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Homogenization of heterogeneous Cauchy-elastic materials leads to Mindlin second-gradient elasticity

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Summary

Through a second-order homogenization procedure, the explicit relation is obtained between the non-local parameters of a second gradient elastic material and the microstructure of a composite material. This result is instrumental for the definition of higher-order models, to be used for the analysis of mechanics at micro- and nano-scale, where size-effects become important.

The obtained relation is valid for both plane and three-dimensional problems and generalizes earlier findings by Bigoni and Drugan (Analytical derivation of Cosserat moduli via homogenization of heterogeneous elastic materials. J. Appl. Mech., 2007, 74, 741753) from several points of view:

i) the result holds for anisotropic phases with spherical or circular ellipsoid of inertia;

ii) the displacement boundary conditions considered in the homogenization procedure is independent of the characteristics of the material;

iii) a perfect energy match is found between heterogeneous and equivalent materials (instead of an optimal bound).

From the obtained solution it follows that the equivalent second-gradient Mindlin elastic solid:

a) is positive definite only when the discrepancy tensor is negative defined;

b) the non-local material symmetries are the same of the discrepancy tensor;

c) the non-local effective behaviour is affected by the shape of the RVE, which does not influence the first-order homogenized response.

Finally, explicit derivations of non-local parameters from heterogeneous Cauchy elastic composites are obtained in particular cases.

Acknowledgements

The constant optimism and great enthusiasm of my advisor, Prof. Davide Bigoni, have been fundamentals during these years. I would like to express profound gratitude to him for his remarks, suggestions and encouragements throughout this research work.

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Finally, I am particularly indebted to all my family and all my friends for supporting me and endured.

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Mattia Bacca

Published papers

The main results presented in this thesis have been summarized in the following papers:

1) M. Bacca, D. Bigoni, F. Dal Corso, D. Veber (2013) ”Homogenization of heterogeneous Cauchy-elastic materials leads to Mindlin second-gradient elasticity Part I: Closed form expression for the effective higher-order constitutive tensor”, (Submitted);

2) M. Bacca, D. Bigoni, F. Dal Corso, D. Veber (2013) ”Homogenization of heterogeneous Cauchy-elastic materials leads to Mindlin second-gradient elasticity Part II: Higher-order constitutive properties and application cases”, (Submitted).

Other results obtained during the PhD are reported in the following publications:

3) M. Gei, S. Roccabianca, M. Bacca (2011) ”Controlling band gap in Electroactive Polymer-based structures”, *IEEE/ASME Transactions on Mechatronics*, vol. 16, p. 102-107. - DOI: 10.1109/TMECH.2010.2090165;

4) M. Gei, D.Bigoni, A.B.Movchan and M. Bacca (2013) ”Band-Gap Properties of Prestressed Structures”, in *Acoustic Metamaterials, Negative Refraction, Imaging, Lensing and Cloaking*, Richard V. Craster, Sebastien Guenneau, p. 61-82.

Contents

Chapter 1

Introduction

In the last century composite materials have become fundamental for advanced applications in aerospatial, bio-mechanical and nano-mechanical engineering. Since size-effects have been experimentally evidenced in the mechanics at the micro- or nano-scale, the local constitutive models are unsuitable for the design of high performance composites.

Despite the formulation of many nonlocal models (Cosserat, 1909; Koiter, 1964; and Mindlin, 1964), the nonlocal constitutive parameters are usually introduced in a phenomenological way and the explicit evaluation between the microstructure and the nonlocal effects has been scarcely investigated.[1]

The main goal of this thesis is to provide through an analytical approach the explicit evaluation of the constitutive non-local parameters (length-scales) from the microstructural description.

In Chapter 2 the kinematic primary quantities and the conjugate statical quantities are introduced for the higher-order material considered, namely, the Second Gradient Elastic material (Mindlin, 1964). Through the principle of virtual work, the governing equations and boundary conditions are obtained and within linear theory, the constitutive response is defined by means of the local $\mathbf{C}$ and non-local $\mathbf{A}$ constitutive tensors.

The second-order homogenization procedure is presented in Chapter 3. While standard (or first-order) homogenization procedures lead to effective homogeneous Cauchy material equivalent only when linear displacement bound-

[1] Theoretical considerations were developed by Wang and Stronge, 1999; Achenbach and Hermann, 1968; and Beran and McCoy, 1970; Pideri and Seppecher, 1997; numerical approaches were given by Forest, 1998; Ostoja-Starzewski et al. 1999; Bouyge et al. 2001; experiments were provided by Anderson and Lakes, 1994; Buechner and Lakes, 2003; Lakes, 1986; Gauthier, 1982.

ary conditions are considered (Fig. 1.1), the proposed procedure extends the energy equivalence to the case of generic quadratic displacement boundary conditions, by taking an higher-order solid as effective material material (Fig. 1.2).

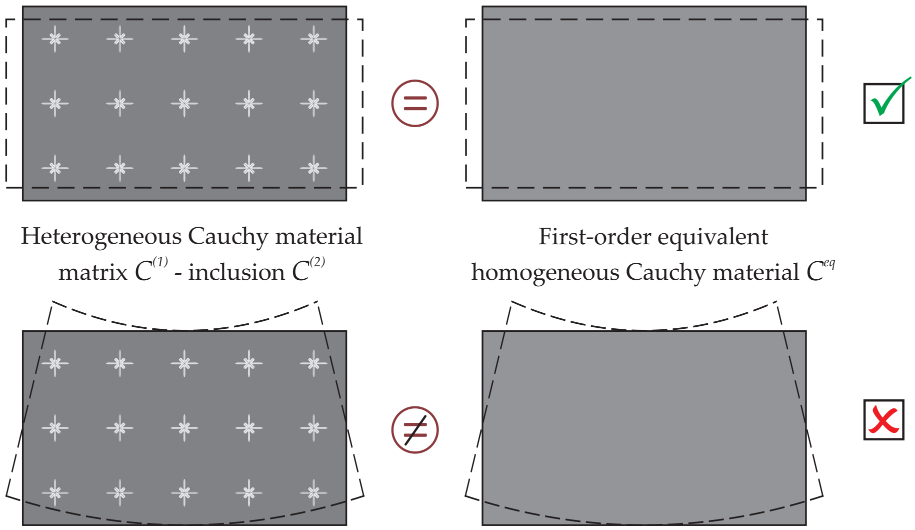

Figure 1.1: Standard (or first-order) homogenization procedure leading to effective homogeneous Cauchy material equivalent only when linear displacement boundary conditions are considered.

Through the second-order homogenization procedure, the equivalent nonlocal constitutive tensor $\mathbf{A}^{eq}$ is explicitly evaluated in Chapter 4 in the case of dilute suspension as

$$\mathbf{A}^{eq}_{ijhlmn} = -f\frac{\rho^2}{4}\left(\tilde{\mathbf{C}}_{ihln}\delta_{jm} + \tilde{\mathbf{C}}_{ihmn}\delta_{jl} + \tilde{\mathbf{C}}_{jhln}\delta_{im} + \tilde{\mathbf{C}}_{jhmn}\delta_{il}\right),$$

where ρ is the radius of the sphere (or circle in 2D) of inertia of the RVE cell, and $\tilde{\mathbf{C}}$ is introduced to define (at first-order in f) the difference between the local constitutive tensors for the effective material $\mathbf{C}^{eq}$ (known from first-order homogenization) and the matrix $\mathbf{C}^{(1)}$, so that

$$\mathbf{C}^{eq} = \mathbf{C}^{(1)} + f\tilde{\mathbf{C}}.$$

Properties of positive definiteness, nonlocal symmetries and influence of the volume and shape of the RVE are derived from the obtained solution.

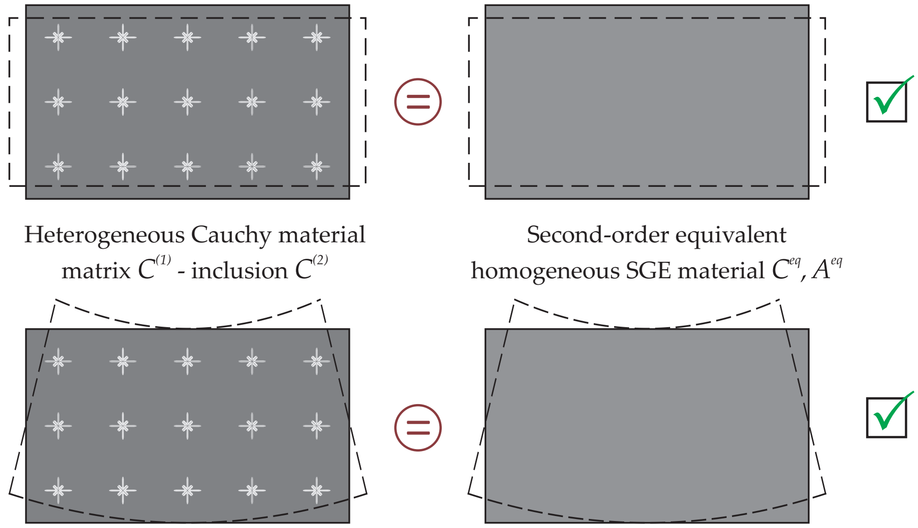

Figure 1.2: The proposed second-order homogenization procedure based on effective higher-order solid. The equivalence between the heterogeneous and the homogeneous material holds for a generic quadratic displacement boundary conditions.

The explicit evaluation of the equivalent nonlocal constitutive tensor $\mathbf{A}^{eq}$ is exploited in Chapter 5 to obtain the non-local parameters in specific application cases. The following cases are considered: (i.) isotropic matrix with spherical elastic inclusions and voids in three-dimensional deformations (Fig. 1.3), (ii.) isotropic matrix with circular elastic inclusions, regular n-polygonal and circular holes in plane strain and (iii) orthotropic matrix with circular holes in plane strain.

Conclusions of this thesis are finally reported in Chapter 6.

The results presented in this thesis have been published in (Bacca et al. 2013, a;b), while complementary results on propagation of elastic waves in periodic composites have been published by Gei et al. (2011; 2013).

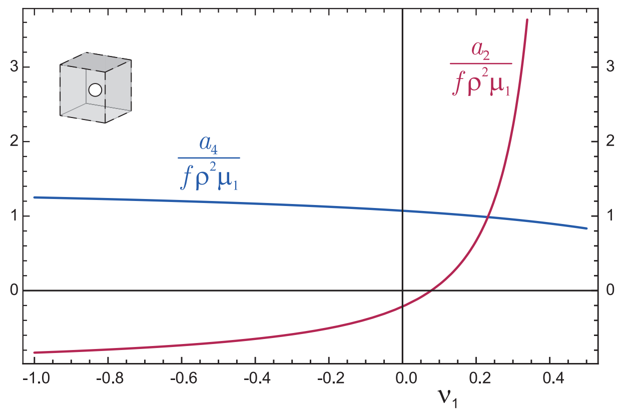

Figure 1.3: Higher-order equivalent constants a_2 and a_4 of the Second Gradient Elastic material equivalent to a composite made up of an isotropic matrix containing a dilute suspension of spherical voids. The constants are made dimensionless through division by parameter $f\rho^2\mu_1$ and are reported as a function of the matrix Poisson's ratio ν_1.

Chapter 2

Preliminaries on Second-Gradient Elasticity

Kinematic variables, principle of virtual work, equilibrium equations and boundary conditions are introduced for a Second-Gradient Elastic material. Within a linear theory, the local $\mathbf{C}$ *and non-local* $\mathbf{A}$ *constitutive tensors are introduced together with the positive definiteness condition (for the case of isotropic materials).*

2.1 Governing equations

The governing equations for the nonlocal material model employed in the homogenization procedure are briefly presented. In particular, the second-gradient elasticity (SGE) model considered is a restriction of the nonlocal model proposed by Mindlin (1964), in which the relative deformation [his eqn (1.11), and therefore also his stress σ_{ij}, eqn $(3.4)_2$] and the coupling between the stress and the curvature [expressed by his fifth-order tensor f_{ijkpq}, eqn $(5.3)_1$] are both assumed to be null.

Considering a quasi-static deformation process, defined by the displacement field $\boldsymbol{u}$ (function of the position $\boldsymbol{x}$), the primary kinematical quantities of the SGE are defined as

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}, \qquad \chi_{ijk} = u_{k,ij}, \tag{2.1}$$

where a comma denotes differentiation, the indices range between 1 and N

(equal to 2 or 3, depending on the space dimensions of the problem considered), and $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ are the (second-order) strain and the (third-order) curvature tensor fields, respectively, satisfying the following symmetry properties

$$\varepsilon_{ij} = \varepsilon_{ji}, \qquad \chi_{ijk} = \chi_{jik}. \tag{2.2}$$

Defining the statical entities Cauchy stress $\sigma_{ij}=\sigma_{ji}$ and double stress $\tau_{ijk}=\tau_{jik}$, respectively work-conjugate to the kinematical entities $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$, eqn (2.1), the principle of virtual work can be written for a solid occupying a domain Ω, with boundary $\partial\Omega$ and set of edges Γ (Fig. 2.1), in the absence of body-force as

$$\int_\Omega (\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\chi_{ijk}) = \int_{\partial\Omega} (t_i\delta u_i + T_i D\delta u_i) + \int_\Gamma \Theta_i\delta u_i, \tag{2.3}$$

where repeated indices are summed, $\boldsymbol{t}$ represents the surface traction (work-conjugate to $\boldsymbol{u}$), while $\mathbf{T}$ and $\boldsymbol{\Theta}$ denote the generalized tractions on the surface $\partial\Omega$ and along the set of edges Γ (work-conjugate respectively to $D\boldsymbol{u}$ and $\boldsymbol{u}$), and $D = n_l\partial_l$ represents the derivative along the outward normal direction to the boundary, $\boldsymbol{n}$ (definite only on $\partial\Omega$ but not on Γ). Through integration

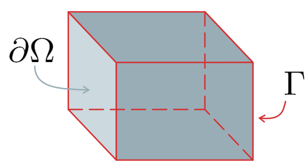

Figure 2.1: Volume region Ω enclosed by the surface $\partial\Omega$ with the set of edges Γ.

by parts, the equilibrium conditions [Mindlin (1964), his eqns (9.30) with null inertia terms], holding for points within the body Ω, can be obtained as

$$\partial_j\left(\sigma_{jk} - \partial_i\tau_{ijk}\right) = 0, \qquad \text{in } \Omega, \tag{2.4}$$

while for points on the boundary $\partial\Omega_p$ and along the set of edges Γ_p, (where statical conditions are prescribed in terms of $\boldsymbol{t}$, $\mathbf{T}$ and $\boldsymbol{\Theta}$) as

$$\begin{cases} n_j\sigma_{jk} - n_in_jD\tau_{ijk} - 2n_jD_i\tau_{ijk} + \left(n_in_jD_ln_l - D_jn_i\right)\tau_{ijk} = t_k, \\ n_in_j\tau_{ijk} = T_k, \end{cases} \qquad \text{on } \partial\Omega_p, \tag{2.5}$$

and

$$[\![e_{mlj} n_i s_m n_l \tau_{ijk}]\!] = \Theta_k, \qquad \text{on } \Gamma_p, \tag{2.6}$$

where e_{mlj} is the Ricci 'permutation' tensor, $D_j = (\delta_{jl} - n_j n_l)\,\partial_l$, $\boldsymbol{s}$ is the unit vector tangent to Γ and $[\![\cdot]\!]$ represents the jump of the enclosed quantity, computed with the normals $\boldsymbol{n}$ defined on the surfaces intersecting at the edge Γ. Finally, kinematical conditions[1] are prescribed for points on the remaining boundary $\partial\Omega_u \equiv \partial\Omega \backslash \partial\Omega_p$ as

$$\begin{cases} u_i = \overline{u}_i, \\ Du_i = \overline{Du}_i, \end{cases} \qquad \text{on } \partial\Omega_u. \tag{2.7}$$

2.2 Linear constitutive response

Introducing the strain energy density $w^{SGE} = w^{SGE}(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$, the $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ fields can be obtained as

$$\sigma_{ij} = \frac{\partial w^{SGE}}{\partial \varepsilon_{ij}}, \qquad \tau_{ijk} = \frac{\partial w^{SGE}}{\partial \chi_{ijk}}, \tag{2.8}$$

so that, within a linear theory (with the above-mentioned assumptions of null relative deformation and no-coupling in the strain energy between strain and curvature), it follows that

$$w^{SGE}(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \underbrace{\frac{1}{2}\mathsf{C}_{ijhk}\varepsilon_{ij}\varepsilon_{hk}}_{w^{SGE,L}(\boldsymbol{\varepsilon})} + \underbrace{\frac{1}{2}\mathsf{A}_{ijklmn}\chi_{ijk}\chi_{lmn}}_{w^{SGE,NL}(\boldsymbol{\chi})}, \tag{2.9}$$

where C and A are the local (fourth-order) and non-local (sixth-order) constitutive tensors, each generating respectively a strain energy density contribution, say 'local', $w^{SGE,L}$ (corresponding to the energy stored in a Cauchy material, $w^{SGE,L} = w^C$) and 'non-local', $w^{SGE,NL}$. Therefore, the linear constitutive equations for the stress and double stress quantities are obtained as

$$\sigma_{ij} = \mathsf{C}_{ijhk}\varepsilon_{hk}, \qquad \tau_{ijk} = \mathsf{A}_{ijklmn}\chi_{lmn}, \tag{2.10}$$

which, from eqns (2.1) and (2.8), have the following symmetries

$$\mathsf{C}_{ijhk} = \mathsf{C}_{jihk} = \mathsf{C}_{ijkh} = \mathsf{C}_{hkij}, \qquad \mathsf{A}_{ijklmn} = \mathsf{A}_{jiklmn} = \mathsf{A}_{ijkmln} = \mathsf{A}_{lmnijk}. \tag{2.11}$$

[1]In the proposed homogenization procedure only kinematical boundary conditions will be imposed ($\partial\Omega_p \equiv \emptyset$, so that $\partial\Omega_u \equiv \partial\Omega$).

In the case of isotropic response, the constitutive elastic tensors $\mathbf{C}$ and $\mathbf{A}$ can be written in the following form

$$
\begin{aligned}
\mathsf{C}_{ijhk} = &\ \lambda\delta_{ij}\delta_{hk} + \mu(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}), \\
\mathsf{A}_{ijhlmn} = &\ \frac{a_1}{2}\left[\delta_{ij}\left(\delta_{hl}\delta_{mn} + \delta_{hm}\delta_{ln}\right) + \delta_{lm}\left(\delta_{in}\delta_{jh} + \delta_{ih}\delta_{jn}\right)\right] \\
&+ \frac{a_2}{2}\left[\delta_{ih}\left(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln}\right) + \delta_{jh}\left(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln}\right)\right] \\
&+ 2\,a_3\left(\delta_{ij}\delta_{hn}\delta_{lm}\right) + a_4\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)\delta_{hn} \\
&+ \frac{a_5}{2}\left[\delta_{in}\left(\delta_{jl}\delta_{hm} + \delta_{jm}\delta_{hl}\right) + \delta_{jn}\left(\delta_{il}\delta_{hm} + \delta_{im}\delta_{hl}\right)\right],
\end{aligned}
\tag{2.12}
$$

where δ_{ij} is the Kronecker delta, λ and μ are the usual Lamé constants, defining the local isotropic behavior, while a_i $(i = 1, ..., 5)$ are the five material constants (with the dimension of a force) defining the nonlocal isotropic behavior. Considering the constitutive isotropic tensors (5.3), the strain energy density (2.9) becomes

$$
w^{SGE}(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \underbrace{\frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij}}_{w^{SGE,L}(\boldsymbol{\varepsilon})} + \underbrace{\sum_{k=1}^{5} a_k \mathcal{I}_k(\boldsymbol{\chi})}_{w^{SGE,NL}(\boldsymbol{\chi})},
\tag{2.13}
$$

where the invariants $\mathcal{I}_k(\boldsymbol{\chi})$ are

$$
\begin{aligned}
&\mathcal{I}_1(\boldsymbol{\chi}) = \chi_{iik}\,\chi_{jkj}(= \chi_{iik}\,\chi_{kjj}), \\
&\mathcal{I}_2(\boldsymbol{\chi}) = \chi_{iki}\,\chi_{jkj}(= \chi_{kii}\,\chi_{jkj} = \chi_{kii}\,\chi_{kjj} = \chi_{iki}\,\chi_{kjj}), \\
&\mathcal{I}_3(\boldsymbol{\chi}) = \chi_{iik}\,\chi_{jjk}, \\
&\mathcal{I}_4(\boldsymbol{\chi}) = \chi_{ijk}\,\chi_{ijk}(= \chi_{jik}\,\chi_{ijk} = \chi_{jik}\,\chi_{jik} = \chi_{ijk}\,\chi_{jik}), \\
&\mathcal{I}_5(\boldsymbol{\chi}) = \chi_{ijk}\,\chi_{kji}(= \chi_{jik}\,\chi_{kji} = \chi_{jik}\,\chi_{jki} = \chi_{ijk}\,\chi_{kji}),
\end{aligned}
\tag{2.14}
$$

so that the linear constitutive relations (2.10) reduce to

$$
\begin{aligned}
\sigma_{ij} = &\ \lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij}, \\
\tau_{ijk} = &\ \frac{a_1}{2}\left(\chi_{lli}\delta_{jk} + 2\chi_{kll}\delta_{ij} + \chi_{llj}\delta_{ik}\right) + a_2\left(\chi_{ill}\delta_{jk} + \chi_{jll}\delta_{ik}\right) + 2a_3\chi_{llk}\delta_{ij} \\
&+ 2a_4\chi_{ijk} + a_5\left(\chi_{kji} + \chi_{kij}\right).
\end{aligned}
\tag{2.15}
$$

Since the invariants defined by eqns (2.14) satisfy the following inequalities

$$
\begin{gathered}
2\mathcal{I}_1(\boldsymbol{\chi}) + \mathcal{I}_2(\boldsymbol{\chi}) + \mathcal{I}_3(\boldsymbol{\chi}) \geq 0, \qquad \mathcal{I}_2(\boldsymbol{\chi}) \geq 0, \qquad \mathcal{I}_3(\boldsymbol{\chi}) \geq 0, \\
\mathcal{I}_4(\boldsymbol{\chi}) \geq 0, \qquad \mathcal{I}_4(\boldsymbol{\chi}) + \mathcal{I}_5(\boldsymbol{\chi}) \geq 0,
\end{gathered}
\tag{2.16}
$$

the positive definiteness condition for the isotropic strain energy density w^{SGE}, eqn (2.13), corresponds to the usual restraints for the local parameters (given by the positive definiteness of $w^{SGE,L}(\boldsymbol{\varepsilon})$)

$$
3\lambda + 2\mu > 0, \quad \mu > 0, \tag{2.17}
$$

which are complemented by the following conditions (Mindlin and Eshel, 1968) on the nonlocal constitutive parameters (given by the positive definiteness of $w^{SGE,NL}(\boldsymbol{\chi})$)

$$
-a_4 < a_5 < 2a_4, \quad e_1 > 0, \quad e_2 > 0, \quad 5e_3^2 < 2e_1e_2, \tag{2.18}
$$

where

$$
\begin{gathered}
e_1 = -4a_1 + 2a_2 + 8a_3 + 6a_4 - 3a_5, \quad e_2 = 5(a_1 + a_2 + a_3) + 3(a_4 + a_5), \\
e_3 = a_1 - 2a_2 + 4a_3.
\end{gathered}
\tag{2.19}
$$

Chapter 3

The second-order homogenization procedure

The second-order homogenization procedure is presented, based on the annihilation of the strain energy mismatch between the heterogeneous Cauchy material and the equivalent Second Gradient Elastic material. Assumptions about geometrical properties of matrix and inclusion phases are introduced.

3.1 Description of the homogenization procedure

The proposed homogenization procedure follows Bigoni and Drugan (2007). In particular, the same[1] (linear and quadratic) displacement is applied on the boundary of both the representative volume element RVE and the homogeneous equivalent SGE material. Then, the equivalent local $\mathbf{C}^{eq}$ and non-local $\mathbf{A}^{eq}$ tensors are obtained imposing the vanishing of the elastic energy mismatch between the two materials. Since the strain energy in the homogeneous SGE material is given only by the local contribution when linear displacement boundary condition are applied (because no strain gradient arises), the equiva-

[1]Bigoni and Drugan (2007) impose a linear and quadratic displacement field on the boundaries of the RVE and of the homogeneous equivalent material, which quadratic part depends on the Poisson's ratio of the material to which the displacement is applied, so that the applied displacements are not exactly equal. Furthermore, the equivalent material considered by Bigoni and Drugan is a non-local Koiter material (1964), which does not permit the annihilation, but only a minimization of the elastic energy mismatch between the RVE and the equivalent material.

lent local tensor C^{eq} corresponds to that obtained with usual homogenization procedures. Thus, the remaining unknown of the equivalent SGE material (namely, the non-local equivalent constitutive tensor A^{eq}) can be obtained by imposing the vanishing mismatch in strain energy when (linear and) quadratic displacement are considered. A chief result in the current procedure is that a perfect match in the elastic energies is achieved, while Bigoni and Drugan (2007) only obtained an 'optimality condition' for the mismatch.

The homogenization procedure is described in the following three steps.

Step 1. Consider a RVE made up of a heterogeneous Cauchy material (C), Fig. 3.1 (left), occupying a region

$$\Omega_{RVE}^{C} \equiv \Omega_1^C \cup \Omega_2^C,$$

where an inclusion, phase '2' (occupying the region Ω_2^C and with elastic tensor $\mathsf{C}^{(2)}$), is fully enclosed in a matrix, phase '1' (occupying the region Ω_1^C and with elastic tensor $\mathsf{C}^{(1)}$), so that the constitutive local tensor $\mathsf{C}(\boldsymbol{x})$ within the RVE can be defined as the piecewise constant function

$$\mathsf{C}(\boldsymbol{x}) = \begin{cases} \mathsf{C}^{(1)} & \boldsymbol{x} \in \Omega_1^C, \\ \\ \mathsf{C}^{(2)} & \boldsymbol{x} \in \Omega_2^C, \end{cases} \tag{3.1}$$

and the volume fraction f of the inclusion phase can be defined as

$$f = \frac{\Omega_2^C}{\Omega_{RVE}^C}. \tag{3.2}$$

The equivalent material is a homogeneous SGE material, Fig. 3.1 (right), occupying the region Ω_{eq}^{SGE}

$$\Omega_{eq}^{SGE} = \Omega_{RVE}^{C}, \tag{3.3}$$

and constitutive elastic tensors C^{eq} (local part) and A^{eq} (nonlocal part). Since the region Ω_{eq}^{SGE} of the equivalent SGE material corresponds by definition to the region Ω_{RVE}^{C} of the heterogeneous RVE, in the following both these domains may be identified as Ω.

Step 2. Impose on the RVE boundary the following second-order (linear and quadratic) displacement field $\overline{\boldsymbol{u}}$, Fig. 3.2 (left)

$$\boldsymbol{u} = \overline{\boldsymbol{u}}, \quad \text{on } \partial\Omega_{RVE}^{C}, \tag{3.4}$$

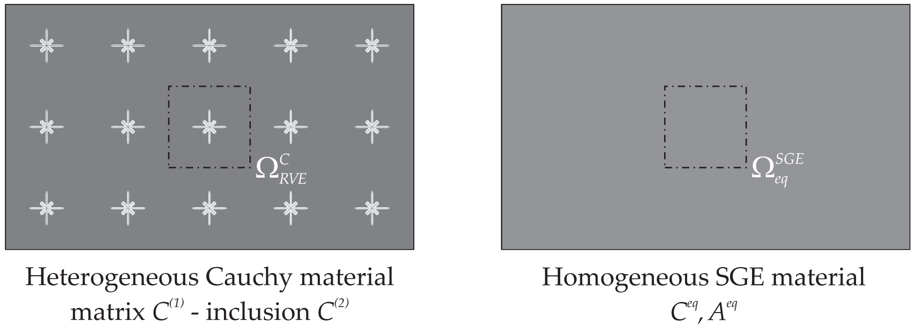

Figure 3.1: Heterogeneous Cauchy material (left) and homogeneous equivalent SGE material (right).

with

$$\overline{u}_i = \underbrace{\alpha_{ij} x_j}_{\overline{u}_i^{\alpha}} + \underbrace{\beta_{ijk} x_j x_k}_{\overline{u}_i^{\beta}}, \tag{3.5}$$

where α_{ij} and β_{ijk} are constant coefficients, the latter having the symmetry β_{ijk}=β_{ikj}.

Impose on the equivalent homogeneous SGE boundary again the displacement (3.5), but together with its normal derivative,[2] Fig. 3.2 (right), so that

$$\begin{cases} \boldsymbol{u} = \overline{\boldsymbol{u}}, \\ D\boldsymbol{u} = D\overline{\boldsymbol{u}}, \end{cases} \quad \text{on } \partial\Omega^{SGE}_{eq}. \tag{3.6}$$

The imposition of the boundary conditions (3.4) on the RVE and (3.6) on the equivalent SGE corresponds, respectively, to the two strain energies

$$\mathcal{W}^{C}_{RVE} = \int_{\Omega^C_1} w^C\big|_{\mathbf{C}^{(1)}} + \int_{\Omega^C_2} w^C\big|_{\mathbf{C}^{(2)}}, \qquad \mathcal{W}^{SGE}_{eq} = \int_{\Omega^{SGE}_{eq}} w^{SGE}\big|_{\mathbf{C}^{eq},\mathbf{A}^{eq}}, \tag{3.7}$$

so that for a generic quadratic displacement field, eqn. (3.5), an energy mismatch (or 'gap') $\mathcal{G}$ between the two materials arises as a function of the unknown equivalent constitutive tensor $\mathbf{A}^{eq}$

$$\mathcal{G}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) = \mathcal{W}^{C}_{RVE} - \mathcal{W}^{SGE}_{eq}. \tag{3.8}$$

[2] It is shown that imposing $D\boldsymbol{u} = D\overline{\boldsymbol{u}}$ is equivalent (in energetic terms at first-order in f) to impose $D\boldsymbol{u} = D\boldsymbol{u}_{RVE}$ (Appendix C).

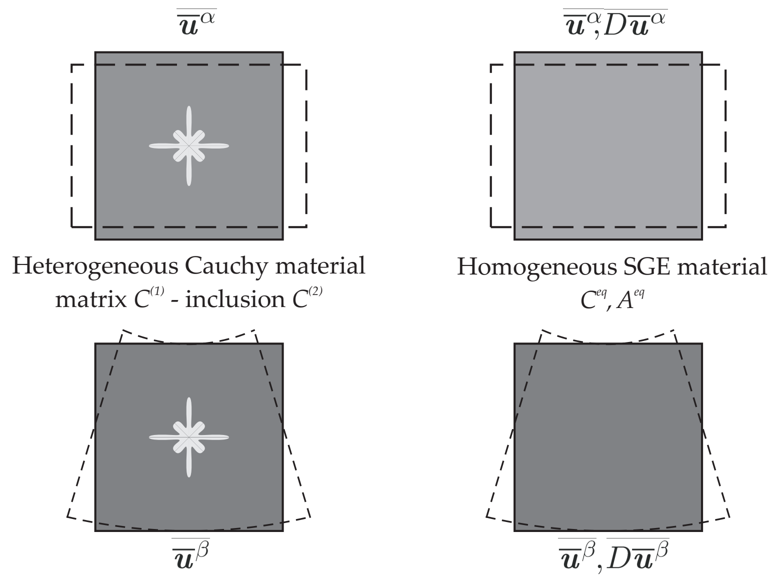

Figure 3.2: Imposition of the same linear (top) and quadratic (bottom) boundary displacement conditions on the heterogeneous Cauchy RVE (left) and on the homogeneous equivalent SGE (right).

Step 3. Find the unknown equivalent constitutive tensor $\mathbf{A}^{eq}$ by imposing a null energy mismatch $\mathcal{G}$

$$\mathcal{G}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) = 0. \tag{3.9}$$

Note that in the case of purely linear displacements ($\boldsymbol{\beta} = \mathbf{0}$) the energy mismatch $\mathcal{G}$ is null by definition of $\mathbf{C}^{eq}$. On the other hand, when quadratic displacements are considered, an energy mismatch $\mathcal{G}$ is different to zero and it can be tuned to vanish by changing the value of the unknown tensor $\mathbf{A}^{eq}$. This gives also a null mean stress mismatching as shown in Appendix D.

The above-procedure is valid for arbitrary concentration (although subsequent calculations will be referred to the dilute approximation) and is a generalization of Bigoni and Drugan (2007) since (i.) the inclusions are of arbitrary shape and, more interestingly, (ii.) the comparison material, a Mindlin elastic second-gradient material, allows a perfect match of the energies (while Bigoni and Drugan (2007) did consider only cylindrical or spherical inclusions and were only able to provide a minimization of energy gap).

3.2 Assumptions about geometrical properties of matrix and inclusion phases

Henceforth the following geometrical properties for both the subsets Ω_1^C and Ω_2^C will be assumed:[3]

GP1) The centroids of the matrix and of the inclusion coincide and correspond to the origin of the x_i–axes, so that both the static moments of the inclusion and of the matrix are null

$$\boldsymbol{S}(\Omega_1^C) = \boldsymbol{0}, \qquad \boldsymbol{S}(\Omega_2^C) = \boldsymbol{0}. \tag{3.12}$$

GP2) The x_i–axes are principal axes of inertia for both the matrix and the inclusion and the ellipsoids of inertia are a sphere (or a circle in 2D)

$$\boldsymbol{E}(\Omega_1^C) = \left[\rho^{(1)}\right]^2 \Omega_1^C \boldsymbol{I}, \qquad \boldsymbol{E}(\Omega_2^C) = \left[\rho^{(2)}\right]^2 \Omega_2^C \boldsymbol{I}, \tag{3.13}$$

where $\boldsymbol{I}$ is the identity second-order tensor and the second-order Euler tensor of inertia $\boldsymbol{E}$ relative to the x_i–axes, defined for a generic solid occupying the region V as

$$E_{ij}(V) = \int_V x_i\, x_j, \tag{3.14}$$

while $\rho^{(1)} = \rho(\Omega_1^C)$ and $\rho^{(2)} = \rho(\Omega_2^C)$ are the radii of the spheres (or circles in 2D) of inertia of the matrix and the inclusion.

GP3) The radius of the sphere of inertia for the inclusion phase vanishes in the limit of null inclusion volume fraction

$$\lim_{f\to 0} \rho^{(2)}(f) = 0, \tag{3.15}$$

[3] Note that, by definition of static moment vector $\boldsymbol{S}$ and Euler tensor of inertia $\boldsymbol{E}$, eqn (3.14), the geometrical properties **GP1**, eqn (3.12) and **GP2**, eqn (3.13), of the subsets Ω_1^C and Ω_2^C are also necessarily satisfied by Ω_{RVE}^C, so that

$$\boldsymbol{S}(\Omega_{RVE}^C) = \boldsymbol{0}, \qquad \boldsymbol{E}(\Omega_{RVE}^C) = \rho^2 \Omega_{RVE}^C \boldsymbol{I}, \tag{3.10}$$

where the radius $\rho = \rho(\Omega_{RVE}^C)$ is related to the radii of the matrix $\rho^{(1)}$ and the inclusion $\rho^{(2)}$ as follows

$$\rho^2 = (1-f)\left[\rho^{(1)}\right]^2 + f\left[\rho^{(2)}\right]^2. \tag{3.11}$$

or, equivalently, all the dimensions of the inclusion (and therefore the radius of the smallest ball containing the inclusion) are zero for $f = 0$.

Examples of two-dimensional RVE, characterized by the geometrical properties **GP1**-**GP2** and **GP3** are reported in Figs. 3.3 and 3.4, respectively.

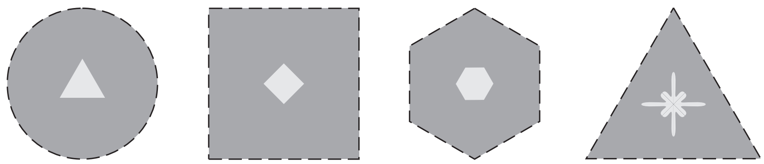

Figure 3.3: Some examples of two-dimensional RVE with the geometrical properties **GP1**, eqn (3.12), and **GP2**, eqn (3.13), for plane strain condition.

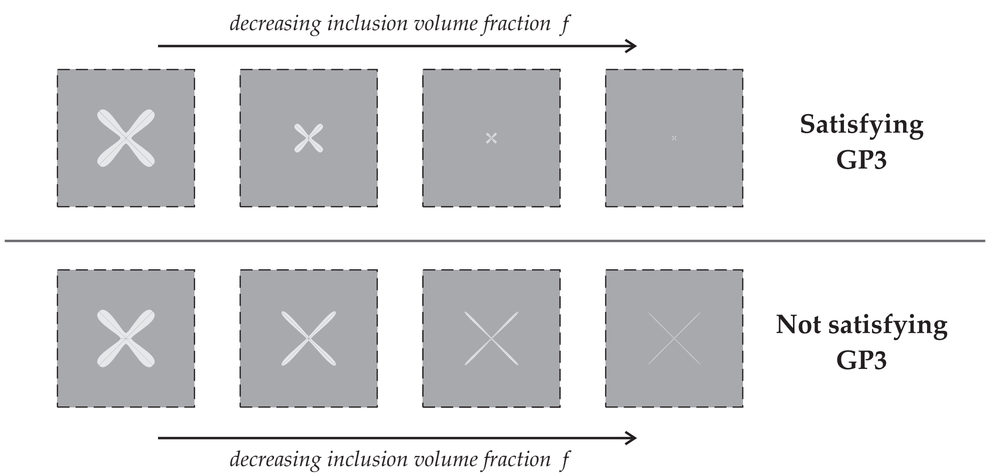

Figure 3.4: Examples of two-dimensional RVE satisfying (upper part) or not (lower part) the geometrical property **GP3**, eqn (3.15).

Chapter 4

Explicit evaluation of the non-local constitutive tensor

The explicit evaluation of the equivalent non-local constitutive tensor is obtained under the geometrical assumptions introduced in Chapter 3 and the dilute approximation. Properties of positive definiteness and non-local symmetries for the equivalent SGE material are derived from the obtained solution.

4.1 Equivalent non-local properties in the dilute case

Homogenization proposition. For a dilute concentration of the inclusion phase ($f \ll 1$) and assuming the geometrical properties **GP1 - GP2 - GP3** for the RVE (Section 3.2), the nonlocal sixth-order tensor $\mathbf{A}^{eq}$ of the equivalent SGE material is evaluated (at first-order in f) as

$$\mathbf{A}^{eq}_{ijhlmn} = -f\frac{\rho^2}{4}\left(\tilde{\mathbf{C}}_{ihln}\delta_{jm} + \tilde{\mathbf{C}}_{ihmn}\delta_{jl} + \tilde{\mathbf{C}}_{jhln}\delta_{im} + \tilde{\mathbf{C}}_{jhmn}\delta_{il}\right) + o(f), \quad (4.1)$$

where ρ is the radius of the sphere (or circle in 2D) of inertia of the RVE cell, and $\tilde{\mathbf{C}}$ is introduced to define (at first-order in f) the difference between the local constitutive tensors for the effective material $\mathbf{C}^{eq}$ and the matrix $\mathbf{C}^{(1)}$, so that

$$\mathbf{C}^{eq} = \mathbf{C}^{(1)} + f\tilde{\mathbf{C}}, \quad (4.2)$$

which is assumed to be known from standard homogenization, performed on linear displacement boundary conditions.

Eqn (4.1) represents the solution of the homogenization problem and is obtained by imposing the vanishing of the energy mismatch $\mathcal{G}$, eqn (3.9), when the same second-order displacement boundary conditions are applied both on the heterogeneous Cauchy material and on the homogeneous equivalent SGE material, eqns (3.4) and (3.6), respectively.

From the solution (4.1), in agreement with Bigoni and Drugan (2007), it can be noted that:

- the equivalent SGE material is positive definite if and only if $\tilde{\mathsf{C}}$ is negative definite;
- the constitutive higher-order tensor A^{eq} is linear in f for dilute concentration.

This properties are treated in Section 4.2.

Proof of the homogenization proposition

i) Consider the second-order (linear and quadratic) displacement boundary condition (3.6) applied on the boundary of a homogeneous SGE material with constitutive tensors C and A. In the absence of body force, $\boldsymbol{b} = \mathbf{0}$, let us consider the extension within the body of the quadratic displacement field $\overline{\boldsymbol{u}}$, eqn (3.5), applied on the boundary

$$u_i = \underbrace{\alpha_{ij} x_j}_{u_i^{\alpha}} + \underbrace{\beta_{ijk} x_j x_k}_{u_i^{\beta}}, \qquad \boldsymbol{x} \text{ in } \Omega, \tag{4.3}$$

providing the following deformation $\boldsymbol{\varepsilon}$ and curvature $\boldsymbol{\chi}$ fields

$$\varepsilon_{ij} = \frac{\alpha_{ij} + \alpha_{ji}}{2} + (\beta_{ijk} + \beta_{jik}) x_k, \qquad \chi_{ijk} = 2\beta_{kij}, \tag{4.4}$$

and the following stress $\boldsymbol{\sigma}$ and double-stress $\boldsymbol{\tau}$ fields,

$$\sigma_{ij} = \mathsf{C}_{ijhk} \alpha_{hk} + 2\mathsf{C}_{ijhk} \beta_{hkl} x_l, \qquad \tau_{ijk} = 2\mathsf{A}_{ijklmn} \beta_{nlm}. \tag{4.5}$$

The stress field (4.5) follows from the displacement field (4.3) and satisfies the equilibrium equation (2.4) if and only if [1]

$$\mathsf{C}_{ijhk} \beta_{hkj} = \mathbf{0}, \tag{4.6}$$

[1] Note that the constraint (4.6) arises independently of whether the material is Cauchy or SGE.

which for isotropic homogeneous materials reduces to the condition obtained by Bigoni and Drugan (2007)

$$\beta_{jji} = -(1-2\nu)\beta_{ikk}, \tag{4.7}$$

(with Poisson's ratio $\nu = \lambda/2(\lambda+\mu)$). Henceforth it is introduced the notation that, when the components of $\boldsymbol{\beta}$ satisfy eqn (4.6) or (4.7) for isotropy, these will be denoted with the superscript $^\diamond$.[2]

ii) Consider an auxiliary material with local constitutive tensor $\mathbf{C}^*$, defined as a first-order perturbation in f to the equivalent local constitutive tensor $\mathbf{C}^{eq}$, namely,

$$\mathbf{C}^* = \mathbf{C}^{eq} + f\left(\hat{\mathbf{C}} - \tilde{\mathbf{C}}\right), \tag{4.8}$$

(where $\hat{\mathbf{C}}$ remains for the moment unspecified together with $\mathbf{C}^*$), so that using eqn (4.2) we can write

$$\mathbf{C}^* = \mathbf{C}^{(1)} + f\hat{\mathbf{C}}. \tag{4.9}$$

By definition, the displacement field

$$u_i^* = \underbrace{\alpha_{ij}x_j}_{u_i^\alpha} + \underbrace{\beta_{ijk}^{\diamond *}x_jx_k}_{u_i^{\beta\diamond *}}, \qquad \boldsymbol{x} \text{ in } \Omega. \tag{4.10}$$

is equilibrated [in other words satisfies eqn (4.6)] in a homogeneous material characterized by the constitutive tensor $\mathbf{C}^*$ and it corresponds to the following quadratic displacement field on the boundary

$$\overline{u}_i^* = \underbrace{\alpha_{ij}x_j}_{\overline{u}_i^\alpha} + \underbrace{\beta_{ijk}^{\diamond *}x_jx_k}_{\overline{u}_i^{\beta\diamond *}}, \qquad \boldsymbol{x} \text{ on } \partial\Omega. \tag{4.11}$$

iii) Apply on the boundary $\partial\Omega_{RVE}^C$ of the heterogeneous Cauchy material (RVE) the displacement boundary condition (4.11),

$$\overline{\boldsymbol{u}}^{RVE} = \overline{\boldsymbol{u}}^*, \qquad \text{on } \partial\Omega_{RVE}^C. \tag{4.12}$$

[2]Following this rule, whenever the third-order tensor $\boldsymbol{\beta}$ is considered and the equilibrium is satisfied, eqn (4.6), the tensor $\boldsymbol{\beta}$ and the arising displacement $\boldsymbol{u}^\beta$ fields are denoted as $\boldsymbol{\beta}^\diamond$ and $\boldsymbol{u}^{\beta^\diamond}$.

According to **Lemma 1** (Section 4.1.1), the strain energy in the RVE is at first-order in f the sum of a strain energy due to the linear ($\boldsymbol{\alpha}$) and nonlinear ($\boldsymbol{\beta}$) fields, and the mutual strain energy, say, the '$\boldsymbol{\alpha}-\boldsymbol{\beta}$ energy term' is null,[3] so that

$$\mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^*\right) = \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^\alpha\right) + \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^{\beta\diamond*}\right) + o(f). \tag{4.14}$$

iv) Apply on the boundary $\partial\Omega^{SGE}_{eq}$ of the homogeneous SGE material the same displacement boundary condition $\overline{\boldsymbol{u}}^*$, eqn (4.11), imposed to the RVE and complemented by the higher-order boundary condition in terms of displacement normal derivative taken equal[4] to $D\overline{\boldsymbol{u}}^*$

$$\begin{cases} \overline{\boldsymbol{u}}^{SGE} = \overline{\boldsymbol{u}}^*, \\ \overline{D\boldsymbol{u}}^{SGE} = D\overline{\boldsymbol{u}}^*, \end{cases} \qquad \text{on } \partial\Omega^{SGE}_{eq}. \tag{4.15}$$

According to the result presented in **Lemma 2** (Section 4.1.1), the $\boldsymbol{\alpha}-\boldsymbol{\beta}$ energy term is null and the strain energy in Ω^{SGE}_{eq} is

$$\mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^*\right) = \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha\right) + \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^{\beta\diamond*}, D\overline{\boldsymbol{u}}^{\beta\diamond*}\right), \tag{4.16}$$

where $D\overline{\boldsymbol{u}}^\alpha$ and $D\overline{\boldsymbol{u}}^{\beta\diamond*}$ are the contributions of the imposed normal derivative depending on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ terms in $D\overline{\boldsymbol{u}}^*$, respectively.

v) The energy minimization procedure, eqn (3.9), can be performed using the energy stored in the heterogeneous Cauchy material $\mathcal{W}^C_{RVE}$, eqn (4.14),

[3] Considering that the RVE satisfies geometrical symmetry conditions, in addition to the geometrical properties **GP1** and **GP2**, it can be proven that the mutual energy is identically null even in the case of non-dilute suspension of inclusion

$$\mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^*\right) = \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^\alpha\right) + \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^{\beta\diamond*}\right), \qquad \forall f. \tag{4.13}$$

[4] The displacement field eqn (4.10) is the solution for a homogeneous SGE when boundary conditions (4.15) are imposed. It can be easily proven that the result of the proposed homogenization procedure holds when the higher-order boundary condition changes as $\overline{D\boldsymbol{u}}^{SGE} = D\overline{\boldsymbol{u}}^{RVE}$ since the strain energy developed in the SGE material is the same at the first order

$$\mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^{RVE}\right) = \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^*\right) + o(f).$$

and in the homogeneous SGE material $\mathcal{W}_{eq}^{SGE}$, eqn (4.16), so that the energy mismatch is given by

$$\begin{aligned}\mathcal{G}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right) = \;& \mathcal{G}^{\alpha}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right) \\ & +\mathcal{G}^{\beta^{\diamond *}}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right),\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}\mathcal{G}^{\alpha}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right) = \;& \mathcal{W}_{RVE}^{C}\left(\overline{\boldsymbol{u}}^{\alpha}\right)-\mathcal{W}_{eq}^{SGE}\left(\overline{\boldsymbol{u}}^{\alpha},D\overline{\boldsymbol{u}}^{\alpha}\right), \\ \mathcal{G}^{\beta^{\diamond *}}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right) = \;& \mathcal{W}_{RVE}^{C}\left(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}\right)-\mathcal{W}_{eq}^{SGE}\left(\overline{\boldsymbol{u}}^{\beta^{\diamond *}},D\overline{\boldsymbol{u}}^{\beta^{\diamond *}}\right).\end{aligned} \tag{4.18}$$

Since only the local contribution (depending on $\mathbf{C}^{eq}$) arises in the SGE strain energy when the linear boundary displacement condition ($\boldsymbol{\beta}^{\diamond *}=\mathbf{0}$ and $\overline{\boldsymbol{u}}^{SGE}=\overline{\boldsymbol{u}}^{\alpha}$, $\overline{D\boldsymbol{u}}^{SGE}=D\overline{\boldsymbol{u}}^{\alpha}$) is imposed (while the non-local contribution depending on $\mathbf{A}^{eq}$ is identically null because higher-order stress and curvature are null), the energy mismatch $\mathcal{G}^{\alpha}$ due to the α terms is null by definition of $\mathbf{C}^{eq}$ (which is known from the first-order homogenization procedure)

$$\mathcal{G}^{\alpha}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right) = \mathcal{G}^{\alpha}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq}\right)=0. \tag{4.19}$$

Therefore, the proposed energy minimization procedure, based on linear and quadratic displacement boundary condition and leading to the definition of $\mathbf{A}^{eq}$, can be performed referring only to the $\beta^{\diamond *}$ terms,

$$\mathcal{G}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right)=\mathcal{G}^{\beta^{\diamond *}}\left(\mathbf{C}^{(1)},\mathbf{C}^{(2)},\mathbf{C}^{eq},\mathbf{A}^{eq}\right). \tag{4.20}$$

vi) Keeping into account the results presented in **Lemma 3** (Section 4.1.1) and **Lemma 4** (Section 4.1.1), the energy mismatch (4.20) is given by the difference of the following two terms

$$\mathcal{W}_{RVE}^{C}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}})=2\rho^2\Omega\mathbf{C}_{ijhk}^{(1)}\beta_{ijl}^{\diamond *}\beta_{hkl}^{\diamond *}+o(f). \tag{4.21}$$

and

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}},D\overline{\boldsymbol{u}}^{\beta^{\diamond *}})=2\Omega\left(\rho^2\mathbf{C}_{ijhk}^{eq}\delta_{lm}+\mathbf{A}_{jlikmh}^{eq}\right)\beta_{ijl}^{\diamond *}\beta_{hkm}^{\diamond *}+o(f). \tag{4.22}$$

vii) Therefore, from eqns (4.2), (4.21) and (4.22), the annihilation of the strain energy gap $\mathcal{G}$, eqn (4.20) (between the real heterogeneous Cauchy and the equivalent homogeneous SGE materials) is represented by the condition

$$\left(f\rho^2\tilde{\mathsf{C}}_{ijhk}\delta_{lm} + \mathsf{A}^{eq}_{jlikmh}\right)\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm} + o(f) = 0. \tag{4.23}$$

viii) The energy annihilation (4.23) has been obtained for a nonlinear displacement field $\boldsymbol{\beta}^{\diamond *}$, in equilibrium within a homogeneous material with local constitutive tensor C^{*}. But, since this tensor is arbitrary, it follows

$$\left(f\rho^2\tilde{\mathsf{C}}_{ijhk}\delta_{lm} + \mathsf{A}^{eq}_{jlikmh}\right)\beta_{ijl}\beta_{hkm} + o(f) = 0, \tag{4.24}$$

where the components of $\boldsymbol{\beta}$ are unrestricted, except for the symmetry $\beta_{ijk}=\beta_{ikj}$. Eventually, the annihilation of energy mismatch $\mathcal{G}$, eqn (4.24), defines the non-local constitutive tensor A^{eq} for the equivalent SGE material as in eqn (4.1). □

4.1.1 Proofs of lemmas 1-4

Lemma 1: Null mutual $\boldsymbol{\alpha}$–$\boldsymbol{\beta}$ energy term for the RVE at the first-order in concentration f

Statement. When a quadratic displacement $\overline{\boldsymbol{u}}^{*}$, eqn (4.11), is applied on the boundary of a RVE satisfying the geometrical property **GP1** and **GP3**, the strain energy at first-order in f is given by eqn (4.14).

Proof. By the superposition principle, the fields originated by the application of $\overline{\boldsymbol{u}}^{*} = \overline{\boldsymbol{u}}^{\alpha} + \overline{\boldsymbol{u}}^{\beta^{\diamond *}}$ are given by the sum of the respective fields originated from the boundary conditions $\overline{\boldsymbol{u}}^{\alpha}$ and $\overline{\boldsymbol{u}}^{\beta^{\diamond *}}$

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{\varepsilon}^{\alpha}(\boldsymbol{x}) + \boldsymbol{\varepsilon}^{\beta^{\diamond *}}(\boldsymbol{x}), \qquad \boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{\sigma}^{\alpha}(\boldsymbol{x}) + \boldsymbol{\sigma}^{\beta^{\diamond *}}(\boldsymbol{x}), \tag{4.25}$$

(the latter calculated through the constitutive eqn $(2.10)_1$) so that the strain energy $(3.7)_1$ becomes

$$\mathcal{W}^{C}_{RVE}(\overline{\boldsymbol{u}}^{*}) = \mathcal{W}^{C}_{RVE}(\overline{\boldsymbol{u}}^{\alpha}) + \mathcal{W}^{C}_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) + \underbrace{\mathcal{W}^{C}_{RVE}(\overline{\boldsymbol{u}}^{\alpha};\overline{\boldsymbol{u}}^{\beta^{\diamond *}})}_{mutual\ energy} \tag{4.26}$$

where

$$\begin{aligned}
\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\alpha}) &= \frac{1}{2}\int_{\Omega_R} \varepsilon^{\alpha}_{ij}(\boldsymbol{x})\mathbf{C}_{ijhk}(\boldsymbol{x})\varepsilon^{\alpha}_{hk}(\boldsymbol{x}),\\
\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) &= \frac{1}{2}\int_{\Omega_R} \varepsilon^{\beta^{\diamond *}}_{ij}(\boldsymbol{x})\mathbf{C}_{ijhk}(\boldsymbol{x})\varepsilon^{\beta^{\diamond *}}_{hk}(\boldsymbol{x}),\\
\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\alpha};\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) &= \int_{\Omega_R} \varepsilon^{\alpha}_{ij}(\boldsymbol{x})\mathbf{C}_{ijhk}(\boldsymbol{x})\varepsilon^{\beta^{\diamond *}}_{hk}(\boldsymbol{x}).
\end{aligned} \tag{4.27}$$

Through two applications of the principle of virtual work[5] the mutual energy $(4.27)_3$ can be computed as

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\alpha};\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) = \alpha_{ij}\int_{\Omega_R} \sigma^{\beta^{\diamond *}}_{ij}(\boldsymbol{x}), \tag{4.29}$$

which, using the constitutive relation $(2.10)_1$ and the symmetries of the local constitutive tensors $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$, can be decomposed as the sum of two contributions

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\alpha};\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) = \alpha_{ij}\mathbf{C}^{(1)}_{ijhk}\int_{\Omega_R} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}) + \alpha_{ij}\left(\mathbf{C}^{(2)}_{ijhk}-\mathbf{C}^{(1)}_{ijhk}\right)\int_{\Omega_{R_2}} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}). \tag{4.30}$$

Through two further applications of the divergence theorem and using the geometrical property **GP1** for the RVE,[6] the first term on the right-hand-side

[5] In the first application, the fields corresponding to the solution (4.25) are considered

$$\int_{\Omega_R} \varepsilon^{\alpha}_{ij}(\boldsymbol{x})\sigma^{\beta^{\diamond *}}_{ij}(\boldsymbol{x}) = \int_{\partial\Omega_R} \overline{u}^{\alpha}_{i}(\boldsymbol{x})t^{\beta^{\diamond *}}_{i}(\boldsymbol{x}), \tag{4.28}$$

while in the second application, the kinematical field generated by the admissible displacement $\boldsymbol{u}^{\alpha}$ (4.10) within the RVE is considered so that the mutual energy (4.29) is obtained.

[6] In the first application of the divergence theorem, $u^{\beta^{\diamond *}} = \overline{u}^{\beta^{\diamond *}}$, eqn (4.11), is considered on the boundary $\partial\Omega_R$, so that

$$\int_{\Omega_R} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}) = \beta^{\diamond *}_{hlm}\int_{\partial\Omega_R} n_k x_l x_m, \tag{4.31}$$

while, in the second application, the kinematically admissible displacement field $\overline{u}^{\beta^{\diamond *}}$, eqn (4.10), is considered within the RVE, yielding

$$\beta^{\diamond *}_{hlm}\int_{\partial\Omega_R} n_k x_l x_m = 2\beta^{\diamond *}_{hlk}\int_{\Omega_R} x_l, \tag{4.32}$$

so that the geometrical property **GP1** for the RVE leads to eqn (4.33).

of eqn (4.30) results to be null

$$\alpha_{ij}\mathsf{C}^{(1)}_{ijhk}\int_{\Omega_R} u^{\beta\diamond *}_{h,k}(\boldsymbol{x}) = 0. \tag{4.33}$$

Introducing the mean value over a domain Ω of the function $f(\boldsymbol{x})$ as

$$\langle f(\boldsymbol{x})\rangle|_{\Omega} = \frac{1}{\Omega}\int_{\Omega} f(\boldsymbol{x}), \tag{4.34}$$

the second term on the right-hand-side of eqn (4.30) can be rewritten as

$$\alpha_{ij}\left(\mathsf{C}^{(2)}_{ijhk} - \mathsf{C}^{(1)}_{ijhk}\right)\Omega_{R_2}\left\langle u^{\beta\diamond *}_{h,k}(\boldsymbol{x})\right\rangle\Big|_{\Omega_{R_2}}. \tag{4.35}$$

Assuming the geometrical property **GP3** for the RVE, the displacement field in the presence of the inclusion is given by the asymptotic expansion in the volume fraction f

$$u^{\beta\diamond *}_i = \beta^{\diamond *}_{ijk}x_j x_k + f^q\,\tilde{u}^{\beta\diamond *}_i + o(f), \tag{4.36}$$

subject to the constraint

$$0 < q \leq 1, \tag{4.37}$$

and considering the geometrical property **GP1** for the RVE, together with the definition of volume fraction f, eqn (3.2), expression (4.35) becomes

$$f^{q+1}\Omega\,\alpha_{ij}\left(\mathsf{C}^{(2)}_{ijhk} - \mathsf{C}^{(1)}_{ijhk}\right)\left\langle \tilde{u}^{\beta\diamond *}_{h,k}(\boldsymbol{x})\right\rangle\Big|_{\Omega_{R_2}}, \tag{4.38}$$

from which, considering the restriction on the power q (C.11), the second term on the right-hand-side of eqn (4.30) is null at first-order in f

$$\alpha_{ij}\left(\mathsf{C}^{(2)}_{ijhk} - \mathsf{C}^{(1)}_{ijhk}\right)\int_{\Omega_{R_2}} u^{\beta\diamond *}_{h,k}(\boldsymbol{x}) = o(f). \tag{4.39}$$

Considering results (4.33) and (4.39), the mutual energy in the RVE $(4.27)_3$ is null at first-order in f and proposition (4.14) follows. □

Lemma 2: Null mutual $\boldsymbol{\alpha}$–$\boldsymbol{\beta}$ energy term for the SGE

Statement. When a quadratic displacement $\overline{\boldsymbol{u}}^*$, eqn (4.11), and the normal component of its derivative $D\overline{\boldsymbol{u}}^*$ are applied on the boundary of a SGE satisfying the geometrical property **GP1**, the strain energy is given by eqn (4.16).

Proof. By the superposition principle, the fields originated by the application of the boundary conditions ($\overline{\boldsymbol{u}}^* = \overline{\boldsymbol{u}}^\alpha + \overline{\boldsymbol{u}}^{\beta\diamond *}$, $D\overline{\boldsymbol{u}}^* = D\overline{\boldsymbol{u}}^\alpha + D\overline{\boldsymbol{u}}^{\beta\diamond *}$) can be obtained as the sum of the respective fields arising from the boundary conditions ($\overline{\boldsymbol{u}}^\alpha$, $D\overline{\boldsymbol{u}}^\alpha$) and ($\overline{\boldsymbol{u}}^{\beta\diamond *}$, $D\overline{\boldsymbol{u}}^{\beta\diamond *}$) in the forms

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{x}) &= \boldsymbol{\varepsilon}^\alpha(\boldsymbol{x}) + \boldsymbol{\varepsilon}^{\beta\diamond *}(\boldsymbol{x}), \qquad & \boldsymbol{\chi}(\boldsymbol{x}) &= \boldsymbol{\chi}^\alpha(\boldsymbol{x}) + \boldsymbol{\chi}^{\beta\diamond *}(\boldsymbol{x}), \\ \boldsymbol{\sigma}(\boldsymbol{x}) &= \boldsymbol{\sigma}^\alpha(\boldsymbol{x}) + \boldsymbol{\sigma}^{\beta\diamond *}(\boldsymbol{x}), \qquad & \boldsymbol{\tau}(\boldsymbol{x}) &= \boldsymbol{\tau}^\alpha(\boldsymbol{x}) + \boldsymbol{\tau}^{\beta\diamond *}(\boldsymbol{x}), \end{aligned} \tag{4.40}$$

(the latter calculated through the constitutive eqn (2.10)) so that the strain energy $(3.7)_2$ becomes

$$\begin{aligned} \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^*) = &\underbrace{\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha) + \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *})}_{direct\ energy} \\ &+ \underbrace{\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *})}_{mutual\ energy} \end{aligned} \tag{4.41}$$

where

$$\begin{aligned} \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha) &= \frac{1}{2}\int_{\Omega_{eq}} \Big[\varepsilon_{ij}^\alpha(\boldsymbol{x})\mathsf{C}_{ijhk}^{eq}\varepsilon_{hk}^\alpha(\boldsymbol{x}) \\ &\quad + \chi_{ijl}^\alpha(\boldsymbol{x})\mathsf{A}_{ijlhkm}^{eq}\chi_{hkm}^\alpha(\boldsymbol{x})\Big], \\ \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}) &= \frac{1}{2}\int_{\Omega_{eq}} \Big[\varepsilon_{ij}^{\beta\diamond *}(\boldsymbol{x})\mathsf{C}_{ijhk}^{eq}\varepsilon_{hk}^{\beta\diamond *}(\boldsymbol{x}) \\ &\quad + \chi_{ijl}^{\beta\diamond *}(\boldsymbol{x})\mathsf{A}_{ijlhkm}^{eq}\chi_{hkm}^{\beta\diamond *}(\boldsymbol{x})\Big], \\ \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}) &= \int_{\Omega_{eq}} \Big[\varepsilon_{ij}^\alpha(\boldsymbol{x})\mathsf{C}_{ijhk}^{eq}\varepsilon_{hk}^{\beta\diamond *}(\boldsymbol{x}) \\ &\quad + \chi_{ijl}^\alpha(\boldsymbol{x})\mathsf{A}_{ijlhkm}^{eq}\chi_{hkm}^{\beta\diamond *}(\boldsymbol{x})\Big]. \end{aligned} \tag{4.42}$$

Application of the boundary condition ($\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha$) on $\partial\Omega_{eq}$ leads to the displacement field $\boldsymbol{u}^\alpha(\boldsymbol{x})$, eqn (4.10), so that $\boldsymbol{\chi}^\alpha(\boldsymbol{x}) = \boldsymbol{0}$ and, considering the symmetries of the equivalent local constitutive tensor C^{eq}, the mutual energy simplifies in the local contribution

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}) = \alpha_{ij}\mathsf{C}_{ijhk}^{eq}\int_{\Omega_{eq}} u_{h,k}^{\beta\diamond *}(\boldsymbol{x}). \tag{4.43}$$

Through two applications of the divergence theorem and using the geometrical property **GP1** of the SGE, the mutual energy (4.43) is null and then proposition (4.16) follows. □

Lemma 3: $\boldsymbol{\beta}$ term in the strain energy $\mathcal{W}^C_{RVE}$

Statement. When a quadratic displacement $\overline{\boldsymbol{u}}^{\beta^{\diamond *}}$, eqn (4.11) with $\boldsymbol{\alpha} = \mathbf{0}$, is applied on the RVE boundary, the strain energy at first-order in the concentration f is given by eqn (4.21).

Proof. The strain energy $\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}})$ stored in the RVE, when a quadratic displacement field $\overline{\boldsymbol{u}}^{\beta^{\diamond *}}$ (4.11) is applied on its boundary $\partial\Omega_{RVE}$, is bounded by (Gurtin, 1972)

$$\int_{\partial\Omega_{RVE}} \sigma^{SA}_{ij} n_i \overline{u}^{\beta^{\diamond *}}_j - \mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA}) \leq \mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) \leq \mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA}), \tag{4.44}$$

where $\boldsymbol{\varepsilon}^{KA}$ is a kinematically admissible (satisfying the kinematic compatibility relation $(2.1)_1$ and the imposed displacement boundary conditions) strain field, $\boldsymbol{\sigma}^{SA}$ is a statically admissible (satisfying the equilibrium condition, eqn (2.4) with $\boldsymbol{\tau} = \mathbf{0}$) stress field, while $\mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA})$ and $\mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA})$ are respectively the following stress and strain energies

$$\begin{aligned} \mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA}) &= \frac{1}{2}\int_{\Omega_R} \sigma^{SA}_{ij}(\boldsymbol{x}) \mathbf{C}^{-1}_{ijhk}(\boldsymbol{x}) \sigma^{SA}_{hk}(\boldsymbol{x}), \\ \mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA}) &= \frac{1}{2}\int_{\Omega_R} \varepsilon^{KA}_{ij}(\boldsymbol{x}) \mathbf{C}_{ijhk}(\boldsymbol{x}) \varepsilon^{KA}_{hk}(\boldsymbol{x}). \end{aligned} \tag{4.45}$$

Considering the kinematically admissible strain field

$$\varepsilon^{KA}_{ij} = (\beta^{\diamond *}_{ijk} + \beta^{\diamond *}_{jik}) x_k, \tag{4.46}$$

and assuming the geometrical properties **GP2** and **GP3**, an estimate for the *upper bound* in eqn (4.44) is the strain energy $\mathcal{W}^C_{RVE}$ given by eqn $(A.5)_1$ (Appendix A.1), so that

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) \leq 2\rho^2 \Omega \mathbf{C}^{(1)}_{ijhk} \beta^{\diamond *}_{ijl} \beta^{\diamond *}_{hkl} + o(f). \tag{4.47}$$

Considering now the statically admissible stress field

$$\sigma^{SA}_{ij} = 2\mathbf{C}^{*}_{ijhk} \beta^{\diamond *}_{hkl} x_l, \tag{4.48}$$

where $\mathbf{C}^{*}$ is a first-order perturbation in f to the material matrix $\mathbf{C}^{(1)}$, eqn (4.9), and assuming the geometrical property **GP2**, the stress energy $\mathcal{U}^C_{RVE}$

is given by eqn (A.5)$_2$ (Appendix A.1). Moreover, since the application of the divergence theorem yields

$$\int_{\partial\Omega_R} \sigma_{ij}^{SA} n_i \overline{u}_j^{\beta\diamond *} = 4\rho^2\Omega \left(\mathsf{C}_{ijhk}^{(1)} + f\hat{\mathsf{C}}_{ijhk} \right) \beta_{ijl}^{\diamond *}\beta_{hkl}^{\diamond *}, \tag{4.49}$$

an estimate is obtained for the *lower bound* in eqn (4.44) as

$$\mathcal{W}_{RVE}^{C}(\overline{\boldsymbol{u}}^{\beta\diamond *}) \geq 2\rho^2\Omega \mathsf{C}_{ijhk}^{(1)}\beta_{ijl}^{\diamond *}\beta_{hkl}^{\diamond *} + o(f), \tag{4.50}$$

which, together with the upper bound (4.47), leads to eqn (4.21). □

Lemma 4: $\boldsymbol{\beta}$ term in the strain energy $\mathcal{W}_{eq}^{SGE}$

Statement. When a quadratic displacement $\overline{\boldsymbol{u}}^{\beta\diamond *}$, eqn (4.11) with $\boldsymbol{\alpha} = \mathbf{0}$, and the normal component of its gradient $D\overline{\boldsymbol{u}}^{\beta\diamond *}$ are imposed on the boundary of the homogeneous SGE equivalent material, the strain energy at first-order in the concentration f is given by eqn (4.22).

Proof. The strain energy $\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *})$ stored in the SGE, when a quadratic displacement field $\overline{\boldsymbol{u}}^{\beta\diamond *}$ (4.11) and the normal component of its gradient $D\overline{\boldsymbol{u}}^{\beta\diamond *}$ are imposed on its boundary $\partial\Omega_{eq}$, is bounded as (Appendix B)

$$\begin{aligned}\int_{\partial\Omega_{eq}} \left(t_i^{SA}\overline{u}_i^{\beta\diamond *} + T_i^{SA} D\overline{u}_i^{\beta\diamond *} \right) + \int_{\Gamma_{eq}} \Theta_i^{SA}\overline{u}_i^{\beta\diamond *} - \mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA}) \leq \\ \leq \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}) \leq \mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA}, \boldsymbol{\chi}^{KA}),\end{aligned} \tag{4.51}$$

with

$$\begin{cases} t_k^{SA} = n_j\sigma_{jk}^{SA} - n_i n_j D\tau_{ijk}^{SA} - 2n_j D_i \tau_{ijk}^{SA} \\ \qquad + \left(n_i n_j D_l n_l - D_j n_i\right)\tau_{ijk}^{SA}, & \text{on } \partial\Omega_{eq}, \\ T_k^{SA} = n_i n_j \tau_{ijk}^{SA}, \end{cases} \tag{4.52}$$

and

$$\Theta_k^{SA} = [[e_{mlj} n_i s_m n_l \tau_{ijk}^{SA}]], \qquad \text{on } \Gamma_{eq}, \tag{4.53}$$

where $\boldsymbol{\varepsilon}^{KA}$ and $\boldsymbol{\chi}^{KA}$ are kinematically admissible strain and curvature fields (satisfying the kinematic compatibility relation (2.1) and the imposed displacement boundary conditions), $\boldsymbol{\sigma}^{SA}$ and $\boldsymbol{\tau}^{SA}$ are statically admissible stress and double-stress fields (satisfying the equilibrium equation (2.4)), while $\mathcal{U}_{eq}^{SGE}$ and $\mathcal{W}_{eq}^{SGE}$ are respectively the stress and the strain energies given by

$$
\begin{aligned}
\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA}) &= \tfrac{1}{2}\int_{\Omega_{eq}} \sigma_{ij}^{SA}(\boldsymbol{x})\mathbf{C}_{ijhk}^{eq^{-1}}\sigma_{hk}^{SA}(\boldsymbol{x}) \\
&\quad + \tfrac{1}{2}\int_{\Omega_{eq}} \tau_{ijh}^{SA}(\boldsymbol{x})\mathbf{A}_{ijhklm}^{eq^{-1}}\tau_{klm}^{SA}(\boldsymbol{x}), \\
\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA}, \boldsymbol{\chi}^{KA}) &= \tfrac{1}{2}\int_{\Omega_{eq}} \varepsilon_{ij}^{KA}(\boldsymbol{x})\mathbf{C}_{ijhk}^{eq}\varepsilon_{hk}^{KA}(\boldsymbol{x}) \\
&\quad + \tfrac{1}{2}\int_{\Omega_{eq}} \chi_{ijh}^{KA}(\boldsymbol{x})\mathbf{A}_{ijhklm}^{eq}\chi_{klm}^{KA}(\boldsymbol{x}).
\end{aligned}
\tag{4.54}
$$

Considering the kinematically admissible strain $\boldsymbol{\varepsilon}^{KA}$ (4.46) and curvature field

$$\chi_{ijk}^{KA} = 2\beta_{kij}^{\diamond *}, \tag{4.55}$$

and assuming geometrical property **GP2**, an estimate for the *upper bound* in eqn (4.51) is the strain energy $\mathcal{W}_{eq}^{SGE}$ given by eqn (A.8)$_1$ (Appendix A.2) as

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) \leq 2\Omega\beta_{ijl}^{\diamond *}\beta_{hkm}^{\diamond *}\left(\rho^2\mathbf{C}_{ijhk}^{eq}\delta_{lm} + \mathbf{A}_{jlikmh}^{eq}\right). \tag{4.56}$$

Considering the statically admissible stress $\boldsymbol{\sigma}^{SA}$ (4.48) and double-stress field

$$\tau_{jli}^{SA} = 2\mathbf{A}_{jlikmh}^{eq}\beta_{hkm}^{\diamond *}, \tag{4.57}$$

where $\mathbf{C}^*$ is a first-order perturbation in f to the material matrix $\mathbf{C}^{eq}$, eqn (4.8) and assuming the geometrical property **GP2**, the stress energy $\mathcal{U}_{eq}^{SGE}$ is given by eqn (A.9) (Appendix A.2). Moreover, since the application of the divergence theorem yields

$$
\begin{aligned}
&\int_{\partial\Omega_{eq}} \left(t_i^{SA}\overline{u}_i^{\beta^{\diamond *}} + T_i^{SA}D\overline{u}_i^{\beta^{\diamond *}}\right) + \int_{\Gamma_{eq}} \Theta_i^{SA}\overline{u}_i^{\beta^{\diamond *}} \\
&= 4\rho^2\Omega\left[\mathbf{C}_{ijhk}^{eq} + f\left(\hat{\mathbf{C}}_{ijhk} - \tilde{\mathbf{C}}_{ijhk}\right)\right]\beta_{ijn}^{\diamond *}\beta_{hkn}^{\diamond *},
\end{aligned}
\tag{4.58}
$$

an estimate is obtained for the *lower bound* in eqn (4.51) as

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) \geq 2\Omega\beta_{ijl}^{\diamond *}\beta_{hkm}^{\diamond *}\left(\rho^2\mathbf{C}_{ijhk}^{eq}\delta_{lm} + \mathbf{A}_{jlikmh}^{eq}\right) + o(f), \tag{4.59}$$

which, together with the upper bound (4.56), leads to eqn (4.22). $\square$

4.2 Some properties of the effective SGE solid

Some properties of the effective SGE solid are obtained below from the definition of the effective higher-order constitutive tensor A^{eq}, eqn (4.1).

4.2.1 Heterogeneous Cauchy RVE leading to positive definite equivalent SGE material

Statement. For constituents characterized by a positive definite strain energy, a positive definite equivalent SGE material is obtained if and only if the first-order discrepancy tensor $\tilde{\mathsf{C}}$ is negative definite.

Proof. For constituents characterized by a positive definite strain energy, the first-order homogenization always leads to a positive definite equivalent fourth-order tensor C^{eq}, so that a positive strain energy (see eqn (9) in Part I) is stored within the equivalent SGE material if and only if

$$\mathsf{A}^{eq}_{ijhlmn}\chi_{ijh}\chi_{lmn} > 0 \qquad \forall\boldsymbol{\chi} \neq \mathbf{0} \text{ with } \chi_{ijk} = \chi_{jik}, \tag{4.60}$$

where the summation convention over repeated indices is used henceforth. Considering the form (4.1) of A^{eq} (note the '$-$' sign), a positive definite equivalent SGE material is obtained when

$$\tilde{\mathsf{C}}_{ijhk}\chi_{lij}\chi_{lhk} < 0 \qquad \forall\boldsymbol{\chi} \neq \mathbf{0} \text{ with } \chi_{ijk} = \chi_{jik}. \tag{4.61}$$

Since the discrepancy tensor has the minor symmetries, $\tilde{\mathsf{C}}_{ijhk} = \tilde{\mathsf{C}}_{jihk} = \tilde{\mathsf{C}}_{ijkh}$, the condition (4.61) can be written as

$$\tilde{\mathsf{C}}_{ijhk}(\chi_{lij} + \chi_{lji})(\chi_{lhk} + \chi_{lkh}) < 0 \qquad \forall\boldsymbol{\chi} \neq \mathbf{0} \text{ with } \chi_{ijk} = \chi_{jik}, \tag{4.62}$$

which corresponds to the negative definite condition for the fourth-order constitutive tensor $\tilde{\mathsf{C}}$, because $\chi_{lij} + \chi_{lji} = \mathbf{0}$ if and only if $\boldsymbol{\chi} = \mathbf{0}$.[7] □

[7] The last statement can be proven as follows. With reference to a third-order tensor ς_{ijk}, symmetric with respect to the first two indices ($\varsigma_{ijk} = \varsigma_{jik}$), we define the tensor γ_{ijk} as

$$\gamma_{ijk} = \varsigma_{ijk} + \varsigma_{ikj}, \tag{4.63}$$

resulting symmetric with respect to the last two indices ($\gamma_{ijk} = \gamma_{ikj}$). Relation (4.63) is invertible, so that

$$\varsigma_{ijk} = \frac{\gamma_{ijk} + \gamma_{jki} - \gamma_{kij}}{2}, \tag{4.64}$$

and therefore $\boldsymbol{\gamma} = \mathbf{0}$ if and only if $\boldsymbol{\varsigma} = \mathbf{0}$.

4.2.2 Higher-order material symmetries for the equivalent SGE solid

Statement. The higher-order material symmetries of the equivalent SGE solid coincide with the material symmetries of the first-order discrepancy tensor $\tilde{\mathsf{C}}$.

Proof. A class of material symmetry corresponds to indifference of a constitutive equations with respect to application of a class of orthogonal transformations represented through an orthogonal tensor $\boldsymbol{Q}$.

An higher-order material symmetry for the equivalent SGE material occurs when

$$\mathsf{A}^{eq}_{ijhlmn} = Q_{ip}Q_{jq}Q_{hr}Q_{ls}Q_{mt}Q_{nu}\mathsf{A}^{eq}_{pqrstu}, \tag{4.65}$$

that, with reference to the form (4.1) for A^{eq} and through the property of orthogonal transformations ($\boldsymbol{Q}\boldsymbol{Q}^T = \mathbf{I}$), is equivalent to the corresponding symmetry condition for the first-order discrepancy tensor,

$$\tilde{\mathsf{C}}_{ijhk} = Q_{ip}Q_{jq}Q_{hr}Q_{ks}\tilde{\mathsf{C}}_{pqrs}, \tag{4.66}$$

so that the higher-order material symmetries for the equivalent SGE solid coincide with the material symmetry of $\tilde{\mathsf{C}}$. $\square$

4.2.3 Influence of the volume and shape of the RVE on the higher-order constitutive response

In addition to the dependence on the shape of the inclusion, typical of first-order homogenization, the representation (4.1) of A^{eq} shows that the higher-order constitutive response in the dilute case depends on the volume and the shape of the RVE through its radius of inertia ρ. This feature distinguishes second-order homogenization from first-order, since in the latter case C^{eq} in the dilute case is independent of the volume and shape of the RVE. Therefore, two composite materials $\mathcal{M}$ and $\mathcal{N}$ differing only in the geometrical distribution of the inclusions correspond to the same equivalent local tensor $\mathsf{C}^{eq}(\mathcal{M}) = \mathsf{C}^{eq}(\mathcal{N})$, but lead to a different higher-order equivalent tensor $\mathsf{A}^{eq}(\mathcal{M}) \neq \mathsf{A}^{eq}(\mathcal{N})$.

An example in 2D is reported in Fig. 4.1 where the hexagonal RVE ($\mathcal{N}$) compared to the squared RVE ($\mathcal{M}$) yields

$$\mathsf{A}^{eq}(\mathcal{M}) = \frac{3\sqrt{3}}{5}\mathsf{A}^{eq}(\mathcal{N}) \sim 1.039\mathsf{A}^{eq}(\mathcal{N}), \tag{4.67}$$

while in the 3D example reported in Fig. 4.2 a truncated-octahedral RVE ($\mathcal{N}$) is compared to a cubic RVE ($\mathcal{M}$) yielding

$$\mathbf{A}^{eq}(\mathcal{M}) = \frac{16\sqrt[3]{2}}{19}\mathbf{A}^{eq}(\mathcal{N}) \sim 1.061\mathbf{A}^{eq}(\mathcal{N}). \tag{4.68}$$

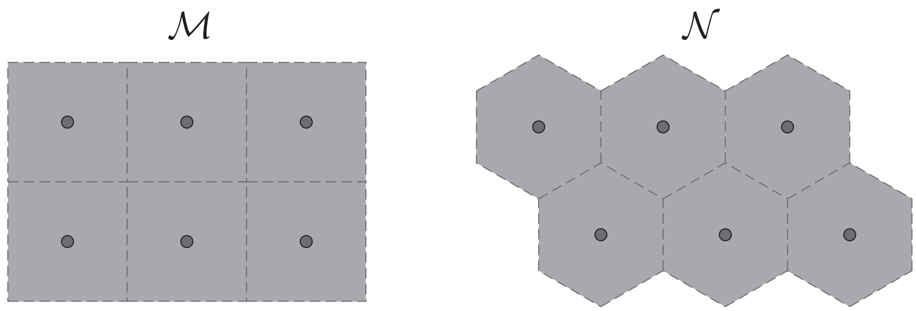

Figure 4.1: Two composite materials $\mathcal{M}$ (squared RVE) and $\mathcal{N}$ (hexagonal RVE) differing only in the geometrical distribution of the inclusions, therefore leading to the same equivalent local tensor, $\mathbf{C}^{eq}(\mathcal{M}) = \mathbf{C}^{eq}(\mathcal{N})$, but to different higher-order equivalent tensors, $\mathbf{A}^{eq}(\mathcal{M}) \neq \mathbf{A}^{eq}(\mathcal{N})$, see eqn (4.67).

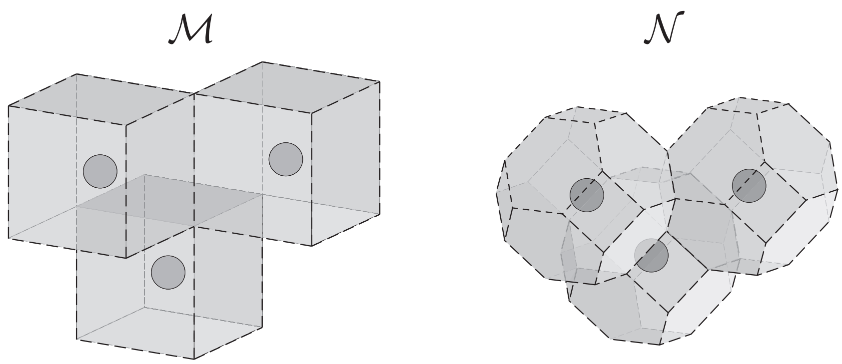

Figure 4.2: Similarly to Fig. 4.1, two composite materials $\mathcal{M}$ (cubic RVE) and $\mathcal{N}$ (truncated-octahedral RVE) leading to the same equivalent local tensor, $\mathbf{C}^{eq}(\mathcal{M}) = \mathbf{C}^{eq}(\mathcal{N})$, but to different higher-order equivalent tensors, $\mathbf{A}^{eq}(\mathcal{M}) \neq \mathbf{A}^{eq}(\mathcal{N})$, see eqn (4.68).

Chapter 5

Application cases

The explicit expression for the non-local constitutive tensor $\mathbf{A}^{eq}$ is exploited to evaluate the non-local parameters in the following cases: (i.) isotropic matrix with spherical elastic inclusions and voids for three-dimensional deformations, (ii.) isotropic matrix with circular elastic inclusions, regular n- polygonal and circular holes in plane strain and (iii) orthotropic matrix with circular holes in plane strain.

Cases where the homogenized material results isotropic are first considered and finally some cases of anisotropic behaviour are presented (cubic and orthotropic symmetries).

5.1 Equivalent isotropic SGE

For an isotropic composite, the first-order discrepancy tensor $\tilde{\mathbf{C}}$ is

$$\tilde{\mathbf{C}}^{iso}_{ijhk} = \tilde{\lambda}\delta_{ij}\delta_{hk} + \tilde{\mu}(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}), \tag{5.1}$$

so that the equivalent sixth-order tensor $\mathbf{A}^{eq}$, eqn (4.1), is given by

$$\begin{aligned} \mathbf{A}^{eq}_{ijhlmn} = \quad & -f\frac{\rho^2}{4}\Big\{\tilde{\lambda}\left[\delta_{ih}\left(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln}\right) + \delta_{jh}\left(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln}\right)\right] \\ & +\tilde{\mu}\left[2\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)\delta_{hn} + \delta_{in}\left(\delta_{jl}\delta_{hm} + \delta_{jm}\delta_{hl}\right)\right. \\ & \left.+\delta_{jn}\left(\delta_{il}\delta_{hm} + \delta_{im}\delta_{hl}\right)\right]\Big\}, \end{aligned} \tag{5.2}$$

which is a special case of isotropic sixth-order tensor

$$
\begin{aligned}
\mathsf{A}^{iso}_{ijhlmn} = \; & \frac{a_1}{2}\left[\delta_{ij}\left(\delta_{hl}\delta_{mn}+\delta_{hm}\delta_{ln}\right)+\delta_{lm}\left(\delta_{in}\delta_{jh}+\delta_{ih}\delta_{jn}\right)\right] \\
& +\frac{a_2}{2}\left[\delta_{ih}\left(\delta_{jl}\delta_{mn}+\delta_{jm}\delta_{ln}\right)+\delta_{jh}\left(\delta_{il}\delta_{mn}+\delta_{im}\delta_{ln}\right)\right] \\
& +2\,a_3\left(\delta_{ij}\delta_{hn}\delta_{lm}\right)+a_4\left(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}\right)\delta_{hn} \\
& +\frac{a_5}{2}\left[\delta_{in}\left(\delta_{jl}\delta_{hm}+\delta_{jm}\delta_{hl}\right)+\delta_{jn}\left(\delta_{il}\delta_{hm}+\delta_{im}\delta_{hl}\right)\right],
\end{aligned}
\tag{5.3}
$$

with the following constants

$$
a_1 = a_3 = 0, \qquad a_2 = -f\frac{\rho^2}{2}\tilde{\lambda}, \qquad a_4 = a_5 = -f\frac{\rho^2}{2}\tilde{\mu}. \tag{5.4}
$$

The related strain energy is positive definite when parameters a_i $(i = 1, ..., 5)$ satisfy eqn (18) of Part I, which for the values (5.4) implies

$$
\tilde{K} < 0, \quad \tilde{\mu} < 0, \tag{5.5}
$$

where $\tilde{K}$ is the bulk modulus, equal to $\tilde{\lambda}+2\tilde{\mu}/3$ in 3D and $\tilde{\lambda}+\tilde{\mu}$ in plane strain, and corresponding to the negative definiteness condition for $\tilde{\mathsf{C}}$, according to our previous results (Section 4.2.1).

An explicit evaluation of the constants (a_2, $a_4 = a_5$) is given now, in the case when an isotropic fourth-order tensor $\tilde{\mathsf{C}}$ is obtained from homogenization of a RVE with both isotropic phases, matrix denoted by '1' (with Lamé constants λ_1 and μ_1) and inclusion denoted by '2' (with Lamé constants λ_2 and μ_2), having a shape leading to an isotropic equivalent constitutive tensor

$$
\mathsf{C}^{eq}_{ijhk} = \lambda_{eq}\delta_{ij}\delta_{hk} + \mu_{eq}(\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh}), \tag{5.6}
$$

where

$$
\lambda_{eq} = \lambda_1 + f\tilde{\lambda}, \qquad \mu_{eq} = \mu_1 + f\tilde{\mu}, \qquad K_{eq} = K_1 + f\tilde{K}. \tag{5.7}
$$

In particular, the following forms of inclusions are considered within an isotropic matrix.

- For 3D deformation:
  - spherical elastic inclusions.
- For plane strain:
  - circular elastic inclusions;

- regular n-polygonal holes with $n \neq 4$ (the case $n = 4$ leads to an orthotropic material and is treated in the next section).

For all of the above cases it is shown that a positive definite equivalent SGE material, eqn (5.5), is obtained only when the inclusion phase is 'softer' than the matrix in terms of *both* shear and bulk moduli,

$$\mu_2 < \mu_1, \quad K_2 < K_1, \tag{5.8}$$

which is always satisfied when the inclusions are voids. The positive definiteness condition (5.8) can be written in terms of the ratio μ_2/μ_1 and the Poisson's ratio of the phases ν_1 and ν_2 [where $\nu_i = \lambda_i/(2(\lambda_i + \mu_i))$] as

$$\frac{\mu_2}{\mu_1} < \min\left\{1; \frac{1-2\nu_2}{1-2\nu_1}\right\}, \tag{5.9}$$

for the case of plane strain, and

$$\frac{\mu_2}{\mu_1} < \min\left\{1; \frac{(1+\nu_1)(1-2\nu_2)}{(1+\nu_2)(1-2\nu_1)}\right\}, \tag{5.10}$$

for three-dimensional case. The regions where a positive definite SGE material is obtained, eqns (5.9) - (5.10), are mapped in the plane $\mu_2/\mu_1 - \nu_1$ for different values of the inclusion Poisson's ratio ν_2 (Fig. 5.1, plane strain on the left and 3D-deformation on the right).

Cylindrical elastic inclusions The elastic constants K_{eq} and μ_{eq} of the isotropic material equivalent to a dilute suspension of parallel isotropic cylindrical inclusions embedded in an isotropic matrix have been obtained by Hashin and Rosen (1964), in our notation

$$\tilde{K} = \frac{(K_2 - K_1)(K_1 + \mu_1)}{K_2 + \mu_1}, \qquad \tilde{\mu} = \frac{2\mu_1(\mu_2 - \mu_1)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)}. \tag{5.11}$$

Exploiting equation (5.4), the equivalent higher-order constants a_i ($i = 1, ..., 5$) can be obtained from the first-order discrepancy quantities, eqn (5.11), so that the non-null constants are evaluated as

$$\begin{aligned} a_2 &= f\frac{\rho^2}{2}\left[\frac{(K_1 - K_2)(K_1 + \mu_1)}{K_2 + \mu_1} - \frac{\mu_1(\mu_1 - \mu_2)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)}\right], \\ a_4 = a_5 &= f\frac{\rho^2}{2}\frac{\mu_1(\mu_1 - \mu_2)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)}. \end{aligned} \tag{5.12}$$

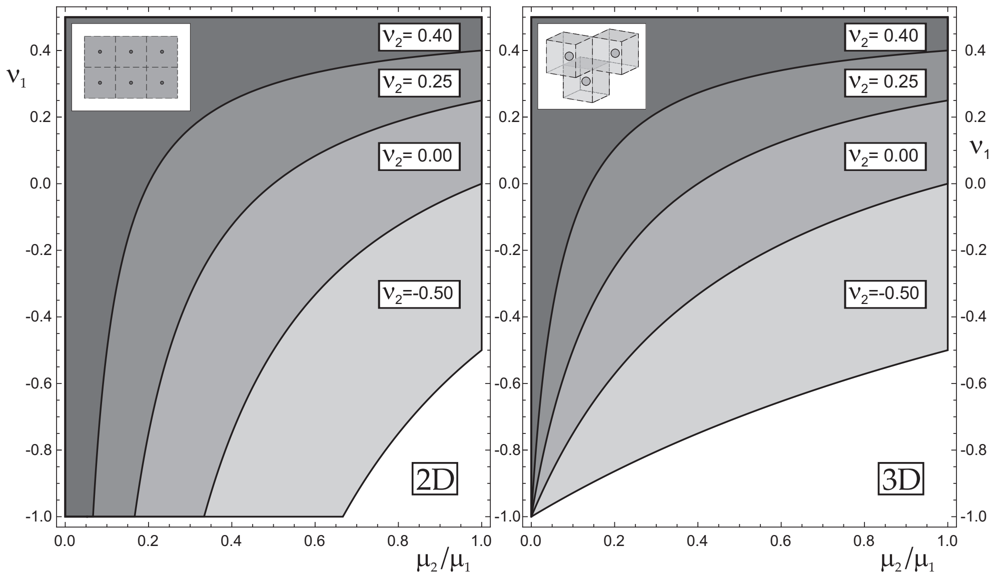

Figure 5.1: Regions in the plane $\mu_2/\mu_1 - \nu_1$ where the higher-order effective constitutive tensor $\mathbf{A}^{eq}$ is positive definite (for different values of ν_2). The regions for the plane strain case, eqn (5.9), are reported on the left, while the case of three-dimensional deformations, eqn (5.10), is reported on the right.

The higher-order equivalent constants a_2 and a_4 given by eqn (5.12) are reported in Figs. 5.2 and 5.3 as a function of the ratio μ_2/μ_1 and for different Poisson's ratios of matrix and inclusion. In all the figures, a red spot denotes the threshold for which the strain energy of the equivalent material looses positive definiteness. The dashed curves refer to regions where this positive definiteness is lost.

With reference to Fig. 5.2, we may note that $a_2 \to \infty$ in the limit $\nu_1 \to 1/2$. Furthermore, a_4 is not affected by the Poisson's ratio of the inclusion ν_2, except that the threshold for positive definiteness condition for the equivalent material strain energy of the changes, eqn (5.9).

Spherical elastic inclusions The equivalent elastic constants K_{eq} and μ_{eq} of the isotropic material equivalent to a dilute suspension of isotropic spherical inclusions within an isotropic matrix have been obtained by Eshelby (1957) and independently by Hashin (1959), in our notation

$$\begin{aligned}\tilde{K} &= \frac{(3K_1+4\mu_1)(K_2-K_1)}{3K_2+4\mu_1},\\ \tilde{\mu} &= \frac{5\mu_1(\mu_2-\mu_1)(3K_1+4\mu_1)}{\mu_1(3K_1+4\mu_2)+2(3K_1+4\mu_1)(\mu_2+\mu_1)},\end{aligned} \tag{5.13}$$

so that, through equation (5.4), the non-null equivalent higher-order constants are given by

$$\begin{aligned}a_2 =\ & f\frac{\rho^2}{2}\left[\frac{(3K_1+4\mu_1)(K_2-K_1)}{3K_2+4\mu_1}\right.\\ & \left.-\frac{2}{3}\frac{5\mu_1(\mu_2-\mu_1)(3K_1+4\mu_1)}{\mu_1(3K_1+4\mu_2)+2(3K_1+4\mu_1)(\mu_2+\mu_1)}\right],\\ a_4 =\ & a_5 = f\frac{\rho^2}{2}\frac{5\mu_1(\mu_2-\mu_1)(3K_1+4\mu_1)}{\mu_1(3K_1+4\mu_2)+2(3K_1+4\mu_1)(\mu_2+\mu_1)},\end{aligned} \tag{5.14}$$

which are reported in Fig. 5.4 and Fig. 5.5 as a function of the shear stiffness ratio μ_2/μ_1 and for different Poisson's ratios of the phases. In these figures the curves become dashed when the strain energy of the equivalent material looses positive definiteness. Moreover, the higher-order constants are reported in Fig. 5.6 as a function of the matrix Poisson's ratio ν_1 in the particular case of spherical voids.

Similar to the case of cylindrical elastic inclusions, $a_2 \to \infty$ in the limit $\nu_1 \to 1/2$ and a_4 is not affected by the Poisson's ratio of the inclusion ν_2, except for the threshold of strain energy's positive definiteness, eqn. (5.10).

Regular n-polygonal holes ($n \neq 4$) The elastic constants μ_{eq} and K_{eq} of the isotropic material equivalent to a dilute suspension of n-polygonal holes ($n \neq 4$) in an isotropic matrix have been obtained by Jasiuk et al. (1994) and Thorpe et al. (1995), from which the first-order discrepancy stiffness can be written in our notation as

$$\tilde{K}(n) = -\mathcal{A}(n)[1-\mathcal{B}(n)]\frac{K_1+\mu_1}{\mu_1}K_1, \qquad \tilde{\mu}(n) = -\mathcal{A}(n)[1+\mathcal{B}(n)]\frac{K_1+\mu_1}{K_1}\mu_1, \tag{5.15}$$

where $\mathcal{A}(n)$ and $\mathcal{B}(n)$ are constants depending on the number of edges n of the regular polygonal hole, which can be approximated through numerical computations, and are reported in Tab. 5.1 for n=$\{3;5;6\}$. In the case of a regular polygon with infinite number of edges, in other words a circle, the value of the constants is $\mathcal{A}(n \to \infty) = 3/2$ and $\mathcal{B}(n \to \infty) = 1/3$, so that the case of cylindrical void inclusion is recovered, eqn (5.11) with $\mu_2 = K_2 = 0$. The equivalent higher-order constants can be obtained from eqn (5.4) by using the first-order discrepancy quantities, eqn (5.15), from which the non-null constants follow

$$
\begin{aligned}
a_2 &= f\frac{\rho^2}{2}\mathcal{A}(n)\left\{[1-\mathcal{B}(n)]K_1^2 - [1+\mathcal{B}(n)]\mu_1^2\right\}\frac{K_1+\mu_1}{\mu_1 K_1}, \\
a_4 = a_5 &= f\frac{\rho^2}{2}\mathcal{A}(n)[1+\mathcal{B}(n)]\frac{K_1+\mu_1}{K_1}\mu_1,
\end{aligned}
\tag{5.16}
$$

and are shown in Fig. 5.7 as functions of the matrix Poisson's ratio ν_1.

		Approximated values	
Polygonal hole	n	$\mathcal{A}(n)$	$\mathcal{B}(n)$
Triangle	3	2.1065	0.2295
Pentagon	5	1.6198	0.3233
Hexagon	6	1.5688	0.3288
Circle	∞	3/2	1/3

Tab. 5.1: Values of the constants $\mathcal{A}(n)$ and $\mathcal{B}(n)$ for triangular ($n = 3$), pentagonal ($n = 5$), hexagonal ($n = 6$), and circular ($n \to \infty$) holes in an isotropic elastic matrix (Thorpe et al., 1995). These values are instrumental to obtain the equivalent properties $\tilde{K}(n)$ and $\tilde{\mu}(n)$, eqn (5.15), of the higher-order material.

5.2 Equivalent cubic SGE

When the first-order discrepancy tensor $\tilde{\mathsf{C}}$ has a cubic symmetry, it can be represented in a cartesian system aligned parallel to the symmetry axes as (Thomas, 1966)

$$
\begin{aligned}
\tilde{\mathsf{C}}^{cub}_{ijhk} = \ & \tilde{\mathsf{C}}^{iso}_{ijhk} + \tilde{\xi}\left[(\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})(\delta_{h2}\delta_{k3} + \delta_{h3}\delta_{k2})\right. \\
& + (\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})(\delta_{h1}\delta_{k3} + \delta_{h3}\delta_{k1}) \\
& \left. + (\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(\delta_{h1}\delta_{k2} + \delta_{h2}\delta_{k1})\right],
\end{aligned}
\tag{5.17}
$$

where $\tilde{\mathbf{C}}^{iso}$ is given by eqn (5.1). The sixth-order tensor $\mathbf{A}^{eq}$ for the equivalent material is obtained using eqn (4.1) in the form

$$\begin{aligned}\mathbf{A}^{eq}_{ijhlmn} = \; & \mathbf{A}^{iso}_{ijhlmn} + \frac{a_6}{2}\left[(\delta_{i1}\delta_{h2} + \delta_{i2}\delta_{h1})(\delta_{l1}\delta_{n2} + \delta_{l2}\delta_{n1})\,\delta_{jm}\right. \\ & + (\delta_{j1}\delta_{h2} + \delta_{j2}\delta_{h1})(\delta_{m1}\delta_{n2} + \delta_{m2}\delta_{n1})\,\delta_{jl} \\ & + (\delta_{j1}\delta_{h2} + \delta_{j2}\delta_{h1})(\delta_{l1}\delta_{n2} + \delta_{l2}\delta_{n1})\,\delta_{im} \\ & \left. + (\delta_{j1}\delta_{h2} + \delta_{j2}\delta_{h1})(\delta_{m1}\delta_{n2} + \delta_{m2}\delta_{n1})\,\delta_{il}\right],\end{aligned} \tag{5.18}$$

with $\mathbf{A}^{iso}$ given by eqn (5.3), parameters a_i $(i = 1, ..., 5)$ by eqn (5.4), and

$$a_6 = -f\frac{\rho^2}{2}\tilde{\xi}. \tag{5.19}$$

According to results presented in sections 4.2.1 and 4.2.2, the effective higher-order tensor $\mathbf{A}^{eq}$ results to be a cubic sixth-order tensor and is positive definite when $\tilde{\mathbf{C}}$, eqn (5.17), is negative definite, namely, eqn (5.5) together with

$$\tilde{\xi} + \tilde{\mu} < 0. \tag{5.20}$$

Aligned square holes within an isotropic matrix There are no results available for the plane strain homogenization of a dilute suspension of square holes periodically distributed (with parallel edges) within an isotropic matrix. Therefore, we have compared with a conformal mapping technique (Misseroni et al. 2013) stress and strain averages, and found the following discrepancy at first-order in the constitutive quantities[1]

$$\begin{aligned}\tilde{\lambda} &= -(1.198K_1^2 - 1.864\mu_1^2)\frac{K_1 + \mu_1}{K_1\mu_1}, \\ \tilde{\mu} &= -1.864\frac{K_1 + \mu_1}{K_1}\mu_1, \\ \tilde{\xi} &= -0.796\frac{K_1 + \mu_1}{K_1}\mu_1,\end{aligned} \tag{5.21}$$

showing that $\tilde{\mathbf{C}}$ is negative definite, eqn (5.20), and therefore the corresponding effective higher-order tensor $\mathbf{A}^{eq}$, eqn (5.18), is positive definite.

[1] Thorpe et al. (1995) give results for composites with a random orientation of square holes, so that the effective behaviour is isotropic and given by eqn (5.15) with $\mathcal{A}(n = 4) = 1.738$ and $\mathcal{B}(n = 4) = 0.306$. This isotropic effective response can be independently obtained by averaging the cubic effective response given by eqn (5.21) over two orientations of the square hole differing by an angle $\pi/4$.

The equivalent higher-order constants a_i $(i = 1, ..., 6)$ can be obtained from the first-order discrepancy quantities, eqn (5.21), so that the non-null constants are evaluated by exploiting eqns (5.4) and (5.19) as

$$
\begin{aligned}
a_2 &= f\rho^2\,(0.599K_1^2 - 0.932\mu_1^2)\frac{K_1+\mu_1}{K_1\mu_1},\\
a_4 &= a_5 = 0.932 f\rho^2\frac{K_1+\mu_1}{K_1}\mu_1,\\
a_6 &= 0.398 f\rho^2\frac{K_1+\mu_1}{K_1}\mu_1.
\end{aligned}
\tag{5.22}
$$

These three independent constants are reported in Fig. 5.8 as functions of the matrix Poisson's ratio ν_1.

5.3 Equivalent orthotropic SGE

When the first-order discrepancy tensor $\tilde{\mathbf{C}}$ is orthotropic, it can be represented in a cartesian system aligned parallel to the symmetry axes as (Spencer, 1982)

$$
\begin{aligned}
\tilde{\mathsf{C}}^{orth}_{ijhk} = \;& \tilde{\mathsf{C}}^{iso}_{ijhk} + \tilde{\xi}_I\,(\delta_{i2}\delta_{j3}+\delta_{i3}\delta_{j2})\,(\delta_{h2}\delta_{k3}+\delta_{h3}\delta_{k2})\\
&+\tilde{\xi}^{II}\,(\delta_{i1}\delta_{j3}+\delta_{i3}\delta_{j1})\,(\delta_{h1}\delta_{k3}+\delta_{h3}\delta_{k1})\\
&+\tilde{\xi}^{III}\,(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})\,(\delta_{h1}\delta_{k2}+\delta_{h2}\delta_{k1})\\
&+\tilde{\omega}_I\delta_{i1}\delta_{j1}\delta_{h1}\delta_{k1}+\tilde{\omega}^{II}\delta_{i3}\delta_{j3}\delta_{h3}\delta_{k3}\\
&+\tilde{\omega}^{III}\,(\delta_{ij}\delta_{h3}\delta_{k3}+\delta_{hk}\delta_{i3}\delta_{j3})\\
&+\tilde{\omega}^{IV}\,(\delta_{i1}\delta_{j1}\delta_{h3}\delta_{k3}+\delta_{i3}\delta_{j3}\delta_{h1}\delta_{k1})\,,
\end{aligned}
\tag{5.23}
$$

where $\tilde{\xi}^{III}$, $\tilde{\omega}^{I}$, $\tilde{\xi}^{I}$, $\tilde{\xi}^{II}$, $\tilde{\omega}^{II}$, $\tilde{\omega}^{III}$ and $\tilde{\omega}^{IV}$ are seven independent constants (in addition to $\tilde{\lambda}$ and $\tilde{\mu}$) defining the orthotropic behaviour in 3D.[2] The in-plane behaviour is defined by groups of four independent constants, which for the x_1–x_2 plane are $\{\tilde{\lambda};\tilde{\mu};\tilde{\xi}^{III};\tilde{\omega}^{I}\}$.

In the case of orthotropic $\tilde{\mathbf{C}}$, eqn (4.1) defining the sixth-order nonlocal tensor $\mathbf{A}^{eq}$ leads to

[2] Note that the cubic representation (5.17) is obtained as a particular case by setting $\tilde{\xi}^{I} = \tilde{\xi}^{II} = \tilde{\xi}^{I} = \tilde{\xi}$ and $\tilde{\omega}^{I} = \tilde{\omega}^{II} = \tilde{\omega}^{III} = \tilde{\omega}^{IV} = 0$.

$$
\begin{aligned}
\mathsf{A}^{eq}_{ijhlmn} = \quad & \mathsf{A}^{iso}_{ijhlmn} \\
& + \frac{a_6}{2}\left[\left(\delta_{i1}\delta_{h2}+\delta_{i2}\delta_{h1}\right)\left(\delta_{l1}\delta_{n2}+\delta_{l2}\delta_{n1}\right)\delta_{jm}\right. \\
& + \left(\delta_{j1}\delta_{h2}+\delta_{j2}\delta_{h1}\right)\left(\delta_{m1}\delta_{n2}+\delta_{m2}\delta_{n1}\right)\delta_{jl} \\
& + \left(\delta_{j1}\delta_{h2}+\delta_{j2}\delta_{h1}\right)\left(\delta_{l1}\delta_{n2}+\delta_{l2}\delta_{n1}\right)\delta_{im} \\
& \left. + \left(\delta_{j1}\delta_{h2}+\delta_{j2}\delta_{h1}\right)\left(\delta_{m1}\delta_{n2}+\delta_{m2}\delta_{n1}\right)\delta_{il}\right] \\
& + \frac{a_7}{2}\left[\left(\delta_{i1}\delta_{h3}+\delta_{i3}\delta_{h1}\right)\left(\delta_{l1}\delta_{n3}+\delta_{l3}\delta_{n1}\right)\delta_{jm}\right. \\
& + \left(\delta_{i1}\delta_{h3}+\delta_{i3}\delta_{h1}\right)\left(\delta_{m1}\delta_{n3}+\delta_{m3}\delta_{n1}\right)\delta_{jl} \\
& + \left(\delta_{j1}\delta_{h3}+\delta_{j3}\delta_{h1}\right)\left(\delta_{l1}\delta_{n3}+\delta_{l3}\delta_{n1}\right)\delta_{im} \\
& \left. + \left(\delta_{j1}\delta_{h3}+\delta_{j3}\delta_{h1}\right)\left(\delta_{m1}\delta_{n3}+\delta_{m3}\delta_{n1}\right)\delta_{il}\right] \\
& + \frac{a_8}{2}\left[\left(\delta_{i2}\delta_{h3}+\delta_{i3}\delta_{h2}\right)\left(\delta_{l2}\delta_{n3}+\delta_{l3}\delta_{n2}\right)\delta_{jm}+\right. \\
& \left(\delta_{i2}\delta_{h3}+\delta_{i3}\delta_{h2}\right)\left(\delta_{m2}\delta_{n3}+\delta_{m3}\delta_{n2}\right)\delta_{jl} \\
& + \left(\delta_{j2}\delta_{h3}+\delta_{j3}\delta_{h2}\right)\left(\delta_{l2}\delta_{n3}+\delta_{l3}\delta_{n2}\right)\delta_{im}+ \\
& \left.\left(\delta_{j2}\delta_{h3}+\delta_{j3}\delta_{h2}\right)\left(\delta_{m2}\delta_{n3}+\delta_{m3}\delta_{n2}\right)\delta_{il}\right] \\
& + \frac{a_9}{2}\left(\delta_{i1}\delta_{h1}\delta_{l1}\delta_{n1}\delta_{jm}+\delta_{i1}\delta_{h1}\delta_{m1}\delta_{n1}\delta_{jl}\right. \\
& \left. +\delta_{j1}\delta_{h1}\delta_{l1}\delta_{n1}\delta_{im}+\delta_{j1}\delta_{h1}\delta_{m1}\delta_{n1}\delta_{il}\right) \\
& + \frac{a_{10}}{2}\left(\delta_{i3}\delta_{h3}\delta_{l3}\delta_{n3}\delta_{jm}+\delta_{j3}\delta_{h3}\delta_{m3}\delta_{n3}\delta_{jl}\right. \\
& \left. +\delta_{j3}\delta_{h3}\delta_{l3}\delta_{n3}\delta_{im}+\delta_{j3}\delta_{h3}\delta_{m3}\delta_{n3}\delta_{il}\right) \\
& + \frac{a_{11}}{2}\left[\left(\delta_{ih}\delta_{l3}\delta_{n3}+\delta_{ln}\delta_{i3}\delta_{h3}\right)\delta_{jm}+\left(\delta_{ih}\delta_{m3}\delta_{n3}+\delta_{mn}\delta_{i3}\delta_{h3}\right)\delta_{jl}\right. \\
& \left. + \left(\delta_{ih}\delta_{l3}\delta_{n3}+\delta_{ln}\delta_{j3}\delta_{h3}\right)\delta_{im}+\left(\delta_{jh}\delta_{m3}\delta_{n3}+\delta_{mn}\delta_{j3}\delta_{h3}\right)\delta_{il}\right] \\
& + \frac{a_{12}}{2}\left[\left(\delta_{i1}\delta_{h1}\delta_{l3}\delta_{n3}+\delta_{i3}\delta_{h3}\delta_{l1}\delta_{n1}\right)\delta_{jm}\right. \\
& + \left(\delta_{i1}\delta_{h1}\delta_{m3}\delta_{n3}+\delta_{i3}\delta_{h3}\delta_{m1}\delta_{n1}\right)\delta_{jl} \\
& + \left(\delta_{j1}\delta_{h1}\delta_{l3}\delta_{n3}+\delta_{j3}\delta_{h3}\delta_{l1}\delta_{n1}\right)\delta_{im} \\
& \left. + \left(\delta_{j1}\delta_{h1}\delta_{m3}\delta_{n3}+\delta_{j3}\delta_{h3}\delta_{m1}\delta_{n1}\right)\delta_{il}\right],
\end{aligned}
\tag{5.24}
$$

with $\mathbf{A}^{iso}$ given by eqn (5.3), parameters a_i $(i = 1, ..., 5)$ by eqn (5.4), and

$$a_6 = -f\frac{\rho^2}{2}\tilde{\xi}^{III}, \qquad a_7 = -f\frac{\rho^2}{2}\tilde{\xi}^{II}, \qquad a_8 = -f\frac{\rho^2}{2}\tilde{\xi}^{I},$$
$$a_9 = -f\frac{\rho^2}{2}\tilde{\omega}^{I}, \qquad a_{10} = -f\frac{\rho^2}{2}\tilde{\omega}^{II}, \qquad a_{11} = -f\frac{\rho^2}{2}\tilde{\omega}^{III}, \qquad a_{12} = -f\frac{\rho^2}{2}\tilde{\omega}^{IV}, \tag{5.25}$$

According to the results presented in sections 4.2.1 and 4.2.2, the effective higher-order tensor $\mathbf{A}^{eq}$ results to be an orthotropic sixth-order tensor, positive definite when $\tilde{\mathbf{C}}$, eqn (5.23), is negative definite, namely

$$\begin{cases} \tilde{\mu} + \tilde{\xi}^{III} < 0, \\ \tilde{\mu} + \tilde{\xi}^{II} < 0, \\ \tilde{\mu} + \tilde{\xi}^{I} < 0, \\ \tilde{\lambda} + 2\tilde{\mu} + \tilde{\omega}_I < 0, \\ 4\tilde{\mu}(\tilde{\lambda} + \tilde{\mu}) + (\tilde{\lambda} + 2\tilde{\mu})\tilde{\omega}_I < 0, \\ 8\tilde{\mu}^3 - \tilde{\omega}_I\tilde{\omega}^{III\,2} + 4\tilde{\mu}^2(\tilde{\omega}^I + \tilde{\omega}^{II} + 2\tilde{\omega}^{III}) \\ +\tilde{\lambda}\left(12\tilde{\mu}^2 + \tilde{\omega}_I\tilde{\omega}^{II} + 4\tilde{\mu}(\tilde{\omega}^I + \tilde{\omega}^{II} - \tilde{\omega}^{IV}) - \tilde{\omega}^{IV\,2}\right) \\ -2\tilde{\mu}\left(2\tilde{\omega}^{III\,2} - \tilde{\omega}_I(\tilde{\omega}^{II} + 2\tilde{\omega}^{III}) + 2\tilde{\omega}^{III}\tilde{\omega}^{IV} + \tilde{\omega}^{IV\,2}\right) < 0, \end{cases} \tag{5.26}$$

while in the case of plane strain, conditions (5.26) become, in the x_1–x_2 plane

$$\begin{cases} \tilde{\mu} + \tilde{\xi}^{III} < 0, \\ \tilde{\lambda} + 2\tilde{\mu} + \tilde{\omega}_I < 0, \\ 4\tilde{\mu}(\tilde{\lambda} + \tilde{\mu}) + (\tilde{\lambda} + 2\tilde{\mu})\tilde{\omega}_I < 0. \end{cases} \tag{5.27}$$

Orthotropic matrix with cylindrical holes We consider the plane strain of an orthotropic matrix containing a dilute suspension of circular holes with centers aligned parallel to the orthotropy symmetry axes. In particular, assuming x_3 as the out-of-plane direction and x_1 and x_2 as the orthotropy axes, the discrepancy tensor has the form (5.23) and is characterized by the following constants [3] (Tsukrov and Kachanov, 2000)

$$
\begin{aligned}
\tilde{\lambda} = \quad & \frac{\gamma(\lambda_1 + 2\mu_1)}{[(-1+\gamma)\lambda_1 + 2\gamma\mu_1]\,(\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)} \times \\
& \left\{ \left[(-1+\gamma)^2 - (1+\gamma)\delta\right] \lambda_1^2 + 2\left[2(-1+\gamma)\gamma - (1+\gamma)\delta\right] \lambda_1\mu_1 + 4\gamma^2\mu_1^2 \right\}, \\
\\
\tilde{\mu} = \quad & \frac{-\,(\lambda_1 + 2\mu_1)}{2\left[(-1+\gamma)\lambda_1 + 2\gamma\mu_1\right](\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)} \times \\
& \Big[\left(-1+\gamma^2\right)(-1+\gamma-\delta)\lambda_1^2 + 2(-1+\gamma)\gamma(2+2\gamma-\delta)\lambda_1\mu_1 \\
& +4\gamma\left(\gamma+\gamma^2+\delta\right)\mu_1^2\Big], \\
\\
\tilde{\xi} = \quad & -\tilde{\mu} - \frac{\delta(1+\gamma+\delta)(\lambda_1+2\mu_1)\left[(-1+\gamma)\lambda_1 + 2\gamma\mu_1\right](\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)}{\left[\left(-2+2\gamma-\delta^2\right)\lambda_1 + 4\gamma\mu_1 - 2\delta^2\mu_1\right]^2}. \\
\\
\tilde{\omega} = \quad & -\tilde{\mu} - \frac{\gamma\,(\lambda_1 + 2\mu_1)}{2\left[(-1+\gamma)\lambda_1 + 2\gamma\mu_1\right](\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)} \times \\
& \Big\{\left(-1+\gamma^2\right)(-1+\gamma+\gamma\delta)\lambda_1^2 + 2(-1+\gamma)\left[\delta + 2\gamma(1+\gamma)(1+\delta)\right]\lambda_1\mu_1 \\
& +4\gamma^2(1+\gamma+\gamma\delta)\mu_1^2\Big\},
\end{aligned}
\tag{5.28}
$$

where

$$
\gamma = \sqrt{\Gamma^2 - \Delta}, \qquad \delta = \sqrt{\Gamma + \sqrt{\Delta}} + \sqrt{\Gamma - \sqrt{\Delta}},
$$

$$
\Gamma = \frac{2\mu_1(\mu_1+\omega_1) + \lambda_1(\mu_1 - \xi_1 + \omega_1)}{(\lambda_1+2\mu_1)(\mu_1+\xi_1)},
$$

$$
\Delta = \frac{[-2\xi_1(\lambda_1 + 2\mu_1 + \xi_1) + (\lambda_1+2\mu_1)\omega_1][2\mu_1(\mu_1+\omega_1) + \lambda_1(2\mu_1+\omega_1)]}{(\lambda_1+2\mu_1)^2(\mu_1+\xi_1)^2}.
\tag{5.29}
$$

The non-null constants a_2, $a_4 = a_5$, a_6, and a_9 defining the effective higher-order tensor $\mathbf{A}^{eq}$ can explicitly be evaluated using eqns (5.4) and (5.25), when a specific orthotropic matrix is considered. With reference to orthotropic

[3] For conciseness, in this section the in-plane orthotropy parameters ξ^{III} and ω^{I} are denoted by ξ and ω, respectively, in the representation of both matrix and discrepancy quantities.

properties of olivine, pine wood, olivinite, marble, and canine femora (which orthotropic constitutive parameters are reported in Tab. 5.2 for the three possible orientations of orthotropy) used as matrix material, the corresponding non-null higher-order constants are given in Tab. 5.3 for a dilute suspension of cylindrical holes with centers aligned parallel to the in-plane orthotropy axes. All the three possible orientations (Or1, Or2, Or3) are considered for the axis of the cylindrical inclusion, defining the out-of-plane direction in the plane strain problem considered.

Matrix material	Orientation	λ_1	μ_1	ξ_1	ω_1
Olivine	Or1	66.000	47.000	−17.000	32.000
	Or2	60.000	106.000	−75.000	−80.000
	Or3	56.000	52.000	−27.500	112.000
Pine (softwood)	Or1	0.740	8.180	−7.590	−15.860
	Or2	0.760	0.515	−0.476	−0.550
	Or3	0.940	8.080	−7.625	−15.310
Olivinite	Or1	93.000	58.500	−21.85	22.000
	Or2	92.000	53.500	−18.05	33.000
	Or3	82.000	64.000	−29.7	−11.000
Marble	Or1	51.000	29.500	−14.65	9.000
	Or2	52.000	26.000	−10.65	15.000
	Or3	47.000	31.500	−15.2	−6.000
Canine femora	Or1	9.730	6.235	−2.900	−3.200
	Or2	11.900	8.900	−6.065	−10.700
	Or3	11.900	5.150	−2.815	7.500

Tab. 5.2: Values of the elastic constants $\lambda_1, \mu_1, \xi_1, \omega_1$ for different orthotropic materials, namely: olivine (Chevrot and Browaeys, 2004), pine wood (Yamai, 1957), olivinite, marble (Aleksandrov, Ryzhove and Belikov, 1968), and canine femora (Cowin and Van Buskirk, 1986). The reported values are in GPa.

Matrix material	Orientation	$\dfrac{a_2}{f\rho^2\mu_1}$	$\dfrac{a_4}{f\rho^2\mu_1}$	$\dfrac{a_6}{f\rho^2\mu_1}$	$\dfrac{a_9}{f\rho^2\mu_1}$
Olivine	Or1	2.426	1.661	3.077	-1.198
	Or2	1.133	2.105	-1.014	-1.804
	Or3	3.254	1.497	0.858	-0.780
Pine wood	Or1	0.269	3.789	-3.754	-3.737
	Or2	10.297	3.551	-3.268	-3.497
	Or3	0.142	3.478	-3.455	-3.399
Olivinite	Or1	3.119	1.644	-0.220	-1.045
	Or2	4.398	1.414	0.804	-0.675
	Or3	4.011	1.481	0.487	-0.782
Marble	Or1	4.023	1.629	-0.257	-1.068
	Or2	5.866	1.389	0.823	-0.768
	Or3	5.080	1.532	0.440	-1.015
Canine femora	Or1	8.279	1.219	2.465	-0.801
	Or2	4.401	2.110	-1.875	-1.788
	Or3	4.273	1.660	-0.690	-1.063

Tab. 5.3: Higher-order equivalent constants a_2, $a_4 = a_5$, a_6, and a_9, eqns (5.4) and (5.25), of the orthotropic SGE material equivalent to an orthotropic matrix containing a dilute suspension of cylindrical holes, collinear to three possible orientations of orthotropy. The constants are made dimensionless through division by parameter $f\rho^2\mu_1$ and are reported for different matrices, which orthotropy parameters are given in Tab. 5.2.

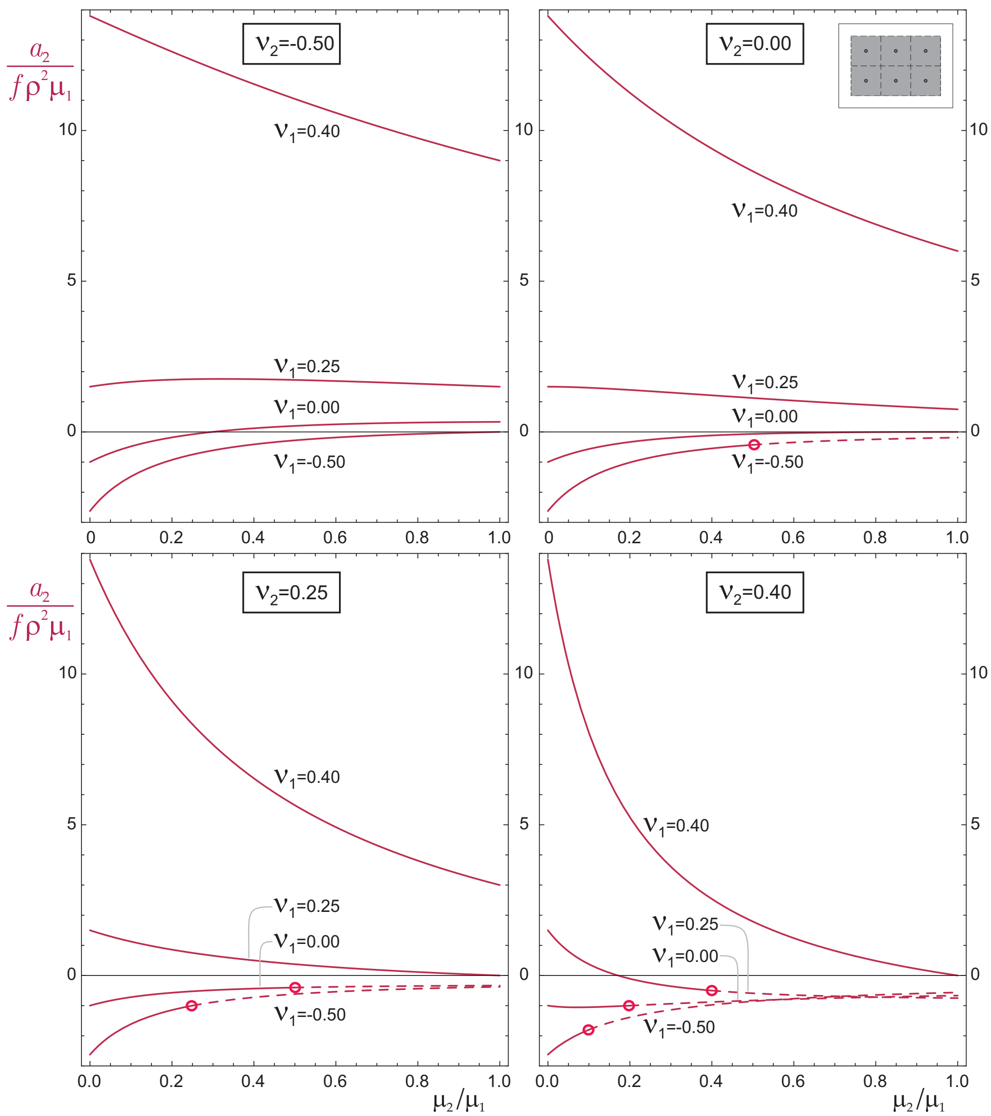

Figure 5.2: Higher-order equivalent constant a_2, eqn $(5.12)_1$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a diluite suspension of cylindrical elastic inclusions, as a function of the ratio μ_2/μ_1, for different values of the Poisson's ratio of the phases $\{\nu_1, \nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_2 is made dimensionless through division by parameter $f\rho^2\mu_1$. The curves are dashed where the strain energy of the equivalent material is not positive definite, a red spot marks where the positive definiteness loss of $\mathbf{A}^{eq}$ occurs.

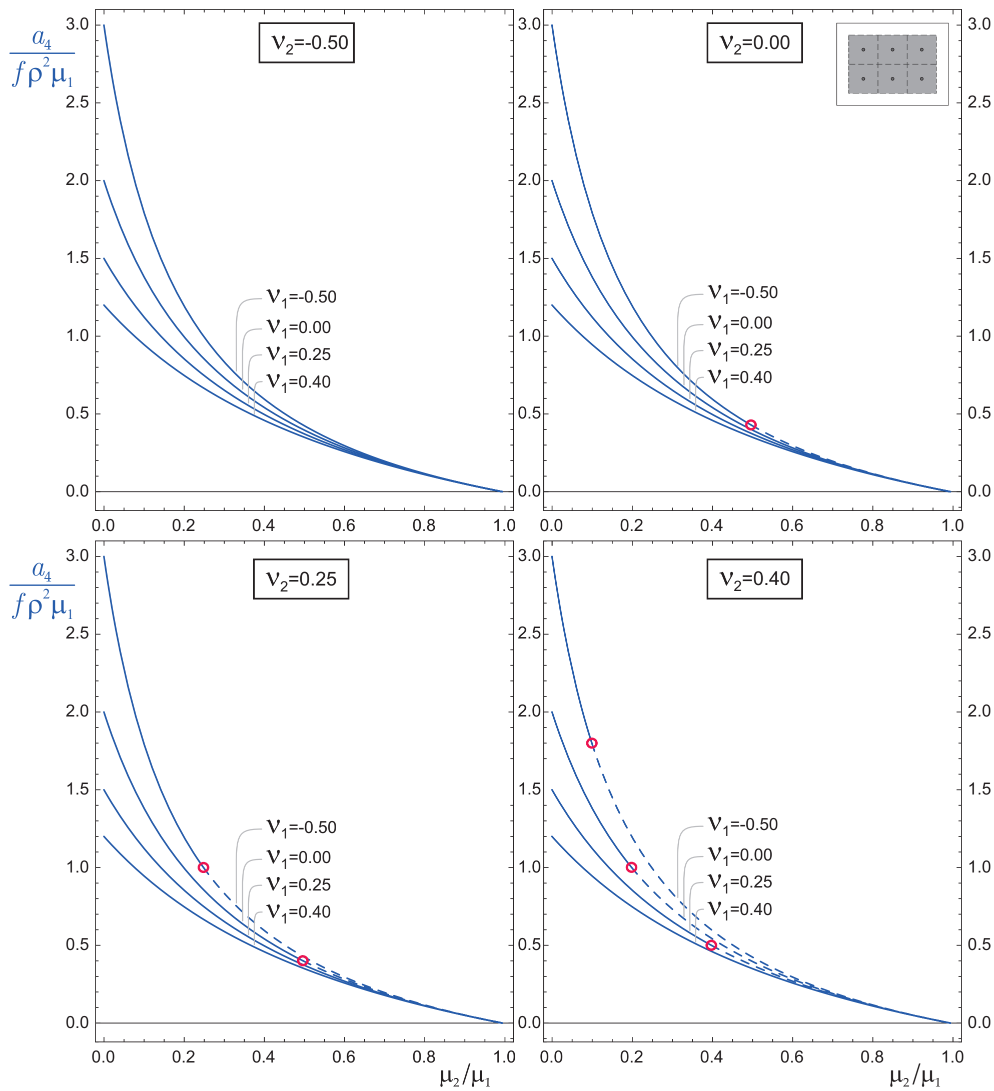

Figure 5.3: Higher-order equivalent constant $a_4 = a_5$, eqn $(5.12)_2$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of cylindrical elastic inclusions, as a function of the ratio μ_2/μ_1, for different values of Poisson's ratio of the phases $\{\nu_1, \nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_4 is made dimensionless through division by parameter $f\rho^2\mu_1$. Note that the curves are not affected by the Poisson's ratio of the inclusion ν_2, except that the threshold (red spot) for positive definiteness of the equivalent material strain energy changes, eqn (5.9). Dashed curve represents values for which the strain energy of the equivalent material is not positive definite.

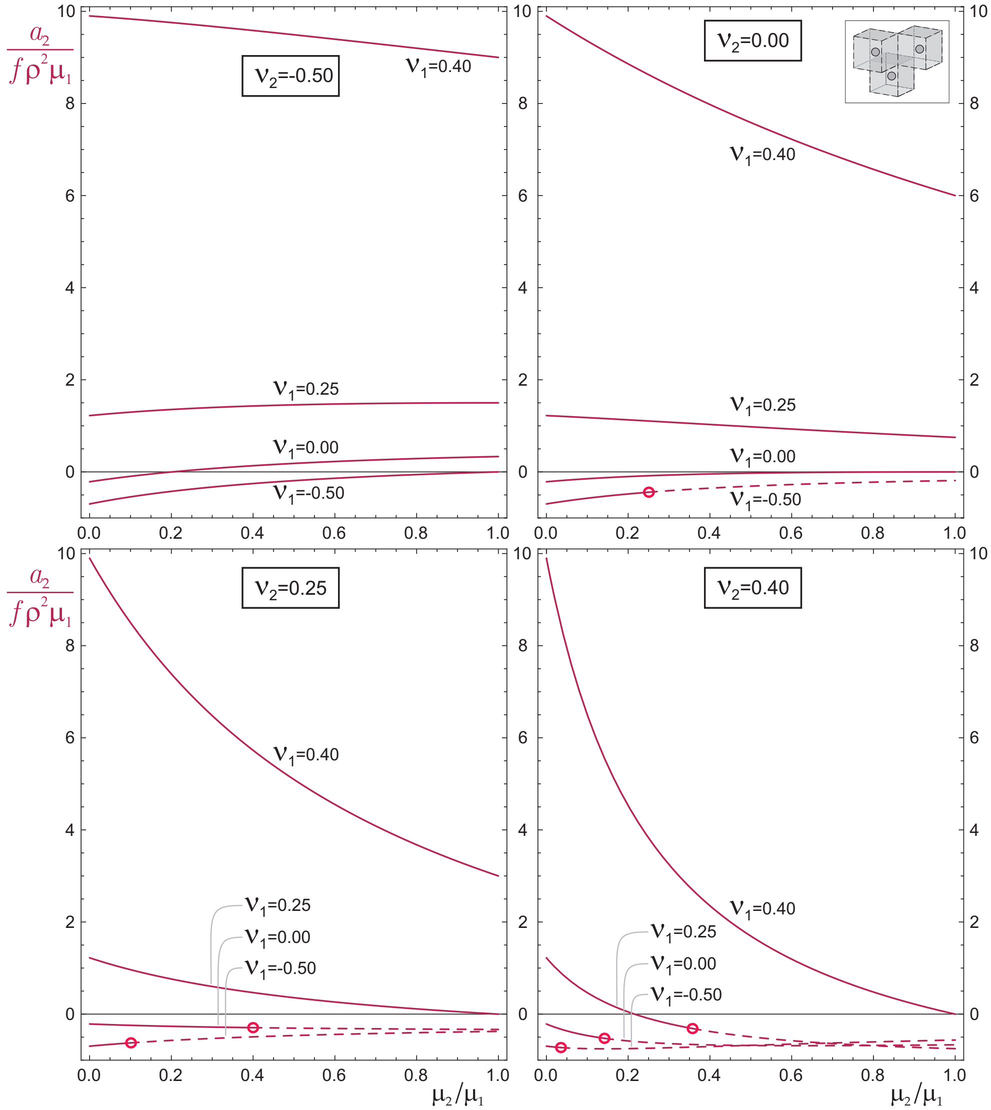

Figure 5.4: Higher-order equivalent constant a_2, eqn $(5.14)_1$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of spherical elastic inclusions as a function of the ratio μ_2/μ_1, for different values of Poisson's ratio of the phases $\{\nu_1, \nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_2 is made dimensionless through division by parameter $f\rho^2\mu_1$. The curves are dashed where the strain energy of the equivalent material is not positive definite, a red spot marks where the positive definiteness loss of $\mathbf{A}^{eq}$ occurs.

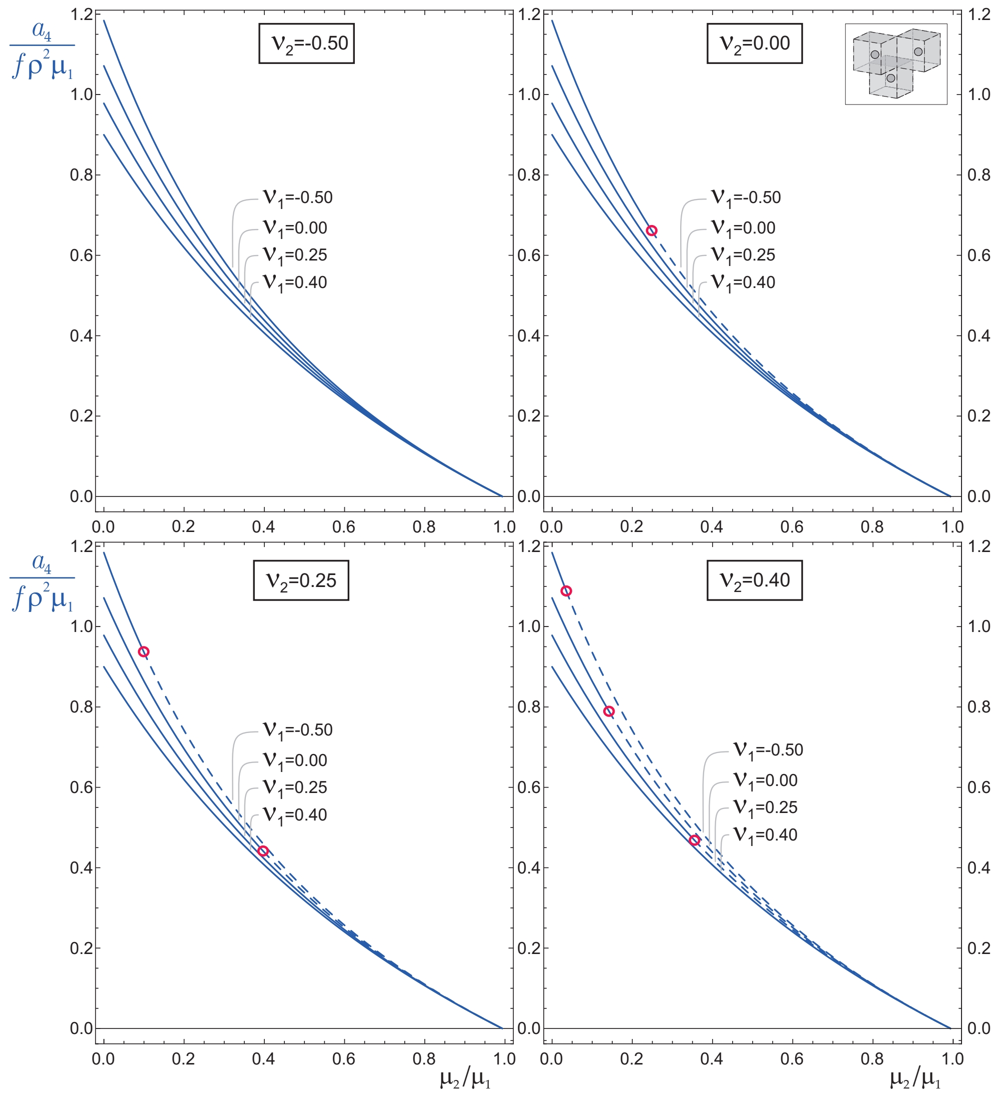

Figure 5.5: Higher-order equivalent constant $a_4 = a_5$, eqn $(5.14)_2$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of cylindrical elastic inclusions as a function of the ratio μ_2/μ_1, for different values of Poisson's ratio of the phases $\{\nu_1, \nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_4 is made dimensionless through division by parameter $f\rho^2\mu_1$. Note that the curves are not affected by the Poisson's ratio of the inclusion ν_2, except that the threshold (red spot) for positive definiteness of the equivalent material strain energy changes, eqn (5.10). Dashed curve represents values for which the strain energy of the equivalent material is not positive definite.

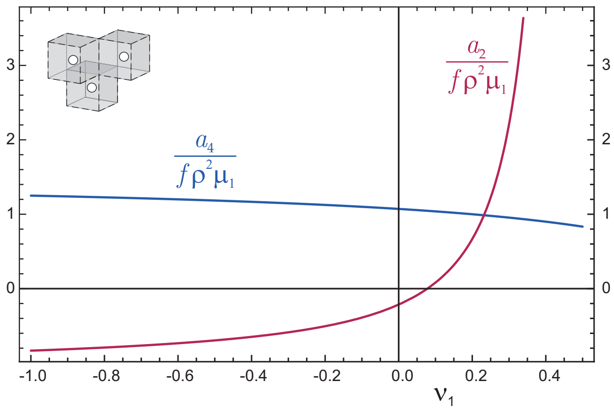

Figure 5.6: Higher-order equivalent constants a_2 and $a_4 = a_5$ of the equivalent SGE material for a composite made up of an isotropic matrix containing a dilute suspension of spherical voids as a function of the matrix Poisson's ratio ν_1, eqn (5.14) with $\mu_2 = K_2 = 0$. The constants are made dimensionless through division by parameter $f\rho^2\mu_1$.

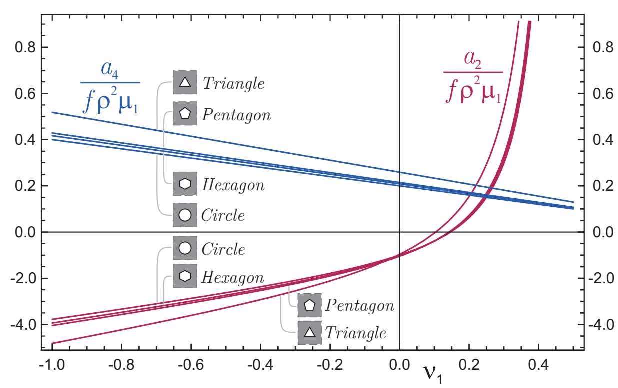

Figure 5.7: Higher-order equivalent constants a_2 and $a_4 = a_5$ of the equivalent SGE material for a diluite suspension of triangular ($n = 3$), pentagonal ($n = 5$), hexagonal ($n = 6$), and circular ($n \to \infty$) holes in an isotropic matrix, as functions of the matrix Poisson's ratio ν_1, eqn (5.16). The constants are made dimensionless through division by parameter $f\rho^2\mu_1$.

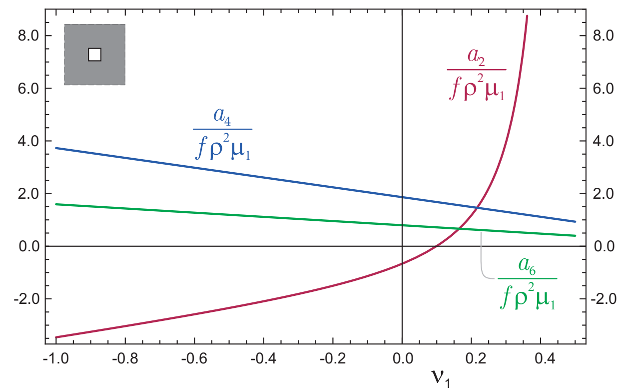

Figure 5.8: Higher-order equivalent constants a_2, $a_4 = a_5$, and a_6 of the equivalent SGE material for the plane strain case of a dilute suspension of periodically-distributed (with parallel edges) square holes in an isotropic matrix, as a function of the matrix Poisson's ratio ν_1, eqn (5.22). The constants are made dimensionless through division by parameter $f\rho^2\mu_1$.

Chapter 6

Conclusions

Micro- or nano-structures embedded in solids introduce internal length-scales and nonlocal effects within the mechanical modelling, leading to higher-order theories. We have provided an analytical approach to the determination of the parameters defining an elastic higher-order (Mindlin) material, as the homogenization of a heterogeneous Cauchy elastic material, eqn (4.1).

This result, obtained through the proposed homogenization procedure, is limited to the dilute approximation (thought we believe it can be generalized to the non-dilute case), but is not restricted to isotropy of the constituents and leaves a certain freedom to the shape of the inclusions. A perfect match between the elastic energies of the heterogeneous and homogeneous materials, for a general class of displacements prescribed on the two respective boundaries, is obtained. However, it has been shown that, to achieve a positive definite strain energy of the equivalent higher-order material, the inclusions have to be less stiff (in a way previously detailed) than the matrix, a situation already found by Bigoni and Drugan (2007) for Cosserat equivalent materials, which limits the applicability of the presented results, but explains the interpretation of previous experiments and results showing nonlocal effects for soft inclusions and 'anti-nonlocal' behaviour for stiff ones.

Appendix A

Elastic energies based on the displacement field $\boldsymbol{u}^{\beta\diamond *}$

In this Appendix it is assumed $\boldsymbol{\alpha} = \mathbf{0}$. The field $\boldsymbol{u}^{\beta\diamond *}$, eqn (4.10), is a kinematically admissible displacement for both boundary conditions $\overline{\boldsymbol{u}}^{\beta\diamond *}$, eqn (4.12), and ($\overline{\boldsymbol{u}}^{\beta\diamond *}$, $D\overline{\boldsymbol{u}}^{\beta\diamond *}$), eqn (4.15), applied on the boundary of the RVE and the SGE, respectively. The related strain and stress energies in the RVE and in the SGE are obtained below.

- In Section A.1 the strain energies are computed with the kinematically admissible deformation $\boldsymbol{\varepsilon}^{KA}$, eqn (4.46), and curvature $\boldsymbol{\chi}^{KA}$, eqn (4.55), originated by the kinematically admissible displacement $\boldsymbol{u}^{\beta\diamond *}$, eqn (4.10);

- In Section A.2 the stress energies are computed with the statically admissible stress $\boldsymbol{\sigma}^{SA}$, eqn (4.48), and double-stress $\boldsymbol{\tau}^{SA}$, eqn (4.57), originated by the above mentioned kinematically admissible fields $\boldsymbol{\varepsilon}^{KA}$ and$\boldsymbol{\chi}^{KA}$ within a homogeneous material with constitutive tensors $\mathbf{C}^*$ and $\mathbf{A}^{eq}$.

A.1 Strain and stress energies in the RVE

The kinematically admissible deformation $\boldsymbol{\varepsilon}^{KA}$, eqn (4.46), and the statically admissible stress $\boldsymbol{\sigma}^{SA}$, eqn (4.48), provide the strain and stress energies (4.45) in the RVE

$$\begin{aligned}\mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \int_\Omega 2\mathsf{C}_{ijhk}(\boldsymbol{x})\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm}x_l x_m,\\ \mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA}) = & \int_\Omega 2\mathsf{C}^*_{ijlm}\mathsf{C}^{-1}_{ijhk}(\boldsymbol{x})\mathsf{C}^*_{hkrs}\beta^{\diamond *}_{lmn}\beta^{\diamond *}_{rst}x_n x_t,\end{aligned} \tag{A.1}$$

which, introducing the definition (3.14) of the Euler tensor of inertia $\boldsymbol{E}$, can be rewritten as

$$\begin{aligned}\mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \; 2\left[\mathsf{C}^{(1)}_{ijhk}E_{lm}(\Omega^C_1) + \mathsf{C}^{(2)}_{ijhk}E_{lm}(\Omega^C_2)\right]\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm},\\ \mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA}) = & \; 2\mathsf{C}^*_{ijlm}\left\{\mathsf{C}^{(1)^{-1}}_{ijhk}E_{nt}(\Omega^C_1) + \mathsf{C}^{(2)^{-1}}_{ijhk}E_{nt}(\Omega^C_2)\right\}\mathsf{C}^*_{hkrs}\beta^{\diamond *}_{lmn}\beta^{\diamond *}_{rst}.\end{aligned} \tag{A.2}$$

Assuming the geometrical property **GP2** and considering the identity (3.11), the strain and stress energies (A.2) simplify as

$$\begin{aligned}\mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \; 2\rho^2\Omega\left\{\mathsf{C}^{(1)}_{ijhk} - f\left(\frac{\rho^{(2)}}{\rho}\right)^2\left[\mathsf{C}^{(1)}_{ijhk} - \mathsf{C}^{(2)}_{ijhk}\right]\right\}\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkl},\\ \mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA}) = & \; 2\rho^2\Omega\mathsf{C}^*_{ijlm}\left\{\mathsf{C}^{(1)^{-1}}_{ijhk} - f\left(\frac{\rho^{(2)}}{\rho}\right)^2\left[\mathsf{C}^{(2)^{-1}}_{ijhk} - \mathsf{C}^{(1)^{-1}}_{ijhk}\right]\right\}\times\\ & \; \mathsf{C}^*_{hkrs}\beta^{\diamond *}_{lmn}\beta^{\diamond *}_{rsn}.\end{aligned} \tag{A.3}$$

Assuming the geometrical property **GP3**

$$\rho^{(2)} = \tilde{\rho}^{(2)}f^r + o(f), \tag{A.4}$$

with $0 < r \leq 1$, and C^* as a first-order perturbation in f to the material matrix $\mathsf{C}^{(1)}$, eqn (4.9), the strain and the stress energies are given in the dilute case ($f \ll 1$) by

$$\begin{aligned}\mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \; 2\rho^2\Omega\mathsf{C}^{(1)}_{ijhk}\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkl} + o(f),\\ \mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA}) = & \; 2\rho^2\Omega\left(\mathsf{C}^{(1)}_{ijhk} + 2f\hat{\mathsf{C}}_{ijhk}\right)\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkl} + o(f).\end{aligned} \tag{A.5}$$

A.2 Strain and stress energies in the SGE

The kinematically admissible deformation and curvature fields [$\boldsymbol{\varepsilon}^{KA}$, eqn (4.46); $\boldsymbol{\chi}^{KA}$, eqn (4.55)] together with the statically admissible stress and double-stress fields [$\boldsymbol{\sigma}^{SA}$, eqn (4.48); $\boldsymbol{\tau}^{SA}$, eqn (4.57)] provide the strain and stress energies (4.54) in the SGE

$$\begin{aligned}
\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}) &= \int_\Omega 2\left[\mathbf{C}_{ijhk}^{eq}x_l x_m + \mathbf{A}_{jlikmh}^{eq}\right]\beta_{ijl}^{\diamond *}\beta_{hkm}^{\diamond *},\\
\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) &= \int_\Omega 2\left\{\mathbf{C}_{ijlm}^{*}\mathbf{C}_{ijhk}^{eq^{-1}}\mathbf{C}_{hkrs}^{*}x_n x_t + \mathbf{A}_{mnlstr}^{eq}\right\}\beta_{lmn}^{\diamond *}\beta_{rst}^{\diamond *},
\end{aligned} \tag{A.6}$$

which, introducing the definition (3.14) for the Euler tensor of inertia $\boldsymbol{E}$, can be rewritten as

$$\begin{aligned}
\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}) &= 2\left[\mathbf{C}_{ijhk}^{eq}E_{lm}(\Omega_{eq}^{SGE}) + \Omega_{eq}^{SGE}\mathbf{A}_{jlikmh}^{eq}\right]\beta_{ijl}^{\diamond *}\beta_{hkm}^{\diamond *},\\
\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) &= 2\left\{\mathbf{C}_{ijlm}^{*}\mathbf{C}_{ijhk}^{eq^{-1}}\mathbf{C}_{hkrs}^{*}E_{nt}(\Omega_{eq}^{SGE})\right.\\
&\quad \left.+\Omega_{eq}^{SGE}\mathbf{A}_{mnlstr}^{eq}\right\}\beta_{lmn}^{\diamond *}\beta_{rst}^{\diamond *}.
\end{aligned} \tag{A.7}$$

Assuming the geometrical property **GP2**, the strain and stress energies (A.7) simplify as

$$\begin{aligned}
\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}) &= 2\Omega\left[\rho^2\mathbf{C}_{ijhk}^{eq}\delta_{lm} + \mathbf{A}_{jlikmh}^{eq}\right]\beta_{ijl}^{\diamond *}\beta_{hkm}^{\diamond *},\\
\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) &= 2\Omega\left\{\rho^2\mathbf{C}_{ijlm}^{*}\mathbf{C}_{ijhk}^{eq^{-1}}\mathbf{C}_{hkrs}^{*}\delta_{nt} + \mathbf{A}_{mnlstr}^{eq}\right\}\beta_{lmn}^{\diamond *}\beta_{rst}^{\diamond *}.
\end{aligned} \tag{A.8}$$

Finally, assuming $\mathbf{C}^*$ as a first-order perturbation in f to the equivalent local tensor $\mathbf{C}^{eq}$, eqn (4.8), the stress energy is given in the dilute case ($f \ll 1$) by

$$\begin{aligned}
\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) = \; & 2\Omega\left\{\rho^2\left[\mathbf{C}_{ijhk}^{eq} + 2f\left(\hat{\mathbf{C}}_{ijhk} - \tilde{\mathbf{C}}_{ijhk}\right)\right]\delta_{nt}\right.\\
& \left.+\mathbf{A}_{mnlstr}^{eq}\right\}\beta_{lmn}^{\diamond *}\beta_{rst}^{\diamond *} + o(f).
\end{aligned} \tag{A.9}$$

Appendix B

Energy bounds for SGE Material

Statement. When boundary displacement conditions $\overline{\boldsymbol{u}}$, $\overline{D\boldsymbol{u}}$ are imposed on the boundary $\partial\Omega_{eq}$ of a SGE, the strain energy $\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}})$ is bounded as

$$
\begin{aligned}
\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}}) &\geq \int_{\partial\Omega_{eq}} \left(t_i^{SA}\overline{u}_i + T_i^{SA}\overline{Du}_i\right) + \int_{\Gamma_{eq}} \Theta_i^{SA}\overline{u}_i - \mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) \\
\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}}) &\leq \mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}),
\end{aligned}
\tag{B.1}
$$

where $\boldsymbol{\varepsilon}^{KA}$ and $\boldsymbol{\chi}^{KA}$ are kinematically admissible strain and curvature fields (satisfying the kinematic compatibility relation (2.1) and the imposed displacement boundary conditions), $\boldsymbol{\sigma}^{SA}$ and $\boldsymbol{\tau}^{SA}$ are statically admissible stress and double-stress fields (satisfying the equilibrium equation (2.4)) and the other statically admissible quantities $\boldsymbol{t}^{SA}$, $\mathbf{T}^{SA}$ and $\boldsymbol{\Theta}^{SA}$ are given by eqns (4.52) and (4.53), while $\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA})$ and $\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA})$ are respectively the stress and the strain energies, eqns $(4.54)_1$ and $(4.54)_2$.

Proof. Considering the displacement field $\boldsymbol{u}^{eq}$ solution to the displacement boundary conditions $\overline{\boldsymbol{u}}$, $\overline{D\boldsymbol{u}}$ and the related statical fields $\boldsymbol{\sigma}^{eq}$ and $\boldsymbol{\tau}^{eq}$ in equilibrium, through the difference fields $\Delta\boldsymbol{\varepsilon}^{KA}$, $\Delta\boldsymbol{\chi}^{KA}$, $\Delta\boldsymbol{\sigma}^{SA}$, $\Delta\boldsymbol{\tau}^{SA}$ the kinematically and statically admissible fields can be defined as

$$
\begin{aligned}
\boldsymbol{\varepsilon}^{KA} &= \boldsymbol{\varepsilon}^{eq} + \Delta\boldsymbol{\varepsilon}^{KA}, \qquad & \boldsymbol{\chi}^{KA} &= \boldsymbol{\chi}^{eq} + \Delta\boldsymbol{\chi}^{KA}, \\
\boldsymbol{\sigma}^{SA} &= \boldsymbol{\sigma}^{eq} + \Delta\boldsymbol{\sigma}^{SA}, \qquad & \boldsymbol{\tau}^{SA} &= \boldsymbol{\tau}^{eq} + \Delta\boldsymbol{\tau}^{SA}.
\end{aligned}
\tag{B.2}
$$

Using the discrepancy fields $\Delta\boldsymbol{\varepsilon}^{KA}$ and $\Delta\boldsymbol{\chi}^{KA}$ the term representing the *upper bound* in eqn (B.1) can be rewritten as

$$\begin{aligned}\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}) = \; & \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}}) + \mathcal{W}_{eq}^{SGE}(\Delta\boldsymbol{\varepsilon}^{KA},\Delta\boldsymbol{\chi}^{KA}) \\ & + \int_{\Omega_{eq}} \left(\mathsf{C}_{ijhk}\varepsilon_{ij}^{eq}\Delta\varepsilon_{hk}^{KA} + \mathsf{A}_{ijklmn}\chi_{ijk}^{eq}\Delta\chi_{lmn}^{KA}\right),\end{aligned} \tag{B.3}$$

which provides a proof to the upper bound, since the strain energy is positive definite and the third term in the RHS of eqn (B.3) is null by the principle of virtual work (2.3) with $\Delta\boldsymbol{u} = \Delta D\boldsymbol{u} = \mathbf{0}$ on the boundary.

Using the discrepancy fields $\Delta\boldsymbol{\sigma}^{KA}$ and $\Delta\boldsymbol{\tau}^{KA}$ the term representing the *lower bound* in eqn (B.1) can be rewritten as

$$\begin{aligned}\int_{\partial\Omega_{eq}} \left(t_i^{SA}\overline{u}_i + T_i^{SA}\overline{Du_i}\right) + \int_{\Gamma_{eq}} \Theta_i^{SA}\overline{u}_i - \mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) = \\ \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}}) - \mathcal{U}_{eq}^{SGE}(\Delta\boldsymbol{\sigma}^{SA},\Delta\boldsymbol{\tau}^{SA})\end{aligned} \tag{B.4}$$

which provides a proof to the lower bound, since the strain energy is positive definite. $\square$

Appendix C

Higher-order displacement boundary condition $D\boldsymbol{u}_{RVE}$

C.1 The asymptotic expansion of the displacement fields

Under the hypothesis of dilute suspension, the relation between the displacement field and the volume fraction of the inclusions can be written in a very simple way. The formulation of this relation is used in order to obtain a proper displacement boundary conditions for the homogeneous SGE material.

C.1.1 $\boldsymbol{\alpha}$ term of the displacement field

For imposed displacements from eqn (3.5), with $\boldsymbol{\beta} = 0$, at the boundary of the RVE, the displacement field in the presence of the inclusion is given by the asymptotic expansion in the volume fraction f

$$u_i^{\alpha}(\boldsymbol{x}) = \alpha_{ij} x_j + f^r \, \tilde{u}_i^{\alpha}(\boldsymbol{x}) + o(f), \tag{C.1}$$

with the restriction

$$0 < r \leq 1, \tag{C.2}$$

and the condition,

$$\tilde{u}_i^{\alpha}(\boldsymbol{x}) = 0 \quad \text{on } \partial\Omega. \tag{C.3}$$

The strain energy stored in the RVE, for the principle of virtual work,[1] can be written as,

$$\begin{aligned}\mathcal{W}^{C}_{\Omega_R}(\overline{\boldsymbol{u}}^{\alpha}) &= \frac{1}{2}\Omega\left[\mathbf{C}^{(1)}_{ijhk} + f\left(\mathbf{C}^{(2)}_{ijhk} - \mathbf{C}^{(1)}_{ijhk}\right)\right]\alpha_{ij}\alpha_{hk} \\ &\quad - f^{2r}\frac{1}{2}\int_{\Omega_R}\mathbf{C}_{ijhk}(\boldsymbol{x})\tilde{u}^{\alpha}_{i,j}(\boldsymbol{x})\tilde{u}^{\alpha}_{h,k}(\boldsymbol{x}) + o(f).\end{aligned} \tag{C.5}$$

For the classical homogenization method, eqn (4.18) and eqn (4.19) gives the following equality,

$$\mathcal{W}^{C}_{\Omega_{RVE}}(\overline{\boldsymbol{u}}^{\alpha}) = \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}) \tag{C.6}$$

where

$$\mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}) = \frac{1}{2}\Omega\,\mathbf{C}^{eq}_{ijhk}\alpha_{ij}\alpha_{hk}. \tag{C.7}$$

Substituting eqn (C.7) and eqn (C.5) into eqn (C.6), we obtain a new restriction for the r parameter,

$$f\,\Omega\left(\mathbf{C}^{(2)}_{ijhk} - \mathbf{C}^{(1)}_{ijhk} - \tilde{\mathbf{C}}_{ijhk}\right)\alpha_{ij}\alpha_{hk} - f^{2r}\int_{\Omega_R}\mathbf{C}_{ijhk}(\boldsymbol{x})\tilde{u}^{\alpha}_{i,j}(\boldsymbol{x})\tilde{u}^{\alpha}_{h,k}(\boldsymbol{x}) = o(f). \tag{C.8}$$

from which it can be concluded that,

$$0 < r \leq \frac{1}{2}. \tag{C.9}$$

C.1.2 $\boldsymbol{\beta}$ term of the displacement field

For imposed displacements from eqn (3.5), with $\boldsymbol{\alpha} = 0$, at the boundary of the RVE, the displacement field in the presence of the inclusion is given by

[1] The strain energy stored in the RVE, for property (C.1) and condition (C.3), can be written as,

$$\begin{aligned}\mathcal{W}^{C}_{\Omega_R}(\overline{\boldsymbol{u}}^{\alpha}) &= \frac{1}{2}\Omega\left[\mathbf{C}^{(1)}_{ijhk} + f\left(\mathbf{C}^{(2)}_{ijhk} - \mathbf{C}^{(1)}_{ijhk}\right)\right]\alpha_{ij}\alpha_{hk} \\ &\quad - f^{2r}\frac{1}{2}\int_{\Omega_R}\mathbf{C}_{ijhk}(\boldsymbol{x})\tilde{u}^{\alpha}_{i,j}(\boldsymbol{x})\tilde{u}^{\alpha}_{h,k}(\boldsymbol{x}) \\ &\quad + f^{r}\int_{\Omega_R}\mathbf{C}_{ijhk}(\boldsymbol{x})u^{\alpha}_{i,j}(\boldsymbol{x})\tilde{u}^{\alpha}_{h,k}(\boldsymbol{x}) + o(f)\end{aligned} \tag{C.4}$$

from property (C.3) the third term of the right hand side of the equation becomes null after application of the principle of virtual work, and eqn (C.5) arises.

the asymptotic expansion in the volume fraction f

$$u_i^{\beta^{\diamond *}}(\boldsymbol{x}) = \beta_{ijk}^{\diamond *} x_j x_k + f^q\, \tilde{u}_i^{\beta^{\diamond *}}(\boldsymbol{x}) + o(f), \tag{C.10}$$

with the restriction

$$0 < q \leq 1, \tag{C.11}$$

and the condition,

$$\tilde{u}_i^{\beta^{\diamond *}}(\boldsymbol{x}) = 0 \quad \text{on } \partial\Omega \tag{C.12}$$

The strain energy stored in the RVE, for the principle of virtual work, [2] can be written as,

$$\begin{aligned} \mathcal{W}_{\Omega_R}^C(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) =& 2\rho^2 \Omega_R \mathbf{C}_{ijhk}^{(1)} \beta_{ijl}^{\diamond *} \beta_{hkl}^{\diamond *} \\ &- f^{2q} \frac{1}{2} \int_{\Omega_R} \mathbf{C}_{ijhk}(\boldsymbol{x}) \tilde{u}_{i,j}^{\beta^{\diamond *}}(\boldsymbol{x}) \tilde{u}_{h,k}^{\beta^{\diamond *}}(\boldsymbol{x}) + o(f). \end{aligned} \tag{C.14}$$

Comparing eqn (C.14) with eqn (4.21), the following equality arises,

$$f^{2q} \frac{1}{2} \int_{\Omega_R} \mathbf{C}_{ijhk}(\boldsymbol{x}) \tilde{u}_{i,j}^{\beta^{\diamond *}}(\boldsymbol{x}) \tilde{u}_{h,k}^{\beta^{\diamond *}}(\boldsymbol{x}) = o(f), \tag{C.15}$$

so that the following condition for q is obtained

$$0 < q < \frac{1}{2}. \tag{C.16}$$

C.2 Energetic equivalence with the higher-order boundary condition $D\overline{\boldsymbol{u}}$

For the displacement problem of a generic SGE, the boundary conditions involves the normal component of the gradient of the displacement. In the

[2]The strain energy stored in the RVE, for property (C.10) and condition (C.12), can be written as,

$$\begin{aligned} \mathcal{W}_{\Omega_R}^C(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) =& 2\rho^2 \Omega_R \mathbf{C}_{ijhk}^{(1)} \beta_{ijl}^{\diamond *} \beta_{hkl}^{\diamond *} \\ &- f^{2q} \frac{1}{2} \int_{\Omega_R} \mathbf{C}_{ijhk}(\boldsymbol{x}) \tilde{u}_{i,j}^{\beta^{\diamond *}}(\boldsymbol{x}) \tilde{u}_{h,k}^{\beta^{\diamond *}}(\boldsymbol{x}) \\ &+ f^{q} \int_{\Omega_R} \mathbf{C}_{ijhk}(\boldsymbol{x}) u_{i,j}^{\beta^{\diamond *}}(\boldsymbol{x}) \tilde{u}_{h,k}^{\beta^{\diamond *}}(\boldsymbol{x}) + o(f). \end{aligned} \tag{C.13}$$

from property (C.12) the third term of the right hand side of the equation becomes null after application of the principle of virtual work, and eqn (C.14) arises.

previous chapters, this boundary condition has been taken equal to the gradient of the displacement field applied on the boundary, eqn (3.6). In order to impose the same boundary conditions on both the RVE and the homogeneous SGE, we apply,

$$\begin{cases} \boldsymbol{u} = \overline{\boldsymbol{u}}, \\ D\boldsymbol{u} = D\boldsymbol{u}_{RVE}, \end{cases} \quad \text{on } \partial\Omega_{eq}^{SGE}. \tag{C.17}$$

where $D\boldsymbol{u}_{RVE}$ is the normal component of the gradient of the displacement of the RVE.

In the case of linear displacements ($\boldsymbol{\beta} = \mathbf{0}$), for eqn (C.1) the higher-order boundary condition becomes,

$$D\boldsymbol{u}^{\alpha}_{RVE} = D\overline{\boldsymbol{u}}^{\alpha} + f^{r} D\tilde{\boldsymbol{u}}^{\alpha}, \tag{C.18}$$

and the energy stored in the homogeneous SGE is,

$$\mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}, D\boldsymbol{u}^{\alpha}_{RVE}) = \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}, D\overline{\boldsymbol{u}}^{\alpha}) + o(f). \tag{C.19}$$

Proof Substituting eqn (C.18) into the boundary conditions (C.17), the energy stored in the SGE becomes,

$$\begin{aligned} \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}, D\boldsymbol{u}^{\alpha}_{RVE}) = & \ \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}, D\overline{\boldsymbol{u}}^{\alpha}) + f^{2r}\mathcal{W}^{SGE}_{\Omega_{eq}}(\mathbf{0}, D\tilde{\boldsymbol{u}}^{\alpha}) \\ & + f^{r}\mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}, D\overline{\boldsymbol{u}}^{\alpha}; \mathbf{0}, D\tilde{\boldsymbol{u}}^{\alpha}), \end{aligned} \tag{C.20}$$

where,

$$\begin{aligned} \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\alpha}, D\overline{\boldsymbol{u}}^{\alpha}; \mathbf{0}, D\tilde{\boldsymbol{u}}^{\alpha}) & = \textstyle\int_{\partial\Omega_{eq}} T^{\alpha}_{i} D\tilde{u}_{i}{}^{\alpha}, \\ \mathcal{W}^{SGE}_{\Omega_{eq}}(\mathbf{0}, D\tilde{\boldsymbol{u}}^{\alpha}) & = \tfrac{1}{2}\textstyle\int_{\partial\Omega_{eq}} \tilde{T}^{\alpha}_{i} D\tilde{u}_{i}{}^{\alpha}. \end{aligned} \tag{C.21}$$

Since the $\boldsymbol{\alpha}$ terms of the displacement field does not activate the non-local effects, we can write

$$T^{\alpha}_{i} = 0 \quad \forall i = 1, ..., N \tag{C.22}$$

and the energy contribution in eqn $(\text{C.21})_1$ becomes null. On the other hand, for eqn $(2.5)_2$, the vector of generalized tractions of eqn $(\text{C.21})_2$ becomes,

$$\tilde{T}^{\alpha}_{k} = n_i n_j \mathbf{A}^{eq}_{ijkhlm} \tilde{\chi}^{\alpha}_{lmn}, \tag{C.23}$$

substituting the sixth-order tensor of eqn (4.1) into eqn (C.23), for property (C.9), the energy contribution (C.21)$_2$, becomes

$$\mathcal{W}^{SGE}_{\Omega_{eq}}(\mathbf{0}, D\tilde{\boldsymbol{u}}^{\alpha}) = o(f), \tag{C.24}$$

and eqn (C.19) arises.

In the case of quadratic displacements, for eqn (C.10) the higher-order boundary condition becomes,

$$D\boldsymbol{u}^{\beta}_{RVE} = D\overline{\boldsymbol{u}}^{\beta} + f^{q} D\tilde{\boldsymbol{u}}^{\beta}, \tag{C.25}$$

and the energy stored in the homogeneous SGE is,

$$\mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\beta}, D\boldsymbol{u}^{\beta}_{RVE}) = \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\beta}, D\overline{\boldsymbol{u}}^{\beta}) + o(f). \tag{C.26}$$

Proof Substituting eqn (C.25) into the boundary conditions (C.17), the energy stored in the SGE becomes,

$$\begin{aligned} \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\beta}, D\boldsymbol{u}^{\beta}_{RVE}) = \; & \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\beta}, D\overline{\boldsymbol{u}}^{\beta}) + f^{2q}\mathcal{W}^{SGE}_{\Omega_{eq}}(\mathbf{0}, D\tilde{\boldsymbol{u}}^{\beta}) \\ & + f^{q}\mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\beta}, D\overline{\boldsymbol{u}}^{\beta}; \mathbf{0}, D\tilde{\boldsymbol{u}}^{\beta}) \end{aligned} \tag{C.27}$$

where,

$$\begin{aligned} \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\beta}, D\overline{\boldsymbol{u}}^{\beta}; \mathbf{0}, D\tilde{\boldsymbol{u}}^{\beta}) \; &= \int_{\partial\Omega_{eq}} T^{\beta}_{i} D\tilde{u}_{i}^{\beta} \\ \mathcal{W}^{SGE}_{\Omega_{eq}}(\mathbf{0}, D\tilde{\boldsymbol{u}}^{\beta}) \; &= \frac{1}{2}\int_{\partial\Omega_{eq}} \tilde{T}^{\beta}_{i} D\tilde{u}_{i}^{\beta} \end{aligned} \tag{C.28}$$

For eqn (2.5)$_2$, the vectors of generalized tractions of eqn (C.28) becomes,

$$\begin{aligned} \tilde{T}^{\beta}_{k} &= n_i n_j \mathbf{A}^{eq}_{ijkhlm} \tilde{\chi}^{\beta}_{lmn}, \\ T^{\beta}_{k} &= n_i n_j \mathbf{A}^{eq}_{ijkhlm} \chi^{\beta}_{lmn}, \end{aligned} \tag{C.29}$$

substituting the sixth-order tensor of eqn (4.1) into eqn (C.29), for property (C.16), the energy contributions (C.28), becomes

$$\begin{aligned} \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^{\beta}, D\overline{\boldsymbol{u}}^{\beta}; \mathbf{0}, D\tilde{\boldsymbol{u}}^{\beta}) \; &= o(f) \\ \mathcal{W}^{SGE}_{\Omega_{eq}}(\mathbf{0}, D\tilde{\boldsymbol{u}}^{\beta}) \; &= o(f) \end{aligned} \tag{C.30}$$

and eqn (C.26) arises.

Appendix D

A different view of the proposed homogenization approach

The result obtained through the homogenization procedure presented in Chapter 3 is based on the annihilation of the elastic energy mismatch $\mathcal{G}$. In this Appendix it is shown that the result leads to the same mean values within the RVE and the SGE.

D.1 Mean values

The mean strain, curvature, stress and double stress values of a generic SGE material are considered as,

$$\begin{aligned} \langle \varepsilon_{ij} \rangle|_\Omega &= \frac{1}{\Omega}\int_\Omega \varepsilon_{ij}, \quad \langle \chi_{ijk} \rangle|_\Omega = \frac{1}{\Omega}\int_\Omega \chi_{ijk}, \\ \langle \sigma_{ij} \rangle|_\Omega &= \frac{1}{\Omega}\int_\Omega \sigma_{ij}, \quad \langle \tau_{ijk} \rangle|_\Omega = \frac{1}{\Omega}\int_\Omega x_i\sigma_{jk} + \tau_{ijk}, \end{aligned} \tag{D.1}$$

for the Lemma of Green, the kinematical quantities become,

$$\langle \varepsilon_{ij} \rangle|_\Omega = \frac{1}{2\Omega}\oint_{\partial\Omega} (n_i u_j + u_i n_j), \quad \langle \chi_{ijk} \rangle|_\Omega = \frac{1}{\Omega}\oint_{\partial\Omega} n_i u_{k,j}. \tag{D.2}$$

From property (D.2), and for an imposed displacement and normal component of the gradient of displacement on the boundary of the RVE and of the homogeneous SGE equivalent [1] such eqn (3.5), it can be observed that the

[1]These are the boundary condition of a generic SGE material for the displacement problem. It corresponds to the imposition of the same displacement field and of the same gradient of displacement on the boundary of the RVE and of the homogeneous SGE equivalent.

prescription of the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ leads to the imposition of the mean strain and mean curvature values [2],

$$\langle \varepsilon_{ij} \rangle|_{\Omega^C_{RVE}} = \langle \varepsilon_{ij} \rangle|_{\Omega^{SGE}_{eq}} \,, \quad \langle \chi_{ijk} \rangle|_{\Omega^C_{RVE}} = \langle \chi_{ijk} \rangle|_{\Omega^{SGE}_{eq}} \,. \tag{D.4}$$

D.2 Mean stress matching

For the classical homogenization method ($\boldsymbol{\beta} = 0$), the energy matching imposes the following equality,

$$\mathcal{W}^C_{\Omega_{RVE}}(\overline{\boldsymbol{u}}^\alpha) = \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^\alpha) \tag{D.5}$$

such equality gives the matching of the mean stress values, through an application of the principle of virtual work [3],

$$\left\langle \sigma^\alpha_{ij} \right\rangle\Big|_{\Omega^C_{RVE}} = \left\langle \sigma^\alpha_{ij} \right\rangle\Big|_{\Omega^{SGE}_{eq}} \tag{D.9}$$

D.3 Second order homogenization with β terms

For the proposed homogenization method, the energy matching imposes the following equality,

$$\mathcal{W}^C_{\Omega_{RVE}}(\overline{\boldsymbol{u}}^\beta, D\overline{\boldsymbol{u}}^\beta) = \mathcal{W}^{SGE}_{\Omega_{eq}}(\overline{\boldsymbol{u}}^\beta, D\overline{\boldsymbol{u}}^\beta) \tag{D.10}$$

[2] Considering eqn (3.5) as a kinematically admissible displacement field, the mean strain and mean curvature values becomes,

$$\langle \varepsilon_{ij} \rangle|_{\Omega} = \frac{\alpha_{ij} + \alpha_{ji}}{2}, \quad \langle \chi_{ijk} \rangle|_{\Omega} = 2\beta_{kij}, \tag{D.3}$$

[3] From eqn (D.5), imposing the same displacement at the boundary and considering the absence of non-local effect in the homogeneous SGE material, the following equality arises,

$$\int_{\partial\Omega} \left(\sigma^\alpha_{ij} - \sigma^\alpha_{eq\ ij} \right) n_i \overline{u}^\alpha_j = 0 \tag{D.6}$$

where σ^α_{ij} and $\sigma^\alpha_{eq\ ij}$ are the stress field in the RVE and in the homogeneous SGE equivalent, respectively. Applying the principle of virtual work, considering the following kinematically admissible displacement field,

$$u^\alpha_i = \alpha_{ij} x_j, \tag{D.7}$$

we obtain,

$$\alpha_{ij} \int_{\Omega} \left(\sigma^\alpha_{ij} - \sigma^\alpha_{eq\ ij} \right) = 0. \tag{D.8}$$

For the arbitrariness of $\boldsymbol{\alpha}$ parameters and eqn for $(\text{D.1})_3$, eqn (D.9) arises.

such equality gives the matching of the mean stress-momentum values, through an application of the principle of virtual work [4],

$$\left\langle \tau^{\beta}_{ijk} \right\rangle\Big|_{\Omega^{C}_{RVE}} = \left\langle \tau^{\beta}_{ijk} \right\rangle\Big|_{\Omega^{SGE}_{eq}} \tag{D.15}$$

[4]From eqn (D.5), imposing the same displacement and the same gradient of the displacement at the boundary, the following equality arises,

$$\int_{\partial\Omega} \left[\left(t^{\beta}_{i} - t^{\beta}_{eq\ i} \right) \overline{u}^{\beta}_{i} + \left(T^{\beta}_{i} - T^{\beta}_{eq\ i} \right) D\overline{u}^{\beta}_{i} \right] + \int_{\Gamma} \left(\Theta^{\beta}_{i} - \Theta^{\beta}_{eq\ i} \right) \overline{u}^{\beta}_{i} = 0 \tag{D.11}$$

where t^{β}_{i}, T^{β}_{i}, Θ^{β}_{i} and $t^{\beta}_{eq\,i}$, $T^{\beta}_{eq\,i}$, $\Theta^{\beta}_{eq\,i}$, are the traction and generalized tractions arising on the boundary and on the edges of the boundary, of the RVE and of the homogeneous SGE, respectively. Since the sixth-order non-local constitutive tensor of the RVE is null, the generalized tractions arising on the boundary of the RVE are null,

$$T^{\beta}_{i} = 0,\ \Theta^{\beta}_{i} = 0, \tag{D.12}$$

Applying the principle of virtual work, considering the following kinematically admissible displacement field,

$$u^{\alpha}_{i} = \beta_{ijk} x_j x_k, \tag{D.13}$$

we obtain,

$$2\beta_{jki} \int_{\Omega} \left[x_i \sigma^{\beta}_{jk} - \left(x_i \sigma^{\beta}_{eq\ jk} + \tau^{\beta}_{eq\ ijk} \right) \right] = 0 \tag{D.14}$$

For the arbitrariness of $\boldsymbol{\beta}$ parameters and eqn for $(D.1)_4$, eqn (D.15) arises.

Nomenclature

$\boldsymbol{x}$ (x_i) : position vector

$\boldsymbol{u}$ (u_i) : displacement vector

$D\boldsymbol{u}$ (Du_i) : derivative of the displacement along the outward normal direction to the boundary $\partial\Omega$

e_{mlj} : Ricci ‘permutation’ tensor

$\boldsymbol{n}$ (n_i) : unit vector normal to $\partial\Omega$

$\boldsymbol{s}$ (s_i) : unit vector tangent to Γ

$\boldsymbol{\varepsilon}$ (ε_{ij}) : strain tensor

$\boldsymbol{\chi}$ (χ_{ijk}) : curvature tensor

$\boldsymbol{\sigma}$ (σ_{ij}) : stess tensor

$\boldsymbol{\tau}$ (τ_{ijk}) : double stess tensor

$\boldsymbol{t}$ (t_i) : traction vector on the surface $\partial\Omega$

$\mathbf{T}$ (T_i) : generalized traction vector on the surface $\partial\Omega$

$\boldsymbol{\Theta}$ (Θ_i) : generalized traction vector along the set of edges Γ

RVE : Representative volume element

SGE : Second Gradient Elastic

Ω^C_{RVE} : RVE made up of a heterogeneous Cauchy material

Ω^C_1 : matrix made up of a homogeneous Cauchy material

Ω_2^C : inclusion made up of a homogeneous Cauchy material

Ω_{eq}^{SGE} : equivalent SGE material

w^{SGE} : strain energy density of a SGE material

$\mathcal{W}_{RVE}^C$: Strain energy stored in the RVE

$\mathcal{W}_{eq}^{SGE}$: Strain energy stored in the SGE material

$\mathcal{G}$: Strain energy mismatch between the RVE and the SGE material

$\mathbf{C}^{(1)}$ ($\mathbf{C}_{ijhk}^{(1)}$) : matrix constitutive tensor

$\mathbf{C}^{(2)}$ ($\mathbf{C}_{ijhk}^{(2)}$) : inclusion constitutive tensor

$\mathbf{C}^{eq}$ ($\mathbf{C}_{ijhk}^{eq}$) : equivalent local (Cauchy) constitutive tensor

$\mathbf{A}^{eq}$ ($\mathbf{A}_{ijhlmn}^{eq}$) : equivalent nonlocal (Mindlin) constitutive tensor

$\tilde{\mathbf{C}}$ ($\tilde{\mathbf{C}}_{ijhk}$) : discrepancy tensor at the first-order in f between the RVE and the SGE

f : volume fraction of the inclusion phase

ρ : radious of inertia of the RVE

$\boldsymbol{Q}$ (Q_{ij}) : orthogonal tensor

$[\![\cdot]\!]$: jump in the relevant argument across the surfaces intersecting at the edge Γ

$\langle\cdot\rangle|_{\Omega}$: mean value of the relevant argument over a domain Ω

SA : Statical admissible

KA : Kinematical admissible

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