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# Tunnelling and Unruh-DeWitt methods in curved spacetimes

*Tesi di Dottorato*

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# Contents

*[...] Out of the cradle*
*onto dry land*
*here it is standing:*
*atoms with consciousness;*
*matter with curiosity.*

*Stands at the sea, wondering: I*
*a universe of atoms*
*an atom in the universe.*

(Richard P. Feynman)

**Αἰὼν παῖς ἐστι παίζων πεσσεύων· παιδὸς ἥ βασιληίη**

(Heraclitus of Ephesus, fragment DK B52)

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# Introduction

The analysis and the results contained in this work are rooted in a first contact between the quantum theory and the general theory of relativity. By first contact it is meant that we are not considering candidates for “unified theories", but rather we focus on aspects of the full quantum theory in changing geometric backgrounds: the analysis of such an interaction already had important applications in cosmology, *e.g.* in the description of the evolution of fields in inflationary scenarios. Another compelling – and still growing – area of application is the study of *thermodynamical properties of gravitional systems*, which covers the main bulk of this thesis.

Since the discovery of quantum black hole thermal radiance by Hawking [1], it became pretty clear that something remarkable concerning the interface of gravity, quantum theory and thermodynamics was at work. In the usual picture, a radiating black hole loses energy and therefore shrinks, evaporating away to a fate which is still debated. Many new ideas came out from the recognition that quantum field theory implied a thermal spectrum: the most impressive probably being 't Hooft's idea of a dimensional reduction in quantum gravity and the associated holographic description (indicating a drastic reduction in the counting of degrees of freedom in finite regions, which scale with area rather than volume) [2, 3], and the principle of black hole complementarity aimed to reconcile the apparent loss of unitarity implied by the Hawking process with the rest of physics as seen by external observers [4]. But there were also other, more practical, issues regarding these matters, some of which bewildered scientists since the very beginning and that have been only partly resolved. A key issue is that the original derivation of Hawking's radiation applied only to stationary black holes, while the picture above uses quasi-stationary arguments. Actually, an evaporating black hole is non-stationary. Nevertheless, a surprising aspect of the semi-classical result is that the radiation caused by the changing metric of the collapsing star approaches a steady outgoing flux at large times, implying a drastic violation of energy conservation. This certainly means that one cannot neglect the back-reaction problem which, however, has not been solved yet in a satisfactory way. A sample of other key issues have to deal with the final state of the evaporation process, the thermal nature and the related information loss paradox, the Bekenstein–Hawking entropy and the associated micro-states counting, the trans-Planckian problem, and so on. It was to address some of these questions that some alternative derivations and descriptions of Hawking's emission process began to appear over the years, one of these being the so-called *tunnelling method*, presented in Chapter 1.

As hinted before, the definition of particles in a changing metric turns out to be non-trivial. Basically, the field quantization in Minkowski spacetime relies on the definition of a vacuum state as the first building block of the "particle content" spectrum of the theory: in more general situations, *i.e.* in the absence of a global time-like Killing vector, the vacuum state is found to be in no way unique on a global context. The vacuum state defined by an observer is a superposition of positive and negative-frequency modes in another observer's frame.[1] An outline regarding the quantization of fields in curved metrics is presented in the first part of Chapter 2. In view of this ambiguity, we can choose to rely on a more pragmatic (and seemingly tautological) definition: a particle is a click in a particle-detector. This definition implies the local character of the measurement process, which is the relevant aspect from the point of view of different observers. The description and the consequences of this approach are made clear in the rest of Chapter 2, where the theory of *Unruh-DeWitt detectors* is outlined.

In Chapter 3 we shift our attention to the relationship between gravitational systems and the laws of thermodynamics: through a simple thought experiment (on the wake of the works by J.D. Bekenstein, H. Bondi and others) we highlight the importance of *tidal effects in general relativity* in order to comply with the laws of thermodynamics. It is a result that points towards Penrose's Weyl Curvature Hypothesis, which conjectures a relationship between the Weyl tensor – responsible for tidal effects – and a possible definition of entropy for the gravitational field.

The results presented in this thesis have been drawn from the following published papers:

- **Chapter 1**: L. Vanzo, G. Acquaviva, R. Di Criscienzo, *Tunnelling Methods and Hawking's radiation: achievements and prospects*, Class. Quant. Grav. **28** (2011) 183001.
- **Chapter 2**: G. Acquaviva, R. Di Criscienzo, M. Tolotti, L. Vanzo and S. Zerbini, *Unruh–DeWitt Detectors in Spherically Symmetric Dy-*

[1] The same applies also to non-inertial observers in Minkowski spacetime: the resulting effect is called *Unruh effect*.

*namical Space-Times*, Int. J. Theor. Phys. **51** (2012) 0020-7748. R. Casadio, S. Chiodini, A. Orlandi, G. Acquaviva, R. Di Criscienzo and L. Vanzo, *On the Unruh effect in de Sitter space*, Mod. Phys. Lett. A **26** (2011) 2149.

- **Chapter 3**: Y. Gaspar and G. Acquaviva, *A gedankenexperiment in gravitation*, Nuovo Cim. B **125** (2010) 1201.

The metric signature is $(-, +, \ldots, +)$.
Unless explicitly stated, we let $c = k_B = G = \hbar = 1$.
Greek indices are used to denote space-time components of a tensor;
mid-Latin indices as $i, j$ are typically used to denote the 0,1 component;
late-Latin indices as $m, n$ are typically used for purely spatial tensors part.
If not otherwise stated, an overdot will represent a time derivative, and a prime a spatial derivative in the (case by case) relevant spatial coordinate.

# 1 Tunnelling method

The tunnelling method has to do with the particle interpretation of the emission process, which would be the most natural way to explain the loss of energy suffered by the radiating black hole, but also poses a problem: while the original derivation of Hawking radiation used the theory of quantum free fields propagating in a fixed gravitational background, for the case at hand a particle interpretation in the near horizon region of a black hole space-time was found to be far from unique and therefore inherently ambiguous. This fact is deeply rooted into the physical nature of the problem, which is that the general covariance of the physical laws allows many inequivalent choices of time, as was explained several times (see, *e.g.* [6, 11]). It was partly to overcome such difficulties that DeWitt launched in the Seventies the "stress-energy-tensor program" aimed to compute the expectation values of the stress tensor and other observables to better describe quantum effects in a curved space-time [12]. Thus, it was found that it is not in general possible to divide $\langle T_{\mu\nu} \rangle_{ren}$ into a real particle contribution and a vacuum polarisation part in an unambiguous way. However, it will always be possible to choose any coordinate system, as long as it is regular across the horizon, and use it to define an observer dependent vacuum relative to which a particle definition is feasible. As we will see, with the tunnelling method we are only concerned with the *probability* that such an observer dependent notion of particle be emitted from the horizon of the black hole. If this probability is a coordinate scalar it will not depend strongly on what particle concept one employs. In fact, we can call it *a click event in a particle detector* if we like, without committing ourselves with the particle concept; concept which remains of great heuristic value however, and will find its realisation in the choice of the particle action used to compute the tunnelling probability. Such an observer dependent notion of particles has been advocated also by Gibbons and Hawking in their treatment of de Sitter radiation [13], on the ground that an observer independent definition of particles is not relevant to what a given observer would measure with a particle detector.

The second aspect we would like to mention is the fact that, as it will be seen, the tunnelling method uses only the classical action of a single massless, spin-less particle and therefore appears to be state independent. It does what it does by relating the particle emission to an imaginary contribution to the classical action localised at the horizon, which only comes from the local geometry. This can be traced back to the work of Damour and Ruffini [14] with contributions of Padmanabhan *et al.* [15–18], Massar and Parentani [19] and Kraus and Wilczek, Parikh and Wilczek [20–22] (see also Kraus's Ph.D. thesis [23]). To what extent this is so was recently studied in [24] where, for a large class of quantum states with Hadamard short distance singularity, it was shown that the two-point correlation function, which is directly related to the tunnelling probability, has a universal thermal appearance whose temperature is, needless to say, the Hawking temperature.

Now all these important findings have the drawback that they have been proved, and one can easily understand why, only for stationary black holes. In that case the tunnelling picture has not much more to say than was already known, at least for the simplest solutions, besides helping to understand black hole radiation from a different viewpoint. Where exact calculations can be done, it typically gives a less detailed picture of the radiation process since it is mainly related to a semi-classical emission rate. The cases where the method is more powerful have to do either with more intricate stationary solutions or else with truly dynamical black holes. We can list here some of its achievements and properties:

1. The estimation of the leading correction to the semi-classical emission rate arising from back-reaction to the background geometry, whose introduction by Kraus and Wilczek, Parikh and Wilczek [20–22] motivated the tunnelling method in a form that will be discussed soon; its extension to Anti–de Sitter (AdS) [25], de Sitter (dS) [26–30] and higher dimensional black holes; the existence of correlations among successive emission events [31–33];

2. The original tunnelling method can be generalised to a Hamilton–Jacobi variant, originated with the work of Padmanabhan and collaborators [15–18] and systematically applied either to stationary or dynamical black holes (see *e.g.* [5, 34–39] for a sample of papers). For dynamical black holes this was particularly important, since even approximate quantum calculations are notoriously hard. Moreover, in this variant it manifestly preserves general covariance;

3. Supplemented with a more precise and more general mathematical definition of a local horizon, the Hamilton–Jacobi variant can be applied to any sort of horizon within this class, and in particular to cosmological and weakly isolated horizons [40] (for definitions see below). It can also be applied to past horizons and white holes, in which case a clear notion of temperature emerges in complete analogy to black holes;

4. The tunnelling method strengthens the connection of the semi-classical rate with the surface gravity of the horizon even for dynamical black holes, provided that opportune definitions of such quantities are employed. In this respect its application to extremal black holes should have something to say;

5. The tunnelling picture can promptly give indications on what occurs in other situations. One example is the WKB approach to Unruh's radiation, reviewed in [41]. It can also be applied to see whether naked singularities are going to emit radiation, or to study the decay of unstable particles that in the absence of gravity would be stable by the action of some conservation law. This is valuable given the great efforts that are normally necessary to analyse quantum effects in the presence of gravity.

It remains to explain how the tunnelling picture works and to what extent it gives a consistent picture, at least from a logical point of view.

## 1.1 Stationary black holes

According to quantum field theory, the physical vacuum – the state with no real particles – is in general a complex entity populated by virtual particles which are constantly created, interact among themselves and are then annihilated in this vacuum. In the absence of external fields, the vacuum is usually stable in the sense that virtual particles are not able to survive long enough to become real. However, it is now well proved that external fields are able to convert short-living particles into real ones just supplying enough energy to the process.

Considering particle creation by a static gravitational field endowed with a Killing vector $\xi^\alpha$ , the energy of a particle is $\omega = -p_\alpha \xi^\alpha$, where $p_\alpha$ is the particle's four-momentum. For massive (massless) particles, the momentum $p_\alpha$ is a future-directed time-like (null) vector. Thus, the energy $\omega$ of a particle is always positive in the region where the Killing vector is also future-directed and time-like. It follows that particle creation in a spacetime region with $\xi^\alpha$ future-directed and time-like is forbidden by energy conservation. Such considerations provide a heuristic argument [42] to conclude that a static gravitational field can create particles only if the spacetime contains a black hole. In fact, when a virtual particle pair is created just inside the horizon, the positive energy virtual particle can tunnel outside – no classical escape route existing – where it materializes as a real particle; in alternative, for a pair created just outside the horizon, the negative energy virtual particle, forbidden outside the horizon, can tunnel inwards [43]. In either cases, the negative energy particle is absorbed by the black hole, resulting in a black hole mass decrease, while the positive energy particle escapes to infinity, visible to distant observers (Hawking's radiation).

In 1974, Hawking [1] proved that indeed black holes are classically stable objects that, due to quantum fluctuations, become unstable with respect to particle emission. Nowadays, several methods are known for deriving the Hawking radiation and calculating its temperature, but none of them has been able to catch the intuitive picture above as the tunnelling method proposed a decade ago by Parikh and Wilczek [22]: if it were not for the inclusion of back-reaction, their method could be considered a semi-classical version of Hartle–Hawking's path integral derivation of black hole radiance [44].

### 1.1.1 Null-geodesic method

The so called *null geodesic method* — as named by Kerner and Mann [36] — derives black hole radiance from few reasonable assumptions, namely: energy conservation, detailed balance and the use of non-singular coordinates across the horizon.
Energy conservation requires fixing the total energy of the space-time before and after particle emission. Since black hole mass and volume are linked together, a mass reduction due to the emission of a particle translates into a size contraction; so one might worry how to deal with quantum fluctuations of the metric originating from such contraction. However, this is not a problem as far as we consider transitions between zero-spin geometries. In this case in fact, no graviton quantisation is involved or, said in other words, passing from different spherically symmetric configurations does not produce gravitational waves. As a consequence, the only degree of freedom remained in the problem is the position of the emitted particle (actually, a thin shell). Thus, to keep things as simple as possible, we can restrict to consider uncharged, static, spherically symmetric black holes emitting neutral matter, referring to next sections treatment of the most general case.

Because of Birkhoff's theorem [45], we are dealing with Schwarzschild geometry which, written in the Schwarzschild frame, reads

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (1.1)$$

where the *gravitational radius* is $r_g = 2M$. Computing the free fall acceleration of a body initially at rest in the Schwarzschild frame, we get

$$a := \sqrt{a_n a^n} = \frac{M}{r^2\sqrt{1 - r_g/r}}\,. \qquad (1.2)$$

The geometry Eq. (1.1) becomes singular at the gravitational radius and the acceleration Eq. (1.2) becomes infinite in the same limit. Nonetheless, near $r_g$ the space-time is regular (*i.e.* neither curvature singularities or geodesic incompleteness): an indication of the fact that the Schwarzschild frame ceases to be defined at $r = r_g$ (For $r \leq r_g$ systems must be non-rigid in the sense that the metric components $g_{\alpha\beta}$ must be functions of time, cf. [42]). It is well-known that the gravitational radius $r_g$ represents the Killing event horizon of Schwarzschild black hole [45]; thus, in order to deal with tunnelling of a particle across it, we need to cover the region $r \sim r_g$ with regular coordinates. A possible choice, not the only one, is represented by Painlevé–Gullstrand coordinates [46]. In order to construct this new regular-across-the-horizon frame, we consider the geodesic equations of a family of observers, freely falling ($dr/d\tau < 0$) in the geometry Eq. (1.1), along a radial trajectory ($\theta, \phi$ constant):

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - \frac{r_g}{r}} \, , \qquad \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_g}{r}\right) = \tilde{E}^2 \tag{1.3}$$

$\tau$ representing observer's proper time and $\tilde{E} = E/m$ is the observer's conserved energy per unit mass. The energy parameter is related to observer's physical three-velocity $v^2 = g_{mn} dx^m dx^n / (-g_{00}) dt^2$: in fact, given the Killing vector $\xi^\mu_{(t)}$ — generating the time-symmetry transformations of Eq. (1.1), the energy reads

$$\tilde{E} = -u_\mu \xi^\mu_{(t)} = -u_0 = -g_{00}\, u^0 = \frac{\sqrt{-g_{00}}}{\sqrt{1 - v^2}} \, . \tag{1.4}$$

We specialise to the case of observers starting at infinity with null initial velocity so that $\tilde{E} = 1$. For this family of observers, there exists a function $t_p$ such that

$$\nabla_\mu t_p = -u_\mu \, . \tag{1.5}$$

Taking Eq. (1.3) into consideration, we get the Painlevé time $t_p$ in terms of Schwarzschild coordinates

$$t_p = t + \int^r dr' \sqrt{\frac{r_g}{r'}} \left(1 - \frac{r_g}{r'}\right)^{-1} = t + 2\sqrt{r_g r} + r_g \ln\left(\frac{\sqrt{r} - \sqrt{r_g}}{\sqrt{r} + \sqrt{r_g}}\right) \, . \tag{1.6}$$

In terms of $(t_p, r, \theta, \phi)$, the Schwarzschild geometry is written as

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt_p^2 + 2\sqrt{\frac{r_g}{r}}\, dt_p\, dr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1.7}$$

At fixed time, the spatial geometry described by Eq. (1.7) is Euclidean; while at any fixed radius, the boundary geometry is the same as that in Eq. (1.1). Most importantly, the metric is no more singular at the gravitational radius $r_g$; the space-time is stationary, but no more static. The $t_p$ coordinate — being the time perceived by infalling observers who reach the curvature singularity $r = 0$ in finite time — remains a valid coordinate as far as it remains finite.

The geodesic for a null s-wave is given by

$$\dot{r} \equiv \frac{dr}{dt_p} = \pm 1 - \sqrt{\frac{r_g}{r}}\,, \tag{1.8}$$

with upper (lower) sign corresponding to outgoing (ingoing) geodesic. The basic idea behind the approach by Parikh and Wilczek [22] is to study the emission of such a null-radial thin shell from the black hole through quantum tunnelling. Imposing energy conservation means that the total space-time energy is fixed and one allows the black hole mass to fluctuate. As showed in [20], the motion of a shell of energy $\omega$ emitted from a black hole of initial mass $M$ is described by Eq.(1.8) in the space-time geometry of Eq.(1.7 with $M$ replaced by $M - \omega$.

Differently by tunnelling in quantum mechanics, where two configurations (classical turning points) separated by a potential barrier are connected by a trajectory in imaginary time, here we have the perception that nothing similar happens. In fact, the problem with black hole radiation is that, as soon as a particle is displaced outside the horizon, it can escape along a classical trajectory. It is worth to mention how the crucial ingredient of the story here is played by energy conservation. As the black hole of mass $M$ and area $16\pi M^2$ emits a particle of energy $\omega$, its mass reduces to $M - \omega$ and its horizon recedes to $2(M - \omega) = r_g - 2\omega$. This amount of contraction, evidently, depends upon the mass of the emitted particle, so that quoting Parikh "it is the tunnelling particle itself that secretly defines the tunnelling barrier" [43].

Before evaluating the black hole emission rate, we can ask to what extent is justified the point-particle approximation. Given that a distant observer detects a wave with frequency $\Omega_{ob}$, this has been emitted with frequency $\Omega_{em} \propto (1 - r_g/r)^{-1/2}\Omega_{ob}$. In the limit of $r \to r_g$, where the emission process occurs, the wavelength vanishes, making the point-particle (WKB) approximation fully reliable.

In the WKB approximation, the tunnelling probability rate $\Gamma_{em}$ turns out to be equal to

$$\Gamma_{em} \sim \exp(-2\ \mathrm{Im}\, I) \tag{1.9}$$

where $I$ is the action for the tunnelling process evaluated to the appropriate order. We shall derive later, in Subsection 1.1.6, the exact correspondence between $\Gamma_{em}$ and the exp–function. Eventually, one would expect to be able to read off the black hole radiation temperature from comparison of the the probability rate Eq.(1.9) with the Boltzmann factor $e^{-\omega/T}$. The imaginary part of the action for a null s-wave outgoing positive energy particle which crosses the horizon outwards from $r_{in}$ to $r_{out}$ can be expressed as

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_{in}}^{r_{out}} dr\, p_r = \mathrm{Im} \int_{r_{in}}^{r_{out}} dr \int_0^{p_r} dp'_r \tag{1.10}$$

Changing the integration from momentum to energy thanks to Hamilton's equation $\dot{r} = dH/dp_r$ and noting that $H = M - \omega'$ with constant $M$ and $0 \leq \omega' \leq \omega$, we have

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_{in}}^{r_{out}} dr \int_{M}^{M-\omega} \frac{dH}{\dot{r}} = \mathrm{Im} \int_{r_{in}}^{r_{out}} dr \int_{0}^{\omega} \frac{(-d\omega')}{\dot{r}}\,. \tag{1.11}$$

Using Eq.(1.8) and switching the order of integration,

$$\mathrm{Im}\, I = \mathrm{Im} \int_{0}^{\omega} (-d\omega') \int_{r_{in}}^{r_{out}} dr \frac{1}{1 - \sqrt{\frac{2(M-\omega')}{r}}}\,. \tag{1.12}$$

With the understanding that the particle starts at $r_{in} = 2M$ and materialises at $r_{out} = 2(M - \omega)$, $r_{in} > r_{out}$, the integral over $r$ is divergent and needs to be appropriately regularised. A tunnelling process corresponds to an energy which is not in the spectrum of the Hamiltonian. We need to continue the energy $\omega$ to complex values, that is $\omega' \to \omega' + i\epsilon$, in order to ensure that positive energy solutions decay in time:

$$\begin{aligned} \mathrm{Im}\, I &= \mathrm{Im} \int_{0}^{\omega} (-d\omega') \int_{r_{in}}^{r_{out}} dr \frac{\sqrt{r}}{\sqrt{r} - \sqrt{2(M-\omega')} + i\epsilon} \\ &= -\mathrm{Im}\, i\pi \int_{0}^{\omega} (-d\omega')\, 4(M - \omega') = +4\pi\omega \left(M - \frac{\omega}{2}\right). \end{aligned} \tag{1.13}$$

Of course, Hawking radiation can also be regarded as pair creation outside the horizon, with the negative energy particle tunnelling into the black hole. The calculation proceeds as above with slight change of signs due to the fact that anti-particles travel back in time (cf. [22] for details). That both particle and anti-particle channels contribute to black hole emission is something which only affects the amplitude of the process, that is something which enters the proportionality factor of Eq.(1.9). In conclusion, the emission rate obtained is

$$\Gamma \sim \exp(-2\,\mathrm{Im}\, I) = e^{-8\pi M\omega\left(1 - \frac{\omega}{2M}\right)}. \tag{1.14}$$

The expected Boltzmann factor $e^{-\omega/T_H}$ is recovered only to order $O(\omega)$: in this order of approximation, the black hole temperature perfectly coincides with standard Hawking's result, $T_H = (8\pi M)^{-1}$. The $O(\omega^2)$ correction arising from the physics of energy conservation, makes higher the effective temperature of the hole as it radiates in agreement with the well-known negative thermal capacity of Schwarzschild black hole.
Neglecting the $\omega(2M)^{-1}$ term in Eq. (1.14) and invoking detailed balance[1], the Planck distribution is recovered.

[1]Consider an ensemble of many identical copies of the same quantum system. Let the energy and the number of accessible states of the system be fixed. The probability that

*A note* – It appears that by neglecting the back-reaction no member of a pair of particles created just inside the event horizon can escape it, by reason of causality. In the terminology to be used in Section 1.2 this correspond to a type-II tunnelling path, the particles created just outside forming instead a type-I path. The inclusion of the back-reaction is truly fundamental for the existence of both possibilities. Since in the dynamical case we will mainly ignore the back-reaction effect, only type-I path will be accessible to the tunnelling probability.

### 1.1.2 Hamilton-Jacobi method

Despite the merits of the seminal work by Parikh and Wilczek, we cannot omit to point out a couple of unpleasant features of their null geodesic method, as the fact that: (i) it strongly relies on a very specific choice of (regular-across-the-horizon) coordinates; and (ii) it turns upside down the relationship between Hawking radiation and back-reaction. As regard the former point, it should be clear how irrelevant is, in the spirit of general relativity, the choice of coordinates: being physical observables invariant with respect to the group of diffeomorphisms (the hole temperature is such an observable), there is no reason why Painlevé–Gullstrand coordinates should be favourable with respect to other (equally well-behaved) coordinates. With respect to the latter, we notice that, in the null geodesic description, apparently there cannot be Hawking radiation without back-reaction: watching carefully, however, it is the discovery of Hawking radiation that justifies back-reaction and makes commendable the treatment of Hawking radiation's self-gravity. The so-called *Hamilton–Jacobi method* can cope with both issues. The intent is to give a particle description of Hawking radiation, ignoring

a system randomly chosen out of the ensemble will be in state $i$ is denoted by $p_i$, with $\sum_i p_i = 1$. The *transition probability* $\lambda_{i\to j}$ denotes the conditional probability per unit time of a system going from state $i$ to state $j$, *i.e.* $p_i\lambda_{i\to j}dt$ is the probability of a system originally being in state $i$ and going to state $j$ within the time interval $dt$ [47]. From this definition, $\sum_j p_i\lambda_{i\to j}dt$ represents the probability of transition from state $i$ to any other possible state of the system in time $dt$; and $\sum_j p_j\lambda_{j\to i}dt$ represents the probability of transition into state $i$ from any other possible state of the system in time $dt$. It follows that, $\dot{p}_i = \sum_j(p_j\lambda_{j\to i} - p_i\lambda_{i\to j})$ is the net rate of change of $p_i$. The *detailed balance condition* requires that,

$$\text{at equilibrium,} \qquad p_j\lambda_{j\to i} = p_i\lambda_{i\to j}\ , \qquad \forall i,j\ . \tag{$*$}$$

In the case at hand, the system is made of "black hole + radiation". The initial state $i$ represents, for example, the black hole with $N-1$ particles (*e.g.* photons). Through the physical mechanism described before, the black hole emits one more particle, so that the state $j$ becomes "black hole + $N$ particles". Neglecting back-reaction, the internal degrees of freedom of the black hole do not change and by ($*$), we have that $\lambda_{i\to j}/\lambda_{j\to i} \equiv \Gamma_{em}/\Gamma_{ab} = p_N/p_{N-1} = e^{-\omega/T}$. This result combined with the classical constraint, $\Gamma_{ab} \mp \Gamma_{em} = |T_l(\omega)|^2$, reproduces Planck (minus sign) and Fermi–Dirac (plus sign) distributions. Here, $T_l(\omega)$ represents the transmission coefficient of the black hole barrier which in general can depend on the energy $\omega$ and the angular momentum $l$ of the particle.

its self-gravitation, under the assumption that the emitted (scalar) particle's action does satisfy the Lorentzian Hamilton–Jacobi equation. Later we will show that the null geodesic method can do the same job using instead the reduced action.
As it will become clear later, this method applies to any well-behaved coordinate system across the horizon; it generalises beyond the assumption of spherical symmetry; it makes possible to include the study of tunnelling by fermionic particles. Some sceptics about Hawking radiation contest the fact that as soon as the black hole starts radiating, the same assumptions on space-time stationarity drops down, invalidating the whole derivation. Given that the departure from perfect stationarity is ridiculously small, nonetheless, as we shall show in Section 1.2, the Hamilton–Jacobi method can prove black hole evaporation even for slowly varying, time dependent, space-times. Finally, we wish to notice that since the methods of tunnelling are intimately related to the physics of (some type of) horizons, we may apply them even to space-times with multiple horizons. In standard computations of Hawking radiation, this is typically a hard task, if not impossible. In fact, it is well known that, as an example, Reissner–Nordström–de Sitter space does not possess regular Euclidean section for general values of mass, electric charge and cosmological constant [50]. This means that for arbitrary values of the parameters, it is not possible to compute the Hawking temperature of event or cosmological horizons by Euclidean continuation, simply because it could be that no Euclidean section of the given Lorentzian space-time exists.

In its minimal formulation, the Hamilton–Jacobi method works according to the following strategy [55]: (a) assume that the tunnelling particle's action satisfies the relativistic Hamilton–Jacobi equation,

$$g^{\mu\nu}\,\partial_\mu I\,\partial_\nu I + m^2 = 0 \tag{1.15}$$

where $m^2$ represents the invariant mass; (b) reconstruct the whole action $I$, starting from the symmetries of the problem and the partial derivatives $\partial_\mu I$, by means of

$$I = \int dx^\mu\,\partial_\mu I\,, \tag{1.16}$$

where the integration is carried along an oriented, null, curve (like for example $\overrightarrow{abc}$ in Fig.(1.1)), to which we refer as the *tunnelling path*, with at least one point on the horizon; (c) split the integration along the null path in two pieces, one along the segment crossing the horizon ($\overrightarrow{ab}$ in Fig.(1.1)), the remaining contribution living in the outer domain of the space-time ($\overrightarrow{bc}$ in Fig.(1.1)); (d) perform a near-horizon approximation in the first integral and regularise its divergence according to Feynman's $i\epsilon$–prescription. Once the procedure is done, the imaginary part of the classical action, relevant for the tunnelling rate Eq. (1.9), reads

$$\mathrm{Im}\ I = \frac{\pi\omega}{\kappa}\,, \tag{1.17}$$

where $\kappa$ and $\omega$ are the surface gravity and particle's energy, respectively.

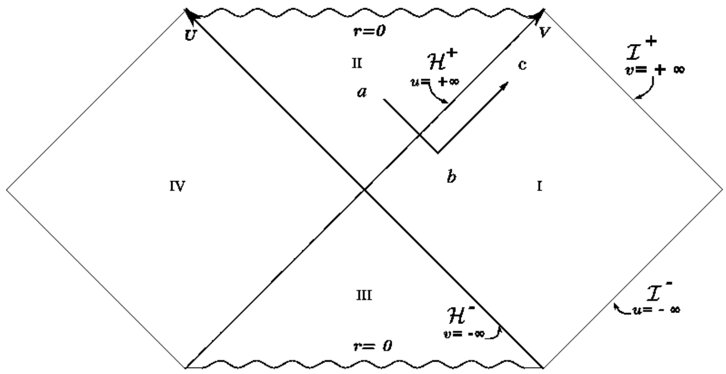


Figure 1.1: Carter–Penrose diagram of the eternal Schwarzschild black hole. $\mathcal{H}^{\pm}$ denote future and past horizon; $\mathcal{I}^{\pm}$ future and past null infinity; $r = 0$ the curvature singularity; $v, u$ represent advanced and retarded null coordinates, respectively. $\overrightarrow{abc}$ is a null piecewise continuous path from inside to outside the hole, with $\overrightarrow{ab}$ running backward in time (classically forbidden trajectory).

As we have already stressed, Hawking radiation is a semi-classical result of quantum field theory in curved space-time. In the spirit of tunnelling, we allow particles to travel along classically forbidden trajectories by starting just behind the horizon onward to infinity. Thus, particles must travel back in time, since the horizon is locally to the future of the static external region. For this reason, we must implement coordinates well defined through regions $I$ and $II$, a requirement which automatically discards Schwarzschild-like or isotropic-like coordinates (it is easy to show that, in Schwarzschild space-time, isotropic coordinates, $(t, \rho, \theta, \phi)$, defined through $r(\rho) := (1+M/2\rho)^2\rho$, cover regions $I$ and $IV$ of Fig.(1.1)). The classical one-particle action becomes complex as an indication of the classical impossibility of the motion and gives, in this way, an imaginary part to the emission rate of the hole. In this sense, we can say that, of the actual path travelled by the tunnelling particle (*e.g.* $\overrightarrow{abc}$), only the infinitesimal region across the horizon plays a crucial role in the whole calculation. Even if null classically forbidden curves like $\overrightarrow{ab}$ do not have anything special with respect to other forbidden paths, their choice is preferable to computational purposes. Also, it will become clearer later the reason why we specifically regularise divergent integrals by Feynman's prescription rather than others. In consideration of our choices, the mass term in Eq. (1.15) is irrelevant to the physics of the horizon (Hawking radiation) but, as we shall show in Section 1.3, under quite general conditions particles masses play an important role in relation to bulk particle creation phenomena.

To show in practice how the Hamilton–Jacobi method works, we are going to derive the emission rate Eq. (1.9) and prove the identity Eq. (1.17) in the case of a scalar, uncharged, particle tunnelling from a Schwarzschild black hole, in complete analogy with what we did in the previous subsection. This will give us the opportunity to unveil other debated points of the method. Later on we shall free the derivation from any dependence by special coordinate systems.

**Painlevé–Gullstrand coordinates**

In this case the space-time geometry is described by Eq. (1.7). Because of Eq. (1.6, expressing the relation between Painlevé and Schwarzschild times, the particle energy associated to a Killing observer is expressed by $\omega = -\partial_{t_{\rm p}} I$ (according to Eq. (1.4), the particle energy is $\omega = -p \cdot \xi$ where $p_\mu = -\partial_\mu I$ is the four-momentum). Since we are dealing with a spherically symmetric space-time, we may neglect without fault the angular coordinates. We consider a null direction $(\Delta t_p, \Delta r)$, for which

$$-\left(1-\frac{r_{\rm g}}{r}\right)\Delta t_{\rm p}^2 + 2\sqrt{\frac{r_{\rm g}}{r}}\,\Delta t_{\rm p}\cdot\Delta r + \Delta r^2 = 0\,. \tag{1.18}$$

In a neighbourhood of the event horizon (*e.g.* $\overrightarrow{ab}$ in Fig. (1.1)), the relation between $\Delta t_{\rm p}$ and $\Delta r$ is uniquely fixed to be $\Delta t_{\rm p} = -\Delta r/2$ since, in this limit, the alternative solution $\Delta t_{\rm p} = -(1-\sqrt{r_{\rm g}/r})^{-1}\,\Delta r$ diverges in contrast to the physical meaning of the coordinate $t_{\rm p}$. It simply means that the segment $\overrightarrow{ab}$ is part of null direction across the horizon. The reduced Hamilton–Jacobi equation

$$-(\partial_{t_{\rm p}} I)^2 + 2\sqrt{\frac{r_{\rm g}}{r}}\,\partial_{t_{\rm p}} I\,\partial_r I + \left(1-\frac{r_{\rm g}}{r}\right)(\partial_r I)^2 = 0 \tag{1.19}$$

can be written as

$$-\omega^2 - 2\,\omega\sqrt{\frac{r_{\rm g}}{r}}\,\partial_r I + \left(1-\frac{r_{\rm g}}{r}\right)(\partial_r I)^2 = 0\,. \tag{1.20}$$

The imaginary part of the action is

$$\begin{aligned}
\operatorname{Im} I &= \operatorname{Im}\int_{a\to b\to c}\left(\partial_r I\,dr + \partial_{t_{\rm p}} I\,dt_{\rm p}\right) && (1.21)\\
&= \operatorname{Im}\int_{a\to b} dr\left(\partial_r I + \frac{1}{2}\omega\right) && (1.22)\\
&= \operatorname{Im}\int_{\searrow} dr\,\partial_r I\,. && (1.23)
\end{aligned}$$

Passing from Eq. (1.21) to Eq. (1.22), we took into consideration that: (i) only the classically forbidden part $\overrightarrow{ab}$ of the whole null path $\overrightarrow{abc}$ contributes

to complexifying the action of the tunnelling particle; (ii) as far as we are concerned with the infinitesimal region of intersection between the horizon and the path $\overrightarrow{ab}$, $dt_{\rm p}$ and $dr$ are related to each other in the way specified above; (iii) the particle's energy goes as $-\partial_{t_{\rm p}} I$. Being $\omega$ a real positive constant, the second term in the integrand of Eq. (1.22) does not enter the imaginary part of the action so it drops down in passing to Eq. (1.23). From now on, integration along the classically forbidden path (*e.g.* $\overrightarrow{ab}$ or, by extension, $\overrightarrow{abc}$) will be denoted simply by $\int_{\searrow}$.
It is remarkable that also the integration over the "time" coordinate enters, in general, the reconstruction process of the particle's action. In this specific case, we see that this "temporal" contribution — being manifestly real — does not affect the result. The next subsection will show a coordinate system where instead the "temporal" contribution vanishes exactly. It should be clear that names such as "temporal" or "spatial" contributions are coordinate dependent terms in no way fundamentally related to the physics of the process, which must be covariant. The general covariance of the tunnelling method will be continuously emphasised as one of the main themes of this chapter.

Inserting the non-manifestly real solution of Eq. (1.20) into (1.23), we get

$$\operatorname{Im} I = 2 \operatorname{Im} \int_{\searrow} dr \, \frac{\omega}{1-\frac{r_{\rm g}}{r}} = 2 \operatorname{Im} \int_{\searrow} dr \, \frac{\omega\, r}{r - r_{\rm g}} \,, \tag{1.24}$$

and regularising the divergent integral according to Feynman's $i\epsilon$–prescription, the imaginary part of the action becomes

$$\operatorname{Im} I = 2 \operatorname{Im} \int_{\searrow} dr \, \frac{\omega\, r}{r - r_{\rm g} - i\epsilon} = 4M\pi\, \omega \,. \tag{1.25}$$

Strictly speaking $\omega$ should be computed on the horizon, because this is how the method of residues works; however this is the same thing as the energy measured at infinity since on the outgoing trajectory $\omega$ is conserved. That said, after the known identification $\kappa = 1/4M$, the identity Eq. (1.17) is fully recovered. Whether we had used the opposite prescription, $r \to r + i\epsilon$, a corresponding change of sign of the imaginary part would have resulted. Its meaning will be explained in Section 1.1.6.

### Eddington–Finkelstein coordinates

Let us introduce another reference frame without singularities on $r_{\rm g}$ originally constructed by Eddington (1924) and Finkelstein (1958), [51]. This frame is fixed to radially moving photons. Since no observer can move together with photons, this new frame is not, strictly speaking, a reference frame. Nevertheless, this system of test photons proves to be very convenient, [42]. According to Eq. (1.1), the equation of motion of a radial incoming photon is $dr/dt = -(1 - r_{\rm g}/r)$. From the viewpoint of a distant observer, the photon, starting from $r_1$ at time $t_1$, arrives in $r$ ($r_{\rm g} < r < r_1$) at time

$$t = r_1 - r - r_{\rm g} \ln \left( \frac{r - r_{\rm g}}{r_1 - r_{\rm g}} \right) + t_1 \,. \tag{1.26}$$

This expression can be opportunely re-written as $v = v_1$, where

$$v := t + r_* \,, \qquad r_* := r + r_{\rm g} \ln \left| \frac{r}{r_{\rm g}} - 1 \right| \tag{1.27}$$

$r_*$ is the so-called *tortoise coordinate* with $v_1$ a constant characterising the initial data of the photon at $(t_1, r_1)$. Because of the logarithm in Eq. (1.26), $r_*$ is defined for any $r > 0$. Chosen a set of photons at fixed $t$, we may label each photon of the set through a number $v$, which will identify uniquely that photon during its whole motion: $v$ rises to the role of a new null coordinate, usually called *advanced time*. After differentiation of Eq. (1.27) and substitution in Eq. (1.1), the line element takes the so called Eddington–Finkelstein form,

$$ds^2 = -\left(1 - \frac{r_{\rm g}}{r}\right) dv^2 + 2\, dv\, dr + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \,. \tag{1.28}$$

Because of Eq. (1.27), the Killing vector field $\xi^\mu = \partial_t$ transforms to $\xi'^\mu = \partial_v$ and particle's energy as measured by such (natural) observer is simply $\omega = -\partial_v I$. Favoured by the spherical symmetry of the problem, we may consider only radial motions. Expanding the trajectory along a null direction in a neighbourhood of the horizon (*e.g.* $\overrightarrow{ab}$ in Fig. (1.1)), gives

$$0 = 2\, \Delta v \cdot \Delta r \,. \tag{1.29}$$

However, along the null path $\overrightarrow{ab}$, the $v$ coordinate is constant, so that

$$\Delta v|_{\searrow} = 0 \tag{1.30}$$

is the right solution. The Hamilton–Jacobi equation,

$$2\, \partial_v I\, \partial_r I + \left(1 - \frac{r_{\rm g}}{r}\right) (\partial_r I)^2 = 0 \tag{1.31}$$

can be re-written as

$$\left(1 - \frac{r_{\rm g}}{r}\right) \partial_r I = 2\omega \,. \tag{1.32}$$

According to the reconstruction assumption,

$$\mathrm{Im}\, I = \mathrm{Im} \int_{a \to b \to c} \left( \partial_r I\, dr + \partial_v I\, dv \right) \,, \tag{1.33}$$

and thanks to Eqs.(1.30) and (1.32),

$$\mathrm{Im}\, I = 2\, \mathrm{Im} \int_{\searrow} dr\, \frac{\omega}{\left(1 - \frac{r_{\rm g}}{r}\right)} + \int_{b \to c} (\dots) \,. \tag{1.34}$$

Neglecting the real contribution coming form the classically allowed path $\overrightarrow{bc}$ and following the same procedure as in Eq. (1.24), we end with the same result as in Eqs. (1.17) and (1.25).

Among the coordinate systems of a certain importance covering the space-time region across the gravitational radius there is the so-called Lemaître frame. In this frame the Schwarzschild space-time appears as truly dynamical. We shall see how the Hamilton–Jacobi method deals with it in Section 1.2.2.

We end this section pointing out the role played by the local observer. As discussed in [84], the particle's energy $\omega = -\partial_t I$ given in the two previous examples is obviously the conserved energy as measured by an observer living at infinity. It follows that the Hawking temperature of a Schwarzschild black hole, $T = 1/8\pi M$, is actually the one measured by a Killing observer at infinity. The particle's energy as detected by a Killing observer at position $\mathbf{x}_{\rm ob}$ is given instead by $\omega_{\rm ob} = \omega/\sqrt{-g_{00}(\mathbf{x}_{\rm ob})}$. Comparing the tunnelling probability derived from Eq. (1.17) with the thermal distribution as measured by the observer $\beta_{\rm ob}\,\omega_{\rm ob} = 2\pi\omega/\kappa$, we obtain

$$T_{\rm ob}\sqrt{-g_{00}(\mathbf{x}_{\rm ob})} = T_\infty = \text{constant}\,, \tag{1.35}$$

which expresses the expected result given by Tolman years ago [52].

### 1.1.3 More general static solutions

Up to now we have only considered the Schwarzschild solution in order to keep the discussion as simple as possible and to illustrate the principles involved without unnecessary complications. However more general static solutions are of interest for a variety of reasons. In order to include a broader class, the metric can be written in a diagonal gauge as

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + C(r)^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{1.36}$$

which describes what Visser [53] termed *dirty black holes*. Black hole solutions are defined by functions $V(r)$ and $W(r)$ having simple and positive zeroes. This is only a necessary condition to have a black hole; we must also require that the domain of outer communication be "outside of the black hole" *i.e.*, it should correspond to values of the radial coordinate larger than the horizon and extending up to spatial infinity. Interesting black holes described by such metrics can be obtained in the Einstein–Maxwell–dilaton coupled system. An example illustrating the feature is the following two-parameter family of solutions

$$ds^2 = -\left(1-\frac{r_+}{r}\right)dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\,d\phi^2)\,. \tag{1.37}$$

The dilaton is $\exp(2\varphi) = (1 - r_-/r)^{-1/2}$; the hole has magnetic charge $q_{\rm m} = 3r_+r_-/16$, horizon radius $r_+ = 2M$ (which defines $M$) and can be extended to a non singular, geodesically complete solution with horizons and asymptotically flat infinities.

As long as $V \neq W$ in Eq. (1.36), it is possible to set $C(r) = r$ by an opportune radius redefinition. But for $V = W$, this is not always the case as can be seen from the following example

$$ds^2 = -\frac{1-2M/r}{\sqrt{\Delta}}dt^2 + \frac{\sqrt{\Delta}}{1-2M/r}dr^2 + \sqrt{\Delta}\, r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ . \quad (1.38)$$

Here $C(r) := \Delta^{1/2} r$ with $\Delta = 1 + 2Mr^{-1}\sinh^2\gamma$ and $\gamma$ a real constant. The metric is asymptotically flat with an event horizon at $r = 2M$ and an electric field due to a charge $Q = M\sinh 2\gamma/2$. This solution can be obtained from dimensional reduction of a boosted Schwarzschild solution in Kaluza–Klein theory.

The metric Eq. (1.36) admits also a Painlevé–Gullstrand representation associated with a redefinition of the time coordinate,

$$t_{\rm p} = t \pm \int dr \sqrt{\frac{1-W(r)}{V(r)W(r)}} \ . \quad (1.39)$$

A simple computation leads to

$$ds^2 = -V(r)dt_{\rm p}^2 \pm 2\sqrt{\frac{1-W(r)}{W(r)}}\cdot V(r)\, dr\, dt_{\rm p} + dr^2 + C(r)^2\,(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (1.40)$$

Turning our attention back to Eq. (1.36) with $C(r) \equiv r$, by computing the Einstein's tensor one sees that the stress-energy tensor must have the following form

$$T^\mu_\nu = \text{diag}\ (-\varrho(r), P(r), S(r), S(r)). \quad (1.41)$$

Thus, in general, the metric Eq. (1.36) has continuously distributed sources and is not asymptotically flat. If the space-time we have in mind is such that

$$\begin{aligned} V(r) &= \frac{(r-r_1)\cdots(r-r_s)}{r^s} \ , \\ W(r) &= \frac{(r-r_1)\cdots(r-r_q)}{r^q} \ , \qquad s \leq q \ , \end{aligned} \quad (1.42)$$

where $r_1 \leq r_2 \leq \cdots \leq r_s \leq r_q$, then

$$V(r)\cdot W(r) > 0 \begin{cases} \forall r > 0 \ , & s = q \ , \\ \forall r > r_q \ , & s < q \ , \end{cases} \quad (1.43)$$

which means that outside the outermost event horizon $r = r_q$, $V(r)$ and $W(r)$ are both positive functions. In this case from Einstein's equations it is possible to prove that [55]

$$V(r) = W(r) \quad \Longleftrightarrow \quad \varrho + P = 0 \quad \text{through space-time} \tag{1.44}$$

which is satisfied only in vacuum space-times with cosmological constant. This condition is completely equivalent to the fact that [54] *in a static space-time,*

$$g_{tt}g_{rr} = -1 \qquad \Longleftrightarrow \qquad T_{\mu\nu}n^\mu n^\nu = 0\,, \tag{1.45}$$

*for all radial null vectors* $n^\mu$. Another important fact regarding such dirty black holes is that as a consequence of the dominant energy condition and the Einstein's equations, if $V(r)$ has a simple zero at some $r = r_H$ then $W(r)$ also has a simple zero at the same point (for a proof of this and other properties see [55]).

The metric Eq. (1.36) is of interest also because it admits two inequivalent definitions of conserved energy: one is the Killing energy, $\omega = -p_\mu \xi^\mu = -\partial_t I$; the other is the Kodama energy which uses the vector field $K = \sqrt{W/V}\,\partial_t$. This vector field comes about because, in spherical symmetry (as spelled out extensively in Section 1.2), it has the amazing property that both $K^\mu$ as well as $J_\nu = -K^\mu T_{\mu\nu}$ are conserved: $\nabla_\mu K^\mu = 0$, $\nabla_\mu J^\mu = 0$. As a consequence it is possible to define two different notions of temperature, depending on which energy one is using.

As we shall prove in detail in Section 1.1.6, even if the metric Eq. (1.36) is singular on the horizon, the tunnelling method works. Using for example the Hamilton–Jacobi version, the massless Hamilton–Jacobi equation for radial motion reads

$$-\frac{1}{V(r)}(\partial_t I)^2 + W(r)(\partial_r I)^2 = 0\,. \tag{1.46}$$

The classical action is given by (plus/minus sign corresponding to outgoing/ingoing particles, respectively)

$$I_\pm = -\omega t \pm \int \omega \frac{dr}{\sqrt{V(r)W(r)}}\,, \tag{1.47}$$

where $\omega = -\partial_t I$ represents the Killing energy. Assuming the near horizon expansion (as noted above, the occurrence of the same zero in $V(r)$ and $W(r)$ is a theorem, given Einstein's equations and some energy condition)

$$V(r) = V'(r_{\rm H})(r - r_{\rm H}) + \dots\,, \qquad W(r) = W'(r_{\rm H})(r - r_{\rm H}) + \dots\,, \tag{1.48}$$

and Feynman's prescription, we readily obtain

$$\mathrm{Im}\int dI_+ - \mathrm{Im}\int dI_- = \frac{\pi\omega}{\kappa} \tag{1.49}$$

where

$$\kappa = \frac{1}{2}\sqrt{V'(r_{\rm H})W'(r_{\rm H})} \tag{1.50}$$

is the Killing surface gravity. One may use the Kodama energy, $\omega_{\rm K} = \sqrt{W/V}\,\omega$, which on the horizon takes the form $\omega_{\rm K} = \sqrt{W'(r_{\rm H})/V'(r_{\rm H})}\,\omega$ via de l'Hôpital rule; in this case one obtains the so called Hayward's surface gravity

$$\kappa_{\rm H} = \frac{1}{2}\,W^{'}(r_{\rm H}) \tag{1.51}$$

and the corresponding temperature $T_{\rm H} = \kappa_{\rm H}/2\pi$. As we shall see in Section 1.2, this is the one that can be generalised naturally to dynamical situations, where there are no more Killing vectors in general. Notice however that $\omega/\kappa = \omega_{\rm K}/\kappa_{\rm H}$, so that the tunnelling probability is invariant.

### 1.1.4 Tunnelling of fermions

At this point of the discussion, one might ask what particles are to be found in the Hawking radiation spectrum of a black hole. Since a black hole has a well defined temperature, in principle it should radiate all the standard model particles like a black body at that temperature (ignoring grey body factors). The emission spectrum, therefore, is expected to contain particles of all spins, in particular fermions as well. The basic reference here is the seminal paper by Kerner and Mann [56], to which others followed for both stationary and dynamical black holes [57–62]. The fact that massless fermions are emitted with the same temperature as massless bosons is not a trivial result, given the fact that fermionic and bosonic vacua are in general distinct. It is important to mention that these expectations have been recently extended to include spin-1 bosons; and that the Hawking temperature does not receive higher order corrections in $\hbar$ beyond the semi-classical one [57, 63, 64] as originally proposed in [65] (see also [66–69] for further extensions).

Now, what do we take as the action of fermionic particles? Undoubtedly the most convenient choice would be to take the phase of the spinor wave function which, as is well known, satisfies the Hamilton–Jacobi equation as for spin-less particles. However, one may also consider the action from another point of view. Virtually, all known variational formulations of the motion of spinning particles have an action of the form [70–72]

$$I_{\rm f} = I_0 + (\text{spin corrections})$$

where $I_0$ is the kinetic term equal to the classical action of scalar particles as considered in previous sections, and the correction terms contain the coupling of the spin degrees of freedom with the spin connection of the manifold. These can either be derived from covariance considerations or more directly by the semi-classical treatment of the Dirac equation itself. Some of these terms give additive corrections to the particle four-momentum but in no case the remaining terms contain horizon singularities, as they are only responsible for spin precession effects. In the following, therefore, we shall ignore them. Next, we neglect any change of angular momentum of the black hole due to the spin of the emitted particle. For zero-angular momentum black holes with mass much larger than the Planck mass ($m_{\rm Pl} \sim 10^{-5}\,$gr) this is a good approximation. Statistically, as many particles with spin in one direction will be emitted as particles with spin in the opposite direction, producing no net change in the black hole angular momentum (although second-order statistical fluctuations will be present in general).

As an example, we consider the tunnelling of fermionic particles across the event horizon of a spherically symmetric, uncharged, static black hole described by the metric Eq. (1.36) with $C(r) = r$; and compute the emission rate in a singular Schwarzschild-like coordinate frame. We refer to the original paper [56] for analogous discussions in generalised Painlevé–Gullstrand and Kruskal–Szekeres coordinates.

Temporarily re-introducing $\hbar$, the Dirac equation in curved space-time reads,

$$\left(\gamma^\mu D_\mu + \frac{m}{\hbar}\right)\Psi(t,r,\theta,\phi) = 0, \tag{1.52}$$

where

$$D_\mu = \partial_\mu + \frac{i}{2}\Gamma^{\alpha}_{\ \mu}{}^{\beta}\,\Sigma_{\alpha\beta}, \tag{1.53}$$

$$\Sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]_- . \tag{1.54}$$

The $\gamma^\mu-$ matrices satisfy the Clifford algebra,

$$[\gamma_\alpha, \gamma_\beta]_+ = 2g_{\alpha\beta}\mathbb{I}, \tag{1.55}$$

where $\mathbb{I}$ is the $(4\times 4)$-identity matrix. For this case, we pick the $\gamma$ matrices

$$\begin{aligned}
\gamma^t &= \frac{i}{\sqrt{V(r)}}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}\end{pmatrix} \quad \gamma^r = \sqrt{W(r)}\begin{pmatrix} \mathbf{0} & \sigma^{\mathbf{3}} \\ \sigma^{\mathbf{3}} & \mathbf{0}\end{pmatrix} \\
\gamma^\theta &= \frac{1}{r}\begin{pmatrix} \mathbf{0} & \sigma^{\mathbf{1}} \\ \sigma^{\mathbf{1}} & \mathbf{0}\end{pmatrix} \quad \gamma^\phi = \frac{1}{r\sin\theta}\begin{pmatrix} \mathbf{0} & \sigma^{\mathbf{2}} \\ \sigma^{\mathbf{2}} & \mathbf{0}\end{pmatrix}
\end{aligned}$$

where $\sigma$'s are the Pauli matrices satisfying usual relations,

$$\sigma_{\mathbf{i}}\sigma_{\mathbf{j}} = \mathbf{1}\delta_{\mathbf{ij}} + \mathbf{i}\varepsilon_{\mathbf{ijk}}\sigma_{\mathbf{k}}, \qquad \mathbf{i},\mathbf{j},\mathbf{k} = \mathbf{1},\mathbf{2},\mathbf{3}\ . \tag{1.56}$$

The matrix for $\gamma^5$ is instead

$$\gamma^5 = i\gamma^t\gamma^r\gamma^\theta\gamma^\phi = i\sqrt{\frac{W(r)}{V(r)}}\frac{1}{r^2\sin\theta}\begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0}\end{pmatrix} .$$

For the spin-up (positive $r$-direction) Dirac field, we employ the *ansatz*

$$\Psi_{\uparrow}(t,r,\theta,\phi) = \begin{pmatrix} H(t,r,\theta,\phi) \\ 0 \\ Y(t,r,\theta,\phi) \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar} I_{\uparrow}(t,r,\theta,\phi)\right] . \tag{1.57}$$

We will only solve the spin-up case explicitly since the spin-down (negative $r$-direction) case is fully analogous. Employing the *ansatz* Eq. (1.57) into the Dirac equation (1.52) results in

$$-\left(\frac{i\,H}{\sqrt{V(r)}}\,\partial_t I_{\uparrow} + Y\sqrt{W(r)}\,\partial_r I_{\uparrow}\right) + Hm = 0 \tag{1.58}$$

$$-\frac{Y}{r}\left(\partial_\theta I_{\uparrow} + \frac{i}{\sin\theta}\,\partial_\phi I_{\uparrow}\right) = 0 \tag{1.59}$$

$$\left(\frac{i\,Y}{\sqrt{V(r)}}\,\partial_t I_{\uparrow} - H\sqrt{W(r)}\,\partial_r I_{\uparrow}\right) + Ym = 0 \tag{1.60}$$

$$-\frac{H}{r}\left(\partial_\theta I_{\uparrow} + \frac{i}{\sin\theta}\,\partial_\phi I_{\uparrow}\right) = 0 \tag{1.61}$$

to leading order in $\hbar$. As we assume that the action takes the form

$$I_{\uparrow} = -\omega\,t + \mathcal{F}(r) + J(\theta,\phi) \tag{1.62}$$

these yield the set of equations

$$\left(\frac{i\,\omega\,H}{\sqrt{V(r)}} - Y\sqrt{W(r)}\,\mathcal{F}'(r)\right) + m\,H = 0 \tag{1.63}$$

$$-\frac{H}{r}\left(J_\theta + \frac{i}{\sin\theta}J_\phi\right) = 0 \tag{1.64}$$

$$-\left(\frac{i\,\omega\,Y}{\sqrt{V(r)}} + H\sqrt{W(r)}\,\mathcal{F}'(r)\right) + Y\,m = 0 \tag{1.65}$$

$$-\frac{H}{r}\left(J_\theta + \frac{i}{\sin\theta}J_\phi\right) = 0 \tag{1.66}$$

Regardless of $H$ and $Y$, Eqs. (1.64) and (1.66) result in $J_\theta + i(\sin\theta)^{-1}J_\phi = 0$, implying that $J(\theta,\phi)$ must be a complex function. The same solution for $J$ is obtained for both the outgoing and incoming cases. Consequently the contribution from $J$ cancels out upon dividing the outgoing probability by the incoming probability as in Eq. (1.91); and we can ignore $J$ from this point. Eqs (1.63) and (1.65) (for $m = 0$) have two possible solutions:

$$H = -i\,Y\,, \qquad \mathcal{F}'(r) \equiv \mathcal{F}'_{\text{out}} = \frac{\omega}{\sqrt{V(r)W(r)}}$$

$$H = i\,Y\,, \qquad \mathcal{F}'(r) \equiv \mathcal{F}'_{\text{in}}(r) = -\frac{\omega}{\sqrt{V(r)W(r)}}$$

where $\mathcal{F}_{\text{out,in}}$ corresponds to outward, inward solutions. The overall tunnelling probability is $\Gamma \sim \exp -2\text{Im}\ (\mathcal{F}_{\text{out}} - \mathcal{F}_{\text{in}})$, with

$$\mathcal{F}_{\text{out}}(r) = -\mathcal{F}_{\text{in}}(r) = \int dr\, \frac{\omega}{\sqrt{V(r)W(r)}}\,. \tag{1.67}$$

Let us remind readers that under the dominant energy condition and Einstein equations, the functions $V$, $W$ have the same zeroes. Therefore near $r_H$ we have, to first order,

$$V(r)W(r) = V^{'}(r_{\text{H}})W^{'}(r_{\text{H}})(r - r_{\text{H}})^2 + \cdots$$

and we see that there is a simple pole with a determined coefficient. Using the Feynman's prescription we obtain

$$\text{Im}\ (\mathcal{F}_{\text{out}} - \mathcal{F}_{\text{in}}) = \text{Im} \int dr\, \frac{2\omega}{\sqrt{V(r)W(r)}} = \frac{\pi\omega}{\kappa} \tag{1.68}$$

where the surface gravity Eq. (1.50) is recovered, namely

$$\kappa = \frac{1}{2}\sqrt{V'(r_{\text{H}})W'(r_{\text{H}})}\,. \tag{1.69}$$

Eq. (1.68) is a special case of a general identity we shall prove in Section 1.1.6. Following a procedure similar to what was done above, we obtain the same result for the Hawking temperature as in the massless case. The spin-down calculation is very similar to the spin-up case discussed here, apart from some changes of signs For both the massive and massless spin-down cases the Hawking temperature $\kappa/2\pi$, with $\kappa$ provided by Eq. (1.69), is obtained, implying that both spin-up and spin-down particles are emitted at the same rate. This is consistent with the initial assumption that there are as many spin-up as spin-down fermions emitted.

With this calculation, we basically end the whole story about tunnelling in static, spherically symmetric black hole space-times. We have seen how the tunnelling picture arose and developed in a variety of different proposals; in Section 1.1.6 we shall confront with more technical aspects as the equivalence between so called null geodesic and Hamilton–Jacobi methods and their mathematical foundations. For now, however, it seems us to be more important to move on and see how the tunnelling method works in conditions where, for example, time starts to play an active role.

### 1.1.5 Axis-symmetric stationary black holes

The generalisation to a less symmetric scenario than the spherical one has not only obvious motivations, being the static spherically symmetric situation not so realistic, but also of fundamental relevance. Accretion processes occurring naturally in astronomical stages are able to spin up a black hole [73, 74] as well as to determine the growth in time of the black hole itself. In this subsection we are not concerned with the latter effect (being substantially a non-stationary process, it will be treated in Section 1.2), but rather we would like to point out some efforts that have been made to extend the tunnelling approach to stationary black hole space-times.

The most immediate generalisation of the Schwarzschild spherical symmetry is the axis-symmetric one, *i.e.* the Kerr solution for a rotating body with mass $M$ and specific angular momentum $a = JM^{-1}$. The subsequent inclusion of electric charge brings us to the Kerr–Newman solution, which still observes the axial symmetry and can be seen as a rotating Reissner–Nordström black hole. We will apply the tunnelling method in order to retrieve the emission probability for scalar and fermionic particles in these cases, highlighting some features that Kerr and Kerr–Newman solutions show.

**Kerr black hole**

The Boyer–Lindquist form for the metric of a stationary rotating black hole — corresponding to the choice of a time-independent reference frame which is asymptotically a Lorentz frame at infinity — is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt\, d\bar{\phi} + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\, \sin^2\theta}{\Sigma} \sin^2\theta\, d\bar{\phi}^2 \tag{1.70}$$

where we have defined,

$$\Sigma := r^2 + a^2\cos^2\theta\,, \qquad \Delta := r^2 + a^2 - 2Mr\,. \tag{1.71}$$

The roots of the equation $\Delta(r) = 0$, $r_\pm = M \pm \sqrt{M^2 - a^2}$, identify the event (outer, plus sign) horizon and the Cauchy (inner, minus sign) horizon.
In order to study the tunnelling process of massless particles from the rotating black hole, however, we consider a metric where the world lines are those of photons with energy $\omega$ moving at infinity with constant $\theta$ and whose projection of the angular momentum on the rotation axis of the black hole is $L_z = a\omega\sin^2\theta$. In these *Kerr ingoing coordinates* [75], obtained from the transformation

$$dv = dt + \frac{(r^2+a^2)}{\Delta} dr \quad , \quad d\phi = d\bar{\phi} + \frac{a}{\Delta} dr \tag{1.72}$$

the line element Eq. (1.70) reads

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dv^2 + 2dv\, dr + \Sigma d\theta^2 - \frac{4aMr\sin^2\theta}{\Sigma} d\phi\, dv + - 2a\sin^2\theta\, d\phi\, dr + \frac{(r^2+a^2)^2 - a^2\Delta\, \sin^2\theta}{\Sigma}\sin^2\theta\, d\phi^2\,. \tag{1.73}$$

We recall that in a rotating black hole the *static limit surface* $r_{\rm st}$, that is the external boundary of the ergosphere given by the equation $g_{tt} = 0$, does not coincide with the event horizon. Some remarks regarding the validity of the semi-classical approach have been raised [76] because the geometrical optical approximation is reliable in proximity of $r_{\rm st}$, not the event horizon, where the tunnelling is supposed to occur. In some works [76,77] the problem is circumvented thanks to a co-rotating change of coordinates $\phi \to \phi - \Omega^{\rm H}\, t$, where $\Omega^{\rm H} = a/(r_+^2 + a^2)$ is the angular velocity of the horizon. In the following we will show that this is not necessary and the right result is recovered automatically.

As fully explained in the previous sections, in the semi-classical approximation, the relevant quantity that plays a role in the calculation of the transition probability through the horizon is the classical action $I$ satisfying the relativistic Hamilton–Jacobi equation. Being the metric Eq. (1.73) independent of $v$ and $\phi$, a suitable *ansatz* for the form of the action is given by $I = -\omega\, v + J\, \phi + \mathcal{F}(\theta, r)$. We are going to see that the imaginary contribution is due to the $r$-dependent part of the action, which will produce a pole in correspondence of the horizon. Expanding the Hamilton–Jacobi equation with the *ansatz* for $I$ above, we obtain

$$\begin{aligned} a^2(m\csc\theta - \omega\sin\theta)^2 + 2\left[a^2 m - \left(a^2 + r^2\right)\omega\right]\mathcal{F}_r + \\ + \left[a^2 + r(-2m + r)\right]\mathcal{F}_r^2 + \mathcal{F}_\theta^2 = 0 \end{aligned} \tag{1.74}$$

where subscripts $r, \theta$ are for $\partial_{r,\theta}$. Solving for $\mathcal{F}_r(\theta, r)$, we immediately note that the function to be integrated is also $\theta$-dependent:

$$\mathcal{F}_r(\theta, r) = -\frac{X(r)}{\Delta(r)} \pm \frac{\sqrt{X(r)^2 - \Delta(r)\left[a^2(M\csc\theta - \omega\sin\theta)^2 + \mathcal{F}_\theta(\theta, r)^2\right]}}{\Delta(r)} \tag{1.75}$$

where $X(r) \equiv a^2 M - \left(a^2 + r^2\right)\omega$. A way to deal with this dependence is to simply fix a constant value $\theta = \theta_0$ and show that eventually the result has no effective dependence on the choice of $\theta_0$ [36]. Actually there is no need to fix $\theta$ because in order to apply the method we consider the regime in which the tunnelling occurs near the horizon, $\Delta(r_+) = 0$: in this situation the term that brings the $\theta$-dependence drops out and the function depends only on $r$. The root of this result is the complete separability of the Hamilton–Jacobi equation in Kerr space-time. Recasting all the expressions in terms of $r_+$ and $r_-$, so that $X(r) = (r_+ + r_-)a^2/2 - \omega(r_+^2 + a^2)$ and $\Delta(r) = (r - r_+)(r - r_-)$, we get

$$\operatorname{Im}\mathcal{F}(\theta, r) = -\operatorname{Im}\int_{\searrow} \frac{X(r) + \sqrt{X^2(r) - (r - r_+)(r - r_-)\,(\dots)}}{(r - r_-)}\frac{dr}{(r - r_+)} \tag{1.76}$$

where $(\dots)$ contains the whole $\theta$-dependence. Regularising the integral by

Feynman's prescription,

$$\mathrm{Im}\, I = -2\pi\, \frac{X(r_+)}{r_+ - r_-} = \pi\left[2\omega\frac{(r_+^2 + a^2)}{(r_+ - r_-)} - \frac{a^2(r_+ + r_-)}{(r_+ - r_-)}\right] . \tag{1.77}$$

Reintroducing $J = a(r_+ + r_-)/2$ and the angular velocity of the horizon $\Omega^{\mathrm{H}}$, we can rearrange the terms and obtain the tunnelling probability rate:

$$\exp\left[-2\,\mathrm{Im}\, I\right] = \exp\left[-\beta_{\mathrm{Kerr}}(\omega - \Omega^{\mathrm{H}} J)\right] \tag{1.78}$$

where the inverse temperature is $\beta_{\mathrm{Kerr}} = 4\pi(r_+^2 + a^2)/(r_+ - r_-)$. In the limit $a \to 0$ we recover the result for the Schwarzschild black hole.

**Kerr–Newman black hole**

The tunnelling of scalar and fermionic, electrically charged, particles in the family of Kerr–Newman space-times does not present new features with respect to previous discussions, the black hole charge $Q$ and specific angular momentum $a$ parameters being treated on the same footing. In the following, we are going to consider first the tunnelling process for a scalar particle of charge $q$ and energy $\omega$, then for a fermion. In both the examples, we implement singular Boyer–Lindquist coordinates $(t, r, \theta, \phi)$

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt^2 - \frac{2(r^2 + a^2 - \Delta)\; a\sin^2\theta}{\Sigma}\, dt\, d\phi + \\ &+ \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\,\sin^2\theta}{\Sigma}\,\sin^2\theta\, d\phi^2 \end{aligned} \tag{1.79}$$

where both incoming and outgoing trajectories are expected to contribute to the emission rate, according to our master equation (1.89). The symbols in Eq. (1.79) are only slightly changed with respect to above: $\Delta := r^2 + a^2 + Q^2 - 2Mr$. The static limit surface is now located at $r_{\mathrm{st}} = M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$, while the event horizon, still the greater root of $\Delta(r) = 0$, is $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$.

**Tunnelling of scalar particles** As in the previous case, the action can be separated as $I = -\omega t + J\phi + \mathcal{F}(\theta, r)$ and further separation of the last terms occurs near the horizon. We explicit the metric functions inside Hamilton–Jacobi equation

$$g^{\mu\nu}\Big(\partial_\mu I - qA_\mu\Big)\Big(\partial_\nu I - qA_\nu\Big) = 0 \tag{1.80}$$

where the vector potential one-form is given by $A = -Q\, r\, \Sigma^{-1}\Big(dt - a\sin^2\theta d\phi\Big)$, to obtain

$$\begin{aligned} -\frac{\left[(r^2+a^2)^2\;\partial_t I + a\;\partial_\phi I - qQr\right]^2}{2\Sigma\Delta} + \frac{\left(\partial_\phi I + a\sin^2\theta\;\partial_t I\right)^2}{2\Sigma\sin^2\theta} + \\ + \frac{\Delta}{2\Sigma}\left(\partial_r I\right)^2 + \frac{\left(\partial_\theta I\right)^2}{2\Sigma} = 0 \end{aligned} \tag{1.81}$$

Making use of the *ansatz* for $I$ and after some manipulations, we solve for $\mathcal{F}_r(\theta, r)$,

$$\partial_r \mathcal{F}_{\text{in,out}}(\theta, r) = \pm \frac{\sqrt{\left[\,(a^2+r^2)\,\omega - a^2 M - qQr\right]^2 - \Delta(r)(\dots)}}{\Delta(r)} \tag{1.82}$$

where the plus (minus) sign corresponds to outgoing (incoming) trajectories. Again, the $(\dots)$ under the square root contains all the $\theta$-dependence and near the horizon it will be negligible. Eq. (1.82) has to be integrated along the path crossing the horizon, so let us write $\Delta(r) = (r - r_+)(r - r_-)$ in order to show explicitly the pole in the integrand and choose the plus sign to select a path that comes out of the black hole. The imaginary part due to the pole, rearranging a little bit the terms, is given by

$$\frac{\pi}{2}\left(\frac{r_+^2 + a^2}{r_+ - M}\right)\left(\omega - \frac{a}{r_+^2 + a^2} M a - \frac{qQr_+}{r_+^2 + a^2}\right) \tag{1.83}$$

where we easily identify the angular velocity $\Omega^{\mathrm{H}}$, the angular momentum parameter $J$ and the term arising from the presence of a charge parameter: this last term is expressible in terms of the electric potential $\Phi = Qr_+/(r_+^2 + a^2)$ of the black hole. In order to retrieve the final expression, we take into account also the contribution coming from the ingoing trajectory, which is given by the very same procedure as before applied to the function Eq. (1.82) with the minus sign. The imaginary contribution is found to be equal and opposite in sign so the tunnelling probability will be given by

$$\frac{\Gamma_{\text{em}}}{\Gamma_{\text{ab}}} = \exp\left[-2\,\mathrm{Im}\,\left(\mathcal{F}_{\text{out}} - \mathcal{F}_{\text{in}}\right)\right] = \; e^{-4\,\mathrm{Im}\,\mathcal{F}_{\text{out}}}\;. \tag{1.84}$$

A more rigorous derivation of this formula will be given in the next section. The final expression for the tunnelling probability can be written as

$$\Gamma_{\text{em}} \propto \exp\Big[-\beta_{\text{KN}}\left(\omega - \Omega^{\mathrm{H}} J - q\Phi\right)\Big] \tag{1.85}$$

from which the temperature can be read $\beta_{\text{KN}}^{-1} = T_{\text{KN}} = (r_+ - r_-)/4\pi(r_+^2 + a^2)$. It is easy to see that two subsequent limits $Q \to 0$ and $a \to 0$ lead us first to the Kerr and then to the Schwarzschild solutions.

**Tunnelling of fermions** As a further extension, it is interesting to consider the tunnelling of gravitinos (spin-3/2 fermions predicted in supergravity theories [78]) in Kerr–Newman black hole space-times [60, 61]. The result is analogous to the scalar case, suggesting again evidence for the universality of black hole radiation, and follows the same line of Section 1.1.4, where now only little more shrewdness in the choice of the representation for the Dirac matrices is needed.

The equation of motion of spin-3/2 particles, namely the Rarita–Schwinger equation, can be written as

$$\begin{aligned} i\,\gamma^{\nu}\left(D_{\nu}+iqA_{\nu}\right)\Psi_{\mu}+\frac{m}{\hbar}\Psi_{\mu}&=0\\ \gamma^{\mu}\,\Psi_{\mu}&=0 \end{aligned}$$

where $\Psi_{\mu}\equiv\Psi_{\mu\,b}$ is a vector valued spinor of charge $q$ and mass $m$; matrices $\gamma^{\mu}$ satisfy the Clifford algebra; and the covariant derivative is defined as in Eqs. (1.53) and (1.54). The Rarita–Schwinger equation presents as a Dirac equation applied to every vector index of the field $\Psi$ and a set of constraints preventing ghosts propagation. In the present context, the *anstatz* for the wave function is $\Psi_{\mu}=(a_{\mu},b_{\mu},c_{\mu},d_{\mu})^{\mathrm{T}}\;e^{iI/\hbar}$, where T is for transposition. It turns out that the Dirac equation can be solved for the action $I$ independently from the components $a_{\mu},...,d_{\mu}$; conversely, $\gamma^{\mu}\Psi_{\mu}=0$ yields a set of constraints for the components $a_{\mu},...,d_{\mu}$, independently by the action $I$. Thus we may conclude that to calculate the Hawking temperature of gravitinos emitted from the Kerr–Newman black hole, we only need to solve for the Dirac equation in precise the same way as in Section 1.1.4; and that, being the action unaffected by the constraints, fermions with every spin will be emitted at the same temperature. We refer the interested reader to the original paper [60] for the precise form of the Dirac matrices suitable to the geometry at hand.

### 1.1.6 Analytic continuation arguments

We have described the Hamilton–Jacobi strategy in a list of four steps, from (a) to (d). It is clear, however, that, at least at first sight, not all of them stay on equal footing: besides some irremissible (*e.g.* postulation of Hamilton–Jacobi equation) or very natural requirements (*e.g.* trajectory splitting as $\overrightarrow{abc}$ into $\overrightarrow{ab}+\overrightarrow{bc}$), we find other less tolerable points. Why, in fact, should it be that particles traveling along classically forbidden trajectories from inside the black hole to outside must follow null paths? And which fundamental principle suggests us to regularise divergent integrals according to one prescription rather than others? By the end of the day, we shall show that — in contrast to the common sense — even the Hamilton–Jacobi equation is an accessory requirement. A certain experience in the field tells us how these points can result in some sense cryptic to the same experts. With the purpose of clarifying some of the points mentioned above, we are going to outline the foundations of the Hamilton–Jacobi method in order to point out what is fundamental and what is only an additional assumption.

**Foundation of Hamilton–Jacobi method** Let us consider the motion of a scalar particle from region $II$ to region $I$ in the eternal version of

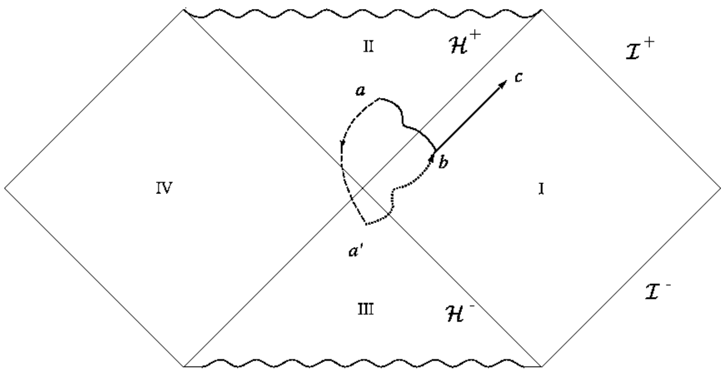


Figure 1.2: A section of the complexified $U, V$ plane of the eternal Schwarzschild black hole. $\theta, \phi$ coordinates are suppressed. $\overrightarrow{abc}$ is a path from inside to outside the hole, with $\overrightarrow{ab}$ a classically forbidden trajectory and $\overrightarrow{a'b}$ representing its reflected trajectory.

Schwarzschild black hole ($\overrightarrow{ab}$ in Fig.(1.2)). This motion is classically forbidden since the particle should travel back in time to follow it. Notice that, in general, nothing we can say about the causal nature of the forbidden path. However, if the coordinates of the starting point $a$ are displaced to complex values, then such an allowed path exists. Thus, the reader must think to Fig.(1.2) as a section of the complexified Schwarzschild plane with coordinates $(\theta, \phi)$ constant and suppressed. By analytically continuing the point $a$ in the complexified Schwarzschild space, the amplitude to propagate to $c$ from a real point $a$ in region $II$ can be related to the amplitude to propagate to $c$ from a reflected point $a'$ in region $III$ [44]. This latter process is just the time-reversed of absorption of a particle by the black hole. In this way, the emission probability for a black hole is related to the probability for it to absorb. All that we need in order to prove the case, is to take as a fundamental assumption the analyticity of the particle action in the complex $(U, V)$ plane as a function of the space-time coordinates of point $a$ [82].

In region $II$ ($III$), the Kruskal–Szekeres coordinates $U, V$ are both positive (negative) definite. So, let us rotate $U$ and $V$ from positive to negative values by posing

$$\tilde{U} = e^{i\vartheta} U \ , \quad \tilde{V} = e^{-i\vartheta} V \ , \qquad \vartheta \in [0, \pi] \ , \tag{1.86}$$

a choice which clearly preserves the invariant product $UV = (1 - r/r_g)e^{r/r_g}$. The integral of $dI$ over the tunnelling path $a \to b \to c$ will now be replaced by the integral over the path (say $\gamma$) consisting of the semi-circle $a \to a'$ with $\vartheta$ ranging from 0 to $\pi$ (over which $|U|$, $|V|$ are constants) plus the integral over the path $a' \to b \to c$ which crosses the past horizon (over which $\vartheta$ is constant). In formulas

$$
\begin{aligned}
I &\equiv \int_{\searrow} dI = \int_\gamma (\partial_{\tilde U} I \, d\tilde U + \partial_{\tilde V} I \, d\tilde V) \\
&= \int^{\nearrow} (\partial_U I \, dU + \partial_V I \, dV) - i \int_0^\pi d\vartheta (\tilde V \, \partial_{\tilde V} - \tilde U \, \partial_{\tilde U}) I \ , \qquad (1.87)
\end{aligned}
$$

where, this time, the upward arrow denotes integration along the classically allowed trajectory $\overrightarrow{a'bc}$. Since in region $I$, $U = -e^{-\kappa u}$ and $V = e^{+\kappa v}$, where $u, v$ are retarded and advanced time coordinates and $\kappa$ is the horizon surface gravity, $\kappa = 1/4M$, we have that $\partial_t = \kappa(V\partial_V - U\partial_U)$. Hence we obtain

$$
\int_{\searrow} dI = \int^{\nearrow} dI - \frac{i\pi}{\kappa} \, \partial_t I = \int^{\nearrow} dI + \frac{i\pi\omega}{\kappa} \ , \qquad (1.88)
$$

with $\omega = -p \cdot \xi = -\partial_t I$ the conserved Killing energy of the emitted particle. Taking into consideration only the imaginary part of the action, which is the relevant part for tunnelling purposes, we get

$$
-\,2 \ \mathrm{Im} \left( \int_{\searrow} dI - \int^{\nearrow} dI \right) = -\frac{2\pi\omega}{\kappa} \ . \qquad (1.89)
$$

Since the geometry is static, it is left invariant by time inversion $\hat T$:

$$
\mathrm{Im} \int^{\nearrow} dI = \hat T \ \ \mathrm{Im} \int^{\nearrow} dI = \ \mathrm{Im} \int_{b \to a} dI \qquad (1.90)
$$

where $b \to a$ is the time inverse of the path $a' \to b$ (Fig.(1.2)). With a justified abuse of terminology, $\int^{\nearrow}$ will denote also the integration along the time-reversed path. From a physical point of view, this is the path followed by an incoming particle absorbed by the black hole, so that exponentiating Eq. (1.89), we can write

$$
\Gamma_{em} = \Gamma_{abs} \, e^{-\frac{2\pi\omega}{\kappa}} \ , \qquad (1.91)
$$

in agreement with the result of Hartle and Hawking [44] derived by path–integral methods.

The rotation of the $(U, V)$ coordinates in the complex plane, Eq. (1.86), has been chosen in the form given because it corresponds to a Wick-like rotation of Schwarzschild time in the lower half complex plane, namely $t \to t - i\vartheta/\kappa$, which is the analyticity region of positive energy solutions of field equations. In fact:

$$
e^{-2\kappa \tilde t} =: \frac{\tilde U}{\tilde V} \overset{(1.86)}{=} e^{2i\vartheta} \, \frac{U}{V} = e^{2i\vartheta} \cdot e^{-2\kappa t} = \exp\left[ -2\kappa \left( t - \frac{i\vartheta}{\kappa} \right) \right] \ . \qquad (1.92)
$$

One might legitimately ask for the meaning of an anti-Wick rotation, $t \to t + i\vartheta/\kappa$. In this case, one rotates $U$ $(V)$ clockwise (counter-clockwise) with consequent change of signs in Eq. (1.86). In turn, Eq. (1.89) changes in favour of

$$- 2\,\mathrm{Im}\left(\int\!\!\!\searrow dI - \int\!\!\!\nearrow dI\right) = \frac{2\pi\omega}{\kappa}\ . \tag{1.93}$$

The plus sign at the right hand side of this equation, tells us that the object we are talking about is not a black hole, but rather its time reverse (*white hole*), for which absorption is exponentially suppressed with respect to emission, $\Gamma_{abs} = \Gamma_{em}\, e^{-\frac{2\pi\omega}{\kappa}}$.

The present discussion can be easily generalized to cover also charged or rotating black hole solutions, with consistent results.

**Equivalence of null geodesic and Hamilton–Jacobi methods** At this point of the discussion, it seems natural to ask what relation exists between the aforementioned null geodesic and Hamilton–Jacobi methods. In literature, (see for instance [36]) we find many specific examples suggesting the essential equivalence between the two methods, but nothing resembling a mathematical proof. To this aim, notice that in the null geodesic approach one starts with

$$\mathrm{Im}\int\!\!\!\searrow p_r dr \tag{1.94}$$

which looks non covariant. However we can use the full Liouville one-form $\varpi = p_\mu dx^\mu$ and write the more general expression,

$$\mathrm{Im}\int\!\!\!\searrow \varpi \tag{1.95}$$

which, without the " Im", is nothing but the reduced action. We can perform the analytic continuation of this integral just as we did for the complete action, first by writing $\varpi = p_U dU + p_V dV$, then rotating $(U, V)$ from zero to $\pi$ and finally integrating along the rotated curve. In this way, the imaginary part will be $i\pi(Up_V - Vp_U)$: but this is $-i\pi\omega/\kappa$, where $\omega = -p_t$ is the Killing energy as measured at infinity. In all we get

$$\mathrm{Im}\int\!\!\!\searrow \varpi = \mathrm{Im}\int\!\!\!\nearrow \varpi + \frac{\pi\omega}{\kappa}\ . \tag{1.96}$$

As another example we consider the line element of the Kerr–Newman solution,

$$
\begin{aligned}
ds^2_{KN} =& \frac{G^2 a^2 \sin^2\theta}{4\kappa_+^2 \Sigma} \frac{(r-r_-)(r+r_+)}{(r^2+a^2)(r_+^2+a^2)} \cdot \\
& \cdot \left[ \frac{\Sigma}{r^2+a^2} + \frac{\Sigma_+}{r_+^2+a^2} \right] (U^2\, dV^2 + V^2\, dU^2) + \\
& + \frac{G(r-r_-)}{2\kappa_+^2 \Sigma} \left[ \frac{\Sigma^2}{(r^2+a^2)^2} + \frac{\Sigma_+^2}{(r_+^2+a^2)^2} \right] dU\, dV + \\
& + \frac{G^2 a^2 \sin^2\theta}{4\kappa_+^2 \Sigma} \cdot \frac{(r+r_+)^2}{(r_+^2+a^2)^2} (U\, dV - V\, dU)^2 + \\
& + \frac{G a \sin^2\theta}{\kappa_+ \Sigma (r_+^2+a^2)} \left[ \Sigma_+ (r-r_-) + (r^2+a^2)(r+r_+) \right] (U dV - V dU) d\phi_+ \\
& + \Sigma\, d\theta^2 + \frac{[(r^2+a^2)^2 - a^2 \sin^2\theta\, \Delta] \sin^2\theta}{\Sigma}\, d\phi_+ \ , \qquad (1.97)
\end{aligned}
$$

where $U, V$ are Kruskal-like null coordinates, defined as,

$$
|U| = \exp \kappa (r_* - t) \ , \qquad |V| = \exp \kappa (r_* + t) \ , \qquad (1.98)
$$

$dr_* = (r^2+a^2) dr/\Delta$ and the surface gravity $\kappa$ can refer to $\kappa_+$ or $\kappa_-$ according to the space-time region of interest. The co-rotating angular coordinate $\phi_+$ is defined as

$$
\phi_+ := \phi - \Omega^H t \qquad (1.99)
$$

with $\Omega^H$ the horizon angular velocity and $G := (r - r_+)/UV$. For all other symbols, the reader is referred to Section 1.1.5 where a complete treatment of axis-symmetric space-times has been given. This metric is particularly important since in the appropriate limits it comprises the maximal analytic extension of Kerr–Newman, Kerr, Reissner–Nordström and Schwarzschild black holes [42].

The analytic continuation of the coordinates in the Euclidean section requires

$$
t \to t - i\,\vartheta/\kappa \qquad \textit{and} \qquad \phi \to \phi - i\,\Omega^H \vartheta/\kappa \ , \qquad \vartheta \in [0, \pi] \qquad (1.100)
$$

where from now on, we consider $\kappa \equiv \kappa_+$. We have $\varpi = p_U dU + p_V dV + p_\phi d\phi$ and therefore

$$
\mathrm{Im} \int_{\searrow} \varpi = \ \mathrm{Im} \int^{\nearrow} \varpi - \frac{\pi \Omega^H}{\kappa}\, p_\phi + \pi (U p_U - V p_V) \ . \qquad (1.101)
$$

But $Up_U - Vp_V = -\kappa^{-1} p_t = \kappa^{-1}\omega$ is the energy and $p_\phi = J$ the conserved angular momentum. We end with our main formula, Eq. (1.89),

$$
2\ \mathrm{Im} \left( \int_{\searrow} \varpi - \int^{\nearrow} \varpi \right) = \frac{2\pi}{\kappa} \left( \omega - \Omega^H J \right) \qquad (1.102)
$$

which proves the equivalence of a *generalised null geodesic method* with Hamilton–Jacobi *ansatz* for all stationary, axis-symmetric, charged black holes. In retrospect, this could not be otherwise because the Hawking effect is an energy conserving process, so that the reduced action is all one needs in a stationary geometry.

## 1.2 Extension to dynamical black holes

In the previous sections we saw that further insight into the Hawking's effect can be obtained by the use of the tunnelling methods. We showed that in some circumstances it is even possible to recover exact results by analytically continuing the integral of the classical action from a tunnelling path to a classical path, and how this possibility provided a foundation to the tunnelling method. But of course everything is strictly true only for stationary black holes.

In this section we shall extend this work to what we shall aptly name *spherically symmetric dynamical black holes*, epitomised by the Vaidya solution [85] and its generalisations by Bardeen [86] and York [87]. As a matter of fact, it has long been felt that the usual semi-classical treatment of stationary black holes had to be extended to cover at least slowly changing black holes. By this expression, we mean black holes that can be still described in terms of few multipole moments such as mass, angular momentum and the charges associated to local gauge symmetries, except that the parameters and the causal structure are allowed to change with time. Although a technical definition of a "slowly varying black hole" can be given in some cases, an example being the Booth–Fairhurst slowly evolving horizon [88], in general this depends on the actual physical processes involved. For example, in the case of Hawking's evaporation, conditions for slowness in the presence of a near-horizon viscous fluid have been given by Brevik [89] in an interesting attempt to generalise 't Hooft's model of the self-screening Hawking atmosphere (quantum corrections to this model can be found in [90]). In general, it is understood that the black hole temperature has to be much smaller than the Planck mass, while in order to study the effects of the expansion, the Hubble rate $H^{-1}$ should be comparable with the black hole emission/absorption rates.

Now, even for the case of a slow evolution, it was pointed out by Fredenhagen and Haag long ago [91] that by letting the mass of the black hole to change with time, the radiation would originate from the surface of the black hole at all times after its formation. This poses the question: *what and where is the surface of a dynamical black hole*? The issue baffled scientists since the beginning of black hole physics and produced several reactions during the Nineties, which eventually culminated with the notion of *outer trapping horizons* by Hayward [92–94] and the isolated and dynamical horizons of

Ashtekar and co-workers [95–98] (an extensive review is in [99]).
One is concerned to study at first the dynamical version of the event horizon and to provide a mathematical definition which is able to capture a useful local notion of it, encompassing the time-lasting textbook definition given in Hawking–Ellis renowned book [45]. Armed with a precise notion of horizon, we shall proceed to study the instabilities occurring near the horizon of the changing black hole. This question looks non trivial since a changing horizon is typically embedded in a dynamical space-time and it is not even expected to be a null hyper-surface, though it is still one of infinite red-shift. Thus we shall start by reviewing what has been done on this question for spherically symmetric metrics. We can anticipate that the arrival point will be the very important local notion of *future trapping horizons* and their associated *surface gravity*, as defined by Hayward.

### 1.2.1 Horizons and surface gravity

Not long after the classical definition of the event (EH) and the apparent horizons (AH) (boundaries of trapped 3-dimensional space-like regions within partial Cauchy surfaces), several quasi-local notions of dynamical horizons were proposed in the literature (another nice review is in [100]), perhaps starting with the notion of perfect horizon due to Hájiček [101]. But this only applied to equilibrium black holes while the apparent horizon, being tied to a partial Cauchy surface, only represents a localisation in time. Moreover, it has proven not possible to formulate thermodynamic laws for AH similar to those holding good for the event horizons.

The first successful attempt to go beyond the limitations imposed either by the instantaneous character of apparent horizons or by the global, teleological nature of event horizons is due to Hayward. His concept of a future outer trapping horizon (to be abbreviated as FOTH) then evolved either into less constrained definitions, like the Ashtekar–Krishnan dynamical horizons (DH), or more specialised ones, like the Booth-Fairhurst slowly evolving FOTH. Although the horizon as defined by Hayward will be central in this thesis, for sake of completeness we shall give an updated list of locally or quasi-locally defined horizons which appeared over the years, each playing some role in the problem of understanding dynamical black hole (more precise definitions will be given soon). At least four types of horizon have been defined over the years:

1. non expanding and perfect horizons (Hájiček [101]);

2. trapping horizons (Hayward [92–94]);

3. dynamical horizons (Ashtekar and Krishnan [97, 98]);

4. isolated and weakly isolated horizons (Ashtekar *et al.* [95, 96]);

5. slowly evolving horizons (Booth and Fairhurst [88]).

Most of these newly defined horizons have very desirable properties: they do not require a space-like hypersurface, no notion of interior and exterior and no conditions referring to infinity, like asymptotic flatness for example (all are non local conditions). Moreover, they are not endowed with teleological features (they do not anticipate the future, so to speak) and, given a solution of Einstein equations, one can find whether they exist or do not exist by purely local computations. Finally, unlike EH they are related to regions endowed by strong gravitational fields and are typically absent in weak field regions.

We recall that the expansion $\theta$ of a bundle of null rays is the rate of change of area transverse to the bundle

$$\frac{dA}{dv} = \int \theta \, d^2 S$$

where $v$ is a parameter along the rays. All quasi-local horizons rely on the local concept of *trapped or marginally trapped surface*: this is a space-like closed two-manifold $\boldsymbol{S}$ such that $\theta_+\theta_- \geq 0$; if $\ell_\pm$ are the future-directed null normals to $\boldsymbol{S}$, normalised to $g_{\mu\nu}\ell_+^\mu \ell_-^\nu \equiv \ell_+ \cdot \ell_- = -1$, then $\theta_+$, $\theta_-$ are the respective expansions or optical scalars, that is the expansions of the two bundles of null rays orthogonal to $\boldsymbol{S}$. It is further assumed that $\ell_+$ is associated to an outgoing null geodesic beam, so that in a region of not too strong gravity $\theta_+ > 0$ and the beam is expanding, as for example within the exterior of a black hole at a safe distance. It follows then that $\ell_-$ is associated to an ingoing null geodesic beam, with $\theta_- < 0$ and the beam contracting along the way. It is always possible to choose double null coordinates $x^\pm$ such that

$$\theta_\pm = \frac{2}{r}\,\partial_\pm r \tag{1.103}$$

where $r$ is the areal radius, defined so that a metric sphere has area $A = 4\pi r^2$. To cover black holes rather than white holes it is further assumed that both expansions are negative (or non positive) on a trapped (marginally trapped) surface. If $\theta_+\theta_- < 0$ the surface is untrapped and marginal if $\theta_+\theta_- = 0$. In spherical symmetry with radial coordinate $r$ this means the co-vector $dr$ is temporal, spatial or null respectively. A further subdivision may be made: a trapped surface is *future* if $\theta_\pm < 0$ and *past* if $\theta_\pm > 0$. A marginal $\boldsymbol{S}$ with $\theta_+ = 0$ is *future* if $\theta_- < 0$, *past* if $\theta_- > 0$, *bifurcating* if $\theta_- = 0$, *outer* if $\partial_-\theta_+ < 0$, *inner* if $\partial_-\theta_+ > 0$ and *degenerate* if $\partial_-\theta_+ = 0$ [92, 94, 102, 103]. Equivalently, the vector field $g^{r\mu}\partial_\mu$ is future causal (it means non-space-like) or past causal, respectively. For a better understanding of these terms we may perhaps note that

$$\partial_r = \partial_+ r\partial_+ + \partial_- r\partial_- = \frac{r}{2}(\theta_+\partial_+ + \theta_-\partial_-)\ .$$

So, for example, on a future trapped surface with $\theta_\pm < 0$ the radial increasing vector is past directed, confirming the intuition that one cannot escape the trapped surface without moving in the past.

So much far for black holes, using the mental picture that we, the observers, are outside it at some radius larger than the radius of the trapped region. For white holes one encounters also marginally trapped surfaces with $\theta_- = 0$ and $\theta_+ > 0$. In cosmology one may encounter again both possibilities, marginally trapped surfaces with either $\theta_- = 0$ or $\theta_+ = 0$, but we defer their description to 1.3.

The abstract definitions will become much more simple and intuitive when referred to spherically symmetric space-times in 1.2.2, where only few of them will be actually used. Here we take the opportunity for few more definitions. The optical scalars can also be defined as follows: the induced metric on each $\boldsymbol{S}$ is

$$q_{\mu\nu} = g_{\mu\nu} + \ell_{+\mu}\ell_{-\nu} + \ell_{-\mu}\ell_{+\nu} \tag{1.104}$$

for in fact $q_{\mu\nu}\ell^\nu_\pm = 0$. Let $q^{\mu\nu} = g^{\mu\nu} + \ell^\mu_+\ell^\nu_- + \ell^\nu_+\ell^\mu_-$, not the inverse of $q_{\mu\nu}$. Then $q^\mu{}_\nu$ is the projection tensor to $T_*(\boldsymbol{S})$, the tangent space to $\boldsymbol{S}$. Associated to the null vector fields $\ell_\pm$ are the projected tensor fields $(\theta_\pm)_{\mu\nu} = q^\alpha_\mu q^\beta_\nu \nabla_\alpha l_{\pm\beta}$ and their decomposition into symmetric, anti-symmetric and trace part. They are tensors on $\boldsymbol{S}$ because $(\theta_\pm)_{\mu\nu}\ell^\nu_\pm = (\theta_\pm)_{\mu\nu}\ell^\nu_\mp = 0$. The twists (anti-symmetric parts) vanish since the geodesic beams are normal to $\boldsymbol{S}$ (we have to assume that $\boldsymbol{S}$ is part of a continuous family of surfaces $\boldsymbol{S}_t$). The expansions are then given by the traces

$$\theta_+ = q_{\mu\nu}\nabla^\mu\ell^\nu_+, \qquad \theta_- = q_{\mu\nu}\nabla^\mu\ell^\nu_- \,. \tag{1.105}$$

Finally, the shear is the trace-free symmetric part (note that $g^{\mu\nu}q_{\mu\nu} = q^{\mu\nu}q_{\mu\nu} = 2$)

$$\sigma^{(\pm)}_{\mu\nu} = \theta_{\pm\mu\nu} - \frac{1}{2}q_{\mu\nu}\theta_\pm \,. \tag{1.106}$$

As before, we indicate the Lie-derivative along a vector field $X$ by the calligraphic symbol $\mathcal{L}_X$ and $\mathcal{L}_\pm = \mathcal{L}_{\ell_\pm}$. Thus, for example, $\mathcal{L}_-\theta_{\mu\nu} = \ell^\alpha_-\partial_\alpha\theta_{\mu\nu} + \partial_\mu\ell^\alpha_-\theta_{\alpha\nu} + \partial_\nu\ell^\alpha_-\theta_{\mu\alpha}$, while on scalars it acts as an ordinary partial derivative. Let us describe the listed horizons in turn, adding comments where appropriate. A black triangle down ▼ will close the definitions.

*Perfect and non-expanding horizons* — A perfect horizon is a smooth three-dimensional *null* sub-manifold $H$ of space-time with null normal $\ell^\mu$ such that its expansion $\theta_\ell = 0$ on $H$ and which intersect space-like hypersurfaces in compact sets. ▼

If in the last clause $H$ is topologically $\mathbb{R} \times \mathbb{S}^2$ and moreover the stress tensor $T_{\mu\nu}$ is such that $-T^\mu_\nu\ell^\nu$ is future causal for any future directed null normal $\ell^\mu$, then $H$ is called a non-expanding horizon .▼

All stationary horizons are perfect, but the converse is not true. These horizons are not very relevant in the present context since they apply only to equilibrium black holes.

*Future outer trapping horizons* — A future outer trapping horizon (FOTH) is a smooth three-dimensional sub-manifold $H$ of space-time which is foliated by closed space-like two-manifolds $\mathbf{S}_t$, $t \in R$, with future-directed null normals $\ell_\pm$ such that: (i) the expansion $\theta_+$ of the null normal $\ell_+$ vanishes; (ii) the expansion $\theta_-$ of $\ell_-$ is negative; (iii) $\mathcal{L}_-\theta_+ < 0$. A future inner trapping horizon (FITH) is as above except that on it $\mathcal{L}_+\theta_- > 0$. ▼

This proved to be the most important definition since forms the basis for almost all other definitions. Condition (i) requires strong fields since certainly $\theta_+ > 0$ in weak fields. The condition (ii) is related to the idea that $H$ is of the future type (e. g. a black hole rather than a white hole); (iii) says that $H$ is of the outer type[2], since a motion of $\mathbf{S}_t$ along $\ell_-$ makes it trapped. It also distinguishes black hole horizons from cosmological ones, which for an expanding universe are FITH. Note that no reference is made to any space-like hypersurface, nor to infinity. Trapping horizons are locally defined and have physical properties such as mass, angular momentum and surface gravity, satisfying conservation laws [97, 104]. They are a geometrically natural generalisation of Killing horizons, which are stationary trapping horizons. A non-stationary trapping horizon is not null, but still has infinite red-shift. Unlike event and apparent horizon they do not require asymptotic flatness.

One can always find a scalar field $C$ on $H$ so that the vector fields

$$V^\mu = \ell_+^\mu - C\ell_-^\mu \quad \text{and} \quad N^\mu = \ell_+^\mu + C\ell_-^\mu \,, \tag{1.107}$$

are respectively tangent and normal to the trapping horizon. Note that $V \cdot V = -N \cdot N = 2C$. Hayward [92, 94] showed that if the *null energy condition* (abbr. NEC) holds then $C \geq 0$ on a FOTH. Thus, the horizon must be either space-like or null, and it is null if and only if the shear $\sigma_{\mu\nu}^{(+)}$ and $T_{\mu\nu}\ell_+^\mu\ell_+^\nu$ both vanish across $H$. Intuitively, $H$ is space-like in the dynamical regime where gravitational radiation and matter are pouring into it and is null when it reaches equilibrium. Conversely, on a FITH $V^\mu$ is either null or time-like.

It is worth mentioning that *the second law of the mechanics of trapping horizons* follows quite easily from this apparatus. Taking the Lie-derivative of $\sqrt{q}$, which is the area density corresponding to the metric $q_{\mu\nu}$ on the cross-sections of $H$, we get

$$\mathcal{L}_V \sqrt{q} = -C\sqrt{q}\,\theta_- \,. \tag{1.108}$$

By definition $\theta_-$ is negative on $H$ and we have just seen that, barring violations of the null energy condition, $C$ also is non-negative. Since $V$ is

[2]For example the Cauchy horizon in the Reissner–Nordström solution is of inner type.

future directed we obtain the local form of the second law: *If the null energy condition holds, then the area element $\sqrt{q}$ on a FOTH is non-decreasing along future directions.*
Integrating over $\boldsymbol{S}_t$ the same law applies to the total area of the trapped sections. As long as the null energy condition is maintained it will be non-decreasing, reaching a constant value if and only if the horizon becomes a null hypersurface.

The main difference between an apparent horizon as defined in Hawking–Ellis and the trapping horizon of Hayward is that the AH represents the instantaneous surface of a black hole, *i.e.* it needs a (partial) Cauchy surface $\Sigma$ and it is very sensitive to the choice of $\Sigma$. To compute the AH one needs only its metric and the second fundamental form of $\Sigma$, namely the initial data for Einstein's equations. Hayward's horizon instead is a null hypersurface $H$ which is insensitive to a choice of $\Sigma$ and does not refer to spatial infinity. The trapping horizon is a foliation of $H$.

The causal character as well as the area law required the validity of the NEC; an evaporating black hole violates NEC, therefore the area law will also be violated and the horizon will be time-like. How it could be that tunnelling along a classically forbidden path is still possible will be seen soon and represents the real possibility of the radiation process even in the temporary absence of a global event horizon.

Next come Ashtekhar and co-workers: they observe that key results, such as the area increase, do not depend on the sign of $\mathcal{L}_-\theta_+$. Hence the following weaker notion was introduced:

*Dynamical horizons* — A smooth three-dimensional, *space-like* sub-manifold $H$ of space-time is a dynamical horizon (DH) if it can be foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, with future-directed null normals $\ell_\pm$ such that: (i) on each leaf the expansion $\theta_+$ of one null normal $\ell_+$ vanishes; (ii) the expansion $\theta_-$ of the other null normal $\ell_-$ is negative. A submanifold $H$ such that $\theta_+ = 0$ and $\theta_- < 0$ is also called a *marginally trapped tube* (abbr. MTT) ▼

Like FOTHs, a DH is a space-time notion defined quasi-locally, it is not related to a space-like hypersurface, it does not refer to infinity, it is not teleological. A space-like FOTH is a DH on which $\mathcal{L}_-\theta_+ < 0$; a DH, which is also a FOTH, will be called a *space-like future outer horizon* (SFOTH). The precise properties of such horizons are fully discussed in [97, 98]. Suitable analogues of the laws of black hole mechanics hold for both FOTHs and DHs. We only note that DHs cannot describe equilibrium black holes since they are space-like by definition. Likewise, DHs cannot describe evaporating black holes since for them the trapping horizon is time-like, but are better suited to describe how a black hole grows in general relativity. In general, if a MTT is space-like it is a DH. If a MTT is time-like it is a time-like membrane. It is of interest that in the Oppenheimer–Volkoff dust collapse the unique MTT on which each *marginally trapped surface* (MTS) is spherical is time-like [105]! This shows that even in spherical symmetry the existence of DH with spherical sections is far from obvious. However, for perfect fluid collapse and spherical scalar field collapse the MTTs are space-like.

If $X$, $Y$ are tangent to a non-expanding horizon we can decompose the covariant derivative

$$\nabla_X Y = D_X Y + N(X,Y)\ell_+ + L(X,Y)\ell_-$$

where $D_X$ is the projection of the vector $\nabla_X Y$ onto the spheres $\boldsymbol{S}_t$ in $H$. If $X$ is tangent to the spheres then $D_X$ is the covariant derivative of the induced metric $q_{\mu\nu}$, and if $X$ is tangent to $H$ one may regard the operator $\widehat{\nabla}_X = D_X + N(X,\cdot)\ell_+$, acting on vector fields, as a connection on $H$. If this connection is "time independent" then the geometry of $H$ is time independent too and we have Ashtekar *et al.* notion of a horizon in isolation.

*Isolated horizons* — A non-expanding horizon with null normal $\ell^\mu$ such that $[\mathcal{L}_\ell, \widehat{\nabla}_X] = 0$ along $H$.▼

These horizons were intended to model black holes that are themselves in equilibrium but possibly in a dynamical space-time. For a detailed description of their mathematical properties we refer the readers to Ashtekar–Krishnan's review [99].

*Slowly evolving horizons* — No matter or radiation can cross an isolated horizon, so the meaning of the first law for them cannot be treated in full generality. Booth and Fairhurst [88] established this law for slowly evolving FOTHs as defined by Hayward by introducing dynamical notions of surface gravity and angular momentum. For this purpose, the concept of a slowly evolving FOTH was then defined.

All the horizons just introduced have their own dynamics governed by Einstein equations. There are for them existence and uniqueness theorems [106], formulation of the first and the second laws [92, 99, 107] and even a "membrane paradigm" analogy. In particular, they carry a momentum density which obey a Navier–Stokes-like equation generalising the classical Damour's equations of EHs, except that the bulk viscosity $\zeta_{FOTH} = 1/16\pi > 0$ [108, 109]. The newly introduced horizons are also mostly space-like or null, therefore the role they may play in the problem of black hole quantum evaporation is unclear. In this connection the following notion can be useful.

*Time-like dynamical horizon* — A smooth three-dimensional, *time-like* submanifold $H$ of space-time is a time-like dynamical horizon (TDH) if it can be foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, with future-directed null normals $\ell_\pm$ such that: (i) on each leaf the expansion $\theta_+$ of one null normal $\ell_+$ vanishes, (ii) the expansion $\theta_-$ of the other null normal is strictly negative.▼

It must be recalled that a FOTH for which the NEC does not hold can be time-like as well. What about non spherical MTS? The Vaidya metric does not admit other, non spherical horizons which also asymptote to the non expanding one. This is general, so a DH is unique, that is a space-like 3-manifold cannot be foliated by two distinct families of MTS each endowed with a structure of a DH. This should also hold for a TH but we do not know whether there is a proof. According to Ashtekhar *et al.* a better control of uniqueness is the most important open issue of the classical theory of dynamical horizons, be trapped or in isolation.

**Surface gravity** The surface gravity associated to an event horizon is a well known concept in black hole physics whose importance can be hardly overestimated. Surprisingly, a number of inequivalent definitions beyond the standard one appeared over the last 15 years or so in the field, with various underlying motivations. We have collected the following (we rely on the review of Nielsen and Yoon [110]):

1. the standard Killing surface gravity (Bardeen *et al.* [111], textbooks);
2. a first definition given by Hayward in [92];
3. the effective surface gravity appearing in Ashtekar–Krishnan [99];
4. the Fodor *et al.* definition for dynamical spherically symmetric space-times [112];
5. the Visser [5] and Nielsen–Visser [39] surface gravity;
6. still one more definition by Hayward [94], using Kodama theory of spherically symmetric space-times [113].

In addition there are some more technical definitions due to Mukohyama and Hayward [114] and to Booth and Fairhurst, the latter related to their notion of evolving horizons [88]. Except for the last item, which is what the tunnelling method leads to, the remaining definitions will be less relevant in this thesis, so we reserve to them only some brief considerations.

The Killing surface gravity is related to the fact that the integral curves of a Killing vector are not affinely parametrized geodesics on the Killing horizon $H$, where the norm $\xi^2 = 0$. Hence

$$\xi^\mu \nabla_\mu \xi_\nu \cong \kappa \xi_\nu$$

defines the Killing surface gravity $\kappa$ on $H$, where $\cong$ means evaluation on the horizon. The Killing field is supposed to be normalised at infinity by $\xi^2 = -1$. The definition can be extended to EHs that are not Killing horizons, by replacing $\xi$ with the null generator of the horizon. However there is no preferred normalisation in this case, and this is one reason of the debating question regarding the value of the surface gravity in dynamical situations.

Hayward's first definition was motivated by the desire to get a proof of the first law for THs. It is defined for a future trapping horizon without appeal to un-affinity of null geodesics, as

$$\kappa \cong \frac{1}{2}\sqrt{-n^\mu \nabla_\mu \theta_\ell}\,. \tag{1.109}$$

This quantity is independent on the parametrisation of the integral curves of the vector field $\ell^\mu$, since the evaluation is on a marginal outer surface where $n \cdot \ell = -1$ and $\theta_\ell = 0$.

Given a weakly isolated horizon $H$, Ashtekar and Krishnan showed that for any vector field $t^a$ along $H$ with respect to which energy fluxes across $H$ are defined, there is an area balance law that takes the form

$$\delta E^t = \frac{\bar{\kappa}}{8\pi G}\delta A_S + \text{work terms}$$

with an effective surface gravity given by

$$\bar{\kappa} = \frac{1}{2R}\frac{dr}{dR}\,.$$

$R$ is the areal radius of the marginally trapped surfaces, *i.e.* $A_S = 4\pi R^2$, the function $r$ is related to a choice of a lapse function and finally $E^t$ is the energy associated with the evolution vector field $t^a$. For a spherically symmetric DH a natural choice would be $r = R$ so $\bar{\kappa} = 1/2R$, just the result for a Schwarzschild black hole. To illustrate the naturalness of this definition, consider a slowly changing spherically symmetric black hole with mass $M(v)$, where $v$ is a time coordinate. Defining the horizon radius at each time by $R = 2M(v)$ and $A_S = 4\pi R^2$, we can differentiate $M$

$$\dot{M} = \frac{\dot{R}}{2} = \frac{1}{2R}\frac{\dot{A}_S}{8\pi}\,.$$

so as to obtain $\delta M = \bar{\kappa}\delta A_S/8\pi$. One recognises the usual area law in differential form with surface gravity $\bar{\kappa} = 1/2R = 1/4M$. Consider, however, the more general possibility where the horizon is the solution of the implicit equation $R = 2M(v, R)$, as it happens for example in the Bardeen–Vaidya metric. The same computation leads to $\dot{M} = \dot{A}_S\,(1 - 2M')/(16\pi R)$, leading to

$$\kappa \cong \frac{1}{4M}\left(1 - 2M'\right) \tag{1.110}$$

a prime denoting the radial derivative. The surface gravity here deduced does not conform to Ashtekar *et al.* definitions, suggesting that its value depends on the definition of the black hole mass one is adopting.

The definition of Fodor *et al.* looks like the Killing form of the surface gravity in that $\kappa\ell^\nu = \ell^\mu \nabla_\mu \ell^\nu$, where now $\ell^\mu$ is an outgoing null vector orthogonal to a trapped or marginally trapped surface. This is because, as a rule, such null vectors are not affinely parametrized, although they can always be parametrized so that $\kappa = 0$. Fodor *et al.* choose to fix the parametrization so that

$$\kappa = -n^{\mu}\ell^{\nu}\nabla_{\nu}\ell_{\mu}$$

with $n^{\mu}$ affinely parametrized and normalised to $n \cdot t = -1$ at space-like infinity, where $t^a$ is the asymptotic Killing field. Note that this definition is non local but looks like a natural generalisation of the Killing surface gravity.

As far as we know, the Visser and Visser–Nielsen surface gravity is only defined in dynamical Painlevé–Gullstrand coordinates, so there is no guarantee that it is a geometrical invariant. Indeed, we shall see that for this class of metrics it is different from the invariant surface gravity computed within the general Hayward formalism.

The last item is a local geometrical definition of the surface gravity for the trapping horizon of a spherically symmetric black hole [94], so we postpone a full discussion to Section 1.2.2. Basically, one introduces local null coordinates $x^{\pm}$ in a tubular neighbourhood of a FOTH and for any sphere of radius $r$ one defines the quantity

$$\kappa = \frac{r}{2}\left(g^{+-}\partial_{-}\theta_{+} + \frac{1}{2}g^{+-}\theta_{+}\theta_{-}\right). \tag{1.111}$$

Evaluated on a trapping horizon, $\theta_{+} = 0$, it will be positive precisely when the horizon is of outer type ($\partial_{-}\theta_{+} < 0$, recall that with our conventions $g^{+-} < 0$) and $\kappa = 0$ if degenerate. The definition may look somewhat artificial, but in fact it can be put in a form that strongly resembles the Killing surface gravity of stationary black holes. To see this, we anticipate a result of the next section according to which, following Kodama [113], any spherically symmetric metric admits a unique (up to normalisation) vector field $K^{\mu}$ such that $\nabla^{\nu}(K^{\mu}\,G_{\mu\nu}) = 0$, where $G_{\mu\nu}$ is the Einstein tensor; for instance, using the double-null form, one finds

$$K = -g^{+-}(\partial_{+}r\partial_{-} - \partial_{-}r\partial_{+}). \tag{1.112}$$

The defining property of $K$ shows that it represents a natural generalisation of the time translation Killing field of a static black hole. Now consider the expression $K_{\mu}\nabla_{[\nu}K_{\mu]}$: it is not hard to see that on H it is proportional to $K_{\nu}$. So one defines the dynamical surface gravity as $K^{\mu}\nabla_{[\nu}K_{\mu]} \cong -\kappa K_{\nu}$. For a Killing vector field $\nabla_{\nu}K_{\mu}$ is anti-symmetric so the definition reduces to the usual one.

### 1.2.2 Spherically symmetric fields

In this section we discuss the general time-dependent spherically symmetric metric that will be the arena of our dynamical tunnelling computations. We have a twofold intent in doing this. If Hawking radiation proceeds by emission of discrete energy quanta, a continuous description of the changing metric would only be possible when their number is so large so as to simulate a continuous streaming of energy. In such a case, a dynamical excreting black hole may be modelled by a continuously differentiable solution of Einstein's equations for most of its history. Given that, it would be quite ironic if there were not evidence of a continuously operating quantum emission process. In other words, if the tunnelling method only worked for the event horizons of stationary black holes then there would be a problem with the use of such dynamical metrics as models of black hole evaporation.

The second aspect has to do with the region where the radiation originates. The tunnelling calculation suggests the the outgoing radiation is emitted from the trapping horizon, not the global event horizon. And it further suggests that the semi-classical probability is related to the horizon surface gravity as defined in Eq. 1.111 [94]. Thus we have first of all to learn how to compute these things in a spherically symmetric, time dependent metric.

### Forms of the metric

The paradigm of a spherically symmetric metric is of course the Schwarzschild vacuum solution of Eq. (1.1). To introduce dynamics one could think to make the replacement $M \to M(t)$; however the resulting metric has a curvature singularity at $r = 2M(t)$; for instance, the scalar curvature is

$$\mathcal{R} = \frac{2r(r-2M)\ddot{M} + 4\dot{M}^2}{(r-2M)^3}$$

so only for special values of the mass function is $\mathcal{R} = 0$; other invariants will diverge though. A more general "Schwarzschild gauge" can be written down

$$ds^2 = -e^{2\Phi(r,t)}\left(1 - \frac{2M(r,t)}{r}\right)dt^2 + \left(1 - \frac{2M(r,t)}{r}\right)^{-1} dr^2 + \\ + r^2(d\theta^2 + \sin^2\theta d\phi) \qquad (1.113)$$

which was studied in depth by Visser in his classic "Dirty Black Holes" [53]. We shall not make use of his metric in the following.

It was discovered by Vaidya [115] that by taking the Schwarzschild metric in advanced Eddington–Finkelstein coordinates $(v, r, \theta, \phi)$ (or retarded, $(u, r, \theta, \phi)$ for other purposes)

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi) \qquad (1.114)$$

and by making the replacement $M \to M(v)$, one obtains a regular solution of Einstein's equations except at the origin $r = 0$, the shining star solution of Vaidya. The metric is still not the most general one allowed by the symmetry:

in fact a spherically symmetric metric can depend at most on two arbitrary functions of the coordinates $(v, r)$. One form is due to Bardeen [86]

$$ds^2 = -e^{2\Phi}\left(1 - \frac{2M}{r}\right)dv^2 + 2e^{\Phi}dvdr + r^2(d\theta^2 + \sin^2\theta d\phi) \qquad (1.115)$$

where $\Phi$ and $M$ are functions of $v$, $r$. Another important form is obtained by passing from Schwarzschild time to the proper time of a radially infalling observer

$$t_p = t + 2\sqrt{2Mr} + 4M\log\left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right). \qquad (1.116)$$

The metric is the Painlevé–Gullstrand metric we encountered in Section 1.1

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt_p^2 + 2\sqrt{\frac{2M}{r}}\,dt_p dr + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi\right). \quad (1.117)$$

Taking now $M = M(t_p)$ we get a dynamical black hole long studied by Lindesay and Brown [116–118] and others [119]. There is a more general form whereby $M$ can also depend on $r$ and a second function $\Phi(\tilde{t}, r)$ is introduced, say

$$ds^2 = -e^{2\Phi(\tilde{t},r)}\left(1 - \frac{2M(\tilde{t}, r)}{r}\right)d\tilde{t}^2 + 2e^{\Phi(\tilde{t},r)}\sqrt{\frac{2M(\tilde{t}, r)}{r}}\,d\tilde{t}\,dr + dr^2 + r^2 d\Omega^2\,, \qquad (1.118)$$

where $d\Omega^2 := d\theta^2 + \sin^2\theta d\phi$. This metric is locally diffeomorphic to the metric Eq. (1.113) under a change of time $t \to \tilde{t}(t, r)$, with the mass treated as a scalar field, *i.e.* $M(\tilde{t}, r) = M(t(\tilde{t}, r), r)$. It was thoroughly studied by Visser and Nielsen in [39]. In this rather impressive work the dynamics and decay of evolving horizons were investigated mainly using a dynamical version of Painlevé–Gullstrand coordinates. The metric Eq. (1.118) is also locally diffeomorphic to the metric Eq. (1.115), under a change $v \to \tilde{t}(v, r)$, where now $M(\tilde{t}, r) = M(v(\tilde{t}, r), r)$.

All these spherically symmetric metrics (though not only these) are special cases of metrics that can locally be expressed in the warped form

$$ds^2 = \gamma_{ij}(x^i)dx^i dx^j + R^2(x^i)d\Omega^2\,, \qquad i, j \in \{0, 1\}\,, \qquad (1.119)$$

where the two-dimensional metric

$$d\gamma^2 = \gamma_{ij}(x^i)dx^i dx^j \qquad (1.120)$$

is referred to as the normal metric (in the space normal to the sphere of symmetry), $x^i$ are associated coordinates and $R(x^i)$ is the areal radius, considered as a scalar field in the normal two-dimensional space. The two-dimensional metric $\gamma_{ij}$ has only one degree of freedom and the function

$R(x^i)$ is the second one; however the two functions on which the metric depends can be moved both to $d\gamma^2$ using the coordinates freedom. That is we can rename $x^1 = r$ and put $R(x^i) = r$. The coordinate $r$ is known as the areal radius, and is a geometrical invariant being the quotient of the area of space-like spheres by $4\pi$. This will be referred to as the *$r$-gauge*. Sometimes the normal metric can be further written, somewhat redundantly, in the expanded form

$$d\gamma^2 = -E(r,t)dt^2 + 2F(r,t)dtdr + G(r,t)dr^2, \quad \gamma = -(EG + F^2) \quad (1.121)$$

especially if one wants to discuss certain limits, but we will always try to get metrics regular across the trapping horizons. If we wish, we can also locally write the line element in a double-null form which is conformally flat

$$ds^2 = -2e^f dx^+ dx^- + r^2(x^+, x^-)d\Omega^2 \quad (1.122)$$

with $f = f(x^+, x^-)$. For instance, starting with Eq. (1.115) we can introduce null coordinates via

$$e^f dx^- = \frac{1}{2}\left(1 - \frac{2m(r,v)}{r}\right)e^{2\Phi}dv - e^{\Phi}dr, \quad dx^+ = dv$$

where $f$ is an integrating factor. The normal section of Eq. (1.115) then takes the form Eq.(1.122). One may use one spatial and one temporal direction on putting $dx^+dx^- = dt^2 - dr^2$, but there is no unique choice of such directions. The remaining coordinate freedom consists of conformal diffeomorphisms (in the language of two-dimensional metrics)

$$x^\pm \to \tilde{x}^\pm(x^\pm)\,.$$

The double-null form will be also referred to as the *conformal gauge.*

Another form we would like to describe is the metric in the so called *synchronous gauge*. Let us consider the Schwarzschild space-time in coordinates $(T, r, \theta, \phi)$ such that the line element can be expressed as

$$ds^2 = -dT^2 + \frac{dr^2}{B} + (r_g B)^2 d\Omega^2\,, \quad (1.123)$$

where $r_g = 2M$ is the usual gravitational radius, and

$$B(T,r) := \left[\frac{3}{2r_g}(r - T)\right]^{\frac{2}{3}}\,. \quad (1.124)$$

We shall refer to these coordinates as the Lemaître–Rylov gauge. This is indeed an interesting (time-dependent) gauge since, contrary for example to isotropic coordinates, $(T, r)$ extend beyond the gravitational radius, $r < r_g$.

Now considering $B(T,r)$ as an arbitrary function and replacing $r_g B$ with a general function $R(r,T)$ we obtain the metric in synchronous gauge

$$ds^2 = -dT^2 + \frac{dr^2}{B(T,r)} + R^2(T,r)d\Omega^2 \tag{1.125}$$

in which the metric is diagonal but the areal radius is a function of $r$ and $T$.

The last form we would like to mention is the *isotropic gauge* in which the spatial part of the metric is conformally flat

$$ds^2 = -A(t,\rho)dt^2 + B(\rho,t)(d\rho^2 + \rho^2 d\Omega^2)\,. \tag{1.126}$$

An important example of this latter form is the McVittie solution describing in author's mind a point mass in a FRW flat cosmology [120]. It reads as in Eq. 1.126 with

$$A(\rho,t) = \left(1 + \frac{M}{2a(t)\rho}\right)^{-2}\left(1 - \frac{M}{2a(t)\rho}\right)^{2} \tag{1.127}$$

$$B(\rho,t) = a(t)^2\left(1 + \frac{M}{2a(t)\rho}\right)^{4}\,. \tag{1.128}$$

When $M = 0$ it reduces to a spatially flat FRW metric with scale factor $a(t)$; when $a(t) = 1$ it reduces to the Schwarzschild metric with mass $M$ in isotropic coordinates. This solution had a strong impact on the general problem of matching the Schwarzschild solution with cosmology, a problem faced also by Einstein and Dirac. Besides McVittie, it has been extensively studied by Nolan in a series of papers [121]. To put the metric in the general $r$-gauge form, the coordinate transformation

$$r = a(t)\rho\left(1 + \frac{M}{2a(t)\rho}\right)^{2}$$

transforms the metric in the so called Nolan gauge, in which it reads

$$ds^2 = -\left(A_s - H^2(t)r^2\right)dt^2 - \frac{H(t)r}{2\sqrt{A_s}}\,dr\,dt + \frac{dr^2}{A_s} + r^2 d\Omega^2 \tag{1.129}$$

where $H(t) = \dot{a(t)}/a(t)$ is the Hubble parameter and, for example, in the charged four-dimensional case, $A_s = 1-2M/r+e^2/r^2$. Unlike the Schwarzschild case, $A_s = 0$, or $r = 2M$ in the neutral case or $\rho = M/2a(t)$, is a curvature singularity rather than a global event horizon. In fact, it represents a Big Bang singularity very similar to $r = 0$ in FRW models. When $H = 0$ one recovers the Schwarzschild solution. For constant $H$, it reduces to the Schwarzschild–de Sitter solution in Painlevé coordinates. As we shall see, the McVittie solution possesses in general black hole and cosmological trapping horizons, and the space-time is dynamical.

Other coordinate systems which can be used to include dynamics in the Schwarzschild metric are discussed in [122]. We may note that Eq. (1.125) contains as a special case the FRW family of metrics. We stress that all forms discussed above are completely equivalent ways to describe the space-time structure of a spherically symmetric field, the use of any particular form being dictated only by computational convenience. We also note that in the warped form Eq. (1.119) the sphere of symmetry can be replaced by anyone among the two-dimensional manifolds with constant curvature which appear in the static topological black hole solutions embedded in Anti–de Sitter space. Only, the rotational isometry group must be replaced by the appropriate isometry group of the surface, which in the static case are torii or compact Riemann surfaces with higher genus.

**Trapping horizon**

To compute the trapping horizons we shall start from the Bardeen–Vaidya form Eq. (1.115). We can take the two null, future directed congruences normal to spheres of constant radius as

$$\ell_+ = \left[1, \frac{1}{2}e^{\psi}\left(1 - \frac{2M}{r}\right), 0, 0\right], \quad \ell_- = [0, -e^{-\psi}, 0, 0] \tag{1.130}$$

where as always $\ell_+ \cdot \ell_- = -1$. A simple computation gives the optical scalars

$$\theta_+ = \frac{1}{r}\left(1 - \frac{2M}{r}\right)e^{\psi}, \quad \theta_- = -\frac{2e^{-\psi}}{r}. \tag{1.131}$$

Trapped or marginally trapped spheres have $\theta_+\theta_- \geq 0$, therefore there are no such surfaces in the region $r > 2M(r, v)$. We also see that $\theta_- < 0$ along the surface defined by $\theta_+ = 0$, which is therefore a trapping horizon of the future type. Its defining equation is $r = 2M(r, v)$, which defines a line $r = r(v)$ in normal space and therefore a hyper-surface with topology $\mathbb{R}^1 \times \mathbb{S}^2$ in space-time.

Consider now $\partial_-\theta_+$, where the symbol $\partial_-$ denotes the directional derivative along $\ell_-$. One obtains, using $M' = \partial_r M$,

$$\partial_-\theta_+ = -\frac{1}{r^2}\left(\frac{2M}{r} - 2M'\right).$$

On the trapping horizon $\partial_-\theta_+ \cong -(1 - 2M')/4M^2$ will be negative if and only if $M' < 1/2$. As anticipated in 1.2.2, this is the condition which ensures the positivity of the surface gravity. Thus in this case $r = 2M$ is a FOTH. The areal radius of the horizon will be denoted by $r_H$ from now on. The signs of $\theta_\pm$ are geometrical invariants, but their actual values are not because the null directions are defined up to an overall scale. An invariant combination is $\chi = 2g^{\pm}\theta_+\theta_-$ or, using Eq. (1.119),

$$\chi = \gamma^{ij}\,\partial_i R\,\partial_j R \equiv g^{\mu\nu}\,\partial_\mu R\,\partial_\nu R\,. \tag{1.132}$$

$\chi = 0$ is then the condition for a trapping horizon in Hayward's sense. For example in any coordinate system where one coordinate is the areal radius $R = r$, the condition is $g^{rr} = 0$. The trapping horizon of McVittie solution can also be computed using Eq. 1.132: using Nolan form it is a solution of the equation $\sqrt{A_s} = Hr_H$, which in turn implies $1 - 2M/r_H = H(t)^2 r_H^2$. As $M$ is a constant this a cubic algebraic equation with a priori more than one real root and in any case at most two positive roots. The situation is similar to de Sitter space, except that the horizon radius is here a function of time, $r_H(t)$. For positive $H$ both horizons are of the future type, but as a rule one is outer (meaning that there is a black hole) while the larger root corresponds to an inner horizon, hence to a FITH, if and only if

$$\frac{M}{r_H^2} - H^2 r_H - \frac{\dot{H}}{2H} > 0\,. \tag{1.133}$$

For $M = 0$, *i.e.* for homogeneous cosmology, only the inner cosmological horizon survives. This is the case, for example, in de Sitter space-time. We will make use of these results in Section 1.3.

### Misner–Sharp–Kodama energy

One special feature of spherically symmetric space-times is the absence of gravitational radiation. This feature makes it possible the existence of a special, privileged notion of energy, the Misner–Sharp mass, which for spheres with areal radius $r$ is the same as the Hawking mass [123] (a general reference for energy in GR is the review [124]). The energy may be defined by [103] (we recall that $r$ is the areal radius)

$$E = \frac{r}{2} - r g^{+-} \partial_+ r \partial_- r = \frac{r}{2} - \frac{r^3}{4}\, g^{+-} \theta_+ \theta_- \tag{1.134}$$

and interpreted as the energy inside a sphere of radius $r$; so, by definition, a metric sphere is trapped if and only if $E > r/2$, marginal if and only if $E = r/2$ and untrapped if and only if $E < r/2$. Note that Eq. (1.134) is a special case of the Hawking mass

$$E(S) = \sqrt{\frac{Area(S)}{16\pi}} \left(1 - \frac{1}{8\pi} \oint g^{-+} \theta_+ \theta_- d^2 S\right) \tag{1.135}$$

and is a geometrical invariant. Two very important properties of $E$ were proved by Hayward [103]: *in an asymptotically flat space-time, $E$ coincides with the Bondi-Sachs scalar energy at null infinity, and with the Arnowitt–Deser–Misner (ADM) mass at spatial infinity.* Using Eq. (1.115) the energy takes the implicit form

$$g^{\mu\nu} \partial_\mu r \partial_\nu r = g^{rr} = 1 - \frac{2E}{r} \tag{1.136}$$

which shows the relation with the Schwarzschild mass. Then $E = M$ and the FOTH is at $r = 2E$. Many important properties of $E$ can be displayed using also the Einstein gravitational field equations. We report two of them which seem remarkable, leaving the interested reader to the literature. Using double-null coordinates the variation of $E$ as determined by Einstein's equations is

$$\partial_\pm E = 2\pi e^{-f} r^3 (T_{+-}\theta_\pm - T_{\pm\pm}\theta_\mp) \qquad (1.137)$$

where we remember that $e^{-f} = -g^{+-}$. These field equations can also be written in the Bardeen–Vaidya form, using Eq. (1.115), where it is seen that $E = M$ and

$$\partial_v M = 4\pi r^2 T^r_{\ v}\,, \quad \partial_r M = -4\pi r^2 T^v_{\ v}\,. \qquad (1.138)$$

Thus in vacuo $E$ is a constant. It can then be shown that the solution is locally isometric to a Schwarzschild solution with energy $E$. This is an improvement of Birkhoff's theorem.

The second result is essentially the area law: *if the NEC holds on a FOTH then $E = r_H/2$ is non-decreasing along the horizon.*

The question arises naturally whether $E$ is the charge associated to a conserved current. It was discovered by Kodama [113] that in spherical symmetry there is a vector field $K^\mu$ such that $\nabla^\mu(G_{\mu\nu}K^\nu) = 0$; by Einstein equations it follows also the conservation equation

$$\nabla_\mu \left(T^{\mu\nu} K_\nu\right) = 0 \qquad (1.139)$$

and a corresponding charge. If we define the two-dimensional Levi–Civita skew tensor

$$\epsilon_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}\, \tau_1^\alpha\, \tau_2^\beta$$

where $\tau_1$, $\tau_2$ are tangent vectors to constant radius spheres, then the Kodama vector may be defined by

$$K^\mu = \epsilon^{\mu\nu}\partial_\nu r\,. \qquad (1.140)$$

It can easily be seen that

$$\nabla_\mu K^\mu = 0\,. \qquad (1.141)$$

Let us give few examples:

(a) for static, non-dirty, black holes it is the Killing field;

(b) for dirty black holes, Eq. 1.36, $K = \sqrt{W/V}\;\partial_t$;

(c) in Bardeen–Vaidya gauge $K = e^{-\Phi}\partial_v$;

(d) for the metrics Eq. (1.117) and Eq. (1.118), $K = \partial_{t_p}$ or $K = e^{-\Phi}\partial_{\tilde{t}}$, respectively;

(e) for the metric Eq. (1.121) $K = (EF + G^2)^{-1/2}\partial_t$;

(f) in conformal gauge $K = -g^{+-}(\partial_+ r \partial_- - \partial_- r \partial_+)$;

(g) in synchronous gauge $K = \sqrt{B}(R'\partial_\tau - \dot{R}\partial_r)$;

(h) for FRW flat cosmology $K = \partial_t - Hr\partial_r$.

In every case it follows that

$$K^2 = \frac{2E}{r} - 1$$

so that $K$ is time-like, space-like or null if and only if $r > 2E$, $r < 2E$ or $r = 2E$, respectively. Let us also define the current

$$j^\mu = -T^{\mu\nu}K_\nu \,. \tag{1.142}$$

We have just seen that $\nabla_\mu j^\mu = 0$, $\nabla_\mu K^\mu = 0$, and therefore there exist two conserved charges

$$Q_j = -\int_\Sigma J^\mu n_\mu d^3V \tag{1.143}$$

$$Q_K = -\int_\Sigma K^\mu n_\mu d^3V \tag{1.144}$$

where $\Sigma$ is a space-like three-dimensional surface with fixed boundary at some constant $r$ and future pointing time-like normal $n_\mu$. Here "conserved" means independent on the choice of $\Sigma$. The charges as defined will be positive in regions where $j^\mu$, $K^\mu$ are both time-like. Using Eq. (1.137) one can easily see that

$$j = \frac{1}{4\pi r^2}(\partial_+ E \partial_- - \partial_- E \partial_+) \,. \tag{1.145}$$

To compute the charges we pass to a synchronous gauge by choosing coordinates $(\tau, \zeta)$ adapted to $\Sigma$, with $\partial_\tau$ normal and $\partial_\zeta$ tangent to $\Sigma$. In these coordinates we can always write the metric in the form

$$ds^2 = -d\tau^2 + e^\lambda d\zeta^2 + r^2(\tau, \zeta) d\Omega^2 \,. \tag{1.146}$$

From Eqs. (1.140) and (1.142) we obtain

$$K^\mu = e^{-\lambda/2}(r', -\dot{r}, 0, 0), \quad j^\mu = \frac{e^{-\lambda/2}}{4\pi r^2}(E', -\dot{E}, 0, 0) \tag{1.147}$$

where $\dot{r} = dr/d\tau$, $r' = dr/d\zeta$. Following Hayward, we shall say that the point $r = 0$ is a *regular centre* if it is a boundary point of the normal space and $E/r \to 0$ as the centre is approached. Otherwise it is a *central singularity.* We now assume that $\Sigma$ extends from a regular centre to some $r > 0$: from Eq. (1.147) and the metric we have the normal $n^\mu = \delta^\mu_0$ and

$$-K^\mu n_\mu = e^{-\lambda/2} r', \quad -j^\mu n_\mu = \frac{e^{-\lambda/2}}{4\pi r^2} E' \,.$$

Therefore integrating over $\Sigma$ with the invariant measure
$dV = e^{\lambda/2} r^2 \sin\theta d\zeta d\theta d\phi$, we finally obtain

$$Q_K = \frac{4\pi}{3}\, r^3 , \qquad Q_j = E\,. \tag{1.148}$$

We conclude that $E$ is indeed the charge associated to a conserved current. The charge $Q_j$ is the definition of energy of Kodama. For static solutions the Kodama vector coincides with the Killing field, the generator of the time translation symmetry.

**The surface gravity**

The properties of the Kodama vector field discussed so far prompt for a natural definition of the surface gravity of a trapping horizon. We have seen that $K$ becomes null precisely on a trapping horizon and space-like within. The time-like integral curves of $K$ are in general not contained within the horizon. Consider the quantity $K^\mu \nabla_{[\nu} K_{\mu]}$: it can be seen that on a trapping horizon it is proportional to $K_\nu$

$$K^\mu \nabla_{[\nu} K_{\mu]} \cong -\kappa K_\nu\,. \tag{1.149}$$

The function $\kappa$ is, by definition, the horizon surface gravity of Hayward. For static black holes $K$ is the Killing field so $\nabla_\mu K_\nu$ is anti-symmetric and the definition reduces to the usual one. A formula to compute $\kappa$ efficiently was found by Hayward [94]. Working in double-null coordinates we have $K_+ = -\partial_+ r$, $K_- = \partial_- r$ (see point (f) of the examples list) therefore

$$\partial_+ K_- = -\partial_- K_+ = \partial_- \partial_+ r = \frac{g_{+-}}{2}\, \Box_\gamma r \tag{1.150}$$

where $\Box_\gamma = 2g^{+-}\nabla_+\nabla_-$ is the two-dimensional Klein–Gordon operator acting on scalars. Computing the left hand side of Eq. (1.149) then gives the wanted formula

$$\kappa = \frac{1}{2}\Box_\gamma r = \frac{1}{2}\, r\left(g^{+-}\partial_-\theta_+ + \frac{1}{2} g^{+-}\theta_+\theta_-\right)\,, \tag{1.151}$$

where the last form is obtained by using $\theta_\pm = 2r^{-1}\partial_\pm r$. That is, $\kappa$ is the "Box" of $r$; when evaluated on the trapping horizon, where $\theta_+ = 0$, it is the surface gravity and is positive if and only if $\partial_-\theta_+ < 0$, that is if the horizon is of outer type. Its invariant character is manifest. Let us give few examples with comparison to other definitions. For the Bardeen–Vaidya metric one obtains

$$\kappa = \frac{1}{4M}(1 - 2M') \tag{1.152}$$

which is also the Visser dynamical surface gravity as defined in [5] in a Painlevé–Gullstrand frame. The first Hayward's definition (see the list in

Section (1.2.1)), invoked without appeal to un-affinity, would give instead

$$\tilde{\kappa} = \frac{1}{4M}\sqrt{1-2M'}$$

which is not even correct for the static Reissner–Nordström solution[3]. The definition of Fodor *et al.* gives

$$\hat{\kappa} = \frac{e^{\Phi}}{4M}(1-2M') + \dot{\Phi}\,.$$

For the dynamic Painlevé–Gullstrand metric Eq. (1.117) with $M = M(t_p, r)$, we obtain

$$\kappa = \frac{1}{4M}(1-2M'+2\dot{M}) \tag{1.153}$$

while the Visser and Visser–Nielsen surface gravity for this kind of metrics would give the same formula without the time derivative term. Incidentally, this shows that the dynamical Bardeen–Vaidya metric is not diffeomorphic to a dynamical Painlevé–Gullstrand metric Eq. (1.117) with the same mass function, as it would happen in the stationary case, and therefore represent physically different gravitational fields. For the dynamic Painlevé–Gullstrand metric Eq. (1.118), with mass function $m(\tilde{t}, r)$, we obtain instead

$$\kappa = \frac{1}{4m}(1-2m'+2\dot{m}e^{-\Phi}) \tag{1.154}$$

again different from the Visser–Nielsen surface gravity for the same mass function, which has no time derivative terms. One may also mention the "effective surface gravity", $\kappa_{eff} = 1/2r_H$ discussed, for example, in [99] and [125].

### 1.2.3 Tunnelling from trapping horizons

We now come to review what the tunnelling method has to say about dynamical, spherically symmetric black holes. We shall start by identifying the dynamical version of the tunnelling path which was displayed and discussed is Section (1.1), see 1.1. This will be accomplished by using a specific and convenient form of the metric, which we start to review a little more than already done. In absence of analytical techniques we shall make explicit use of the Hamilton–Jacobi equation, the null geodesic method being quite inconvenient to treat truly dynamical metrics (but see Clifton [119]). The covariance of the method will be stressed throughout.

[3] For which $M' = q^2/2(2M^2 - q^2 + 2M\sqrt{M^2-q^2})$.

**Metrics to be used**

We shall use for the time being the Bardeen–Vaidya (BV) metric Eq. (1.115), which we recall here

$$ds^2 = -e^{2\Phi(r,v)}\left(1 - \frac{2M(r,v)}{r}\right)dv^2 + 2e^{\Phi(r,v)}dvdr + r^2 d\Omega^2 \,. \tag{1.155}$$

The sphere of symmetry will not play any role here, though. To illustrate the covariance of the results we shall occasionally make use of the metric in Painlevé–Gullstrand (PG) form, either Eq. (1.117) or Eq. 1.118). Few things about metric (1.155) will help with the understanding. The field equations read

$$\frac{\partial M}{\partial v} = 4\pi r^2 T^r_{\ v}, \quad \frac{\partial M}{\partial r} = -4\pi r^2 T^v_{\ v}, \quad \frac{\partial \Phi}{\partial r} = 4\pi r e^{\Phi} T^v_{\ r}\,, \tag{1.156}$$

and the stress tensor can be written as

$$T_{\mu\nu} = \frac{\dot{M}}{4\pi r^2}\nabla_\mu v \nabla_\nu v - \frac{M'}{2\pi r^2}\nabla_{(\mu} r \nabla_{\nu)} v\,. \tag{1.157}$$

If $M$ only depends on $v$, it describes a null fluid obeying the dominant energy condition for $\dot{M} > 0$. For the excreting black hole $\dot{M} < 0$ so the null energy condition will also be violated. We already know that $r = 2M(r,v)$ is a trapping horizon, that is a FOTH, if and only if $2M' < 1$, which we shall assume from now on. Putting $r = 2M$ into the metric gives

$$ds^2 = e^{2\Phi}\left(\frac{4e^{-\Phi}\dot{M}}{1-2M'} - 1\right)dv^2$$

so we conclude that the horizon is certainly time-like if $\dot{M} < 0$; we know from general results that it will be space-like or null if $\dot{M} > 0$, even if it is not evident from this expression. The Hayward surface gravity is

$$\kappa = \frac{1}{4M}(1 - 2M') \tag{1.158}$$

and is positive under our assumptions. As we extensively explained, the Misner–Sharp mass, or energy for short, is the value $E(v)$ taken by $M(v,r)$ on the trapping horizon, *i.e.*

$$E(v) = M(v, r_H(v)) = \frac{r_H(v)}{2}\,. \tag{1.159}$$

Using Eq. (1.156) one can show that an observer at rest at $r \gg r_H$ sees a quasi-static geometry with a luminosity $L = -dE/dv$. We shall not assume, initially, that $L$ has the Hawking form $L = \hbar\mathcal{N}E^{-2}$, with $\mathcal{N}$ a constant proportional to the number of massless species radiated from the black hole.

For sake of completeness, we note that a cosmological constant can be introduced via

$$ds^2 = -e^{2\Phi(r,v)} \left(1 - \frac{2M(r,v)}{r} - \frac{\Lambda r^2}{3}\right) dv^2 + 2e^{\Phi(r,v)} dv dr + r^2 d\Omega^2 \,. \quad (1.160)$$

We shall admit that $\Lambda > 0$; if $M$ is constant and $\Phi = 0$ one recognises the Schwarzschild–de Sitter metric in advanced coordinates. Let us call $C$ the expression in round brackets above; the null normals and their expansions are

$$\ell_+ = \left[1, \frac{1}{2}e^{\Phi}C, 0, 0\right], \qquad \theta_+ = \frac{1}{r}e^{\Phi}C \quad (1.161)$$

$$\ell_- = [0, -e^{-\Phi}, 0, 0], \qquad \theta_- = -\frac{2e^{-\Phi}}{r} \,. \quad (1.162)$$

Hence the horizons are located in correspondence of the roots of the equation $C = 0$; the tractable case is $M = M(v)$. Then this becomes a cubic equation which for $0 < 9\Lambda M^2 < 1$, as is well known, admits precisely two real positive roots $r_c$, $r_b$, with $r_c > r_b$ by definition. We see that $\theta_+$ vanishes at both roots and $\theta_- < 0$, therefore the horizon spheres $r = r_b$, $r = r_c$, are marginally trapped surfaces of the future type which foliate a black hole and a cosmological trapping horizon, respectively. One can easily show that

$$\partial_- \theta_+ = -\frac{1}{r}\,\partial_r C \,.$$

Computing the radial derivatives at both horizons we see that this is negative at $r_b$ and positive at $r_c$. The cosmological horizon is therefore an example of a trapping horizon of inner type, the black hole horizon at $r = r_b$ remaining of the outer type.

**Rays tracing**

The most important features of a dynamical black hole of the kind discussed here are: (i) the existence of the irremovable space-like singularity at the origin $r = 0$ of the coordinate system; (ii) the possible existence of a global event horizon ($\mathcal{H}$ in Fig. (1.3)); and (iii) the time-like future trapping horizon ($TH$ in Fig. (1.3)).
We have seen that the BV form and the PG form are locally diffeomorphic to each other with the mass function transforming as a scalar field, so the causal structure of both solutions must be the same. A visual picture of the trapping horizon for an evaporating black hole is displayed in Fig. (1.3). The horizontal line represents the space-like singularity $r = 0$. Consider the radial outgoing null rays: in BV form they obey the differential equation

$$\dot{r} \equiv \frac{dr}{dv} = \frac{1}{2}e^{\Phi}\left(1 - \frac{2M}{r}\right) \quad (1.163)$$

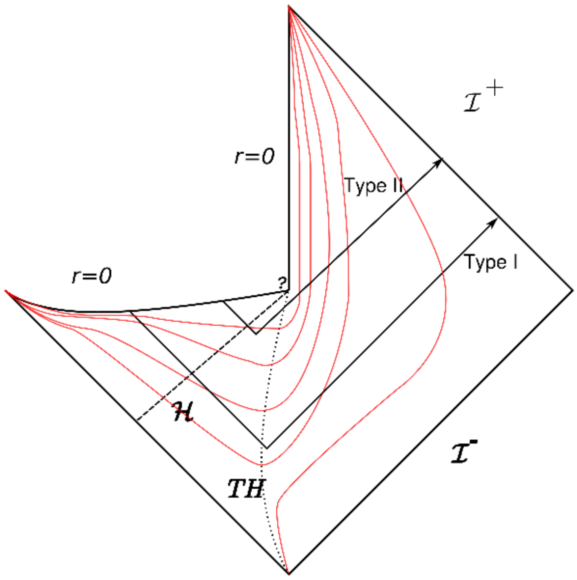


Figure 1.3: Evaporating black hole: the trapping horizon is at points where the radial outgoing rays have zero speed (dotted line). The event horizon $\mathcal{H}$ is represented by a dashed line. Red lines represent curves of constant $r$.

while ingoing rays simply are $v = v_0$, for some constant $v_0$. For comparison, in the PG form Eq. (1.117) the same equation takes the form

$$\dot{r} = \left( \pm 1 - \sqrt{\frac{2m}{r}} \right) e^{\Phi} \tag{1.164}$$

where $m(\tilde{t}, r) = M(v, r)$ and the plus (minus) sign referring to outgoing (ingoing) rays. Returning to the BV form, the event horizon is an outgoing null surface,

$$\dot{r}_{EH} = \frac{1}{2} \left( 1 - \frac{2M}{r_{EH}} \right) e^{\Phi} < 0 \tag{1.165}$$

so that $r_{EH} < r_H = 2M$. The acceleration close to the trapping horizon is

$$\ddot{r} = -e^{\Phi} \frac{\dot{M}}{2M} + \frac{e^{\Phi}}{4M}(1 - 2M')\dot{r} \; \simeq \; \frac{e^{\Phi_H}}{r_H} L + \frac{e^{\Phi}}{4M}(1 - 2M')\dot{r} > 0 \, . \tag{1.166}$$

We see that in both cases at the TH, which we recall is the line $r = 2M(v,r)$ for BV or $r = 2\tilde{M}(\tilde{t},r)$ for PG, photons are only momentarily at rest, subsequently escaping on a dynamical time scale $\kappa^{-1}$, where $\kappa$ is the surface gravity Eq. (1.158). Therefore outgoing photons (massless particles) have $\dot{r} < 0$ before reaching the trapping horizon and $r$ must decrease on going to the horizon. This can be seen from the lines of constant $r$ in Fig. (1.3). Implicit in the Figure is also the idea that the Misner–Sharp mass has a non zero limit as $v \to -\infty$. More complicated dynamics are possible in general depending on the mass function, including the possibility of horizon formation at finite times (some of these are discussed in Kodama's original paper [113]).

*The fact that $\dot{r} = 0$ at the trapping horizon and only there is one of the most important facts at the root of the quantum tunnelling phenomenon.* Now we would like to consider the case where $M(r,v)$ goes to a constant at very large negative advanced time and the trapping horizon extends to past infinity, since this is the region when the black hole does not radiate yet. In this case all radial null geodesics emerging from the trapping horizon seem to do so at $v = -\infty$, because the speed vanishes at $r = 2M$ (or $2\tilde{M}$). For particles with non zero angular momentum the trapping horizon is not a surface of momentarily zero speed. For instance, in the equatorial plane the velocity is given by

$$\dot{r} = \frac{1}{2}\left(1 - \frac{2M}{r}\right)e^{\Phi} - \frac{J^2}{2r^2} \tag{1.167}$$

where $J$ is the conserved angular momentum per unit mass. The essential point is that $\dot{r} < 0$ at $r = 2M$, so that for a rotating particle it will even be harder to escape quantum mechanically than to a non rotating one. We will see in the next section that the presence of angular momentum will not affect the main conclusion that the radiation originates close to the trapping horizon, not the global event horizon. Actually, the global event horizon cannot even be computed without knowledge of the dynamics near the evaporation end point. A practical definition was given by Bardeen [86]: one looks for photons that can only reach large $r$ in a time comparable to the evaporation time $\sim E/L \gg E$. Thus, the event horizon is located by the unaccelerated photons with $\ddot{r} = 0$. From Eqs. (1.163) and (1.166) it follows then that

$$\dot{r}_{EH} \simeq -2L/r_H\kappa, \qquad r_{EH} \simeq 2E\left(1 - \frac{2L}{r_H\kappa}\right) \simeq 2E(1 - 4L) \tag{1.168}$$

should be a solution to Eq. (1.165) to first order in the black hole luminosity $L$ (which is very small for large black holes) and ignoring contributions due to $\Phi$ and $M'$ which are known to be negligible to first order in $L$ [87]. This is the result of Bardeen–York.

**Tunnelling paths**

We propose now to compute the probability amplitude for a quantum particle to cross a trapping horizon. We do this by making use of the Hamilton–Jacobi method. Recall that, in the spherically symmetric dynamical case, it is possible to introduce the Kodama vector field $K$ such that $(K^{\alpha}G_{\alpha\beta})^{;\beta}=0$, actually its defining property. Given the metric Eq. (1.119), the Kodama vector components are

$$K^i(x)=\frac{1}{\sqrt{-\gamma}}e^{ij}\partial_j R\,,\qquad K^{\theta}=0=K^{\phi}\,. \tag{1.169}$$

where $e^{ij}$ is the numerical skew tensor with $e^{01}=1$. The Kodama vector gives a preferred flow of time and in this sense it generalises the flow of time given by the Killing vector in the static case (see [126] for a detailed study of the space-time foliation it determines). The conserved charges associated to $K$, Eqs. (1.143) and (1.144), are respectively the volume and the Misner–Sharp mass of space-time. We may also use the Kodama flow to define the invariant energy associated with a particle by means of the scalar quantity on the normal space

$$\omega=-K^i\partial_i I\,, \tag{1.170}$$

where $I$ denotes the classical action of the massless particle, which we assume to satisfy the reduced Hamilton–Jacobi equation

$$\gamma^{ij}\partial_i I\partial_j I=0\,. \tag{1.171}$$

Thus, for example, $\omega=-e^{-\Phi}\partial_v I$ will be the particle energy on an extremal in a BV gauge; we may note that this gauge is only fixed up to diffeomorphisms $v\to v^{'}=v^{'}(v)$, under which the field $\Phi(v,r)$ transforms as a conformal field, $\Phi(v,r)\to\Phi^{'}(v^{'},r)=\Phi(v,r)+\log|dv^{'}/dv|$. Therefore $\omega$ will not depend on the choice of advanced time which respect to the BV form of the metric. Similarly, in Painlevé–Gullstrand gauge the invariant energy is $\omega=e^{-\Phi}\partial_{\tilde{t}}I$, and is gauge invariant under time re-parametrisation $\tilde{t}\to\bar{t}=\bar{t}(\tilde{t})$.

To illustrate the method we shall work in s-wave and omit the angular dependence of $I$. We stress the importance to have at disposal an invariant definition of energy. Eq. (1.170) certainly satisfies this requirement if the action is a scalar. In the following our aim will be to show that there is a precise invariant prescription to deal with the imaginary part of the action, in case there is one, which is valid for all solutions in all coordinate systems which are regular across the horizon. The task is to compute

$$\mathrm{Im}\int_{\searrow} dI \tag{1.172}$$

along a tunnelling path, say $\gamma$. A priori there are two types of such tunnelling paths, which we shall call type-I and type-II, because a pair may form on either sides of the trapping horizon. A type-I path corresponds to a pair forming outside and therefore will be a backward null ray coming out from the future singularity at $r = 0$, crossing the trapping horizon and subsequently escaping to infinity, as illustrated in Fig. (1.3). The figure is the dynamical analogue of the corresponding figure in Schwarzschild space-time, except that it is smooth at $\dot{r} = 0$, namely at the trapping horizon. A type-II path corresponds to a pair forming in the trapped region, so we may take a backward null ray crossing the event horizon from the singularity till some interior point, which subsequently escapes to infinity along a null ray crossing the trapping horizon at some other event. In some cases such a path (namely an outgoing light ray emerging from the trapping horizon) seems to emerge from the trapping horizon at $v = -\infty$ in advanced time, therefore in this case, type-II paths are those with exactly one point, the starting point, lying on the horizon. Type-II paths are absent in static geometries, since there is no region in between the two horizons. The segment of type-II paths crossing the horizon outward seems nevertheless an allowed classical path so one may wonder whether they can contribute to an emission amplitude: does the pole disappear? We will see indeed that on such paths the radial momentum vanishes, basically for the reason that the speed is zero on the trapping horizon — and only there — so in fact they do not contribute semi-classically.

We shall illustrate these results in some of the gauges we discussed above for spherically symmetric space-times, having two main purposes: to illustrate the intrinsic covariance of the result and to display how contributions to the tunnelling amplitude may or may not come from all terms in the differential form $dI = \partial_i I dx^i$. We also have the opportunity to test the formalism and to see how different can be the prosaic description of the same physical effect from the perspective of other coordinate systems, a fact well known in special and general relativity.

*The BV gauge*

Here the metric is

$$ds^2 = -e^{2\Phi} C dv^2 + 2e^{\Phi} dv dr + r^2 d\Omega^2 \tag{1.173}$$

where $M = (1 - C)r/2$ is the Misner–Sharp mass discussed in 1.2.2. The Kodama vector and the invariant particle energy assume the simple expressions given by $K = (e^{-\Phi}, \mathbf{0})$ and $\omega = -e^{-\Phi} \partial_v I$, while the invariant surface gravity is just given by $\kappa_H = \partial_r C_H / 2$. As noted above $\Phi$ transforms as an ordinary Liouville field, *i.e.* $\Phi \to \Phi + \log |\partial \tilde{v} / \partial v|$, under $v \to \tilde{v}(v)$, making $\omega$ invariant under re-parametrisation of the advanced time coordinate.

Type-II tunnelling paths are easily dispensed for: they cross the horizon along a null direction which is outward with zero speed at the horizon, so

$\dot{r} = 0$ at and in a neighbourhood of the horizon, or $dr = 0$ along the null direction; therefore on a small segment crossing the trapping horizon we have

$$\int \partial_r I \, dr = 0 \, .$$

On a real trajectory the energy is real and finite everywhere, which shows that the temporal $v$-contribution does not play any role, too, in what concerns us at the moment (the evaluation of $\mathrm{Im}\, I$). Thus $\mathrm{Im}\, I = 0$ on type-II paths.

For a type-I path the ray crosses the horizon along a null ingoing ray but in the backward direction; such a ray has $v = v_0$ (some constant) and $\delta v = 0$ exactly, therefore again the $\partial_v I dv$ term will vanish at the horizon and $\delta r > 0$. From the Hamilton–Jacobi equation we see that, for outgoing modes of the kind we consider,

$$C(\partial_r I) = 2\omega \, . \tag{1.174}$$

Thus one has

$$\begin{aligned} \mathrm{Im}\, I &= \mathrm{Im} \int_\gamma (\partial_r I dr + \partial_v I dv) = \mathrm{Im} \int_\gamma dr \, \frac{2\omega}{C} \\ &= 2 \ \mathrm{Im} \int_\gamma dr \, \frac{\omega}{\partial_r C\big|_H (r - r_H - i\epsilon)} = \frac{\pi \omega_H}{\kappa_H} \, , \end{aligned} \tag{1.175}$$

where $\omega_H$ is $\omega$ evaluated on the horizon, the quantity $C$ has been expanded around the horizon pole, that is

$$C(v, r) = \partial_r C\Big|_H \Delta r + \dots \tag{1.176}$$

and Feynman's $i\epsilon$-prescription has been implemented in order to deal with the simple pole. $\kappa_H = \partial_r C\big|_H / 2$ is the horizon surface gravity and coincides with our geometrical expectations. Unlike the stationary black holes, where $\omega$ is a constant of motion, here we get the local energy and temperature as measured near the horizon, which are connected to the quantities at infinity by the corresponding red-shift factors. We see that, in a BV coordinate system, the temporal integration does not give any contribution to the imaginary part of the action of particles tunnelling through the trapping horizon.

It will be very important to discuss the physical meaning of this result but we prefer to postpone this discussion to the end of the section, after we will have shown that the tunnelling amplitude is not an artefact due to a special coordinate system but holds as good as in any other.

A coordinate system where the temporal contribution to the action plays an essential role is the general diagonal form of a spherically symmetric metric, which reads

$$ds^2 = e^{\psi(t,r)}\left(-dt^2 + dr^2\right) + R^2(t,r)d\Omega^2\,. \tag{1.177}$$

In this form, the normal metric is conformally related to the two-dimensional Minkowski space-time. The $\chi$ function simply reads

$$\chi = e^{-\psi}\left[-(\partial_t R)^2 + (\partial_r R)^2\right]\,, \tag{1.178}$$

If $R(t,r)$ is a monotonically increasing function of $r$ (the normal case) then this leads to the future trapped horizon condition

$$(\partial_t R)_H + (\partial_r R)_H = 0\,. \tag{1.179}$$

If instead $R(r,t)$ is decreasing as a function of $r$, then we have to take the opposite relative sign, because we expect that an evaporating horizon should have a negative time derivative, $(\partial_t R)_H < 0$. A computation of the expansion scalars will confirm this fact. The Kodama vector and the associated invariant energy are

$$K = e^{-\psi}\left(\partial_r R, -\partial_t R, 0, 0\right)\,, \tag{1.180}$$

$$\omega = e^{-\psi}\left(-\partial_r R\partial_t I + \partial_t R\partial_r I\right)\,. \tag{1.181}$$

The dynamical surface gravity reads

$$\kappa_H = \frac{1}{2}e^{-\psi_H}\left(-\partial_t^2 R + \partial_r^2 R\right)\Big|_H\,. \tag{1.182}$$

Due to conformal invariance, the Hamilton–Jacobi equation is the same as in two-dimensional Minkowski space-time, namely using double null coordinates $\partial_+ I\partial_- I = 0$, and for the outgoing particle we may take

$$\partial_+ I = \partial_t I + \partial_r I = 0\,, \tag{1.183}$$

since the radial momentum $p_r = \partial_r I > 0$ and $\partial_t I < 0$ for a real outgoing particle. Given that for a type-II path the null expansion condition leads to $\delta x^- = \delta t - \delta r = 0$, we get $dI = \partial_+ I dx^+ + \partial_- I dx^- = 0$ on account of the Hamilton–Jacobi equation, and there is no imaginary part. On reflection, this result seems to violate our basic tenet that there is no amplitude because the photon speed vanishes at the horizon. In fact, the coordinate speed $\dot{r} = \pm 1$ everywhere in conformal gauge and it never vanishes. However, the speed of the wave front is

$$\dot{R} = \partial_r R\,\dot{r} + \partial_t R$$

and this vanishes on $H$ for an outgoing photon with $\dot{r} = 1$, on account of the horizon equation Eq. (1.179). Thus, what really matters is that the speed of the areal radius should vanish at the horizon.

But there are also the type-I tunnelling paths, for which the null expansion gives $\delta x^+ = \delta t + \delta r = 0$. Therefore in this case

$$I = \int_\gamma (dr\, \partial_r I + dt\, \partial_t I) = 2 \int_\gamma dr\, \partial_r I\,. \tag{1.184}$$

Furthermore, due to Eqs. (1.181) and (1.183), one has

$$\partial_r I = \frac{\omega}{e^{-\psi}(\partial_r R + \partial_t R)}\,. \tag{1.185}$$

and we have a pole at the horizon. Making use of near horizon approximation along the null direction, from Eqs. (1.179) and (1.182), one has $(\partial_r R)_H + (\partial_t R)_H = 0$, $\delta t + \delta r = 0$, thus with the understanding that we shall put $\delta r = r - r_H$, we get

$$\begin{aligned} e^{-\psi}[\partial_r R + \partial_t R] &= e^{-\psi_H} \left(\partial^2_{rr} R - \partial^2_{rt} R - \partial^2_{tt} R + \partial^2_{tr} R\right)\Big|_H (r - r_H) + \ldots \\ &= 2\kappa_H (r - r_H) + \ldots \end{aligned} \tag{1.186}$$

and the imaginary part follows. If $R(t, r)$ is a decreasing function of $r$ then we shall have to take $p_r = \partial_r I < 0$ but again $\partial_t I < 0$ so the Hamilton–Jacobi equation implies now $\partial_- I = 0$ along the tunnelling path. Moreover since $r$ is decreasing as $R$ increases (*i.e.* toward infinity), on an outgoing (ingoing) null direction we have $\delta t + \delta r = 0$ ($\delta t - \delta r = 0$) and the horizon condition becomes

$$(\partial_t R - \partial_r R)\Big|_H = 0$$

instead of Eq. (1.179). It is now simple to see that a tunnelling type-II path gives no amplitude at all, but that a type-I does produce an imaginary part with the right magnitude and sign. In fact in this case we obtain first

$$\partial_r I = -\frac{\omega}{e^{-\psi}(\partial_r R - \partial_t R)}\,.$$

then, recalling $\partial_- I = 0$,

$$dI = \partial_+ I dx^+ = 2\partial_r I dr = -2\frac{\omega}{e^{-\psi}(\partial_r R - \partial_t R)} dr \cong -2\frac{\omega}{\kappa_H (r - r_H)} dr\,.$$

The integral of $dI$ has to be done on the segment of the path crossing the horizon along decreasing $r$, say from $r_2$ to $r_1$ with $r_1 < r_2$, and this is of course minus the usual integral in increasing order; reversing the orientation gives back the usual result

$$\mathrm{Im} \fint_{\searrow} dI = \frac{\pi\omega}{\kappa_H}\,.$$

Another coordinate system where the coordinate speed does not vanish at the trapping horizon is described by the line element

$$ds^2 = -dt^2 + \frac{1}{B(r,t)}dr^2 + R^2(r,t)d\Omega^2 = d\gamma^2 + R^2(r,t)d\Omega^2 \,, \qquad (1.187)$$

in which the metric is diagonal, but $R$ is a function of $r$ and $t$. In this case, one has

$$\chi = -(\partial_t R)^2 + B(\partial_r R)^2 \,, \qquad (1.188)$$

If $R(r,t)$ is an increasing function of $r$, the horizon $\chi_H = 0$ should be defined by

$$(\partial_t R)_H = -\sqrt{B_H}(\partial_r R)_H \,, \qquad (1.189)$$

in which we are assuming again a regular coordinate system on the horizon, namely that $B_H$ and the partial derivatives are non-vanishing. For $R$ decreasing with $r$ one should change the sign to the right of Eq. (1.189). The Kodama vector reads

$$K = (\sqrt{B}\partial_r R, -\sqrt{B}\partial_t R, 0, 0) \,, \qquad (1.190)$$

and the invariant energy

$$\omega = \sqrt{B}(\partial_r R \partial_t I - \partial_t R \partial_r I) \,. \qquad (1.191)$$

The dynamical surface gravity is evaluated to be

$$\kappa_H = \frac{1}{4}\left(-2\partial_t^2 R_H + 2B_H \partial_r^2 R_H + \frac{1}{B_H}\partial_t R_H \partial_t B_H + \partial_r R_H \partial_r B_H\right) \,. \qquad (1.192)$$

Making use of the horizon condition, we may rewrite it

$$\kappa_H = \frac{1}{4}\left(-2\partial_t^2 R_H + 2B_H \partial_r^2 R_H - \frac{1}{\sqrt{B_H}}\partial_r R_H \partial_t B_H + \partial_r R_H \partial_r B_H\right) \,. \qquad (1.193)$$

In this case, the Hamilton–Jacobi equation reads simply

$$-(\partial_t I)^2 + B(\partial_r I)^2 = 0 \qquad (1.194)$$

and for an outgoing particle we have to choose the root with $\partial_r I > 0$. Therefore, in a type-II path crossing the trapping horizon we have $\sqrt{B_H}\delta t = \delta r$, in a neighbourhood of TH, because $\dot{r} = \sqrt{B}$. As a consequence the outgoing temporal contribution will cancel the radial one. Type-II paths will then give no tunnelling amplitude. On reflection, this happens because although the coordinate speed is non vanishing at the trapping horizon, the areal velocity indicating the speed of the wave front

$$\dot{R} = \partial_r R \,\dot{r} + \partial_t R$$

does vanish on TH for an outgoing photon, which has $\dot{r} = \sqrt{B}$, on account of Eq. (1.189). On a type-I path instead $\sqrt{B_H}\Delta t = -\Delta r$, and the two contributions add so that

$$I = 2\int_\gamma dr\partial_r I\,. \tag{1.195}$$

The Hamilton–Jacobi equation and the expression for the invariant energy lead to

$$\partial_r I = \frac{\omega}{B\partial_r R + \sqrt{B}\partial_t R}\,. \tag{1.196}$$

Making the expansion along the null curve, for which $\sqrt{B_H}\Delta t + \Delta r = 0$, in the near-horizon approximation, one gets

$$\text{Im}\;\, I = 2\cdot\text{Im}\,\int_\gamma dr \frac{\omega}{2\kappa_H(r - r_H - i\epsilon)} = \frac{\pi\omega_H}{\kappa_H}\,, \tag{1.197}$$

leading to the desired amplitude. But notice that in this gauge, the temporal contribution is essential in order to obtain the correct result.

*The $r$-gauge*

The fact that type-II paths never contributed to the tunnelling amplitude suggest that it is a general fact. Our last example will use a general metric in the $r$-gauge to confirm this. The normal metric here is non-diagonal, but as in BV gauge $R = r$. We have

$$ds^2 = d\gamma^2 + r^2 d\Omega^2\,, \tag{1.198}$$

where the reduced normal metric is now taken in the (redundant) form

$$d\gamma^2 = -E(r,t)dt^2 + 2F(r,t)dtdr + G(r,t)dr^2\,, \qquad F \neq 0\,. \tag{1.199}$$

The horizon is located at the zeroes of

$$\chi(t,r) = \gamma^{ij}\,\partial_i R\,\partial_j R = \gamma^{rr}(t,r) = \frac{E}{EG + F^2} = 0 \tag{1.200}$$

*i.e.* at $E_H = E(t, r_H) = 0$, provided $F_H \neq 0$. The Misner–Sharp mass inside a sphere of radius $r$ is

$$M = \frac{r}{2}\left(1 - \frac{E}{EG + F^2}\right) \tag{1.201}$$

and its value on the horizon is the black hole mass by definition, $M = r_H/2$. The other ingredient, the Kodama vector, reads

$$K = \left(\frac{1}{\sqrt{F^2 + EG}}, 0, 0, 0\right)\,, \tag{1.202}$$

and the invariant particle energy follows

$$\omega = -\frac{\partial_t I}{\sqrt{F^2 + EG}} \,. \tag{1.203}$$

The dynamical surface gravity is computed to be

$$\kappa_H = \frac{1}{2}\Box_\gamma r\Big|_H = \left[\frac{1}{2F^3}\left(E'F - \frac{1}{2}\dot{E}G\right)\right]\Big|_H \,, \tag{1.204}$$

where an overdot and a prime denote differentiation with respect to $t$ and $r$, respectively. Now the horizon will be future and of outer type provided $F > 0$ and $\kappa_H > 0$. We expect the mass within the sphere to be a decreasing function of $r$; also, for an excreting black hole the time derivative $\dot{M}$ should be negative as well. From Eq. (1.201) this will be true if $E' > 0$ and $\dot{E} > 0$, although these are only sufficient conditions. It follows that for $F < 0$, $\kappa_H > 0$ always but the horizon is foliated by past marginally trapped surfaces ($G > 0$ of course), while for $F > 0$ it is future and outer if the inequality

$$E'\Big|_H > \left(\frac{\dot{E}G}{2F}\right)\Big|_H \tag{1.205}$$

is true, which we shall assume from now on. The outgoing null rays at the horizon satisfy either $\dot{r} = 0$ or, if $F < 0$, $\dot{r} = -2F/G$. Only the first type will concern us, since photons from the past horizon should always be allowed classically. In fact, from the Kodama energy expression (1.203) and Eq. (1.200) as well, the Hamilton–Jacobi equation reads

$$\chi(\partial_r I)^2 - 2\frac{\omega F}{\sqrt{EG + F^2}}\partial_r I - \omega^2 G = 0 \,, \tag{1.206}$$

Solving this, one sees that $\partial_r I$ has no pole at $H$ (which is past for $F < 0$) and therefore no imaginary part is found. Returning to our case with $F > 0$, on a type-II path $\dot{r} = 0$ at the horizon therefore, as before, the radial momentum vanishes and no imaginary term can come from $\int \partial_r I \, dr$.

It remains to consider type-I paths. From the metric the null radial expansion across the segment crossing the horizon outward gives $\delta t = -(G/2F)|_H \delta r$, now with $F > 0$, so we can expand the $\chi$ function along this ingoing null direction to first order in $\delta r = r - r_H$. We obtain easily

$$\chi \cong \dot{\chi}\delta t + \chi'\delta r = \left(\chi' - \frac{G}{2F}\dot{\chi}\right)\Big|_H (r - r_H) + \cdots = 2\kappa_H(r - r_H) + O\left((r - r_H)^2\right) \tag{1.207}$$

where $\cong$ means evaluation on the horizon and Eq. (1.204) has been used. Also, $\partial_t I = -F_H\omega$ from definition Eq. (1.203) and the horizon condition

$E = 0$. Hence we end up with $I$ given by the sum of a real term and a possibly imaginary part coming from the horizon:

$$I = \int_\gamma (dr\partial_r I + dt\partial_t I) = \int_\gamma dr \left[\partial_r I + \frac{1}{2} G_H \omega_H\right] . \tag{1.208}$$

What is remarkable is that in this gauge the temporal part is present but, being regular, it does not contribute to the imaginary part of the action. From the Hamilton–Jacobi equation Eq. (1.206) we get, choosing the solution with positive radial momentum,

$$\partial_r I = \frac{\omega F}{\sqrt{EG + F^2}\ \chi} \left(2 + O(\chi)\right) . \tag{1.209}$$

and we see that this has a pole at $\chi = 0$. Making use of this equation, Feynman's prescription and Eq. (1.207), one has once more

$$\text{Im } I = \text{Im} \int_\gamma dr\, \partial_r I = \text{Im} \int_\gamma dr \,\frac{\omega F}{\sqrt{F^2 + EG}} \cdot \frac{1 + \sqrt{1 + O(\chi)}}{2\, \kappa_H\, (r - r_H - i\epsilon)} = \frac{\pi \omega_H}{\kappa_H} . \tag{1.210}$$

We have shown in general that type-II paths will not contribute a tunnelling amplitude to future trapping horizons ($F > 0$), because $\delta r = 0$ to first order in $\chi$ and we are assuming the energy to be real and finite. As to type-I paths the situation also is clear, they give a amplitude fully controlled by the value of the geometrical surface gravity $\kappa_H$. This example also shows the possibility of a quantum tunnelling through past trapping horizons of outer type, as for $F < 0$ there is an amplitude for a photon to cross with negative radial momentum, *i.e.* for a photon to be absorbed (we are using the mental picture of a past horizon as one from which it is impossible to enter but very easy to escape). The fact that there is no imaginary part on type-II paths either because the coordinate speed of a photon vanishes or because the temporal contribution cancels the radial one was the source of much confusion in the past literature. For instance, Belinski ventured to say that there is no Hawking radiation from black holes just from this fact [127], since he apparently forgot about the existence of type-I paths.

We end this section by noticing once more the special role of the trapping horizon. The imaginary part being produced on crossing it, it should be natural to think that this is the place where most of radiation forms. This has been confirmed in [119], where it is shown that asymptotic observers register a radiation flux that starts increasing at a time they see the collapsing shell crosses the surface $r = 2M$ of the model used (a special case of Painlevé-Gullstrand model with a time dependent mass, see Eq. (1.117)). The stress-tensor for Vaidya space-time has been considered in [128, 129] and should be consistent with this result.

*A note* — In the previous computations various choices of signs have been applied in such a way that it may seem they were chosen somewhat *ad hoc* in order to get the wanted result. This is not so. Once the future sheet of the trapping horizon has been chosen, and the sign of the Kodama vector determined so that it is future directed, no other sign uncertainties will occur for either outgoing or ingoing particles if one uses consistently the Hamilton–Jacobi equation for outgoing particles on every segment of the tunnelling path; furthermore, questions such as the right form of the equations determining the horizon, the orientation of the space-time coordinates and the sign of the metric coefficients are carefully established once for all.

On the other hand, if there exists a past sheet in the trapping horizon then using the tunnelling picture we may as well compute the action along an inward directed curve at the horizon (the ambiguity inherent in this and analogous terms is easily resolved if the manifold is asymptotically flat). Then there will be again a non-vanishing imaginary part, but we can interpret it as a small absorption probability, as we did for the static case in relation to white holes.

## 1.3 Cosmological horizons, decays, naked singularities

In this section we review the formalism of horizon tunnelling as applied to cosmological horizons, decay of unstable particles that in the absence of gravity would be otherwise stable and radiation from naked singularities. Especially the subject of horizon tunnelling in cosmology attracted much interest recently, leading to a related stream of papers. A sample of these articles which seemed relevant to us can be found in [130–132], although we rely mainly on the papers [55, 133] and the references cited therein. We shall follow mainly the Hamilton–Jacobi version of the tunnelling picture, although the null geodesic method can be still applied. Results in this direction appeared recently in [134], where the Parikh–Wilczek and Hamilton–Jacobi methods are compared and showed to agree. An early study of the evolution of evaporating black holes in inflationary cosmology is in [135].

### 1.3.1 The FRW space-time

Consider a generic FRW space-time, namely one with constant curvature spatial sections. Its line element can be written as

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}\,dr^2 + [a(t)r]^2 d\Omega^2 \tag{1.211}$$

where $r$ is measured in units of the curvature radius and, as usual, $k = 0, -1, +1$ labels flat, open and closed three-geometries, respectively. In this gauge, the normal reduced metric is diagonal and the horizon is implicitly given by the equation $\chi = 0$, where

$$\chi(t,r) = 1 - [a(t)r]^2 \left[H^2(t) + \frac{k}{a^2(t)}\right] , \tag{1.212}$$

namely

$$R_H := a(t)r_H = \frac{1}{\sqrt{H^2(t) + \frac{k}{a^2(t)}}} , \quad \text{with} \quad H(t) = \frac{\dot{a}(t)}{a(t)} , \tag{1.213}$$

provided the space-time energy density $\varrho(t)$ is positive. As always it corresponds to the vanishing of the wave front velocity whose radius is $R = a(t)r$. The surface $R_H(t)$ coincides also with the Hubble radius as defined by astronomers for vanishing curvature, but we shall call it Hubble radius in any case. The important case $k = 0$ deserves special attention. The horizon is the surface $r\dot{a}(t) = 1$; substituting into the metric reveals the causal character of the horizon: it will be null if and only if either $p = \varrho/3$ (radiation) or $\varrho = -p$ (vacuum energy), $\varrho$ and $p$ being the energy density and pressure, respectively. It will be time-like or space-like according to whether $|(\varrho+3p)/2\varrho|$ is less or greater than one. In the former case $-\varrho < p < \varrho/3$, which covers almost the totality of cosmological models, while the space-like character can only be achieved if $p < -\varrho$ (dubbed phantom energy) or $p > \varrho/3$, which includes stiff matter. These results actually hold for non zero curvature too. In Section 1.2.1 we defined future outer trapping horizons. The situation in flat cosmology is easily spelled out: the horizon is a surface with $\theta_- = 0$, $\theta_+ > 0$ that for $-\varrho < p < \varrho/3$ satisfies also $\partial_+\theta_- > 0$. In the terminology of Section 1.2.1 it can be classified as a past inner trapping horizon (abbr. PITH), because all metric spheres at larger radii are trapped with $\theta_\pm > 0$. It means that given a comoving observer, a spherical light beam approaching him from cosmic distance will have increasing area due to cosmological expansion. One has to be careful here that the areal radius is $R = a(t)r$, not $r$, so the area of the wave front is $4\pi R^2$ and the horizon can be written as the condition $HR_H = 1$. Examples of a FITH are provided by the de Sitter horizon and the collapsing Vaidya solution in de Sitter space, Eq. (1.160).

The dynamical surface gravity reads

$$\kappa_H = \frac{1}{2}\Box_\gamma[a(t)r] = -\left(H^2(t) + \frac{1}{2}\dot{H}(t) + \frac{k}{2a^2(t)}\right) R_H(t) < 0 , \tag{1.214}$$

and the minus sign refers to the fact the Hubble horizon is of the inner type. Similarly, the Kodama vector is

$$K = \sqrt{1 - kr^2}(\partial_t - rH(t)\partial_r) \tag{1.215}$$

so that the invariant Kodama energy of a particle is equal to

$$\omega = \sqrt{1 - kr^2}(-\partial_t I + rH(t)\partial_r I) \equiv \sqrt{1 - kr^2}\,\tilde{\omega} . \tag{1.216}$$

Notice that $K$ is space-like for $ra > (H^2 + k/a^2)^{-1/2}$, *i.e.* beyond the horizon, so that particles will eventually tunnel from outside to the inner region, $r < r_H$.

The next ingredient will be the reduced Hamilton–Jacobi equation for a relativistic particle with mass parameter $m$, which reads

$$- (\partial_t I)^2 + \frac{(1 - kr^2)}{a^2(t)} (\partial_r I)^2 + m^2 = 0 . \tag{1.217}$$

Solving Eqs. (1.216) and (1.217) for $\partial_r I$ and $\partial_t I$ we obtain

$$\partial_t I = -\tilde{\omega} + rH\partial_r I \tag{1.218}$$

$$\partial_r I = -\frac{aH\tilde{\omega}(ar) \pm a\sqrt{\omega^2 - m^2 + m^2 \left(H^2 + \frac{k}{a^2}\right) (ar)^2}}{1 - \left(H^2 + \frac{k}{a^2}\right) (ar)^2} , \tag{1.219}$$

with the sign chosen according to which direction we think the particle is propagating. The effective mass here defines two important and complementary energy scales: if one is interested in the horizon tunnelling then only the pole matters (since the denominator vanishes), and we may neglect to all the extents the mass parameter setting $m = 0$ (since its coefficient vanishes on the horizon). On the opposite, in investigating other effects in the bulk away from the horizon, such as the decay rate of composite particles, the role of the effective mass becomes relevant as the energy of the particle can be smaller than the energy scale settled by $m$, and the square root can possibly acquire a branch cut singularity. Eq. (1.219) will be the starting point of our considerations since it embraces all semi-classical quantum effects we are interested in.

### 1.3.2 Cosmic horizon tunnelling

Using Eq. (1.219) we may derive, following [55], the cosmic horizon tunnelling rate. To this aim, as we have anticipated, the energy scale is such that near the horizon, we may neglect the particle's mass, and note that radially moving massless particles follow a null direction. The horizon region being a region with strong gravity we expect indeed some "particle creation". To a pair created near the horizon in the inner region $r < r_H$ corresponds a type-I path, one segment crossing the horizon backward in cosmological time. To a pair created outside the horizon corresponds a type-II path, the segment crossing the horizon inward being classically allowed. As in Section 1.2.3 only type-I paths will contribute to horizon tunnelling.

Take then a null radial direction crossing the horizon to the inner region: we have

$$\delta t = \frac{a(t)}{\sqrt{1 - kr^2}} \delta r . \tag{1.220}$$

The action for particles coming out of the horizon towards the inner region is to be integrated on a type-I path: then we must choose the solution with negative radial momentum, $\partial_r I < 0$ and we obtain

$$I = \int dt\, \partial_t I + \int dr\, \partial_r I = 2 \int dr \partial_r I \,, \tag{1.221}$$

upon solving the Hamilton–Jacobi equation (1.217) with zero mass and using Eq. (1.220). Note that the time derivative of the action contributes to the total imaginary part which, as we have seen, is a general feature of the tunnelling method. For $\partial_r I$ we use now Eq. (1.219), which exhibits a pole at the vanishing of the function $F(r,t) := 1-(a^2H^2+\hat{k})r^2$, defining the horizon position. Expanding $F(r,t)$ again along the null direction Eq. (1.220) we obtain

$$F(r,t) \approx +4\kappa_H a(t)(r-r_H) + \dots \,, \tag{1.222}$$

where $\kappa_H$ given in Eq. (1.214) represents the dynamical surface gravity associated with the horizon. In order to deal with the simple pole in the integrand, we implement Feynman's $i\epsilon$ – prescription. In the final result, beside a real (irrelevant) contribution, we obtain the following imaginary part

$$\mathrm{Im}\, I = -\frac{\pi\omega_H}{\kappa_H} \,. \tag{1.223}$$

This imaginary part is usually interpreted as arising because of a non-vanishing tunnelling probability rate of massless particles across the cosmological horizon,

$$\Gamma \sim \exp\left(-2\mathrm{Im}\, I\right) \sim e^{-\frac{2\pi}{(-\kappa_H)}\cdot\omega_H} \,. \tag{1.224}$$

Notice that, since $\kappa_H < 0$ and $\omega_H > 0$ for physical particles, (1.223) is positive definite. Due to the invariant character of the quantities involved, we may interpret the scalar $T = -\kappa_H/2\pi$ as the dynamical temperature parameter associated to FRW space-times. In particular, this gives naturally a positive temperature for de Sitter space-time, a long debated question years ago, usually resolved by changing the sign of the horizon's energy. It should be noted that in literature, the dynamical temperature is usually given in the form $T = H/2\pi$ (exceptions are the papers [136]). Of course this is the expected result for de Sitter space in inflationary coordinates, but it ceases to be correct in any other coordinate system. In this regard, the $\dot{H}$ and $k$ terms are crucial in order to get an invariant temperature. The horizon's temperature and the ensuing heating of matter was foreseen several years ago in the interesting paper [137].

### 1.3.3 Decay rate of unstable particles

We consider the decay rate of a composite particle in a regime where the energy of the decaying particle is lower than the proper mass $m$ of the decayed products. A crucial point is to identify the energy of the particle before the decay with the Kodama energy. We also denote $m$ the effective mass parameter of one of the decay products (recall it may contain a curvature terms). The relevant contribution to the action comes from the radial momentum given by Eq. (1.219). If we introduce the instantaneous radius $r_0$ by

$$[a(t)r_0]^2 = R_0^2 := \left(1 - \frac{\omega^2}{m^2}\right) R_H^2 \,, \tag{1.225}$$

where $R_H$ is the horizon radius given by Eq. (1.213), then the classically forbidden region is $0 < r < r_0$. From Eq. (1.219), we see that for the unstable particle sitting at rest at the origin of the comoving coordinates, one has an imaginary part of the action as soon as the decay product is tunnelling into this region to escape beyond $r_0$,

$$\operatorname{Im} I = mR_H \int_0^{R_0} dR \frac{\sqrt{R_0^2 - R^2}}{R_H^2 - R^2} \,. \tag{1.226}$$

The integral can be computed exactly at fixed $t$, and the result is

$$\operatorname{Im} I = \frac{\pi}{2}\, R_H\,(m - \omega) > 0\,, \tag{1.227}$$

leading to a rate which, assuming a two-particle decay, takes the form

$$\Gamma = \Gamma_0 e^{-2\pi\, R_H\,(m-\omega)}\,, \tag{1.228}$$

where $\Gamma_0$ is an unknown pre-factor depending on the coupling constant of the interaction responsible of the decay (for instance, for a $\lambda\phi^3$ interaction one should have $\Gamma_0 \sim \lambda^2$). Of course, each newly produced particle will itself decay, leading possibly to the instability mechanism first discussed by Myhrvold [139] in de Sitter space. Since the tunnelling process locally conserves energy one should put $\omega = m/2$, so that the tunnelled particle will emerge in the classical region at $r = r_0$ with vanishing momentum. Furthermore, the result is again invariant against coordinate changes, since both $\omega$ and $R_H$ are invariantly defined quantities.

A particularly interesting case is represented by de Sitter space. Taking for example the line element in the static patch

$$ds^2 = -(1 - H_0^2 r^2)dt^2 + \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 d\Omega^2\,, \tag{1.229}$$

for the imaginary part (1.228) we obtain

$$\operatorname{Im} I = \frac{\pi}{2H_0}(m - \omega) \tag{1.230}$$

a result actually independent of the coordinate system in use. Putting $\omega = m/2$, the above result has been obtained by Volovik [138] using the so-called "fluid" static form of de Sitter space

$$ds^2 = -dt^2 + (dR - H_0 R dt)^2 + R^2 d\Omega^2 \,. \tag{1.231}$$

in agreement with the asymptotic approximation of the exact result due to [140].

### 1.3.4 Particle creation by black holes singularities

One may also use the tunnelling formalism to investigate whether particle creation in the bulk of space-time is possible due to the presence of space-time singularities, for example due to static black holes. With regard to this, we consider the exterior region of a spherically symmetric static black hole space-time and repeat the same argument. Quite generally, we can write the line element as

$$ds^2 = -e^{2\Phi(r)} C(r) dt^2 + C^{-1}(r) dr^2 + r^2 d\Omega^2. \tag{1.232}$$

From the Hamilton–Jacobi equation, the radial momentum turns out to be

$$\int dr\, \partial_r I = \int dr\, \frac{\sqrt{\omega^2 - m^2 C(r) e^{2\Phi(r)}}}{C(r) e^{\Phi(r)}} \,. \tag{1.233}$$

If we are interested in particle creation we should set $\omega = 0$: in fact, according to the interpretation of the Kodama energy we gave before, this approximation simulates the vacuum condition. Then

$$\int dr\, \partial_r I = m \int_{r_1}^{r_2} dr \frac{1}{\sqrt{-C(r)}} \,, \tag{1.234}$$

where the integration is performed in every interval $(r_1, r_2)$ in which $C(r) > 0$. Eq. (1.234) shows that, under very general conditions, in static black hole space-times there could be a production rate whenever a region where $C(r) > 0$ exists.

As a first example, let us analyse the Schwarzschild black hole. For the exterior (static) solution, one has $C(r) = 1 - 2M/r > 0$ and $\Phi(r) = 0$, thus the imaginary part diverges since the integral has an infinite range. We conclude that the space-like singularity does not create particles in the semi-classical regime. In the interior the Kodama vector is space-like, thus no energy can be introduced. A similar conclusion has been obtained also for the Big Bang cosmic singularity, the only scale factor leading to particle emission being $a(t) \sim t^{-1}$. This is like a big rip in the past.

The situation is different when a naked singularity is present. Consider a neutral particle in the Reissner–Nordström solution with mass $M$ and charge

$Q > 0$ (for definiteness) given by the spherically symmetric line element

$$ds^2 = -\frac{(r-r_-)(r-r_+)}{r^2}dt^2 + \frac{r^2}{(r-r_-)(r-r_+)}\,dr^2 + r^2 d\Omega^2\,. \qquad (1.235)$$

Here $r_\pm = M \pm \sqrt{M^2 - Q^2}$ are the horizon radii, connected to the black hole mass and charge by the relations

$$M = \frac{r_+ + r_-}{2}, \qquad Q = \sqrt{r_+ r_-}\,. \qquad (1.236)$$

The Kodama energy coincides with the usual Killing energy and

$$C(r) = \frac{(r-r_-)(r-r_+)}{r^2}\,. \qquad (1.237)$$

The metric function $C(r)$ is negative in between the two horizons, where the Kodama vector is space-like, so there the action is real. On the other hand, it is positive within the outer communication domain, $r > r_+$, and also within the region contained in the inner Cauchy horizon, that is $0 < r < r_-$. Thus, because of Eq. (1.234) and assuming the particles come created in pairs, we obtain

$$\mathrm{Im}\, I = -m\int_0^{r_-} \frac{r}{\sqrt{(r_- - r)(r_+ - r)}}\,dr = mq - \frac{mM}{2}\ln\left(\frac{M+Q}{M-Q}\right)\,. \qquad (1.238)$$

Modulo the pre-factor over which we have nothing to say, with the usual interpretation there is a probability

$$\Gamma \sim \exp(-2Im\, I) = \left(\frac{M-Q}{M+Q}\right)^{mM} \quad e^{-2mQ}\,. \qquad (1.239)$$

Pleasantly, Eq. (1.239) vanishes in the extremal limit $M = Q$. Being computed for particles with zero energy, we can interpret this as an effect of particle creation by the strong gravitational field near the singularity. Since the electric field is of order $Q/r^2$ near $r = 0$, there should be also a strong Schwinger's effect. In that case one should write the Hamilton–Jacobi equation for charged particles.

The processes just discussed should bear a bit on the question of the stability of the Cauchy horizons. Due to infinite blue-shift of perturbations coming in from the asymptotically flat exterior regions both sheets of the Cauchy horizon ($H_F$ and $H_P$ in Fig. (1.4)) are believed to be classically unstable. Of course, if the naked singularity is formed from collapse of charged matter, one asymptotically flat region (say, the left one) disappears. Taking into account particle creation, it can be easily seen that escaping particles will reach the future portion $H_F$ of the inner horizon with infinite blue-shift, or infinite Kodama energy, as measured by an observer on a Kodama trajectory. Hence, the future sheet will probably become unstable, this time by quantum effects. On the other hand, the particle reaching the singularity will do so with infinite red-shift, that is with zero energy, giving a negligible back-reaction and substantially not changing the nature of the singularity (its time-like character, for example). Thus, the only process potentially capable of modifying the singularity would be its conjectured screening by absorption of charged particles. Of course the mass inflation phenomenon is an issue here but, in the context of perfect spherical symmetry, exact solutions of the Einstein–Maxwell system exist having both event and Cauchy horizons and describing collapse of spherical shells. For these what we said should apply. The present formalism also predicts the absence of particle creation in the region in between the two horizons, despite the metric there is dynamical. The possibility that the naked singularity itself radiates away its mass will be taken up in the next section.

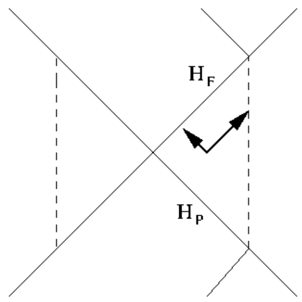


Figure 1.4: A pair of particles is created in the compact region inside the inner horizon.

### 1.3.5 Naked singularities

A general reference on the physics of naked singularities is [141]. Particle emission from naked singularities in higher dimensions has been studied also in [142]. Sticking to two-dimensional models for simplicity, consider the following metric [143]

$$ds^2 = \sigma^{-1} dx^+ dx^-, \quad \sigma = \lambda^2 x^+ x^- - a(x^+ - x_0^+)\Theta(x^+ - x_0^+) \tag{1.240}$$

where $\lambda$ is related to the cosmological constant by $\Lambda = -4\lambda^2$. This metric arises as a solution of two-dimensional dilaton gravity coupled to a bosonic field with stress tensor $T_{++} = 2a\delta(x^+ - x_0^+)$, describing a shock wave. A look at Fig. (1.5) reveals that $\sigma = 0$ is a naked singularity partly to the future of a flat space region, usually named the linear dilaton vacuum. The heavy arrow represents the history of the shock wave responsible for the existence of the time-like singularity. The Hamilton–Jacobi equation implies either $\partial_+ I = 0$ or $\partial_- I = 0$, $I$ being the action. To find the ingoing flux we integrate along $x^+$ till we encounter the naked singularity, using $\partial_- I = 0$, so that

$$I = \int dx^+ \partial_+ I = \int \omega \frac{dx^+}{2\sigma} = \int \frac{\omega\, dx^+}{2(\lambda^2 x^- - a)(x^+ + a x_0^+/C - i\epsilon)} \tag{1.241}$$

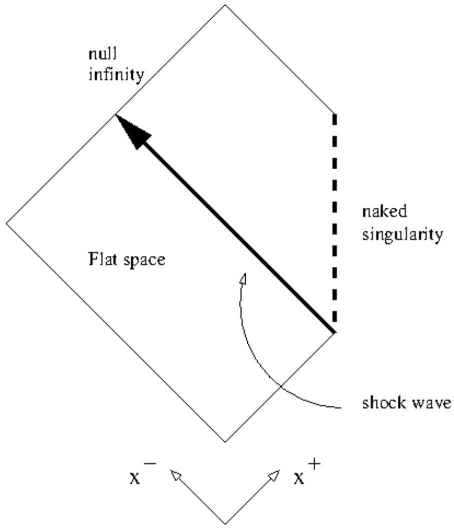


Figure 1.5: The naked singularity formed by the shock wave.

where $C = C(x^-) := (\lambda^2 x^- - a)$ and $\omega = 2\sigma\partial_+ I$ is the familiar Kodama's energy. The imaginary part immediately follows on using $(x - i\epsilon)^{-1} = P\frac{1}{x} + i\pi\delta(x)$, giving the absorption probability as a function of retarded time

$$\Gamma(\omega) = \Gamma_0 \; e^{-2Im\, I} = \Gamma_0 \; e^{-\pi\omega/C(x^-)} \; , \tag{1.242}$$

$\Gamma_0$ being some pre-factor of order one. The flux is computed by integrating the probability over the coordinate frequency (that is, the variable conjugated to the coordinate time) $\hat{\omega} = \omega/\sigma$, with the density of states measure $d\hat{\omega}/2\pi$, which gives

$$T_{++} = \frac{\Gamma_0}{2\pi} \int \Gamma(\sigma\hat{\omega}) \, \hat{\omega} \, d\hat{\omega} = \Gamma_0 \frac{(\lambda^2 x^- - a)^2}{2\pi^3 \sigma^2} \; . \tag{1.243}$$

Similarly, in order to find the outgoing flux we integrate along $x^-$ starting from the naked singularity, this time using $\partial_+ I = 0$. A similar calculation gives

$$\mathrm{Im} I = \frac{\pi\omega}{2\lambda^2 x^+} \; , \tag{1.244}$$

then, integrating the probability over the coordinate frequency, we obtain

$$T_{--} = \Gamma_0 \frac{\lambda^4 (x^+)^2}{2\pi^3 \sigma^2} \; . \tag{1.245}$$

The outgoing flux is $2(T_{++} - T_{--})$. The conservation equations

$$\sigma\partial_+ T_{--} + \partial_-(\sigma T_{+-}) = 0, \qquad \sigma\partial_- T_{++} + \partial_+(\sigma T_{+-}) = 0 \tag{1.246}$$

determine the components only up to arbitrary functions $B(x^-)$ and $A(x^+)$, something which corresponds to the freedom in the choice of a vacuum. For instance, requiring the fluxes to vanish in the linear dilaton vacuum fixes them uniquely. As it is well known, $T_{+-}$ is given by the conformal anomaly: $T = 4\sigma T_{+-} = R/24\pi$ (for one bosonic degree of freedom). Matching to the anomaly gives the pre-factor $\Gamma_0 = \pi^2/24$, of order one indeed. These results agree with the one-loop calculation to be found in [143]. Note that the stress tensor diverges while approaching the singularity, indicating that its resolution will not be possible within classical gravity but requires quantum gravity instead [144, 145].

We return now to the Reissner–Nordström solution. Could it be that the naked singularity emits particles? In the four-dimensional case one easily sees that the action has no imaginary part along null trajectories either ending or beginning at the singularity. Formally this is because the Kodama energy coincides with the Killing energy in such a static manifold and there is no infinite red-shift from the singularity to infinity. Even considering the metric as a genuinely two-dimensional solution, this would lead to an integral for $I$

$$I = \int \omega \, \frac{(r - r_+)(r - r_-)}{r^2} \, dx^+ \tag{1.247}$$

where $x^{\pm} = t \pm r_*$, with

$$r_* = r + \frac{r_+^2}{r_+ - r_-} \ln\left(\frac{r_+ - r}{r_+}\right) - \frac{r_+^2}{r_+ - r_-} \ln\left(\frac{r_- - r}{r_-}\right) = \frac{x^+ - x^-}{2} \, . \tag{1.248}$$

But close to the singularity

$$r^2 = \left[\frac{3r_+ r_-}{2}(x^+ - x^-)\right]^{2/3} + \cdots \tag{1.249}$$

not a simple pole, rather, an integrable singularity. It is fair to say that the Reissner–Nordström naked singularity will not emit particles in this approximation. This seems to be coherent with quantum field theoretical results. With the customary $u = x^-$ and $v = x^+$, there is a map $u \to v = G(u)$ which gives the family of ingoing null geodesics, characterised by constant values of $v$, which after reflection in the origin emerge as the the family of outgoing null geodesics with constant $u$. According to [146], the radiated s-wave power of a minimally coupled scalar field is given in terms of the map $G(u)$ by the Schwarzian derivative

$$\mathcal{W} = \frac{1}{24\pi}\left[\frac{3}{2}\left(\frac{G''}{G'}\right)^2 - \frac{G'''}{G'}\right] \, . \tag{1.250}$$

The $(u, v)$ section of the Reissner–Nordström metric is conformally flat, hence the above map is trivial (*i.e.* linear) and $\mathcal{W} = 0$.

# 2 Unruh-DeWitt detectors

## 2.1 Quantized fields in curved metrics

As a foreword to this chapter we shall introduce some concepts regarding the quantization of fields living in generic curved metric. The topic has been extensively covered in the literature (see *e.g.* [6]), hence we will recall only the aspects that will be useful later.

In order to quantize a field in a generic 1+3 curved metric we will follow the standard procedure used in Minkowski spacetime, then we will highlight the features of gravity that impose a different interpretation for the particle content of the field. The first step in translating the formalism from flat to curved metrics involves the substitutions

$$\begin{aligned} \eta_{\mu\nu} &\rightarrow g_{\mu\nu} \\ \partial_\mu &\rightarrow \nabla_\mu \end{aligned}$$

It has to be noted, however, that the covariant derivative of a scalar field coincides with the usual derivative. Making use of these tools, we can construct the generalization of the Lagrangian density for the real scalar field $\phi(x)$ with mass $m$ coupled to a metric $g_{\mu\nu}$,

$$\mathcal{L}(x) = \frac{1}{2\sqrt{-g}} \left[ g^{\mu\nu}\, \phi_{,\mu}\, \phi_{,\nu} - \left(m^2 + \xi\, R\right)\, \phi^2 \right] \tag{2.1}$$

$R$ is the Ricci scalar and the term $\xi R\phi^2$ is the coupling between the scalar field and the metric: a choice of $\xi = 0$ corresponds to a *minimal coupling*; the case $\xi = 1/6$ instead is called *conformal coupling* because, if $m = 0$ and if the field transforms as $\phi \rightarrow \Omega^{-1}(x)\phi$, then the action and the field equation are invariant under the conformal transformation

$$g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu} \tag{2.2}$$

where $\Omega(x)$ is a continuous and non-vanishing real function.

Keeping $\xi$ generic, the vanishing variation of the action obtained from Eq.(2.1) gives the field equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)+\left(m^2+\xi R\right)\phi=0 \tag{2.3}$$

The mode solutions $u_i$ of Eq.(2.3) satisfy orthonormality conditions on constant spacelike hypersurfaces $\Sigma$, so that every solution $\phi$ can be expanded in terms of this orthonormal set as

$$\phi(x)=\sum_i\Big(a_i\,u_i(x)+a_i^\dagger\,u_i^*(x)\Big) \tag{2.4}$$

where $a_i$ and $a_i^\dagger$ are creation and annihilation operators for the set of quantities labeled by $i$. The quantization procedure itself involves then the imposition of commutation relations between these operators,

$$\left[a_i,a_j^\dagger\right]=\delta_{ij}\qquad\left[a_i,a_j\right]=\left[a_i^\dagger,a_j^\dagger\right]=0 \tag{2.5}$$

and the subsequent definition of the vacuum state of the Fock space as the state annihilated by $a_i,\forall i$. The complications due to the fact that spacetime is curved arise here: in order to understand why, let's analyze the two cases.

**Minkowski spacetime.** The group that leaves the Minkowski line element unchanged is the Poincaré group, which naturally selects the rectangular coordinate system $(t,x,y,z)$. In particular, Minkowski spacetime has a Killing vector $\frac{\partial}{\partial t}$ which is everywhere orthogonal to the spacelike hypersurfaces of constant $t$. The modes that satisfy the flat-space version of Eq. (2.3) are always positive frequency with respect to $t$, *i.e.* are eigenfunctions of the Killing vector:

$$\frac{\partial}{\partial t}u_k=-i\,\omega\,u_k\quad,\quad\omega>0 \tag{2.6}$$

The vacuum state in this case is invariant under Poincaré transformations or, equivalently, this is the vacuum on which all inertial observers agree.

**Curved spacetime.** Poincaré group is not a symmetry of the line element if spacetime is generically curved: specifically, the general-relativistic request of diffeomorphism invariance is translated in the physical irrelevance of coordinate systems. In this case no general Killing vector can be found[1] in order to define positive frequency solutions of Eq.(2.3) and $\phi$ has no "natural" decomposition: this ineherent ambiguity in the choice of mode decomposition leads to the non-uniqueness of annihilation operators and hence to the non-uniqueness of the vacuum state.

[1]One can find particular spacetimes whose symmetries allow a definition of Killing vectors and hence a coordinate system naturally associated to them. On the other hand general covariance reminds us that the coordinate system in question is not to be considered special.

## 2.2 Modelling a detector

A way to deal with this intrinsic ambiguity is to face it in a more pragmatic way, following Unruh and DeWitt's definition of particle as *something detected by a particle detector* (see [6] and [149, 150]).

The basic idea of a Unruh-DeWitt detector is fairly simple and involves two ingredients:

1. a real scalar field $\hat{\phi}(x^0, \mathbf{x})$;

2. a localized detector, *i.e.* a two-level quantum system, endowed with a trajectory $(x^0(\tau), \mathbf{x}(\tau))$ and a set of eigenstates $\{|E_0\rangle, |E_1\rangle\}$ with eigenvalues $E_0 < E_1$.

Both the field and the detector live in a generic metric $g_{\mu\nu}$. All the dynamics will be parametrized by the proper time $\tau$ along the trajectory.
The detector's Hamiltonian of interaction with the scalar field is

$$\hat{H}_I = \lambda \, \hat{M}(\tau) \, \hat{\phi}(\tau) \tag{2.7}$$

where $\lambda$ is a small coupling constant and $\hat{M}(\tau)$ is the monople moment of the detector, whose evolution is given by $\hat{H}_d$, the detector's Hamiltonian:

$$\hat{M}(\tau) = e^{i \, \hat{H}_d \, \tau} \, \hat{M}(0) \, e^{-i \, \hat{H}_d \, \tau} \tag{2.8}$$

Let the initial state of the system at time $\tau_0$ be given by the product $|0\rangle|E_0\rangle$, where $|0\rangle$ is the vacuum state for the field: we want to know the probability for the detector to be found in the state $|E_1\rangle$ at a subsequent time $\tau_1 > \tau_0$, irrespective of the state of the field $\phi$. In order to calculate this quantity we make use of the interaction picture: we let operators evolve according to the free Hamiltonian, while states will evolve according to the interaction Hamiltonian. The transition amplitude we are interested in is given by

$$\langle \phi, E_1 | 0, E_0 \rangle = \langle \phi, E_1 | \, \hat{T} \, \exp \left[ -i \int_{\tau_0}^{\tau_1} \hat{H}_I \, d\tau \right] \, |0, E_0\rangle \tag{2.9}$$

where $\hat{T}$ is the time-ordering operator. We then expand to first order in $\lambda$ the right-hand side of Eq.(2.9):

$$\begin{aligned} \langle \phi, E_1 | \hat{\mathbb{I}} | 0, E_0 \rangle - i \, \lambda \, \langle \phi, E_1 | \int_{\tau_0}^{\tau_1} e^{i \, \hat{H}_d \, \tau} \, \hat{M}(0) \, e^{-i \, \hat{H}_d \, \tau} \, \hat{\phi}(\tau) \, d\tau \; |0, E_0\rangle = \\ = -i \, \lambda \, \langle E_1 | \hat{M}(0) | E_0 \rangle \int_{\tau_0}^{\tau_1} e^{-i \, (E_1 - E_0) \tau} \, \langle \phi | \hat{\phi}(\tau) | 0 \rangle \end{aligned} \tag{2.10}$$

In order to obtain the probability for the detector to make the transition $|E_0\rangle \to |E_1\rangle$, we square Eq.(2.10) and sum over all possible final states of the field:

$$\sum_\phi |\langle\phi, E_1|0, E_0\rangle|^2 =$$
$$= \lambda^2 \, |\langle E_1|\hat{M}(0)|E_0\rangle \, |^2 \int_{\tau_0}^{\tau_1} d\tau \int_{\tau_0}^{\tau_1} d\tau' \, e^{-i\,(E_1-E_0)(\tau'-\tau)} \langle 0|\hat{\phi}(\tau)\hat{\phi}(\tau')|0\rangle$$

The term in front of the integrals depends on the details of the detector. In the following analysis we will need to focus only on the so-called *response function*, which is universal, in the sense that it does not depend on the internal structure of the particular detector:

$$F_{\tau_0,\tau_1}(E) = \int_{\tau_0}^{\tau_1} d\tau \int_{\tau_0}^{\tau_1} d\tau' \, e^{-i\,E(\tau'-\tau)} \langle 0|\hat{\phi}(\tau)\hat{\phi}(\tau')|0\rangle \tag{2.11}$$

where $E = E_1 - E_0$ is the energy gap of the detector (it can be positive or negative, depending on whether we take into account excitations or decays). Now change variables of integration to $u = \tau$ and $s = \tau - \tau'$ if $\tau > \tau'$, or $u = \tau'$ and $s = \tau' - \tau$ if $\tau' > \tau$: using the fact that $\langle 0|\hat{\phi}(\tau)\hat{\phi}(\tau')|0\rangle = \langle 0|\hat{\phi}(\tau')\hat{\phi}(\tau)|0\rangle^*$, the response function can be recast in the form

$$F_{\tau_0,\tau_1}(E) = 2\int_{\tau_0}^{\tau_1} du \int_0^{u-\tau_0} ds \, \mathrm{Re}\left[e^{-i\,E\,s}\langle 0|\hat{\phi}(\tau)\hat{\phi}(\tau-s)|0\rangle\right] \tag{2.12}$$

This equation can be differentiated with respect to $\tau_1$,

$$\dot{F}_{\tau_0,\tau_1}(E) = 2\int_0^{\tau-\tau_0} ds \, \mathrm{Re}\left[e^{-i\,E\,s}\langle 0|\hat{\phi}(\tau)\hat{\phi}(\tau-s)|0\rangle\right] \tag{2.13}$$

where again $u = \tau$. Eq.(2.13) is the *transition rate* of the detector.

The quantity $W(\tau, s) = \langle 0|\hat{\phi}(\tau)\hat{\phi}(\tau-s)|0\rangle$, the positive frequency Wightman function, is calculated by solving the equation of motion Eq.(2.3) and expanding the field on the modes as schematically shown in Eq.(2.4). In the following sections we will deal with conformally flat metrics[2], obtained in general by defining a conformal time $\eta$

$$g_{\mu\nu} = \Omega^2(x)\left(-d\eta^2 + d\mathbf{x}^2\right) \tag{2.14}$$

so that a comfortable choice will be a conformally coupled ($\xi = 1/6$) massless scalar field. In this case the Wightman function is related to the Minkowskian one simply through the conformal factor:

$$W(\tau, s) = \frac{1}{\Omega(\tau)\Omega(\tau-s)} \cdot \frac{1/4\pi^2}{|\mathbf{x}(\tau)-\mathbf{x}(\tau-s)|^2 - |\eta(\tau)-\eta(\tau-s)|^2} \tag{2.15}$$

[2]In the case of non-conformally flat metrics, however, the choice of radial trajectories in spherical symmetry allows to reduce the dimensionality of the problem from $1+4$ to $1+1$.

Our last effort before analyzing some specific examples regards the regularization of this quantity, which has a double pole in the coincidence limit $s \to 0$. The $i\epsilon$ prescription in this case has been shown by Takagi [7] and Schlicht [8] to lead to unphysical results with regard to instantaneous proper-time rate in Minkowski spacetime, so that a better choice would be [9, 10]

$$W(\tau, s) = \frac{1}{\Omega(\tau)\Omega(\tau - s)} \cdot \frac{1/4\pi^2}{(x(\tau) - x(\tau - s))^2 - i\epsilon(\dot{x}(\tau) - \dot{x}(\tau - s))^2} \tag{2.16}$$

which involves also the four-velocity $\dot{x}$ along the trajectory. Although the covariant $i\epsilon$ prescription is necessary in order to deal with the second order pole, one may try to avoid the awkward limit $\epsilon \to 0^+$ by omitting the $\epsilon$-terms and subtracting the leading pole at $s = 0$. In this way Louko and Satz [9] give the following expression for the transition rate:

$$\dot{F}_\tau(E) = -\frac{E}{4\pi} + \frac{1}{2\pi^2 \Delta\tau} + \frac{1}{2\pi^2} \int_0^{\Delta\tau} \Big( \frac{\cos(Es)}{\sigma^2(\tau, s)} + \frac{1}{s^2} \Big) ds \tag{2.17}$$

where $\sigma^2(\tau, s) = \Omega(\tau)\Omega(\tau - s)\,(x(\tau) - x(\tau - s))^2$. It's easy to see that this expression will be different from zero even for an inertial detector in Minkowski spacetime: the first term on the right-hand side represents the spontaneous emission contribution.

A more suitable form can be obtained, from eq.(2.17), by simple manipulations:

$$\begin{aligned} \dot{F}_\tau(E) &= -\frac{E}{4\pi} + \frac{1}{2\pi^2 \tau} + \\ &\quad + \frac{1}{2\pi^2} \left[ \int_0^\infty \cos(Es) \Big( \frac{1}{\sigma^2(\tau, s)} + \frac{1}{s^2} \Big) ds - \int_0^\infty \frac{\cos(Es) - 1}{s^2} \right] + \\ &\quad - \frac{1}{2\pi^2} \left\{ \int_\tau^\infty \frac{\cos(Es)}{\sigma^2(\tau, s)} ds + \int_\tau^\infty \frac{ds}{s^2} \right\} \\ &= -\frac{E}{2\pi}\theta(-E) \; + \; \frac{1}{2\pi^2} \int_0^\infty \cos(Es) \Big( \frac{1}{\sigma^2(\tau, s)} + \frac{1}{s^2} \Big) ds \; + \\ &\quad - \frac{1}{2\pi^2} \int_{\Delta\tau}^\infty \frac{\cos(Es)}{\sigma^2(\tau, s)} ds \end{aligned} \tag{2.18}$$

This is the transition rate that will be used in the following: in the last expression, the first term contributes only for $E < 0$ (spontaneous emission term) and it will be discarded in the following; the second term is an integration over infinite time and it's therefore an *asymptotic contribution* (simply $\dot{F}$ in the following); the last term instead takes into account that a realistic detector has a "switch-on" and "switch-off" time, so it constitutes a *finite-time tail* ($J_\tau$ in the following) which vanishes in the limit $\Delta\tau \to \infty$ and, as we will see, in the case of asymptotically stationary situations controls how fast the thermal equilibrium is reached.

## 2.3 Stationary metrics

### 2.3.1 Static black hole

The general metric for a static black hole reads

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + r^2 d\Omega^2 \,, \tag{2.19}$$

where, for sake of simplicity, we shall assume $W(r) = V(r)$, with $V(r)$ having just simple poles in order to describe what we might call a *nice black hole* (suggested by Hayward). Let $r_H$ be the (greatest) solution of $V(r) = 0$, the general formalism tells us that the horizon is located at $r = r_H$; the Kodama vector coincides with the usual Killing vector $(1, \mathbf{0})$; and the Hayward surface gravity is the Killing surface gravity, namely $\kappa_H = \kappa = V'_H/2$. We now introduce the Kruskal-like gauge associated with this static black hole solution. The first step consists in introducing the tortoise coordinate

$$r^*(r) = \int^r \frac{d\tilde{r}}{V(\tilde{r})} \,. \tag{2.20}$$

One has $-\infty < r^* < \infty$ and

$$ds^2 = V(r^*)[-dt^2 + (dr^*)^2] + r^2(r^*) d\Omega^2_{(2)} \,. \tag{2.21}$$

The Kruskal-like coordinates are

$$R = \frac{1}{\kappa} e^{\kappa r^*} \cosh(\kappa t) \,, \quad T = \frac{1}{\kappa} e^{\kappa r^*} \sinh(\kappa t) \,, \tag{2.22}$$

so that

$$-T^2 + R^2 = \frac{1}{\kappa^2} e^{2\kappa r^*} \,, \tag{2.23}$$

and the line element becomes

$$\begin{aligned} ds^2 &= e^{-2\kappa r^*} V(r^*)[-dT^2 + dR^2] + r^2(T,R) d\Omega^2 \\ &\equiv e^{\Psi(r^*)}(-dT^2 + dR^2) + r^2(T,R) d\Omega^2 \end{aligned} \tag{2.24}$$

where now the coordinates are $T$ and $R$, $r^* = r^*(T,R)$, $e^{\Psi(r*)} := V(r^*)e^{-2\kappa r^*}$.

The key point to recall here is that in the Kruskal gauge (2.24) the normal metric – the important one for radial trajectories – is conformally related to two dimensional Minkoswki space-time. The second observation is that Kodama observers are defined by the integral curves associated with the Kodama vector, thus the areal radius $r(T,R)$ and $r^*$ are *constant.* As a consequence, the proper time along Kodama trajectories reads

$$\begin{aligned} d\tau^2 = V(r^*)dt^2 &= e^{\Psi(r^*)}(dT^2 - dR^2) \\ &= a^2(r^*)(dT^2 - dR^2) \end{aligned} \tag{2.25}$$

so that $t = \tau/\sqrt{V(r^*)}$ and

$$R(\tau) = \frac{1}{\kappa} e^{\kappa r^*} \cosh\left(\kappa \frac{\tau}{\sqrt{V(r^*)}}\right)$$
$$T(\tau) = \frac{1}{\kappa} e^{\kappa r^*} \sinh\left(\kappa \frac{\tau}{\sqrt{V(r^*)}}\right) . \tag{2.26}$$

The geodesic distance reads

$$\sigma^2(\tau, s) = e^{\Psi(r^*)} \Big[ - (T(\tau) - T(\tau - s))^2 + + (R(\tau) - R(\tau - s))^2 \Big] , \tag{2.27}$$

and one gets, using (2.26),

$$\sigma^2(\tau, s) = -\frac{4V(r^*)}{\kappa^2} \sinh^2\left(\frac{\kappa\, s}{2\sqrt{V(r^*)}}\right) . \tag{2.28}$$

Since $\sigma^2(\tau, s) = \sigma^2(s) = \sigma^2(-s)$, we can use eq.(2.18) in the limit when $\Delta\tau$ goes to infinity:

$$\dot{F} = \frac{\kappa}{8\pi^2\sqrt{V^*}} \int_{-\infty}^{\infty} dx e^{-\frac{2i\sqrt{V^*}Ex}{\kappa}} \left[ -\frac{1}{\sinh^2 x} + \frac{1}{x^2} \right] . \tag{2.29}$$

The integral can be evaluated using the contour as in fig.(2.1) and by the theorem of residues the final result is

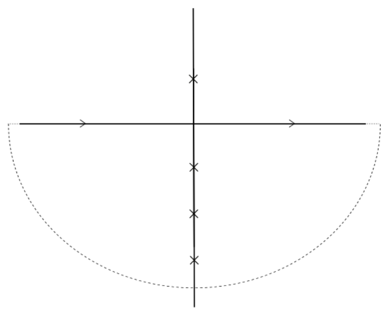

Figure 2.1: The periodic poles ($\times$) of the regularized integrand in eq.(2.29) in the complex $x$-plane and the contour of integration: the dashed path is sent to infinity.

$$\dot{F} = \frac{1}{2\pi} \frac{E}{\exp\left(\frac{2\pi\sqrt{V^*}E}{\kappa}\right) - 1} . \tag{2.30}$$

Since the transition rate exhibits the characteristic Planck distribution, it means that the Unruh–DeWitt thermometer in the generic spherically symmetric black hole space-time detects a quantum system in thermal equilibrium at the local temperature

$$T = \frac{\kappa}{2\pi\sqrt{V^*}} . \tag{2.31}$$

With regard to the factor $\sqrt{V^*} = \sqrt{-g_{00}}$, recall Tolman's theorem which states that, for a gravitational system at thermal equilibrium, $T\sqrt{-g_{00}}$ = constant. For asymptotically flat black hole space-times, one obtains the "intrinsic" constant temperature of the Hawking effect, i.e.

$$T_H = \frac{\kappa}{2\pi} = \frac{V'_H}{4\pi}\,. \tag{2.32}$$

We would like to point out that this is a quite general result, valid for a large class of nice black holes, as for example Reissner-Nordström and Schwarzschild-AdS black holes. On the other hand, the Schwarzschild-dS black hole cannot be included, due to the presence of two horizons. However, as an important particular case, we may consider the static patch of de Sitter space, with a metric defined by

$$V(r) = 1 - H_0^2 r^2\,, \qquad H_0^2 = \frac{\Lambda}{3}\,. \tag{2.33}$$

The unique horizon is located at $r_H = H_0^{-1}$ and the Gibbons–Hawking temperature is [13] $T_H = H_0/2\pi$. In the next Section, we will present a derivation of this well known result in another gauge.

We conclude this subsection making some remarks on de Sitter and anti-de Sitter black holes. First, we observe that in a static space-time, namely the one corresponding to a nice black hole, the Killing–Kodama observers with $r = K$ constant have an invariant acceleration

$$A^2 = g_{\mu\nu}A^\mu A^\nu = \frac{V'^2(K)}{4V(K)}\,, \tag{2.34}$$

where $A^\mu = u^\nu \nabla_\nu u^\mu$, $u^\mu$ being the observer's four-velocity, that is the (normalized) tangent vector to the integral curves of the Kodama vector field. In the case of de Sitter black hole, one has

$$A^2 = \frac{H_0^4 K^2}{1 - H_0^2 K^2}\,. \tag{2.35}$$

As a result,

$$A^2 + H_0^2 = \frac{H_0^2}{1 - H_0^2 K^2}\,, \tag{2.36}$$

and the de Sitter local temperature felt by the Unruh detector,

$$T_{dS} = \frac{H_0}{2\pi}\frac{1}{\sqrt{1 - H_0^2 K^2}} \tag{2.37}$$

can be re-written as [151, 170]

$$T_{dS} = \frac{1}{2\pi}\sqrt{A^2 + H_0^2} = \sqrt{T_U^2 + T_{GH}^2}\,. \tag{2.38}$$

A similar result was also obtained for AdS in [151], and it reads

$$T_{AdS} = \frac{1}{2\pi}\sqrt{A^2 - H_0^2}\,. \tag{2.39}$$

We would like to show that it is a particular case of our general formula (2.31). In fact, it is sufficient to apply it to the four-dimensional topological black hole with hyperbolic horizon manifold found in [80,81,153,154], which is a nice black hole with

$$V(r) = -1 - \frac{C}{r} + H_0^2 r^2\,, \tag{2.40}$$

where $C$ is a constant of integration related to the black hole mass. The space-time is a solution of Einstein equation with negative cosmological constant $\Lambda = -3H_0^2$, which is asymptotically Anti-de Sitter. When the constant of integration goes to zero, one has still a black hole solution, and calculation similar to the one valid for de Sitter space-time gives

$$T_{AdS} = \frac{H_0}{2\pi}\frac{1}{\sqrt{-1 + H_0^2 K^2}} = \frac{1}{2\pi}\sqrt{A^2 - H_0^2}\,, \tag{2.41}$$

which is Deser *et al.* result [151]. Thus, for spherically symmetric space-times with constant curvature one has that the local temperature felt by the Kodama–Unruh–DeWitt detector can be written as

$$T = \sqrt{T_U^2 + \alpha T_{GH}^2}\,, \tag{2.42}$$

where $T_U$ is the Unruh temperature associated with the acceleration of the Kodama observer, $T_{GH}$ is the Gibbons–Hawking temperature and $\alpha = 1$ for the de Sitter space-time, $\alpha = 0$ for Minkowski space-time (this is the original Unruh effect) and $\alpha = -1$ for the “massless" AdS topological black hole. This formula may help to understand better the relation between the Unruh-like effects and the genuine presence of a thermal bath and shows that, in general, the Kodama–Unruh detector gives an intricate relation between Killing–Hayward temperature and other invariant temperatures such as the Unruh's one. Note that $T$ in Eq. (2.31) is greater than $T_U = A(r)/2\pi$ for $r > r_H$, where $A$ is the local acceleration of an observer following a Killing trajectory in the black hole space-time, a fact that has been interpreted as a violation of the equivalence principle [152]. We prefer to interpret this effect as due to the additional presence of the Hawking radiation over the vacuum thermal Unruh's noise.

### 2.3.2 de Sitter spacetime

To apply the Unruh–DeWitt detector formalism to cosmology we consider a generic FRW spatially flat space-time. This case has been investigated also in [156] (see also [158]). Recall that here the areal radius is $R = ra(t)$. Thus, for the Kodama observer, one has

$$r(t) = \frac{K}{a(t)}, \tag{2.43}$$

with constant $K$. For a radial trajectory, the proper time in FRW is

$$d\tau^2 = a(\eta)(d\eta^2 - dr^2)\,. \tag{2.44}$$

As a function of the proper time, the conformal time along a Kodama trajectory is

$$\begin{aligned}\eta(\tau) &= -\int d\tau\, \frac{1}{a(\eta)\sqrt{1-K^2H^2(\tau)}} \\ &\equiv -\int d\tau\, \frac{1}{a(\tau)\sqrt{V(\tau)}}\,,\end{aligned} \tag{2.45}$$

$H(\tau)$ being the Hubble parameter as a function of proper time. In general, we may use Eq. (2.18) in which, for radial Kodama observer, one has

$$\begin{aligned}x(\tau) &= (\eta(\tau), r(\tau), 0, 0) \\ &= \left(-\int \frac{1}{a(\tau)\sqrt{V(\tau)}}d\tau, \frac{K}{a(\tau)}, 0, 0\right).\end{aligned} \tag{2.46}$$

As a warm up, we first revisit the well known example of FRW space is the stationary flat de Sitter expanding (contracting) space-time, which in the FRW context is defined by considering $a(t) = e^{H_0 t}$. Thus,

$$ds^2 = -dt^2 + e^{2H_0 t}(dr^2 + r^2 d\Omega^2)\,. \tag{2.47}$$

Here $H(t) = H_0$ is constant as well as $V = V_0 = 1 - H_0^2 K^2$. For Kodama observers

$$\tau = \sqrt{V_0}\, t\,, \qquad a(\tau) = e^{\frac{H_0}{\sqrt{V_0}}\tau}\,, \tag{2.48}$$

and

$$\eta(\tau) = -\frac{1}{H_0}e^{-\frac{H_0}{\sqrt{V_0}}\tau}\,, \quad r(\tau) = K\, e^{-\frac{H_0}{\sqrt{V_0}}\tau}\,, \tag{2.49}$$

thus, the geodesic distance is

$$\sigma^2_{dS}(\tau, s) = -\frac{4\,V_0}{H_0^2}\sinh^2\left(\frac{H_0\, s}{2\sqrt{V_0}}\right). \tag{2.50}$$

This result is formally equal to the one obtained for the generic static black hole (2.28). Since again $\sigma^2(\tau, s) = \sigma^2(s) = \sigma^2(-s)$, for $E > 0$ and in the infinite time limit

$$\dot{F} = \frac{H_0}{8\pi^2\sqrt{V_0}}\int_{-\infty}^{\infty} dx\, e^{-\frac{2i\sqrt{V_0}Ex}{H_0}}\left[\frac{1}{x^2} - \frac{1}{\sinh^2 x}\right] \tag{2.51}$$

Again, we arrive at

$$\frac{dF_{dS}}{d\tau} = \frac{1}{2\pi} \frac{E}{\exp\left(\frac{2\pi\sqrt{V_0}E}{H_0}\right) - 1}\,, \tag{2.52}$$

which shows again that the Unruh–DeWitt thermometer in the FRW de Sitter space detects a quantum system in thermal equilibrium at a temperature $T = H_0/2\pi\sqrt{V_0}$. Here, the Tolman factor takes the form a Lorentz $\gamma$-factor, which represents the Unruh acceleration part. In fact, we recall that the four-acceleration of a Kodama observer in a FRW space-time has the expression

$$A^2 = A^\mu A_\mu = K^2 \left[\frac{\dot{H}(t) + (1 - H^2(t)K^2)H^2(t)}{(1 - H^2(t)K^2)^{\frac{3}{2}}}\right]^2 . \tag{2.53}$$

As a result, for dS space in a time dependent spatially flat patch we have

$$A^2 = \frac{K^2 H_0^4}{1 - K^2 H_0^2}\,, \tag{2.54}$$

showing that

$$\frac{H_0}{\sqrt{1 - H_0^2 K^2}} = \sqrt{H_0^2 + A^2}\,, \tag{2.55}$$

in agreement with the dS static calculation. When $K = 0$, that is when the detector is co-moving, one has $V_0 = 1$ and the classical Gibbons–Hawking result $T_{dS} = H_0/2\pi$ is recovered.

**Finite-time effects.** We now present a brief discussion of finite-time effects which will be relevant to the following discussion on asymptotic behaviour: how is the thermal distribution of the response function reached in the limit of very large times? In the case of non inertial particle detector in Minkowski space-time, see [155], and for de Sitter FRW space see [157].

To answer this, we consider the finite time contribution due to the tail $J_\tau$ in Eq. (2.18) for de Sitter or black hole cases compared to the thermal value given by the time-independent part. A direct calculation of $J_\tau$ for both black holes and dS and using the fact that

$$\text{csch}^2(x) = 4 \sum_{n=1}^{\infty} n\, e^{-2nx} \tag{2.56}$$

gives

$$
\begin{aligned}
J_\tau &= \frac{\kappa_l^2}{8\pi^2}\int_{\Delta\tau}^{\infty} ds\, \frac{\cos(E\, s)}{\sinh^2\left(\frac{\kappa_l s}{2}\right)} \\
&= \frac{E}{2\pi^2}\sum_{n=1}^{\infty}\frac{n e^{-2\pi n T_H \Delta\tau}}{n^2+E^2/4\pi^2 T_H^2}\times \\
&\times\left(\frac{2\pi T_H}{E} n\cos(E\Delta\tau)-\sin(E\Delta\tau)\right)
\end{aligned} \tag{2.57}
$$

where $\kappa_l$ is the rescaled surface gravity and $T_H = \kappa_l/2\pi \equiv \kappa/2\pi\sqrt{V}$ the local Hawking temperature. We recall that $\kappa = H_0$ for de Sitter space and $\kappa = V'(r_H)/2$ for the black hole: these quantities in fact determine the characteristic time-scales the thermalization time has to be compared to.

We consider as before the peculiar Kodama observer for which $V = 1$, so that $T_H = \kappa/2\pi$. As a general feature, the fluctuating tail term drops out exponentially for large $\Delta\tau$, that is for long proper time intervals in which the detector stays on. In order to analyze the approaching to an equilibrium condition of the response function, we consider the ratio between the finite-time expression – the sum of $\dot{F}$ and the tail $J_\tau$ – and $\dot{F}$ alone, with the agreement that equilibrium is attained whenever

$$
R_{eq} = \frac{\dot{F}+J_\tau}{\dot{F}} \sim O(1)\ .
$$

Looking at (2.57) one easily sees that the equilibrium value, $R_{eq} = 1$, is reached sooner if $E/\kappa \ll 1$. To be more precise, irrespective of the absolute value of $\kappa$, *a detector that is switched on for a time much shorter than the characteristic time-scale $\Delta\tau \ll \kappa^{-1}$, detects a thermal bath only with particles whose energies are $E \ll \kappa$; on the other hand, a thermal equilibrium for particles with energies $E \gg \kappa$ is registered only if the detector lifetime is* $\Delta\tau \gg \kappa^{-1}$, which is the age of the universe. The Hubble scale corresponds to an extremely small energy scale of order $10^{-42}$Gev, therefore $E \gg \kappa$ is the physical region.

It easy to see that if the factor $V < 1$, the thermalization time decreases for every energy scale.

**Stability of Kodama trajectories.** It is easy to see that a detector moving along a Kodama trajectory in an expanding de Sitter universe, has the same dynamics of a particle which moves along the separatrix in the potential of an inverted harmonic oscillator. From the equation of motion Eq.(2.43), we can therefore introduce the effective Lagrangian

$$
L_{iho} = m\left(\left(\frac{dr}{dt}\right)^2 + H^2\, r^2\right) \tag{2.58}
$$

where $m$ is a parameter with mass dimension whose relation with physical quantities will be clarified later. Our detector is initially (at $t = 0$) represented by a Gaussian wave packet of size $b$ peaked around $r = K$,

$$\psi(0,r) = \frac{\exp\left[-i\frac{H\,K\,m\,r}{\hbar} - \frac{(r-K)^2}{2\,b^2}\right]}{\sqrt{b\,\sqrt{\pi}}} \tag{2.59}$$

which is then propagated to later times by the propagator obtained from the Lagrangian Eq.(2.58) (see, e.g., [147]),

$$G(t,r;0,r') = \sqrt{\frac{i\,H\,m}{2\,\pi\,\hbar\,\sinh(H\,t)}}\,\exp\left[i\,\frac{H\,m\left[(r^2-r'^2)\,\cosh(H\,t) - 2\,r\,r'\right]}{2\,\hbar\,\sinh(H\,t)}\right] \tag{2.60}$$

The complete expression of the detector's propagated wavefunction is rather cumbersome, however we notice that its square modulus reads

$$|\psi(t,r)|^2 \simeq \exp\left[-\frac{2\,b^2\,H^2\,m^2\,(r-K\,e^{-H\,t})^2}{b^4\,H^2\,m^2 - \hbar^2 + (\hbar^2 + b^4\,H^2\,m^2)\,\cosh(H\,t)}\right] \tag{2.61}$$

which represents a Gaussian wave-packet peaked along the classical trajectory Eq.(2.43) and spreading in time. The classical behaviour is properly recovered in the limit $\hbar \to 0$ followed by $b \to 0$ [147], in which the detector's wavefunction $\psi(r,t)$ reproduces the usual Dirac $\delta$-function peaked on the classical trajectory.

Now, in order to study the probability for the detector to absorb a scalar quantum and make a transition between two different trajectories (parameterized by different $m_i$ and $K_i$), one needs to compute the transition amplitude for finite $b$ and $\hbar$ (otherwise the result would automatically vanish, the response function involving the product of two Dirac $\delta$'s peaked on different trajectories). The detector also interacts with the quantized scalar field $\varphi = \varphi(t,r)$ according to

$$\mathcal{L}_{\rm int} = \frac{1}{2}\,Q\,(\psi_2^*\,\psi_1 + \psi_2\,\psi_1^*)\,\varphi \tag{2.62}$$

where $Q$ is a coupling constant and $\psi_i = \psi_i(t,r)$ two possible states of the detector corresponding to different trajectories $r_i = K_i\,e^{-H\,t}$ and mass parameters $m_i$.[3] We assume the difference between the two states is small,

$$\begin{cases} K_1 = K - \frac{1}{2}\,\delta K \\ \\ K_2 = K + \frac{1}{2}\,\delta K \end{cases} \quad , \quad \begin{cases} m_1 = m - \frac{1}{2}\,\delta m \\ \\ m_2 = m + \frac{1}{2}\,\delta m\ , \end{cases} \tag{2.63}$$

[3] A fundamental difference with respect to the Unruh effect analyzed in Ref. [148] is that the acceleration parameter $H$ is not varied here, since it is a property of the background space-time. A change $\delta K$ implies a change in the detector's acceleration.

and expand to leading order in $\delta K$ and $\delta m$ and, subsequently, for short times ($H\,t \sim H\,t' \ll 1$), keeping $\hbar$ and $b$ finite. In particular, one obtains

$$\psi_2^* \,\psi_1(t)\, \psi_1^* \,\psi_2(t') \simeq \exp\left[-i\,\frac{H^2\,K^2}{\hbar}\,\delta m\,(t-t') + O(b)\right] \tag{2.64}$$

in which we have evaluated the phase (in the saddle-point approximation) with $r$ along the average trajectory between $r_1$ and $r_2$ [148].

Like the Unruh detector in [148], the above exponential does not contain a real (quadratic in $\delta K$) part, contrary to the case of a geodetic observer, which implies that the transition amplitude will not vanish in the point-like limit $b \to 0$. Upon comparing with the result obtained for the point-like case, we immediately recognize that

$$H^2\,K^2\,m = E\,\sqrt{1 - H^2\,K^2} \tag{2.65}$$

where $E$ is the detector's proper energy and

$$\begin{aligned}\psi_2^* \,\psi_1(t)\, \psi_1^* \,\psi_2(t') \simeq \exp\Bigg[&-\frac{i}{\hbar}\,\delta E\,\sqrt{1-H^2\,K^2}\,(t-t') + \\ &+\frac{i}{\hbar}\,\frac{2-3H^2\,K^2}{K\sqrt{1-K^2\,H^2}}\,E\;\delta K\,(t-t') + \mathcal{O}(b)\Bigg]\end{aligned} \tag{2.66}$$

We can now take the limit $b \to 0$, as part of the point-like limit in which one would not expect the second term in the above exponential. In [148], the analogue of the second term above was required to vanish and the equation of motion for a uniformly accelerated detector in Minkowski space-time was obtained, namely $m\,a = f$ and constant. Following the same line of reasoning, we now obtain the equation of motion

$$\delta K = 0 \tag{2.67}$$

This can be interpreted as meaning the Kodama trajectory is stable against thermal emission of scalar quanta in the de Sitter background.

## 2.4 Non-stationary metric

Let us come to consider the more realistic scenario of a truly dynamical space-time of cosmological interest. From previous considerations, our basic formulas for the transition rate of the detector Eq.(2.18) are manageable – in the sense that we can extract quantitative information – only in the few highly symmetrical circumstances mentioned in Sec. 2.3. As it will be clear at the end of this section, any departure from those models is responsible for significant difficulties.

For instance, let us take on the case of homogeneous, spatially flat, universe dominated by cold matter and cosmological constant. The scale factor is (see *e.g.* [159])

$$a(t) = a_0 \sinh^{2/3}\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right) \tag{2.68}$$

where $a_0 = (\Omega_m/\Omega_\Lambda)^{1/3}$ and $\Omega_m + \Omega_\Lambda = 1$; $H_0 = \sqrt{8\pi\rho_{cr}/3}$ and $\Omega_A$ represents the relative density of matter (if $A = m$) or cosmological constant(if $A = \Lambda$). Setting $h \equiv \sqrt{\Omega_\Lambda}H_0$ for simplicity, its current value is of order $h \approx 2\times 10^{-18} sec^{-1}$. Upon integration the conformal time becomes

$$\begin{aligned}\eta(t) = \frac{1}{a_0 h}\Bigg\{ & \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{4}{3}\right)}{\sqrt{\pi}} - \operatorname{sech}^{2/3}\left(\frac{3}{2}ht\right)\times \\ & \times {}_2F_1\left(\frac{5}{6},\frac{1}{3},\frac{4}{3};\operatorname{sech}^2\left(\frac{3}{2}ht\right)\right)\Bigg\}\end{aligned} \tag{2.69}$$

where ${}_2F_1(a,b,c;z)$ is a hypergeometric function and the constant has been opportunely chosen so that at the Big Bang $\eta(t=0) = 0$. The detector's proper time is related to the cosmic time through a manageable expression only if we limit ourselves to consider co-moving detectors: $\tau(t) - \tau_0 = \int dt\sqrt{1 - H^2(t)K^2}$ so that for $K = 0$, $\Delta\tau(t) = t$, $\Delta\tau$ being the proper time interval during which the detector is turned on. Unlike the stationary cases analyzed previously, this model presents a Big Bang singularity at the origin of the time coordinate, so that the detector must be switched on at some $\tau_0 > 0$. In particular, the Big Bang prevents taking the limit as $\tau_0 \to -\infty$. By the same reason, the scale factor Eq.(2.68) is defined only for positive values of the argument: a new feature with respect to what we have seen in the previous Sections. As a consequence, $a(t-s)$ is defined as in Eq.(2.68) only for $t - s > 0$ and trivially continued outside the interval in order to make well defined the transition rate Eq.(2.18).

We obtain the following response function

$$\begin{aligned}\dot{F}_\tau = {} & \dot{F}_{dS} + J_{dS,\tau} + \\ & - \frac{h^2}{2\pi^2}\sum_{n=1}^{\infty}\sum_{k=1}^{3n-1} g(n,k)\, e^{-3n\, h\Delta\tau}\times \\ & \frac{e^{hk\Delta\tau}\Big(hk\cos(E\Delta\tau) + E\sin(E\Delta\tau)\Big) - hk}{h^2k^2 + E^2}\end{aligned} \tag{2.70}$$

in which $\dot{F}_{dS}$ is the De Sitter $\tau$ independent contribution given by Eq.(2.52) but with effective Hubble constant $h = \sqrt{\Omega_\Lambda}H_0$ and $J_{dS,\tau}$ is the related tail given by Eq.(2.57). The numerical coefficients $g(n,k)$ can in principle be computed but enter in a tail which decays exponentially fast in the switching time and which also contains oscillating terms. We may take the limit

$\Delta\tau \to \infty$ and observe that every $\tau$ dependent term of this expression goes to zero.

To summarize, we may say that the detector clicks close to a de Sitter response and reaches thermalization (possibly, through decaying oscillations) as $\Delta\tau$ is sufficiently large. In fact, as far as the regime $h\Delta\tau \gg 1$ is concerned, de Sitter space-time is recovered. We may think of this as describing a de Sitter thermal noise continuously perturbed by the expansion (or contraction) of the universe. In particular, insofar as we can speak of temperature, in this large-time regime the detector registers the de Sitter temperature $h/2\pi$, equal to the large-time limit of the horizon temperature parameter given by the surface gravity, which in the present case has the exact but slow long-time evolution

$$T_H = \frac{h}{2\pi} \left[\coth(3ht/2) - 3/4 \, \mathrm{sech}(3ht/2)\mathrm{csch}(3ht/2)\right]$$

It is worth noting that, while in the stationary phase $h\Delta\tau \gg 1$ the limiting result is consistent with the limiting value of the surface gravity, in the non-stationary regime it seems less trivial to compare the results of the two methods, because it has not been possible to extract a temperature parameter from the transition rate of the detector, but asymptotically.

# 3 *Gedankenexperiment* in gravity

In [163] Einstein imagined an ideal experiment (the Einstein light-box) in order to probe the consistency of quantum mechanics: in this experiment a box full of light is placed in a gravitational field and his goal was to show that in such a system a violation of the uncertainty principle occurs. Niels Bohr, making use of the equivalence principle, ultimately showed that there is no violation of the tenets of quantum mechanics. Thought experiments continue to play an important guiding role in theoretical physics: the ideal experiment conceived by J. Bekenstein [164] in order to elucidate black hole entropy or the thought experiment imagined by S. Hawking to tackle the problem of unitarity violation in black hole space-times are some examples. F. Dyson [163] argues that if no conceivable thought experiment can show effects of quantum gravity, then the latter looses its physical meaning. Thus the search for ideal experiments analysing the features of new fundamental physics theories is of great relevance.

The thought experiments that we consider here are formulated in the context of classical general relativity. An interesting link between thermodynamics and gravitation is shown and as a consequence a relation might exist with the Weyl Curvature Hypothesis of R. Penrose [165]. All the arguments presented throughout the Chapter have a heuristic nature, typical of thought experiment analysis, and will need further rigorous developments. The following physical principles are assumed to be true:

1. The positivity of mass and the equivalence between mass and energy, as predicted by special relativity, $E = mc^2$
2. the existence of single photons having energy proportional to frequency, $E = h\nu$ and carrying momentum $p = E/c$
3. the equivalence between inertial and gravitational mass
4. Newton's theory of gravitation for weak fields, Einstein's general theory of relativity and in particular the phenomenon of gravitational redshift

5. The principles of thermodynamics.

In particular, as a consequence of these principles, we assume that light rays can be bended in a gravitational field and that photons have a weight.

In Sec. 3.1 we will explain the construction of the thought experiment and show how it violates the principles of thermodynamics if tidal effects of gravity are not included and that this violation would occur if one performs the experiment in a linearly accelerated frame.

In Sec. 3.2 the experiment is analysed under a minimal requirement of the presence of tidal effects.

## 3.1 The thought experiment

### 3.1.1 The structure

Consider the system represented in Fig.(3.1), namely a tall tower supporting a wide and frictionless scale or balance.
The tilting part of the balance has the following properties:

1. it is made of a perfectly reflecting optical fiber structured as a *closed path*. Perfect reflection is also considered by H. Bondi in his thought experiments [166], which shows how to avoid a violation of the first principle of thermodynamics when considering photons interacting with atoms in a gravitational field

2. a small aperture (A) near the fulcrum of the balance allows incoming photons to enter the fiber and begin to travel through it;

3. the structure of the balance is such that the time spent by these photons in the terminals (C and D) is greater than the time spent in the straight central part (B); this can be achieved, for example, by coiling a long portion of the fiber inside the external loops. Another way to achieve this could be to build the terminals as perfectly reflecting cavities, so that the photon will enter and will remain inside for a (statistically) long time before it finds the way out again;

4. the masses of the terminal parts are equal and very large compared to the mass of the straight central part.

Now let a sufficiently localized wave packet (photon) enter the fiber through the aperture A and start to travel in it. The structure allows the wave packet to travel back and forth inside the fiber, from a coil to the other. Note that the ideal optical properties of the fiber prevent any dispersion that might occur to the wave packet during its journey.

Let the photon travel along B in a time $t_1$ and inside the coil D in a time $t_2 \gg t_1$. Because of the equivalence between mass and energy and because of the equivalence between inertial and gravitational mass, the presence of a photon in coil D increases the gravitational mass of that portion of the balance: as a result, the balance will undergo a tilt towards D and a consequent torque around the fulcrum. After the time $t_2$, the situation is as represented in Fig.(3.2).

Waiting a period of time $2t_1$ after the run inside coil D, we would find the photon entering the coil C and spend another $t_2$ travelling in it, such that the gravitational mass of the C-portion will be increased and as a result, the system will undergo a movement towards the side of the C coil (see Fig.(3.3)). Once out again, the photon can start a new cycle.

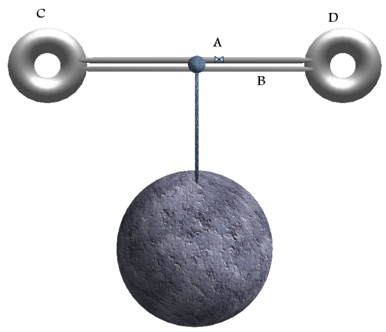


Figure 3.1: structure of the balance. (A) a small aperture, which allows photons to enter the (B) straight segments of the optical fiber, towards (C and D) the external loops containing coiling of optical fiber.

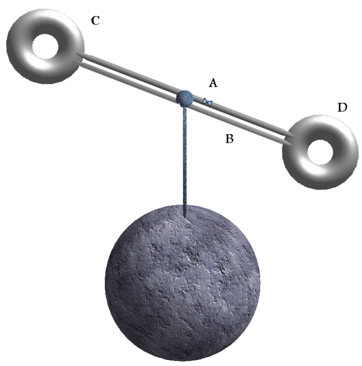


Figure 3.2: After a time $t_1 + t_2$, the coil D is heavier than coil C due to the presence of the photon traveling inside D.

### 3.1.2 Thermodynamics

Let us now examine what happens if the experiment is performed in a linearly accelerated frame. The equivalent gravitational field perceived by an observer in this frame has no tidal effects. The scale or balance seems to be able to describe an infinity of oscillations around the fulcrum: indeed the cycle described so far in the previous section can repeat itself *ad infinitum*. This leads to the possibility to extract an infinite amount of energy from the movement of the balance. In fact, introducing a friction needed to extract energy, one initially observes a decrease in the amplitude of the oscillations, but there would be no continuous damping of the oscillations of the structure: after each cycle, the photon looses no energy. This implies that *the*

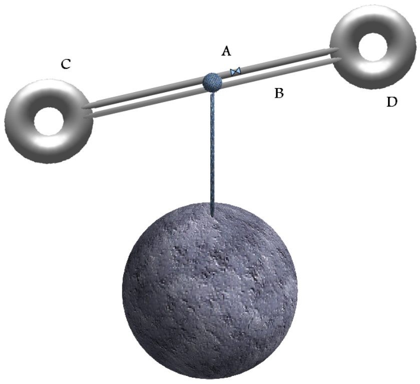


Figure 3.3: After a time $3t_1 + 2t_2$, the coil C is heavier than coil D due to the presence of the photon traveling inside C.

*first principle of thermodynamics is violated.* A similar violation occurs in the thought experiment conceived by H. Bondi [165]: in this case the inconsistency with the principle of energy conservation was resolved by taking into account the redshift of the photon when travelling upwards in a gravitational field. In our thought experiment, the redshift that the photon would undergo when travelling back, say from D to C in Fig.(3.2), would cause only a finite reduction of the photon energy. After each cycle, when the photon returns to the lowest part D of the balance, it regains the reshifted energy. Additionally, one can restrict the gravitational redshift by limiting the rotation angle around the fulcrum of the balance suitably.

### 3.1.3 Photon redshift

For a spherical body of mass M, such as our fictious planet, we find the following relation between the shift in frequency of the photon and the gravitational potential difference during the travel:

$$\frac{\nu_r - \nu_e}{\nu_e} = -\frac{G_N M}{c^2}\left(\frac{1}{r_r} - \frac{1}{r_e}\right)$$

Suppose that the altitude and the width of the balance have the same value as the planet radius $R$. A quick calculation, considering a planet with the mass of the Sun ($M \simeq 1.98874 \cdot 10^{30}\ kg$) but with the radius of the Earth ($R \simeq 6371\ km$) and non-rotating, leads to

$$\frac{\nu_r - \nu_e}{\nu_e} = \frac{G_N\ 1.98874 \cdot 10^{30}}{c^2}\left(\frac{1}{14246} - \frac{1}{10073}\right)$$

so that

$$1 - \frac{\nu_r}{\nu_e} = 0.0429476$$

In an ideal system as the one described above, such a non-zero ratio between frequencies signals a *periodic variation of the gravitational mass distribution in the scale*, leading to an oscillation of the same.

Therefore, it appears that gravitational redshift is not sufficient to prevent violation of the first principle of thermodynamics. Furthermore, *a violation of the second principle of thermodynamics* occurs when the scale is placed in a linearly accelerated frame. Imagine that some sufficiently hot body looses heat by emitting light. A given scale could absorb a single photon. The possibility to observe complete oscillations of the balance shows that all the absorbed heat has as unique effect the production of work. Thus one has a contradiction with Kelvin's formulation of the second law of thermodynamics.

## 3.2 Tidal effects

Let us now examine what would happen if the scale were placed in a gravitational field characterised by *non-vanishing tidal components*. If we assume that the scale is at a given instant of time in the configuration depicted in Fig.(3.2), and that the gravity is sufficiently weak to allow the use of a Newtonian approximation in an almost Euclidean space. Tidal effects can prevent oscillations of the balance if the gravitational acceleration at the lower level D is greater than the acceleration at the higher level C, even if the latter part of the balance contains the travelling photon and possesses a greater total gravitational mass. If $m_D$ and $m_C$ are respectively the gravitational masses of part D and C of the scale and if $g_C$ and $g_D$ are respectively the gravitational accelerations at the levels of C and D, then the oscillation will not occur if

$$m_D g_D > m_C g_C \tag{3.1}$$

Since the coil C contains the photon, its mass can be expressed as:

$$m_C = m_D + \frac{h\nu}{c^2}$$

with $\nu$ being the photon frequency. Rewriting the acceleration $g_D$ as $g_D = g_C + \Delta g$, with $\Delta g = g_D - g_C$, the inequality becomes

$$m_D(g_C + \Delta g) > \left(m_D + \frac{h\nu}{c^2}\right) g_C$$

which leads to

$$m_D \Delta g > \left(\frac{h\nu}{c^2}\right) g_C \tag{3.2}$$

The frequency can be expressed as $\nu = 1/T$, with $T$ being the period, and the acceleration $g_C$ can be written as

$$g_C = \frac{GM}{R^2}$$

where $M$ is the mass of the body that generates the gravitational field and $R$ is the distance from the coil C to the center of the body.
As a result the inequality Eq.(3.2) becomes

$$\Delta g \; T > \frac{hG}{c^2} \frac{1}{R^2} \frac{M}{m_D}$$

The Planck area is defined as $A_{pl} = \frac{Gh}{2\pi c^3}$, and $R^2$ can be expressed in terms of the area A of the surface of a sphere of radius $R$ as $R^2 = A/4\pi$. With these notations the above inequality becomes

$$\Delta g \; T > 8\pi^2 \frac{A_{pl}}{A} \frac{M}{m_D} c \tag{3.3}$$

Therefore, according to Eq.(3.3), given the photon period, the distance from the lowest part of the balance to the centre of a gravitation-generating body and given the masses of the coils of the scale and of the source of the gravitational field, *the relative gravitational acceleration between the levels of two coils of the balance can not be arbitrarily small*, otherwise a violation of the principles of thermodynamics might occur. Furthermore, *from the occurrence of the constants h, G and c in the inequality Eq.(3.3), it appears that the origin of this constraint on the value of tidal effects might be linked to quantum effects in a gravitational field.* More precisely, Eq.(3.3) is reflecting a quantum gravity effect, in the same sense that Hawking's black hole entropy formula [167] reflects properties of quantum gravity, because of the simultaneous occurrence of the constants h, G and c. Indeed, if the factor $hG/c^3$ tends to zero, then the minimal tidal effect requirement would not be applicable and the result would break down in this limit, which includes:

- special relativity (G and h tend to zero)

- classical general relativity (h tends to zero)

- classical Newtonian gravity (h tends to zero and c tends to infinity)

- relativistic quantum physics (only G tends to zero, while h remains non-zero and c finite)

- non-relativistic quantum mechanics in a Newtonian gravitational field (c tends to infinity).

# 4 Conclusions

The first part of this thesis dealt with the description of the mechanism of tunnelling through horizons: we list here the results which seem to us more robust and have a chance to last longer.
Results that were obtained in Sec.1.1 comprise:

1. The ability of the method to include first order back-reaction effects in order to comply with energy conservation and the predicted correlations among successive emission events is one of the key results.

2. In the stationary case, it is possible to provide a foundation of tunnelling methods using analytic continuation of the classical action alone, avoiding in this way the seemingly impossible task of crossing the horizon. The analytic continuation could be extended to cover white holes and to associate with them a well defined geometric temperature, a point not always appreciated in literature.

3. In the same vein, one can prove the complete equivalence of the null geodesic method with the Hamilton–Jacobi method for stationary fields. We also gave a "covariantised" version of the null geodesic method, which allows to avoid questions such as covariance or canonical invariance.

4. Many special cases have been considered, including rotating black holes, Gödel space-times, higher dimensional solutions, de Sitter and anti–de Sitter spaces, Taub and Taub–NUT solutions; also, we considered some supergravity solutions and fermion tunnelling, including gravitinos. The temperature of these solutions were computed and showed to agree with those obtained by geometric methods, confirming the universal character and kinematics of the Hawking effect. It should also be noticed that the Hawking temperature computed from the tunnelling method does not receive higher order corrections in $\hbar$, contrary to past suggestions.

In Sec.1.2, using a local notion of horizon and dynamical surface gravity for time varying spherically symmetric black holes, the main results comprise:

1. The application of the tunnelling method to non stationary, spherically symmetric black holes and the fact that it naturally selects a trapping horizon of the future non degenerate type, either inner or outer. It can also be applied to past horizons under the same condition and by interchanging emission with absorption. In the limiting situation where the surface gravity vanishes it is consistent with the idea that extremal black holes have zero temperature.

2. One of the principal consequences of the tunnelling method is the observation that the radiation seems to originate near the local trapping horizon, not the global event horizon.

3. Using the Kodama–Hayward theory of spherically symmetric space-times, and noting that one can possibly associate an observable temperature to a black hole only if it is a coordinate scalar, we showed that the tunnelling probability depends on the ratio, $2\pi E/\kappa$, of the Kodama invariant charge taken as energy and the geometrical surface gravity as defined by Hayward.

4. One can include non interacting fermions in the formalism, as demonstrated in several papers. Also noticeable is the fact that any mass term gets strongly suppressed by the horizon pole, which dominates the rate well over mass contributions.

5. The time-like nature of the trapping horizon of an excreting black hole is consistent with the physical interpretation of the formalism. It is true that it can be crossed either ways, but it remains the fact that there is a class of paths for which an imaginary part exists, and these paths precisely correspond in Feynman diagram language to particle creation.

In Section 1.3 we addressed other themes – cosmology, decays and naked singularities. Since FRW spaces are spherically symmetric, the general theory outlined in Section 1.2.2 applies. We showed that, with obvious modifications, the tunnelling picture works equally well. In particular the type-II paths never contribute, while type-I paths do. We may consider these findings for cosmological horizons as a generalisation of the well known facts discovered for de Sitter space-time, in complete analogy with the parallel extension of the theory from stationary to dynamical black holes. We also considered the decay of particles which in the absence of gravity would not occur, and find that the Hamilton–Jacobi equation can manage them through the presence of a branch point singularity in the radial momentum, that would not be present in absence of an external field. From this point of view the tunnelling method has universally valid features. Again, its value is to be found where it provides approximate results when exact calculations are in general impossible, at the same time being consistent with the exact calculations when available. Such is the case of de Sitter space. We showed also that the presence of strong fields in a certain space-time region, even in the absence of horizons, can result in particle production by means of a tunnelling process "from nothing". It can be considered as the gravitational analogue of the Schwinger effect in a strong external electric field. Finally, it is of interest that the tunnelling method can also be applied to naked singularities, although in this case there is nothing to tunnel through, the singularity being the boundary of space-time. We showed nevertheless that the emerging picture is quite in agreement with quantum field theory in two-dimensional models, making one confident of its general validity.

As regard the drawbacks, even if there is little doubt that the method is correct, still it remains of a hypothetical nature. However, we think that also in case of a failure due to some internal inconsistency, that should be nonetheless an important message. In fact, tunnelling processes are predictions of quantum theory, so a failure of them in gravity theory would certainly signal something interesting. The tunnelling method is and remains an essentially semi-classical procedure carrying along with it all the limitations inherent to its nature. In particular, it applies only to free particles while in principle quantum field theory methods certainly have a wider scope.

The tunnelling method is not a closed subject. For instance, in striking contrast with the stationary case, an important missing point of the dynamical case is the absence of an extension of Kodama–Hayward's theory to dynamical axis-symmetric black holes, to be used as a tool for studying tunnelling. We mention here the papers [168, 169] where a tentative theory is developed. We feel that this is perhaps the most important missing point. The first back-reaction corrections are also of great interest, as they are not so well developed in dynamical situations. Processes where small black holes are emitted by large ones should also be within the range of problems where the tunnelling picture could work, and that also is an important "to do".

In cosmology, the physical interpretation of Hayward's dynamical surface gravity as a temperature is not as clear as in de Sitter or black hole case, but it is strongly favoured by the tunnelling picture. In particular, an Unruh–DeWitt detector moving in de Sitter or black hole space-time always thermalises at the corresponding Gibbons–Hawking temperature even if it is not on a geodesic path, while for cosmology there is not an analogous result.

This leads us to the comparison with the results of Sec.2. For black holes and pure de Sitter space the QFT analysis is consistent with the tunnelling picture and even predicts the dependence of the temperature on position or acceleration. Moreover, the analysis of the finite-time oscillating tail has been extended to stationary black holes.

For cosmology and away from de Sitter space the thermal interpretation, strictly speaking, is lost but the detector still gets excited by the expansion of the universe. By accepting the surface gravity versus temperature paradigm we would expect a quasi-thermal excitation rate of the form

$$\dot{F} \sim E \exp(-E/T_H(t))$$

That is, although in a generic FRW space-time the thermal interpretation breaks down in most of the cases because of the time-dependence of the background, still this time dependence of the transition rate could be expected to mainly reside in an effective temperature parameter. But using a comoving detector this is not what we have found. For instance, in the Einstein-de Sitter regime there seems to be excitations of non-thermal type and we showed that the scale factor of the flat $\Lambda$CDM-cosmology has no other temperatures in action than the de Sitter one.

It remains to see whether there is any non trivial (quasi-)thermal effect on more general Kodama trajectories. In the affirmative case, that would mean that the horizon surface gravity and temperature should be associated more likely to vacuum correlations than to particle creation and this forces, in our view, a different interpretation of the tunneling picture. In this respect, the classical Parker's papers on particle creation [160, 161] are certainly relevant. One possibility is that the horizon surface gravity could represent an intrinsic property of the horizon itself, leading to some kind of holographic description, while the detector in the bulk simply clicks because it is embedded in a changing geometry. In fact, we would expect the clicks in almost any changing geometry, even for those lacking a trapping horizon.

As regards the last part of this thesis, in Chapter 3 we considered a thought experiment which generalises an ideal experiment discussed by H. Bondi [166]. The connection between thermodynamical concepts and gravity is tackled in a somewhat different perspective: in a system that comprises the simultaneous validity of thermodynamic principles and (special and general) relativistic *tenet*s, one is lead to contradictions unless specific features are supposed to arise.

It appears, as far as our ideal experiment is concerned and differently from the original analysis by Bondi, that gravitational redshift is not sufficient to prevent the violations of the first and the second principles of thermodynamics. In particular, these contradictions can occur if the experiment is performed in a linearly accelerated frame. On the other hand, if minimal tidal effects inherent to a gravitational field are taken into account, then the contradictions with the principles of thermodynamics can be avoided.

The heuristic analysis of the tidal effect in a gravitational field shows through inequality (3.3) that the necessary existence of a minimal non-vanishing relative gravitational acceleration might be related to quantum gravitational effects.

An interesting point is that the arguments developed might also be related to the Weyl Curvature Hypothesis (WCH) of R. Penrose [165]. The

Weyl curvature essentially describes the tidal effects of a gravitational field, and if some invariant function constructed with the components of the Weyl tensor is linked to the gravitational entropy, then the WCH states that the Weyl curvature associated to a cosmological space-time should have a very small value initially, close to the Big Bang singularity, whereas at late-times in the far future the solution ought to be characterised by a large value of the Weyl curvature. The cosmological space-time solution exhibits in this way a time-asymmetric evolution, compatible with the second principle of thermodynamics.

Our analysis suggests that tidal effects are necessary and essential features of gravity, which guarantee compatibility with the principles of thermodynamics. Thus the evolution of cosmological space-time solutions possessing such features will exhibit a time-asymmetric evolution implied by the second principle of thermodynamics. As R. Penrose argues [165], this time-asymmetry might ultimately be related to quantum gravity.

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