $d = |\mathcal{B}_D|$ (see Remark 5.1.10). Let $\epsilon\colon T' \to T$ be the blow-up of T in these points. We get

$$K_{T'}^2 = K_T^2 - 2d = K_Y^2 - (k(\mathcal{B}(Y)) + 2d) = 2(K_X^2 - k(\mathcal{B}) - d) . \qquad (5.10)$$

Since ζ lifts to an involution ζ' on T' that fixes the exceptional divisor of ϵ, the quotient T'/ζ' is smooth and isomorphic to S; $\tilde{\pi}\colon T' \to S$ is a double cover branched over $F = F_1 + \ldots + F_{2d}$, where the F_i are rational curves and $F_i.F_j = 0$ if $i \neq j$. In particular, we get (see [CD89, pages 13-14]):

$$K_{T'} = \tilde{\pi}^* \left(K_S + \frac{F}{2} \right) .$$

We note that $(K_S + F_i).F_i = \deg K_{F_i} = -2$, and by construction $F_i^2 = -2$ for all i and so $K_S.F = 0$, it follows that

$$K_{T'}^2 = 2\left(K_S + \frac{F}{2} \right)^2 = 2\left(K_S^2 + \frac{-4d}{4} \right) = 2(K_S^2 - d) . \qquad (5.11)$$

From equations (5.10) and (5.11), we get:

$$K_S^2 = K_X^2 - k(\mathcal{B}) = \frac{8(g-1)^2}{|G|} - k(\mathcal{B}) .$$

To prove (5.9), we argue as follows: let $X^0 := X \setminus \mathrm{Sing}(X)$ be the smooth locus of X. Let x be a point of type $C_{n,a}$, then $\rho^{-1}(x)$ is a tree of l (the length of the continued fraction n/a) rational curves and so $e(\rho^{-1}(x)) = l+1$, while for a point x of type $D_{n,a}$: $e(\rho^{-1}(x)) = m+4$, where m is as in Definition 5.2.2; therefore

$$e(S) = e(X^0) + \sum_{x \in \mathcal{B}_C} (l_x + 1) + \sum_{x \in \mathcal{B}_D} (m_x + 4)\,.$$

Let $Z^0 := (C \times C) \setminus ((\pi \circ \sigma)^{-1}(\mathrm{Sing}(X)))$, so $Z^0 \to X^0$ is étale, hence

$$\begin{aligned} e(X^0) &= \frac{e(Z^0)}{|G|} = \frac{e(C \times C) - |(\pi \circ \sigma)^{-1}(\mathrm{Sing}(X))|}{|G|} \\ &= \frac{e(C \times C)}{|G|} - \sum_{x \in \mathcal{B}_C} \frac{|(\pi \circ \sigma)^{-1}(x)|}{|G|} - \sum_{x \in \mathcal{B}_D} \frac{|(\pi \circ \sigma)^{-1}(x)|}{|G|} \\ &= \frac{e(C \times C)}{|G|} - \sum_{x \in \mathcal{B}_C} \frac{1}{n_x} - \sum_{x \in \mathcal{B}_D} \frac{1}{2n_x} \end{aligned}$$

It follows that

$$\begin{aligned} e(S) &= \frac{e(C \times C)}{|G|} - \sum_{x \in \mathcal{B}_C} \left(l_x + 1 - \frac{1}{n_x} \right) - \sum_{x \in \mathcal{B}_D} \left(m_x + 4 - \frac{1}{2n_x} \right) \\ &= \frac{4(g-1)^2}{|G|} + e(\mathcal{B}) \end{aligned}$$

□

Corollary 5.2.5. *Let $S \to X = (C \times C)/G$ be the minimal resolution of singularities of X, and let $\mathcal{B}$ be the basket of singularities of X. Then*

$$K_S^2 = 8\chi(S) - \frac{1}{3} B(\mathcal{B})\,.$$

Proof. By Proposition 5.2.4 we have

$$e(S) = e(\mathcal{B}) + \frac{K_S^2 + k(\mathcal{B})}{2} = \frac{K_S^2 + B(\mathcal{B})}{2}\,.$$

Using Noether's formula we get

$$12\chi(S) = K_S^2 + e(S) = \frac{3K_S^2 + B(\mathcal{B})}{2}\,.$$

□

Since $B(\mathcal{B}) \geq 0$, it follows that

Corollary 5.2.6.

$$K_S^2 \leq 8\chi(S)\,.$$

Lemma 5.2.7. *Let X be a mixed quasi-étale surface. Let $\rho\colon S \to X$ be the minimal resolution of X and let $\lambda\colon T \to Y$ be the minimal resolution of Y. Let d be the number of branch points for π, then*

$$p_g(T) = 2p_g(S) + 1 - \frac{d}{2},$$

in particular d is even.

Proof. The involution μ on T has an even number of fixed points: $2d$ with $d = |\mathcal{B}_D|$; let $\epsilon\colon T' \to T$ be the blow-up of T in these points. Moreover μ lifts to an involution μ' on T' that fixes only the exceptional divisor of ϵ, hence the quotient T'/μ' is smooth and isomorphic to S; we have that $\tilde{\pi}\colon T' \to S$ is a double cover branched over $F = F_1 + \ldots + F_{2d}$, where the F_i are rational curves, and $K_{T'}^2 = 2(K_S^2 - d)$. Since $\tilde{\pi}$ is branched along $2d$ rational curve ($e(\mathbb{P}^1) = 2$) we have that $e(T') = 2e(S) - 4d$, we also note that $e(T) = e(T') - 2d = 2e(S) - 6d$. By the proof of Proposition 5.2.4, we get $K_T^2 = 2K_S^2$.
Since T is smooth, Noether's formula applies and

$$\begin{aligned}\chi(\mathcal{O}_T) &= \frac{1}{12}(K_T^2 + e(T)) = \frac{1}{12}(2K_S^2 + 2e(S) - 6d) \\ &= 2\chi(\mathcal{O}_S) - \frac{d}{2}\end{aligned}$$

By Lemma 4.3.15, since $T \to X$ is a product-quotient surface, we have $q(T) = 2g(C/G^0) = 2q(S)$, hence

$$\begin{aligned}p_g(T) &= 2 + 2p_g(S) - 2q(S) - \frac{d}{2} + q(T) - 1 \\ &= 2p_g(S) + 1 - \frac{d}{2}.\end{aligned}$$

□

Noting that the branch points of $\pi\colon Y \to X$ are exactly the singular points of X of type $D_{n,a}$, the next statement follows:

Corollary 5.2.8. *The number d of singular points of type $D_{n,a}$ of a mixed q.e. surface X is even and*

$$\frac{d}{2} \leq 2p_g(S) + 1,$$

where $S \to X$ is the minimal resolution of X.

5.2.1 Determining the signatures

Definition 5.2.9. Let S be the minimal resolution of the mixed q.e. surface $X = (C \times C)/G$. Let $\psi\colon \mathbb{T}(g'; m_1, \dots, m_r) \longrightarrow G^0$ be the appropriate orbifold homomorphism induced by $c\colon C \longrightarrow C/G^0$. Let $\mathcal{B}$ be the basket of singularities of X. Then we define the following numbers:

$$\Theta := 2q(S) - 2 + \sum_{i=1}^{r} \left(\frac{m_i - 1}{m_i} \right),$$

$$\beta := \frac{12\chi(\mathcal{O}_S) + k(\mathcal{B}) - e(\mathcal{B})}{3\Theta},$$

$$\xi := 4\chi(\mathcal{O}_S) + \frac{k(\mathcal{B}) - e(\mathcal{B})}{3}.$$

Remark 5.2.10. We note that $\xi = \beta \cdot \Theta$. Moreover by Noether's formula we have $12\chi(\mathcal{O}_S) = K_S^2 + e(S)$ and so we get:

$$12\chi = \frac{8(g-1)^2}{|G|} - k + \frac{4(g-1)^2}{|G|} + e$$

hence

$$\xi = \frac{1}{3}(12\chi + k - e) = \frac{4(g-1)^2}{|G|} = \frac{K_S^2 + k(\mathcal{B})}{2}.$$

In particular ξ depends only on K_S^2 and on the basket of singularities.

Definition 5.2.11 (see [Rei87])**.** The minimal positive integer I_x such that $I_x K_X$ is Cartier in a neighborhood of $x \in X$ is called the *index of the singularity* x.
The *index* of a normal variety X is the minimal positive integer I such that IK_X is Cartier. In particular, $I = \mathrm{lcm}_{x \in \mathrm{Sing}(X)} I_x$.

Lemma 5.2.12. *The index of a singularity of type $C_{n,a}$ is*

$$I_x = \frac{n}{\gcd(n, a+1)}.$$

Proof. Let $[b_1, \dots, b_l] := \dfrac{n}{a}$; let $\mu_0 = n$, $\mu_1 = a$ and $\mu_{i+1} = b_i\mu_i - \mu_{i-1}$ for $i = 2, \dots, l$. Let $\lambda_0 = 0$, $\lambda_1 = 1$ and $\lambda_{i+1} = \lambda_i b_i - \lambda_{i-1}$ for $i = 2, \dots, l$. By Lemma 4.2.10, in a neighborhood $X = \mathbb{C}^2/H$ of a singular point of type $C_{n,a}$,

$$K_S = \rho^* K_X + \sum_{i=1}^{l} r_i E_i,$$

where $r_i = \dfrac{\lambda_i + \mu_i - n}{n}$ for $i = 1, \dots, l$.

By [Mat67, Remark 4-6-26],

$$I_x = \min\{e \in \mathbb{N} \mid er_i \in \mathbb{Z}\,\forall i\}\,.$$

We claim that $\alpha := \dfrac{n}{\gcd(n, a+1)} = I_x$.
Let $i = 0$, then $r_0 = (0 + n - n)/n = 0$ and $\alpha \cdot r_0 \in \mathbb{Z}$.
Let $i = 1$, then $r_1 = \dfrac{a+1}{n} - 1$ and

$$\alpha \cdot r_1 = \frac{n}{\gcd(n, a+1)} \cdot \frac{a+1}{n} - 1 \in \mathbb{Z}\,.$$

By definition, α is the minimal integer such that $\alpha \cdot r_1 \in \mathbb{Z}$.
To complete the proof it is enough to prove that $\alpha \cdot r_i \in \mathbb{Z}$ for $i \geq 2$. We prove it by induction on i:

$$r_{i+1} = \frac{\lambda_{i+1} + \mu_{i+1} - n}{n} = \frac{(\lambda_i + \mu_i)b_i}{n} - \frac{\lambda_{i-1} + \mu_{i-1}}{n} - 1\,.$$

Hence

$$\alpha \cdot r_{i+1} = \alpha \cdot \frac{(\lambda_i + \mu_i)}{n} b_i - \alpha \cdot \frac{\lambda_{i-1} + \mu_{i-1}}{n} - \alpha \in \mathbb{Z}\,,$$

since, by inductive hypothesis, we are summing three integers. □

For fixed K_S^2, $p_g(S)$, $q(S)$ and $\mathcal{B}$, we want to bound the possibilities for $(m_1, \ldots, m_r)$ and $|G|$ for a group G acting on $C \times C$ giving rise to a surface S as minimal resolution of the mixed q.e. surface $X = (C \times C)/G$, where S has these invariants.

Proposition 5.2.13 (cf. [BP10, Proposition 1.13]). *Let S be the minimal resolution of the singularities of the mixed q.e. surface $X = (C \times C)/G$. Let $\psi\colon \mathbb{T}(g'; m_1, \ldots, m_r) \to G^0$ be the appropriate orbifold homomorphism induced by $c\colon C \to C/G^0$. Let $\mathcal{B} = \mathcal{B}_C \cup \mathcal{B}_D$ be the basket of singularities of X. Then*

a) $\Theta > 0$ *and* $\beta = g(C) - 1$;

b) $|G| = \frac{8\beta^2}{K_S^2 + k(\mathcal{B})}$;

c) $r \leq \frac{K_S^2 + k(\mathcal{B})}{\beta} + 4(1-q)$;

d) each m_i *divides* $2\beta I$ *where* I *is the index of* Y;

e) there are at most $N := |\mathcal{B}_C| + |\mathcal{B}_D|/2$ *indices* i *such that* m_i *does not divide* β;

f) if $r \neq 0$, $m_i \leq \dfrac{1 + I(K_S^2 + k(\mathcal{B}))}{M}$, *where* $M := \max\left\{\frac{1}{6}, \frac{r-3+4q}{2}\right\}$*; moreover, except for at most* N *indexes* i*, we have the stronger inequality* $m_i \leq \dfrac{1}{M}\left(1 + \frac{K_S^2 + k(\mathcal{B})}{2}\right)$*;*

Proof. a) Let g be the genus of C. Since $q(S) = g(C/G^0)$ (Lemma 5.0.15), Hurwitz's formula says that

$$2(g-1) = |G^0| \cdot \Theta\,,$$

hence $\Theta = \dfrac{2(g-1)}{|G^0|} > 0$, since $g \geq 2$. Let $k := k(\mathcal{B})$ and $B := B(\mathcal{B})$.
By Corollary 5.2.5, we get

$$\beta = \frac{24\chi + 3k - B}{6\Theta} = \frac{K_S^2 + k}{2\Theta}\,,$$

and by Proposition 5.2.4 and Hurwitz's formula:

$$\beta = \frac{8(g-1)^2}{4\Theta|G^0|} = \frac{8(g-1)^2}{8(g-1)} = g - 1\,.$$

b) $|G| = \dfrac{8(g-1)^2}{K_X^2} = \dfrac{8\beta^2}{K_S^2 + k}$.

c) We note that $\Theta \geq 2q - 2 + \frac{r}{2} = \frac{r + 4(q-1)}{2}$, hence

$$r \leq 2\Theta - 4(q-1) = \frac{K_S^2 + k}{\beta} + 4(1-q)\,.$$

d) Each m_i is the branch index of a branch point p_i of $c\colon C \to C'$. Let F_i be the fiber over p_i of the map $Y = (C \times C)/G^0 \to C/G^0$. Then $F_i = m_i W_i$ for some irreducible Weil divisor W_i, moreover F_i is isomorphic to C (see Remark 4.3.5), then

$$2\beta = 2g(C) - 2 = K_Y.F_i + F_i^2 = K_Y.F_i = m_i K_Y.W_i\,.$$

Therefore

$$\frac{2\beta I}{m_i} = (IK_Y)W_i \in \mathbb{Z}\,.$$

e) By Theorem 4.3.8, if F_i contains a singular point of Y, then it contains at least 2 singular points. Therefore there are at most $|\mathcal{B}(Y)|/2 = |\mathcal{B}_C| + |\mathcal{B}_D|/2 = N$ indexes i such that $F_i \cap \mathrm{Sing}(Y) \neq \emptyset$, here $\mathcal{B}(Y)$ is the basket of singularities of Y.

For all other indexes j we have $F_j \cap \mathrm{Sing}(Y) = \emptyset$. Then W_j is Cartier and K_Y is Cartier in a neighborhood of W_j. In particular,

$$\frac{\beta}{m_i} = \frac{K_Y W_j}{2} \in \mathbb{Z}\,,$$

since $K_Y W_j$ is even.

f) We distinguish two cases: $q = 0$ and $q \geq 1$.

If $q = 0$, then $r \geq 3$, and if $r = 3$ at most one m_i can be equal to 2. Hence we have:

$$\Theta + \frac{1}{m_i} = -1 + \sum_{j=1, i \neq j}^{r} \left(1 - \frac{1}{m_j}\right) \geq \frac{1}{6}.$$

If $r > 3$, since $\Theta = (r-2) - \sum_{j=1}^{r} \frac{1}{m_j}$, we have that

$$\begin{aligned} \Theta + \frac{1}{m_i} &= (r-2) - \sum_{j=1, i \neq j}^{3} \frac{1}{m_j} \\ &\geq (r-2) - \frac{r-1}{2} = \frac{r-3}{2} = \frac{r-3+4q}{2} > \frac{1}{6}. \end{aligned}$$

If $q \geq 1$, we have that $\Theta = 2q - 2 + r - \sum_{j=1}^{r} \frac{1}{m_j}$, hence

$$\begin{aligned} \Theta + \frac{1}{m_i} &= 2q - 2 + r - \sum_{j=1, i \neq j}^{r} \frac{1}{m_j} \\ &\geq 2q - 2 + r - \frac{r-1}{2} = \frac{r-3+4q}{2} > \frac{1}{6}. \end{aligned}$$

It follows that $\Theta + \frac{1}{m_i} \geq \max\left\{\frac{1}{6}, \frac{r-3+4q}{2}\right\} =: M$. Since $m_i \leq 2\beta I = \frac{K_S^2 + k}{\Theta} I$, we get

$$m_i \leq \frac{1}{M}\left(1 + \Theta \cdot m_i\right) \leq \frac{1}{M}\left(1 + \Theta \cdot \frac{K_S^2 + k}{\Theta} I\right) = \frac{1}{M}\left(1 + (K_S^2 + k) I\right).$$

Except for at most N indices, $m_i \leq \beta$ and so we get

$$m_i \leq \frac{1}{M}\left(1 + \Theta \cdot \frac{K_S^2 + k}{2\Theta}\right) = \frac{1}{M}\left(1 + \frac{K_S^2 + k}{2}\right).$$

$\square$

Remark 5.2.14. By Proposition 5.2.13 we have that

$$|G^0| = \frac{4\beta^2}{K_S^2 + k(\mathcal{B})} = \frac{K_S^2 + k(\mathcal{B})}{\Theta^2}$$

are strictly positive integers.

Under the same assumption of Proposition 5.2.13, let

$$\mathbb{T}(g'; m_1, \dots, m_r) := \langle a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r \mid \\ c_1^{m_1}, \dots, c_r^{m_r}, \prod_{i=1}^{g'} [a_i, b_i] \cdot c_1 \cdots c_r \rangle ,$$

and let $h_i := \psi(c_i)$, in particular $\operatorname{ord}(h_i) = m_i$.

Lemma 5.2.15. *Under the same assumptions of Proposition 5.2.13,*

$$m_i \le 2\left(\frac{2\xi}{\Theta} + 3\right).$$

Proof. We have that $m_i = \operatorname{ord}(h_i)$ and $\langle h_i \rangle$ is a cyclic group acting on C that has genus $g \ge 2$. Theorem 2.2.18 applies and we get

$$m_i \le 4g + 2 = 2(2\beta + 3) = 2\left(\frac{2\xi}{\Theta} + 3\right),$$

since $g = \beta + 1$ and $\xi = \beta \cdot \Theta$. □

Proposition 5.2.16. *Under the same assumptions of Proposition 5.2.13, let $R := r - 3 + 4q$. For all $i \in \{1, \dots, r\}$ we have*

i) if $q(S) = 0$ and $r = 3$ then

$$m_i \le 12(2\xi + 1)$$

ii) otherwise

$$m_i \le 6 + \frac{(8\xi + 2)}{R};$$

Proof. Arguing as in point f) of the proof of Proposition 5.2.13, we get

$$\Theta + \frac{1}{m_i} \ge \begin{cases} \dfrac{1}{6} & \text{if } q = 0,\ r = 3 \\ \dfrac{R}{2} & \text{otherwise} \end{cases}$$

i) If $q = 0$ and $r = 3$, we have $\Theta \ge \dfrac{m_i - 6}{6m_i}$; since $12(2\xi + 1) > 12$ we can assume $m_i > 6$ and so $\dfrac{1}{\Theta} \le \dfrac{6m_i}{m_i - 6}$.
By Lemma 5.2.15 we get:

$$m_i \le 2\left(\frac{2\xi}{\Theta} + 3\right) \le 2\left(\frac{12\xi m_i}{m_i - 6} + 3\right)$$

hence $(m_i - 6)^2 \le 24\xi m_i$ and so

$$m_i^2 - 12m_i(1 + 2\xi) + 36 \le 0.$$

It follows that

$$\begin{aligned} m_i &\leq 6(1+2\xi)+\sqrt{36(1+2\xi)^2-36} \leq 6[(1+2\xi)+\sqrt{(1+2\xi)^2-1}\,] \\ &< 12(1+2\xi)\,. \end{aligned}$$

ii) If $q \neq 0$ or $r > 3$, we have $\Theta \geq \dfrac{Rm_i - 2}{2m_i}$ and $R \in \mathbb{N}_{>0}$; since $6 + \dfrac{8\xi+2}{R} > 6$ we can assume $m_i > 2$ and so $\dfrac{1}{\Theta} \leq \dfrac{2m_i}{Rm_i-2}$. By Lemma 5.2.15 we get:

$$m_i \leq 2\left(\frac{2\xi}{\Theta}+3\right) \leq 2\left(\frac{4\xi m_i}{Rm_i-2}+3\right)$$

hence $m_i^2 R - 2m_i(3R+4\xi+1)+12 \leq 0$. It follows that

$$\begin{aligned} m_i &\leq \frac{(3R+4\xi+1)+\sqrt{(3R+4\xi+1)^2-12R}}{R} \\ &< \frac{(3R+4\xi+1)+(3R+4\xi+1)}{R} = 6+\frac{8\xi+2}{R}\,. \end{aligned}$$

□

Lemma 5.2.17. *Under the same assumptions of Proposition 5.2.13, let $\mathcal{B}(Y)$ be the basket of singularities of $Y = (C \times C)/G^0$. Then for each $C_{n,a} \in \mathcal{B}(Y)$ there exists m_i such that n divides m_i, in particular $n \leq m_i$.*

Proof. Let $\sigma(x,y)$ be a singular point of Y of type $C_{n,a}$. We have that $\mathrm{Stab}_{G^0}(x,y) = \langle\eta\rangle$ and has order n, in particular $\eta(x,y) = (\eta(x), \varphi(\eta)(y)) = (x,y)$, that is $\eta \in \mathrm{Stab}_{G^0}(x)$, hence x is a ramification point of c, let $p := c(x)$. By Lemma 2.3.4 there is a bijection

$$c^{-1}(p) \longleftrightarrow \{gH\}$$

where $g \in G^0$ and $H = \langle h_i\rangle$ for some i. By Lemma 2.3.5 $\mathrm{Stab}_{G^0}(gH) = (gHg^{-1})$, and so $\eta = gh_i^{\alpha}g^{-1}$ for some $\alpha \in \{1,\dots,m_i-1\}$, then $n \mid m_i$. □

Chapter 6

An algorithm to classify regular mixed quasi-étale surfaces

In this chapter we give an algorithm to classify regular surfaces occurring as minimal resolution of the singularities of a mixed q.e. surface, with fixed values of the invariants K^2 and p_g. As an application of this algorithm we will obtain the classification of these surfaces with $K^2 > 0$ and $p_g = 0$.

6.1 The classification

In this section we give a complete classification of the regular surfaces S with $K_S^2 > 0$ and $p_g(S) = 0$ occurring as minimal resolution $\rho\colon S \to X$ of the singularities of a mixed q.e. surface $X := (C \times C)/G$.
We make a systematic computer search of the surfaces that satisfy these assumptions. As output we get the following theorem:

Theorem 6.1.1. *Let S be the minimal resolution of the singularities of a mixed q.e. surface X with $p_g(S) = q(S) = 0$ and $K_S^2 > 0$, then*

1. *S is minimal and of general type.*

2. *S belongs to one of the 17 families collected in Table 6.1.*

This chapter is dedicated to proving the second part of this statement; the first part is proved in the next chapter.

The first column of Table 6.1 gives K_S^2 of the surfaces, Sing(X) represents the basket of singularities of X. The column Type gives the type of the set of spherical generators in a compacted notation, e.g. $2^3, 4 = (2, 2, 2, 4)$. The columns G and G^0, obviously, give the group and its index two subgroup. The groups denoted by G(a,b) are groups of order a, while b is the MAGMA identifier of the group, as described in Section 7.3. The last column gives the reference. Some groups are given as semidirect products $H \rtimes \mathbb{Z}_r$; to specify them, we should indicate the image of the generator of $\mathbb{Z}_r$ in $\mathrm{Aut}(H)$. For lack of space in the table, we explain in Section 7.3 which is the automorphism. The column b_2 gives the second Betti number of X.

K_S^2	Sing(X)	Type	G^0	G	b_2	Label
1	$2\,C_{2,1} + 2\,D_{2,1}$	$2^3, 4$	$D_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$	1	7.3.1
2	$6\,C_{2,1}$	2^5	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	2	7.3.2
2	$6\,C_{2,1}$	4^3	$(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$	G(64, 82)	2	7.3.3
2	$C_{2,1} + 2\,D_{2,1}$	$2^3, 4$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$	1	7.3.4
2	$C_{2,1} + 2\,D_{2,1}$	$2^2, 3^2$	$\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$	$\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$	1	7.3.5
2	$2\,C_{4,1} + 3\,C_{2,1}$	$2^3, 4$	G(64, 73)	G(128, 1535)	3	7.3.6
2	$2\,C_{3,1} + 2\,C_{3,2}$	$3^2, 4$	G(384, 4)	G(768, 1083540)	2	7.3.7
2	$2\,C_{3,1} + 2\,C_{3,2}$	$3^2, 4$	G(384, 4)	G(768, 1083541)	2	7.3.8
3	$C_{8,3} + C_{8,5}$	$2^3, 8$	G(32, 39)	G(64, 42)	2	7.3.9
4	$4\,C_{2,1}$	2^5	$D_4 \times \mathbb{Z}_2$	$D_{2,8,5} \rtimes \mathbb{Z}_2$	2	7.3.10
4	$4\,C_{2,1}$	2^5	$\mathbb{Z}_2^4$	$(\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$	2	7.3.11
4	$4\,C_{2,1}$	4^3	G(64, 23)	G(128, 836)	2	7.3.12
8	$\emptyset$	2^5	$D_4 \times \mathbb{Z}_2^2$	$(D_{2,8,5} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$	2	7.3.13
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	7.3.14
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	7.3.15
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	7.3.16
8	$\emptyset$	4^3	G(128, 36)	G(256, 3679)	2	7.3.17

Table 6.1: The surfaces. G(a,b) denotes the bth group of order a in the MAGMA database of finite groups. See Section 7.3 for a detailed description.

Remark 6.1.2. It is automatic that $b_0(X) = b_4(X) = 1$ and $b_1(X) = b_3(X) = 0$ (see (3.2) and (3.3)). If in addition $b_2(X) = 1$ then X is a $\mathbb{Q}$-homology projective planes, i.e. normal projective complex surface with the same Betti numbers of $\mathbb{P}^2$, as studied in [HK11] and [Keu10].

Remark 6.1.3. Let $S \to X$ be the minimal resolution of the singularities of a mixed q.e. surface, with $q(S) = 0$. By Theorem 2.4.3, Theorem 4.4.4 and Lemma 5.0.15, X is completely determined by the following data:

- a finite group G;
- a spherical system of generators $(h_1, \ldots, h_r)$ of type $(m_1, \ldots, m_r)$ of an index two subgroup $G^0 \triangleleft G$ such that $1 \to G^0 \to G \to \mathbb{Z}_2 \to 1$ does not split;
- an ordered set of r points $p_1, \ldots, p_r$ in $\mathbb{P}^1$.

Once we fix G and $(h_1, \ldots, h_r)$ as above, by Theorem 2.4.3 we get a curve C such that the Galois covering $c\colon C \to C/G^0 \cong \mathbb{P}^1$ is branched over

$\{p_1, \ldots, p_r\} \subseteq \mathbb{P}^1$. Using Theorem 4.4.4 we define a mixed action on $C \times C$ and by Theorem 5.0.12 the quotient map is quasi-étale.

We note that, given a spherical set of generators, the mixed q.e. surface is determined up to the choice of r points in $\mathbb{P}^1$, hence we get a family of surfaces depending on $r-3$ parameters. We do not known the dimension of its image in the moduli space.

Remark 6.1.4. We observe that the basket of singularities of each output contains either zero or two points of type $D_{n,a}$.

We already knew this fact. Indeed it follows by Corollary 5.2.8:

Lemma 6.1.5. *Let S be the minimal resolution of the singularities of the mixed q.e. surface $X = (C \times C)/G$. Let T be the minimal resolution of the singularities of $Y = (C \times C)/G^0$. If $p_g(S) = 0$, then the map $\pi \colon Y \to X$ is*

- *either étale and $p_g(T) = 1$;*
- *or branched exactly in 2 points and $p_g(T) = 0$.*

6.1.1 Finiteness of the classification

If in Definition 5.2.9 we assume that S is a regular surface, then $\chi(\mathcal{O}_S) = 1 + p_g(S)$. Thus Θ and β are so defined:

$$\Theta := -2 + \sum_{i=1}^{r} \left(\frac{m_i - 1}{m_i} \right) \qquad \beta := \frac{12(1 + p_g(S)) + k(\mathcal{B}) - e(\mathcal{B})}{3\Theta}.$$

From Proposition 5.2.13 it follows immediately:

Proposition 6.1.6. *Let S be the minimal resolution of the regular mixed q.e. surface $X = (C \times C)/G$. Let $\psi \colon \mathbb{T}(g'; m_1, \ldots, m_r) \to G^0$ be the appropriate orbifold homomorphism induced by $c \colon C \to C/G^0$. Let $\mathcal{B} = \mathcal{B}_C \cup \mathcal{B}_D$ be the basket of singularities of X. Then*

- $\Theta > 0$ *and* $\beta = g(C) - 1$;
- $|G| = \frac{8\beta^2}{K_S^2 + k(\mathcal{B})}$;
- $r \leq \frac{K_S^2 + k(\mathcal{B})}{\beta} + 4$;
- *each m_i divides $2\beta I$, where I is the index of Y;*
- *there are at most $N := |\mathcal{B}_C| + |\mathcal{B}_D|/2$ indices i such that m_i does not divide β;*

- $m_i \leq \frac{1+I(K_S^2+k(\mathcal{B}))}{M}$, *where* $M := \max\left\{\frac{1}{6}, \frac{r-3}{2}\right\}$;
  moreover, except for at most N *indices* i *we have the stronger inequality*
$$m_i \leq \frac{1}{M}\left(\frac{K_S^2+k(\mathcal{B})}{2}+1\right).$$

Remark 6.1.7. By Corollary 5.2.5 and Hurwitz's formula, it follows that
$$\beta = \frac{K^2+k(\mathcal{B})}{2\Theta}\,.$$

Once we have fixed the values of $p_g(S)$ and $q(S)$, by the standard inequalities (see Section 3.6) and Corollary 5.2.6, we get a finite number of possible values for K_S^2: $2p_g(S)-4 \leq K_S^2 \leq 8\chi(\mathcal{O}_S)$; if $q>0$ the stronger inequality $2p_g(S) \leq K_S^2$ holds. By Corollary 5.2.5, we get $B(\mathcal{B})$.

Lemma 6.1.8. *Let* $H \in \mathbb{Q}$. *Then there are finitely many baskets* $\mathcal{B}$ *such that*
$$B(\mathcal{B}) = H\,,$$
in particular:

1. $|\mathcal{B}| \leq H/3$;
2. *if* $\eta \times C_{n,a} \in \mathcal{B}$ *and* $n/a = [b_1, \dots, b_l]$ *then* $\eta \sum b_i \leq H$;
3. *if* $\zeta \times D_{n,a} \in \mathcal{B}$ *and* $n/a = [b_1, \dots, b_l]$ *then* $\frac{\zeta}{2}(\sum b_i + 12) \leq H$.

Proof. We note that $B(C_{n,a}) = \frac{a+a'}{n} + \sum b_i \geq 3$, while $B(D_{n,a}) = \frac{B(C_{n,a})}{2} + 6 \geq 15/2$. It follows that $H = \sum_{x\in\mathcal{B}} B_x \geq 3|\mathcal{B}|$, this prove the first point. The second is obvious, while the third follows by $H \geq \zeta(\frac{1}{2}\sum b_i + 6)$. □

Remark 6.1.9. If $\mathcal{B}$ is the basket of singularities of a mixed q.e. surface X then, by Corollary 5.2.8, it contains either no points of type $D_{n,a}$ or at least two. In the latter case we have that $B(D_{n_1,a_1}) + B(D_{n_2,a_2}) \leq B(\mathcal{B})$; that is
$$B(\mathcal{B}) - \frac{15}{2} \geq B(D_{n_1,a_1}) = \frac{\frac{2a_1}{n_1} + \sum b_i}{2} + 6$$
hence
$$2B(\mathcal{B}) - 15 \geq 2B(D_{n_1,a_1}) = \frac{2a_1}{n_1} + \sum b_i + 12\,.$$
That is
$$\sum b_i \ < 2B(\mathcal{B}) - 27\,.$$

By Lemma 6.1.8, we have only finitely many baskets with assigned B. Fixing K_S^2 and $\mathcal{B}$, by Proposition 5.2.13, we have only finitely many types, and for each type only finitely many groups.

6.1.2 How to compute the singularities

We need to understand how to compute the singularities on Y and on X starting from the algebraic data.

Let $(h_1, \ldots, h_r)$ be a spherical system of generators for the group G^0, of type $(m_1, \ldots, m_r)$. By Theorem 2.4.3 we get the Galois covering $c\colon C \to C/G^0 \cong \mathbb{P}^1$ branched over $\{p_1, \ldots, p_r\} \subset \mathbb{P}^1$. Let $Q\colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$ the map $Q(\sigma(x, y)) = (c(x), c(y))$.
We recall the following commutative diagram:

$$\begin{tikzcd}
 & & C \times C \arrow[dll] \arrow[drr] \arrow[dd, "\sigma"] & & \\
C \arrow[dd, "c"'] & & & & C \arrow[dd, "c"] \\
 & & Y = (C \times C)/G^0 \arrow[dll] \arrow[drr] \arrow[dd, "Q"] \arrow[ddr, "\pi"] & & \\
\mathbb{P}^1 & & & & \mathbb{P}^1 \\
 & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[ull] \arrow[urr] & X = (C \times C)/G &
\end{tikzcd} \tag{6.1}$$

Remark 6.1.10. We recall that the points in $c^{-1}(p_i)$ are the only ones with non-trivial stabilizer with respect to the action of G^0 on C and they are in bijection with the left cosets $\{gK_i\}$, where $K_i = \langle h_i \rangle$ is the stabilizer of a point of the fibre (see Lemma 2.4.1). We recall that the point gK_i has stabilizer gK_ig^{-1} and that $|c^{-1}(p_i)| = \dfrac{|G^0|}{m_i}$. Moreover, each point $(x, y) \in C \times C$ such that $(Q \circ \sigma)(x, y) = (p_i, p_j)$, is associated to a pair of left cosets: $(gK_i, g'K_j)$ (see Lemma 2.3.4).
Let $\xi \in G^0$ and assume that $\xi(x, y) = (\xi x, \varphi(\xi) y) = (x, y)$:

$$\xi(gK_i, g'K_j) = (gK_i, g'K_j) \Longleftrightarrow \begin{cases} \xi \in gK_ig^{-1} \\ \varphi(\xi) \in g'K_jg'^{-1} \end{cases}$$

That is $\xi \in gK_ig^{-1} \cap \varphi^{-1}(g'K_jg'^{-1})$. Hence the singular points of Y are the points $\sigma(u, v)$ such that

$$\mathrm{Stab}(u, v) := \mathrm{Stab}_{G^0}(u) \cap \varphi^{-1}(\mathrm{Stab}_{G^0}(v)) \neq \{1\}\,.$$

Lemma 6.1.11 (cf. [BP10, Proposition 1.16])**.** *Let* i, $j \in \{1, \ldots, r\}$. *Then*

i) there is a G^0-equivariant bijection $(Q \circ \sigma)^{-1}(p_i, p_j) \to G^0/K_i \times G^0/K_j$, where the G^0-action on the target is given by left multiplication (simultaneously on both factors)[1]*;*

[1] G^0 acts as follows: $g(aK_i, bK_j) = (gaK_i, \varphi(g)bK_j)$

ii) there is a K_i-equivariant bijection between the orbits of the above G^0-action on $G^0/K_i \times G^0/K_j$ with the orbits of the K_i-action on $\{\overline{1}\} \times G^0/K_j$.

Proof. i) By Lemma 2.4.1 we have a G^0-equivariant bijection

$$c^{-1}(p_i) \longleftrightarrow \{gK_i\}$$

for each $i = 1, \dots, r$. Hence there is a G^0-equivariant bijection between $(Q \circ \sigma)^{-1}(p_i, p_j)$ and $G^0/K_i \times G^0/K_j$.

ii) We note that the G^0-orbits of $G^0/K_i \times G^0/K_j$ are in one-to-one correspondence with the points $\sigma((Q \circ \sigma)^{-1}(p_i, p_j))$.

Observe that

- (k_iK_i, k_jK_j) is in the same G^0-orbit as $(K_i, \varphi(k_i^{-1})k_jK_j)$.
- (K_i, k_1K_j) is in the same G^0-orbit as (K_i, k_2K_j), if and only if there exists $\alpha \in K_i$ such that $k_2 = \varphi^{-1}(\alpha)k_1$.

□

We have to determine the types of the singularities:

Proposition 6.1.12 (cf. [BP10, Proposition 1.18])**.**
An element $[g] \in \{\overline{1}\} \times G^0/K_j$ corresponds to a point $\frac{1}{n}(1, a)$ where $n = |K_i \cap \varphi^{-1}(gK_jg^{-1})|$, and a is given as follows: let δ_i be the minimal positive integer such that there exists $1 \leq \gamma_j \leq \operatorname{ord}(h_j)$ with $h_i^{\delta_i} = g\varphi^{-1}(h_j^{\gamma_j})g^{-1}$. Then $a = \dfrac{n\gamma_j}{\operatorname{ord}(h_j)}$.

Proof. $[g] \in \{\overline{1}\} \times G^0/K_j$ corresponds to a (singular) point of type $\frac{1}{n}(1, a)$ with $n = |\operatorname{Stab}(q_i, gq_j)| = |\operatorname{Stab}(q_i) \cap \varphi^{-1}(\operatorname{Stab}(gq_j))| = |K_i \cap \varphi^{-1}(gK_jg^{-1})|$, where q_i is the unique point of $c^{-1}(p_i)$ with stabilizer K_i and q_j is the unique point of $c^{-1}(p_j)$ with stabilizer K_j.

Let δ be the minimal positive integer such that there is $\gamma \in \{1, \dots, \operatorname{ord}(h_j)\}$ such that $h_i^\delta = g\varphi^{-1}(h_j^\gamma)g$. Then $\langle h_i^\delta \rangle = \operatorname{Stab}(q_i, gq_j)$.
Therefore $\operatorname{ord}(h_i) = n\delta$. In local analytic coordinate $(x, y) \in C \times C$, h_i^δ acts as

$$e^{\frac{2\pi i}{n}} = e^{\frac{2\pi i\delta}{\operatorname{ord}(h_i)}}$$

on the variable x and as

$$e^{\frac{2\pi ia}{n}} = e^{\frac{2\pi i\gamma}{\operatorname{ord}(h_j)}}$$

on the variable y. This shows that $a = \frac{n\gamma}{\operatorname{ord}(h_j)}$. □

Using Lemma 6.1.11 and Proposition 6.1.12 we can compute the singularities of Y. We have to do the same for X. Since we already know that the quotient by an involution of a singular point $C_{n,a}$ is a singular point of type

$D_{n,a}$ we only need to know which singular points of Y are also ramification point for $\pi\colon Y \to X$. We start computing "where" the ramification points can be.

Lemma 6.1.13. *Let $y \in Y$ be a ramification point for π. Then $Q(y) = (p_i, p_i)$ for some i.*

Proof. Let $h \in G^0$, we recall that $\tau' h$ acts in this way: $\tau' h(x,y) = (\varphi(h)y, \tau h x)$. Let $\sigma(x,y) \in Y$ be a ramification point for π then $\sigma(x,y) = \sigma(\tau' h(x,y))$, for some $h \in G^0$. Suppose that $(Q \circ \sigma)(x,y) = (c(x), c(y))(p_i, p_j)$ then $(Q \circ \sigma)(\varphi(h)y, \tau h x) = (c(\varphi(h)y), c(\tau h x)) = (c(y), c(x)) = (p_j, p_i)$ since $\varphi(h)$, $\tau h \in G^0$. Hence $p_i = p_j \in \mathbb{P}^1$. We get that every branch point belongs to $Q^{-1}(p_i, p_i)$, for some i. $\square$

Proposition 6.1.14. *An element $[g] \in \{\bar{1}\} \times G^0/K_j$ corresponds to a singular point that is also a ramification point for $\pi\colon Y \to X$ if and only if there exists an element $\tau' h \in G \setminus G^0$ such that:*

$$\begin{cases} \varphi(h)\tau h \in K_i \\ \varphi(h) g \in K_i \end{cases}$$

Proof. The point (K_i, gK_i) corresponding to $[g]$ is a ramification point for π if and only if there exists an element $\tau' h \in G \setminus G^0$ such that $(K_i, gK_i) = \tau' h(K_i, gK_i) = (\varphi(h) g K_i, \tau h K_i)$, that is

$$\begin{cases} \varphi(h) g K_i = K_i \\ gK_i = \tau h K_i \end{cases} \Longleftrightarrow \begin{cases} \varphi(h) g K_i = K_i \\ \varphi(h) \tau h K_i = (\tau' h)^2 K_i = K_i \end{cases}$$

$\square$

6.1.3 Hurwitz moves

Let G be a finite group. Let $(h_1, \ldots, h_r)$ and $(h'_1, \ldots, h'_r)$ be spherical systems of generators of type $(m_1, \ldots, m_r)$ of G^0 and G^0_1, index two subgroups of G such that

$$\begin{array}{l} 1 \to G^0 \to G \to \mathbb{Z}_2 \to 1 \quad \text{and} \\ 1 \to G^0_1 \to G \to \mathbb{Z}_2 \to 1 \end{array}$$

do not split.

In the following we investigate this problem: "when do two sets of spherical generators give the same Galois covering C of $\mathbb{P}^1$ (up to isomorphism)? And so isomorphic surfaces?"

Following the solution to the problem given in [BCG08, Section 1-2] (see also [BCP06, Section 5.1-5.2]), we start defining the *braid group* $\mathbf{B}_r$, for $r \in \mathbb{N}$:

$$\mathbf{B}_r := \left\langle \sigma_1, \ldots, \sigma_{r-1} \;\middle|\; \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \ldots, r-2 \end{array} \right\rangle .$$

Consider now the *Hurwitz action* of $\mathbf{B}_r$ on the set of r-tuples of G corresponding to the standard embedding of $\mathbf{B}_r$ into the automorphism group of a free group on r generators.

Let $T = (g_1, \dots, g_r)$ be a r-tuple of elements of G and $1 \leq i \leq r-1$. Define $\sigma_i(T)$ by

$$\sigma_i(T) := [g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_i, g_{i+2}, \dots, g_r]\,.$$

It is easy to prove that the braid relations are satisfied and that the group $\mathbf{B}_r$ maps set of spherical generators to set of spherical generators preserving the type.

Also the automorphism group $\mathrm{Aut}(G)$ of G acts on the sets of spherical generators by simultaneous application of an automorphism to each element.

Given $(\gamma, \eta) \in \mathbf{B}_r \times \mathrm{Aut}(G)$ and $T = (g_1, \dots, g_r)$ a set of spherical generators of $G^0 \triangleleft_2 G$, we set:

$$(\gamma, \eta) \cdot (G^0, T) := (\eta(G^0), \eta(\gamma(T)))\,. \tag{6.2}$$

Now, assume to have a Galois covering $C \to \mathbb{P}^1$ with Galois group G^0. Let $\{p_1, \dots, p_r\} \subset \mathbb{P}^1$ be the branch locus of the covering. Choose a base point $p \in \mathbb{P}^1$ distinct from them. Choose a geometric basis $\gamma_1 \dots, \gamma_r$ of $\pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}, p)$ (γ_i is a simple counterclockwise loop around p_i, and they follow each other by counterclockwise ordering around the base point). Notice that $\gamma_1 \cdots \gamma_r = 1$. Choose a monodromy representation, i.e., a surjective homomorphism $\psi \colon \pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}) \to G^0$. Notice that only the kernel of ψ is uniquely determined by the covering. Then the elements $\psi(\gamma_i), \dots, \psi(\gamma_r)$ form a spherical system of generators of G^0.

The mapping class group of the sphere $\pi_0(Diff((\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}, p)))$ (see [BCP06, Definition 17]), which is a quotient of the braid group $\mathbf{B}_r$, operates on such homomorphisms, and their orbits are called Hurwitz equivalence classes of spherical systems of generators. This action is the one described in (6.2).

6.2 The algorithm

Using the results of the previous sections we have implemented a MAGMA script to find all the regular surfaces that satisfy our assumptions.
We explain here the strategy of the program and the most important scripts; we attach a commented version of the program in Section 6.4.

The algorithm follows closely the algorithms in [BCGP08] and [BP10]. We have adapted them to the mixed q.e. case and we have improved the computational complexity.

First of all we fix a value of K_S^2 and of p_g. By assumption $q = 0$ so $\chi = 1 + p_g$.

Step 1: the script **Baskets** lists all the possible baskets of singularities for K_S^2 and p_g, accordingly to Corollary 5.2.5 and Lemma 6.1.8.

Step 2: once we fix K_S^2, p_g and a possible basket of singularities $\mathcal{B}(X)$ there are finitely many possible signatures satisfying the condition of Proposition 6.1.6. **ListOfTypes** computes them. The input are K_S^2 and p_g, so this script first computes $Baskets(K_S^2, p_g)$ and returns a list of pairs: the first entry is a possible basket and the second is the list with all the possible signatures.

Step 3: if we know the signature, by Proposition 6.1.6, we can compute the order of G^0. **ListGroups**, whose inputs are K_S^2 and p_g, searches, for every element in the output of $ListOfTypes$, if among the groups of the right order there are groups having at least one set of spherical generators of the prescribed type. Further it checks if these groups have a pair of sets of spherical generators that give the prescribed basket of singularities on $Y = (C \times C)/G^0$. Once it finds a group G^0 with the right properties, it searches among all the groups of order $2|G^0|$, if there are groups which are unsplit extensions of G^0.

For each positive answer to these two questions it stores the fourtuple $(basket, type, id(G^0), \{id(G)\})$, where $id(G^0)$ is the MAGMA identifier for G^0, while $\{id(G)\}$ is the set of the MAGMA identifiers of the groups that are non split extensions of G^0.

The script has some conditional instructions:

- if one of the signatures is $(2, 3, 7)$, then G^0, being a quotient of $\mathbb{T}(2, 3, 7)$, is perfect. MAGMA knows all perfect groups of order ≤ 50000, and then *ListGroups* checks first if there are perfect group of the right order: if not, this case cannot occur.
- If the expected order of the group G^0 is 1024 or bigger than 2000, since MAGMA does not have a list of the finite groups of this order, then *ListGroups* just stores these cases in a list, third output of the script.
- If the order of G^0 is in $\{1001, \ldots, 2000\}$, since MAGMA does not have a list of the groups of order bigger than 2000, we cannot check if there exist unsplit extensions of G^0; so we make the other tests and if a group passes these tests, then we collect it in a list, second output of the script.

To save RAM memory, when the script has to make a search among a big class of groups (e.g. the groups of order 576), *ListGroups* uses "SmallGroupProcess", which is a bit slow, but does not need to store the whole class of groups under consideration.

Step 4: **ExistingSurfaces** takes the output of $ListGroups(K_S^2)$ and throws away all 4-tuples $(basket, type, id(G^0), id(G))$ that do not give a surface with the expected singularities.

Step 5: each fourtuple in the output of $ExistingSurfaces(K_S^2)$ gives many surfaces, one for each spherical systems of generators. Two different spherical systems of generators can give isomorphic surfaces: this is taken into account by declaring that two spherical systems of generators are equivalent if and only if they are in the same orbit of the natural action of $\mathrm{Aut}(G)$ and of the respective braid groups (see Section 6.1.3). The script **FindSurfaces** produces one representative for each equivalence class.

Step 6: **Pi1** computes the fundamental group of the surfaces constructed using Armstrong's results (see [Arm65] and [Arm68]), as we will see in section 7.1.

Remark 6.2.1. The principal computational improvement in our script is in the first part of *ListGroups*, in particular in the search of which groups have at least a set of spherical generators of the prescribed type.
If the group G^0 has a set of spherical generators of type $(m_1, \dots, m_r)$, then there exists an appropriate orbifold homomorphism

$$\psi \colon \mathbb{T}(m_1, \dots, m_r) \to G^0 .$$

The map ψ induces a surjective morphism $\overline{\psi} \colon \mathbb{T}^{ab} \to {G^0}^{ab}$ between their abelianizations, hence ${G^0}^{ab}$ is isomorphic to a quotient of $\mathbb{T}^{ab}$.

Differently from the analogous scripts in [BCGP08] and [BP10], our script checks first (by the script **Test**) which groups have abelianization isomorphic to a quotient of the suitable $\mathbb{T}^{ab}$ and only for the groups that pass this test if they have a set of spherical generators of the right type.

In the following table we compare the execution times of the program with and without Test for high values of K_S^2.

K^2	3	4	5	6	7	8
Time with Test	18.67	50.18	36.29	226.61	4.36	4205.85
Time without Test	1470.14	1128.35	3117.02	262.63	6.16	26431.57

Table 6.2: Eexecution times (in minutes) for high values of K_S^2.

6.3 Skipped cases for $p_g = 0$ and $K^2 > 0$

We run the script for $p_g = 0$ and $K^2 = 1, \dots, 8$; for each value of K^2, the MAGMA scripts *ListGroups* returns 3 output: the first is processed by

other functions of the program that possibly return some surfaces. All the surfaces constructed are collected in Section 7.3.
We want to prove that they are all the mixed q.e. surfaces whose minimal resolution of the singularities is a regular surface of general type with $p_g = 0$ and $K^2 > 0$. In order to do this, we have to show that all the cases stored by the script in the second and third output do not occur.

One of the main tools here is the script Test (or TestBAD in some cases), which checks, given a signature and an order, if there exist groups of that order and with a spherical system of generators of that signature.

For $p_g = 0$ and all the values $1 \leq K_S^2 \leq 8$, we have that the second output is empty, while the cases stored in the third outputs are collected in Table 6.3:

K_S^2	Sing$X$	type	$\|G^0\|$
1	$2 \times C_{8,1} + C_{4,1}$	2,3,8	6336
1	$3 \times C_{4,1} + C_{4,3}$	2,3,8	2304
1	$C_{8,1} + C_{4,1} + C_{8,5}$	2,3,8	4032
1	$4 \times C_{4,1} + C_{2,1}$	2,3,8	2880
1	$2 \times C_{8,3} + C_{4,1} + C_{2,1}$	2,3,8	2304
1	$2 \times C_{2,1} + C_{8,3} + C_{8,1}$	2,3,8	3744
2	$2 \times C_{8,3} + C_{4,1}$	2,3,8	2880
2	$C_{8,3} + C_{8,1} + C_{2,1}$	2,3,8	4320
2	$4 \times C_{4,1}$	2,4,5	2400
2	$4 \times C_{4,1}$	2,3,8	3456
2	$C_{8,3} + C_{8,5} + C_{2,1}$	2,3,8	2016
2	$2 \times C_{4,1} + 3 \times C_{2,1}$	2,3,8	2304
2	$2 \times C_{4,1} + C_{3,1} + C_{3,2}$	2,3,8	2496
3	$2 \times C_{4,1} + 2 \times C_{2,1}$	2,3,8	2880
3	$C_{8,3} + C_{8,1}$	2,3,8	4896
3	$2 \times C_{4,1} + C_{5,3}$	2,4,5	2160
3	$C_{8,3} + C_{8,5}$	2,3,8	2592
3	$C_{4,3} + C_{4,1} + C_{2,1}$	2,3,8	2304

K_S^2	Sing$X$	type	$\|G^0\|$
4	$C_{4,3} + C_{4,1}$	2,3,8	2880
4	$4 \times C_{2,1}$	2,3,8	2304
4	$C_{3,1} + C_{3,2} + C_{2,1}$	2,3,8	2496
4	$2 \times C_{4,1} + C_{2,1}$	2,4,5	2400
4	$2 \times C_{4,1} + C_{2,1}$	2,3,8	3456
5	$C_{5,2} + C_{2,1}$	2,4,5	2160
5	$3 \times C_{2,1}$	2,3,8	2880
5	$C_{3,1} + C_{3,2}$	2,3,8	3072
5	$2 \times C_{4,1}$	2,4,5	2800
5	$2 \times C_{4,1}$	2,3,8	4032
6	$2 \times C_{2,1}$	2,4,5	2400
6	$2 \times C_{2,1}$	2,3,8	3456
6	$2 \times C_{5,3}$	2,4,5	2560
7	$C_{2,1}$	2,3,9	2268
7	$C_{2,1}$	2,4,5	2800
7	$C_{2,1}$	2,3,8	4032
8	$\emptyset$	2,3,9	2592
8	$\emptyset$	2,4,5	3200
8	$\emptyset$	2,3,8	4608

Table 6.3: The skipped cases for $p_g = 0$ and $K^2 > 0$

In the following we will sometimes need the number of perfect groups of a given order; we compute it by the MAGMA function:

```
NumberOfGroups(PerfectGroupDatabase(),order);
```

while the other functions that we use are in the MAGMA script reported in Section 6.4.

6.3.1 Non generation results

Lemma 6.3.1. *No group of order 2016, 2304, 2496, 2592, 2880, 3456 or 3744 has a spherical system of generators of type* $[2, 3, 8]$.

Proof. Assume that G^0 is a group of order 2016 (2304, 2496, 2592, 2880, 3456, 3744 resp.) admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\text{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2016 (2304, 2496, 2592, 2880, 3456, 3744 resp.), the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1008 (1152, 1248, 1296, 1440, 1728, 1872 resp.) and it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8)] \cong \mathbb{T}(3,3,4)$. The following MAGMA computations

```
>  Test([3,3,4], 1008);
{}
>
> TestBAD([3,3,4], 1152);
{}
>
>  Test([3,3,4], 1248);
{}
>
>  Test([3,3,4], 1296);
{}
>
> Test([3,3,4], 1440);
{}
>
> Test([3,3,4], 1728);
{}
>
>  Test([3,3,4], 1872);
{}
>
```

show that there are no groups of order 1008 (1152, 1248, 1296, 1440, 1728, 1827 resp.) with a spherical system of generators of type $[3,3,4]$, a contradiction. $\square$

Lemma 6.3.2. *No group of order 4608 or 6336 has a spherical system of generators of type* $[2,3,8]$.

Proof. Assume that G^0 is a group of order 4608 (6336 resp.) admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\text{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 4608 (6336 resp.), the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2304 (3168 resp.) and it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8] \cong \mathbb{T}(3,3,4)$.

Since $\mathbb{T}(3,3,4)^{\text{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2304 (3168 resp.), the commutator subgroup $G^{0\prime\prime} = [G^{0\prime}, G^{0\prime}]$ of $G^{0\prime}$ has order 768 (1056 resp.) and it is a quotient of $[\mathbb{T}(3,3,4), \mathbb{T}(3,3,4)] \cong \mathbb{T}(4,4,4)$. The following MAGMA computations

```
> TestBAD([4,4,4], 768);
{}
>
> Test([4,4,4], 1056);
{}
>
```

show that there are no groups of order 768 (1056 resp.) with a spherical system of generators of type $[4,4,4]$, a contradiction. $\square$

Lemma 6.3.3. *No group of order 2400, 2800 or 3200 has a spherical system of generators of type* $[2,4,5]$.

Proof. Assume that G^0 is a group of order 2400 (2800, 3200 resp.) admitting a surjective homomorphism $\mathbb{T}(2,4,5) \to G^0$.

Since $\mathbb{T}(2,4,5)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2400 (2800, 3200 resp.), the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1200 (1400, 1600 resp.) and it is a quotient of $[\mathbb{T}(2,4,5), \mathbb{T}(2,4,5)] \cong \mathbb{T}(2,5,5)$. The following MAGMA computations

```
> Test([2,5,5], 1200);
{}
>
> Test([2,5,5], 1400);
{}
>
> Test([2,5,5], 1600);
{}
```

show that there are no groups of order 1200 (1400, 1600 resp.) with a spherical system of generators of type $[2,5,5]$, a contradiction. $\square$

Lemma 6.3.4. *No group of order 2268 has a spherical system of generators of type* $[2,3,9]$.

Proof. Assume that G^0 is a group of order 2268 admitting a surjective homomorphism $\mathbb{T}(2,3,9) \to G^0$.

Since $\mathbb{T}(2,3,9)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2268, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 756 and is a quotient of $[\mathbb{T}(2,3,9), \mathbb{T}(2,3,9] \cong \mathbb{T}(2,2,2,3)$. The following MAGMA computation

```
> Test([2,2,2,3], 756);
{}
>
```

shows that there are no groups of order 756 with a spherical system of generators of type $[2,2,2,3]$, a contradiction. ☐

Lemma 6.3.5. *No group of order 2160 has a spherical system of generators of type* $[2,4,5]$.

Proof. Assume that G^0 is a group of order 2160 admitting a surjective homomorphism $\mathbb{T}(2,4,5) \to G^0$. It holds $\mathbb{T}(2,4,5)^{\mathrm{ab}} \cong \mathbb{Z}_2$.
There is only one perfect group of order 2160, we denote it by H. $H = 6.\mathcal{A}_6$ has the following MAGMA representation:

```
>  F<w>:=GF(9);
>
>  x:=CambridgeMatrix(1,F,6,[
>  010000,
>  200000,
>  000100,
>  002000,
>  000001,
>  000020]);
>
>  y:=CambridgeMatrix(1,F,6,[
>  300000,
>  550000,
>  007000,
>  126600,
>  000030,
>  240155]);
>  H<x,y>:=MatrixGroup<6,F|x,y>;
>
> #H;
2160
> IsPerfect(H);
true
```

The following MAGMA computation

```
> ExSphGens(H,[2,4,5]);
false
```

shows that H does not have a spherical system of generators of type $[2,4,5]$

If G^0 is a group of order 2160 with a spherical system of generators of type $[2,4,5]$, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order

1080 and it is a quotient of $[\mathbb{T}(2,4,5), \mathbb{T}(2,4,5)] \cong \mathbb{T}(2,5,5)$. The following MAGMA computation

```
> Test([2,5,5], 1080);
{}
>
```

shows that there are no groups of order 1080 with a spherical system of generators of type $[2,5,5]$, a contradiction. $\square$

Lemma 6.3.6. *No group of order 4320 has spherical system of generators of type* $[2,3,8]$.

Proof. Assume that G^0 is a group of order 4320 admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 4320, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2160 and it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8] \cong \mathbb{T}(3,3,4)$.

Now $\mathbb{T}(3,3,4)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and there is only one perfect group of order 2160: the $H = 6.\mathcal{A}_6$ in the proof of Lemma 6.3.5. The following MAGMA computation

```
> ExSphGens(H,[3,3,4]);
false
```

shows that H does not have a spherical system of generators of type $[3,3,4]$

If G^0 is a group of order 2160 with a spherical system of generators of type $[3,3,4]$, the commutator subgroup $G^{0\prime\prime} = [G^{0\prime}, G^{0\prime}]$ of $G^{0\prime}$ has order 720 and it is a quotient of $[\mathbb{T}(3,3,4), \mathbb{T}(3,3,4)] \cong \mathbb{T}(4,4,4)$. The following MAGMA computation

```
> Test([4,4,4], 720);
{ 584, 585, 763, 765, 766, 773, 776 }
>
```

shows that only the groups $G(720, j)$[2] with $j \in \{584, 585, 763, 765, 766, 773, 776\}$ have a spherical system of generators of type $[4,4,4]$.

Assume that $G^{0\prime}$ has a spherical system of generators of type $(3,3,4)$. Let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & \mathbb{T}(3,3,4) \arrow[r, "q"] \arrow[d, "p"] & \mathbb{T}(3,3,4)^{ab} = \mathbb{Z}_3 \arrow[d, equal] & \\
1 \arrow[r] & G^{0\prime\prime} \arrow[r] & G^{0\prime} \arrow[r, "f"] & \mathbb{Z}_3 \arrow[r] & 1
\end{tikzcd}$$

[2] G(a,b) denotes the bth group of order a in the MAGMA database of finite groups.

where $q(c_i) = d_i$. Let

$$\begin{aligned} \mathbb{T}(3,3,4) &= \langle c_1, c_2, c_3 \mid c_1^3, c_2^3, c_3^4, c_1c_2c_3 \rangle \\ \\ \mathbb{T}(3,3,4)^{ab} &= \langle d_1, d_2, d_3 \mid d_1^3, d_2^3, d_3^4, d_1d_2d_3, [d_i, d_j]_{1 \le i,j \le 3} \rangle \\ &= (\mathbb{Z}_3 d_1 \times \mathbb{Z}_3 d_2 \times \mathbb{Z}_4 d_3)/\langle (1,1,1) \rangle \end{aligned}$$

since $[d_1] = (1,0,0) \notin \langle (1,1,1) \rangle$, then $[d_1] \neq [0]$; so we have $q(c_1) \neq [0]$, and $f(p(c_1)) = f(g_1) \neq 0$. We have found an element of $G^{0'}$ of order 3 that does not belong to $G^{0''}$, this means that the following exact sequence

$$1 \longrightarrow G^{0''} \longrightarrow G^{0'} \xrightarrow{f} \mathbb{Z}_3 \longrightarrow 1$$

splits with map

$$\begin{aligned} \alpha : \mathbb{Z}_3 &\longrightarrow G^{0'} \\ d_1 &\longmapsto g_1 \end{aligned}$$

and so $G^{0'} \cong G^{0''} \rtimes \mathbb{Z}_3$.

The next claim, that we do not prove, is a standard result about semidirect products.

Claim 4. *Let L be a finite group and let K be a cyclic group of order p. Let $\varphi_1, \varphi_2 \colon K \to \mathrm{Aut}(L)$ such that $\varphi_1(K)$ and $\varphi_2(K)$ are conjugated. Then $L \rtimes_{\varphi_1} K \cong L \rtimes_{\varphi_2} K$.*

This means that, in order to build up the group $G^{0'}$, we have only to look at the conjugacy classes of elements of order 3 in $\mathrm{Aut}(G^{0''})$ and at $\mathrm{Id}(\mathrm{Aut}(G^{0''}))$. The function `ConjugCl(A,n)`(see Section 6.4) returns a representative of each conjugacy class of elements of A of order n.

The following MAGMA script

```
>  v:={ 584, 585, 763, 765, 766, 773, 776 };
>  for j in  v do
for> H2:=SmallGroup(720, j);
for> Aut2:=AutGr(H2);
for> A2:=AutomorphismGroup(H2);
for> R2:=ConjugCl(Aut2,3);
for> C3:=CyclicGroup(3);
for> R2[1+#R2]:=Id(A2);
for> f2:=[]; for i in [1..#R2] do
for|for> f2[i]:=hom<C3->A2|R2[i]>;end for;
for> h1:=[]; for i in [1..#R2] do
for|for> h1[i]:=SemidirectProduct(H2,C3,f2[i]);
for|for> j, i, ExSphGens(h1[i],[3,3,4]); end for; end for;
```

```
584 1 false
584 2 false
585 1 false
585 2 false
773 1 false
773 2 false
763 1 false
763 2 false
765 1 false
765 2 false
776 1 false
776 2 false
766 1 false
766 2 false
>
```

shows that no group isomorphic to $G^{0\prime} = G^{0\prime\prime} \rtimes \mathbb{Z}_3$ has a spherical system of generators of type $[3, 3, 4]$. □

6.3.2 Non existence results

Remark 6.3.7. Let $X = (C \times C)/G^0$ be a mixed q.e. surface given by a set spherical system of generators $(h_1, \dots, h_r)$ of $G^0 \subseteq G$, we have seen that in order to compute the basket of singularities we have to compare $(h_1, \dots, h_r)$ with its conjugate by $\tau' \in G \setminus G^0$. Note that $(\tau' h_1 \tau'^{-1}, \dots, \tau' h_r \tau'^{-1})$ is a spherical system of generators of $G^0 \subseteq G$ of the same type.
Hence, if a group has a set of spherical generators of the required type, we check if this group has a pair of set of spherical generators that give the right singularities (on Y). If this is not the case surely a set of spherical generators and its conjugated by τ' in G cannot give the required singularities.

Lemma 6.3.8. *No group of order 4032 has a pair of spherical system of generators of type* $[2, 3, 8]$ *which give the expected singularities on* Y*, i.e. either* $\{2 \times C_{8,1}, 2 \times C_{4,1}, 2 \times C_{8,5}\}$ *or* $\{4 \times C_{4,1}\}$ *or* $\{2 \times C_{2,1}\}$.

Proof. Assume that G^0 is a group of order 4032 admitting a surjective homomorphism $\mathbb{T}(2, 3, 8) \to G^0$.

Since $\mathbb{T}(2, 3, 8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 4032, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2016 and it is a quotient of $[\mathbb{T}(2, 3, 8), \mathbb{T}(2, 3, 8)] \cong \mathbb{T}(3, 3, 4)$.

Since $\mathbb{T}(3, 3, 4)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2016, the commutator subgroup $G^{0\prime\prime} = [G^{0\prime}, G^{0\prime}]$ of $G^{0\prime}$ has order 672 and it is a quotient of $[\mathbb{T}(3, 3, 4), \mathbb{T}(3, 3, 4)] \cong \mathbb{T}(4, 4, 4)$. The following MAGMA computation

```
> Test([4,4,4], 672);
{ 1046, 1255 }
>
```

shows that only the groups $G(672, v)$ with $v \in \{1046, 1255\}$ have a spherical system of generators of type $[4, 4, 4]$.

Now the proof continues exactly as the proof of Lemma 6.3.6: we have that $G^{0'} = G^{0''} \rtimes \mathbb{Z}_3$ and we construct all the groups of this form up to isomorphism.

The following MAGMA script shows that no group isomorphic to $G^{0'} = G^{0''} \rtimes \mathbb{Z}_3$, with $G^{0''} = G(672, 1046)$, has a spherical system of generators of type $[3, 3, 4]$:

```
> H2:=SmallGroup(672,1046);
> A2:=AutomorphismGroup(H2);
> Aut2:=AutGr(H2);
> R2:=ConjugCl(Aut2,3);
> C3:=CyclicGroup(3);
> R2[1+#R2]:=Id(A2);
> f2:=[]; for i in [1..#R2] do
for> f2[i]:=hom<C3->A2|R2[i]>;end for;
> h1:=[]; for i in [1..#R2] do
for> h1[i]:=SemidirectProduct(H2,C3,f2[i]);
for> i, ExSphGens(h1[i],[3,3,4]); end for;
1 false
2 false
>
```

The following MAGMA script shows that two extensions $G^{0'} = G^{0''} \rtimes \mathbb{Z}_3$, with $G^{0''} = G(672, 1255)$, have a spherical system of generators of type $[3, 3, 4]$; moreover this two extensions are isomorphic.

```
> H2:=SmallGroup(672,1255);
> A2:=AutomorphismGroup(H2);
> Aut2:=AutGr(H2);
> R2:=ConjugCl(Aut2,3);
> C3:=CyclicGroup(3);
> R2[1+#R2]:=Id(A2);
> f2:=[]; for i in [1..#R2] do
for> f2[i]:=hom<C3->A2|R2[i]>;end for;
> h1:=[]; for i in [1..#R2] do
for> h1[i]:=SemidirectProduct(H2,C3,f2[i]);
for> i, ExSphGens(h1[i],[3,3,4]); end for;
1 true
2 false
```

```
3 true
4 false
> IsIsomorphic(h1[1],h1[3]);
true  Homomorphism of ...
>H1:=h1[1];
```

It can be proved, in a similar way as for $G^{0'} \cong G^{0''} \rtimes \mathbb{Z}_3$, that G^0 is isomorphic to a semidirect product $G^{0'} \rtimes \mathbb{Z}_2$.
The following MAGMA script (that continues the previous one) shows that $G^{0'}$=h1[1] has only one extension $G^{0'} \rtimes \mathbb{Z}_2$ with a spherical system of generators of type $(2,3,8)$:

```
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> R1:=ConjugCl(Aut1,2);
> R1[1+#R1]:=Id(A1);
> C2:=CyclicGroup(2);
> f1:=[]; for i in [1..#R1] do
for> f1[i]:=hom<C2->A1|R1[i]>;end for;
> h:=[]; for i in [1..#R1] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExSphGens(h[i],[2,3,8]); end for;
1 false
2 false
3 false
4 true
5 false
6 false
7 false
8 false
9 false
>
> H:=h[4];
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,3,8]$ of G^0=h[1], the singularity test fails, and so also this case does not occur.

```
> SingularitiesY([{*1/8,1/4,5/8*},{**}],H,[2,3,8]);
false
>
> SingularitiesY([{*1/4^^2*},{**}],H,[2,3,8]);
false
>
> SingularitiesY([{*1/2*},{**}],H,[2,3,8]);
```

```
false
```

□

Lemma 6.3.9. *No group of order 2560 has a pair of spherical system of generators of type* $[2,4,5]$ *which give the expected singularities on* Y*, i.e.* $\{2 \times C_{5,3}\}$.

Proof. Assume that G^0 is a group of order 2560 admitting a surjective homomorphism $\mathbb{T}(2,4,5) \to G^0$.

Since $\mathbb{T}(2,4,5)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2560, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1280 and it is a quotient of $[\mathbb{T}(2,4,5), \mathbb{T}(2,4,5)] \cong \mathbb{T}(2,5,5)$.

The following MAGMA computation

```
> Test([2,5,5], 1280);
{ 1116310 }
>
```

shows that only the group $G(1280, 1116310)$ has a spherical system of generators of type $[2,5,5]$.

Now the proof continues exactly as the proof of Lemma 6.3.6: we have that $G^0 = G^{0\prime} \rtimes \mathbb{Z}_2$ and we construct all the groups of this form up to isomorphism. Among these groups only one has a spherical system of generators of type $[2,4,5]$ as the following MAGMA script shows:

```
> H1:=SmallGroup(1280,1116310);
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> C2:=CyclicGroup(2);
> R:=ConjugCl(Aut1,2);
> R[1+#R]:=Id(A1);
> f1:=[]; for i in [1..#R] do
for> f1[i]:=hom<C2->A1|R[i]>; end for;
> h:=[]; for i in [1..#R] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExSphGens(h[i],[2,4,5]); end for;
1 true
2 false
3 false
4 false
5 false
6 false
7 false
8 false
9 false
10 false
```

```
11 false
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,4,5]$ of G^0=h[1], the singularities test fails, and so also this case does not occur.

```
> H:=h[1];
> SingularitiesY([{* 3/5 *},{* *}],H,[2,4,5]);
false
>
```

□

Lemma 6.3.10. *No group of order 3072 has a pair of spherical system of generators of type* $[2,3,8]$ *which give the expected singularities on* Y*, i.e.* $\{2 \times C_{3,1}, 2 \times C_{3,2}\}$.

Proof. Assume that G^0 is a group of order 3072 admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 3072, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1536 and it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8] \cong \mathbb{T}(3,3,4)$.

The following MAGMA computation

```
> TestBAD([3,3,4], 1536);
{ 408526602 }
>
```

shows that only the group $G(1536, 408526602)$ has a spherical system of generators of type $[3,3,4]$.
Now the proof is the same of Lemma 6.3.9: we have that $G^0 = G^{0\prime} \rtimes \mathbb{Z}_2$ and we construct all the groups of this form up to isomorphism. Among these groups only one has a spherical system of generators of type $[2,3,8]$ as the following MAGMA script shows:

```
> H1:=SmallGroup(1536,408526602);
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> C2:=CyclicGroup(2);
> R:=ConjugCl(Aut1,2);
> R[1+#R]:=Id(A1);
> f1:=[]; for i in [1..#R] do
for> f1[i]:=hom<C2->A1|R[i]>; end for;
> h:=[]; for i in [1..#R] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExSphGens(h[i],[2,3,8]); end for;
1 false
```

```
2 true
3 false
4 false
5 false
6 false
7 false
8 false
9 false
10 false
11 false
12 false
13 false
14 false
15 false
16 false
>
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,3,8]$ of G^0=h[2], the singularities test fails, and so also this case does not occur.

```
> H:=h[2];
> SingularitiesY([{* 1/3, 2/3 *}, {* *}], H, [2,3,8]);
false
>
```

□

Lemma 6.3.11. *No group of order 4896 has a pair of spherical system of generators of type* $[2,3,8]$ *which give the expected singularities on* Y*, i.e.* $\{2 \times C_{8,1}, 2 \times C_{8,3}\}$.

Proof. Assume that G^0 is a group of order 4896 admitting a surjective homomorphism $\mathbb{T}(2,3,8) \to G^0$.

It holds $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$. There is only one perfect group of order 4896, we denote it by H. $H = 2.L_2(17)$ has the following MAGMA representation:

```
> F<w>:=GF(9);
>
> x:=CambridgeMatrix(1,F,8,[
> 01000000,
> 20000000,
> 00010000,
> 00200000,
> 00000100,
> 00002000,
```

```
> 83300083,
> 37420004]);
>
> y:=CambridgeMatrix(1,F,8,[
> 62000000,
> 00100000,
> 48300000,
> 00001000,
> 00000010,
> 00000001,
> 00010000,
> 46466262]);
>
> H<x,y>:=MatrixGroup<8,F|x,y>;
> IsPerfect(H);
true
> #H;
4896
>
```

The following MAGMA computation

```
> ExSphGens(H,[2,4,5]);
false
```

shows that H does not have a spherical system of generators of type $[2,3,8]$ If G^0 is a group of order 4896 with a spherical system of generators of type $[2,3,8]$, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2448 and it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8)] \cong \mathbb{T}(3,3,4)$.

It holds $\mathbb{T}(3,3,4)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and there is only one perfect group of order 2448, we denote it by H', and we will analyze it later.

If G^0 is a group of order 2448 ($G^0 \neq H'$) with a spherical system of generators of type $[3,3,4]$, the commutator subgroup $G^{0\prime\prime} = [G^{0\prime}, G^{0\prime}]$ of $G^{0\prime}$ has order 816 and it is a quotient of $[\mathbb{T}(3,3,4), \mathbb{T}(3,3,4)] \cong \mathbb{T}(4,4,4)$. The following MAGMA computation

```
> Test([4,4,4], 816);
{}
>
```

shows that there are no groups of order 816 with a spherical system of generators of type $[4,4,4]$.

Now we go back to H'. $H' = 2.L_2(17)$ has the following MAGMA representation:

```
> F:=GF(17);
>
```

```
> x:=CambridgeMatrix(3,F,3,\[
> 1,0,0,
> 3,16,0,
> 3,0,16]);
>
> y:=CambridgeMatrix(3,F,3,\[
> 0,1,0,
> 0,0,1,
> 1,0,0]);
>
> H1<x,y>:=MatrixGroup<3,F|x,y>;
> IsPerfect(H1);
true
> #H1;
2448
>
```

The following MAGMA script

```
> ExSphGens(H1,[3,3,4]);
true
>
```

shows that this group has a spherical system of generators of type $[3,3,4]$.

Now the proof continues exactly as the proof of Lemma 6.3.9: we have that $G^0 = G^{0'} \rtimes \mathbb{Z}_2$ and we construct all the groups of this form up to isomorphism. Among these groups only one has a spherical system of generators of type $[2,3,8]$ as the following MAGMA script shows:

```
> A1:=AutomorphismGroup(H1);
> Aut1:=AutGr(H1);
> C2:=CyclicGroup(2);
> R:=ConjugCl(Aut1,2);
> R[1+#R]:=Id(A1);
> f1:=[]; for i in [1..#R] do
for> f1[i]:=hom<C2->A1|R[i]>; end for;
> h:=[]; for i in [1..#R] do
for> h[i]:=SemidirectProduct(H1,C2,f1[i]);
for> i, ExSphGens(h[i],[2,3,8]); end for;
1 false
2 true
3 true
> IsIsomorphic(h[2],h[3]);
true  Homomorphism of ...
> H:=h[2];
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,3,8]$ of G^0=h[2], the singularity test fails, and so also this case does not occur.

```
> SingularitiesY([{* 3/5 *},{* *}],H,[2,4,5]);
false
>
```

□

Definition 6.3.12. We say that a pair of spherical systems generators (T_1, T_2) is *disjoint* if

$$\Sigma(T_1) \cap \Sigma(T_2) = \{1\}.$$

We note that a pair of spherical system of generators are disjoint if and only if the basket of singularities that they induce is empty.

Lemma 6.3.13. *No group of order 2592 has a disjoint pair of spherical systems of generators of type* $[2,3,9]$.

Proof. Assume that G^0 is a group of order 2592 admitting a surjective homomorphism $\mathbb{T}(2,3,9) \to G^0$.

Since $\mathbb{T}(2,3,9)^{\text{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2592, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 864 and it is a quotient of $[\mathbb{T}(2,3,9), \mathbb{T}(2,3,9] \cong \mathbb{T}(2,2,2,3)$. The following MAGMA computation

```
> Test([2,2,2,3], 864);
{2225, 4175}
>
```

shows that only the groups $G(864, v)$ with $v \in \{2225, 4175\}$ have a spherical system of generators of type $[2,2,2,3]$.

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are a disjoint pair of spherical system of generators of type $[2,3,9]$ for G^0, then $(a_i,\ b_i a_i b_i^{-1},\ b_i^2 a_i b_i^{-2},\ c_i^3)$, for $i = 1,2$, are spherical system of generators of type $[2,2,2,3]$ for $G^{0\prime} = [G^0, G^0]$; moreover these two systems are disjoint (see [BCG08, Lemma 4.3, page 574]).

The following MAGMA computations

```
> SingularitiesY([{**},{**}],SmallGroup(864,2225),[2,2,2,3]);
false
> SingularitiesY([{**},{**}],SmallGroup(864,4175),[2,2,2,3]);
false
```

show that the groups $G(864, 2225)$ and $G(864, 4175)$ do not have a disjoint pair of spherical system of generators of type $[2,2,2,3]$, a contradiction. □

6.4 The MAGMA script

In this section we report a commented version of the MAGMA script that we used to find the surfaces in Table 6.1.

```
// Input: Ksquare and p_g; we are assuming q=0.
//
// Step 1: the baskets.
//
// We start finding, for each K^2 and p_g,
// what are the possible baskets of
// singularities of X=(CxC)/G. By Lemma 5.2.5 the sum
// of the invariants B  of the singularities must
// be  equal to 3(8-K^2).
//
// We will represent the singular points of type
// C_{n,a} or D_{n,a} by the  rational number
// a/n in two different multisets;
// a basket of singularities will be a pair of multisets
// of rational numbers.
//
// Remembering that cyclic quotient singularities C_{n,a}
// and C_{n,a'} are isomorphic if a*a'=1 mod n, we consider
// rational numbers in (0,1) modulo the equivalence
// relation a/n ~ a'/n.
//
// We see the entries of the continuous fraction of n/a
// as the sequence [b_1,...,b_r]. Note that the continuous
// fraction of n/a' is the  sequence [b_r,...,b_1].
//
// This can be seen as a bijection between rational numbers
// in (0,1) and sequences of integers strictly bigger than 1.
// We make this bijiection explicit by the following scripts.

ContFrac:=function(s)
  CF:=[ ]; r:=1/s;
  while not IsIntegral(r) do
    Append(~CF, Ceiling(r)); r:=1/(Ceiling(r)-r);
  end while;
  return Append(CF, r);
end function;

Nq:=func<cf|#cf eq 1 select cf[1] else cf[1]-
                   1/$$(Remove(cf,1))>;
```

```
RatNum:=func<seq|1/Nq(seq)>;

// "Wgt" computes the weight of a sequence.
// It bounds strictly from below B of the corresponding
// singular point of type C_{n,a}; and 2*B-12 for D_{n,a}.

Wgt:=function(seq)
  w:=0; for i in seq do w+:=i; end for; return w;
end function;

// The next script computes all rational number whose
// continuous fraction has small weight.

RatNumsWithSmallWgt:=function(maxW)
  S:={ }; T:={}; setnums:={RationalField()| };
  for i in [2..maxW] do Include(~S, [i]); end for;
  for i in [1..Floor(maxW/2)-1] do
  for seq in S do
    if #seq eq i then
    if maxW-Wgt(seq) ge 2 then
    for k in [2..maxW-Wgt(seq)] do
     Include(~S,Append(seq, k));
    end for; end if; end if;
  end for; end for;
  for seq in S do
  if Reverse(seq) notin T then Include(~T,seq);
  end if; end for;
  for seq in T do Include(~setnums, RatNum(seq)); end for;
  return setnums;
end function;

// The next 4 scripts compute the invariants B and e
// of singular points of type C and D respectively (r=a/n).

InvBC:=func<r|Wgt(ContFrac(r))+r+RatNum(Reverse(ContFrac(r)))>;

InveC:=func<r|#ContFrac(r)+1-1/Denominator(RationalField()!r)>;

InvBD:=func<r|InvBC(r)/2 +6>;

InveD:=func<r|InveC(r)/2 +3>;

// The next two scripts compute the invariants B and e of
```

```
// a pair of multisets of rational numbers
// (corresponding to a basket of singular points).

InvBSet:= function(basketC, basketD)
  B:=0; for r in basketC do B+:=InvBC(r); end for;
        for r in basketD do B+:=InvBD(r); end for;
   return B;
end function;

InveSet:= function(basketC, basketD)
  e:=0; for r in basketC do e+:=InveC(r); end for;
        for r in basketD do e+:=InveD(r); end for;
  return e;
end function;

// Here is the invariant k of the basket:

InvkSet:=func<r,s|InvBSet(r,s)-2*InveSet(r,s)>;

// The next script computes all rational numbers with
// weight bounded from above by maxW, as computed by
// RatNumsWithSmallWgt, and returns them in a sequence
// ordered by the value of their invariant B,
// starting from the one with biggest B.

OrderedRatNums:=function(maxW)
   seq:=[RationalField()| ]; seqB:=[RationalField()| ];
   set:=RatNumsWithSmallWgt(Floor(maxW));
     for r in set do i:=1;
       for s in seqB do
       if s gt InvBC(r) then i+:=1;
       else break s;
       end if; end for;
     Insert(~seq, i, r); Insert(~seqB, i, InvBC(r));
     end for;
  return seq;
end function;

// The next one, CutSeqByB, takes a sequence "seq" and
// recursively removes the first element if its invariant B
// is at least maxB.

CutSeqByB:=function(seq,maxB)
  Seq:=seq;
```

```
  while #Seq ge 1 and InvBC(Seq[1]) gt maxB do
    Remove(~Seq,1); end while;
  return Seq;
end function;

// Now we have a way to compute the set of rationals with
// B bounded by the integer maxB, ordered by B:
// CutSeqByB(OrderedRatNums(maxB-1),maxB)
//
// The next script takes a sequence of rational numbers
// ordered by B and computes the baskets with invariant
// exactly B that use only these rationals.
// The function is as follows:
// - first it removes the elements with B too big to be
//   in a basket;
// - then it takes the first element, say r, if B(r)=B,
//   it stores {* r *};
// - else it attaches it to each basket with invariant
//   B-B(r) (computed recalling the function with the
//   same sequence) and store the result;
// - now we have all baskets containing r: remove r
//  from the sequence and repeat the procedure until
//   the sequence is empty.

BasketsWithSeqAndB:=function(seq,B)
  ratnums:=CutSeqByB(seq,B); baskets:={ };
  while #ratnums gt 0 do
    bigguy:=ratnums[1];
    if InvBC(bigguy) eq B then
    Include(~baskets,{* bigguy *});
    else for basket in $$(ratnums, B-InvBC(bigguy)) do
      Include(~baskets, Include(basket, bigguy));
    end for; end if;
    Remove(~ratnums,1);
  end while;
  return baskets;
end function;

// Now we can compute all the "C-parts" (of baskets) with
// a given B:

PartsOfTypeC:=func<B|BasketsWithSeqAndB(OrderedRatNums(B),B)>;

// Next script computes all the possible "D-parts"
```

```
// with a given B and p_g:

PartsOfTypeD:=function(B,pg)
 singD:={ }; basketD:={ };
 D:=RatNumsWithSmallWgt(2*B-27);
 for r in D do
     if InvBD(r)le B then
     if IsIntegral(Denominator(RationalField()!r)/2) then
     if ContFrac(r) eq Reverse(ContFrac(r)) then
     if IsIntegral((#ContFrac(r)+1)/2) then
     if IsIntegral(ContFrac(r)[IntegerRing()!((#ContFrac(r)+1)/2)]/2) then
     Include(~singD,r);
     end if;end if;end if;end if;end if;
 end for;
 for d in { 2*x: x in { 0..(2*pg+1) }} do
 for s in Multisets({ x: x in singD},d) do
    if InvBSet({* *},s) le B then
    Include(~basketD,s);
    end if;
 end for; end for;
 return basketD;
 end function;

// We do not need all these baskets, since most of them
// violate Corollary 5.1.14 or Lemma 5.2.16.
// The next scripts take care of this:
// "BasketOfY" computes the basket of the surface Y starting
// from the basket of X.
// "TestBasket" checks if a basket violates Corollary 5.1.14;
// "TestDen"  checks if a basket respects Lemma 5.2.16;
//
// "Basket" constructs all the basket with given B and
// removes all the baskets which violate the conditions.

BasketOfY:=function(basketX)
basketY:={**};
for r in basketX[1] do
Include(~basketY, r);
Include(~basketY, RatNum(Reverse(ContFrac(r))));
end for;
for r in basketX[2] do Include(~basketY, r); end for;
return basketY;
end function;
```

```
TestBasket:=function(basketC, basketD)
  S:=0; test:=false;
 for r in BasketOfY([basketC, basketD]) do
     S+:=  r; end for;
   if IsIntegral(S) then test:=true;
    end if;
  return test;
end function;

 TestDen:=function(chi, BC,BD)
 test:=true; xi:= 4*chi+(InvkSet(BC,BD)-InveSet(BC,BD))/3;
 for r in Set(BC join BD) do
     if Denominator(RationalField()!r) ge 12*(2*xi+1) then
     test:=false; break r;
 end if; end for;
  return test;
 end function;

Baskets:=function(Ksquare,pg)
baskets:=[**];  chi:=1+pg;
B:=3*(8*chi-Ksquare);
for partD in PartsOfTypeD(B,pg) do
 if (InvBSet({**},partD) eq B) and TestBasket({* *}, partD)
    then Append(~baskets, [{* *}, partD]); end if;
 for partC in PartsOfTypeC(B-InvBSet({* *},partD)) do
    if TestBasket(partC, partD) then
    if TestDen(chi,partC,partD) then
    Append(~baskets, [partC, partD]); end if; end if;
end for; end for;
return baskets;
end function;

// Step 2: the signatures
//
// Now we have found, for each K^2, a finite  number of
// possible baskets. Proposition 5.2.13 says that once
// we fix K^2, p_g and a basket of singularities,
// there are finitely many possible signatures satisfying
// all the condition of the proposition.
//
// The next step is to compute, for each basket, the
// signatures. We will represent a signature as
// a multiset of natural numbers {* m_i *}.
//
```

```
// We first define the index of a basket of singularities
// as the lowest common multiple of the indexes of the
// singularities of type C_{n,a} in BasketOfY.

GI:=func<r|Denominator(r)/GCD(Numerator(r)+1,Denominator(r))>;

GorInd:= function(bas)
  I:=1;
  for r in bas do I:=LCM(IntegerRing()!I,IntegerRing()!GI(r));
  end for;  return I;
end function;

// We define the invariants Theta and Beta:

Theta:=function(sig)
a:=-2;
for m in sig do a+:=(1-1/m); end for;
return a;
end function;

Beta:=func<K, B, T | (K+InvkSet(B[1],B[2]))/(2*T)>;

// These two scripts transform a multiset, resp. a tuple
// into a sequence.

MsetToSeq:=function(mset)
seq:=[ ];
while #mset ne 0 do Append(~seq, Minimum(mset));
Exclude(~mset, Minimum(mset)); end while;
return seq;
end function;

TupleToSeq:=function(tuple)
seq:=[];
for el in Tuplist(tuple) do
Append(~seq,el);
end for;
return seq;
end function;

// Next script computes all the divisor (different from 1)
// of a natural number:

Divisors:=function(n)
```

```
set:={};
for i in { 2.. n} do
if n/i in IntegerRing() then
Include(~set, i);
end if; end for;
return set;
end function;

// The input of the next script are 5 numbers: CardBasket,
// Length, SBound, HBound (SBound<=HBound) and n,
// and its output are all signatures with
// #signature=Length such that (let M:=max(1/6,(Length-3)/2)
// 1) each m_i is smaller than HBound/M;
// 2) most m_i are smaller than SBound/M, the number of
//   exceptions is bounded from above by half of CardBasket.
//
// For sparing time, the script first checks if the length
// is smaller than the number of possible exceptions,
// in which case only the inequality 1 is to consider.
// Moreover, to spare time, since m_i divides n=2*Beta*I,
// the script looks for the m_i's only among the divisors of n.

CandTypes:=function(CardBasketY,Length,S,H,n)
  D:=Divisors(n);
  Exc:=Floor(CardBasketY/2);
  if Length le Exc then
   Types:=Multisets({x: x in D | x in { 2..H}},Length);
  else Types:=Multisets({x: x in D | x in { 2..S}},Length);
    for k in [1..Exc] do
    for TypeBegin in Multisets({x: x in D | x in { 2..S}},Length-k) do
    for TypeEnd in Multisets({x: x in D | x in  {S+1..H}},k) do
      Include(~Types, TypeBegin join TypeEnd);
    end for; end for; end for;
  end if;
  return Types;
end function;

// The function ListTypes calculates all the types that
// fulfill the conditions imposed by Proposition 5.2.13:

ListTypes:=function(Ksquare,pg, basketX)
list:=[];    chi:=1+pg;
BC:=basketX[1]; BD:=basketX[2];
BY:=BasketOfY(basketX);
```

```
den:={};
for r in BY do
Include(~den,Denominator(RationalField()!r)); end for;
xi:= 4*chi+(InvkSet(BC,BD)-InveSet(BC,BD))/3;
I:=GorInd(BY); k:=InvkSet(BC,BD);
Rmin:=3; Tmin:=1/42;
Rmax:=Floor((Ksquare + k) +4);
BetaMax:=Floor(Beta(Ksquare,basketX,Tmin));
for R in [Rmin..Rmax] do
if  R eq 3 then
    top:=Floor(12*(2*xi+1));
    else top:=Floor(6+(8*xi+2)/(R-3));
end if;
    M:=Max(1/6,(R-3)/2);
    SB:=Min(top, Floor((1/M)*(1+(Ksquare+k)/2)));
    HB:=Min(top, Floor((1/M)*(1+I*(Ksquare+k))));
    for b in { 1..BetaMax} do  n:=2*b*I;
    for cand in CandTypes(#BY,R,SB,HB,n) do ;
     if forall{n : n in den |
      exists{m: m in cand| m/n in IntegerRing()}} then
     T:=Theta(cand);
    if (T le (Ksquare+k)/2) and (T gt 0) then
     beta:=Beta(Ksquare,basketX,T);
     if IsIntegral(beta) and beta eq b then
     if IsIntegral((Ksquare+k)/(T^2)) then
     if IsIntegral((4*beta^2)/(Ksquare+k)) then bads:=0;
     for n in cand do
     if not IsIntegral(beta/n) then bads +:=1;
         if bads gt #BY/2 then  break cand; end if;
     end if; end for;
     Append(~list,cand);
     end if;end if;end if;end if;end if;
    end for;
    end for;
end for;
return list;
end function;

// ListOfTypes returns, for given K^2 and p_g, all possible
// baskets (using Baskets) and for each basket all the
// possible types (using ListTypes).

ListOfTypes:=function(Ksquare,pg)
list:=[**];   chi:=1+pg;
```

```
B:=3*(8*chi-Ksquare);
for basket in Baskets(Ksquare,pg) do
L:=ListTypes(Ksquare,pg, basket);
if not IsEmpty(L) then
Append(~list,[* basket, L*]);
end if;
end for;
return list;
end function;

// Step 3: calculating the groups.
//
// Fixed K^2, p_g, the basket and the signature,
// using Proposition 5.2.13 we can compute the order
// of the group G^0.
// We search among the group of this order which groups
// have a prescribed set of spherical generators.

ElsOfOrd:=func<group, order | {g: g in group| Order(g) eq order}>;

// TuplesOfGivenOrder creates a sequence of the same length
// as the input sequence seq, whose entries are subsets
// of the group in the input, and precisely the subsets
// of elements of order  the corresponding entry of seq.

TuplesOfGivenOrders:=function(group,seq)
SEQ:=[];
for i in [1..#seq] do
if IsEmpty(ElsOfOrd(group,seq[i])) then SEQ:=[]; break i;
else Append(~SEQ,ElsOfOrd(group,seq[i]));
end if; end for;
return SEQ;
end function;

// This script says if a group has a set
// of spherical generators of the given type:

ExSphGens:=function(group,type)
test:=false;
SetCands:=TuplesOfGivenOrders(group,Prune(type));
if not IsEmpty(SetCands) then
for cands in CartesianProduct(SetCands) do
    if Order(&*cands) eq type[#type] then
    if #sub<group|TupleToSeq(cands)> eq #group then
```

```
    test:=true; break cands;
end if; end if; end for; end if;
return test;
end function;

// Polygonal builds the polygonal group of the type given by seq

Polygonal:=function(seq)
F:=FreeGroup(#seq);  Rel:={F![1..#seq]};
for i in [1..#seq] do  Include(~Rel,F.i^(seq[i])); end for;
return quo<F|Rel>;
end function;

// Test and TestBAD search among all the groups of
// the order in input which groups have a spherical
// system of generators of the type in input.
// These function work in two steps (see Remark 6.2.1):
// i) they check  which  groups  have abelianization
//  isomorphic to a quotient of the abelianization
//   of the polygonal group given by the type;
// ii) if a group passes the first test the scripts
//   check if it has a spherical system of generators
//   of the type in input.
// These two scripts make exactly the same controls, and
// we use Test in general, but in some cases there are too
// much isomorphism classes of groups of the given order
// and we use TestBAD because, SmallGroupProcess is slower
// than SmallGroups but it uses less memory.

Test:=function(type, order)
group:=AbelianQuotient(Polygonal(type));
checked:={}; quo:={}; set:={};
for g in Subgroups(group) do
Include(~quo, group/(g`subgroup)); end for;
for h in quo do Include(~set,#h); end for;
i:=1;
for H in SmallGroups(order: Warning:=false) do
    if #AbelianQuotient(H) in set then
    for p in quo do
    if IsIsomorphic(p, AbelianQuotient(H)) then
    if ExSphGens(H,type) then
    Include(~checked,i); end if;
    break p;end if;  end for; end if;
i+:=1; end for;
```

```
return checked;
end function;

TestBAD:=function(type, order)
group:=AbelianQuotient(Polygonal(type));
checked:={}; quo:={}; set:={};
for g in Subgroups(group) do
Include(~quo, group/(g`subgroup)); end for;
for h in quo do Include(~set,#h); end for;
i:=1; P:= SmallGroupProcess(order);
repeat
H := Current(P);
  if #AbelianQuotient(H) in set then
  for p in quo do
  if IsIsomorphic(p, AbelianQuotient(H)) then
  if ExSphGens(H,type) then
  Include(~checked,i); end if;
  break p;end if; end for; end if;
i+:=1; Advance(~P);
until IsEmpty(P);
return checked;
end function;

// The next script takes a sequence of elements of a group
// and a further element g and conjugates each element
// of the sequence with g.

Conjug:=function(seq,el)
output:=[];
for h in seq do Append(~output,h^el); end for;
return output;
end function;

// SphGenUpToConj computes all possible sets of spherical
// generators of a group of a prescribed type and return
// (to spare memory) only one of these sets for each
// conjugacy class.

SphGenUpToConj:=function(group,seq)
Set:={ }; Rep:={ };
SetCands:=TuplesOfGivenOrders(group,Prune(seq));
if not IsEmpty(SetCands) then
for cands in CartesianProduct(SetCands) do
  if Order(&*cands) eq seq[#seq] then
```

```
  if Append(TupleToSeq(cands),(&*cands)^-1) notin Set
   then if sub<group|TupleToSeq(cands)> eq group then
     Include(~Rep, Append(TupleToSeq(cands),(&*cands)^-1));
     for g in group do
     Include(~Set, Conjug(Append(TupleToSeq(cands),(&*cands)^-1),g));
end for;  end if; end if; end if; end for; end if;
return Rep;
end function;

// If a group has a set of spherical generators of the
// right type before to look for an unsplit extension,
// we check if the group has a pair of sets of
// spherical generators that give the right singularities.
// If this is not the case surely a set of spherical
// generators and its conjugation by tau' in G cannot give
// the right singularities.
//
// Given two sets of spherical generators,
// next script computes the singular points
// coming from a fixed pair (g1,g2), where
// - g_1 is a generator of the first set;
// - g_2 is a generator of the second set;
//  and 1<=n_1<=ord(g_1); 1<=n_2<=ord(g_2);
// Moreover, it returns the element g such that
// g_1^n_1= (g_2^n_2)^g.

BasketByAPairOfGens:= function(group,gen1,gen2)
   ord1 := Order(gen1);     ord2 := Order(gen2);
   basket := [ ];  els:=[];
   delta := GCD(ord1, ord2);
   if delta eq 1 then return {* *}; end if;
   alpha2 := ord2 div delta;
   H := sub<group | gen2>;    K := sub<group | gen1>;
   if Type(H) eq GrpPC then
      RC := Transversal(group, H, K);
   else  RC := DoubleCosetRepresentatives(group, H, K);
   end if;
   for g in RC do
       HgK := H^g meet K;
       ord_HgK := #HgK;
       if ord_HgK eq 1 then continue g; end if;
       x := gen1^(ord1 div ord_HgK);
       y := (gen2^(ord2 div ord_HgK))^g;
       if exists(i){i:i in [1..delta] | y^i eq x} then
```

```
          d2 := (i*(ord2 div ord_HgK)) div alpha2;
          Append(~basket, d2/delta);
          Append(~els,g);
        end if;
    end for;
    return basket,els;
end function;

// CheckSingsH checks if a pair of set of spherical
// generators of groupH gives a surface Y=(CxC)/G^0
// with the expected singularities.
//
// It only checks if, given two sets of spherical
// generators and a "candidate" basket, the resulting
// surface has the prescribed basket. The advantage is that
// in the wrong cases, the script stops when it finds a
// "forbidden" singularity, without losing time computing
// all the other singular points.

CheckSingsH:=function(basket,gens1,gens2,group)
test:=true; bas:=basket;
for i in [1..#gens1] do gen1:=gens1[i];
for j in [1..#gens2] do gen2:=gens2[j];
    pb:=BasketByAPairOfGens(group,gen1,gen2);
      for r in pb do r1:=RatNum(Reverse(ContFrac(r)));
      if r in bas then Exclude(~bas,r);
      elif r1 in bas then Exclude(~bas,r1);
      else test:=false; break i;
      end if; end for;
end for; end for;
return test and IsEmpty(bas);
end function;

// These function checks if a group has a pair of sets
// of spherical generators that give the expected
// singularities

SingularitiesY:=function(basketX,groupH,type)
BY:=BasketOfY(basketX);
s:=SetToSequence(SphGenUpToConj(groupH,type));
c:=1; test:= false;
for i in [1..#s] do   gens1:=s[i];
for j in [c..#s] do   gens2:=s[j];
    if CheckSingsH(BY,gens1,gens2, groupH) then
```

```
    test:=true; break i;
end if;  end for; c+:=1; end for;
return test;
end function;

// Now we check if a given group G has a set of
// spherical generators for a group isomorphic to G^0
// in the group G of prescribed type.

ExistSphGens:=function(groupG, idH, type)
test:=false;
SetCands:=TuplesOfGivenOrders(groupG,Prune(type));
if not IsEmpty(SetCands) then
for cands in CartesianProduct(SetCands) do
    if Order(&*cands) eq type[#type] then
    if IdentifyGroup(sub<groupG|TupleToSeq(cands)>)
       eq idH then  test:=true; break cands;
end if; end if; end for; end if;
return test;
end function;


// GroupExtension checks if the given group "groupH"=G^0
// has some unsplit extension of degree 2, and returns
// all the groups G which are unsplit extension of groupH.
//
// If the order of the group is "bad", it uses
// SmallGroupProcess instead of SmallGroups.

GroupExtension:=function(groupH,type, badorders)
ordG:= 2*Order(groupH); ext:=[* *];
idH:=IdentifyGroup(groupH);
card:=#{x: x in groupH | Order(x) eq 2};
if ordG notin badorders then
for G in SmallGroups(ordG: Warning := false) do
    if  #{x: x in G | Order(x) eq 2} eq card then
    if ExistSphGens(G, idH, type) then
           Append(~ext, IdentifyGroup(G));
    end if; end if;
end for;
else
P:= SmallGroupProcess(ordG);
repeat
G := Current(P);
```

```
  if  #{x: x in G | Order(x) eq 2} eq card then
  if ExistSphGens(G, idH,type) then
   Append(~ext, IdentifyGroup(G));
 end if;end if;
Advance(~P); until IsEmpty(P);
end if;
return ext;
end function;

// ListGroups lists in checked all possible fourtuples
// (basket, type, subgroup G^0, {groups G}).
// It lists in limbo the triples
// basket, type, group G^0, where G^0 has
// a pair of sets of spherical generators of groupH
// gives a surface Y=(CxC)/G^0 with the expected
// singularities, but we cannot check the extensions,
// since the order of the group is too big.
// It lists in tocheck the triples basket, type, order G^0,
// if order G^0 is bigger than 2000 or it is 1024.

ListGroups:=function(Ksquare, pg:
                  badorders1:={ 256, 384, 512, 576, 768},
                  badorders2:={ 1152,1280,1536,1920})
checked:=[* *]; tocheck:=[* *]; limbo:=[* *];
for pair in ListOfTypes(Ksquare, pg) do
basket:=pair[1]; setoftypes:=pair[2];
for type in setoftypes do
ordH:=IntegerRing()!((Ksquare+InvkSet(basket[1],basket[2]))/
((Theta(type))^2));
if {*2,3,7*} eq type and
      NumberOfGroups(PerfectGroupDatabase(),ordH) eq 0 then ;
elif (ordH gt 2000) or (ordH eq 1024) then
     Append(~tocheck, [* basket, type, ordH *]);
elif ordH in { 1001..2000} and
             (ordH in Include(badorders2)) then
     type1:=MsetToSeq(type);
     for p in TestBAD(type1, ordH)  do
     H:=SmallGroup(ordH,p);
     if SingularitiesY(basket,H,type1) then
     Append(~limbo, [* basket, type, <ordH, p>*]); end if;
     end for;
elif ordH in { 1001..2000} and
             (ordH notin Include(badorders2)) then
     type1:=MsetToSeq(type);
```

```
          for p in Test(type1, ordH)  do
          H:=SmallGroup(ordH,p);
          if SingularitiesY(basket,H,type1) then
          Append(~limbo, [*basket, type, <ordH, p>*]); end if;
          end for;
     elif ordH in Include(badorders1,512) then
          type1:=MsetToSeq(type);
          for p in TestBAD(type1, ordH)  do
          H:=SmallGroup(ordH,p);
          if SingularitiesY(basket,H,type1) then
          extensions:=GroupExtension(H,type1, badorders1 join badorders2);
          if not IsEmpty(extensions) then
          Append(~checked, [* basket, type, IdentifyGroup(H), extensions *]);
          end if;end if;  end for;
     else type1:=MsetToSeq(type);
          for p in Test(type1, ordH)  do
          H:=SmallGroup(ordH,p);
          if SingularitiesY(basket,H,type1) then
          extensions:=GroupExtension(H,type1, badorders1 join badorders2);
          if not IsEmpty(extensions) then
          Append(~checked, [* basket, type, IdentifyGroup(H), extensions *]);
          end if; end if;  end for;
     end if;  end for; end for;
     return checked, limbo, tocheck;
     end function;

     // Step 4: existence of surfaces
     //

     // First we create all the sets of spherical generators
     // of a prescribed type that generate a
     // group isomorphic to G^0 in the group G.

     SphGens:=function(groupG, idH, type)
     Gens:={ };
     SetCands:=TuplesOfGivenOrders(groupG,Prune(type));
     if not IsEmpty(SetCands) then
     for cands in CartesianProduct(SetCands) do
       if Order(&*cands) eq type[#type] then
       if IdentifyGroup(sub<groupG|TupleToSeq(cands)>) eq idH then
       Include(~Gens, Append(TupleToSeq(cands),(&*cands)^-1));
     end if; end if; end for; end if;
     return Gens;
     end function;
```

```
// CheckSingsG checks if a set of elements of groupG that
// is a system of spherical generators of groupH gives
// a surface X=(CxC)/G with the expected singularities.
//
// First it checks if the singularities of Y=(CxC)/G^0
// are the expected ones.
// If this is the case it checks if the ramification
// points are right.

CheckSingsG:=function(basket, gens, groupG)
groupH:= sub<groupG|gens>;
tp:=[g: g in groupG | g notin groupH][1];
gens2:=[]; BY:=BasketOfY(basket); BD:=basket[2];
for i in [1..#gens] do Append(~gens2, gens[i]^tp);
    end for;
test:=CheckSingsH(BY,gens,gens2,groupH);
if test then
for k in [1..#gens] do gen:=gens[k]; gen2:=gen^tp;
    sing,els:=BasketByAPairOfGens(groupH,gen,gen2);
    S:=sub<groupH|gen>;
    for j in [1..#sing] do
        r:=sing[j]; g:=tp*(els[j]^(-1))*tp^(-1);
         if exists{h: h in groupH | ((tp*h)^2 in S)
                    and ((tp*h*tp^-1)*g in S) } then
          if r in BD then Exclude(~BD,r);
          else test:= false; break k;
           end if;   end if;
end for;    end for; end if;
if not IsEmpty(BD) then test:=false; end if;
return test;
end function;

// ExistingSurfaces returns all the fourtuples
// (basket, type,  G^0, G) that give at least
// a surface with the correct singularities.

ExistingSurfaces:=function(Ksquare, pg)
M:=[* *];
list,limbo,tocheck:=ListGroups(Ksquare, pg);
for quadruple in list do
basket:=quadruple[1]; type:=quadruple[2];
idH:=quadruple[3]; listOfG:=quadruple[4];
for idgroupG in listOfG do  test:=false;
```

```
    G:=SmallGroup(idgroupG[1], idgroupG[2]);
    SetGens:=SphGens(G,idH,MsetToSeq(type));
    for gens in SetGens do
    if CheckSingsG(basket, gens, G) then test:=true;
    break gens;  end if; end for;
if test then
Append(~M, [* basket, type, idH, idgroupG *]);
end if;
end for; end for;
return  M, limbo, tocheck;
end function;

// Step 5: to find all the surfaces.
//
// We still have not found all possible surfaces.
// In fact the output of ExistingSurfaces(a, b)
// gives all possible fourtuples (basket, type , G^0, G)
// which give AT LEAST a surface with p_g=b and K^2=a,
// but there could be more than one. In fact, there is
// a surface for each set of spherical generators of the
// prescribed types which passes the singularity test,
// but they are often isomorphic. More precisely,
// they are isomorphic if the sets of spherical generators
// are equivalent for the equivalence relation generated
// by Hurwitz moves and the automorhisms of the group.
// We need to construct orbits for this equivalence relation.

// The next scripts create the Automorphism Group of a group
// as an explicit set.

AutGr:= function(gr)
Aut:=AutomorphismGroup(gr); A:={ Aut!1 };
repeat
for g1 in Generators(Aut) do
for g2 in A do
Include(~A,g1*g2);
end for; end for;
until  #A eq #Aut;
return A;
end function;

// The next one create the Hurwitz moves:

HurwitzMove:= func<seq,idx|Insert(Remove(seq,idx),
```

```
idx+1,seq[idx]^seq[idx+1])>;

// This one, starting from a sequence of elements of a group,
// creates all sequences of elements which are equivalent to
// the given one for the equivalence relation generated by
// the Hurwitz moves, and return (to spare memory) only
// the ones whose entries have never decreasing order.

HurwitzOrbit:=function(seq)
orb:={ }; shortorb:={  }; Trash:={ seq };
repeat
    ExtractRep(~Trash,~gens); Include(~orb, gens);
    for k in [1..#seq-1] do newgens:=HurwitzMove(gens,k);
    if newgens notin orb then Include(~Trash, newgens);
    end if; end for;
until IsEmpty(Trash);
for gens in orb do  test:=true;
for k in [1..#seq-1] do
    if Order(gens[k]) gt Order(gens[k+1]) then
    test:=false; break k;
    end if; end for;
    if test then Include(~shortorb, gens); end if;
end for;
return shortorb;
end function;

// Finally we can find all surfaces. The next program
// finds all surfaces with a given groups, type and basket.

FindSurfaces:=function(K, basket, type,idH, idG)
Good:=[* *]; Surfaces:={ };  All:={ };
G:=SmallGroup(idG[1], idG[2]);
AutG:=AutGr(G);
NumberOfCands:=#SphGens(G,idH,MsetToSeq(type));
printf "To Find= %o\n", NumberOfCands;
for gens in SphGens(G,idH,MsetToSeq(type)) do
if gens notin All then
    printf "A new one! ";
    Include(~Surfaces, gens); H:=sub<G|gens>;
    if CheckSingsG(basket, gens, G) then
    S:=[* basket, type, gens, idH, idG*];
    printf " and right singularities!\n";
    printf "A REALLY NEW SURFACE!!!\n";
    Append(~Good, S);
```

```
        else printf " but wrong singularities!\n";
        end if;
        orb:=HurwitzOrbit(gens);
        for g1 in orb do
            if g1 notin All then
            for phi in AutG do Include(~All, phi(g1));
            if #All eq  NumberOfCands then
            printf "#Surfs= %o\n", #Surfaces; break gens;
        end if; end for; end if; end for;
        printf "#Surfs= %o, To Find= %o\n", #Surfaces, NumberOfCands-#All;
    end if; end for;
    return Good;
    end function;

    // Next script calls the previous scripts
    // and stores the data of the surfaces in
    // a text file.

    Output:=function(Ksquare,pg)
    t:=Realtime();
    New:=[* *];
    M, limbo, tocheck:=ExistingSurfaces(Ksquare,pg);
    for m in M do
        basket:=m[1]; type:=m[2]; idH:=m[3]; idgroup:=m[4];
        printf "\n Checking news %o \n", m[4];
        Surf:=FindSurfaces(Ksquare, basket, type, idH, idgroup);
        for surf in Surf do Append(~New, surf);
    end for; end for;
    F:= Open("OUTPUT_WITH_Ks" cat IntegerToString(Ksquare)
    cat "_pg" cat IntegerToString(pg) cat  ".txt","w");
    fprintf F, "K^2=%o\n\n\n", Ksquare;
    if #New ne 0 then
    fprintf F, "NEW SURFACES: %o\n", #New;
    fprintf F, "basket,  type, gens, Id(H), Id(G)\n\n";
    for new in New do fprintf F, "%o\n\n", new; end for;
    fprintf F, "\n\n";
    end if;
    if #limbo ne 0 then
    fprintf F, "PARTIALLY TO CHECK CASES: %o\n", #limbo;
    for L in limbo do fprintf F, "%o\n\n", L; end for;
    fprintf F, "\n\n";
    end if;
    if #tocheck ne 0 then
    fprintf F, " TO CHECK CASES: %o\n", #tocheck;
```

```
for T in tocheck do fprintf F, "%o\n\n", T; end for;
end if;
printf "Time: %o\n", Realtime(t);
return "K^2=",Ksquare,", #New surf=",#New;
end function;

// Step 6: the fundamental group
//
// Next scripts allow us to calculate the topological
// fundamental group of the surfaces we constructed.
// We use the description of the fundamental
// given in Theorem 7.1.2 and Proposition 7.1.8.

// Poly constructs the polygonal group and the
// appropriate orbifold homomorphism.

Poly:=function(seq, gr)
F:=FreeGroup(#seq);  Rel:={F![1..#seq]};
for i in [1..#seq] do
    Include(~Rel,F.i^Order(seq[i])); end for;
P:=quo<F|Rel>;
return P, hom<P->gr|seq>;
end function;

// DirProd(A,B) returns the direct product between
// the groups A and B, and the corresponding injections
// and projections.

DirProd:=function(G1,G2)
G1xG2:=DirectProduct(G1,G2); vars:=[];
n:=[NumberOfGenerators(G1),NumberOfGenerators(G2)];
for i in [1..(n[1]+n[2])] do
    Append(~vars,G1xG2.i); end for;
SplittedVars:=Partition(vars,n);
injs:=[hom< G1->G1xG2 | SplittedVars[1]>,
hom< G2->G1xG2 | SplittedVars[2]>];
vars1:=[]; vars2:=[];
for i in [1..n[1]] do
     Append(~vars1,G1.i); Append(~vars2,G2!1); end for;
for i in [1..n[2]] do
     Append(~vars1,G1!1); Append(~vars2,G2.i); end for;
projs:=[hom< G1xG2->G1 | vars1>,hom< G1xG2->G2 | vars2>];
return G1xG2, injs, projs;
end function;
```

```
// MapProd computes given two maps f,g:A->B the map product
// induced by the product on B

MapProd:=function(map1,map2)
seq:=[];
A:=Domain(map1); B:=Codomain(map1);
if Category(A) eq GrpPC then n:=NPCgens(A);
else n:=NumberOfGenerators(A); end if;
for i in [1..n] do
     Append(~seq, map1(A.i)*map2(A.i)); end for;
return hom<A->B|seq>;
end function;

// Pi1 uses a sequence of spherical generators for G^0
// inside G to construct the corresponding polygonal group
// and the group HH that acts on the universal cover of CxC.
// Then it constructs the degree 2 extension GG.
// Finally it takes the quotient by Tors(GG).

Pi1:=function(seq, G)
H:=sub<G|seq>; REL:=[]; TorsG:=[]; Sing:=;
el:=random{g: g in G | g notin H};
phi1:=hom<H->H| x:-> x^el>;
T,f1:=Poly(seq,H);  t:=(el^2)@@f1;
TxT,inT,proT:=DirProd(T,T);
HxH,inH:=DirectProduct(H,H);
Diag:=MapProd(inH[1],inH[2])(H);
f:=MapProd(proT[1]*f1*inH[1],proT[2]*f1*phi1*inH[2]);
bigH:=Rewrite(TxT,Diag@@f); tt:=inT[1](t)*inT[2](t);
PHI:=hom<bigH->bigH| x:-> inT[1](proT[2](x))*inT[2](t*proT[1](x)*(t^-1))>;
genH:=SetToSequence(Generators(bigH)); relH:=Relations(bigH);
F:=FreeProduct(bigH,FreeGroup(1)); im:=[];
for i in [1..#genH] do Append(~im,F.i);  end for;
map:=hom<bigH->F|im>; tau:=map(tt);
ul:=F.(#Generators(F)); Append(~REL, ul^2*(tau^-1));
for i in [1..#genH] do
  Append(~REL, map(PHI(genH[i]))* ul * map(genH[i]^-1 )*(ul^-1));
end for;
bigG,pr:=quo<F|REL>;
for i in [1..#seq] do gen1:=seq[i];
for j in [1..#seq] do gen2:=seq[j];
for o1 in [1..Order(gen1)-1] do
for o2 in [1..Order(gen2)-1] do
```

```
test,v:=IsConjugate(H,gen1^o1, phi1(gen2^o2));
if test then  Include(~Sing, [i,j]);
for d in Centralizer(H, gen1^o1) do
    Append(~TorsG, pr(map(TxT.i^o1 *
        ((TxT.(j+#seq)^o2)^(inT[2]((el *d^-1*v*el^-1)@@f1)^-1)))));
end for; end if; end for; end for; end for; end for;
for i in [1..#seq] do gen:=seq[i];
if [i,i] in Sing then
for o in [1..Order(gen)-1] do
for h in H do
test, v:= IsConjugate(H, (el*h)^2, gen^o);
if test then
for d in Centralizer(H, gen^o) do
w:=(v*d)@@f1;  h1:=h@@f1; h2:= (el*h*(el^-1))@@f1; s:=h2*t*h1;
k:=(s^-1)*((T.i^o)^(w^-1));
Append(~TorsG, pr(ul*(map(inT[1](h1)*inT[2](k*h2)))));
end for;
 end if; end for; end for;end if; end for;
     return Simplify(quo<bigG|TorsG>);
end function;

// Next function is an additional function
// that we used to exclude some skipped cases.
// It returns a representative of each conjugacy class
// of elements of the given order.

ConjugCl:=function(group, order)
     Set:={}; Rep:=[];
     list:=[x: x in group | Order(x) eq order];
     for el in list do
         if el notin Set then
            for a in group do
                Include(~Set, el^a);
         end for;  Append(~Rep, el);
     end if; end for;
return Rep;
end function;
```

Chapter 7

Regular mixed quasi-étale surfaces with $p_g = 0$ and $K^2 > 0$

In this chapter we study the surfaces constructed in the preceding chapter.

In Section 7.1 we explain how to compute the fundamental group of a regular surface which is the minimal resolution of a mixed q.e. surface; in particular we compute the fundamental group of the surfaces we have constructed, see Table 7.1. In Section 7.2, we determine the minimal model of the surfaces that we have constructed, proving that they are all minimal.

So we prove the first part of Theorem 6.1.1. Finally in Section 7.3, we give a detailed description of the surfaces.

7.1 The fundamental group

In this section we show how to compute the fundamental group of the surfaces that we have constructed. To calculate the fundamental groups we will follow the idea developed in [BCGP08] (see also [DP10]) for the unmixed case, and we adapt it to the mixed case.

Let $X = (C \times C)/G$ be a regular mixed q.e. surface determined by the appropriate orbifold homomorphism $\psi\colon \mathbb{T}(m_1, \ldots, m_r) \to G^0$. Let

$$\mathbb{T} := \mathbb{T}(m_1, \ldots, m_r) = \langle c_1, \ldots, c_r \mid c_1^{m_1}, \ldots, c_r^{m_r}, c_1 \cdots c_r \rangle \,.$$

By Lemma 2.5.1, the kernel of ψ is isomorphic to the fundamental group $\pi_1(C)$ and the sequence

$$1 \longrightarrow \pi_1(C) \longrightarrow \mathbb{T} \xrightarrow{\psi} G^0 \longrightarrow 1 \tag{7.1}$$

is exact. By Lemma 2.5.2, the action of $\pi_1(C)$ on the universal cover Δ of C extends to a discontinuous action of $\mathbb{T}$. Let $u\colon \Delta \to C$ be the covering map, it is ψ-equivariant, i.e $u(g(x)) = \psi(g)u(x)$ for all $x \in C$ and $g \in \mathbb{T}$; and so $C/G^0 \cong \Delta/\mathbb{T} \cong \mathbb{P}^1$. Let $\mathcal{U} := (u,u)\colon \Delta \times \Delta \to C \times C$.

$$\begin{tikzcd}
 & \Delta \times \Delta \arrow[dl] \arrow[d, "\mathcal{U}"] \arrow[dr] & \\
\Delta \arrow[d, "u"] & C \times C \arrow[dl] \arrow[d, "\sigma"] \arrow[dr] & \Delta \arrow[d, "u"] \\
C \arrow[d] & (C \times C)/G^0 \arrow[dl] \arrow[d, "\pi"] \arrow[dr] & C \arrow[d] \\
\mathbb{P}^1 & (C \times C)/G & \mathbb{P}^1
\end{tikzcd}$$

Fix $\tau' \in G \setminus G^0$; let $\tau = \tau'^2 \in G^0$ and let $\varphi \in \operatorname{Aut}(G^0)$ defined by $\varphi(h) := \tau' h \tau'^{-1}$. Let

$$\mathbb{H} := \{(t_1, t_2) \in \mathbb{T} \times \mathbb{T} \mid \psi(t_1) = \varphi^{-1}(\psi(t_2))\}.$$

It embeds in $\operatorname{Aut}(\Delta \times \Delta)$ as follows:

$$(h_1, h_2) \cdot (x, y) = (h_1 \cdot x, h_2 \cdot y) \qquad \text{for } (h_1, h_2) \in \mathbb{H}.$$

Choose $t \in \psi^{-1}(\tau)$, since ψ is surjective and $\varphi(\tau) = \tau$, then $\tilde{\tau} := (t,t) \in \mathbb{H}$. We define

$$\begin{array}{rccc} \tilde{\tau}'\colon & \Delta \times \Delta & \longrightarrow & \Delta \times \Delta \\ & (x,y) & \longmapsto & (y, t \cdot x) \end{array}$$

it is an element of $\operatorname{Aut}(\Delta \times \Delta)$ that exchanges the factors and $(\tilde{\tau}')^2 = \tilde{\tau}$; we further define $\tilde{\varphi}\colon \mathbb{H} \to \mathbb{H}$ as $\tilde{\varphi}(t_1, t_2) := (t_2, t \cdot t_1 \cdot t^{-1})$, it is the conjugation by $\tilde{\tau}'$.

Let $\mathbb{H} = \langle gen(\mathbb{H}) \mid rel(\mathbb{H}) \rangle$ be a presentation of $\mathbb{H}$, and let $REL := \{\tilde{\varphi}(h)\tilde{\tau}' h^{-1} \tilde{\tau}'^{-1} \mid h \in gen(\mathbb{H})\}$. We define $\mathbb{G}$ as follows:

$$\mathbb{G} := \langle gen(\mathbb{H}), \tilde{\tau}' \mid rel(\mathbb{H}), (\tilde{\tau}')^2 \tilde{\tau}^{-1}, REL \rangle.$$

Definition 7.1.1. Let H be a group. Its *torsion subgroup* $\operatorname{Tors}(H)$ is the normal subgroup generated by all elements of finite order in H.

Theorem 7.1.2. *Let $X = (C \times C)/G$ be a mixed q.e. surface. Let $S \to X$ be its minimal resolution of the singularities and assume $q(S) = 0$. Then*

$$\pi_1(S) \cong \pi_1\left(\frac{C \times C}{G}\right) \cong \frac{\mathbb{G}}{\operatorname{Tors}(\mathbb{G})}.$$

We recall that the minimal resolution $S \to X$ of X replace each singular point by a tree of smooth rational curves, hence, by van Kampen's theorem, $\pi_1(S) = \pi_1(X)$.
To prove the second part of the theorem we need some lemmas.

$\mathbb{H}$ is an index 2 subgroup of $\mathbb{G}$ and every element $g \in \mathbb{G}$ either is in $\mathbb{H}$ or there exists $h \in \mathbb{H}$ such that $g = \tilde{\tau}' h$. We define a left action of $\mathbb{G}$ on $\Delta \times \Delta$, as follows:

$$\begin{array}{rcl} (h_1, h_2) \cdot (x, y) & = & (h_1 \cdot x, h_2 \cdot y) \\ \tilde{\tau}'(h_1, h_2) \cdot (x, y) & = & (h_2 \cdot y, (t \cdot h_1) \cdot x) \end{array} \qquad \text{for } (h_1, h_2) \in \mathbb{H}\,. \tag{7.2}$$

hence $\mathbb{G}$ embeds in $\mathrm{Aut}(\Delta \times \Delta)$.
We also define a group homomorphism $\vartheta\colon \mathbb{G} \to G$:

$$\begin{array}{rcl} \vartheta(h_1, h_2) & = & \psi(h_1) = \varphi^{-1}\psi(h_2) \\ \vartheta(\tilde{\tau}'(h_1, h_2)) & = & \tau'\psi(h_1) = \tau'\varphi^{-1}(\psi(h_2)) \end{array} \qquad \text{for } (h_1, h_2) \in \mathbb{H}\,.$$

Remembering the relations between τ', τ and φ, it is easy to prove that ϑ is a group homomorphism.

Lemma 7.1.3. *$\mathcal{U} = (u, u)\colon \Delta \times \Delta \to C \times C$ is ϑ-equivariant.*

Proof. Let $g = (h_1, h_2) \in \mathbb{H}$. Remembering that u is ψ-equivariant, we get

$$\begin{array}{rcl} \mathcal{U}(g(x, y)) & = & \mathcal{U}(h_1 x, h_2 y) = (u(h_1 x), u(h_2 y)) \\ & = & (\psi(h_1)u(x), \psi(h_2)u(y)) = (\psi(h_1)u(x), \varphi(\psi(h_1))u(y)) \end{array}$$

while

$$\vartheta(g)\mathcal{U}(x, y) = \psi(h_1)(u(x), u(y)) = (\psi(h_1)u(x), \varphi(\psi(h_1))u(y))\,.$$

Let $g = \tilde{\tau}'(h_1, h_2) \in \mathbb{H}$, we get

$$\begin{array}{rcl} \mathcal{U}(g(x, y)) & = & \mathcal{U}(\tilde{\tau}'(h_1 x, h_2 y)) = \mathcal{U}(h_2 y, t h_1 x) \\ & = & (u(h_2 y), u(t h_1 x)) = (\psi(h_2)u(y), \psi(t h_1)u(x)) \\ & = & (\psi(h_2)u(y), \tau\psi(h_1)u(x)) \end{array}$$

while

$$\begin{array}{rcl} \vartheta(g)\mathcal{U}(x, y) & = & \tau'\psi(h_1)(u(x), u(y)) = \tau'(\psi(h_1)u(x), \varphi(\psi(h_1))u(y)) \\ & = & (\psi(h_2)u(y), \tau\psi(h_1)u(x))\,. \end{array}$$

□

It follows that
$$\frac{\Delta \times \Delta}{\mathbb{G}} \cong \frac{C \times C}{G}\,.$$

Lemma 7.1.4. *The following short sequence:*

$$1 \longrightarrow \pi_1(C \times C) \longrightarrow \mathbb{G} \xrightarrow{\vartheta} G \longrightarrow 1$$

is exact.

Proof. We have to prove that $\ker\vartheta \cong \pi_1(C \times C)$.
If $g = \tilde{\tau}'(h_1, h_2) \in \mathbb{G} \setminus \mathbb{H}$, then $\vartheta(g) = \tau'\psi(h_1) \neq 1$, so $\ker\vartheta \subseteq \mathbb{H}$.
If $g = (h_1, h_2) \in \mathbb{H}$, then $\vartheta(g) = \psi(h_1) = \varphi^{-1}(\psi(h_2)) = 1$ if and only if $h_1, h_2 \in \ker(\psi)$, hence

$$\ker\vartheta = \{(h_1, h_2) \in \mathbb{G} \mid h_1,\, h_2 \in \ker(\psi)\} = \ker\psi \times \ker\psi \cong \pi_1(C) \times \pi_1(C)\,.$$

□

Remark 7.1.5. The $\pi_1(C \times C)$-action on $\Delta \times \Delta$ is free, so $\pi_1(C \times C) \cap \mathrm{Stab}(x) = \{1\}$, this gives that the restriction of ϑ to the stabilizer of a point $x \in \Delta \times \Delta$ is an isomorphism onto $\mathrm{Stab}_G(\mathcal{U}(x))$.

Lemma 7.1.6. *The $\mathbb{G}$-action on $\Delta \times \Delta$ is discontinuous (see Definition 1.1.5).*

Proof. (i) By Remark 7.1.5, the restriction of ϑ to the stabilizer of x is injective, and so $\mathrm{Stab}(x)$ is finite since G is finite.

(ii) Let $x \in \Delta \times \Delta$ and let $y := \mathcal{U}(x) \in C \times C$; since G is finite and $C \times C$ is Hausdorff, there exists a neighborhood U' of y such that for any element $g \in G$ not in the stabilizer of y: $g(U') \cap U' = \emptyset$. Let V' be the connected component of $\mathcal{U}^{-1}(U')$ that contains x. Since $\mathcal{U}\colon \Delta \times \Delta \to C \times C$ is a covering, there is a connected neighborhood $V \subseteq V'$ of x which is mapped isomorphically by $\mathcal{U}$ onto its image. Shrinking it if necessary, we can assume that $\mathcal{U}(V) =: U \subseteq U'$ is $\mathrm{Stab}(y)$-invariant, and so V is $\mathrm{Stab}(x)$-invariant.
Let $\overline{g} \in \mathbb{G} \setminus \mathrm{Stab}(x)$. We claim that $\overline{g}(V) \cap V = \emptyset$:

$$\begin{aligned} \mathcal{U}(\overline{g}(V) \cap V) &\subseteq \mathcal{U}(\overline{g}(V)) \cap \mathcal{U}(V) \\ &= \vartheta(\overline{g})\mathcal{U}(V) \cap \mathcal{U}(V) \\ &= \vartheta(\overline{g})U \cap U \end{aligned}$$

Then we have $\vartheta(\overline{g}) \in \mathrm{Stab}(y)$, by Remark 7.1.5, there exists a unique $\overline{g}' \in \mathrm{Stab}(x)$ such that $\vartheta(\overline{g}') = \vartheta(\overline{g})$.
By assumption, $\overline{g} = k\overline{g}'$, with $k \in \pi_1(C \times C) \setminus \{1\}$, we get:

$$\begin{aligned} \overline{g}(V) \cap V &= k\overline{g'}(V) \cap V \\ &= k(V) \cap V \\ &= \emptyset \end{aligned}$$

□

Lemma 7.1.7. *The normal subgroup $\mathbb{G}'$ of $\mathbb{G}$ generated by the elements which have non-empty fixed-point set is exactly* $\mathrm{Tors}(\mathbb{G})$.

Proof. To prove our claim we show that each element $g \in \mathbb{G}$ of finite order has non-empty fixed-point set, and vice versa. We distinguish two cases:

(i) $g \in \mathbb{H} \subset \mathbb{G}$. Let $g = (h_1, h_2)$ for some $h_1, h_2 \in \mathbb{T}$ that fixes a point $(x, y) \in \Delta \times \Delta$:

$$(h_1, h_2)(x, y) = (x, y) \iff \begin{cases} h_1 = \alpha c_i^{n_i} \alpha^{-1} \\ h_2 = \beta c_j^{n_j} \beta^{-1} \end{cases} \iff (h_1, h_2) \text{ has finite order;}$$

the first equivalence follows by Lemma 2.5.4, while for the second see [Bea83].

(ii) $g \in \mathbb{G} \setminus \mathbb{H}$. Let $g = \tilde{\tau}'(h_1, h_2)$ for some $h_1, h_2 \in \mathbb{T}$.
If g fixes a point $(x, y) \in \Delta \times \Delta$, also $g^2 \in \mathbb{H}$ fixes the point, by (i) it has finite order, then g has finite order.
Conversely, if g has finite order, $g^2(x, y) = (x, y)$ for some $(x, y) \in \Delta \times \Delta$ since $g^2 \in \mathbb{H}$ has finite order:

$$(x, y) = g^2(x, y) = (\tilde{\tau}'(h_1, h_2))^2(x, y) = ((h_2 t h_1) \cdot x, (t h_1 h_2) \cdot y)\,,$$

hence $(t h_1)x = (h_2^{-1})x$. It follows that $g(x, (h_2^{-1})x) = (x, (h_2^{-1})x)$.

□

Proof of Theorem 7.1.2. Because of Lemma 7.1.6, the main theorem in [Arm68] applies and we get:

$$\pi_1\left(\frac{C \times C}{G}\right) \cong \pi_1\left(\frac{\Delta \times \Delta}{\mathbb{G}}\right) \cong \frac{\mathbb{G}}{\mathbb{G}'}$$

where $\mathbb{G}'$ is the normal subgroup of $\mathbb{G}$ generated by the elements which have non-empty fixed-point set, which is exactly $\mathrm{Tors}(\mathbb{G})$ by Lemma 7.1.7:

$$\pi_1\left(\frac{C \times C}{G}\right) \cong \frac{\mathbb{G}}{\mathrm{Tors}(\mathbb{G})}\,.$$

□

In order to write a MAGMA script that calculates the fundamental group, we have to find a finite set of generators of $\mathrm{Tors}(\mathbb{G})$.

Proposition 7.1.8. *Let $X = (C \times C)/G$ be a regular mixed q.e. surface determined by the spherical system of generators $(h_1, \ldots, h_r)$ and let*

$$\psi\colon \mathbb{T}(m_1, \ldots, m_r) \to G^0$$

be the appropriate orbifold homomorphism. Fix $\tau' \in G \backslash G^0$; let $\tau = \tau'^2 \in G^0$ and let $\varphi \in \mathrm{Aut}(G^0)$ defined by $\varphi(h) := \tau' h \tau'^{-1}$. Then

$$\pi_1(X) \cong \frac{\mathbb{G}}{\mathrm{Tors}(\mathbb{G})}$$

and $\mathbb{G}' = \mathrm{Tors}(\mathbb{G})$ is normally generated by the finite set $T_1 \cup T_2$ where:

- *$T_1 \subset \mathbb{H}$: for every $i, j \in \{1, \ldots, r\}$, $1 \leq \alpha \leq m_i - 1$ and $1 \leq \beta \leq m_j - 1$, if h_i^α is conjugated to $\varphi^{-1}(h_j^\beta)$, then we choose an element $v \in G^0$ such that $v\, h_i^\alpha\, v^{-1} = \varphi^{-1}(h_j^\beta)$. Then for every element d in the finite group $Z(h_i^\alpha)$ we choose an element $w \in \psi^{-1}(v \cdot d)$ and we include $(w\, c_i^\alpha\, w^{-1}, c_j^\beta)$ in T_1.*
- *$T_2 \subset \mathbb{G} \setminus \mathbb{H}$: for every $i, \in \{1, \ldots, r\}$, $1 \leq \alpha \leq m_i - 1$ and $h \in G^0$, if $(\tau' h)^2$ is conjugated to h_i^α, then we choose an element $v \in G^0$ such that $v\, h_i^\alpha\, v^{-1} = (\tau' h)^2$ and we choose $g_1 \in \psi^{-1}(h)$ and $g_2 \in \psi^{-1}(\varphi(h))$. Then for every element d in the finite group $Z(h_i^\alpha)$ we choose an element $w \in \psi^{-1}(v \cdot d)$, and we include $\tilde{\tau}'(g_1, k g_2)$ in T_2 , where $k := (g_2\, t\, g_1)^{-1} w c_i^\alpha w^{-1}$.*

Proof. Let $(g_1, g_2) \in \mathbb{H} \subset \mathbb{G}$ and assume that there exist $x, y \in \Delta$ such that $(g_1, g_2)(x, y) = (x, y)$. We have that $(g_1, g_2) = (a c_i^\alpha a^{-1}, b c_j^\beta b^{-1})$ for some $a, b \in \mathbb{T}$. Since there is an element in $\mathbb{H}$ of the form (f, b), we can say that every element that stabilizes some points is conjugate to an element of the form $(z c_i^\alpha z^{-1}, c_j^\beta)$.
The elements $z \in \mathbb{T}$ such that $(z c_i^\alpha z^{-1}, c_j^\beta) \in \mathbb{H}$ are infinite, but

$$\vartheta(z c_i^\alpha z^{-1}, c_j^\beta) = v h_i^\alpha v^{-1} = \varphi^{-1}(h_j^\beta)$$

for some $v \in G^0$. Let v be a fixed element of G^0 such that

$$v h_i^\alpha v^{-1} = \varphi^{-1}(h_j^\beta)\,,$$

the other $v' \in G^0$ with $v' h_i^\alpha v'^{-1} = \varphi^{-1}(h_j^\beta)$ are of the form $v \cdot d$ for some $d \in Z(h_i^\alpha)$.

Let w be a preimage via ψ of $v \cdot d$ then $(w c_i^\alpha w^{-1}, c_j^\beta) \in \mathbb{H}$; if we pick another preimage w' of $v \cdot d$, then $w = k w'$ with $k \in \ker \psi$, but $(k, 1) \in \mathbb{H}$, so $(w c_i^\alpha w^{-1}, c_j^\beta)$ and $(w' c_i^\alpha w'^{-1}, c_j^\beta)$ are conjugated in $\mathbb{H}$, so it suffices to take a preimage w of $v \cdot d$ for each $d \in Z(h_i^\alpha)$ that are finitely many.

We note that every element in $\mathbb{H}$ that stabilizes some points in $\Delta \times \Delta$ belongs to the subgroup of $\mathbb{H}$ generated by T_1 that is $\mathrm{Tors}(\mathbb{H})$.

Let $h \in G^0$ such that $\tau' h(x, y) = (x, y)$ for some $(x, y) \in C \times C$, i.e.

$$\tau' h(x,y) = (x,y) \Longleftrightarrow \begin{cases} x = \varphi(h)\, y \\ y = \tau h\, x \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} x = (\tau' h)^2 x \\ y = \tau h\, x \end{cases}$$

So $\tau' h$ stabilizes some points in $C \times C$ if and only if $(\tau' h)^2$ is conjugated to h_i^α for some $1 \le i \le r$ and $1 \le \alpha \le m_i - 1$.
Fix $g_1 \in \psi^{-1}(h)$ and $g_2 \in \psi^{-1}(\varphi(h))$, so the preimages of $\tau' h$ are of the form $\tilde{\tau}'(g_1 k_1, g_2 k_2)$, where k_1, $k_2 \in \ker \psi$, but up to coniugation with $(k_1, 1) \in \mathbb{H}$, we can assume that the preimages are of the form $\tilde{\tau}'(g_1, k g_2)$ with $k \in \ker \psi$.
We claim that for each point $(x,y) \in C \times C$ stabilized by $\tau' h$, there exists $k \in \ker \psi$ such that $\tilde{\tau}'(g_1, k g_2)(x_0, y_0) = (x_0, y_0)$ for some $(x_0, y_0) \in \Delta \times \Delta$, i.e.

$$\begin{cases} x_0 = k g_2\, y_0 \\ y_0 = t g_1\, x_0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} x_0 = k g_2\, y_0 \\ x_0 = k g_2 t g_1\, x_0 \end{cases}$$

Let $s := g_2 t g_1 \in \mathbb{T}$, we have that

$$\psi(s) = (\tau' h)^2 = v h_i^\alpha v^{-1}$$

for some $v \in G^0$. For any $d \in Z(h_i^\alpha)$, let w be a preimage of $v \cdot d$ via ψ, so $s = w c_i^\alpha w^{-1} k'$ where $k' \in \ker \psi$. We define

$$k := (k')^{-1} = s^{-1} w c_i^m w^{-1}\,,$$

hence ks is conjugated to c_i^α and so it stabilizes some point $x_0 \in \Delta$ and the same goes for $\tilde{\tau}'(g_1, k g_2)$ that stabilizes $(x_0, (k g_2)^{-1} x_0) \in \Delta \times \Delta$, moreover $\mathcal{U}(x_0, (k g_2)^{-1} x_0) = (x,y)$. We include $\tilde{\tau}'(g_1, k g_2)$ in T_2.

To complete the proof we have to show that every element in $\mathbb{G} \setminus \mathbb{H}$ that stabilizes some points in $\Delta \times \Delta$ belongs to the subgroup normally generated by $T_1 \cup T_2$

Let $\tilde{\tau}'(h_1, h_2) \in \mathbb{G}$ be an element that stabilizes a point $(x_1, y_1) \in \Delta \times \Delta$, so $\theta(\tilde{\tau}'(h_1, h_2))$ stabilizes the point $\mathcal{U}(x_1, y_1) \in C \times C$. By the above construction, there exists $g \in T_2$ such that $g(x_0, y_0) = (x_0, y_0)$ with $\mathcal{U}(x_0, y_0) = \mathcal{U}(x_1, y_1)$; by construction, there exists $g' \in \mathbb{G}$ such that $g'(x_0, y_0) = (x_1, y_1)$. It follows that $g' g g'^{-1}(x_1, y_1) = (x_1, y_1)$, and so $\tilde{\tau}'(h_1, h_2)$ and $g' g g'^{-1} \in \mathrm{Stab}(x_1, y_1) \setminus \mathbb{H}$. By remark 7.1.5, there exists $h \in \mathrm{Stab}_{\mathbb{H}}(x_1, y_1) \subset \mathrm{Tors}(\mathbb{H})$ such that $\tilde{\tau}'(h_1, h_2) = h g' g g'^{-1}$. Noting that $\mathrm{Tors}(\mathbb{H})$ is normally generated by T_1, we are done. $\square$

In order to compute the fundamental group of the surfaces we have constructed, we have developed a MAGMA script (see Section 6.4) that implements these results. We have run it on the constructed surfaces, the outputs are collected in Table 7.1.
In the first column we report the value K_S^2 of the self-intersection on the

canonical class of the surface, Sing(X) represents the basket of singularities of X. The column Type gives the type of the set of spherical generators of G^0 (see Section 2.3) in a compacted way, e.g. $2^3, 4 = (2, 2, 2, 4)$. The columns G and G^0 give the group and its index two subgroup. The column $b_2(X)$, $H_1(S, \mathbb{Z})$, and $\pi_1(S)$ give respectively the second Betti number of X, the first homology group and the fundamental group of S. The last column gives a label, referring to a subsection of Section 7.3, where we give more details on each construction.

Remark 7.1.9. All the smooth surfaces in Table 7.1 have non trivial topological fundamental group and $K^2 > 0$, so they are surfaces of general type.

Remark 7.1.10. We point out that the surfaces 7.3.4 and 7.3.7 are numerical Campedelli surfaces ($K_S^2 = 2$) with topological fundamental group (and therefore algebraic fundamental group) $\mathbb{Z}_4$. We discussed the importance of these surfaces in Section 3.6.1.

Remark 7.1.11. We have constructed 2 new topological types of surfaces of general type with $p_g = 0$. These surfaces are tagged by 7.3.10 and 7.3.12.

Remark 7.1.12. The surface tagged by 7.3.11 has $K_S^2 = 4$ and the same fundamental group of a Keum-Naie surface (see [Nai94] and [BC11]), as the following MAGMA script shows:

```
> G:=SmallGroup(32,22);
> seq:=[G.2*G.5, G.2*G.3, G.2*G.4,G.2*G.3*G.5,G.4];
> P:=Pi1(seq,G);
> F<a,b,c,d,s,t>:=FreeGroup(6);
>  rel:=[(a,b),(a,c),(a,d),(b,c),(b,d),(c,d),s^2,t^2*b^-1,
> (t,a),(t,b), (s^-1,a^-1)*a^2, (s^-1,b^-1)*b^2,
> (s^-1,c^-1)*c^2, (s^-1,d^-1)*d^2,(t^-1,c^-1)*c^2,
> (t^-1,d^-1)*d^2, (t^-1,s^-1)*(d^-1)*(b^-1)];
> E:=Simplify(quo<F|rel>);
> SearchForIsomorphism(E,P,5);
true Homomorphism of GrpFP: E into GrpFP: P induced by
    E.1 |--> P.3
    E.2 |--> P.2
    E.3 |--> P.1
    E.4 |--> P.4
Homomorphism of GrpFP: P into GrpFP: E induced by
    P.1 |--> E.3
    P.2 |--> E.2
    P.3 |--> E.1
    P.4 |--> E.4
>
```

We expect that this surface belongs to the family studied in [BC11] but we have not proved it.

Remark 7.1.13. There has been a growing interest for surfaces of general type with $p_g = 0$ having an involution, see [CCML07], [CMLP08], [Rit10a] and [LS10].

K_S^2	Sing(X)	Type	G^0	G	$b_2(X)$	$H_1(S,\mathbb{Z})$	$\pi_1(S)$	Label
1	$2\,C_{2,1}+2\,D_{2,1}$	$2^3\,,4$	$D_4\times\mathbb{Z}_2$	$\mathbb{Z}_2^3\rtimes\mathbb{Z}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$	7.3.1
2	$6\,C_{2,1}$	2^5	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\rtimes\mathbb{Z}_4$	2	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_4$	7.3.2
2	$6\,C_{2,1}$	4^3	$(\mathbb{Z}_2\times\mathbb{Z}_4)\rtimes\mathbb{Z}_4$	G(64, 82)	2	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	7.3.3
2	$C_{2,1}+2\,D_{2,1}$	$2^3\,,4$	$\mathbb{Z}_2^4\rtimes\mathbb{Z}_2$	$\mathbb{Z}_2^4\rtimes\mathbb{Z}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$	7.3.4
2	$C_{2,1}+2\,D_{2,1}$	$2^2\,,3^2$	$\mathbb{Z}_3^2\rtimes\mathbb{Z}_2$	$\mathbb{Z}_3^2\rtimes\mathbb{Z}_4$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$	7.3.5
2	$2\,C_{4,1}+3\,C_{2,1}$	$2^3\,,4$	G(64, 73)	G(128, 1535)	3	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	7.3.6
2	$2\,C_{3,1}+2\,C_{3,2}$	$3^2\,,4$	G(384, 4)	G(768, 1083540)	2	$\mathbb{Z}_4$	$\mathbb{Z}_4$	7.3.7
2	$2\,C_{3,1}+2\,C_{3,2}$	$3^2\,,4$	G(384, 4)	G(768, 1083541)	2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	7.3.8
3	$C_{8,3}+C_{8,5}$	$2^3\,,8$	G(32, 39)	G(64, 42)	2	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_4$	7.3.9
4	$4\,C_{2,1}$	2^5	$D_4\times\mathbb{Z}_2$	$D_{2,8,5}\rtimes\mathbb{Z}_2$	2	$\mathbb{Z}_2\times\mathbb{Z}_8$	$\mathbb{Z}_2^2\rtimes\mathbb{Z}_8$	7.3.10
4	$4\,C_{2,1}$	2^5	$\mathbb{Z}_2^4$	$(\mathbb{Z}_2^2\rtimes\mathbb{Z}_4)\times\mathbb{Z}_2$	2	$\mathbb{Z}_2^3\times\mathbb{Z}_4$	K-N	7.3.11
4	$4\,C_{2,1}$	4^3	G(64, 23)	G(128, 836)	2	$\mathbb{Z}_2^3$	$\mathbb{Z}_4^2\rtimes\mathbb{Z}_2$	7.3.12
8	$\emptyset$	2^5	$D_4\times\mathbb{Z}_2^2$	$(D_{2,8,5}\rtimes\mathbb{Z}_2)\times\mathbb{Z}_2$	2	$\mathbb{Z}_2^3\times\mathbb{Z}_8$	$1\to\Pi_{17}\times\Pi_{17}\to\pi_1\to G\to 1$	7.3.13
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	$\mathbb{Z}_4^3$	$1\to\Pi_9\times\Pi_9\to\pi_1\to G\to 1$	7.3.14
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	$\mathbb{Z}_2^4\times\mathbb{Z}_4$	$1\to\Pi_9\times\Pi_9\to\pi_1\to G\to 1$	7.3.15
8	$\emptyset$	4^3	G(128, 36)	G(256, 3678)	2	$\mathbb{Z}_2^2\times\mathbb{Z}_4^2$	$1\to\Pi_9\times\Pi_9\to\pi_1\to G\to 1$	7.3.16
8	$\emptyset$	4^3	G(128, 36)	G(256, 3679)	2	$\mathbb{Z}_2^2\times\mathbb{Z}_4^2$	$1\to\Pi_9\times\Pi_9\to\pi_1\to G\to 1$	7.3.17

Table 7.1: The surfaces and their fundamental group. See Section 7.3 for a detailed description.

The "intermediate" surface $Y = (C \times C)/G^0$ has an involution induced by the action of G. The surface Y has $p_g = 0$ in the cases 7.3.1, 7.3.4 and 7.3.5, and $p_g = 1$ in the others, see Lemma 6.1.5.

The numerical Godeaux surface ($K_S^2 = 1$) tagged by 7.3.1 is obtained as minimal desingularization $S \to X$ of the mixed q.e. surface $X = (C \times C)/G$. The surface $Y = (C \times C)/G^0$ has 6 nodes and $K_Y^2 = 2$, moreover its desingularization T inherits an involution ν from the involution acting on Y and has $K_T^2 = 2$, hence we have a numerical Campedelli surface with an involution. By construction, the involution fixes 4 points on T, by [CMLP08, Proposition 2.3] in this case the involution is not composed with the bicanonical map $\varphi\colon T \to \mathbb{P}^2$. By construction S is also the desingularization of $T/\langle \nu \rangle$, this means that S is an example of the case (i) of [CMLP08, Proposition 4.3].

In the cases 7.3.4 and 7.3.5, Y is a surface with $K^2 = 4$, $p_g = 0$ and 4 nodes. These surfaces are the quotient models of two product-quotient surfaces constructed in [BCGP08].

7.2 Determining the minimal model

In this section we want to determine the minimal model of the surfaces we have constructed, we follow the ideas of [BP10, Section 4]. We recall the following diagram:

$$
\begin{tikzcd}
& C \times C \arrow[dl, "p_1"'] \arrow[dd, "\sigma"] \arrow[dr, "p_2"] & \\
C \arrow[dd, "c"'] & & C \arrow[dd, "c"] \\
& Y = (C \times C)/G^0 \arrow[dl] \arrow[dr] \arrow[dd, "\pi"] & \\
C/G^0 & & C/G^0 \\
& X = (C \times C)/G &
\end{tikzcd}
\tag{7.3}
$$

Assume that $\Gamma \subset X$ is a (possibly singular) rational curve. Let $\Gamma' := (\pi \circ \sigma)^*(\Gamma) = \sum_1^k n_i \Gamma_i$ be the decomposition in irreducible components of its pull back to $C \times C$. We observe that $n_i = 1$ $\forall i$ (since $\pi \circ \sigma$ is quasi-étale), and that G acts transitively on the set $\{\Gamma_i \mid i = 1, \dots, k\}$. Hence there is a subgroup $H \leq G$ of index k acting on Γ_1 such that $\pi(\sigma(\Gamma_1)) = \Gamma_1/H = \Gamma$.

Normalizing Γ_1 and Γ, we get the following commutative diagram:

$$\begin{tikzcd} \tilde{\Gamma}_1 \arrow[r, "\alpha"] \arrow[d, "f"'] & \Gamma_1 \arrow[r, "\beta"] \arrow[d] & C \times C \arrow[d] \\ \mathbb{P}^1 \arrow[r, "\nu"] & \Gamma \arrow[r, hook] & X \end{tikzcd}$$

Since each automorphism lifts to the normalization, H acts on $\tilde{\Gamma}_1$ and f is the quotient map $\tilde{\Gamma}_1 \to \tilde{\Gamma}_1/H \cong \mathbb{P}^1$.

Moreover we have that $p_i \circ \beta \circ \alpha \colon \Gamma_1 \to C$ is surjective for at least one $i \in \{1, 2\}$, otherwise $\beta(\alpha(\Gamma_1))$ is a point. Hence we have that $g(\Gamma_1) \geq g(C) \geq 2$ and so by Corollary 2.2.14 f is branched in at least 3 points.

Lemma 7.2.1. *Let p be a branch point of f, then $\nu(p)$ is a singular point of X.*

Proof. Let $p' \in f^{-1}(p) \subset \tilde{\Gamma}_1$ be a ramification point of f, then $\mathrm{Stab}_H(p') := H_1 \neq \{1\}$ and so $\mathrm{Stab}_G(\beta(\alpha(p'))) \supseteq H_1$. Hence $\nu(f(p')) = \nu(p) \in \mathrm{Sing}(X)$. □

Corollary 7.2.2. *Any rational curve in X passes at least 3 times through singular points.*

Lemma 7.2.3 (cf. [Bom73, Proposition 1])**.** *On a smooth surface S of general type every irreducible curve C satisfies $K_S.C \geq -1$.*

Proof. If an irreducible curve C on a surface S satisfies $C^2 \geq 0$, it is clear that $C.D \geq 0$ for every effective divisor D of S. Since $|mK_S|$ is not empty for m large, there exists an effective divisor E linearly equivalent to mK_S and so $mK_S.C = E.C \geq 0$. Hence if $K_S.C < 0$ then $C^2 < 0$ and so:

$$2g(C) - 2 = K_S.C + C^2 \leq -2\,.$$

Since C is irreducible we get

$$g(C) = 0 \qquad C^2 = -1 \text{ and } K_S.C = -1\,.$$

□

Lemma 7.2.4 ([BP10, Remark 4.3])**.** *On a smooth surface S of general type every irreducible curve C with $K_S.C \leq 0$ is smooth and rational.*

Proof. Consider the morphism $f \colon S \to M$ to its minimal model. Assume that there is an irreducible curve $C \subset S$ with $K_S.C \leq 0$ which is either singular or irrational. Then C is not contracted by f and $C' := f(C)$ is a still singular resp. irrational curve with $K_M.C' \leq K_S.C \leq 0$ which implies (see [Bom73, Proposition 1]) that C' is a smooth rational curve of self-intersection (-2), a contradiction. □

Proposition 7.2.5. *Let S be a smooth surface of general type. Assume that E is a (-1)-curve in S, then $C.E \leq 1$ for every rational curve C in S with $C^2 \in \{-2, -3, -4\}$.*

Proof. Assume that $C.E = n \geq 2$ and $C^2 = -\alpha \in \{-2, -3, -4\}$. Let $f\colon S \to S'$ be the blow-down given by the contraction of E and let $C' := f(C)$ that is a singular curve since $n \geq 2$.
Since $K_S.C = -C^2 - 2 = \alpha - 2$, $K_S.E = -2 - E^2 = -1$ and $\deg f = 1$, we get

$$\begin{aligned} K_{S'}.C' &= f^*(K_S).f^*(C') = (K_S - E).(C + nE) \\ &= K_S.C - C.E + nK_S.E + nE^2 \\ &= \alpha - 2 - n \end{aligned}$$

Since $n \geq 2$ then $K_{S'}.C' \leq 0$ and so C' is smooth, a contradiction. □

Lemma 7.2.6. *Let S be a smooth surface of general type. Assume that E is a (-1)-curve in S, then E intersects at most one (-2) curve.*

Proof. Suppose E intersects two (-2) curves, contracting E we get two (-1) curves intersecting in a point. Pick one of these curves and contract it, we get a surface S' and a (0)-curve C' such that $K_{S'}.C' + C'^2 = -2$ and so $K_{S'}.C' = -2$, but C' is irreducible and so $K_{S'}.C' \geq -1$, a contradiction. □

Proposition 7.2.7. *Let S be a surface of general type that is the minimal resolution of singularities of the mixed q.e. surface X. If X has only R.D.P. as singularities then S is minimal.*

Proof. We recall that the minimal resolution of a R.D.P. is a tree of (-2)-curves. If S is not minimal there is a (-1)-curve, and this curve intersects three different (-2)-curves by Corollary 7.2.2, but this contradicts Lemma 7.2.6. □

We need the following classical results.

Theorem 7.2.8 (see [Bom73, Proposition 1])**.** *If S is a minimal surface of general type, then the (-2)-curves form a finite set and they are numerically independent on S.*

Lemma 7.2.9 (see [BHPV04, Proposition VII.2.5])**.** *If S is a minimal surface of general type, then the intersection form restricted to the (-2) curves is negative-definite.*

Definition 7.2.10 (see [Bom73] and [BCP11, Definition 3.7])**.** The *canonical model* of a surface S of general type is the normal surface S_{can} obtained from the minimal model S_{min} of S contracting all the (-2)-curves.

Proposition 7.2.11. *Let S be a surface of general type. Let $E_1, \ldots, E_n$ be (-2)-curves on S and let $M = (m_{ij})_{i,j}$ be the matrix given by $m_{ij} = E_i.E_j$. Then M is negative-definite.*

Proof. Let S_{min} be the minimal model of S and let S_{can} its canonical model; hence

$$S \xrightarrow{\pi} S_{min} \xrightarrow{\rho} S_{can},$$

where π is a birational morphism and ρ is the contraction of all the (-2)-curves of S.

Up to relabel the curves, we can assume that $\pi(E_i) = \{pt.\}$ for $i \leq r$, while $\pi(E_i) = F_i$ is a curve for $i > r$.

Let $i > r$, then F_i is a (-2)-curves; indeed if E_i intersects at least an exceptional curve of π, then $K_{S_{min}}.F_i < K_S.E_i = 0$, hence the canonical divisor is not nef and S_{min} is not minimal, a contradiction. In particular $\pi^*(F_i) = E_i$. Moreover $E_i.E_j = 0$ if $i \leq r$ and $j > r$.
We note that $\{E_1, \ldots, E_r\}$ are independent in $H^2(S)$ since they are the exceptional curves of π (see Proposition 3.3.4).
Since S_{min} is minimal, by Theorem 7.2.8, $\{F_r, \ldots, F_n\}$ are independent in $H^2(S_{min})$. Since $H^2(S_{min}) \overset{\pi^*}{\hookrightarrow} H^2(S)$ we get that $\{E_r, \ldots, E_n\}$ are independent in $H^2(S)$.

The intersection form is non-degenerate, $\{E_1, \ldots, E_r\}$ and $\{E_r, \ldots, E_n\}$ are independent and $E_i.E_j = 0$ if $i \leq r$ and $j > r$, hence $\{E_1, \ldots, E_n\}$ is independent and they form a basis for $V := Span(E_1, \ldots, E_r) \subseteq H^2(S)$.

Since $(\pi^* K_{S_{min}})^2 > 0$, by Algebraic Index Theorem (see [BHPV04, Corollary IV.2.16]), we get that the intersection form restricted to V is negative-definite. $\square$

Corollary 7.2.12. *Let E_1 and E_2 be two (-2)-curves on a surface S of general type, then $E_1.E_2 \leq 1$.*

Proof. If $E_1.E_2 \geq 2$ then

$$\det \begin{pmatrix} E_1^2 & E_1.E_2 \\ E_1.E_2 & E_2^2 \end{pmatrix} = 4 - 2E_1.E_2 \leq 0$$

and so the intersection form is not negative definite, a contradiction. $\square$

Proposition 7.2.13. *Let S be a smooth surface of general type. Assume that E is a (-1)-curve in S, then E cannot intersect a (-2)-curve and two (-3)-curves.*

Proof. Aiming for a contradiction, let us assume that E intersects two (-3)-curves and a (-2)-curve E'. We contract E and then E', so we get two (-1)-curves E_1 and E_2, with $E_1.E_2 = 2$ on the surface S', moreover $K_{S'}.E_i = -1$.

Contracting one of them, say E_1, we get a curve E_2' on S'' such that

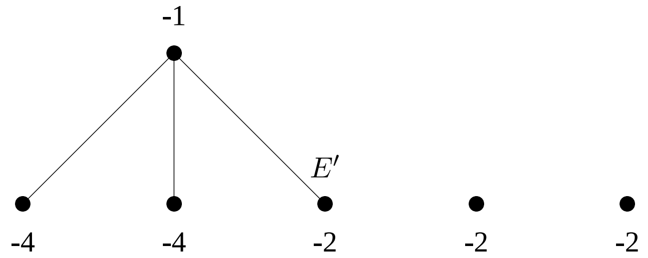
$$\begin{aligned} K_{S''}.E_2' \leq (K_{S'} - E_1).(E_2 + 2E_1) &= K_{S'}.E_2 - E_1.E_2 + 2K_{S'}.E_1 - 2E_1^2 \\ &= -1 - 2 - 2 + 2 = -3\,, \end{aligned}$$

but E_2' is irreducible and so $K_{S''}.E_2' \geq -1$, a contradiction. $\square$

Proposition 7.2.14. *Let S be a surface of general type that is the minimal resolution of singularities of the mixed q.e. surface X.*
If $\mathcal{B}(X) = \{2 \times C_{4,1}, 3 \times C_{2,1}\}$ then S is minimal.

Proof. The minimal resolution of the singularities in X is given by two (-4)-curves and three (-2)-curves that do not intersect each other.

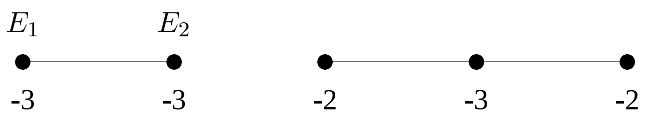
Assume that E is a (-1)-curve in S, since it has to intersect at least three exceptional curves and by Lemma 7.2.6 it cannot intersect more that one (-2)-curve. There is only one possible configuration of rational curves on S; its dual graph is:



After the contraction of the (-1)-curve we get that E' is a (-1)-curve. Contracting it we get two (-2)-curves E_1 and E_2, with $E_1.E_2 = 2$ on the surface S', contradicting Corollary 7.2.12. $\square$

Proposition 7.2.15. *Let S be a surface of general type that is the minimal resolution of singularities of the mixed q.e. surface X.*
If $\mathcal{B}(X) = \{C_{8,3}, C_{8,5}\}$ then S is minimal.

Proof. The minimal resolution of the singularities in X is given by two (-3)-curves intersecting in a point and two (-2)-curves intersecting in two different points a further (-3)-curve. The dual graph is:



Assume that E is a (-1)-curve in S, it has to intersect at least three exceptional curves and it cannot intersect more that one (-2)-curve.

Moreover, by Proposition 7.2.13, E cannot intersect a (-2)-curve and two (-3)-curves, so E intersects the three (-3)-curves. We claim that this is not possible; indeed contracting E we get two (-2)-curves E_1 and E_2 with $E_1.E_2 = 2$, contradicting Corollary 7.2.12. $\square$

Proposition 7.2.16. *Let S be a surface of general type that is the minimal resolution of singularities of the mixed q.e. surface X.*

If $\mathcal{B}(X) = \{2 \times C_{3,1},\ 2 \times C_{3,2}\}$ then S is minimal.

Proof. The minimal resolution of the singularities in X is given by two (-3)-curves and two pairs of (-2)-curves intersecting in a point. The dual graph is:

Assume that E is a (-1)-curve in S, it has to intersect at least three exceptional curves and it cannot intersect more that one (-2)-curve, thank to Proposition 7.2.5 and Lemma 7.2.6. So, the only possibility left is that E intersect both the (-3)-curves and just one of the four (-2)-curves, contradicting Proposition 7.2.13. □

Corollary 7.2.17. *If S is the minimal resolution S of the singularities of a mixed q.e. surface X with $p_g(S) = q(S) = 0$ and $K_S^2 > 0$, then S is minimal.*

7.3 The surfaces

In this section we give a detailed description of the surfaces collected in Table 6.1. We will follow the scheme below:
G: the Galois group.
G^0: the index 2 subgroup of the elements that do not exchanges the factors.
In the follow $\mathfrak{S}_n$ will denote the symmetric group in n letters, $D_{p,q,r}$ the generalized dihedral group with presentation: $D_{p,q,r} = \langle x, y | x^p, y^q, xyx^{-1}y^{-r} \rangle$ and $D_n := D_{2,n,-1}$ is the usual dihedral group of order $2n$.
T: the type of the system of spherical generators.
L: here we list the set of elements of G that is a spherical generators system for G^0 that gives the curve C.
H_1: the first homology group of the surface.
π_1: the fundamental group of the surface.

$K^2 = 1$, basket $\{2 \times C_{2,1},\ 2 \times D_{2,1}\}$

7.3.1. Galois group $(\mathbb{Z}_2)^3 \rtimes_\varphi \mathbb{Z}_4 \,:\, \varphi(1) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

G: $\langle (2,5,6,8)(3,7),\ (1,2)(3,5)(4,6)(7,8),\ (1,3)(2,5)(4,7)(6,8),$
$(2,6)(5,8),\ (1,4)(2,6)(3,7)(5,8) \rangle < \mathfrak{S}_8$

G^0: $D_4 \times \mathbb{Z}_2$

T: $(2,2,2,4)$

L: $(1,8)(2,7)(3,6)(4,5)$, $(1,7)(2,8)(3,4)(5,6)$, $(1,3)(2,8)(4,7)(5,6)$, $(1,5,4,8)(2,7,6,3)$

H_1: $\mathbb{Z}_4$

π_1: $\mathbb{Z}_4$

$K^2 = 2$, basket $\{6 \times C_{2,1}\}$

7.3.2. Galois group $(\mathbb{Z}_2)^2 \rtimes_\varphi \mathbb{Z}_4$: $\varphi(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

G: $\langle(1,2,4,6)(3,5,7,8),\ (2,5)(6,8),\ (1,3)(2,5)(4,7)(6,8),\ (1,4)(2,6)(3,7)(5,8)\rangle < \mathfrak{S}_8$

G^0: $(\mathbb{Z}_2)^3$

T: $(2,2,2,2,2)$

L: $(1,3)(4,7)$, $(1,7)(2,6)(3,4)(5,8)$, $(1,3)(2,5)(4,7)(6,8)$, $(2,5)(6,8)$, $(1,7)(2,6)(3,4)(5,8)$

H_1: $\mathbb{Z}_2 \times \mathbb{Z}_4$

π_1: $\mathbb{Z}_2 \times \mathbb{Z}_4$

7.3.3. Galois group: $G(64,82)$: Sylow 2-subgroup of the Suzuki group $Sz(8)$,

G: $\langle g_1,\ g_2,\ g_3 \mid g_3^4,\ g_2^4,\ g_1^4,\ g_1g_3g_1^{-1}g_3g_2^2,\ g_2^{-2}g_3^{-1}g_1^{-1}g_3^{-1}g_1,\ g_2g_3g_1^2g_2g_3^{-1},\ g_1^{-1}g_3^2g_2g_1g_2^{-1},\ g_2^{-1}g_3^2g_2g_3^2,\ g_1^{-2}g_3^{-1}g_2g_3g_2\rangle$

G^0: $G(32,2)$: $\langle h_1, h_2 \mid h_1^4,\ h_2^4,\ h_2^{-1}h_1^{-2}h_2h_1^{-2},\ h_2^{-2}h_1h_2^{-2}h_1^{-1},\ (h_1h_2h_1^{-1}h_2)^2,\ (h_2^{-1}h_1h_2h_1)^2,\ h_1^{-2}h_2^-3h_1^{-2}h_2^{-1},\ (h_2, h_1^{-1})^2\rangle$

it is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes_\varphi \mathbb{Z}_4$ where $\varphi(1) = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$

T: $(4,4,4)$

L: g_3^{-1}, $g_1g_3^{-2}$, $g_1g_3g_2^{-2}g_3^2g_2^2g_1^{-2}$

H_1: $(\mathbb{Z}_2)^3$

π_1: $(\mathbb{Z}_2)^3$

$K^2 = 2$, basket $\{C_{2,1},\ 2 \times D_{2,1}\}$

7.3.4. Galois group: $(\mathbb{Z}_2)^4 \rtimes_\varphi \mathbb{Z}_4$: $\varphi(1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

G: $\langle (2,6,7,12)(3,9,10,16)(4,11)(8,14,15,13),$
$(1,2)(3,6)(4,7)(5,8)(9,13)(10,14)(11,15)(12,16),$
$(1,3)(2,6)(4,9)(5,10)(7,13)(8,14)(11,16)(12,15),$
$(2,7)(3,10)(6,12)(8,15)(9,16)(13,14),$
$(1,4)(2,7)(3,9)(5,11)(6,13)(8,15)(10,16)(12,14),$
$(1,5)(2,8)(3,10)(4,11)(6,14)(7,15)(9,16)(12,13)\rangle < \mathfrak{S}_{16}$

G^0: $(\mathbb{Z}_2)^4 \rtimes_\psi \mathbb{Z}_2$, $\psi(1) = \varphi(2)$

T: $(2,2,2,4)$

L: $(2,7)(3,10)(6,12)(8,15)(9,16)(13,14),$
$(1,16)(2,12)(3,11)(4,10)(5,9)(6,15)(7,14)(8,13),$
$(1,14)(2,10)(3,8)(4,12)(5,6)(7,16)(9,15)(11,13),$
$(1,2,4,7)(3,14,9,12)(5,8,11,15)(6,16,13,10)$

H_1: $\mathbb{Z}_4$

π_1: $\mathbb{Z}_4$

7.3.5. Galois group: $(\mathbb{Z}_3)^2 \rtimes_\varphi \mathbb{Z}_4$: $\varphi(1) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

G: $\langle (1,2)(3,4,5,6),\ (3,5)(4,6),\ (2,4,6),\ (1,3,5)(2,4,6)\rangle < \mathfrak{S}_6$

G^0: $(\mathbb{Z}_3)^2 \rtimes_\psi \mathbb{Z}_2$, $\psi(1) = \varphi(2)$

T: $(2,2,3,3)$

L: $(3,5)(4,6),\ (2,6)(3,5),\ (1,3,5),\ (1,5,3)(2,4,6)$

H_1: $\mathbb{Z}_3$

π_1: $\mathbb{Z}_3$

$K^2 = 2$, basket $\{2 \times C_{4,1},\ 3 \times C_{2,1}\}$

7.3.6. Galois group: $G(128, 1535)$

G: $\langle g_1,\ g_2,\ g_3,\ g_4 \mid g_1^{-1}g_4g_1g_4,\ g_4^4,\ (g_2^{-1}g_3^{-1})^2,\ g_2^4,\ (g_3, g_4^{-1}),\ (g_3^{-1}g_2)^2,$
$g_2^{-1}g_4g_2^{-1}g_4^{-1},\ g_1^{-1}g_2^{-1}g_1g_2^{-1},\ g_1^{-1}g_3^{-1}g_1^2g_3g_1^{-1},\ g_3^{-2}g_1g_3^2g_1^{-1},$
$g_4^{-2}g_1g_3g_2^2g_1^{-1}g_3^{-1},\ g_4^{-2}g_3^{-1}g_1g_3g_1^{-1}g_2^2,\ g_4^2g_1^{-2}g_3^{-1}g_2^2g_3^{-1},$

$g_4^{-1} g_1^{-1} g_2 g_3 g_2^{-1} g_4 g_3^{-1} g_1^{-1},\ g_4^{-2} g_1^3 g_4^{-2} g_1 \rangle$

G^0: $G(64,73)$: $\langle h_1,\ h_2,\ h_3 \mid h_1^2,\ h_2^2,\ h_3^2,\ (h_1h_3)^4,\ (h_1h_2)^4,$
$(h_2h_3)^4,\ (h_2h_3h_2h_1h_3)^2,\ (h_1h_2h_3h_1h_3)^2,\ (h_2h_1h_3)^4 \rangle$

T: $(2,2,2,4)$

L: $g_1g_3g_4^{-1}g_2^2,\ g_1g_3g_2^{-2}g_3^{-2}g_2^2,\ g_2g_3,\ g_2g_3g_4g_2^{-2}g_4^{-2}g_2^2g_3^{-2}g_2^2$

H_1: $(\mathbb{Z}_2)^3$

π_1: $(\mathbb{Z}_2)^3$

$K^2 = 2$, basket $\{2 \times C_{3,1},\ 2 \times C_{3,2}\}$

7.3.7. Galois group: $G(768, 1083540)$

G: $\langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9 \mid g_1^3,\ g_2^2(g_5g_6g_7)^{-1},\ g_3^2(g_5g_6),\ g_4^2(g_5)^{-1},$
$g_5^2,\ g_6^2,\ g_7^2,\ g_8^2,\ g_9^2,\ (g_2,\ g_1)(g_4g_6g_7g_9)^{-1},\ (g_3,\ g_1)(g_3g_7g_9)^{-1},$
$(g_3,\ g_2)g_5^{-1},\ (g_4,\ g_1)(g_8g_9)^{-1},\ (g_4,\ g_2)g_6^{-1},\ (g_4,\ g_3)g_7^{-1},$
$(g_5,\ g_1)(g_6g_7)^{-1},\ (g_5,\ g_2)g_8^{-1},\ (g_5,\ g_3)g_9^{-1},\ (g_6,\ g_1)g_8^{-1},$
$(g_6,\ g_2) = g_8g_9,\ (g_6,\ g_3)g_9^{-1},\ (g_6,\ g_4)g_8^{-1},\ (g_7,\ g_1)g_9^{-1},\ (g_7,\ g_2)g_9^{-1},$
$(g_7,\ g_3)g_8^{-1},\ (g_7,\ g_4)g_9^{-1},\ (g_8,\ g_1)g_9^{-1},\ (g_9,\ g_1)\rangle$

G^0: $G(384,4)$: $\langle h_1,\ h_2 \mid h_1^3,\ h_2^4,\ (h_2^{-1}h_1)^3,\ (h_2^{-1}h_1^{-1})^6,\ (h_2,h_1)^4,$
$h_1^{-1}h_2^{-2}h_1h_2^{-2}h_1^{-1}h_2^{-1}h_1^{-1}h_2h_1^{-1}h_2^{-1},$
$h_2^{-1}h_1h_2h_1h_2^{-1}h_1^{-1}h_2h_1h_2h_1^{-1}h_2^{-1}h_1^{-1}h_2h_1h_2^{-1}h_1^{-1},\ \rangle$

T: $(3,3,4)$

L: $g_1^2g_4g_9,\ g_1g_6g_7g_9,\ g_2g_5g_8$

H_1: $\mathbb{Z}_4$

π_1: $\mathbb{Z}_4$

7.3.8. Galois group: $G(768, 1083541)$

G: $\langle g_1,\ g_2,\ g_3 \mid g_1^3,\ g_3^4,\ g_2^4,\ g_2g_3g_1g_2^{-1}g_1^{-1}g_3,\ g_3^2g_2^2g_3^{-2}g_2^2,$
$g_3g_2^{-1}g_3g_1^{-1}g_2g_3^{-2}g_1,\ g_1^{-1}g_3^{-2}g_1g_2g_3g_2^{-1}g_3^{-1},$
$g_2g_3g_2^{-1}g_1g_2g_1^{-1}g_2^{-1}g_3,\ g_3g_2^2g_3g_1^{-1}g_2g_1g_2,\ (g_3^{-1}g_2^{-1}g_3g_2^{-1})^2,$
$g_2^{-1}g_3^{-1}g_1^{-1}g_3^2g_1g_2g_3,\ (g_3^{-1}g_2)^4,\ g_3g_1g_2^{-2}g_3^{-1}g_2^{-1}g_3^{-1}g_1^{-1}g_2^{-1}g_3,$
$g_3g_2^2g_3^{-1}g_1^{-1}g_3g_2^{-2}g_3^{-1}g_1,\ g_2g_1^{-1}g_2g_1g_3^{-1}g_2g_3g_1g_2g_1^{-1},$
$g_3^{-1}g_2^2g_1^{-1}g_3^{-1}g_1g_3^{-1}g_1^{-1}g_3g_1,\ g_3^{-1}g_2g_3^2g_2g_1^{-1}g_2^2g_1g_3^{-1},$
$g_1^{-1}g_2g_3^{-1}g_2g_3^{-1}g_1g_3^{-2}g_2^2,\ g_3g_1^{-1}g_3g_1g_3^{-1}g_2^{-2}g_3g_2^{-1}g_1g_3g_1^{-1},$
$g_2^{-1}g_1^{-1}g_2g_3^{-1}g_1g_3^{-1}g_1^{-1}g_3g_2^{-2}g_1g_3,$
$g_3^{-1}g_1^{-1}g_2^{-1}g_3^{-1}g_2^{-1}g_1g_3g_2^{-1}g_3^{-1}g_1g_3^{-2}g_2g_3^{-1}g_1^{-1}g_2^{-1}\rangle$

G^0: $G(384,4)$, as above.

T: $(3,3,4)$

L: $g_1^2g_2g_3g_2^{-2}g_3g_2^2g_3g_2^2g_1g_3^{-1}g_2^2g_3g_2^2g_1^{-1}$, $g_1g_2^3g_3g_2^2g_1g_3^{-1}g_2^2g_3g_2^2g_1^{-1}$,
$g_2g_3^{-1}g_1^{-1}g_3g_1g_2^{-1}g_3^{-2}g_2^{-1}g_3^{-1}g_2^3g_3g_2^2$

H_1: $(\mathbb{Z}_2)^2$

π_1: $(\mathbb{Z}_2)^2$

$K^2 = 3$, basket $\{C_{8,3},\, C_{8,5}\}$

7.3.9. Galois group: $G(64,42)$:

G: $\langle(1,2,3,5,8,13,6,10)(4,7,11,14,15,16,9,12),$
$(2,4)(3,6)(5,9)(7,12)(10,11)(13,15)(14,16)\rangle < \mathfrak{S}_{16}$

G^0: $G(32,39)$: $\langle(2,4)(5,7)(6,8)(9,11)(10,12)(13,15),$
$(1,2)(3,5)(4,6)(7,9)(8,10)(11,13)(12,14)(15,16),$
$(1,3)(2,5)(4,7)(6,9)(8,11)(10,13)(12,15)(14,16)\rangle < \mathfrak{S}_{16}$

T: $(2,2,2,8)$

L: $(2,13)(4,15)(5,10)(9,11)$,
$(1,7)(2,5)(3,12)(4,15)(6,14)(8,16)(10,13)$,
$(2,15)(3,6)(4,13)(5,11)(7,12)(9,10)(14,16)$,
$(1,7,3,14,8,16,6,12)(2,15,10,11,13,4,5,9)$

H_1: $\mathbb{Z}_2 \times \mathbb{Z}_4$

π_1: $\mathbb{Z}_2 \times \mathbb{Z}_4$

$K^2 = 4$, basket $\{4 \times C_{2,1}\}$

7.3.10. Galois group: $D_{2,8,5} \rtimes_\varphi \mathbb{Z}_2$, $\varphi(1) = \begin{cases} x \mapsto x \\ y \mapsto yxy^4 \end{cases}$

G: $\langle(1,2,3,6,4,5,7,8),\,(2,5)(3,7),\,(2,5)(6,8),\,(1,3,4,7)(2,6,5,8),$
$(1,4)(2,5)(3,7)(6,8)\rangle < \mathfrak{S}_8$

G^0: $D_4 \times \mathbb{Z}_2$

T: $(2,2,2,2,2)$

L: $(2,5)(6,8), (1,7)(2,6)(3,4)(5,8), (1,4)(2,5), (1,4)(2,5),$
$(1,7)(2,8)(3,4)(5,6)$

H_1: $\mathbb{Z}_2 \times \mathbb{Z}_8$

π_1: $(\mathbb{Z}_2)^2 \rtimes_\psi \mathbb{Z}_8$, $\psi(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

7.3.11. Galois group: $((\mathbb{Z}_2)^2 \rtimes_\varphi \mathbb{Z}_4) \times \mathbb{Z}_2$, $\varphi(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

G: $\langle(1,2,5,8)(3,7,10,14)(4,6,11,13)(9,12,15,16),$
$(2,6)(7,12)(8,13)(14,16),$
$(1,3)(2,7)(4,9)(5,10)(6,12)(8,14)(11,15)(13,16),$
$(1,4)(2,6)(3,9)(5,11)(7,12)(8,13)(10,15)(14,16),$
$(1,5)(2,8)(3,10)(4,11)(6,13)(7,14)(9,15)(12,16)\rangle < \mathfrak{S}_{16}$

G^0: $\mathbb{Z}_2^4$

T: $(2,2,2,2,2)$

L: $(1,5)(2,13)(3,10)(4,11)(6,8)(7,16)(9,15)(12,14),$
$(1,3)(2,12)(4,9)(5,10)(6,7)(8,16)(11,15)(13,14),$
$(1,4)(3,9)(5,11)(10,15),$
$(1,10)(2,16)(3,5)(4,15)(6,14)(7,13)(8,12)(9,11),$
$(1,4)(2,6)(3,9)(5,11)(7,12)(8,13)(10,15)(14,16)$

H_1: $(\mathbb{Z}_2)^3 \times \mathbb{Z}_4$

π_1: $\langle p_1, p_2, p_3, p_4 \mid p_1^2, p_3^2, (p_3p_2)^2, (p_1p_2^{-1})^2, p_4p_2^{-1}p_4^{-1}p_2^{-1},$
$p_4p_1p_3p_4^{-1}p_3p_1, (p_1p_4^2)^2, (p_4^{-2}p_3)^2\rangle$

7.3.12. Galois group: Sylow 2-subgroup of a double cover of the Suzuki group $Sz(8)$

G: $G(128,836)$, $\langle(2,4,9,13)(3,7,12,15)(8,10)(11,16),$
$(1,2,5,9)(3,6)(4,10,13,8)(7,11)(12,14)(15,16),$
$(1,3,8,7)(2,6,4,11)(5,12,10,15)(9,14,13,16)\rangle < \mathfrak{S}_{16}$

G^0: $G(64,23)$: $\langle(2,3,5,8)(6,10)(7,11,12,13)(14,16),$
$(1,2,4,7)(3,6,11,14)(5,9,12,15)(8,10,13,16)\rangle < \mathfrak{S}_{16}$

T: $(4,4,4)$

L: $(1,12,8,15)(2,14,4,16)(3,10,7,5)(6,13,11,9),$
$(1,13,5,4)(2,8,9,10)(3,11)(6,7)(12,16)(14,15),$
$(1,14,8,16)(2,3,13,15)(4,7,9,12)(5,6,10,11)$

H_1: $(\mathbb{Z}_2)^3$

π_1: $(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes_\psi \mathbb{Z}_2$, $\psi(1) = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

$K^2 = 8$, basket $\emptyset$

7.3.13. Galois group: $(D_{2,8,5} \rtimes_{\varphi} \mathbb{Z}_2) \times \mathbb{Z}_2$, $\varphi(1) = \begin{cases} x \mapsto x \\ y \mapsto yxy^4 \end{cases}$

G: $\langle(1,2,4,8,5,9,12,16)(3,7,10,15,11,6,13,14),$
$(2,6)(4,12)(7,9)(8,15)(10,13)(14,16),$
$(1,3)(2,7)(4,10)(5,11)(6,9)(8,15)(12,13)(14,16),$
$(1,3)(2,6)(4,10)(5,11)(7,9)(8,14)(12,13)(15,16),$
$(1,4,5,12)(2,8,9,16)(3,10,11,13)(6,14,7,15),$
$(1,5)(2,9)(3,11)(4,12)(6,7)(8,16)(10,13)(14,15)\rangle < \mathfrak{S}_{16}$

G^0: $D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

T: $(2,2,2,2,2)$

L: $(1,5)(2,7)(3,11)(6,9)(8,14)(15,16),$
$(2,7)(4,12)(6,9)(8,14)(10,13)(15,16),$
$(1,13)(2,8)(3,12)(4,11)(5,10)(6,14)(7,15)(9,16),$
$(1,4)(2,14)(3,10)(5,12)(6,8)(7,16)(9,15)(11,13),$
$(1,3)(2,6)(4,10)(5,11)(7,9)(8,14)(12,13)(15,16)$

H_1: $(\mathbb{Z}_2)^3 \times \mathbb{Z}_8$

π_1: $1 \to \Pi_{17} \times \Pi_{17} \to \pi_1 \to G \to 1$

7.3.14. Galois group: $G(256, 3678)$

G: $\langle g_1,\, g_2,\, g_3 \mid g_1^4,\, g_2^4,\, g_3^4,\, g_1g_2g_3^2g_1^{-1}g_2^{-1},$
$g_2^{-1}g_1^2g_3^{-1}g_2^{-1}g_3,\, g_3g_1^{-1}g_2^{-1}g_3^{-1}g_1^{-1}g_2,\, g_1g_2g_3g_2^{-1}g_1g_3,$
$g_3g_1^{-1}g_2^{-1}g_1g_2g_3,\, g_2^2g_3g_1^{-1}g_3g_1,\, g_3g_1g_2^{-1}g_3^{-1}g_2^{-1}g_3^{-1}g_1g_3,$
$g_2^{-1}g_1g_2g_1^2g_3^{-2}g_1,\, g_1g_2^2g_1g_3^{-1}g_1g_3^{-1}g_1,\, g_2^{-2}g_1^{-1}g_3^{-1}g_1g_3^3,$
$g_3^{-1}g_1g_2^{-1}g_3^{-2}g_1^{-1}g_3^2g_2g_3^{-1}\rangle$

G^0: $G(128,36)$: $\langle h_1, h_2 \mid h_2^4,\, h_1^4,\, h_1h_2^2h_1^{-2}h_2^{-2}h_1,\, (h_2^{-1}h_1h_2h_1)^2,\, (h_1,h_2)^2,$
$(h_1^{-1}h_2^{-1}h_1h_2^{-1})^2,\, (h_1^{-1}h_2^{-1}h_1^{-2}h_2h_1^{-1})^2,\, (h_2^2h_1^{-1}h_2^2h_1)^2\rangle$

T: $(4,4,4)$

L: $g_2g_3,\, g_3g_2^{-1}g_3^{-1}g_2g_3g_1^{-1}g_3g_1g_3g_2g_3^{-2}g_2g_3^2,\, g_2^{-1}g_3^2g_2^{-1}g_3^{-1}g_2g_3$

H_1: $(\mathbb{Z}_4)^3$

π_1: $1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$

7.3.15. Galois group: $G(256, 3678)$

G: as above

G^0: $G(128,36)$, as above

T: $(4,4,4)$

L: $g_1g_3^{-2}g_1^{-1}g_3g_1g_3g_2^2$, $g_3g_2^{-2}g_3^2g_2^{-1}g_3^{-1}g_2g_3$, $g_1g_3^{-1}g_2^{-2}g_3^2g_1^{-1}g_3g_1g_3g_2g_3^{-2}g_2g_3^2$

H_1:$(\mathbb{Z}_2)^4 \times \mathbb{Z}_4$

π_1: $1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$

7.3.16. Galois group: $G(256, 3678)$

G: as above

G^0: $G(128, 36)$, as above

T: $(4,4,4)$

L: $g_1g_3g_2^{-2}g_3^2g_2^{-1}g_3^{-2}g_2g_3^2$, $g_1g_2g_3g_1^{-1}g_3g_1g_3g_2^2$, $g_2g_3^{-2}g_1^{-1}g_3g_1g_3g_2^2$

H_1: $(\mathbb{Z}_2)^2 \times (\mathbb{Z}_4)^2$

π_1: $1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$

7.3.17. Galois group: $G(256, 3679)$

G: $\langle g_1,\, g_2,\, g_3 \mid g_3^4,\, g_1^4,\, g_2^4,\, g_2g_3^2g_1^{-1}g_2^{-1}g_1,\, g_3^{-1}g_2^{-1}g_3^{-1}g_1g_2^{-1}g_1,$
$g_3^{-1}g_2g_3g_2g_1^2,\, g_2^{-1}g_1g_2^{-1}g_3^{-1}g_1^{-1}g_3,\, g_1^2g_2^{-1}g_3^{-1}g_2^{-1}g_3,\, g_2^{-1}g_3g_2g_1g_3g_1,$
$g_1^{-1}g_2^{-1}g_1^2g_3g_1^{-1}g_3^{-1}g_2^{-1},\, g_3^{-1}g_2g_3g_2^{-1}g_1^{-2}g_2^{-2},\, (g_3^{-1}g_2)^4,$
$g_2^{-1}g_1g_2^{-1}g_1g_3^{-1}g_1g_3g_1,\, \rangle$

G^0: $G(128, 36)$, as above

T: $(4,4,4)$

L: g_2g_3, $g_3g_2^{-1}g_3^{-1}g_2g_3g_1g_3^2g_1^{-1}g_3^{-2}g_2^{-1}g_3^{-2}g_2g_3^{-2}$, $g_2^{-1}g_3^2g_2^{-1}g_3^{-1}g_2g_3$

H_1: $(\mathbb{Z}_2)^2 \times (\mathbb{Z}_4)^2$

π_1: $1 \to \Pi_9 \times \Pi_9 \to \pi_1 \to G \to 1$

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