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PhD Thesis

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Extended-order algebras and some applications

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Contents

Introduction

In recent years there has been a big development of many-valued logic, and more generally, of many-valued mathematics, encouraged by their usefulness for the applications in the context of different scientific disciplines.
The growing interest for the reasoning under uncertainty and partial knowledge conditions [10, 45, 74] urged producing a considerable variety of approaches to non-classical logics and to related mathematics; the in-depth study of algebraic structures associated to logic, that has led to various modifications, settlements and evolutions, is due to the fact that these algebras are strictly related to fundamental aspects of mathematics, like set theory and topology [59].
The algebras that occur in classical logic essentially depend on the notion of order and on the related operations that characterize various types of lattices, including the well-known boolean and Heyting algebras, whose structure is, in fact, determined by operations strongly connected with the connectives of classical and intuitionistic logic.
Considerable work has been done to investigate in detail the algebraic operations directly or indirectly connected with the logical connectives and with the lattice structures and to study interrelationships between these different operations; moreover, the possibility to deduce all the operations from one or some of them has been exploited and it depends on the properties of these and on the underlying lattice structure. The main tools allowing to clarify interrelations and to construct new binary operations from old ones are undoubtedly isotone and antitone classical Galois connections.
The connective generally used as a starting point is the logical conjunction, whose algebraic counterpart provides a monoidal structure, that has interesting connections and similarities with analogous structures, related to several mathematical contexts: important examples are residuated lattices and quantales, originally considered in relation with the ideals of rings of operators [28, 29, 68, 75], and t-norms, originally connected to the theory of probabilistic metric spaces [63, 64]. Moreover, this kind of structures has been moved into the context of category theory, to build a categorical approach to logic (see [65]).
In such monoidal structures it has become usual to drop the commutativity, besides the idempotency of the operation, though for applications of logics in the classical context these are fundamental conditions.
On the other hand the associativity condition, whose motivations are not clear enough, in our opinion, seems to cannot be given up, due to its permanent use in logics and in

various other contexts. The unique exception, to our knowledge, is the trend, developed recently, to consider semi-quantales (actually, quantales without associativity and distributivity conditions) or arbitrary varieties of algebras as reference structures for categories of L-sets, L-topological spaces and topological systems (see [21, 23, 79, 81]); this trend however seems to aim essentially the goal of generality. The monoidal associative, and distributive as well, structures have been explicitly considered as fundamental by J. A. Goguen, one of the pioneers of fuzzy sets theory [44].
The second connective most frequently used as a reference and starting point to build algebras for many-valued logics is the implication: so called implication algebras, among which the Hilbert algebras, have been studied since the fifties (see [26]) and they are widely described in [74], in their algebraic aspects and in their application to non-classical logics; however, the "implication" operation that characterizes these structures turns out to be the one connected, by classical adjunction, to the meet operation $\wedge$, whenever the underlying structure is also a complete lattice with respect to the induced order, which strongly reduces the application fields.
Nowadays, residuated lattices (a kind of lattice ordered, both monoidal and implicative structure) first considered in [28, 29] are commonly used (see [9, 58]); they have two operations (the multiplication-conjunction and the implication) that form an adjunction (i.e. an isotonic Galois connection). An equivalent approach to these structures and to implication algebras, as well, is preferred by those moving in the universe of (pseudo-)BCK algebras (an extensive approach and a detailed description can be found in [56, 57]). However, these structures, based on a pair of operations, are partially justified in absence of completeness of the underlying ordered set, whose assumption, instead, would allow to obtain either operation from the other one by the quite standard procedures of residuation and deresiduation. What is important to us is to note that the multiplication is always assumed to be associative and if it is not commutative then the existence of two implications, both connected, though in two symmetrical ways, with the multiplication, is assumed.
This approach includes structures that satisfy additional conditions of the two operations, such as the prelinearity or the divisibility, or the double negation principle (see [12, 52, 56, 57]).
Recently it has been proposed by C. Guido and P. Toto in [50] an order-theoretic approach to residuated structures, going from the implicative structures of lower level (implicative algebras in [74], *w-eo algebras* in [50]) to various kinds of residuated algebras, in particular to integral residuated lattices, that, in theirs turn, include most algebras used in logic, either classical or not.
The basic idea is rather suggestive, even if simple: once considered that every lattice structure is completely determined by the underlying (crisp) order relation (so that a fixed partially ordered set either is or is not a lattice of some kind) one can believe that any extension (not necessarily crisp) of an order relation can determine completely the algebraic structures mostly used in logics, including many-valued logic.
More explicitly, given any ordered set L with a greatest element $\top$, $(L, \leq, \top)$, an extension of $\leq$ is a function

$$\to : L \times L \to L$$

such that

$$a \to b = \top \Leftrightarrow a \le b, \text{ for all } a, b \in L.$$

The structure $(L, \to, \top)$, that is an implicative algebra according to the terminology of [74], is called *weak extended-order algebra* in [50], due to its motivating idea.
The first part of this thesis, mainly Chapter 2, describes in details the study of weak extended-order algebras started in [50], continued in my master thesis and fully developed in the first year of my PhD studies, including some applications of weak extended-order algebras to the theory of fuzzy implicators; the obtained results have been published in [13, 14]. This approach to the algebras of logics produces a clear framework to understand which are the fundamental conditions that allow to define and study the various operations and the associated algebraic structures related to logics.
First of all, in a weak extended-order algebra the following two axioms are considered:
(o_5) $a \to b = \top \Rightarrow (c \to a) \to (c \to b) = \top$ (weak isotonic condition in the second variable);
(o_5') $a \to b = \top \Rightarrow (b \to c) \to (a \to c) = \top$ (weak antitonic condition in the first variable).
If these axioms are satisfied, the algebra is called *extended-order algebra*.
Such conditions on $\to$ are needed to obtain an embedding of the algebra $(L, \to, \top)$ into its Dedekind-MacNeille completion, which, moreover, preserves further properties possibly assumed on L. Hence, this result allows to consider the completeness requirement not restrictive for extended-order algebras.
Properties stronger than those expressed by the axioms (o_5) and (o_5') are the right-distributivity, the left-distributivity and the distributivity. The results of [50] and our studies lead to consider the right-distributivity as a key-condition: in fact, it allows to define on L, by using the adjoint functor theorem, an operation $\otimes : L \times L \to L$, called adjoint product of L, that can be viewed as a logical connective of conjunction.
This multiplication has good properties, usually asked to classical conjunction, except the idempotency, the commutativity and the associativity, which are not obtained, in general. On the contrary, it is proved in [50] that the associativity of the multiplication is characterized in a quite general context by a condition that is, actually, a second order formula, which shows its weakness and its unpleasant character.
Furthermore, the distributivity condition allows to build further operations, starting from $\otimes$ and using the implication $\to$ that can be interpreted as disjunction connectives. Also these operations, as well as the negation, which we have defined and studied, have a good behavior in a quite general context of distributive extended-order algebras; we prove further good properties by adding on L assumptions like the double negation principle.
The algebras considered, unlike residuated lattices and (pseudo-)BCK algebras, are not necessarily symmetrical, since the product has not the same properties on both arguments; this is in fact one of the main motivations of [50].
We investigate the question of symmetry by considering algebras that are not symmetrical and characterize those which are. More precisely, we say that a weak extended-order algebra $(L, \to, \top)$ is *symmetrical* if there exists a binary operation $\rightsquigarrow : L \times L \to L$, called

dual implication, such that $(L, \rightsquigarrow, \top)$ is a weak extended-order algebra, $\to$ and $\rightsquigarrow$ induce the same order and $a \le b \rightsquigarrow c \Leftrightarrow b \le a \to c$, for all $a, b, c \in L$.
The introduction of the dual implication allows to define a dual negation and, if the distributivity is assumed, the dual product and the dual disjunctions. In the context of symmetrical extended-order algebras, the various operations introduced undergo improvements, without assuming commutativity and associativity. We note also, that the symmetry condition is a weakening of the commutativity; in fact, $(L, \to, \top)$ is commutative if and only if it is symmetrical and $\rightsquigarrow = \to$.
Particular attention has been devoted to the possibility of dropping the associativity assumption, which is instead a well established requirement in all the approaches to structures related to logical connectives, including residuated lattices [9, 28, 29, 52, 58], (pseudo-)BCK algebras [56, 57], quantales [75], t-norms [30, 52, 63, 64] and (monoidal) closed categories [32, 65, 68, 89]. The effective power of associativity turns out rather to allow the strong version of several properties, provided that those are satisfied in their weak version. For instance, if L is symmetrical, the associativity is equivalent to the equality $a \to (b \rightsquigarrow c) = a \rightsquigarrow (b \to c)$, for all $a, b, c \in L$, that is the strong version of that Galois connection which is the required link between the two implications considered (see further details in Subsections 2.1.5 and 2.1.6).
An immediate application of weak extended-order algebras occurs in the study of implicators; we have reconsidered the notion as it is presented in the literature, taking into account the point of view of extended-order algebras.
Usually, an implicator is defined as a map $\mathcal{I} : [0,1] \times [0,1] \to [0,1]$ that extends the classical logic implication; in fact an implicator has to satisfy the boundary conditions

$$(b)\ \mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1 \text{ and } \mathcal{I}(1,0) = 0.$$

As already explained, the implication of a weak extended-order algebra satisfies the equivalence $a \le b \Leftrightarrow a \to b = \top$, for all $a, b \in L$; this criterion states exactly when the implication $a \to b$ has value "true" (the top element $\top$), but it says almost nothing, in general, on when the implication is false, even if the existence of the value "false" (the least element $\bot$) is assumed in the algebra. Moreover, to obtain the condition $\mathcal{I}(\top, \bot) = \bot$ it is necessary to assume the symmetry condition on $(L, \to, \top)$.
Hence, we have proposed an alternative definition of implicator and a different arrangement of the axioms usually assumed on it. The obtained results are described in Section 2.2 and have been published in [14].
Starting from the second year of the Doctoral School we have directed our work toward the study of the fundamental mathematical tool of binary relations between sets, exploiting in this field the effect of using extended-order algebras as the range for the evaluation of the relationships between related elements.
Though non-classical logics give the main motivation for our work, our study is developed essentially in the framework of classical mathematics being, in this respect, in the line of abstract algebraic logic. This is quite usual also for fuzzy logic (see [52]), and even when a purely logical-theoretic approach on relations in fuzzy class theory is developed (see [4]), a corresponding description of the results in classical mathematical terms is

explained, to allow an easier reading and a more effective usefulness in their possible applications.
Our interest for (many-valued) binary relations has a twofold link with the first part of our work. On one hand, it is natural to start using the structure we propose for the algebra of the truth values in one of the most basic tools useful to develop a qualitative approach to mathematics and its applications. On the other hand, binary relations have a fundamental role, both in the origin of extended-order algebras, in particular for the order-theoretic approach, and in their development, based essentially on Galois connections.
So, most of our work is devoted to evaluate the application of extended-order algebras in the context of both homogeneous and heterogeneous binary relations, either in the classical case or in the many-valued one; binary relations are fundamental tools to describe any kind of situations under uncertain and incomplete knowledge and their treatment involves their composition and the powerset operators they generate (see also [4]).
In Chapter 3 it is developed a non-commutative and non-associative approach to many-valued relations taking values in some kind of extended-order algebras; most considered notions in the algebras of relations are already known and well developed (see for instance [4, 6, 87]), but our framework is more general than the usual ones; hence it has been highlighted the advantages to consider weaker structures in order to describe the manifold aspects of algebras of relations with values in extended-order algebras. In particular, in Section 3.3, we have considered some extensions of the notion of function among many-valued relations and we have checked under which conditions the considered generalizations characterize, in fact, functions. These notions are related to the structure of the underlying algebra: in fact, a first notion is given in the context of complete lattices, without assuming the existence of other structures on it, while the second is studied enriching the lattice with the structure of distributive extended-order algebra and the third involves symmetrical, possibly associative, algebras.
As already remarked, fundamental tools in the context of residuated structures are the isotone and antitone Galois connections. These have been first introduced while studying the properties of the powerset operators, now also called Birkhoff operators, associated to binary relations, in particular to functions (see [8]). Then, Galois connections have reached a wider framework within posets [70] and, more generally, in category theory where they, in fact, correspond to adjunctions [1, 51]. Meanwhile, the notion of conjugated pairs (of functions) have been introduced, once more dealing with powerset operators, by Tarski [83] and formalized in the framework of boolean algebras [60, 61]. This latter notion, being related by a sort of duality to Galois connections in the classical framework of boolean algebras, have received not much attention until the big development of Galois connections within fuzzy set theory, seemed to have broken the above mentioned duality. Then, G. Georgescu and A. Popescu considered in [42] fuzzy Galois connections and fuzzy conjugated pairs, looking for a possible duality that they could not find, as the title of their paper (*Non dual fuzzy connections*) suggests.
In Chapter 4 we restore a perfect duality between Galois connections and conjugated pairs, thanks to a quite general approach and a new, more appropriate view of the topic.

In fact, we have considered an approach to *global* and *relative connections* between weakly structured sets, giving a unified and elegant framework for Galois connections and conjugated pairs, which we have called *Tarski connections*. This common framework allows to clarify the interrelationships between these notions, which find out to be order-dual.
In order to understand the sense of these notions, it is useful to eliminate what seems to be redundant in the considered context, leaving only the (pre)order structure in the right place.
Even more, looking only at the essence of the structure, it is possible to observe that both Galois and Tarski connections are connections of the same type. We remark that, once more, binary many-valued relations have a fundamental role in Chapter 4.
Having in our hands a unified approach to all types of connections, which have important roles both for algebraic residuated structures and for the algebras of binary relations, in Chapter 5 we introduce a quite simple, but very effective tool, consisting in suitable diagrams, namely *relational* and *double relational triangles of relations*. These have been used to describe and characterize the structure of the extended-order algebras using the implication $\rightarrow$ and its derived operations. Moreover, these triangles give information on the algebra of many-valued relations, involving not only the algebraic operations lifted from the algebra of truth values, but also the richer structure given by the compositions of many-valued relations. As just remarked, connections, in particular Galois and Tarski connections, described in Chapter 4 have a fundamental role in this context, too.
We note, moreover, that thanks to this particular constructions we can synthesize also the properties of powerset operators associated to many-valued relations, considered up to now in less general contexts than the symmetrical distributive extended-order algebras we use.
In Chapter 6 we present a categorical accommodation of the main topics we have treated that involve binary, possibly many-valued, relations, moving into two directions. In both directions we fall into the need to weaken the requirements usually asked to categorical structures.
First, as previously noted, the lack of associativity has highlighted that the notion of category may be not appropriate to describe the class of relations taking values in some kind of extended-order algebras. So, we have proposed the notion of *pseudo-category* that misses the associativity of the composition of morphisms and the existence of identity morphisms, too. This might be useful to construct abstract models for algebras and categories of many-valued relations, more general than Dedekind categories [37, 69] and MV-relation algebras [71, 72].
The second followed direction leads to devote considerable attention to some kinds of *relational systems*, in particular of *tied relational systems* and to their morphisms. These notions are closely related to most concepts we have developed in the previous Chapters, since many objects we have considered, among which weak extended-order algebras, are particular types of relational systems, while connections considered in Chapter 4 are special types of morphisms of suitable relational systems.
A comprehensive view of these aspects of the topics we have treated leads to a de-

scription of a class of relational systems largely used in several contexts, including formal concept analysis [39, 40, 85], Chu spaces [73], topological and interchange systems [23, 24, 25, 79, 81, 82] and, more closely related to our work, implicative groupoids and extended-order algebras [13, 49, 50]. Doing this we can provide meaningful examples of semicategories and precategories, which have been introduced and studied in [53, 54, 76] as particular multiplicative graphs in the sense of C. Ehresman [31].
We note, in particular, that tied relational systems considered in Section 6.4 are generalizations of weak extended-order algebras; looking at the semicategory having those as objects, one can see that morphisms between them are not functions, in the style of universal algebra as they have been considered in [49, 80], but pairs of functions which form global connections. We have not exploited the usefulness of this kind of morphisms, which are also mentioned in [49] and would be the topic of our future researches.
Summing up, one could say that the path gone along the last three Sections of Chapter 6 closes the circle including the whole content of this thesis in a coherent and comprehensive framework.

Chapter 1

Preliminary classical notions and results

1.1 Preliminaries on order and lattices

To fix notation, which is slightly different from the usual one, we recall some notions and properties of classical binary relations (a classical reference is [67]) and, in particular of order and preorder relations and the lattice structures determined by them (for these topics we refer to [8, 34, 43, 59].
Let X, Y be two sets. A binary relation from X to Y, denoted by $r : X \rightharpoonup Y$, is a subset of the cartesian product, that is $r \subseteq X \times Y$; if $(x, y) \in r$, then we also write xry. The empty subset of $X \times Y$ is called *empty relation* from X to Y and it is denoted by $\perp\!\!\!\perp_{XY}$; $X \times Y$ is the *universal relation* denoted by $\top\!\!\!\top_{XY}$.
A *function* (or *map*) from X to Y, denoted by $f : X \to Y$, is a relation from X to Y such that for any $x \in X$, a unique $y \in Y$ exists such that $(x, y) \in f$; in this case we write $(x)f = y$. Y^X is the set of all functions from X to Y. We shall identify a subset $A \subseteq X$ with its characteristic function, that we shall denote by $A : X \to \mathbf{2}$ ($\mathbf{2} = \{\perp, \top\}$ is the trivial boolean algebra), where $(x)A = \top \Leftrightarrow x \in A$. Thus, the powerset $\mathcal{P}(X)$ of a set X is also denoted by $\mathbf{2}^X$.
We shall identify the binary relation $r : X \rightharpoonup Y$, as a subset, with its characteristic function, that we shall denote by $r : X \times Y \to \mathbf{2}$; so, $(x, y) \in r$, xry and $(x, y)r = \top$ are equivalent notations.
The *identity relation* $i_X : X \rightharpoonup X$ is defined by $(x, x')i_X = \top$ if and only if $x = x'$, for all $x, x' \in X$ (of course, i_X is a function). The *opposite* or *reverse relation* of r is $r_- : Y \rightharpoonup X$ defined by $(y, x)r_- = (x, y)r$. Obviously $(r_-)_- = r$.
A *preorder relation* in X is a function $\leq: X \times X \to \mathbf{2}$ such that $x = y \Rightarrow (x, y)\leq = (y, x)\leq = \top$ and $(x, y)\leq = (y, z)\leq = \top \Rightarrow (x, z)\leq = \top$, while an *order relation* requires the converse implication $(x, y)\leq = (y, x)\leq = \top \Rightarrow x = y$, too.
A set with a preorder relation is called *preordered set* and it is denoted by $(L, \leq)$; an ordered set is also called *poset*, which stands for partially ordered set.
The elements a and b in a preordered set are called *comparable* if either $a \leq b$ or $b \leq a$.

An ordered set $(L, \leq)$ in which all elements are comparable is called *totally ordered* or *linearly ordered* or it is said to be a *chain* and $\leq$ is a *total order relation*.
An *equivalence relation* on X is a function $\equiv: X \times X \to \mathbf{2}$ such that $(x,x)\equiv\, = \top$, $(x,y)\equiv = \top \Rightarrow (y,x)\equiv\, = \top$ and $(x,y)\equiv\, = (y,z)\equiv\, = \top \Rightarrow (x,z)\equiv\, = \top$.
For every $E \subseteq X$, we denote by $\langle E \rangle_{\equiv} = \{x \in X | \exists a \in E : x \equiv a\}$ the *saturation* of E with respect to $\equiv$. We note that the map

$$\langle \cdot \rangle_{\equiv} : \mathbf{2}^X \to \mathbf{2}^X, \ E \mapsto \langle E \rangle_{\equiv}$$

is a topological closure operator, that, moreover, commutes with arbitrary unions and intersections. The fixed points of such operator are the *saturated* subsets of L.
If $(L, \leq)$ is a preordered set, the binary relation $\equiv$ on L, defined, for all $x, y \in L$, by

$$x \equiv y \Leftrightarrow x \leq y \wedge y \leq x$$

is an equivalence relation on L which is said to be *induced* by $\leq$.
The map

$$x \mapsto \langle x \rangle_{\equiv} \in L_{\equiv}$$

is the *canonical surjection* of L onto the quotient set $L_{\equiv}$. Obviously, for all $x, x', y, y' \in L$, it follows from $x \equiv x', y \equiv y', x \leq y$ that $x' \leq y'$; hence, the relation $\leq$ on L induces on $L_{\equiv}$ an order relation $\preceq$ defined, for all $x, y \in L$, by $\langle x \rangle_{\equiv} \preceq \langle y \rangle_{\equiv} \Leftrightarrow x \leq y$. The relation $\preceq$ on $L_{\equiv}$ is called the *order relation induced* on $L_{\equiv}$ by the preorder $\leq$ and $(L_{\equiv}, \preceq)$ is called the *induced ordered set* by $(L, \leq)$.
Let $(L, \leq)$ be a preordered set and $X \subseteq L$. An element $a \in L$ is a *lower bound* for X if the following implication is true

$$x \in X \Rightarrow a \leq x.$$

We denote by LbX the set of all lower bounds of X.
$a \in L$ is a *greatest lower bound* or an *infimum* for X if $a \in LbX$ and the following implication holds

$$a' \in LbX \Rightarrow a' \leq a.$$

We denote by $InfX$ or by $\bigwedge X$ the set of all infima of X. If $\bigwedge X \neq \emptyset$, one says that X has infimum.
Moreover, $a \in L$ is a *minimum* for X if $a \in X \cap LbX$. We denote by $MinX = X \cap LbX$ the set of all minima for X. If $MinX \neq \emptyset$, one says that X has minimum.
An element $a \in L$ is an *upper bound* for X if the following implication is true

$$x \in X \Rightarrow x \leq a.$$

We denote by UbX the set of all upper bounds of X.
$a \in L$ is a *least upper bound* or a *supremum* for X if $a \in UbX$ and the following implication holds

$$a' \in UbX \Rightarrow a \leq a'.$$

We denote by $SupX$ or by $\bigvee X$ the set of all suprema of X. If $\bigvee X \neq \emptyset$, one says that X has supremum.

Moreover, $a \in L$ is a *maximum* for X if $a \in X \cap UbX$. We denote by $MaxX = X \cap UbX$ the set of all maxima for X. If $MaxX \neq \emptyset$, one says that X has maximum.

Remark 1.1.1. 1. For any subset $X \subseteq L$, $UbX, LbX, \bigwedge X$ and $\bigvee X$ are saturated, while $MinX$ and $MaxX$ need not to be so, unless X itself is saturated. The elements of $\bigwedge X$ and those of $\bigvee X$, $MinX$, $MaxX$, as well, are equivalent to each other.

2. Elements equivalent to a minimum or to a maximum of X need not to be minima or maxima, since they might not be in X but, clearly, X has a minimum if and only if $\langle X \rangle_{\equiv} \cap LbX \neq \emptyset$ and X has a maximum if and only if $\langle X \rangle_{\equiv} \cap UbX \neq \emptyset$.

3. For $X \subseteq L$ one has $\bigwedge X = Max(LbX)$ and $\bigvee X = Min(UbX)$, while $MaxX = X \cap \bigvee X$ and $MinX = X \cap \bigwedge X$. In particular, we note that $Ub\emptyset = Lb\emptyset = L$ and, hence, $\bigvee \emptyset$ is non-empty if and only if $MinL$ is non-empty and they coincide; similarly, $\bigwedge \emptyset$ is non-empty if and only if $MaxL$ is non-empty and they coincide.

4. If $(L, \leq)$ is an ordered set, there exist at most one infimum and at most one supremum for $X \subseteq L$. Similarly, there exist at most one minimum and one maximum for $X \subseteq L$.
   Even more, the uniqueness of infimum, or supremum, characterizes orders among preorders, by the equivalent statements:

   (a) $\leq$ is an order relation;

   (b) for every $X \subseteq L$: $|\bigwedge X| \leq 1$;

   (c) for every $X \subseteq L$: $|\bigvee X| \leq 1$.

Definition 1.1.2. *A preordered set $(L, \leq)$ is a bounded prelattice (in the sequel only prelattice) if, for all finite subsets $F \subseteq L$, $\bigvee F$ and $\bigwedge F$ are non-empty. If $\leq$ is an order relation, $(L, \leq)$ is called bounded lattice.*

In the sequel we omit the attribute bounded for bounded lattice.
In case of a lattice, it is well known that the operations $\vee$ and $\wedge$ defined by

$$a \vee b = \bigvee \{a, b\} \text{ and } a \wedge b = \bigwedge \{a, b\}$$

are associative, commutative, idempotent, satisfy the absorption conditions and have $\bot$ and $\top$ as units, respectively; conversely, these properties characterize a lattice among the algebras $(L, \vee, \wedge, \bot, \top)$ of signature $(2, 2, 0, 0)$.
From the above definition it follows that, in every lattice $(L, \leq)$, there exist $\bot = \bigvee \emptyset = MinL$ and $\top = \bigwedge \emptyset = MaxL$. Of course, $|L| \geq 2$ if and only if $\top \neq \bot$.
If $\top = \bot$ the lattice is said to be *inconsistent*; otherwise it is *consistent*.

Definition 1.1.3. *A preordered (ordered) set $(L, \leq)$ is a complete prelattice (lattice) if, for all subsets $F \subseteq L$, $\bigvee F$ and $\bigwedge F$ are non-empty.*

Definition 1.1.4. *A lattice* $(L, \leq)$ *is distributive if the following equalities hold (which are equivalent to each other)*
$(D1)$ $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$, *for all* $a, x, y \in L$;
$(D2)$ $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$, *for all* $a, x, y \in L$.

Definition 1.1.5. *A (complete) distributive lattice* $(L, \leq)$ *is a boolean lattice, usually called (complete) boolean algebra, if there is a unary operation* $\neg : L \to L$, $a \mapsto a^{\neg}$ *such that* $a^{\neg} \wedge a = \bot$ *and* $a^{\neg} \vee a = \top$, *for every* $a \in L$. *The function* $[\cdot]^{\neg}$ *(uniquely determined by the order relation* $\leq$*) is called complementation and* $a^{\neg}$ *is the complement of* a.

Proposition 1.1.6. *If* $(L, \leq)$ *is a boolean lattice, then the following hold.*

1. $[\cdot]^{\neg}$ *is an involution;*

2. $[\cdot]^{\neg}$ *reverses order.* $\square$

Of course, as for every order reversing involution, the De Morgan laws are satisfied, then in particular $[\cdot]^{\neg}$ satisfies the De Morgan laws, both in the finite and in the infinite form, the latter in case of complete boolean lattices.

Definition 1.1.7. *A (complete) lattice* $(L, \leq)$ *is an Heyting lattice, usually called (complete) Heyting algebra, if, for all* $a, b \in L$, $max\,\{x \in L | x \wedge a \leq b\}$ *exists.*

The binary operation $\to$ that can be defined in an Heyting lattice as follows

$$\to: L \times L \to L,\ (a, b) \mapsto a \to b = max\,\{x \in L | x \wedge a \leq b\},$$

is called *implication*.

Remark 1.1.8. An Heyting lattice may be defined as a lattice with a binary operation $\to$ that satisfies the condition

$$x \leq a \to b \Leftrightarrow a \wedge x \leq b, \text{ for all } a, b, x \in L.$$

Proposition 1.1.9. *If* $(L, \leq)$ *is an Heyting lattice, then, for all* $a, b, c \in L$, *the following hold*

1. $a \to a = \top$;

2. $a \wedge (a \to b) = a \wedge b$;

3. $b \wedge (a \to b) = b$;

4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$. $\square$

Proposition 1.1.10. *If* $(L, \leq)$ *is a lattice with a binary operation* $\to: L \times L \to L$ *that satisfies the properties (1), (2), (3) and (4) of Proposition 1.1.9, then the following hold*

1. $b \leq b' \Rightarrow a \to b \leq a \to b'$, *for all* $a \in L$;

2. $b \leq a \to b$. $\square$

Proposition 1.1.11. *A lattice* $(L, \leq)$ *with a binary operation* $\to$ *is an Heyting lattice if and only if* $\to$ *satisfies the conditions (1)-(4) of Proposition 1.1.9.* $\square$

Proposition 1.1.12. *Every Heyting lattice is distributive.* $\square$

Corollary 1.1.13. *If* $(L, \leq)$ *is an Heyting lattice, then the following properties hold, for all* $a, a', b \in L$

1. $(a \vee a') \to b = (a \to b) \wedge (a' \to b)$;
2. $a \leq a' \Rightarrow a' \to b \leq a \to b$. $\square$

Proposition 1.1.14. *Every boolean lattice* L *is an Heyting lattice, whose implication is defined by* $a \to b = a^{\neg} \vee b$, *for all* $a, b \in L$. $\square$

Definition 1.1.15. *Let* $(L, \leq)$ *be an Heyting lattice. The unary operation*

$$[\cdot]^{\neg} : L \to L,\ a \mapsto a^{\neg} = a \to \bot$$

is called pseudo-complementation or negation.

Proposition 1.1.16. *In any Heyting lattice* $(L, \leq)$ *the negation satisfies the following properties, for all* $a, b \in L$:

1. $a \wedge a^{\neg} = \bot$;
2. $[\cdot]^{\neg}$ *reverses order;*
3. $(a \vee b)^{\neg} = a^{\neg} \wedge b^{\neg}$;
4. $\bot^{\neg} = \top$ *and* $\top^{\neg} = \bot$. $\square$

Proposition 1.1.17. *Let* $(L, \leq)$ *be an Heyting lattice and* $[\cdot]^{\neg}$ *its negation. Then*

$$(L, \leq) \text{ is a boolean lattice} \Leftrightarrow [\cdot]^{\neg} \text{ is an involution. } \square$$

Definition 1.1.18. *Let* $(L, \leq)$ *be a complete lattice.*
$(L, \leq)$ *satisfies the first infinite distributivity law if it satisfies the condition*
$(1DL\infty)$ $a \wedge (\bigvee S) = \bigvee \{a \wedge s | s \in S\}$, *for all* $a \in L$ *and* $S \subseteq L$.
$(L, \leq)$ *satisfies the second infinite distributivity law if it satisfies the condition*
$(2DL\infty)$ $a \vee (\bigwedge S) = \bigwedge \{a \vee s | s \in S\}$, *for all* $a \in L$ *and* $S \subseteq L$.
A complete lattice that verifies $(1DL\infty)$, *(*$(2DL\infty)$*, respectively), is called frame, (coframe, respectively).*

Proposition 1.1.19. *Every complete Heyting lattice is a frame.* $\square$

Proposition 1.1.20. *If* $(L, \leq)$ *is a frame, then it is an Heyting lattice.* $\square$

Proposition 1.1.21. *If* $(L, \leq)$ *is a complete boolean lattice, then it is a frame and a coframe.* $\square$

1.2 Galois connections and conjugated pairs

In this Section we describe, in their classical version, the notion of *Galois connections* and *conjugated pairs* that are fundamental tools for the study of residuated structures, within which are framed the algebras studied in this thesis.
Galois connections, also known as *adjunctions*, have been considered, with a different terminology by G. Birkhoff already in 1940 in the first edition of [8] and by O. Ore [70]; the *conjugated pairs* have been introduced by A. Tarski in [83] and subsequently studied by B. Jónsson and A. Tarski in [61].
An unitary and more general study of these notions has been developed in [18, 19].

Definition 1.2.1. *An adjuction, also called isotonic Galois connection, between partially ordered sets L and M, denoted by $f \dashv g : (L, \leq) \to (M, \leq)$ or simply by $f \dashv g$, is a pair of maps $f : L \to M$ and $g : M \to L$ satisfying the condition*

$$\forall x \in L, y \in M\colon\ x \leq (y)g \Leftrightarrow (x)f \leq y.$$

The map f is called left adjoint of g and g right adjoint of f.

Remark 1.2.2. Equivalently, $f \dashv g$ if and only if f and g are isotonic and satisfy the adjoint inequalities:
$(AD1)$ $x \leq ((x)f)g$, for any $x \in L$;
$(AD2)$ $((y)g)f \leq y$, for any $y \in M$.
In the following we denote $(A)f = \{(a)f \in M \mid a \in A\}$ and $(B)f^{-} = \{a \in L \mid (a)f \in B\}$, for every function $f : L \to M$ and subsets $A \subseteq L$, $B \subseteq M$.

Proposition 1.2.3. *(Adjoint functor theorem) Let $(L, \leq)$ and $(M, \leq)$ be posets. If $f : L \to M$ and $g : M \to L$ are maps such that $f \dashv g$, then the following hold:*

1. *for any $A \subseteq L$, if $\bigvee A$ exists, then $(\bigvee A)f = \bigvee (A)f$;*
2. *for any $B \subseteq M$, if $\bigwedge B$ exists, then $(\bigwedge B)g = \bigwedge (B)g$;*
3. *for any $x \in L$, $(x)f = \bigwedge \{y \in M \mid x \leq (y)g\}$ and f is the unique left adjoint of g;*
4. *for any $y \in M$, $(y)g = \bigvee \{x \in L \mid (x)f \leq y\}$ and g is the unique right adjoint of f.* □

Proposition 1.2.4. *Let $(L, \leq)$ and $(M, \leq)$ be posets.*

1. *If $(L, \leq)$ is a complete lattice and $f : L \to M$ preserves $\bigvee$, then the function*

$$g : M \to L,\ y \mapsto (y)g = \bigvee \{x \in L \mid (x)f \leq y\}$$

is the unique right adjoint of f.

2. *If $(M, \leq)$ is a complete lattice and $g : M \to L$ preserves $\bigwedge$, then the function*

$$f : L \to M,\ x \mapsto (x)f = \bigwedge \{y \in M \mid x \leq (y)g\}$$

is the unique left adjoint of g. □

Definition 1.2.5. *A Galois connection, also called antitonic Galois connection, between two posets $(L,\leq)$ and $(M,\leq)$, denoted by $[f,g]:(L,\leq)\leftrightarrow(M,\leq)$ or simply by $[f,g]$, is a pair of maps $f:L\to M$ and $g:M\to L$ satisfying the condition*

$$\forall x\in L, y\in M\colon\ x\leq (y)g \Leftrightarrow y\leq (x)f.$$

Remark 1.2.6. Equivalently, $[f,g]$ if and only if f and g are antitonic and satisfy the inequalities:
$(GC1)$ $x\leq ((x)f)g$, for any $x\in L$;
$(GC2)$ $y\leq ((y)g)f$, for any $y\in M$.
Of course $[g,f]$ is a Galois connection if and only if $[f,g]$ is.

Proposition 1.2.7. *Let $[f,g]$ be a Galois connection between the posets $(L,\leq)$ and $(M,\leq)$. Then, for every $A\subseteq L$ the following hold:*

1. *if $\bigvee A$ exists, then $(\bigvee A)f = \bigwedge(A)f$;*
2. *if $\bigwedge A$ exists, then $(\bigwedge A)f\in Ub((A)f)$;*
3. *for any $y\in M$, $(y)g=\bigvee\{x\in L\mid y\leq (x)f\}$ and g is the unique function such that $[f,g]$ is a Galois connection.* □

Proposition 1.2.8. *Let $(L,\leq)$ be a complete lattice, $(M,\leq)$ a poset and $f:L\to M$ a function such that $(\bigvee A)f=\bigwedge(A)f$, for every $A\subseteq L$. The function $g:M\to L$ defined by*

$$(y)g=\bigvee\{x\in L\mid y\leq (x)f\},\ \textit{for any } y\in M,$$

is the unique function such that $[f,g]$ is a Galois connection. □

Let X and Y be two sets and $f:X\to Y$ be a function. The following are two classical and important examples of Galois connection and adjunction.

1. Denoting by $Lb:\mathcal{P}(X)\to\mathcal{P}(X)$ the function that to each subset $S\subseteq X$ associates LbS and the function $Ub:\mathcal{P}(X)\to\mathcal{P}(X)$ the function that to each subset $S\subseteq X$ associates UbS, we have that

$$[Lb,Ub]:(\mathcal{P}(X),\subseteq)\leftrightarrow(\mathcal{P}(Y),\subseteq) \text{ for any } X,Y.$$

2. Denoting again with f the function that to each subset $A\subseteq X$ associates its image by f and with f^- the function that to each subset $B\subseteq Y$ associates its reciprocal image, we have that

$$f\dashv f^-:(\mathcal{P}(X),\subseteq)\to(\mathcal{P}(Y),\subseteq) \text{ for any } X,Y,\ f:X\to Y.$$

Now we consider an boolean algebra $(A,+,0,\cdot,1)$ and the notion of conjugated pair introduced in [83].

Definition 1.2.9. *Let $A = (A, +, 0, \cdot, 1)$ be a boolean algebra. Two functions $f, g : A \to A$ form a conjugated pair, denoted by $\langle f, g\rangle$ if, for all $a, b \in A$, the following equivalence holds:*

$$(a)f \cdot b = 0 \Leftrightarrow (b)g \cdot a = 0.$$

If, in particular, a function f is a conjugated of itself, then we call f self-conjugated.

Theorem 1.2.10. *Let $A = (A, +, 0, \cdot, 1)$ be a boolean algebra.*

1. *If f and g are two functions from A to A such that $\langle f, g\rangle$, then $\langle g, f\rangle$.*
2. *Let $f : A \to A$ be a function. Then there exists at most one $g : A \to A$ such that $\langle g, f\rangle$. If such g exists, it can be determined, for every $y \in A$, by the formula $(y)g = \bigvee \{x^{\neg} |\ (x)f \cdot y = 0\}$.*
3. *Let $f : A \to A$ be a function. There exists $g : A \to A$ such that $\langle g, f\rangle$ if and only if $(\bigvee S)f = \bigvee(Sf)$, for every $S \subseteq A$.* □

1.3 MacNeille completion of posets

In [66], MacNeille presents a completion process of ordered sets that generalizes the Dedekind completion of rational numbers, by using the sections or cuts of ordered set, after adding to it a maximum and a minimum, if it is devoid of them.
In this Section, we recall the MacNeille completion of a poset with a greatest element as it has been described in [50]. This process is a variant of the Dedekind-MacNeille completion proposed in [66] which requires only the existence of the maximum in the ordered set and which allows to build a completion that is isomorphic to that of MacNeille.
Let $(L, \leq, \top)$ be a partially ordered set with greatest element $\top$.
Define in $\mathcal{P}(L)$ the equivalence relation

$$A \approx A' \text{ if } Lb(A) = Lb(A'), \text{ for all } A, A' \subseteq L$$

and denote by K the quotient set, i.e. $K = \frac{\mathcal{P}(L)}{\approx}$, and by $[A]$ the equivalence class of $A \subseteq L$; $[a]$ stands for $[\{a\}]$, if $a \in L$.

Lemma 1.3.1. *[50]*

1. *If $\alpha = [A_i]$, for every $i \in I$, then $\alpha = [\bigcup_{i\in I} A_i]$;*
2. *if $\alpha = [A]$, then $Ub(Lb(A)) = \bigcup \{X |\ [X] = \alpha\}$ is the largest representative of α;*
3. *if $a = \bigwedge A \in L$ exists, then $[A] = [a] = [\uparrow a]$;*
4. *if $[a] = [a']$, then $a = a'$;*
5. *for every $\alpha \in K$ there exists $A \neq \varnothing$: $\alpha = [A]$.* □

Consider the binary relation in K defined, for all $\alpha, \beta \in K$, $\alpha = [A]$, $\beta = [B]$, by

$$\alpha \leq_K \beta \text{ if and only if } Lb(A) \subseteq Lb(B).$$

We note that

- $[A] \leq_K [B]$ if $B \subseteq A$; moreover, if B is the largest representative of β, then $\alpha \leq_K \beta \Leftrightarrow B \subseteq A$.
- The map $\varphi : L \to K$, $a \mapsto (a)\varphi = [a]$ is an embedding of posets.
- $(K, \leq_K)$ is a complete lattice with greatest element $\top_K = [\top]$, least element $\bot_K = [L]$ and, moreover, for any $S \subseteq K$,

$$\vee_K S = [\cap \{Ub(Lb(A))|\ [A] \in S\}]$$

and

$$\wedge_K S = [Ub(\cap \{Lb(A)|\ [A] \in S\})].$$

- It is clear that $(K, \leq_K)$ is isomorphic to the MacNeille completion of the poset $(L, \leq)$ as it is described in [66].

Remark 1.3.2. [50]

1. If $B \subseteq K$, then it follows from the properties of the Galois connection $[Lb, Ub]$ that

$$\bigwedge_K B = [Ub(Lb(\bigcup \{A|\ [A] \in B\}))] = [\bigcup \{A|\ [A] \in B\}]$$

and

$$\bigvee_K B = [Ub(\bigcup \{Lb(A)|\ [A] \in B\})].$$

2. If $A \subseteq L$, then $\wedge_K (A)\varphi = \wedge_K \{[a] \mid a \in A\} = [Ub(Lb(A))] = [A]$.
Hence $\wedge_K A = \wedge_K A'$ if and only if $[A] = [A']$.
Similarly, if $B \subseteq L$, then $\vee_K (B)\varphi = [\cap \{Ub(Lb(b))|\ b \in B\}] = [\cap \{Ub(b)|\ b \in B\}] = [Ub(\cup \{\{b\} \mid b \in B\})] = [Ub(B)]$.

1.4 Classical binary relations and powerset operators

As just recalled in the previous Section, a binary relation from the set X to the set Y, denoted by $r : X \rightharpoonup Y$, is a subset of their cartesian product, that is $r \subseteq X \times Y = \{(x, y)|\ x \in X,\ y \in Y\}$.
We can give an alternative definition, by considering the set $\{0, 1\}$ that is generally denoted by **2**. It is well known that **2** is a boolean algebra, with respect to the obvious order relation $\leq$ such that $0 \leq 1$.

Let $X, Y \in |\mathbf{Set}|$. The binary relation r from X to Y is a map $r : X \times Y \to \mathbf{2}$ such that, for all $x \in X, y \in Y$:

$$(x,y)r = \begin{cases} 1 & \text{if } xry \\ 0 & \text{otherwise} \end{cases} .$$

As usual, for all sets X, Y, we define:

1. the identity relation $i_X : X \rightharpoonup X$: $(x,x')i_X = 1$ if and only if $x = x'$, for all $x, x' \in X$;
2. the universal relation $\top\!\!\top_{XY}$: $X \rightharpoonup Y$: $(x,y)\top\!\!\top_{XY} = 1$, for all $x \in X, y \in Y$;
3. the zero or empty relation $\perp\!\!\perp_{XY}$: $X \rightharpoonup Y$: $(x,y)\perp\!\!\perp_{XY} = 0$, for all $x \in X, y \in Y$.

We note that the empty relation is the unique relation from X to Y if and only if either X or Y is empty, that is if and only if $\perp\!\!\perp_{XY} = \varnothing$. Moreover, we note that for every $X \in |\mathbf{Set}|$ $\top\!\!\top_{XX} = i_X$ if and only if $X = \{x\}$.
In the following we consider the class of all binary relations between two arbitrary sets and we denote it by $\mathbf{R_2} = \{r : X \rightharpoonup Y|\ \forall X, Y \in |\mathbf{Set}|\}$. Moreover, for all $X, Y \in |\mathbf{Set}|$ we denote by $\mathbf{R_2}(X,Y) = \{r : X \rightharpoonup Y\}$ the set of all binary relation from X to Y.
Let $X, Y \in |\mathbf{Set}|$ and let $r \in \mathbf{R_2}(X,Y)$. The opposite relation of r is $r_- : Y \rightharpoonup X$ defined by $(y,x)r_- = (x,y)r$. Obviously $(r_-)_- = r$.
Let $X \in |\mathbf{Set}|$. A binary relation $r \in \mathbf{R_2}(X,X)$ is called preorder if it satisfies the following conditions:

1. $(x,x)r = 1$, for every $x \in X$ (reflexivity condition);
2. if $(x,y)r = 1$ and $(y,z)r = 1$, then $(x,z)r = 1$, for all $x, y, z \in X$ (transitivity condition).

An order relation $r \in \mathbf{R_2}(X,X)$ is a preorder that verifies the following condition: if $(x,y)r = 1$ and $(y,x)r = 1$, then $x = y$, for all $x, y \in X$ (antisymmetry condition).
An equivalence relation $r \in \mathbf{R_2}(X,X)$ is a preorder that verifies the following condition: if $(x,y)r = 1$ then $(y,x)r = 1$, for all $x, y \in X$ (symmetry condition).

Remark 1.4.1. 1. The opposite relation of an order relation is an order, too.

2. Every symmetrical relation, and hence every equivalence, coincides with its opposite.
3. The identity is the unique relation that is both an order and an equivalence.
4. The empty relation $\varnothing$ from X to X is an order and an equivalence if and only if $X = \varnothing$.

Definition 1.4.2. *Let $X, Y \in |\boldsymbol{Set}|$ and let $r \in \mathbf{R_2}(X,Y)$. For all $A \subseteq X, B \subseteq Y$ we define the set*

$$Ar = \{y \in Y | \exists x \in A : \ (x,y)r = 1\}$$

as the direct cone or image of A with respect to r and the set

$$rB = \{x \in X | \exists y \in B : \ (x, y)r = 1\}$$

as the inverse cone or inverse image of B with respect to r.
In particular, for all $\{x\} \subseteq X, \{y\} \subseteq Y$ we define the set

$$\{x\}\, r = xr = \{y \in Y | \ (x, y)r = 1\}$$

as the direct cone or image of x with respect to r and the set

$$r\,\{y\} = ry = \{x \in X | \ (x, y)r = 1\}$$

as the inverse cone or inverse image of y with respect to r.

Note that $rB = Br_{-}$ and $Ar = \{y \in Y | \ ry \cap A \neq \varnothing\}$.
Moreover, their characteristic functions are $(y)Ar = \bigvee_{x \in X} (x, y)r \wedge (x)A$ and $(x)rB = \bigvee_{y \in Y} (x, y)r \wedge (y)B$.

Definition 1.4.3. *Let $X, Y \in |\boldsymbol{Set}|$ and let $r \in \mathbf{R_2}(X, Y)$. We say that:*

1. *r is left total if and only if for every $x \in X$: $xr \neq \varnothing$;*
2. *r is right total or surjective if and only if for every $y \in Y$: $ry \neq \varnothing$;*
3. *r is left univocal if and only if for all $x, x' \in X$, $x \neq x'$: $xr \cap x'r = \varnothing$;*
4. *r is right univocal if and only if for all $y, y' \in Y$, $y \neq y'$: $ry \cap ry' = \varnothing$;*
5. *r is injective if and only if for all $x, x' \in X$, $x \neq x'$: $xr \neq x'r$;*
6. *r is back injective if and only if for all $y, y' \in Y$, $y \neq y'$: $ry \neq ry'$;*
7. *r is bijective if and only if r is injective and surjective.*
8. *r is function if and only if r is right univocal and left total (in this case we denote r by $r : X \to Y$);*
9. *r is biunivocal if and only if r is an injective function.*

Proposition 1.4.4. *Let $X, Y \in |\boldsymbol{Set}|$ and let $r \in \mathbf{R_2}(X, Y)$. The following hold:*

1. *r is left univocal if and only if r_{-} is right univocal;*
2. *r is left total if and only if r_{-} is right total;*
3. *r is back injective if and only if r_{-} is injective;*
4. *if r is a function, then r_{-} is function if and only if r is injective and surjective (in this case we denote r_{-} by r^{-1});*

5. *if r is a function, then r is injective if and only if r is left univocal.* □

Remark 1.4.5. 1. The biunivocality is not equivalent to the bijectivity.

2. The identity relation is bijective.

There are different ways to compose two relations (see [4, 6]).

Definition 1.4.6. *Let $X, Y, Z \in |\boldsymbol{Set}|$; let $r \in \mathbf{R_2}(X,Y)$ and $s \in \mathbf{R_2}(Y,Z)$. For all $x \in X, z \in Z$, we define the following relations compositions:*

1. $x(r \cdot s)z \Leftrightarrow xr \cap sz \neq \varnothing$;
2. $x(r \oslash s)z \Leftrightarrow xr \subseteq sz$;
3. $x(r \varobslash s)z \Leftrightarrow xr \supseteq sz$;
4. $x(r \ominus s)z \Leftrightarrow xr = sz$.

Remark 1.4.7. 1. The relations compositions $\oslash$ and $\varobslash$ are related by the following equality $r \varobslash s = (s_- \oslash r_-)_-$.

2. If $r \in \mathbf{R_2}(X,Y)$ and $s \in \mathbf{R_2}(Y,Z)$ are functions, then $r \cdot s = r \oslash s$.

Proposition 1.4.8. *Let $X, Y, Z, W \in |\boldsymbol{Set}|$ and let $r \in \mathbf{R_2}(X,Y)$, $s \in \mathbf{R_2}(Y,Z)$ and $t \in \mathbf{R_2}(Z,W)$. The following hold.*

1. $i_X \cdot r = r \cdot i_Y = r$;
2. $r \cdot (s \cdot t) = (r \cdot s) \cdot t$;
3. $r \cdot \perp\!\!\!\perp_{YZ} = \perp\!\!\!\perp_{XY} \cdot s = \perp\!\!\!\perp_{XZ}$;
4. *s is right total if and only if* $\top\!\!\!\top_{XY} \cdot s = \top\!\!\!\top_{XZ}$;
5. *r is left total if and only if* $r \cdot \top\!\!\!\top_{YZ} = \top\!\!\!\top_{XZ}$;
6. *r is right (left, respectively) univocal if and only if $r_- \cdot r \leq i_Y$ ($r \cdot r_- \leq i_X$, respectively);*
7. *r is right (left, respectively) total if and only if $r_- \cdot r \geq i_Y$ ($r \cdot r_- \geq i_X$, respectively);*
8. *r is a function if and only if $r \cdot r_- \geq i_X$ and $r_- \cdot r \leq i_Y$.* □

Remark 1.4.9. The Proposition 1.4.8 allows to say that the triple $\mathbf{R_2^{\cdot}} = (|\mathbf{Set}|, \mathbf{R_2}, \cdot)$ where the objects are the sets, the morphisms are the binary relations, $\cdot$ is the composition above defined and the identity morphisms are the identity relations, is a category.

Since $r \varobslash s = (s_- \oslash r_-)_-$, for all $X, Y, Z \in |\mathbf{Set}|$, $r \in \mathbf{R_2}(X,Y)$ and $s \in \mathbf{R_2}(Y,Z)$, we list the properties involving the composition $\oslash$, from which are deduced those of $\varobslash$.

Proposition 1.4.10. *Let $X, Y, Z \in |\boldsymbol{Set}|$; let $r \in \mathbf{R_2}(X,Y)$ and $s \in \mathbf{R_2}(Y,Z)$. The following hold.*

1. *$i_X \oslash r = r$;*
2. *$r \oslash \top\!\!\top_{YZ} = \top\!\!\top_{XZ}$;*
3. *$\perp\!\!\perp_{XY} \oslash s = \top\!\!\top_{XZ}$;*
4. *$\top\!\!\top_{XY} \oslash s = \top\!\!\top_{XZ}$ if and only if $s = \top\!\!\top_{YZ}$;*
5. *$r \oslash \perp\!\!\perp_{YZ} = \top\!\!\top_{XZ}$ if and only if $r = \perp\!\!\perp_{XY}$;*
6. *$\top\!\!\top_{XX} \oslash r \le r$;*
7. *$\top\!\!\top_{XY} \oslash i_Y = \begin{cases} \top\!\!\top_{XY} & \text{if } Y = \{y\} \\ \perp\!\!\perp_{XY} & \text{otherwise} \end{cases}$*
8. *r is right (left, respectively) univocal if and only if $r \le r \oslash i_Y$ ($r \le i_X \oslash r$, respectively);*
9. *r is left total if and only if $r \oslash \perp\!\!\perp_{YZ} = \perp\!\!\perp_{XZ}$;*
10. *r is a function if and only if $r \oslash \perp\!\!\perp_{YZ} = \perp\!\!\perp_{XZ}$ and $r \le r \oslash i_Y$.* □

Proposition 1.4.11. *Let $X, Y, Z \in |\boldsymbol{Set}|$; let $r \in \mathbf{R_2}(X,Y)$ and $s \in \mathbf{R_2}(Y,Z)$. The following hold.*

1. *$r \ominus \top\!\!\top_{YZ} = \top\!\!\top_{XZ}$ if and only if $r = \top\!\!\top_{XY}$;*
2. *$\perp\!\!\perp_{XY} \ominus s = \top\!\!\top_{XZ}$ if and only if $s = \top\!\!\top_{YZ}$;*
3. *$r \ominus \perp\!\!\perp_{YZ} = \top\!\!\top_{XZ}$ if and only if $r = \perp\!\!\perp_{XY}$;*
4. *$\perp\!\!\perp_{XY} \ominus s = \top\!\!\top_{XZ}$ if and only if $s = \perp\!\!\perp_{YZ}$;*
5. *r is right (left, respectively) univocal if and only if $r \ominus i_Y = r$ ($i_X \ominus r = r$, respectively);*
6. *r is right (left, respectively) total if and only if $\perp\!\!\perp_{ZX} \ominus r = \perp\!\!\perp_{ZY}$ ($r \ominus \perp\!\!\perp_{YZ} = \perp\!\!\perp_{XZ}$, respectively);*
7. *r is a function if and only if $r \ominus \perp\!\!\perp_{YZ} = \perp\!\!\perp_{XZ}$ and $r \ominus i_Y = r$.* □

We note that $\oslash$, $\otimes$ and $\ominus$ are not associative, in general.

Definition 1.4.12. *The correspondence mapping any subset $A \subseteq X$ to its direct image is a function called weak (or existential or angelic) forward powerset operator, which we denote by $r^{\wedge} : \mathbf{2}^X \to \mathbf{2}^Y$; hence $(A)r^{\wedge} = Ar$.*
The strong (or universal or demonic) left forward powerset operators ${}^{\le}r$ may be defined by $(y)(A)^{\le}r = \bigwedge_{x \in X}((x)A, (x,y)r)\le$. In more usual, classical terms, $(A)^{\le}r = \{y \in Y|\ A \subseteq ry\}$.
The strong (or universal or demonic) right forward powerset operators $r^{\le}$ may be defined by $(y)(A)r^{\le} = \bigwedge_{x \in X}((x,y)r, (x)A)\le$. In more usual, classical terms, $(A)r^{\le} =$

$\{y \in Y \mid ry \subseteq A\}$.
The backward powerset operators of r *are the corresponding forward powerset operators of* r_-.

We recall that it is possible to characterize functions among binary relations by means of their backward powerset operators as follows.

Proposition 1.4.13. *Let* $r : X \rightharpoonup Y$. *Then* r *is a function if and only if* $r_-^{\wedge} = r_-^{\leq}$. □

Chapter 2

Extended-order algebras and the implicators

2.1 Extended-order algebras

In this Section we develop the study of extended-order algebras, recently introduced by C. Guido and P. Toto, which are implication algebras that generalize all the widely considered integral residuated structures. Particular care is devoted to the requirement of completeness that can be obtained by the MacNeille completion process. Associativity, commutativity and symmetry assumptions are characterized and their role is discussed toward the structure of the algebra and of its completion. As an application, further operations corresponding to the logical connectives of conjunction, negation and disjunction are considered and their properties are investigated, either assuming or excluding the additional conditions of associativity, commutativity and symmetry. An overlook is also devoted to the relationship with other similar structures already considered such as implication algebras (in particular Heyting algebras), BCK algebras, quantales, residuated lattices and closed categories.

2.1.1 Basic definition and results

In this Subsection we give the basic definitions, examples and results concerning the algebraic structures we shall deal with all along this thesis. In fact, for the sake of completeness we recall the basic notions from [50] where extended-order algebras have been first considered: we only change slightly the approach to the distributivity conditions, which allows to strengthen most results of [50] related to distributivity.

Definition 2.1.1. *[50] Let L be a non-empty set, $\rightarrow: L \times L \rightarrow L$ a binary operation and $\top$ a fixed element of L. The triple $(L, \rightarrow, \top)$ is a* ***weak extended-order algebra****, shortly* ***w-eo algebra****, if for all $a, b, c \in L$ the following conditions are satisfied:*
(o_1) *$a \rightarrow \top = \top$ (upper bound condition);*
(o_2) *$a \rightarrow a = \top$ (reflexivity condition);*

(o_3) $a \to b = \top$ *and* $b \to a = \top \Rightarrow a = b$ *(antisymmetry condition);*
(o_4) $a \to b = \top$ *and* $b \to c = \top \Rightarrow a \to c = \top$ *(weak transitivity condition).*

Proposition 2.1.2. *[50] For every w-eo algebra* $(L, \to, \top)$ *the relation determined by the operation* $\to$*, by means of the equivalence*

$$a \leq b \text{ if and only if } a \to b = \top, \text{ for all } a, b \in L$$

is an order relation in L*. Moreover* $\top$ *is the greatest element in* $(L, \leq)$*. This order relation is called the* ***natural ordering*** *in* L*.*
Conversely, if $(L, \leq, \top)$ *is a partially ordered set with a greatest element* $\top$ *and* $\to: L \times L \to L$ *extends* $\leq$*, i.e.* $a \to b = \top \Leftrightarrow a \leq b$*, for all* $a, b \in L$*, then* $(L, \to, \top)$ *is a w-eo algebra.* □

Weak extended-order algebras are called *implicative algebras* in [74].
The next two stronger axioms on L have been also considered in [50]:
(o_5) $a \to b = \top \Rightarrow (c \to a) \to (c \to b) = \top$ (*weak isotonic condition in the second variable*);
(o_5') $a \to b = \top \Rightarrow (b \to c) \to (a \to c) = \top$ (*weak antitonic condition in the first variable*).

We note that in a w-eo algebra (o_5) and (o_5') are independent of each other and that (o_4) does not imply any of them, even if (o_1), (o_2) and (o_3) are also assumed. It is easy to show that (o_5) and (o_5') imply (o_4), if (o_1) and (o_3) are assumed (see [50] and the structures 24, 25 and 26 of Appendix).

Definition 2.1.3. *[50]* $(L, \to, \top)$ *is a* ***right w-eo algebra*** *if it satisfies the axioms* (o_1)*,* (o_2)*,* (o_3) *and* (o_5)*.*
$(L, \to, \top)$ *is a* ***left w-eo algebra*** *if it satisfies the axioms* (o_1)*,* (o_2)*,* (o_3) *and* (o_5')*.*
$(L, \to, \top)$ *is an* ***extended-order algebra****, shortly* ***eo algebra****, if it is a right and a left w-eo algebra.*

In the following we denote $P \to Q = \{p \to q | \; p \in P, q \in Q\}$, for any $P, Q \subseteq L$.

Definition 2.1.4. *Let* $(L, \to, \top)$ *be a w-eo algebra.*
$(L, \to, \top)$ *is* ***right-distributive*** *if it satisfies the right distributivity condition:*
(d_r) *for any* $a \in L, B, B' \subseteq L$*, if* $Lb(B) = Lb(B')$ *then* $Lb(a \to B) = Lb(a \to B')$*.*
$(L, \to, \top)$ *is* ***left-distributive*** *if it satisfies the left distributivity condition:*
(d_l) *for any* $A, A' \subseteq L, b \in L$*, if* $Ub(A) = Ub(A')$ *then* $Lb(A \to b) = Lb(A' \to b)$*.*
$(L, \to, \top)$ *is* ***distributive*** *if it satisfies the distributivity condition:*
(d) *for any* $A, A', B, B' \subseteq L$*, if* $Ub(A) = Ub(A')$ *and* $Lb(B) = Lb(B')$ *then* $Lb(A \to B) = Lb(A' \to B')$*.*

We also say that the operation $\to$ is (right-)(left-)distributive if $(d_r)(d_l)d$ is satisfied.
We adopt the short notation *deo* for distributive extended-order algebras.
Of course $(L, \to, \top)$ is distributive if and only if it is left-distributive and right-distributive.
Moreover it is easily seen that a right-distributive (left-distributive) w-eo algebra is a

right (left) w-eo algebra hence a deo algebra is an eo algebra. All the above defined and the subsequently considered algebras $(L, \rightarrow \top)$ are said to be *complete* if L with the natural ordering $\leq$ induced by $\rightarrow$ is a complete lattice; so we shall consider, with a short obvious notation, *w-ceo* algebras, *ceo* algebras and *cdeo* algebras. The result below extends Proposition 21 of [50] and has a similar proof.

Proposition 2.1.5. *Let $(L, \rightarrow \top)$ be a w-ceo algebra. Then*

- *the condition (d_r) is equivalent to*
  (d'_r) for any $a \in L, B \subseteq L$: $a \rightarrow \bigwedge B = \bigwedge(a \rightarrow B)$;
- *the condition (d_l) is equivalent to*
  (d'_l) for any $b \in L, A \subseteq L$: $(\bigvee A) \rightarrow b = \bigwedge(A \rightarrow b)$;
- *the condition (d) is equivalent to*
  (d') for any $A, B \subseteq L$: $\bigvee A \rightarrow \bigwedge B = \bigwedge(A \rightarrow B)$. $\square$

Remark 2.1.6. We note that in any w-ceo algebra $(L \rightarrow, \top)$ the condition (d), or equivalently (d'), can be expressed by means of the two conditions (d'_r) and (d'_l).

The inception of fuzzy set theory and the consequent incredible development of theory and applications of lattice-valued mathematics and logics have produced a big interest in several kinds of lattice-ordered algebraic structures. Our work, following [50], proposes an approach to these structures based on the notion of order and its extension to the many-valued case: such an extension, described as an implication operator, completely determines the algebraic structure, in a similar way as the order determines completely the lattice structure.
It is quite evident that in many-valued mathematics the underlying lattice structure and its completeness are fundamental requirements for most applications; so eo algebras have been studied since [50] in connection with the completeness condition of their natural ordering. The Dedekind-MacNeille completion process has turned out to suit well for getting the completeness requirement of eo algebras.
The next results we present in this Subsection come from [50] except for a few of them on right-distributive eo algebras.

Definition 2.1.7. *[50] Let $(L, \rightarrow, \top)$ be a w-eo algebra and let $(K, \leq_K)$ be the MacNeille completion of $(L, \leq)$. For all $\alpha = [A], \beta = [B] \in K$, define*

$$\alpha \rightarrow_K \beta = [Lb(A) \rightarrow Ub(Lb(B))].$$

Proposition 2.1.8. *[50] With the above notation the following hold.*

1. *$\rightarrow_K$ is an extension of $\leq_K$, i.e. for all $\alpha, \beta \in K$ $\alpha \rightarrow_K \beta = \top_K \Leftrightarrow \alpha \leq_K \beta$.*
2. *$(K, \rightarrow_K, \top_K)$ is a complete w-eo algebra.* $\square$

We call $(K, \rightarrow_K, \top_K)$ the *MacNeille completion* of the w-eo algebra $(L, \rightarrow, \top)$.
The next lemma and proposition are extensions of similar results obtained in [50].

Lemma 2.1.9. *Let $(L, \rightarrow, \top)$ be a w-eo algebra. With the above notation the following hold.*

1. *If $\alpha = [A], \beta = [B] \in K$ and B is the largest representative of β, then $\alpha \rightarrow_K \beta = [Lb(A) \rightarrow B]$.*

2. *If $(L, \rightarrow, \top)$ is right-distributive, then for all $\alpha = [A], \beta = [B] \in K$ one has $\alpha \rightarrow_K \beta = [Lb(A) \rightarrow B]$.*

3. *If $(L, \rightarrow, \top)$ is an eo algebra, then $\rightarrow_K$ is an extension of $\rightarrow$, i.e. $[a] \rightarrow_K [b] = [a \rightarrow b]$, for all $a, b \in L$.*

Proof. We only prove item (2) since (1) is trivial and (3) is proved in [50].
In fact, it follows by the assumption and by the equality $Lb(Ub(Lb(B))) = Lb(B)$ that $Lb(Lb(A) \rightarrow Ub(Lb(B))) = Lb(Lb(A) \rightarrow B)$ which proves (2). □

It is shown in [50] that the assumption that $(L, \rightarrow, \top)$ be a left or a right w-eo algebra would be not enough to prove that $\rightarrow_K$ is an extension of $\rightarrow$ (see the structure 27 of Appendix).

Proposition 2.1.10. *Let $(L, \rightarrow, \top)$ be an eo algebra. Then, assuming the above notation, the following hold.*

1. *$(K, \rightarrow_K, \top_K)$ is a complete eo algebra and the map $\varphi : L \rightarrow K$ defined by $\varphi(a) = [a]$ is an embedding of eo algebras;*

2. *$(L, \rightarrow, \top)$ is right-distributive if and only if $(K, \rightarrow_K, \top_K)$ is right-distributive;*

3. *$(L, \rightarrow, \top)$ is left-distributive if and only if $(K, \rightarrow_K, \top_K)$ is left-distributive;*

4. *$(L, \rightarrow, \top)$ is distributive if and only if $(K, \rightarrow_K, \top_K)$ is distributive.*

Proof. 1. It is proved in [50].

2. Assume $(L, \rightarrow, \top)$ to be right-distributive and let $\alpha = [A], T \subseteq K$.
   It follows from $\bigwedge_K T = [\bigcup\{B| \ [B] \in T\}]$ and from Lemma 2.1.9 that $\alpha \rightarrow_K \bigwedge_K T = [Lb(A) \rightarrow \bigcup\{B| \ [B] \in T\}]$. On the other hand $\alpha \rightarrow_K T = \{\alpha \rightarrow_K [B] | \ [B] \in T\} = \{[Lb(A) \rightarrow B] | \ [B] \in T\}$. Therefore $\bigwedge_K (\alpha \rightarrow_K T) = [\bigcup\{Lb(A) \rightarrow B| \ [B] \in T\}]$.
   Clearly $\bigcup\{Lb(A) \rightarrow B| \ [B] \in T\} = Lb(A) \rightarrow \bigcup\{B| \ [B] \in T\}$, hence $\alpha \rightarrow_K \bigwedge_K T = \bigwedge_K (\alpha \rightarrow_K T)$.
   The converse follows by (1) and the equivalence $(d_r) \Leftrightarrow (d'_r)$ in $(K, \rightarrow_K, \top_K)$.

3. Assume now that $(L, \rightarrow, \top)$ be left-distributive. For all $\beta = [B] \in K, S \subseteq K$, we now prove that $\bigvee_K S \rightarrow_K \beta = \bigwedge_K (S \rightarrow_K \beta)$.
   Assume that B is the largest representative of β.
   Since $\bigvee_k S = [Ub(\bigcup\{Lb(A)| \ [A] \in S\})]$, then, by Lemma 2.1.9, one has that:
   $\bigvee_K S \rightarrow_K \beta = [Lb(Ub(\bigcup\{Lb(A)| \ [A] \in S\})) \rightarrow B]$.
   On the other hand $S \rightarrow_K \beta = \{[A] \rightarrow_K \beta| \ [A] \in S\} = \{[Lb(A) \rightarrow B] | \ [A] \in S\}$.

Thence $\bigwedge_K(S \to_K \beta) = [\bigcup\{Lb(A) \to B|\ [A] \in S\}]$.
Since $Ub(Lb(Ub(\bigcup\{Lb(A)|\ [A] \in S\}))) = Ub(\bigcup\{Lb(A)|\ [A] \in S\})$, it follows by the distributivity assumption on L that:
$\bigvee_K S \to_K \beta = [Lb(Ub(\bigcup\{Lb(A)|\ [A] \in S\})) \to B] = [\bigcup\{Lb(A)|\ [A] \in S\} \to B] = [\bigcup\{Lb(A) \to B|[A] \in S\}] = \bigwedge_K(S \to_K \beta)$.
The converse follows by (1) and the equivalence $(d_l) \Leftrightarrow (d'_l)$ in $(K, \to_K, \top_K)$.

4. It follows trivially from (2) and (3).

□

2.1.2 The adjoint product and the idempotency conditions

The algebras we consider in this thesis correspond to an approach to many-valued logic based on the fundamental connective of implication which can generate other connectives, more precisely other operations in our algebraic context.
In other approaches, more frequently addressed, a multiplication corresponding to conjunction is the basic tool, for instance in t-norm based logics (see [45] and [52]).
The implication-based approach to non-classical logics, however, goes back to long time ago (see [74]) but it has been developed by means of quite strong assumptions that characterize (positive) implication algebras and lead to the meet operation as the only possible conjunction.
Our approach, instead, though starting from the common base of implicative algebras, i.e. w-eo algebras described in the previous Subsection, produces a big variety of algebras for many-valued logics, having as a main tool the isotonic and antitonic Galois connections.
In this Subsection we define, following once more [50], a product which is the left adjoint of the implication. The results proved in the previous Subsection allow to consider the completeness condition a not too strong requirement for eo algebras thanks to the embedding described in Proposition 2.1.10 and to realize that the right-distributivity and the left-distributivity conditions of eo algebras are not invalidated by the completion process leading to the completeness condition.
However, we remark that for w-eo algebras the completeness assumption has to be considered quite restrictive. This is why we shall devote our interest in (right-)distributive ceo algebras that one can see as the algebraic counterpart of a logical system.
Nevertheless some of our constructions only need to assume $(L, \to, \top)$ to be a right-distributive w-ceo algebra; stronger assumptions will be specified when they will be necessary.

Let $(L, \to, \top)$ be a right-distributive w-ceo algebra. Thus, by adjunction applied to the function

$$g_a : L \to L,\ y \mapsto (y)g_a = a \to y$$

that preserves $\bigwedge$, we consider the left adjoint f_a and define the product $\otimes : L \times L \to L$ by

$$a \otimes x = (x)f_a = \bigwedge\{t \in L|\ x \leq a \to t\}.$$

Then $\otimes$ and $\to$ form an adjoint pair, i.e. for all $x, y, z \in L$:

$$x \otimes y \leq z \Leftrightarrow y \leq x \to z.$$

This operation $\otimes$ is called **adjoint product** of the right-distributive w-ceo algebra; the properties listed in the following Proposition have been already proved in [50], where the stronger distributivity condition (d) is assumed; their proof in the more general case we consider is similar, hence it is omitted.

Proposition 2.1.11. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra and let $\otimes$ be the adjoint product. The following properties hold, for all $a, b, c \in L, A, B \subseteq L$.*

1. $a \otimes b \leq a$;
2. $a \otimes \top = a$;
3. $a \otimes \bot = \bot$;
4. $\bot \otimes a = \bot$;
5. $\top \otimes a \leq a$ *if and only if* $\top \to a \geq a$;
6. $\top \otimes a \geq a$ *if and only if* $(\forall x \in L : \top \to x \geq a \Rightarrow x \geq a)$;
7. $\top \otimes a = a$ *if and only if* $(\forall x \in L : \top \to x \geq a \Leftrightarrow x \geq a)$;
8. $a \otimes (\bigvee B) = \bigvee(a \otimes B)$;
9. *if* $b \leq c$, *then* $a \otimes b \leq a \otimes c$;
10. $a \otimes (a \to b) \leq b \leq a \to (a \otimes b)$.

If, moreover, L is a ceo algebra, then

11. *if* $a \leq b$, *then* $a \otimes c \leq b \otimes c$;
12. $(\bigvee A) \otimes b \geq \bigvee(A \otimes b)$. $\square$

Remark 2.1.12. We note that the equivalence $\forall x \in L : \top \to x \geq a \Leftrightarrow x \geq a$ considered in the item (7) of the Proposition 2.1.11 is equivalent to the equality $\forall x \in L : \top \to x = x$. This statement is an easy consequence of the criterion for which, for all $x, y \in L$ $x = y$ if and only if $\forall a \in L$, $x \geq a \Leftrightarrow y \geq a$.

The above results enlighten the main differences between (right-)distributive complete weak extended-order algebras and complete residuated lattices.
We recall from [9, 58] that a complete integral residuated lattice is a complete lattice L with additional binary operations $\otimes$, $\to$, $\rightsquigarrow$ such that $\otimes$ gives a monoidal structure whose unit is the top element $\top$ and such that, for all $a, b, c \in L$:

$$a \leq b \rightsquigarrow c \Leftrightarrow a \otimes b \leq c \Leftrightarrow b \leq a \to c.$$

If $\otimes$ is commutative, then clearly $\rightarrow$ and $\rightsquigarrow$ coincide.
As it is well known the product $\otimes$ is distributive over arbitrary joins (and hence isotonic) in both arguments in every complete residuated lattice. So this product has a symmetrical character (the same properties on both arguments) even when it is not commutative. We are not aware of good motivations for the strong assumptions of associativity and symmetry of the product; on the contrary, our results show that (right-)distributive (w-)ceo algebras are a well motivated framework where such conditions do not necessarily occur.
We shall discuss the symmetrical character of our algebras in the next Subsection. Associativity and commutativity conditions will be considered in Subsection 2.1.5.

Now we approach the **idempotency condition** in eo algebras.

Definition 2.1.13. *A w-eo algebra $(L, \rightarrow, \top)$ is* ***idempotent*** *if the following equivalence holds, for all $a, b \in L$:*

$$a \rightarrow b = \top \Leftrightarrow a \rightarrow (a \rightarrow b) = \top.$$

Note that the implication $a \rightarrow b = \top \Rightarrow a \rightarrow (a \rightarrow b) = \top$ is trivially satisfied, so the above definition consists, in fact, in requiring the converse implication.

Proposition 2.1.14.

1. *The MacNeille completion of an idempotent w-eo algebra is idempotent.*
2. *An eo-algebra is idempotent if and only if its MacNeille completion is idempotent.*

Proof.

1. Let $(L, \rightarrow, \top)$ be an idempotent w-eo algebra, $(K, \rightarrow_K, \top_K)$ its MacNeille completion. We need to prove that for all $\alpha, \beta \in K$: $\alpha \leq_K \alpha \rightarrow_K \beta \Rightarrow \alpha \leq_K \beta$, i.e. that for all non-empty subsets $A, B \subseteq L, B = Ub(Lb(B))$ the implication

$$Lb(A) \subseteq Lb(Lb(A) \rightarrow B) \Rightarrow Lb(A) \subseteq Lb(B)$$

is true.
So, let $x \in Lb(A)$. Then $x \leq a' \rightarrow b, \forall a' \in Lb(A), b \in B$, hence $x \leq x \rightarrow b, \forall b \in B$ and, by assumption, $x \leq b, \forall b \in B$, i.e. $x \in Lb(B)$.

2. We only need to prove that $(L, \rightarrow, \top)$ is idempotent if its MacNeille completion is idempotent, which follows easily from Proposition 2.1.10 (1).

□

The following result explains why we have used the attribute "idempotent" in Definition 2.1.13.

Proposition 2.1.15. *A right-distributive w-ceo algebra is idempotent if and only if its adjoint product is idempotent.*

Proof. Let $(L, \rightarrow, \top)$ be a right-distributive w-ceo algebra and $\otimes$ the adjoint product.
If the algebra is idempotent, then the following equalities are true, for every $a \in L$:
$a \otimes a = \bigwedge \{t \in L | \ a \leq a \rightarrow t\} = \bigwedge \{t \in L | \ a \leq t\} = a$.
Conversely, if $\otimes$ is idempotent, then, for all $a, b \in L$, one has the following equivalences:
$\top = a \rightarrow (a \rightarrow b) \Leftrightarrow a \leq a \rightarrow b \Leftrightarrow a \otimes a \leq b \Leftrightarrow a \leq b \Leftrightarrow \top = a \rightarrow b$. □

Now we consider the **strong idempotency condition**, i.e. $a \to (a \to b) = a \to b$, for all $a, b \in L$; then the following holds.

Proposition 2.1.16. *Let $(L, \to, \top)$ be a right-distributive eo algebra and $(K, \to_K, \top_K)$ its MacNeille completion. Assuming that $(L, \leq)$ is a $\bigvee$-semilattice with respect to the natural ordering in L, then the following are equivalent:*

1. *$a \to (a \to b) = a \to b$, for all $a, b \in L$;*

2. *$\alpha \to_K (\alpha \to_K \beta) = \alpha \to_K \beta$, for all $\alpha, \beta \in K$.*

Proof. "(1) $\Rightarrow$ (2)". Let $\alpha, \beta \in K$ and let $A, B \subseteq L$ be representatives of α and β, i.e. $\alpha = [A]$ and $\beta = [B]$. We prove that $[Lb(A) \to (Lb(A) \to B)] = [Lb(A) \to B]$.
Let $x \in Lb(Lb(A) \to (Lb(A) \to B))$; then $x \leq a \to (a' \to b)$, for all $a, a' \in Lb(A), b \in B$ and in particular, $x \leq a \to (a \to b)$, for all $a \in Lb(A), b \in B$. By assumption $x \leq a \to b$, for all $a \in Lb(A), b \in B$, hence $x \in Lb(Lb(A) \to B)$.
Conversely, let $x \in Lb(Lb(A) \to B)$; then $x \leq t \to b$, for all $t \in Lb(A), b \in B$.
By assumption $x \leq t \to (t \to b)$, for all $t \in Lb(A), b \in B$. Now for all $a, a' \in Lb(A)$, $a \vee a' \in Lb(A)$ hence $x \leq (a \vee a') \to ((a \vee a') \to b) \leq a \to (a' \to b)$ for all $a, a' \in Lb(A), b \in B$. Thus, $x \in Lb(Lb(A) \to (Lb(A) \to B))$.
"(2) $\Rightarrow$ (1)". (1) follows trivially from (2), thanks to Proposition 2.1.10 (1), taking $\alpha = [a]$ and $\beta = [b]$. $\square$

2.1.3 Symmetrical extended-order algebras

In this Subsection we shall see that the symmetrical character of the adjoint product partially bridges the gap between cdeo algebras and complete residuated lattice, as already remarked in the previous Subsection.
However we shall approach the symmetry condition in the quite general context of w-eo algebras and give an internal characterization of **symmetrical** w-ceo algebras in terms of the implication only; the behavior toward the MacNeille completion will be discussed, too.

Definition 2.1.17. *A w-eo algebra $(L, \to, \top)$ is called **symmetrical** if there exists a binary operation $\rightsquigarrow: L \times L \to L$ such that $(L, \rightsquigarrow, \top)$ is a w-eo algebra, $\to$ and $\rightsquigarrow$ induce the same order and $y \leq x \rightsquigarrow b \Leftrightarrow x \leq y \to b$, for all $b, x, y \in L$. The w-eo algebras $(L, \to, \top)$, $(L, \rightsquigarrow, \top)$ and their implications $\to$, $\rightsquigarrow$ are said to be dual to each other and they form a Galois pair $[\to, \rightsquigarrow]$.*

The above Definition has a symmetrical character, so, with the above notation, $(L, \rightsquigarrow, \top)$ is symmetrical if and only if $(L, \to, \top)$ is symmetrical.

Lemma 2.1.18. *If $(L, \to, \top)$ is a symmetrical w-eo algebra and $\rightsquigarrow$ is the dual implication, then for all $a, b \in L$ one has:*

$$a \leq (a \rightsquigarrow b) \to b \text{ and } a \leq (a \to b) \rightsquigarrow b.$$

Proof. The inequalities are equivalent to the obvious inequalities $a \rightsquigarrow b \leq a \rightsquigarrow b$ and $a \to b \leq a \to b$, respectively. □

Proposition 2.1.19. *Let $(L, \to, \top)$ and $(L, \rightsquigarrow, \top)$ be symmetrical w-eo algebras dual to each other. Then:*

1. *each of them is left-distributive;*
2. *for all $a, x \in L : a \leq \top \to x \Leftrightarrow a \leq x \Leftrightarrow a \leq \top \rightsquigarrow x$;*
3. *if one of them is a right w-eo algebra then each of them is an eo algebra;*
4. *if one of them is right-distributive then each of them is distributive.*

Proof.
1. Let A, A' be subsets of L, $b \in L$ and $Ub(A) = Ub(A')$.
Then $x \in Lb(A \rightsquigarrow b) \Leftrightarrow \forall a \in A : x \leq a \rightsquigarrow b \Leftrightarrow \forall a \in A : a \leq x \to b \Leftrightarrow x \to b \in Ub(A) = Ub(A') \Leftrightarrow \forall a' \in A' : a' \leq x \to b \Leftrightarrow \forall a' \in A' : x \leq a' \rightsquigarrow b \Leftrightarrow x \in Lb(A' \rightsquigarrow b)$. It can be proved similarly that $(L, \to, \top)$ is left-distributive.
2. Since $(L, \rightsquigarrow, \top)$ is a w-eo algebra it follows from the equivalence $y \leq x \rightsquigarrow b \Leftrightarrow x \leq y \to b$, for all $b, x, y \in L$, that $a \leq \top \to x \Leftrightarrow \top \leq a \rightsquigarrow x \Leftrightarrow a \leq x$, for all $a, x \in L$. The second equivalence can be proved similarly.
3. Assume $(L, \to, \top)$ to be a right w-eo algebra and let $x, y \in L$, $x \leq y$. Then for every $a \in L$ one has $a \leq (a \rightsquigarrow x) \to x \leq (a \rightsquigarrow x) \to y$; hence $a \rightsquigarrow x \leq a \rightsquigarrow y$.
The statement then follows by also considering (1), since $(d_l) \Rightarrow (o_5')$.
4. Assume $(L, \to, \top)$ to be a right-distributive w-eo algebra and prove that $(L, \rightsquigarrow, \top)$ is right-distributive.
Let $a \in L$ and B, B' be subsets of L and assume $Lb(B) = Lb(B')$.
Then $x \in Lb(a \rightsquigarrow B) \Leftrightarrow \forall b \in B : x \leq a \rightsquigarrow b \Leftrightarrow \forall b \in B : a \leq x \to b \Leftrightarrow a \in Lb(x \to B) = Lb(x \to B') \Leftrightarrow \forall b' \in B' : a \leq x \to b' \Leftrightarrow \forall b' \in B' : x \leq a \rightsquigarrow b' \Leftrightarrow x \in Lb(a \rightsquigarrow B')$.

□

Since $\to$ and $\rightsquigarrow$ form a Galois pair, each of them is uniquely determined by the other one. So one gets the following, by also considering item (3) of the above Proposition.

Corollary 2.1.20. *A w-eo algebra (eo algebra, respectively) $(L, \to, \top)$ is symmetrical if and only if there exists a unique symmetrical w-eo algebra (eo algebra, respectively) $(L, \rightsquigarrow, \top)$ such that $\to$ and $\rightsquigarrow$ induce the same order and $y \leq x \rightsquigarrow b \Leftrightarrow x \leq y \to b$, for all $b, x, y \in L$.* □

The following Theorem gives an internal characterization of symmetrical w-ceo algebras.

Theorem 2.1.21. *Let $(L, \to, \top)$ be a w-ceo algebra. L is symmetrical if and only if it is left-distributive and the following equality $\top \to x = x$ holds, for every $x \in L$.*

Proof. Assume $(L, \rightarrow, \top)$ to be a symmetrical w-ceo algebra. By Proposition 2.1.19 (1) and (2) and by Remark 2.1.12 it follows trivially that the required conditions are necessary.
Conversely, assume $(L, \rightarrow, \top)$ to be a left-distributive w-ceo algebra such that $\top \rightarrow x = x$, for every $x \in L$, that is equivalent, by Remark 2.1.12, to the equivalence $a \leq \top \rightarrow x \Leftrightarrow a \leq x$, for all $a, x \in L$.
From Proposition 1.2.8 applied to the function defined by $(y)h_b = y \rightarrow b$, there exists a unique function g_b such that $[h_b, g_b]$ is a Galois connection; denote $(x)g_b = x \rightsquigarrow b$.
Then the equivalence $y \leq x \rightsquigarrow b \Leftrightarrow x \leq y \rightarrow b$ holds, for all $b, x, y \in L$.
Moreover, by the assumption that $(L, \rightarrow, \top)$ be a w-ceo algebra and from the equivalence $a \leq \top \rightarrow x \Leftrightarrow a \leq x$, for all $a, x \in L$, it follows that $\rightsquigarrow$ is an extension of the partial order $\leq$, since the equivalence $\top \leq a \rightsquigarrow b \Leftrightarrow a \leq \top \rightarrow b \Leftrightarrow a \leq b$ holds, for all $a, b \in L$. This proves that $(L, \rightsquigarrow, \top)$ is a complete weak extended-order algebra, hence the assertion follows. $\square$

We observe that in the above Theorem we need the completeness condition only to prove that the required conditions are sufficient.

Theorem 2.1.22. *Let $(L, \rightarrow, \top)$ be a w-eo algebra. If $(L, \rightarrow, \top)$ is symmetrical, then its MacNeille completion $(K, \rightarrow_K, \top_K)$ is symmetrical, too.*

Proof. Assume $(L, \rightarrow, \top)$ to be a symmetrical w-eo algebra. By definition, there exists $\rightsquigarrow : L \times L \rightarrow L$ such that $(L, \rightsquigarrow, \top)$ is a w-eo algebra, $\rightarrow$ and $\rightsquigarrow$ induce the same order $\leq$ and $y \leq x \rightsquigarrow b \Leftrightarrow x \leq y \rightarrow b$, for all $x, y, b \in L$. Then from Proposition 2.1.8 (2) $(K, \rightsquigarrow_K, \top_K)$ is a w-eo algebra. Since $\rightsquigarrow$ extends $\leq$ then $\rightsquigarrow_K$ extends $\leq_K$ as $\rightarrow_K$ does. So we only need to prove that $\gamma \leq_K \alpha \rightsquigarrow_K \beta \Leftrightarrow \alpha \leq_K \gamma \rightarrow_K \beta$, for all $\alpha, \beta, \gamma \in K$.
Let X, Y be representatives of α and γ, respectively, and let B be the largest representative of β, i.e. $\alpha = [X]$, $\beta = [B]$ and $\gamma = [Y]$.
Suppose $[Y] \leq_K [X] \rightsquigarrow_K [B]$, i.e. $Lb(Y) \subseteq Lb(Lb(X) \rightsquigarrow B)$ and prove the inequality: $[X] \leq_K [Y] \rightarrow_K [B]$, i. e. $Lb(X) \subseteq Lb(Lb(Y) \rightarrow B)$.
Let $p \in Lb(X)$; then, for all $q \in Lb(Y), b \in B$ one has that $q \leq p \rightsquigarrow b$. Hence $p \leq q \rightarrow b$, for all $q \in Lb(Y), b \in B$. Then $p \in Lb(Lb(Y) \rightarrow B)$.
Conversely, assuming that $Lb(X) \subseteq Lb(Lb(Y) \rightarrow B)$ it can be proved in a similar way that $Lb(Y) \subseteq Lb(Lb(X) \rightsquigarrow B)$. $\square$

We are not able to prove the converse of the implication in the above Theorem even restricting to the case of eo algebras. Nevertheless we can characterize those eo algebras whose MacNeille completion is symmetrical. First we prove the following.

Lemma 2.1.23. *Let $(L, \rightarrow, \top)$ be an eo algebra and $(K, \rightarrow_K, \top_K)$ its MacNeille completion. Then the following are equivalent:*

1. $a \leq \top \rightarrow x \Leftrightarrow a \leq x, \ \forall a, x \in L$;

2. $\alpha \leq_K \top_K \rightarrow_K \gamma \Leftrightarrow \alpha \leq_K \gamma, \ \forall \alpha, \gamma \in K$.

Proof. "(1) $\Rightarrow$ (2)". Let $\alpha = [A]$ and let X be the largest representative of γ; then $\top_K \rightarrow_K \gamma = [L \rightarrow X]$. By the assumptions the following holds, for every $a \in L$:
$a \in Lb(L \rightarrow X) \Leftrightarrow a \leq t \rightarrow x,\ \forall t \in L, \forall x \in X \Leftrightarrow a \leq \top \rightarrow x,\ \forall x \in X \Leftrightarrow a \leq x,\ \forall x \in X \Leftrightarrow a \in Lb(X)$.
Hence $Lb(L \rightarrow X) = Lb(X)$ and (2) is true since $Lb(A) \subseteq Lb(L \rightarrow X) \Leftrightarrow Lb(A) \subseteq Lb(X)$.
"(2) $\Rightarrow$ (1)". (1) follows trivially from (2), thanks to Proposition 2.1.10 (1), taking $\alpha = [a]$ and $\gamma = [x]$. □

Proposition 2.1.24. *An eo algebra $(L, \rightarrow, \top)$ has a symmetrical MacNeille completion if and only if it is left-distributive and satisfies the condition $a \leq \top \rightarrow x \Leftrightarrow a \leq x, \forall a, x \in L$.*

Proof. It follows easily from Proposition 2.1.10, Theorem 2.1.21 and Lemma 2.1.23. □

Corollary 2.1.25. *If $(L, \rightarrow, \top)$ is a symmetrical eo algebra, then it is left-distributive and the following equivalence holds $a \leq \top \rightarrow x \Leftrightarrow a \leq x$, for all $a, x \in L$.*

Proof. It follows from the above Theorem 2.1.22 that the MacNeille completion $(K, \rightsquigarrow_K, \top_K)$ is symmetrical. Then the assertion follows from Proposition 2.1.24. □

Proposition 2.1.26. *If $(L, \rightarrow, \top)$ is a symmetrical right-distributive w-ceo algebra, $\rightsquigarrow$ the dual implication and $\otimes$ the adjoint product then:*

1. *$(L, \rightarrow, \top)$ is a cdeo algebra;*
2. *$(L, \rightsquigarrow, \top)$ is a cdeo algebra;*
3. *$\top \otimes a = a$ for every $a \in L$;*
4. *$(\bigvee B) \otimes a = \bigvee(B \otimes a)$, for all $a \in L, B \subseteq L$.*

Proof. (1) and (2) These follow by the assumption and from Proposition 2.1.19 (4).
(3) It is a trivial consequence of Propositions 2.1.19 and 2.1.11 (7).
(4) By symmetry and by the adjunction between $\otimes$ and $\rightarrow$, the following equivalences hold: $x \leq a \rightsquigarrow y \Leftrightarrow a \leq x \rightarrow y \Leftrightarrow x \otimes a \leq y$, for all $a, x, y \in L$.
In particular $x \leq a \rightsquigarrow y \Leftrightarrow x \otimes a \leq y$, for all $a, x, y \in L$; from this equivalence it follows that the functions defined by $(x)f_a = x \otimes a$ and $(y)g_a = a \rightsquigarrow y$ form an adjunction $f_a \dashv g_a$. By this adjunction it follows that the map f_a preserves sups, i. e. $(\bigvee B) \otimes a = \bigvee(B \otimes a)$, for all $a \in L, B \subseteq L$. □

Remark 2.1.27. With the assumption of the above Proposition, the adjoint product of the cdeo algebra $(L, \rightsquigarrow, \top)$ is the opposite $\otimes^{op}$ of $\otimes$, i. e. $a \otimes^{op} b = b \otimes a$. Of course $\otimes$ and $\rightsquigarrow$ are related by the equivalence $a \leq b \rightsquigarrow c \Leftrightarrow a \otimes b \leq c$.

One cannot say that the cdeo algebras are symmetrical. However the structures described in 17 and 18 of Appendix and subsequent results show that what cdeo algebras miss is that $\top$ is a left unit of the adjoint product.

Proposition 2.1.28. *If $(L, \rightarrow, \top)$ is a cdeo algebra and $\otimes$ is the adjoint product, then $(\bigvee B) \otimes a = \bigvee(B \otimes a)$, for all $a \in L, B \subseteq L$.*

Proof. By left-distributivity, from Proposition 1.2.8 applied to the function h_b defined by $(y)h_b = y \to b$, there exists a function g_b such that $[h_b, g_b]$. Then the binary operation $\rightsquigarrow$ defined by $(x)g_b = x \rightsquigarrow b$ is such that $x \le y \rightsquigarrow b \Leftrightarrow y \le x \to b$, for all $b, x, y \in L$.
Since the adjoint product is such that $a \otimes x \le y \Leftrightarrow x \le a \to y$, for all $a, x, y \in L$, then the following equivalences hold $x \le a \rightsquigarrow y \Leftrightarrow a \le x \to y \Leftrightarrow x \otimes a \le y$, for all $a, x, y \in L$.
So the function f_a defined by $(x)f_a = x \otimes a$ is left adjoint to the function g'_a defined by $(y)g'_a = a \rightsquigarrow y$ and hence it preserves $\bigvee$. Then, $(\bigvee B) \otimes a = (\bigvee B)f_a = \bigvee(B)f_a = \bigvee(B \otimes a)$, for all $a \in L, B \subseteq L$. □

The following Proposition characterizes the symmetrical cdeo algebras in terms of the adjoint product.

Proposition 2.1.29. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra and let $\otimes$ be the adjoint product. Then L is a symmetrical cdeo algebra if and only if the following conditions hold:*
(s_1) $(\bigvee B) \otimes a = \bigvee(B \otimes a)$, *for all $a \in L, B \subseteq L$;*
(s_2) $\top \otimes b = b$, *for all $b \in L$.*

Proof. Assume that $(L, \to, \top)$ be a symmetrical cdeo algebra. Then, (s_1) follows from Proposition 2.1.28.
From Theorem 2.1.21 the following equivalence holds

$$b \le \top \to x \Leftrightarrow x \ge b, \text{ for all } b, x \in L$$

which is equivalent to (s_2), by Proposition 2.1.11 (7).
Conversely, assume that $(L, \to, \top)$ be a right-distributive w-ceo algebra such that (s_1) and (s_2) hold. From Proposition 2.1.11 (7), (s_2) is equivalent to the equivalence

$$b \le \top \to x \Leftrightarrow x \ge b, \text{ for all } b, x \in L.$$

Moreover, from (s_1) it follows that the map f_a defined by $(x)f_a = x \otimes a$ preserves $\bigvee$. Then there exists a right adjoint g_a that determines a binary operation $\rightsquigarrow$, defined by $(y)g_a = a \rightsquigarrow y$, such that $x \le a \rightsquigarrow y \Leftrightarrow x \otimes a \le y$, for all $a, x, y \in L$. By also considering the adjunction between $\otimes$ and $\to$ the following equivalences hold $x \le y \rightsquigarrow b \Leftrightarrow x \otimes y \le b \Leftrightarrow y \le x \to b$, for all $b, x, y \in L$. Thus, the maps defined by $(y)g'_b = y \rightsquigarrow b$ and $(x)h_b = x \to b$ form a Galois connection $\left[g'_b, h_b\right]$. Therefore, from Proposition 1.2.7 it follows that $(\bigvee A) \to b = (\bigvee A)h_b = \bigwedge(A)h_b = \bigwedge(A \to b)$, for any $b \in L$, $A \subseteq L$, i.e. $\to$ is left-distributive. From Theorem 2.1.21 and Proposition 2.1.26 it follows that $(L, \to, \top)$ is a symmetrical cdeo algebra. □

As already claimed in the Introduction and in the previous Subsections, we are interested in right-distributive w-ceo algebras, where an adjoint product can be defined. So, the above Proposition 2.1.26 shows that the symmetry condition, when needed, has to be considered directly in cdeo algebras. The possible further assumption of associativity condition, that will be discussed in Subsection 2.1.5, would produce exactly the structure of an integral residuated lattice as it is defined in [9, 58].

The results of this Subsection also show that the completeness assumption in residuated lattices is not a too strong restriction, by also considering that the associativity is preserved by the MacNeille completion as it is shown in [50] (see also Subsection 2.1.6 below for further details).

Proposition 2.1.30. *Let* $(L, \rightarrow, \top)$ *be a symmetrical cdeo algebra,* $\rightsquigarrow$ *the dual implication and* $\otimes$ *the adjoint product. Then, for all* $a, b, c \in L$*, the following properties hold.*

1. $a \otimes b \leq b$;
2. $a \otimes b \leq a \wedge b$;
3. $(a \rightsquigarrow b) \otimes a \leq b \leq a \rightsquigarrow (b \otimes a)$;
4. $(a \rightsquigarrow b) \otimes a \leq a \wedge b$;
5. $b \leq a \rightarrow b$;
6. $b \leq a \rightsquigarrow b$;
7. $\top \rightarrow a = a$;
8. $\top \rightsquigarrow a = a$;
9. $a \leq (a \rightarrow b) \rightsquigarrow b$;
10. $a \leq (a \rightsquigarrow b) \rightarrow b$;
11. $a \rightarrow (b \rightarrow c) = \top \Leftrightarrow b \rightsquigarrow (a \rightsquigarrow c) = \top$;
12. $a \rightsquigarrow (b \rightarrow c) = \top \Leftrightarrow b \rightarrow (a \rightsquigarrow c) = \top$.

Proof.

1. From (o_2) it follows that $a \leq b \rightsquigarrow b$, hence $a \otimes b \leq b$.
2. From (1) and Proposition 2.1.11 (1) the inequality trivially follows.
3. From $a \rightsquigarrow b \leq a \rightsquigarrow b$ it follows $(a \rightsquigarrow b) \otimes a \leq b$; from $b \otimes a \leq b \otimes a$ it follows that $b \leq a \rightsquigarrow (b \otimes a)$.
4. It follows easily by (1) and (3).
5. It follows from (1) by adjunction, according to Remark 2.1.27.
6. It follows from Proposition 2.1.11 (1) by adjunction.
7. From (5) $a \leq \top \rightarrow a$; moreover the following equivalences hold $\top \rightarrow a \leq \top \rightarrow a \Leftrightarrow \top \otimes (\top \rightarrow a) \leq a \Leftrightarrow \top \rightarrow a \leq a$.
8. Similarly to (7).
9. See Lemma 2.1.18.

10. See Lemma 2.1.18.
11. It follows easily by the equivalence $a \le b \to c \Leftrightarrow b \le a \rightsquigarrow c$.
12. Similarly to (11).

$\square$

Remark 2.1.31. Let $(L, \to, \top)$ be a w-ceo algebra. Consider the *coimplication* operation defined as follows, for all $a, b \in L$: $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. If $(L, \to, \top)$ is a symmetrical cdeo algebra and $\otimes$ is its adjoint product, then, for all $a, b \in L$, the following equivalence holds:

$$a \otimes b \le a \leftrightarrow b \Leftrightarrow a \wedge b \le a \leftrightarrow b.$$

In fact, by Proposition 2.1.30 (2), we have that $a \wedge b \le a \leftrightarrow b \Rightarrow a \otimes b \le a \wedge b \le a \leftrightarrow b$, for all $a, b \in L$.
Conversely, by Proposition 2.1.30 (5), for all $a, b \in L$: $a \wedge b \le a \le b \to a$ and $a \wedge b \le b \le a \to b$.

One can see, by means of the above Proposition, that symmetrical cdeo algebras have the main properties of residuated lattices, except the associativity of the product. As we shall see in Subsection 2.1.5, the assumption of associativity (and commutativity, as well) will add not so much to our investigations.
So we argue that the associativity and commutativity assumptions in residuated lattices have to be considered restrictive and, maybe, not necessary; in any case we have not found good motivations for associativity and commutativity coming from the structure of the algebra we consider, as we have done instead for the symmetry condition, naturally arising by means of Galois connections. Nevertheless the role and the strength of the associativity condition will be discussed in Subsections 2.1.5 and 2.1.6.

2.1.4 The negation and the disjunction connectives

It is quite natural to define a negation $\neg$ in a bounded implicative structure $(L, \to, \top)$ by setting $\neg a = a \to \bot$. This of course can be done in any w-ceo algebra but, taking into account the symmetry condition and the consequent topics discussed in the previous Subsection we state the following definition and notation.

Definition 2.1.32. *Let $(L, \to, \top)$ a w-ceo algebra. We define the following unary operation*

$$[\cdot]^- : L \to L, x \mapsto x^- = x \to \bot.$$

If $(L, \to, \top)$ is a symmetrical w-ceo algebra, then we can define a further unary operation

$$[\cdot]^\sim : L \to L, x \mapsto x^\sim = x \rightsquigarrow \bot$$

Both these operations are called ***negations*** *and they are said to be dual to each other.*
The negation $[\cdot]^-$ ($[\cdot]^\sim$, respectively) is ***involutive*** *if $x^{--} = x$ ($x^{\sim\sim} = x$, respectively), for every $x \in L$.*
The negations $[\cdot]^-$ and $[\cdot]^\sim$, and the symmetrical w-ceo algebra as well, are said to be

cross-involutive *if* $x^{\sim -} = x^{-\sim} = x$, *for every* $x \in L$.
A symmetrical w-ceo algebra is said to be ***good*** *if* $x^{\sim -} = x^{-\sim}$, *for every* $x \in L$.

Notation and the term "good" in the above Definition are borrowed from [12, 56, 57] where, indeed, the cross-involutivity condition is expressed in terms of the so called "double negation properties" $x^{\sim -} = x$ and $x^{-\sim} = x$.
The following Proposition states basic properties of the negation in a quite general context; all the properties we shall state for a negation also hold for the dual negation, under the assumption of symmetry, in a dual form which consists in interchanging $[\cdot]^-$ with $[\cdot]^\sim$, $\rightarrow$ with $\rightsquigarrow$ and each term of the product $\otimes$ with the other one.

Proposition 2.1.33. *Let* $(L, \rightarrow, \top)$ *be a left w-ceo algebra. Then the following hold, for all* $x, y \in L, \{x_i\}_{i\in I} \subseteq L$.

1. $\bot^- = \top$;
2. $x \le y \Rightarrow y^- \le x^-$;
3. $(\bigvee_{i\in I} x_i)^- \le \bigwedge_{i\in I} x_i^-$;
4. $(\bigwedge_{i\in I} x_i)^- \ge \bigvee_{i\in I} x_i^-$;
5. *if* $(L, \rightarrow, \top)$ *is a ceo algebra, then* $x^- \le x \rightarrow y$.

   If the negation $[\cdot]^-$ *is involutive, then:*

6. $\top^- = \bot$;
7. $(\bigvee_{i\in I} x_i)^- = \bigwedge_{i\in I} x_i^-$;
8. $(\bigwedge_{i\in I} x_i)^- = \bigvee_{i\in I} x_i^-$.

Proof. 1. From (o_1) $\bot^- = \bot \rightarrow \bot = \top$.

2. Let $x \le y$; from (o_5') it follows that $y \rightarrow \bot \le x \rightarrow \bot$.
3. From (o_5') it follows that $(\bigvee_{i\in I} x_i)^- = (\bigvee_{i\in I} x_i) \rightarrow \bot \le x_i \rightarrow \bot$, for any $i \in I$. Then $(\bigvee_{i\in I} x_i)^- \le \bigwedge_{i\in I}(x_i \rightarrow \bot) = \bigwedge_{i\in I} x_i^-$.
4. From (o_5') it follows that $(\bigwedge_{i\in I} x_i)^- = (\bigwedge_{i\in I} x_i) \rightarrow \bot \ge x_i \rightarrow \bot$, for any $i \in I$. Then $(\bigwedge_{i\in I} x_i)^- \ge \bigvee_{i\in I}(x_i \rightarrow \bot) = \bigvee_{i\in I} x_i^-$.
5. From (o_5) it follows that $x^- = x \rightarrow \bot \le x \rightarrow y$.
6. By assumption $\bot^{--} = \bot$; from (1) it follows that $\bot = \bot^{--} = (\bot^-)^- = \top^-$.
7. From (2), (4) and by the assumption one has $\bigwedge_{i\in I} x_i^- = ((\bigwedge_{i\in I} x_i^-)^-)^- \le (\bigvee_{i\in I} x_i^{--})^- = (\bigvee_{i\in I} x_i)^-$.

8. From (2), (3) and by the assumption it follows that $\bigvee_{i \in I} x_i^- = ((\bigvee_{i \in I} x_i^-)^-)^- \geq (\bigwedge_{i \in I} x_i^{--})^- = (\bigwedge_{i \in I} x_i)^-$.

□

It is clear that the properties of the negation depend essentially on the behavior of the implication with respect to the first argument; so the left-distributivity condition has further effects on the negation.
We exploit such effects by also assuming the right-distributivity condition, which provides the adjoint product, in the next Proposition; of course this means that we consider cdeo algebras.

Proposition 2.1.34. *If $(L, \rightarrow, \top)$ is a cdeo algebra, then the following hold, for all $x, y \in L, \{x_i\}_{i \in I} \subseteq L$.*

1. $x \otimes x^- = \bot$;
2. $x \leq y^- \Leftrightarrow y \otimes x = \bot$;
3. $(\bigvee_{i \in I} x_i)^- = \bigwedge_{i \in I} x_i^-$.

Proof. 1. From Proposition 2.1.11 (10) it follows that $x \otimes (x \rightarrow \bot) \leq \bot$.

2. It follows by adjunction.

3. From (d_l) it follows that $(\bigvee_{i \in I} x_i)^- = (\bigvee_{i \in I} x_i) \rightarrow \bot = \bigwedge_{i \in I} (x_i \rightarrow \bot) = \bigwedge_{i \in I} x_i^-$.

□

Eventually, we list and prove further properties involving the pair of dual negations of a symmetrical cdeo algebra, by omitting the dual version of those already considered in Propositions 2.1.33 and 2.1.34.

Proposition 2.1.35. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. Then the following hold, for all $x, y \in L$.*

1. $\top^- = \bot$, $\top^\sim = \bot$;
2. $x \leq x^{-\sim}$, $x \leq x^{\sim -}$;
3. $x \leq y^\sim \Leftrightarrow y \leq x^-$;
4. $x \leq x^- \rightsquigarrow y$, $x \leq x^\sim \rightarrow y$;
5. $x^{-\sim -} = x^-$, $x^{\sim - \sim} = x^\sim$.

Proof. 1. It follows from Proposition 2.1.30 (7) and (8).

2. From adjunction and from Proposition 2.1.34 (1), the equivalences $x \leq x^{-\sim} \Leftrightarrow x \leq x^- \rightsquigarrow \bot \Leftrightarrow x \otimes x^- \leq \bot$ hold; hence the first part of (2) is true and the second part follows by duality.

3. Since $\rightarrow$ and $\rightsquigarrow$ form a Galois connection the following equivalence $x \le y \rightsquigarrow \bot \Leftrightarrow y \le x \rightarrow \bot$ holds.

4. The first part follows from the true equivalence $x \le x^{-} \rightsquigarrow y \Leftrightarrow x \otimes x^{-} = \bot \le y$. The second part is dual.

5. From (4) and Proposition 2.1.33 (2) it follows that $x^{-\sim-} \le x^{-}$. Moreover the inequality $(x^{-} \rightsquigarrow \bot) \otimes x^{-} \le \bot$ is true, by Proposition 2.1.30 (3); hence by adjunction $x^{-} \le (x^{-} \rightsquigarrow \bot) \rightarrow \bot$. So $x^{-\sim-} = x^{-}$. The second equality follows by duality.

□

Proposition 2.1.36. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra and assume that it is good.*

1. *If the negation $[\cdot]^{-}$ is involutive, then the dual negations are cross-involutive.*

2. *If the negation $[\cdot]^{\sim}$ is involutive, then the dual negations are cross-involutive.*

3. *If both the dual negations are involutive, then they coincide.*

Proof. 1. By assumption it follows from Proposition 2.1.35 (5) that $x^{\sim-} = x^{-\sim} = (x^{-\sim})^{--} = (x^{-\sim-})^{-} = x$, for every $x \in L$.

2. It follows similarly from Proposition 2.1.35 (5).

3. By assumption and Proposition 2.1.35 (5), the following implications hold $x^{--} = x = x^{\sim\sim} \Rightarrow x^{--\sim} = x^{\sim\sim\sim} \Rightarrow x^{-\sim-} = x^{\sim\sim\sim} \Rightarrow x^{-} = x^{\sim}$, for every $x \in L$.

□

The disjunction connective $\oplus$ in the algebras frequently used in many-valued logics is usually derived from the conjunction $\otimes$ and the negation $\neg$ by means of the formula

$$a \oplus b = \neg(\neg a \otimes \neg b)$$

as it is done for instance in MV-algebras (see [10] and [52]). Something similar has been done in pseudo-MTL algebras, that are, in our terminology, symmetrical associative (see Subsection 2.1.5 for the notion of associativity) cdeo algebras with the pseudo-prelinearity condition (see [12]). In fact the disjunction operation is defined in [12] by

$$a \oplus b = (b^{\sim} \otimes a^{\sim})^{-}$$

under the additional property of goodness, i.e. assuming $a^{\sim-} = a^{-\sim}$.
The above formulas are motivated by the classical equality, in a boolean algebra,

$$A \vee B = -(-A \wedge -B)$$

that, indeed, holds thanks to the double negation property $- - A = A$.
In this sense we find correct what is done in MV-algebras where, in fact, the double negation law holds; on the contrary we think that the definition of disjunction given in [12] is not well motivated.
We prefer to consider the classical equivalences

$$A \subseteq B \bigcup C \Leftrightarrow A \smallsetminus B \subseteq C \Leftrightarrow A \bigcap -B \subseteq C \Leftrightarrow -B \bigcap A \subseteq C$$

whose first and forth terms form indeed an adjunction (considering $f_B = -B \bigcap [\cdot]$ and $g_B = B \bigcup [\cdot]$, the equivalence becomes $A \subseteq (C)g_B \Leftrightarrow (A)f_B \subseteq C$), as a motivating suggestion to get disjunction connectives through conjunction and negation.
In fact, this can be done in quite general algebras (right-distributive w-ceo algebras) by simply applying the adjoint condition (in particular Proposition 1.2.4); this can be done in several, different but closely related ways, mainly when the symmetry condition is assumed.
Doing so, we shall not need to assume the algebra to be either commutative or associative: only the symmetry assumption and the involutive and cross-involutive conditions will be useful to prove that the disjunction connectives have quite good properties, resembling those which are satisfied in the classical context.

Let $(L, \to, \top)$ be a right-distributive w-ceo algebra.
Then we can consider the operator $a^- \otimes [\cdot]$; it preserves $\bigvee$ so it has a unique right adjoint that is

$$(y)h_a^- = \bigvee \{x \in L |\ a^- \otimes x \le y\} = \bigvee \{x \in L |\ x \le a^- \to y\} = a^- \to y$$

which can be meant as disjunction either of a and y or of y and a.

Definition 2.1.37. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra. The following binary operations*

$$\uplus_{\to}^{(-)} : L \times L \to L,\ (a,b) \mapsto a \uplus_{\to}^{(-)} b = b^- \to a;$$
$${}^{(-)}\uplus_{\to} : L \times L \to L,\ (a,b) \mapsto a^{(-)} \uplus_{\to} b = a^- \to b$$

*are called **disjunction operations** or simply **disjunctions**.*

Remark 2.1.38. With the above notation, for all $a, b \in L$ the following equality holds:

$$a^{(-)} \uplus_{\to} b = b \uplus_{\to}^{(-)} a.$$

So we shall say that these operations are opposite to each other. We devote our investigation mainly to one of the two disjunctions, namely to $\uplus_{\to}^{(-)}$; corresponding results will hold for ${}^{(-)}\uplus_{\to}$, that we shall not list explicitly.

Proposition 2.1.39. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra. The following properties hold, for all $a, b, c \in L$, $\{b_i\}_{i \in I} \subseteq L$.*

1. $\bot \uplus_{\to}^{(-)} a = a^{--}$;
2. $a \uplus_{\to}^{(-)} \bot = \top \to a$;
3. $\top \uplus_{\to}^{(-)} a = \top$;
4. $a^- \uplus_{\to}^{(-)} a = \top$;

5. *$a \uplus_{\rightarrow}^{(-)} a^{-} = \top \Leftrightarrow a^{--} \leq a$;*

6. *if $a \leq b$, then $a \uplus_{\rightarrow}^{(-)} c \leq b \uplus_{\rightarrow}^{(-)} c$;*

7. *$(\bigwedge_{i\in I} b_i) \uplus_{\rightarrow}^{(-)} a = \bigwedge_{i\in I}(b_i \uplus_{\rightarrow}^{(-)} a)$;*

8. *$(\bigvee_{i\in I} b_i) \uplus_{\rightarrow}^{(-)} a \geq \bigvee_{i\in I}(b_i \uplus_{\rightarrow}^{(-)} a)$.*

If the negation $[\cdot]^{-}$ is involutive, then:

9. *$\bot \uplus_{\rightarrow}^{(-)} a = a$;*

10. *$a \uplus_{\rightarrow}^{(-)} b \geq b$;*

11. *$a \uplus_{\rightarrow}^{(-)} \top = \top$;*

12. *$a \uplus_{\rightarrow}^{(-)} a^{-} = \top$.*

Proof. We note that L satisfies the axiom (o_5).

1. By definition $\bot \uplus_{\rightarrow}^{(-)} a = a^{-} \rightarrow \bot = a^{--}$.

2. From Proposition 2.1.33 (1) it follows easily that $a \uplus_{\rightarrow}^{(-)} \bot = \bot^{-} \rightarrow a = \top \rightarrow a$.

3. From (o_1) $\top \uplus_{\rightarrow}^{(-)} a = a^{-} \rightarrow \top = \top$.

4. From (o_2) $a^{-} \uplus_{\rightarrow}^{(-)} a = a^{-} \rightarrow a^{-} = \top$.

5. From Proposition 2.1.2 it follows trivially that $a \uplus_{\rightarrow}^{(-)} a^{-} = a^{--} \rightarrow a = \top \Leftrightarrow a^{--} \leq a$.

6. Let $a \leq b$; from (o_5) it follows that $a \uplus_{\rightarrow}^{(-)} c = c^{-} \rightarrow a \leq c^{-} \rightarrow b = b \uplus_{\rightarrow}^{(-)} c$.

7. From (d_r) it follows that $(\bigwedge_{i\in I} b_i) \uplus_{\rightarrow}^{(-)} a = a^{-} \rightarrow (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a^{-} \rightarrow b_i) = \bigwedge_{i\in I}(b_i \uplus_{\rightarrow}^{(-)} a)$.

8. From (o_5) it follows that $(\bigvee_{i\in I} b_i) \uplus_{\rightarrow}^{(-)} a = a^{-} \rightarrow (\bigvee_{i\in I} b_i) \geq a^{-} \rightarrow b_i$, for every $i \in I$. Then $(\bigvee_{i\in I} b_i) \uplus_{\rightarrow}^{(-)} a \geq \bigvee_{i\in I}(a^{-} \rightarrow b_i) = \bigvee_{i\in I}(b_i \uplus_{\rightarrow}^{(-)} a)$.

9. It follows trivially by assumption and from (1).

10. It follows from (6) and (9) that $a \uplus_{\rightarrow}^{(-)} b \geq \bot \uplus_{\rightarrow}^{(-)} b = b$.

11. From Proposition 2.1.33 (6) it follows that $a \uplus_{\rightarrow}^{(-)} \top = \top^{-} \rightarrow a = \bot \rightarrow a = \top$.

12. By assumption it follows that $a \uplus_{\rightarrow}^{(-)} a^{-} = a^{--} \rightarrow a = a \rightarrow a = \top$.

□

Proposition 2.1.40. *Let $(L, \rightarrow, \top)$ be a right-distributive ceo algebra. The following properties hold, for all $a, b, c \in L, \{b_i\}_{i \in I} \subseteq L$.*

1. *if $a \leq b$, then $c \uplus_{\rightarrow}^{(-)} a \leq c \uplus_{\rightarrow}^{(-)} b$;*

2. *$a \uplus_{\rightarrow}^{(-)} (\bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I}(a \uplus_{\rightarrow}^{(-)} b_i)$;*

3. *$a \uplus_{\rightarrow}^{(-)} (\bigvee_{i \in I} b_i) \geq \bigvee_{i \in I}(a \uplus_{\rightarrow}^{(-)} b_i)$.*

Proof. 1. Let $a \leq b$; from (o_5') and from Proposition 2.1.33 (2) it follows that $c \uplus_{\rightarrow}^{(-)} a = a^- \rightarrow c \leq b^- \rightarrow c = c \uplus_{\rightarrow}^{(-)} b$.

2. It follows easily from (1).

3. It follows easily from (1).

□

Proposition 2.1.41. *Let $(L, \rightarrow, \top)$ be a cdeo algebra. If the negation $[\cdot]^-$ is involutive, then the following property holds, for all $a \in L, \{b_i\}_{i \in I} \subseteq L$:*

$$a \uplus_{\rightarrow}^{(-)} (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I}(a \uplus_{\rightarrow}^{(-)} b_i).$$

Proof. By assumption and left-distributivity of L, it follows that $a \uplus_{\rightarrow}^{(-)} (\bigwedge_{i \in I} b_i) = (\bigwedge_{i \in I} b_i)^- \rightarrow a = (\bigvee_{i \in I} b_i^-) \rightarrow a = \bigwedge_{i \in I}(b_i^- \rightarrow a) = \bigwedge_{i \in I}(a \uplus_{\rightarrow}^{(-)} b_i)$. □

Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. Then, in addition to the disjunction operations already defined, the above discussed motivation allows to consider three further pairs of such operations as follows.
We consider the operator $[\cdot] \otimes b^-$; since, by Proposition 2.1.29, $\otimes$ preserves $\bigvee$ on the left side, we can consider its right adjoint

$$(y)g_b^- = \bigvee \{x \in L | \ x \otimes b^- \leq y\} = \bigvee \{x \in L | \ x \leq b^- \rightsquigarrow y\} = b^- \rightsquigarrow y.$$

Similarly, we consider the operator $[\cdot] \otimes b^{\sim}$ which preserves $\bigvee$ and then its right adjoint

$$(y)g_b^{\sim} = \bigvee \{x \in L | \ x \otimes b^{\sim} \leq y\} = \bigvee \{x \in L | \ x \leq b^{\sim} \rightsquigarrow y\} = b^{\sim} \rightsquigarrow y.$$

If we consider the operator $a^{\sim} \otimes [\cdot]$ then, it preserves $\bigvee$ and its right adjoint is

$$(y)h_a^{\sim} = \bigvee \{x \in L | \ a^{\sim} \otimes x \leq y\} = \bigvee \{x \in L | \ x \leq a^{\sim} \rightarrow y\} = a^{\sim} \rightarrow y.$$

We list all the disjunction operations that can be considered in a symmetrical cdeo algebra, including those already considered in Definition 2.1.37.

Definition 2.1.42. *Let $(L, \rightarrow, \top)$ be a symmetrical right-distributive w-ceo algebra. The following binary operations*

$$
\begin{aligned}
\uplus_{\rightsquigarrow}^{(-)} &: L \times L \to L,\ (a,b) \mapsto a \uplus_{\rightsquigarrow}^{(-)} b = b^{-} \rightsquigarrow a;\\
{}^{(-)}\uplus_{\rightsquigarrow} &: L \times L \to L,\ (a,b) \mapsto a^{(-)} \uplus_{\rightsquigarrow} b = a^{-} \rightsquigarrow b;\\
\uplus_{\to}^{(-)} &: L \times L \to L,\ (a,b) \mapsto a \uplus_{\to}^{(-)} b = b^{-} \to a;\\
{}^{(-)}\uplus_{\to} &: L \times L \to L,\ (a,b) \mapsto a^{(-)} \uplus_{\to} b = a^{-} \to b;\\
\uplus_{\rightsquigarrow}^{(\sim)} &: L \times L \to L,\ (a,b) \mapsto a \uplus_{\rightsquigarrow}^{(\sim)} b = b^{\sim} \rightsquigarrow a;\\
{}^{(\sim)}\uplus_{\rightsquigarrow} &: L \times L \to L,\ (a,b) \mapsto a^{(\sim)} \uplus_{\rightsquigarrow} b = a^{\sim} \rightsquigarrow b;\\
\uplus_{\to}^{(\sim)} &: L \times L \to L,\ (a,b) \mapsto a \uplus_{\to}^{(\sim)} b = b^{\sim} \to a;\\
{}^{(\sim)}\uplus_{\to} &: L \times L \to L,\ (a,b) \mapsto a^{(\sim)} \uplus_{\to} b = a^{\sim} \to b
\end{aligned}
$$

*are called **disjunction operations** or simply **disjunctions** of the algebra* $(L, \to, \top)$.

Remark 2.1.43. With the above notation, the following equalities hold, for all $a, b \in L$

1. $a^{(-)} \uplus_{\rightsquigarrow} b = b \uplus_{\rightsquigarrow}^{(-)} a$;
2. $a^{(-)} \uplus_{\to} b = b \uplus_{\to}^{(-)} a$;
3. $a^{(\sim)} \uplus_{\rightsquigarrow} b = b \uplus_{\rightsquigarrow}^{(\sim)} a$;
4. $a^{(\sim)} \uplus_{\to} b = b \uplus_{\to}^{(\sim)} a$.

The above equalities and the already discussed symmetry of $\to$ and $\rightsquigarrow$ allows clearly to get quite similar, or at least related, properties of these operations: in this sense one would not need to list all of them, but only one of each pair of opposite disjunctions. It has to be expected, by the symmetrical character of $\to$ and $\rightsquigarrow$, that $\uplus_{\to}^{(-)}$ and $\uplus_{\rightsquigarrow}^{(\sim)}$ have similar properties; $\uplus_{\to}^{(\sim)}$ and $\uplus_{\rightsquigarrow}^{(-)}$ have similar properties to each other, too.
So, one could say in some sense that in a symmetrical cdeo algebra there are two types of disjunctions, said the **linear disjunctions** (which are $\uplus_{\to}^{(-)}$, $\uplus_{\rightsquigarrow}^{(\sim)}$, ${}^{(-)}\uplus_{\to}$ and ${}^{(\sim)}\uplus_{\rightsquigarrow}$) and the **cross disjunctions** (which are $\uplus_{\to}^{(\sim)}$, $\uplus_{\rightsquigarrow}^{(-)}$, ${}^{(\sim)}\uplus_{\to}$ and ${}^{(-)}\uplus_{\rightsquigarrow}$).
Our choice is to list and prove the properties of the linear disjunction $\uplus_{\to}^{(-)}$ (partly already considered in Propositions 2.1.39, 2.1.40 and 2.1.41) and those of the cross disjunction $\uplus_{\rightsquigarrow}^{(-)}$. It would be an easy exercise to express and prove corresponding properties of the other disjunctions.

Proposition 2.1.44. *Let $(L, \to, \top)$ be a symmetrical cdeo algebra. The following properties hold, for all $a, b, c \in L, \{b_i\}_{i \in I} \subseteq L$.*

1. $a \uplus_{\rightsquigarrow}^{(-)} b \geq a$;
2. $a \uplus_{\rightsquigarrow}^{(-)} b \geq b$;
3. $a \uplus_{\rightsquigarrow}^{(-)} b \geq a \vee b$;
4. $a \uplus_{\rightsquigarrow}^{(-)} \bot = a$;
5. $\bot \uplus_{\rightsquigarrow}^{(-)} a = a^{-\sim}$;

6. $a \uplus^{(-)}_{\rightsquigarrow} \top = \top$;

7. $\top \uplus^{(-)}_{\rightsquigarrow} a = \top$;

8. $a^{-} \uplus^{(-)}_{\rightsquigarrow} a = \top$;

9. $a \uplus^{(-)}_{\rightsquigarrow} a^{-} = \top \Leftrightarrow a^{--} \leq a$;

10. $a^{\sim} \uplus^{(-)}_{\rightsquigarrow} a = \top \Leftrightarrow a \leq a^{--}$;

11. $a \uplus^{(-)}_{\rightsquigarrow} a^{\sim} = \top \Leftrightarrow a^{\sim -} \leq a$;

12. *if* $a \leq b$, *then* $a \uplus^{(-)}_{\rightsquigarrow} c \leq b \uplus^{(-)}_{\rightsquigarrow} c$;

13. *if* $a \leq b$, *then* $c \uplus^{(-)}_{\rightsquigarrow} a \leq c \uplus^{(-)}_{\rightsquigarrow} b$;

14. $a \uplus^{(-)}_{\rightsquigarrow} (\bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I}(a \uplus^{(-)}_{\rightsquigarrow} b_i)$;

15. $(\bigwedge_{i \in I} b_i) \uplus^{(-)}_{\rightsquigarrow} a = \bigwedge_{i \in I}(b_i \uplus^{(-)}_{\rightsquigarrow} a)$;

16. $a \uplus^{(-)}_{\rightsquigarrow} (\bigvee_{i \in I} b_i) \geq \bigvee_{i \in I}(a \uplus^{(-)}_{\rightsquigarrow} b_i)$;

17. $(\bigvee_{i \in I} b_i) \uplus^{(-)}_{\rightsquigarrow} a \geq \bigvee_{i \in I}(b_i \uplus^{(-)}_{\rightsquigarrow} a)$.

If the negations $[\cdot]^{\sim}$ *and* $[\cdot]^{-}$ *are cross-involutive, then:*

18. $\bot \uplus^{(-)}_{\rightsquigarrow} a = a$;

19. $a \uplus^{(-)}_{\rightsquigarrow} a^{\sim} = \top$.

If the negation $[\cdot]^{-}$ *is involutive, then:*

20. $a \uplus^{(-)}_{\rightsquigarrow} (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I}(a \uplus^{(-)}_{\rightsquigarrow} b_i)$;

21. $a \uplus^{(-)}_{\rightsquigarrow} a^{-} = \top$;

22. $a^{\sim} \uplus^{(-)}_{\rightsquigarrow} a = \top$.

Proof. We note that, from Proposition 2.1.26 (2) it follows that $(L, \rightsquigarrow, \top)$ is distributive. Then $(L, \rightsquigarrow, \top)$ satisfies the axioms (o_5) and (o_5').

1. From Proposition 2.1.30 (6) it follows trivially that $a \uplus^{(-)}_{\rightsquigarrow} b = b^{-} \rightsquigarrow a \geq a$.

2. From Proposition 2.1.35 (4) it follows trivially that $a \uplus^{(-)}_{\rightsquigarrow} b = b^{-} \rightsquigarrow a \geq b$.

3. It follows easily from (1) and (2).

4. From Propositions 2.1.33 (1) and 2.1.30 (8) it follows that $a \uplus^{(-)}_{\rightsquigarrow} \bot = \bot^{-} \rightsquigarrow a = \top \rightsquigarrow a = a$.

5. By definition $\bot \uplus^{(-)}_{\rightsquigarrow} a = a^{-} \rightsquigarrow \bot = a^{-\sim}$.

6. From Proposition 2.1.35 (1) it follows easily that $a \uplus^{(-)}_{\rightsquigarrow} \top = \top^{-} \rightsquigarrow a = \bot \rightsquigarrow a = \top$.

7. From (o_1) it follows that $\top \uplus^{(-)}_{\rightsquigarrow} a = a^{-} \rightsquigarrow \top = \top$.

8. From (o_2) it follows that $a^{-} \uplus^{(-)}_{\rightsquigarrow} a = a^{-} \rightsquigarrow a^{-} = \top$.

9. From Proposition 2.1.2 it follows trivially that $a \uplus^{(-)}_{\rightsquigarrow} a^{-} = a^{--} \rightsquigarrow a = \top \Leftrightarrow a^{--} \leq a$.

10. From Proposition 2.1.2 and Definition 2.1.17 the following equivalences hold:
$a^{\sim} \uplus^{(-)}_{\rightsquigarrow} a = a^{-} \rightsquigarrow a^{\sim} = \top \Leftrightarrow a^{-} \leq a^{\sim} \Leftrightarrow a^{-} \leq a \rightsquigarrow \bot \Leftrightarrow a \leq a^{-} \rightarrow \bot \Leftrightarrow a \leq a^{--}$.

11. From Proposition 2.1.2 it follows trivially that $a \uplus^{(-)}_{\rightsquigarrow} a^{\sim} = a^{\sim -} \rightsquigarrow a = \top \Leftrightarrow a^{\sim -} \leq a$.

12. Let $a \leq b$; from (o_5) it follows that $a \uplus^{(-)}_{\rightsquigarrow} c = c^{-} \rightsquigarrow a \leq c^{-} \rightsquigarrow b = b \uplus^{(-)}_{\rightsquigarrow} c$.

13. Let $a \leq b$; from (o_5') and from Proposition 2.1.33 (2) it follows that:
$c \uplus^{(-)}_{\rightsquigarrow} a = a^{-} \rightsquigarrow c \leq b^{-} \rightsquigarrow c = c \uplus^{(-)}_{\rightsquigarrow} b$.

14. From distributivity, (o_5') and from Proposition 2.1.33 (4) it follows that:
$a \uplus^{(-)}_{\rightsquigarrow} (\bigwedge_{i\in I} b_i) = (\bigwedge_{i\in I} b_i)^{-} \rightsquigarrow a \leq (\bigvee_{i\in I} b_i^{-}) \rightsquigarrow a = \bigwedge_{i\in I}(b_i^{-} \rightsquigarrow a) = \bigwedge_{i\in I}(a \uplus^{(-)}_{\rightsquigarrow} b_i)$.

15. From (d_r) it follows that $(\bigwedge_{i\in I} b_i) \uplus^{(-)}_{\rightsquigarrow} a = a^{-} \rightsquigarrow (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a^{-} \rightsquigarrow b_i) = \bigwedge_{i\in I}(b_i \uplus^{(-)}_{\rightsquigarrow} a)$.

16. From Proposition 2.1.34 (3) and (o_5') it follows that:
$a \uplus^{(-)}_{\rightsquigarrow} (\bigvee_{i\in I} b_i) = (\bigvee_{i\in I} b_i)^{-} \rightsquigarrow a = (\bigwedge_{i\in I} b_i^{-}) \rightsquigarrow a \geq b_i^{-} \rightsquigarrow a$, for every $i \in I$.
Then $a \uplus^{(-)}_{\rightsquigarrow} (\bigvee_{i\in I} b_i) \geq \bigvee_{i\in I}(b_i^{-} \rightsquigarrow a) = \bigvee_{i\in I}(a \uplus^{(-)}_{\rightsquigarrow} b_i)$.

17. From (o_5) it follows that $(\bigvee_{i\in I} b_i) \uplus^{(-)}_{\rightsquigarrow} a = a^{-} \rightsquigarrow (\bigvee_{i\in I} b_i) \geq a^{-} \rightsquigarrow b_i$, for any $i \in I$.
Then $(\bigvee_{i\in I} b_i) \uplus^{(-)}_{\rightsquigarrow} a \geq \bigvee_{i\in I}(a^{-} \rightsquigarrow b_i) = \bigvee_{i\in I}(b_i \uplus^{(-)}_{\rightsquigarrow} a)$.

18. By assumption and from (5), it follows trivially.

19. By assumption and from (11) it follows that $a \uplus^{(-)}_{\rightsquigarrow} a^{\sim} = a^{\sim -} \rightsquigarrow a = a \rightsquigarrow a = \top$.

20. By assumption and left-distributivity of L, it follows that:
$a \uplus^{(-)}_{\rightsquigarrow} (\bigwedge_{i\in I} b_i) = (\bigwedge_{i\in I} b_i)^{-} \rightsquigarrow a = (\bigvee_{i\in I} b_i^{-}) \rightsquigarrow a = \bigwedge_{i\in I}(b_i^{-} \rightsquigarrow a) = \bigwedge_{i\in I}(a \uplus^{(-)}_{\rightsquigarrow} b_i)$.

21. By assumption it follows that $a \uplus^{(-)}_{\rightsquigarrow} a^{-} = a^{--} \rightsquigarrow a = a \rightsquigarrow a = \top$.

22. By assumption and from (10) it follows trivially.

$\square$

We note that the properties of the operator $\uplus_{\to}^{(-)}$ already listed in Propositions 2.1.39, 2.1.40 and 2.1.41 hold, obviously, in a symmetrical cdeo algebra.
Now we list and prove the additional properties that we can obtain, by assumption of symmetry.

Proposition 2.1.45. *Let $(L, \to, \top)$ be a symmetrical cdeo algebra. The following properties hold, for all $a, b \in L$.*

1. $a \uplus_{\to}^{(-)} b \geq a$;
2. $a \uplus_{\to}^{(-)} \bot = a$;
3. $a \uplus_{\to}^{(-)} \top = \top$;
4. $a^{\sim} \uplus_{\to}^{(-)} a = \top \Leftrightarrow a \leq a^{--}$;
5. $a \uplus_{\to}^{(-)} a^{\sim} = \top \Leftrightarrow a^{\sim-} \leq a$;
6. *if the negation $[\cdot]^{-}$ is involutive, then* $a^{\sim} \uplus_{\to}^{(-)} a = \top$;
7. *if the negations $[\cdot]^{-}$ and $[\cdot]^{\sim}$ are cross-involutive, then* $a \uplus_{\to}^{(-)} a^{\sim} = \top$.

Proof.

1. From Proposition 2.1.30 (5) it follows trivially that $a \uplus_{\to}^{(-)} b = b^{-} \to a \geq a$.
2. From Propositions 2.1.33 (1) and 2.1.30 (7) it follows that $a \uplus_{\to}^{(-)} \bot = \bot^{-} \to a = \top \to a = a$.
3. From Proposition 2.1.35 (1) it follows that $a \uplus_{\to}^{(-)} \top = \top^{-} \to a = \bot \to a = \top$.
4. From Proposition 2.1.2 and Definition 2.1.17 the following equivalences hold: $a^{\sim} \uplus_{\to}^{(-)} a = a^{-} \to a^{\sim} = \top \Leftrightarrow a^{-} \leq a^{\sim} \Leftrightarrow a^{-} \leq a \rightsquigarrow \bot \Leftrightarrow a \leq a^{-} \to \bot \Leftrightarrow a \leq a^{--}$.
5. From Proposition 2.1.2 it follows trivially that $a \uplus_{\to}^{(-)} a^{\sim} = a^{\sim-} \to a = \top \Leftrightarrow a^{\sim-} \leq a$.
6. It follows trivially by assumption and from (4).
7. It follows trivially by assumption and from (5).

$\square$

Remark 2.1.46.

1. We have already quoted, at the beginning of this Subsection, a different approach to disjunction done in [12]; the associativity condition there assumed on the product allows to prove that the (unique defined there) disjunction $\oplus$ is associative too.
We remark that the associativity is the only property of $\oplus$ proved in [12] that we cannot obtain for our disjunctions; all the other properties of $\oplus$ are satisfied by our types of disjunctions under suitable involutivity assumptions on the negations: nevertheless we do not need either associativity or pseudo-prelinearity which are assumed in [12].

2. It is also relevant that the disjunction $\oplus$ considered in [12], under the conditions there assumed on the algebra, can be expressed also by the equalities

$$x \oplus y = x^{-} \rightsquigarrow y^{-\sim} = y^{\sim} \to x^{-\sim}$$

which shows that $\oplus$ coincides with two of our cross disjunctions in case the negations are cross-involutive.

3. If all the involutivity conditions are satisfied in a symmetrical cdeo algebra, then by Proposition 2.1.36 there is a unique negation, whose basic properties can be deduced by the above Propositions.

2.1.5 Commutativity and associativity of w-eo algebras

In this Subsection we shall deal with the **commutativity** and **associativity** conditions of the various kinds of w-eo algebras we have considered, in a more general setting than in [50]; we shall slightly improve most results given in [50] so, in fact, we restate all of those replacing the distributivity condition, assumed there, by the right-distributivity condition. We omit the proofs of these results since they are similar to those given in [50]. New results will be stated and proved mainly in connection with the symmetry condition.
What we are mainly interested to note is that the commutativity and the associativity assumptions do not allow much better results than those obtained under the symmetry assumption: commutativity adds essentially the equality $a^{-} \otimes a = \bot$ while associativity adds a lot of technical but somehow unessential results. Nevertheless it will become evident which is the fundamental role of associativity, that we shall discuss in the concluding Subsection.

Definition 2.1.47. *[50] A w-eo algebra $(L, \to, \top)$ is* ***commutative*** *if and only if it satisfies the weak exchange condition:*
(c) $a \to (b \to c) = \top \Leftrightarrow b \to (a \to c) = \top$, for all $a, b, c \in L$.

Proposition 2.1.48. *The MacNeille completion of a commutative and right-distributive w-eo algebra is commutative.* □

Corollary 2.1.49. *If $(L, \to, \top)$ is a right-distributive eo algebra, then $(L, \to, \top)$ is commutative if and only if its MacNeille completion $(K, \to_K, \top_K)$ is.* □

Proof. It follows trivially by the embedding described in Proposition 2.1.10 and by the above Proposition. □

Proposition 2.1.50. *A right-distributive w-ceo algebra $(L, \to, \top)$ is commutative if and only if its adjoint product $\otimes$ is commutative.* □

Proposition 2.1.51. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra. If it is commutative the following hold.*

1. $(L, \rightarrow, \top)$ *is symmetrical;*

2. $\rightsquigarrow$ *coincides with* $\rightarrow$*.*

Proof. Assume that $(L, \rightarrow, \top)$ to be a right-distributive commutative w-ceo algebra.

1. From Proposition 2.1.11 (2),(8) and by assumption it follows that:
   (s_1) $(\bigvee B) \otimes a = a \otimes (\bigvee B) = \bigvee(a \otimes B) = \bigvee(B \otimes a)$, for all $a \in L, B \subseteq L$;
   (s_2) $\top \otimes b = a \otimes \top = b$, for all $b \in L$.
   Then, from Proposition 2.1.29 $(L, \rightarrow, \top)$ is symmetrical.

2. We consider the map $f_a(x) = x \otimes a$; its right adjoint is $h_a(y) = a \rightsquigarrow y$. The map $f'_a(x) = a \otimes x$ has right adjoint $g_a(y) = a \rightarrow y$. By commutativity, $f_a = f'_a$ and, hence, by uniqueness of adjoint, $g_a = h_a$.

□

Corollary 2.1.52. *If* $(L, \rightarrow, \top)$ *is a right-distributive commutative w-ceo algebra, then it is a commutative cdeo algebra.*

Proof. From Proposition 2.1.51 $(L, \rightarrow, \top)$ is symmetrical. Hence, from Proposition 2.1.26 (1) it is a commutative cdeo algebra. □

From the previous results it follows that the commutative cdeo algebras and the related connectives have all the properties that are satisfied in symmetrical cdeo algebras, in particular those we have obtained in the previous Subsections.
We can obtain only a few additional properties that we list below, among which the commutativity condition itself.

Proposition 2.1.53. *If* $(L, \rightarrow, \top)$ *is a commutative cdeo algebra, then the following hold, for all* $a, b \in L$*:*

1. $a \otimes b = b \otimes a$*;*

2. $a^{-} \otimes a = \bot$*.*

Proof. (1) is a trivial consequence of Proposition 2.1.50. Thence (2) follows by also using Proposition 2.1.34 (1). □

Definition 2.1.54. *[50] A w-eo algebra* $(L, \rightarrow, \top)$ *is* ***associative*** *if and only if it satisfies the following condition:*
(a) $Lb(Lb(A) \rightarrow (Lb(B) \rightarrow C)) = Lb(Lb(\{x|\ Lb(A) \subseteq Lb(Lb(B) \rightarrow x)\}) \rightarrow C)$*, for all* $A, B, C \subseteq L$*.*

Lemma 2.1.55. *If* $(L, \rightarrow, \top)$ *is a right-distributive w-ceo algebra, then the condition* (a) *can be formulated in the following equivalent way:*
(a') $a \rightarrow (b \rightarrow c) = (\bigwedge \{x|\ a \rightarrow (b \rightarrow x) = \top\}) \rightarrow c$*, for all* $a, b, c \in L$*.* □

Proposition 2.1.56. *Let $(L, \to, \top)$ be a right-distributive w-eo algebra and let $(K, \to_K, \top_K)$ be its MacNeille completion, with the above notation. Then K is associative if and only if L is associative.*

Proof. It is proved in [50], Proposition 38, that the associativity condition is preserved by the MacNeille completion under the assumption of distributivity of L; however, such a proof only needs the right-distributivity.
The converse is also claimed in [50], Corollary 44, without any proof, once more assuming distributivity. Now, we give a proof in our case. So, let $(K, \to_K, \top_K)$ be associative, let $A, B, C \subseteq L$ and assume $Lb(A), Lb(B), C$ be non-empty (otherwise, the proof is trivial). By Lemma 2.1.55 the assumption implies that:

$$[A] \to_K ([B] \to_K [C]) = (\bigwedge_K \{[D] \mid D \subseteq L,\ [A] \leq_K [B] \to_K [D]\}) \to_K [D]$$

hence, by Remark 1.3.2 and Lemma 2.1.9 (2)

$$\begin{aligned}[Lb(A) \to (Lb(B) \to C)] &= [\bigcup\{D \subseteq L \mid Lb(A) \subseteq Lb(Lb(B) \to D)\}] \to_K [C] \\ &= [\{x \in L \mid Lb(A) \subseteq Lb(Lb(B) \to x)\}] \to_K [C] \\ &= [Lb(\{x \in L \mid Lb(A) \subseteq LB(Lb(B) \to x)\} \to C]\,.\end{aligned}$$

So the condition of Definition 2.1.54 is satisfied. □

Remark 2.1.57. Let $(L, \to, \top)$ be a right-distributive w-ceo algebra. If $\otimes$ denotes the adjoint product, then the associativity condition (a) is equivalent to:
(a'') $a \to (b \to c) = (b \otimes a) \to c$, for all $a, b, c \in L$.

Proposition 2.1.58. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra. Its adjoint product $\otimes$ is associative if and only if L is associative.* □

Corollary 2.1.59. *Let $(L, \to, \top)$ be a symmetrical cdeo algebra and $\rightsquigarrow$ the dual implication. Then $(L, \to, \top)$ is associative if and only if $(L, \rightsquigarrow, \top)$ is associative.* □

Proposition 2.1.60. *Let $(L, \to, \top)$ be a cdeo algebra and let $\otimes$ be its adjoint product. If L is associative, then, for all $a, b, c \in L$:*

1. $(a \to b) \otimes (b \to c) \leq a \to c$;
2. $(b \to c) \leq (a \to b) \to (a \to c)$;
3. $a \to b^- = (b \otimes a)^-$;
4. $a^- \underset{\to}{\uplus}^{(-)} b = (a \otimes b^-)^-$.

Proof. (1) and (2) are proved in [50], Proposition 42.
(3) From (a'') it follows that $a \to b^- = a \to (b \to \bot) = (b \otimes a) \to \bot = (b \otimes a)^-$.
(4) It follows easily from (3).

□

Proposition 2.1.61. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. If it is associative, then the following properties hold, for all $a, b, c \in L$.*

1. $(b \rightsquigarrow c) \otimes (a \rightsquigarrow b) \leq a \rightsquigarrow c$;
2. $(b \rightsquigarrow c) \leq (a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c)$;
3. $a \rightsquigarrow (b \rightsquigarrow c) = (a \otimes b) \rightsquigarrow c$;
4. $a \rightsquigarrow b^{\sim} = (a \otimes b)^{\sim}$;
5. $a^{\sim} \uplus^{(\sim)}_{\rightsquigarrow} b = (b^{\sim} \otimes a)^{\sim}$.

Proof. 1. By associativity it follows from Proposition 2.1.30 (3) that:
$[(b \rightsquigarrow c) \otimes (a \rightsquigarrow b)] \otimes a = (b \rightsquigarrow c) \otimes [(a \rightsquigarrow b) \otimes a] \leq (b \rightsquigarrow c) \otimes b \leq c$.
Then the statement follows by adjunction.

2. It is clearly equivalent to (1).

3. The first inequality is true since the following equivalence holds:
$a \rightsquigarrow (b \rightsquigarrow c) \leq (a \otimes b) \rightsquigarrow c \Leftrightarrow (a \rightsquigarrow (b \rightsquigarrow c)) \otimes (a \otimes b) \leq c$.
In fact, by assumption and from Proposition 2.1.30(3) we have that:
$(a \rightsquigarrow (b \rightsquigarrow c)) \otimes (a \otimes b) = [(a \rightsquigarrow (b \rightsquigarrow c)) \otimes a] \otimes b \leq (b \rightsquigarrow c) \otimes b \leq c$.
The second inequality $(a \otimes b) \rightsquigarrow c \leq a \rightsquigarrow (b \rightsquigarrow c)$ follows from the equivalences $(a \otimes b) \rightsquigarrow c \leq a \rightsquigarrow (b \rightsquigarrow c) \Leftrightarrow [(a \otimes b) \rightsquigarrow c] \otimes a \leq b \rightsquigarrow c \Leftrightarrow \{[(a \otimes b) \rightsquigarrow c] \otimes a\} \otimes b \leq c$.
In fact, by assumption and from Proposition 2.1.30 (3) $\{[(a \otimes b) \rightsquigarrow c] \otimes a\} \otimes b = [(a \otimes b) \rightsquigarrow c] \otimes (a \otimes b) \leq c$.

4. From (3), it follows that $a \rightsquigarrow b^{\sim} = (a \otimes b)^{\sim} \Leftrightarrow a \rightsquigarrow (b \rightsquigarrow \perp) = (a \otimes b) \rightsquigarrow \perp$.

5. From (4) it follows that $a^{\sim} \uplus^{(\sim)}_{\rightsquigarrow} b = b^{\sim} \rightsquigarrow a^{\sim} = (b^{\sim} \otimes a)^{\sim}$.

□

Proposition 2.1.62. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. If it is associative, then the following properties hold, for all $a, b, c \in L$.*

1. $a \rightsquigarrow b \leq b^{\sim} \rightarrow a^{\sim}$, $a \rightarrow b \leq b^{-} \rightsquigarrow a^{-}$;
2. $a \rightsquigarrow b^{-} = b \rightarrow a^{\sim}$, $a \rightarrow b^{\sim} = b \rightsquigarrow a^{-}$;
3. $a \rightsquigarrow a^{-} = a \rightarrow a^{\sim}$;
4. $b^{\sim} \rightarrow a^{\sim} = a^{\sim -} \rightsquigarrow b^{\sim -} = a \rightsquigarrow b^{\sim -}$, $b^{-} \rightsquigarrow a^{-} = a^{- \sim} \rightarrow b^{- \sim} = a \rightarrow b^{- \sim}$;
5. $a \rightsquigarrow (b \rightarrow c) = b \rightarrow (a \rightsquigarrow c)$;
6. $a \rightarrow b \leq (b \rightarrow c) \rightsquigarrow (a \rightarrow c)$;
7. $a \rightsquigarrow b^{-} = b^{- \sim} \rightarrow a^{\sim} = a^{\sim -} \rightsquigarrow b^{-}$;

8. *$a \to b^{\sim} = b^{\sim -} \rightsquigarrow a^{-} = a^{-\sim} \to b^{\sim}$;*

9. *$(a \rightsquigarrow b^{-\sim})^{-\sim} = a \rightsquigarrow b^{-\sim}$, $(a \to b^{\sim -})^{\sim -} = a \to b^{\sim -}$;*

10. *if the negations $[\cdot]^{\sim}$ and $[\cdot]^{-}$ are cross-involutive, then $b^{\sim} \to a^{\sim} = a \rightsquigarrow b$, $b^{-} \rightsquigarrow a^{-} = a \to b$.*

Proof. 1. From Proposition 2.1.61 (1) the following equivalence holds:
$a \rightsquigarrow b \leq b^{\sim} \to a^{\sim} \Leftrightarrow a \rightsquigarrow b \leq (b \rightsquigarrow \bot) \to (a \rightsquigarrow \bot) \Leftrightarrow (b \rightsquigarrow \bot) \otimes (a \rightsquigarrow b) \leq a \rightsquigarrow \bot$.
From Proposition 2.1.60 (1) it follows that:
$a \to b \leq b^{-} \rightsquigarrow a^{-} \Leftrightarrow a \to b \leq (b \to \bot) \rightsquigarrow (a \to \bot) \Leftrightarrow (a \to b) \otimes (b \to \bot) \leq a \to \bot$.

2. The first inequality $a \rightsquigarrow b^{-} \leq b \to a^{\sim}$ holds since this equivalence is true $a \rightsquigarrow b^{-} \leq b \to a^{\sim} \Leftrightarrow a \rightsquigarrow b^{-} \leq b \to (a \rightsquigarrow \bot) \Leftrightarrow b \otimes (a \rightsquigarrow b^{-}) \leq a \rightsquigarrow \bot \Leftrightarrow [b \otimes (a \rightsquigarrow b^{-})] \otimes a \leq \bot$. In fact, from Propositions 2.1.30 (3) and 2.1.34 (1) and by assumption it follows that $[b \otimes (a \rightsquigarrow b^{-})] \otimes a = b \otimes [(a \rightsquigarrow b^{-}) \otimes a] \leq b \otimes b^{-} = \bot$.
The second inequality $b \to a^{\sim} \leq a \rightsquigarrow b^{-}$ holds since this equivalence is true $b \to a^{\sim} \leq a \rightsquigarrow (b \to \bot) \Leftrightarrow (b \to a^{\sim}) \otimes a \leq b \to \bot \Leftrightarrow b \otimes [(b \to a^{\sim}) \otimes a] \leq \bot$. In fact, from Propositions 2.1.11(10) and 2.1.34(1) and by assumption it follows that $b \otimes [(b \to a^{\sim}) \otimes a] = [b \otimes (b \to a^{\sim})] \otimes a \leq a^{\sim} \otimes a = \bot$. It follows from the first equality, changing a with b.

3. From (2), with $a = b$.

4. From (2) it follows that $b^{\sim} \to a^{\sim} = a \rightsquigarrow b^{\sim -}$. From (2) and from Proposition 2.1.35 (5) $a^{\sim -} \rightsquigarrow b^{\sim -} = b^{\sim} \to a^{\sim -\sim} = b^{\sim} \to a^{\sim}$. From (2) $b^{-} \rightsquigarrow a^{-} = a \to b^{-\sim}$. From (2) and from Proposition 2.1.35 (5) $a^{-\sim} \to b^{-\sim} = b^{-} \rightsquigarrow a^{-\sim -} = b^{-} \rightsquigarrow a^{-}$.

5. The first inequality $a \rightsquigarrow (b \to c) \leq b \to (a \rightsquigarrow c)$ holds since this equivalence is true $a \rightsquigarrow (b \to c) \leq b \to (a \rightsquigarrow c) \Leftrightarrow b \otimes [a \rightsquigarrow (b \to c)] \leq a \rightsquigarrow c \Leftrightarrow \{b \otimes [a \rightsquigarrow (b \to c)]\} \otimes a \leq c$. In fact, by assumption and from Propositions 2.1.11 (10) and 2.1.30 (3) it follows that $\{b \otimes [a \rightsquigarrow (b \to c)]\} \otimes a = b \otimes \{[a \rightsquigarrow (b \to c)] \otimes a\} \leq b \otimes (b \to c) \leq c$.
The second inequality $b \to (a \rightsquigarrow c) \leq a \rightsquigarrow (b \to c)$ holds since the following equivalence is true $b \to (a \rightsquigarrow c) \leq a \rightsquigarrow (b \to c) \Leftrightarrow [b \to (a \rightsquigarrow c)] \otimes a \leq b \to c \Leftrightarrow b \otimes \{[b \to (a \rightsquigarrow c)] \otimes a\} \leq c$.
In fact, by assumption and from Propositions 2.1.11 (10) and 2.1.30 (3) it follows that $b \otimes \{[b \to (a \rightsquigarrow c)] \otimes a\} = \{b \otimes [b \to (a \rightsquigarrow c)]\} \otimes a \leq (a \rightsquigarrow c) \otimes a \leq c$.

6. It follows from Proposition 2.1.60 (2) by symmetry.

7. From Proposition 2.1.35 (5) and from (4) $(b^{-})^{\sim} \to a^{\sim} = a^{\sim -} \rightsquigarrow (b^{-})^{\sim -} = a^{\sim -} \rightsquigarrow b^{-} = a \rightsquigarrow (b^{-})^{\sim -} = a \rightsquigarrow b^{-}$.

8. From (4) and from Proposition 2.1.35 (5) $(b^{\sim})^{-} \rightsquigarrow a^{-} = a^{-\sim} \to (b^{\sim})^{-\sim} = a^{-\sim} \to b^{\sim} = a \to (b^{\sim})^{-\sim} = a \to b^{\sim}$.

9. From Proposition 2.1.35 (2) it follows that $a \rightsquigarrow b^{-\sim} \leq (a \rightsquigarrow b^{-\sim})^{-\sim}$. The inequality $(a \rightsquigarrow b^{-\sim})^{-\sim} \leq a \rightsquigarrow b^{-\sim}$ follows from (5), (8) and the following equalities:
$\top = (a \rightsquigarrow b^{-\sim}) \rightarrow (a \rightsquigarrow b^{-\sim}) = a \rightsquigarrow [(a \rightsquigarrow b^{-\sim}) \rightarrow b^{-\sim}] = a \rightsquigarrow [(a \rightsquigarrow b^{-\sim})^{-\sim} \rightarrow b^{-\sim}] =$
$(a \rightsquigarrow b^{-\sim})^{-\sim} \rightarrow (a \rightsquigarrow b^{-\sim})$.
From Proposition 2.1.35 (2) it follows that $a \rightarrow b^{\sim-} \leq (a \rightarrow b^{\sim-})^{\sim-}$.
The inequality $(a \rightarrow b^{\sim-})^{\sim-} \leq a \rightarrow b^{\sim-}$ follows from (8), (10) and the following equalities: $\top = (a \rightarrow b^{\sim-}) \rightsquigarrow (a \rightarrow b^{\sim-}) = a \rightarrow [(a \rightarrow b^{\sim-}) \rightsquigarrow b^{\sim-}] = a \rightarrow [(a \rightarrow b^{\sim-})^{\sim-} \rightsquigarrow b^{\sim-}] = (a \rightarrow b^{\sim-})^{\sim-} \rightsquigarrow (a \rightarrow b^{\sim-})$.

10. It follows trivially from (4) and by assumption. It follows trivially from (4) and by assumption.

□

The property (5) of the above Proposition, which is the "strong" version of the symmetry condition of Definition 2.1.17, has a very important feature, in fact it characterizes the associativity condition of symmetrical cdeo algebras, as follows.

Proposition 2.1.63. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. Then it is associative if and only if for all $a, b, c \in L$ one has*

$$a \rightsquigarrow (b \rightarrow c) = b \rightarrow (a \rightsquigarrow c).$$

Proof. Of course, we only have to prove that the given condition is sufficient.
So, let $a, b, c \in L$. Clearly, $a \otimes (b \otimes c) \leq (a \otimes b) \otimes c \Leftrightarrow b \leq c \rightsquigarrow (a \rightarrow ((a \otimes b) \otimes c)) \Leftrightarrow b \leq a \rightarrow (c \rightsquigarrow ((a \otimes b) \otimes c))$ and the latter inequality is true thanks to Propositions 2.1.11 (10) and 2.1.30 (3); in fact $a \rightarrow (c \rightsquigarrow ((a \otimes b) \otimes c)) \geq a \rightarrow (a \otimes b) \geq b$.
The inequality $(a \otimes b) \otimes c \leq a \otimes (b \otimes c)$ can be proved similarly. □

Corollary 2.1.64. *A commutative cdeo algebra satisfies the strong exchange condition if and only if it is associative.* □

Proposition 2.1.65. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. If it is associative, then the following properties hold, for all $a, b \in L$.*

1. $a^{-} \uplus^{(\sim)}_{\rightarrow} b = (a \otimes b^{\sim})^{-}$;

2. $a^{\sim} \uplus^{(-)}_{\rightsquigarrow} b = (b^{-} \otimes a)^{\sim}$;

3. $a^{\sim} \uplus^{(-)}_{\rightarrow} b = a \rightsquigarrow b^{--}$;

4. $a^{\sim} \uplus^{(\sim)}_{\rightarrow} b = a^{\sim-} \rightsquigarrow b^{\sim-} = a \rightsquigarrow b^{\sim-}$;

5. $a^{-} \uplus^{(-)}_{\rightsquigarrow} b = a^{-\sim} \rightarrow b^{-\sim} = a \rightarrow b^{-\sim}$;

6. $a^{-} \uplus^{(\sim)}_{\rightsquigarrow} b = a \rightarrow b^{\sim\sim}$;

7. *if the negation $[\cdot]^{-}$ is involutive, then* $a^{\sim} \uplus^{(-)}_{\rightarrow} b = a \rightsquigarrow b$;

8. *if the negation* $[\cdot]^{\sim}$ *is involutive, then* $a^{-} \uplus_{\rightsquigarrow}^{(\sim)} b = a \rightarrow b$.

If the negations $[\cdot]^{\sim}$ *and* $[\cdot]^{-}$ *are cross-involutive, then:*

9. $a^{\sim} \uplus_{\rightarrow}^{(\sim)} b = a \rightsquigarrow b$;

10. $a^{-} \uplus_{\rightsquigarrow}^{(-)} b = a \rightarrow b$.

Proof. 1. From Proposition 2.1.60 (3) it follows that $a^{-} \uplus_{\rightarrow}^{(\sim)} b = b^{\sim} \rightarrow a^{-} = (a \otimes b^{\sim})^{-}$.

2. From Proposition 2.1.61 (4) it follows that $a^{\sim} \uplus_{\rightsquigarrow}^{(-)} b = b^{-} \rightsquigarrow a^{\sim} = (b^{-} \otimes a)^{\sim}$.

3. From Proposition 2.1.62 (5) it follows that $a^{\sim} \uplus_{\rightarrow}^{(-)} b = b^{-} \rightarrow a^{\sim} = b^{-} \rightarrow (a \rightsquigarrow \bot) = a \rightsquigarrow (b^{-} \rightarrow \bot) = a \rightsquigarrow b^{--}$.

4. From Proposition 2.1.62 (4) it follows easily that $a^{\sim} \uplus_{\rightarrow}^{(\sim)} b = b^{\sim} \rightarrow a^{\sim} = a^{\sim -} \rightsquigarrow b^{\sim -} = a \rightsquigarrow b^{\sim -}$.

5. From Proposition 2.1.62 (4) it follows easily that $a^{-} \uplus_{\rightsquigarrow}^{(-)} b = b^{-} \rightsquigarrow a^{-} = a^{-\sim} \rightarrow b^{-\sim} = a \rightarrow b^{-\sim}$.

6. From Proposition 2.1.62 (5) it follows easily that $a^{-} \uplus_{\rightsquigarrow}^{(\sim)} b = b^{\sim} \rightsquigarrow a^{-} = b^{\sim} \rightsquigarrow (a \rightarrow \bot) = a \rightarrow (b^{\sim} \rightsquigarrow \bot) = a \rightarrow b^{\sim\sim}$.

7. It follows easily by assumption and from (3).

8. It follows easily by assumption and from (6).

9. It follows easily by assumption and from (4).

10. It follows easily by assumption and from (5).

□

Remark 2.1.66. Of course, all the statements we have proved that involve the dual implications and negations and the conjunction in a symmetrical associative cdeo algebra still hold in any commutative and associative cdeo algebra with respect to the (unique) implication, the conjunction and the (unique) negation: these are already known properties in commutative complete residuated lattices. The results involving the disjunctions are new, though in good pseudo-MTL algebras these lead to similar results as in [12].

2.1.6 Some remarks on extended-order algebras

Lattice ordered, in particular residuated structures have been introduced and considered in several kinds of contexts that have determined different approaches giving motivation for various sets of axioms. The point of view of implicative algebras is motivated by the study of algebraic aspects of logics, in particular of deductive systems, and it has also been put into the context of category theory to develop a categorical approach to logic. The detailed study of extended-order algebras aims to motivate and justify or to critically discuss basic requirements of residuated structures and further conditions such as order-completeness, associativity, commutativity and symmetry frequently assumed within lattice valued mathematics and logics.

It is noteworthy that just like every lattice-structure on a set L is completely determined by the underlying order relation $\leq$ in L, it can be seen, since [50], that the properties and even the existence of an integral residuated structure on the upper bounded poset $(L, \leq, \top)$ are determined by the way of the order relation is extended to get an implication in L with true value $\top$.
The fundamental result that every extended-order algebra (w-eo algebra whose implication is antitonic in the first and isotonic in the second argument) can be embedded into its MacNeille completion allows to consider the completeness condition a not too strong assumption, which is very important in most applications. Moreover it is possible to recognize since the first step of extending the order relation of $(L, \leq, \top)$ whether the obtained implication $\to$ originates a complete residuated structure, in particular an adjoint product $\otimes$, and which properties they have. Particular attention is devoted to associativity, commutativity and symmetry (as a good substitute of commutativity) of the product with a critical view of their motivation.
Relevant discussion and results aimed to exploit the possibility of dropping the associativity assumption, which is instead a well established requirement in all the approaches to structures related to logical connectives, including residuated lattices, (pseudo-)BCK algebras, quantales, t-norms and (monoidal) closed categories.
The effective power of associativity turns out rather to allow the "strong" version of several properties, provided that those are satisfied in their "weak" version: examples are given in [50], in the previous Subsections (see, for instance, Propositions 2.1.60, 2.1.61 and 2.1.63) and in the next Claim, where we observe that the *strong idempotency condition* is a consequence of the *weak idempotency condition* under the assumption of associativity, as follows.

Claim 2.1.67. *Let $(L, \to, \top)$ be an idempotent right-distributive w-ceo algebra. If it is associative, then it satisfies the strong idempotency condition.*

In fact, by assumption the following equalities hold:
$$\begin{aligned} a \to (a \to b) &= \bigvee \{t \in L |\ a \otimes t \leq a \to b\} \\ &= \bigvee \{t \in L |\ a \otimes (a \otimes t) \leq b\} \\ &= \bigvee \{t \in L |\ a \otimes t \leq b\} \\ &= \bigvee \{t \in L |\ t \leq a \to b\} = a \to b. \end{aligned}$$

As further most relevant examples, if $(L, \to, \top)$ is complete with a distributive implication, the adjoint product $\otimes$ and, possibly, the dual implication $\rightsquigarrow$ are related to $\to$ by "weak" conditions expressed by the equivalences

(i) $a \to (b \to c) = \top \ \Leftrightarrow\ (b \otimes a) \to c = \top$;

(ii) $a \rightsquigarrow (b \to c) = \top \ \Leftrightarrow\ b \to (a \rightsquigarrow c) = \top$.

Then it can be seen that the associativity assumption is equivalent to ask that the "strong" version of (i) and (ii) are satisfied; more precisely:

Claim 2.1.68. *The following are equivalent for a right-distributive, possibly symmetrical, ceo algebra*

1. $(L, \rightarrow, \top)$ *is associative;*
2. *for all* $a, b, c \in L$*:* $a \rightarrow (b \rightarrow c) = (b \otimes a) \rightarrow c$*;*
3. *for all* $a, b, c \in L$*:* $a \rightsquigarrow (b \rightarrow c) = b \rightarrow (a \rightsquigarrow c)$*.*

As a consequence the following is true.

Claim 2.1.69. *A right distributive eo algebra is associative and commutative if and only if it satisfies the strong exchange condition:* $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$*.*

In fact it has been shown in [50] that the condition is necessary. Conversely, the strong exchange condition implies commutativity (hence symmetry) and, by Proposition 2.1.63, associativity, too.

We close this Subsection with a few remarks that relate some of our results with other residuated structures.
We have already explained that w-eo algebras are exactly the implicative algebras considered in [74] that have been specialized to get positive implication algebras, also called Hilbert algebras; so our results on idempotency allow to characterize eo algebras whose MacNeille completion are Heyting algebras as follows.

Claim 2.1.70. *A right-distributive eo algebra* $(L, \rightarrow, \top)$ *is symmetrical and idempotent if and only if its MacNeille completion* $(K, \rightarrow_K, \top_K)$ *is a complete Heyting algebra.*

In fact if K is a complete Heyting algebra, then it is idempotent and commutative, so by Proposition 2.1.14 and Corollary 2.1.49 L is idempotent and commutative, hence symmetrical. Conversely, if L is idempotent and symmetrical then the adjoint product $\otimes$ in its MacNeille completion is the meet operation $\wedge$ since clearly $a \wedge b = (a \wedge b) \otimes (a \wedge b) \leq a \otimes b$ and $a \otimes b \leq a \wedge b$ (see Proposition 2.1.30).

As a special case of *closed categories* (see [32]) *implication structures* $(L, e, \rightarrow)$ have been considered in [89] assuming that L is a poset (ordered category), e is an element of L and for all $a, b, c \in L$ the following conditions are satisfied

1. $a \rightarrow b \leq a \rightarrow c$, if $b \leq c$;
2. $a \rightarrow c \leq b \rightarrow c$, if $a \geq b$;
3. $a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$;
4. $e \leq a \rightarrow b \Leftrightarrow a \leq b$;
5. $e \rightarrow a \leq a$.

Also, a structure of *adjoint implications* $(L, e, \to, \rightsquigarrow)$, where $\to$ and $\rightsquigarrow$ are implications related to the same order in L and form a Galois pair, has been considered in [89]. Such a structure is said to be *symmetric if* $\to$ and $\rightsquigarrow$ coincide.

It is evident that this approach, which is closely related to ordered categories and to *quantales* as well, is quite similar to ours. Nevertheless there are fundamental and important differences (which concern quantale and closed category theory, too) that can be summarized as follows.

On one hand, the closed-category and quantale-like approach is more general than extended-order algebras approach since the former does not assume that the identity (or unit) e is necessarily the greatest element of L. Such an assumption, i.e. considering integral structures, would render (adjoint) implication structures a special kind of (symmetrical) eo algebras. Also, note that the integral symmetric (in the sense of [89]) adjoint implication structures are commutative eo algebras.

On the other hand, taking into consideration the integral structures, w-eo algebras are by far more general since these reduce to assuming the above condition (4) of implication structures only; eo algebras would add conditions (1) and (2).
But what is most relevant is that all along the development of the study of (w-)eo algebras, since [50] through our works [13, 14] and the present thesis, we have not assumed conditions of the "strong-type" such as the above condition (3) that correspond to the composition law of closed categories; this is called "strong isotonic condition" in [50], where it is shown to be a consequence of the "weak isotonic condition" (1) under the assumption of associativity (see the above discussion about this feature of the associativity axiom, that allows the extension of most conditions assumed in a "weak" form to their "strong" version).
These remarks confirm that, roughly speaking, the associativity condition of the product-conjunction allows the strong version of conditions once these are satisfied in their weak version and moreover it seems to be required as a fundamental tool within the context of category theory. A new evidence comes from [33], where it is shown that fuzzy Galois connections (in fact, "strong versions" of the classical Galois connections) and functorial conditions of powerset operators, determined by cdeo algebra-valued relations, require associativity, and from [48], where good examples of cdeo algebra-valued categories benefit of the associativity assumption.

Our last comment concerns residuated lattices (see [9, 58]), that are ordered algebraic structures $(L, \wedge, \vee, \cdot, e, \to, \rightsquigarrow)$ such that

1. $(L, \wedge, \vee)$ is a lattice;
2. $(L, \cdot, e)$ is a monoid;
3. $\to$ and $\rightsquigarrow$ are binary operations for which the equivalences $a \cdot b \leq c \Leftrightarrow a \leq b \rightsquigarrow c \Leftrightarrow b \leq a \to c$ hold, for all $a, b, c \in L$.

If we reduce, once more, to the integral case, then the following is trivially true.

Claim 2.1.71. *The complete integral residuated lattices are exactly the symmetrical associative cdeo algebras.*

Hence also the relationship in the non-complete case can be stated.

Claim 2.1.72. *Every integral residuated lattice is a symmetrical associative deo algebra.*

In fact, if $(L, \wedge, \vee, \cdot, \top, \to, \rightsquigarrow)$ is a residuated lattice, then clearly $(L, \to, \top)$ is a symmetrical, hence left-distributive eo algebra (see Proposition 2.1.19); moreover, if $a \in L$, $B, B' \subseteq L$ and $Lb(B) = Lb(B')$, then the right-distributivity follows by the equivalences $x \in Lb(a \to B) \Leftrightarrow a \otimes x \in Lb(B) = Lb(B') \Leftrightarrow x \in Lb(a \to B')$.
Now, the MacNeille completion $(K, \to_K, \top_K)$ of $(L, \to, \top)$ is symmetrical, with dual implication $\rightsquigarrow_K$, and it is associative since for all $\alpha = [A], \beta = [B], \gamma = [X] \in K$ it is true that

$$\begin{aligned}\alpha \to_K (\beta \rightsquigarrow_K \gamma) &= [Lb(A) \to (Lb(B) \rightsquigarrow X)]\\ &= [Lb(B) \rightsquigarrow (Lb(A) \to X)]\\ &= \beta \rightsquigarrow_K (\alpha \to_K \gamma).\end{aligned}$$

In fact, by the associativity of the product $\cdot$, we have that:
$t \in Lb(Lb(A) \to (Lb(B) \rightsquigarrow X)) \Leftrightarrow (a' \cdot t) \cdot b' \leq x, \forall a' \in Lb(A), b' \in Lb(B), x \in X \Leftrightarrow a' \cdot (t \cdot b') \leq x, \ \forall a' \in Lb(A), b' \in Lb(B), x \in X \Leftrightarrow t \in Lb(Lb(B) \rightsquigarrow (Lb(A) \to X))$.
As a conclusion $(L, \to, \top)$ is associative, by Proposition 2.1.56.

Eventually, we remark that assuming the completeness condition (and consequently the distributivity of the product over joins on both arguments) for integral residuated lattices is not a strong restriction since the following holds, which justify the use of such a condition in most applications to lattice-valued mathematics and logics.

Claim 2.1.73. *1. The MacNeille completion of an integral residuated lattice $(L, \wedge, \vee, \cdot, \top, \to, \rightsquigarrow)$ is a complete integral residuated lattice.*

2. $(L, \wedge, \vee, \cdot, e, \to, \rightsquigarrow)$ is embedded in its MacNeille completion.

In fact, under the given assumptions, $(L, \to, \top)$ is a symmetrical associative deo algebra then it is embedded in its MacNeille completion $(K, \to_K, \top_K)$ that has a structure of a symmetrical associative cdeo algebra, i.e. is a complete integral residuated lattice and the dual implication $\rightsquigarrow_K$ extends $\rightsquigarrow$. Eventually, it can be seen that the product $\otimes_K$ in K extends the one given in L. In fact, for all $a, b \in L$:

$$[a \cdot b] = [\bigwedge\{t \in L |\ b \leq a \to t\}] = [\{t \in L |\ b \leq a \to t\}],$$

$$\begin{aligned}[a] \otimes_K [b] &= \bigwedge_K \{\gamma \in K |\ [b] \leq_K [a] \to_K \gamma\}\\ &= [\bigcup\{X \subseteq L |\ \downarrow b \subseteq Lb(\downarrow a \to X)\}]\\ &= [\bigcup\{X \subseteq L |\ b \leq a \to x, \forall x \in X\}]\end{aligned}$$

and since clearly $\{t \in L |\ b \leq a \to t\} = \bigcup\{X \subseteq L |\ b \leq a \to x, \forall x \in X\}$, then $[a \cdot b] = [a] \otimes_K [b]$.

2.2 Implicators

This Section deals with a first application of extended-order algebras in connection with the fundamental concepts of implicators and conjunctors largely used and studied in many contexts (see, for instance, [2, 3, 30, 77, 78]).
The notion of implicator will be reconsidered in the light of the conditions that the implication of an extended-order algebra satisfies, and the notion of relative implication is also introduced.
Moreover, we introduce and study the notion of **conditional conjunction**, which is strictly related to the divisibility condition of BL-algebras.

2.2.1 Fuzzy implicators and the conditional implication

In the literature the term fuzzy implication operator, shortly fuzzy implicator, usually is meant as an extension of the implication defined in classical logic; in fact in the most general sense it is defined as a map $\mathcal{I} : [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary conditions

$$(b)\ (0,0)\mathcal{I} = (0,1)\mathcal{I} = (1,1)\mathcal{I} = 1 \text{ and } (1,0)\mathcal{I} = 0.$$

Further properties are considered and assumed on $\mathcal{I}$ in different theoretical approaches and applications; a detailed list of the most important (some of which imply the equalities in the boundary condition (b)) is given in [78] as follows:
(fi_1) $a \leq b \Rightarrow (b,c)\mathcal{I} \leq (a,c)\mathcal{I}$, for all $a,b,c \in [0,1]$;
(fi_2) $a \leq b \Rightarrow (c,a)\mathcal{I} \leq (c,b)\mathcal{I}$, for all $a,b,c \in [0,1]$;
(fi_3) $(0,b)\mathcal{I} = 1$, for every $b \in [0,1]$;
(fi_4) $(a,1)\mathcal{I} = 1$, for every $a \in [0,1]$;
(fi_5) $(1,0)\mathcal{I} = 0$;
(fi_6) $(1,b)\mathcal{I} = b$, for every $b \in [0,1]$;
(fi_7) $(a,(b,c)\mathcal{I})\mathcal{I} = (b,(a,c)\mathcal{I})\mathcal{I}$, for all $a,b,c \in [0,1]$;
(fi_8) $(a,b)\mathcal{I} = 1 \Leftrightarrow a \leq b$, for all $a,b \in [0,1]$;
(fi_9) the map N', defined as $(a)N' = (a,0)\mathcal{I}$, for every $a \in [0,1]$, is an involutive fuzzy negation;
(fi_{10}) $(a,b)\mathcal{I} \geq b$, for all $a,b \in [0,1]$;
(fi_{11}) $(a,a)\mathcal{I} = 1$, for every $a \in [0,1]$;
(fi_{12}) $(a,b)\mathcal{I} = ((b)N,(a)N)\mathcal{I}$, for every $a \in [0,1]$, where N is an involutive fuzzy negation;
(fi_{13}) I is a continuous mapping.

Evidently there are several interrelationships among these axioms. In [78] there is a complete view of these: there, taking (fi_1)-(fi_5) as the basic system of axioms for a fuzzy implicator the authors investigate the dependence and independence of the other axioms, given the first five that, in their turn, implies the boundary condition (b).
In this Subsection we reconsider the above conditions and we give a definition of an internal implicator in a complete lattice L, taking into account the point of view of the

implication operation of extended-order algebras.
This leads to a different arrangement and grouping of the basic requirements a fuzzy implicator should satisfy, according to the interpretation of the implication operation in the semantic of many-valued logic that motivated the introduction [50] and development of extended-order algebras, strictly related to implicative algebras considered in [74].
We recalled in Section 2.1 that the implication of a w-eo algebra (i.e. implicative algebra) is any internal extension of the order relation of any poset with a greatest element (true value $\top$), as stated in Proposition 2.1.2. This criterion states exactly in which cases the implication $a \to b$ is true saying nothing, in general, on when the implication is false, even if the existence of the value "false" (the least element $\bot$) in the algebra is assumed. So our feeling is that we should agree with the requirement $(0,0)\mathcal{I} = (0,1)\mathcal{I} = (1,1)\mathcal{I} = 1$ without asking to $(1,0)\mathcal{I}$ nothing but to be different from $\top$, in general, which makes the implication of a w-ceo algebra a bounded implicator, as we shall see. Nevertheless, we shall weaken further the requirements of an *implicator*, making it more general than the implication operation of a w-ceo algebra.

Definition 2.2.1. *Let $(L, \leq)$ be a complete lattice, with greatest element $\top$ and least element $\bot$. A map $\mathcal{I} : L \times L \to L$ is an **implicator** in L if it satisfies the axiom:*
(i) $(a,b)\mathcal{I} = \top \Rightarrow a \leq b \Rightarrow (b,a)\mathcal{I} \leq (a,b)\mathcal{I}$, for all $a, b \in L$.
*The implicator $\mathcal{I}$ is **bounded** if it satisfies the following axioms:*
(i_1) $(\bot, b)\mathcal{I} = \top$, for every $b \in L$;
(i_2) $(a, \top)\mathcal{I} = \top$, for every $a \in L$.
*The implicator $\mathcal{I}$ is **weak-ordered** if it satisfies the axiom:*
(i_3) $(a,b)\mathcal{I} = \top \Leftrightarrow a \leq b$, for all $a, b \in L$.
*The implicator $\mathcal{I}$ is **isotonic** if it satisfies the following axioms:*
(i_4) $a \leq b \Rightarrow (c,a)\mathcal{I} \leq (c,b)\mathcal{I}$, for all $a, b, c \in L$;
(i_5) $a \leq b \Rightarrow (b,c)\mathcal{I} \leq (a,c)\mathcal{I}$, for all $a, b, c \in L$.
*The implicator $\mathcal{I}$ is **crisp-bounded** if it satisfies the axiom:*
(b) $(\bot,\bot)\mathcal{I} = (\bot,\top)\mathcal{I} = (\top,\top)\mathcal{I} = \top$ and $(\top,\bot)\mathcal{I} = \bot$.
*The implicator $\mathcal{I}$ is **ordered** if it satisfies the axioms (i_3), (i_4) and (i_5).*
*The implicator $\mathcal{I}$ is **distributive** if it satisfies the following axioms:*
(i_6) $(a, \bigwedge_{i\in I} b_i)\mathcal{I} = \bigwedge_{i\in I}((a,b_i)\mathcal{I})$, for all $a \in L, \{b_i\}_{i\in I} \subseteq L$;
(i'_6) $(\bigvee_{i\in I} a_i, b)\mathcal{I} = \bigwedge_{i\in I}((a_i,b)\mathcal{I})$, for all $b \in L, \{a_i\}_{i\in I} \subseteq L$.
*The implicator $\mathcal{I}$ is **continuous** if it satisfies (i_6), (i'_6) and the following axioms:*
(i_7) $(a, \bigvee_{i\in I} b_i)\mathcal{I} = \bigvee_{i\in I}((a,b_i)\mathcal{I})$, for all $a \in L, \{b_i\}_{i\in I} \subseteq L$;
(i'_7) $(\bigwedge_{i\in I} a_i, b)\mathcal{I} = \bigvee_{i\in I}((a_i,b)\mathcal{I})$, for all $b \in L, \{a_i\}_{i\in I} \subseteq L$.
*The implicator $\mathcal{I}$ is **commutative** or **interchanging** if it satisfies the axiom:*
(i_8) $(a,(b,c)\mathcal{I})\mathcal{I} = \top \Leftrightarrow (b,(a,c)\mathcal{I})\mathcal{I} = \top$, for all $a, b, c \in L$.
*The implicator $\mathcal{I}$ is **involutive** if it satisfies the axiom:*
(i_9) the map N, defined by $(a)N = (a,\bot)\mathcal{I}$, for every $a \in L$, is an order reversing involution.
*The implicator $\mathcal{I}$ is **contrapositive** if it satisfies the axiom:*
(i_{10}) $(a,b)\mathcal{I} = ((b)N,(a)N)\mathcal{I}$, for all $a, b \in L$.

Eventually we list the other axioms for a fuzzy implicator considered in [78].
(i_{11}) $(\top, b)\mathcal{I} \le b$, for every $b \in L$;
(i_{12}) $(\top, b)\mathcal{I} \ge b$, for every $b \in L$;
(i_{13}) $(a, b)\mathcal{I} \ge b$, for all $a, b \in L$.

Some examples of implicators can be found in Appendix of this thesis.
The following Proposition shows in the item (1) that the implication of every one of the complete algebras we have considered in Section 2.1 is a weak-ordered, hence bounded, implicator; the subsequent statements (2)-(11) only state the further properties it satisfies depending on which kind of w-ceo algebra is considered.

Proposition 2.2.2. *1. If $(L, \to, \top)$ is a w-ceo algebra, then the operation $\to$ is a bounded weak-ordered implicator.*

2. *If $(L, \to, \top)$ is a right w-ceo algebra, then the implicator $\to$ satisfies the axiom (i_4), too.*

3. *If $(L, \to, \top)$ is a left w-ceo algebra, then the implicator $\to$ satisfies the axiom (i_5), too.*

4. *If $(L, \to, \top)$ is a ceo algebra, then $\to$ is an ordered implicator, too.*

5. *If $(L, \to, \top)$ is a right-distributive w-ceo algebra, then the implicator $\to$ satisfies the axioms (i_4) and (i_6), too.*

6. *If $(L, \to, \top)$ is a left-distributive w-ceo algebra, then the implicator $\to$ satisfies the axioms (i_5) and (i'_6), too.*

7. *If $(L, \to, \top)$ is a cdeo algebra, then $\to$ is an ordered distributive implicator, too.*

8. *If $(L, \to, \top)$ is a symmetrical cdeo algebra, then both $\to$ and $\rightsquigarrow$ are ordered, distributive and crisp-bounded implicators that satisfy the axioms (i_{11}), (i_{12}) and (i_{13}).*

9. *If $(L, \to, \top)$ is an involutive left w-ceo algebra, then $\to$ is an involutive implicator, too.*

10. *If $(L, \to, \top)$ is a commutative cdeo algebra, then $\to$ is an ordered, distributive, crisp-bounded and commutative implicator that satisfies the axioms (i_{11}), (i_{12}) and (i_{13}).*

11. *If $(L, \to, \top)$ is a commutative, associative and involutive cdeo algebra, then $\to$ is an ordered, distributive, crisp-bounded, commutative and contrapositive implicator that satisfies (i_{11}), (i_{12}) and (i_{13}).*

Proof. 1. From Proposition 2.1.2 $\to$ satisfies the following equivalence $a \to b = \top \Leftrightarrow a \le b$ and, in particular, it follows trivially that $a \to b = \top \Leftrightarrow a \le b \Rightarrow b \to a \le a \to b$, for all $a, b \in L$. Hence it satisfies the axioms (i) and (i_3).
Moreover the following equivalences are true $\bot \le b \Leftrightarrow \bot \to b = \top$, $a \le \top \Leftrightarrow a \to \top = \top$, for all $a, b \in L$. Therefore, $\to$ satisfies (i_1) and (i_2).

2. From (o_5) it follows that $a \leq b \Leftrightarrow a \to b = \top \Rightarrow c \to a \leq c \to b$, for all $a, b, c \in L$. Then $\to$ satisfies (i_4).

3. From (o'_5) it follows that $a \leq b \Leftrightarrow a \to b = \top \Rightarrow b \to c \leq a \to c$, for all $a, b, c \in L$. Then $\to$ satisfies (i_5).

4. It follows easily from (2) and (3).

5. Since (d_r) implies (o_5), it follows that $(L, \to, \top)$ is right w-ceo algebra.
Then from (2) one has that $\to$ satisfies (i_4). From (d_r) it follows that $a \to (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \to b_i)$; hence $\to$ satisfies (i_6).

6. Since (d_l) implies (o'_5) it follows that $(L, \to, \top)$ is left w-ceo algebra. Then from (3) one has that $\to$ satisfies (i_5).
From (d_l) it follows that $(\bigvee_{i\in I} b_i) \to a = \bigwedge_{i\in I}(a \to b_i)$; hence $\to$ satisfies (i'_6).

7. Since (d) implies (o_5) and (o'_5), it follows that $(L, \to, \top)$ is a ceo algebra. Then the statement follows from (4), (5) and (6).

8. By assumption and from Proposition 2.1.26 it follows that $(L, \to, \top)$ and $(L, \rightsquigarrow, \top)$ are cdeo algebras. Hence, from (7) $\to$ and $\rightsquigarrow$ are ordered and distributive implicators.
From Proposition 2.1.30 (5) it follows that $\to$ and $\rightsquigarrow$ satisfy (i_{11}), (i_{12}) and (b). Moreover, from Proposition 2.1.30 (4) it follows that $\to$ and $\rightsquigarrow$ satisfy (i_{13}).

9. It follows trivially from Definitions 2.1.32, 2.2.1 and Proposition 2.1.33 (2).

10. From Proposition 2.1.50 it follows that $(L, \to, \top)$ is a symmetrical cdeo algebra. Then from (8) $\to$ is an ordered, distributive and crisp-bounded implicator that satisfies (i_{11}), (i_{12}) and (i_{13}). Moreover it is commutative by Definition 2.2.1.

11. From (10) it follows that $\to$ is an ordered, distributive, commutative and crisp-bounded implicator that satisfies (i_{11}), (i_{12}) and (i_{13}). Moreover, by assumption and from Propositions 2.1.50 and 2.1.62 (4) it follows that $\to$ is a contrapositive implicator.

□

The above Proposition explains and suggests the possibility to think of an implicator as nothing but the implication operation of some w-ceo algebra. In particular, if L is a complete lattice and $\mathcal{I} : L \times L \to L$, then

- $\mathcal{I}$ is a weak-ordered implicator if and only if $(L, \mathcal{I}, \top)$ is a w-ceo algebra;

- $\mathcal{I}$ is an ordered implicator if and only if $(L, \mathcal{I}, \top)$ is a ceo algebra.

Fuzzy implicators have been mostly considered, since [3], to express and to evaluate the inclusion between fuzzy sets. Classically, crisp inclusion between fuzzy (or lattice-valued) sets has been considered to be the order relation point-wisely induced by the

lattice-order of L in the L-powerset L^X of any set X.
In order to determine the inclusion degree between two L-sets A, B with respect to an implicator $\rightarrow$ in L it seems reasonable, of course, to evaluate the extended-order relationship between A and B; this is usually done by means of the so called *subsethood degree* $(A,B)\mathcal{S} = \bigwedge_x (x)A \rightarrow (x)B$, that extends to the many-valued context the following classical equivalence:

$$\forall A, B \subseteq L : A \subseteq B \Leftrightarrow (a \in A \Rightarrow a \in B).$$

Here we further propose the possibility to evaluate the inclusion of A in B by seeing to which extent A *is included in the conjunction of* A *and* B (or *how much of* A *is in* B). In terms of the right-distributive (w-)ceo algebra structure this purpose can be realized using the relative implication given by the following Definition; this operator extends to the many-valued context the following classical equivalence:

$$\forall A, B \subseteq L : A \subseteq B \Leftrightarrow (a \in A \Rightarrow a \in A \cap B).$$

Definition 2.2.3. *Let* $(L, \rightarrow, \top)$ *be a right-distributive w-ceo algebra. The* ***relative implication*** *is the binary operation*

$$\begin{gathered} \rightarrow_{\otimes}: L \times L \rightarrow L \\ (a,b) \mapsto a \rightarrow_{\otimes} b = a \rightarrow (a \otimes b), \end{gathered}$$

where $\otimes$ *is the adjoint product of* L*.*

Lemma 2.2.4. *If* $(L, \rightarrow, \top)$ *is a right-distributive w-ceo algebra, then the relative implication satisfies the axioms* (i_1)*,* (i_2)*,* (i_4) *and* (i_{12})*.*

Proof. From (o_2) and Proposition 2.1.11 (3) it follows that $\bot \rightarrow_{\otimes} b = \bot \rightarrow (\bot \otimes b) = \bot \rightarrow \bot = \top$.
From (o_2) and Proposition 2.1.11 (2) one has that $a \rightarrow_{\otimes} \top = a \rightarrow (a \otimes \top) = a \rightarrow a = \top$.
From (o_5) and Proposition 2.1.11 (6) it follows that $a \le b \Rightarrow c \otimes a \le c \otimes b \Rightarrow c \rightarrow (c \otimes a) \le c \rightarrow (c \otimes b)$; hence $c \rightarrow_{\otimes} a \le c \rightarrow_{\otimes} b$.
From Proposition 2.1.11 (7) one has that $\top \rightarrow_{\otimes} b = \top \rightarrow (\top \otimes b) \ge b$.
Hence $\rightarrow_{\otimes}$ satisfies the axioms (i_1), (i_2), (i_4) and (i_{12}). □

Proposition 2.2.5. *If* $(L, \rightarrow, \top)$ *is a symmetrical cdeo algebra that satisfies the condition* $a \le b \Rightarrow b \otimes a \le a \otimes b$*, then the relative implications* $\rightarrow_{\otimes}$ *and* $\rightsquigarrow_{\otimes}$ *are bounded and crisp-bounded implicators that satisfy the axioms* (i_4)*,* (i_{11}) (i_{12})*.*

Proof. From Proposition 2.1.30 (2), (o_5) and the assumption it follows that $a \rightarrow_{\otimes} b = \top \Leftrightarrow a \le a \otimes b \le b \Rightarrow a \le b \Rightarrow b \rightarrow_{\otimes} a \le a \rightarrow_{\otimes} b$. Hence $\rightarrow_{\otimes}$ is an implicator; trivially, by the above Lemma $\rightarrow_{\otimes}$ is bounded and satisfies (i_4) and (i_{12}).
Moreover, from Propositions 2.1.29 and 2.1.30 (5) it follows that $\top \rightarrow_{\otimes} b = \top \rightarrow (\top \otimes b) = \top \rightarrow b = b$, for every $b \in L$ and, in particular, $\top \rightarrow_{\otimes} \bot = \top \rightarrow (\top \otimes \bot) = \top \rightarrow \bot = \bot$. Hence $\rightarrow_{\otimes}$ satisfies the axioms (i_{11}) and (b). Similarly for $\rightsquigarrow_{\otimes}$. □

Clearly a commutative cdeo algebra satisfies the assumption of Proposition 2.2.5; but even the symmetry condition, and hence the commutativity are not necessary for $\rightarrow_\otimes$ to be an implicator, as the structure 22 of Appendix shows.

Proposition 2.2.6. *Let* $(L, \rightarrow, \top)$ *be a* cdeo *algebra. Then* $\rightarrow_\otimes$ *is a weak-ordered implicator if and only if the equivalence* $a \leq b \iff a = a \otimes b$ *holds, for all* $a, b \in L$.

Proof. Assume $\rightarrow_\otimes$ to be a weak-ordered implicator. Then it follows from $a \leq b$ that $a \rightarrow (a \otimes b) = \top$, hence $a \leq a \otimes b \leq a$. Conversely, from $a = a \otimes b$ one has $a \rightarrow_\otimes b = \top$, hence $a \leq b$. Now assume the stated equivalence.
Then the equivalence $a \rightarrow_\otimes b = \top \Leftrightarrow a \leq a \otimes b \leq a \Leftrightarrow a \leq b$ proves that $\rightarrow_\otimes$ is a weak-ordered implicator. □

Corollary 2.2.7. *If* $(L, \rightarrow, \top)$ *is a symmetrical cdeo algebra, then the dual relative implication* $\rightsquigarrow_\otimes$ *defined by*

$$a \rightsquigarrow_\otimes b = a \rightsquigarrow (b \otimes a), \text{ for all } a, b \in L$$

is a weak-ordered implicator if and only if the equivalence $a \leq b \iff a = b \otimes a$ *holds, for all* $a, b \in L$.

Proof. It follows trivially by the above Proposition since, under the assumption, $(L, \rightsquigarrow, \top)$ is a cdeo algebra with adjoint product $\tilde{\otimes}$ defined by $a \tilde{\otimes} b = b \otimes a$. □

Corollary 2.2.8. *If* $(L, \rightarrow, \top)$ *is a symmetrical cdeo algebra and* $\otimes$ *is idempotent, then* $\rightarrow_\otimes$ *and* $\rightsquigarrow_\otimes$ *are weak-ordered implicators.*

Proof. Under the assumption, for all $a \leq b$ in L one has $a = a \otimes a \leq a \otimes b \leq a$ and $a = a \otimes a \leq b \otimes a \leq a$. Conversely, it follows from $a = a \otimes b$ that $a = a \otimes b \leq b$ and from $a = b \otimes a$ that $a = b \otimes a \leq b$.
Now the statement follows from Proposition 2.2.6 and Corollary 2.2.7. □

Remark 2.2.9. The idempotency condition is necessary for each of $\rightarrow_\otimes$ and $\rightsquigarrow_\otimes$ to be a weak-ordered implicator. In fact, if this is the case it follows from either Proposition 2.2.6 or Corollary 2.2.7 and from $a \leq a$ that $a = a \otimes a$, for every $a \in L$.
However idempotency is not necessary for $\rightarrow_\otimes$ to be an implicator (see structure 22 of Appendix) and the symmetry condition is not necessary for $\rightarrow_\otimes$ to be a weak-ordered implicator, as the structure 23 of Appendix shows.

2.2.2 The conditional conjunction

In this last Subsection we consider a new binary operator that can be read as: "*a*" *and* "*b*, *given a*" which motivates the term we have chosen to denote it.

Definition 2.2.10. *Let* $(L, \rightarrow, \top)$ *be a right-distributive w-ceo algebra. The related* ***conditional conjunction*** *is the binary operation*

$$\begin{aligned} \otimes_{\rightarrow} : L \times L \rightarrow L \\ (a, b) \mapsto a \otimes_{\rightarrow} b = a \otimes (a \rightarrow b), \end{aligned}$$

where $\otimes$ *is the adjoint product of* L.

Proposition 2.2.11. *Let* $(L, \rightarrow, \top)$ *be a right-distributive w-ceo algebra. The conditional conjunction has the following properties, for all* $a, b \in L, \{b_i\} \subseteq L$.

1. $a \leq b \Rightarrow x \otimes_{\rightarrow} a \leq x \otimes_{\rightarrow} b,\ \forall x \in L$;
2. $\bot \otimes_{\rightarrow} b = \bot$;
3. $b \otimes_{\rightarrow} \bot = \bot$;
4. $a \otimes_{\rightarrow} \top = a$;
5. $a \otimes_{\rightarrow} x \leq b \otimes_{\rightarrow} x, \forall x \in L \Rightarrow a \leq b$;
6. $\top \otimes_{\rightarrow} b \leq b$;
7. $a \otimes_{\rightarrow} a = a$;
8. $a \otimes_{\rightarrow} b \leq a \wedge b$;
9. $a \leq b \Leftrightarrow a \otimes_{\rightarrow} b = a$;
10. $a \otimes_{\rightarrow} x \leq b \otimes_{\rightarrow} x, \forall x \in L \Rightarrow a \otimes_{\rightarrow} b = b \otimes_{\rightarrow} a = a$;
11. $a \otimes_{\rightarrow} b = \top \Leftrightarrow a = b = \top$;
12. $(a \otimes b) \otimes_{\rightarrow} a = a \otimes b$;
13. $a \otimes_{\rightarrow} (a \otimes b) = a \otimes b$;
14. $a \otimes_{\rightarrow} (a \otimes_{\rightarrow} b) = a \otimes_{\rightarrow} b$;
15. $(a \otimes_{\rightarrow} b) \otimes_{\rightarrow} b = a \otimes_{\rightarrow} b$.

    If moreover $(L, \rightarrow, \top)$ *is a right-distributive ceo algebra, then:*

16. $\top \rightarrow x \geq x, \forall x \in L \Rightarrow a \otimes b \leq a \otimes_{\rightarrow} b$;
17. *if* $\otimes$ *is distributive over* $\bigwedge$ *on the right side, then* $a \otimes_{\rightarrow} (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \otimes_{\rightarrow} b_i)$.

    If $(L, \rightarrow, \top)$ *is a symmetrical cdeo algebra, then:*

18. $\top \otimes_{\rightarrow} b = b$;
19. $a \otimes b \leq a \otimes_{\rightarrow} b$;
20. $a \leq b \Leftrightarrow x \otimes_{\rightarrow} a \leq x \otimes_{\rightarrow} b, \forall x \in L$.

    If $\rightarrow$ *satisfies the axiom* (i_7), *then:*

21. $a \otimes_{\to} (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \otimes_{\to} b_i)$.

Proof. 1. Let $a \leq b$; from (o_5) $x \to a \leq x \to b$. Then from Proposition 2.1.11 (6) one has $x \otimes_{\to} a = x \otimes (x \to a) \leq x \otimes (x \to b) = x \otimes_{\to} b$.

2. From Proposition 2.1.11 (3) it follows that $\bot \otimes_{\to} b = \bot \otimes (\bot \to b) = \bot$.

3. From Proposition 2.1.11 (7) one has that $b \otimes_{\to} \bot = b \otimes (b \to \bot) \leq \bot$.

4. From (o_1) and Proposition 2.1.11 (2) it follows that $a \otimes_{\to} \top = a \otimes (a \to \top) = a \otimes \top = a$.

5. Taking $x = \top$, the assumption gives $a \leq b$ by (4).

6. From Proposition 2.1.11 (7) one has that $\top \otimes_{\to} b = \top \otimes (\top \to b) \leq b$.

7. From Proposition 2.1.11 (2) and (o_2) it follows that $a \otimes_{\to} a = a \otimes (a \to a) = a \otimes \top = a$.

8. From Proposition 2.1.11 (1) $a \otimes_{\to} b \leq a$; moreover, from Proposition 2.1.11 (9) $a \otimes_{\to} b \leq b$. Hence $a \otimes_{\to} b \leq a \wedge b$.

9. Let $a \leq b$. Then $a \to b = \top$ and hence, from Proposition 2.1.11 (2) $a \otimes_{\to} b = a \otimes (a \to b) = a \otimes \top = a$. Conversely, from Proposition 2.1.11 (7) $a = a \otimes_{\to} b = a \otimes (a \to b) \leq b$.

10. The assumption gives $a \leq b$ by (5); then $a \otimes_{\to} b = a$ by (9).
Moreover, taking $x = a$, by the assumption, (7) and (8) one has $a \leq b \otimes_{\to} a \leq a$.

11. Assume $a \otimes_{\to} b = \top$; from (8) it follows that $\top \leq a \wedge b$ and hence $a = b = \top$.
Conversely, it is clear that $\top \otimes_{\to} \top = \top$.

12. From Proposition 2.1.11 (1),(2) one has that $(a \otimes b) \otimes_{\to} a = (a \otimes b) \otimes [(a \otimes b) \to a] = (a \otimes b) \otimes \top = (a \otimes b)$.

13. From Proposition 2.1.11 (7) $a \to (a \otimes b) \geq b$; hence, from Proposition 2.1.11 (6) $a \otimes_{\to} (a \otimes b) = a \otimes [a \to (a \otimes b)] \geq a \otimes b$. The converse inequality follows from (8), so $a \otimes_{\to} (a \otimes b) = a \otimes b$.

14. From Proposition 2.1.11 (6),(7) one has that $a \otimes_{\to} (a \otimes_{\to} b) = a \otimes [a \to (a \otimes (a \to b))] \geq a \otimes (a \to b) = a \otimes_{\to} b$. The converse inequality follows from (8).

15. By (8) $a \otimes_{\to} b \leq b$, hence by (9) $(a \otimes_{\to} b) \otimes_{\to} b = a \otimes_{\to} b$.

16. From (o_5') and Proposition 2.1.11 (6) and by assumption the following inequality holds $a \otimes_{\to} b = a \otimes (a \to b) \geq a \otimes (\top \to b) \geq a \otimes b$.

17. By assumption and condition (d_r) it follows that $a \otimes_{\to} (\bigwedge_{i\in I} b_i) = a \otimes [a \to (\bigwedge_{i\in I} b_i)] = a \otimes [\bigwedge_{i\in I}(a \to b_i)] = \bigwedge_{i\in I} [a \otimes (a \to b_i)] = \bigwedge_{i\in I}(a \otimes_{\to} b_i)$.

18. From Propositions 2.1.29 and 2.1.30 (5) it follows that $\top \otimes_{\to} b = \top \otimes (\top \to b) = \top \otimes b = b$.

19. By assumption the equality $\top \to x = x, \forall x \in L$, follows from Proposition 2.1.30 (5), hence the statement follows by (16).

20. The implication " $\Rightarrow$ " comes from (1); the converse implication follows by (7) and (8), in fact by the assumption for $x = a$ one has $a = a \otimes_{\to} a \leq a \otimes_{\to} b \leq b$.

21. By assumption and Proposition 2.1.11 (5) it follows that:
$a \otimes_{\to} (\bigvee_{i\in I} b_i) = a \otimes [a \to (\bigvee_{i\in I} b_i)] = a \otimes [\bigvee_{i\in I}(a \to b_i)] = \bigvee_{i\in I}(a \otimes_{\to} b_i)$.

□

The properties listed in the above Proposition show that $\otimes_{\to}$ satisfies most conditions usually asked to a conjunction operator and moreover it is well related to the operator $\otimes$, the main conjunction operator of a right-distributive w-ceo algebra, and to the classical conjunction operator $\wedge$; in fact $x \otimes_{\to} y \leq x \wedge y$ in any case and if $(L, \to, \top)$ is a symmetrical cdeo algebra then

$$x \otimes y \leq x \otimes_{\to} y \leq x \wedge y, \text{ for all } x, y \in L.$$

We also recall that the equality $x \wedge y = x \otimes_{\to} y$, for all $x, y \in L$ is just the so called *divisibility condition* that is assumed in BL-algebras and in MV-algebras, as well (see for instance [52]).
We can show that whenever L is a chain the divisibility condition is necessary for the conditional conjunction $\otimes_{\to}$ to be isotonic in the first argument; more precisely the only conditional conjunction in a right-distributive w-ceo chain that is isotonic in the first argument is the meet operation. In fact, the following holds.

Proposition 2.2.12. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra and let L be a chain in the natural ordering. If the related conditional conjunction is isotonic in the first argument, then $a \otimes_{\to} b = a \wedge b$, for all $a, b \in L$.*

Proof. Consider $a, b \in L$ and assume $a \wedge b = a$. Then $a \leq b$ and by the assumption the inequality $a \otimes_{\to} x \leq b \otimes_{\to} x$, for every $x \in L$, holds.
Then from Proposition 2.2.11 (10) $a \otimes_{\to} b = b \otimes_{\to} a = a = a \wedge b$. □

Corollary 2.2.13. *A right-distributive w-ceo chain is a BL-algebra if and only if the related conditional conjunction is isotonic in the first argument.* □

Chapter 3

Many-valued relations

3.1 L-relations

In Chapter 1 we have recalled some basic notions concerning binary relations in the classical mathematical context.
Now we approach the study of binary many-valued relations taking values in some kind of extended-order algebras.
Though the considered notions are already known and well developed, our framework, based on extended-order algebras, is more general than the usual ones and allows a non-commutative and non-associative approach, which has been already considered only in [33] where, however, the structure of symmetrical cdeo algebras has not been exploited in its full power.

3.1.1 Algebras of L-relations

For any complete lattice $(L, \leq)$, and any set X, *L-sets* on X, or L-subsets of X, are maps from X to L (see [44]), which generalize Zadeh's fuzzy sets; L^X has the complete lattice structure pointwisely induced by L; we follow notation of [35, 44], in particular, for any subset $Y \subseteq X$ we denote by α_Y the L-set on X that takes value $\alpha \in L$ on Y and value $\perp$ elsewhere; α_x stands for $\alpha_{\{x\}}$ and it is called *L-point* with support x and value α.
The **binary many-valued relations** from a set X to a set Y taking values in L are functions $\mathcal{R} : X \times Y \to L$, i.e. L-sets on $X \times Y$. The **opposite** or **reverse** $\mathcal{R}_- : Y \times X \to L$ of $\mathcal{R}$ is defined by $(y, x)\mathcal{R}_- = (x, y)\mathcal{R}$, for all $x \in X, y \in Y$. An L-relation is **crisp** if $\top$ and $\perp$ are the only values it may take.
Most researchers using L-valued relations for several kinds of applications (for instance [4, 6, 11, 20, 35]) assume L to be a "complete residuated lattice", which is a simplified notation for "complete, integral, commutative residuated lattice", that is a commutative, associative cdeo algebra, as shown in Subsection 2.1. In particular, [37] and [87] deal with complete Heyting algebras which are idempotent, commutative, associative cdeo algebras.

In this chapter, we shall assume L to be, in any case, a complete lattice, possibly with a w-ceo algebra structure $(L, \rightarrow, \top)$ with further specified properties; in this last case, the set $L^{X \times Y}$ of L-valued relations has a w-ceo algebra structure of the same kind as $(L, \rightarrow, \top)$, getting all the needed elements and properties pointwisely from L, as we shall specify below.
We start considering $\mathbf{R_L}$, the class of all L-relations between two sets taking values in a w-ceo algebra $(L, \rightarrow, \top)$:

$$\mathbf{R_L} = \{\mathcal{R} : X \times Y \rightarrow L | \; X, Y \in |\mathbf{Set}|\} ;$$

sometimes we denote $\mathcal{R} : X \times Y \rightarrow L$ by $\mathcal{R} : X \rightharpoondown Y$ and $(x, y)\mathcal{R}$ by $x\mathcal{R}y$, for all $x \in X$, $y \in Y$.
We recall the following notation and pointwise definitions in $\mathbf{R_L}$.

Definition 3.1.1. *Let $(L, \rightarrow, \top)$ be a w-ceo algebra. For all $X, Y \in |\boldsymbol{Set}|$, $\mathcal{R}, \mathcal{R}' : X \rightharpoondown Y$, $\{\mathcal{R}_j : X \rightharpoondown Y\}_{j \in J}$ and for all $x, x' \in X, y \in Y$ one has:*

1. $\mathcal{R} \leq \mathcal{R}' \Leftrightarrow (x, y)\mathcal{R} \leq (x, y)\mathcal{R}'$;
2. $\perp\!\!\!\perp_{XY} : X \rightharpoondown Y \;:\; (x, y)\perp\!\!\!\perp_{XY} = \perp$;
3. $\top\!\!\!\top_{XY} : X \rightharpoondown Y \;:(x, y)\top\!\!\!\top_{XY} = \top$;
4. $\mathcal{I}_X : X \rightharpoondown X \;:\; (x, x')\mathcal{I}_X = \begin{cases} \top & \text{if } x = x' \\ \perp & \text{otherwise} \end{cases}$;
5. $\bigvee_{j \in J} \mathcal{R}_j : X \rightharpoondown Y :\; (x, y)(\bigvee_{j \in J} \mathcal{R}_j) = \bigvee_{j \in J} (x, y)\mathcal{R}_j$;
6. $\bigwedge_{j \in J} \mathcal{R}_j : X \rightharpoondown Y :\; (x, y)(\bigwedge_{j \in J} \mathcal{R}_j) = \bigwedge_{j \in J} (x, y)\mathcal{R}_j$;
7. $\mathcal{R} \rightarrow \mathcal{R}' : X \rightharpoondown Y :\; (x, y)(\mathcal{R} \rightarrow \mathcal{R}') = \mathcal{R}(x, y) \rightarrow (x, y)\mathcal{R}'$;
8. $\mathcal{R}^- : X \rightharpoondown Y :\; (x, y)\mathcal{R}^- = \mathcal{R}(x, y) \rightarrow \perp$.

If we consider $\mathbf{R_L}(X, Y)$, the set of all L-relations between two arbitrary fixed sets X, Y, then the triple $(\mathbf{R_L}(X, Y), \rightarrow, \top\!\!\!\top_{XY})$ is a w-ceo algebra, with respect to the implication $\rightarrow$ defined above, which has as a top element the constant relation $\top\!\!\!\top_{XY}$.
Indeed, this algebra of L-relations inherits all the properties which $(L, \rightarrow, \top)$ has. More precisely, the following hold.

Proposition 3.1.2. *The following equivalences are true.*

1. *$(L, \rightarrow, \top)$ is a w-ceo algebra if and only if $(\mathbf{R_L}(X, Y), \rightarrow, \top\!\!\!\top_{XY})$ is a w-ceo algebra, for all $X, Y \in |\boldsymbol{Set}|$;*
2. *$(L, \rightarrow, \top)$ is a (right)(left)(w-)ceo algebra if and only if $(\mathbf{R_L}(X, Y), \rightarrow, \top\!\!\!\top_{XY})$ is a (right)(left)(w-)ceo algebra, for all $X, Y \in |\boldsymbol{Set}|$;*

3. $(L, \rightarrow, \top)$ *is a (right-)(left-) distributive (w-)ceo algebra if and only if* $(\mathbf{R_L}(X,Y), \rightarrow, \top\!\!\top_{XY})$ *is a (right-)(left-) distributive (w-)ceo algebra, for all* $X, Y \in |\boldsymbol{Set}|$*;*

4. $(L, \rightarrow, \top)$ *is a symmetrical w-ceo algebra if and only if* $(\mathbf{R_L}(X,Y), \rightarrow, \top\!\!\top_{XY})$ *is a symmetrical w-ceo algebra, for all* $X, Y \in |\boldsymbol{Set}|$*;*

5. $(L, \rightarrow, \top)$ *is a symmetrical cdeo algebra if and only if* $(\mathbf{R_L}(X,Y), \rightarrow, \top\!\!\top_{XY})$ *is a symmetrical cdeo algebra, for all* $X, Y \in |\boldsymbol{Set}|$*;*

6. $(L, \rightarrow, \top)$ *is a commutative w-ceo algebra if and only if* $(\mathbf{R_L}(X,Y), \rightarrow, \top\!\!\top_{XY})$ *is a commutative w-ceo algebra, for all* $X, Y \in |\boldsymbol{Set}|$*;*

7. $(L, \rightarrow, \top)$ *is an associative w-ceo algebra if and only if* $(\mathbf{R_L}(X,Y), \rightarrow, \top\!\!\top_{XY})$ *is an associative w-ceo algebra, for all* $X, Y \in |\boldsymbol{Set}|$*.* □

Remark 3.1.3. If either $X = \varnothing$ or $Y = \varnothing$, then there is only the empty relation from X to Y, that is the inclusion function $\varnothing \hookrightarrow L$; hence $\mathbf{R_L}(X,Y) = \{\varnothing\}$ is the inconsistent order algebra. In previous works any kind of w-eo algebra has been assumed to be consistent, i.e. such that the carrier L has at least two elements; now we do not exclude to consider the trivial case $(\{\top\}, \rightarrow, \top)$, in order to cover the algebra of L-relations from X to Y in case either X or Y is the empty set $\varnothing$, that is, in fact the inconsistent w-eo algebra $(\{\varnothing\}, \rightarrow, \varnothing)$.
Note that, for all X, Y, the w-ceo algebra of L-relations from X to Y is inconsistent if L is.

Obviously, the properties stated in the Subsection 2.1 (see also [13, 50]) hold in the algebra of L-relations.

3.1.2 Compositions of L-relations

Let X, Y, Z, L be sets and let $\chi : L \times L \rightarrow L$ be an operation on L. Starting from χ it is possible to consider the following partial compositions, for all $\mathcal{R} : X \rightharpoondown Y$, $\mathcal{S} : Y \rightharpoondown Z$:

1. the **universal** (L, χ)**-composition** $\mathcal{R}\hat{\chi}\mathcal{S} : X \rightharpoondown Z$, defined by

$$(x,y)(\mathcal{R}\hat{\chi}\mathcal{S}) = \bigwedge_{y \in Y} ((x,y)\mathcal{R}, (y,z)\mathcal{S})\chi, \ \forall x \in X, z \in Z;$$

2. the **existential** (L, χ)**-composition** $\mathcal{R}\check{\chi}\mathcal{S} : X \rightharpoondown Z$, defined by

$$(x,y)(\mathcal{R}\check{\chi}\mathcal{S}) = \bigvee_{y \in Y} ((x,y)\mathcal{R}, (y,z)\mathcal{S})\chi, \ \forall x \in X, z \in Z.$$

In the context of w-ceo algebras, we can introduce in $\mathbf{R_L}$ the following partial compositions (that are particular universal or existential (L, χ)-compositions) between L-relations in a similar way as it has been done in [4, 6]. We adopt specific notations.

Definition 3.1.4. *Let $(L,\to,\top)$ be a w-ceo algebra. For all $X,Y,Z \in |\boldsymbol{Set}|$, $\mathcal{R}: X \rightharpoonup Y$, $\mathcal{S}: Y \rightharpoonup Z$ define:*

1. $\mathcal{R}\cdot\mathcal{S}: X \rightharpoonup Z$: $(x,y)(\mathcal{R}\cdot\mathcal{S}) = \bigvee_{y\in Y}(x,y)\mathcal{R}\wedge(y,z)\mathcal{S}$, $\forall x\in X, z\in Z$;

2. $\mathcal{R}\oslash\mathcal{S}: X \rightharpoonup Z$: $(x,z)(\mathcal{R}\oslash\mathcal{S}) = \bigwedge_{y\in Y}(x,y)\mathcal{R}\to(y,z)\mathcal{S}$, $\forall x\in X, z\in Z$;

3. $\mathcal{R}\otimes\mathcal{S}: X \rightharpoonup Z$: $(x,z)(\mathcal{R}\otimes\mathcal{S}) = \bigwedge_{y\in Y}(y,z)\mathcal{S}\to(x,y)\mathcal{R}$, $\forall x\in X, z\in Z$;

4. $\mathcal{R}\ominus\mathcal{S}: X \rightharpoonup Z$: $(x,z)(\mathcal{R}\ominus\mathcal{S}) = \bigwedge_{y\in Y}(x,y)\mathcal{R}\leftrightarrow(y,z)\mathcal{S}$, $\forall x\in X, z\in Z$.

Remark 3.1.5. It is easy to show that, for all $X,Y,Z\in|\mathbf{Set}|$, $\mathcal{R}: X\rightharpoonup Y$, $\mathcal{S}: Y\rightharpoonup Z$: $(\mathcal{R}\ominus\mathcal{S}) = (\mathcal{R}\oslash\mathcal{S})\wedge(\mathcal{R}\otimes\mathcal{S})$. Hence the properties of $\ominus$ can be deduced fro the properties of $\oslash$ and $\otimes$.

Proposition 3.1.6. *Let $(L,\to,\top)$ be a ceo algebra. Then the following hold, for all $X,Y,Z\in|\boldsymbol{Set}|$, $\mathcal{R},\mathcal{R}': X\rightharpoonup Y$ and $\mathcal{S},\mathcal{S}': Y\rightharpoonup Z$.*

1. $(\mathcal{R}_-)_- = \mathcal{R}$;

2. *if* $\mathcal{R}\le\mathcal{R}'$, *then* $\mathcal{R}_-\le\mathcal{R}'_-$;

3. $(\mathcal{R}\to\mathcal{R}')_- = \mathcal{R}_-\to\mathcal{R}'_-$;

4. $\mathcal{R}\oslash \top\!\!\!\top_{YZ} = \top\!\!\!\top_{XZ}$;

5. $\top\!\!\!\top_{XY}\oslash\mathcal{S} = \top\!\!\!\top_{XZ} \Leftrightarrow \mathcal{S} = \top\!\!\!\top_{YZ}$;

6. $\perp\!\!\!\perp_{XY}\oslash\mathcal{S} = \top\!\!\!\top_{XZ}$;

7. $\mathcal{R}\oslash \perp\!\!\!\perp_{YZ} = \top\!\!\!\top_{XZ} \Leftrightarrow \mathcal{R} = \perp\!\!\!\perp_{XY}$;

8. *if* $\mathcal{R}'\le\mathcal{R}$, *then* $\mathcal{R}\oslash\mathcal{S}\le\mathcal{R}'\oslash\mathcal{S}$;

9. *if* $\mathcal{S}\le\mathcal{S}'$, *then* $\mathcal{R}\oslash\mathcal{S}\le\mathcal{R}\oslash\mathcal{S}'$;

10. $\mathcal{R}\oslash\mathcal{R}_-\ge\mathcal{I}_X$;

11. $\mathcal{R}_-\oslash\mathcal{R}\ge\mathcal{I}_Y$;

12. $\mathcal{R}\otimes\mathcal{S} = (\mathcal{S}_-\oslash\mathcal{R}_-)_-$;

13. *if* $\top\to a = a$, *for every* $a\in L$, *then* $\mathcal{I}_X\oslash\mathcal{R} = \mathcal{R} = \mathcal{R}\otimes\mathcal{I}_Y$.

Proof. 1. $\forall x\in X, y\in Y$: $(x,y)((\mathcal{R}_-)_-) = (y,x)\mathcal{R}_- = (x,y)\mathcal{R}$.

2. $\forall x\in X, y\in Y$: $(y,x)\mathcal{R}_- = (x,y)\mathcal{R}\le(x,y)\mathcal{R}' = (y,x)\mathcal{R}'_-$.

3. $\forall x\in X, y\in Y$: $(y,x)(\mathcal{R}\to\mathcal{R})_- = (x,y)(\mathcal{R}\to\mathcal{R}') = (x,y)\mathcal{R}\to(x,y)\mathcal{R}' = (y,x)\mathcal{R}_-\to(y,x)\mathcal{R}'_- = (x,y)(\mathcal{R}_-\to\mathcal{R}'_-)$.

4. $\forall x \in X, z \in Z$: $(x,z)(\mathcal{R}\oslash \top\!\!\top_{YZ}) = \bigwedge_{y\in Y}(x,y)\mathcal{R} \to (y,z)\top\!\!\top_{YZ} = \bigwedge_{y\in Y}(x,y)\mathcal{R} \to \top = \bigwedge_{y\in Y}\top = \top = (x,z)\top\!\!\top_{XZ}$.

5. $\forall x \in X, z \in Z$, we have that:

$$\begin{aligned}(x,z)(\top\!\!\top_{XY}\oslash\mathcal{S}) = (x,z)\top\!\!\top_{XZ} &\Leftrightarrow \bigwedge_{y\in Y}(x,y)\top\!\!\top_{XY} \to (y,z)\mathcal{S} = (x,z)\top\!\!\top_{XZ}\\ &\Leftrightarrow \bigwedge_{y\in Y}\top \to (y,z)\mathcal{S} = \top\\ &\Leftrightarrow \forall y\in Y, z\in Z\ \top \to (y,z)\mathcal{S} = \top\\ &\Leftrightarrow \forall y\in Y, z\in Z\ (y,z)\mathcal{S} = \top\\ &\Leftrightarrow \forall y\in Y, z\in Z\ (y,z)\mathcal{S} = (y,z)\top\!\!\top_{YZ}.\end{aligned}$$

6. $\forall x \in X, z \in Z$: $(x,z)(\perp\!\!\perp_{XY}\oslash\mathcal{S}) = \bigwedge_{y\in Y}(x,y)\perp\!\!\perp_{XY} \to (y,z)\mathcal{S} = \bigwedge_{y\in Y}\perp \to (y,z)\mathcal{S} = \bigwedge_{y\in Y}\top = \top = (x,z)\top\!\!\top_{XZ}$.

7. $\forall x \in X, z \in Z$, one has that:

$$\begin{aligned}(x,z)(\mathcal{R}\oslash\perp\!\!\perp_{YZ}) = (x,z)\top\!\!\top_{XZ} &\Leftrightarrow \forall x\in X, z\in Z\ \bigwedge_{y\in Y}(x,y)\mathcal{R} \to (y,z)\perp\!\!\perp_{YZ} = \top\\ &\Leftrightarrow \forall x\in X, z\in Z\ \bigwedge_{y\in Y}(x,y)\mathcal{R} \to \perp = \top\\ &\Leftrightarrow \forall x\in X, y\in Y\ (x,y)\mathcal{R} \to \perp = \top\\ &\Leftrightarrow \forall x\in X, y\in Y\ (x,y)\mathcal{R} = \perp\\ &\Leftrightarrow \forall x\in X, y\in Y\ (x,y)\mathcal{R} = (x,y)\perp\!\!\perp_{XY}\end{aligned}$$

8. If $\mathcal{R}' \le \mathcal{R}$, then $\forall x\in X, y\in Y\ (x,y)\mathcal{R}' \le (x,y)\mathcal{R}$. Hence, from (o_5') it follows that $\forall x\in X, z\in Z\ (x,z)(\mathcal{R}\oslash\mathcal{S}) = \bigwedge_{y\in Y}(x,y)\mathcal{R} \to (y,z)\mathcal{S} \le \bigwedge_{y\in Y}(x,y)\mathcal{R}' \to (y,z)\mathcal{S} = (x,z)(\mathcal{R}'\oslash\mathcal{S})$.

9. If $\mathcal{S} \le \mathcal{S}'$, then $\forall y\in Y, z\in Z\ (y,z)\mathcal{S} \le (y,z)\mathcal{S}'$. Hence, from (o_5) it follows that $\forall x\in X, z\in Z\ (x,z)(\mathcal{R}\oslash\mathcal{S}) = \bigwedge_{y\in Y}(x,y)\mathcal{R} \to (y,z)\mathcal{S} \le \bigwedge_{y\in Y}(x,y)\mathcal{R} \to (y,z)\mathcal{S}' = (x,z)(\mathcal{R}\oslash\mathcal{S}')$.

10. For every $x \in X$ we have that: $(x,x)(\mathcal{R}\oslash\mathcal{R}_-) = \bigwedge_{y\in Y}(x,y)\mathcal{R} \to (y,x)\mathcal{R}_- = \bigwedge_{y\in Y}(x,y)\mathcal{R} \to (x,y)\mathcal{R} = \top = (x,x)\mathcal{I}_X$; moreover, if $x \ne x'$ $(x,x')(\mathcal{R}\oslash\mathcal{R}_-) \ge \perp = (x,x')\mathcal{I}_X$.

11. This proof is similar to (10).

12. $\forall x\in X, z\in Z$: $(x,z)(\mathcal{R}\otimes\mathcal{S}) = \bigwedge_{y\in Y}(y,z)\mathcal{S} \to (x,y)\mathcal{R} = \bigwedge_{y\in Y}(z,y)\mathcal{S}_- \to (y,x)\mathcal{R}_- = (z,x)(\mathcal{S}_-\oslash\mathcal{R}_-) = (x,z)(\mathcal{S}_-\oslash\mathcal{R}_-)_-$.

13. $\forall x \in X, y \in Y$, one has that: $(x,y)(\mathcal{I}_X\oslash\mathcal{R}) = \bigwedge_{x'\in X}(x,x')\mathcal{I}_X \to (x',y)\mathcal{R} = (\bigwedge_{x'\ne x\in X}\perp \to (x',y)\mathcal{R}) \wedge (\top \to (x,y)\mathcal{R}) = \top \wedge (x,y)\mathcal{R} = (x,y)\mathcal{R}$. Similarly, we can prove that $\mathcal{R} = \mathcal{R}\otimes\mathcal{I}_Y$.

$\square$

Remark 3.1.7. We note that in the above Proposition the assumption on L to satisfy (o_5) and (o'_5) is needed only to prove the properties (7) and (8), respectively.

Now we consider on L the right-distributivity condition, that allows to define the adjoint product and the disjunction connectives in L, as it is done in the previous Chapter.
By using these, we can give pointwisely further operations on $\mathbf{R_L}$.

Definition 3.1.8. *Let $(L, \rightarrow, \top)$ be a right-distributive ceo algebra. For all $X, Y \in |\boldsymbol{Set}|$, $\mathcal{R}, \mathcal{R}' : X \rightharpoondown Y$, define:*

1. $\mathcal{R} \otimes \mathcal{R}' : X \rightharpoondown Y$*:* $(x,y)(\mathcal{R} \otimes \mathcal{R}') = (x,y)\mathcal{R} \otimes (x,y)\mathcal{R}'$*,* $x \in X, y \in Y$*;*
2. $\mathcal{R} \uplus_{\rightarrow}^{(-)} \mathcal{R}' : X \rightharpoondown Y$*:* $(x,y)(\mathcal{R} \uplus_{\rightarrow}^{(-)} \mathcal{R}') = (x,y)\mathcal{R} \uplus_{\rightarrow}^{(-)} (x,y)\mathcal{R}'$*,* $x \in X, y \in Y$*;*
3. $\mathcal{R}^{(-)} \uplus_{\rightarrow} \mathcal{R}' : X \rightharpoondown Y$*:* $(x,y)(\mathcal{R}^{(-)} \uplus_{\rightarrow} \mathcal{R}') = (x,y)\mathcal{R}^{(-)} \uplus_{\rightarrow} (x,y)\mathcal{R}'$*,* $x \in X, y \in Y$*.*

Also in this case the properties satisfied in L are inherited by the pointwise operations defined in $\mathbf{R_L}$. For instance, the constant relation $\top\!\!\top_{XY}$ is a right unit with respect to $\otimes$ and the reverse operation $[]_-$ commutes with $\otimes$, $\uplus_{\rightarrow}^{(-)}$ and ${}^{(-)}\uplus_{\rightarrow}$.
The adjoint product of L allows to define as follows another partial composition that has also been considered by Bělohlávek in [6], in the less general context of commutative integral residuated lattices.

Definition 3.1.9. *Let $(L, \rightarrow, \top)$ be a right-distributive ceo algebra. For all $X, Y, Z \in |\boldsymbol{Set}|$, $\mathcal{R} : X \rightharpoondown Y$, $\mathcal{S} : Y \rightharpoondown Z$, we can define the following composition:*

1. $\mathcal{R} \odot \mathcal{S} : X \rightharpoondown Z$*:* $(x,z)(\mathcal{R} \odot \mathcal{S}) = \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (y,z)\mathcal{S}$*,* $x \in X, z \in Z$*;*
2. $\mathcal{R} \odot^{op} \mathcal{S} : X \rightharpoondown Z$*:* $(x,z)(\mathcal{R} \odot^{op} \mathcal{S}) = \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes^{op} (y,z)\mathcal{S}$*,* $x \in X, z \in Z$*.*

These compositions have good properties, even if the associativity and commutativity conditions are not assumed on L.

Proposition 3.1.10. *Assume $(L, \rightarrow, \top)$ to be a right-distributive ceo algebra. The following hold, for all $X, Y, Z, W \in |\boldsymbol{Set}|$, $\mathcal{R}, \mathcal{R}', \mathcal{R}_i : X \rightharpoondown Y$, $\mathcal{S}, \mathcal{S}', \mathcal{S}_i : Y \rightharpoondown Z$, $\mathcal{T} : X \rightharpoondown Z$, $Q : Z \rightharpoondown W$ and for every $i \in I$.*

1. $\mathcal{R} \odot \mathcal{I}_Y = \mathcal{R}$*;*
2. $\mathcal{R} \odot \perp\!\!\perp_{YZ} = \perp\!\!\perp_{XY}$*;*
3. $\perp\!\!\perp_{XY} \odot \mathcal{S} = \perp\!\!\perp_{XZ}$*;*
4. *if* $\mathcal{R} \leq \mathcal{R}'$*, then* $\mathcal{R} \odot \mathcal{S} \leq \mathcal{R}' \odot \mathcal{S}$*;*
5. *if* $\mathcal{S} \leq \mathcal{S}'$*, then* $\mathcal{R} \odot \mathcal{S} \leq \mathcal{R} \odot \mathcal{S}'$*;*
6. $\mathcal{R} \odot (\bigwedge_{i \in I} \mathcal{S}_i) \leq \bigwedge_{i \in I} (\mathcal{R} \odot \mathcal{S}_i)$*;*

7. $(\bigwedge_{i\in I}\mathcal{R}_i)\odot\mathcal{S}\leq\bigwedge_{i\in I}(\mathcal{R}_i\odot\mathcal{S})$;
8. $\mathcal{R}\odot(\bigvee_{i\in I}\mathcal{S}_i)=\bigvee_{i\in I}(\mathcal{R}\odot\mathcal{S}_i)$;
9. $\mathcal{T}\leq\mathcal{R}\oslash\mathcal{S}\Leftrightarrow\mathcal{R}_{-}\odot\mathcal{T}\leq\mathcal{S}$;
10. *if moreover* L *is left-distributive, then* $(\bigvee_{i\in I}\mathcal{R}_i)\odot\mathcal{S}=\bigvee_{i\in I}(\mathcal{R}_i\odot\mathcal{S})$;
11. *if moreover* L *is associative, then* $\mathcal{R}\odot(\mathcal{S}\odot\mathcal{Q})=(\mathcal{R}\odot\mathcal{S})\odot\mathcal{Q}$;
12. *if moreover* L *is associative and commutative, then the following hold:*
    (a) $(\mathcal{R}\otimes\mathcal{R}')\odot(\mathcal{S}\otimes\mathcal{S}')\leq(\mathcal{R}\odot\mathcal{S}')\otimes(\mathcal{R}'\odot\mathcal{S})$;
    (b) $(\mathcal{R}\otimes\mathcal{R}')\odot(\mathcal{S}\otimes\mathcal{S}')\leq(\mathcal{R}'\odot\mathcal{S})\otimes(\mathcal{R}\odot\mathcal{S}')$.

Proof. 1. $\forall x\in X,y\in Y$, $(x,y)(\mathcal{R}\odot\mathcal{I}_Y)=\bigvee_{y'\in Y}(x,y)\mathcal{R}'\otimes(y',y)\mathcal{I}_Y=(x,y)\mathcal{R}\otimes\top=(x,y)\mathcal{R}$.

2. $\forall x\in X,z\in Z$, $(x,z)(\mathcal{R}\odot\amalg_{YZ})=\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(y,z)\amalg_{YZ}=\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes\bot=\bigvee_{y\in Y}\bot=\bot=(x,z)\amalg_{XZ}$.

3. $\forall x\in X,z\in Z$, $(x,z)(\amalg_{XY}\odot\mathcal{S})=\bigvee_{y\in Y}(x,y)\amalg_{XY}\otimes(y,z)\mathcal{S}=\bigvee_{y\in Y}\bot\otimes(y,z)\mathcal{S}=\bigvee_{y\in Y}\bot=\bot=(x,z)\amalg_{XZ}$.

4. Let $\mathcal{R}\leq\mathcal{R}'$; then for all $x\in X,y\in Y$ $(x,y)\mathcal{R}\leq(x,y)\mathcal{R}'$. So, for all $x\in X,y\in Y,z\in Z$ we have that $(x,z)(\mathcal{R}\odot\mathcal{S})=\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(y,z)\mathcal{S}\leq(x,y)\mathcal{R}'\otimes(y,z)\mathcal{S}\leq\bigvee_{y\in Y}(x,y)\mathcal{R}'\otimes(y,z)\mathcal{S}=(x,z)(\mathcal{R}'\odot\mathcal{S})$.

5. This proof is similar to above.

6. $\forall x\in X,z\in Z,i\in I$, $(x,z)(\mathcal{R}\odot(\bigwedge_{i\in I}\mathcal{S}_i))=\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(y,z)(\bigwedge_{i\in I}\mathcal{S}_i)=\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(\bigwedge_{i\in I}(y,z)\mathcal{S}_i)\leq\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(y,z)\mathcal{S}_i$. Then $(x,z)(\mathcal{R}\odot(\bigwedge_{i\in I}\mathcal{S}_i))\leq\bigwedge_{i\in I}(\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(y,z)\mathcal{S}_i)=\bigwedge_{i\in I}(x,z)(\mathcal{R}\odot\mathcal{S}_i)$.

7. This proof is similar to (6).

8. $\forall x\in X,z\in Z$, we have that:

$$\begin{aligned}(x,z)(\mathcal{R}\odot(\bigvee_{i\in I}\mathcal{S}_i))&=\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes[(y,z)(\bigvee_{i\in I}\mathcal{S}_i)]\\&=\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes[\bigvee_{i\in I}(y,z)\mathcal{S}_i]\\&=\bigvee_{y\in Y}\bigvee_{i\in I}(x,y)\mathcal{R}\otimes(y,z)\mathcal{S}_i\\&=\bigvee_{i\in I}\bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(y,z)\mathcal{S}_i\\&=\bigvee_{i\in I}(x,z)(\mathcal{R}\odot\mathcal{S}_i)\\&=(x,z)(\bigvee_{i\in I}(\mathcal{R}\odot\mathcal{S}_i)).\end{aligned}$$

9. $\forall x \in X, z \in Z$, we have that:

$$\begin{aligned}
(x,z)\mathcal{T} \le (x,y)(\mathcal{R} \oslash \mathcal{S}) &\Leftrightarrow \forall x \in X, \forall z \in Z\ (x,z)\mathcal{T} \le \bigwedge_{y \in Y} (x,y)\mathcal{R} \to (y,z)\mathcal{S} \\
&\Leftrightarrow \forall x \in X, \forall y \in Y, \forall z \in Z\ (x,z)\mathcal{T} \le (x,y)\mathcal{R} \to (y,z)\mathcal{S} \\
&\Leftrightarrow \forall x \in X, \forall y \in Y, \forall z \in Z\ (x,y)\mathcal{R} \otimes (x,z)\mathcal{T} \le (y,z)\mathcal{S} \\
&\Leftrightarrow \forall x \in X, \forall y \in Y, \forall z \in Z\ (y,x)\mathcal{R}_{-} \otimes (x,z)\mathcal{T} \le (y,z)\mathcal{S} \\
&\Leftrightarrow \forall y \in Y, \forall z \in Z\ \bigvee_{x \in X} (y,x)\mathcal{R}_{-} \otimes (x,z)\mathcal{T} \le (y,z)\mathcal{S} \\
&\Leftrightarrow \forall y \in Y, \forall z \in Z\ (y,z)(\mathcal{R}_{-} \odot \mathcal{T}) \le (y,z)\mathcal{S}.
\end{aligned}$$

10. This proof is similar to (8).

11. $\forall x \in X, w \in W$: $(x,w)(\mathcal{R} \odot (\mathcal{S} \odot \mathcal{Q})) = \bigvee_{y \in Y}(x,y)\mathcal{R} \otimes (y,w)(\mathcal{S} \odot \mathcal{Q})$= $\bigvee_{y \in Y}(x,y)\mathcal{R} \otimes \bigvee_{z \in Z}(y,z)\mathcal{S} \otimes (z,w)\mathcal{Q} = \bigvee_{y \in Y}\bigvee_{z \in Z}(x,y)\mathcal{R} \otimes [(y,z)\mathcal{S} \otimes (z,w)\mathcal{Q}] = \bigvee_{y \in Y}\bigvee_{z \in Z}[(x,y)\mathcal{R} \otimes (y,z)\mathcal{S}] \otimes (z,w)\mathcal{Q} = \bigvee_{z \in Z}\bigvee_{y \in Y}[(x,y)\mathcal{R} \otimes (y,z)\mathcal{S}] \otimes (z,w)\mathcal{Q} = \bigvee_{z \in Z}(x,z)(\mathcal{R} \odot \mathcal{S}) \otimes (z,w)\mathcal{Q} = (x,w)((\mathcal{R} \odot \mathcal{S}) \odot \mathcal{Q})$.

12. (a) $\forall x \in X, z \in Z$, one has that:

$$\begin{aligned}
(x,z)\left[(\mathcal{R} \otimes \mathcal{R}') \odot (\mathcal{S} \otimes \mathcal{S}')\right] &= \bigvee_{y \in Y} (x,y)(\mathcal{R} \otimes \mathcal{R}') \otimes (y,z)(\mathcal{S} \otimes \mathcal{S}') \\
&= \bigvee_{y \in Y} \left[(x,y)\mathcal{R} \otimes (x,y)\mathcal{R}'\right] \otimes \left[(y,z)\mathcal{S} \otimes (y,z)\mathcal{S}'\right] \\
&= \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes \left[(x,y)\mathcal{R}' \otimes (y,z)\mathcal{S}\right] \otimes (y,z)\mathcal{S}' \\
&\le \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes [\bigvee_{y' \in Y} (x,y')\mathcal{R}' \otimes (y',z)\mathcal{S}] \otimes (y,z)\mathcal{S}' \\
&= \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (x,z)(\mathcal{R}' \odot \mathcal{S}) \otimes (y,z)\mathcal{S}' \\
&= \bigvee_{y \in Y} (x,z)(\mathcal{R}' \odot \mathcal{S}) \tilde{\otimes} (x,y)\mathcal{R} \otimes (y,z)\mathcal{S}' \\
&= (x,z)(\mathcal{R}' \odot \mathcal{S}) \otimes \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (y,z)\mathcal{S}' \\
&= (x,z)(\mathcal{R}' \odot \mathcal{S}) \otimes (x,z)(\mathcal{R} \odot \mathcal{S}') \\
&= (x,z)\left[(\mathcal{R}' \odot \mathcal{S}) \otimes (\mathcal{R} \odot \mathcal{S}')\right].
\end{aligned}$$

(b) This follows from (a) and commutativity of $\otimes$.

□

Now we consider the L-relations taking values in a symmetrical cdeo algebra. So, we consider $\mathbf{R_L}$, the class of binary L-valued relations, where $(L, \to, \top)$ is a symmetrical cdeo algebra. Then, in $\mathbf{R_L}$ we can define further pointwise operations.

Definition 3.1.11. *Let $(L, \to, \top)$ be a symmetrical cdeo algebra. For all $X, Y \in |\boldsymbol{Set}|$, $\mathcal{R}, \mathcal{R}' : X \rightharpoonup Y$, define:*

1. $\mathcal{R} \rightsquigarrow \mathcal{R}' : X \rightharpoonup Y$: $(x,y)(\mathcal{R} \rightsquigarrow \mathcal{R}') = (x,y)\mathcal{R} \rightsquigarrow (x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$;
2. $\mathcal{R}\tilde{\otimes}\mathcal{R}' : X \rightharpoonup Y$: $(x,y)(\mathcal{R}\tilde{\otimes}\mathcal{R}') = (x,y)\mathcal{R}\tilde{\otimes}(x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$;
3. $\mathcal{R} \uplus_{\rightsquigarrow}^{(\sim)} \mathcal{R}' : X \rightharpoonup Y$: $(x,y)(\mathcal{R} \uplus_{\rightsquigarrow}^{(\sim)} \mathcal{R}') = (x,y)\mathcal{R} \uplus_{\rightsquigarrow}^{(\sim)} (x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$;
4. $\mathcal{R}^{(\sim)} \uplus_{\rightsquigarrow} \mathcal{R}' : X \rightharpoonup Y$: $(x,y)(\mathcal{R}^{(\sim)} \uplus_{\rightsquigarrow} \mathcal{R}') = (x,y)\mathcal{R}^{(\sim)} \uplus_{\rightsquigarrow} (x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$;
5. $\mathcal{R} \uplus_{\rightsquigarrow}^{(-)} \mathcal{R}' : X \rightharpoonup Y$:$(x,y)(\mathcal{R} \uplus_{\rightsquigarrow}^{(-)} \mathcal{R}') = (x,y)\mathcal{R} \uplus_{\rightsquigarrow}^{(-)} (x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$;
6. $\mathcal{R}^{(-)}\uplus_{\rightsquigarrow} : X \rightharpoonup Y$: $(x,y)(\mathcal{R}^{(-)} \uplus_{\rightsquigarrow} \mathcal{R}') = (x,y)\mathcal{R}^{(-)} \uplus_{\rightsquigarrow} (x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$;
7. $\mathcal{R} \uplus_{\rightarrow}^{(\sim)} \mathcal{R}' : X \rightharpoonup Y$: $(x,y)(\mathcal{R} \uplus_{\rightarrow}^{(\sim)} \mathcal{R}') = (x,y)\mathcal{R} \uplus_{\rightarrow}^{(\sim)} (x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$;
8. $\mathcal{R}^{(\sim)} \uplus_{\rightarrow} \mathcal{R}' : X \rightharpoonup Y$: $(x,y)(\mathcal{R}^{(\sim)} \uplus_{\rightarrow} \mathcal{R}') = (x,y)\mathcal{R}^{(\sim)} \uplus_{\rightarrow} (x,y)\mathcal{R}'$, $\forall x \in X, y \in Y$.

Also in this case all the properties proved on L in Section 2.1 hold for the pointwise operations defined on $\mathbf{R_L}$. For example, the constant relation $\top\!\!\top_{XY}$ become also a left unit with respect to $\otimes$ and the reverse operation $[\,]_-$ commutes with $\tilde{\otimes}$, $\uplus_{\rightsquigarrow}^{(\sim)}$, ${}^{(\sim)}\uplus_{\rightsquigarrow}$, $\uplus_{\rightsquigarrow}^{(-)}$, ${}^{(-)}\uplus_{\rightsquigarrow}$, $\uplus_{\rightarrow}^{(\sim)}$ and ${}^{(\sim)}\uplus_{\rightarrow}$.

Moreover, the symmetry assumption on L allows to define further partial compositions.

Definition 3.1.12. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. For all $X, Y \in |\boldsymbol{Set}|$, $\mathcal{R} : X \rightharpoonup Y$, $\mathcal{S} : Y \rightharpoonup Z$, define:*

1. $\mathcal{R}\tilde{\oslash}\mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R}\tilde{\oslash}\mathcal{S}) = \bigwedge_{y\in Y}(x,y)\mathcal{R} \rightsquigarrow (y,z)\mathcal{S}$, $\forall x \in X, z \in Z$;
2. $\mathcal{R}\tilde{\obslash}\mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R}\tilde{\obslash}\mathcal{S}) = \bigwedge_{y\in Y}(y,z)\mathcal{S} \rightsquigarrow (x,y)\mathcal{R}$, $\forall x \in X, z \in Z$;
3. $\mathcal{R}\tilde{\ominus}\mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R}\tilde{\ominus}\mathcal{S}) = \bigwedge_{y\in Y}(x,y)\mathcal{R} \leftrightsquigarrow (y,z)\mathcal{S}$, $\forall x \in X, z \in Z$;
4. $\mathcal{R}\tilde{\odot}\mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R}\tilde{\odot}\mathcal{S}) = \bigvee_{y\in Y}(x,y)\mathcal{R}\tilde{\otimes}(y,z)\mathcal{S}$, $\forall x \in X, z \in Z$;
5. $\mathcal{R}\tilde{\odot}^{op}\mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R}\tilde{\odot}^{op}\mathcal{S}) = \bigvee_{y\in Y}(y,z)\mathcal{S}\tilde{\otimes}(x,y)\mathcal{R}$, $\forall x \in X, z \in Z$.

Remark 3.1.13. Under symmetry assumption on $(L, \rightarrow, \top)$, an immediate verification allows to show that $\otimes^{op} = \tilde{\otimes}$ and $\tilde{\otimes}^{op} = \otimes$. Moreover, we recall that $(L, \rightsquigarrow, \top)$ is a symmetrical cdeo algebra if and only if $(L, \rightarrow, \top)$ is and that $\tilde{\otimes}$ is the adjoint product of $(L, \rightsquigarrow, \top)$. So $\rightsquigarrow$ and $\tilde{\otimes}$ have exactly the same properties as $\rightarrow$ and $\otimes$. This allow to extend most result we have considered up to now for the algebra of L-relations.

Proposition 3.1.14. *Consider a symmetrical cdeo algebra $(L, \rightarrow, \top)$. The following hold, for all $X, Y, Z \in |\boldsymbol{Set}|$, $\mathcal{R}, \mathcal{R}', \mathcal{R}_i : X \rightharpoonup Y$, $\mathcal{S}, \mathcal{S}', \mathcal{S}_i : Y \rightharpoonup Z$, $\mathcal{T} : X \rightharpoonup Z$, $\mathcal{Q} : Z \rightharpoonup W$ and $i \in I$.*

1. $\mathcal{R}\tilde{\odot}\mathcal{I}_Y = \mathcal{R}$, $\mathcal{I}_X\tilde{\odot}\mathcal{R} = \mathcal{R}$ *and* $\mathcal{I}_X \odot \mathcal{R} = \mathcal{R}$;
2. $\mathcal{R}\tilde{\odot}\perp\!\!\perp_{YZ} = \perp\!\!\perp_{XY}$ *and* $\perp\!\!\perp_{XY}\tilde{\odot}\mathcal{S} = \perp\!\!\perp_{XZ}$;

3. $\mathcal{R}\tilde{\odot}(\bigwedge_{i\in I}\mathcal{S}_i) \le \bigwedge_{i\in I}(\mathcal{R}\tilde{\odot}\mathcal{S}_i)$ *and* $(\bigwedge_{i\in I}\mathcal{R}_i)\tilde{\odot}\mathcal{S} \le \bigwedge_{i\in I}(\mathcal{R}_i\tilde{\odot}\mathcal{S})$;

4. $\mathcal{R}\tilde{\odot}(\bigvee_{i\in I}\mathcal{S}_i) = \bigvee_{i\in I}(\mathcal{R}\tilde{\odot}\mathcal{S}_i)$ *and* $(\bigvee_{i\in I}\mathcal{R}_i)\tilde{\odot}\mathcal{S} = \bigvee_{i\in I}(\mathcal{R}_i\tilde{\odot}\mathcal{S})$;

5. $(\mathcal{R}\odot\mathcal{S})_- = \mathcal{S}_-\tilde{\odot}\mathcal{R}_-$ *and* $(\mathcal{R}\tilde{\odot}\mathcal{S})_- = \mathcal{S}_-\odot\mathcal{R}_-$;

6. $\mathcal{T} \le \mathcal{R}\tilde{\oslash}\mathcal{S} \Leftrightarrow \mathcal{R}_-\tilde{\odot}\mathcal{T} \le \mathcal{S}$;

7. $\mathcal{T} \le \mathcal{R}\tilde{\oslash}\mathcal{S} \Leftrightarrow \mathcal{R} \le \mathcal{T}\oslash\mathcal{S}_-$ *and* $\mathcal{T} \le \mathcal{R}\oslash\mathcal{S} \Leftrightarrow \mathcal{R} \le \mathcal{T}\tilde{\oslash}\mathcal{S}_-$;

8. $\mathcal{R}\tilde{\otimes}\mathcal{S} = (\mathcal{S}_-\tilde{\otimes}\mathcal{R}_-)_-$;

9. *if moreover L is associative, then the following hold:*

   (a) $\mathcal{R}\tilde{\odot}(\mathcal{S}\tilde{\odot}\mathcal{Q}) = (\mathcal{R}\tilde{\odot}\mathcal{S})\tilde{\odot}\mathcal{Q}$;

   (b) $\mathcal{R}\oslash(\mathcal{S}\oslash\mathcal{Q}) = (\mathcal{R}\tilde{\odot}\mathcal{S})\oslash\mathcal{Q}$;

   (c) $\mathcal{R}\tilde{\oslash}(\mathcal{S}\tilde{\oslash}\mathcal{Q}) = (\mathcal{R}\odot\mathcal{S})\tilde{\oslash}\mathcal{Q}$;

   (d) $(\mathcal{R}\odot\mathcal{S})\otimes\mathcal{T} = \perp\!\!\!\perp_{XZ} \Leftrightarrow (\mathcal{T}\tilde{\odot}\mathcal{S}_-)\tilde{\otimes}\mathcal{R} = \perp\!\!\!\perp_{XY}$.

Proof. Most items are immediate consequences of corresponding results already listed in previous statements, by means of Remark 3.1.13. So, we only verify explicitly the remaining items.
(5) $\forall x \in X, z \in Z$, we have that: $(z,x)(\mathcal{R}\odot\mathcal{S})_- = (x,z)(\mathcal{R}\odot\mathcal{S}) = \bigvee_{y\in Y}(x,y)\mathcal{R}\otimes(y,z)\mathcal{S} = \bigvee_{y\in Y}(y,x)\mathcal{R}_-\otimes(z,y)\mathcal{S}_- = \bigvee_{y\in Y}(z,y)\mathcal{S}_-\tilde{\otimes}(y,x)\mathcal{R}_- = (z,x)(\mathcal{S}_-\tilde{\odot}\mathcal{R}_-)$. The second equality can be similarly proved.
(7) $\forall x \in X, z \in Z$, we have that:

$$\begin{aligned}
(x,z)\mathcal{T} \le (x,z)(\mathcal{R}\tilde{\oslash}\mathcal{S}) &\Leftrightarrow \forall x \in X, z \in Z\ (x,z)\mathcal{T} \le \bigwedge_{y\in Y}(x,y)\mathcal{R} \rightsquigarrow (y,z)\mathcal{S} \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z\ (x,z)\mathcal{T} \le (x,y)\mathcal{R} \rightsquigarrow (y,z)\mathcal{S} \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z\ (x,y)\mathcal{R} \le (x,z)\mathcal{T} \to (y,z)\mathcal{S} \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z\ (x,y)\mathcal{R} \le (x,z)\mathcal{T} \to (z,y)\mathcal{S}_- \\
&\Leftrightarrow \forall x \in X, y \in Y\ (x,y)\mathcal{R} \le \bigwedge_{z\in Z}(x,z)\mathcal{T} \to (z,y)\mathcal{S}_- \\
&\Leftrightarrow \forall x \in X, y \in Y\ (x,y)\mathcal{R} \le (x,y)(\mathcal{T}\oslash\mathcal{S}_-).
\end{aligned}$$

The second equivalence can be similarly proved.
(8) $\forall x \in X, z \in Z$: $(x,z)(\mathcal{R}\tilde{\otimes}\mathcal{S}) = \bigwedge_{y\in Y}(y,z)\mathcal{S} \rightsquigarrow (x,y)\mathcal{R} = \bigwedge_{y\in Y}(z,y)\mathcal{S}_- \rightsquigarrow (y,x)\mathcal{R}_- = (z,x)(\mathcal{S}_-\tilde{\oslash}\mathcal{R}_-) = (x,z)(\mathcal{S}_-\tilde{\oslash}\mathcal{R}_-)_-$.

$(9)(b)$ $\forall x \in X, w \in W$, one has that:

$$\begin{aligned}
(x,w)(\mathcal{R} \oslash (\mathcal{S} \oslash \mathcal{Q})) &= \bigwedge_{y \in Y} (x,y)\mathcal{R} \to (y,w)(\mathcal{S} \oslash \mathcal{Q}) \\
&= \bigwedge_{y \in Y} (x,y)\mathcal{R} \to (\bigwedge_{z \in Z} (y,z)\mathcal{S} \to (z,w)\mathcal{Q}) \\
&= \bigwedge_{y \in Y} \bigwedge_{z \in Z} (x,y)\mathcal{R} \to [(y,z)\mathcal{S} \to (z,w)\mathcal{Q}] \\
&= \bigwedge_{y \in Y} \bigwedge_{z \in Z} [(y,z)\mathcal{S} \otimes (x,y)\mathcal{R}] \to (z,w)\mathcal{Q} \\
&= \bigwedge_{y \in Y} \bigwedge_{z \in Z} [(x,y)\mathcal{R} \tilde{\otimes} (y,z)\mathcal{S}] \to (z,w)\mathcal{Q} \\
&= \bigwedge_{z \in Z} \bigwedge_{y \in Y} [(x,y)\mathcal{R} \tilde{\otimes} (y,z)\mathcal{S}] \to (z,w)\mathcal{Q} \\
&= \bigwedge_{z \in Z} (\bigvee_{y \in Y} (x,y)\mathcal{R} \tilde{\otimes} (y,z)\mathcal{S}) \to (z,w)\mathcal{Q} \\
&= \bigwedge_{z \in Z} (x,z)(\mathcal{R} \tilde{\odot} \mathcal{S}) \to (z,w)\mathcal{Q} \\
&= (x,w)((\mathcal{R} \tilde{\odot} \mathcal{S}) \oslash \mathcal{Q}).
\end{aligned}$$

$(9)(c)$ The proof is similar to the above.
$(9)(d)$ $\forall x \in X, \forall z \in Z$, we have that:

$$\begin{aligned}
(x,z)[(\mathcal{R} \odot \mathcal{S}) \otimes \mathcal{T}] = (x,z)\amalg_{XZ} &\Leftrightarrow \forall x \in X, z \in Z \ (x,z)(\mathcal{R} \odot \mathcal{S}) \otimes (x,z)\mathcal{T} = \bot \\
&\Leftrightarrow \forall x \in X, z \in Z \ [\bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (y,z)\mathcal{S}] \otimes (x,z)\mathcal{T} = \bot \\
&\Leftrightarrow \forall x \in X, z \in Z \ \bigvee_{y \in Y} \{[(x,y)\mathcal{R} \otimes (y,z)\mathcal{S}] \otimes (x,z)\mathcal{T}\} = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z \ [(x,y)\mathcal{R} \otimes (y,z)\mathcal{S}] \otimes (x,z)\mathcal{T} = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z \ (x,y)\mathcal{R} \otimes [(y,z)\mathcal{S} \otimes (x,z)\mathcal{T}] = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z \ (x,y)\mathcal{R} \otimes [(x,z)\mathcal{T} \tilde{\otimes} (y,z)\mathcal{S}] = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z \ [(x,z)\mathcal{T} \tilde{\otimes} (y,z)\mathcal{S}] \tilde{\otimes} (x,y)\mathcal{R} = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y, z \in Z \ [(x,z)\mathcal{T} \tilde{\otimes} (z,y)\mathcal{S}_-] \tilde{\otimes} (x,y)\mathcal{R} = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y \ [\bigvee_{z \in Z} (x,z)\mathcal{T} \tilde{\otimes} (z,y)\mathcal{S}_-] \tilde{\otimes} (x,y)\mathcal{R} = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y \ (x,y)(\mathcal{T} \tilde{\odot} \mathcal{S}_-) \tilde{\otimes} (x,y)\mathcal{R} = \bot \\
&\Leftrightarrow \forall x \in X, y \in Y \ (x,y)[(\mathcal{T} \tilde{\odot} \mathcal{S}_-) \tilde{\otimes} \mathcal{R}] = (x,y)\amalg_{XY}.
\end{aligned}$$

□

3.2 L-relations and associated operators on L-sets

In this Section we reconsider well known concepts concerning L-relations and some operators they induce, either directly or by means of the algebraic structure on L, in the

L-powersets of involved sets. First we introduce the notion of many-valued (pre)order, based on extended-order algebras; however, its weakest version may be simply based on (pre)ordered sets.

3.2.1 L-order, subsethood and intersection degree

L-preorder and L-order on X have been approached usually assuming L to have some kind of lattice ordered algebraic structure and assuming X to be a set equipped with an L-equality (possibly the usual crisp equality, which will be the case we shall consider). In the spirit of weak extended-order algebras and of the more general implicative groupoids considered in [49] and described later in Section 6.4, we begin with a weak notion of L-preorder and of L-order on a set X based on an ordered set $(L, \leq)$, though a preordered set could be enough.

Definition 3.2.1. *Let $(L, \leq)$ be an ordered set, X a set, $F \subseteq L$ an upperset. An L-relation $\alpha : X \times X \to L$ is a* ***weak (L,F)-preorder*** *if the following conditions are satisfied, for all $x, y, z \in X$:*
$(r)_F$ $(x,x)\alpha \in F$;
$(wt)_F$ $(x,y)\alpha \in F, (y,z)\alpha \in F \Rightarrow (x,z)\alpha \in F$;
α is a ***weak (L,F)-order*** *if, moreover, the following holds, for all $x, y \in X$:*
$(as)_F$ $(x,y)\alpha \in F, (y,x)\alpha \in F \Rightarrow x = y$.

Of course, if the top element $\top$ exists in $(L, \leq)$, then $\{\top\}$ is an upperset and we may consider the above conditions $(r)_\top$, $(wt)_\top$ and $(as)_\top$ in the specific case $F = \{\top\}$, speaking of weak $(L, \top)$-preorder and weak $(L, \top)$-order.
In fact, one can say that α is a weak (L, F)-(pre)order if and only if it induces on X a (pre)order $\preceq$ by means of the equivalence

$$x \preceq y \Leftrightarrow (x,y)\alpha \in F$$

as it happens for relational groupoids (see Section 6.4) and, in case $F = \{\top\}$, for weak extended-order algebras (see Subsection 2.1.1).
Stronger versions of L-(pre)order require stronger assumptions on L.

Definition 3.2.2. *Let $(L, \leq)$ be an $\wedge$-semilattice, X a set and $\alpha : X \times X \to L$. Then α is a* ***strong L-preorder*** *if the following hold, for all $x, y, z \in X$:*
(r) $(x,x)\alpha = \top$;
$(t)_\wedge$ $(x,y)\alpha \wedge (y,z)\alpha \leq (x,z)\alpha$;
α is a ***strong L-order*** *if, moreover, it satisfies the condition, for all $x, y \in X$:*
(as) $(x,y)\alpha = (y,x)\alpha = \top \Rightarrow x = y$.

Definition 3.2.3. *Let $(L, \to, \top)$ be a right-distributive w-ceo algebra, X a set and $\alpha : X \times X \to L$. Then α is an* ***(L,⊗)-preorder*** *if it satisfies (r) and the following, for all $x, y, z \in X$:*
$(t)_\otimes$ $(x,y)\alpha \otimes (y,z)\alpha \leq (x,z)\alpha$;
α is a ***(L,⊗)-order*** *if it satisfies (r), $(t)_\otimes$ and (as).*

Remark 3.2.4. The attributes "weak" and "strong" are only partially motivated by the relationships between the above Definitions. In fact, despite the attribute "weak", the condition $(as)_F$ is stronger than (as), whenever $F \neq \{\top\}$.
However, the following results restore the right meaning of the above attributes.

Proposition 3.2.5. *Let X be any set, $(L, \leq)$ an $\wedge$-semilattice and $\alpha : X \times X \to L$. Then*

1. *if α is a strong L-order, then α is an $(L, \top)$-order;*
2. *α is a strong L-preorder if and only if α is a (L, F)-preorder, for every filter F in L;*
3. *if $(L, \to, \top)$ is a symmetrical cdeo algebra and α is a strong L-(pre)order, then α is an $(L, \otimes)$-(pre)order.*

Proof. 1. Clearly $(r)_\top$ and $(as)_\top$ are exactly (r) and (as). From $(t)_\wedge$ follows easily $(wt)_\top$.

2. If α is a strong L-preorder and F is a filter of L, then, for every $x \in X$, $(x, x)\alpha = \top \in F$.
Moreover, if $(x, y)\alpha \in F$ and $(y, z)\alpha \in F$, then $(x, z)\alpha \geq (x, y)\alpha \wedge (y, z)\alpha \in F \Rightarrow (x, z)\alpha \in F$.
Conversely, since $\{\top\}$ is a filter, (r) holds since $(r)_\top$ holds. Now, let $x, y, z \in X$ and consider the filter $F = ((x, y)\alpha \wedge (y, z)\alpha)^{[\uparrow]}$; evidently, $(x, y)\alpha, (y, z)\alpha \in F$ so, by assumption, $(x, z)\alpha \in F$, i.e. $(x, y)\alpha \wedge (y, z)\alpha \leq (x, z)\alpha$.

3. The proof follows easily by the inequality $a \otimes b \leq a \wedge b$, which holds for all $a, b \in L$. □

Remark 3.2.6. The notion of $\otimes$-filter has been sometimes used in connection with many-valued order; we recall that a **$\otimes$-filter** F in a right-distributive w-ceo algebra L is an upperset $F \subseteq L$ which is closed under $\otimes$.
Now, if α is an $(L, \otimes)$-(pre)order and F is a $\otimes$-filter in L, then α is an (L, F)-(pre)order, which can be proved in a similar way as in the first part of the proof in the item (2) of the above proposition.

Consider, now, the following L-relations, defined by using an operation on L.

Definition 3.2.7. *Let L be a set and let $\chi : L \times L \to L$ an operation on it. For every set X, we call **universal** (L, χ)**-aggregation** the L-relation $\mathcal{R}^{\chi,\wedge} : L^X \times L^X \to L$ defined by $(A, B)\mathcal{R}_X^{\chi,\wedge} = \bigwedge_{x \in X}((x)A, (x)B)\chi$, for all $A, B \in L^X$ and **existential** (L, χ)**-aggregation** the L-relation $\mathcal{R}_X^{\chi,\vee} : L^X \times L^X \to L$ defined, for all $A, B \in L^X$, by $(A, B)\mathcal{R}^{\chi,\vee} = \bigvee_{x \in X}((x)A, (x)B)\chi$.*

Particular (L, χ)-aggregations have already been considered in the literature; the following definition recall the one, well known, used to evaluate the inclusion between L-sets.

Definition 3.2.8. *Let X be a set and let $(L, \to, \top)$ be a w-ceo algebra. We consider the L-relation $\mathcal{S}_X : L^X \times L^X \to L$, called **subsethood L-relation** and defined by $(A, B)\mathcal{S}_X = \bigwedge_{x \in X}(x)A \to (x)B$, for all $A, B \in L^X$; $(A, B)\mathcal{S}_X$ is called **subsethood degree** of A and B.*
*If $(L, \to, \top)$ is a symmetrical cdeo algebra, we also consider the **dual subsethood L-relation**, $\tilde{\mathcal{S}}_X : L^X \times L^X \to L$ such that $(A, B)\tilde{\mathcal{S}}_X = \bigwedge_{x \in X}(x)A \rightsquigarrow (x)B$, for all $A, B \in L^X$.*

Of course, $\mathcal{S}_X = \mathcal{R}_X^{\to,\wedge}$ and we shall denote $\mathcal{S}_X^{\vee} = \mathcal{R}_X^{\to,\vee}$ and, if L is symmetrical, $\tilde{\mathcal{S}}_X = \mathcal{R}_X^{\rightsquigarrow,\wedge}$ and $\tilde{\mathcal{S}}_X^{\vee} = \mathcal{R}_X^{\rightsquigarrow,\vee}$.
We note that $\mathcal{S}_X^{\vee}$ is not frequently used, since it is trivial in some cases; for instance, we have that it is sufficient that there is at least $\bar{x} \in X$ such that $(\bar{x})A \to (\bar{x})B = \top$, i.e. $(\bar{x})A \leq (\bar{x})B$, to conclude that $(A, B)\mathcal{S}_X^{\vee} = \top$.

Proposition 3.2.9. *The following statements are true.*

1. *If $(L, \to, \top)$ is a w-ceo algebra, then the following hold, for all $A, B, C \in L^X$:*
   - *(a)* $(A, A)\mathcal{S}_X = \top$;
   - *(b)* $(A, B)\mathcal{S}_X = (B, A)\mathcal{S}_X = \top \Rightarrow A = B$;
   - *(c)* $(A, B)\mathcal{S}_X = (B, C)\mathcal{S}_X = \top \Rightarrow (A, C)\mathcal{S}_X = \top$.
2. *If L is a right-distributive ceo algebra, then $(A, B)\mathcal{S}_X = \top \Rightarrow (C, A)\mathcal{S}_X \to (C, B)\mathcal{S}_X = (B, C)\mathcal{S}_X \to (A, C)\mathcal{S}_X = \top$, for all $A, B, C \in L^X$.*
3. *If L is a symmetrical cdeo algebra, then the following hold, for all $A, B, C \in L^X$:*
   - *(a)* $(A, A)\tilde{\mathcal{S}}_X = \top$;
   - *(b)* $(A, B)\tilde{\mathcal{S}}_X = (B, A)\tilde{\mathcal{S}}_X = \top \Rightarrow A = B$;
   - *(c)* $(A, B)\tilde{\mathcal{S}}_X = (B, C)\tilde{\mathcal{S}}_X = \top \Rightarrow (A, C)\tilde{\mathcal{S}}_X = \top$.
   - *(d)* $(A, B)\tilde{\mathcal{S}}_X = \top \Rightarrow (C, A)\tilde{\mathcal{S}}_X \to (C, B)\tilde{\mathcal{S}}_X = (B, C)\tilde{\mathcal{S}}_X \to (A, C)\tilde{\mathcal{S}}_X = \top$.
4. *If L is an associative and right-distributive ceo algebra, then the following hold, for all $A, B, C \in L^X$:*
   - *(a)* $(A, B)\mathcal{S}_X \otimes (B, C)\mathcal{S}_X \leq (A, C)\mathcal{S}_X$ *and* $(B, C)\mathcal{S}_X \leq (A, B)\mathcal{S}_X \to (A, C)\mathcal{S}_X$;
   - *(b) if, moreover, L is symmetrical, then* $(B, C)\tilde{\mathcal{S}}_X \otimes (A, B)\tilde{\mathcal{S}}_X \leq (A, C)\tilde{\mathcal{S}}_X$.

Proof. 1. Let $(L, \to, \top)$ be a w-ceo algebra and let $A, B, C \in L^X$.

(a) From (o_2) it follows easily that $(A, A)\mathcal{S}_X = \bigwedge_{x \in X}(x)A \to (x)A = \top$.

(b) By assumption $(A, B)\mathcal{S}_X = \bigwedge_{x \in X}(x)A \to (x)B = \top$ and $(B, A)\mathcal{S}_X = \bigwedge_{x \in X}(x)B \to (x)A = \top$; hence, for every $x \in X$ $(x)A \to (x)B = (x)B \to (x)A = \top$ that implies, thanks to the antisymmetry condition (o_3) that $(x)A = (x)B$, for every $x \in X$, i.e. $A = B$.

(c) By assumption $(A,B)\mathcal{S}_X = \bigwedge_{x\in X}(x)A \to (x)B = \top$ and $(B,C)\mathcal{S}_X = \bigwedge_{x\in X}(x)B \to (x)C = \top$; so, for every $x \in X$ $(x)A \to (x)B = (x)B \to (x)C = \top$. Hence, from the weak transitivity condition (o_4) it follows that, for every $x \in X$, $(x)A \to (x)C = \top$ and then $(A,C)\mathcal{S}_X = \bigwedge_{x\in X}(x)A \to (x)C = \top$, i.e. $(A,C)\mathcal{S}_X = \top$.

2. Let $(L,\to,\top)$ be a right-distributive ceo algebra and let $A,B,C \in L^X$ such that $(A,B)\mathcal{S}_X = \top$. By using the right-distributivity assumption and the axioms (o_2), (o_5) and $(o_5)'$ we can deduce the following:

$$\begin{aligned}(C,A)\mathcal{S}_X \to (C,B)\mathcal{S}_X &= (\bigwedge_{x\in X}(x)C \to (x)A) \to (\bigwedge_{x'\in X}(x')C \to (x')B)\\ &= \bigwedge_{x'\in X}((\bigwedge_{x\in X}(x)C \to (x)A) \to ((x')C \to (x')B))\\ &\geq \bigwedge_{x'\in X}(((x')C \to (x')A) \to ((x')C \to (x')B))\\ &= \top.\end{aligned}$$

So, the first statement is proved.
Similarly, we can prove that $(B,C)\mathcal{S}_X \to (A,C)\mathcal{S}_X = \top$.

3. By Proposition 2.1.19 we have that $(L,\rightsquigarrow,\top)$ is a cdeo algebra, too. So the items (a), (b), (c) (d) of (3) follow from the items (1) and (2).

4. Let $(L,\to,\top)$ be an associative and right-distributive ceo algebra and let $A,B,C \in L^X$.

(a) By Propositions 2.1.11 (9), (10) and 2.1.60 (1), the following holds: $(A,B)\mathcal{S}_X \otimes (B,C)\mathcal{S}_X = (\bigwedge_{x\in X}(x)A \to (x)B) \otimes (\bigwedge_{x'\in X}(x')B \to (x')C) \leq ((x)A \to (x)B) \otimes ((x)B \to (x)C) \leq (x)A \to (x)C$.
Hence, $(A,B)\mathcal{S}_X \otimes (B,C)\mathcal{S}_X \leq \bigwedge_{x\in X}(x)A \to (x)C = (A,C)\mathcal{S}_X$.
The second inequality is equivalent to the first one.

(b) The proof is similar to the above and it can be given recalling that $\tilde{\otimes} = \otimes^{op}$ and by using the Proposition 2.1.61 (1).

□

Corollary 3.2.10. *1. If $(L,\to,\top)$ is a w-ceo algebra, then $\to$ is an $(L,\top)$-order in L and $\mathcal{S}_X$ is a weak $(L,\top)$-order in L^X. Under symmetry assumption, $\rightsquigarrow$ and $\tilde{\mathcal{S}}_X$ are weak $(L,\top)$-orders, too.*

2. If $(L,\to,\top)$ is an associative cdeo algebra, then $\to$ is an $(L,\otimes)$-order in L and $\mathcal{S}_X$ is an $(L,\otimes)$-order in L^X; moreover, if L is symmetrical and associative cdeo algebra, then $\rightsquigarrow$ and $\tilde{\mathcal{S}}_X$ are $(L,\otimes)$-orders, too. □

In the following definition we recall a further particular (L,χ)-aggregations used in many-valued mathematics to evaluate the intersection between L-sets.

Definition 3.2.11. *Let X be a set and let $(L, \rightarrow, \top)$ be a w-ceo algebra. The* ***intersection L-relation*** *$\mathcal{T}_X : L^X \times L^X \rightarrow L$ is defined by $(A, B)\mathcal{T}_X = \bigvee_{x \in X}(x)A \otimes (x)B$, for all $A, B \in L^X$.*
If $(L, \rightarrow, \top)$ is a symmetrical cdeo algebra, we can define the ***dual intersection L-relation*** *$\tilde{\mathcal{T}}_X : L^X \times L^X \rightarrow L$ such that $(A, B)\tilde{\mathcal{T}}_X = \bigvee_{x \in X}(x)A\tilde{\otimes}(x)B$, for all $A, B \in L^X$.*

Of course $\mathcal{T}_X = \mathcal{R}_X^{\otimes,\vee}$ and we use the notation $\mathcal{T}_X^{\wedge} = \mathcal{R}_X^{\otimes,\wedge}$ and, if L is symmetrical, we denote $\tilde{\mathcal{T}}_X = \mathcal{R}_X^{\tilde{\otimes},\vee}$, $\tilde{\mathcal{T}}_X^{\wedge} = \mathcal{R}_X^{\tilde{\otimes},\wedge}$ and we call $\mathcal{T}_X^{\wedge}$ and $\tilde{\mathcal{T}}_X^{\wedge}$ **overlap** and **dual overlap L-relation**, respectively.
We note that $\mathcal{T}_X^{\wedge}$ is not frequently used, since it is trivial in some cases; in fact, for instance, we have that $(A, B)\mathcal{T}_X^{\wedge} = \top \Leftrightarrow (x)A = (x)B = \top,\ \forall x \in X$.

Proposition 3.2.12. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. Then the following hold, for all $A, B, C \in L^X$.*

1. $(A, B)\mathcal{T}_X = (B, A)\tilde{\mathcal{T}}_X$;
2. $(A, \perp\!\!\!\perp_X)\mathcal{T}_X = (\perp\!\!\!\perp_X, A)\mathcal{T}_X = \perp$;
3. $(A, \perp\!\!\!\perp_X)\tilde{\mathcal{T}}_X = (\perp\!\!\!\perp_X, A)\tilde{\mathcal{T}}_X = \perp$;
4. *L is associative if and only if one of the following conditions holds:*
   (a) $(A, B \otimes C)\mathcal{T}_X = (A \otimes C, B)\mathcal{T}_X,\ \forall X \in |\boldsymbol{Set}|$;
   (b) $(A, B \otimes C)\tilde{\mathcal{T}}_X = (A \otimes C, B)\tilde{\mathcal{T}}_X,\ \forall X \in |\boldsymbol{Set}|$;
   (c) $(A, B\tilde{\otimes}C)\mathcal{T}_X = (A\tilde{\otimes}C, B)\mathcal{T}_X,\ \forall X \in |\boldsymbol{Set}|$;
   (d) $(A, B\tilde{\otimes}C)\mathcal{T}_X = (C\tilde{\otimes}A, B)\mathcal{T}_X,\ \forall X \in |\boldsymbol{Set}|$;
   (e) $(A, B\tilde{\otimes}C)\mathcal{T}_X = (B, A \otimes C)\tilde{\mathcal{T}}_X,\ \forall X \in |\boldsymbol{Set}|$;
   (f) $(A, B \otimes C)\tilde{\mathcal{T}}_X = (C \otimes A, B)\tilde{\mathcal{T}}_X,\ \forall X \in |\boldsymbol{Set}|$;
   (g) $(A, B \otimes C)\tilde{\mathcal{T}}_X = (B, A\tilde{\otimes}C)\mathcal{T}_X,\ \forall X \in |\boldsymbol{Set}|$.

Proof. 1. The equality follows easily from the definition of $\mathcal{T}_X$ and $\tilde{\mathcal{T}}_X$ and recalling that $\tilde{\otimes} = \otimes^{op}$.

The items (2) and (3) follows from Proposition 2.1.11 (2), (3) and by $\tilde{\otimes} = \otimes^{op}$. 4. We prove the equivalence between the associativity of L and the equality of item (a); the other equivalences can be similarly proved, recalling $\tilde{\otimes} = \otimes^{op}$.
Assuming L to be associative, let $A, B, C \in L^X$; then the following holds:
$(A, B \otimes C)\mathcal{T} = \bigvee_{x \in X}(x)A \otimes ((x)B \otimes (x)C) = \bigvee_{x \in X}((x)A \otimes (x)B) \otimes (x)C = (A \otimes B, C)\mathcal{T}$.
The converse implication can be easily proved considering a singleton $X = \{x\}$. □

3.2.2 L-powerset operators

L-powerset operators associated to functions or to L-valued relations are somehow defined in [4, 6, 16, 22, 25, 33, 35, 45, 90] assuming in any case that L is a complete lattice, sometimes with a lattice ordered algebraic structure. We adopt notation similar to those of [33, 35], omitting the attribute "powerset", with weaker assumptions on the algebraic structure of L.

Definition 3.2.13. *[35] Let $(L, \rightarrow, \top)$ be a right-distributive ceo algebra and $\mathcal{R} : X \rightharpoonup Y$ be an L-relation, with $X, Y \in |\boldsymbol{Set}|$.*
The ***right forward weak operator*** *of $\mathcal{R}$ is $\mathcal{R}^{\otimes} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)\mathcal{R}^{\otimes} = \bigvee_{x \in X} (x,y)\mathcal{R} \otimes (x)A.$$

The ***left forward weak operator*** *of $\mathcal{R}$ is ${}^{\otimes}\mathcal{R} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)^{\otimes}\mathcal{R} = \bigvee_{x \in X} (x)A \otimes (x,y)\mathcal{R}.$$

The ***right forward strong operator*** *of $\mathcal{R}$ is $\mathcal{R}^{\rightarrow} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)\mathcal{R}^{\rightarrow} = \bigwedge_{x \in X} (x,y)\mathcal{R} \rightarrow (x)A.$$

The ***left forward strong operator*** *of $\mathcal{R}$ is ${}^{\rightarrow}\mathcal{R} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)^{\rightarrow}\mathcal{R} = \bigwedge_{x \in X} (x)A \rightarrow (x,y)\mathcal{R}.$$

If $\mathcal{R}_{-}$ is the reverse L-relation of $\mathcal{R}$, we denote the ***backward operators*** *of $\mathcal{R}$ by the following notation.*

The ***right backward weak operator*** *of $\mathcal{R}$ is $\mathcal{R}_{-}^{\otimes} = (\mathcal{R}_{-})^{\otimes} : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)\mathcal{R}_{-}^{\otimes} = \bigvee_{y \in Y} (y,x)\mathcal{R}_{-} \otimes (y)B.$$

The ***left backward weak operator*** *of $\mathcal{R}$ is ${}^{\otimes}\mathcal{R}_{-} =^{\otimes} (\mathcal{R}_{-}) : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)^{\otimes}\mathcal{R}_{-} = \bigvee_{y \in Y} (y)B \otimes (y,x)\mathcal{R}_{-}.$$

The ***right backward strong operator*** *of $\mathcal{R}$ is $\mathcal{R}_{-}^{\rightarrow} = (\mathcal{R}_{-})^{\rightarrow} : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)\mathcal{R}_{-}^{\rightarrow} = \bigwedge_{y \in Y} (y,x)\mathcal{R}_{-} \rightarrow (y)B.$$

The ***left backward strong operator*** *of $\mathcal{R}$ is ${}^{\rightarrow}\mathcal{R}_{-} =^{\rightarrow} (\mathcal{R}_{-}) : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)^{\rightarrow}\mathcal{R}_{-} = \bigwedge_{y \in Y} (y)B \rightarrow (y,x)\mathcal{R}_{-}.$$

With further assumption on L we can define more powerset operators as follows.

Definition 3.2.14. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra and $\mathcal{R} : X \rightharpoonup Y$ be an L-relation, with $X, Y \in |\boldsymbol{Set}|$.*
The ***dual right forward weak operator*** *of $\mathcal{R}$ is $\mathcal{R}^{\tilde{\otimes}} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)\mathcal{R}^{\tilde{\otimes}} = \bigvee_{x \in X} (x, y)\mathcal{R} \tilde{\otimes} (x)A.$$

The ***dual left forward weak operator*** *of $\mathcal{R}$ is ${}^{\tilde{\otimes}}\mathcal{R} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)^{\tilde{\otimes}}\mathcal{R} = \bigvee_{x \in X} (x)A \tilde{\otimes} (x, y)\mathcal{R}.$$

The ***dual right forward strong operator*** *of $\mathcal{R}$ is $\mathcal{R}^{\rightsquigarrow} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)\mathcal{R}^{\rightsquigarrow} = \bigwedge_{x \in X} (x, y)\mathcal{R} \rightsquigarrow (x)A.$$

The ***dual left forward strong operator*** *of $\mathcal{R}$ is ${}^{\rightsquigarrow}\mathcal{R} : L^X \to L^Y$, such that, for every $A \in L^X$ and $y \in Y$:*

$$(y)(A)^{\rightsquigarrow}\mathcal{R} = \bigwedge_{x \in X} (x)A \rightsquigarrow (x, y)\mathcal{R}.$$

If $\mathcal{R}_-$ is the reverse L-relation of $\mathcal{R}$, we denote the ***dual backward operators*** *of $\mathcal{R}$ by the following notation.*

The ***dual right backward weak operator*** *of $\mathcal{R}$ is $\mathcal{R}_-^{\tilde{\otimes}} = (\mathcal{R}_-)^{\tilde{\otimes}} : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)\mathcal{R}_-^{\tilde{\otimes}} = \bigvee_{y \in Y} (y, x)\mathcal{R}_- \tilde{\otimes} (y)B.$$

The ***dual left backward weak operator*** *of $\mathcal{R}$ is ${}^{\tilde{\otimes}}\mathcal{R}_- = {}^{\tilde{\otimes}}(\mathcal{R}_-) : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)^{\tilde{\otimes}}\mathcal{R}_- = \bigvee_{y \in Y} (y)B \tilde{\otimes} (y, x)\mathcal{R}_-.$$

The ***dual right backward strong operator*** *of $\mathcal{R}$ is $\mathcal{R}_-^{\rightsquigarrow} = (\mathcal{R}_-)^{\rightsquigarrow} : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)\mathcal{R}_-^{\rightsquigarrow} = \bigwedge_{y \in Y} (y, x)\mathcal{R}_- \rightsquigarrow (y)B.$$

The ***dual left backward strong operator*** *of $\mathcal{R}$ is ${}^{\rightsquigarrow}\mathcal{R}_- = {}^{\rightsquigarrow}(\mathcal{R}_-) : L^Y \to L^X$, such that, for every $B \in L^Y$ and $x \in X$:*

$$(x)(B)^{\rightsquigarrow}\mathcal{R}_- = \bigwedge_{y \in Y} (y)B \rightsquigarrow (y, x)\mathcal{R}_-.$$

Remark 3.2.15. It is easy to show that in case of a symmetrical cdeo algebra, $\mathcal{R}^{\tilde{\otimes}} = {}^{\otimes}\mathcal{R}$ and ${}^{\tilde{\otimes}}\mathcal{R} = \mathcal{R}^{\otimes}$. Hence, the results proved for ${}^{\otimes}\mathcal{R}$ and $\mathcal{R}^{\otimes}$ can be restated for $\mathcal{R}^{\tilde{\otimes}}$ and ${}^{\tilde{\otimes}}\mathcal{R}$.
Clearly, if L is a commutative cdeo algebra, then $\rightsquigarrow = \rightarrow$ and $\tilde{\otimes} = \otimes$; hence $\mathcal{R}^{\tilde{\otimes}} = \mathcal{R}^{\otimes} = {}^{\otimes}\mathcal{R} = {}^{\tilde{\otimes}}\mathcal{R}$, $\mathcal{R}^{\rightsquigarrow} = \mathcal{R}^{\rightarrow}$ and ${}^{\rightsquigarrow}\mathcal{R} = {}^{\rightarrow}\mathcal{R}$.

The following extends some results of [33].

Proposition 3.2.16. *1. Let $(L, \rightarrow, \top)$ be a right-distributive ceo algebra. The following hold, for every $\mathcal{R} : X \rightharpoonup Y$ for all $A, B \in L^X$.*

(a) $\mathcal{R}^{\otimes}$, ${}^{\otimes}\mathcal{R}$ and $\mathcal{R}^{\rightarrow}$ are isotonic;

(b) ${}^{\rightarrow}\mathcal{R}$ is antitonic.

2. Let $(L, \rightarrow, \top)$ be a symmetrical right-distributive ceo algebra. The following hold, for every $\mathcal{R} : X \rightharpoonup Y$ for all $A, B \in L^X$.

(a) $\mathcal{R}^{\tilde{\otimes}}$, ${}^{\tilde{\otimes}}\mathcal{R}$ and $\mathcal{R}^{\rightsquigarrow}$ are isotonic;

(b) ${}^{\rightsquigarrow}\mathcal{R}$ is antitonic.

Proof. 1. Let $(L, \rightarrow, \top)$ be a right-distributive ceo algebra, $\mathcal{R} : X \rightharpoonup Y$ and $A, B \in L^X$ such that $A \subseteq B$, i.e. $(x)A \leq (x)B$, for every $x \in X$.

(a) By Proposition 2.1.11 (9), (10) and by assumption, for every $y \in Y$, we have that $(y)(A)\mathcal{R}^{\otimes} = \bigvee_{x \in X}(x, y)\mathcal{R} \otimes (x)A \leq \bigvee_{x \in X}(x, y)\mathcal{R} \otimes (x)B = (y)(B)\mathcal{R}^{\otimes}$; similarly we can prove that ${}^{\tilde{\otimes}}\mathcal{R}$ is isotonic.
From (o_5) and by assumption, it follows that:
$(y)(A)\mathcal{R}^{\rightarrow} = \bigvee_{x \in X}(x, y)\mathcal{R} \rightarrow (X)A \leq \bigvee_{x \in X}(x, y)\mathcal{R} \rightarrow (X)B = (y)(B)\mathcal{R}^{\rightarrow}$.

(b) From $(o_5)'$ and by assumption, it follows that:
$(y)(A)^{\rightarrow}\mathcal{R} = \bigvee_{x \in X}(x)A \rightarrow (x, y)\mathcal{R} \geq \bigvee_{x \in X}(x)B \rightarrow (x, y)\mathcal{R} = (y)(B)^{\rightarrow}\mathcal{R}$.

2. The proof of the item (a) and (b) of (2) is similar to above.

□

Proposition 3.2.17. *1. Let $(L, \rightarrow, \top)$ be a right-distributive ceo algebra. For every $\mathcal{R} : X \rightharpoonup Y$, the following hold.*

(a) $\mathcal{R}^{\otimes} \dashv \mathcal{R}^{\rightarrow}_{-}$;

(b) $\mathcal{R}^{\otimes}_{-} \dashv \mathcal{R}^{\rightarrow}$.

2. Let $(L, \rightarrow, \top)$ be a symmetrical right-distributive ceo algebra. For every $\mathcal{R} : X \rightharpoonup Y$, the following hold.

(a) $\mathcal{R}^{\tilde{\otimes}} \dashv \mathcal{R}^{\rightsquigarrow}_{-}$;

(b) $\mathcal{R}^{\tilde{\otimes}}_{-} \dashv \mathcal{R}^{\rightsquigarrow}$.

Proof. 1. Let $(L, \to, \top)$ be a symmetrical right-distributive ceo algebra and let $A \in L^X$ and $B \in L^Y$.

(a) Let $x \in X$; by Proposition 2.1.11 (11) and from (o_5), the following hold: $(x)((A)^{\otimes}\mathcal{R})\mathcal{R}_-^{\rightharpoonup} = \bigwedge_{y \in Y} ((x,y)\mathcal{R} \to (\bigvee_{x' \in X}((x',y)\mathcal{R} \otimes (x')A)) \geq \bigwedge_{y \in Y}(x,y)\mathcal{R} \to ((x,y)\mathcal{R} \otimes (x)A) \geq (x)A$.
Now, let $y \in Y$; by Proposition 2.1.11 (9) (11), we have that $(y)((B)\mathcal{R}_-^{\rightharpoonup})\mathcal{R}^{\otimes} = \bigvee_{x \in X}(x,y)\mathcal{R} \otimes \left(\bigwedge_{y' \in Y}(x,y')\mathcal{R} \to (y')B\right) \leq \bigvee_{x \in X}(x,y)\mathcal{R} \otimes ((x,y)\mathcal{R} \to (y)B) \leq (y)B$.

(b) This statement follows by applying the item (a) to the reverse L-relation $\mathcal{R}_-$.

2. The items (a) and (b) of (2) can be proved similarly.

□

The equalities stated in Remark 3.2.15 allow to prove the following.

Corollary 3.2.18. *If $(L, \to, \top)$ is a symmetrical cdeo algebra and $\mathcal{R} : X \times Y \to L$, then one has the following adjunctions:*

1. ${}^{\otimes}\mathcal{R} \dashv \mathcal{R}_-^{\rightsquigarrow}$;
2. ${}^{\tilde{\otimes}}\mathcal{R} \dashv \mathcal{R}_-^{\to}$. □

Proposition 3.2.19. *Let $(L, \to, \top)$ be a symmetrical cdeo algebra and let $\mathcal{R} : X \times Y \to L$. The following hold.*

1. $[{}^{\to}\mathcal{R}, {}^{\rightsquigarrow}\mathcal{R}_-]$;
2. $[{}^{\rightsquigarrow}\mathcal{R}, {}^{\to}\mathcal{R}_-]$.

Proof. 1. Let $A \in L^X$ and $x \in X$; by Propositions 2.1.26 (2) and 2.1.30 (9) and by applying $(o_5)'$, we have that $(x)((A)^{\to}\mathcal{R})^{\rightsquigarrow}\mathcal{R}_- = \bigwedge_{y \in Y} ((\bigwedge_{x' \in X}(x')A \to (x',y)\mathcal{R}) \rightsquigarrow (x,y)\mathcal{R}) \geq \bigwedge_{y \in Y}((x)A \to (x,y)\mathcal{R}) \rightsquigarrow (x,y)\mathcal{R} \geq (x)A$.
Now, let $B \in L^Y$ and $y \in Y$; by Propositions 2.1.26 (2) and 2.1.30 (10) and by $(o_5)'$, similarly we can prove that $(y)((B)^{\rightsquigarrow}\mathcal{R}_-)^{\to}\mathcal{R} \geq (y)B$. So, by Proposition 3.2.16, we have that ${}^{\to}\mathcal{R}$ and ${}^{\rightsquigarrow}\mathcal{R}_-$ are antitonic and satisfy the inequalities $A \leq ((A)^{\to}\mathcal{R})^{\rightsquigarrow}\mathcal{R}_-$ and $B \leq ((B)^{\rightsquigarrow}\mathcal{R}_-)^{\to}\mathcal{R}$. Hence, by Remark 1.2.6 it follows that $[{}^{\to}\mathcal{R}, {}^{\rightsquigarrow}\mathcal{R}_-]$.

2. The proof is similar to above.

□

We note that, if $(L, \to, \top)$ is a commutative cdeo algebra (and hence, $\to = \rightsquigarrow$), we have also that $[{}^{\to}\mathcal{R}, {}^{\to}\mathcal{R}_-]$. Moreover, this condition characterizes the commutativity of cdeo algebra, as the following result shows.

Proposition 3.2.20. *Let $(L, \to, \top)$ be a right-distributive ceo algebra. The following are equivalent.*

(i) L is commutative;

(ii) for all X, Y non-empty sets and for every $\mathcal{R} : X \rightharpoonup Y$: $[{}^{\rightarrow}\mathcal{R}, {}^{\rightarrow}\mathcal{R}_{-}]$.

Proof. $(i) \Rightarrow (ii)$ Thanks to commutativity assumption, we have that $\rightarrow = \rightsquigarrow$; hence (ii) follows from Proposition 3.2.19.
$(ii) \Rightarrow (i)$ If $[{}^{\rightarrow}\mathcal{R}, {}^{\rightarrow}\mathcal{R}_{-}]$, then ${}^{\rightarrow}\mathcal{R}$ is uniquely determined by ${}^{\rightarrow}\mathcal{R}_{-}$ and hence, by Proposition 3.2.19 we have that ${}^{\rightarrow}\mathcal{R} = {}^{\rightsquigarrow}\mathcal{R}$.
Consider now the set $X = \{x\}$, the L-point $A = a_X$ and the L-relation $\mathcal{R} : X \rightharpoonup Y$ defined by $(x, x)\mathcal{R} = b$, for some $b \in L$. Then, we have that $a \rightarrow b = {}^{\rightarrow}\mathcal{R} = {}^{\rightsquigarrow}\mathcal{R} = a \rightsquigarrow b$, for all $a, b \in L$. Hence $\rightarrow = \rightsquigarrow$, i.e. L is commutative. □

Proposition 3.2.21. *1. If $(L, \rightarrow, \top)$ is a cdeo algebra, then, for every set X: ${}^{\otimes}\mathcal{I}_X = \mathcal{I}_{L^X}$.*

2. If $(L, \rightarrow, \top)$ is a symmetrical cdeo algebra, for every set X:

(a) $\mathcal{I}_X^{\otimes} = \mathcal{I}_{L^X}$;

(b) ${}^{\tilde{\otimes}}\mathcal{I}_X = \mathcal{I}_{L^X}$;

(c) $\mathcal{I}_X^{\tilde{\otimes}} = \mathcal{I}_{L^X}$.

Proof. 1. Let $(L, \rightarrow, \top)$ be a cdeo algebra; for all $A \in L^X$ and $x \in X$, the following equalities hold: $(x)(A)^{\otimes}\mathcal{I}_X = \bigvee_{x' \in X}(x')A \otimes (x', x)\mathcal{I} = (x)A \otimes \top = (x)A$.

2. Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra.

   (a) Since $\top$ is left neutral with respect to the adjoint product $\otimes$, for all $A \in L^X$ and $x \in X$, the following equalities hold: $(x)(A)\mathcal{I}_X^{\otimes} = \bigvee_{x' \in X}(x, x')\mathcal{I} \otimes (x')A = \top \otimes (x)A = (x)A$.

   The items (b) and (c) of (2) follows easily recalling that $\tilde{\otimes} = \otimes^{op}$.

□

Lemma 3.2.22. *1. Let $(L, \rightarrow, \top)$ be a cdeo algebra. L is associative if and only if ${}^{\otimes}(\mathcal{R} \odot \mathcal{S}) = {}^{\otimes}\mathcal{R} \cdot {}^{\otimes}\mathcal{S}$, for all $\mathcal{R} : X \times Y \to L$ and $\mathcal{S} : Y \times Z \to L$.*

2. Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. L is associative if and only if ${}^{\tilde{\otimes}}(\mathcal{R} \tilde{\odot} \mathcal{S}) = {}^{\tilde{\otimes}}\mathcal{R} \cdot {}^{\tilde{\otimes}}\mathcal{S}$, for all $\mathcal{R} : X \times Y \to L$ and $\mathcal{S} : Y \times Z \to L$.

Proof. 1. “ $\Rightarrow$ ” By Propositions 2.1.11 (8) and 2.1.28, applying the associativity condition, the following equalities hold, for all $\mathcal{R} : X \times Y \to L$, $\mathcal{S} : Y \times Z \to L$, $A \in L^X$ and $z \in Z$: $(z)(A)^{\otimes}(\mathcal{R} \odot \mathcal{S}) = \bigvee_{x \in X}(x)A \otimes \left(\bigvee_{y \in Y}(x, y)\mathcal{R} \otimes (y, z)\mathcal{S}\right) = \bigvee_{x \in X}\bigvee_{y \in Y}(x)A \otimes ((x, y)\mathcal{R} \otimes (y, z)\mathcal{S}) = \bigvee_{y \in Y}\bigvee_{x \in X}((x)A \otimes (x, y)\mathcal{R}) \otimes (y, z)\mathcal{S} = \bigvee_{y \in Y}((\bigvee_{x \in X}(x)A \otimes (x, y)\mathcal{R}) \otimes (y, z)\mathcal{S}) = (z)(A)({}^{\otimes}\mathcal{R} \cdot {}^{\otimes}\mathcal{S})$.
“ $\Leftarrow$ ” Consider now the L-point $A = a_X$, with $a \in L$, $\mathcal{R} : X \times Y \to L$ defined by $(x, y)\mathcal{R} = b$, for all $x \in X, y \in Y$, with $b \in L$ and $\mathcal{S} : Y \times Z \to L$ defined by $(y, z)\mathcal{S} = c$, for all $y \in Y, z \in Z$, with $c \in L$. By assumption, the following hold, for every $z \in Z$: $(z)(A)^{\otimes}(\mathcal{R} \odot \mathcal{S}) = (z)(A)({}^{\otimes}\mathcal{R} \cdot {}^{\otimes}\mathcal{S}) \Rightarrow a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

2. The proof of the equivalence is similar to above, recalling that $\tilde{\otimes} = \otimes^{op}$.

$\square$

Proposition 3.2.23. *If $(L, \rightarrow, \top)$ is an associative and symmetrical cdeo algebra, then*

1. *the maps*
$$X \in |\boldsymbol{Set}| \mapsto (X)^{\otimes}[\cdot] = L^X$$
$$\mathcal{R} \in \mathbf{R_L}(X, Y) \mapsto (\mathcal{R})^{\otimes}[\cdot] = ^{\otimes} \mathcal{R} : L^X \to L^Y$$
*define a functor $^{\otimes}[\cdot]$ between the categories $\mathbf{R}_{\boldsymbol{L}}^{\odot} = (|\boldsymbol{Set}|, \mathbf{R_L}, \odot)$ and $\bigvee$-**CSLat**.*

2. *the maps*
$$X \in |\boldsymbol{Set}| \mapsto (X)^{\tilde{\otimes}}[\cdot] = L^X$$
$$\mathcal{R} \in \mathbf{R_L}(X, Y) \mapsto (\mathcal{R})^{\tilde{\otimes}}[\cdot] = ^{\tilde{\otimes}} \mathcal{R} : L^X \to L^Y$$
*define a functor $^{\tilde{\otimes}}[\cdot]$ between the categories $\mathbf{R}_{\boldsymbol{L}}^{\tilde{\odot}} = (|\boldsymbol{Set}|, \mathbf{R_L}, \tilde{\odot})$ and $\bigvee$-**CSLat**.*

Proof. First we note that, if $(L, \rightarrow, \top)$ is an associative and symmetrical cdeo algebra, then both $\mathbf{R}_{\mathbf{L}}^{\odot} = (|\mathbf{Set}|, \mathbf{R_L}, \odot)$ and $\mathbf{R}_{\mathbf{L}}^{\tilde{\odot}} = (|\mathbf{Set}|, \mathbf{R_L}, \tilde{\odot})$ are categories, by Propositions 3.1.10 and 3.1.14. The statement of both items follows from Lemma 3.2.22. $\square$

We note that, in general, $(\mathcal{R} \odot \mathcal{S})^{\otimes} \neq \mathcal{R}^{\otimes} \cdot \mathcal{S}^{\otimes}$ and $(\mathcal{R} \tilde{\odot} \mathcal{S})^{\tilde{\otimes}} \neq \mathcal{R}^{\tilde{\otimes}} \cdot \mathcal{S}^{\tilde{\otimes}}$. But, the following results hold.

Lemma 3.2.24. *Let $(L, \rightarrow, \top)$ be a cdeo algebra and $\mathcal{R} : X \times Y \to L$ and $\mathcal{S} : Y \times Z \to L$. $(\mathcal{R} \odot \mathcal{S})^{\otimes} = \mathcal{R}^{\otimes} \cdot \mathcal{S}^{\otimes}$ if and only if $(a \otimes b) \otimes c = b \otimes (c \otimes a)$, for all $a, b, c \in L$.*

Proof. The proof can be given with the similar arguments of Lemma 3.2.22. $\square$

Proposition 3.2.25. *If $(L, \rightarrow, \top)$ is an associative and symmetrical cdeo algebra, the following are equivalent:*

1. *the maps*
$$X \in |\boldsymbol{Set}| \mapsto (X)[\cdot]^{\otimes} = L^X$$
$$\mathcal{R} \in \mathbf{R_L}(X, Y) \mapsto (\mathcal{R})[\cdot]^{\otimes} = \mathcal{R}^{\otimes} : L^X \to L^Y$$
*define a functor $[\cdot]^{\otimes}$ between the categories $\mathbf{R}_{\boldsymbol{L}}^{\odot} = (|\boldsymbol{Set}|, \mathbf{R_L}, \odot)$ and $\bigvee$-**CSLat**;*

2. *$(a \otimes b) \otimes c = b \otimes (c \otimes a)$, for all $a, b, c \in L$;*

3. *L is commutative;*

4. *$^{\otimes}[\cdot] = [\cdot]^{\otimes}$.*

Proof. "(1) $\Rightarrow$ (2)" This implication follows from Lemma 3.2.24.
"(2) $\Rightarrow$ (3)" The commutativity of $\otimes$ follows easily from (2), choosing $c = \top$.
"(3) $\Rightarrow$ (4)" This implication follows easily from Remark 3.2.15.
"(4) $\Rightarrow$ (1)" It follows from Proposition 3.2.23. $\square$

Lemma 3.2.26. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra and $\mathcal{R} : X \times Y \rightarrow L$ and $\mathcal{S} : Y \times Z \rightarrow L$. $(\mathcal{R}\tilde{\odot}\mathcal{S})^{\tilde{\otimes}} = \mathcal{R}^{\tilde{\otimes}} \cdot \mathcal{S}^{\tilde{\otimes}}$ if and only if $(a \otimes b) \otimes c = (a \otimes c) \otimes b$, for all $a, b, c \in L$.*

Proof. The proof is an immediate consequence of Remark 3.1.13 and Lemma 3.2.22. □

Proposition 3.2.27. *If $(L, \rightarrow, \top)$ is an associative and symmetrical cdeo algebra, the following are equivalent:*

1. *the maps*
$$X \in |\boldsymbol{Set}| \mapsto (X)[\cdot]^{\tilde{\otimes}} = L^X$$
$$\mathcal{R} \in \mathbf{R_L}(X, Y) \mapsto (\mathcal{R})[\cdot]^{\tilde{\otimes}} = \mathcal{R}^{\tilde{\otimes}} : L^X \rightarrow L^Y$$
define a functor $[\cdot]^{\tilde{\otimes}}$ between the categories $\mathbf{R}^{\odot}_{\boldsymbol{L}} = (|\boldsymbol{Set}|, \mathbf{R_L}, \odot)$ and $\bigvee$–$\mathbf{CSLat}$;
2. *$a \otimes (b \otimes c) = (a \otimes c) \otimes b$, for all $a, b, c \in L$;*
3. *L is commutative;*
4. *${}^{\tilde{\otimes}}[\cdot] = [\cdot]^{\tilde{\otimes}}$.*

Proof. The equivalences follow soon by Remark 3.1.13 and Proposition 3.2.25. □

Proposition 3.2.28. *If $(L, \rightarrow, \top)$ is a symmetrical cdeo algebra, then, for every non-empty set X, the following hold.*

1. *$\mathcal{I}_X^{\rightarrow} = \mathcal{I}_{L^X}$;*
2. *$\mathcal{I}_X^{\rightsquigarrow} = \mathcal{I}_{L^X}$.*

Proof.
1. Let X be a non-empty set and let $A \in L^X$. Then, for every $x \in X$, by Proposition 2.1.30 (7), the following hold: $(x)(A)\mathcal{I}_X^{\rightarrow} = \bigwedge_{x' \in X}(x', x)\mathcal{I}_X \rightarrow (x')A = (\bigwedge_{x' \neq x \in X} \bot \rightarrow (x')A) \wedge (\top \rightarrow (x)A) = \top \rightarrow (x)A = (x)A$.
2. The proof is similar, by using Proposition 2.1.30 (8).

□

Lemma 3.2.29.
1. *Let $(L, \rightarrow, \top)$ be a cdeo algebra. L is associative if and only if $(\mathcal{R} \odot \mathcal{S})^{\rightarrow} = \mathcal{R}^{\rightarrow} \cdot \mathcal{S}^{\rightarrow}$, for all $\mathcal{R} : X \times Y \rightarrow L$ and $\mathcal{S} : Y \times Z \rightarrow L$.*
2. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. L is associative if and only if $(\mathcal{R}\tilde{\odot}\mathcal{S})^{\rightsquigarrow} = \mathcal{R}^{\rightsquigarrow} \cdot \mathcal{S}^{\rightsquigarrow}$, for all $\mathcal{R} : X \times Y \rightarrow L$ and $\mathcal{S} : Y \times Z \rightarrow L$.*

Proof.
1. “ $\Rightarrow$ ” By distributivity and associativity conditions, for every $z \in Z$, the following hold: $(z)(A)(\mathcal{R} \cdot \mathcal{S})^{\rightarrow} = \bigwedge_{x \in X} \left(\left(\bigvee_{y \in Y} (x, y)\mathcal{R} \otimes (y, z)\mathcal{S} \right) \rightarrow (x)A \right) = \bigwedge_{x \in X} \bigwedge_{y \in Y} ((x, y)\mathcal{R} \otimes (y, z)\mathcal{S}) \rightarrow (x)A = \bigwedge_{x \in X} \bigwedge_{y \in Y} ((y, z)\mathcal{S} \rightarrow ((x, y)\mathcal{R} \rightarrow (x)A)) = \bigwedge_{y \in Y} ((y, z)\mathcal{S} \rightarrow (\bigwedge_{x \in X} (x, y)\mathcal{R} \rightarrow (x)A)) = (z)(A)(\mathcal{R}^{\rightarrow} \cdot \mathcal{S}^{\rightarrow})$.
“ $\Leftarrow$ ” Consider now the L-point $A = c_X$, the L-relations $\mathcal{R} : X \times Y \rightarrow L$ defined by $(x, y)\mathcal{R} = a$, for all $x \in X, y \in Y$ and $\mathcal{S} : Y \times Z \rightarrow L$ defined by$(y, z)\mathcal{S} = b$, for all $y \in Y, z \in Z$, where $a, b, c \in L$. Then, for all non-empty sets X, Y, Z, by assumption the following hold: $(z)(A)(\mathcal{R} \odot \mathcal{S})^{\rightarrow} = (z)(A)(\mathcal{R}^{\rightarrow} \cdot \mathcal{S}^{\rightarrow}) \Rightarrow (a \otimes b) \rightarrow c = b \rightarrow (a \rightarrow c)$. Hence $\otimes$ is associative, i.e. L is associative, too.

2. The proof is similar, by using Proposition 2.1.61 (3). □

Proposition 3.2.30. *If* $(L, \to, \top)$ *is an associative and symmetrical cdeo algebra, the maps*

$$X \in |\boldsymbol{Set}| \mapsto (X)[\cdot]^{\to} = L^X$$

$$\mathcal{R} \in \mathbf{R_L}(X, Y) \mapsto (\mathcal{R})[\cdot]^{\to} = \mathcal{R}^{\to} : L^X \to L^Y$$

define a functor $[\cdot]^{\to}$ *between the categories* $\mathbf{R}_{\boldsymbol{L}}^{\odot} = (|\boldsymbol{Set}|, \mathbf{R_L}, \odot)$ *and* $\bigwedge$−**CSLat**.

Proof. Since $(L, \to, \top)$ is an associative and symmetrical cdeo algebra, $\mathbf{R}_{\mathbf{L}}^{\odot} = (|\mathbf{Set}|, \mathbf{R_L}, \odot)$ is a category, by Propositions 3.1.10 and 3.1.14. The statement follows from Lemma 3.2.29. □

Proposition 3.2.31. *If* $(L, \to, \top)$ *is an associative and symmetrical cdeo algebra, the maps*

$$X \in |\boldsymbol{Set}| \mapsto (X)[\cdot]^{\rightsquigarrow} = L^X$$

$$\mathcal{R} \in \mathbf{R_L}(X, Y) \mapsto (\mathcal{R})[\cdot]^{\rightsquigarrow} = \mathcal{R}^{\rightsquigarrow} : L^X \to L^Y$$

define a functor $[\cdot]^{\rightsquigarrow}$ *between the categories* $\mathbf{R}_{\boldsymbol{L}}^{\odot} = (|\boldsymbol{Set}|, \mathbf{R_L}, \odot)$ *and* $\bigwedge$−**CSLat**.

Proof. Similarly to the above Proposition. □

3.2.3 Powerset operators as compositions

Developing an original hint, which can be found in [44], the authors of [4] show how the composition of many-valued relations may be used to express many concepts involving L-sets. In fact, one can represent L-sets as L-relations and, then, the powerset operators can be expressed using the compositions considered in the Subsection 3.1.2. For this goal, let $(L, \to, \top)$ be a right-distributive w-ceo algebra and consider an L-set A, i.e. $A : X \to L$, for any set X. Fixing an element p, consider the singleton $P = \{p\}$; then we can see an L-set on X $A : X \to L$ as an L-relation $A : P \times X \to L$ from P to X, identifying them by means of the following bijective correspondence:

$$L^X \to \mathbf{R_L}(P, X)$$

$$A : X \to L \mapsto A : P \rightharpoondown X, \; such\, that\; (x)A = (p, x)A, \; for\; every\; x \in X.$$

In this way it is possible to express the powerset operators in terms of compositions between L-relations. More precisely, the following results hold, which show that notation adopted in [33] for powerset operators anticipated their link with the compositions of relations.

Proposition 3.2.32. *Let* $(L, \to, \top)$ *be a right-distributive ceo algebra,* $\mathcal{R} : X \rightharpoondown Y$ *be an* L*-relation and* A *an* L*-set on* X*, with* $X, Y \in |\boldsymbol{Set}|$*. The following equalities hold:*

1. $(A)\mathcal{R}^{\otimes} = (\mathcal{R}_{-} \odot A_{-})_{-}$;

2. $(A)^{\otimes}\mathcal{R} = A \odot \mathcal{R}$;
3. $(A)\mathcal{R}^{\rightarrow} = (\mathcal{R}_- \oslash A_-)_-$;
4. $(A)^{\rightarrow}\mathcal{R} = A \oslash \mathcal{R}$.

Proof. 1. For every $y \in Y$, $(y)(A)\mathcal{R}^{\otimes} = \bigvee_{x\in X}(x,y)\mathcal{R} \otimes (x)A = \bigvee_{x\in X}(x,y)\mathcal{R} \otimes (i,x)A = \bigvee_{x\in X}(y,x)\mathcal{R}_- \otimes (x,p)A_- = (y,p)(\mathcal{R}_- \odot A_-) = (p,y)((\mathcal{R}_- \odot A_-)_-) = (y)(\mathcal{R}_- \odot A_-)_-$.

2. For every $y \in Y$, $(y)(A)^{\otimes}\mathcal{R} = \bigvee_{x\in X}(x)A \otimes (x,y)\mathcal{R} = \bigvee_{x\in X}(p,x)A \otimes (x,y)\mathcal{R} = (p,y)(A \odot \mathcal{R}) = (y)(A \odot \mathcal{R})$.

3. For every $y \in Y$, $(y)(A)\mathcal{R}^{\rightarrow} = \bigwedge_{x\in X}(x,y)\mathcal{R} \rightarrow (x)A = \bigwedge_{x\in X}(x,y)\mathcal{R} \rightarrow (p,x)A = \bigwedge_{x\in X}(y,x)\mathcal{R}_- \rightarrow (x,p)A_- = (y,p)(\mathcal{R}_- \oslash A_-) = (p,y)((\mathcal{R}_- \oslash A_-)_-) = (y)(\mathcal{R}_- \oslash A_-)_-$.

4. For every $y \in Y$, $(y)(A)^{\rightarrow}\mathcal{R} = \bigwedge_{x\in X}(x)A \rightarrow (x,y)\mathcal{R} = \bigwedge_{x\in X}(p,x)A \rightarrow (x,y)\mathcal{R} = (p,y)(A \oslash \mathcal{R}) = (y)(A \oslash \mathcal{R})$.

□

Proposition 3.2.33. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra, $\mathcal{R} : X \rightharpoonup Y$ be an L-relation and A an L-set on X, with $X, Y \in |\boldsymbol{Set}|$. The following equalities hold:*

1. $(A)\mathcal{R}^{\tilde{\otimes}} = (\mathcal{R}_- \tilde{\odot} A_-)_- = A \odot \mathcal{R}$;
2. $(A)^{\tilde{\otimes}}\mathcal{R} = A \tilde{\odot} \mathcal{R}$;
3. $(A)\mathcal{R}^{\rightsquigarrow} = (\mathcal{R}_- \tilde{\oslash} A_-)_-$;
4. $(A)^{\rightsquigarrow}\mathcal{R} = A \tilde{\oslash} \mathcal{R}$.

Proof. This proof is similar to above. □

We shall consider again this notation for powerset operators in a more general view of the algebra of L-relations, their compositions and properties in Chapter 5.

3.3 Functorial many-valued relations

Since the inception of fuzzy set theory [90] and, more generally, of lattice-valued mathematics, through the big development of many-valued logic there have been several attempts to present and study suitable generalizations of the concept of function within the universe of many-valued relations.
In the many-valued setting it is useful to recognize whether the involved many-valued relations are, or at least resemble functions, for instance, in case one has to check whether a predicate transformer is deterministic or not (see [27]); as another example, functional relations are used to characterize the points that allow a representation of Dedekind categories as categories of relations [37].
It is not the purpose of this thesis to discuss and compare the numerous generalizations of functions and to present a list of them, even; we only quote some papers where this

topic has been addressed, mainly [6, 11, 20, 35, 37, 45, 87, 89].
Our purpose is to present some generalizations of functions, among many-valued relations and to check under which conditions these generalizations characterize, in fact, functions. This is done in classical mathematical terms, using nothing but sets, whose elements are related, and extended-order algebras (in the first step, simply complete lattices), where the relations take their values. We shall obtain different results, depending on the properties the extended-order algebra satisfies.

3.3.1 Functionality by cuts

In this Subsection we consider L-relations taking values in a complete lattice L.

Let $\mathcal{R} : X \times Y \to L$ be an L-relation. If $x \in X, y \in Y, \alpha \in L$ denote

$$x\mathcal{R}^{[\alpha]} = \{y \in Y |\ (x,y)\mathcal{R} \geq \alpha\}\,;$$

$$x\mathcal{R}^{(\alpha)} = \{y \in Y |\ (x,y)\mathcal{R} \nleq \alpha\}\,;$$

$$\mathcal{R}^{[\alpha]}y = \{x \in X |\ (x,y)\mathcal{R} \geq \alpha\}\,;$$

$$\mathcal{R}^{(\alpha)}y = \{x \in X |\ (x,y)\mathcal{R} \nleq \alpha\}\,.$$

The above described sets are widely used types of α**-cuts** of the $\mathcal{R}$**-cone** of $x \in X$, that is the L-set $(x,-)\mathcal{R}$ on Y, and of the $\mathcal{R}$-cone of $y \in Y$, that is the L-set $(-,y)\mathcal{R}$ on X.

Definition 3.3.1. *With the above notation, we state the following:*

1. *$\mathcal{R}$ is **left** $[\alpha]$**-univocal** ((α)**-univocal**) if $y \in Y \Rightarrow \left|\mathcal{R}^{[\alpha]}y\right| \leq 1$ $\left(\left|\mathcal{R}^{(\alpha)}y\right| \leq 1\right)$;*
2. *$\mathcal{R}$ is right $[\alpha]$**-univocal** ((α)**-univocal**) if $x \in X \Rightarrow \left|x\mathcal{R}^{[\alpha]}\right| \leq 1$ $\left(\left|x\mathcal{R}^{(\alpha)}\right| \leq 1\right)$;*
3. *$\mathcal{R}$ is **left** $[\alpha]$**-total** ((α)**-total**) if $x \in X \Rightarrow x\mathcal{R}^{[\alpha]} \neq \varnothing$ $(x\mathcal{R}^{(\alpha)} \neq \varnothing)$;*
4. *$\mathcal{R}$ is **right** $[\alpha]$**-total** ((α)**-total**) if $y \in Y \Rightarrow \mathcal{R}^{[\alpha]}y \neq \varnothing$ $(\mathcal{R}^{(\alpha)}y \neq \varnothing)$.*

From now on, we use the term **function** to denote any crisp L-relation $\mathcal{R} : X \times Y \to L$ that satisfies the condition:

$$x \in X \Rightarrow \exists | y \in Y \text{ such that } (x,y)\mathcal{R} = \top.$$

In case $L = \mathbf{2}$, this means that we identify a function, as a subset $\mathcal{R} \subseteq X \times Y$, with its characteristic function.

Remark 3.3.2. It can be easily seen that an L-relation $\mathcal{R}$ is a function if and only if it is right $(\bot)$-univocal and left $[\top]$-total.

Since we are interested in functions and their generalizations, we simplify notation of the above Definition dropping the attribute right for right $[\alpha]$-(or (α)-)univocal relations and the attribute left for left $[\alpha]$-(or (α)-) total relations; also, we may avoid speaking at all of left univocality and right totality, since $\mathcal{R}$ is left $[\alpha]$-(or (α)-)univocal (right $[\alpha]$-(or (α)-)total, respectively) if and only if its reverse relation $\mathcal{R}_{-}$ is right $[\alpha]$-(or (α)-)univocal (left $[\alpha]$-(or (α)-)total, respectively).
An easy inspection shows how the above defined notions depend on the considered value $\alpha \in L$. First we note that, for $x \in X$, $x\mathcal{R}^{(\top)} = \varnothing$ and $x\mathcal{R}^{[\bot]} = Y$, so, it is clear that $\mathcal{R}$ is $[\bot]$-univocal if and only if $|Y| \leq 1$ and it is $(\top)$-total if and only if $X = \varnothing$. Moreover, every $\mathcal{R}$ is $(\top)$-univocal and it is $[\bot]$-total, unless $Y = \varnothing$ and $X \neq \varnothing$.
For arbitrary sets X and Y, the following results hold.

Lemma 3.3.3. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation. With the above notation, the following is true:*

1. *$\mathcal{R}$ is (α)-univocal ($[\alpha]$-univocal) if and only if $\mathcal{R}$ is (β)-univocal ($[\beta]$-univocal) for every $\beta \geq \alpha$;*
2. *$\mathcal{R}$ is (α)-total ($[\alpha]$-total) if and only if $\mathcal{R}$ is (γ)-total ($[\gamma]$-total) for every $\gamma \leq \alpha$;*
3. *if $\mathcal{R}$ is (α)-univocal, then $\mathcal{R}$ is $[\alpha']$-univocal for every $\alpha' \nleq \alpha$;*
4. *if $\mathcal{R}$ is $[\alpha]$-total, then $\mathcal{R}$ is (α'')-total for every $\alpha \nleq \alpha''$.*

Proof. 1. The sufficiency is trivial. Conversely, if $X \neq \varnothing$ and $\beta \geq \alpha$, then clearly $x \in X \Rightarrow x\mathcal{R}^{(\beta)} \subseteq x\mathcal{R}^{(\alpha)}$ and $x\mathcal{R}^{[\beta]} \subseteq x\mathcal{R}^{[\alpha]}$. So, by assumption, the implication $x \in X \Rightarrow \left|x\mathcal{R}^{(\beta)}\right| \leq \left|x\mathcal{R}^{(\alpha)}\right| \leq 1$ ($\left|x\mathcal{R}^{[\beta]}\right| \leq \left|x\mathcal{R}^{[\alpha]}\right| \leq 1$, respectively) is true.

2. Once more, the sufficiency is trivial. Conversely, by the same argument used in the proof of the item (1), it follows from the assumption that the implication $x \in X \Rightarrow x\mathcal{R}^{(\gamma)} \supseteq x\mathcal{R}^{(\alpha)} \neq \varnothing$ ($x\mathcal{R}^{[\gamma]} \supseteq x\mathcal{R}^{[\alpha]} \neq \varnothing$, respectively) is true, which proves the stated assertion.

3. If $x \in X$ and $\alpha' \nleq \alpha$, then the inclusion $x\mathcal{R}^{[\alpha']} \subseteq x\mathcal{R}^{(\alpha)}$ is easily proved. Then, by assumption, the implication $x \in X \Rightarrow \left|x\mathcal{R}^{[\alpha']}\right| \leq \left|x\mathcal{R}^{(\alpha)}\right| \leq 1$ is true.

4. The assumption and the inclusion considered in the item (3) show that the implication $x \in X \Rightarrow x\mathcal{R}^{(\alpha'')} \supseteq x\mathcal{R}^{[\alpha]} \neq \varnothing$ is true.

□

As a consequence we can remark that if $\mathcal{R}$ is a function then it is (α)-univocal and $[\alpha]$-total, for every $\alpha \in L$, $[\gamma]$-univocal for every $\gamma \neq \bot$ and (β)-total, for every $\beta \neq \top$.
It is easy to see that in case L is a chain every $[\alpha]$-univocal L-relation is (α)-univocal and every (α)-total L-relation is $[\alpha]$-total, while the converse need not be true. In the general case, (α)-univocality((α)-totality, respectively) and $[\alpha]$-univocality ($[\alpha]$-totality, respectively) are independent of each other, as the following Example 3.3.4 shows.

Example 3.3.4. Let X, Y be non-empty sets, assume $\alpha, \beta \in L$ be incomparable and consider the L-relation $\mathcal{R}$ from X to Y with the constant value α. Then, clearly, $\mathcal{R}$ is $[\alpha]$-total and (β)-total but it is neither (α)-total nor $[\beta]$-total.
If Y has at least two elements, then $\mathcal{R}$ is (α)-univocal and $[\beta]$-univocal, but it is neither $[\alpha]$-univocal nor (β)-univocal.

There are several ways to extend the concept of function within the context of relations between two sets X and Y, with values in a complete lattice L, which depend either on some additional tools on the sets X and Y or on some lattice-ordered algebraic structure on L. We shall consider only the case when X and Y are nothing but simply sets, while L is a complete lattice, possibly with a structure $(L, \to, \top)$ of right distributive ceo algebra.
The first extension of functions we consider in this Subsection, is based on the above generalizations of univocality and totality and requires only the lattice structure on L. In the next subsections we shall consider further possibilities assuming $(L, \to, \top)$ to be a right-distributive ceo algebra.

Definition 3.3.5. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation and let $\alpha \in L$.*

1. *$\mathcal{R}$ is (α)-**functional**, $\alpha \neq \top$, if it is (α)-univocal and (α)-total.*
2. *$\mathcal{R}$ is $[\alpha]$-**functional**, $\alpha \neq \bot$, if it is $[\alpha]$-univocal and $[\alpha]$-total.*

We note that $\bot$-functional and $\top$-functional L-relations considered in [35] are, in fact, $(\bot)$-functional and $[\top]$-functional, respectively. $(\bot)$-functional L-relations are called simply **functional** in [45], a term that we shall use for a different notion in the next Subsection.

Example 3.3.6. No relationship between (α)-functionality and $[\alpha]$-functionality exists in general. In fact, for X non-empty set and $\alpha, \beta \in L$ incomparable, the L-relation $\mathcal{R}$ from X to Y defined by $\mathcal{R} = \alpha_{X\times X} \wedge \mathcal{I}_X$ is $[\alpha]$-functional and (β)-functional but it is neither (α)-functional nor $[\beta]$-functional.

It is clear that for every L-relation $\mathcal{R}$ and for all $\alpha \neq \top$, $\beta \neq \bot$, if $\mathcal{R}$ is a function, then it is (α)-functional and $[\beta]$-functional; so, the latter notions are more general than functions. For crisp L-relations, each of (α)-functionality, $\alpha \neq \top$, and $[\beta]$-functionality, $\beta \neq \bot$, characterizes functions but, in general, there is no $\alpha \in L$ such that either (α)-functional or $[\alpha]$-functional L-relations need to be functions. In fact, assuming $X \neq \varnothing$, $L \neq \{\bot, \top\}$ and $\bot \neq \alpha \neq \top$, $\alpha_{X\times X} \wedge \mathcal{I}_X$ is $[\alpha]$-functional while $\alpha_{X\times X} \vee \mathcal{I}_X$ is (α)-functional, but none of them is a function.
Even more, the case $|L| = 3$ shows easily that $\mathcal{R}$ may be (α)-functional, for every $\alpha \neq \top$, and $[\beta]$-functional, for every $\beta \neq \bot$ without being a function.
So, the notion of function cannot be gotten as a particular case of any of the notions considered in Definition 3.3.5, unless $L = \{\bot, \top\}$. It seems that a better suited approximate notion of function is provided by the following Definition.

Definition 3.3.7. *Let X, Y be arbitrary sets and $\alpha, \beta \in L$, $\beta \nleq \alpha$. The L-relation $\mathcal{R} : X \times Y \to L$ is α-β* ***functional*** *if it is (α)-univocal and $[\beta]$-total.*

Of course functions are just $\bot$-$\top$ functional relations; so, the above Definition gives a generalization of functions which, moreover, is proper; in fact, the L-relation $\alpha_{X\times X} \wedge \mathcal{I}_X$ of Example 3.3.6 is β-α functional but not a function. Also note that every α-β functional L-relation is (α)-functional and $[\beta]$-functional, as a consequence of Lemma 3.3.3.
The following results give further explanation of the feature of α-β functionality.

Proposition 3.3.8. *Let us consider $\mathcal{R} : X \times Y \to L$. Then:*

1. *If $\mathcal{R}$ is α-β functional, $\alpha \leq \alpha', \beta' \leq \beta, \beta' \nleq \alpha'$, then $\mathcal{R}$ is α'-β' functional.*
2. *If $\mathcal{R}$ is a function, $\beta \nleq \alpha$, then $\mathcal{R}$ is α-β functional.*
3. *For every $(\alpha, \beta) \in L^2 \backslash \{(\bot, \top)\}$, $\beta \nleq \alpha$, there exist α-β functional L-relations that are not functions.*

Proof. Item (1) follows easily from Lemma 3.3.3 and consequently (2) holds, too, since a function is $\bot$-$\top$ functional.
As for (3), if $X \neq \varnothing$ and $\beta \neq \top$, then $\beta_{X\times X} \wedge \mathcal{I}_X$ is α-β functional but not a function, while, in case $\alpha \neq \bot$, $\alpha_{X\times X} \vee \mathcal{I}_X$ is, once more, α-β functional but not a function. □

The functionality conditions considered in this Subsection characterize L-relations having a suitable approximation that is a function, as follows.

Definition 3.3.9. *Let $\alpha, \beta \in L$, $\beta \nleq \alpha$, $\mathcal{R} : X \times Y \to L$. The (α)-**approximation** $\mathcal{R}_{(\alpha)}$, the $[\beta]$-**approximation** $\mathcal{R}_{[\beta]}$ and the α-β **approximation** $\mathcal{R}_{\alpha-\beta}$ of $\mathcal{R}$ are defined by*

$$(x,y)\mathcal{R}_{(\alpha)} = \begin{cases} \bot & \textit{if } (x,y)\mathcal{R} \leq \alpha \\ \top & \textit{otherwise} \end{cases}$$

$$(x,y)\mathcal{R}_{[\beta]} = \begin{cases} \top & \textit{if } (x,y)\mathcal{R} \geq \beta \\ \bot & \textit{otherwise} \end{cases}$$

$$(x,y)\mathcal{R}_{\alpha-\beta} = \begin{cases} \bot & \textit{if } (x,y)\mathcal{R} \leq \alpha \\ \top & \textit{if } (x,y)\mathcal{R} \geq \beta \\ (x,y)\mathcal{R} & \textit{otherwise} \end{cases}$$

It is easy to prove how the above defined approximations are related to the considered functionality conditions.

Proposition 3.3.10. *For any $\mathcal{R} : X \times Y \to L$ the following equivalences hold, for all $\alpha, \beta \in L$, $\beta \nleq \alpha$:*

1. *$\mathcal{R}$ is (α)-functional if and only if $\mathcal{R}_{(\alpha)}$ is a function;*
2. *$\mathcal{R}$ is $[\beta]$-functional if and only if $\mathcal{R}_{[\beta]}$ is a function;*
3. *$\mathcal{R}$ is α-β functional if and only if $\mathcal{R}_{\alpha-\beta}$ is a function.* □

3.3.2 Functionality by composition

The characterization of classical relations that are functions, by means of composition (see Proposition 1.4.8), motivates another way to generalize the notion of function (we explain details in Definition 3.3.11, below). To this extent, in this Subsection we consider L-relations taking values in a right-distributive ceo algebra. Further conditions considered on L, when needed, will be explicitly marked.

Definition 3.3.11. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation.*

1. *$\mathcal{R}$ is **(right) univocal** if $\mathcal{R}_{-} \odot \mathcal{R} \leq \mathcal{I}_Y$.*
2. *$\mathcal{R}$ is **(left) total** if $\mathcal{R} \odot \mathcal{R}_{-} \geq \mathcal{I}_X$.*
3. *$\mathcal{R}$ is **functional** if it is (right) univocal and (left) total.*

Left univocal and **right total** L-relations can be defined interchanging $\mathcal{R}$ and $\mathcal{R}_{-}$; for the use of the attributes left and right, or, rather, for their omission, we refer to the discussion expressed after Definition 3.3.1.
Functional L-relations above defined, as particular morphisms of Dedekind or Goguen categories, are called functions in [37, 87]; as remarked soon after Definition 3.3.1, we do not follow such a terminology.

Lemma 3.3.12. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation.*

1. *$\mathcal{R}$ is univocal if and only if the implication*

$$x \in X,\ y, y' \in Y, y \neq y' \Rightarrow (x,y)\mathcal{R} \otimes (x,y')\mathcal{R} = \bot$$

is true;

2. *$\mathcal{R}$ is total if and only if the implication*

$$x \in X \Rightarrow \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (x,y)\mathcal{R} = \top$$

is true.

Proof. We note that an L-relation $\mathcal{R}$ from X to Y, including the trivial cases when either X or Y is empty, is univocal if and only if the implication

$$y, y' \in Y, y \neq y' \Rightarrow (y,y')(\mathcal{R}_{-} \odot \mathcal{R}) = \bot$$

is true, while $\mathcal{R}$ is total if and only if the implication

$$x \in X \Rightarrow (x,x)(\mathcal{R} \odot \mathcal{R}_{-}) = \top$$

is true.
Hence, the stated conditions follow easily since $(y,y')(\mathcal{R}_{-} \odot \mathcal{R}) = \bigvee_{x \in X} (x,y)\mathcal{R} \otimes (x,y')\mathcal{R}$, for all $y, y' \in Y$ and $(x,x)(\mathcal{R} \odot \mathcal{R}_{-}) = \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (x,y)\mathcal{R}$, for every $x \in X$. □

Similar characterizations as in Lemma 3.3.12 can be proved for right totality and left univocality, switching the above conditions from $\mathcal{R}$ to $\mathcal{R}_-$.

Remark 3.3.13. Of course, every function is a functional L-relation and every functional crisp L-relation is a function, but there are functional L-relations that are not functions as the following Example 3.3.14 shows.

Example 3.3.14. Consider the sets $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the boolean algebra $L = \{\bot, a, b, \top\}$. Let $\mathcal{R} : X \times Y \to L$ be the L-relation described as follows: $(x_1, y_1)\mathcal{R} = (x_2, y_2)\mathcal{R} = a$, $(x_1, y_2)\mathcal{R} = (x_2, y_1)\mathcal{R} = b$. $\mathcal{R}$ is a functional L-relation, but it is neither $[\top]$-total nor $(\bot)$-univocal. This example also shows that a functional L-relation need not be either $(\bot)$-univocal or $[\top]$-total.

To give a characterization of functions, involving functional L-relations, we prove the following results, for any L-relation $\mathcal{R} : X \times Y \to L$.

Lemma 3.3.15. *$\mathcal{R}$ is $(\bot)$-univocal $\Rightarrow$ $\mathcal{R}$ is univocal $\Rightarrow$ for every $\alpha \in L$, $\alpha \otimes \alpha \neq \bot$, $\mathcal{R}$ is $[\alpha]$-univocal.*

Proof. From $(\bot)$-univocality of $\mathcal{R}$, if $y, y' \in Y$, $y \neq y'$, then, in case $X \neq \varnothing$, for every $x \in X$, either $(x, y)\mathcal{R} = \bot$ or $(x, y')\mathcal{R} = \bot$, so $(y, y')(\mathcal{R}_- \odot \mathcal{R}) = \bigvee_{x \in X}(x, y)\mathcal{R} \otimes (x, y')\mathcal{R} = \bot$; in case $X = \varnothing$, $(y, y')(\mathcal{R}_- \odot \mathcal{R}) = \bigvee_{x \in \varnothing}(y, x)\mathcal{R}_- \otimes (x, y')\mathcal{R} = \bigvee \varnothing = \bot$. Hence in any case $\mathcal{R}_- \odot \mathcal{R} \leq \mathcal{I}_Y$.
Assuming $\mathcal{R}$ to be not $[\bar{\alpha}]$-univocal, for some $\bar{\alpha} \in L$, there exist $\bar{x} \in X$, $y, y' \in Y$, $y \neq y'$, such that $(\bar{x}, y)\mathcal{R} \wedge (\bar{x}, y')\mathcal{R} \geq \bar{\alpha}$; then, since $\mathcal{R}$ is univocal, $\bot = (y, y')(\mathcal{R}_- \odot \mathcal{R}) \geq (\bar{x}, y)\mathcal{R} \otimes (\bar{x}, y')\mathcal{R} \geq \bar{\alpha} \otimes \bar{\alpha}$. So, the second implication holds, too. $\square$

Example 3.3.16. 1. Let $L = [0, 1]$ be the Łukasievicz algebra (i.e. $\alpha \to \beta = max\{\alpha + \beta - 1, 0\}$, and $\alpha \otimes \beta = min\{1 - \alpha + \beta, 1\}$, for all $\alpha, \beta \in [0, 1]$), $X \neq \varnothing$ and $|Y| > 1$. Consider the L-relation $\mathcal{R} : X \times Y \to L$ with the constant value $\frac{1}{2}$. $\mathcal{R}$ is univocal, but it is not $(\bot)$-univocal. We note, moreover, that $\mathcal{R}$ is $[\alpha]$-univocal, for every $\alpha \in L$, $\alpha \otimes \alpha \neq \bot$ (i.e. for every $\alpha > \frac{1}{2}$), but it is not $[\alpha']$-univocal, if $\alpha' \leq \frac{1}{2}$.
2. Let L be the Łukasievicz algebra, $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$. Consider the L-relation $\mathcal{R} : X \times Y \to L$ such that $(x_1, y_1)\mathcal{R} = (x_2, y_2)\mathcal{R} = \top$ and $(x_1, y_2)\mathcal{R} = (x_2, y_1)\mathcal{R} = \frac{1}{2}$. $\mathcal{R}$ is $[\alpha]$-univocal, for every $\alpha \in L$, $\alpha \otimes \alpha \neq \bot$, but it is not univocal.

Lemma 3.3.17. *$\mathcal{R}$ is $[\top]$-total $\Rightarrow$ $\mathcal{R}$ is total $\Rightarrow$ for every $\alpha \neq \top$, $\mathcal{R}$ is (α)-total.*

Proof. For the first implication, note that, by assumption, if $x \in X$, then there exists $y_x \in Y$ such that $(x, y_x)\mathcal{R} = \top$; then one has that: $(x, x)\mathcal{R} \odot \mathcal{R}_- = \bigvee_{y \in Y}(x, y)\mathcal{R} \otimes (y, x)\mathcal{R}_- \geq (x, y_x)\mathcal{R} \otimes (x, y_x)\mathcal{R} = \top$; hence $\mathcal{R}$ is total.
Now, in case $\mathcal{R}$ is not $(\bar{\alpha})$-total, for some $\bar{\alpha} \in L$, one can find $\bar{x} \in X$ such that $(\bar{x}, y)\mathcal{R} \leq \bar{\alpha}$, for every $y \in Y$. As a consequence, by assumption one has $\top = (\bar{x}, \bar{x})(\mathcal{R} \odot \mathcal{R}_-) = \bigvee_{y \in Y}(\bar{x}, y)\mathcal{R} \otimes (\bar{x}, y)\mathcal{R} \leq \bar{\alpha} \otimes \bar{\alpha} \leq \bar{\alpha}$. So, the second implication is true. $\square$

Example 3.3.18. 1. Let $X = \{x\}$, $Y = \mathbb{N}$ and $L = [0,1]$ with the product algebra structure (i.e. $\alpha \to \beta = \begin{cases} 1 & if\ \alpha \leq \beta \\ \frac{\beta}{\alpha} & otherwise \end{cases}$ and $\alpha \otimes \beta = \alpha \cdot \beta$, for all $\alpha, \beta \in [0,1]$). Consider the L-relation $\mathcal{R} : X \times Y \to L$ defined by $(x,n)\mathcal{R} = \frac{n}{n+1}$, for $n \in \mathbb{N}$.
One has $\bigvee_{n\in\mathbb{N}} (x,n)\mathcal{R} \otimes (x,n)\mathcal{R} = \bigvee_{n\in\mathbb{N}} \frac{n}{n+1} \cdot \frac{n}{n+1} = 1$, hence $\mathcal{R}$ is total, by Lemma 3.3.12.
$\mathcal{R}$ is not $[1]$-total, since $\{n \in \mathbb{N} |\ (x,n)\mathcal{R} \geq 1\} = \left\{n \in \mathbb{N} |\ \frac{n}{n+1} \geq 1\right\} = \varnothing$.
2. Let $X = \{x\}$, $Y = \{y_1, y_2\}$ and let $L = \{\bot, a, b, \top\}$ be the right-distributive ceo algebra with implication and product described in 19 of Appendix. Consider the L-relation $\mathcal{R} : X \times Y \to L$ such that $(x,y_1)\mathcal{R} = a$ and $(x,y_2)\mathcal{R} = b$. $\mathcal{R}$ is (α)-total, for every $\alpha \neq \top$, but it is not total.

Remark 3.3.19. A necessary condition for an L-relation $\mathcal{R} : X \times Y \to L$, with X and Y non-empty, to be (α)-total, for every $\alpha \in L$, $\alpha \neq \top$, is the following

$$\forall x \in X : \bigvee_{y\in Y} (x,y)\mathcal{R} = \top;$$

in fact, if $x \in X$, no $\alpha \in L$, $\alpha \neq \top$, can be an upper bound for the set $\{(x,y)\mathcal{R}|\ y \in Y\}$, since, by the assumption, $y_{x,\alpha}$ exists such that $(x, y_{x,\alpha})\mathcal{R} \nleq \alpha$; note also that, to the same extent, the equality $\bigvee L \backslash \{\top\} = \top$ is needed, under the further assumption for $\mathcal{R}$ to be not $[\top]$-total.
If the lattice L satisfies the implication

$$S \subseteq L,\ \bigvee S = \top \Rightarrow \bigvee_{a\in S} a \otimes a = \top,$$

the second implication in Lemma 3.3.17 is reversible; in such a case in fact, for every $x \in X$, one has $\bigvee_{y\in Y} (x,y)\mathcal{R} = \top \Rightarrow \bigvee_{y\in Y} (x,y)\mathcal{R} \otimes (x,y)\mathcal{R} = \top$, which implies that $\mathcal{R}$ is total, if it is (α)-total, for every $\alpha \in L$, $\alpha \neq \top$.

Now, the following characterization is an immediate consequence.

Proposition 3.3.20. *For any L-relation $\mathcal{R} : X \times Y \to L$, the following are equivalent.*

(i) $\mathcal{R}$ is a function;

(ii) $\mathcal{R}$ is univocal and $[\top]$-total;

(iii) $\mathcal{R}$ is $(\bot)$-univocal and total.

Proof. Since (i) clearly implies both (ii) and (iii), we only need to prove the converse implications; in both cases, by Remark 3.3.13 and the first implication of Lemmas 3.3.15 and 3.3.17 it is enough to prove that $\mathcal{R}$ is crisp, whenever $X \neq \varnothing$.
So, assume $\mathcal{R}$ univocal and $[\top]$-total, hence for every $x \in X$ there exists $y_x \in Y$ such that $(x, y_x)\mathcal{R} = \top$; then, for every $y' \in Y$, $y' \neq y_x$, it follows from $\mathcal{R}_- \odot \mathcal{R} \leq \mathcal{I}_Y$ that $\bot = \bigvee_{x'\in X} (x', y')\mathcal{R} \otimes (x', y_x)\mathcal{R} \geq (x,y')\mathcal{R} \otimes (x,y_x)\mathcal{R} = (x,y')\mathcal{R} \otimes \top = (x,y')\mathcal{R}$; so, $(x,y)\mathcal{R} \in \{\bot, \top\}$, for all $x \in X, y \in Y$.
Now assume $\mathcal{R}$ $(\bot)$-univocal and total and let $x \in X$; then it follows from $(x,x)(\mathcal{R} \odot \mathcal{R}_-) =$

$\top$ that $Y \neq \emptyset$; now consider $y \in Y$ and assume $(x,y)\mathcal{R} \neq \bot$; since $\mathcal{R}$ is $(\bot)$-univocal, $(x,y')\mathcal{R} = \bot$, for every $y' \neq y$, and, by $\mathcal{R} \odot \mathcal{R}_{-} \geq \mathcal{I}_X$, one gets $\top = (x,x)(\mathcal{R} \odot \mathcal{R}_{-}) = [(x,y)\mathcal{R} \otimes (x,y)\mathcal{R}] \vee \left[\bigvee_{y \neq y' \in Y}(x,y')\mathcal{R} \otimes (x,y')\mathcal{R}\right] = (x,y)\mathcal{R} \otimes (x,y)\mathcal{R} \leq (x,y)\mathcal{R}$. Hence $\mathcal{R}$ is crisp. □

Remark 3.3.21. The first implication in Lemma 3.3.15 and in Lemma 3.3.17, as well, still hold under the weaker assumption for L to be a right-distributive w-ceo algebra.

Now, we consider a further composition of relations, just defined in Subsection 3.1.2, according to what is given in the context of Dedekind Categories in [37], with respect to the L-relations taking values in a complete Heyting algebra. However, such a composition can be considered in case L is any complete lattice. Recall its definition.

Definition 3.3.22. *Let L be a complete lattice. For all $\mathcal{R} : X \times Y \to L$, $\mathcal{S} : Y \times Z \to L$ define $\mathcal{R} \cdot \mathcal{S} : X \rightharpoonup Z$ by setting $(x,z)(\mathcal{R} \cdot \mathcal{S}) = \bigvee_{y \in Y}(x,y)\mathcal{R} \wedge (y,z)\mathcal{S}$, for all $x \in X, y \in Y, z \in Z$.*

All the notions and statements of this Subsection can be rephrased with respect to the composition $\cdot$; in fact one can call $\mathcal{R}$ ***$\wedge$-univocal*** or ***$\wedge$-total*** if $\mathcal{R}_{-} \cdot \mathcal{R} \leq \mathcal{I}_Y$ or $\mathcal{R} \cdot \mathcal{R}_{-} \geq \mathcal{I}_X$, respectively, and say that $\mathcal{R}$ is ***$\wedge$-functional*** if it is ***$\wedge$-univocal*** and ***$\wedge$-total***.
All the results already obtained in this Subsection by means of the composition $\odot$ still hold for these new notions for any complete lattice L; even more, the restriction on α in Lemma 3.3.15 can be relaxed to $\alpha \neq \bot$. In particular, we note that if L is any complete lattice, the following holds.

Proposition 3.3.23. *If L is a complete lattice, for any L-relation $\mathcal{R}$ the following are equivalent*

1. *$\mathcal{R}$ is a function;*
2. *$\mathcal{R}$ is $\wedge$-univocal and $[\top]$-total;*
3. *$\mathcal{R}$ is $(\bot)$-univocal and $\wedge$-total.* □

For relations with values in a right-distributive w-ceo algebra, univocality, totality, functionality can be considered together with their $\wedge$-related versions . Of course, in case of relations with values in a complete Heyting algebra, the compositions $\cdot$ and $\odot$ coincide (we get new notions only when $\otimes \neq \wedge$). We clarify relationships of these different notions.

First we note that, since $a \otimes a \leq a$, for every $a \in L$, in any right-distributive w-ceo algebra L, one has that if $\mathcal{R}$ is total, then $\mathcal{R}$ is $\wedge$-total, for every $\mathcal{R} : X \times Y \to L$.
Moreover, if L is a symmetrical cdeo algebra, then the inequality $\alpha \otimes \beta \leq \alpha \wedge \beta$ holds, for all $\alpha, \beta \in L$, which implies clearly that if $\mathcal{R}$ is $\wedge$-univocal then $\mathcal{R}$ is univocal, for every $\mathcal{R} : X \times Y \to L$.
However, in general, there is no relationship between functionality and $\wedge$-functionality.

Example 3.3.24. 1. Example 3.3.18 (2) provides an L-relation that is $\wedge$-functional but not functional.

2. A functional L-relation that is not $\wedge$-functional can be obtained as follows. Consider $L = \{\bot, a, b, c, \top\}$ with the structure of right-distributive w-ceo algebra described in 20 of Appendix and consider $X = \{x\}$, $Y = \{y_1, y_2\}$. It is easy to check, by means of Lemma 3.3.12, that the L-relation $\mathcal{R} : X \times Y \to L$ such that $(x, y_1)\mathcal{R} = b$ and $(x, y_2)\mathcal{R} = c$ is functional; instead, it is not $\wedge$-functional since $(x, y_1)R \wedge (x, y_2)R = a \neq \bot$, though it is $\wedge$-total.

The above discussion allows to slightly improve the results of Propositions 3.3.20 and 3.3.23.

Proposition 3.3.25. *If L is a right-distributive ceo algebra, for any L-relation $\mathcal{R}$ the following are equivalent*

1. *$\mathcal{R}$ is a function;*
2. *$\mathcal{R}$ is univocal and $[\top]$-total;*
3. *$\mathcal{R}$ is $(\bot)$-univocal and $\wedge$-total.* □

3.3.3 Power-functionality

In this Subsection we consider L-relations taking values in a symmetrical cdeo algebra, unless otherwise stated. Further conditions on L, when needed, will be explicitly marked.
In fact, we need two adjoint operations $\otimes$ and $\to$, possibly with the properties they have in a symmetrical cdeo algebra. So, the notions treated in this Subsection, unlike the corresponding one considered in Subsections 3.3.1 and 3.3.2, cannot be expressed in terms of the complete lattice structure only, unless the meet operation distributes over arbitrary joins; in this last case, the complete lattice is a frame (according to the terminology of [59]) and the residual implication $\to$ of the meet operation $\wedge$ gives a structure of Heyting algebra, i.e. of a commutative (hence symmetrical) cdeo algebra, with adjoint product $\wedge$, which is, moreover, associative and idempotent.
Most result we shall obtain, however, do not require either idempotency or associativity, which motivates the more general framework we consider.
The following result for crisp L-relations, which can be found in [35], motivate a further generalization of the notions of univocality, totality and functionality.

Proposition 3.3.26. *[35] Let $(L, \to, \top)$ be a symmetrical cdeo algebra. If $\mathcal{R} : X \times Y \to L$ is a crisp L-relation, then the following equivalences hold.*

1. *$\mathcal{R}$ is univocal if and only if $\mathcal{R}^{\otimes}_{-} \leq \mathcal{R}^{\to}_{-}$;*
2. *$\mathcal{R}$ is total if and only if $\mathcal{R}^{\otimes}_{-} \geq \mathcal{R}^{\to}_{-}$;*
3. *$\mathcal{R}$ is a function if and only if $\mathcal{R}^{\otimes}_{-} = \mathcal{R}^{\to}_{-}$.* □

Hence, taking into account the above results, we give the following definition.

Definition 3.3.27. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation.*

1. *$\mathcal{R}$ is **right (left) power-univocal** if $\mathcal{R}^{\otimes}_{-} \le \mathcal{R}^{\to}_{-}$ ($\mathcal{R}^{\otimes} \le \mathcal{R}^{\to}$).*

2. *$\mathcal{R}$ is **left (right) power-total** if $\mathcal{R}^{\otimes}_{-} \ge \mathcal{R}^{\to}_{-}$ ($\mathcal{R}^{\otimes} \ge \mathcal{R}^{\to}$).*

3. *$\mathcal{R}$ is **power-functional** if $\mathcal{R}^{\otimes}_{-} = \mathcal{R}^{\to}_{-}$.*

Once more, we are interested and may reduce to consider only right power-univocal and left power-total L-relations, for which, as stated after Definition 3.3.1, we shall also omit the attributes right and left.
We shall see that these notions play the same role as those introduced in Definition 3.3.11, to which they are somehow related, in the characterization of functions among L-relations. We start showing a general relationship.

Proposition 3.3.28. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation.*

1. *If $\mathcal{R}$ is power-univocal, then $\mathcal{R}$ is univocal.*

2. *If $\mathcal{R}$ is power-total, then $\mathcal{R}$ is total.*

3. *If $\mathcal{R}$ is power-functional, then $\mathcal{R}$ is functional.*

Proof. 1. If $x \in X$, $y_1, y_2 \in Y$, $y_1 \ne y_2$, then applying the assumption $\mathcal{R}^{\otimes}_{-} \le \mathcal{R}^{\to}_{-}$ to the crisp L-point $\top_{y_2}$, one has the following:
$(x, y_2)\mathcal{R} = \bigvee_{y \in Y}(x, y)\mathcal{R} \otimes (y)\top_{y_2} \le \bigwedge_{y \in Y}(x, y)\mathcal{R} \to (y)\top_{y_2} \le (x, y_1)\mathcal{R} \to (y_1)\top_{y_2} = (x, y_1)\mathcal{R} \to \bot$; hence $(x, y_1)\mathcal{R} \otimes (x, y_2)\mathcal{R} = \bot$ and it follows by arbitrariness of x, y_1, y_2 and Lemma 3.3.12 that $\mathcal{R}$ is univocal.

2. If $\mathcal{R}$ is power-total, then for all $B \in L^Y$, $x \in X$, one has that: $\bigvee_{y \in Y}(x, y)\mathcal{R} \otimes (y)B \ge \bigwedge_{y \in Y}(x, y)\mathcal{R} \to (y)B$.
If, in particular, for every $x \in X$, we consider the L-set $B_x : Y \to L$ defined by: $(y)B_x = (x, y)\mathcal{R}$, for every $y \in Y$, then $\bigvee_{y \in Y}(x, y)\mathcal{R} \otimes (y)B_x \ge \bigwedge_{y \in Y}(x, y)\mathcal{R} \to (y)B_x$, that is $\bigvee_{y \in Y}(x, y)\mathcal{R} \otimes (x, y)\mathcal{R} \ge \bigwedge_{y \in Y}(x, y)\mathcal{R} \to (x, y)\mathcal{R}$; hence $\bigvee_{y \in Y}(x, y)\mathcal{R} \otimes (x, y)\mathcal{R} \ge \top$ for every $x \in X$ and then, from Lemma 3.3.12, it follows that $\mathcal{R}$ is total.

3. The stated implication follows trivially from (1) and (2).

□

Note that the statement and the proof of the above Proposition do not require the symmetry assumption, but only require that L be a right-distributive ceo algebra. Under such an assumption, the converse implication of that given in the item (3) of the above Proposition does not hold, as the following example shows.

Example 3.3.29. Let $(L, \to, \top)$ be the right-distributive ceo algebra with the set $L = \{\bot, b_1, b_2, b_3, c_1, c_2, d, \top\}$, implication and adjoint product described in 21 of Appendix; consider $X = \{x\}$, $Y = \{y_1, y_2\}$. The L-relation $\mathcal{R} : X \times Y \to L$ such that $(x, y_1)\mathcal{R} = b_1$ and $(x, y_2)\mathcal{R} = b_3$ is functional, but it is not power-functional; in fact, for $B \in L^Y$, $(y_1)B = (y_2)B = b_2$, one has $(x)(B)\mathcal{R}^{\otimes}_{-} = (b_1 \otimes b_2) \vee (b_3 \otimes b_2) = b_1 \vee b_3 = \top$ and $(x)(B)\mathcal{R}^{\to}_{-} = (b_1 \to b_2) \wedge (b_3 \to b_2) = b_3 \wedge b_1 = \bot$, hence R is not power-univocal.

The following statement strengthens some results of [35] and of Lemmas 3.3.15, 3.3.17.

Lemma 3.3.30. *With the above notation, the following results hold:*

1. *$\mathcal{R}$ is $(\bot)$-univocal $\Rightarrow$ $\mathcal{R}$ is power-univocal $\Rightarrow$ $\mathcal{R}$ is univocal $\Rightarrow$ for every $\alpha \in L$, $\alpha \otimes \alpha \neq \bot$, $\mathcal{R}$ is $[\alpha]$-univocal.*

2. *$\mathcal{R}$ is $[\top]$-total $\Rightarrow$ $\mathcal{R}$ is power-total $\Rightarrow$ $\mathcal{R}$ is total $\Rightarrow$ for every $\alpha \in L$, $\alpha \neq \top$, $\mathcal{R}$ is (α)-total.*

Proof.

1. The first implication is Proposition 8.5 in [35]; the second implication is Proposition 3.3.28(1); the third implication is in Lemma 3.3.15.

2. The first implication is Proposition 8.11 in [35]; the second implication is Proposition 3.3.28(2); the third implication is in Lemma 3.3.17.

□

Remark 3.3.31. The results of Proposition 3.3.26 and pieces of Lemma 3.3.30 are proved in [35] assuming L to be a complete residuated lattice (i.e. a commutative, associative cdeo algebra); however, their proofs do not need associativity, while symmetry replaces commutativity.

Now a similar result as in Proposition 3.3.20 holds trivially, using Propositions 3.3.26, 3.3.28; recall that, now, L is assumed to be symmetrical.

Corollary 3.3.32. *For any L-relation $\mathcal{R} : X \times Y \to L$, the following are equivalent.*

1. *$\mathcal{R}$ is a function;*

2. *$\mathcal{R}$ is $(\bot)$-univocal and power-total;*

3. *$\mathcal{R}$ is power-univocal and $[\top]$-total.* □

If L is an associative symmetrical cdeo algebra, the reciprocal relationships of those described in Proposition 3.3.28 hold, too.

Proposition 3.3.33. *Let L be an associative symmetrical cdeo algebra, $\mathcal{R} : X \times Y \to L$ an L-relation. The following hold:*

1. *if $\mathcal{R}$ is univocal, then $\mathcal{R}$ is power-univocal;*

2. *if $\mathcal{R}$ is total, then $\mathcal{R}$ is power-total.*

Proof. 1. In order to prove that, for all $x \in X$, $B \in L^Y$, $(x)(B)\mathcal{R}^{\otimes}_{-} \leq (x)(B)\mathcal{R}^{\rightarrow}_{-}$ it is enough to verify that for all $x \in X, y, y' \in Y$ $(x,y)\mathcal{R} \otimes (y)B \leq (x,y')\mathcal{R} \to (y')B$. If $y = y'$, then for every $x \in X$ the inequality $(x,y)\mathcal{R} \otimes (y)B \leq (x,y)\mathcal{R} \to (y)B$ follows from Proposition 2.1.30 (1). If $y \neq y'$, by adjunction the following equivalence $(x,y)\mathcal{R} \otimes (y)B \leq (x,y')\mathcal{R} \to (y')B \Leftrightarrow (x,y')\mathcal{R} \otimes ((x,y)\mathcal{R} \otimes (y)B) \leq (y')B$ is true, for every $x \in X$.
Under associativity assumption on L and univocality of $\mathcal{R}$, it follows from Lemma 3.3.12 (1) that $(x,y')\mathcal{R} \otimes ((x,y)\mathcal{R} \otimes (y)B) = ((x,y')\mathcal{R} \otimes (x,y)\mathcal{R}) \otimes (y)B = \bot \otimes (y)B = \bot \leq (y')B$, for every $x \in X$, thence the assertion follows.

2. To prove that for all $x \in X$, $B \in L^Y$, $(x)(B)\mathcal{R}^{\rightarrow}_{-} \leq (x)(B)\mathcal{R}^{\otimes}_{-}$, we verify that if $x \in X$, $B \in L^Y$, $k \in L$, then $k \leq (x)(B)\mathcal{R}^{\rightarrow}_{-} \Rightarrow k \leq (x)(B)\mathcal{R}^{\otimes}_{-}$. So, let $k \in L$ and $B \in L^Y$; for every $x \in X$ one has that $k \leq (x)(B)\mathcal{R}^{\rightarrow}_{-} \Leftrightarrow k \leq (x,y)\mathcal{R} \to (y)B$, for every $y \in Y \Leftrightarrow (x,y)\mathcal{R} \otimes k \leq (y)B$, for every $y \in Y$; then, by associativity and symmetry of L and totality of $\mathcal{R}$, from Lemma 3.3.12 (2) one has, for every $x \in X$: $(x)(B)\mathcal{R}^{\otimes}_{-} = \bigvee_{y \in Y}(x,y)\mathcal{R} \otimes (y)B \geq \bigvee_{y \in Y}(x,y)\mathcal{R} \otimes [(x,y)\mathcal{R} \otimes k] = \left[\bigvee_{y \in Y}(x,y)\mathcal{R} \otimes (x,y)\mathcal{R}\right] \otimes k = \top \otimes k = k$.

□

Of course, every functional L-relation is power-functional, if L is an associative, symmetrical cdeo algebra, as a consequence of the above Proposition. Nevertheless, we give an alternative, more elegant proof of this relationship, using directly the properties of powerset operators described in Section 3.2; this makes the proof holds in an abstract approach to relations as in [37, 87].

Proposition 3.3.34. *Let L be an associative symmetrical cdeo algebra, $\mathcal{R} : X \times Y \to L$ an L-relation. If $\mathcal{R}$ is functional, then ${}^{\otimes}\mathcal{R}_{-} = \mathcal{R}^{\rightsquigarrow}_{-}$ and ${}^{\tilde{\otimes}}\mathcal{R}_{-} = \mathcal{R}^{\rightarrow}_{-}$.*

Proof. By Proposition 3.2.22 and the inequalities $\mathcal{R} \odot \mathcal{R}_{-} \geq \mathcal{I}_X$ and $\mathcal{R}_{-} \odot \mathcal{R} \leq \mathcal{I}_Y$, for every $A \in L^X$, $B \in L^Y$ one has $({}^{\otimes}\mathcal{R} \odot^{\otimes} \mathcal{R}_{-})(A) \geq A$ and $({}^{\otimes}\mathcal{R}_{-} \odot^{\otimes} \mathcal{R})(B) \leq B$. Then, since both ${}^{\otimes}\mathcal{R}$ and ${}^{\otimes}\mathcal{R}_{-}$ are isotonic, it is easy to check that ${}^{\otimes}\mathcal{R} \dashv {}^{\otimes}\mathcal{R}_{-}$. On the other hand, by Corollary 3.2.18, ${}^{\otimes}\mathcal{R} \dashv \mathcal{R}^{\rightsquigarrow}_{-}$ and, by uniqueness of right adjoint, it follows that ${}^{\otimes}\mathcal{R}_{-} = \mathcal{R}^{\rightsquigarrow}_{-}$. The second equality can be proved in a similar way. □

The short discussion after Definitions 3.2.13 and 3.2.14 allows to state the following consequence.

Corollary 3.3.35. *Let L be an associative symmetrical cdeo algebra, $\mathcal{R} : X \times Y \to L$ an L-relation. If $\mathcal{R}$ is functional, then $\mathcal{R}^{\otimes}_{-} = \mathcal{R}^{\rightarrow}_{-}$, $\mathcal{R}^{\tilde{\otimes}}_{-} = \mathcal{R}^{\rightsquigarrow}_{-}$ and, in particular, $\mathcal{R}$ is power-functional.* □

We can further clarify relationships between the considered notions, under suitable assumptions on the lattice or on the algebraic structure of L.
To this extent we need to extend classical notions usually considered in complete lattices, as follows.

Definition 3.3.36. *Let* $(L, \rightarrow, \top)$ *be a right-distributive ceo algebra,* $a, b \in L$*. Then*

- a *is* $\otimes$***-prime*** *if, for all* $x, y \in L$ *such that* $x \otimes y = y \otimes x = a$*, one has either* $x = a$ *or* $y = a$*;*
- b *is* ***completely*** $\otimes$***-coprime*** *if, for every* $S \subseteq L$ *such that* $\bigvee_{s \in S} s \otimes s = b \Rightarrow b \in S$ *(recall that we assume* $\top \neq \bot$*).*

Note that for $\otimes = \wedge$ the above Definition gives classical, well known notions in (completely) distributive lattices. These generalizations are motivated by the following results.

Proposition 3.3.37. *Let* $\mathcal{R} : X \times Y \rightarrow L$ *be an* L*-relation. If* $\bot$ *is* $\otimes$*-prime, the following are equivalent:*

1. $\mathcal{R}$ *is univocal;*
2. $\mathcal{R}$ *is* $(\bot)$*-univocal;*
3. $\mathcal{R}$ *is power-univocal.*

Proof. By Lemma 3.3.30 (1) and Proposition 3.3.28 (1) we only need to prove the implication "(1) ⇒ (2)". If $\mathcal{R}$ is univocal, then, by Lemma 3.3.12 (2) the implication $x \in X$, $y, y' \in Y, y \neq y' \Rightarrow (x, y)\mathcal{R} \otimes (x, y')\mathcal{R} = (x, y')\mathcal{R} \otimes (x, y)\mathcal{R} = \bot$ is true. Since $\bot$ is $\otimes$-prime, it follows that, for all $x \in X$, $y, y' \in Y$, $y \neq y'$ either $(x, y)\mathcal{R} = \bot$ or $(x, y')\mathcal{R} = \bot$. Hence, for every $x \in X$: $\left|\mathcal{R}^{(\bot)} y\right| \leq 1$, that is $\mathcal{R}$ is $(\bot)$-univocal. □

Proposition 3.3.38. *Let* $\mathcal{R} : X \times Y \rightarrow L$ *be an* L*-relation. If* $\top$ *is completely* $\otimes$*-coprime, the following are equivalent:*

1. $\mathcal{R}$ *is total;*
2. $\mathcal{R}$ *is* $[\top]$*-total;*
3. $\mathcal{R}$ *is power-total.*

Proof. By Lemma 3.3.30 (2) and Proposition 3.3.28 (2) it is enough to prove "(1) ⇒ (2)". By assumption, from Lemma 3.3.30 (2) one has that $\mathcal{R}$ is (α)-total, $\forall \alpha \in L$, $\alpha \neq \top$; moreover, $\bigvee_{y \in Y} (x, y)\mathcal{R} \otimes (x, y)\mathcal{R} = \top$, for every $x \in X$. Since $\top$ is completely $\otimes$-coprime, for every $x \in X$ there exists $y_x \in Y$ such that $(x, y_x)\mathcal{R} = \top$; so the set $x\mathcal{R}^{[\alpha]}$ is non-empty and hence $\mathcal{R}$ is $[\top]$-total. □

Combining the above Propositions 3.3.37 and 3.3.38 with previous results one can obtain sufficient conditions on L under which power-functional and, possibly, functional L-relations are functions (for instance requiring either $\bot$ to be $\otimes$-prime or $\top$ to be completely $\otimes$-coprime and, possibly, L to be associative). However, better results can be obtained by means of the following Lemma (recall that an element a in a lattice L is said to be $\vee$-*irreducible* or *coprime*, if for all $x, y \in L$, if $x \vee y \geq a$, then either $x \geq a$ or $y \geq a$).

Lemma 3.3.39. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation. If $\top$ is coprime, then the following implication holds:*

$$\mathcal{R} \textit{ is power-functional} \Rightarrow \mathcal{R} \textit{ is a crisp } L\textit{-relation.}$$

Proof. Excluding the trivial cases, let us assume X, Y to be non-empty sets and $\mathcal{R}$ to be power-functional. Let $(\bar{x}, \bar{y})\mathcal{R} = \bar{b} \neq \top$ for some $\bar{x} \in X, \bar{y} \in Y$. Then by Remark 3.3.19, Lemma 3.3.30 (2) and by assumption that $\top$ is coprime, X has at least two elements and $\top = \bigvee_{y \in Y}(\bar{x}, y)\mathcal{R} = (\bar{x}, \bar{y})\mathcal{R} \vee \bigvee_{\bar{y} \neq y \in Y}(\bar{x}, y)\mathcal{R}$; since $\top$ is coprime, it follows that $\bigvee_{\bar{y} \neq y \in Y}(\bar{x}, y)\mathcal{R} = \top$.
Now consider the crisp L-point $\top_{\bar{y}}$; then

$$(x)(\top_{\bar{y}})\mathcal{R}^{\otimes}_{-} = \bigvee_{y \in Y}(\bar{x}, y)\mathcal{R} \otimes (y)\top_{\bar{y}} = (\bar{x}, \bar{y})\mathcal{R} \otimes \top = \bar{b}$$

and

$$\begin{aligned}(x)(\top_{\bar{y}})\mathcal{R}^{\to}_{-} &= \bigwedge_{y \in Y}(\bar{x}, y)\mathcal{R} \to (y)\top_{\bar{y}} = ((\bar{x}, \bar{y})\mathcal{R} \to \top) \wedge \bigwedge_{\bar{y} \neq y \in Y}((\bar{x}, y)\mathcal{R} \to \bot) \\ &= \top \wedge ((\bigvee_{\bar{y} \neq y \in Y}(\bar{x}, y)\mathcal{R}) \to \bot) = \top \wedge (\top \to \bot) = \bot.\end{aligned}$$

By the assumption, $\bar{b} = \bot$. □

Proposition 3.3.40. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation. If either $\bot$ is $\otimes$-prime or $\top$ is coprime, then*

$$\mathcal{R} \textit{ is power-functional} \Leftrightarrow \mathcal{R} \textit{ is a function.}$$

Proof. In the first case, by Corollary 3.3.32 and Proposition 3.3.37

$$\mathcal{R} \text{ is power-functional} \Leftrightarrow \mathcal{R} \text{ is } (\bot)\text{-univocal and power-total} \Leftrightarrow \mathcal{R} \text{ is a function.}$$

In the second case, the stated equivalence follows by Proposition 3.3.26 (3), Definition 3.3.27 (3) and Lemma 3.3.39. □

The above results allow to prove easily the following.

Corollary 3.3.41. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation. If $\bot$ is $\otimes$-prime and $\top$ is completely $\otimes$-coprime, then the following are equivalent:*

1. *$\mathcal{R}$ is a function;*
2. *$\mathcal{R}$ is functional;*
3. *$\mathcal{R}$ is power-functional.*

Proof. The implications "(1) ⇒ (2)" and "(1) ⇒ (3)" are obvious.
The converse implications "(2) ⇒ (1)" and "(3) ⇒ (1)" follows easily by Remark 3.3.2 and Propositions 3.3.37, 3.3.38. □

Corollary 3.3.42. *Let $\mathcal{R} : X \times Y \to L$ be an L-relation, L an associative symmetrical cdeo algebra. If either $\bot$ is $\otimes$-prime or $\top$ is coprime, then the following are equivalent:*

1. *$\mathcal{R}$ is a function;*
2. *$\mathcal{R}$ is functional;*
3. *$\mathcal{R}$ is power-functional.*

Proof. "(1) $\Rightarrow$ (2)" Obvious.
"(2) $\Rightarrow$ (3)" See Corollary 3.3.35.
"(3) $\Rightarrow$ (1)" See Corollary 3.3.40. □

3.3.4 Some remarks on functorial L-relations

To illustrate the usefulness of the characterizations stated in Corollary 3.3.42 we remark that the conditions there required are satisfied by all the so called *fuzzy relations* (i.e. relations with values in the unit interval $[0,1]$). In particular, we note the following.

- Every complete chain L is completely distributive hence, in particular, it is a complete Heyting algebra, i.e. an associative and commutative cdeo algebra. Such a structure is in fact the so called *Gödel algebra* in case $L = [0,1]$, with implication $a \to_G b = \begin{cases} 1 & if\ a \le b \\ b & otherwise \end{cases}$ and adjoint product $a \otimes_G b = a \wedge b$. Evidently 0 is $\otimes_G$-prime and 1 is coprime.

- The *product algebra* has the same carrier $[0,1]$; the implication $a \to_P b = \begin{cases} 1 & if\ a \le b \\ \frac{b}{a} & otherwise \end{cases}$ gives a structure of associative and commutative cdeo algebra whose adjoint product $\otimes_P$ is the ordinary product. This is not an Heyting algebra, but 0 is $\otimes_P$-prime and 1 is coprime in this case, too.

- The *Łukasievicz algebra* on $L = [0,1]$, with implication $a \to_L b = max\{a + b - 1, 0\}$, and adjoint product $a \otimes_L b = min\{1 - a + b, 1\}$, for all $\alpha, \beta \in [0,1]$, is, once more, an associative and commutative cdeo algebra, not an Heyting algebra. Of course, 1 is coprime though 0 is not $\otimes_L$-prime, in this case.

These are the algebraic structures frequently used for fuzzy relations; in all the three cases, functional and power-functional fuzzy relations are functions.
The equivalences of Proposition 3.3.37 are not true for relations taking values in the Łukasievicz algebra while, independently of which of the three algebraic structures is considered, total fuzzy relations need not be $[\top]$-total. Indeed, we can prove the following results, assuming $(L, \to, \top)$ to be a symmetrical cdeo algebra.

Remark 3.3.43. The implication

$$\mathcal{R} : X \times Y \to L,\ \mathcal{R} \text{ power-univocal} \Rightarrow \mathcal{R}\ (\bot)\text{-univocal}$$

forces $\perp$ to be $\otimes$-prime.
In fact, let $\alpha, \beta \in L$, $\alpha \otimes \beta = \beta \otimes \alpha = \perp$; consider $X = \{x\}, Y = \{y, y'\}$ and the L-relation $\mathcal{R} : X \times Y \to L$ defined by $(x,y)\mathcal{R} = \alpha, (x,y')\mathcal{R} = \beta$.
Obviously, $(x,y)\mathcal{R} \otimes (y)B \leq (x,y)\mathcal{R} \to (y)B$ and $(x,y')\mathcal{R} \otimes (y')B \leq (x,y')\mathcal{R} \to (y')B$, for every $B \in L^Y$. Since $\beta \otimes \alpha = \perp$, one has that $\alpha \leq \beta \to \perp$.
So, for every $B \in L^Y$ it follows that $(x,y)\mathcal{R} \otimes (y)B = \alpha \otimes (y)B \leq \alpha \leq \beta \to \perp \leq \beta \to (y')B = (x,y')\mathcal{R} \to (y')B$; from the equivalence $\alpha \otimes \beta = \perp \Leftrightarrow \beta \leq \alpha \to \perp$, it follows that, for every $B \in L^Y$, $(x,y')\mathcal{R} \otimes (y')B \leq \beta \otimes (y')B \leq \beta \leq \alpha \to \perp \leq \alpha \to (y)B = (x,y)\mathcal{R} \to (y)B$. Hence $\mathcal{R}^{\otimes}_{-} \leq \mathcal{R}^{\to}_{-}$, i.e. $\mathcal{R}$ is power-univocal; by assumption $\mathcal{R}$ is $(\perp)$-univocal and then either $\alpha = \perp$ or $\beta = \perp$.

Remark 3.3.44. Of course, by Proposition 3.3.28 (1), one can argue easily that the implication

$$\mathcal{R} : X \times Y \to L,\ \mathcal{R} \text{ univocal} \Rightarrow \mathcal{R}\ (\perp)\text{-univocal}$$

forces $\perp$ to be $\otimes$-prime, too.

Up to now, we are not able to prove a similar result as in Remark 3.3.43 for the notions of power-totality and $[\top]$-totality; however, the weaker version, as in Remark 3.3.44, holds.

Remark 3.3.45. The implication

$$\mathcal{R} : X \times Y \to L,\ \mathcal{R} \text{ total} \Rightarrow \mathcal{R}\ [\top]\text{-total}$$

forces $\top$ to be completely $\otimes$-coprime.
In fact, let $S \subseteq L$, $\bigvee_{s \in S} s \otimes s = \top$; then $S \neq \varnothing$, since $\bigvee \varnothing = \perp \neq \top$. Consider the sets $X = \{x\}, Y = S$ and the L-relation $\mathcal{R} : X \times Y \to L$, defined by $(x,s)\mathcal{R} = s, \forall s \in S$. It follows from Lemma 3.3.12 that $\mathcal{R}$ is total and then, by assumption, $\mathcal{R}$ is $[\top]$-total, that is $x\mathcal{R}^{[\top]} \neq \varnothing$. Hence, there exists $s \in S$ such that $s = (x,s)\mathcal{R} = \top$; by arbitrariness of S, $\top$ is completely $\otimes$-coprime.

Chapter 4

Galois and Tarski connections

4.1 Lower and upper hulls in preordered sets

In this Section we recall basic notions on preordered sets and, in particular, on order filters and ideals. In fact we wish to give a detailed description of the folklore on these topics, which are only seldom found in textbooks and monographs; we do this taking into account the concepts and results needed in the sequel of our work. We refer to Section 1.1 for preliminary notions and notation.

Definition 4.1.1. *Let $(L, \leq)$ be a preordered set and X a subset of L. X is an **upperset** (**lowerset**, respectively) or **preorder filter** (**preorder ideal**, respectively) of L if, for all $x, y \in L$, the following implication is true:*

$$x \in X, x \leq y \Rightarrow y \in X \ (x \in X, y \leq x \Rightarrow y \in X, \ respectively).$$

The set of all uppersets (lowersets, respectively) of a preordered set $(L, \leq)$ is denoted by $\mathcal{F}(L)$ ($\mathcal{I}(L)$, respectively).

In the following $X^{\neg}$ stands for $L \backslash X$, for any $X \subseteq L$.

Proposition 4.1.2. *Let $(L, \leq)$ be a preordered set and X a subset of L. The following equivalences hold.*

X is an upperset (lowerset) of L if and only if $X^{\neg}$ is a lowerset (upperset) of L.

Proof. Assume X to be an upperset and let $x \in X^{\neg}$, $y \in L, y \leq x$. Then $y \in X^{\neg}$. In fact, if we suppose by contradiction that $y \notin X^{\neg}$, we have that $y \in X$ and, since $y \leq x$ and X is an upperset, it follows that $x \in X$, which contradicts the assumption $x \notin X$.
The other part of the statement can be similarly proved. □

Proposition 4.1.3. *Let $(L, \leq)$ be a preordered set and $X_i \subseteq L$, for every $i \in I$. The following hold.*

1. *If X_i is an upperset of L, for every $i \in I$, then $\bigcup_{i \in I} X_i$ and $\bigcap_{i \in I} X_i$ are uppersets of L.*

2. *If X_i is a lowerset of L, for every $i \in I$, then $\bigcup_{i \in I} X_i$ and $\bigcap_{i \in I} X_i$ are lowersets of L.*

Proof. 1. The proof is trivial for $I = \varnothing$, since $\varnothing$ and L are uppersets and lowersets in L. Assume every X_i to be an upperset of L. If $x \in \bigcup_{i \in I} X_i, y \in L$ and $x \leq y$, then, there exists $j \in I$ such that $x \in X_j$, $x \leq y$. Hence, by assumption, $y \in X_j \subseteq \bigcup_{i \in I} X_i$. If $x \in \bigcap_{i \in I} X_i, y \in L$ and $x \leq y$, then, for every $i \in I$, $x \in X_i$, $x \leq y$. By assumption, $y \in X_i$, for every $i \in I$ and hence $y \in \bigcap_{i \in I} X_i$.

2. Similarly to item (1).

□

The above Proposition allows to give the following Definition.

Definition 4.1.4. *Let $(L, \leq)$ be a preordered set and E a subset of L.*

1. *The **including upper** (**lower**, respectively) **hull** of E, denoted by $E^{[\uparrow]}$ ($E^{[\downarrow]}$, respectively) is the smallest upperset (lowerset, respectively) containing E.*

2. *The **excluding upper** (**lower**, respectively) **hull** of E, denoted by $E^{(\uparrow)}$ ($E^{(\downarrow)}$, respectively) is the largest upperset (lowerset, respectively) disjoint from E.*

Remark 4.1.5. Let $(L, \leq)$ be a preordered set.

- For any $E \subseteq L$ one has $UbE \subseteq E^{[\uparrow]}$ and $LbE \subseteq E^{[\downarrow]}$.

- The operators $[\cdot]^{[\uparrow]}$ and $[\cdot]^{[\downarrow]}$ are isotonic, while the operators $[\cdot]^{(\uparrow)}$ and $[\cdot]^{(\downarrow)}$ are antitonic.

- For any $A \subseteq L$ one has that A is an upperset if and only if $A = A^{[\uparrow]}$, hence for any $E \subseteq L$: $E^{(\uparrow)} = (E^{(\uparrow)})^{[\uparrow]}$ and $E^{[\uparrow]} = (E^{[\uparrow]})^{[\uparrow]}$.

- The notions of including (excluding) upper and lower hull of $E \subseteq L$ are order-dual to each other. So, the properties which can be proved for $E^{[\uparrow]}$ ($E^{(\uparrow)}$) have a dual version for $E^{[\downarrow]}$ ($E^{(\downarrow)}$), too.

- In case of singletons, it is useful to consider the following notation and equalities, for every $e \in L$:

$$e^{[\uparrow]} = \{x \in L |\ e \leq x\}\,;\ e^{[\downarrow]} = \{x \in L |\ x \leq e\}\,;\ e^{(\uparrow)} = \{x \in L |\ x \nleq e\}\,;\ e^{(\downarrow)} = \{x \in L |\ e \nleq x\}\,.$$

Proposition 4.1.6. *Let $(L, \leq)$ be a preordered set and I any, possibly empty, set. Then for all subsets E, F and E_i, $i \in I$, of L one has:*

1. $E \subseteq F \Rightarrow E^{[\uparrow]} \subseteq F^{[\uparrow]}$;

2. $E \subseteq E^{[\uparrow]}$;

3. $(E^{[\uparrow]})^{[\uparrow]} = E^{[\uparrow]}$;

4. $(\bigcup_{i \in I} E_i)^{[\uparrow]} = \bigcup_{i \in I} E_i^{[\uparrow]}$.

Proof. 1. See Remark 4.1.5.

2. By definition, it follows trivially that $E \subseteq E^{[\uparrow]}$.

3. See Remark 4.1.5.

4. From item (2) we have that $E_i \subseteq \bigcup_{i \in I} E_i, \forall i \in I \Rightarrow E_i^{[\uparrow]} \subseteq (\bigcup_{i \in I} E_i)^{[\uparrow]}, \forall i \in I \Rightarrow \bigcup_{i \in I} E_i^{[\uparrow]} \subseteq (\bigcup_{i \in I} E_i)^{[\uparrow]}$. Furthermore, by definition, $(\bigcup_{i \in I} E_i)^{[\uparrow]}$ is the smallest upperset containing $\bigcup_{i \in I} E_i$ and, so, it is an upperset containing E_i, $\forall i \in I$. Moreover, $\bigcup_{i \in I} E_i^{[\uparrow]}$ is an upperset, since it is the join of the uppersets $E_i^{[\uparrow]}$ and $E_i \subseteq E_i^{[\uparrow]} \subseteq \bigcup_{i \in I} E_i^{[\uparrow]}$, $\forall i \in I$. So $\bigcup_{i \in I} E_i^{[\uparrow]}$ is an upperset containing $\bigcup_{i \in I} E_i$; hence, by definition, $(\bigcup_{i \in I} E_i)^{[\uparrow]} \subseteq \bigcup_{i \in I} E_i^{[\uparrow]}$.

□

Corollary 4.1.7. *If $(L, \leq)$ is a preordered set, the maps*

$$[\cdot]^{[\uparrow]} : \mathit{2}^L \to \mathit{2}^L,\ E \mapsto E^{[\uparrow]} \text{ and } [\cdot]^{[\downarrow]} : \mathit{2}^L \to \mathit{2}^L,\ E \mapsto E^{[\downarrow]}$$

are topological closure operators. The fixed points of each of such operators, i.e. the uppersets of L in the first case and the lowersets in the second case, are the closed sets of an ***Alexandroff topological space*** *on L.*

Corollary 4.1.8. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$. The following equalities hold:*

1. $E^{[\uparrow]} = \bigcup_{e \in E} e^{[\uparrow]}$;

2. $E^{[\downarrow]} = \bigcup_{e \in E} e^{[\downarrow]}$.

Proof. It is an easy consequence of Proposition 4.1.6 (4) and of its dual version, since $E = \bigcup_{e \in E} \{e\}$. □

Corollary 4.1.9. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$.*

1. $E^{[\uparrow]} = \{x \in L |\ \exists e \in E : \ e \leq x\}$;

2. $E^{[\downarrow]} = \{x \in L |\ \exists e \in E : \ x \leq e\}$.

Proof. 1. Since $E^{[\uparrow]} = \bigcup_{e \in E} e^{[\uparrow]}$, one has that $y \in E^{[\uparrow]} \Leftrightarrow \exists e_y \in E \ni' y \in e_y^{[\uparrow]} \Leftrightarrow \exists e_y \in E \ni' e_y \leq y \Leftrightarrow y \in \{x \in L |\ \exists e \in E : \ e \leq x\}$.

2. Dual to item (1).

□

Lemma 4.1.10. *Let $(L, \leq)$ be a preordered set and $X, E \subseteq L$. If X is a lowerset (upperset) such that $X \cap E = \varnothing$, then $X \cap E^{[\uparrow]} = \varnothing$ ($X \cap E^{[\downarrow]} = \varnothing$).*

Proof. Suppose, by contradiction, that there exists $\bar{x} \in X \cap E^{[\uparrow]}$; then, $\bar{x} \in X$ and $\bar{x} \in E^{[\uparrow]}$, so, there exists $e_{\bar{x}} \in E$ such that $e_{\bar{x}} \leq \bar{x}$. Since X is a lowerset, it follows that $e_{\bar{x}} \in X$, that is in contradiction with the assumption $X \cap E = \varnothing$. □

Proposition 4.1.11. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$.*

1. *$E^{(\downarrow)}$ and $E^{[\uparrow]}$ are complement of each other, that is $[E]^{(\downarrow)} = [E]^{[\uparrow]\neg}$ and $[E]^{[\uparrow]} = [E]^{(\downarrow)\neg}$;*

2. *$E^{(\uparrow)}$ and $E^{[\downarrow]}$ are complement of each other, that is $[E]^{(\uparrow)} = [E]^{[\downarrow]\neg}$ and $[E]^{[\downarrow]} = [E]^{(\uparrow)\neg}$.*

Proof.
1. We prove that $(E^{[\uparrow]})^{\neg}$ is the largest lowerset disjoint from E. In fact, $(E^{[\uparrow]})^{\neg}$ is a lowerset disjoint from E, since it is the complement of the upperset $E^{[\uparrow]}$ that contains E. If, moreover, $B \subseteq L$ is a lowerset of L such that $B \cap E = \varnothing$, then by the above Lemma 4.1.10, it follows that $B \cap E^{[\uparrow]} = \varnothing$; so $B \subseteq (E^{[\uparrow]})^{\neg}$.

2. Dual to item (1).

□

Corollary 4.1.12. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$.*

1. *$E^{(\uparrow)} = \{x \in L|\ \forall e \in E:\ x \nleq e\}$;*

2. *$E^{(\downarrow)} = \{x \in L|\ \forall e \in E:\ e \nleq x\}$.*

Proof.
1. Since $E^{(\uparrow)} = (E^{[\downarrow]})^{\neg}$, $y \in E^{(\uparrow)} \Leftrightarrow y \notin E^{[\downarrow]} \Leftrightarrow \forall e \in E : y \nleq e \Leftrightarrow y \in \{x \in L|\ \forall e \in E:\ x \nleq e\}$.

2. Dual to item (1).

□

Proposition 4.1.13. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$.*

1. *$E^{(\downarrow)} = \bigcap_{e \in E} e^{(\downarrow)} = \bigcup_{e \notin E^{[\uparrow]}} e^{[\downarrow]}$;*

2. *$E^{(\uparrow)} = \bigcap_{e \in E} e^{(\uparrow)} = \bigcup_{e \notin E^{[\downarrow]}} e^{[\uparrow]}$.*

Proof.
1. By the Propositions 4.1.11 (1) and 4.1.8 (1), the following equalities hold: $E^{(\downarrow)} = (E^{[\uparrow]})^{\neg} = (\bigcup_{e \in E} e^{[\uparrow]})^{\neg} = \bigcap_{e \in E} (e^{[\uparrow]})^{\neg} = \bigcap_{e \in E} e^{(\downarrow)}$.
   Moreover, $E^{(\downarrow)} = (E^{[\uparrow]})^{\neg} = ((E^{[\uparrow]})^{\neg})^{[\downarrow]} = \bigcup_{e \notin E^{[\uparrow]}} e^{[\downarrow]}$.

2. Dual to item (1).

□

Proposition 4.1.14. *Let $(L, \leq)$ be a preordered set, $E \subseteq L$. Then, for all $e, d \in L$:*

1. *$e \in \langle MinE \rangle_{\equiv} \Leftrightarrow e^{[\uparrow]} = E^{[\uparrow]}$;*

2. $d \in \langle MaxE \rangle_{\equiv} \Leftrightarrow d^{[\downarrow]} = E^{[\downarrow]}$.

Proof. 1. " $\Rightarrow$ " Let $e \in \langle MinE \rangle_{\equiv} = \langle E \rangle_{\equiv} \cap LbE$, i.e. there exists $\bar{e} \in E$ such that $e \equiv \bar{e}$ and $e \leq e'$, for every $e' \in E$. Then:
– $x \in e^{[\uparrow]} \Rightarrow e \leq x \Rightarrow \bar{e} \leq x \Leftrightarrow x \in E^{[\uparrow]}$;
– $x \in E^{[\uparrow]} \Rightarrow \exists e' \in E \ni' e' \leq x \Rightarrow e \leq e' \leq x \Rightarrow x \in e^{[\uparrow]}$.
" $\Leftarrow$ " Since $e \in e^{[\uparrow]} = E^{[\uparrow]}$, there exists $e' \in E : e' \leq e$. Moreover, $e' \in E \subseteq E^{[\uparrow]} = e^{[\uparrow]}$ implies that $e \leq e'$; so, $e \in \langle E \rangle_{\equiv}$. Furthermore, $x \in E \subseteq E^{[\uparrow]} = e^{[\uparrow]} \Rightarrow e \leq x$, i.e. $e \in LbE$.

2. Dual to item (1).

□

The uppersets and the lowersets of L, which are the fixed points of the operators $[\cdot]^{[\uparrow]}$ and $[\cdot]^{[\downarrow]}$, respectively, can be also characterized as follows.

Proposition 4.1.15. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$. The following are equivalent.*

1. *E is an upperset;*
2. *$E = (E^{\neg})^{(\uparrow)}$;*
3. *$(E^{\neg})^{(\uparrow)} = (E^{(\downarrow)})^{\neg}$.*

Proof. "(1) $\Rightarrow$ (2)" Since E is an upperset, then $E^{\neg}$ is a lowerset, so $(E^{\neg})^{[\downarrow]} = E^{\neg}$. Then, by Proposition 4.1.11, $(E^{\neg})^{(\uparrow)} = ((E^{\neg})^{[\downarrow]})^{\neg} = (E^{\neg})^{\neg} = E$.
"(2) $\Rightarrow$ (3)" It follows by assumption and Proposition 4.1.11 that $((E)^{(\downarrow)})^{\neg} = E^{[\uparrow]} = ((E^{\neg})^{(\uparrow)})^{[\uparrow]} = (E^{\neg})^{(\uparrow)}$.
"(3) $\Rightarrow$ (1)" Of course, $E \subseteq E^{[\uparrow]}$. Conversely, it follows by assumption and by Proposition 4.1.11 that $E^{[\uparrow]} = (E^{(\downarrow)})^{\neg} = (E^{\neg})^{(\uparrow)} \subseteq E$. □

Proposition 4.1.16. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$. The following are equivalent.*

1. *E is a lowerset;*
2. *$E = (E^{\neg})^{(\downarrow)}$;*
3. *$(E^{\neg})^{(\downarrow)} = (E^{(\uparrow)})^{\neg}$.*

Proof. Dual to Proposition 4.1.15. □

Corollary 4.1.17. *Let $(L, \leq)$ be a preordered set and $E \subseteq L$. Then*

1. *$E^{(\uparrow)} = ((E)^{[\downarrow]})^{(\uparrow)}$;*
2. *$E^{[\uparrow]} = (E^{(\downarrow)})^{(\uparrow)}$;*
3. *$E^{(\downarrow)} = (E^{[\uparrow]})^{(\downarrow)}$;*

4. $E^{[\downarrow]} = (E^{(\uparrow)})^{(\downarrow)}$.

Proof. Since $E^{(\uparrow)}$ is an upperset, by Propositions 4.1.11 and 4.1.15 $E^{(\uparrow)} = ((E^{(\uparrow)})^{\neg})^{(\uparrow)} = (E^{[\downarrow]})^{(\uparrow)}$, then (1) is true. Similarly, since $E^{[\uparrow]}$ is an upperset, $E^{[\uparrow]} = ((E^{[\uparrow]})^{\neg})^{(\uparrow)} = (E^{(\downarrow)})^{(\uparrow)}$, then (2) is true.
(3) and (4) can be proved by duality. □

Remark 4.1.18. - The item (2) of Propositions 4.1.15 and 4.1.16 show that every upperset (lowerset) is an excluding upper (lower) hull. As a consequence, the operators $[\cdot]^{(\uparrow)}$ and $[\cdot]^{(\downarrow)}$ have the same image as $[\cdot]^{[\uparrow]}$ and $[\cdot]^{[\downarrow]}$, respectively; more explicitly one has

$$\mathscr{F}(L) = \left\{E^{[\uparrow]} \mid E \in \mathbf{2}^L\right\} = \left\{E^{(\uparrow)} \mid E \in \mathbf{2}^L\right\}$$

and

$$\mathscr{I}(L) = \left\{E^{[\downarrow]} \mid E \in \mathbf{2}^L\right\} = \left\{E^{(\downarrow)} \mid E \in \mathbf{2}^L\right\}.$$

- Of course, there are subsets of $(L, \leq)$ which are both uppersets and lowersets: $\varnothing$ and L are among these subsets.
  However, in quite general situations $\varnothing$ and L are the only uppersets that are lowersets, too. The following example describes two cases.

Example 4.1.19. If $(L, \leq)$ satisfies one of the following conditions:

(i) there exists either a minimum or a maximum in L;

(ii) for all $x, y \in L$, either $x \vee y \neq \varnothing$ or $x \wedge y \neq \varnothing$ in L,

then $\varnothing$ and L are the only subsets that are both lowersets and uppersets.
In fact, assume $\varnothing \neq X \subseteq L$ to be both a lowerset and an upperset. If L has a minimum $\perp$, then $\perp \in X$, hence $L = \perp^{[\uparrow]} \subseteq X^{[\uparrow]} = X$; dually, if L has a maximum, then $X = L$, too. If (ii) holds, consider a fixed element $x \in X$ and any $y \in L$. If $z \in x \vee y$ exists in L, then $z \in X$ and $y \leq z$ hence $y \in X$; dually, $y \in X$ if $t \in x \wedge y$ exists. By arbitrariness of y, $X = L$.

Remark 4.1.20. - As already remarked, the uppersets of a preordered set $(L, \leq)$ are the closed sets of a topology on L, which we call **lower Alexandroff topology** of $(L, \leq)$, since its open sets are the lowersets.
  Dually, the **upper Alexandroff topology** of $(L, \leq)$ can be defined, whose closed sets are the lowersets.
  The statements we list below concern the lower Alexandroff topology and each of them has a dual statement for the upper Alexandroff topology.

- Since the closed sets of the lower Alexandroff topological space on $(L, \leq)$ are the uppersets of L, one has, by Proposition 4.1.11, that the open sets are the excluding lower hulls $E^{(\downarrow)}$, for every $E \subseteq L$, i.e. the lowersets of L. Then, by Remark 4.1.18 and Corollary 4.1.8, $E \subseteq L$ is an open set if and only if $E = E^{[\downarrow]} = \bigcup_{e \in E} e^{[\downarrow]}$. So, the family of subsets $\left\{e^{[\downarrow]} \mid e \in E\right\}$ is a basis of the lower Alexandroff space.

- It is clear that the closure operator of the lower Alexandroff topology is $cl_l[\cdot] = [\cdot]^{[\uparrow]}$; hence, by Proposition 4.1.11, the interior operator is $[\cdot]^{\neg[\uparrow]\neg} = [\cdot]^{\neg(\downarrow)}$.
  As a consequence, by Corollary 4.1.8 and Proposition 4.1.13, it is possible to express the closure and the interior of any subset $X \subseteq L$ as follows:
  - $cl_l X = \bigcup\left\{x^{[\uparrow]} \mid x \in X\right\}$;
  - $int_l X = \bigcap\left\{y^{(\downarrow)} \mid y \notin X\right\} = \bigcup\left\{z^{[\downarrow]} \mid z \notin X^{\neg[\uparrow]}\right\} = \bigcup\left\{z^{[\downarrow]} \mid z^{[\downarrow]} \subseteq X\right\}$.

4.2 Equivalences and connections

Let L be a set and let $\equiv$ be an equivalence relation on L.
For any set Z we consider in L^Z the equivalence induced pointwisely by $\equiv$, i.e. we recall that the functions $h, k : Z \to L$ are equivalent with respect to $\equiv$ and write $h \equiv k$ if $(z)h \equiv (z)k$, for every $z \in Z$.
We may also specify that in such a case h and k are globally equivalent, or that there is a global equivalence between them. However, we also consider much weaker equivalences, namely **relative equivalences** in L^Z, with respect to a subset of L; these notions will allow to give a unified presentation of all types of Galois connections considered up to now (see [35, 42, 51]) together with their dual forms, which we shall call **Tarski connections**, since they extend conjugated pairs introduced by Tarski in [83] and also studied in [42, 61].

Definition 4.2.1. *Let Z, L be two sets, $h, k : Z \to L$ two functions, $\equiv$ an equivalence relation on L and $E \subseteq L$. We say that h and k are E-**equivalent** and write $h \equiv_E k$ if for every $z \in Z$, the following equivalence holds:*

$$(z)h \in \langle E\rangle_{\equiv} \Leftrightarrow (z)k \in \langle E\rangle_{\equiv}.$$

If $a \in L$, we write $h \equiv_a k$ for $h \equiv_{\{a\}} k$.
Of course, $h \equiv_E k$ if and only if $\langle E\rangle_{\equiv} h_{-} = \langle E\rangle_{\equiv} k_{-}$. Note also that $h \equiv_E k$ if and only if $h \equiv_{\langle E\rangle_{\equiv}} k$, by the idempotency of the closure operator $\langle\cdot\rangle_{\equiv}$.

Proposition 4.2.2. *Let Z be a set, $h, k : Z \to L$ be functions.*
Then the following are equivalent:

1. $h \equiv k$;
2. $h \equiv_E k$, $\forall E \subseteq L$;
3. $h \equiv_a k$, $\forall a \in L$.

Proof. "(1) $\Rightarrow$ (2)" Assume $h \equiv k$, i.e. $(z)h \equiv (z)k$, for every $z \in Z$. Then for all $E \subseteq L$ and $z \in Z$ one has $(z)h \in \langle E\rangle_{\equiv} \Leftrightarrow \exists a \in E : a \equiv (z)h \equiv (z)k \Leftrightarrow (z)k \in \langle E\rangle_{\equiv}$.
"(2) $\Rightarrow$ (3)" Obvious.
"(3) $\Rightarrow$ (1)" For every $z \in Z$ it follows that $(z)h \in \langle\{(z)h\}\rangle_{\equiv}$, hence, by assumption, $(z)k \in \langle\{(z)h\}\rangle_{\equiv}$, i.e. $(z)h \equiv (z)k$. $\square$

Remark 4.2.3. We note that every subset $E \subseteq L$ determines, by means of Definition 4.2.1, an equivalence $\equiv_E$ in L^Z which is coarser than $\equiv$, in the sense that, as subsets of $L^Z \times L^Z$, $\equiv \subseteq \equiv_E$; moreover, Proposition 4.2.2 states that $\equiv = \bigcap_{E \subseteq L} \equiv_E = \bigcap_{a \in L} \equiv_a$.
Sometimes we shall call $\equiv$ the global equivalence and each of $\equiv_E$ the equivalence relative to E; more generally, we say that $\sim$ is a relative equivalence (with respect to $\equiv$) if there is $E \subseteq L$ such that $\sim = \equiv_E$.
Note also that $\equiv$ coincides with the (global) equivalence pointwisely induced in L^Z by the equivalence on L whose classes are either E or $L \backslash E$.

We shall deal mostly with a preordered set $(L, \leq)$, in which case we consider the equivalence relation $\equiv$ induced by $\leq$ on L, which is the equality if $\leq$ is an order.
In this case, with the above notation, we give the following definitions.

Definition 4.2.4. *Let $(L, \leq)$ be a preordered set, $E \subseteq L$ and let $h, k : Z \to L$ be two functions.*
*– h and k are **upper (lower) equivalent** with respect to $\equiv$ if there exists an upperset $F \subseteq L$ (a lowerset $I \subseteq L$) such that $h \equiv_F k$ ($h \equiv_I k$).*
*– h and k are E**-including** (E**-excluding**) **upper equivalent** with respect to $\equiv$ if for every $z \in Z$ one has $(z)h \in E^{[\uparrow]} \Leftrightarrow (z)k \in E^{[\uparrow]}$ ($(z)h \in E^{(\uparrow)} \Leftrightarrow (z)k \in E^{(\uparrow)}$).*
*– Dually, h and k are E**-including** (E**-excluding**) **lower equivalent** with respect to $\equiv$ if for every $z \in Z$ one has $(z)h \in E^{[\downarrow]} \Leftrightarrow (z)k \in E^{[\downarrow]}$ ($(z)h \in E^{(\downarrow)} \Leftrightarrow (z)k \in E^{(\downarrow)}$).*

Remark 4.2.5. Definitions 4.2.1 and 4.2.4 do not exclude the trivial cases when E is either empty or the whole set L.
However, one can note that if $\Gamma \in \{\varnothing, L\}$ then each of $\langle \Gamma \rangle_\equiv$, $\Gamma^{[\uparrow]}$, $\Gamma^{(\uparrow)}$, $\Gamma^{[\downarrow]}$ and $\Gamma^{(\downarrow)}$ belongs to $\{\varnothing, L\}$.
As a consequence, for $\Gamma \in \{\varnothing, L\}$, any two functions $h, k : Z \to L$ are trivially Γ-equivalent, Γ-including and Γ-excluding upper and lower equivalent.
In the subsequent discussions, statements and proofs we may consider these trivial cases without mentioning.

Remark 4.2.6.

1. Since every upperset and every lowerset set is saturated, it is clear that E-including or E-excluding, upper or lower equivalences may be denoted by one of the symbols $\equiv_{E^{[\uparrow]}}$, $\equiv_{E^{(\uparrow)}}$, $\equiv_{E^{[\downarrow]}}$ and $\equiv_{E^{(\downarrow)}}$, among which the right one can be easily understood.

2. If F is an upperset, then $F = F^{[\uparrow]} = (F^{\neg})^{(\uparrow)}$. So any upper equivalence is both an E-including and E'-excluding upper equivalence, for suitable $E, E' \subseteq L$; then, by Corollary 4.1.17, it is an $E^{(\downarrow)}$-excluding and an $E'^{(\uparrow)}$-including upper equivalence, too.
   The dual statement for lower equivalences hold, too.

3. As a consequence of Lemma 4.1.14, if $E \subseteq L$ and either $e \in \langle MinE \rangle_\equiv$ or $d \in \langle MaxE \rangle_\equiv$, then either $\equiv_{E^{[\uparrow]}} = \equiv_{e^{[\uparrow]}}$ or $\equiv_{E^{[\downarrow]}} = \equiv_{d^{[\downarrow]}}$.

Looking at the proof of Proposition 4.2.2, it is easy to realize the following.

Proposition 4.2.7. *Let $(L, \leq)$ be a preordered set and let $h, k : Z \to L$ be functions. With the above notation, the following statements are equivalent:*

1. $h \equiv k$;

2. $\forall E \subseteq L : h \equiv_{E[\uparrow]} k$;

2'. $\forall e \in L : h \equiv_{e[\uparrow]} k$. □

Dually, the following proposition holds.

Proposition 4.2.8. *Let $(L, \leq)$ be a preordered set and let $h, k : Z \to L$ be functions. With the above notation, the following statements are equivalent:*

1. $h \equiv k$;

2. $\forall E \subseteq L : h \equiv_{E[\downarrow]} k$;

2'. $\forall d \in L : h \equiv_{d[\downarrow]} k$. □

On the base of equivalences of functions with values in a preordered set $(L \leq)$, in particular of L-relations, we shall present a quite general approach to Galois connections, including their classification in four types, related to each other by analogies and dualities, as in [42, 51]. Our general approach will further allow to include the classical notion of conjugated pairs of functions introduced by Tarski in [83] and developed in [61]; this will be the topic of Section 4.3 where, not only the notion of conjugated pairs (also considered recently in [30, 33, 42]) will be extended, but it will be also obtained their classification into four types (we shall call them **Tarski connections**) that are perfectly order-dual to Galois connections.
Both Galois and Tarski connections will be defined by considering two special cases, dual to each other, of the relative connections we consider below.

Let $(L, \leq)$ be a preordered set, (X, α), (Y, β) be two sets, each equipped with a fixed binary L-relation on it and $f : X \to Y$, $g : Y \to X$ be two functions. Consider the L-relations from X to Y defined as follows:

$$\mathcal{R}_{\beta f} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{\beta f} = (y, (x)f)\beta;$$

$$\mathcal{R}_{f\beta} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{f\beta} = ((x)f, y)\beta;$$

$$\mathcal{R}_{\alpha g} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{\alpha g} = (x, (y)g)\alpha;$$

$$\mathcal{R}_{g\alpha} : X \times Y \to L,\ (x, y) \mapsto (x, y)\mathcal{R}_{g\alpha} = ((y)g, x)\alpha.$$

Remark 4.2.9. Note that

- $(\mathcal{R}_{\beta f})_{-} = \mathcal{R}_{f\beta_{-}}$;

- $(\mathcal{R}_{f\beta})_{-} = \mathcal{R}_{\beta_{-} f}$;

- $(\mathcal{R}_{\alpha g})_{-} = \mathcal{R}_{g\alpha_{-}}$;

- $(\mathcal{R}_{g\alpha})_- = \mathcal{R}_{\alpha_- g}$.

With the above notations, we state the following.

Definition 4.2.10. - *f and g form a **type I (global) connection** from (X,α) to (Y,β), denoted by $[f\text{-}g] : (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{\beta f} \equiv \mathcal{R}_{\alpha g}$.*

- *f and g form a **type II (global) connection** from (X,α) to (Y,β), denoted by $]f\text{-}g[: (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{f\beta} \equiv \mathcal{R}_{g\alpha}$.*

- *f and g form a **type III (global) connection** from (X,α) to (Y,β), denoted by $(f,g) : (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{f\beta} \equiv \mathcal{R}_{\alpha g}$.*

- *f and g form a **type IV (global) connection** from (X,α) to (Y,β), denoted by $)f,g(: (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{\beta f} \equiv \mathcal{R}_{g\alpha}$.*

Proposition 4.2.11. *With the above notation, the following are equivalent:*

1. $[f\text{-}g] : (X,\alpha) \to (Y,\beta)$;
2. $[g\text{-}f] : (Y,\beta) \to (X,\alpha)$;
3. $]f\text{-}g[: (X,\alpha_-) \to (Y,\beta_-)$;
4. $]g\text{-}f[: (Y,\beta_-) \to (X,\alpha_-)$;
5. $(f,g) : (X,\alpha) \to (Y,\beta_-)$;
6. $(g,f) : (Y,\beta) \to (X,\alpha_-)$;
7. $)f,g(: (X,\alpha_-) \to (Y,\beta)$;
8. $)g,f(: (Y,\beta_-) \to (X,\alpha)$.

Proof. The equivalence between all the items can be obtained by the following equivalences, that are true thanks to Remark 4.2.9 and the symmetry of $\equiv$, for all considered elements: $\mathcal{R}_{\beta f} \equiv \mathcal{R}_{\alpha g} \Leftrightarrow \mathcal{R}_{\alpha g} \equiv \mathcal{R}_{\beta f} \Leftrightarrow \mathcal{R}_{f\beta_-} \equiv \mathcal{R}_{g\alpha_-} \Leftrightarrow \mathcal{R}_{g\alpha_-} \equiv \mathcal{R}_{f\beta_-} \Leftrightarrow \mathcal{R}_{f\beta_-} \equiv \mathcal{R}_{\alpha g} \Leftrightarrow \mathcal{R}_{g\alpha_-} \equiv \mathcal{R}_{\beta f} \Leftrightarrow \mathcal{R}_{\beta f} \equiv \mathcal{R}_{g\alpha_-} \Leftrightarrow \mathcal{R}_{\alpha g} \equiv \mathcal{R}_{f\beta_-}$. ☐

Relative connections from (X,α) to (Y,β) can be also considered as follows using, once more, the above notation, with respect to any subset $E \subseteq L$.

Definition 4.2.12. - *f and g form a **type I E-connection** from (X,α) to (Y,β), denoted by $[f\text{-}g]_E : (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{\beta f} \equiv_E \mathcal{R}_{\alpha g}$.*

- *f and g form a **type II E-connection** from (X,α) to (Y,β), denoted by $]f\text{-}g[_E : (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{f\beta} \equiv_E \mathcal{R}_{g\alpha}$.*

- *f and g form a **type III E-connection** from (X,α) to (Y,β), denoted by $(f,g)_E : (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{f\beta} \equiv_E \mathcal{R}_{\alpha g}$.*

- *f and g form a* ***type IV** E**-connection*** *from (X,α) to (Y,β), denoted by $)f,g(_E\colon (X,\alpha) \to (Y,\beta)$, if $\mathcal{R}_{\beta f} \equiv_E \mathcal{R}_{g\alpha}$.*

Roughly speaking, a pair of maps is a relative connection of one of the four types if there exists $E \subseteq L$ such that those maps form an E-connection of that type.
It is clear that the properties proved in Proposition 4.2.11 hold for all types of (relative) E-connections; in fact, the equivalences proved for global connections only depend on the symmetry of the considered equivalence relation on L and on the links between the L-relations $\mathcal{R}_{\beta f}$, $\mathcal{R}_{f\beta}$, $\mathcal{R}_{\alpha g}$ and $\mathcal{R}_{g\alpha}$ and their opposite observed in Remark 4.2.9.
Also note that we shall sometimes simplify notation into $[f\text{-}g]_E$, $]f\text{-}g[_E$, $(f,g)_E$, $)f,g(_E$ if the two connected sets (X,α), (Y,β) are clearly meant.

Proposition 4.2.13. *With the above notation, the following are equivalent:*

1. $[f\text{-}g]_E : (X,\alpha) \to (Y,\beta)$;
2. $[g\text{-}f]_E : (Y,\beta) \to (X,\alpha)$;
3. $]f\text{-}g[_E\colon (X,\alpha_-) \to (Y,\beta_-)$;
4. $]g\text{-}f[_E\colon (Y,\beta_-) \to (X,\alpha_-)$;
5. $(f,g)_E : (X,\alpha) \to (Y,\beta_-)$;
6. $(g,f)_E : (Y,\beta) \to (X,\alpha_-)$;
7. $)f,g(_E\colon (X,\alpha_-) \to (Y,\beta)$;
8. $)g,f(_E\colon (Y,\beta_-) \to (X,\alpha)$.

Proof. The proof uses the same equivalences as Proposition 4.2.11. □

The classification of connections of Definitions 4.2.10 and 4.2.12 has been arranged in such a way as to agree with a similar classification done in [51] for fuzzy Galois connections, to which our notions are closely related, as will be clear going on with the development of the next Sections.

4.3 Galois connections

In this Section we define Galois connections between (possibly structured) sets as upper connections, i.e. connections considered in Definition 4.2.12, relative to uppersets; we apply, as well, the classification there considered.
According to the discussion already done for upper equivalences in Section 4.2, every Galois connection is both an E-including and an E'-excluding upper connection, for suitable subsets $E, E' \subseteq L$.
Conversely, every subset $E \subseteq L$ determines two classes of Galois connections by means

of the upper hulls $E^{[\uparrow]}$ and $E^{(\uparrow)}$. To involve explicitly the interrelation between a Galois connection and any subset may determine it, we give the following Definition in a form that may seem redundant but will give practical advantages.

Definition 4.3.1. *Let $E \subseteq L$ be a subset of the preordered set $(L, \leq)$, (X, α), (Y, β) be two possibly structured sets, each with a fixed binary L-relation on it and $f : X \to Y$, $g : Y \to X$ be two functions.*

- *f and g form a **type I** E**-including** (E**-excluding**) **Galois connection** from (X, α) to (Y, β), denoted by $[f\text{-}g]_{E^{[\uparrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($[f\text{-}g]_{E^{(\uparrow)}} : (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{\beta f} \equiv_{E^{[\uparrow]}} \mathcal{R}_{\alpha g}$ ($\mathcal{R}_{\beta f} \equiv_{E^{(\uparrow)}} \mathcal{R}_{\alpha g}$).*
- *f and g form a **type II** E**-including** (E**-excluding**) **Galois connection** from (X, α) to (Y, β), denoted by $]f\text{-}g[_{E^{[\uparrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($]f\text{-}g[_{E^{(\uparrow)}} \colon (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{f\beta} \equiv_{E^{[\uparrow]}} \mathcal{R}_{g\alpha}$ ($\mathcal{R}_{f\beta} \equiv_{(E)^{\uparrow}} \mathcal{R}_{g\alpha}$).*
- *f and g form a **type III** E**-including** (E**-excluding**) **Galois connection** from (X, α) to (Y, β), denoted by $(f, g)_{E^{[\uparrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($(f, g)_{E^{(\uparrow)}} \colon (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{f\beta} \equiv_{E^{[\uparrow]}} \mathcal{R}_{\alpha g}$ ($\mathcal{R}_{f\beta} \equiv_{E^{(\uparrow)}} \mathcal{R}_{\alpha g}$).*
- *f and g form a **type IV** E**-including** (E**-excluding**) **Galois connection** from (X, α) to (Y, β), denoted by $)f, g(_{E^{[\uparrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($)f, g(_{E^{(\uparrow)}} \colon (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{\beta f} \equiv_{E^{[\uparrow]}} \mathcal{R}_{g\alpha}$ ($\mathcal{R}_{\beta f} \equiv_{E^{(\uparrow)}} \mathcal{R}_{g\alpha}$).*

If $F \subseteq L$ is an upperset, we shall call F**-Galois connection** (of any of the four types) any upper connection relative to F.
As already remarked in the previous Section, the classification of connections of Definition 4.2.10, and of Definition 4.3.1 as well, has been arranged in such a way as to agree with a similar classification done in [51] for fuzzy Galois connections.
Of course, most statements concerning relative upper equivalences and connections in Section 4.2 might be restated now for Galois connections, including those listed in Propositions 4.2.11 and 4.2.13.
Among Galois connections, we highlight those that can be determined by singletons in L and that we call **point-including** or **point-excluding** depending on whether they are relative to $e^{[\uparrow]}$ or to $e^{(\uparrow)}$, for some $e \in L$.
Some results stated in Section 4.1 allow to realize when a Galois connection is point-including or point-excluding, as follows.

Proposition 4.3.2. *With the above notation and terminology, the following hold, for any type of Galois connections.*

1. *If E has a minimum, then any E-including Galois connection is point-including.*
2. *If $E^{(\downarrow)}$ has a maximum, then any E-including Galois connection is point-excluding.*
3. *If $E^{(\uparrow)}$ has a minimum, any E-excluding Galois connection is point-including.*
4. *If E has a maximum, then any E-excluding Galois connection is point-excluding.*

Proof.

1. If E has a minimum, then there exists $e \in \langle MinE \rangle_{\equiv}$; so, by Proposition 4.1.14, $e^{[\uparrow]} = E^{[\uparrow]}$ and hence, every E-including Galois connection is point-including, in particular it is e-including.

2. If $E^{(\downarrow)}$ has a maximum, then there exists $e \in MaxE^{(\downarrow)} = \left\langle MaxE^{(\downarrow)} \right\rangle_{\equiv}$; so, by Proposition 4.1.14 and since $E^{(\downarrow)}$ is a lower hull, we have that $e^{[\downarrow]} = (E^{(\downarrow)})^{[\downarrow]} = E^{(\downarrow)}$. Moreover, by Corollary 4.1.17, it follows that $E^{[\uparrow]} = (E^{(\downarrow)})^{(\uparrow)} = (e^{[\downarrow]})^{(\uparrow)} = e^{(\uparrow)}$. Hence, any E-including Galois connection is point-excluding, more precisely, it is e-excluding.

3. If $E^{(\uparrow)}$ has a minimum, then there exists $e \in MinE^{(\uparrow)} = \left\langle MinE^{(\uparrow)} \right\rangle_{\equiv}$; so, by Proposition 4.1.14 $e^{[\uparrow]} = (E^{(\uparrow)})^{[\uparrow]} = E^{(\uparrow)}$. Hence, every E-excluding Galois connection is point-including, more precisely, it is e-including.

4. If E has a maximum, then there exists $e \in \langle MaxE \rangle_{\equiv}$; so, by Proposition 4.1.14, $e^{[\downarrow]} = E^{[\downarrow]}$. Moreover, by Corollary 4.1.17, it follows that $E^{(\uparrow)} = (E^{[\downarrow]})^{(\uparrow)} = (e^{[\downarrow]})^{(\uparrow)} = e^{(\uparrow)}$. Hence, any E-excluding Galois connection is point-excluding, more precisely, it is e-excluding.

□

Remark 4.3.3. We note that if $e \in L$, then the type I e-including and e-excluding Galois connections can be expressed as follows, with the above notation.

- $[f\text{-}g]_{e^{[\uparrow]}}$, if and only if for all $x \in X, y \in Y$ $(x,y)\mathcal{R}_{\beta f} \geq e \Leftrightarrow (x,y)\mathcal{R}_{\alpha g} \geq e$;
- $[f\text{-}g]_{e^{(\uparrow)}}$, if and only if for all $x \in X, y \in Y$ $(x,y)\mathcal{R}_{\beta f} \nleq e \Leftrightarrow (x,y)\mathcal{R}_{g\alpha} \nleq e$.

The other types can be expressed similarly.

Our approach to Galois connections and to their dual Tarski connections takes its main motivations from a more general reading of the conditions expressed in the above Remark and of the corresponding dual conditions.
Galois connections arise in several, different contexts and we believe that the algebraic structure of logic is the one which motivates and illustrates all their fundamental aspects; the implicative structure of logic and, in particular, their order-theoretic approach developed in [13, 50] give a suitable framework to explain details of our viewpoint.
In the classical approaches to Galois connections (X, α) and (Y, β) are posets, possibly complete, and $L = \mathbf{2}$. However, it appears clearly from our general approach that Galois connections from (X, α) to (Y, β) do not require α and β to be order (or preorder) relations but, rather, they require a (pre)order in the set L of evaluation of such connections (which is ensured in the classical case, where $L = \mathbf{2}$).
Classical Galois connections are, in our terminology, 1-including Galois connections between posets; more precisely, antitonic Galois connections are 1-including Galois connections of type I while isotonic Galois connections are 1-including Galois connections of type III (the above Propositions show that those of type II and IV are closely related to the ones of type I and III, respectively).

More generally, the classical formulation of Galois connections can be obtained assuming $(L, \leq)$ to be a preordered set and α, β binary L-relations on X and Y, respectively, under the following notations.
Let $F \subseteq L$ be a non-empty upperset of L and let (X, α) and (Y, β) be sets, each equipped with an L-relation and consider the relations $\leq_{\alpha,F}$ and $\leq_{\beta,F}$ defined, for all $x_1, x_2 \in X, y_1, y_2 \in Y$, by $x_1 \leq_{\alpha,F} x_2 \Leftrightarrow (x_1, x_2)\alpha \in F$ and $y_1 \leq_{\beta,F} y_2 \Leftrightarrow (y_1, y_2)\beta \in F$.
Under such assumptions, the four types of F-Galois connections from (X, α) to (Y, β) can be expressed as follows.

Proposition 4.3.4. *With the above notation, for all $f : X \to Y$, $g : Y \to X$, the following hold:*

- $[f\text{-}g]_F : (X, \alpha) \to (Y, \beta) \Leftrightarrow \forall x \in X, y \in Y\colon\ y \leq_{\beta,F} (x)f \Leftrightarrow x \leq_{\alpha,F} (y)g$;
- $]f\text{-}g[_F : (X, \alpha) \to (Y, \beta) \Leftrightarrow \forall x \in X, y \in Y\colon\ (x)f \leq_{\beta,F} y \Leftrightarrow (y)g \leq_{\alpha,F} x$;
- $(f, g)_F : (X, \alpha) \to (Y, \beta) \Leftrightarrow \forall x \in X, y \in Y\colon\ (x)f \leq_{\beta,F} y \Leftrightarrow x \leq_{\alpha,F} (y)g$;
- $)f, g(_F : (X, \alpha) \to (Y, \beta) \Leftrightarrow \forall x \in X, y \in Y\colon\ y \leq_{\beta,F} (x)f \Leftrightarrow (y)g \leq_{\alpha,F} x$. □

In the classical case $\leq_{\alpha,F}$ and $\leq_{\beta,F}$ are (pre)order relations; in such a case the four types of Galois connections may be characterized as pairs of "functors" that form "adjunctions" (as in [51]), i.e. as isotonic or antitonic maps that satisfy suitable *adjoint inequalities*. However, the only assumption needed to characterize Galois connections by means of such kind of inequalities is the transitivity of the relations $\leq_{\alpha,F}$ and $\leq_{\beta,F}$.
One may think of such an assumption and extend the usual notation saying that a function $f : (X, \leq_\alpha) \to (Y, \leq_\beta)$ is *isotonic* if f preserves relations, i.e.

$$\forall x, y \in X :\ x \leq_\alpha y \Rightarrow (x)f \leq_\beta (y)f.$$

Similarly, f is said to be *antitonic* if f reverses relations, i.e.

$$\forall x, y \in X :\ x \leq_\alpha y \Rightarrow (y)f \leq_\beta (x)f.$$

Then one has easily the following characterization.

Proposition 4.3.5. *With the above notation, assuming $\leq_{\alpha,F}$ and $\leq_{\beta,F}$ to be transitive relations, the following hold.*

1. *$[f\text{-}g]_F : (X, \alpha) \to (Y, \beta)$ if and only if f, g are antitonic and satisfy the following Galois conditions:*
   (GC_1) $x \leq_{\alpha,F} ((x)f)g$, for every $x \in X$;
   (GC_2) $y \leq_{\beta,F} ((y)g)f$, for every $y \in Y$.

2. *$]f\text{-}g[_F : (X, \alpha) \to (Y, \beta)$ if and only if f, g are antitonic and satisfy the following Galois conditions:*
   (GC_1^{op}) $((x)f)g \leq_{\alpha,F} x$, for every $x \in X$;
   (GC_2^{op}) $((y)g)f \leq_{\beta,F} y$, for every $y \in Y$.

3. $(f,g)_F : (X,\alpha) \to (Y,\beta)$ *if and only if* f, g *are isotonic and satisfy the following Galois conditions:*
   (GC_1) $x \leq_{\alpha,F} ((x)f)g$, *for every* $x \in X$;
   (GC_2^{op}) $((y)g)f \leq_{\beta,F} y$, *for every* $y \in Y$.

4. $)f,g(_F\colon (X,\alpha) \to (Y,\beta)$ *if and only if* f, g *are isotonic and satisfy the following Galois conditions:*
   (GC_1^{op}) $((x)f)g \leq_{\alpha,F} x$, *for every* $x \in X$;
   (GC_2) $y \leq_{\beta,F} ((y)g)f$, *for every* $y \in Y$.

Proof. (1) " $\Rightarrow$ " If $[f\text{-}g]_F$, from Proposition 4.3.4 the following hold:
(GC_1) $(x)f \leq_{\beta,F} (x)f \Rightarrow x \leq_{\alpha,F} ((x)f)g$, for every $x \in X$;
(GC_2) $(y)g \leq_{\alpha,F} (y)g \Rightarrow y \leq_{\beta,F} ((y)g)f$, for every $y \in Y$.
Moreover, let $a, b \in X$ be such that $a \leq_{\alpha,F} b$; by (GC_1), we have that:
$a \leq_{\alpha,F} b \leq_{\alpha,F} ((b)f)g \Rightarrow (b)f \leq_{\beta,F} (a)f$.
Similarly, we can prove that g is antitonic, too.
" $\Leftarrow$ " Let $f, g : L \to L$ be two antitonic functions that satisfy (GC_1) and (GC_2).
So, for all $x \in X, y \in Y$ the following implications hold:

$$y \leq_{\beta,F} (x)f \Rightarrow ((x)f)g \leq_{\alpha,F} (y)g \Rightarrow x \leq_{\alpha,F} ((x)f)g \leq_{\alpha,F} (y)g \Rightarrow x \leq_{\alpha,F} (y)g$$

$$x \leq_{\alpha,F} (y)g \Rightarrow ((y)g)f \leq_{\beta,F} (x)f \Rightarrow y \leq_{\beta,F} ((y)g)f \leq_{\beta,F} (x)f \Rightarrow y \leq_{\beta,F} (x)f.$$

Hence, for all $x \in X, y \in Y : y \leq_{\beta,F} (x)f \Leftrightarrow x \leq_{\alpha,F} (y)g$, i.e. $[f\text{-}g]_F$.
The items (2), (3) and (4) are proved dually by using Proposition 4.2.13. □

With the above notation and assuming $\leq_{\alpha,F}$ and $\leq_{\beta,F}$ to be preorders, each term of a Galois connection can be determined by the other one, up to equivalence in X or in Y, as follows.

Proposition 4.3.6. *With the above notation, assuming* $\leq_{\alpha,F}$ *and* $\leq_{\beta,F}$ *to be preorder relations, the following hold, if the needed sups and infs are non-empty.*

1. *If* $[f\text{-}g]_F : (X,\alpha) \to (Y,\beta)$, *then*

$$\forall x \in X, y \in Y : (y)g \in \bigvee \left\{x \middle|\ y \leq_{\beta,F} (x)f\right\} \text{ and } (x)f \in \bigvee \{y|\ x \leq_{\alpha,F} (y)g\}.$$

2. *If* $]f\text{-}g[_F\colon (X,\alpha) \to (Y,\beta)$, *then*

$$\forall x \in X, y \in Y : (y)g \in \bigwedge \left\{x \middle|\ (x)f \leq_{\beta,F} y\right\} \text{ and } (x)f \in \bigwedge \{y|\ (y)g \leq_{\alpha,F} x\}.$$

3. *If* $(f,g)_{[F} : (X,\alpha) \to (Y,\beta)$, *then*

$$\forall x \in X, y \in Y : (y)g \in \bigvee \left\{x \middle|\ (x)f \leq_{\beta,F} y\right\} \text{ and } (x)f \in \bigwedge \{y|\ x \leq_{\alpha,F} (y)g\}.$$

4. $)f,g(_F\colon (X,\alpha) \to (Y,\beta)$, *then*

$$\forall x \in X, y \in Y : (y)g \in \bigwedge \left\{x|\ y \leq_{\beta,F} (x)f\right\} \text{ and } (x)f \in \bigvee \{y|\ (y)g \leq_{\alpha,F} x\}.$$

Proof. (1) If $[f\text{-}g]_F : (X,\alpha) \to (Y,\beta)$, then $\forall x \in X, y \in Y$: $(y,(x)f)\beta \in F \Leftrightarrow (x,(y)g)\alpha \in F$, i.e. $\forall x \in X, y \in Y$: $y \leq_{\beta,F} (x)f \Leftrightarrow x \leq_{\alpha,F} (y)g$.
Hence, for every $y \in Y$: $(y)g \in Ub\left\{x|\ y \leq_{\beta,F} (x)f\right\}$. Moreover, let $x' \in X, x \leq_{\alpha,F} x'$, for every $x \in X$ such that $y \leq_{\beta,F} (x)f$, i.e. such that $x \leq_{\alpha,F} (y)g$. In particular, for $x = (y)g$, we have that $(y)g = x \leq_{\alpha,F} x'$ and hence $(y)g \in \bigvee\left\{x|\ y \leq_{\beta,F} (x)f\right\}$. Similarly, or recalling Proposition 4.2.13, we can prove that $(x)f \in \bigvee\{y|\ x \leq_{\alpha,F} (y)g\}$.

(2) If $]f\text{-}g[_F$: $(X,\alpha) \to (Y,\beta)$, then $\forall x \in X, y \in Y$: $((x)f,y)\beta \in F \Leftrightarrow ((y)g,x)\alpha \in F$, i.e. $\forall x \in X, y \in Y$: $(x)f \leq_{\beta,F} y \Leftrightarrow (y)g \leq_{\alpha,F} x$.
Hence, for every $y \in Y$ $(y)g \in Lb\left\{x|\ (x)f \leq_{\beta,F} y\right\}$. Moreover, let $x' \in X, x' \leq_{\alpha,F} x$, for every $x \in X$ such that $(x)f \leq_{\beta,F} y$, i.e. such that $(y)g \leq_{\alpha,F} x$. In particular, for $x = (y)g$, we have that $x' \leq_{\alpha,F} x = (y)g$ and hence $(y)g \in \bigwedge\left\{x|\ (x)f \leq_{\beta,F} y\right\}$. Similarly, or recalling Proposition 4.2.13, we can prove that $(x)f \in \bigwedge\left\{y|\ (y)g \leq_{\beta,F} x\right\}$.
The items (3) and (4) are proved dually or by using Proposition 4.2.13. □

From Corollary 4.1.17, we have that $E^{[\uparrow]} = (E^{(\downarrow)})^{(\uparrow)}$ and hence f and g form a (type I, II, III or IV) E-including Galois connection if and only if f and g form a (type I, II, III or IV) $E^{(\downarrow)}$-excluding Galois connection.
Moreover, since $E^{(\uparrow)}$ is an upperset and hence $E^{(\uparrow)} = (E^{(\uparrow)})^{[\uparrow]}$, we have that f and g form a (type I, II, III or IV) E-excluding Galois connection if and only if f and g form a (type I, II, III or IV) $E^{(\uparrow)}$-including Galois connection.
With the above notations and thanks to these properties we can extend the results of Proposition 4.3.6 to E-including (E-excluding, respectively) Galois connections, assuming that the relations $\leq_{\alpha,E^{[\uparrow]}}$ and $\leq_{\beta,E^{[\uparrow]}}$ ($\leq_{\alpha,E^{(\uparrow)}}$ and $\leq_{\beta,E^{(\uparrow)}}$, respectively), defined as above, are preorders.
With the same assumptions as in Proposition 4.3.6, functions that may be terms of Galois connections can be characterized by their behavior with respect to infs and sups as stated in the following Proposition, which extends well known results of classical Galois connections. To simplify notation and proofs, we add the completeness assumption and denote both $\leq_{\alpha,F}$ and $\leq_{\beta,F}$ by $\leq$ and their infs and sups by $\bigwedge$ and $\bigvee$.

Proposition 4.3.7. *With the above notation and assumptions, let $f : X \to Y$ be a map and let $(X,\leq)$ be a complete prelattice.*

1. *There exists a map $g : Y \to X$ such that $[f\text{-}g]_F : (X,\alpha) \to (Y,\beta)$ if and only if $(\bigvee S)f \subseteq \bigwedge(Sf)$, for every $S \subseteq X$.*

2. *There exists a map $g : Y \to X$ such that $]f\text{-}g[_F$: $(X,\alpha) \to (Y,\beta)$ if and only if $(\bigwedge S)f \subseteq \bigvee(Sf)$, for every $S \subseteq X$.*

3. *There exists a map $g : Y \to X$ such that $(f,g)_F : (X,\alpha) \to (Y,\beta)$ if and only if $(\bigvee S)f \subseteq \bigvee(Sf)$, for every $S \subseteq X$.*

4. *There exists a map $g : Y \to X$ such that $)f,g(_F$: $(X,\alpha) \to (Y,\beta)$ if and only if $(\bigwedge S)f \subseteq \bigwedge(Sf)$, for every $S \subseteq X$.*

Proof. (1) " $\Rightarrow$ " Let $a \in \bigvee S$; hence $(a)f \in (\bigvee S)f$ and $b \leq a$, for every $b \in S$. Since $[f\text{-}g]_F$, by Proposition 4.3.5, we have that f is antitonic and then $(a)f \leq (b)f$, for every $b \in S$. So, $(a)f \in Lb(Sf)$. In order to see that $(a)f \in \bigwedge(Sf)$, we prove the following implication $z \leq (b)f, \forall b \in S \Rightarrow z \leq (a)f$. Let $z \leq (b)f$; since $[f\text{-}g]_F$ the following holds: $z \leq (b)f \Leftrightarrow b \leq (z)g, \forall b \in S$. Hence $(z)g \in UbS$; since $a \in \bigvee S$, we have that $a \leq (z)g$, hence $z \leq (a)f$.

" $\Leftarrow$ " For any $y \in Y$, chose $(y)g$ in $\bigvee\{t|\ y \leq (t)f\}$. By definition of g, we have easily that, for all $x \in X, y \in Y : y \leq (x)f \Rightarrow x \leq (y)g$. Moreover, by the assumed condition saying that f is antitonic, we have that $x \leq (y)g \Rightarrow ((y)g)f \leq (x)f$, for all $x \in X, y \in Y$. Since $(y)g \in \bigvee\{t|\ y \leq (t)f\}$, for every $y \in Y$ and f maps sups into infs by assumption, it is clear that $((y)g)f \in \bigwedge\{(t)f|\ y \leq (t)f\}$ and hence, $y \leq ((y)g)f$, for every $y \in Y$. So, $x \leq (y)g \Rightarrow y \leq ((y)g)f \leq (x)f \Rightarrow y \leq (x)f$, for all $x \in X, y \in Y$. Hence, for all $x \in X, y \in Y : y \leq (x)f \Leftrightarrow x \leq (y)g$, i.e. $[f\text{-}g]_F$.

Items (2), (3) and (4) can be proved by using Proposition 4.2.13 and taking into account that $\leq_{\alpha_-,F} = (\leq_{\alpha,F})_-$ and $\leq_{\beta_-,F} = (\leq_{\beta,F})_-$. □

Proposition 4.3.8. *With the same notation and assumptions recalled in Proposition 4.3.7, let $g : Y \to X$ be a map and let $(Y, \leq)$ be a complete prelattice.*

1. *There exists a map $f : X \to Y$ such that $[f\text{-}g]_F : (X, \alpha) \to (Y, \beta)$ if and only if $(\bigvee S)g \subseteq \bigwedge(Sg)$, for every $S \subseteq L$.*

2. *There exists a map $f : X \to Y$ such that $]f\text{-}g[_F$: $(X, \alpha) \to (Y, \beta)$ if and only $(\bigwedge S)g \subseteq \bigvee(Sg)$, for every $S \subseteq L$.*

3. *There exists a map $f : X \to Y$ such that $(f, g)_F : (X, \alpha) \to (Y, \beta)$ if and only if $(\bigwedge S)g \subseteq \bigwedge(Sg)$, for every $S \subseteq L$.*

4. *There exists a map $f : X \to Y$ such that $)f, g(_F$: $(X, \alpha) \to (Y, \beta)$ if and only if $(\bigvee S)g \subseteq \bigvee(Sg)$, for every $S \subseteq L$.*

Proof. (1) Since by Proposition 4.2.13 the equivalence $[f\text{-}g]_F : (X, \alpha) \to (Y, \beta) \Leftrightarrow [g\text{-}f]_F : (Y, \beta) \to (X, \alpha)$ is true, the stated assertion follows easily by Proposition 4.3.7 interchanging f and g.

Items (2), (3) and (4) can be proved similarly by using Propositions 4.2.13 and 4.3.7. □

We can give, as an example, a quite general framework where the conditions required in the Propositions 4.3.6 and 4.3.7 are satisfied, considering a w-ceo algebra.
Let $(L, \to, \top)$ be a w-ceo algebra and let $e \in L$ be an element that satisfies the following condition:

$$(n)\ x \leq y \Leftrightarrow e \leq x \to y,\ \forall x, y \in L.$$

We note that $\top$ is such an element and, moreover, the following characterization holds.

Lemma 4.3.9. *If L is right-distributive and $\otimes$ is its adjoint product, then $e \in L$ satisfies (n) if and only if $x \otimes e = x$, for every $x \in L$ (i.e. e is a right-unit with respect to the adjoint product of L).*

Proof. Assuming (n) one has that for every $x \in L$: $x \leq x \otimes e \Leftrightarrow e \leq x \to (x \otimes e) \Leftrightarrow x \otimes e \leq x \otimes e$, which is true. The inequality $x \otimes e \leq x$ follows from Proposition 2.1.11. Conversely, assume $x \otimes e = x$, for every $x \in L$. Then $a \leq b \Rightarrow a \to b = \top \geq e$ and $e \leq a \to b \Rightarrow a \otimes e \leq b \Rightarrow a \leq b$. □

Of course, in a symmetrical cdeo algebra, $\top$ is a unit, hence it is the only element satisfying (n).
Now, if $e \in L$ satisfies (n) it is an easy exercise to express the conditions listed in Remark 4.3.3 for e-including Galois connections, assuming $X = Y = L$ and $\alpha = \beta = \to$, exactly in the classical form, with respect to the natural ordering $\leq$ of $(L, \to, \top)$. Moreover, we can restate easily all the properties seen in the general case, getting, in fact, the main classical results. For example, the following holds, for type I e-including Galois connections.

Proposition 4.3.10. *Let $(L, \to, \top)$ be a w-ceo algebra and let $e \in L$ satisfy (n). The following hold.*

1. *$f, g : L \to L$ are two functions such that $[f, g]_{e[\uparrow]} : (L, \to) \to (L, \to)$ if and only if f, g are antitonic and satisfy the following Galois conditions:*
   (GC_1) $x \leq ((x)f)g$, for every $x \in X$;
   (GC_2) $y \leq ((y)g)f$, for every $y \in Y$.
2. *If $f, g : L \to L$ are two functions such that $[f, g]_{e[\uparrow]} : (L, \to) \to (L, \to)$, then $(x)f = \bigvee \{y |\ x \leq (y)g\}$ and $(y)g = \bigvee \{x |\ y \leq (x)f\}$.*
3. *Let $f : L \to L$ be a function. There exists a map $g : L \to L$ such that $[f, g]_{e[\uparrow]} : (L, \to) \to (L, \to)$ if and only if $(\bigvee S)f = \bigwedge (Sf)$, for every $S \subseteq L$.*
4. *Let $g : L \to L$ be a function. There exists a map $f : L \to L$ such that $[f, g]_{e[\uparrow]} : (L, \to) \to (L, \to)$ if and only if $(\bigvee S)g = \bigwedge (Sg)$, for every $S \subseteq L$.* □

Similar results can be stated for the other types of e-including Galois connections.

4.4 Tarski connections

In this Section we consider the notion which is the order-dual of Galois connections, by means of relative lower equivalences. We call the new notion **Tarski connection** because in some special cases, that we shall describe below, they become conjugated pairs introduced by Tarski in [83] and studied in [61].
Most discussions already done in the preceding Section for Galois connections with respect to the more general notion of global and relative connections are still appropriate for Tarski connections, for which, moreover, technical results dual to those proved for Galois connections hold.

What will be completely new is the way Tarski connections are related to the classical notion of conjugated pairs.

Definition 4.4.1. *Let $E \subseteq L$ be a subset of the preordered set $(L, \leq)$, (X, α), (Y, β) be two sets, each with a fixed binary L-relation on it and $f : X \to Y$, $g : Y \to X$ be two functions.*

- *f and g form a **type I** E**-including** (E**-excluding**) **Tarski connection** from (X, α) to (Y, β), denoted by $[f\text{-}g]_{E^{[\downarrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($[f\text{-}g]_{E^{(\downarrow)}} \colon (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{\beta f} \equiv_{E^{[\downarrow]}} \mathcal{R}_{\alpha g}$ ($\mathcal{R}_{\beta f} \equiv_{E^{(\downarrow)}} \mathcal{R}_{\alpha g}$).*

- *f and g form a **type II** E**-including** (E**-excluding**) **Tarski connection** from (X, α) to (Y, β), denoted by $]f\text{-}g[_{E^{[\downarrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($]f\text{-}g[_{E^{(\downarrow)}} \colon (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{f\beta} \equiv_{E^{[\downarrow]}} \mathcal{R}_{g\alpha}$ ($\mathcal{R}_{f\beta} \equiv_{E^{(\downarrow)}} \mathcal{R}_{g\alpha}$).*

- *f and g form a **type III** E**-including** (E**-excluding**) **Tarski connection** from (X, α) to (Y, β), denoted by $(f, g)_{E^{[\downarrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($(f, g)_{E^{(\downarrow)}} \colon (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{f\beta} \equiv_{E^{[\downarrow]}} \mathcal{R}_{\alpha g}$ ($\mathcal{R}_{f\beta} \equiv_{E^{(\downarrow)}} \mathcal{R}_{\alpha g}$).*

- *f and g form a **type IV** E**-including** (E**-excluding**) **Tarski connection** from (X, α) to (Y, β), denoted by $)f, g(_{E^{[\downarrow]}} \colon (X, \alpha) \to (Y, \beta)$ ($)f, g(_{E^{(\downarrow)}} \colon (X, \alpha) \to (Y, \beta)$), if $\mathcal{R}_{\beta f} \equiv_{E^{[\downarrow]}} \mathcal{R}_{g\alpha}$ ($\mathcal{R}_{\beta f} \equiv_{E^{(\downarrow)}} \mathcal{R}_{g\alpha}$).*

If $I \subseteq L$ is a lowerset, we shall call I**-Tarski connection** (of any of the four types) any, of course lower, connection relative to I.
Of course, most statements concerning relative lower equivalences and connections in Section 4.2 might be restated now for Tarski connections, including those listed in Propositions 4.2.11 and, in particular, 4.2.13.
From Corollary 4.1.17, we have that $E^{[\downarrow]} = (E^{(\uparrow)})^{(\downarrow)}$; so, f and g form a (type I, II, III or IV) E-including Tarski connection if and only if f and g form a (type I, II, III or IV) $E^{(\uparrow)}$-excluding Tarski connection.
Moreover, since $E^{(\downarrow)}$ is a lowerset, we have that $E^{(\downarrow)} = (E^{(\downarrow)})^{[\downarrow]}$, hence f and g form a (type I, II, III or IV) E-excluding Tarski connection if and only if f and g form a (type I, II, III or IV) $E^{(\downarrow)}$-including Tarski connection.
Among Tarski connections, similarly to what has been done for Galois connections in the previous Section, we underline those determined by singletons in L, which we call **point-including** or **point-excluding** depending on whether they are relative to $d^{[\downarrow]}$ or to $d^{(\downarrow)}$, for some $d \in L$.

Some results stated in Section 4.1 allow to realize when a given Tarski connection is point-including or point-excluding, as follows.

Proposition 4.4.2. *With the above notation and terminology, the following hold, for any type of Tarski connection.*

1. *If E has a maximum, then any E-including Tarski connection is point-including.*

2. *If $E^{(\downarrow)}$ has a maximum, then any E-excluding Tarski connection is point-including.*

3. *If $E^{(\uparrow)}$ has a minimum, then any E-including Tarski connection is point-excluding.*

4. *If E has a minimum, then any E-excluding Tarski connection is point-excluding.*

Proof. It is easy to dualize the proof of Proposition 4.3.2. □

Remark 4.4.3. We note that if $E = d^{[\downarrow]}$ or $E = d^{(\downarrow)}$, $d \in L$, the definition of type I Tarski connections can be restated in this way.

- $[f\text{-}g]_{d^{[\downarrow]}}$, if and only if for all $x \in X, y \in Y$ $(x,y)\mathcal{R}_{f\beta} \le d \Leftrightarrow (x,y)\mathcal{R}_{g\alpha} \le d$;
- $[f\text{-}g]_{d^{(\downarrow)}}$, if and only if for all $x \in X, y \in Y$ $d \nleq (x,y)\mathcal{R}_{\beta f} \Leftrightarrow d \nleq (x,y)\mathcal{R}_{\alpha g}$.

The other types can be described similarly.

As already claimed, Tarski connections generalize conjugated pairs whose classical formulation in [60, 61, 83] required $(L,\le)$ to be a boolean algebra, $(X,\alpha) = (Y,\beta) = (L,\wedge)$ and $I = \{\bot\}$. In fact, Tarski's conjugated pairs are $\bot$-including Tarski connections from a boolean algebra equipped with the meet operation to itself.
To give more details of this relationship we consider now Tarski connections in a quite general context where most results on conjugated pairs in [61] can be extended. Eventually, such a context is shown to be provided by symmetrical cdeo algebras, making these algebras the most general framework where the main features of both Galois connection (as shown in the previous Section) and conjugated pairs hold.

With the general notation already used, consider the preordered set $(L,\le)$, a non-empty lower set $I \subseteq L$, the pairs (X,α), (Y,β), with $\alpha : X\times X \to L$, $\beta : Y\times Y \to L$; assume each of X and Y to be equipped with a bijection onto itself, in both cases denoted by $[\cdot]^-$, while both the inverse bijections are denoted by $[\cdot]^\sim$; eventually, assume that for all $x_1,x_2 \in X, y_1,y_2 \in Y$ the following equivalences hold:

$$(x_1,x_2^-)\alpha \in I \Leftrightarrow (x_2^\sim,x_1)\alpha \in I$$

and

$$(y_1,y_2^-)\beta \in I \Leftrightarrow (y_2^\sim,y_1)\beta \in I.$$

With respect to the fixed pairs of bijections $([\cdot]^-,[\cdot]^\sim)$, both in X and in Y, define binary relations in X and in Y denoted by $\le_{\alpha,I}$ and $\le_{\beta,I}$ as follows:

$$x_1 \le_{\alpha,I} x_2 \Leftrightarrow (x_1,x_2^-)\alpha \in I \;\Leftrightarrow (x_2^\sim,x_1)\alpha \in I,\;\; \forall x_1,x_2 \in X$$

and

$$y_1 \le_{\beta,I} y_2 \Leftrightarrow (y_1,y_2^-)\beta \in I \;\Leftrightarrow (y_2^\sim,y_1)\beta \in I,\;\; \forall y_1,y_2 \in Y.$$

Remark 4.4.4. 1. Under the stated assumptions, it is easy to prove that $[\cdot]^-$ and $[\cdot]^\sim$ are antiisomorphisms inverse of each other both in $(X,\le_{\alpha,I})$ and $(Y,\le_{\beta,I})$. In fact, by definition and since $[\cdot]^-$ and $[\cdot]^\sim$ are inverse of each other, we have that $x_1 \le_{\alpha,I} x_2 \Leftrightarrow (x_1,x_2^-)\alpha \in I \Leftrightarrow (x_1^{-\sim},x_2^-)\alpha \in I \Leftrightarrow x_2^- \le_{\alpha,I} x_1^-$, for all $x_1,x_2 \in X$. In a similar way, we can prove that $[\cdot]^\sim$ reverses $\le_{\alpha,I}$ and the same happens in Y.

2. It is also useful to note that the following equivalences hold, for all $x_1, x_2 \in X$:
   - $x_1 \leq_{\alpha,I} x_2 \Leftrightarrow (x_1, x_2^{\sim})\alpha_- \in I \Leftrightarrow (x_2^{-}, x_1)\alpha_- \in I$;
   - $x_1 \leq_{\alpha,I} x_2^{-} \Leftrightarrow x_2 \leq_{\alpha,I} x_1^{\sim}$.

   Similar equivalences hold in Y with respect to $\beta, \beta_-, \leq_{\beta,I}$.

Under such notations and assumptions, the four types of E-including Tarski connections from (X,α) to (Y,β) may be characterized as follows (note that, despite the notation resembling that used for orderings, $\leq_{\alpha,I}$ and $\leq_{\beta,I}$ are nothing but binary relations).

Proposition 4.4.5. *With the above notation and assumptions, the following are equivalent.*

1. $[f\text{-}g]_I : (X,\alpha) \to (Y,\beta)$.
2. *For all* $x \in X, y \in Y$: $(x)f \leq_{\beta,I} y^{-} \Leftrightarrow (y)g \leq_{\alpha,I} x^{-}$.
3. *For all* $x \in X, y \in Y$: $(x)f \leq_{\beta,I} y^{-} \Leftrightarrow x \leq_{\alpha,I} (y)g^{\sim}$.
4. *For all* $x \in X, y \in Y$: $y \leq_{\beta,I} (x)f^{\sim} \Leftrightarrow x \leq_{\alpha,I} (y)g^{\sim}$.
5. *For all* $x \in X, y \in Y$: $y \leq_{\beta,I} (x)f^{\sim} \Leftrightarrow (y)g \leq_{\alpha,I} x^{-}$.

Proof. With the above notation, $[f\text{-}g]_I : (X,\alpha) \to (Y,\beta)$ if and only if for all $x \in X, y \in Y$: $(y,(x)f)\beta \in I \Leftrightarrow (x,(y)g)\alpha \in I$. Then the statement follows easily from the equivalences $(x)f \leq_{\beta,I} y^{-} \Leftrightarrow (y,(x)f)\beta \in I \Leftrightarrow y \leq_{\beta,I} (x)f^{\sim}$ and $(y)g \leq_{\alpha,I} x^{-} \Leftrightarrow (x,(y)g)\alpha \in I \Leftrightarrow x \leq_{\alpha,I} (y)g^{\sim}$. □

Proposition 4.4.6. *With the above notation, the following are equivalent.*

1. $]f\text{-}g[_I : (X,\alpha) \to (Y,\beta)$.
2. *For all* $x \in X, y \in Y$: $y \leq_{\beta,I} (x)f^{-} \Leftrightarrow x \leq_{\alpha,I} (y)g^{-}$.
3. *For all* $x \in X, y \in Y$: $y \leq_{\beta,I} (x)f^{-} \Leftrightarrow (y)g \leq_{\alpha,I} x^{\sim}$.
4. *For all* $x \in X, y \in Y$: $(x)f \leq_{\beta,I} y^{\sim} \Leftrightarrow (y)g \leq_{\alpha,I} x^{\sim}$.
5. *For all* $x \in X, y \in Y$: $(x)f \leq_{\beta,I} y^{\sim} \Leftrightarrow x \leq_{\alpha,I} (y)g^{-}$.

Proof. The proof of these equivalences is similar to that of Proposition 4.4.5. □

Proposition 4.4.7. *With the above notation, the following are equivalent.*

1. $(f,g)_I : (X,\alpha) \to (Y,\beta)$.
2. *For all* $x \in X, y \in Y$: $y \leq_{\beta,I} (x)f^{-} \Leftrightarrow (y)g \leq_{\alpha,I} x^{-}$.
3. *For all* $x \in X, y \in Y$: $y \leq_{\beta,I} (x)f^{-} \Leftrightarrow x \leq_{\alpha,I} (y)g^{\sim}$.

4. *For all* $x \in X, y \in Y$*:* $(x)f \leq_{\beta,I} y^{\sim} \Leftrightarrow x \leq_{\alpha,I} (y)g^{\sim}$.

5. *For all* $x \in X, y \in Y$*:* $(x)f \leq_{\beta,I} y^{\sim} \Leftrightarrow (y)g \leq_{\alpha,I} x^{-}$.

Proof. The proof of these equivalences is similar to that of Proposition 4.4.5. □

Proposition 4.4.8. *With the above notation, the following are equivalent.*

1. $)f,g(_{[I]^{\downarrow}}: (X,\alpha) \to (Y,\beta)$.

2. *For all* $x \in X, y \in Y$*:* $(x)f \leq_{\beta,I} y^{-} \Leftrightarrow x \leq_{\alpha,I} (y)g^{-}$.

3. *For all* $x \in X, y \in Y$*:* $(x)f \leq_{\beta,I} y^{-} \Leftrightarrow (y)g \leq_{\alpha,I} x^{\sim}$.

4. *For all* $x \in X, y \in Y$*:* $y \leq_{\beta,I} (x)f^{\sim} \Leftrightarrow (y)g \leq_{\alpha,I} x^{\sim}$.

5. *For all* $x \in X, y \in Y$*:* $y \leq_{\beta,I} (x)f^{\sim} \Leftrightarrow x \leq_{\alpha,I} (y)g^{-}$.

Proof. The proof of these equivalences is similar to that of Proposition 4.4.5. □

Proposition 4.4.9. *With the above notation and assumptions, if* $\leq_{\alpha,I}$ *and* $\leq_{\beta,I}$ *are transitive, the following hold.*

1. $[f\text{-}g]_I : (X,\alpha) \to (Y,\beta)$ *if and only if* f, g *are isotonic and they satisfy the following Tarski conditions:*
   $(TC_1^{\sim\sim})$ $x \leq_{\alpha,I} ((x)f^{\sim})g^{\sim}$, *for every* $x \in X$*;*
   $(TC_2^{\sim\sim})$ $y \leq_{\beta,I} ((y)g^{\sim})f^{\sim}$, *for every* $y \in Y$.

2. $]f\text{-}g[_I: (X,\alpha) \to (Y,\beta)$ *if and only if* f, g *are isotonic and they satisfy the following Tarski conditions:*
   (TC_1^{--}) $x \leq_{\alpha,I} ((x)f^{-})g^{-}$, *for every* $x \in X$*;*
   (TC_2^{--}) $y \leq_{\beta,I} ((y)g^{-})f^{-}$, *for every* $y \in Y$.

3. $(f,g)_I : (X,\alpha) \to (Y,\beta)$ *if and only if* f, g *are isotonic and they satisfy the following Tarski conditions:*
   $(TC_1^{-\sim})$ $x \leq_{\alpha,I} ((x)f^{-})g^{\sim}$, *for every* $x \in X$*;*
   $(TC_2^{\sim-})$ $y \leq_{\beta,I} ((y)g^{\sim})f^{-}$, *for every* $y \in Y$.

4. $)f,g(_I: (X,\alpha) \to (Y,\beta)$ *if and only if* f, g *are isotonic and they satisfy the following Tarski conditions:*
   $(TC_1^{\sim-})$ $x \leq_{\alpha,I} ((x)f^{\sim})g^{-}$, *for every* $x \in X$*;*
   $(TC_2^{-\sim})$ $y \leq_{\beta,I} ((y)g^{-})f^{\sim}$, *for every* $y \in Y$.

Proof. (1) " $\Rightarrow$ " By assumptions and the Proposition 4.4.5 (3), for every $x \in X$: $(x)f = (x)f^{\sim-} \Rightarrow x \leq_{\alpha,I} ((x)f^{\sim})g^{\sim}$, i.e. $(TC_1^{\sim\sim})$. Similarly, by Proposition 4.4.5 (5), for every $y \in Y$: $(y)g = (y)g^{\sim-} \Rightarrow y \leq_{\beta,I} ((y)g^{\sim})f^{\sim}$, i.e. $(TC_2^{\sim\sim})$.
Now, let $a, b \in X$, $a \leq_{\alpha,I} b$; then $a \leq_{\alpha,I} b \leq_{\alpha,I} ((b)f^{\sim})g^{\sim} \Rightarrow (a)f \leq_{\beta,I} (b)f^{\sim-} = (b)f$. Similarly, $(a)g \leq_{\alpha,I} (b)g$.

" $\Leftarrow$ " Let $f, g : L \to L$ be isotonic functions that satisfy $(TC_1^{\sim\sim})$ and $(TC_2^{\sim\sim})$. Then $f^{\sim}$ and $g^{\sim}$ are antitonic and, for all $x \in X, y \in Y$ the following implications hold:

$$(x)f \leq_{\beta,I} y^{-} \Rightarrow y \leq (x)f^{\sim} \Rightarrow x \leq ((x)f^{\sim})g^{\sim} \leq (y)g^{\sim} \Rightarrow x \leq_{\alpha,I} (y)g^{\sim}$$

$$x \leq_{\alpha,I} (y)g^{\sim} \Rightarrow y \leq_{\beta,I} ((y)g^{\sim})f^{\sim} \leq_{\beta,I} (x)f^{\sim} \Rightarrow y \leq_{\beta,I} (x)f^{\sim} \Rightarrow (x)f \leq_{\beta,I} y^{-}.$$

Hence, for all $x \in X, y \in Y : (x)f \leq_{\beta,I} y^{-} \Leftrightarrow x \leq_{\alpha,I} (y)g^{\sim}$, i.e. $[f\text{-}g]_I$.

The items (2), (3) and (4) can be proved in a similar way. □

The following results generalize those proved in [61]. In these results we assume $\leq_{\alpha,I}$ and $\leq_{\beta,I}$ to be preorders and denote both of them by $\leq$ to simplify notation. $\bigvee$ and $\bigwedge$ are referred to the appropriate preorder and are assumed to be non-empty, when this is needed.

Proposition 4.4.10. *With the above notation and specified assumptions, the following hold, if the needed infs and sups are non-empty.*

1. *If* $[f\text{-}g]_I : (X, \alpha) \to (Y, \beta)$, *then*

$$\forall x \in X, y \in Y : (x)f \in \bigwedge\{y^{-} \mid (y)g \leq x^{-}\} \text{ and } (y)g \in \bigwedge\{x^{-} \mid (x)f \leq y^{-}\}.$$

2. *If* $]f\text{-}g[_I : (X, \alpha) \to (Y, \beta)$, *then*

$$\forall x \in X, y \in Y : (x)f \in \bigwedge\{y^{\sim} \mid (y)g \leq x^{\sim}\} \text{ and } (y)g \in \bigwedge\{x^{\sim} \mid (x)f \leq y^{\sim}\}.$$

3. *If* $(f, g)_I : (X, \alpha) \to (Y, \beta)$, *then*

$$\forall x \in X, y \in Y : (x)f \in \bigwedge\{y^{\sim} \mid (y)g \leq x^{-}\} \text{ and } (y)g \in \bigwedge\{x^{-} \mid (x)f \leq y^{\sim}\}.$$

4. $)f, g(_I : (X, \alpha) \to (Y, \beta)$, *then*

$$\forall x \in X, y \in Y : (x)f \in \bigwedge\{y^{-} \mid (y)g \leq x^{\sim}\} \text{ and } (y)g \in \bigwedge\{x^{\sim} \mid (x)f \leq y^{-}\}.$$

Proof. (1) If $[f\text{-}g]_I : (X, \alpha) \to (Y, \beta)$, then $\forall x \in X, y \in Y$: $(y, (x)f)\beta \in I \Leftrightarrow (x, (y)g)\alpha \in I$, i.e. $(x)f \leq y^{-} \Leftrightarrow (y)g \leq x^{-}$. So, $\forall x \in X, y \in Y$ $(y)g$ is a lower bound of the set $\{x^{-} \mid (x)f \leq y^{-}\}$. Let $x' \in X$, such that $x' \leq x^{-}$, for every $x \in X$, such that $(x)f \leq y^{-}$, i.e. such that $x \leq (y)g^{\sim}$. For $x = (y)g^{\sim}$, we have that $x' \leq ((y)g^{\sim})^{-} = (y)g$. So, $(y)g \in \bigwedge\{x^{-} \mid (x)f \leq y^{-}\}$. Similarly, we can prove that $(x)f \in \bigwedge\{y^{-} \mid (y)g \leq x^{-}\}$, $\forall x \in X, y \in Y$.

The items (2), (3) and (4) can be proved similarly. □

Proposition 4.4.11. *With the notation and assumption already recalled in Proposition 4.4.10, let* $f : X \to Y$ *be a map and* $(X, \leq)$ *a complete prelattice. Then the following hold.*

1. *There exists a map* $g : Y \to X$ *such that* $[f\text{-}g]_I : (X,\alpha) \to (Y,\beta)$ *if and only if* $(\bigvee S)f \subseteq \bigvee(Sf)$*, for every* $S \subseteq X$*.*

2. *There exists a map* $g : Y \to X$ *such that* $]f\text{-}g[_I: (X,\alpha) \to (Y,\beta)$ *if and only if* $(\bigvee S)f \subseteq \bigvee(Sf)$*, for every* $S \subseteq X$*.*

3. *There exists a map* $g : Y \to X$ *such that* $(f,g)_I : (X,\alpha) \to (Y,\beta)$ *if and only if* $(\bigvee S)f \subseteq \bigvee(Sf)$*, for every* $S \subseteq X$*.*

4. *There exists a map* $g : Y \to X$ *such that* $)f,g(_I: (X,\alpha) \to (Y,\beta)$ *if and only if* $(\bigvee S)f \subseteq \bigvee(Sf)$*, for every* $S \subseteq X$*.*

Proof. (1) " $\Rightarrow$ " Let $x \in \bigvee S$. For any $s \in S$ one has $s \le x$ and $(s)f \le (x)f$, so $(x)f \in Ub((S)f)$. Now, let $y \in Y$ belongs to $Ub((S)f)$. Then, for every $s \in S$, $(s)f \le_{\beta,I} y = y^{\sim -}$; hence $s \le (y^{\sim})g^{\sim}$. So, $(y^{\sim})g^{\sim} \in Ub(S)$ and since $x \in \bigvee S$, we have that $x \le (y^{\sim})g^{\sim}$ and hence, by Proposition 4.4.5 (3), for every $y \in Y$, $(x)f \le y^{\sim -} = y$.
" $\Leftarrow$ " For any $y \in Y$, chose $(y)g$ in $\bigwedge\{a^-|\ (a)f \le y^-\}$. By definition of g, we have easily that, for all $x \in X, y \in Y : (x)f \le y^- \Rightarrow (y)g \le x^-$. Moreover, it can be seen that f is isotonic. In fact, for all $x_1, x_2 \in X$, it follows by assumption that $x_1 \le x_2 \Rightarrow x_2 \in \bigvee\{x_1, x_2\} \Rightarrow (x_2)f \in \bigvee\{(x_1)f, (x_2)f\} \Rightarrow (x_1)f \le (x_2)f$. Now assume $(y)g \le x^-$, i.e. $x \le (y)g^{\sim}$; then $(x)f \le ((y)g^{\sim})f$, for all $x \in X, y \in Y$.
Since $(y)g \in \bigwedge\{a^-|\ (a)f \le y^-\}$, for every $y \in Y$, by Remark 4.4.4 we have that $(y)g^{\sim} \in \bigvee\{a^{-\sim}|\ (a)f \le y^-\} = \bigvee\{a|\ (a)f \le y^-\}$, for every $y \in Y$. So, by the assumption, $((y)g^{\sim})f \in \bigvee\{(a)f|\ (a)f \le y^-\}$ and then $((y)g^{\sim})f \le y^-$, for every $y \in Y$, since $y^- \in Ub\{(a)f|\ (a)f \le y^-\}$. Hence, $(y)g \le x^- \Rightarrow (x)f \le ((y)g^{\sim})f \le y^- \Rightarrow (x)f \le y^-$, for all $x \in X, y \in Y$. So, for all $x \in X, y \in Y : (x)f \le y^- \Leftrightarrow (y)g \le x^-$, i.e. $[f\text{-}g]_I$.
Items (2), (3) and (4) can be proved similarly. □

Proposition 4.4.12. *With the notation and assumptions already recalled in Propositions 4.4.10 and 4.4.11, let* $g : Y \to X$ *be a map and let* $(Y, \le)$ *be a complete prelattice.*

1. *There exists a map* $f : X \to Y$ *such that* $[f\text{-}g]_I : (X,\alpha) \to (Y,\beta)$ *if and only if* $(\bigvee S)g \subseteq \bigvee(Sg)$*, for every* $S \subseteq Y$*.*

2. *There exists a map* $f : X \to Y$ *such that* $]f\text{-}g[_I: (X,\alpha) \to (Y,\beta)$ *if and only* $(\bigvee S)g \subseteq \bigvee(Sg)$*, for every* $S \subseteq Y$*.*

3. *There exists a map* $f : X \to Y$ *such that* $(f,g)_I : (X,\alpha) \to (Y,\beta)$ *if and only if* $(\bigvee S)g \subseteq \bigvee(Sg)$*, for every* $S \subseteq Y$*.*

4. *There exists a map* $f : X \to Y$ *such that* $)f,g(_I: (X,\alpha) \to (Y,\beta)$ *if and only if* $(\bigvee S)g \subseteq \bigvee(Sg)$*, for every* $S \subseteq Y$*.*

Proof. For items (1) and (2), note that, by Propositions 4.2.13, 4.4.11, there is a map f such that $[f\text{-}g]_I : (X,\alpha) \to (Y,\beta)$ or $]f\text{-}g[_I: (X,\alpha) \to (Y,\beta)$ if and only if there is a map f such that $[g\text{-}f]_I : (Y,\beta) \to (X,\alpha)$ or $]g\text{-}f[_I: (Y,\beta) \to (X,\alpha)$, respectively, which holds if and only if $(\bigvee S)g \subseteq \bigvee(Sg)$, for every $S \subseteq Y$.

For items (3) and (4) consider that, by Proposition 4.2.13, the statements hold if and only if there is a map f such that $(g,f)_I : (Y,\beta_-) \to (X,\alpha_-)$ or $)g,f(_I : (Y,\beta_-) \to (X,\alpha_-)$, respectively.
Now, it can be seen that the pair of bijections $([\cdot]^\sim,[\cdot]^-)$, both in X and in Y, has the same role with respect to α_- and β_- as the pair$([\cdot]^-,[\cdot]^\sim)$ has with respect to α and β. In fact, the following equivalences hold, for all $x_1,x_2 \in X, y_1,y_2 \in Y$:

$$(x_1,x_2^\sim)\alpha_- \in I \Leftrightarrow (x_2^\sim,x_1)\alpha \in I \Leftrightarrow (x_1,x_2^-)\alpha \in I \Leftrightarrow (x_2^-,x_1)\alpha_- \in I;$$

$$(y_1,y_2^\sim)\beta_- \in I \Leftrightarrow (y_2^\sim,y_1)\beta \in I \Leftrightarrow (y_1,y_2^-)\beta \in I \Leftrightarrow (y_2^-,y_1)\beta_- \in I.$$

So, the binary relations $\leq_{\alpha,I}$ and $\leq_{\beta,I}$ may be characterized as follows, for all $x_1,x_2 \in X, y_1,y_2 \in Y$:

$$x_1 \leq_{\alpha,I} x_2 \Leftrightarrow (x_1,x_2^\sim)\alpha_- \in I \Leftrightarrow (x_2^-,x_1)\alpha_- \in I$$

and

$$y_1 \leq_{\beta,I} y_2 \Leftrightarrow (y_1,y_2^\sim)\beta_- \in I \Leftrightarrow (y_2^-,y_1)\beta_- \in I.$$

With the simplified notation of Proposition 4.4.11, one gets a map f such that $(g,f)_I : (Y,\beta_-) \to (X,\alpha_-)$ if and only if $(\bigvee S)g \subseteq \bigvee(Sg)$, for every $S \subseteq Y$. □

As for Galois connections, extended-order algebras give a context where the requirements are satisfied, which allows to extend the classical results of conjugated pairs to Tarski connections. We generalize the notion of negation, just introduced in Section 2.1, speaking of **d-negation**, i.e. of negation relative to an element d of the considered algebra.
In the following Definition we list and explain terms and symbols most of which are widely used by people working with algebraic aspects of many-valued logics.

Definition 4.4.13. *Let $(L,\to,\top)$ a w-eo algebra. For every $d \in L$ the **d-negation** is the unary operation $[\cdot]^{-d} : L \to L$, $x \mapsto x^{-d} = x \to d$, for every $x \in L$.*
*If L is symmetrical, we can define a **dual** **d-negation** $[\cdot]^{\sim d} : L \to L$, $x \mapsto x^{\sim d} = x \rightsquigarrow d$, for every $x \in L$.*
*The d-negation $[\cdot]^{-d}$ ($[\cdot]^{\sim d}$, respectively) is said to be **involutive** if $x^{-d-d} = x$ ($x^{\sim d\sim d} = x$), for every $x \in L$.*
*The d-negations $[\cdot]^{-d}$ and $[\cdot]^{\sim d}$ are said to be **cross-involutive** if $x^{\sim d-d} = x^{-d\sim d} = x$, for every $x \in L$.*
*An element $d \in L$ that determines cross-involutive d-negations, i.e. that satisfies the condition $x^{\sim d-d} = x^{-d\sim d} = x$, for every $x \in L$, is called **dualizing** (see also [51]). A symmetrical w-eo algebra is said to be d-**good** if $x^{\sim d-d} = x^{-d\sim d}$, for every $x \in L$.*

Remark 4.4.14. 1. We recall that, if L has the minimum $\bot$, the $\bot$-negations $[\cdot]^{-\bot}$ and $[\cdot]^{\sim\bot}$ are simply called **negations** and are denoted by $[\cdot]^-$ and $[\cdot]^\sim$.
A symmetrical w-eo algebra is called **good** if $x^{\sim-} = x^{-\sim}$, for every $x \in L$ and it is called **cross-involutive** if the negations $[\cdot]^-$ and $[\cdot]^\sim$ are.

2. In the classical case of boolean algebras, the involutivity of the complementation is equivalent to require that the element $\bot$ is dualizing with respect to the $\bot$-negation (the usual complementation), that is self-dual, due to the commutativity assumption which implies $\rightarrow = \rightsquigarrow$.

Now consider a symmetrical cdeo algebra $(L, \rightarrow, \top)$ and let $d \in L$ be a dualizing element. Looking at notation used above for Tarski connections, let us reduce to the case where $(L, \leq)$ has the natural ordering of the algebra, $X = Y = L$ and $\alpha = \beta = \otimes$.
The d-negations $[\cdot]^{-d}$ and $[\cdot]^{\sim d}$ are bijections inverse of each other, thanks to the cross-involutivity assumption. Moreover, for all $x_1, x_2 \in L$ the following equivalences hold: $(x_1, x_2^{-d})\otimes \in d^{[\downarrow]} \Leftrightarrow x_1 \otimes x_2^{-d} \leq d \Leftrightarrow x_1 \otimes (x_2 \rightarrow d) \leq d \Leftrightarrow x_1 \leq (x_2 \rightarrow d) \rightsquigarrow d \Leftrightarrow x_1 \leq x_2 \Leftrightarrow x_1 \leq (x_2 \rightsquigarrow d) \rightarrow d \Leftrightarrow x_2^{\sim d} \otimes x_1 \leq d \Leftrightarrow (x_2^{\sim d}, x_1)\otimes \in d^{[\downarrow]}$. Eventually, the binary relation induced on L by $\otimes$ and $d^{[\downarrow]}$ is the natural ordering $\leq$; in fact it is determined, for all $x_1, x_2 \in L$, by:

$$x_1 \leq_{\otimes, d^{[\downarrow]}} x_2 \Leftrightarrow (x_1, x_2^{-})\otimes \in d^{[\downarrow]} \Leftrightarrow x_1 \otimes (x_2 \rightarrow d) \leq d \Leftrightarrow x_1 \leq (x_2 \rightarrow d) \rightsquigarrow d \Leftrightarrow x_1 \leq x_2.$$

So, all the requirements needed to let the statements of Propositions 4.4.5-4.4.12 be true are satisfied. Moreover, with the above notation, all the results proved in the general case can be proved for Tarski connections between symmetrical cdeo algebras, now involving the adjoint product which corresponds to the boolean product of the classical case. We only give as an example, the following statement, which is expressed in the shape of a similar result in [61].

Proposition 4.4.15. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra, with adjoint product $\otimes$ and let d be a dualizing element of L.*

1. *If $f, g : L \rightarrow L$ are two functions such that $[f, g]_{d^{[\downarrow]}} : (L, \otimes) \rightarrow (L, \otimes)$, then*

$$(y)g = \bigwedge \{x^{\sim d} | (x)f \otimes y \leq d\} = (\bigvee \{x | (x)f \leq y^{\sim d}\})^{\sim d}$$

and

$$(x)f = \bigwedge \{y^{\sim d} | (y)g \otimes x \leq d\} = (\bigvee \{y | x \leq (y)g^{-d}\})^{\sim d}.$$

2. *If $f : L \rightarrow L$ is any function, then there exists a map $g : L \rightarrow L$ such that $[f, g]_{d^{[\downarrow]}} : (L, \otimes) \rightarrow (L, \otimes)$ if and only if f preserves $\bigvee$.* □

Remark 4.4.16. Assuming $(L, \rightarrow, \top)$ to be a symmetrical cross-involutive cdeo algebra, the $\bot$-including Tarski connections from $(L, \otimes)$ to $(L, \otimes)$ generalize the conjugated pairs considered in [61] and recalled in Definition 1.2.9.
Nevertheless, unlike in the classical case of boolean algebras, there are the following four types of $\bot$-including Tarski connections from $(L, \otimes)$ to $(L, \otimes)$:

- $[f, g]_{\bot^{[\downarrow]}} : (L, \otimes) \rightarrow (L, \otimes)$: $y \otimes (x)f = \bot \Leftrightarrow x \otimes (y)g = \bot$, for all $x, y \in L$;
- $]f, g[_{\bot^{[\downarrow]}}: (L, \otimes) \rightarrow (L, \otimes)$: $(x)f \otimes y = \bot \Leftrightarrow (y)g \otimes x = \bot$, for all $x, y \in L$;
- $(f, g)_{\bot^{[\downarrow]}} : (L, \otimes) \rightarrow (L, \otimes)$: $(x)f \otimes y = \bot \Leftrightarrow x \otimes (y)g = \bot$, for all $x, y \in L$;

- $)f,g(_{\perp[\downarrow]}: (L,\otimes) \to (L,\otimes)$: $y \otimes (x)f = \perp \Leftrightarrow (y)g \otimes x = \perp$, for all $x, y \in L$.

These four types of Tarski connections coincide if $\otimes$ is commutative. Moreover the four equivalent characterizations of each type described in Propositions 4.4.5-4.4.8 coincide if the negation is involutive. For instance, reducing to refer to Proposition 4.4.5, the following equivalent characterizations of $[f,g]_{\perp[\downarrow]} : (L,\otimes) \to (L,\otimes)$:

- $(x)f \le y^- \Leftrightarrow (y)g \le x^-$, for all $x, y \in L$;
- $(x)f \le y^- \Leftrightarrow x \le (y)g^-$, for all $x, y \in L$;
- $y \le (x)f^- \Leftrightarrow x \le (y)g^-$, for all $x, y \in L$;
- $y \le (x)f^- \Leftrightarrow (y)g \le x^-$, for all $x, y \in L$;

coincide under involutivity assumption of negation, which, in fact, implies that $(x)f \le y^- \Leftrightarrow y \le (x)f^-$ and $(y)g \le x^- \Leftrightarrow x \le (y)g^-$, for all $x, y \in L$.
So, for instance, conjugated pairs behave in MV-algebras just like in boolean algebras.

4.5 Some remarks on connections

In the classical mathematical context Galois connections (two types, the one isotonic and the other antitonic) and conjugated pairs (one type only) have been considered in quite specific contexts (see Section 1.2 for these classical notions). The original motivating framework for both concepts is related to the powerset operators (various types of image and preimage operators) determined by functions or, more generally, by binary relations [8, 83]. These operators are functions between the powersets of the domain and of the codomain of the relation originating them, i.e. are functions between boolean algebras. It is well known how, in such a framework, each Galois connection determines a Galois connection of the other type or a conjugated pair and how to proceed conversely, by means of the complementation in the boolean algebras (see details, for instance, in [42]).
In fact, let X, Y two sets and consider the functions $f : \mathbf{2}^X \to \mathbf{2}^Y$ and $g : \mathbf{2}^Y \to \mathbf{2}^X$. Assume that f and g form an antitonic Galois connection; the condition that characterizes it, that is

$$\forall A \in \mathbf{2}^X, B \in \mathbf{2}^Y : \; A \subseteq (B)g \Leftrightarrow B \subseteq (A)f$$

is equivalent to the following

$$\forall A \in \mathbf{2}^X, B \in \mathbf{2}^Y : \; A \cap (B)g^\neg = \varnothing \Leftrightarrow B \cap (A)f^\neg = \varnothing.$$

So the functions $h : \mathbf{2}^X \to \mathbf{2}^Y, A \mapsto (A)h = (A)f^\neg$ and $k : \mathbf{2}^Y \to \mathbf{2}^X : B \mapsto (B)k = (B)g^\neg$ form a conjugated pair.
Furthermore, if f and g form an isotonic Galois connection, the functions $h : \mathbf{2}^X \to \mathbf{2}^Y, A \mapsto (A)h = (A)f^\neg$ and $k : \mathbf{2}^Y \to \mathbf{2}^X : B \mapsto (B)k = (B^\neg)g$ form an antitonic Galois connection.

This process establishes a bijection between the classes of the two types of Galois connections and between each of them and the class of conjugated pairs.
Also, the class of any type of connections as well as the class of conjugated pairs, in this framework, is in a bijective correspondence with the class of binary relations.
In fact, the correspondence

$$r \in \mathbf{2}^{X\times Y} \mapsto (f_r, g_r)$$

where $f_r : \mathbf{2}^X \to \mathbf{2}^Y$ and $g_r : \mathbf{2}^Y \to \mathbf{2}^X$ are defined by

$$\forall A \in \mathbf{2}^X : \ (A)f_r = \{y|\ \forall x \in X : x \in A \Rightarrow \ xry\}$$

$$\forall B \in \mathbf{2}^Y : \ (B)g_r = \{x|\ \forall y \in Y : y \in B \Rightarrow \ xry\}$$

is a bijection between the class of binary relations and the class of antitonic Galois connections.
Moreover, the correspondence

$$r \in \mathbf{2}^{X\times Y} \mapsto (f'_r, g'_r)$$

where $f'_r : \mathbf{2}^X \to \mathbf{2}^Y$ and $g'_r : \mathbf{2}^Y \to \mathbf{2}^X$ are defined by

$$\forall A \in \mathbf{2}^X : \ (A)f'_r = \{y|\ \exists x \in X : x \in A,\ xry\}$$

$$\forall B \in \mathbf{2}^Y : \ (B)g'_r = \{x|\ \forall y \in Y : \ xry \Rightarrow y \in B\}$$

is a bijection between the class of binary relations and the class of isotonic Galois connections.
Similarly, the correspondence

$$r \in \mathbf{2}^{X\times Y} \mapsto (h_r, k_r)$$

where $h_r : \mathbf{2}^X \to \mathbf{2}^Y$ and $k_r : \mathbf{2}^Y \to \mathbf{2}^X$ are defined by

$$\forall A \in \mathbf{2}^X : \ (A)h_r = \{y|\ \exists x \in X : x \in A,\ xry\}$$

$$\forall B \in \mathbf{2}^Y : \ (B)k_r = \{x|\ \exists y \in Y : x \in B,\ xry\}$$

is a bijection between the class of binary relations and the class of conjugated pairs.
The development of fuzzy set theory and lattice-valued mathematics has changed the algebraic feature of the main framework where connections are considered, moving from classical logic to many-valued logics and, consequently, from boolean algebras to more general lattice-ordered algebras. This has not only changed the algebraic aspects of the framework but has also led to a more general formulation of the concepts, which are widely known as fuzzy Galois connections.
In the classical approach we have recalled in Section 1.2, which extends that briefly discussed above, Galois connections, unlike conjugated pairs, have been described as pairs of maps between posets.
This general framework of posets or, as one should say, of preordered sets clearly relates Galois connections to the important mathematical concept of adjunctions between

categories and, in fact, isotonic Galois connections are also called *adjunctions*, since the isotonic maps that form such connections are functors that form an adjunction between the skeletal categories determined by the involved posets. Antitonic Galois connections may be reduced to the same framework, by means of opposite categories (see also [1]).
This point of view has been further developed in the fuzzy version in [51] where the authors introduce and classify Galois connections as adjunctions between some trivial kinds of categories enriched over suitable monoidal-closed categories. This and that considered in [33] are the only approaches to Galois connections which use explicitly some kinds of lattice-ordered implicative structure: we do the same to relate our general approach to the classical frameworks (see the last part of Sections 4.3 and 4.4).
Among the so many approaches to Galois connections, [33, 35, 42] are the only, to our knowledge, that consider also conjugated pairs, without clarifying, however, how these are related to Galois connections. Even if G. Georgescu and A. Popescu have arranged in [42] a common framework for both concepts, they claim that such a framework is rather "artificious" and does not establish any "duality" or, as they should say, any "triality" linking the apparently different notions of isotonic Galois connections, antitonic ones and conjugated pairs.
It seems that G. Georgescu and A. Popescu tried to find such a duality without success, as the title of their paper (*Non-dual fuzzy connections*) proves. Our general approach, instead, shows clearly how the whole of conjugated pairs (Tarski connections, in our terminology) is perfectly order-dual to the whole of Galois connections.
Even more, we can say that the essence of both Galois and Tarski connections is the same, not only because in their global versions both are nothing else than global connections but also because, yet in their relative versions, each Galois connection is a Tarski connection, too, and conversely, as the following result shows.

Proposition 4.5.1. *Let $E \subseteq L$ be a subset of the preordered set $(L, \leq)$, (X, α), (Y, β) be two possibly structured sets, each with a fixed binary L-relation on it and $f : X \to Y$, $g : Y \to X$ be two functions. Then, for any $\Gamma \in \{I, II, III, IV\}$, f and g form a type Γ E-including (E-excluding) Galois connection from (X, α) to (Y, β) if and only if f and g form a type Γ E-excluding (E-including) Tarski connection from (X, α) to (Y, β).*

Proof. Considering only the case $\Gamma = I$, the statement follows from the equivalences $[f\text{-}g]_{E^{[\uparrow]}} : (X, \alpha) \to (Y, \beta) \Leftrightarrow \mathcal{R}_{\beta f} \equiv_{E^{[\uparrow]}} \mathcal{R}_{\alpha g} \Leftrightarrow (\mathcal{R}_{\beta f} \in E^{[\uparrow]} \Leftrightarrow \mathcal{R}_{\alpha g} \in E^{[\uparrow]}) \Leftrightarrow (\mathcal{R}_{\beta f} \notin E^{[\uparrow]} \Leftrightarrow \mathcal{R}_{\alpha g} \notin E^{[\uparrow]}) \Leftrightarrow (\mathcal{R}_{\beta f} \in (E^{[\uparrow]})^{\neg} \Leftrightarrow \mathcal{R}_{\alpha g} \in (E^{[\uparrow]})^{\neg}) \Leftrightarrow (\mathcal{R}_{\beta f} \in E^{(\downarrow)} \Leftrightarrow \mathcal{R}_{\alpha g} \in E^{(\downarrow)}) \Leftrightarrow [f\text{-}g]_{E^{(\downarrow)}} : (X, \alpha) \to (Y, \beta)$.
Similarly we can prove the second equivalence. □

Here, we have to remark that the above equivalence is due to the classical-mathematical character of relative connections, for which for any two elements, both belong to a subset if and only if none of them belongs to the complement of that subset.
However, to restore the meaningful distinction between Galois and Tarski connections, we remark that it is also important for each of them the way it is related to a subset that may determine it. In this respect we have to note that with respect to a subset

$E \subseteq L$, a relative E-connection cannot be, in general, both a Galois and a Tarski connection and cannot be simultaneously E-including or E-excluding since, in general, there is no non-trivial subset which is both a lowerset and an upperset (see Remark 4.1.18 and Example 4.1.19).

Another important aspect of Galois and Tarski connections that has to be clarified is the two-fold approach to their classical, or "crisp", and their "fuzzy" version: the latter have been seen, up to now, as nothing but generalizations of the former, on the base of a point of view that is not-appropriate, in our opinion.

The distinction between Galois connections and conjugated pairs has been focused on the structures in the sets related by the considered connections: the order relation (inclusion) in the powersets, for Galois connection, the meet operation in the boolean algebra, for conjugated pairs. The structure of the set of the truth values of the order relation and of the meet operation, in both cases a boolean algebra, has not been considered; even more, the peculiarity of the subsets that determine Galois connections (the singleton $\{\top\}$) and conjugated pairs (the singleton $\{\bot\}$) has not been enlightened.

This inappropriate point of view is confirmed by the main steps of the generalization of the framework in the big development of Galois connections: since [70], these have been extended from powersets to ordered sets and, in their so called "fuzzy" approach the classical powersets with the inclusion relation have been replaced by the L-powersets with the subsethood L-relation which, as seen in Subsection 3.2.1, is an L-(pre)order relation.

The recent approach based on enriched categories moves from posets to special kinds of enriched categories and from order preserving, or reversing, maps to functors that form adjunctions. This includes and generalizes the case of the so called fuzzy Galois connections most widely considered between the L-powersets of two sets equipped with the subsethood degree relations, where L is usually assumed to be a commutative complete integral residuated lattice, i.e. an associative and commutative cdeo algebra (see [6, 37]); the non-commutative case is considered in [32, 33, 41, 42, 50].

Fuzzy conjugated pairs are only considered in [33, 36, 42] between L-powersets equipped with the intersection degree relation, which, evidently, extend to the L-powersets the meet operation of boolean algebras (see Subsection 3.2.1).

In our approach, both fuzzy Galois connections and fuzzy conjugated pairs are global connections.

Moreover, in our approach to connections, we have remarked that no special kinds of relations are needed in the sets (X, α) and (Y, β) linked by the connections; the notion of Galois and Tarski connections and their distinctive character depend simply on the existence of a (pre)order relation in the set L of truth values for the relations α and β. Even this weak structure in L is not needed for global connections.

We remark that one can consider connections, in the shape we have defined those, between any kind of structures that are characterized by binary relations taking values in a fixed (either preordered or not) set L; these include metric spaces, measurable spaces, probability spaces, Hilbert spaces, Riemannian manifolds and others.

As well known example we may consider two vector spaces $(V;\cdot)$ and $(W;\cdot)$ with

their scalar product and homomorphisms between those spaces. Two homomorphisms $f : V \to W$ and $g : W \to V$ are **adjoint** (or **transpose**) to each other if, for all $x \in V, y \in W$ the equality $(x)f \cdot y = x \cdot (y)g$ holds, i.e. $(f, g) : (V; \cdot) \to (W; \cdot)$ is a global connection of type III; in particular, it is well known that an endomorphism h of $(V; \cdot)$ is said to be **symmetric** if, for all $x, y \in V$, the equality $(x)h \cdot y = x \cdot (y)h$ holds, i.e. $(h, h) : (V; \cdot) \to (V; \cdot)$ is a type III (global) connection or, equivalently, h is self adjoint.
We close this discussion remarking that we have developed in this Chapter an approach to global and relative connections, hence to Galois and Tarski connections, too, that is fixed basis (we consider the preordered set $(L, \leq)$ as the basis in question). Moreover, we have looked at properties of each single connection, without relating two of them either of the same or of different types.
In the last Chapter, Section 6.2, we shall see how connections can be considered as morphisms of suitable categories and of more general semicategories and precategories.
It will be also the case that a variable basis approach to connections is closely related to special kinds of **relational systems** and morphisms between them, which form once more suitable categories and semicategories.

Chapter 5

The structure of many-valued relations

In Chapter 3 we have approached the study of binary relations with values in a w-ceo algebra $(L, \to, \top)$, possibly with further properties. We have already remarked that the structure on L can be lifted, pointwisely, on each set $\mathbf{R_L}(X, Y)$, for each pair (X, Y) of sets. Moreover, we have started to look at the structure of the whole class of L-valued binary relations $\mathbf{R_L}$, in particular recalling several compositions that the operations of the algebra $(L, \to, \top)$, possibly symmetrical, allow to define.
In this chapter we shall see how the structure of the algebra $(L, \to, \top)$, including the interconnections of the operation $\to$ and the derived further operations, can be lifted in the class of binary L-relations.
To this extent, we introduce a quite simple, but very effective tool, consisting in suitable triangles. On one hand such triangles characterize the structure of the algebra $(L, \to, \top)$, including the derived operations, on the other hand they describe fundamental aspects of the class of binary L-relations involving not only the algebraic operations lifted on each $\mathbf{R_L}(X, Y)$ by L but also the composite structure given by the compositions of L-relations.
Connections, in particular Galois and Tarski connections, described in Chapter 4 will have a fundamental role.
Further details on the description of structures for many-valued relations will be given in Chapter 6, where suitable generalizations of categories, some of which already well known, will be considered.

5.1 Relational triangles

Definition 5.1.1. *Let L be a possibly structured set and (A, α), (B, β) and (C, γ) be sets, each equipped with a fixed binary L-relation on it. The diagram*

$$\begin{tikzcd} & (C,\gamma) \arrow[dr, "\chi"] & \\ (A,\alpha) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (B,\beta) \end{tikzcd}$$

*with $\varphi \in \mathbf{R_C}(A,B)$, i.e. φ is a C-valued binary relation from A to B, $\psi \in \mathbf{R_B}(A,C)$ and $\chi \in \mathbf{R_A}(C,B)$, is called **triangle of L-relations** or **L-triangle**.*

Definition 5.1.2. *Let L be a set equipped with an equivalence relation $\equiv$ on it. The triangle of L-relations*

$$\begin{tikzcd} & (C,\gamma) \arrow[dr, "\chi"] & \\ (A,\alpha) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (B,\beta) \end{tikzcd}$$

*is called **relational L-triangle** if for all $a \in A, b \in B, c \in C$ the following conditions hold:*

$$(a,(c,b)\chi)\alpha \equiv ((a,c)\psi,b)\beta \equiv (c,(a,b)\varphi)\gamma.$$

Remark 5.1.3. It is clear that the equivalences of the above Definition can be expressed, for all $a \in A$, $b \in B$, $c \in C$, as follows

$$((c,b)\chi,a)\alpha_- \equiv (b,(a,c)\psi)\beta_- \equiv ((a,b)\varphi,c)\gamma_-.$$

Proposition 5.1.4. *Let L be a set and assume $\equiv$ to be an equivalence relation on it. Each of the following triangles of L-relations is a relational L-triangle if and only if any one of them is.*

1. $$\begin{tikzcd} & (C,\gamma) \arrow[dr, "\chi"] & \\ (A,\alpha) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (B,\beta) \end{tikzcd}$$

2. $$\begin{tikzcd} & (C,\gamma) \arrow[dr, "\psi_-"] & \\ (B,\beta_-) \arrow[ur, "\chi_-"] \arrow[rr, "\varphi_-"'] & & (A,\alpha_-) \end{tikzcd}$$

3. $$\begin{tikzcd} & (A,\alpha) \arrow[dr, "\varphi"] & \\ (C,\gamma) \arrow[ur, "\psi_-"] \arrow[rr, "\chi"'] & & (B,\beta) \end{tikzcd}$$

4.
$$\begin{tikzcd} & (B,\beta_-) \arrow[dr, "\varphi_-"] & \\ (C,\gamma) \arrow[ur, "\chi"] \arrow[rr, "\psi_-"'] & & (A,\alpha_-) \end{tikzcd}$$

5.
$$\begin{tikzcd} & (B,\beta_-) \arrow[dr, "\chi_-"] & \\ (A,\alpha) \arrow[ur, "\varphi"] \arrow[rr, "\psi"'] & & (C,\gamma_-) \end{tikzcd}$$

6.
$$\begin{tikzcd} & (A,\alpha) \arrow[dr, "\psi"] & \\ (B,\beta_-) \arrow[ur, "\varphi_-"] \arrow[rr, "\chi_-"'] & & (C,\gamma_-) \end{tikzcd}$$

Proof. Consider the equivalences:

1. $(a,(c,b)\chi)\alpha \equiv ((a,c)\psi,b)\beta \equiv (c,(a,b)\varphi)\gamma$;
2. $(b,(c,a)\psi)\beta_- \equiv ((b,c)\chi_-,a)\alpha_- \equiv (c,(b,a)\varphi_-)\gamma$;
3. $(c,(a,b)\varphi)\gamma \equiv ((c,a)\psi_-,b)\beta \equiv (a,(c,b)\chi)\alpha$;
4. $(c,(b,a)\varphi_-)\gamma \equiv ((c,b)\chi,a)\alpha_- \equiv (b,(c,a)\psi_-)\beta_-$;
5. $(a,(b,c)\chi_-)\alpha \equiv ((a,b)\varphi,c)\gamma_- \equiv (b,(a,c)\psi)\beta_-$;
6. $(b,(a,c)\psi)\beta_- \equiv ((b,a)\varphi_-,c)\gamma_- \equiv (a,(b,c)\chi_-)\alpha$,

that in each item characterize the relational L-triangle described in the corresponding item. Then, it is easily seen that the equivalences in the item (1) hold if and only if the equivalences in each of the other items (2), (3), (4), (5), and (6) hold, too, by using the obvious equalities

- $(a,(c,b)\chi)\alpha = (a,(b,c)\chi_-)\alpha = ((c,b)\chi,a)\alpha_- = ((b,c)\chi_-,a)\alpha_-$;
- $((a,c)\psi,b)\beta = ((c,a)\psi_-,b)\beta = (b,(a,c)\psi)\beta_- = (b,(c,a)\psi_-)\beta_-$;
- $(c,(a,b)\varphi)\gamma = (c,(b,a)\varphi_-)\gamma = ((a,b)\varphi,c)\gamma_- = ((b,a)\varphi_-,c)\gamma_-$.

□

Definition 5.1.5. *Let L be a set equipped with an equivalence relation $\equiv$ and let E be a subset of L. The triangle of L-relations*

$$\begin{tikzcd} & (C,\gamma) \arrow[dr, "\chi"] & \\ (A,\alpha) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (B,\beta) \end{tikzcd}$$

*is called E **relational L-triangle** or **relational L-triangle relative to** E, if for all $a \in A, b \in B, c \in C$ the following conditions hold:*

$$(a,(c,b)\chi)\alpha \equiv_E ((a,c)\psi,b)\beta \equiv_E (c,(a,b)\varphi)\gamma.$$

Of course, the E relational L-triangles of the above Definition may be expressed in six equivalent forms, according to Proposition 5.1.4.
The whole terminology we have introduced in Chapter 4 for equivalences and, in particular, for connections may be borrowed for relational L-triangles; so, assuming the needed requirements on L, we shall speak of

- **relative relational L-triangles** (E relational L-triangles, for some $E \subseteq L$);
- **upper** or **lower relational L-triangles** (relational L-triangles relative to some upperset or to some lowerset);
- **E-including** and **E-excluding upper** or **lower relational L-triangles**;
- **point-including** and **point-excluding upper** or **lower relational L-triangles**.

We shall also call every kind of upper relational L-triangle **Galois L-triangle** of that kind and every kind of lower relational L-triangle **Tarski L-triangle** of that kind. One can restate for L-relational triangles most results proved for equivalences and for connections in Sections 4.2, 4.3 and 4.4 , except those involving the classification into four types of connections. In fact, relational L-triangles have the feature to gather all together the four types of connections as the following results shows. To give details of this feature we note that each arrow of a triangle of L-relations determines two families of maps from each of the two vertices adjacent to the arrow into the opposite vertex; so, with notation of Definition 5.1.2, for any $a \in A, b \in B, c \in C$ we consider

$$\varphi_a = (a,-)\varphi : B \to C,\ \ \psi_a = (a,-)\psi : C \to B;$$

$$\varphi_b = (-,b)\varphi : A \to C,\ \ \chi_b = (-,b)\chi : C \to A;$$

$$\psi_c = (-,c)\psi : A \to B,\ \ \chi_c = (c,-)\chi : B \to A.$$

Then in case of relational L-triangles one has four families of connections of the four different types; in fact, the assumed equivalences of Definition 5.1.2 give soon

$$\{)\varphi_a,\psi_a(: (B,\beta) \to (C,\gamma)|\ a \in A\}\,;$$

$$\{[\varphi_b\text{-}\chi_b] : (A,\alpha) \to (C,\gamma)|\ b \in B\}\,;$$

$$\{(\psi_c,\chi_c) : (A,\alpha) \to (B,\beta)|\ c \in C\}$$

and the forth family of connections comes from Remark 5.1.3 (of course, Proposition 4.2.11 could be invoked, too)

$$\{]\varphi_b\text{-}\chi_b[: (A,\alpha_-) \to (C,\gamma_-)|\ b \in B\}\,.$$

Similarly, every E relational L-triangle, or every F Galois L-triangle, $F \subseteq L$ any upperset, or every I Tarski L-triangle, $I \subseteq L$ any lowerset, gives four families of relative or Galois or Tarski L-connections of the four different types.

The results of Section 4.2 on equivalences allow to state the following result, where we assume, for items (3), (3'), (4), (4') that $\equiv$ is induced by a preorder relation in L.

Proposition 5.1.6. *With the already stated notation, for the triangle of L-relations*

$$\begin{tikzcd}
& (C,\gamma) \arrow[dr, "\chi"] & \\
(A,\alpha) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (B,\beta)
\end{tikzcd}$$

the following are equivalent:

(1) it is a relational L-triangle;

(2) it is an E relational L-triangle, for every $E \subseteq L$;

(2') it is an $\{x\}$ relational L-triangle, for every $x \in L$;

(3) it is an F Galois L-triangle, for every upper set $F \subseteq L$;

(3') it is an x-including Galois L-triangle, for every $x \in L$;

(4) it is an I Tarski L-triangle, for every lower set $I \subseteq L$;

(4') it is an x-including Tarski L-triangle, for every $x \in L$.

5.1.1 Relational triangles and cdeo algebras

In this Subsection we show how the crucial properties of distributivity, symmetry, associativity and commutativity for w-ceo algebras may be characterized by suitable relational triangles.
In order to clarify the results and their proofs that we present in the following, we make a few preliminary comments.
Consider the structured sets (X,α) and (X,β), where $\alpha,\beta : X \times X \to L$, L is a poset with the equality $=$ and maximum and minimum $\top$ and $\perp$, respectively.

- If $(i_X,i_X)_{\top[\uparrow]} : (X,\alpha) \to (X,\beta)$ is a type III $\top$-including Galois connection, then $\top \le (x,y)\alpha \Leftrightarrow \top \le (x,y)\beta$, for all $x,y \in X$;

- If $(i_X,i_X)_{\perp(\downarrow)} : (X,\alpha) \to (X,\beta)$ is a type III $\perp$-excluding Tarski connection, then $(x,y)\alpha \ne \perp \Leftrightarrow (x,y)\beta \ne \perp$, for all $x,y \in X$;

  $(i_X,i_X) : (X,\alpha) \to (X,\beta)$ is a type III global connection, then $(x,y)\alpha = (x,y)\beta$, for all $x,y \in X$, i.e. $\alpha = \beta$.

Proposition 5.1.7. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is distributive if and only if there exist and are unique $\psi, \chi : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ is a type III $\top$-including Galois connection, for $j = 1, 2, 3$, the diagram*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle.

Proof. " $\Rightarrow$ " From right-distributivity of L, it follows that there exists the (unique) adjoint product $\otimes$; so, fixing $a \in L$ we can define $\psi_a : L \to L, (c) \mapsto (c)\psi_a = a \otimes c$ and we have that $c \le a \to b \Leftrightarrow (c)\psi_a \le b$, for all $b, c \in L$.
Moreover, from left-distributivity of L, fixing $b \in L$, there exists (a unique) $\chi_b : L \to L$ such that $c \le a \to b \Leftrightarrow a \le (c)\chi_b$, for all $a, c \in L$.
Now consider the functions $\psi : L \times L \to L, (a, b) \mapsto (b)\psi_a$ and $\chi : L \times L \to L, (a, b) \mapsto (a)\chi_b$. Hence, by arbitrariness of $a, b \in L$ we have the following equivalences: $c \le a \to b \Leftrightarrow (a, c)\psi \le b \Leftrightarrow a \le (c, b)\chi$, for all $a, b, c \in L$.
This equivalences can be rewritten in this way $\top \le (c, (a, b) \to) \to \Leftrightarrow \top \le ((a, c)\psi, b) \to \Leftrightarrow \top \le (a, (c, b)\chi) \to$, for all $a, b, c \in L$ and, hence, by assumption on α_j, for $j = 1, 2, 3$, we have that $\top \le (c, (a, b) \to)\alpha_3 \Leftrightarrow \top \le ((a, c)\psi, b)\alpha_2 \Leftrightarrow \top \le (a, (c, b)\chi)\alpha_1$, for all $a, b, c \in L$, i.e. the diagram

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\otimes"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle.
" $\Leftarrow$ " Consider $\alpha_1 = \alpha_2 = \alpha_3 = \to : L \times L \to L$; obviously, $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \to)$ and hence, by assumption there exist and are unique $\psi, \chi : L \times L \to L$ such that the diagram

$$\begin{tikzcd} & (L,\to) \arrow[dr, "\chi"] & \\ (L,\to) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\to) \end{tikzcd}$$

is a $\top$-including Galois L-triangle. Then, by definition, we have that for all $a, b, c \in L$: $\top \le (a, (c, b)\chi) \to \Leftrightarrow \top \le ((a, c)\psi, b) \to \Leftrightarrow \top \le (c, (a, b) \to) \to$. Fixing $\bar{a} \in L$, the following equivalence holds: $c \le \bar{a} \to b \Leftrightarrow (\bar{a}, c)\psi \le b$.
So, the functions $\psi_{\bar{a}} : L \to L, c \mapsto (c)\psi_{\bar{a}} = (\bar{a}, c)\psi$ and $\varphi_{\bar{a}} : L \to L, b \mapsto (b)\varphi_{\bar{a}} = \bar{a} \to b$ form an adjunction and, hence, $\varphi_{\bar{a}}$ preserves infs, i.e. $\to$ is right-distributive. Similarly, fixing $\bar{b} \in L$, we have that, for all $a, c \in L$ the following holds: $a \le (c)\chi_{\bar{b}} \Leftrightarrow c \le (a)\varphi_{\bar{b}}$, where $\varphi_{\bar{b}} : L \to L, a \mapsto (a)\varphi_{\bar{b}} = a \to \bar{b}$ and $\chi_{\bar{b}} : L \to L, c \mapsto (c)\chi_{\bar{b}} = (c, \bar{b})\chi$; so the functions

$\varphi_{\bar{b}}$ and $\chi_{\bar{b}}$ form a Galois connection and hence $\varphi_{\bar{b}}$ transforms infs into sups, i.e. $\to$ is left-distributive. $\square$

Remark 5.1.8. 1. Let $(L, \to, \top)$ be a w-ceo algebra.
The requirement $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha)$ can be rewritten by means of the equivalence $\top \leq a \to b \Leftrightarrow \top \leq (a,b)\alpha$, for all $a, b \in L$. Hence, $a \leq b \Leftrightarrow \top \leq (a,b)\alpha$, for all $a, b \in L$, i.e. α, just like $\to$, is an extension of the natural ordering of the algebra $(L, \to, \top)$.

2. Since, obviously, $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \to)$, by Proposition 5.1.7, the diagram

$$\begin{tikzcd}
& (L,\to) \arrow[dr, "\chi"] & \\
(L,\to) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\to)
\end{tikzcd}$$

is a $\top$-including Galois L-triangle if and only if $(L, \to, \top)$ is a cdeo algebra.

3. If $(L, \to, \top)$ is a symmetrical cdeo algebra, then, by definition, the dual implication $\rightsquigarrow : L \times L \to L$ induces the same order as $\to$, i.e. $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \rightsquigarrow)$. Hence, by Proposition 5.1.7, in the diagram of the above item (2), in each vertex of the triangle $\to$ may be replaced by $\rightsquigarrow$; moreover, $\chi = \rightsquigarrow$, so further $\top$-including Galois triangles can be obtained, among which the following

$$\begin{tikzcd}
& (L,\rightsquigarrow) \arrow[dr, "\rightsquigarrow"] & \\
(L,\rightsquigarrow) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\rightsquigarrow)
\end{tikzcd}$$

It is possible to characterize the symmetry of cdeo algebra by means of the following result.

Proposition 5.1.9. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is a symmetrical cdeo algebra if and only if there exist and are unique $\psi, \chi : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ is type III $\top$-including Galois connection, for $j = 1, 2, 3$, the diagram*

$$\begin{tikzcd}
& (L,\alpha_3) \arrow[dr, "\chi"] & \\
(L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2)
\end{tikzcd}$$

is a $\top$-including Galois L-triangle and $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \chi)$ is a type III $\top$-including Galois connection.

Proof. " $\Rightarrow$ " Since, by assumption, L is a symmetrical cdeo algebra, then there exists $\otimes : L\times L \to L$ and $\rightsquigarrow : L\times L \to L$ such that the following equivalences hold, for all $a,b,c \in L$: $c \le a \to b \Leftrightarrow a \otimes c \le b \Leftrightarrow a \le c \rightsquigarrow b$, i.e. $\top \le (c,(a,b) \to) \to \Leftrightarrow \top \le ((a,c)\otimes, b) \to \Leftrightarrow \top \le (a,(c,b) \rightsquigarrow) \to$.
Hence, for all $\alpha_1,\alpha_2,\alpha_3 : L\times L \to L$ such that $(i_L,i_L)_{\top[\uparrow]} : (L,\to) \to (L,\alpha_j)$ are type III $\top$-including Galois connections, for $j=1,2,3$, by Remark 5.1.8 (1), it follows that: $\top \le (c,(a,b)\to)\alpha_3 \Leftrightarrow \top \le ((a,c)\otimes,b)\alpha_2 \Leftrightarrow \top \le (a,(c,b)\rightsquigarrow)\alpha_1$, for all $a,b,c \in L$.
Therefore, the diagram

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\rightsquigarrow"] & \\ (L,\alpha_1) \arrow[ur, "\otimes"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle and, by Remark 5.1.8 (1), $(i_L,i_L)_{\top[\uparrow]} : (L,\to) \to (L,\rightsquigarrow)$ is a type III $\top$-including Galois connection, since $\rightsquigarrow$ induces the same order as $\to$.
" $\Leftarrow$ " By Proposition 5.1.7, it follows that L is a cdeo algebra; in particular, we have that, for all $a,b,c \in L$: $c \le a \to b \Leftrightarrow (a,c)\psi \le b \Leftrightarrow a \le (c,b)\chi$.
By assumption $(i_L,i_L)_{\top[\uparrow]} : (L,\to) \to (L,\chi)$ and by using the Remark 5.1.8 (1), the following equivalences hold, for all $b,c \in L$: $c \le \top \to b \Leftrightarrow \top \le (c,b)\chi \Leftrightarrow b \le c$; hence, for all $b,c \in L$: $c \le \top \to b \Leftrightarrow b \le c$, i.e. L is symmetrical. □

Proposition 5.1.10. *Let $L = (L,\to,\top)$ be a w-ceo algebra. L is an associative cdeo algebra if and only if there exist and are unique $\psi,\chi : L\times L \to L$ such that, for all $\alpha_1,\alpha_2,\alpha_3 : L\times L \to L$ such that $(i_L,i_L)_{\top[\uparrow]} : (L,\to) \to (L,\alpha_j)$ is type III $\top$-including Galois connection, for $j=1,2,3$, the diagram*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle and, for every $b \in L$, the functions $\psi_b : L \to L$, $a \mapsto (a)\psi_b = (a,b)\psi$ and $\to_b : L \to L, a \mapsto (a)\to_b = b \to a$ form a type III global connection $(\psi_b,\to_b) : (L,\to) \to (L,\to)$.

Proof. " $\Rightarrow$ " If L is an associative cdeo algebra, then, from Proposition 5.1.7, it follows that there exist and are unique $\psi,\chi : L\times L \to L$ such that, for all $\alpha_1,\alpha_2,\alpha_3 : L\times L \to L$ such that $(i_L,i_L)_{\top[\uparrow]} : (L,\to) \to (L,\alpha_j)$ is type III $\top$-including Galois connection, for $j=1,2,3$, the diagram

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle and, in particular, $\psi = \otimes$.
Moreover, from associativity condition, the following equality holds, for all $a, b, c \in L$: $(b \otimes a) \to c = a \to (b \to c)$.
Fixing $b \in L$, the equality becomes: $((a)\otimes_b, c) \to = (a, (c) \to_b) \to$, where $\to_b$: $L \to L$, $c \mapsto (c) \to_b = b \to c$ and $\otimes_b : L \to L, a \mapsto (a)\otimes_b = b \otimes a$. Hence the functions $\psi_b = \otimes_b$ and $\to_b$ form a type III global connection from $(L, \to)$ to $(L, \to)$.
" $\Leftarrow$ " By Proposition 5.1.7, it follows that L is a cdeo algebra and $\psi = \otimes$. Moreover, since, by assumption, $(\otimes_b, \to_b) : (L, \to) \to (L, \to)$ is a type III global connection, for every $b \in L$, we have that, for all $a, c \in L$ $((a)\otimes_b, c) \to = (a, (c) \to_b) \to$. Hence, for all $a, b, c \in L$, $(b \otimes a) \to c = a \to (b \to c)$, that is equivalent to the associativity for cdeo algebras, by Remark 2.1.57. $\square$

As a consequence of the above propositions, we state the following result.

Corollary 5.1.11. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is an associative symmetrical cdeo algebra if and only if there exist and are unique $\psi, \chi : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ is a type III $\top$-including Galois connection, for $j = 1, 2, 3$, the diagram*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle and the following conditions hold:

(a) for every $b \in L$, $(\psi_b, \to_b) : (L, \to) \to (L, \to)$ is a type III global connection;

(b) $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \chi)$ is a type III $\top$-including Galois connection. $\square$

Moreover, we can give an alternative characterizations of associative symmetrical cdeo algebras. In fact, the following hold.

Proposition 5.1.12. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is an associative symmetrical cdeo algebra if and only if there exist and are unique $\psi, \chi : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ is a type III $\top$-including Galois connection, for $j = 1, 2, 3$, the diagram*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle and the following conditions hold:

(a) for every $b \in L$, $[\to_b \text{-} \chi_b] : (L, \chi) \to (L, \to)$ is a type I global connection from (L, χ) to $(L, \to)$;

(b) $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \chi)$ *is a type III* $\top$*-including Galois connection.*

Proof. " $\Rightarrow$ " By Proposition 5.1.9, there exist and are unique $\psi, \chi : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ is a type III $\top$-including Galois connections, for $j = 1, 2, 3$, the diagram

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle, $\psi = \otimes$, $\chi = \rightsquigarrow$ and $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \rightsquigarrow)$ is a type III $\top$-including Galois connection.
Moreover, by Proposition 2.1.62 the associativity of a symmetrical cdeo algebra is equivalent to the following equality: $c \to (a \rightsquigarrow b) = a \rightsquigarrow (c \to b)$, for all $a, b, c \in L$, which, fixing $b \in L$, can be rewritten as follows: $(c, (a) \rightsquigarrow_b) \to = (a, (c) \to_b) \rightsquigarrow$, for all $a, c \in L$, i. e. $[\to_b - \rightsquigarrow_b] : (L, \rightsquigarrow) \to (L, \to)$, for all $b \in L$.
" $\Leftarrow$ " By Proposition 5.1.9, L is a symmetrical cdeo algebra and, in particular, $\chi = \rightsquigarrow$; hence, for every $b \in L$ $\rightsquigarrow_b = \chi_b$.
Moreover, by assumption, we have $[\to_b - \rightsquigarrow_b] : (L, \rightsquigarrow) \to (L, \to)$, for every $b \in L$. So it follows that $(a, (c, b) \rightsquigarrow) \to = (c, (a, b) \to) \rightsquigarrow$, for all $a, b, c \in L$, i.e. $a \to (c \rightsquigarrow b) = c \rightsquigarrow (a \to b)$, for all $a, b, c \in L$, that is equivalent to the associativity of a symmetrical cdeo algebra L, by Proposition 2.1.62. □

Now we shall see that turning from the $\top$-including Galois connection in the characterization of symmetrical cdeo algebras of Proposition 5.1.9 to the corresponding global connection we get a characterization of commutative cdeo algebras. To this intent it will be useful the following remark.

Remark 5.1.13. Let $L = (L, \to, \top)$ be a w-ceo algebra. We note that for any $\alpha : L \times L \to L$ the requirement $(i_L, i_L) : (L, \to) \to (L, \alpha)$ is a type III global connection forces α to coincide with $\to$. In fact, (i_L, i_L) is a type III global connection from $(L, \to)$ to (L, α) if and only if, for all $a, b \in L$, $((a)i_L, b)\alpha = (a, (b)i_L) \to$, i.e. $a\alpha b = a \to b$, for all $a, b \in L$.

Proposition 5.1.14. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra.* L *is a commutative cdeo algebra if and only if there exist and are unique* $\psi, \chi : L \times L \to L$ *such that, for all* $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ *such that* $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ *is a type III* $\top$*-including Galois connection, for* $j = 1, 2, 3$*, the diagram*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$*-including Galois* L*-triangle and* $(i_L, i_L) : (L, \to) \to (L, \chi)$ *is a type III global connection.*

Proof. The proof is an easy consequence of Proposition 5.1.9 if one considers that the L-relation χ has to coincide with $\rightsquigarrow$, hence the assumption on of the type III global connection $(i_L, i_L) : (L, \to) \to (L, \chi)$ is equivalent to the equality $a \to b = a \rightsquigarrow b$, for all $a, b \in L$, which, in turn, is equivalent to commutativity in symmetrical cdeo algebras. $\square$

Now, the following result is an immediate consequence of Corollary 5.1.11 and Proposition 5.1.14.

Corollary 5.1.15. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is an associative commutative cdeo algebra if and only if there exist and are unique $\psi, \chi : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ is a type III $\top$-including Galois connection, for $j = 1, 2, 3$, the diagram*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle and the following conditions hold:

(a) for every $b \in L$, $(\psi_b\text{-}\to_b) : (L, \to) \to (L, \to)$ is a type III global connection;

(b) $(i_L, i_L) : (L, \to) \to (L, \chi)$ is a type III global connection. $\square$

Similarly, Proposition 5.1.12 can be modified to have yet another characterization of associative, commutative cdeo algebras.

Corollary 5.1.16. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is an associative, commutative cdeo algebra if and only if there exist and are unique $\psi, \chi : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \to) \to (L, \alpha_j)$ is a type III $\top$-including Galois connection, for $j = 1, 2, 3$, the diagram*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\chi"] & \\ (L,\alpha_1) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\alpha_2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle and the following conditions hold:

(a) for every $b \in L$, $[\to_b\text{-}\chi_b] : (L, \to) \to (L, \to)$ is a type I global connection;

(b) $(i_L, i_L) : (L, \to) \to (L, \chi)$ is a type III global connection. $\square$

The following result shows that commutativity and associativity of w-ceo algebra $(L, \to, \top)$ can be characterized by a suitable relational L-triangle.

Proposition 5.1.17. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is an associative, commutative cdeo algebra if and only if there exist a unique $\psi : L \times L \to L$ such that the diagram*

$$\begin{tikzcd} & (L,\to) \arrow[dr, "\to"] & \\ (L,\to) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\to) \end{tikzcd}$$

is a relational L-triangle.

Proof. " $\Rightarrow$ " The results in Subsection 2.1.5 allow to verify easily that the diagram

$$\begin{tikzcd} & (L,\to) \arrow[dr, "\to"] & \\ (L,\to) \arrow[ur, "\otimes"] \arrow[rr, "\to"'] & & (L,\to) \end{tikzcd}$$

is a relational L-triangle.
" $\Leftarrow$ " If there exists $\psi : L \times L \to L$ such that the diagram

$$\begin{tikzcd} & (L,\to) \arrow[dr, "\to"] & \\ (L,\to) \arrow[ur, "\psi"] \arrow[rr, "\to"'] & & (L,\to) \end{tikzcd}$$

is a relational L-triangle, we have, in particular, that it is a $\top$-including Galois L-triangle. Hence, by Proposition 5.1.7, L is a cdeo algebra and $\psi = \otimes$. Then, the relational L-triangle gives the equality $c \to (a \to b) = (a \otimes c) \to b$ and $c \to (a \to b) = a \to (c \to b)$, for all $a, b, c \in L$; therefore L is associative and commutative, as remarked in Subsection 2.1.5. □

Further relational L-triangles (either relative or global) can be obtained, which involve the product of a right-distributive w-ceo algebra; also, these triangles can be used to characterize further properties of the algebra.

Proposition 5.1.18. *Let $(L, \to, \top)$ be a right-distributive ceo algebra.*

1. *If $(L, \to, \top)$ is a symmetrical cdeo algebra, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \otimes) \to (L, \alpha_j)$, for $j = 1, 2, 3$, the triangle of L-relations*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\otimes"] & \\ (L,\alpha_1) \arrow[ur, "\otimes"] \arrow[rr, "\otimes"'] & & (L,\alpha_2) \end{tikzcd}$$

is both a $\top$-including Galois L-triangle and a $\top$-excluding Tarski L-triangle.

2. *If $\perp$ is an element $\otimes$-irreducible of L, for all $\alpha_1, \alpha_2, \alpha_3 : L \times L \to L$ such that $(i_L, i_L)_{\perp(\uparrow)} : (L, \otimes) \to (L, \alpha_j)$, for $j = 1, 2, 3$, the triangle of L-relations*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\otimes"] & \\ (L,\alpha_1) \arrow[ur, "\otimes"] \arrow[rr, "\otimes"'] & & (L,\alpha_2) \end{tikzcd}$$

is $\perp$-including Tarski L-triangle and a $\perp$-excluding Galois L triangle.

3. $(L,\to,\top)$ *is an associative and commutative cdeo algebra if and only if for all* $\alpha_1,\alpha_2,\alpha_3 : L\times L\to L$ *such that* $(i_L,i_L) : (L,\otimes)\to(L,\alpha_j)$, *for* $j = 1,2,3$, *the triangle of L-relations*

$$\begin{tikzcd} & (L,\alpha_3) \arrow[dr, "\otimes"] & \\ (L,\alpha_1) \arrow[ur, "\otimes"] \arrow[rr, "\otimes"'] & & (L,\alpha_2) \end{tikzcd}$$

is a relational L-triangle.

Proof. 1. Let $\alpha_1,\alpha_2,\alpha_3 : L\times L\to L$ be such that $(i_L,i_L)_{\top[\uparrow]} : (L,\otimes)\to(L,\alpha_j)$, for $j = 1,2,3$; hence, for all $a,b\in L$: $\top\le(a,b)\alpha_j \Leftrightarrow \top\le a\otimes b$.
Since, under symmetry assumption on the cdeo algebra, one has that $x\otimes(z\otimes y) = \top \Leftrightarrow x = \top, y = \top, z = \top$, for all $x,y,z\in L$, it follows that $\top\le x\otimes(z\otimes y)\Leftrightarrow\top\le(x\otimes z)\otimes y\Leftrightarrow\top\le z\otimes(x\otimes y)$, for all $x,y,z\in L$. By assumption on α_j, for $j = 1,2,3$, we have that $\top\le(x,((z,y)\otimes)\alpha_1\Leftrightarrow\top\le((x,z)\otimes,y)\alpha_2\Leftrightarrow\top\le(z,(x,y)\otimes)\alpha_3$, for all $x,y,z\in L$, i.e. the diagram in the statement is a $\top$-including Galois L-triangle.
As already seen in Section 4.5, this is equivalent to say that the triangle is a $\top$-excluding Tarski L-triangle.

2. Let $\alpha_1,\alpha_2,\alpha_3 : L\times L\to L$ be such that $(i_L,i_L)_{\perp(\downarrow)} : (L,\otimes)\to(L,\alpha_j)$, for $j = 1,2,3$; hence, for all $a,b\in L$: $(a,b)\alpha_j\ne\perp\Leftrightarrow a\otimes b\ne\perp$.
Since $\perp$ is $\otimes$-irreducible, one has that $x\otimes(z\otimes y)\ne\perp\Leftrightarrow x\ne\perp, y\ne\perp, z\ne\perp$, for all $x,y,z\in L$. So, by assumption on α_j, for $j = 1,2,3$, we have the equivalences:
$(x,((z,y)\otimes)\alpha_1\ne\perp\Leftrightarrow((x,z)\otimes,y)\alpha_2\ne\perp\Leftrightarrow(z,(x,y)\otimes)\alpha_3\ne\perp$, for all $x,y,z\in L$, i.e. the diagram in the statement is a $\perp$-excluding Galois L-triangle and, consequently, it is a $\perp$-including Tarski L-triangle, too.

3. " $\Rightarrow$ " Assuming L to be an associative and commutative cdeo algebra, let $\alpha_1,\alpha_2,\alpha_3 : L\times L\to L$ be such that $(i_L,i_L) : (L,\otimes)\to(L,\alpha_j)$, for $j = 1,2,3$; hence, for all $a,b\in L$: $(a,b)\alpha_j = (a,b)\otimes$. So, the assumed condition forces $\alpha_j = \otimes$.
Moreover, if L is associative and commutative $x\otimes(z\otimes y) = y\otimes(x\otimes z) = z\otimes(x\otimes y)$, for all $x,y,z\in L$ and hence, by assumption on α_j, we have that $(x,(z,y)\otimes)\alpha_1 = (y,(x,z)\otimes)\alpha_2 = (z,(x,y)\otimes)\alpha_3$, for all $x,y,z\in L$, i.e. the diagram in the statement is a relational L-triangle.
" $\Leftarrow$ " If the diagram is a relational L-triangle, since, by assumption, $\alpha_j = \otimes$, for $j = 1,2,3$, we have that, for all $x,z\in L$: $x\otimes z = x\otimes(z\otimes\top) = z\otimes(x\otimes\top) = z\otimes x$,

hence $\otimes$ is commutative; moreover, by using the assumption on the L-triangle, we have that, for all $x, y, z \in L$: $x \otimes (z \otimes y) = (x \otimes z) \otimes y$, i.e. $\otimes$ is associative. So, L is an associative and commutative cdeo algebra.

□

Remark 5.1.19. 1. The assumptions $(i_L, i_L)_{\top[\uparrow]} : (L, \otimes) \to (L, \alpha_j)$ and $(i_L, i_L)_{\bot(\uparrow)} : (L, \otimes) \to (L, \alpha_j)$ are equivalent to $(i_L, i_L)_{\top[\uparrow]} : (L, \alpha_j) \to (L, \otimes)$ and $(i_L, i_L)_{\bot(\uparrow)} : (L, \alpha_j) \to (L, \otimes)$ and they are satisfied in case $\alpha_j = \otimes$ and, if $(L, \to, \top)$ is symmetrical, in case $\alpha_j = \tilde{\otimes}$, too.
The assumption $(i_L, i_L) : (L, \otimes) \to (L, \alpha_j)$ which is equivalent to $(i_L, i_L) : (L, \alpha_j) \to (L, \otimes)$, is satisfied only in case $\alpha_j = \otimes$.

2. Under symmetry assumption, the diagram

$$\begin{tikzcd}
 & (L, \tilde{\otimes}) \arrow[dr, "\tilde{\otimes}"] & \\
(L, \tilde{\otimes}) \arrow[ur, "\tilde{\otimes}"] \arrow[rr, "\tilde{\otimes}"'] & & (L, \tilde{\otimes})
\end{tikzcd}$$

is

- both a $\top$-including Galois L-triangle and a $\top$-excluding Tarski L-triangle;
- both a $\bot$-including Tarski L-triangle and a $\bot$-excluding Galois L triangle, if $\bot$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible);
- a relational L-triangle if and only if $(L, \to, \top)$ is an associative and commutative cdeo algebra;

in fact, similarly to $\otimes$, $\tilde{\otimes}$ satisfies the equivalences $x \tilde{\otimes} (y \tilde{\otimes} z) = \top \Leftrightarrow x = \top, y = \top, z = \top$ and $x \tilde{\otimes} (y \tilde{\otimes} z) \neq \bot \Leftrightarrow x \neq \bot, y \neq \bot, z \neq \bot$.

3. Since $x \tilde{\otimes} (y \otimes z) = \top \Leftrightarrow x \otimes (y \tilde{\otimes} z) = \top \Leftrightarrow x = \top, y = \top, z = \top$ and $x \tilde{\otimes} (y \otimes z) \neq \bot \Leftrightarrow x \otimes (y \tilde{\otimes} z) \neq \bot \Leftrightarrow x \neq \bot, y \neq \bot, z \neq \bot$, taking into account the above item (1) one can realize that all the diagrams that can be obtained considering arbitrarily either $(L, \otimes)$ or $(L, \tilde{\otimes})$ in each vertex and either $\otimes$ or $\tilde{\otimes}$ on each side, are

- both a $\top$-including Galois L-triangle and a $\top$-excluding Tarski L-triangle;
- both a $\bot$-including Tarski L-triangle and a $\bot$-excluding Galois L triangle, if $\bot$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible);
- a relational L-triangle if and only if $(L, \to, \top)$ is an associative and commutative cdeo algebra (in fact, under commutativity assumption, $\otimes = \tilde{\otimes}$, i.e. $(i_L, i_L) : (L, \otimes) \to (L, \tilde{\otimes})$).

5.1.2 Relational triangles and many-valued relations

In this Subsection we shall see how to obtain relational L-triangles, either relative or global, involving relations and their compositions; these L-triangles may be used to characterize the structure of L and, moreover, give informations on the more composite structures of the class of L-relations may have.

Let X, Y, Z be sets, $(L, \to, \top)$ be a symmetrical cdeo algebra and let $\mathcal{R} : X \times Y \to L$, $\mathcal{S} : Y \times Z \to L$ be L-relations. We recall the compositions between L-relations we have defined in Section 3.1 as follows, $\forall x \in X, z \in Z$:

1. $\mathcal{R} \odot \mathcal{S} : X \rightharpoonup Z$: $(x,y)(\mathcal{R} \cdot \mathcal{S}) = \bigvee_{y \in Y} (x,y)\mathcal{R} \otimes (y,z)\mathcal{S}$;
2. $\mathcal{R} \tilde{\odot} \mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R} \tilde{\odot} \mathcal{S}) = \bigvee_{y \in Y} (x,y)\mathcal{R} \tilde{\otimes} (y,z)\mathcal{S}$;
3. $\mathcal{R} \oslash \mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R} \oslash \mathcal{S}) = \bigwedge_{y \in Y} (x,y)\mathcal{R} \to (y,z)\mathcal{S}$;
4. $\mathcal{R} \tilde{\oslash} \mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R} \tilde{\oslash} \mathcal{S}) = \bigwedge_{y \in Y} (x,y)\mathcal{R} \rightsquigarrow (y,z)\mathcal{S}$;
5. $\mathcal{R} \obslash \mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R} \obslash \mathcal{S}) = \bigwedge_{y \in Y} (y,z)\mathcal{S} \to (x,y)\mathcal{R}$;
6. $\mathcal{R} \tilde{\obslash} \mathcal{S} : X \rightharpoonup Z$: $(x,z)(\mathcal{R} \tilde{\obslash} \mathcal{S}) = \bigwedge_{y \in Y} (y,z)\mathcal{S} \rightsquigarrow (x,y)\mathcal{R}$.

Moreover, we recall that, starting from an arbitrary operation on L, $\chi_L : L \times L \to L$, we have defined in Section 3.1 the **universal** and the **existential** (L, χ_L)**-compositions** of L-relations defined, respectively, as follows, $\forall x \in X, z \in Z$:

1. $\mathcal{R} \hat{\chi}_L \mathcal{S} : X \rightharpoonup Z$: $(x,y)(\mathcal{R} \hat{\chi}_L \mathcal{S}) = \bigwedge_{y \in Y} ((x,y)\mathcal{R}, (y,z)\mathcal{S})\chi_L$;
2. $\mathcal{R} \check{\chi}_L \mathcal{S} : X \rightharpoonup Z$: $(x,y)(\mathcal{R} \check{\chi}_L \mathcal{S}) = \bigvee_{y \in Y} ((x,y)\mathcal{R}, (y,z)\mathcal{S})\chi_L$.

Conversely, if $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Z)$ is a composition between L-relations, we can induce an operation on L, $\chi_L : L \times L \to L$, defined by $(a,b)\chi_L = a_{X \times Y} \chi b_{Y \times Z}$, where $a_{X \times Y} \in \mathbf{R_L}(X,Y), b_{Y \times Z} \in \mathbf{R_L}(Y,Z)$ are the constant L-relations, with values $a, b \in L$, respectively, provided that the result does not depend on the choice of the triple of sets.
Moreover, let X be a set and let $(L, \to, \top)$ be a w-ceo algebra.
From Section 3.2, we recall that the **subsethood** relation $\mathcal{S}_X : L^X \times L^X \to L$ is defined by $(A,B)\mathcal{S}_X = \bigwedge_{x \in X} (x)A \to (x)B$, for all $A, B \in L^X$.
More generally, we have considered in Section 3.2 the (L, χ_L)-relation $\mathcal{S}_X^{\chi_L, \wedge} : L^X \times L^X \to L$ defined by $(A,B)\mathcal{S}_X^{\chi_L, \wedge} = \bigwedge_{x \in X} ((x)A, (x)B)\chi_L$, for all $A, B \in L^X$.
In particular $\mathcal{S}_X^{\rightsquigarrow, \wedge} = \tilde{\mathcal{S}}_X$ is the **dual subsethood** relation determined by the dual implication of a symmetrical cdeo algebra.

Proposition 5.1.20. *Let $L = (L, \to, \top)$ be a w-ceo algebra.*
L is distributive if and only if for any triple of sets (X, Y, Z) there exist and are unique $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ such that, for all $\mathcal{S}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{S}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ and

$\mathcal{S}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ *such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^3)$,

are type III $\top$*-including Galois connections, the diagram*

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y), \mathcal{S}^3) \arrow[dr, "\chi"] & \\ (\mathbf{R_L}(Z,X), \mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}^2) \end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, *for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, *is a* $\top$*-including Galois* L*-triangle.*

Proof. " $\Rightarrow$ " Assume L to be a cdeo algebra and let (X,Y,Z) be a triple of sets.
The assumed conditions on $\mathcal{S}^1$, $\mathcal{S}^2$ and $\mathcal{S}^3$ imply that, for all $\rho,\rho' \in \mathbf{R_L}(Z,X), \sigma,\sigma' \in \mathbf{R_L}(Y,Z), \tau,\tau' \in \mathbf{R_L}(X,Y)$:

(a) $\top \le (\rho,\rho')\mathcal{S}^1 \Leftrightarrow \top \le (\rho,\rho')\mathcal{S}_{Z\times X}$;

(b) $\top \le (\sigma,\sigma')\mathcal{S}^2 \Leftrightarrow \top \le (\sigma,\sigma')\mathcal{S}_{Y\times Z}$;

(c) $\top \le (\tau,\tau')\mathcal{S}^3 \Leftrightarrow \top \le (\tau,\tau')\mathcal{S}_{X\times Y}$.

From distributivity assumption on L, similarly to Proposition 5.1.7, it follows that there exist the adjoint product $\otimes$ and a function $\chi_L : L \times L \to L$ such that $c \le a \to b \Leftrightarrow a \otimes c \le b \Leftrightarrow a \le (c,b)\chi_L$, for all $a,b,c \in L$.
Hence, for all $x \in X, y \in Y, z \in Z$ and for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$, the following equivalences hold: $(x,y)\tau \le (z,x)\rho \to (y,z)\sigma \Leftrightarrow (z,x)\rho \otimes (x,y)\tau \le (y,z)\sigma \Leftrightarrow (z,x)\rho \le ((x,y)\tau,(y,z)\sigma)\chi_L$ that is $\top \le (x,y)\tau \to ((z,x)\rho \to (y,z)\sigma) \Leftrightarrow \top \le ((z,x)\rho \otimes (x,y)\tau) \to (y,z)\sigma \Leftrightarrow \top \le (z,x)\rho \to ((x,y)\tau,(y,z)\sigma)\chi_L$.
Hence, by arbitrariness of x,y,z:
$\top \le \bigwedge_{x\in X y\in Y z\in Z}((x,y)\tau \to ((z,x)\rho \to (y,z)\sigma)) \Leftrightarrow$
$\top \le \bigwedge_{x\in X y\in Y z\in Z}(((z,x)\rho \otimes (x,y)\tau) \to (y,z)\sigma) \Leftrightarrow$
$\top \le \bigwedge_{x\in X y\in Y z\in Z}((z,x)\rho \to ((x,y)\tau,(y,z)\sigma)\chi_L)$.
By the distributivity of L, it follows that:
$\top \le \bigwedge_{x\in X y\in Y}((x,y)\tau \to (\bigwedge_{z\in Z}(z,x)\rho \to (y,z)\sigma)) \Leftrightarrow$
$\top \le \bigwedge_{y\in Y z\in Z}(\bigvee_{x\in X}((z,x)\rho \otimes (x,y)\tau) \to (y,z)\sigma) \Leftrightarrow$
$\top \le \bigwedge_{x\in X z\in Z}\left((z,x)\rho \to \bigwedge_{y\in Y}((x,y)\tau,(y,z)\sigma)\chi_L\right)$
that is
$\top \le \bigwedge_{x\in X y\in Y}((x,y)\tau \to ((x,y)(\rho_- \oslash \sigma_-)) \Leftrightarrow$
$\top \le \bigwedge_{y\in Y z\in Z}((y,z)(\rho \odot \tau)_- \to (y,z)\sigma) \Leftrightarrow$
$\top \le \bigwedge_{x\in X z\in Z}((z,x)\rho \to ((z,x)((\tau\chi\sigma)_-)) = \top$.
Hence, denoting $(\tau,\sigma)\chi = (\tau\chi\sigma)_-$, $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$ and $(\rho,\tau)\psi = (\rho \odot \tau)_-$, we have that, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$, the following equivalences hold:

$$\top \le (\tau,(\rho,\sigma)\varphi)\mathcal{S}_{X\times Y} \Leftrightarrow \top \le ((\rho,\tau)\psi,\sigma)\mathcal{S}_{Y\times Z} \Leftrightarrow \top \le (\rho,(\tau,\sigma)\chi)\mathcal{S}_{Z\times X}.$$

Thanks to the conditions (a), (b) and (c) we have for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$, the following equivalences:

$$\top \leq (\tau, (\rho,\sigma)\varphi)\mathcal{S}^3 \Leftrightarrow \top \leq ((\rho,\tau)\psi,\sigma)\mathcal{S}^2 \Leftrightarrow \top \leq (\rho,(\tau,\sigma)\chi)\mathcal{S}^1,$$

i.e. the diagram

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y),\mathcal{S}^3) \arrow[dr, "\chi"] & \\ (\mathbf{R_L}(Z,X),\mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}^2) \end{tikzcd}$$

is a $\top$-including Galois L-triangle.
"$\Leftarrow$" Let (X,Y,Z) be a triple of sets and consider $\psi : \mathbf{R_L}(Z,X)\times\mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi : \mathbf{R_L}(X,Y)\times\mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ that gives the $\top$-including Galois L-triangle

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y),\mathcal{S}^3) \arrow[dr, "\chi"] & \\ (\mathbf{R_L}(Z,X),\mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}^2) \end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X)\times\mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ is defined by $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$ and $\mathcal{S}^1,\mathcal{S}^2,\mathcal{S}^3$ satisfy the assumed requirements. So, the following conditions hold, for all $\rho,\rho' \in \mathbf{R_L}(Z,X), \sigma,\sigma' \in \mathbf{R_L}(Y,Z), \tau,\tau' \in \mathbf{R_L}(X,Y)$:

- $\top \leq (\tau,(\rho,\sigma)\varphi)\mathcal{S}^3 \Leftrightarrow \top \leq ((\rho,\tau)\psi,\sigma)\mathcal{S}^2 \Leftrightarrow \top \leq (\rho,(\tau,\sigma)\chi)\mathcal{S}^1$;
- $\top \leq (\rho,\rho')\mathcal{S}^1 \Leftrightarrow \top \leq (\rho,\rho')\mathcal{S}_{Z\times X}$;
- $\top \leq (\sigma,\sigma')\mathcal{S}^2 \Leftrightarrow \top \leq (\sigma,\sigma')\mathcal{S}_{Y\times Z}$;
- $\top \leq (\tau,\tau')\mathcal{S}^3 \Leftrightarrow \top \leq (\tau,\tau')\mathcal{S}_{X\times Y}$.

Hence, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$

$$\top \leq (\tau,(\rho,\sigma)\varphi)\mathcal{S}_{X\times Y} \Leftrightarrow \top \leq ((\rho,\tau)\psi,\sigma)\mathcal{S}_{Y\times Z} \Leftrightarrow \top \leq (\rho,(\tau,\sigma)\chi)\mathcal{S}_{X\times Y}.$$

Choosing for any $a,b,c \in L$ the constant L-relations defined by $(z,x)\rho = a$, $(y,z)\sigma = b$ and $(x,y)\tau = c$, for all $x \in X, y \in Y, z \in Z$, we have that the following equivalences hold: $\top \leq c \to (a \to b) \Leftrightarrow \top \leq (a,c)\psi \to b \Leftrightarrow \top \leq a \to (c,b)\chi$, i.e. $c \leq a \to b \Leftrightarrow (a,c)\psi \leq b \Leftrightarrow a \leq (c,b)\chi$. So, there exist $\psi_L, \chi_L : L\times L \to L$ induced from ψ and χ, respectively, such that $c \leq a \to b \Leftrightarrow (a,c)\psi_L \leq b \Leftrightarrow a \leq (c,b)\chi_L$, for all $a,b,c \in L$. Then, as in Proposition 5.1.7, we can prove the distributivity of L and, in particular, one has that ψ_L is the adjoint product $\otimes$ of L.
Note that ψ_L (i.e. $\otimes$) and χ_L do not depend on the chosen sets X,Y,Z, being uniquely determined by $\to$. $\square$

Remark 5.1.21. 1. Let $(L, \rightarrow, \top)$ be a w-ceo algebra, X, Y be two sets and let $\hat{\mathcal{S}} : L^{X\times Y} \times L^{X\times Y} \rightarrow L$ be an L-relation.
If $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \rightarrow (\mathbf{R_L}(X,Y), \hat{\mathcal{S}})$ is type III $\top$-including Galois connection, we have that, for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$:
$\top \leq (\alpha, \alpha')\hat{\mathcal{S}}_{X\times Y} \Leftrightarrow \top \leq (\alpha, \alpha')\mathcal{S}_{X\times Y}$.
Moreover, the following equivalences, for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$ hold:
$\top \leq (\alpha, \alpha')\mathcal{S}_{X\times Y} \Leftrightarrow \top \leq \bigwedge_{x\in X y\in Y}(x,y)\alpha \rightarrow (x,y)\alpha' \Leftrightarrow \top \leq (x,y)\alpha \rightarrow (x,y)\alpha', \forall x \in X, y \in Y \Leftrightarrow (x,y)\alpha \leq (x,y)\alpha' \Leftrightarrow \alpha \leq \alpha', \forall x \in X, y \in Y$.
Hence, $\top \leq (\alpha, \alpha')\hat{\mathcal{S}} \Leftrightarrow \alpha \leq \alpha'$, i.e. $\hat{\mathcal{S}}$ induce the same order as $\mathcal{S}_{X\times Y}$ on $\mathbf{R_L}(X,Y)$, that is the natural ordering of the w-ceo algebra $(\mathbf{R_L}(X,Y), \rightarrow, \top\!\!\top_{XY})$.

2. Let $(L, \rightarrow, \top)$ be a w-ceo algebra.
Obviously, $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \rightarrow (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y})$ is type III $\top$-including Galois connection, for all X, Y and then the diagram

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z})
\end{tikzcd}$$

with φ, ψ, χ determined as in the proof of Proposition 5.1.20, is a $\top$-including Galois L-triangle if and only if L is distributive.

3. If $(L, \rightarrow, \top)$ is a symmetrical cdeo algebra, the dual implication $\rightsquigarrow$ induces the same order as $\rightarrow$. Let X, Y be two sets and let $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$ be two L-relations; hence for all $x \in X, y \in Y$: $\top \leq (x,y)\alpha \rightarrow (x,y)\alpha' \Leftrightarrow \top \leq (x,y)\alpha \rightsquigarrow (x,y)\alpha'$. So $\top \leq \bigwedge_{x\in X y\in Y}(x,y)\alpha \rightarrow (x,y)\alpha' \Leftrightarrow \top \leq \bigwedge_{x\in X y\in Y}(x,y)\alpha \rightsquigarrow (x,y)\alpha'$, i.e. $\top \leq (\alpha,\alpha')\mathcal{S}_{X\times Y} \Leftrightarrow \top \leq (\alpha,\alpha')\tilde{\mathcal{S}}_{X\times Y}$, for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$.
Therefore, $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \rightarrow (\mathbf{R_L}(X,Y), \tilde{\mathcal{S}}_{X\times Y})$ is a type III $\top$-including Galois connection.
Hence, the diagram

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \tilde{\mathcal{S}}_{X\times Y}) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X), \tilde{\mathcal{S}}_{Z\times X}) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \tilde{\mathcal{S}}_{Y\times Z})
\end{tikzcd}$$

with φ, ψ, χ determined as in the proof of Proposition 5.1.20 and all the diagrams that can be obtained by considering, arbitrarily, in each vertex either the subsethood degree L-relation or the dual subsethood degree L-relation are $\top$-including Galois L-triangle.

The symmetrical cdeo algebras can be characterized as follows.

Proposition 5.1.22. *Let* $\mathbf{L} = (L, \to, \top)$ *be a w-ceo algebra.* $\mathbf{L}$ *is a symmetrical cdeo algebra if and only if for any triple of sets* (X, Y, Z) *there exist and are unique* $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that, for all* $\mathcal{S}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{S}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ *and* $\mathcal{S}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ *such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^3)$,

are type III $\top$*-including Galois connections, the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y), \mathcal{S}^3) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}^2)
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, *for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, *is a* $\top$*-including Galois* L*-triangle and the following condition holds:*
(a) $\forall X, Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y})$, *where* $\chi_L : \times L \to L, (a,b) \mapsto (a,b)\chi_L = (a_{X\times Y}, b_{Y\times Z})\chi$.

Proof. " $\Rightarrow$ " The $\top$-including Galois L-triangle may be determined as in the proof of Proposition 5.1.20; moreover one can see that χ is defined, for any $\sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$, by $(\tau,\sigma)\chi = (\tau \tilde{\oslash} \sigma)_-$ and, if $\rho \in \mathbf{R_L}(Z,X)$, $(\rho,\tau)\psi = (\rho \odot \tau)_-$. In fact, $\chi_L = \rightsquigarrow$ and consequently $\mathcal{S}^{\chi_L}_{X\times Y} = \tilde{\mathcal{S}}_{X\times Y}$. Now, since $\to$ and $\rightsquigarrow$ induce the same order on L, the following hold, for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$: $\top \le (x,y)\alpha \to (x,y)\alpha' \Leftrightarrow \top \le (x,y)\alpha \rightsquigarrow (x,y)\alpha'$, for all $x \in X, y \in Y$.
Hence, $\top \le \bigwedge_{x\in X y\in Y} (x,y)\alpha \to (x,y)\alpha' \Leftrightarrow \top \le \bigwedge_{x\in X y\in Y} (x,y)\alpha \rightsquigarrow (x,y)\alpha'$, that is equivalent to $\top \le (\alpha,\alpha')\mathcal{S}_{X\times Y} \Leftrightarrow \top \le (\alpha,\alpha')\tilde{\mathcal{S}}_{X\times Y}$.
So, $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \tilde{\mathcal{S}}_{X\times Y})$.
" $\Leftarrow$ " As in the proof of Proposition 5.1.20, ψ and χ induce on L two operations ψ_L and χ_L such that, for all $a,b,c \in L$: $c \le a \to b \Leftrightarrow (a,c)\psi_L \le b \Leftrightarrow a \le (c,b)\chi_L$. In fact, L is a cdeo algebra and $\psi_L = \otimes$. Now, the condition (a) gives the equivalence $\top \le (\tau,\tau')\mathcal{S}^{\chi_L}_{X\times Y} \Leftrightarrow \top \le (\tau,\tau')\mathcal{S}_{X\times Y}$, as in Remark 5.1.21. Then, for any $b,c \in L$ choosing $(x,y)\tau = c$ and $(x,y)\tau' = b$, for all $x \in X, y \in Y$, we have $\top \le (c,b)\chi_L \Leftrightarrow \top \le c \to b \Leftrightarrow c \le b$. Using the above equivalences, for $a = \top$ and for all $b,c \in L$, the following hold: $c \le \top \to b \Leftrightarrow \top \le (c,b)\chi_L \Leftrightarrow c \le b$. So, by Remark 2.1.12 and Theorem 2.1.21, it follows that L is symmetrical. □

Proposition 5.1.23. *Let* $\mathbf{L} = (L, \to, \top)$ *be a w-ceo algebra.* $\mathbf{L}$ *is an associative cdeo algebra if and only if for any triple of sets* (X, Y, Z) *there exist and are unique* $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that, for all* $\mathcal{S}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{S}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ *and* $\mathcal{S}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ *such that:*

- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^3)$,

are type III $\top$*-including Galois connections, the diagram*

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y), \mathcal{S}^3) \arrow[dr, "\chi"] & \\ (\mathbf{R_L}(Z,X), \mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}^2) \end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois* L*-triangle and the following condition holds:*

(b) $\forall \rho \in \mathbf{R_L}(Z,X)$*:* $(\psi_\rho\text{-}\varphi_\rho) : (\mathbf{R_L}(X,Y), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}_{X\times Y})$ *is a type III global connection, where* $\psi_\rho : \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z), \tau \mapsto (\tau)\psi_\rho = (\rho,\tau)\psi$ *and* $\varphi_\rho : \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y), \sigma \mapsto (\sigma)\varphi_\rho = (\rho,\sigma)\varphi$*.*

Proof. " $\Rightarrow$ " The proof runs as in Proposition 5.1.20 to get the required triangle, where, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$ $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ is defined by $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$ and $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Z,Y)$ is defined by $(\rho,\tau)\psi = (\rho \oslash \tau)_-$, is a $\top$-including Galois L-triangle. Since L is an associative cdeo algebra, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y), x \in X, y \in Y, z \in Z$, $((z,x)\rho \otimes (x,y)\tau) \to (y,z)\sigma = (x,y)\tau \to ((z,x)\rho \to (y,z)\sigma)$.
Hence, we have that:
$\bigwedge_{x\in X y\in Y z\in Z}((z,x)\rho \otimes (x,y)\tau) \to (y,z)\sigma = \bigwedge_{x\in X y\in Y z\in Z}(x,y)\tau \to ((z,x)\rho \to (y,z)\sigma)$.
By using the distributivity assumption, the following equality holds: $\bigwedge_{y\in Y z\in Z}(\bigvee_{x\in X}(z,x)\rho \otimes (x,y)\tau) \to (y,z)\sigma = \bigwedge_{x\in X y\in Y}(x,y)\tau \to (\bigwedge_{z\in Z}(z,x)\rho \to (y,z)\sigma)$, i.e. $\bigwedge_{y\in Y z\in Z}(y,z)(\rho \odot \tau)_- \to (y,z)\sigma = \bigwedge_{x\in X y\in Y}(x,y)\tau \to (x,y)(\rho_- \oslash \sigma_-)$.
Fixing $\rho \in \mathbf{R_L}(Z,X)$ and considering $\psi_\rho : \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z), \tau \mapsto (\tau)\psi_\rho = (\rho,\tau)\psi = (\rho \odot \tau)_-$ and $\varphi_\rho : \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y), \sigma \mapsto (\sigma)\chi_\rho = (\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, we have the following, for all $\sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$ $((\tau))\psi_\rho, \sigma)\mathcal{S}_{Y\times Z} = (\tau, (\sigma)\varphi_\rho)\mathcal{S}_{X\times Y}$, i.e. $(\psi_\rho\text{-}\varphi_\rho) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z})$.
" $\Leftarrow$ " By Proposition 5.1.20, it follows that L is a cdeo algebra and in particular the operation ψ_L induced from ψ is the adjoint product. Since, by assumption, $(\psi_\rho\text{-}\varphi_\rho) : (\mathbf{R_L}(X,Y), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}_{X\times Y})$, for all $\rho \in \mathbf{R_L}(Z,X)$, choosing, in particular, the constant L-relations ρ, σ, τ defined by $(z,x)\rho = b, (x,y)\tau = a, (y,z)\sigma = c$, with $a,b,c \in L$, for all $x \in X, y \in Y, z \in Z$ we have that $(b \otimes a) \to c = a \to (b \to c)$. Hence, by arbitrariness of $a,b,c \in L$, it follows that L is associative. $\square$

The following result is an easy consequence of the above propositions.

Corollary 5.1.24. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra.* L *is an associative symmetrical cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exist and are unique* $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$*,* $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that, for all* $\mathcal{S}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$*,* $\mathcal{S}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ *and*

$\mathcal{S}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ *such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^3)$,

are type III $\top$*-including Galois connections, the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}^3) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}^2)
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, *for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, *is a* $\top$*-including Galois L-triangle and the following conditions hold:*

(a) $\forall X,Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y})$.

(b) $\forall \rho \in \mathbf{R_L}(Z,X)$, $(\psi_\rho\text{-}\varphi_\rho) : (\mathbf{R_L}(X,Y), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}_{X\times Y})$.

Proof. The statement is an easy consequence of Propositions 5.1.22 and 5.1.23. □

Moreover, it is possible to characterize the associative symmetrical cdeo algebras as follows.

Proposition 5.1.25. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra. L is an associative symmetrical cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exist and are unique* $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that, for all* $\mathcal{S}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{S}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ *and* $\mathcal{S}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ *such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^3)$,

are type III $\top$*-including Galois connections, the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}^3) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}^2)
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, *for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, *is a* $\top$*-including Galois L-triangle and the following conditions hold:*

(a) $\forall X,Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y})$.

(c) $\forall \sigma \in \mathbf{R_L}(Y,Z)$: $[\varphi_\sigma\text{-}\chi_\sigma] : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y})$ *is a type I global connection, where* $\varphi_\sigma : \mathbf{R_L}(Z,X) \to \mathbf{R_L}(X,Y), \rho \mapsto (\rho)\varphi_\sigma = (\rho,\sigma)\varphi$, $\chi_\sigma : \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Z,X), \tau \mapsto (\tau)\chi_\sigma = (\tau,\sigma)\chi$.

Proof. By Propositions 5.1.20 and 5.1.22 the symmetry and distributivity of L is equivalent to the existence of the $\top$-including Galois L-triangle described in the statement and to the condition (a). So, we only need to prove that the associativity of L is equivalent to the condition (c).
" $\Rightarrow$ " If L is associative, then, for all $a, b, c \in L$: $a \to (b \rightsquigarrow c) = b \rightsquigarrow (a \to c)$. Then, for all sets X, Y, Z, for all L-relations $\rho \in \mathbf{R_L}(Z, X), \sigma \in \mathbf{R_L}(Y, Z), \tau \in \mathbf{R_L}(X, Y)$ and for all $x \in X, y \in Y, z \in Z$, the following equality holds:
$(z, x)\rho \to ((x, y)\tau \rightsquigarrow (y, z)\sigma) = (x, y)\tau \rightsquigarrow ((z, x)\rho \to (z, y)\sigma)$.
So, $\bigwedge_{x\in X y\in Y z\in Z}(z, x)\rho \to ((x, y)\tau \rightsquigarrow (y, z)\sigma) = \bigwedge_{x\in X y\in Y z\in Z}(x, y)\tau \rightsquigarrow ((z, x)\rho \to (z, y)\sigma)$.
From distributivity, we have that:
$\bigwedge_{x\in X z\in Z}(z, x)\rho \to (\bigwedge_{y\in Y}(x, y)\tau \rightsquigarrow (y, z)\sigma) = \bigwedge_{x\in X y\in Y}(x, y)\tau \rightsquigarrow (\bigwedge_{z\in Z}(z, x)\rho \to (z, y)\sigma)$,
i.e. $\bigwedge_{x\in X z\in Z}(z, x)\rho \to ((z, x)(\tau \tilde{\oslash} \sigma)_-) = \bigwedge_{x\in X y\in Y}(x, y)\tau \rightsquigarrow ((x, y)(\rho_- \oslash \sigma_-)$.
Hence, for all $\sigma \in \mathbf{R_L}(Y, Z)$, $[\varphi_\sigma\text{-}\chi_\sigma] : (\mathbf{R_L}(Z, X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(X, Y), \tilde{\mathcal{S}}_{X\times Y})$ is a type I global connection, where $\varphi_\sigma : \mathbf{R_L}(Z, X) \to \mathbf{R_L}(X, Y), \rho \mapsto (\rho)\varphi_\sigma = \rho_- \oslash \sigma_-$, $\chi_\sigma : \mathbf{R_L}(X, Y) \to \mathbf{R_L}(Z, X), \tau \mapsto (\tau)\chi_\sigma = (\tau \tilde{\oslash} \sigma)_-$ and $\tilde{\mathcal{S}}_{X\times Y} : \mathbf{R_L}(X, Y) \times \mathbf{R_L}(X, Y) \to L$ is the dual subsethood degree.
" $\Leftarrow$ " The symmetry of L assures the existence and uniqueness of $\rightsquigarrow$; hence $\chi_L = \rightsquigarrow$ and $\mathcal{S}^{\chi_L}_{X\times Y} = \tilde{\mathcal{S}}_{X\times Y}$. By assumption (c), choosing the constant L-relations $\rho \in \mathbf{R_L}(Z, X), \sigma \in \mathbf{R_L}(Y, Z), \tau \in \mathbf{R_L}(X, Y)$ defined as follows $(z, x)\rho = a$, $(y, z)\sigma = b$ and $(x, y)\tau = c$, with $a, b, c \in L$, we have that $a \to (b \rightsquigarrow c) = b \rightsquigarrow (a \to c)$. By arbitrariness of elements $a, b, c \in L$, it follows that L is associative. □

Remark 5.1.26. We note that the condition (a) of Proposition 5.1.25 can be replaced by an equivalent condition, to characterize the symmetry condition.
In fact, by definition, if $(L, \to, \top)$ is symmetrical cdeo algebra, there exists $\rightsquigarrow: L \times L \to L$ such that $(L, \rightsquigarrow, \top)$ is a w-ceo algebra, $\to$ and $\rightsquigarrow$ form a Galois connection and they induce the same order; more explicitly, the latter condition is the following: $a \le b \Leftrightarrow \top \le a \to b \Leftrightarrow \top \le a \rightsquigarrow b$, for all $a, b \in L$.
Consider a triple of sets (X, Y, Z) and the L-relation $\rho \in \mathbf{R_L}(Z, X), \sigma \in \mathbf{R_L}(Y, Z)$.
Then the following equivalence hold, for all $x \in X, y \in Y, z \in Z$: $\top \le (z, x)\rho \to (y, z)\sigma \Leftrightarrow \top \le (z, x)\rho \rightsquigarrow (y, z)\sigma$, i.e. $\top \to ((z, x)\rho \to (y, z)\sigma) = \top \Leftrightarrow \top \to ((z, x)\rho \rightsquigarrow (y, z)\sigma) = \top$.
So, $\bigwedge_{x\in X y\in Y z\in Z} \top \to ((z, x)\rho \to (y, z)\sigma) = \top \Leftrightarrow \bigwedge_{x\in X y\in Y z\in Z} \top \to ((z, x)\rho \rightsquigarrow (y, z)\sigma) = \top$.
Thanks to right-distributivity, one has that:
$\bigwedge_{x\in X y\in Y} \top \to (\bigwedge_{z\in Z}(z, x)\rho \to (y, z)\sigma) = \top \Leftrightarrow \bigwedge_{x\in X y\in Y} \top \to (\bigwedge_{z\in Z}(z, x)\rho \rightsquigarrow (y, z)\sigma) = \top$,
that is equivalent to
$\bigwedge_{x\in X y\in Y} \top \to (x, y)(\rho_- \oslash \sigma_-) = \top \Leftrightarrow \bigwedge_{x\in X y\in Y} \top \to (x, y)(\rho_- \tilde{\oslash} \sigma_-) = \top$.
Hence, it follows that:
$\bigwedge_{x\in X y\in Y}(x, y)\top\!\!\!\top_{XY} \to (x, y)(\rho_- \oslash \sigma_-) = \top \Leftrightarrow \bigwedge_{x\in X y\in Y}(x, y)\top\!\!\!\top_{XY} \to (x, y)(\rho_- \tilde{\oslash} \sigma_-) = \top$,
which means $(\top\!\!\!\top_{XY}, \rho_- \oslash \sigma_-)\mathcal{S}_{X\times Y} = \top \Leftrightarrow (\top\!\!\!\top_{XY}, \rho_- \tilde{\oslash} \sigma_-)\mathcal{S}_{X\times Y} = \top$.

Now we see that we can characterize commutativity of a cdeo algebra replacing the Galois connection of the condition (a) of Proposition 5.1.22 by the corresponding global connection.

Proposition 5.1.27. *Let $L = (L, \rightarrow, \top)$ be a w-ceo algebra. L is a commutative cdeo algebra if and only if for any triple of sets (X, Y, Z) there exist and are unique $\psi : \mathbf{R_L}(Z, X) \times \mathbf{R_L}(X, Y) \rightarrow \mathbf{R_L}(Y, Z)$, $\chi : \mathbf{R_L}(X, Y) \times \mathbf{R_L}(Y, Z) \rightarrow \mathbf{R_L}(Z, X)$ such that, for all $\mathcal{S}^1 : \mathbf{R_L}(Z, X) \times \mathbf{R_L}(Z, X) \rightarrow L$, $\mathcal{S}^2 : \mathbf{R_L}(Y, Z) \times \mathbf{R_L}(Y, Z) \rightarrow L$ and $\mathcal{S}^3 : \mathbf{R_L}(X, Y) \times \mathbf{R_L}(X, Y) \rightarrow L$ such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z, X), \mathcal{S}_{Z\times X}) \rightarrow (\mathbf{R_L}(Z, X), \mathcal{S}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y, Z), \mathcal{S}_{Y\times Z}) \rightarrow (\mathbf{R_L}(Y, Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X, Y), \mathcal{S}_{X\times Y}) \rightarrow (\mathbf{R_L}(X, Y), \mathcal{S}^3)$,
are type III $\top$-including Galois connections, the diagram

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}^3) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}^2)
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z, X) \times \mathbf{R_L}(Y, Z) \rightarrow \mathbf{R_L}(X, Y)$ is defined by $(\rho, \sigma)\varphi = \rho_- \oslash \sigma_-$, for all $\rho \in \mathbf{R_L}(Z, X), \sigma \in \mathbf{R_L}(Y, Z)$, is a $\top$-including Galois L-triangle and the following condition holds:
(a') $\forall X, Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X, Y), \mathcal{S}_{X\times Y}) \rightarrow (\mathbf{R_L}(X, Y), \mathcal{S}^{\chi_L}_{X\times Y})$, where $\chi_L : \times L \rightarrow L, (a, b) \mapsto (a, b,)\chi_L = (a_{X\times Y}, b_{Y\times Z})\chi$.

Proof. Taking into account the proof of Proposition 5.1.22 we only need to prove that the stronger requirement here assumed is equivalent to the equality $\rightarrow = \rightsquigarrow$, where $\rightsquigarrow = \chi_L$. In fact, it follows from this equality that for all $\tau, \tau' \in \mathbf{R_L}(X, Y)$ and for all $x \in X, y \in Y$: $(x, y)\tau \rightarrow (x, y)\tau' = (x, y)\tau \rightsquigarrow (x, y)\tau' = (x, y)\tau\chi_L(x, y)\tau'$; whence $(\tau, \tau')\mathcal{S}_{X\times Y} = (\tau, \tau')\mathcal{S}^{\chi_L}_{X\times Y}$.
Conversely, the latter equality applied to the constant relations τ, τ' with values $a, b \in L$, gives the equality $a \rightarrow b = a \rightsquigarrow b$, for all $a, b \in L$. $\square$

Corollary 5.1.28. *Let $L = (L, \rightarrow, \top)$ be a w-ceo algebra. L is an associative commutative cdeo algebra if and only if for any triple of sets (X, Y, Z) there exist and are unique $\psi : \mathbf{R_L}(Z, X) \times \mathbf{R_L}(X, Y) \rightarrow \mathbf{R_L}(Y, Z)$, $\chi : \mathbf{R_L}(X, Y) \times \mathbf{R_L}(Y, Z) \rightarrow \mathbf{R_L}(Z, X)$ such that, for all $\mathcal{S}^1 : \mathbf{R_L}(Z, X) \times \mathbf{R_L}(Z, X) \rightarrow L$, $\mathcal{S}^2 : \mathbf{R_L}(Y, Z) \times \mathbf{R_L}(Y, Z) \rightarrow L$ and $\mathcal{S}^3 : \mathbf{R_L}(X, Y) \times \mathbf{R_L}(X, Y) \rightarrow L$ such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z, X), \mathcal{S}_{Z\times X}) \rightarrow (\mathbf{R_L}(Z, X), \mathcal{S}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y, Z), \mathcal{S}_{Y\times Z}) \rightarrow (\mathbf{R_L}(Y, Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X, Y), \mathcal{S}_{X\times Y}) \rightarrow (\mathbf{R_L}(X, Y), \mathcal{S}^3)$,
are type III $\top$-including Galois connections, the diagram

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}^3) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}^2)
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois L-triangle and the following conditions hold:*

(a') $\forall X,Y \in |\boldsymbol{Set}|$*:* $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y})$.

(b) $\forall \rho \in \mathbf{R_L}(Z,X)$*:* $(\psi_\rho\text{-}\varphi_\rho) : (\mathbf{R_L}(X,Y), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}_{X\times Y})$. □

Corollary 5.1.29. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra. L is an associative commutative cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exist and are unique* $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$*,* $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that, for all* $\mathcal{S}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$*,* $\mathcal{S}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ *and* $\mathcal{S}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ *such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^1)$*;*
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^2)$*;*
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^3)$*,*

are type III $\top$*-including Galois connections, the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}^3) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}^2)
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois L-triangle and the following conditions hold:*

(a') $\forall X,Y \in |\boldsymbol{Set}|$*:* $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y})$.

(b) $\forall \sigma \in \mathbf{R_L}(Y,Z)$*:* $[\varphi_\sigma\text{-}\chi_\sigma] : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y}$. □

Remark 5.1.30. We note that the condition required in the above proposition $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y})$ is equivalent to $(\beta,\beta')\mathcal{S}_{X\times Y} = (\beta,\beta')S^{\chi_L}_{X\times Y}$, for all $\beta,\beta' \in \mathbf{R_L}(X,Y)$. Hence, choosing, in particular, the constant L-relations $\beta = b_{X\times Y}$ $\beta' = b'_{X\times Y}$ and for all $x \in X, y \in Y$, we have that: $\bigwedge_{x\in X y\in Y} b \to b' = \bigwedge_{x\in X y\in Y}(b,b')\chi_L \Leftrightarrow\; b \to b' = (b,b')\chi_L$. By arbitrariness of $b, b' \in L$, it follow that $\to = \chi_L$.
Hence, the global connection contained in the condition (a') of the above proposition forces χ_L to be equal to $\to$.

Proposition 5.1.31. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra. L is an associative and commutative cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exist and are unique* $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$*,* $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that, for all* $\mathcal{S}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$*,* $\mathcal{S}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ *and* $\mathcal{S}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ *such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)}) : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^1)$*;*

- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)}) : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^3)$,

are type III global connections, the diagram

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y), \mathcal{S}^3) \arrow[dr, "\chi"] & \\ (\mathbf{R_L}(Z,X), \mathcal{S}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}^2) \end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, *for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, *is a relational* L*-triangle and the following condition holds:*
(a') *for all sets* X, Y*:* $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{XL}_{X\times Y})$.

Proof. We note that the relational L-triangle in the statement is equivalent to the global connections of conditions (b) and (c) of Corollaries 5.1.28 and 5.1.29.
Then, the proof of the statement follows easily from such Corollaries. □

Corollary 5.1.32. *Let* $(L, \to, \top)$ *be a right-distributive w-ceo algebra.* $(L, \to, \top)$ *is associative and commutative if and only if, for any triple of sets* (X, Y, Z)*, the diagram*

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \arrow[dr, "\chi"] & \\ (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \end{tikzcd}$$

where φ, ψ, χ *are defined, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$ *by:*

- $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$;
- $(\rho,\tau)\psi = (\rho \odot \tau)_-$;
- $(\tau,\sigma)\chi = (\tau \oslash \sigma)_-$

is a relational L*-triangle.* □

Further examples of the above defined L-triangles can be obtained involving the **intersection** and the **overlap degree** L-relations defined in Section 3.2 instead of the subsethood degree L-relations.
We recall from Section 3.2 that given a set X, a rigth-distributive w-ceo algebra $(L, \to, \top)$ with adjoint product $\otimes$, the **intersection** L-relation $\mathcal{T}_X : L^X \times L^X \to L$ is defined by $(A,B)\mathcal{T}_X = \bigvee_{x\in X}(x)A \otimes (x)B$, for all $A, B \in L^X$. If $(L, \to, \top)$ is a symmetrical cdeo algebra, we can also consider the **dual intersection** L-relation $\tilde{\mathcal{T}}_X : L^X \times L^X \to L$ such that $(A,B)\tilde{\mathcal{T}}_X = \bigvee_{x\in X}(x)A \tilde{\otimes} (x)B$, for all $A, B \in L^X$. We notice that the equality $(A,B)\mathcal{T}_X = (B,A)\tilde{\mathcal{T}}_X$ holds for all $A, B \in L^X$.

Remark 5.1.33. 1. Let X, Y be two sets and consider the L-relation from $\mathbf{R_L}(X,Y)$ to itself $\mathcal{T}'_{X\times Y} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ such that:
$(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{T}'_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y})$ is a type III $\top$-including Galois connection.
Hence, for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$: $\top \leq (\alpha,\alpha')\mathcal{T}'_{X\times Y} \Leftrightarrow \top \leq (\alpha,\alpha')\mathcal{T}_{X\times Y}$.
Moreover, if $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\perp(\uparrow)} : (\mathbf{R_L}(X,Y), \mathcal{T}'_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y})$, we have that for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$: $(\alpha,\alpha')\mathcal{T}'_{X\times Y} \neq \perp \Leftrightarrow (\alpha,\alpha')\mathcal{T}_{X\times Y} \neq \perp$.
If $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{T}'_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y})$ is a global connection, then, obviously, for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$: $(\alpha,\alpha')\mathcal{T}'_{X\times Y} = (\alpha,\alpha')\mathcal{T}_{X\times Y}$, and hence, $\mathcal{T}'_{X\times Y} = \mathcal{T}_{X\times Y}$.

2. Let X, Y be two sets and consider $\mathcal{T}^{\psi_L,\vee}_{X\times Y} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ such that $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}^{\psi_L,\vee}_{X\times Y})$ is a type III $\top$-including Galois connection.
Hence, for all $\alpha, \alpha' \in \mathbf{R_L}(X,Y)$: $\top \leq (\alpha,\alpha')\mathcal{T}^{\psi_L,\vee}_{X\times Y} \Leftrightarrow \top \leq (\alpha,\alpha')\mathcal{T}_{X\times Y}$.
Choosing, in particular, the constant L-relations $\alpha = a_{X\times Y}$ and $\alpha' = b_{X\times Y}$, it is easy to seen that $\top \leq (a,b)\psi_L \Leftrightarrow \top \leq a \otimes b$; by arbitrariness of $a, b \in L$, we have that $\top \leq (a,b)\psi_L \Leftrightarrow \top \leq a \otimes b$, for all $a, b, \in L$.
If $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}^{\psi_L,\vee}_{X\times Y})$ is a global connection, then, obviously, $(a,b)\psi_L = a \otimes b$, for all $a, b, \in L$ and hence, this condition forces ψ_L to be $\otimes$.

Proposition 5.1.34. *Let $L = (L, \to, \top)$ be a cdeo algebra.*

1. *If $\perp$ is an element $\otimes$-irreducible of L, for any triple of sets (X,Y,Z) there exist $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ such that, for all $\mathcal{T}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{T}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ and $\mathcal{T}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ such that:*
 - $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\perp(\downarrow)} : (\mathbf{R_L}(Z,X), \mathcal{T}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{T}^1)$;
 - $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\perp(\downarrow)} : (\mathbf{R_L}(Y,Z), \mathcal{T}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{T}^2)$;
 - $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\perp(\downarrow)} : (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}^3)$,
 are type III $\perp$-excluding Tarski connections, the diagram

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y), \mathcal{T}^3) \arrow[dr, "\chi"] & \\ (\mathbf{R_L}(Z,X), \mathcal{T}^1) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z), \mathcal{T}^2) \end{tikzcd}$$

 where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ is defined by $(\rho,\sigma)\varphi = \rho_- \odot \sigma_-$, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, is both a $\perp$-including Tarski L-triangle and $\perp$-excluding Galois L-triangle.

2. *$(L, \to, \top)$ is an associative and commutative cdeo algebra if and only if for any triple of sets (X,Y,Z) there exist $\psi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ such that the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{T}_{X\times Y}) \arrow[dr, "\chi"] & \\
(\mathbf{R_L}(Z,X),\mathcal{T}_{Z\times X}) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\mathbf{R_L}(Y,Z),\mathcal{T}_{Y\times Z})
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \odot \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a relational* L*-triangle.*

Proof. 1. If $\perp$ is $\otimes$-irreducible, one has that $a \otimes (c \otimes b) = \perp \Leftrightarrow a = \perp, b = \perp, c = \perp$, for all $a,b,c \in L$. Hence the following equivalences hold, for all $a,b,c \in L$:

$$a \otimes (c \otimes b) = \perp \Leftrightarrow (a \otimes c) \otimes b = \perp \Leftrightarrow c \otimes (a \otimes b) = \perp.$$

So, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$ and for all $x \in X, y \in Y, z \in Z$:
$(z,x)\rho \otimes ((x,y)\tau \otimes (y,z)\sigma) = \perp \Leftrightarrow$
$((z,x)\rho \otimes (x,y)\tau) \otimes (y,z)\sigma = \perp \Leftrightarrow$
$(x,y)\tau \otimes ((z,x)\rho \otimes (y,z)\sigma) = \perp$.
Hence, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$:
$\bigvee_{x\in X, y\in Y, z\in Z}(z,x)\rho \otimes ((x,y)\tau \otimes (y,z)\sigma) = \perp \Leftrightarrow$
$\bigvee_{x\in X, y\in Y, z\in Z}((z,x)\rho \otimes (x,y)\tau) \otimes (y,z)\sigma = \perp \Leftrightarrow$
$\bigvee_{x\in X, y\in Y, z\in Z}(x,y)\tau \otimes ((z,x)\rho \otimes (y,z)\sigma) = \perp$.
From distributivity condition on L,it follows that:
$\bigvee_{x\in X, z\in Z}(z,x)\rho \otimes (\bigvee_{y\in Y}(x,y)\tau \otimes (y,z)\sigma) = \perp \Leftrightarrow$
$\bigvee_{y\in Y, z\in Z}(\bigvee_{x\in X}(z,x)\rho \otimes (x,y)\tau) \otimes (y,z)\sigma = \perp \Leftrightarrow$
$\bigvee_{x\in X, y\in Y}(x,y)\tau \otimes (\bigvee_{z\in Z}(z,x)\rho \otimes (y,z)\sigma) = \perp$.
Then, we have that:
$\bigvee_{x\in X, z\in Z}(z,x)\rho \otimes (z,x)(\tau \odot \sigma)_- = \perp \Leftrightarrow$
$\bigvee_{y\in Y, z\in Z}(y,z)(\rho \odot \tau)_- \otimes (y,z)\sigma = \perp \Leftrightarrow$
$\bigvee_{x\in X, y\in Y}(x,y)\tau \otimes (x,y)(\rho_- \odot \sigma_-)\sigma) = \perp$,
that is

$$(\rho, ((\tau \odot \sigma)_-)\mathcal{T}_{Z\times X} = \perp \Leftrightarrow ((\rho \odot \tau)_-, \sigma)\mathcal{T}_{Y\times Z} = \perp \Leftrightarrow (\tau, (\rho_- \odot \sigma_-)\mathcal{T}_{X\times Y} = \perp.$$

Hence, for all $\mathcal{T}^1 : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{T}^2 : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$ and $\mathcal{T}^3 : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ such that:
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\perp(\downarrow)} : (\mathbf{R_L}(Z,X), \mathcal{T}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{T}^1)$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\perp(\downarrow)} : (\mathbf{R_L}(Y,Z), \mathcal{T}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{T}^2)$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\perp(\downarrow)} : (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}^3)$,
by Remark 5.1.33 we have that

$$(\rho, ((\tau \odot \sigma)_-)\mathcal{T}^1 = \perp \Leftrightarrow ((\rho \odot \tau)_-, \sigma)\mathcal{T}^2) = \perp \Leftrightarrow (\tau, (\rho_- \odot \sigma_-)\mathcal{T}^3 = \perp.$$

Hence the diagram in the statement, where $\psi \in \mathbf{R}(\mathbf{R_L}(Y,Z))(\mathbf{R}(Z,X), \mathbf{R}(X,Y))$ and $\chi \in \mathbf{R}(\mathbf{R_L}(Z,X))(\mathbf{R}(Y,Z), \mathbf{R}(X,Y))$ are defined, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$ by:

- $(\rho,\tau)\psi = (\rho \odot \tau)_-$;
- $(\tau,\sigma)\chi = (\tau \odot \sigma)_-$

is a $\perp$-including Tarski L-triangle. As already seen in Section 4.5, this is equivalent to say that the diagram is a $\perp$-excluding Galois L-triangle, too.

2. Associativity and commutativity of L are equivalent to the equality, for all $a,b,c \in L$, $a \otimes (c \otimes b) = (a \otimes c) \otimes b = c \otimes (a \otimes b)$. So, we can prove easily the statement, with the similar arguments used in the item (1) and in Proposition 5.1.18, using Remark 5.1.33.

□

Remark 5.1.35. We note that the result in the item (1) of Proposition 5.1.18 cannot be extended to many-valued relations in the shape of the item (1) of the above Proposition, even if L is symmetrical.
In fact, on one side we have that, for all $a,b,c \in L$:

$$a \otimes (c \otimes b) = \top \Leftrightarrow (a \otimes c) \otimes b = \top \Leftrightarrow c \otimes (a \otimes b) = \top$$

and, hence, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$ and for all $x \in X, y \in Y, z \in Z$ we have that: $(z,x)\rho \otimes ((x,y)\tau \otimes (y,z)\sigma) = \top \Leftrightarrow ((z,x)\rho \otimes (x,y)\tau) \otimes (y,z)\sigma = \top \Leftrightarrow (x,y)\tau \otimes ((z,x)\rho \otimes (y,z)\sigma) = \top$.
On the other side, however the equivalences:
$\bigvee_{x\in X, z\in Z}(z,x)\rho \otimes (\bigvee_{y\in Y}(x,y)\tau \otimes (y,z)\sigma) = \top \Leftrightarrow$
$\bigvee_{y\in Y, z\in Z}(\bigvee_{x\in X}(z,x)\rho \otimes (x,y)\tau) \otimes (y,z)\sigma = \top \Leftrightarrow$
$\bigvee_{x\in X, y\in Y}(x,y)\tau \otimes (\bigvee_{z\in Z}(z,x)\rho \otimes (y,z)\sigma) = \top$
need not be satisfied, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$, unless $\top$ is completely $\vee$-irreducible.
Nevertheless, we remark that under the stronger assumption of associativity and commutativity of $(L,\rightarrow,\top)$, the diagram of the item (2) of the above Proposition is, of course, an F-Galois L-triangle, for every upperset $F \subseteq L$, so, in particular, a $\top$-including and $\perp$-excluding Galois L-triangle.

Remark 5.1.36. Of course, under symmetry assumption we can consider the dual intersection degree L-relation $\tilde{\mathcal{T}}_X$; recalling the equality $(A,B)\mathcal{T}_X = (B,A)\tilde{\mathcal{T}}_X$, for all $A,B \in L^X$, we have that the diagram

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\tilde{\mathcal{T}}_{X\times Y}) \arrow[dr, "\tilde{\chi}"] & \\
(\mathbf{R_L}(Z,X),\tilde{\mathcal{T}}_{Z\times X}) \arrow[ur, "\tilde{\psi}"] \arrow[rr, "\tilde{\varphi}"'] & & (\mathbf{R_L}(Y,Z),\tilde{\mathcal{T}}_{Y\times Z})
\end{tikzcd}$$

where φ,ψ,χ are defined for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$ by:

- $(\rho,\sigma)\tilde{\varphi} = (\sigma \tilde{\odot} \rho) = \rho_- \odot \sigma_-$;

- $(\rho,\tau)\tilde{\psi} = \tau_{-}\tilde{\odot}\rho_{-} = (\rho\odot\tau)_{-}$;

- $(\tau,\sigma)\tilde{\chi} = \sigma_{-}\tilde{\odot}\tau_{-} = (\tau\odot\sigma)_{-}$,

is both a $\perp$-including Tarski L-triangle and a $\perp$-excluding Galois L triangle, if $\perp$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible), since $\tilde{\mathcal{T}}_X$ have the similar properties of $\mathcal{T}_X$, as $\tilde{\otimes}$ has the similar properties of $\otimes$.
Moreover, all the diagrams that can be obtained considering arbitrarily the intersection L-relation and the dual intersection in the vertices and ψ,χ,φ or $\tilde{\psi},\tilde{\chi},\tilde{\varphi}$ on the sides, are both $\perp$-including Tarski L-triangles and $\perp$-excluding Galois L triangles, if $\perp$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible).

5.2 Double relational triangles

The results of Propositions 5.1.17, 5.1.18, 5.1.31 and 5.1.34 show how relational L-triangles may be used to characterize the simultaneous occurrence of associativity and commutativity of the algebra $(L,\to,\top)$.
To deal with the symmetrical non-commutative case it is more convenient to consider double L-triangles, as we shall see in this Section.

Definition 5.2.1. *Let L be a possibly structured set and let (A,α_1,α_2), (B,β_1,β_2) and (C,γ_1,γ_2) be sets, each equipped with two fixed binary L-relations on it. The diagram*

$$\begin{tikzcd} & (C,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (A,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (B,\beta_1,\beta_2) \end{tikzcd}$$

*with $\varphi,\tilde{\varphi}\in\mathbf{R_C}(A,B)$, $\psi,\tilde{\psi}\in\mathbf{R_B}(A,C)$ and $\chi,\tilde{\chi}\in\mathbf{R_A}(C,B)$ is called **double triangle of L-relations** or, simply, **double L-triangle**.*

Definition 5.2.2. *Let L be a set and let $\equiv$ be an equivalence relation on it. The double triangle of L-relations*

$$\begin{tikzcd} & (C,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (A,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (B,\beta_1,\beta_2) \end{tikzcd}$$

*is **called double relational L-triangle** if for all $a\in A, b\in B, c\in C$ the following equivalences hold:*

1. $((a,c)\psi,b)\beta_1 \equiv (c,(a,b)\varphi)\gamma_1$;

2. $((a,c)\tilde{\psi},b)\beta_2 \equiv (c,(a,b)\tilde{\varphi})\gamma_2$;

3. $(a,(c,b)\chi)\alpha_1 \equiv ((a,c)\tilde{\psi},b)\beta_1$;

4. $(a,(c,b)\tilde{\chi})\alpha_2 \equiv ((a,c)\psi,b)\beta_2$;

5. $(a,(c,b)\tilde{\chi})\alpha_1 \equiv (c,(a,b)\varphi)\gamma_2$;

6. $(a,(c,b)\chi)\alpha_2 \equiv (c,(a,b)\tilde{\varphi})\gamma_1$.

Double relational L-triangles are an enrichment of relational L-triangles. In fact, there is no choice of the relation in each vertex and of the relation on each side that give rise to a relational L-triangle; so, a double relational L-triangle cannot be obtained by merging two relational L-triangles.
Conversely, the class of relational L-triangles may be embedded in that of double ones; in fact, to any given relational L-triangle as in Definition 5.1.2 one can associate the following

$$\begin{tikzcd} & (C,\gamma,\gamma) \arrow[dr, "{(\chi,\chi)}"] & \\ (A,\alpha,\alpha) \arrow[ur, "{(\psi,\psi)}"] \arrow[rr, "{(\varphi,\varphi)}"'] & & (B,\beta,\beta) \end{tikzcd}$$

which is a double relational L-triangle.

Remark 5.2.3. In a similar way as in Remark 5.1.3 and Proposition 5.1.4, one can get several equivalent descriptions of the double relational L-triangle of Definition 5.2.2 as well as of those of the Definition 5.2.4 below.

Definition 5.2.4. *With the above notation, the double triangle of L-relations*

$$\begin{tikzcd} & (C,\gamma_1,\gamma_2) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\ (A,\alpha_1,\alpha_2) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\varphi,\tilde{\varphi})}"'] & & (B,\beta_1,\beta_2) \end{tikzcd}$$

is called ***double E relational L-triangle*** *if for all $a \in A, b \in B, c \in C$ the following equalities hold:*

1. $((a,c)\psi,b)\beta_1 \equiv_E (c,(a,b)\varphi)\gamma_1$;

2. $((a,c)\tilde{\psi},b)\beta_2 \equiv_E (c,(a,b)\tilde{\varphi})\gamma_2$;

3. $(a,(c,b)\chi)\alpha_1 \equiv_E ((a,c)\tilde{\psi},b)\beta_1$;

4. $(a,(c,b)\tilde{\chi})\alpha_2 \equiv_E ((a,c)\psi,b)\beta_2$;

5. $(a,(c,b)\tilde{\chi})\alpha_1 \equiv_E (c,(a,b)\varphi)\gamma_2$;

6. $(a,(c,b)\chi)\alpha_2 \equiv_E (c,(a,b)\tilde{\varphi})\gamma_1$.

Of course, also in this case, the terminology introduced in Chapter 4 on equivalences and connections may be borrowed for double relational L-triangles; so, assuming the needed requirements on L, we shall speak of

- **relative double relational L-triangles**;
- **upper** and **lower double relational L-triangles**;
- **E-including** and **E-excluding upper** or **lower double relational L-triangles**;
- **point-including** and **point-excluding upper** or **lower double relational L-triangles**.

We shall also call every kind of upper double relational L-triangle **Galois double L-triangle** of that kind and every kind of lower double relational L-triangle **Tarski double L-triangle** of that kind.
One can restate for double relational L-triangles most results proved for equivalences and connections in Sections 4.2, 4.3 and 4.4 , except those involving the classification into four types of equivalences and connections. In fact, double relational L-triangles have the feature to gather all together the four types of connections as the following results shows. To give details of this feature, we note that each arrow of a double triangle of L-relations determines four maps defined on the vertices adjacent to the arrow and taking values in the opposite vertex; so, with notation of Definition 5.2.2, for any $a \in A, b \in B, c \in C$ we consider

$$\varphi_a = (a,-)\varphi : B \to C,\ \psi_a = (a,-)\psi : C \to B;$$
$$\tilde{\varphi}_a = (a,-)\tilde{\varphi} : B \to C,\ \tilde{\psi}_a = (a,-)\tilde{\psi} : C \to B;$$
$$\varphi_b = (-,b)\varphi : A \to C,\ \chi_b = (-,b)\chi : C \to A;$$
$$\tilde{\varphi}_b = (-,b)\tilde{\varphi} : A \to C,\ \tilde{\chi}_b = (-,b)\tilde{\chi} : C \to A;$$
$$\psi_c = (-,c)\psi : A \to B,\ \chi_c = (c,-)\chi : B \to A.$$
$$\tilde{\psi}_c = (-,c)\tilde{\psi} : A \to B,\ \tilde{\chi}_c = (c,-)\tilde{\chi} : B \to A.$$

Then, in case of a double relational L-triangle, one has four families of connections of the four different types; in fact, the assumed equivalences give soon

$$\left\{(\tilde{\psi}_c, \chi_c) : (A,\alpha_1) \to (B,\beta_1) |\ c \in C\right\}$$
$$\{)\tilde{\chi}_c, \psi_c(: (B,\beta_2) \to (A,\alpha_2) |\ c \in C\} ;$$
$$\{[\tilde{\chi}_b\text{-}\varphi_b] : (C,\gamma_2) \to (A,\alpha_1) |\ b \in B\} ;$$

and the forth family comes from Proposition 4.2.11 and from Remark 5.2.3

$$\{]\tilde{\chi}_b\text{-}\varphi_b[: (C,(\gamma_2)_-) \to (A,(\alpha_1)_-) |\ b \in B\} .$$

Similarly, every E double relational L-triangle, or every F double Galois L-triangle, $F \subseteq L$ any upperset, or every I double Tarski L-triangle, $I \subseteq L$ any lowerset, gives four families of corresponding connections of the four different types.

The results of Section 4.2 on equivalences allow to state the following result, where we assume, for items (3), (3'), (4), (4') that $\equiv$ is induced by a preorder relation in L.

Proposition 5.2.5. *With the already stated notation, for the double triangle of L-relations*

$$\begin{tikzcd} & (C,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (A,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (B,\beta_1,\beta_2) \end{tikzcd}$$

the following are equivalent:

(1) it is a double relational L-triangle;

(2) it is a double E relational L-triangle, for every $E \subseteq L$;

(2') it is a double $\{x\}$ relational L-triangle, for every $x \in L$;

(3) it is a double F relational L-triangle, for every upper set $F \subseteq L$;

(3') it is a double $x^{[\uparrow]}$ relational L-triangle, for every $x \in L$;

(4) it is a double I relational L-triangle, for every lower set $I \subseteq L$;

(4') it is a double $x^{[\downarrow]}$ relational L-triangle, for every $x \in L$.

5.2.1 Double relational triangles and cdeo algebras

Proposition 5.2.6. *Let $L = (L, \rightarrow, \top)$ be a w-ceo algebra.*
L is distributive if and only if there exist $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \rightarrow L$ such that, for all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \rightarrow L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \gamma_j)$, for $j = 1, 2$, the diagram

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\rightarrow,\tilde{\varphi})"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$-including double Galois L-triangle.

Proof. " $\Rightarrow$ " As in Proposition 5.1.7, if L is distributive, then there exist $\tilde{\varphi} : L \times L \rightarrow L$ and $\otimes : L \times L \rightarrow L$ such that $c \leq a \rightarrow b \Leftrightarrow a \otimes c \leq b \Leftrightarrow a \leq (c, b)\tilde{\varphi}$, for all $a, b, c \in L$. Thanks to these equivalences, we can deduce the following conditions, which hold for all $a, b, c \in L$:

1. $\top \leq ((a,c)\otimes, b) \rightarrow \Leftrightarrow \top \leq (c, (a,b) \rightarrow) \rightarrow$;

2. $\top \leq ((c,a)\otimes, b) \rightarrow \Leftrightarrow \top \leq (c, (a,b)\tilde{\varphi}) \rightarrow$;

3. $\top \leq (a, (c,b) \rightarrow) \rightarrow \Leftrightarrow \top \leq ((c,a)\otimes, b) \rightarrow$;

4. $\top \leq (a, (c,b)\tilde{\varphi}) \rightarrow \Leftrightarrow \top \leq ((a,c)\otimes, b) \rightarrow$;

5. $\top \leq (a, (c,b)\tilde{\varphi}) \rightarrow \Leftrightarrow \top \leq (c, (a,b) \rightarrow) \rightarrow$;

6. $\top \leq (a, (c,b) \rightarrow) \rightarrow \Leftrightarrow \top \leq (c, (a,b)\tilde{\varphi}) \rightarrow$.

Hence, for all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \gamma_j)$, for $j = 1, 2$, there exist $\chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$ determined uniquely, for all $a, b, c \in L$, by:

- $(c,b)\chi = c \to b$;
- $(c,b)\tilde{\chi} = (c,b)\tilde{\varphi}$;
- $(a,c)\psi = a \otimes c$;
- $(a,c)\tilde{\psi} = c \otimes a = a \otimes^{op} c$

such that the following conditions hold:

1. $\top \leq ((a,c)\psi, b)\beta_1 \Leftrightarrow \top \leq (c, (a,b)\varphi)\gamma_1$;

2. $\top \leq ((a,c)\tilde{\psi}, b)\beta_2 \Leftrightarrow \top \leq (c, (a,b)\tilde{\varphi})\gamma_2$;

3. $\top \leq (a, (c,b)\chi)\alpha_1 \Leftrightarrow \top \leq ((a,c)\tilde{\psi}, b)\beta_1$;

4. $\top \leq (a, (c,b)\tilde{\chi})\alpha_2 \Leftrightarrow \top \leq ((a,c)\psi, b)\beta_2$;

5. $\top \leq (a, (c,b)\tilde{\chi})\alpha_1 \Leftrightarrow \top \leq (c, (a,b)\varphi)\gamma_2$;

6. $\top \leq (a, (c,b)\chi)\alpha_2 \Leftrightarrow \top \leq (c, (a,b)\tilde{\varphi})\gamma_1$.

Hence, the diagram in the statement is a $\top^{[\uparrow]}$-including double Galois L-triangle.
" $\Leftarrow$ " By assumption that the diagram in the statement is a $\top^{[\uparrow]}$-including double Galois L-triangle and by the conditions (1) and (5) of Definition 5.2.4, it follows that, for all $a, b, c \in L$:

- $\top \leq ((a,c)\psi, b)\beta_1 \Leftrightarrow \top \leq (c, (a,b) \rightarrow)\gamma_1$;
- $\top \leq (a, (c,b)\tilde{\chi})\alpha_1 \Leftrightarrow \top \leq (c, (a,b)\varphi)\gamma_2$.

Since, by assumption, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \gamma_j)$, for $j = 1, 2$, we have that, for all $a, b, c \in L$:

- $(a,c)\psi \leq b \Leftrightarrow c \leq a \to b$;

- $a \le (c,b)\tilde{\chi} \Leftrightarrow c \le a \to b$.

So, as in Proposition 5.1.7, we can deduce that L is distributive. $\square$

Remark 5.2.7. 1. As in Remark 5.1.8, using the type III connection $(i_L, i_L)_{\top[\uparrow]} : (L,\to) \to (L,\to)$, by Proposition 5.2.6 one can deduce that the diagram

$$\begin{tikzcd} & (L,\to,\to) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\ (L,\to,\to) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\to,\tilde{\varphi})}"'] & & (L,\to,\to) \end{tikzcd}$$

is a $\top^{[\uparrow]}$-including double Galois L-triangle if and only if $(L,\to,\top)$ is a cdeo algebra.

2. If $(L,\to,\top)$ is a symmetrical cdeo algebra, $\rightsquigarrow : L \times L \to L$ is the dual implication, hence it induces the same order as $\to$, then $(i_L, i_L)_{\top[\uparrow]} : (L,\to) \to (L,\rightsquigarrow)$, and $c \le a \to b \Leftrightarrow a \le c \rightsquigarrow b$, for all $a,b,c \in L$. Hence, by Proposition 5.2.6, the diagram

$$\begin{tikzcd} & (L,\rightsquigarrow,\rightsquigarrow) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\ (L,\rightsquigarrow,\rightsquigarrow) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\to,\rightsquigarrow)}"'] & & (L,\rightsquigarrow,\to) \end{tikzcd}$$

is a $\top^{[\uparrow]}$-including double Galois L-triangle.
Further L-triangles of the same kind can be obtained from this one by a free choice, between $\to$ and $\rightsquigarrow$, of the two operations (e.g. L-relations) in each vertex.

Proposition 5.2.8. *Let $L = (L,\to,\top)$ be a w-ceo algebra.*
L is a symmetrical cdeo algebra if and only if there exist $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L,\to) \to (L,\alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L,\to) \to (L,\beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L,\to) \to (L,\gamma_j)$, for $j = 1,2$, the diagram

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\to,\tilde{\varphi})}"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$-including double Galois L-triangle and the following condition holds:
(a) $(i_L, i_L)_{\top[\uparrow]} : (L,\to) \to (L,\tilde{\chi})$.

Proof. " $\Rightarrow$ " As in Proposition 5.1.9, if L is a symmetrical cdeo algebra, then there exist $\tilde{\varphi} = \rightsquigarrow : L \times L \to L$, $\otimes, \tilde{\otimes} : L \times L \to L$ such that $c \le a \to b \Leftrightarrow a \otimes c \le b \Leftrightarrow a \le c \rightsquigarrow b$, for all $a,b,c \in L$, $\to$ and $\rightsquigarrow$ induce the same order, i.e.$(i_L, i_L)_{\top[\uparrow]} : (L,\to) \to (L,\rightsquigarrow)$ and $\otimes$ and $\tilde{\otimes}$ are opposite to each other.
Thanks to these properties, we can deduce the following conditions, which hold for all $a,b,c \in L$:

1. $\top \leq ((a,c)\otimes, b) \rightarrow \Leftrightarrow \top \leq (c, (a,b) \rightarrow) \rightarrow$;
2. $\top \leq ((a,c)\tilde{\otimes}, b) \rightarrow \Leftrightarrow \top \leq (c, (a,b) \rightsquigarrow) \rightarrow$;
3. $\top \leq (a, (c,b) \rightarrow) \rightarrow \Leftrightarrow \top \leq ((a,c)\tilde{\otimes}, b) \rightarrow$;
4. $\top \leq (a, (c,b) \rightsquigarrow) \rightarrow \Leftrightarrow \top \leq ((a,c)\otimes, b) \rightarrow$;
5. $\top \leq (a, (c,b) \rightsquigarrow) \rightarrow \Leftrightarrow \top \leq (c, (a,b) \rightarrow) \rightarrow$;
6. $\top \leq (a, (c,b) \rightarrow) \rightarrow \Leftrightarrow \top \leq (c, (a,b) \rightsquigarrow) \rightarrow$.

Hence, for all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \gamma_j)$, for $j = 1, 2$, there exist $\chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$ determined uniquely, for all $a, b, c \in L$, by:

- $(c,b)\chi = c \rightarrow b$;
- $(c,b)\tilde{\chi} = c \rightsquigarrow b$;
- $(a,c)\psi = a \otimes c$;
- $(a,c)\tilde{\psi} = c \otimes a = a \tilde{\otimes} c$

such that the following conditions hold:

1. $\top \leq ((a,c)\psi, b)\beta_1 \Leftrightarrow \top \leq (c, (a,b)\varphi)\gamma_1$;
2. $\top \leq ((a,c)\tilde{\psi}, b)\beta_2 \Leftrightarrow \top \leq (c, (a,b)\tilde{\varphi})\gamma_2$;
3. $\top \leq (a, (c,b)\chi)\alpha_1 \Leftrightarrow \top \leq ((a,c)\tilde{\psi}, b)\beta_1$;
4. $\top \leq (a, (c,b)\tilde{\chi})\alpha_2 \Leftrightarrow \top \leq ((a,c)\psi, b)\beta_2$;
5. $\top \leq (a, (c,b)\tilde{\chi})\alpha_1 \Leftrightarrow \top \leq (c, (a,b)\varphi)\gamma_2$;
6. $\top \leq (a, (c,b)\chi)\alpha_2 \Leftrightarrow \top \leq (c, (a,b)\tilde{\varphi})\gamma_1$.

Hence, the diagram in the statement is a $\top^{[\uparrow]}$-including double Galois L-triangle.
" $\Leftarrow$ " By assumption that the diagram in the statement is a $\top^{[\uparrow]}$-including double Galois L-triangle and by Proposition 5.2.6, we can deduce that L is distributive. Moreover, as in Proposition 5.1.9, we can prove that the condition $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \tilde{\chi})$ implies that L is symmetrical. □

Proposition 5.2.9. *Let $L = (L, \rightarrow, \top)$ be a w-ceo algebra.*
L is an associative cdeo algebra if and only if there exist $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$ such that, for all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \to L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \gamma_j)$, for $j = 1, 2$, there exist $\chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$, the diagram

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\rightarrow,\tilde{\varphi})"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$-including double Galois L-triangle and the following condition holds:
(b) *for every $b \in L$, $(\psi_b\text{-} \rightarrow_b) : (L, \rightarrow) \rightarrow (L, \rightarrow)$.*

Proof. By Proposition 5.2.6, the distributivity of L is equivalent to all the conditions required in the statement, except the type I connection $(\psi_b\text{-} \rightarrow_b) : (L, \rightarrow) \rightarrow (L, \rightarrow)$.
As in Proposition 5.1.10, we can prove that the associativity of L is equivalent to the condition (b). □

Corollary 5.2.10. *Let $\mathbf{L} = (L, \rightarrow, \top)$ be a w-ceo algebra. $\mathbf{L}$ is an associative symmetrical cdeo algebra if and only if there exist $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \rightarrow L$ such that, for all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \rightarrow L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \gamma_j)$, for $j = 1, 2$, the diagram*

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\rightarrow,\tilde{\varphi})"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$-including double Galois L-triangle and the following conditions hold:

(a) $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \tilde{\chi})$;

(b) for every $b \in L$, $(\psi_b\text{-} \rightarrow_b) : (L, \rightarrow) \rightarrow (L, \rightarrow)$.

Proof. The statement is an easy consequence of Propositions 5.2.8 and 5.2.9. □

An alternative characterization of associative symmetrical cdeo algebra can be given as follows.

Proposition 5.2.11. *Let $\mathbf{L} = (L, \rightarrow, \top)$ be a w-ceo algebra. $\mathbf{L}$ is an associative symmetrical cdeo algebra if and only if there exist $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \rightarrow L$ such that, for all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \rightarrow L$ such that $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \rightarrow (L, \gamma_j)$, for $j = 1, 2$, the diagram*

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\rightarrow,\tilde{\varphi})"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$-including double Galois L-triangle and the following conditions hold:

(a) $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \tilde{\chi})$;

(c) for every $b \in L$, $[\rightarrow_b \text{-} \tilde{\chi}_b] : (L, \tilde{\chi}) \to (L, \rightarrow)$.

Proof. The proof is similar to that of Proposition 5.1.12. □

Proposition 5.2.12. *Let* $L = (L, \rightarrow, \top)$ *be a w-ceo algebra.*
L is a commutative cdeo algebra if and only if there exist $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$ *such that, for all* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \to L$ *such that* $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \gamma_j)$, *for* $j = 1, 2$, *the diagram*

$$\begin{tikzcd} & (L, \gamma_1, \gamma_2) \arrow[dr, "(\chi, \tilde{\chi})"] & \\ (L, \alpha_1, \alpha_2) \arrow[ur, "(\psi, \tilde{\psi})"] \arrow[rr, "(\rightarrow, \tilde{\varphi})"'] & & (L, \beta_1, \beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$*-including double Galois L-triangle and the following condition holds:*
(a') $(i_L, i_L) : (L, \rightarrow) \to (L, \tilde{\chi})$.

Proof. The proof is similar to that of Proposition 5.1.14. □

Corollary 5.2.13. *Let* $L = (L, \rightarrow, \top)$ *be a w-ceo algebra. L is an associative and commutative cdeo algebra if and only if there exist* $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$ *such that, for all* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \to L$ *such that* $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \gamma_j)$, *for* $j = 1, 2$, *the diagram*

$$\begin{tikzcd} & (L, \gamma_1, \gamma_2) \arrow[dr, "(\chi, \tilde{\chi})"] & \\ (L, \alpha_1, \alpha_2) \arrow[ur, "(\psi, \tilde{\psi})"] \arrow[rr, "(\rightarrow, \tilde{\varphi})"'] & & (L, \beta_1, \beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$*-including double Galois L-triangle and the following conditions hold:*

(a') $(i_L, i_L) : (L, \rightarrow) \to (L, \tilde{\chi})$;

(b) for every $b \in L$, $(\psi_b \text{-} \rightarrow_b) : (L, \rightarrow) \to (L, \rightarrow)$.

Proof. The statement is an easy consequence of Corollary 5.2.10 and Proposition 5.2.12. □

Proposition 5.2.14. *Let* $L = (L, \rightarrow, \top)$ *be a w-ceo algebra. L is an associative commutative cdeo algebra if and only if there exist* $\tilde{\varphi}, \chi, \tilde{\chi}, \psi, \tilde{\psi} : L \times L \to L$ *such that for all* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \to L$ *such that* $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \rightarrow) \to (L, \gamma_j)$, *for* $j = 1, 2$, *the diagram*

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\rightarrow,\tilde{\varphi})"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is a $\top^{[\uparrow]}$*-including double Galois* L*-triangle and the following conditions hold:*

(a') $(i_L, i_L) : (L, \rightarrow) \rightarrow (L, \tilde{\chi})$;

(b) for every $b \in L$, $[\rightarrow_b \text{-} \chi_b] : (L, \chi) \rightarrow (L, \rightarrow)$.

Proof. The statement is an easy consequence of Corollary 5.1.12 and Proposition 5.2.12. □

As anticipated in the introduction to this section, unlike what happens with L-triangles, it is possible to characterize associativity of symmetrical cdeo algebra by means of the existence of a particular double relational L-triangle, without getting the commutativity condition. In fact, the following holds.

Proposition 5.2.15. *Let* $L = (L, \rightarrow, \top)$ *be a symmetrical cdeo algebra.* L *is associative if and only if the diagram*

$$\begin{tikzcd} & (L,\rightarrow,\rightsquigarrow) \arrow[dr, "(\rightarrow,\rightsquigarrow)"] & \\ (L,\rightarrow,\rightsquigarrow) \arrow[ur, "(\otimes,\tilde{\otimes})"] \arrow[rr, "(\rightarrow,\rightsquigarrow)"'] & & (L,\rightarrow,\rightsquigarrow) \end{tikzcd}$$

is a double relational L*-triangle.*

Proof. " ⇒ " If L is a symmetrical associative cdeo algebra, then it has been proved in Subsection 2.1.5 that the following equalities hold, for all $a, b, c \in L$:

1. $((a, c)\otimes, b) \rightarrow = (c, (a, b) \rightarrow) \rightarrow$;
2. $((a, c)\tilde{\otimes}, b) \rightsquigarrow = (c, (a, b) \rightsquigarrow) \rightsquigarrow$;
3. $(a, (c, b) \rightarrow) \rightarrow = ((a, c)\tilde{\otimes}, b) \rightarrow$;
4. $(a, (c, b) \rightsquigarrow) \rightsquigarrow = ((a, c)\otimes, b) \rightsquigarrow$;
5. $(a, (c, b) \rightsquigarrow) \rightarrow = (c, (a, b) \rightarrow) \rightsquigarrow$;
6. $(a, (c, b) \rightarrow) \rightsquigarrow = (c, (a, b) \rightsquigarrow) \rightarrow$.

So, the diagram in the statement is a double relational L-triangle.
" ⇐ " It has been seen in Subsection 2.1.5 how the first condition (1) that, by Definition 5.2.2, characterize the double relational L-triangles implies that L is associative. □

Proposition 5.2.16. *Let* $(L, \rightarrow, \top)$ *be a right-distributive w-ceo algebra.*

1. *If* $(L, \rightarrow, \top)$ *is a symmetrical cdeo algebra, for all* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \rightarrow L$ *such that* $(i_L, i_L)_{\top[\uparrow]} : (L, \otimes) \rightarrow (L, \alpha_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \otimes) \rightarrow (L, \beta_j)$, $(i_L, i_L)_{\top[\uparrow]} : (L, \otimes) \rightarrow (L, \gamma_j)$, *for* $j = 1, 2$, *the triangle of* L*-relations*

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\otimes,\otimes)"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\otimes,\otimes)"] \arrow[rr, "(\otimes,\otimes)"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is both a $\top$*-including double Galois* L*-triangle and a* $\top$*-excluding double Tarski* L*-triangle.*

2. *If* $\bot$ *is* $\otimes$*-irreducible, for all* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \rightarrow L$ *such that* $(i_L, i_L)_{\bot(\downarrow)} : (L, \otimes) \rightarrow (L, \alpha_j)$, $(i_L, i_L)_{\bot(\downarrow)} : (L, \otimes) \rightarrow (L, \beta_j)$, $(i_L, i_L)_{\bot(\downarrow)} : (L, \otimes) \rightarrow (L, \gamma_j)$, *for* $j = 1, 2$, *the triangle of* L*-relations*

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\otimes,\otimes)"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\otimes,\otimes)"] \arrow[rr, "(\otimes,\otimes)"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is $\bot$*-including double Tarski* L*-triangle and a* $\bot$*-excluding double Galois* L *triangle.*

3. $(L, \rightarrow, \top)$ *is an associative and commutative cdeo algebra if and only if for all* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 : L \times L \rightarrow L$ *such that* $(i_L, i_L) : (L, \otimes) \rightarrow (L, \alpha_j)$, $(i_L, i_L) : (L, \otimes) \rightarrow (L, \beta_j)$, $(i_L, i_L) : (L, \otimes) \rightarrow (L, \gamma_j)$, *for* $j = 1, 2$, *the triangle of* L*-relations*

$$\begin{tikzcd} & (L,\gamma_1,\gamma_2) \arrow[dr, "(\otimes,\otimes)"] & \\ (L,\alpha_1,\alpha_2) \arrow[ur, "(\otimes,\otimes)"] \arrow[rr, "(\otimes,\otimes)"'] & & (L,\beta_1,\beta_2) \end{tikzcd}$$

is a double relational L*-triangle.* □

Remark 5.2.17. Of course, under symmetry assumption, the diagram

$$\begin{tikzcd} & (L,\tilde{\otimes},\tilde{\otimes}) \arrow[dr, "(\tilde{\otimes},\tilde{\otimes})"] & \\ (L,\tilde{\otimes},\tilde{\otimes}) \arrow[ur, "(\tilde{\otimes},\tilde{\otimes})"] \arrow[rr, "(\tilde{\otimes},\tilde{\otimes})"'] & & (L,\tilde{\otimes},\tilde{\otimes}) \end{tikzcd}$$

is

1. both a $\top$-including double Galois L-triangle and a $\top$-excluding double Tarski L-triangle;

2. both a $\bot$-including double Tarski L-triangle and a $\bot$-excluding double Galois L triangle, if $\bot$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible);

3. a double relational L-triangle if and only if $(L, \to, \top)$ is an associative and commutative cdeo algebra.

Moreover, all the diagrams that can be obtained considering arbitrarily $(L, \otimes, \tilde{\otimes})$ or $(L, \tilde{\otimes}, \otimes)$ in the vertices and $(\otimes, \tilde{\otimes})$ or $(\tilde{\otimes}, \otimes)$ on the sides, are

1. both a $\top$-including double Galois L-triangle and a $\top$-excluding double Tarski L-triangle;

2. both a $\bot$-including double Tarski L-triangle and a $\bot$-excluding double Galois L triangle, if $\bot$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible);

3. a double relational L-triangle if and only if $(L, \to, \top)$ is an associative and commutative cdeo algebra (in fact, under commutativity assumption, $\otimes = \tilde{\otimes}$, i.e. $(i_L, i_L) : (L, \otimes) \to (L, \tilde{\otimes})$).

Proposition 5.2.18. *Let $(L, \to, \top)$ be a symmetrical cdeo algebra. L is associative if and only if the triangle of L-relations*

$$\begin{tikzcd}
& (L, \otimes, \tilde{\otimes}) \arrow[dr, "(\otimes, \tilde{\otimes})"] & \\
(L, \otimes, \tilde{\otimes}) \arrow[ur, "(\tilde{\otimes}, \otimes)"] \arrow[rr, "(\otimes, \tilde{\otimes})"'] & & (L, \otimes, \tilde{\otimes})
\end{tikzcd}$$

is a double relational L-triangle.

Proof. The statement is equivalent to the following equalities that hold, for all $a, b, c \in L$, if and only if L is an associative symmetrical cdeo algebra:

1. $(a \tilde{\otimes} c) \otimes b = c \otimes (a \otimes b)$;

2. $(a \otimes c) \tilde{\otimes} b = c \tilde{\otimes} (a \tilde{\otimes} b)$;

3. $a \otimes (c \otimes b) = (a \otimes c) \otimes b$;

4. $a \tilde{\otimes} (c \tilde{\otimes} b) = (a \tilde{\otimes} c) \tilde{\otimes} b$;

5. $a \otimes (c \tilde{\otimes} b) = c \tilde{\otimes} (a \otimes b)$;

6. $a \tilde{\otimes} (c \otimes b) = c \otimes (a \tilde{\otimes} b)$.

$\square$

5.2.2 Double relational triangles and many-valued relations

Results similar to those viewed for relational L-triangles can be given for double relational L-triangles. The proof of the following propositions use similar arguments as in Subsection 5.1.2.

Proposition 5.2.19. *Let $L = (L, \to, \top)$ be a w-ceo algebra. L is distributive if and only if for any triple of sets (X, Y, Z) there exist $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ such that, for all $\mathcal{S}^1_{Z\times X} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{S}^2_{Z\times X} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{S}^1_{Y\times Z} : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$, $\mathcal{S}^2_{Y\times Z} : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$, $\mathcal{S}^1_{X\times Y} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ and $\mathcal{S}^2_{X\times Y} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ such that:*
- $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)})_{\top[\uparrow]} : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{S}^j_{Z\times X})$;
- $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\top[\uparrow]} : (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}^j_{Y\times Z})$;
- $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^j_{X\times Y})$,
for $j = 1, 2$, are type III $\top$-including Galois connections, the diagram

$$
\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{S}^1_{X\times Y}, \mathcal{S}^2_{X\times Y}) \arrow[dr, "{(\chi, \tilde{\chi})}"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}^1_{Z\times X}, \mathcal{S}^2_{Z\times X}) \arrow[ur, "{(\psi, \tilde{\psi})}"] \arrow[rr, "{(\varphi, \tilde{\varphi})}"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}^1_{Y\times Z}, \mathcal{S}^2_{Y\times Z})
\end{tikzcd}
$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ is defined by $(\rho, \sigma)\varphi = \rho_- \oslash \sigma_-$, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, is a $\top$-including double Galois L-triangle.

Proof. The proof runs as in Proposition 5.1.20, after defining the functions $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ as follows:

- $(\rho, \sigma)\tilde{\varphi} = \rho_- \tilde{\varphi} \sigma_-$;
- $(\rho, \tau)\psi = (\rho \odot \tau)_-$;
- $(\rho, \tau)\tilde{\psi} = \tau_- \odot \rho_- = (\rho \odot^{op} \tau)_-$;
- $(\tau, \sigma)\chi = (\tau \oslash \sigma)_-$;
- $(\tau, \sigma)\tilde{\chi} = (\tau \tilde{\varphi} \sigma)_-$

by using the operations $\otimes, \tilde{\varphi}_L : L \times L \to L$ such that $c \le a \to b \Leftrightarrow a \otimes c \le b \Leftrightarrow a \le (c, b)\tilde{\varphi}_L$, for all $a, b, c \in L$, whose existence is equivalent to the distributivity of L. □

Remark 5.2.20. 1. The conditions assumed in the above Proposition such as

$$(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^j_{X\times Y})$$

mean that each of $\mathcal{S}^j_{X\times Y}$ and the other L-relations in the vertices of the L-triangle induce on the set where they are defined, e.g. $\mathbf{R_L}(X,Y)$ or $\mathbf{R_L}(Z,X)$ or $\mathbf{R_L}(Y,Z)$,

the same order as the corresponding subsethood degree does, i.e. the natural ordering of the w-ceo algebra pointwisely induced by $(L, \to, \top)$.
This means that each of $\mathcal{S}^j_{Z\times X}$, $\mathcal{S}^j_{Y\times Z}$, $\mathcal{S}^j_{X\times Y}$ is a weak L-order which is compatible with the subsethood L-relation.
Of course, each of the subsethood L-relation, and the dual subsethood L-relation, in case of a symmetrical cdeo algebra, as well, satisfies the above condition, so each of them may replace each of the L-relation in the vertices of the L-triangle in Proposition 5.2.19.
Moreover, we remark that the statement of such Proposition holds if and only if it is true for any single, free choice of the L-orders in each vertex of the double L-triangle, so we shall formulate next statements only using either the subsethood or the dual subsethood L-relation.
It has to be meant, however, that in the subsequent statements, except in Propositions 5.2.28 and 5.2.29, each of the subsethood or dual subsethood L-relation in the vertices of the double L-triangles may be replaced by corresponding relations that, in fact, induce the same order as the subsethood L-relation does, i.e. by corresponding compatible weak L-order on the appropriate set.

2. Now, we also restate the above Proposition, according to item (1) of this remark. Let $(L, \to, \top)$ be a w-ceo algebra. L is distributive if and only if the diagram

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \mathcal{S}_{X\times Y}) \arrow[dr, "(\chi, \tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \mathcal{S}_{Z\times X}) \arrow[ur, "(\psi, \tilde{\psi})"] \arrow[rr, "(\varphi, \tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \mathcal{S}_{Y\times Z})
\end{tikzcd}$$

is a $\top$-including Galois L-triangle, where $\varphi, \tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ are defined as in the above Proposition.
We also remark that, if L is a symmetrical cdeo algebra, all the diagrams that can be obtained by the one above described, considering, arbitrarily, in each vertex one of the corresponding pairs $(\mathcal{S}, \tilde{\mathcal{S}})$, $(\tilde{\mathcal{S}}, \mathcal{S})$, $(\tilde{\mathcal{S}}, \tilde{\mathcal{S}})$ are $\top$-including Galois L-triangles.

Proposition 5.2.21. *Let $L = (L, \to, \top)$ be a w-ceo algebra.*
L is symmetrical cdeo algebra if and only if for any triple of sets (X, Y, Z) there exist $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ such that the diagram

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \mathcal{S}_{X\times Y}) \arrow[dr, "(\chi, \tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \mathcal{S}_{Z\times X}) \arrow[ur, "(\psi, \tilde{\psi})"] \arrow[rr, "(\varphi, \tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \mathcal{S}_{Y\times Z})
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ is defined by $(\rho, \sigma)\varphi = \rho_- \oslash \sigma_-$, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, is a $\top$-including Galois L-triangle and the following condition

holds:
(*a*) $\forall X, Y \in |\boldsymbol{Set}|$*:* $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]} : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\tilde{\chi}_L}_{X\times Y})$*,* *where* $\tilde{\chi}_L : L \times L \to L, (a,b) \mapsto (a,b,)\tilde{\chi}_L = (a_{X\times Y}, b_{Y\times Z})\tilde{\chi}$*.*

Proof. The proof runs as in Proposition 5.1.22, after defining the functions $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ as follows:

- $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$;
- $(\rho,\sigma)\tilde{\varphi} = \rho_- \tilde{\oslash} \sigma_-$;
- $(\rho,\tau)\psi = (\rho \odot \tau)_-$;
- $(\rho,\tau)\tilde{\psi} = (\rho \tilde{\odot} \tau)_-$;
- $(\tau,\sigma)\chi = (\tau \oslash \sigma)_-$;
- $(\tau,\sigma)\tilde{\chi} = (\tau \tilde{\oslash} \sigma)_-$.

□

Proposition 5.2.22. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra.*
L is an associative cdeo algebra if and only if for any triple of sets (X,Y,Z) there exists $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$*,* $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$*,* $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that the diagram*

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \mathcal{S}_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \mathcal{S}_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \mathcal{S}_{Y\times Z})
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a $\top$-including Galois L-triangle and the following condition hold:*
(*b*) $\forall \rho \in \mathbf{R_L}(Z,X)$*:* $(\psi_\rho\text{-}\varphi_\rho) : (\mathbf{R_L}(X,Y), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}_{X\times Y})$ *is a type I global connection, where* $\psi_\rho : \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z), \tau \mapsto (\tau)\psi_\rho = (\rho,\tau)\psi$ *and* $\varphi_\rho : \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y), \sigma \mapsto (\sigma)\varphi_\rho = (\rho,\sigma)\varphi$*.*

Proof. The proof runs as in Proposition 5.1.23, after defining the functions $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ as in Proposition 5.2.19. □

Corollary 5.2.23. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra. L is a symmetrical associative cdeo algebra if and only if for any triple of sets (X,Y,Z) there exist* $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$*,* $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$*,* $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}_{X\times Y},\mathcal{S}_{X\times Y}) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}_{Z\times X},\mathcal{S}_{Z\times X}) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\varphi,\tilde{\varphi})}"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}_{Y\times Z},\mathcal{S}_{Y\times Z})
\end{tikzcd}$$

where $\varphi:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi=\rho_{-}\oslash\sigma_{-}$*, for all* $\rho\in\mathbf{R_L}(Z,X),\sigma\in\mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois L-triangle and the following conditions hold:*
(a) $\forall X,Y\in|\boldsymbol{Set}|$*:* $(\mathcal{I}_{\mathbf{R_L}(X,Y)},\mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]}:(\mathbf{R_L}(X,Y),\mathcal{S}_{X\times Y})\to(\mathbf{R_L}(X,Y),\mathcal{S}^{\tilde{\chi}_L}_{X\times Y})$*;*
(b) $\forall\rho\in\mathbf{R_L}(Z,X)$*:* $(\psi_\rho\text{-}\varphi_\rho):(\mathbf{R_L}(X,Y),\mathcal{S}_{Y\times Z})\to(\mathbf{R_L}(Y,Z),\mathcal{S}_{X\times Y})$*.*

Proof. The statement is an easy consequence of the Proposition 5.2.21 and 5.2.22. □

Proposition 5.2.24. *Let* $L=(L,\to,\top)$ *be a w-ceo algebra.* L *is symmetrical associative cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exists* $\tilde{\varphi}:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(X,Y)$*,* $\psi,\tilde{\psi}:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(X,Y)\to\mathbf{R_L}(Y,Z)$*,* $\chi,\tilde{\chi}:\mathbf{R_L}(X,Y)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(Z,X)$ *such that the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}_{X\times Y},\mathcal{S}_{X\times Y}) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}_{Z\times X},\mathcal{S}_{Z\times X}) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\varphi,\tilde{\varphi})}"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}_{Y\times Z},\mathcal{S}_{Y\times Z})
\end{tikzcd}$$

where $\varphi:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi=\rho_{-}\oslash\sigma_{-}$*, for all* $\rho\in\mathbf{R_L}(Z,X),\sigma\in\mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois L-triangle and the following conditions hold:*
(a) $\forall X,Y\in|\boldsymbol{Set}|$*:* $(\mathcal{I}_{\mathbf{R_L}(X,Y)},\mathcal{I}_{\mathbf{R_L}(X,Y)})_{\top[\uparrow]}:(\mathbf{R_L}(X,Y),\mathcal{S}_{X\times Y})\to(\mathbf{R_L}(X,Y),\mathcal{S}^{\tilde{\chi}_L}_{X\times Y})$*.*
(c) $\forall\sigma\in\mathbf{R_L}(Y,Z)$*:* $[\varphi_\sigma\text{-}\tilde{\chi}_\sigma]:(\mathbf{R_L}(Z,X),\mathcal{S}_{Z\times X})\to(\mathbf{R_L}(X,Y),\mathcal{S}^{\chi_L}_{X\times Y})$ *is a type I global connection, where* $\varphi_\sigma:\mathbf{R_L}(Z,X)\to\mathbf{R_L}(X,Y),\rho\mapsto(\rho)\varphi_\sigma=(\rho,\sigma)\varphi$*,* $\tilde{\chi}_\sigma:\mathbf{R_L}(X,Y)\to\mathbf{R_L}(Z,X),\tau\mapsto(\tau)\chi_\sigma=(\tau,\sigma)\chi$*.*

Proof. The proof runs as in Proposition 5.1.25, after defining the functions $\tilde{\varphi}:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(X,Y)$, $\psi,\tilde{\psi}:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(X,Y)\to\mathbf{R_L}(Y,Z)$, $\chi,\tilde{\chi}:\mathbf{R_L}(X,Y)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(Z,X)$ as in Proposition 5.2.21. □

Proposition 5.2.25. *Let* $L=(L,\to,\top)$ *be a w-ceo algebra.*
L *is commutative cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exist* $\tilde{\varphi}:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(X,Y)$*,* $\psi,\tilde{\psi}:\mathbf{R_L}(Z,X)\times\mathbf{R_L}(X,Y)\to\mathbf{R_L}(Y,Z)$*,* $\chi,\tilde{\chi}:\mathbf{R_L}(X,Y)\times\mathbf{R_L}(Y,Z)\to\mathbf{R_L}(Z,X)$ *such that the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y),\mathcal{S}_{X\times Y},\mathcal{S}_{X\times Y}) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\
(\mathbf{R_L}(Z,X),\mathcal{S}_{Z\times X},\mathcal{S}_{Z\times X}) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\varphi,\tilde{\varphi})}"'] & & (\mathbf{R_L}(Y,Z),\mathcal{S}_{Y\times Z},\mathcal{S}_{Y\times Z})
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois L-triangle and the following condition holds:*
(a') $\forall X,Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\tilde{\chi}_L}_{X\times Y})$*,*
where $\chi_L : \times L \to L, (a,b) \mapsto (a,b,)\tilde{\chi}_L = (a_{X\times Y}, b_{Y\times Z})\tilde{\chi}$.

Proof. The proof runs as in Proposition 5.1.27, after defining the functions $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ as in Proposition 5.2.19. $\square$

Corollary 5.2.26. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra.* L *is an associative commutative cdeo algebra if and only if for any triple of sets* (X,Y,Z) *ther exists* $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y), \mathcal{S}^1_{X\times Y}, \mathcal{S}^2_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}^1_{Z\times X}, \mathcal{S}^2_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}^1_{Y\times Z}, \mathcal{S}^2_{Y\times Z})
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois L-triangle and the following conditions hold:*
(a') $\forall X,Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\tilde{\chi}_L}_{X\times Y})$*,*
where $\chi_L : \times L \to L, (a,b) \mapsto (a,b,)\tilde{\chi}_L = (a_{X\times Y}, b_{Y\times Z})\tilde{\chi}$.
(b) $\forall \rho \in \mathbf{R_L}(Z,X)$: $(\psi_\rho\text{-}\varphi_\rho) : (\mathbf{R_L}(X,Y), \mathcal{S}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{S}_{X\times Y})$.

Proof. The statement is an easy consequence of the Propositions 5.2.22 and 5.2.25. $\square$

Corollary 5.2.27. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra.* L *is an associative commutative cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exist* $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that the diagram*

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \mathcal{S}_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \mathcal{S}_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \mathcal{S}_{Y\times Z})
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$*, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$*, is a* $\top$*-including Galois L-triangle and the following conditions hold:*
(a') $\forall X,Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\tilde{\chi}_L}_{X\times Y})$*,*
where $\chi_L : \times L \to L, (a,b) \mapsto (a,b,)\tilde{\chi}_L = (a_{X\times Y}, b_{Y\times Z})\tilde{\chi}$.
(c) $\forall \sigma \in \mathbf{R_L}(Y,Z)$, $[\varphi_\sigma\text{-}\tilde{\chi}_\sigma] : (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\chi_L}_{X\times Y}$ *is a type*

I global connection, where $\varphi_\sigma : \mathbf{R_L}(Z,X) \to \mathbf{R_L}(X,Y), \rho \mapsto (\rho)\varphi_\sigma = (\rho,\sigma)\varphi$, $\tilde{\chi}_\sigma : \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Z,X), \tau \mapsto (\tau)\chi_\sigma = (\tau,\sigma)\chi$.

Proof. The statement is an easy consequence of the Propositions 5.2.24 and 5.2.25, recalling that a commutative cdeo algebra is, of course, symmetrical. □

Proposition 5.2.28. *Let* $L = (L, \to, \top)$ *be a symmetrical cdeo algebra. L is associative if and only if for any triple of sets* (X,Y,Z) *the diagram*

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \tilde{\mathcal{S}}_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \tilde{\mathcal{S}}_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \tilde{\mathcal{S}}_{Y\times Z})
\end{tikzcd}$$

where the functions $\varphi, \tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *are defined as follows, for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$:

- $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$;
- $(\rho,\sigma)\tilde{\varphi} = \rho_- \tilde{\oslash} \sigma_-$;
- $(\rho,\tau)\psi = (\rho \odot \tau)_-$;
- $(\rho,\tau)\tilde{\psi} = (\rho \tilde{\odot} \tau)_-$;
- $(\tau,\sigma)\chi = (\tau \oslash \sigma)_-$;
- $(\tau,\sigma)\tilde{\chi} = (\tau \tilde{\oslash} \sigma)_-$

is a double relational L-triangle. □

Proposition 5.2.29. *Let* $L = (L, \to, \top)$ *be a w-ceo algebra. L is an associative commutative cdeo algebra if and only if for any triple of sets* (X,Y,Z) *there exist* $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *such that the diagram*

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \mathcal{S}_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \mathcal{S}_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \mathcal{S}_{Y\times Z})
\end{tikzcd}$$

where $\varphi : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$ *is defined by* $(\rho,\sigma)\varphi = \rho_- \oslash \sigma_-$, *for all* $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z)$, *is a double relational L-triangle and the following condition hold:*
(a') $\forall X,Y \in |\boldsymbol{Set}|$: $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)}) : (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{S}^{\tilde{\chi}_L}_{X\times Y})$, *where* $\chi_L : \times L \to L, (a,b) \mapsto (a,b,)\tilde{\chi}_L = (a_{X\times Y}, b_{Y\times Z})\tilde{\chi}$.

Proof. The proof runs as in Proposition 5.1.31, after defining the functions $\tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ as in Proposition 5.2.19. □

Proposition 5.2.30. *Let $L = (L, \to, \top)$ be a cdeo algebra.*

1. *If $\bot$ is $\otimes$-irreducible, for any triple of sets (X,Y,Z) and for all $\mathcal{T}^1_{Z\times X} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{T}^2_{Z\times X} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Z,X) \to L$, $\mathcal{T}^1_{Y\times Z} : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$, $\mathcal{T}^2_{Y\times Z} : \mathbf{R_L}(Y,Z) \times \mathbf{R_L}(Y,Z) \to L$, $\mathcal{T}^1_{X\times Y} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ and $\mathcal{T}^2_{X\times Y} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(X,Y) \to L$ such that:*
   - $(\mathcal{I}_{\mathbf{R_L}(Z,X)}, \mathcal{I}_{\mathbf{R_L}(Z,X)}))_{\bot(\downarrow)} : (\mathbf{R_L}(Z,X), \mathcal{T}_{Z\times X}) \to (\mathbf{R_L}(Z,X), \mathcal{T}^j_{Z\times X})$;
   - $(\mathcal{I}_{\mathbf{R_L}(Y,Z)}, \mathcal{I}_{\mathbf{R_L}(Y,Z)})_{\bot(\downarrow)} : (\mathbf{R_L}(Y,Z), \mathcal{T}_{Y\times Z}) \to (\mathbf{R_L}(Y,Z), \mathcal{T}^j_{Y\times Z})$;
   - $(\mathcal{I}_{\mathbf{R_L}(X,Y)}, \mathcal{I}_{\mathbf{R_L}(X,Y)})_{\bot(\downarrow)} : (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}) \to (\mathbf{R_L}(X,Y), \mathcal{T}^j_{X\times Y})$,

   for $j = 1, 2$, are type III $\top$-including Galois connections, the diagram

$$\begin{tikzcd}[column sep=small]
 & (\mathbf{R_L}(X,Y), \mathcal{T}^1_{X\times Y}, \mathcal{T}^2_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{T}^1_{Z\times X}, \mathcal{T}^2_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{T}^1_{Y\times Z}, \mathcal{T}^2_{Y\times Z})
\end{tikzcd}$$

   where the functions $\varphi, \tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ are defined as follows, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$:

   - $(\rho,\sigma)\varphi = \rho_- \odot \sigma_-$;
   - $(\rho,\sigma)\tilde{\varphi} = (\sigma \odot \rho)_-$;
   - $(\rho,\tau)\psi = \tau_- \odot \rho_-$;
   - $(\rho,\tau)\tilde{\psi} = (\rho \odot \tau)_-$;
   - $(\tau,\sigma)\chi = (\tau \odot \sigma)_-$
   - $(\tau,\sigma)\tilde{\chi} = \sigma_- \odot \tau_-$

   is a $\bot$-including double Tarski L-triangle and a $\bot$-excluding double Galois L-triangle.

2. *L is an associative cdeo algebra if and only if for any triple of sets (X,Y,Z), the diagram*

$$\begin{tikzcd}[column sep=small]
 & (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}, \tilde{\mathcal{T}}_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{T}_{Z\times X}, \tilde{\mathcal{T}}_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{T}_{Y\times Z}, \tilde{\mathcal{T}}_{Y\times Z})
\end{tikzcd}$$

where the functions $\varphi, \tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ *are defined as in item (1) is a double relational L-triangle.*

Proof. The proof runs as in Proposition 5.1.34 using the same equalities as in Proposition 5.2.18, taking into account $\tilde{\otimes} = \otimes^{op}$. □

Remark 5.2.31. Under symmetry assumption, the diagram

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y), \tilde{\mathcal{T}}_{X\times Y}, \tilde{\mathcal{T}}_{X\times Y}) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\
(\mathbf{R_L}(Z,X), \tilde{\mathcal{T}}_{Z\times X}, \tilde{\mathcal{T}}_{Z\times X}) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\varphi,\tilde{\varphi})}"'] & & (\mathbf{R_L}(Y,Z), \tilde{\mathcal{T}}_{Y\times Z}, \tilde{\mathcal{T}}_{Y\times Z})
\end{tikzcd}$$

where the functions $\varphi, \tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ are defined as follows, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$:

- $(\rho,\sigma)\varphi = \rho_- \odot \sigma_-$;
- $(\rho,\sigma)\tilde{\varphi} = \rho_- \tilde{\odot} \sigma_-$;
- $(\rho,\tau)\psi = (\rho \odot \tau)_-$;
- $(\rho,\tau)\tilde{\psi} = (\rho \tilde{\odot} \tau)_-$;
- $(\tau,\sigma)\chi = (\tau \odot \sigma)_-$;
- $(\tau,\sigma)\tilde{\chi} = (\tau \tilde{\odot} \sigma)_-$

is

1. both a $\perp$-including Tarski L-triangle and a $\perp$-excluding Galois L triangle, if $\perp$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible);
2. a relational L-triangle if and only if $(L, \to, \top)$ is an associative and commutative cdeo algebra,

since $\tilde{\mathcal{T}}_X$ have the similar properties of $\mathcal{T}_X$, as $\tilde{\otimes}$ has the similar properties of $\otimes$.
Moreover, the diagram

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}, \tilde{\mathcal{T}}_{X\times Y}) \arrow[dr, "{(\chi,\tilde{\chi})}"] & \\
(\mathbf{R_L}(Z,X), \mathcal{T}_{Z\times X}, \tilde{\mathcal{T}}_{Z\times X}) \arrow[ur, "{(\psi,\tilde{\psi})}"] \arrow[rr, "{(\varphi,\tilde{\varphi})}"'] & & (\mathbf{R_L}(Y,Z), \mathcal{T}_{Y\times Z}, \tilde{\mathcal{T}}_{Y\times Z})
\end{tikzcd}$$

where the functions $\varphi, \tilde{\varphi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(X,Y)$, $\psi, \tilde{\psi} : \mathbf{R_L}(Z,X) \times \mathbf{R_L}(X,Y) \to \mathbf{R_L}(Y,Z)$, $\chi, \tilde{\chi} : \mathbf{R_L}(X,Y) \times \mathbf{R_L}(Y,Z) \to \mathbf{R_L}(Z,X)$ are defined as follows, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$:

- $(\rho,\sigma)\varphi = \rho_- \odot \sigma_-$;
- $(\rho,\sigma)\tilde{\varphi} = \rho_- \tilde{\odot} \sigma_-$;
- $(\rho,\tau)\psi = (\rho \tilde{\odot} \tau)_-$;
- $(\rho,\tau)\tilde{\psi} = (\rho \odot \tau)_-$;
- $(\tau,\sigma)\chi = (\tau \odot \sigma)_-$
- $(\tau,\sigma)\tilde{\chi} = (\tau \tilde{\odot} \sigma)_-$

is

1. both a $\perp$-including Tarski L-triangle and a $\perp$-excluding Galois L-triangle, if $\perp$ is $\otimes$-irreducible (equivalently, $\tilde{\otimes}$-irreducible);
2. a relational L-triangle if and only if $(L, \rightarrow, \top)$ is an associative cdeo algebra.

5.3 Relational triangles, connections and powerset operators

In this Section we show how the relational L-triangles contain all the informations on powerset operators and related notions considered in other works (see, in particular, [33, 35, 42]).
We recall that powerset operators of an L-relation have been considered and some of their properties have been studied in Section 3.2, where, after identifying the L-sets with particular L-relations, we have described the possibility of expressing the powerset operators by means of the compositions between L-relations considered in Subsection 3.1.2.
So we shall follow notation already introduced, identifying an L-set of X $A : X \rightarrow L$ with the L-relation $A : \{p\} \times X \rightarrow L$, by setting $(p,x)A = (x)A$, for every $x \in X$. We shall consider the singleton $P = \{p\}$ fixed, from now on, and identify $P \times X$, with X and L^X with $L^{P\times X}$.
First we consider relations with values in a symmetrical cdeo algebra assuming subsequently associativity and/or commutativity of such algebra.
So, let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra and consider the $\top$-including double Galois L-triangle of Proposition 5.2.21

$$\begin{tikzcd} & (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \mathcal{S}_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\ (\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \mathcal{S}_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \mathcal{S}_{Y\times Z}) \end{tikzcd}$$

where X, Y, Z are sets and for all $A \in \mathbf{R_L}(Z,X), B \in \mathbf{R_L}(Y,Z), C \in \mathbf{R_L}(X,Y)$ the following equalities hold:

- $(A,B)\varphi = A_- \oslash B_- = ((B)A^{\rightharpoondown})_- = (A_-)^{\rightharpoondown}B_-$;
- $(A,B)\tilde{\varphi} = A_-\tilde{\oslash}B_- = ((B)A^{\rightsquigarrow})_- = (A_-)^{\rightsquigarrow}B_-$;
- $(A,C)\psi = (A \odot C)_- = ((A)^{\otimes}C)_- = (C_-)^{\tilde{\otimes}}A_-$;
- $(A,C)\tilde{\psi} = (A\tilde{\odot}C)_- = ((A)^{\tilde{\otimes}}C)_- = (C_-)^{\otimes}A_-$;
- $(C,B)\chi = (C \oslash B)_- = (B_-)C_-^{\rightharpoondown} = ((C)^{\rightharpoondown}B)_-$;
- $(C,B)\tilde{\chi} = (C\tilde{\oslash}B)_- = (B_-)C_-^{\rightsquigarrow} = ((C)^{\rightsquigarrow}B)_-$.

We note that in the third and forth columns we assume, in turn, that:

- either X is the singleton P and then A_- and C are L-sets on Z and Y, respectively;
- or Y is the singleton P and then B and C_- are L-sets on Z and X, respectively;
- or Z is the singleton P and then A and B_- are L-sets on X and Y, respectively.

Now, recalling Definition 5.2.4 and taking into account that for any set D the subsethood L-relation $\mathcal{S}_D$ on L^D satisfies the equivalence $(H,K)\mathcal{S}_D = \top \Leftrightarrow H \leq K$, for all $H,K \in L^D$, we get the equivalences, for A,B,C as above,

1. $((A,C)\psi, B)\mathcal{S}_{Y\times Z} = \top \Leftrightarrow (C,(A,B)\varphi)\mathcal{S}_{X\times Y} = \top$;
2. $((A,C)\tilde{\psi}, B)\mathcal{S}_{Y\times Z} = \top \Leftrightarrow (C,(A,B)\tilde{\varphi})\mathcal{S}_{X\times Y} = \top$;
3. $(A,(C,B)\chi)\mathcal{S}_{Z\times X} = \top \Leftrightarrow ((A,C)\tilde{\psi}, B)\mathcal{S}_{Y\times Z} = \top$;
4. $(A,(C,B)\tilde{\chi})\mathcal{S}_{Z\times X} = \top \Leftrightarrow ((A,C)\psi, B)\mathcal{S}_{Y\times Z} = \top$;
5. $(A,(C,B)\tilde{\chi})\mathcal{S}_{Z\times X} = \top \Leftrightarrow (C,(A,B)\varphi)\mathcal{S}_{X\times Y} = \top$;
6. $(A,(C,B)\chi)\mathcal{S}_{Z\times X} = \top \Leftrightarrow (C,(A,B)\tilde{\varphi})\mathcal{S}_{X\times Y} = \top$.

Assuming Y to be the singleton P and considering B and C_- as L-sets on Z and X, respectively, we get the following type III $\top$-including Galois connections, made by the weak left backward operators and the strong right forward operators on any L-relation A,

$$({}^{\tilde{\otimes}}A_-, A^{\rightharpoondown})_{\top[\uparrow]} : (L^X, \mathcal{S}_X) \to (L^Z, \mathcal{S}_Z) \textit{ and } ({}^{\otimes}A_-, A^{\rightsquigarrow})_{\top[\uparrow]} : (L^X, \mathcal{S}_X) \to (L^Z, \mathcal{S}_Z);$$

in fact, from (1) and (2) we get the equivalences $(C_-)^{\tilde{\otimes}}A_- \leq B \Leftrightarrow C_- \leq (B)A^{\rightharpoondown}$ and $(C_-)^{\otimes}A_- \leq B \Leftrightarrow C_- \leq (B)A^{\rightsquigarrow}$.
Assuming Z to be the singleton P and considering A and B_- as L-sets on X and Y, respectively, we get the following type III $\top$-including Galois connections, made by the weak left forward operators and the strong right backward operators of any L-relation C,

$$({}^{\tilde{\otimes}}C, C_-^{\rightharpoondown})_{\top[\uparrow]} : (L^X, \mathcal{S}_X) \to (L^Y, \mathcal{S}_Y) \textit{ and } ({}^{\otimes}C, C_-^{\rightsquigarrow})_{\top[\uparrow]} : (L^Y, \mathcal{S}_Y) \to (L^X, \mathcal{S}_X);$$

in fact, from (3) and (4) we get the equivalences $(A)^{\tilde{\otimes}}C \leq B_- \Leftrightarrow A \leq (B_-)C_-^{\rightarrow}$ and $(A)^{\otimes}C \leq B_- \Leftrightarrow A \leq (B)C_-^{\rightsquigarrow}$.
Assuming X to be the singleton P and considering C and A_- as L-sets on Y and Z, respectively, we get the following type I $\top$-including Galois connections, made by the strong left forward operators and the strong left backward operators on any L-relation B,

$$[{}^{\rightsquigarrow}B, {}^{\rightarrow}B_-]_{\top[\uparrow]} : (L^Y, \mathcal{S}_Y) \to (L^Z, \mathcal{S}_Z) \textit{ and } [{}^{\rightarrow}B, {}^{\rightsquigarrow}B_-]_{\top[\uparrow]} : (L^Y, \mathcal{S}_Y) \to (L^Z, \mathcal{S}_Z);$$

in fact, from (5) and (6) we get the equivalences $A_- \leq (C)^{\rightsquigarrow}B \Leftrightarrow C \leq (A_-)^{\rightarrow}B_-$ and $A_- \leq (C)^{\rightarrow}B \Leftrightarrow C \leq (A_-)^{\rightsquigarrow}B_-$.

Remark 5.3.1. The above obtained $\top$-including Galois connections of type I and III, of course, determine corresponding $\top$-including connections of type II and IV, respectively, according to Proposition 4.2.11.
We note also that the equalities ${}^{\tilde{\otimes}}\mathcal{R} = \mathcal{R}^{\otimes}$ and ${}^{\otimes}\mathcal{R} = \mathcal{R}^{\tilde{\otimes}}$ allow to express all the above type III connections using the suitable weak right operators instead of the corresponding left operators.
Moreover, we remark that in each pair of connections we have obtained, the second one is the dual to the first one and that the connections of first pair are essentially the same as those of the second pair.

Consider now a symmetrical and associative cdeo algebra and the double relational L-triangle of Proposition 5.2.28

$$\begin{tikzcd}
& (\mathbf{R_L}(X,Y), \mathcal{S}_{X\times Y}, \tilde{\mathcal{S}}_{X\times Y}) \arrow[dr, "(\chi, \tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{S}_{Z\times X}, \tilde{\mathcal{S}}_{Z\times X}) \arrow[ur, "(\psi, \tilde{\psi})"] \arrow[rr, "(\varphi, \tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{S}_{Y\times Z}, \tilde{\mathcal{S}}_{Y\times Z})
\end{tikzcd}$$

where X, Y, Z are sets and $\varphi, \tilde{\varphi}\chi, \tilde{\chi}, \psi, \tilde{\psi}$ are defined as above, for all $A \in \mathbf{R_L}(Z,X), B \in \mathbf{R_L}(Y,Z), C \in \mathbf{R_L}(X,Y)$.
Now, recalling Definition 5.2.2 and considering, for any set D, the dual subsethood L-relation $\tilde{\mathcal{S}}_D$ on L^D, we get the following equivalences, for A, B, C as above,

1. $((A,C)\psi, B)\mathcal{S}_{Y\times Z} = (C, (A,B)\varphi)\mathcal{S}_{X\times Y}$;
2. $((A,C)\tilde{\psi}, B)\tilde{\mathcal{S}}_{Y\times Z} = (C, (A,B)\tilde{\varphi})\tilde{\mathcal{S}}_{X\times Y}$;
3. $(A, (C,B)\chi)\mathcal{S}_{Z\times X} = ((A,C)\tilde{\psi}, B)\mathcal{S}_{Y\times Z}$;
4. $(A, (C,B)\tilde{\chi})\tilde{\mathcal{S}}_{Z\times X} = ((A,C)\psi, B)\tilde{\mathcal{S}}_{Y\times Z}$;
5. $(A, (C,B)\tilde{\chi})\mathcal{S}_{Z\times X} = (C, (A,B)\varphi)\tilde{\mathcal{S}}_{X\times Y}$;
6. $(A, (C,B)\chi)\tilde{\mathcal{S}}_{Z\times X} = (C, (A,B)\tilde{\varphi})\mathcal{S}_{X\times Y}$.

Assuming Y to be the singleton P and considering B and C_{-} as L-sets on Z and X, respectively, from the equalities (1) and (2) we deduce the following type III global connections, made by the weak left backward operators and the strong right forward operators on any L-relation A,

$$({}^{\tilde{\otimes}}A_{-}, A^{\rightarrow}) : (L^X, \mathcal{S}_X) \to (L^Z, \mathcal{S}_Z) \textit{ and } ({}^{\otimes}A_{-}, A^{\rightsquigarrow}) : (L^X, \tilde{\mathcal{S}}_X) \to (L^Z, \tilde{\mathcal{S}}_Z).$$

Assuming Z to be the singleton P and considering A and B_{-} as L-sets on X and Y, respectively, from the equalities (3) and (4) we deduce the following type III global connections, made by the weak left forward operators and the strong right backward operators of any L-relation C,

$$({}^{\tilde{\otimes}}C, C_{-}^{\rightarrow}) : (L^X, \mathcal{S}_X) \to (L^Y, \mathcal{S}_Y) \textit{ and } ({}^{\otimes}C, C_{-}^{\rightsquigarrow}) : (L^Y, \tilde{\mathcal{S}}_Y) \to (L^X, \tilde{\mathcal{S}}_X).$$

Assuming X to be the singleton P and considering C and A_{-} as L-sets on Y and Z, respectively, from the equalities (5) and (6) we deduce the following type I global connections, made by the strong left forward operators and the strong left backward operators on any L-relation B,

$$[{}^{\rightsquigarrow}B, {}^{\rightarrow}B_{-}] : (L^Y, \tilde{\mathcal{S}}_Y) \to (L^Z, \mathcal{S}_Z) \textit{ and } [{}^{\rightarrow}B, {}^{\rightsquigarrow}B_{-}] : (L^Y, \mathcal{S}_Y) \to (L^Z, \tilde{\mathcal{S}}_Z).$$

Remark 5.3.2. 1. Similarly to what was observed in Remark 5.3.1, the above obtained global connections of type I and III, of course, determine corresponding global connections of type II and IV, respectively.
Obviously, also in this case the equalities ${}^{\tilde{\otimes}}\mathcal{R} = \mathcal{R}^{\otimes}$ and ${}^{\otimes}\mathcal{R} = \mathcal{R}^{\tilde{\otimes}}$ allow to express all the above type III global connections using the suitable weak right operators instead of the corresponding left operators.
We note, moreover, that each second term of the pairs of the global connections obtained is the dual of the corresponding first term. Note that unlike in the commutative case mostly considered dealing with fuzzy Galois connection (the only exceptions are [33, 51]) both powerset operators and their dual, as well as both the subsethood and the dual subsethood L-relations are involved in the above obtained connections.
In [51] and elsewhere, the isotonic connections are also called **axialities** while the antitonic ones are also called **polarities**.

2. Of course, the double relational L-triangle is a $\perp$-including double Tarski L-triangle, too; hence, from the above considered diagram we can deduce type I, II, III and IV $\perp$-including Tarki connections. In particular, we note that if $A, B \in L^X$: $(A, B)\mathcal{S}_X = \perp$ if there exists $\bar{x} \in X$ such that $(\bar{x})A = \top$ and $(\bar{x})B = \perp$, i.e. $A \nsubseteq B$. So, the equality $(A, B)\mathcal{S}_X = \perp$ is useful to state the "non-inclusion" $A \nsubseteq B$ between L-sets, considering that, in the classical case, $A \nsubseteq B \Leftrightarrow (A, B)\mathcal{S}_X = 0$.

3. Clearly, if L is a commutative cdeo algebra, then $\rightarrow = \rightsquigarrow$, $\otimes = \tilde{\otimes}$ and $\mathcal{S}_D = \tilde{\mathcal{S}}_D$, for any set D; hence, we obtain, from the above double relational L-triangle the following connections between the powerset operators:

- $({}^{\otimes}A_{-}, A^{\rightarrow}) : (L^X, \mathcal{S}_X) \to (L^Z, \mathcal{S}_Z)$;
- $({}^{\otimes}C, C_{-}^{\rightarrow}) : (L^X, \mathcal{S}_X) \to (L^Y, \mathcal{S}_Y)$;
- $[{}^{\rightarrow}B, {}^{\rightarrow}B_{-}] : (L^Y, \mathcal{S}_Y) \to (L^Z, \mathcal{S}_Z)$.

By using Proposition 5.2.30 and Remark 5.2.31, and, hence, by assuming that $\perp$ is $\otimes$-irreducible, we can deduce the corresponding $\perp$-including Tarski, or equivalently the $\perp$-excluding Galois connections between appropriate L-powersets L^D equipped with the intersection L-relation $\mathcal{T}_D$.
So, in general, we can not obtain the results corresponding to those made using the subsethood L-relations.
Nevertheless, assuming associativity and symmetry on L, we could drop in Proposition 5.2.30 the condition on $\perp$ and we could obtain global connections involving the intersection L-relation an its dual. Obviously, starting from these global connections, we can deduce, in particular, $\perp$-including Tarski and $\top$-excluding Galois connections.

We recall the result of Proposition 5.2.30, using both compositions $\odot$ and $\tilde{\odot}$ and considering the equality $\alpha\tilde{\odot}\beta = (\beta_{-} \odot \alpha_{-})_{-}$ for any composable relation α, β. So the diagram

$$\begin{tikzcd}
 & (\mathbf{R_L}(X,Y), \mathcal{T}_{X\times Y}, \tilde{\mathcal{T}}_{X\times Y}) \arrow[dr, "(\chi,\tilde{\chi})"] & \\
(\mathbf{R_L}(Z,X), \mathcal{T}_{Z\times X}, \tilde{\mathcal{T}}_{Z\times X}) \arrow[ur, "(\psi,\tilde{\psi})"] \arrow[rr, "(\varphi,\tilde{\varphi})"'] & & (\mathbf{R_L}(Y,Z), \mathcal{T}_{Y\times Z}, \tilde{\mathcal{T}}_{Y\times Z})
\end{tikzcd}$$

where X, Y, Z are sets and $\varphi, \tilde{\varphi}\chi, \tilde{\chi}, \psi, \tilde{\psi}$ are defined as follows, for all $A \in \mathbf{R_L}(Z,X), B \in \mathbf{R_L}(Y,Z), C \in \mathbf{R_L}(X,Y)$:

- $(A,B)\varphi = A_{-} \odot B_{-}$;
- $(A,B)\tilde{\varphi} = A_{-}\tilde{\odot}B_{-}$;
- $(A,C)\psi = (A\tilde{\odot}C)_{-}$;
- $(A,C)\tilde{\psi} = (A \odot C)_{-}$;
- $(C,B)\chi = (C \odot B)_{-}$;
- $(C,B)\tilde{\chi} = (C\tilde{\odot}B)_{-}$

is a double relational L-triangle.

Then, we deduce the following equalities:

1. $((A,C)\psi, B)\mathcal{T}_{Y\times Z} = (C, (A,B)\varphi)\mathcal{T}_{X\times Y}$;
2. $((A,C)\tilde{\psi}, B)\tilde{\mathcal{T}}_{Y\times Z} = (C, (A,B)\tilde{\varphi})\tilde{\mathcal{T}}_{X\times Y}$;

3. $(A,(C,B)\chi)\mathcal{T}_{Z\times X} = ((A,C)\tilde{\psi},B)\mathcal{T}_{Y\times Z}$;
4. $(A,(C,B)\tilde{\chi})\tilde{\mathcal{T}}_{Z\times X} = ((A,C)\psi,B)\tilde{\mathcal{T}}_{Y\times Z}$;
5. $(A,(C,B)\tilde{\chi})\mathcal{T}_{Z\times X} = (C,(A,B)\varphi)\tilde{\mathcal{T}}_{X\times Y}$;
6. $(A,(C,B)\chi)\tilde{\mathcal{T}}_{Z\times X} = (C,(A,B)\tilde{\varphi})\mathcal{T}_{X\times Y}$;

Assuming Y to be the singleton P and considering B and C_- as L-sets on Z and X, respectively, we deduce from the equalities (1) and (2) the following type III (global) connections made by the weak operators of any L-relation A

$$({}^{\otimes}A_-, {}^{\tilde{\otimes}}A) : (L^X, \mathcal{T}_X) \to (L^Z, \mathcal{T}_Z) \textit{ and } ({}^{\tilde{\otimes}}A_-, {}^{\otimes}A) : (L^X, \tilde{\mathcal{T}}_X) \to (L^Z, \tilde{\mathcal{T}}_Z).$$

In fact, from (1) and (2) we get the equivalences:
- $((A\tilde{\odot}C)_-,B)\mathcal{T} = (C,A_-\odot B_-)\mathcal{T} \Leftrightarrow (C_-\odot A_-,B)\mathcal{T} = (C_-,(B\tilde{\odot}A)_-)\mathcal{T} \Leftrightarrow ((C_-)^{\otimes}A_-,B)\mathcal{T} = (C_-,(B)^{\tilde{\otimes}}A)\mathcal{T}$;
- $((A\odot C)_-,B)\tilde{\mathcal{T}} = (C_-,(A_-\tilde{\odot}B_-)_-)\tilde{\mathcal{T}} \Leftrightarrow (C_-\tilde{\odot}A_-,B)\tilde{\mathcal{T}} = (C_-,B\odot A)\tilde{\mathcal{T}} \Leftrightarrow ((C_-)^{\tilde{\otimes}}A_-,B)\tilde{\mathcal{T}} = (C_-,(B)^{\otimes}A)\tilde{\mathcal{T}}$.

Assuming Z to be the singleton P and considering A and B_- as L-sets on X and Y, respectively, we deduce from the equalities (3) and (4) the following type III (global) connections made by the weak operators of any L-relation C

$$({}^{\otimes}C, {}^{\tilde{\otimes}}C_-) : (L^X, \mathcal{T}_X) \to (L^Y, \mathcal{T}_Y) \textit{ and } ({}^{\tilde{\otimes}}C, {}^{\otimes}C_-) : (L^X, \tilde{\mathcal{T}}_X) \to (L^Y, \tilde{\mathcal{T}}_Y).$$

In fact, from (3) and (4) we get the following equivalences:
- $(A,(C\odot B)_-)\mathcal{T} = ((A\odot C)_-,B)\mathcal{T} \Leftrightarrow (A,B_-\tilde{\odot}C_-)\mathcal{T} = (A\odot C,B_-)\mathcal{T} \Leftrightarrow (A,(B_-)^{\tilde{\otimes}}C_-)\mathcal{T} = ((A)^{\otimes}C,B_-)\mathcal{T}$;
- $(A,(C\tilde{\odot}B)_-)\tilde{\mathcal{T}} = ((A\tilde{\odot}C)_-,B)\tilde{\mathcal{T}} \Leftrightarrow (A,B_-\odot C_-)\tilde{\mathcal{T}} = (A\tilde{\odot}C,B_-)\tilde{\mathcal{T}} \Leftrightarrow (A,(B_-)^{\otimes}C_-)\tilde{\mathcal{T}} = ((A)^{\tilde{\otimes}}C,B_-)\tilde{\mathcal{T}}$.

Assuming X to be the singleton P and considering A_- and C as L-sets on Z and Y, respectively, we deduce from the equalities (5) and (6) the following type I (global) connections made by the weak operators of any L-relation B

$$[{}^{\tilde{\otimes}}B, {}^{\otimes}B_-] : (L^Y, \mathcal{T}_Y) \to (L^Z, \tilde{\mathcal{T}}_Z) \textit{ and } [{}^{\otimes}B, {}^{\tilde{\otimes}}B_-] : (L^Y, \tilde{\mathcal{T}}_Y) \to (L^Z, \mathcal{T}_Z).$$

In fact, from (5) and (6) we get the following equivalences:
- $(A,(C\tilde{\odot}B)_-)\mathcal{T} = (C,A_-\odot B_-)\tilde{\mathcal{T}} \Leftrightarrow (A_-,C\tilde{\odot}B)\mathcal{T} = (C,A_-\odot B_-)\tilde{\mathcal{T}} \Leftrightarrow (A_-,(C)^{\tilde{\otimes}}B)\mathcal{T} = (C,(A_-)^{\otimes}B_-)\tilde{\mathcal{T}}$;
- $(A,(C\odot B)_-)\tilde{\mathcal{T}} = (C,A_-\tilde{\odot}B_-)\mathcal{T} \Leftrightarrow (A_-,C\odot B)\tilde{\mathcal{T}} = (C,A_-\tilde{\odot}B_-)\mathcal{T} \Leftrightarrow (A_-,(C)^{\otimes}B)\tilde{\mathcal{T}} = (C,(A_-)^{\tilde{\otimes}}B_-)\mathcal{T}$.

We close this discussion observing that we can deduce many other connections, starting from the previous ones; this is due to the link between the adjoint product $\otimes$ and the dual one $\tilde{\otimes}$ of a symmetrical cdeo algebra $(L,\to,\top)$ ($a\tilde{\otimes}b = b\otimes a$, for all $a,b\in L$) and to the consequent connection between the intersection and the dual intersection L-relations $\mathcal{T}$ and $\tilde{\mathcal{T}}$.

Chapter 6

Structures for many-valued relations

In this Chapter we will be interested in a categorical accommodation of the main topics we have treated that involve binary, possibly many-valued, relations. We shall move into two directions.
The first involves directly the class of binary relations, between sets, taking values in a w-ceo algebra, which may satisfy further conditions; the obtained results are described in Section 6.1 and its subsections.
The second concerns some class of relational systems which are closely related to most concepts we have developed, from the algebras described in Chapter 2 up to the connections introduced and studied in Chapter 4; the results obtained in this direction are the main context of Sections 6.2, 6.3 and 6.4.
In any case it will be clear that, unless in the corresponding classical cases, dealing with many-valued relations, the classical notion of category has too strong requirements that need to be weakened according to the objects and morphisms to be considered. The categorical-like structures we shall consider either are already known (in which case we provide new, relevant examples) or are quite similar to other already considered structures, with some new and important aspects.
The structure of extended-order algebras, described in Chapter 2, the general approach to connections introduced in Chapter 4 and the relational triangles defined in Chapter 5 will have a crucial role.

6.1 Pseudo-categories

We have started in Chapter 3 the study of the structure of the class of binary relations assuming that these take values in a w-ceo algebra, possibly right-distributive and, may be, symmetrical; moreover, we have seen how (double) relational triangles, which may be used to characterize various classes of w-ceo algebras, give plenty of information on the class of relations and their associated operators, all of which may be obtained by means of suitable compositions.

A categorical accommodation of this matter, where binary many-valued relations are morphisms and the associated operators depend on various compositions defined between relations has to take into account that none of those compositions is associative, in general. In fact, the unique composition of binary relations that is associative in any case, namely the one denoted by $\cdot$ in Definition 3.1.4, is not relevant in case of relations taking values in a w-ceo algebra; the main composition that corresponds to the usual composition of binary relations in the classical case, i.e. the one denoted $\odot$ in Definition 3.1.9, is associative if and only if the algebra is associative.
So, categorical structures for many-valued relations should drop the requirement of associativity on the composition and, moreover, might be equipped with more that one composition only.
In the present section we propose an extended notion of category, called **pseudo-category**, which misses associativity and identities but may have additional tools that are useful to deal with categories of relations.

6.1.1 Reversible and ordered pseudo-categories

Definition 6.1.1. *A* ***pseudo-category*** $\boldsymbol{C} = (\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ)$ *consists in the following data:*

1. *a class of* ***objects*** $\mathcal{O}bj(\boldsymbol{C})$*, also denoted by* $|\boldsymbol{C}|$*;*
2. *for each pair of objects* (X, Y)*, a class* $\boldsymbol{C}(X, Y)$*, whose elements, denoted by* $X \xrightarrow{f} Y$ *or by* $f : X \to Y$ *are called* ***morphisms*** *from* X *(**domain**) to* Y *(**codomain**);* $\mathcal{M}or(\boldsymbol{C})$ *is the disjoint union of such classes of morphisms;*
3. *a* ***partial morphisms composition****, shortly* ***composition****, that is a univocal relation* $\circ : \mathcal{M}or(\boldsymbol{C}) \times \mathcal{M}or(\boldsymbol{C}) \rightharpoonup \mathcal{M}or(\boldsymbol{C})$ *that assigns to each* $f \in \boldsymbol{C}(X, Y)$ *and* $g \in \boldsymbol{C}(Y, Z)$*, their composition* $f \circ g \in \boldsymbol{C}(X, Z)$*.*

Definition 6.1.2. *A pseudo-category* $\boldsymbol{C} = (\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ)$ *is said to be* ***right (left****, respectively) if for every* $X \in \mathcal{O}bj(\boldsymbol{C})$ *there exists a unique* $r_X \in \boldsymbol{C}(X, X)$ *(a unique* $l_X \in \boldsymbol{C}(X, X)$*, respectively), called* ***right (left****, respectively)* ***identity morphism*** *on* X*, such that, for every* $f \in \boldsymbol{C}(X, Y)$ *one has* $f \circ r_Y = f$ *(*$l_X \circ f = f$*, respectively).*

Of course, if the pseudo-category **C** is right and left, then the right and the left identity morphisms coincide, for each object of **C**. So, we give a further definition.

Definition 6.1.3. *A pseudo-category* $\boldsymbol{C} = (\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ)$ *is called* ***unital*** *if for every* $X \in \mathcal{O}bj(\boldsymbol{C})$ *there exists* $i_X \in \boldsymbol{C}(X, X)$*, called* ***identity morphism*** *on* X*, such that, for every* $f \in \boldsymbol{C}(X, Y)$ *one has* $i_X \circ f = f \circ i_Y = f$*.*

We observe that, with respect to the well-known notion of category, a pseudo-category misses the existence of the identity morphisms, unless it is unital, and the associativity of partial composition.

Definition 6.1.4. *The pseudo-category* $\boldsymbol{C} = (\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ)$ *is said to be* ***associative*** *if the partial morphisms composition is associative.*

Of course, an associative unital pseudo-category is a category in the usual sense. The notions of (**full**) **pseudo-subcategory** and **pseudo-functor** are obvious extensions of the corresponding notions for categories.

Definition 6.1.5. *A* ***pseudo-subcategory*** $\boldsymbol{D} = (\mathcal{O}bj(\boldsymbol{D}), \mathcal{M}or(\boldsymbol{D}), \circ_{\boldsymbol{D}})$ *of the pseudo-category* $\boldsymbol{C} = (\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ)$ *, denoted by* $\boldsymbol{D} \subseteq \boldsymbol{C}$ *is a pseudo-category such that*

1. $\mathcal{O}bj(\boldsymbol{D}) \subseteq \mathcal{O}bj(\boldsymbol{C})$*;*
2. *for all* $X, Y \in \mathcal{O}bj(\boldsymbol{D})$*,* $\boldsymbol{D}(X,Y) \subseteq \boldsymbol{C}(X,Y)$*;*
3. $\circ_{\boldsymbol{D}} = \circ_{|\mathcal{M}or(\boldsymbol{D})}$*;*
4. *for every* $X \in \mathcal{O}bj(\boldsymbol{D})$*, if* r_X *(*l_X*,* i_X*, respectively) is the right identity (left identity, identity, respectively) morphism in* $\boldsymbol{C}$*, then it is in* $\boldsymbol{D}(X,X)$*.*

The pseudo-subcategory $\boldsymbol{D}$ *is said to be* ***full*** *if* $\boldsymbol{D}(X,Y) = \boldsymbol{C}(X,Y)$*, for all* $X, Y \in \mathcal{O}bj(\boldsymbol{D})$*.*

The following are examples of pseudo-categories; we shall specify which of them are categories.

Example 6.1.6. 1. The triple $(\{X\}, \mathbb{N}, exp)$ is a right pseudo-category where the only object is X, the set of morphism is the set of natural numbers, the composition is defined by $n \circ m = exp(n,m) = n^m$ and the right unit is 1.

2. The triple $\mathbf{R_2^{\cdot}} = (|\mathbf{Set}|, \mathbf{R_2}, \cdot)$, where the objects are the sets, the morphisms are the binary relations and $\cdot$ is the composition defined in Section 1.4, is an associative, unital pseudo-category, hence it is a category in the usual sense; of course, the identity morphism on X is the equality $\mathcal{I}_X$.

3. The triple $\mathbf{R_2^{\oslash}} = (|\mathbf{Set}|, \mathbf{R_2}, \oslash)$ where the objects are the sets, the morphisms are the binary relations and $\oslash$ is the composition defined in the Section 1.4, is a left pseudo-category; the left unit of $\mathbf{R_2^{\oslash}}(X,X)$ is $\mathcal{I}_X$.

4. The triple $\mathbf{R_2^{\otimes}} = (|\mathbf{Set}|, \mathbf{R_2}, \otimes)$ where the objects are the sets, the morphisms are the binary relations and $\otimes$ is the composition defined in Section 1.4, is a right pseudo-category; $\mathcal{I}_X$ is the right unit of $\mathbf{R_2^{\otimes}}(X,X)$.

5. The triple $\mathbf{R_2^{\ominus}} = (|\mathbf{Set}|, \mathbf{R_2}, \ominus)$ where the objects are the sets, the morphisms are the binary relations and $\ominus$ is the composition defined in Section 1.4, is a pseudo-category, without left or right identity morphism, for most $X \in |\mathbf{Set}|$; in fact i_X is neither a left nor a right unit, unless X is a singleton or the empty set.

6. Of course, a pseudo-subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is a category, too; moreover, a pseudo-subcategory of a pseudo-category $\mathbf{C}$ may be a category, even when $\mathbf{C}$ is not, as the following example shows (trivial examples may also be produced).

Consider the pseudo-subcategory of $\mathbf{R_2^\ominus}$ that have the class $|\mathbf{Set}|$ as objects and the injective functions as morphisms. The identity relation i_X on X is, of course, an injective function and it is the unit, for every non-empty set X; in fact, for every $r \in \mathbf{R_2}(X,Y)$ and for all $x \in X, y \in Y$, we have that:

- $x(r \ominus i_Y)y \Leftrightarrow xr = i_Y y \Leftrightarrow xr = \{y\} \Leftrightarrow xry$;
- $x(i_X \ominus)y \Leftrightarrow xi_X = ry \Leftrightarrow \{x\} = ry \Leftrightarrow xry$.

Moreover, for all non-empty sets X, Y, Z, $r \in \mathbf{R_2^\ominus}(X,Y)$ and $s \in \mathbf{R_2^\ominus}(Y,Z)$, if r and s are injective functions, then $r \ominus s \in \mathbf{R_2^\ominus}(X,Z)$ is an injective function, too. In fact, for all $x, x' \in X$, $z, z' \in Z$, the following hold:

- since r and s are functions, $x \in X \Rightarrow \exists | y \in Y : xr = \{y\} \Rightarrow \exists | z \in Z : ys = \{z\}$ and, since s is injective, $\{y\} = sz$. Then $x \in X \Rightarrow \exists z : xr = \{y\} = sz \Rightarrow xr \ominus sz$; so, $r \ominus s$ is left total.
- $(x(r \ominus s)z$ and $x(r \ominus s)z') \Leftrightarrow (xr = sz$ and $xr = sz')$; hence $sz = sz'$ and then, since s is a function, we have that $z = z'$; so, $r \ominus s$ is right univocal;
- $(x(r \ominus s)z$ and $x'(r \ominus s)z) \Leftrightarrow (xr = sz$ and $x'r = sz)$; hence $xr = x'r$ and then, since r is injective, it follows that $x = x'$; so, $r \ominus s$ is left univocal;.

So, $r \ominus s$ is an injective function; moreover, for such relations, the composition induced by $\ominus$ is associative.

In the next Section we shall consider the notion of **multiplicative graph** [69], which is more general than that of pseudo-category, and that of **composition graph** [76].

Definition 6.1.7. *Let $\boldsymbol{C} = (\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ_{\boldsymbol{C}})$ and $\boldsymbol{D} = (\mathcal{O}bj(\boldsymbol{D}), \mathcal{M}or(\boldsymbol{D}), \circ_{\boldsymbol{D}})$ be two pseudo-categories. A **pseudo-functor** $F : \boldsymbol{C} \to \boldsymbol{D}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ is a pair of maps $F = (F_1, F_2)$ such that:*

1. *$F_1 : \mathcal{O}bj(\boldsymbol{C}) \to \mathcal{O}bj(\boldsymbol{D})$ and $F_2 : \mathcal{M}or(\boldsymbol{C}) \to \mathcal{M}or(\boldsymbol{D})$;*
2. *for all $X, Y \in \mathcal{O}bj(\boldsymbol{C})$, if $f \in \boldsymbol{C}(X,Y)$, then $(f)F_2 \in \boldsymbol{D}((X)F_1, (Y)F_1)$;*
3. *for all $X, Y, Z \in \mathcal{O}bj(\boldsymbol{C})$, if $f \in \boldsymbol{C}(X,Y)$ and $g \in \boldsymbol{C}(Y,Z)$, then $(f \circ_{\boldsymbol{C}} g)F_2 = (f)F_2 \circ_{\boldsymbol{D}} (g)F_2 \in \boldsymbol{D}((X)F_1, (Z)F_1)$.*

Definition 6.1.8. *A pseudo-functor $F : \boldsymbol{C} \to \boldsymbol{D}$ between two pseudo-categories $\boldsymbol{C}$ and $\boldsymbol{D}$ is said to be a **functor** if the following conditions hold, for every $X \in \mathcal{O}bj(\boldsymbol{C})$:*

1. *if there exists the right identity morphism $r_X \in \boldsymbol{C}(X,X)$ on X, then $r_{(X)F_1} = (r_X)F_2$ is a right identity morphism on $(X)F_1$ in $\boldsymbol{D}$;*
2. *if there exists the left identity morphism $l_X \in \boldsymbol{C}(X,X)$ on X, then $l_{(X)F_1} = (l_X)F_2$ is a left identity morphism on $(X)F_1$ in $\boldsymbol{D}$.*

Of course, if $F : \mathbf{C} \to \mathbf{D}$ is a functor and if there exists the identity morphism $i_X \in \mathbf{C}(X, X)$, then $i_{(X)F_1} = (i_X)F_2$ is the identity morphism of $(X)F_1$ in $\mathbf{D}$.
If $F : \mathbf{C} \to \mathbf{D}$ is a functor between two pseudo-categories $\mathbf{C}$ and $\mathbf{D}$ and $\mathbf{C}$ is a right (left, unital, respectively) pseudo-category, or a category, then the full pseudo-subcategory of $\mathbf{D}$ with class of objects $\{(X)F_1 |\ X \in \mathcal{Obj}(\mathbf{C})\}$ is a right (left, unital, respectively) pseudo-category, or a category, too.

The following are trivial examples of functors and pseudo-functors.

Example 6.1.9. 1. For every pseudo-category $\mathbf{C}$ the identity pseudo-functor $Id_{\mathbf{C}} = (i_{\mathcal{Obj}(\mathbf{C})}, i_{\mathcal{Mor}(\mathbf{C})})$ is a functor.

2. Let $\mathbf{C}$ and $\mathbf{D}$ be two pseudo-categories; if there exist $D \in \mathcal{Obj}(\mathbf{D})$ and $k_D \in \mathbf{D}(D, D)$ such that $k_D \circ k_D = k_D$, we can consider the constant pseudo-functor $k_D = (k_D, k_D)$ defined as follows

   - $k_D : \mathcal{Obj}(\mathbf{C}) \to \mathcal{Obj}(\mathbf{D})$: $X \mapsto (X)k_D = D$;
   - $k_D : \mathcal{Mor}(\mathbf{C}) \to \mathcal{Mor}(\mathbf{D})$: $f \in \mathbf{C}(X, Y) \mapsto (f)k_D = k_D \in \mathbf{D}(D, D)$.

The composition $\cdot$ of pseudo-functors is obviously defined by component-wise composition of functions.

Definition 6.1.10. *A pseudo-functor $F : \boldsymbol{C} \to \boldsymbol{D}$ between two pseudo-categories $\boldsymbol{C}$ and $\boldsymbol{D}$ is an* ***isomorphism*** *if there exists a pseudo-functor $F^{-1} : \boldsymbol{D} \to \boldsymbol{C}$ such that $F \cdot F^{-1} = Id_{\boldsymbol{C}}$ and $F^{-1} \cdot F = Id_{\boldsymbol{D}}$.*

It is easily seen the following.

- A pseudo-functor $F = (F_1, F_2) : \mathbf{C} \to \mathbf{D}$ is an isomorphism if and only if F_1 and F_2 are bijective;

- the identity functor $Id_{\mathbf{C}}$ of a pseudo-category $\mathbf{C}$ is an isomorphism;

- every isomorphism is a functor.

Definition 6.1.11. *Let $\boldsymbol{C} = (\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \circ)$ be a pseudo-category. The* ***opposite pseudo-category*** *$\boldsymbol{C}^{op} = (\mathcal{Obj}(\boldsymbol{C}^{op}), \mathcal{Mor}(\boldsymbol{C}^{op}), \circ^{op})$ of $\boldsymbol{C}$ consists in the following data:*

1. *$\mathcal{Obj}(\boldsymbol{C}^{op}) = \mathcal{Obj}(\boldsymbol{C})$.*

2. *$\mathcal{Mor}(\boldsymbol{C}^{op}) = \mathcal{Mor}(\boldsymbol{C})$ and, for all $X, Y \in \mathcal{Obj}(\boldsymbol{C})$, $\boldsymbol{C}^{op}(X, Y) = \boldsymbol{C}(Y, X)$.*

3. *For all $X, Y, Z \in \mathcal{Obj}(\boldsymbol{C})$ and for all $f \in \boldsymbol{C}^{op}(X, Y)$, $g \in \boldsymbol{C}^{op}(Y, Z)$: $f \circ^{op} g = g \circ f$.*

As usual, a morphism in $\mathbf{C}(X, Y)$ is denoted f^{op} whenever it is considered as a morphism, $f^{op} \in \mathbf{C}^{op}(Y, X)$ in $\mathbf{C}^{op}$, and conversely, so that $(f^{op})^{op} = f$; this allows to denote by $\circ$ the composition of $\mathbf{C}^{op}$, so that if $f \circ g$ exists in $\mathbf{C}$ or in $\mathbf{C}^{op}$ then $(f \circ g)^{op} = g^{op} \circ f^{op}$. Note that if $\mathbf{C}$ is a right (left, unital, respectively) pseudo-category, then $\mathbf{C}^{op}$ is a left (right, unital, respectively) pseudo-category and the obvious notation $(r_X^{\mathbf{C}})^{op} = l_X^{\mathbf{C}^{op}}$, $(l_X^{\mathbf{C}})^{op} = r_X^{\mathbf{C}^{op}}$, $(i_X^{\mathbf{C}})^{op} = i_X^{\mathbf{C}^{op}}$ can be used, for every $X \in \mathcal{Obj}(\mathbf{C})$.

Definition 6.1.12. *An* ***ordered pseudo-category*** *is a pair* $(\boldsymbol{C}; \leq)$*, where* $\boldsymbol{C}$ *is a pseudo-category and* $\leq$ *is an order relation in the class* $\mathcal{M}or(\boldsymbol{C})$*, called* ***natural order*** *of* $\boldsymbol{C}$*, such that:*

> (i) *for all* $f, g \in \mathcal{M}or(\boldsymbol{C})$*, if* $f \leq g$*, then there exist* $X, Y \in \mathcal{O}bj(\boldsymbol{C})$ *such that* $f, g \in \boldsymbol{C}(X,Y)$*;*
>
> (ii) *for all* $X, Y \in \mathcal{O}bj(\boldsymbol{C})$*:* $(\boldsymbol{C}(X,Y), \leq)$ *is a non-empty ordered set;*
>
> (iii) *the composition is isotonic in the second argument, i.e. for all* $f \in \boldsymbol{C}(X,Y)$*,* $g, g' \in \boldsymbol{C}(Y,Z)$*, if* $g \leq g'$*, then one has* $f \circ g \leq f \circ g'$*.*

Moreover, we speak of ***bounded ordered***, ***lattice ordered***, ***complete ordered*** *pseudo-category in case when, for all* $X, Y \in \mathcal{O}bj(\boldsymbol{C})$*,* $(\boldsymbol{C}(X,Y), \leq)$ *is a bounded poset or a lattice or a complete lattice, respectively, in the natural order of* $\boldsymbol{C}$*. In these cases the top element is denoted by* $\top\!\!\top_{XY}$ *and the bottom element is denoted by* $\perp\!\!\perp_{XY}$*.*
$\top\!\!\top_{XY}$ *and* $\perp\!\!\perp_{XY}$ *need not be distinct.*
If $|\boldsymbol{C}(X,Y)| = 1$*, for all* $X, Y \in \mathcal{O}bj(\boldsymbol{C})$*, then* $\boldsymbol{C}$ *is said to be* ***degenerate****.*
In a non-degenerate ordered pseudo-category $\boldsymbol{C}$*, an object* X *is said to be* ***empty*** *if* $|\boldsymbol{C}(X,X)| = 1$*.*

Example 6.1.13.

1. $(\mathbf{R}_{\mathbf{2}}^{\cdot}; \leq)$ and $(\mathbf{R}_{\mathbf{2}}^{\oslash}; \leq)$ are complete ordered pseudo-categories with respect to the order relation induced pointwisely on $\mathbf{R_2}$, since $\cdot$ and $\oslash$ are isotonic in the second argument. The top element and the bottom element in the set of morphisms from X to Y are, respectively, the universal relation $\top\!\!\top_{X \times Y}$ and the zero relation $\perp\!\!\perp_{X \times Y}$.
2. $(\mathbf{R}_{\mathbf{2}}^{\oslash}; \leq)$ and $(\mathbf{R}_{\mathbf{2}}^{\ominus}; \leq)$ are not ordered, since $\oslash$ and $\ominus$, in general, are not isotonic in the second argument.

Definition 6.1.14. *A* ***reversible pseudo-category*** *is a pair* $(\boldsymbol{C}; [\cdot]_{-})$*, where* $\boldsymbol{C}$ *is a pseudo-category and* $[\cdot]_{-} : \mathcal{M}or(\boldsymbol{C}) \to \mathcal{M}or(\boldsymbol{C})$ *is an involution such that for every* $f \in \boldsymbol{C}(X,Y)$*,* $(f)[\cdot]_{-} = f_{-} \in \boldsymbol{C}(Y,X)$*.* f_{-} *is the reverse morphism of* f*.* $f \in \boldsymbol{C}(X,X)$ *is* ***symmetric*** *if* $f_{-} = f$*.*

Definition 6.1.15. *The* ***reverse pseudo-category*** *of a reversible pseudo-category* $(\boldsymbol{C}; [\cdot]_{-})$ *is the reversible pseudo-category* $\boldsymbol{C}_{-} = (\boldsymbol{C}_{-}; [\cdot]_{-})$*, where* $\boldsymbol{C}_{-}$ *has the same objects and morphisms as* $\boldsymbol{C}$*, in particular* $\boldsymbol{C}_{-}(X,Y) = \boldsymbol{C}(Y,X)$*, for all* $X, Y \in \mathcal{O}bj(\boldsymbol{C})$*, the same involution* $[\cdot]_{-} : \mathcal{M}or(\boldsymbol{C}_{-}) \to \mathcal{M}or(\boldsymbol{C}_{-})$ *and the composition* $\circ_{-}$ *defined by*

$$f \circ_{-} g = (g_{-} \circ f_{-})_{-} \text{ for all } f \in \boldsymbol{C}_{-}(X,Y),\ g \in \boldsymbol{C}_{-}(Y,Z).$$

$\boldsymbol{C}$ *and* $\boldsymbol{C}_{-}$ *are said to be* ***reverse*** *to each other.*
$\boldsymbol{C}$ *is* ***self-reverse*** *if* $\boldsymbol{C}_{-} = \boldsymbol{C}$*, i.e. if* $(g_{-} \circ f_{-})_{-} = f \circ g$*, for all* $f \in \boldsymbol{C}(X,Y), g \in \boldsymbol{C}(Y,Z)$*.*

The following are examples of reversible pseudo-categories.

Example 6.1.16. 1. $(\mathbf{R_2}; [\cdot]_-)$, $(\mathbf{R_2^{\oslash}}; [\cdot]_-)$, $(\mathbf{R_2^{\otimes}}; [\cdot]_-)$ and $(\mathbf{R_2^{\ominus}}; [\cdot]_-)$ are reversible pseudo-categories, since the function $[\cdot]_- : \mathbf{R_2} \to \mathbf{R_2}$, $r \mapsto r_-$ is an involution, mapping relations from X to Y into relations from Y to X.

2. $(\mathbf{R_2^{\oslash}}; [\cdot]_-)$ and $(\mathbf{R_2^{\otimes}}; [\cdot]_-)$ are reverse of each other.

3. $(\mathbf{R_2^{\ominus}}; [\cdot]_-)$ is self-reverse.

Proposition 6.1.17. *Let $(\boldsymbol{C}; [\cdot]_-)$ be a reversible pseudo-category. Then:*

1. *$(\boldsymbol{C}; [\cdot]_-)$ is the reverse pseudo-category of $(\boldsymbol{C}_-; [\cdot]_-)$;*

2. *leaving fixed the objects and mapping any morphism f to its reverse f_- one obtains an isomorphism*
$$Rev_{\boldsymbol{C}} = (id_{\mathcal{O}bj(\boldsymbol{C})}, [\cdot]_-) : \boldsymbol{C} \to \boldsymbol{C}_-^{op}$$
whose inverse, $Rev_{\boldsymbol{C}_-^{op}} : \boldsymbol{C}_-^{op} \to \boldsymbol{C}$ is determined by the same maps;

3. *if r_X (l_X, respectively) is the right (left, respectively) identity morphism in $\boldsymbol{C}$, then $(r_X)_-$ ($(l_X)_-$, respectively) is the left (right, respectively) identity in $\boldsymbol{C}_-$.*

Proof. 1. Obviously, $(\mathbf{C}_-; [\cdot]_-)$ is reversible and its reverse is $((\mathbf{C}_-)_-; [\cdot]_-) = (\mathbf{C}; [\cdot]_-)$. In fact, by definition, $((\mathbf{C}_-)_-; [\cdot]_-)$ has the same objects, morphism and involution as $(\mathbf{C}_-; [\cdot]_-)$ and hence the same as $(\mathbf{C}; [\cdot]_-)$. Moreover, for all $X, Y \in \mathcal{O}bj(\mathbf{C})$ one has $(\mathbf{C}_-)_-(X, Y) = \mathbf{C}_-(Y, X) = \mathbf{C}(X, Y)$ and, for all $f \in \mathbf{C}(X, Y)$, $g \in \mathbf{C}(Y, Z)$ one has $f(\circ_-)_- g = (g_- \circ_- f_-)_- = [(f_-)_- \circ (g_-)_-]_- = f \circ g$.

2. Clearly, $Rev_{\mathbf{C}} \circ Rev_{\mathbf{C}_-^{op}} = (id_{\mathcal{O}bj(\mathbf{C})} \circ id_{\mathcal{O}bj(\mathbf{C}_-^{op})}, [\cdot]_- \circ [\cdot]_-) = (id_{\mathcal{O}bj(\mathbf{C})}, id_{\mathcal{M}or(\mathbf{C})}) = Id_{\mathbf{C}}$. Similarly $Rev_{\mathbf{C}_-^{op}} \circ Rev_{\mathbf{C}} = Id_{\mathbf{C}_-^{op}}$.

3. For every $X \in \mathcal{O}bj(\mathbf{C})$, let r_X be the right identity morphism in $\mathbf{C}$. Then, for every $f \in \mathcal{M}or(\mathbf{C})$, it follows from definition of $\circ_-$ that $(r_X)_- \circ_- f_- = (f \circ r_X)_- = f_-$, i.e. $(r_X)_-$ is the left identity morphism in $\mathbf{C}_-$.

□

6.1.2 Relational pseudo-categories

Definition 6.1.18. *A* ***relational pseudo-category*** *is a reversible ordered pseudo-category $(\boldsymbol{C}; \leq, [\cdot]_-)$ such that:*

1. *the involution $[\cdot]_-$ preserves the order of morphisms;*

2. *all the existing (right, left) identity morphisms coincide and are symmetric.*

Definition 6.1.19. *A relational pseudo-category is called* ***multiplicative (implicative****, respectively) if the composition is isotonic (antitonic, respectively) in the first argument.*

Of course, the involution $[\cdot]_-$ also reflects the order, hence for all $f, f' \in \mathbf{C}(X,Y)$: $f \leq f'$ if and only if $f_- \leq f'_-$.
It follows from the definition of $\circ_-$ that the isotonic (antitonic) conditions of the composition turn from one side in $\mathbf{C}$ to the other side in $\mathbf{C}_-$. Then, the reverse of a multiplicative relational pseudo-category is a multiplicative relational pseudo-category, but a similar result does not hold for implicative relational pseudo-category whose reverse need not be a relational pseudo-category.

Example 6.1.20. 1. From Proposition 1.4.10 it follows that $\mathbf{R}_\mathbf{2}^{;}$ is a relational multiplicative category.

2. $\mathbf{R}_\mathbf{2}^{\oslash}$, $\mathbf{R}_\mathbf{2}^{\odot}$ and $\mathbf{R}_\mathbf{2}^{\ominus}$ are not categories, since they miss the associativity of partial compositions and the existence of identity morphisms. In particular, $\mathbf{R}_\mathbf{2}^{\oslash}$ is a relational implicative pseudo-category, $\mathbf{R}_\mathbf{2}^{\odot}$ and $\mathbf{R}_\mathbf{2}^{\ominus}$ are pseudo-categories, but not relational pseudo-category, since they are not ordered. Nevertheless, we note that $\odot$ is antitonic in the first and in the second argument and $\ominus$ is antitonic in the first argument.

Proposition 6.1.21. *Let $F : \mathbf{C} \to \mathbf{D}$ be an isomorphism of pseudo-categories. Then the following hold.*

1. *If $(\mathbf{C}; [\cdot]_-)$ is reversible, then a unique involution $[\cdot]_{-\mathbf{D}}$ in $\mathcal{M}or(\mathbf{D})$ exists such that $(\mathbf{D}; [\cdot]_{-\mathbf{D}})$ is reversible and F preserves and reflects the involution.*

2. *If $(\mathbf{C}; \leq)$ is ordered, then a unique order $\leq_\mathbf{D}$ in $\mathcal{M}or(\mathbf{D})$ exists such that $(\mathbf{D}; \leq_\mathbf{D})$ is ordered and F preserves and reflects the order.*

3. *If $(\mathbf{C}; \leq, [\cdot]_-)$ is an implicative (multiplicative, respectively) relational pseudo-category, then $(\mathbf{D}; \leq_\mathbf{D}, [\cdot]_{-\mathbf{D}})$ is, too.*

Proof. 1. Map any $h \in \mathbf{D}(X,Y)$ to $h_{-\mathbf{D}} = (((h)F^{-1})_-)F \in \mathbf{D}(Y,X)$. Then $(h_{-\mathbf{D}})_{-\mathbf{D}} = (((h_{-\mathbf{D}})F^{-1})_-)F = ((((((h)F^{-1})_-)F)F^{-1})_-)F = (((((h)F^{-1})_-)F \circ F^{-1})_-)F = ((((h)F^{-1})_-)_-)F = ((h)F^{-1})F = h$.
If $[\cdot]_\sim$ is an involution in $\mathcal{M}or(\mathbf{D})$ that commutes with F and $[\cdot]_-$, then, for any $h \in \mathcal{M}or(\mathbf{D})$, $h_\sim = (((h)F^{-1})F)_\sim = (((h)F^{-1})_-)F = (((h)F^{-1})F)_{-\mathbf{D}} = h_{-\mathbf{D}}$.

2. For all $h, k \in \mathbf{D}(X,Y)$, define $h \leq_\mathbf{D} k \Leftrightarrow (h)F^{-1} \leq (k)F^{-1}$.
Then, for all $A \in \mathcal{O}bj(\mathbf{D})$ and $g \in \mathbf{D}(A,X)$ one has $h \leq_\mathbf{D} k \Rightarrow (h)F^{-1} \leq (k)F^{-1} \Rightarrow (g)F^{-1} \circ (h)F^{-1} \leq (g)F^{-1} \circ (k)F^{-1} \Rightarrow (g \circ h)F^{-1} \leq (g \circ k)F^{-1} \Rightarrow g \circ h \leq_\mathbf{D} g \circ k$.
If $\preceq$ is an order relation and F preserves and reflects $\leq$ and $\preceq$, then $h \leq_\mathbf{D} k \Leftrightarrow (h)F^{-1} \leq (k)F^{-1} \Leftrightarrow h \preceq k$.

3. $(\mathbf{D}; \leq_\mathbf{D}, [\cdot]_{-\mathbf{D}})$ is an ordered reversible pseudo-category by items (1) and (2).
Then, for all $h, k \in \mathbf{D}(X,Y)$, we have that $h \leq_\mathbf{D} k \Rightarrow (h)F^{-1} \leq (k)F^{-1} \Rightarrow ((h)F^{-1})_- \leq ((k)F^{-1})_- \Rightarrow h_{-\mathbf{D}} = (((h)F^{-1})_-)F \leq (((k)F^{-1})_-)F = k_{-\mathbf{D}}$.
If $X \in \mathcal{O}bj(\mathbf{D})$ and r_X is a right identity, then $(r_X)F^{-1}$ is a right identity, hence $(r_X)_{-\mathbf{D}} = (((r_X)F^{-1})_-)F = ((r_X)F^{-1}))F = r_X$.

That the implicative or multiplicative character is inherited by $\mathbf{D}(X,Y)$ can be shown with a similar argument as in item (2).

□

Of course, since any reversible pseudo-category is isomorphic to the opposite of its reverse (see Proposition 6.1.17 (2)), one has that the opposite of the reverse of a relational pseudo-category of some kind is a relational pseudo-category of the same kind.

Proposition 6.1.22. *Let $(\boldsymbol{C};[\cdot]_{-})$ be a relational pseudo-category. The following hold:*

1. *for all $X,Y \in \mathcal{O}b\!j(\boldsymbol{C})$, $(\top\!\!\!\top_{XY})_{-} = \top\!\!\!\top_{YX}$, if $\boldsymbol{C}(Y,X)$ has the top;*
2. *for all $X,Y \in \mathcal{O}b\!j(\boldsymbol{C})$, $(\perp\!\!\!\perp_{XY})_{-} = \perp\!\!\!\perp_{YX}$, if $\boldsymbol{C}(Y,X)$ has the bottom.*

Proof. To prove the first item, let $X,Y \in \mathcal{O}b\!j(\mathbf{C})$ and let $f \in \mathbf{C}(Y,X)$; then $f_{-} \in \mathbf{C}(X,Y)$ and hence $f_{-} \leq \top\!\!\!\top_{XY}$, since $\top\!\!\!\top_{XY}$ is the top element. So, $f = (f_{-})_{-} \leq (\top\!\!\!\top_{XY})_{-}$, since $[\cdot]_{-}$ is an involution that preserves the order. For the uniqueness of maximum, it follows that $(\top\!\!\!\top_{XY})_{-} = \top\!\!\!\top_{YX}$. The proof of the second item is similar. □

We note that, in the above Proposition, we use only the compatibility of the involution $[\cdot]_{-}$ with the order, required in the definition of relational pseudo-category; in fact, the condition (2) of Definition 6.1.18 is not used.

Proposition 6.1.23. *Let $\boldsymbol{C}$ be a bounded ordered pseudo-category and assume $\boldsymbol{C}(X,X)$, $\boldsymbol{C}(X,Y)$, $\boldsymbol{C}(Y,Y)$ have their top element.*

1. *If $\boldsymbol{C}$ is a right, ordered pseudo-category, then $\top\!\!\!\top_{XY} = \top\!\!\!\top_{XY} \circ \top\!\!\!\top_{YY}$.*
2. *If $\boldsymbol{C}$ is a left, multiplicative relational pseudo-category, then $\top\!\!\!\top_{XY} = \top\!\!\!\top_{XX} \circ \top\!\!\!\top_{XY}$.*

Proof.

1. $\top\!\!\!\top_{XY} \geq \top\!\!\!\top_{XY} \circ \top\!\!\!\top_{YY}$, since $\top\!\!\!\top_{XY}$ is the top element.
   Conversely, $\top\!\!\!\top_{XY} = \top\!\!\!\top_{XY} \circ r_Y \leq \top\!\!\!\top_{XY} \circ \top\!\!\!\top_{YY}$, since $\circ$ is isotonic in the second argument, by assumption.
2. $\top\!\!\!\top_{XY} \geq \top\!\!\!\top_{XX} \circ \top\!\!\!\top_{XY}$, since $\top\!\!\!\top_{XY}$ is a top element.
   Conversely, $\top\!\!\!\top_{XY} = l_X \circ \top\!\!\!\top_{XX} \leq \top\!\!\!\top_{XX} \circ \top\!\!\!\top_{XY}$, since $\circ$ is isotonic in the first argument, by assumption.

□

6.1.3 Pseudo-n-categories

To describe the structure of the class of many-valued relations, it will be also useful to consider altogether some of the many partial compositions we have defined in Section 3.1. So, we arrange the following Definitions that will be useful in Subsection 6.1.5.

Definition 6.1.24. *For $n \in \mathbb{N}$, a **pseudo-n-category** is a tuple*

$$\boldsymbol{C} = (\mathcal{O}b\!j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ_1, \circ_2, \dots, \circ_n)$$

such that, for every $i \in \{1,2,\dots,n\}$, $(\mathcal{O}b\!j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ_i)$ is a pseudo-category.

Hence, a pseudo-category above defined is a pseudo-1-category; a pseudo-2-category and a pseudo-3-category are also called **pseudo-bicategory** and **pseudo-tricategory**.

Of course, it is meaningful to consider pseudo-n-categories when suitable relationships between (some of) the partial compositions of morphisms are satisfied, or required. Quite typical are, in case an involution and an order are assigned in the class of morphisms, the requirement of some kinds of connections that extend typical conditions of residuated algebraic structures. The following Definition includes a few types of pseudo-n-categories.

Definition 6.1.25. *A pseudo-bicategory $\boldsymbol{C} = ((\mathcal{O}b j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ, *); \leq, [\cdot]_{-})$, with an order $\leq$ and an involution $[\cdot]_{-}$ in the class of morphisms, is said to be* ***residuated*** *if $\boldsymbol{C}^{\circ} = ((\mathcal{O}b j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ); \leq, [\cdot]_{-})$ and $\boldsymbol{C}^{*} = ((\mathcal{O}b j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), *); \leq, [\cdot]_{-})$ are relational pseudo-categories such that for all $X, Y, Z \in \mathcal{O}b j(\boldsymbol{C})$ and for all $f \in \boldsymbol{C}(Z, X), g \in \boldsymbol{C}(Z, Y), h \in \boldsymbol{C}(X, Y)$, the equivalence $h \leq f_{-} \circ g \Leftrightarrow f * h \leq g$ holds.*

Remark 6.1.26.

1. The required condition that for all $X, Y, Z \in \mathcal{O}b j(\mathbf{C})$ and for all $f \in \mathbf{C}(Z, X), g \in \mathbf{C}(Z, Y), h \in \mathbf{C}(X, Y)$: $h \leq f_{-} \circ g \Leftrightarrow f * h \leq g$ is satisfied, means that for every $f \in \mathbf{C}(Z, X)$, the functions $h \mapsto f * h$ and $g \mapsto f_{-} \circ g$ form a type III $\top$-including Galois connection $(f * [\cdot], f_{-} \circ [\cdot])_{\top[\uparrow]} : (\mathbf{C}(X, Y), \leq) \to (\mathbf{C}(Z, Y), \leq)$.
2. Since the order $\leq$ is a **2**-relation, we have that, the relative connection of item (1) is indeed a type III global connection $(f * [\cdot], f_{-} \circ [\cdot]) : (\mathbf{C}(X, Y), \leq) \to (\mathbf{C}(Z, Y), \leq)$.

Proposition 6.1.27. *Let $\boldsymbol{C} = ((\mathcal{O}b j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ, *); \leq, [\cdot]_{-})$ be a residuated pseudo-bicategory. With the above notation, the following hold.*

1. *l_X is the left identity of the object X in $\boldsymbol{C}^{*}$ if and only if it is the left identity of the object X in $\boldsymbol{C}^{\circ}$.*
2. *$\boldsymbol{C}^{*}$ is left if and only if $\boldsymbol{C}^{\circ}$ is left.*

Proof.

1. Let l_X be the left identity of the object X of $\mathbf{C}^{*}$; then $(l_X)_{-} = l_X$ and, for all $X, Y \in \mathcal{O}b j(\mathbf{C})$ and for every $g \in \mathbf{C}(X, Y)$, one has that $l_X * g = g$. Since $\mathbf{C}$ is residuated, by definition it follows that $l_X * g \leq g \Leftrightarrow g \leq l_X \circ g$ and $l_X \circ g \leq l_X \circ g \Leftrightarrow l_X * (l_X \circ g) \leq g \Leftrightarrow l_X \circ g \leq g$. Hence $l_X \circ g = g$. The converse implication can be similarly proved.
2. It is an easy consequence of the item (1) and of the assumption on $\mathbf{C}^{*}$ and $\mathbf{C}^{\circ}$.

□

Definition 6.1.28. *A pseudo-bicategory $\boldsymbol{C} = ((\mathcal{O}b j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ, \bullet); \leq, [\cdot]_{-})$, with an order $\leq$ and an involution $[\cdot]_{-}$ in the class of morphisms, is said to be* ***symmetrical*** *if $\boldsymbol{C}^{\circ} = ((\mathcal{O}b j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ); \leq, []_{-})$ and $\boldsymbol{C}^{\bullet} = ((\mathcal{O}b j(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \bullet); \leq, []_{-})$ are relational pseudo-categories such that for all $X, Y, Z \in \mathcal{O}b j(\boldsymbol{C})$ and and for all $f \in \boldsymbol{C}(X, Z), g \in \boldsymbol{C}(Z, Y), h \in \boldsymbol{C}(X, Y)$, the equivalence $h \leq f \circ g \Leftrightarrow f \leq h \bullet g_{-}$ holds.*

Remark 6.1.29. 1. As observed in Remark 6.1.26, the condition requiring that for all $X, Y, Z \in \mathcal{O}bj(\mathbf{C})$ and for all $f \in \mathbf{C}(X, Z), g \in \mathbf{C}(Z, Y), h \in \mathbf{C}(X, Y)$ the equivalence $h \leq f \circ g \Leftrightarrow f \leq h \bullet g_-$ is satisfied, means that, for every $g \in \mathbf{C}(Z, Y)$ the functions $f \mapsto f \circ g$ and $h \mapsto h \bullet g_-$ form a type I $\top$-including Galois connection $[[\cdot] \circ g, [\cdot] \bullet g_-]_{\top[\uparrow]} : (\mathbf{C}(X, Z), \leq) \to (\mathbf{C}(X, Y), \leq)$.

2. As noted for a residuated pseudo-bicategory in Remark 6.1.26, the Galois connection $[[\cdot] \circ g, [\cdot] \bullet g_-]_{\top[\uparrow]} : (\mathbf{C}(X, Z), \leq) \to (\mathbf{C}(X, Y), \leq)$ is indeed a type I global connection $[[\cdot] \circ g, [\cdot] \bullet g_-] : (\mathbf{C}(X, Z), \leq) \to (\mathbf{C}(X, Y), \leq)$.

Proposition 6.1.30. *Let $\boldsymbol{C} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ, \bullet); \leq, [\cdot]_-)$ be a pseudo-bicategory. If $\boldsymbol{C}$ is symmetrical, then both $\boldsymbol{C}^{\circ} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ); \leq, []_-)$ and $\boldsymbol{C}^{\bullet} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \bullet); \leq, []_-)$ are implicative relational pseudo-categories.*

Proof. Since $\mathbf{C}$ is a symmetrical pseudo-bicategory, then $h \leq f \circ g \Leftrightarrow f \leq h \bullet g_-$, for all $X, Y, Z \in \mathcal{O}bj(\mathbf{C})$ and for all $f \in \mathbf{C}(X, Z), g \in \mathbf{C}(Z, Y), h \in \mathbf{C}(X, Y)$. This means that the functions $[\cdot] \circ g$ and $[\cdot] \bullet g_-$ form a (classical) Galois connection and hence they are antitonic. So, the composition of morphisms $\circ$ and $\bullet$ are antitonic in the first argument; then $\mathbf{C}^{\circ} = ((\mathcal{O}bj(\mathbf{C}), \mathcal{M}or(\mathbf{C}), \circ); \leq, []_-)$ and $\mathbf{C}^{\bullet} = ((\mathcal{O}bj(\mathbf{C}), \mathcal{M}or(\mathbf{C}), \bullet); \leq, []_-)$ are implicative relational pseudo-categories. □

Definition 6.1.31. *Let $\boldsymbol{C}$ be a symmetrical pseudo-bicategory and let $h \in \boldsymbol{C}(X, Z)$. We say that h is **crisp** in $\boldsymbol{C}$ if it satisfies the condition:*
(cr) $h \leq f \circ g \Leftrightarrow h \leq f \bullet g$, for all $f \in \boldsymbol{C}(X, Y), g \in \boldsymbol{C}(Y, Z)$.

Remark 6.1.32. Clearly, all morphisms of a symmetrical pseudo-bicategory $\mathbf{C} = ((\mathcal{O}bj(\mathbf{C}), \mathcal{M}or(\mathbf{C}), \circ, \bullet); \leq, [\cdot]_-)$ are crisp if and only if $\circ = \bullet$. In fact, if $\circ = \bullet$, obviously $h \leq f \circ g \Leftrightarrow h \leq f \bullet g$, for all $f \in \mathbf{C}(X, Y), g \in \mathbf{C}(Y, Z)$. Conversely, if for all $h \in \mathbf{C}(X, Z)$, $f \in \mathbf{C}(X, Y), g \in \mathbf{C}(Y, Z)$ we have that $h \leq f \circ g \Leftrightarrow h \leq f \bullet g$, then $\circ = \bullet$.

Example 6.1.33. 1. In each bounded symmetrical pseudo-bicategory $\mathbf{C} = ((\mathcal{O}bj(\mathbf{C}), \mathcal{M}or(\mathbf{C}), \circ, \bullet); \leq, [\cdot]_-)$, $\perp\!\!\!\perp_{XY}$ is a crisp morphisms, for all $X, Y \in \mathcal{O}bj(\mathbf{C})$; in fact, since it is the bottom element, $\perp\!\!\!\perp_{XY} \leq f \circ g \Leftrightarrow \perp\!\!\!\perp_{XY} \leq f \bullet g$, for all $f : X \to Z, g : Z \to Y$.

2. In $(\mathbf{R}_{\mathbf{2}}^{\oslash}; \leq, [\cdot]_-)$ all (crisp) relations are crisp morphism, obviously.

Proposition 6.1.34. *Let $\boldsymbol{C} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ, \bullet), \leq, [\cdot]_-)$ be a symmetrical pseudo-bicategory. If $\boldsymbol{C}$ has left identity l_X for one of the compositions $\circ$ and $\bullet$, then l_X is a left identity for the other composition if and only if it is a crisp element.*

Proof. Let $l_X \in \mathbf{C}(X, X)$ be left identity for both compositions $\circ$ and $\bullet$.
So, for all $f \in \mathbf{C}(X, Y), g \in \mathbf{C}(Y, X)$, one has that $l_X \leq f \circ g \Leftrightarrow f \leq l_X \bullet g_- \Leftrightarrow f \leq g_- \Leftrightarrow f \leq l_X \circ g_- \Leftrightarrow l_X \leq f \bullet g$. Hence, l_X is crisp.
Conversely, suppose that l_X is a crisp left identity morphism for $\circ$, that is $l_X \leq f \circ g \Leftrightarrow l_X \leq f \bullet g$ and $l_X \circ g_- = g_-$, for all $f \in \mathbf{C}(X, Y), g \in \mathbf{C}(Y, X)$. By assumption the following is true: $g_- \leq l_X \circ g_- \Rightarrow l_X \leq g_- \bullet g \Rightarrow l_X \leq g_- \circ g \Rightarrow g_- \leq l_X \bullet g$; moreover, we have that

$l_X \bullet g_- \leq l_X \bullet g_- \Rightarrow l_X \leq (l_X \bullet g_-) \circ g \Rightarrow l_X \leq (l_X \bullet g_-) \bullet g \Rightarrow l_X \bullet g_- \leq l_X \circ g_- \Rightarrow l_X \bullet g_- \leq g_-$. So $l_X \bullet g_- = g_-$ and hence l_X is a left identity with respect to the composition $\bullet$. $\square$

Remark 6.1.35. By the above proposition if l_X is a left identity both for $\mathbf{C}^{\circ}$ and $\mathbf{C}^{\bullet}$, the it is a crisp morphism.
The converse does not hold, in general; in fact, in $(\mathbf{R}_{\mathbf{2}}^{\oslash}; \leq, [\cdot]_-)$ all relations are crisp morphism, but not all are left identities, of course.

Definition 6.1.36. *A pseudo-tricategory $\boldsymbol{C} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \circ, *, \bullet); \leq, [\cdot]_-)$, with an order $\leq$ and an involution $[\cdot]_-$ in the class of morphisms, is said to be* ***triangular*** *if $\boldsymbol{C}^{\circ} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \circ); \leq, [\cdot]_-)$, $\boldsymbol{C}^{*} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), *); \leq, [\cdot]_-)$ and $\boldsymbol{C}^{\bullet} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \bullet); \leq, [\cdot]_-)$ are relational pseudo-categories such that, for all $X, Y, Z \in \mathcal{Obj}(\boldsymbol{C})$, the diagram*

$$\begin{tikzcd}
& (\boldsymbol{C}(X,Y), \leq) \arrow[dr, "\chi"] & \\
(\boldsymbol{C}(Z,X), \leq) \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & (\boldsymbol{C}(Y,Z), \leq)
\end{tikzcd}$$

where $\varphi : \boldsymbol{C}(Z,X) \to \boldsymbol{C}(Y,Z)$, $\chi : \boldsymbol{C}(X,Y) \to \boldsymbol{C}(Y,Z)$ and $\psi : \boldsymbol{C}(Z,X) \to \boldsymbol{C}(X,Y)$ are defined as follows, for all $f \in \boldsymbol{C}(Z,X)$, $g \in \boldsymbol{C}(Y,Z)$ and $h \in \boldsymbol{C}(X,Y)$:

- $(f,g)\varphi = f_- \circ g_-$;
- $(h,g)\chi = (h \bullet g)_-$;
- $(f,h)\psi = (f * h)_-$

is a $\top$-including Galois $\mathbf{2}$-triangle.

Remark 6.1.37.
1. Comments similar to those made in Remarks 6.1.26 and 6.1.29 allows to say that, in the above definition, requiring that the diagram is a $\top$-including Galois $\mathbf{2}$-triangle is equivalent to require that it is a relational $\mathbf{2}$-triangle.
2. The diagram in the above definition is a $\top$-including Galois $\mathbf{2}$-triangle if and only if for all $X, Y, Z \in \mathcal{Obj}(\mathbf{C})$ and for all $f \in \mathbf{C}(Z,X)$, $g \in \mathbf{C}(Y,Z)$ and $h \in \mathbf{C}(X,Y)$ the following equivalences hold: $f \leq (h \bullet g)_- \Leftrightarrow (f * h)_- \leq g \Leftrightarrow h \leq f_- \circ g_-$.

Proposition 6.1.38. *Let $\boldsymbol{C} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \circ, *, \bullet); \leq, [\cdot]_-)$ be a pseudo-tricategory, with an order $\leq$ and an involution $[\cdot]_-$ that preserves the order in the class of morphisms.*

1. *$\boldsymbol{C}$ is triangular if and only if $\boldsymbol{C}^{\circ,*} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \circ, *); \leq, [\cdot]_-)$ is a residuated pseudo-bicategory and $\boldsymbol{C}^{\circ,\bullet} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \circ, \bullet); \leq, [\cdot]_-)$ is a symmetrical pseudo-bicategory.*
2. *If $\boldsymbol{C}$ is triangular, then $\boldsymbol{C}^{\bullet,*_-} = ((\mathcal{Obj}(\boldsymbol{C}), \mathcal{Mor}(\boldsymbol{C}), \bullet, *_-); \leq, [\cdot]_-)$ is a residuated pseudo-bicategory.*

Proof. 1. " $\Rightarrow$ " Let $\mathbf{C} = ((\mathcal{Obj}(\mathbf{C}), \mathcal{Mor}(\mathbf{C}), \circ, *, \bullet); \leq, [\cdot]_-)$ be a triangular pseudo-tricategory. For all $X, Y, Z \in \mathcal{Obj}(\mathbf{C})$ and for all $f \in \mathbf{C}(Z, X)$, $g \in \mathbf{C}(Y, Z)$ and $h \in \mathbf{C}(X, Y)$ the equivalences: $f \leq (h \bullet g)_- \Leftrightarrow (f * h)_- \leq g \Leftrightarrow h \leq f_- \circ g_-$ are true. Hence, the following hold, for all considered elements:

(a) $(h \leq f_- \circ g_- \Leftrightarrow (f * h)_- \leq g) \Leftrightarrow (h \leq f_- \circ g_- \Leftrightarrow f * h \leq g_-)$; so, for all $f \in \mathbf{C}(Z, X)$, $g' = g_- \in \mathbf{C}(Z, Y)$ and $h \in \mathbf{C}(X, Y)$, one has that $h \leq f_- \circ g' \Leftrightarrow f * h \leq g'$, i.e. $\mathbf{C}^{\circ, *}$ is a residuated pseudo-category.

(b) $(f \leq (h \bullet g)_- \Leftrightarrow h \leq f_- \circ g_-) \Leftrightarrow (f \leq h \bullet g \Leftrightarrow h \leq f_- \circ g_-)$; so, for all $f' = f_- \in \mathbf{C}(X, Z)$, $g' = g_- \in \mathbf{C}(Z, Y)$ and $h \in \mathbf{C}(X, Y)$, one has that $f' \leq h \bullet g'_- \Leftrightarrow h \leq f' \circ g'$, i.e. $\mathbf{C}^{\circ, \bullet}$ is a symmetrical pseudo-category.

" $\Leftarrow$ " Assume that $\mathbf{C}^{\circ, *} = ((\mathcal{Obj}(\mathbf{C}), \mathcal{Mor}(\mathbf{C}), \circ, *); \leq, [\cdot]_-)$ is a residuated pseudo-bicategory and $\mathbf{C}^{\circ, \bullet} = ((\mathcal{Obj}(\mathbf{C}), \mathcal{Mor}(\mathbf{C}), \circ, \bullet); \leq, [\cdot]_-)$ is a symmetrical pseudo-bicategory. Then, for all $X, Y, Z \in \mathcal{Obj}(\mathbf{C})$ and for all $f \in \mathbf{C}(Z, X)$, $g \in \mathbf{C}(Z, Y)$ and $h \in \mathbf{C}(X, Y)$ the following equivalences hold:

- $h \leq f_- \circ g \Leftrightarrow f * h \leq g$;
- $h \leq f \circ g \Leftrightarrow f \leq h \bullet g_-$.

Then, for all $f \in \mathbf{C}(Z, X)$, $g' \in \mathbf{C}(Y, Z)$ and $h \in \mathbf{C}(X, Y)$ one has that $h \leq f_- \circ g'_- \Leftrightarrow f * h \leq g'_- \Leftrightarrow f_- \leq h \bullet g'$; since the involution $[\cdot]_-$ preserves the order, it follows that $h \leq f_- \circ g'_- \Leftrightarrow (f * h)_- \leq g' \Leftrightarrow f \leq (h \bullet g')_-$. Hence the pseudo-tricategory $\mathbf{C} = ((\mathcal{Obj}(\mathbf{C}), \mathcal{Mor}(\mathbf{C}), \circ, *, \bullet); \leq, [\cdot]_-)$ is triangular.

2. By assumption, for all $f \in \mathbf{C}(Z, X)$, $g \in \mathbf{C}(Y, Z)$ and $h \in \mathbf{C}(X, Y)$, we have that: $((f \leq (h \bullet g)_- \Leftrightarrow (f * h)_- \leq g) \Leftrightarrow (f_- \leq h \bullet g \Leftrightarrow f * h \leq g_-)$; so, for all $h' = f_- \in \mathbf{C}(X, Z)$, $f' = h_- \in \mathbf{C}(Y, X)$ and $g \in \mathbf{C}(Y, Z)$, one has $(h' \leq f'_- \bullet g \Leftrightarrow h'_- * f'_- \leq g_-) \Leftrightarrow (h' \leq f'_- \bullet g \Leftrightarrow (f' *_- h')_- \leq g_-) \Leftrightarrow (h' \leq f'_- \bullet g \Leftrightarrow f' *_- h' \leq g)$, i.e. $\mathbf{C}^{\bullet, *_-}$ is a residuated pseudo-bicategory.

$\square$

Proposition 6.1.39. *Let $\mathbf{C} = ((\mathcal{Obj}(\mathbf{C}), \mathcal{Mor}(\mathbf{C}), \circ, *, \bullet); \leq, [\cdot]_-)$ be a triangular pseudo-tricategory. Then, the following hold, for every $X \in \mathcal{Obj}(\mathbf{C})$:*

1. *l_X is a left identity in $\mathbf{C}^*$ if and only if it is the left identity in $\mathbf{C}^\circ$.*
2. *l_X is a left identity in $\mathbf{C}^{*_-}$ if and only if l_X is a left identity in $\mathbf{C}^\bullet$.*
3. *If l_X is a left identity for one of the compositions $\circ$ and $\bullet$, then l_X is a left identity for the other composition if and only if it is a crisp morphism.*
4. *l_X is a left identity in $\mathbf{C}^\circ$ and in $\mathbf{C}^\bullet$ if and only if l_X is an identity in $\mathbf{C}^*$.*
5. *If l_X is an identity in $\mathbf{C}^*$, then it is a crisp morphism.*

Proof. The proof of the items (1), (2) and (3) is an easy consequence of Propositions 6.1.27, 6.1.34 and 6.1.38.
(4) If l_X is a left identity of the object X in $\mathbf{C}^\circ$ and $\mathbf{C}^\bullet$, then, by item (1), l_X is a left identity of the object X of $\mathbf{C}^*$; moreover, by item (2) we have that l_X is a left identity in $\mathbf{C}^{*-}$, i.e. l_X is a right identity in $\mathbf{C}^*$. Hence, l_X is an identity of $\mathbf{C}^*$.
Conversely, if l_X is an identity in $\mathbf{C}^*$, in particular is a left identity and hence, by item (1) it is a left identity in $\mathbf{C}^\circ$. Moreover, if l_X is an identity, in particular a right identity in $\mathbf{C}^*$, then it is a left identity in $\mathbf{C}^{*-}$; hence, by item (2), l_X is a left identity in $\mathbf{C}^\bullet$.
(5) By item (4) if l_X is an identity in $\mathbf{C}^*$, then l_X is a left identity in $\mathbf{C}^\circ$ and $\mathbf{C}^\bullet$. Hence, by Proposition 6.1.34, l_X is a crisp morphism. □

Proposition 6.1.40. *Let $\boldsymbol{C} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ, *, \bullet); \leq, [\cdot]_-)$ be a bounded triangular pseudo-tricategory. Then $\perp\!\!\!\perp_{XY}$ and $\top\!\!\!\top_{XY}$ are crisp morphisms.*

Proof. The statement for the bottom element $\perp\!\!\!\perp_{XY}$ is already stated in Example 6.1.33. For the top morphism, by Remark 6.1.37 the following equivalences hold, for all $f : X \to Z, g : Z \to Y$: $\top\!\!\!\top_{XY} \leq f \circ g \Leftrightarrow (f_- * \top\!\!\!\top_{XY})_- \leq g_- \Leftrightarrow \top\!\!\!\top_{XY} \leq f \bullet g$. □

6.1.4 Special elements in relational pseudo-categories

In this Subsection we introduce, in the more general framework of relational pseudo-categories, a few notions, which have been already considered for **Dedekind categories** in [37, 87], where they have been revealed to be particularly useful for many purposes, among which the characterization of those Dedekind categories that are isomorphic to the category of binary relations with values in a Heyting algebra.
We note that notions corresponding to those considered for Dedekind categories have been used in the context of MV-algebras by A. Popescu in [71, 72] in order to describe and characterize MV-relation algebras that are an abstract model for homogeneous many-valued relations taking values in a MV-algebra.
We start recalling the definition of **Dedekind category**.

Definition 6.1.41. *[37] A **Dedekind category** $\mathcal{D} = (\mathrm{Obj}(\mathcal{D}), \boldsymbol{Mor}(\mathcal{D}), *)$ is a category such that:*
(D_1) $\mathcal{D}(X,Y) = (\mathcal{D}(X,Y), \subseteq, \cup, \cap, \Rightarrow, 0_{XY}, \nabla_{XY})$, is an Heyting algebra, $\forall X, Y \in Obj(\mathcal{D})$, where

1. $\alpha \subseteq \beta \Leftrightarrow \alpha = \alpha \cap \beta \Leftrightarrow \beta = \alpha \cup \beta$;
2. $\alpha \Rightarrow \beta$ *is a **pseudo-complementation** of α **relatively** to β i.e.*
$$\gamma \subseteq \alpha \Rightarrow \beta \Leftrightarrow \alpha \cap \gamma \subseteq \beta;$$
3. 0_{XY} *and ∇_{XY} are respectively the minimum and the maximum of the lattice.*

(D_2) There exists an unary operation $\# : \mathcal{D}(X,Y) \to \mathcal{D}(X,Y)$ such that, $\forall \alpha, \alpha' : X \rightharpoonup Y$, $\forall \beta : Y \rightharpoonup Z$ the following hold:

1. $(\alpha * \beta)^{\#} = \beta^{\#} * \alpha^{\#}$;
2. $(\alpha^{\#})^{\#} = \alpha$;
3. *if* $\alpha \subseteq \alpha'$, *then* $\alpha^{\#} \subseteq \alpha'^{\#}$.

(D_3) $\forall \alpha : X \rightharpoonup Y, \beta : Y \rightharpoonup Z, \gamma : X \rightharpoonup Z$ *the* ***Dedekind formula*** *or* ***modular law*** *holds:*

$$\alpha * \beta \cap \gamma \subseteq \alpha * (\beta \cap \alpha^{\#} * \gamma).$$

(D_4) $\forall \alpha : X \rightharpoonup Y, \beta : Y \rightharpoonup Z$ *the* ***residual composition*** $\alpha \circ \beta : X \rightharpoonup Z$ *is such that:*

$$\delta \subseteq \alpha \circ \beta \Leftrightarrow \alpha^{\#} * \delta \subseteq \beta\ ,\ \forall \delta : X \rightharpoonup Z.$$

It is easy to see that $(\mathit{Obj}(\mathcal{D}), \mathbf{Mor}(\mathcal{D}), *, \circ); \leq, [\cdot]^{\#})$ is a residuated pseudo-bicategory.

Example 6.1.42. The bicategory $\mathbf{R}_{\mathbf{2}}^{*,\circ}$ is a Dedekind category.

In this context, fundamental tools are the following.

Definition 6.1.43. *[37] Let $\mathcal{D}$ be a Dedekind category.*

1. *An object J is called **unit** if it satisfies the following condition:*
   (a) $\nabla_{JJ} = id_J \neq 0_{JJ}$;
   (b) $\nabla_{XJ} * \nabla_{JX} = \nabla_{XX}$, *for every* $X \in \mathrm{Obj}(\mathcal{D})$.
2. *A morphism $\alpha : X \rightharpoonup Y$ is called **ideal** if $\nabla_{XY} * \alpha * \nabla_{YY} = \alpha$.*

Remark 6.1.44. 1. if J is a unit of a Dedekind category $\mathcal{D}$ and $\alpha \in \mathcal{D}(J, J)$, then α satisfies the condition $\alpha \subseteq id_J$ and it called **partial identity** in [37, 87] and a **subdiagonal** morphism in [71, 72].

2. For all $X, Y \in \mathit{Obj}(\mathcal{D})$: $\nabla_{XJ} * \nabla_{JY} = \nabla_{XY}$.
3. Units are abstract versions of singleton sets, that are the objects of the Dedekind category **Rel** of sets and crisp binary heterogeneous relations and, also, of the Dedekind category Rel_L of sets and L-valued binary heterogeneous relations taking values in a Heyting algebra.

In [37] the notions of ideals, introduced by B. Jónsson and A. Tarski in [61], has been revised and the one that has been proposed is useful to characterize the set L of truth values that allows a representation of Dedekind categories as categories of L-relations. For instance, the least element 0_{XY} and the greatest one ∇_{XY} are the unique ideals of $\mathbf{R}_{\mathbf{2}}^{*,\circ}$. Moreover, in the Dedekind category Rel_L, the sublattice of ideals in each hom-lattice $\mathit{Rel}_L(X, Y)$ is isomorphic to the Heyting lattice L. In particular, it is easy to prove that an L-relation $\alpha : X \rightharpoonup Y$ is an ideal of Rel_L if and only if α is constant.

In the more general context of pseudo-categories the existence of (right, left) identity morphism is not assumed; moreover, since the associativity of morphisms composition is not required, the condition $\nabla_{XY} * \alpha * \nabla_{YY} = \alpha$ which characterizes the ideals of a Dedekind category is susceptible of to two different readings: $\nabla_{XY} * (\alpha * \nabla_{YY}) = \alpha$ and $(\nabla_{XY} * \alpha) * \nabla_{YY} = \alpha$.
Nevertheless, in a Dedekind category, thanks to the associativity of composition, the equality $\nabla_{XY} * \alpha * \nabla_{YY} = \alpha$ is equivalent to $\nabla_{XY} * \alpha = \alpha * \nabla_{YY} = \alpha$.
So, taking into account the above comments, we propose the following definition in the context of pseudo-categories.

Definition 6.1.45. *Let $\boldsymbol{C} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ); \leq, [\cdot]_-)$ be a bounded ordered pseudo-category. A morphism $f : X \rightharpoonup Y$ is called*

- ***right ideal*** *if $\top\!\!\top_{XX} \circ f = f$;*
- ***left ideal*** *if $f \circ \top\!\!\top_{YY} = f$;*
- ***ideal*** *if it is a right and left ideal.*

For all $X, Y \in \mathcal{O}bj(\boldsymbol{C})$ we denote by $\boldsymbol{I}(X,Y)$ the set of all ideals from X to Y.

Remark 6.1.46. In any bounded ordered reversible pseudo-category $\mathbf{C}$, if $f \in \mathbf{C}(X,Y)$ is an ideal, then the equality $\top\!\!\top_{XX} \circ f \circ \top\!\!\top_{YY} = f$ is consistent and true since clearly $(\top\!\!\top_{XX} \circ f) \circ \top\!\!\top_{YY} = \top\!\!\top_{XX} \circ (f \circ \top\!\!\top_{YY}) = f$.

Under associativity of the composition we get similar results as in Dedekind categories.

Proposition 6.1.47. *Let $\boldsymbol{C} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ); \leq, [\cdot]_-)$ be a bounded ordered associative pseudo-category. Then*

1. *$f \in \boldsymbol{C}(X,Y)$ is an ideal if and only if $\top\!\!\top_{XX} \circ f \circ \top\!\!\top_{YY} = f$;*
2. *if $f \in \boldsymbol{I}(X,Y)$ and $g \in \boldsymbol{I}(Y,Z)$, then $f \circ g \in \boldsymbol{I}(X,Z)$.*

Proof. 1. The necessity of the condition is stated in the above remark.
As for the sufficiency, by Proposition 6.1.23 and by assumption we have that: $\top\!\!\top_{XX} \circ f = \top\!\!\top_{XX} \circ (\top\!\!\top_{XX} \circ f \circ \top\!\!\top_{YY}) = (\top\!\!\top_{XX} \circ \top\!\!\top_{XX}) \circ f \circ \top\!\!\top_{YY} = \top\!\!\top_{XX} \circ f \circ \top\!\!\top_{XX} = f$; similarly, $f \circ \top\!\!\top_{XX} = f$.

2. Let $f \in \mathbf{I}(X,Y)$ and $g \in \mathbf{I}(Y,Z)$. By assumption, the following equalities hold: $\top\!\!\top_{XX} \circ (f \circ g) \circ \top\!\!\top_{YY} = (\top\!\!\top_{XX} \circ f) \circ (g \circ \top\!\!\top_{ZZ}) = f \circ g$. Hence, by item (1), $f \circ g \in \mathbf{I}(X,Z)$. ☐

Definition 6.1.48. *Let $\boldsymbol{C} = ((\mathcal{O}bj(\boldsymbol{C}), \mathcal{M}or(\boldsymbol{C}), \circ); \leq)$ be a left (right, unital) bounded ordered pseudo-category. An object J of $\boldsymbol{C}$ is called* ***singleton*** *if:*

1. *$\top\!\!\top_{JJ} = l_J \neq \perp\!\!\perp_{JJ}$ ($\top\!\!\top_{JJ} = r_J \neq \perp\!\!\perp_{JJ}$, $\top\!\!\top_{JJ} = i_J \neq \perp\!\!\perp_{JJ}$);*

2. $\top\!\!\!\top_{XJ} \circ \top\!\!\!\top_{JX} = \top\!\!\!\top_{XX}$, *for every* $X \in \mathcal{O}bj(\boldsymbol{C})$.

Proposition 6.1.49. *Let J be a singleton of a left (right, unital) bounded ordered pseudo-category $\boldsymbol{C}$. The followings hold.*

1. *If $f \in \boldsymbol{C}(J, J)$ then f is subdiagonal.*

2. *If $\boldsymbol{C}$ is associative, then $\top\!\!\!\top_{XY} = \top\!\!\!\top_{XJ} \circ \top\!\!\!\top_{JY}$, for all $X, Y \in \mathcal{O}bj(\boldsymbol{C})$.*

Proof. 1. Assuming l_J to be the left (right, unital) identity of J; since J is a singleton, from definition we have that $\top\!\!\!\top_{JJ} = l_J$; obviously $f \le l_J = \top\!\!\!\top_{JJ}$, because $\top\!\!\!\top_{JJ}$ is the top element.

2. $\top\!\!\!\top_{XY} \ge \top\!\!\!\top_{XJ} \circ \top\!\!\!\top_{JY}$, since $\top\!\!\!\top_{XY}$ is the top element.
Conversely, by Proposition 6.1.23 we have that:
$\top\!\!\!\top_{XY} = \top\!\!\!\top_{XY} \circ \top\!\!\!\top_{YY} = \top\!\!\!\top_{XY} \circ (\top\!\!\!\top_{YJ} \circ \top\!\!\!\top_{JY}) = (\top\!\!\!\top_{XY} \circ \top\!\!\!\top_{YJ}) \circ \top\!\!\!\top_{JY} \le \top\!\!\!\top_{XJ} \circ \top\!\!\!\top_{JY}$.

□

It follows clearly from Proposition 6.1.23 that in some bounded pseudo-categories the top morphisms are left or right ideals. Moreover, the following hold.

Proposition 6.1.50. *Let $\boldsymbol{C}$ be a bounded ordered reversible pseudo-category.*

1. *If $\boldsymbol{C}$ is a unital multiplicative relational pseudo-category, then $\top\!\!\!\top_{XY} \in \boldsymbol{I}(X, Y)$.*

2. *If $\boldsymbol{C}$ is a left implicative relational pseudo-category, then $\bot\!\!\!\bot_{XY}$ is right ideal.*

3. *If $\boldsymbol{C}$ is a unital bounded ordered pseudo-category and J is a singleton of $\boldsymbol{C}$, then $\boldsymbol{C}(J, J) = \boldsymbol{I}(J, J)$.*

Proof. 1. The proof follows easily by Proposition 6.1.23.

2. $\bot\!\!\!\bot_{XY} \le \top\!\!\!\top_{XX} \circ \bot\!\!\!\bot_{XY}$, since $\bot\!\!\!\bot_{XY}$ is the bottom element. Moreover, since $\top\!\!\!\top_{XX}$ is a top element and **C** is left and implicative, we have that $\top\!\!\!\top_{XX} \circ \bot\!\!\!\bot_{XY} \le l_X \circ \bot\!\!\!\bot_{XY} = \bot\!\!\!\bot_{XY}$. Hence $\bot\!\!\!\bot_{XY} \in \mathbf{I}(X, Y)$.

3. Let **C** be a unital relational pseudo-category. Trivially, $\mathbf{C}(J, J) \supseteq \mathbf{I}(J, J)$. Let $f \in \mathbf{C}(J, J)$; we have $\top\!\!\!\top_{JJ} \circ f = i_J \circ f = f$ and $f \circ \top\!\!\!\top_{JJ} = f \circ i_J = f$.

□

The following notion, introduced by H. Furusawa in [36] and also considered by M. Winter in [86, 87, 88], has been used before the concept of that of ideal were defined.

Definition 6.1.51. *[37] Let $\mathcal{D}$ be a Dedekind category. A morphism $\kappa : X \rightharpoonup X$ is called* ***scalar*** *if $\kappa \subseteq id_X$ and $\top\!\!\!\top_{XX} * \kappa = \kappa * \top\!\!\!\top_{XX}$.*

It is easy to prove that if $\alpha : X \rightharpoonup X$ is an ideal, then $\kappa = \alpha \cap id_X$ is a scalar morphism. Conversely, if $\kappa : X \rightharpoonup X$ is a scalar, then $\alpha = \kappa * \top\!\!\!\top_{XX}$ is an ideal.
In the Dedekind category Rel_L, an L-relation $\alpha : X \rightharpoonup X$ is a scalar if and only if $(x, x')\alpha = 0$, if $x \ne x'$ and there exists $p \in L$ such that $(x, x)\alpha = p$, for every $x \in X$. Also in our context, we can define the corresponding notion of scalar.

Definition 6.1.52. *Let $\boldsymbol{C}$ be a left (right, unital) bounded ordered pseudo-category. For every $X \in \mathcal{O}bj(\boldsymbol{C})$, a morphism $f \in \boldsymbol{C}(X,X)$ is called **scalar** if it satisfies the following conditions:*

1. $f \leq r_X$ $(f \leq l_X,\ f \leq i_X)$;
2. $\top\!\!\top_{XX} \circ f = f \circ_- \top\!\!\top_{XX}$.

Finally, we recall that in a Dedekind category a notion has been considered that is an abstract description of element of an object; in fact, an element can be determined by mapping the element of a singleton to that element. This idea is formalized as follows.

Definition 6.1.53. *[37] Let $\mathcal{D}$ be a Dedekind category and $X, J \in \mathcal{O}bj(\mathcal{D})$ such that J is a unit. A morphism $\alpha : J \rightharpoonup X$ is called **J-point** of X if it satisfies the conditions $\alpha^{\#} * \alpha \subseteq id_X$ and $id_J \subseteq \alpha * \alpha^{\#}$.*

The condition $0_{JJ} \neq id_J \subseteq \alpha * \alpha^{\#}$ assures that every J-point is non-zero. Hence, every empty object X, i.e. an object X such that $\nabla_{XX} = 0_{XX}$, has no J-point. Moreover, note that id_J is the only J-point of J. In the context of a pseudo-category, we give the following.

Definition 6.1.54. *Let $\boldsymbol{C}$ be a left (right, unital, respectively) bounded ordered pseudo-category and $X, J \in \mathcal{O}bj(\boldsymbol{C})$ such that J is a singleton. A morphism $f \in \boldsymbol{C}(J,X)$ is called J-point if $(f_- \circ f \leq l_X$ and $f \circ f_- \geq l_J)((f_- \circ f \leq r_X$ and $f \circ f_- \geq r_J)$, $(f_- \circ f \leq i_X$ and $f \circ f_- \geq i_J)$, respectively).*

In the complete ordered unital pseudo-category $(\mathbf{R}_{\mathbf{L}}^{\odot}; \leq)$ of binary relations with values in a symmetrical cdeo algebra $(L, \to, \top)$, the J-points of X are the functional relations from J to X, according to the definition given in Subsection 3.3.2.

6.1.5 Examples and remarks

At this point, we have the data and the informations necessary to summarize the categorical aspects of the class $\mathbf{R}_{\mathbf{L}}$ of relations with values in some kind of extended-order algebra. In fact, we can consider $\mathbf{R}_{\mathbf{L}}^{\circ_1,\dots,\circ_n} = (\mathcal{O}bj(\mathbf{R}_{\mathbf{L}}^{\circ_1,\dots,\circ_n}), \mathcal{M}or(\mathbf{R}_{\mathbf{L}}^{\circ_1,\dots,\circ_n}), \circ_1, \dots, \circ_n)$, where $\mathcal{O}bj(\mathbf{R}_{\mathbf{L}}^{\circ_1,\dots,\circ_n}) = |\mathbf{Set}|$, $\mathcal{M}or(\mathbf{R}_{\mathbf{L}}^{\circ_1,\dots,\circ_n}) = \mathbf{R}_{\mathbf{L}}$ and $\circ_i$ is one of partial compositions defined in Section 3.1, for $i = 1, \dots, n$.
The properties just listed and proved allow to obtain these results.

Proposition 6.1.55.

1. *If $(L, \to, \top)$ is a w-ceo algebra, then:*
   (a) *$\boldsymbol{R}_{\boldsymbol{L}}^{\oslash}$, $\boldsymbol{R}_{\boldsymbol{L}}^{\otimes}$ and $\boldsymbol{R}_{\boldsymbol{L}}^{\ominus}$ are pseudo-categories, neither right nor left, in general;*
   (b) *$(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash}; [\cdot]_-)$ and $(\boldsymbol{R}_{\boldsymbol{L}}^{\otimes}; [\cdot]_-)$ are reversible pseudo-categories, one the reverse of the other, where the involution is the unary operation of opposite L-relation.*
2. *If $(L, \to, \top)$ is a w-ceo algebra such that $\top \to a = a$, for every $a \in L$, then:*

(a) $\boldsymbol{R}_{\boldsymbol{L}}^{\oslash}$ *is a left pseudo-category, where the identity morphisms are the identity L-relations* $\mathcal{I}_X$*, for every* $X \in |\boldsymbol{Set}|$*;*

(b) $\boldsymbol{R}_{\boldsymbol{L}}^{\otimes}$ *is a right pseudo-category, where the identity morphisms are the identity L-relations* $\mathcal{I}_X$*, for every* $X \in |\boldsymbol{Set}|$*.*

3. *If* $(L, \rightarrow, \top)$ *is a right w-ceo algebra, then* $(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash}; \leq)$ *is a complete ordered pseudo-category, where the top and the bottom element are* $\top\!\!\top_{XY}$ *and* $\perp\!\!\perp_{XY}$*, for all* $X, Y \in |\boldsymbol{Set}|$*.*

4. *If* $(L, \rightarrow, \top)$ *is a ceo algebra, then:* $(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash}; \leq, [\cdot]_{-})$ *is a complete implicative relational pseudo-category.*

5. *If* $(L, \rightarrow, \top)$ *is a right-distributive w-ceo algebra, then:*

   (a) $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot}; [\cdot]_{-})$*,* $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot^{op}}; [\cdot]_{-})$ *are respectively right and left reversible pseudo-categories where the identity morphisms are the identity L-relations and where the involution is the unary operation of opposite L-relation; moreover, they are reverse of the other;*

   (b) $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot}; \leq)$ *is a complete ordered pseudo-category, where the top and the bottom element are* $\top\!\!\top_{XY}$ *and* $\perp\!\!\perp_{XY}$*, for all* $X, Y \in |\boldsymbol{Set}|$*;*

   (c) $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot}; \leq, [\cdot]_{-})$ *is a multiplicative relational pseudo-category;*

   (d) $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot,\oslash}; \leq, [\cdot]_{-})$ *is a residuated pseudo-bicategory;*

   (e) *if, moreover,* $\top \rightarrow a = a$*, for every* $a \in L$*, then* $\boldsymbol{R}_{\boldsymbol{L}}^{\odot}$ *is a unital pseudo-category.*

6. *If* $(L, \rightarrow, \top)$ *is a right-distributive ceo algebra, then:*

   (a) $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot^{op}}; \leq)$ *is a complete ordered pseudo-category, where the top and the bottom element are* $\top\!\!\top_{XY}$ *and* $\perp\!\!\perp_{XY}$*, for all* $X, Y \in |\boldsymbol{Set}|$*;*

   (b) $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot^{op}}; \leq, [\cdot]_{-})$ *is a multiplicative relational pseudo-category.*

7. *If* $(L, \rightarrow, \top)$ *is a symmetrical cdeo algebra, then:*

   (a) $(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash}; \leq, [\cdot]_{-})$ *and* $(\boldsymbol{R}_{\boldsymbol{L}}^{\tilde{\oslash}}; \leq, [\cdot]_{-})$ *are complete implicative relational left pseudo-categories; their reverse* $(\boldsymbol{R}_{\boldsymbol{L}}^{\otimes}; \leq, [\cdot]_{-})$ *and* $(\boldsymbol{R}_{\boldsymbol{L}}^{\tilde{\otimes}}; \leq, [\cdot]_{-})$ *are right reversible pseudo-categories;*

   (b) $(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash,\tilde{\oslash}}; \leq, [\cdot]_{-})$ *is a symmetrical pseudo-bicategory;*

   (c) $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot}; \leq, [\cdot]_{-})$ *and* $(\boldsymbol{R}_{\boldsymbol{L}}^{\tilde{\odot}}; \leq, [\cdot]_{-})$ *are complete multiplicative relational unital pseudo-categories, the one reverse of the other;*

   (d) $(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash,\odot,\tilde{\oslash}}; \leq, [\cdot]_{-})$ *is a triangular pseudo-tricategory.*

8. *If* $(L, \rightarrow, \top)$ *is a symmetrical associative cdeo algebra, then* $\boldsymbol{R}_{\boldsymbol{L}}^{\odot}$ *and* $\boldsymbol{R}_{\boldsymbol{L}}^{\tilde{\odot}}$ *are categories.*

Proof. The proof of all statements is an easy consequence of properties proved in Propositions 3.1.6, 3.1.10 and 3.1.14. □

Proposition 6.1.56. *Let $(L, \rightarrow, \top)$ be a symmetrical cdeo algebra. All crisp L-relations are crisp morphisms in $(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash,\tilde{\oslash}}; \leq, [\cdot]_-)$. In particular, $\perp\!\!\!\perp_{XY}$ and $\top\!\!\!\top_{XY}$ are crisp morphisms, for all $X, Y \in |\boldsymbol{Set}|$.*

Proof. Let $r \in \mathbf{R_L}(X, Z)$, r crisp, i.e. $(x, z)r \in \{\perp, \top\}$, for all $x \in X, z \in Z$. Obviously, if $(x, z)r = \perp$, the equivalence $(x, z)r \leq (x, z)(\mathcal{R} \oslash \mathcal{S}) \Leftrightarrow (x, z)r \leq (x, z)(\mathcal{R}\tilde{\oslash}\mathcal{S})$ is true, for all $\mathcal{R} \in \mathbf{R_L}(X, Y), \mathcal{S} \in \mathbf{R_L}(Y, Z)$.
Moreover, recalling that $\rightarrow$ and $\rightsquigarrow$ satisfy the condition $\top \leq a \rightarrow b \Leftrightarrow \top \leq a \rightsquigarrow b$, for all $a, b \in L$ if $(x, z)r = \top$, we have that:
$\top \leq (x, z)(\mathcal{R} \oslash \mathcal{S}) \Leftrightarrow \top \leq \bigwedge_{y \in Y}(x, y)\mathcal{R} \rightarrow (y, z)\mathcal{S} \Leftrightarrow \top \leq (x, y)\mathcal{R} \rightarrow (y, z)\mathcal{S}, \forall y \in Y \Leftrightarrow$ $\top \leq (x, y)\mathcal{R} \rightsquigarrow (y, z)\mathcal{S}, \forall y \in Y \Leftrightarrow \top \leq \bigwedge_{y \in Y}(x, y)\mathcal{R} \rightsquigarrow (y, z)\mathcal{S} \Leftrightarrow \top \leq (x, z)(\mathcal{R}\tilde{\oslash}\mathcal{S})$. Hence, the statement is proved. □

Proposition 6.1.57. *Let $(L, \rightarrow, \top)$ be a w-ceo algebra. The following hold.*

1. *$\top\!\!\!\top_{XY}$ is an ideal of the bounded ordered reversible pseudo-category $(\boldsymbol{R}_L^{\oslash}; \leq, [\cdot]_-)$.*
2. *If $\top \rightarrow a = a$, for every $a \in L$, then:*
   - (a) *the constant L-relations in the left bounded ordered reversible pseudo-category $(\boldsymbol{R}^{\oslash}; \leq, [\cdot]_-)$ are right ideals. In particular, $\perp\!\!\!\perp_{XY}$ is a right ideal.*
   - (b) *The singletons of $(\boldsymbol{R}_{\boldsymbol{L}}^{\oslash}; \leq, [\cdot]_-)$ are the singletons sets.*
   - (c) *A relation $\mathcal{R} : \{j\} \times X \rightarrow L$ is a $\{j\}$-point if and only if $(j, x)\mathcal{R} \rightarrow (j, x')\mathcal{R} = \perp$, for all $x, x' \in X, x \neq x'$.*

Proof. Let $(L, \rightarrow, \top)$ be a w-ceo algebra.

1. Let $X, Y \in |\mathbf{Set}|$; for all $x \in X, y \in Y$, we have that: $(x, y)(\top\!\!\!\top_{XX} \oslash \top\!\!\!\top_{XY}) = \bigwedge_{x' \in X}(x, x')\top\!\!\!\top_{XX} \rightarrow (x', y)\top\!\!\!\top_{XY} = \bigwedge_{x' \in X} \top \rightarrow \top = \top = (x, y)\top\!\!\!\top_{XY}$.
   Similarly we can prove $\top\!\!\!\top_{XY} \oslash \top\!\!\!\top_{YY} = \top\!\!\!\top_{XY}$.
2. Assume that $\top \rightarrow a = a$, for every $a \in L$. By Proposition 6.1.55, $(\mathbf{R}_{\mathbf{L}}^{\oslash}; \leq, [\cdot]_-)$ is a left bounded ordered reversible pseudo-category.
   - (a) Let $X, Y \in |\mathbf{Set}|$ and let $\mathcal{R} : X \times Y \rightarrow L$ be a relation such that, for all $x \in X, y \in Y$ $(x, y)\mathcal{R} = a$, $a \in L$.
     Hence $(x, y)(\top\!\!\!\top_{XX} \oslash \mathcal{R}) = \bigwedge_{x' \in X} \top \rightarrow a = \bigwedge_{x' \in X} a = a = (x, y)\mathcal{R}$.
   - (b) Let $J = \{j\}$; hence $(j, j)\top\!\!\!\top_{JJ} \leq (j, j)\mathcal{I}_J$. Moreover, it is clear that for all $X \in |\mathbf{Set}|, x, x' \in X$: $(x, x')(\top\!\!\!\top_{XJ} \oslash \top\!\!\!\top_{JX}) = (x, x')\top\!\!\!\top_{XX}$.
     Conversely, if J is a singleton object, then, for all $x, x' \in J$ one has $\top = (x, x')\top\!\!\!\top_{JJ} \leq (x, x')\mathcal{I}_J \Rightarrow x = x'$, hence J has to be a singleton set.
   - (c) Let $\mathcal{R} : \{j\} \times X \rightarrow L$ be an L-relation such that $(j, x)\mathcal{R} \rightarrow (j, x')\mathcal{R} = \perp$, for all $x, x' \in X, x \neq x'$.
     For all $x, x' \in X$, the following hold:

- $(x,x')(\mathcal{R}_- \oslash \mathcal{R}) = \bigwedge_{j\in J}(x,j)\mathcal{R}_- \to (j,x')\mathcal{R} = (j,x)\mathcal{R} \to (j,x')\mathcal{R}$; if $x = x'$, $(j,x)\mathcal{R} \to (j,x')\mathcal{R} = \top$, while, by assumption, if $x \neq x'$, $(j,x)\mathcal{R} \to (j,x')\mathcal{R} = \bot$.
  In any case, we have that $(x,x')(\mathcal{R}_- \oslash \mathcal{R}) \leq \mathcal{I}_X$.
- $(j,j)(\mathcal{R} \oslash \mathcal{R}_-) = \bigwedge_{x\in X}(j,x)\mathcal{R} \to (x,j)\mathcal{R}_- = \bigwedge_{x\in X}(j,x)\mathcal{R} \to (j,x)\mathcal{R} = \top \geq (j,j)\mathcal{I}_J$.

Hence $\mathcal{R}$ is a J-point.
The converse can be easily proved.

□

Proposition 6.1.58. *Let $(L,\to,\top)$ be a right-distributive ceo algebra. The following hold.*

1. *$\perp\!\!\!\perp_{XY}$ and $\top\!\!\!\top_{XY}$ are ideals of the right bounded ordered reversible pseudo-category $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot};\leq,[\cdot]_-)$.*

2. *The constant relations are left ideals of the right bounded ordered reversible pseudo-category $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot};\leq,[\cdot]_-)$.*

3. *If $\top \to a = a$, for every $a \in L$, then:*

   (a) the constant L-relation of the unital bounded ordered reversible pseudo-category $(\boldsymbol{R}_{\boldsymbol{L}}^{\odot};\leq,[\cdot]_-)$ are ideals.
   (b) The singleton of $(\boldsymbol{R}_{L}^{\odot};\leq,[\cdot]_-)$ are the singleton sets.
   (c) A relation $\mathcal{R} : \{j\} \times X \to L$ is a $\{j\}$-point if $(j,x)\mathcal{R} \otimes (j,x')\mathcal{R} = \bot$, for all $x,x' \in X, x \neq x'$ and there exists $\bar{x} \in X$ such that $(j,\bar{x})\mathcal{R} = \top$.

Proof. Let $(L,\to,\top)$ be a right-distributive ceo algebra.

1. Let $X,Y \in |\mathbf{Set}|$; for all $x \in X, y \in Y$ we have that: $(x,y)(\top\!\!\!\top_{XX}\odot\top\!\!\!\top_{XY}) = \bigvee_{x'\in X}(x,x')\top\!\!\!\top_{XX}\otimes(x',y)\top\!\!\!\top_{XY} = \bigvee_{x'\in X}\top\otimes\top = \top = (x,y)\top\!\!\!\top_{XY}$. So, $\top\!\!\!\top_{XY}$ is a right ideal; similarly we can prove that $\top\!\!\!\top_{XY}$ is a left ideal, too.
   Moreover, $(x,y)(\top\!\!\!\top_{XX}\odot\perp\!\!\!\perp_{XY}) = \bigvee_{x'\in X}(x,x')\top\!\!\!\top_{XX}\otimes(x',y)\perp\!\!\!\perp_{XY} = \bigvee_{x'\in X}\top\otimes\bot = \bot = (x,y)\perp\!\!\!\perp_{XY}$. So, $\perp\!\!\!\perp_{XY}$ is a right ideal; similarly, we can prove that $\perp\!\!\!\perp_{XY}$ is a left ideal, too.

2. Let $X,Y \in |\mathbf{Set}|$ and let $\mathcal{R} : X \times Y \to L$ be a relation such that, for all $x \in X, y \in Y$ $(x,y)\mathcal{R} = a$, $a \in L$. Hence $(x,y)(\mathcal{R}\odot\top\!\!\!\top_{YY}) = \bigvee_{y'\in Y} a \otimes \top = \bigwedge_{x'\in X} a = a = (x,y)\mathcal{R}$. Similarly, we can prove the other equality that characterizes the left ideal, recalling that a reverse of constant relation is constant, too.

3. Assume that the $\top \to a = a$ is satisfied for every $a \in L$. This is equivalent to say that $\top$ is a left and right neutral with respect to the adjoint product. By Proposition 6.1.55, $(\mathbf{R}_{L}^{\odot};\leq,[\cdot]_-)$ is a unital bounded ordered reversible pseudo-category.

(a) Let $X, Y \in |\mathbf{Set}|$ and let $\mathcal{R} : X \times Y \to L$ be a relation such that, for all $x \in X, y \in Y$ $(x,y)\mathcal{R} = a$, $a \in L$. Hence $(x,y)(\Pi_{XX} \odot \mathcal{R}) = \bigwedge_{x' \in X} \top \otimes a = \bigwedge_{x' \in X} a = a = (x,y)\mathcal{R}$. Similarly, we can prove the other equality that characterizes the ideal.

(b) Let $J = \{j\}$; hence $(j,j)\Pi_{JJ} \leq (j,j)\mathcal{I}_J$. Moreover, it is clear that for every $X \in |\mathbf{Set}|, x, x' \in X$: $(x,x')(\,\Pi_{XJ} \odot \Pi_{JX}) = (x,x')\Pi_{XX}$.

(c) Let $\mathcal{R} : \{j\} \times X \to L$ be an L-relation such that $(j,x)\mathcal{R} \otimes (j,x')\mathcal{R} = \bot$, for all $x, x' \in X, x \neq x'$ and there exists $\bar{x} \in X$ such that $(j,\bar{x})\mathcal{R} = \top$. For all $x, x' \in X$, the following hold:

- $(x,x')(\mathcal{R}_{-} \odot \mathcal{R}) = \bigvee_{j \in J}(x,j)\mathcal{R}_{-} \otimes (j,x')\mathcal{R} = (j,x)\mathcal{R} \otimes (j,x')\mathcal{R}$.
  If $x = x'$, $(j,x)\mathcal{R} \to (j,x')\mathcal{R} \leq \top$, while, by assumption, if $x \neq x'$, $(j,x)\mathcal{R} \otimes (j,x')\mathcal{R} = \bot$. In any case, we have that $(x,x')(\mathcal{R}_{-} \odot \mathcal{R}) \leq \mathcal{I}_X$.
- By assumption, we have that:
  $(j,j)(\mathcal{R} \odot \mathcal{R}_{-}) = \bigvee_{x \in X}(j,x)\mathcal{R} \otimes (x,j)\mathcal{R}_{-} = \bigvee_{x \in X}(j,x)\mathcal{R} \otimes (j,x)\mathcal{R} \geq (j,\bar{x})\mathcal{R} \otimes (j,\bar{x})\mathcal{R} = \top \geq (j,j)\mathcal{I}_J$.

Hence $\mathcal{R}$ is a J-point.

□

Remark 6.1.59. By using one of the equivalences that characterize the $\top$-including Galois L-triangle of the Proposition 5.1.7, we can deduce the property (D_4) of Dedekind categories with respect to the partial compositions $\odot$ and $\oslash$. In fact, since the diagram of Definition 6.1.36 is a $\top$-including Galois L-triangle, then the following equivalence holds, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$:

$$(\tau, \rho_{-} \oslash \sigma_{-})\mathcal{S}_{X \times Y} = \top \Leftrightarrow ((\rho \odot \tau)_{-}, \sigma)\mathcal{S}_{Y \times Z} = \top$$

that is equivalent to

$$\tau \leq \rho_{-} \oslash \sigma_{-} \Leftrightarrow (\rho \odot \tau)_{-} \leq \sigma.$$

So, for all $\rho \in \mathbf{R_L}(Z,X), \sigma \in \mathbf{R_L}(Y,Z), \tau \in \mathbf{R_L}(X,Y)$:

$$\tau \leq \rho_{-} \oslash \sigma_{-} \Leftrightarrow \rho \odot \tau \leq \sigma_{-}$$

that can be rewritten, considering $\alpha = \rho_{-}$, $\beta = \sigma_{-}$ and $\gamma = \tau$ in this way

$$\gamma \leq \alpha \oslash \beta \Leftrightarrow \alpha_{-} \odot \gamma \leq \beta.$$

Remark 6.1.60. We recall the notions of **fuzzy-relation algebra** and **MV-relation algebra**, as defined in [71, 72].
A **fuzzy-relation algebra** (FRA) is a structure $(A, \vee, \wedge, \odot, \to, 0, 1, ;, ^{\smile}, \Delta)$ such that:
(A_0) $(A, \vee, \wedge, \odot, \to, 0, 1)$is a BL-algebra;
(A_1) $(A, ;, \Delta)$ is a monoid;
(A_2) the **cycle law** holds: $\forall x, y, z \in A$, $(x;y) \odot z = 0 \Leftrightarrow (x^{\smile}; z) \odot y = 0 \Leftrightarrow (z; y^{\smile}) \odot x = 0$.
Moreover, an **MV-relation algebra** (MVRA) is a structure $(A, \oplus, \odot, ^{-}, 0, 1, ;, ^{\smile}, \Delta)$ such

that:
(A_0) $(A, \oplus, \odot, ^-, 0, 1)$ is an MV-algebra;
(A_1) $(A, ;, \Delta)$ is a monoid;
(A_2) the **cycle law** holds: $\forall x, y, z \in A,\ (x; y) \odot z = 0 \Leftrightarrow (x^{\smile}; z) \odot y = 0 \Leftrightarrow (z; y^{\smile}) \odot x = 0$;
(A_3) $(x \oplus y)^{\smile} = x^{\smile} \oplus y^{\smile}$;
(A_4) $(a \odot x); (b \odot y) \leq (a; b) \odot (x; y)$;
(A_5) $\Delta \odot \Delta = \Delta$.

Assuming that $\perp$ is $\otimes$-irreducible, by means of the equivalences that characterize the $\perp$-including Tarski L-triangle of the Proposition 5.1.18 (2), we can deduce the generalization of the cycle law required in an MV-relation algebra and in a fuzzy-relation algebra. In fact, since the previous diagram is a $\perp$-including Tarski L-triangle, the following hold, for all $\rho \in \mathbf{R_L}(Z, X), \sigma \in \mathbf{R_L}(Y, Z), \tau \in \mathbf{R_L}(X, Y)$:

$$(\rho, (\tau \odot \sigma)_-)\mathcal{T}_{Z\times X} = \perp \Leftrightarrow ((\rho \odot \tau)_-, \sigma)\mathcal{T}_{Y\times Z} = \perp \Leftrightarrow (\tau, \rho_- \odot \sigma_-)\mathcal{T}_{X\times Y} = \perp.$$

Hence, for all $\rho \in \mathbf{R_L}(Z, X), \sigma \in \mathbf{R_L}(Y, Z), \tau \in \mathbf{R_L}(X, Y)$:

$$\rho \otimes (\tau \odot \sigma)_- = \perp \Leftrightarrow (\rho \odot \tau)_- \otimes \sigma = \perp \Leftrightarrow \tau \otimes (\rho_- \odot \sigma_-) = \perp$$

that are equivalent to the following, for all $\rho \in \mathbf{R_L}(Z, X), \sigma \in \mathbf{R_L}(Y, Z), \tau \in \mathbf{R_L}(X, Y)$:

$$\rho_- \otimes (\tau \odot \sigma) = \perp \Leftrightarrow (\tau_- \tilde{\odot} \rho_-) \otimes \sigma = \perp \Leftrightarrow \tau \otimes (\rho_- \odot \sigma_-) = \perp.$$

So, considering $\alpha = \tau$, $\beta = \sigma$ and $\gamma = \rho_-$ the following equivalences hold, for all $\gamma \in \mathbf{R_L}(X, Z), \beta \in \mathbf{R_L}(Y, Z), \alpha \in \mathbf{R_L}(X, Y)$:

$$\gamma \otimes (\alpha \odot \beta) = \perp \Leftrightarrow (\alpha_- \tilde{\odot} \gamma) \otimes \beta = \perp \Leftrightarrow \alpha \otimes (\gamma \odot \beta_-) = \perp.$$

The algebra of relation traditionally depends on the principle of the logics involved in the underlying set (or class) theory and reflects the structures assumed in the set of the corresponding truth values. In classical logic, relation algebras have been studied and characterized as algebras of binary homogeneous relations, that are boolean algebras with operators (see [61]). An approach that deals directly with heterogeneous binary relations has also developed starting from [69]. Of course, the big development of many-valued logics and mathematics in the last decades and the relevance of relations in pure and applied sciences have favored and increased new appropriate approaches to many-valued binary relations, their algebra and theirs structures including the corresponding operators (see [6, 87]). An abstract approach to algebras of homogeneous relations taking values in MV-algebras is developed in [71, 72], while an abstract approach to heterogeneous relation with values in Heyting algebras is developed in [37, 87]. More generally, the algebra of relations, their compositions and corresponding operators have been approached in more general contexts dealing with complete residuated lattices [6, 4, 35]. This thesis extends the framework to the more general context of w-eo algebras, which is intended as a first step toward a characterization of algebras and categories of binary relations in such more general context.

As we have already remarked, the concept introduced and studied in this Chapter are intended as a general framework where we would like to develop an abstract approach to categories of relations with values in some kind of extended-order algebra; this would be the topic of our future work.

6.2 Composition graphs and connections

Structures made up of objects, morphisms and composition operation that in general are neither categories nor pseudo-categories are studied, among others, by C. Ehrsmann [31], H. Herrlich and L. Schöder [53, 54, 76]. In the following we recall the basic notions of these structures considered in the works of these authors. Now, we recall the basic notions according to the approach and terminology considered in [76].

Definition 6.2.1. *A **composition graph** is a triple* $\mathscr{C} = (\mathcal{O}bj(\mathscr{C}), \mathcal{M}or(\mathscr{C}), \cdot)$ *defined by the following.*

1. $\mathcal{O}bj(\mathscr{C})$ *is a class of objects denoted by capital letters* $A, B, ...$*;*
2. $\mathcal{M}or(\mathscr{C})$ *is a class of morphism and there exist a function* $\mathscr{C} : \mathcal{O}bj(\mathscr{C}) \times \mathcal{O}bj(\mathscr{C}) \to \mathcal{P}(\mathcal{M}or(\mathscr{C}))$ *that maps every pair of objects* (A, B) *into a set* $\mathscr{C}(A, B)$*, whose elements are called morphisms and are denoted by* $f : A \to B$*; then the object* $A = domf$ *is called domain of* f *and the object* $B = codomf$ *is called codomain of* f*. Moreover,* $\{\mathscr{C}(A, B) | A, B \in \mathcal{O}bj(\mathscr{C}), \mathscr{C}(A, B) \neq \varnothing\}$ *is a partition of* $\mathcal{M}or(\mathscr{C})$*.*
3. *A function* $id : \mathcal{O}bj(\mathscr{C}) \to \mathcal{M}or(\mathscr{C})$ *that associates to each object* A *a morphism* $i_A \in \mathscr{C}(A, A)$*.*
4. *A subclass* $\mathscr{K}$ *of* $\mathcal{M}or(\mathscr{C}) \times \mathcal{M}or(\mathscr{C})$ *and an operation of composition* $\cdot : \mathscr{K} \to \mathcal{M}or(\mathscr{C})$ *that maps any pair* $(f, g) \in \mathscr{K}$ *into* $f \cdot g \in \mathcal{M}or(\mathscr{C})$*. Moreover, the following implication holds:* $(f, g) \in \mathscr{K} \Rightarrow domg = codomf$*,* $domf \cdot g = domf$*,* $codomf \cdot g = codomg$*.*

We note that in a category, the converse implication of the last condition (4) holds too.
We observe that no axiom concerning the identity law and the associativity law are assumed, in general, in a composition graph, but some of the following conditions may be considered.

Definition 6.2.2. *Let* $\mathscr{C}$ *be a composition graph. Then the following conditions may be satisfied.*
(wi) *If* $f \in \mathscr{C}(A, B)$*,* $f \cdot i_B$ *and* $i_A \cdot f$ *are defined in* $\mathscr{C}$ *, then* $f \cdot i_B = i_A \cdot f = f$ *(**weak identity law**).*
(si) *If* $f \in \mathscr{C}(A, B)$*, then both* $f \cdot i_B$ *and* $i_A \cdot f$ *are defined in* $\mathscr{C}$ *and* $f \cdot i_B = i_A \cdot f = f$ *(**strong identity law**).*
(wa) *If* $f, g, h \in \mathcal{M}or(\mathscr{C})$*,* $f \cdot g$ *and* $g \cdot h$ *are defined in* $\mathscr{C}$ *and if one of the compositions* $f \cdot (g \cdot h)$ *or* $(f \cdot g) \cdot h$ *is defined in* $\mathscr{C}$*, then there exists also the other one and* $(f \cdot g) \cdot h =$

$f \cdot (g \cdot h)$ *(**weak associativity law**).*
(sa) *If $f, g, h \in \mathcal{M}or(\mathscr{C})$, $f \cdot g$ and $g \cdot h$ are defined in $\mathscr{C}$, then both $f \cdot (g \cdot h)$ and $(f \cdot g) \cdot h$ are defined in $\mathscr{C}$ and $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ (**strong associativity law**).*

Remark 6.2.3. 1. It is possible to consider two weaker versions of the weak associativity law; in fact, we may speak of **right weak associativity law**, if we consider the condition:
$(wa)_r$ if $f, g, h \in \mathcal{M}or(\mathscr{C})$ and $f \cdot g$, $g \cdot h$ and $f \cdot (g \cdot h)$ are defined in $\mathscr{C}$, then there exists $(f \cdot g) \cdot h$ and $(f \cdot g) \cdot h = f \cdot (g \cdot h)$
and of **left weak associativity law** if we consider the condition:
$(wa)_l$ if $f, g, h \in \mathcal{M}or(\mathscr{C})$ and $f \cdot g$, $g \cdot h$ and $(f \cdot g) \cdot h$ are defined in $\mathscr{C}$, then there exists $f \cdot (g \cdot h)$ and $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

2. A composition graph that satisfies the strong identity law is a **multiplicative graph** considered by Ehresmann [31].

3. Each of the strong versions of the condition considered in Definition 6.2.2 implies its weak version.

Proposition 6.2.4. *A composition graph is a category if and only if it satisfies (si) and (sa).*

Proof. The conditions (si) and (sa) are clearly necessary. Conversely, assume that those are satisfied. Then, for $f : A \to B$ and $g : B \to C$, consider $i_B \in \mathscr{C}(B, B)$. By (si), both $f \cdot i_B$ and $i_B \cdot g$ exist and $f \cdot i_B = f$ and $i_B \cdot g = g$. Then, by (sa), $(f \cdot i_B) \cdot g = f \cdot g$ exists. Now, it becomes clear that the composition is associative and every identity morphism is neutral for the composition. □

Definition 6.2.5. *[76] A composition graph is called **precategory** if it satisfies the strong identity law and the weak associativity law and it is called **semicategory** if it satisfy the weak identity law and the strong associativity law.*

Both precategories and semicategories are proper generalizations of categories, as subsequent examples we shall describe show.
First, we recall some of the examples described in [76].

Example 6.2.6. 1. Let $\mathbf{A}$ and $\mathbf{B}$ two partial algebras of the same signature; a homomorphism $f : \mathbf{A} \to \mathbf{B}$ is called *full* if, for every operation σ of arity n in the signature, the definedness of $((x_1)f, \ldots, (x_n)f)\sigma = (x)f$ in $\mathbf{B}$ implies that there exist $x'_1, \ldots, x'_n$ and x' such that $(x'_1, \ldots, x'_n)\sigma = x'$ is defined in $\mathbf{A}$, $(x')f = (x)f$ and $(x'_i)f = (x_i)f$, for $i = 1, \ldots, n$. We note that in [76] it is shown that the composition of full homomorphisms need not be full. The structure where the objects are the partial algebras, the morphisms are the full homomorphisms equipped with the composition of functions is a precategory.

2. Let ρ and σ be two n-ary relations on sets X and Y, respectively. A relation morphism $f : (X, \rho) \to (Y, \sigma)$ is called *full* if, whenever $(x_1, \ldots, x_n)f \in \sigma$, there

exists $(x'_1, \dots, x'_n) \in \rho$ such that $(x'_i)f = (x_i)f$, for $i = 1, \dots, n$. The structure, denoted by **fRel(n)**, whose objects are the structured sets (X, ρ), for every set X and every $\rho \subseteq X^n$, whose morphisms are the full relation morphisms and whose composition is the composition of functions is a precategory.

3. Let X and Y be two probability spaces; a function $f : X \to Y$ is called *probability presentation morphism* if $P([B]f^{-1}) = P(B|[X]f)$, for every $B \subset Y$ where P denotes the probability measure on X as well as on Y. The structure **Prob** where the objects are the probability spaces, the morphisms are the probability presentation morphisms with the composition of functions is a precategory.

4. Let **C** be a category and let $\sim$ be a congruence on **A** (i.e. if $f \sim g$, for $f : A \to B$ and $g : C \to D$, then $A \sim C$ and $B \sim D$, and if $f \sim f'$, $g \sim g'$ and fg and $f'g'$ are both defined, then $fg \sim f'g'$). Then the equivalence classes modulo $\sim$ equipped with the induced domain and codomain assignments and the induced composition form a composition graph, that, in general, is neither a precategory nor a semicategory.

Now, we reconsider the notions of global and relative connections introduced in Section 4.2, to obtain new, meaningful examples of precategories and semicategories.
First of all, we note that if one considers the set L with an equivalence $\equiv$ and the structured set (X, α), where X is a set and α is an L-relation on X, then the following connections $(i_X, i_X) : (X, \alpha) \to (X, \alpha)$ and $)i_X, i_X(: (X, \alpha) \to (X, \alpha)$ are obtained. Moreover, if α is a symmetrical L-relation (i.e. for every $x, x' \in X$, $(x, x')\alpha = (x', x)\alpha$), then further connections $[i_X\text{-}i_X] : (X, \alpha) \to (X, \alpha)$ and $]i_X\text{-}i_X[: (X, \alpha) \to (X, \alpha)$ are obtained, too.
In fact, for every $x, x' \in X$: $((x)i_X, x')\alpha = (x, x')\alpha = (x, (x')i_X)\alpha$, i.e. one has (i_X, i_X) and $(x', (x)i_X)\alpha = (x', x)\alpha = ((x')i_X, x)\alpha$, which implies $)i_X, i_X($; moreover, the equalities $(x', (x)i_X)\alpha = (x, (x')i_X)\alpha$ and $((x)i_X, x')\alpha = ((x')i_X, x)\alpha$, for all $x, x' \in X$, can be obtained, if α is symmetrical, which provides the connections $[i_X\text{-}i_X]$ and $]i_X\text{-}i_X[$, respectively.
Clearly, similar statements hold for relative connections; hence, for every $E \subseteq L$, $(i_X, i_X)_E : (X, \alpha) \to (X, \alpha)$ and $)i_X, i_X(_E : (X, \alpha) \to (X, \alpha)$, while $[i_X\text{-}i_X]_E : (X, \alpha) \to (X, \alpha)$ and $]i_X\text{-}i_X[_E : (X, \alpha) \to (X, \alpha)$, if α is symmetrical.
Now we consider, for a fixed set $(L, \equiv)$ with an equivalence relation, the class of pairs $\mathcal{O}bj(\mathbf{Con}) = \{(X, \alpha) |\ X \in |\mathbf{Set}|,\ \alpha \in L^{X \times X}\}$, whose elements we consider as objects. Connections may be considered as morphisms between such a kind of objects; denote $\mathcal{M}or(\mathbf{Con})$ the class of such morphisms and $\mathbf{Con}((X, \alpha), (Y, \beta))$ the set of connections of any kind which are suitable pairs of functions satisfying the required conditions.
To proceed, we approach the composition between connections. For this goal, we consider the usual composition of functions. It is clear that given three structured sets (X, α), (Y, β) and (Z, γ), $f : X \to Y$ and $g : Y \to X$ maps that form a connection from (X, α) to (Y, β) of some kind and $h : Y \to Z$ and $k : Z \to Y$ maps that form a connection from (Y, β) to (Z, γ) of some kind, we can compose f with h and k with g, but the pair $\langle f \cdot h, k \cdot g \rangle$ is not necessarily a connection of any kind.

In the following Propositions we describe all the compositions that lead surely to some kind of connection by means of component-wise composition of functions.

Proposition 6.2.7. *With the above notation, assuming* $[f\text{-}g] : (X\alpha) \to (Y,\beta)$*, the following hold.*

1. *If* $]h\text{-}k[: (Y,\beta) \to (Z,\gamma)$*, then there exists* $[f\text{-}g]\circ]h\text{-}k[: (X,\alpha) \to (Z,\gamma)$ *and* $[f\text{-}g]\circ]h\text{-}k[= (f\cdot h, k\cdot g) \in \mathbf{Con}((X,\alpha),(Z,\gamma))$.

2. *If* $)h,k(: (Y,\beta) \to (Z,\gamma)$*, then there exists* $[f\text{-}g]\circ)h,k(: (X,\alpha) \to (Z,\gamma)$ *and* $[f\text{-}g]\circ)h,k(= [f\cdot h, k\cdot g] \in \mathbf{Con}((X,\alpha),(Z,\gamma))$.

Proof. These statements can be easily proved, but we shall give an explicit verification in the general case of relational systems in the next Section. □

Similarly, the following results can be stated.

Proposition 6.2.8. *With the above notation, assuming* $]f\text{-}g[: (X\alpha) \to (Y,\beta)$*, the following hold.*

1. *If* $[h\text{-}k] : (Y,\beta) \to (Z,\gamma)$*, then there exists* $]f\text{-}g[\circ[h\text{-}k] : (X\alpha) \to (Z,\gamma)$ *and* $]f\text{-}g[\circ[h\text{-}k] =)f\cdot h, k\cdot g(\in \mathbf{Con}((X,\alpha),(Z,\gamma))$.

2. *If* $(h,k) : (Y,\beta) \to (Z,\gamma)$*, then there exists* $]f\text{-}g[\circ(h,k) : (X\alpha) \to (Z,\gamma)$ *and* $]f\text{-}g[\circ(h,k) =]f\cdot h, k\cdot g[\in \mathbf{Con}((X,\alpha),(Z,\gamma))$. □

Proposition 6.2.9. *With the above notation, assuming* $(f\text{-}g) : (X\alpha) \to (Y,\beta)$*, the following hold.*

1. *If* $[h\text{-}k] : (Y,\beta) \to (Z,\gamma)$*, then there exists* $(f,g)\circ[h\text{-}k] : (X\alpha) \to (Z,\gamma)$ *and* $(f,g)\circ[h\text{-}k] = [f\cdot h, k\cdot g] \in \mathbf{Con}((X,\alpha),(Z,\gamma))$.

2. *If* $(h,k) : (Y,\beta) \to (Z,\gamma)$*, then there exists* $(f,g)\circ(h,k) : (X\alpha) \to (Z,\gamma)$ *and* $(f,g)\circ(h,k) = (f\cdot h, k\cdot g) \in \mathbf{Con}((X,\alpha),(Z,\gamma))$. □

Proposition 6.2.10. *With the above notation, assuming* $)f\text{-}g(: (X\alpha) \to (Y,\beta)$*, the following hold.*

1. *If* $]h\text{-}k[: (Y,\beta) \to (Z,\gamma)$*, then there exists* $)f,g(\circ]h\text{-}k[: (X\alpha) \to (Z,\gamma)$ *and* $)f,g(\circ]h\text{-}k[=]f\cdot h, k\cdot g[\in \mathbf{Con}((X,\alpha),(Z,\gamma))$.

2. *If* $)h,k(: (Y,\beta) \to (Z,\gamma)$*, then there exists* $)f,g(\circ)h,k(: (X\alpha) \to (Z,\gamma)$ *and* $)f,g(\circ)h,k(=)f\cdot h, k\cdot g(\in \mathbf{Con}((X,\alpha),(Z,\gamma))$. □

Clearly, the same results hold for relative E-connections, for any $E \subseteq L$, with respect to the same componet-wise composition of pairs of functions.
Hence, the composition is a partial operation in the class of global and relative connections.

Remark 6.2.11. We note that given $(X\alpha)$, (Y,β), (Z,γ) and two pairs of functions $< f, g >$ and $< h, k >$ that form connections of any type from $(X\alpha)$ to (Y,β) and from (Y,β) to (Z,γ), respectively, the pair $< f \cdot h, k \cdot g >$ may be a connection of some type even if the condition of the above Propositions 6.2.7-6.2.10 do not occur; in fact, such Propositions give sufficient conditions that are not necessary, in general, to let two connections be composed.

Now the triple $\mathbf{Con} = (\mathcal{O}bj(\mathbf{Con}), \mathcal{M}or(\mathbf{Con}), \circ)$ with the class of objects and of morphisms and composition above defined is a composition graph, for which the function $id : \mathcal{O}bj(\mathbf{Con}) \to \mathcal{M}or(\mathbf{Con})$ is defined by $(X,\alpha)id = \langle i_X, i_X\rangle$.
Similarly, the composition graph $\mathbf{E\text{-}Con} = (\mathcal{O}bj(\mathbf{E\text{-}Con}), \mathcal{M}or(\mathbf{E\text{-}Con}), \circ)$ of connections relative to any subset $E \subseteq L$ can be defined, where $\mathcal{O}bj(\mathbf{E\text{-}Con}) = \mathcal{O}bj(\mathbf{Con})$, $\mathbf{E\text{-}Con}((X,\alpha),(Y,\beta)) = \{\langle f,g\rangle \mid f : X \to Y, g : Y \to X \ form\ an\ E-connection\ of\ any\ type\}$, $\circ$ is the component-wise composition and id is defined as for **Con**.
It is clear that for any $(X,\alpha) \in \mathcal{O}bj(\mathbf{Con})$ the pair of functions $\langle i_X, i_X\rangle$ is both a type III and a type IV connection.
Unlike the case of arbitrary morphisms, the composition of all types of connections having (X,α) either as domain or as codomain with the pair of identities functions $\langle i_X, i_X\rangle$ is still a connection of the same type.
More precisely, if the pair $\langle f,g\rangle$ is a connection of some kind from (X,α) to (Y,β), then the composition of such pair either on the left with the pair $\langle i_X, i_X\rangle$ or on the right with the pair $\langle i_Y, i_Y\rangle$ give the pair $\langle f,g\rangle$ itself, hence exactly the same connection.
Clearly, the same results hold for relative E-connections, for any $E \subseteq L$.
Hence, for all types of global and relative connections each pair of identity functions on any set is neutral with respect to the partial composition, so the strong identity law is satisfied.

Now, it is easy to show that the strong associativity law is not satisfied in **Con**. In fact, if we consider the connections $[f\text{-}g] : (X,\alpha) \to (Y,\beta)$, $(i_Y, i_Y) : (Y,\beta) \to (Y,\beta)$ and $[h\text{-}k] : (Y,\beta) \to (Z,\gamma)$, we have that $[f\text{-}g] \circ (i_Y, i_Y) = [f\text{-}g]$ and $(i_Y, i_Y) \circ [h\text{-}k] = [h\text{-}k]$ are defined, but neither $[f\text{-}g] \circ ((i_Y, i_Y) \circ [h\text{-}k])$ nor $([f\text{-}g] \circ (i_Y, i_Y)) \circ [h\text{-}k]$ are defined, in general. Similarly, **E-Con** does not satisfy the strong associativity law.
Instead, the weak associativity law holds.
To prove this, let us denote by $\langle f,g\rangle$ any connection, either global or relative, of some kind from (X,α) to (Y,β), where $f \in Y^X$, $g \in X^Y$ evidently satisfy the required conditions. Then the following holds.

Proposition 6.2.12. *With the above notation, let* $\langle f,g\rangle : (X,\alpha) \to (Y,\beta)$, $\langle h,k\rangle : (Y,\beta) \to (Z,\gamma)$, $\langle l,m\rangle : (Z,\gamma) \to (W,\delta)$ *be connections of any kind.*
If one of the compositions $(\langle f,g\rangle \circ \langle h,k\rangle) \circ \langle l,m\rangle$ *and* $\langle f,g\rangle \circ (\langle h,k\rangle \circ \langle l,m\rangle)$ *exists in* $\mathcal{M}or(\mathbf{Con})$*, then the other also exists and both compositions are equal to each other.*

Proof. Assuming that $(\langle f,g\rangle \circ \langle h,k\rangle) \circ \langle l,m\rangle$ exists means that the functions $(f \cdot h) \cdot l$ and $m \cdot (k \cdot g)$ form a connection $\langle (f \cdot h) \cdot l, m \cdot (k \cdot g)\rangle$ of some kind; by the associativity of the composition of functions this is equivalent to saying that $f \cdot (h \cdot l)$ and $(m \cdot k) \cdot g$ form the same connection, which is the composition $\langle f, g\rangle \circ (\langle h, k\rangle \circ \langle l, m\rangle)$. □

Corollary 6.2.13. **Con** *satisfies the* (wa) *condition.*

Proof. If $\langle f, g\rangle \circ \langle h, k\rangle$ and $\langle h, k\rangle \circ \langle l, m\rangle$ exist, then the codomain of $\langle f, g\rangle$, say (Y, β), coincides with the domain of $\langle h, k\rangle$ and the codomain of $< h, k >$, say (Z, γ), coincides with the domain of $\langle l, m\rangle$.
Denoting by (X, α) the domain of $\langle f, g\rangle$ and by (W, δ) the codomain of $\langle l, m\rangle$, the above Proposition gives the assertion. □

Clearly, the same results hold for relative E-connections, for any $E \subseteq L$.
The above results show that $\mathbf{Con} = (\mathcal{Obj}(\mathbf{Con}), \mathcal{Mor}(\mathbf{Con}), \circ)$ is a composition graph that satisfies the strong identity law and the weak associativity law; hence $\mathscr{C}$ is a precategory.
Similarly, for any fixed $E \subseteq L$, the triple $\mathbf{E\text{-}Con} = (\mathcal{Obj}(\mathbf{E\text{-}Con}), \mathcal{Mor}(\mathbf{E\text{-}Con}), \circ)$, that has the same objects and the same composition of **Con**, but whose morphisms are the pairs of maps that are relative E-connections of any type, as in Definition 4.2.12, is a precategory.
We can consider, moreover, $\mathbf{Con}_{\exists E} = (\mathcal{Obj}(\mathbf{Con}_{\exists E}), \mathcal{Mor}(\mathbf{Con}_{\exists E}), \circ)$, where $\mathcal{Obj}(\mathbf{Con}_{\exists E}) = \mathcal{Obj}(\mathbf{Con})$ is the class of all structured sets $\{(X, \alpha) | \forall X \in |\mathbf{Set}|, \alpha \in \mathbf{R_L}(X, X)\}$, $\mathcal{Mor}(\mathbf{Con}_{\exists E})$ is the class of pairs of functions that are relative E-connections of some type between two structured sets, for some $E \subseteq L$ and $\circ$ is the partial composition above considered.
Clearly, two of their morphisms can be composed according to the rule stated in Propositions 6.2.7-6.2.10, provided that subsets $E, E' \subseteq L$ exist such that those morphisms are a relative E-connection and a relative E'-connection, respectively, and E and E' satisfy the following condition, for all $X \in |\mathbf{Set}|, \alpha \in \mathbf{R_L}(X, X), x, x' \in X$: $(x, x')\alpha \in E \Leftrightarrow (x, x')\alpha \in E'$.
Then, it can be proved that $\mathbf{Con}_{\exists E}$ is a precategory and that **Con** and every **E-Con**, with $E \subseteq L$, are subprecategories of $\mathbf{Con}_{\exists E}$.
Now, we proceed in the opposite direction, restricting the class of morphisms. First we consider the class of objects $\mathcal{Obj}(\mathbf{Con})$ and restrict the class of morphisms to the connections of type III, then the restriction of the partial composition $\circ$ of **Con** determines a subprecategory that we denote by $\mathbf{Con}_{III}$. By Proposition 6.2.8 (2) it is clear that $\mathbf{Con}_{III} = (Ob(\mathbf{Con}_{III}), \mathcal{Mor}(\mathbf{Con}_{III}), \circ)$ is a category.
Similarly, by Proposition 6.2.9 (2) the subprecategory of **Con** whose morphisms are the connections of type IV is a category $\mathbf{Con}_{IV} = (Ob(\mathbf{Con}_{IV}), \mathcal{Mor}(\mathbf{Con}_{IV}), \circ)$.
Similar categories of relative E-connections of type III and categories of relative E-connections of type IV can be considered, for any fixed $E \subseteq L$.

Remark 6.2.14. A special instance of the category $\mathbf{E\text{-}Con}_{III}$ has been already constructed in [79], where type III Galois connections between posets have been considered.

A different arrangement of the structure of connections, either global or relative, allows to give interesting examples of semicategories.

Now, we need to consider connections not simply as pairs of functions satisfying suitable conditions, but as labeled pairs of functions whose label, running from I to IV, corresponds to a type of connection and requires the corresponding conditions on the functions of the labeled pairs. So, with obvious notation, we denote type Γ connections by $\langle f, g\rangle_\Gamma$ and type Γ E-connections by E-$\langle f, g\rangle_\Gamma$, where $\Gamma \in \{I, II, III, IV\}$ and $E \subseteq L$. To compose connections $\langle f, g\rangle_\Gamma : (X, \alpha) \to (Y, \beta)$ and $\langle h, k\rangle_\Gamma : (Y, \beta) \to (Z, \gamma)$ we require the condition (arising from Propositions 6.2.7-6.2.10)

$$(\Gamma, \Gamma') \in \{(I, II), (I, IV), (II, I), (II, III), (III, I), (III, III), (IV, II), (IV, IV)\}$$

then we compose the function f by h and the function k by g getting a connection of appropriate type, according to the table below, where $\hat{\circ}$ is the symbol to denote the obtained composition of labeled connections.

- $< f, g >_I \hat{\circ} < h, k >_{II} = < f \cdot h, k \cdot g >_{III}$;
- $< f, g >_I \hat{\circ} < h, k >_{IV} = < f \cdot h, k \cdot g >_I$;
- $< f, g >_{II} \hat{\circ} < h, k >_I = < f \cdot h, k \cdot g >_{IV}$;
- $< f, g >_{II} \hat{\circ} < h, k >_{III} = < f \cdot h, k \cdot g >_{II}$;
- $< f, g >_{III} \hat{\circ} < h, k >_I = < f \cdot h, k \cdot g >_I$;
- $< f, g >_{III} \hat{\circ} < h, k >_{III} = < f \cdot h, k \cdot g >_{III}$;
- $< f, g >_{IV} \hat{\circ} < h, k >_{II} = < f \cdot h, k \cdot g >_{II}$;
- $< f, g >_{IV} \hat{\circ} < h, k >_{IV} = < f \cdot h, k \cdot g >_{IV}$.

One can have a sample of the different behavior of the compositions $\circ$ of **Con** and the composition $\hat{\circ}$ noting that $< i_X, i_X >_{III}$ and $< i_X, i_X >_{IV}$ cannot be composed by $\hat{\circ}$ with all types of connections, as, instead, it happens for the composition $\circ$ previously considered. The following Proposition list all the possible compositions with the connections made by the identity functions.

Proposition 6.2.15. *With the above notation, the following holds.*

1. $< f, g >_I \hat{\circ} < i_Y, i_Y >_{IV} = < f, g >_I$;
2. $< f, g >_{II} \hat{\circ} < i_Y, i_Y >_{III} = < f, g >_{II}$;
3. $< f, g >_{III} \hat{\circ} < i_Y, i_Y >_{III} = < f, g >_{III}$;
4. $< f, g >_{IV} \hat{\circ} < i_Y, i_Y >_{IV} = < f, g >_{IV}$;
5. $< i_X, i_X >_{III} \hat{\circ} < f, g >_I = < f, g >_I$;
6. $< i_X, i_X >_{III} \hat{\circ} < f, g >_{III} = < f, g >_{III}$;

7. $< i_X, i_X >_{IV} \hat{\circ} < f, g >_{II} = < f, g >_{II}$;

8. $< i_X, i_X >_{IV} \hat{\circ} < f, g >_{IV} = < f, g >_{IV}$.

Proof. The proof of all statements follows easily by definition of $\hat{\circ}$. □

Obviously, similar results hold for relative E-connections, $E \subseteq L$.
Of course, to obtain a composition graph whose morphisms are the labeled connections we need to fix, for every object (X, α), the identity $(X, \alpha)id$. We have two alternative choices, the one assigning $(X, \alpha)id = < i_X, i_X >_{III}$, the other assigning $(X, \alpha)id = < i_X, i_X >_{IV}$: denote by **LCon3** the composition graph obtained in the first case and by **LCon4** the composition graph obtained in the second case. We shall see that both these composition graphs are semicategories, as a consequence of the following results.

Proposition 6.2.16. *The weak identity condition* (wi) *holds in* **LCon3** *and in* **LCon4**.

Proof. We prove the statement in **LCon3**; the other case will be similar.
Let $\langle f, g\rangle_\Gamma : (X, \alpha) \to (Y, \beta)$ be any connection from (X, α) to (Y, β).
If $\Gamma \in \{I, II, IV\}$, then either $< i_X, i_X >_{III} \hat{\circ} \langle f, g\rangle_\Gamma$ or $\langle f, g\rangle_\Gamma \circ < i_Y, i_Y >_{III}$ or both are not defined.
If $\Gamma = III$, then $< i_X, i_X >_{III} \hat{\circ} \langle f, g\rangle_{III} = \langle f, g\rangle_{III} \hat{\circ} < i_Y, i_Y >_{III} = \langle f, g\rangle_{III}$. So, in any case (wi) is satisfied. □

Proposition 6.2.17. *The strong associativity condition* (sa) *holds in* **LCon3** *and in* **LCon4**.

Proof. The proof is the same in both cases, since it does not depend on the choice of the identity morphism; so, we consider only **LCon3**.
Let $\langle f, g\rangle_\Gamma \hat{\circ} \langle h, k\rangle_{\Gamma'}$ and $\langle h, k\rangle_{\Gamma'} \hat{\circ} \langle l, m\rangle_{\Gamma''}$ be defined.
Then $(X, \alpha), (Y, \beta), (Z, \gamma), (W, \delta) \in |\mathbf{LCon3}|$ exist such that $\langle f, g\rangle_\Gamma : (X, \alpha) \to (Y, \beta)$, $\langle h, k\rangle_{\Gamma'} : (Y, \beta) \to (Z, \gamma)$, $\langle l, m\rangle_{\Gamma''} : (Z, \gamma) \to (W, \delta)$. Moreover, $(\Gamma, \Gamma'), (\Gamma', \Gamma'') \in \{(I, II), (I, IV), (II, I), (II, III), (III, I), (III, III), (IV, II), (IV, IV)\}$. Hence

$$\begin{aligned}(\Gamma, \Gamma', \Gamma'') \in \{ & (I, II, I), (I, II, III), (I, IV, II), (I, IV, IV),\\ & (II, I, II), (II, I, IV), (II, III, I), (II, III, III),\\ & (III, I, II), (III, I, IV), (III, III, I), (III, III, III),\\ & (IV, II, I), (IV, II, III), (IV, IV, II), (IV, IV, IV) \}.\end{aligned}$$

Using Propositions 6.2.7-6.2.10, it is easy to check that, in any case, both $(\langle f, g\rangle_\Gamma \hat{\circ} \langle h, k\rangle_{\Gamma'}) \hat{\circ} \langle l, m\rangle_{\Gamma''}$ and $\langle f, g\rangle_\Gamma \hat{\circ} (\langle h, k\rangle_{\Gamma'} \hat{\circ} \langle l, m\rangle_{\Gamma''})$ are defined, have the same type and, of course, coincide being in both case determined by the functions $f \cdot h \cdot l$ and $m \cdot k \cdot g$. □

Similar conclusions can be drawn for the relative E-connections, for any $E \subseteq L$, so obtaining the semicategories **E-LCon3**, **E-LCon4**, with obvious notations.
Of course, the category $\mathbf{Con}_{III}$ is isomorphic to a subsemicategory of **LCon3** and the category $\mathbf{Con}_{IV}$ is isomorphic to a subsemicategory of **LCon4**.

Remark 6.2.18. Semicategories are the main topic of the papers [53, 54], where, indeed, only artificious examples are considered. Composition graphs, precategories and semicategories are also the considered in [76] but, even there, there is no meaningful example of semicategories, since examples of composition graphs there considered are not semicategories. So **LCon** and **E-LCon** give a relevant contribution as interesting examples of precategories.

6.3 Relational systems

Toward the end of the present thesis we have a look at some of the fundamental ideas we have developed.
It should be quite clear the relevance of binary, possibly many-valued, relations in all the topics we have treated. On one side we have tried to enlighten the role of extended-order algebras in the study of (many valued) binary relations. Conversely, we have seen that the structure itself of such algebras is essentially based on binary relations, either directly or not; in fact, the primitive operation of implication is defined as an extension of an order relation and, moreover, the fundamental tools that allow to enrich the structure of the algebra, namely isotone and antitone Galois connections, have been shown in Chapter 4 to be properly described and studied by means of (many-valued) relations.
An overall view of the interconnection of most topics we have considered suggests the opportunity of gathering all together such topics getting considerable advantage in the comprehension of the whole matter and improvements of the results already obtained.
The last Sections of this thesis are thus devoted to the description of suitable relational systems that summarize and widen in a unified framework most topics we have approached and developed.
Of course, relational systems are very general structures largely used in mathematics, logics and their applications and it will be not surprising that the systems we shall describe are closely related to some of those well established and studied in quite different contexts.
We think, in particular, of the *formal contexts* of formal concept analysis [85], of the *Chu spaces* [38, 73] and of the *topological systems* and the closely related *attachment structures* [46, 47].
A unified view of these different contexts, approached by quite similar relational systems, has been favored by the development of their many-valued version and by their categorical-theoretic description [82].
We start recalling some basic notions to describe already known relational systems.
Formal concept analysis (FCA) has been introduced by R. Wille [85] and it consists of many methods for data analysis and knowledge representation; (FCA) is developed by means of Galois connections between powersets, determined by relations between underlying sets of objects and of attributes which the objects can have.

Definition 6.3.1. *[25] Let X and Y be two sets and $r : X \times Y \to \mathbf{2}$ be a (crisp) binary relation from X to Y. The triple $c = (X, Y, r)$ is called* ***formal context****, X the* ***set of objects*** *and Y the* ***set of attributes****.*

We note that the notion of **Chu space** over **2** [73] is equivalent to that of formal context.

Definition 6.3.2. *[73] A* ***Chu space over 2*** *is defined as a triple* $A = (X, r, Y)$*, consisting of a set* X *of* ***points****, a set* Y *of* ***states*** *and a function* $r : X \times Y \to \mathbf{2}$*.*

Though identical in their formulation and requirements, these two notions have been independently considered in different contexts and with different purposes. The different terminology chosen depends on the primary intended field of application. Chu spaces are usually represented as matrices with rows and columns made by sequences of terms 0 and 1, whose interrelations describe the structure of the space; in this way most mathematical structures, either geometrical or algebraic or topological, may be represented as Chu spaces (see [73], Chapters 2 and 3).
It is quite well known the motivation of (FCA), taking its original idea from the study of concepts in the sense of Port-Royal logic, which requires to lift the formal context (X, Y, r) to the second order level getting a Galois connection between the powersets of both the objects and the attributes sets; this connection is given by the Birkhoff operators [8] of r, which are the strong forward and backward left operators we have described in Section 3.2. Similar structures, which are triples with suitable restrictions on the second and third terms, were introduced by S. Vickers in 1989.

Definition 6.3.3. *[84] A* ***topological system*** *is a triple* (X, Y, r)*, where* X *is a set,* Y *is a frame and* $r : X \times Y \to \mathbf{2}$ *is a binary relation from* X *to* Y *such that* $(x, \cdot)r : Y \to \mathbf{2}$ *is a frame map, for every* $x \in X$*.*

One of the motivations for the introduction of topological systems has been the possibility they offer to put in a unified framework topological spaces and locales, so merging point set and point free topology. Similar structures, with the different purpose of providing topological transformations, have been introduced by C. Guido [47] as follows.

Definition 6.3.4. *[47] An* ***attachment*** *in a frame* L *is a triple* $(L, L, \mathcal{A})$*, where* $\mathcal{A} : L \to \boldsymbol{QFrm}(L, \mathbf{2})$ *is a function which associates to any* $x \in L$ *a quasi-frame map (i.e. a function that preserves arbitrary joins and finite non-empty meets) from* L *to* $\mathbf{2}$*.*

We also remark that the notion of *interchange system* is considered in [25], which, however, coincides with that of Chu space being, in fact, a "topological" system "made" by two sets and one relation without restrictions.
Morphisms of Chu spaces (called *Chu transforms* in [73]), of their equivalent interchange systems and of topological systems (called *continuous maps* in [84]) have been considered as follows (we have unified different notations already used)

$$(f, g^{op}) : (X_1, Y_1, r_1) \to (X_2, Y_2, r_2)$$

where $f \in \mathbf{Set}(X_1, X_2)$, $g^{op} \in \mathbf{Set}^{op}(Y_1, Y_2)$ (for Chu spaces and for interchange systems) or $g^{op} \in \mathbf{Loc}(Y_1, Y_2) = \mathbf{Frm}^{op}(Y_1, Y_2)$ (for topological systems) satisfy the condition

$$(x_1, (y_2)g)r_1 = ((x_1)f, y_2)r_2, \ for\ all\ x_1 \in X_1,\ y_2 \in Y_2.$$

The corresponding categories have been denoted $\mathbf{Chu_2}$, **IntSys** and **TopSys** (see [25, 73, 82, 84]).
Morphisms of $\mathbf{Chu_2}$ and of **IntSys**, as well, have been also considered, without any categorical-theoretic settlement, in [38] where they are described as generalized Galois connections, which have also been further generalized to get the notion of **relational Galois connection** as follows.

Definition 6.3.5. *[38] A **relational Galois connection** from an interchange system (X_1, Y_1, r_1) to another one (X_2, Y_2, r_2) is a pair (φ, ψ) of binary relations $\varphi : X_1 \times Y_2 \to \mathit{2}$, $\psi : X_2 \times Y_1 \to \mathit{2}$ that satisfy the conditions:*

1. *for all $x_1 \in X_1, x_2 \in X_2$: $x_2\psi \le x_1 r_1 \Leftrightarrow x_1\varphi \le x_2 r_2$;*
2. *if $\varphi' : X_1 \times Y_2 \to \mathit{2}$, $\psi' : X_2 \times Y_1 \to \mathit{2}$ are binary relations such that, for all $x_1 \in X_1, x_2 \in X_2$:*

   - $x_2\psi \le x_1 r_1 \Leftrightarrow x_1\varphi' \le x_2 r_2$;
   - $x_2\psi' \le x_1 r_1 \Leftrightarrow x_1\varphi \le x_2 r_2$,

   then $\varphi' \le \varphi$ and $\psi' \le \psi$.

It is quite clear that the full subcategory of both $\mathbf{Chu_2}$ and **IntSys** whose objects are of the form (X, X, r), is isomorphic to the category **LCon3** of global **2**-connections of type III we have described in Section 6.2. Reducing to the case when r is an order relation, one obtains a full subcategory which is denoted **GalCon** in [82] having posets as objects and classical isotonic Galois connections as morphisms.
Note that classical isotonic Galois connections, which are type III $\top$-including Galois connections in our terminology, may be, indeed, identified with type III global **2**-connections. As already stated in Definition 6.3.1, the formal contexts of formal concept analysis are, in fact, interchange systems, but categories arranged in such a framework are quite different from **IntSys** since they involve the Galois connections determined by each formal context. A description of a category **FCI** is given in [25], whose objects are formal contexts but a morphisms from (X_1, Y_1, r_1) to (X_2, Y_2, r_2) is a pair of maps $f : \mathbf{2}^{X_1} \to \mathbf{2}^{X_2}$, $g : \mathbf{2}^{Y_2} \to \mathbf{2}^{Y_1}$ such that the diagrams

$$\begin{tikzcd} \mathbf{2}^{X_1} \arrow[r, "f"] \arrow[d, "H_1"'] & \mathbf{2}^{X_2} \arrow[d, "H_2"] \\ \mathbf{2}^{Y_1} & \mathbf{2}^{Y_2} \arrow[l, "g"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{2}^{X_1} \arrow[r, "f"] \arrow[d, "K_1"'] & \mathbf{2}^{X_2} \arrow[d, "K_2"] \\ \mathbf{2}^{Y_1} & \mathbf{2}^{Y_2} \arrow[l, "g"] \end{tikzcd}$$

commute, where H_i and K_i are the Birkhoff operators determined by the binary relations r_i.
The extension to the L-valued case, where L is a, possibly structured, fixed set is almost straightforward and has been considered as well. Moreover, in [82] the variable basis case, where L is not fixed but runs in the category **Quant** of quantales is considered; we recall that a quantale is a complete lattice with a structure of monoid, whose product distributes on arbitrary sups or both sides.
Dropping any structure on X and Y, any condition on the L-relation $\eta : X \times Y \to L$ and assuming, at most, an equivalence $\equiv$ in L, we call any quadruple (X, Y, L, η) **relational system**. Morphisms between relational systems are generalized L-connections which can be defined in several different ways, two of which have been considered in the framework of the variable basis approach of [82] to L-interchange systems, i.e. L-topological systems free of any structure in their context.
In [82], among others, the following categories are defined, all of which have the same objects, that are relational systems, but the morphisms from (X_1, Y_1, L_1, η_1) to (X_2, Y_2, L_2, η_2) are differently defined:

1. $\mathbf{S\text{-}FC}^C$: the morphisms are triples $(f, g, \varphi) \in \mathbf{Set}(X_1, X_2) \times \mathbf{Set}^{op}(Y_1, Y_2) \times \mathbf{Set}^{op}(L_1, L_2)$ such that $(x_1, (y_2)g^{op})\eta_1 = (((x_1)f, y_2)\eta_2)\varphi^{op}$, for all $x_1 \in X_1, y_2 \in Y_2$;

2. $\mathbf{S\text{-}FC}^C_m$: the morphisms are triples $(f, g, \varphi) \in \mathbf{Set}(X_1, X_2) \times \mathbf{Set}^{op}(Y_1, Y_2) \times \mathbf{Set}(L_1, L_2)$ such that $((x_1, (y_2)g^{op})\eta_1)\varphi = ((x_1)f, y_2)\eta_2$, for all $x_1 \in X_1, y_2 \in Y_2$.

Here, we propose a comprehensive approach that, on one side, takes into account all the four types of connections, so enlarging the framework considered in [82], on the other side, it unifies the two alternative choices of [82] in one context only, considering as the third term of any morphism a binary relation that need not be a function.
We consider as **morphisms** from $S_1 = (X_1, Y_1, L_1, \eta_1)$ to $S_2 = (X_2, Y_2, L_2, \eta_2)$ the triples (f, g, φ) which are morphisms of $\mathbf{Set} \times \mathbf{Set} \times \mathbf{R_2}$ such that $\varphi \in \mathbf{R_2}(L_1, L_2)$ and at least one of the following conditions is satisfied.

1. $f : X_1 \to Y_2$, $g : X_2 \to Y_1$ and, for all $x_1 \in X_1, x_2 \in X_2$: $(x_2, (x_1)f)\eta_2\varphi(x_1, (x_2)g)\eta_1$. In this case, the triple (f, g, φ) is said to be a **type I morphism** from S_1 to S_2 and it may be labeled and denoted by $< f, g, \varphi >_I : S_1 \to S_2$.

2. $f : Y_1 \to X_2$, $g : Y_2 \to X_1$ and, for all $y_1 \in Y_1, y_2 \in Y_2$: $((y_1)f, y_2)\eta_2\varphi((y_2)g, y_1)\eta_1$. In this case, the triple (f, g, φ) is said to be a **type II morphism** from S_1 to S_2 and it may be labeled and denoted by $< f, g, \varphi >_{II} : S_1 \to S_2$.

3. $f : X_1 \to X_2$, $g : Y_2 \to Y_1$ and, for all $x_1 \in X_1, y_2 \in Y_2$ $((x_1)f, y_2)\eta_2\varphi(x_1, (y_2)g)\eta_1$. In this case, the triple (f, g, φ) is said to be a **type III morphism** from S_1 to S_2 and it may be labeled and denoted by $< f, g, \varphi >_{III} : S_1 \to S_2$.

4. $f : Y_1 \to Y_2$, $g : X_2 \to X_1$ and, for all $x_2 \in X_2, y_1 \in Y_1$ $(x_2, (y_1)f)\eta_2\varphi((x_2)g, y_1)\eta_1$. In this case, the triple (f, g, φ) is said to be a **type IV morphism** from S_1 to S_2 and it may be labeled and denoted by $< f, g, \varphi >_{IV} : S_1 \to S_2$.

We remark that a labeled triple has to be considered as a morphism of one type only (the one corresponding to the label), while a triple (f, g, φ) may be a morphism of more that one type. For instance, consider a relational system $S = (X, X, L, \eta)$, the identity function $i_X : X \to X$ and the equality relation $i_L \subseteq L \times L$. The triple (i_X, i_X, i_L) is a morphism from S to itself, both of type III and of type IV; in fact, the following hold:

- $\forall x, y \in X,\ ((x)i_X, y)\eta i_L(x, (y)i_X)\eta$,
- $\forall x, y \in X,\ (x, (y)i_X)\eta i_L((x)i_X, y)\eta$.

Moreover, (i_X, i_X, i_L) is a morphism of type I and of type II, too, from S to itself if and only if η is symmetrical. In fact, in this case, the following conditions hold:

- $\forall x, y \in X,\ (x, (y)i_X)\eta i_L(y, (x)i_X)\eta$,
- $\forall x, y \in X,\ ((x)i_X, y)\eta i_L((y)i_X, x)\eta$,

if and only if η is symmetrical.

Now we consider the class of relational systems $\{S = (X, Y, L, \eta) |\ X, Y, L \in |\mathbf{Set}|,\ \eta \in L^{X \times Y}\}$, the class of labeled relational systems morphisms and define the composition between relational systems morphisms; for this goal, we consider the usual composition of functions, according to the following rules.
Let $S_1 = (X_1, Y_1, L_1, \eta_1)$, $S_2 = (X_2, Y_2, L_2, \eta_2)$ and $S_3 = (X_3, Y_3, L_3, \eta_3)$ be relational systems; if $< f, g, \varphi >_\Gamma : (X_1, Y_1, L_1, \eta_1) \to (X_2, Y_2, L_2, \eta_2)$, $< h, k, \psi >_\Delta : (X_2, Y_2, L_2, \eta_2) \to (X_3, Y_3, L_3, \eta_3)$ are morphisms of type Γ and Δ, respectively, then we define the composition $< f, g, \varphi >_\Gamma \circ < h, k, \psi >_\Delta = < f \cdot h, k \cdot g, \varphi \cdot \psi >_\Sigma$ whenever $(\Gamma, \Delta, \Sigma) \in \{(I, II, III), (I, IV, I), (II, I, IV), (II, III, II), (III, I, I), (III, III, III), (IV, II, II), (IV, IV, IV)\}$. In fact, it can be easily checked that for any triple (Γ, Δ, Σ) included in the above list, the needed compositions of maps and relations can be done and give a triple which is a morphism of type Σ.
The following results say that, for any relational system $S = (X, Y, L, \eta)$ both the type III morphism $< i_X, i_Y, i_L >: S \to S$ and the type IV morphism $< i_Y, i_X, i_L >: S \to S$ satisfy the weak identity law (wi).

Proposition 6.3.6. *With the above notation, the following hold.*

1. *Assuming* $< f, g, \varphi >_I : S_1 \to S_2$, *one has the compositions*
   (a) $< f, g, \varphi >_I \circ < i_{Y_2}, i_{X_2}, i_{L_2} >_{IV} = < f, g, \varphi >_I$;
   (b) $< i_{X_1}, i_{Y_1}, i_{L_1} >_{III} \circ < f, g, \varphi >_I \circ = < f, g, \varphi >_I$.
2. *Assuming* $< f, g, \varphi >_{II} : S_1 \to S_2$, *one has the compositions*
   (a) $< f, g, \varphi >_{II} \circ < i_{X_2}, i_{Y_2}, i_{L_2} >_{III} = < f, g, \varphi >_{II}$;
   (b) $< i_{X_1}, i_{Y_1}, i_{L_1} >_{IV} \circ < f, g, \varphi >_{II} = < f, g, \varphi >_{II}$.

3. *Assuming* $< f, g, \varphi >_{III}: S_1 \to S_2$*, one has the compositions*

   (a) $< f, g, \varphi >_{III} \circ < i_{X_2}, i_{Y_2}, i_{L_2} >_{III} = < f, g, \varphi >_{III}$;

   (b) $< i_{X_1}, i_{Y_1}, i_{L_1} >_{IV} \circ < f, g, \varphi >_{III} = < f, g, \varphi >_{III}$.

4. *Assuming* $< f, g, \varphi >_{IV}: S_1 \to S_2$*, one has the compositions*

   (a) $< f, g, \varphi >_{IV} \circ < i_{Y_2}, i_{X_2}, i_{L_2} >_{IV} = < f, g, \varphi >_{IV}$;

   (b) $< i_{X_1}, i_{Y_1}, i_{L_1} >_{IV} \circ < f, g, \varphi >_{IV} = < f, g, \varphi >_{IV}$. □

Now, we consider the class of all relational systems as the class of objects and the class of all labeled morphisms of any type as morphisms. We also consider the composition of morphisms above defined. We have two choices for identities: the one given by $(X, Y, L, \eta)id_3 = < i_X, i_Y, i_L >_{III}$, the other given by $(X, Y, L, \eta)id_4 = < i_Y, i_X, i_L >_{IV}$, for any relational system.
We denote by **LRelSys3** the composition graph obtained in the first case and by **LRelSys4** the composition graph that can be obtain in the second case. Similarly to the composition graph of connections between structured sets considered in Section 6.2, we can prove the following results.

Proposition 6.3.7. *The weak identity law* (wi) *hold in* **LRelSys3** *and in* **LRelSys4**. □

Proposition 6.3.8. *The strong associativity law* (sa) *hold in* **LRelSys3** *and in* **LRelSys4**. □

Hence, **LRelSys3** and **LRelSys4** are semicategories.
Of course, the subsemicategories $\mathbf{RelSys}_{III}$ and $\mathbf{RelSys}_{IV}$, obtained restricting the class of morphisms to those of type III and of type IV, respectively, and leaving unchanged the objects, are categories.
One can restrict the class of morphisms between relational systems to those whose third term is a relation φ that is a function or to those for which it is the opposite of a function. We call **functional** and **co-functional**, respectively, these two kinds of morphisms. In this case, all types of relational system morphisms can be described as a triple of functions and the conditions that they have to satisfy can be reformulated as follows (we describe only morphisms of type I; the others may be described similarly).

1. With the above notation, if $f : X_1 \to Y_2$, $g : X_2 \to Y_1$ and $h : L_1 \to L_2$ are functions, the triple (f, g, h) is a **type I functional morphism** from S_1 to S_2 and it is denoted by $< f, g, h >_I$: $S_1 \to S_2$ if, for all $x_1 \in X_1, x_2 \in X_2$: $((x_1, (x_2)g)\eta_1)h = (x_2, (x_1)f)\eta_2$.

2. With the above notation, if $f : X_1 \to Y_2$, $g : X_2 \to Y_1$ and $k : L_2 \to L_1$ are functions, the triple (f, g, k) is a **type I co-functional morphism** from S_1 to S_2 and it is denoted by $< f, g, k >_I$: $S_1 \to S_2$ if, for all $x_1 \in X_1, x_2 \in X_2$: $((x_2, (x_1)f)\eta_2)k = (x_1, (x_2)g)\eta_1$.

It is clear that the composition of functional (co-functional, respectively) relational system morphisms are still functional (co-functional, respectively) and that all the identity morphisms are both functional and co-functional. So, these two classes determine subsemicategories of both **LRelSys3** and **LRelSys4** that we denote, with clearly meant symbols, **FLRelSys3**, **FLRelSys4**, **co-FLRelSys3** and **co-FLRelSys4**.
Similarly, we can get the subcategories $\mathbf{FRelSys}_{III}$, $\mathbf{co\text{-}FRelSys}_{III}$, $\mathbf{FRelSys}_{IV}$, $\mathbf{co\text{-}FRelSys}_{IV}$.

We note, in particular, that the subsemicategory **FLRelSys3** generalizes the category $\mathbf{S\text{-}FC}^C$ considered in [82] and above recalled, which is, in fact, isomorphic to $\mathbf{FRelSys}_{III}$.
Similarly, the subsemicategory **co-FLRelSys3** generalize the category $\mathbf{S\text{-}FC}^C_m$ of [82], which is isomorphic to $\mathbf{co\text{-}FRelSys}_{III}$.

Remark 6.3.9. We note that the structured set (X, α), where X is a set and α is an L-relation on X can be viewed as a relational system (X, X, L, α), with $X = Y$.
The conditions that define the four types of morphisms between relational systems are exactly those used to define, in the previous Section, the global and relative connections if and only if $L = L'$ and $\varphi = \equiv$ is a fixed equivalence.
Hence, the relational systems and their morphisms can be viewed as a variable basis generalization of the structured sets and of the global connections.
Hence the semicategory **LCon** is isomorphic to the subsemicategory $(L, \equiv)$-**LRelSys** of **LRelSys**, whose objects are the relational systems (X, X, L, η), with L fixed, and the morphisms are the triples $< f, g, \equiv>_\Gamma$, $\Gamma \in \{I, II, III, IV\}$, that are labeled relational morphisms.
Similarly, the categories $\mathbf{Con}_{III}$, isomorphic to a subsemicategory of **LCon3**, and $\mathbf{Con}_{IV}$, isomorphic to a subsemicategory of **LCon4**, are isomorphic to the fixed-basis subsemicategories $(L, \equiv)$-**LRelSys3** and $(L, \equiv)$-**LRelSys4**, of **LRelSys3** and **LRelSys4**, respectively.

6.4 Tied relational systems

The relational systems we have considered in the previous Section and the semicategories and categories having those as objects give the most general unified framework for a variable basis and many-valued approach to the basic structures for formal concept analysis, i.e. formal contexts, for Chu spaces, for interchange systems and for topological systems. Within their objects, apart from the more composite structures that consider the Galois connections determined by formal contexts, our framework includes the one developed in [82], as we have remarked in the previous Section. We have also shown that our framework constitutes a comprehensive, variable basis approach to Galois connections and to conjugated pairs, which are the morphisms of the semicategory of relational systems; more precisely, these morphisms are variable basis global connections.
However, all the many-valued versions of the above listed basic structures, already considered in [24, 25, 39, 40] and the more general variable basis approaches considered in [79, 81, 82] and in the previous Section may be considered simply as generalizations of

the corresponding classical contexts, which occur as special cases of the corresponding more general situations.

In the variable basis cases, of course, the many-valued and the classical (corresponding to **2**-valued) contexts are related to each other within the framework of the quite general categories or semicategories where they have been included as objects; nevertheless, the link relating the many-valued and the classical systems lies outside each of them.

In the lattice-valued case there is a natural way to determine uniquely a classical relational system (i.e. a formal context) from a relational system (X, Y, L, η), where L is a bounded lattice (possibly with a richer structure), taking the so called crisp counterpart of the structure: a formal context (X, Y, r_η) may be determined by $x r_\eta y \Leftrightarrow (x, y)\eta = \top$, for all $x \in X, y \in Y$. Something similar may be done by means of α-cuts: $x r_\eta y \Leftrightarrow (x, y)\eta \geq \alpha$, for all $x \in X, y \in Y$.

In the general case we remark that any subset $E \subseteq L$ determines a formal context (X, Y, r_η) by $x r_\eta y \Leftrightarrow (x, y)\eta \in E$, for all $x \in X, y \in Y$.

If we think of a relational system as a description of an uncertain relationship between elements of a set X of objects and elements of a set Y of attributes, giving an evaluation of the relationship between any object and any attribute, within a set L of values, then in the above construction of the binary relation r_η, E may be meant as the set of values "true".

This point of view is particularly interesting to us. In fact the main topic of this thesis, i.e. the structure of extended-order algebra, may be seen as a special case of the general situation above described; it has been the original motivation for the introduction of such algebras in [50] to describe a context where both terms of the context coincide with the set of the values that the statements in a logical system may have and the investigated relationship is the implication between two statements, which of course has to be evaluated in the same set of values.

The most basic structure of many-valued logics, namely w-eo algebra $(L, \rightarrow, \top)$, which is called implicative algebra in [74], consists in fact in a relational system $(L, L, L, \rightarrow)$ that, with respect to a singleton $\{\top\} \subseteq L$ determines, in the above described way, a formal context $(L, L, r_\rightarrow)$, where $r_\rightarrow$ is an order relation in L, with maximum $\top$. Conversely, as explained in Section 2.1, the implication of w-eo algebra is an extension of an order relation.

The work of H. Rasiowa [74] and the results we have described in Chapter 2 (see also [13]) show the relevant role of these algebras in many-valued logics. In particular, we have seen how the class of such algebras includes all the integral residuated lattices. Not all residuated lattices are w-eo algebras, but this seems to be not so relevant from the viewpoint of many-valued logics, since most algebras of such logics are, basically, integral residuated lattices. Instead, this seems to be a gap of w-eo algebras from the mathematical point of view, if one looks at the original context where residuated lattices have been considered, which was related to the study of ideals of rings [29].

To close this gap, a more general approach to implicative and, subsequently, to residuated structures has been developed in [49], which involves the approach to tied relational systems we are going to describe in this Section; we mean relational systems, as defined

in the previous Section, enriched by the addiction of a subset that has the role to allow a formal context to be uniquely determined as a basic background.
We start giving the main definition.

Definition 6.4.1. *Let X, Y and L be sets, $\eta : X \times Y \to L$ an L-relation from X to Y and let E be a subset of L. The quintuple $S = (X, Y, L, \eta, E)$ is called* ***tied relational system****.*
The pair (X, Y) is called ***context*** *of the system S, where X and Y are said to be, respectively, the* ***first*** *and the* ***second term*** *of the context (X, Y).*
L is called ***set of values*** *and its elements are called* ***truth values****. The function η is called* ***evaluation****. The subset $E \subseteq L$ is called* ***true set*** *and its elements* ***true values****.*

From a tied relational system S we can define uniquely a formal context associated to it. In fact, let $S = (X, Y, L, \eta, E)$ be a tied relational system and define the crisp binary relation $r_S \subseteq X \times Y$ by means of the following equivalence:

$$\forall x \in X, y \in Y \ x r_S y \Leftrightarrow (x, y)\eta \in E.$$

$c_S = (X, Y, r_S)$ is called **natural context** of the tied relational system S or **formal context induced** by S and r_S is called **natural relation E-induced** by η.

Of course, it is possible to build, not uniquely, a tied relational system from a formal context (X, Y, r). In fact, considering the set L and a subset E of L, we can define an L-relation $\eta_r : X \times Y \to L$ which satisfies the following equivalence:

$$\forall x \in X, y \in Y, \ (x, y)\eta_r \in E \Leftrightarrow xry.$$

$S = (X, Y, L, \eta_r, E)$ is a tied relational system associated to the formal context (X, Y, r) and it is called **E-extension** to L of the formal context (X, Y, r); η_r is called **E-extension** to L of the binary relation r.
Obviously, any tied relational system is an E-extension of its natural context.

Definition 6.4.2. *If $X = Y = L$, we denote $S = (L, L, L, \eta, E)$ simply by the triple $G = (L, \eta, E)$ which we call* ***relational groupoid*** *and we may specify that it is E-**extended**; its natural context is denoted by (L, r_G). Moreover:*

1. *if the natural relation r_G E-induced by η is a (pre)order, $G = (L, \eta, E)$ is called* ***(pre)implicative*** *groupoid;*

2. *if the natural relation r_G E-induced by η satisfies the conditions:*

   (a) there exists an element i in L, such that $i r_G i$;

   (b) for all $x, y \in L$, if $x r_G y$ then $y r_G x$ (symmetry condition);

   (c) for all $x, y, z \in L$, if $x r_G y$ and $y r_G z$, then $x = z$ (univocality condition),

   then $G = (L, \eta, E)$ is called ***quasi-group****.*

Example 6.4.3.

1. Any w-eo algebra $(L, \rightarrow, \top)$ is an implicative groupoid. Its natural context is a partially ordered set $(L, \leq)$ with maximum $\top$.
2. Let $M = (M, \odot, 1)$ be a commutative monoid. It is a relational groupoid and its natural relation can be determined as follows: $xr_My \Leftrightarrow x \odot y = 1 \Leftrightarrow y = x^{-1}$. Clearly, the following hold:
   (a) $1r_M1$, since $1 \odot 1 = 1$;
   (b) $xr_My \Leftrightarrow x \odot y = 1 \Leftrightarrow y \odot x = 1 \Leftrightarrow yr_Mx$;
   (c) $xr_My, yr_Mz \Leftrightarrow y = x^{-1}, z = y^{-1} \Leftrightarrow z = y^{-1} = (x^{-1})^{-1} = x$.

   Hence, M is a quasi-group.
3. Let $L = (L, \rightarrow, \top)$ be a right-distributive w-ceo algebra and let $\otimes$ be its adjoint product; $(L, \otimes, \top)$ is a relational groupoid, but in general it is not a quasi-group. If L is a symmetrical cdeo algebra, then $a \otimes b = \top \Leftrightarrow a = b = \top$; hence $r_L = \{(\top, \top)\}$ and $(L, \otimes, \top)$ is a quasi-group.

The relational groupoids are a generalizations of the extended-order algebras, which are the main topic of this thesis. A detailed study of some class of relational groupoids, which are defined below, and of their properties is developed in [49], where, among others, the main result shows that the class of implicative groupoids contains the class of residuated lattices.

Definition 6.4.4. *[49] A (pre)implicative groupoid (X, η, E), is called **(pre)implicative ub$E$-extended groupoid**, if $E \subseteq MaxL$; **(pre)implicative us$E$-extended groupoid**, if E is an upperset.*

Definition 6.4.5. *Let $S_1 = (X_1, Y_1, L_1, \eta_1, E_1)$ and $S_2 = (X_2, Y_2, L_2, \eta_2, E_2)$ be two tied relational systems. S and S' are said to be **correlated** if they have the same natural relation E-induced by η.*

Remark 6.4.6. Clearly, if the tied relational systems $S_1 = (X_1, Y_1, L_1, \eta_1, E_1)$ and $S_2 = (X_2, Y_2, L_2, \eta_2, E_2)$ are correlated, i.e. $r_{S_1} = r_{S_2}$, then $X_1 = X_2$, $Y_1 = Y_2$ and for all $x \in X_1, y \in Y_1$ $(x, y)\eta_1 \in E_1 \Leftrightarrow xr_{S_1}y \Leftrightarrow xr_{S_2}y \Leftrightarrow (x, y)\eta_2 \in E_2$. Hence, S_1 and S_2 are correlated if and only they have the same context and the following equivalence holds, for all $x \in X_1, y \in Y_1$: $(x, y)\eta_1 \in E_1 \Leftrightarrow (x, y)\eta_2 \in E_2$.

Example 6.4.7. Any symmetrical w-eo algebra $(L, \rightarrow, \top)$ and its dual $(L, \rightsquigarrow, \top)$ are correlated implicative groupoid. In fact, by Definition 2.1.17, the two implications induce the same order relation on L.

Definition 6.4.8. *Let $S_1 = (X_1, Y_1, L_1, \eta_1, E_1)$ and $S_2 = (X_2, Y_2, L_2, \eta_2, E_2)$ be tied relational systems. S_1 and S_2 are said to be **equivalent** if they are correlated and have the same evaluation, that is $\eta_1 = \eta_2$.*

Remark 6.4.9. The previous Definitions of correlated and equivalent tied relational systems determine two equivalence relations in the class of tied relational systems.

Remark 6.4.10. Clearly, if the tied relational systems $S_1 = (X_1, Y_1, L_1, \eta_1, E_1)$ and $S_2 = (X_2, Y_2, L_2, \eta_2, E_2)$ are equivalent, then $X_1 = X_2$, $Y_1 = Y_2$, $L_1 = L_2$; moreover, for all $x \in X_1, y \in Y_1$ the equality $(x,y)\eta_1 = (x,y)\eta_2$ holds and $r_{S_1} = r_{S_2}$, though in general $E_1 \neq E_2$.
Moreover, in this case, for all $x \in X_1, y \in Y_1$ we have the following equivalences: $x r_{S_1} y \Leftrightarrow (x,y)\eta_1 \in E_1 \Leftrightarrow (x,y)\eta_2 \in E_1$ and $x r_{S_2} y \Leftrightarrow (x,y)\eta_2 \in E_2 \Leftrightarrow (x,y)\eta_1 \in E_2$.
Hence, S_1 and S_2 are equivalent systems if and only if they have the same context (X,Y), the same set of values, the same evaluation function η and the following equivalence hold, for all $x \in X, y \in Y$: $(x,y)\eta \in E_1 \Leftrightarrow (x,y)\eta \in E_2$.

In particular, two relational groupoids $G_1 = (L_1, \eta_1, E_1)$ and $G_2 = (L_2, \eta_2, E_2)$ are equivalent if and only if $L_1 = L_2 = L$, $\eta_1 = \eta_2$ and the equivalence $(x,y)\eta \in E_1 \Leftrightarrow (x,y)\eta \in E_2$ is true, for all $x, y \in L$; in this case we also say that E_1 and E_2 are **equivalent**, with respect to η.

Definition 6.4.11. *Let $c = (X,Y,r)$ be a formal context and let L be a set. We say that c is* ***extendible*** *to L if there exist $E \subseteq L$ and $\eta : X \times Y \to L$ such that $S = (X,Y,L,\eta,E)$ is a tied relational system whose natural context is c. Moreover, η is called* ***extension*** *to L of r or, more explicitly, an E-**extension** to L of r. Also, (X,Y,L,η,E) is said to be an L-**extension** of c.*

Remark 6.4.12. Let $c = (X,Y,r)$ be a formal context and let L be a set. If $|L| \geq 2$, then c is surely extendible to L. If $|L| = 1$, then c is extendible to L if and only if either $r = X \times Y$ or $r = \emptyset$. If $L = \emptyset$, then c is extendible to L if and only if $X \times Y = \emptyset$.

Definition 6.4.13. *Let $c = (X,Y,r)$ be a formal context and let L be a set. We denote by $\mathcal{S}(c;L)$ the class of all tied relational systems that are E-extensions to L of c, for some $E \subseteq L$, that is*

$$\mathcal{S}(c;L) = \left\{(X,Y,L,\eta,E) \mid E \subseteq L, \eta \in L^{X\times Y} \ni' \forall x \in X, y \in Y : (x,y)\eta \in E \Leftrightarrow xry\right\}.$$

Moreover, let $E \subseteq L$; we denote by $\mathcal{S}(c;L,E)$ the class of all tied relational systems that are E-extension to L of c, that is

$$\mathcal{S}(c;L,E) = \left\{(X,Y,L,\eta,E) \mid \eta \in L^{X\times Y} \ni' \forall x \in X, y \in Y : (x,y)\eta \in E \Leftrightarrow xry\right\}.$$

Similarly, let $\eta \in L^{X\times Y}$; we denote by $\mathcal{S}(c;\eta)$ the class of all tied relational systems that are E-extension to L of c, for some $E \subseteq L$, that have η as evaluation function, i.e.

$$\mathcal{S}(c;\eta) = \left\{(X,Y,L,\eta,E) \mid E \subseteq L \ni' \forall x \in X, y \in Y : (x,y)\eta \in E \Leftrightarrow xry\right\}.$$

Remark 6.4.14. It is easy to see that

$$\mathcal{S}(c;L) = \bigcup_{E\subseteq L} \mathcal{S}(c;L,E) = \bigcup_{\eta\in L^{X\times Y}} \mathcal{S}(c;\eta).$$

The tied relational systems contained in $\mathcal{S}(c;L)$ are all correlated to each other, so $\mathcal{S}(c;L)$ is the equivalence class with respect to the correlation relation between tied

relational systems.
The tied relational systems contained in $\mathcal{S}(c; L, \eta)$ are all equivalent to each other, so $\mathcal{S}(c; L, \eta)$ is the equivalence class with respect to the equivalence relation between tied relational systems.

Remark 6.4.15. Let $c = (X, Y, r)$ be any fixed context. It follows clearly from Remark 6.4.12 that $\mathcal{S}(c; L)$ is empty if and only if either $|L| = 1$, $r \neq \varnothing$ and $r \neq \top\!\!\top_{XY}$, or $L = \varnothing$ and $X \times Y \neq \varnothing$.
If $|L| < 2$, then $|\mathcal{S}(c; L)| = 1$ so there exist a unique $E \subseteq L$ and a unique $\eta \in L^{X \times Y}$ such that (X, Y, L, η, E) is an L-extension of c.

Definition 6.4.16. *Let $c = (X, Y, r)$ be a formal context and let L be a set. A function $\eta : X \times Y \to L$ is said to be c-**compatible** if there exists $E \subseteq L$ such that (X, Y, L, η, E) is an E-extension to L of c.*
We denote by $\mathcal{V}(c; L)$ the set of all the c-compatible extensions to L of r, that is

$$\mathcal{V}(c; L) = \{\eta : X \times Y \to L | \exists E \subseteq L \ni' \forall x \in X, y \in Y : (x, y)\eta \in E \Leftrightarrow xry\}.$$

Definition 6.4.17. *Let $c = (X, Y, r)$ be a formal context, L be a set and $E \subseteq L$. A function $\eta : X \times Y \to L$ is said to be $(c; E)$-**compatible** if (X, Y, L, η, E) is an E-extension to L of c.*
We denote by $\mathcal{V}(c; L, E)$ the class of all E-extension to L of r, i.e.

$$\mathcal{V}(c; L, E) = \{\eta \in L^{X \times Y} | \forall x \in X, y \in Y : \eta(x, y) \in E \Leftrightarrow xry\}.$$

Remark 6.4.18. 1. $\mathcal{V}(c; L) = \bigcup_{E \subseteq L} \mathcal{V}(c; L, E)$.

2. Two L-relations that belong to $\mathcal{V}(c; L)$ are also said to be c-**compatible** to each other.

3. Two L-relations that belong to $\mathcal{V}(c; L, E)$ are also said to be (c, E)-**compatible** to each other.

Definition 6.4.19. *Let $c = (X, Y, r)$be a formal context and L be a set. A subset E of L is said to be c-**compatible** if there exists $\eta : X \times Y \to L$ such that η is an E-extension to L of r.*
We denote by $\mathcal{E}(c; L)$ the set of all the subsets of L trough which c can be extended to L, that is

$$\mathcal{E}(c; L) = \{E \subseteq L | \exists \eta \in L^{X \times Y} \ni' \forall x \in X, y \in Y : (x, y)\eta \in E \Leftrightarrow xry\}.$$

Definition 6.4.20. *Let $c = (X, Y, r)$ be a formal context, L a set and $\eta : X \times Y \to L$. A subset E of L is said to be $(c; \eta)$-**compatible** if η is an E-extension to L of r.*
We denote by $\mathcal{E}(c; \eta)$ the set of all the subsets of L trough which η extends r to L, i.e.

$$\mathcal{E}(c; \eta) = \{E \subseteq L | \forall x \in X, y \in Y : (x, y)\eta \in E \Leftrightarrow xry\}.$$

If, moreover, L is a (pre)ordered set, we denote by $us\mathcal{E}(c; \eta)$ the set of all the uppersets of L trough which η extends r to L, that is

$$us\mathcal{E}(c; \eta) = \{E \in \mathcal{E}(c; \eta) | E \ upperset \ of \ L\}.$$

Remark 6.4.21. 1. It is easy to prove that $\mathcal{E}(c;L) = \bigcup_{\eta\in L^{X\times Y}} \mathcal{E}(c;\eta)$.

2. Let $c = (X,Y,r)$ be a formal context, L be a set, E be a subset of L and consider $\eta : X\times Y \to L$. Hence, one has that: $\eta \in \mathcal{V}(c;L,E) \Leftrightarrow (X,Y,L,\eta,E) \in \mathcal{S}(c;L,E) \Leftrightarrow (X,Y,L,\eta,E) \in \mathcal{S}(c;\eta) \Leftrightarrow E \in \mathcal{E}(c;\eta)$.

Remark 6.4.22. 1. Let $c = (X,Y,r)$ be a formal context, L a set, E a subset of L and $\eta : X\times Y \to L$. $\mathcal{E}(c;\eta)$ is closed for non-empty unions and non-empty intersections. In fact, let $\mathcal{U} \subseteq \mathcal{E}(c;\eta)$, $\mathcal{U} \neq \varnothing$. For every $x \in X$, $y \in Y$ we have $(x,y)\eta \in \bigcup\mathcal{U} \Leftrightarrow \exists E \in \mathcal{U} : (x,y)\eta \in E \Leftrightarrow xry$. Moreover, for all $x \in X, y \in Y$: $\eta(x,y) \in \bigcap\mathcal{U} \Rightarrow \forall E \in \mathcal{U} : (x,y)\eta \in E \Rightarrow \exists E \in \mathcal{U}, (x,y)\eta \in E \Rightarrow xry$; conversely, $xry \Rightarrow \forall E \in \mathcal{U}, \eta(x,y) \in E \Rightarrow (x,y)\eta \in \bigcap\mathcal{U}$. Hence, $\bigcup\mathcal{U}$, $\bigcap\mathcal{U} \in \mathcal{E}(c;\eta)$.
This means that for any tied relational system (X,Y,L,η,E), the subset E may be replaced by any other subset varying in the interval $[\bigcap\mathcal{E}(c;\eta), \bigcup\mathcal{E}(c;\eta)]$ of the powerset $\mathcal{P}(L)$, allowing an equivalent system.

2. A similar discussion may be done to show that $us\mathcal{E}(c;\eta)$ is an interval in the complete lattice of the uppersets of L. For further details we refer to [49].

Tied relational systems are many-valued extensions of a formal context to which they are intrinsically connected by means of a subset of true values. Truth values of two tied relational systems may vary in different sets, but morphisms between them need not include any morphism between the sets of the respective truth values sets. So the variable basis machinery is not necessary.
Nevertheless, one gets again a semicategory structure as in the case we have already described dealing with (global) connections (see Section 6.2) and with their variable basis extensions given by morphisms of relational systems (see Section 6.3).
We propose the following definition.

Definition 6.4.23. *Let $S = (X_1,Y_1,L_1,\eta_1,E_1)$ and $S_2 = (X_2,Y_2,L_2,\eta_2,E_2)$ be tied relational systems. We define **morphism** from S_1 to S_2 any pair of maps (f,g) such that at least one of the following conditions hold. More precisely:*

1. *$f : X_1 \to Y_2$, $g : X_2 \to Y_1$ and, for all $x_1 \in X_1, x_2 \in X_2$: $(x_2,(x_1)f)\eta_2 \in E_2 \Leftrightarrow (x_1,(x_2)g)\eta_1 \in E_1$. In this case, the pair (f,g) is said to be a **type I morphism** from S_1 to S_2 and it may be labeled and denoted by $< f,g >_I: S_1 \to S_2$.*

2. *$f : Y_1 \to X_2$, $g : Y_2 \to X_1$ and, for all $y_1 \in Y_1, y_2 \in Y_2$: $((y_1)f,y_2)\eta_2 \in E_2 \Leftrightarrow ((y_2)g,y_1)\eta_1 \in E_1$. In this case the pair (f,g) is said to be a **type II morphism** from S_1 to S_2 and it may be labeled and denoted by $< f,g >_{II}: S_1 \to S_2$.*

3. *$f : X_1 \to X_2$, $g : Y_2 \to Y_1$ and, for all $x_1 \in X_1, y_2 \in Y_2$: $((x_1)f,y_2)\eta_2 \in E_2 \Leftrightarrow (x_1,(y_2)g)\eta_1 \in E_1$. In this case the pair (f,g) is said to be a **type III morphism** from S_1 to S_2 and it may be labeled and denoted by $< f,g >_{III}: S_1 \to S_2$.*

4. *$f : Y_1 \to Y_2$, $g : X_2 \to X_1$ and, for all $x_2 \in X_2, y_1 \in Y_2$: $(x_2,(y_1)f)\eta_2 \in E_2 \Leftrightarrow ((x_2)g,y_1)\eta_1 \in E_1$. In this case the pair (f,g) is said to be a **type IV morphism** from S_1 to S_2 and it may be labeled and denoted by $< f,g >_{IV}: S_1 \to S_2$.*

Remark 6.4.24. The morphisms above defined may be seen as generalizations of the relative connections described in Section 4.2. In fact, let us consider the particular relational systems, where the first and the second terms of the contexts coincide, i.e. $X_1 = Y_1$ and $X_2 = Y_2$, the truth sets are the same in each system, i.e. $L_1 = L_2 = L$; moreover, let us assume the true sets coincide, i.e. $E_1 = E_2 = E$ and they are saturated. Under this assumption, a type I morphism $< f, g >_I$ from (X_1, X_1, L, η_1, E) to (X_2, X_2, L, η_2, E) is characterized by the conditions $f : X_1 \to X_2$, $g : X_2 \to X_1$ and for all $x_1 \in X_1, x_2 \in X_2$: $(x_2, (x_1)f)\beta \in E \Leftrightarrow (x_1, (x_2)g)\alpha \in E$. This is equivalent to say that we have the type I E-connection $[f\text{-}g]_E : (X, \alpha) \to (Y, \beta)$, where $X = X_1$, $Y = X_2$, $\alpha = \eta_1$ and $\beta = \eta_2$.
The others types of morphisms can are similarly related to the corresponding relative connections considered in Section 4.2.

Remark 6.4.25. The definition of tied relational systems and of their morphisms suggests an intermediate generalization of relative connections between structured sets.
In fact, we can consider the following extension.
Consider the sets L, $E_1, E_2 \subseteq L$ and the pairs $(X, \alpha), (Y, \beta)$, where X, Y are sets and α, β are L-relations, respectively, on X and Y. Let $f : X \to Y$ and $g : Y \to X$ be two functions.

1. f and g form a **type I** (E_1, E_2)**-connection** from (X, α) to (Y, β), denoted by $[f\text{-}g]_{E_1,E_2} : (X, \alpha) \to (Y, \beta)$ if, $\forall x \in X, y \in Y$: $(y, (x)f)\beta \in E_2 \Leftrightarrow (x, (y)g)\alpha \in E_1$.
2. f and g form a **type II** (E_1, E_2)**-connection** from (X, α) to (Y, β), denoted by $]f\text{-}g[_{E_1,E_2}$: $(X, \alpha) \to (Y, \beta)$, if, $\forall x \in X, y \in Y$: $((x)f, y)\beta \in E_2 \Leftrightarrow ((y)g, x)\alpha \in E_1$.
3. f and g form a **type III** (E_1, E_2)**-connection** from (X, α) to (Y, β), denoted by $(f, g)_{E_1,E_2} : (X, \alpha) \to (Y, \beta)$, if, $\forall x \in X, y \in Y$: $((x)f, y)\beta \in E_2 \Leftrightarrow (x, (y)g)\alpha \in E_1$.
4. f and g form a **type IV** (E_1, E_2)**-connection** from (X, α) to (Y, β), denoted by $)f, g(_{E_1,E_2}$: $(X, \alpha) \to (Y, \beta)$, if, $\forall x \in X, y \in Y$: $(y, (x)f)\beta \in E_2 \Leftrightarrow ((y)g, x)\alpha \in E_1$.

Consider the class of tied relational systems and the class of tied relational system morphisms.
As for relational system morphisms, the composition between tied relational system morphisms is defined by using the usual composition of functions. Also in this case, it is not possible to compose each other all types considered of morphisms; in fact, we can define the composition using similar rules as in the previous Section.
If $< f, g >_\Gamma$: $(X_1, Y_1, L_1, \eta_1) \to (X_2, Y_2, L_2, \eta_2)$, $< h, k >_\Delta$: $(X_2, Y_2, L_2, \eta_2) \to (X_3, Y_3, L_3, \eta_3)$ are morphisms of type Γ and Δ, respectively, then we may define the composition $< f, g >_\Gamma \circ < h, k >_\Delta = < f \cdot h, k \cdot g >_\Sigma$ whenever $(\Gamma, \Delta, \Sigma) \in \{(I, II, III), (I, IV, I), (II, I, IV), (II, III, II), (III, I, I), (III, III, III), (IV, II, II), (IV, IV, IV)\}$. In fact, it can be easily checked that for any triple (Γ, Δ, Σ) included in the above list, the above compositions of maps can be done and give a pair which is a morphism of type Σ.
Now, we consider the relational system $S = (X, Y, L, \eta, E)$ and the identity functions $i_X : X \to X$ and $i_Y : Y \to Y$. We have that:

1. for all $x \in X, y \in Y$: $((x)i_X, y)\eta \in E \Leftrightarrow (x,y)\eta \in E \Leftrightarrow (x,(y)i_Y)\eta \in E$;
2. for all $x \in X, y \in Y$: $(x,(y)i_Y)\eta \in E \Leftrightarrow (x,y)\eta \in E \Leftrightarrow ((x)i_X, y)\eta \in E$.

Hence, the pair (i_X, i_Y) is a type III morphism from S to itself and the pair (i_Y, i_X) is a type IV morphism from S to itself.
Clearly, if $X = Y$ and η is symmetrical, the pair (i_X, i_X) is a type I and a type II morphism from (X, X, L, η, E) to itself, too.
The following results say that, for any tied relational system $S = (X, Y, L, \eta, E)$ both the type III morphism $< i_X, i_Y >_{III} : S \to S$ and the type IV morphism $< i_Y, i_X >_{IV} : S \to S$ satisfy the weak identity law (wi).

Proposition 6.4.26. *With the above notation, the following hold.*

1. *Assuming* $< f, g >_I : S_1 \to S_2$, *one has the compositions*
   (a) $< f, g >_I \circ < i_{Y_2}, i_{X_2} >_{IV} = < f, g >_I$;
   (b) $< i_{X_1}, i_{Y_1} >_{III} \circ < f, g >_I \circ = < f, g >_I$.
2. *Assuming* $< f, g >_{II} : S_1 \to S_2$, *one has the compositions*
   (a) $< f, g >_{II} \circ < i_{X_2}, i_{Y_2} >_{III} = < f, g >_{II}$;
   (b) $< i_{X_1}, i_{Y_1} >_{IV} \circ < f, g >_{II} = < f, g >_{II}$.
3. *Assuming* $< f, g >_{III} : S_1 \to S_2$, *one has the compositions*
   (a) $< f, g >_{III} \circ < i_{X_2}, i_{Y_2} >_{III} = < f, g >_{III}$;
   (b) $< i_{X_1}, i_{Y_1} >_{IV} \circ < f, g >_{III} = < f, g >_{III}$.
4. *Assuming* $< f, g >_{IV} : S_1 \to S_2$, *one has the compositions*
   (a) $< f, g >_{IV} \circ < i_{Y_2}, i_{X_2} >_{IV} = < f, g >_{IV}$;
   (b) $< i_{X_1}, i_{Y_1} >_{IV} \circ < f, g >_{IV} = < f, g >_{IV}$. □

Now, we consider the class of all tied relational systems as the class of objects and the class of all labeled morphisms of any type between them as morphisms. We also consider the composition of morphisms above defined. Also in this case, we have two choices for identities: the one given by $(X, Y, L, \eta, E)id_3 = < i_X, i_Y >_{III}$, the other given by $(X, Y, L, \eta, E)id_4 = < i_Y, i_X >_{IV}$, for any tied relational system.
We denote by **LTRelSys3** the composition graph obtained in the first case and by **LTRelSys4** the composition graph that can be obtain in the second case. Similarly to composition graph of connections between structured sets considered in Section 6.2 and similar to the relational system case discussed in the previous Section, we can prove the following results.

Proposition 6.4.27. *The weak identity law* (wi) *hold in* **LTRelSys3** *and in* **LTRelSys4**. □

Proposition 6.4.28. *The strong associativity law* (sa) *hold in* **LTRelSys3** *and in* **LTRelSys4**. $\square$

Hence, **LTRelSys3** and **LTRelSys4** are semicategories.
Of course, the subsemicategories $\mathbf{LTRelSys}_{III}$ and $\mathbf{LTRelSys}_{IV}$, obtained restricting the class of morphisms to those of type III and of type IV, respectively, and leaving unchanged the objects, are categories.
We have to remark that while composing two morphisms $< f, g >_\Gamma: (X_1, Y_1, L_1, \eta_1, E_1) \to (X_2, Y_2, L_2, \eta_2, E_2)$, $< h, k >_\Delta: (X_2, Y_2, L_2, \eta_2, E_2) \to (X_3, Y_3, L_3, \eta_3, E_3)$ the requirement that the domain of the second morphism coincides with the codomain of the first may be considerably relaxed. In fact, any two subset E_2', E_2'' of L that belong to $\mathcal{E}(c_2; \eta_2)$, where c_2 is the natural context of $(X_2, Y_2, L_2, \eta_2, E_2)$, may replace E_2, respectively, in the codomain of $< f, g >_\Gamma$ and in the domain of $< h, k >_\Delta$.
Such replacements do not invalidate the conditions the morphisms have to satisfy, since for all $x_2 \in X_2, y_2 \in Y_2$ one has $(x_2, y_2)\eta_2 \in E_2 \Leftrightarrow (x_2, y_2)\eta_2 \in E_2' \Leftrightarrow (x_2, y_2)\eta_2 \in E_2''$; moreover, the compositions $f \cdot h$ and $k \cdot g$ do not depend at all on the subsets E_2, E_2', E_2''.

Appendix

In this Appendix we summarize a description of the algebras involved in examples and remarks all along this thesis, recalling the conditions they satisfy.

Algebra 1. Let $L = [0,1]$ be the real unit interval equipped with the implication, called **Kleene-Dienes implicator**, defined by $a \rightarrow_1 b = max(1-a, b)$, for all $a, b \in L$. $(L, \rightarrow_1, 1)$ is not a w-eo algebra, since the reflexivity condition does not hold. In fact, $a \rightarrow_1 a = max(1-a, a) \neq 1$, if $a \neq 1$ and $a \neq 0$. Moreover, $\rightarrow_1$ is an implicator, in the sense of the Definition 2.2.1 and it is an implicator in the usual sense.

Algebra 2. Let $L = [0,1]$ be the real unit interval equipped with the implication, called **Reichenbach implicator**, defined by $a \rightarrow_2 b = 1 - a + ab$, for all $a, b \in L$. $(L, \rightarrow_2, 1)$ is not a w-eo algebra, since the reflexivity condition does not hold. In fact, $a \rightarrow_2 a = 1 - a + a^2 \neq 1$, if $a \neq 1$ and $a \neq 0$. Moreover, $\rightarrow_2$ is a bounded implicator, in the sense of the Definition 2.2.1.

Algebra 3. Let $L = [0,1]$ be the real unit interval equipped with the implication, called **most strict implicator**, defined, for all $a, b \in L$ by $a \rightarrow_3 b = \begin{cases} 1 & if\ a = 0 \\ b & otherwise \end{cases}$. $(L, \rightarrow_3, 1)$ is not a w-eo algebra, since the reflexivity condition does not hold. In fact, $a \rightarrow_3 a \neq 1$, if $a \neq 0$ and $a \neq 1$. Moreover, $\rightarrow_3$ is an isotonic implicator, in the sense of the Definition 2.2.1.

Algebra 4. Let $L = [0,1]$ be the real unit interval equipped with the implication, called **least strict implicator**, defined, for all $a, b \in L$ by $a \rightarrow_4 b = \begin{cases} b & if\ a = 1 \\ 1 & otherwise \end{cases}$. $(L, \rightarrow_4, 1)$ is not a w-eo algebra, since the antisymmetry condition does not hold. In fact, for instance, $0,2 \rightarrow_4 0,3 = 0,3 \rightarrow_4 0,2 = 1$, but $0,2 \neq 0,3$. $\rightarrow_4$ is not an implicator in the sense of Definition 2.2.1, but it is an implicator in the usual sense.

Algebra 5. Let $L = [0,1]$ be the real unit interval equipped with the implication defined, for all $a, b \in L$ by $a \rightarrow_5 b = \begin{cases} 1 & if\ a \leq b \\ max((x)N, y) & otherwise \end{cases}$, where $N : L \rightarrow L, a \mapsto (a)N$ is an involutive unary operation that reverses the order of L. $(L, \rightarrow_5, 1)$ is a ceo algebra. Moreover, $\rightarrow_5$ is an implicator in the sense of Definition 2.2.1, but it is not an implicator in the usual sense.

Algebra 6. Let $L = [0, 1]$ be the unit real interval equipped with the implication, **Gödel implication**, defined, for all $a, b \in L$ by $a \to_6 b = \begin{cases} 1 & if\ a \leq b \\ b & otherwise \end{cases}$. $(L, \to_6, 1)$ is a cdeo algebra and its adjoint product is exactly the meet operation. So, $(L, \to_6, 1)$ is an associative and commutative cdeo algebra. Moreover, $\to_6$ is a distributive implicator, in the sense of the Definition 2.2.1.

Algebra 7. Let $L = [0, 1]$ be the unit real interval equipped with the implication, called **Goguen implicator**, defined, for all $a, b \in L$ by $a \to_7 b = \begin{cases} 1 & if\ a \leq b \\ \frac{b}{a} & otherwise \end{cases}$. $(L, \to_7, 1)$ is a cdeo algebra and its adjoint product is exactly the usual product on L. So, $(L, \to_7, 1)$ is an associative and commutative cdeo algebra. Moreover, $\to_7$ is a commutative implicator, in the sense of the Definition 2.2.1.

Algebra 8. Let $L = [0, 1]$ be the unit real interval equipped with the implication, called **early Zadeh implicator**, defined by $a \to_8 b = max(1 - a, min(a, b))$, for all $a, b \in L$. $(L, \to_8, 1)$ is not a w-ceo algebra, since misses, for instance, the upper bound condition; in fact,$a \to_8 1 = max(1 - a, min(a, 1)) = max(1 - a, a)) \neq 1$, if $a \neq 0$ and $a \neq 1$. Moreover, $\to_8$ is an involutive implicator, in the sense of the Definition 2.2.1, that satisfies the axioms (i_{11}), (i_{12}) and (i_{13}).

Algebra 9. Let $L = [0, 1]$ be the unit real interval equipped with the implication, called **Klir and Yuan 1 implicator**, defined by $a \to_9 b = 1 - a + a^2 a^2 b$, for all $a, b \in L$. $(L, \to_9, 1)$ is not a w-ceo algebra, since misses, for instance, the upper bound condition; in fact,$a \to_9 1 = 1 - a + a^2 \neq 1$, if $a \neq 0$ and $a \neq 1$. Moreover, $\to_9$ is not an implicator in the usual sense.

Algebra 10. Let $L = [0, 1]$ be the unit real interval equipped with the implication, called **Klir and Yuan 2 implicator**, defined, for all $a, b \in L$ by for all $a, b \in L$ by $a \to_{10} b = \begin{cases} b & if\ a = 1 \\ 1 - a & if\ a \neq 1\ and\ b \neq 1 \\ 1 & if\ a \neq 1\ and\ b = 1 \end{cases}$. $(L, \to_{10}, 1)$ is not a w-eo algebra, since the reflexivity condition not holds. In fact, $a \to_{10} a \neq 1$, if $a \neq 1$ and $a \neq 0$. Moreover, $\to_{10}$ is a contrapositive implicator that satisfies (i_{11}) and (i_{12}), in the sense of the Definition 2.2.1.

Algebra 11. Let $L = [0, 1]$ be the unit real interval equipped with the implication defined, for all $a, b \in L$ by $a \to_{11} b = \begin{cases} 0 & if\ a = 1\ and\ b = 0 \\ 1 & otherwise \end{cases}$. $(L, \to_{11}, 1)$ is not a w-ceo algebra, since the antisymmetry condition not holds. In fact, for instance, $0,1 \to_{11} 0,2 = 0,2 \to_{11} 0,1 = 1$, but $0,2 \neq 0,1$. Moreover, $\to_{11}$ is not an implicator in the usual sense.

Algebra 12. Let $L = [0, 1]$ be the unit real interval equipped with the implication defined, for all $a, b \in L$ by $a \to_{12} b = \begin{cases} 1 & if\ a \leq b \\ \sqrt{1 - (a - b)^2} & otherwise \end{cases}$. $(L, \to_{12}, 1)$ is an cdeo algebra with adjoint product defined, for all $a, b \in L$ by $a \otimes_{12} b = max(a - \sqrt{1 - b^2}, 0)$.

We note that $(L, \rightarrow_{12}, 1)$ is neither associative nor commutative. Moreover, $\rightarrow_{12}$ is not a commutative implicator, in the sense of the Definition 2.2.1.

Algebra 13. Let $L = [0,1]$ be the unit real interval equipped with the implication defined, for all $a, b \in L$ by $a \rightarrow_{13} b = \begin{cases} 1 & if\ a \leq b \\ \sqrt{1-a+b} & otherwise \end{cases}$. $(L, \rightarrow_{13}, 1)$ is a cdeo algebra with adjoint product defined, for all $a, b \in L$ by $a \otimes_{13} = max(b^2 + a - 1, 0)$. We note that $(L, \rightarrow_{13}, 1)$ is neither associative nor commutative. Moreover, $\rightarrow_{13}$ is a not contrapositive implicator, in the sense of the Definition 2.2.1.

Algebra 14. Let $L = [0,1]$ be the unit real interval equipped with the implication defined, for all $a, b \in L$ by $a \rightarrow_{14} b = \begin{cases} 1 & if\ a^2 \leq b \\ 1 - a^2 + b & otherwise \end{cases}$. $(L, \rightarrow_{14}, 1)$ is not a w-ceo algebra, since the antisymmetry condition not holds. In fact, for instance, $0,1 \rightarrow_{14} 0,2 = 0,2 \rightarrow_{14} 0,1 = 1$, but $0,2 \neq 0,1$. Moreover, $\rightarrow_{14}$ is not an implicator in the sense of the Definition 2.2.1, but it is an implicator in the usual sense.

Algebra 15. Let $L = [0,1]$ be the unit real interval equipped with the implication defined, for all $a, b \in L$ by $a \rightarrow_{15} b = \begin{cases} 1 & if\ a \leq b \\ \frac{(1-\sqrt{1-a})b}{a} + \sqrt{1-a} & otherwise \end{cases}$. $(L, \rightarrow_{15}, 1)$ is a w-ceo algebra. Moreover, $\rightarrow_{15}$ is an implicator not commutative, involutive and contrapositive, in the sense of the Definition 2.2.1.

Algebra 16. Let $L = \{a, b, c, d\}$, $E = \{b, c\}$ and $\rightarrow$ be defined by

$\rightarrow$	a	b	c	d
a	c	c	c	d
b	a	c	a	c
c	a	a	c	c
d	a	a	a	c

Then (o_2), (o_3), (o_5) and $(o_5)'$ hold, while (o_1) and (o_4) fail to be true.

Algebra 17. Let $L = \{\bot, a, b, \top\}$ and a and b be incomparable element; the operations $\rightarrow$ defined as follows gives to L a structure of cdeo algebra that is not symmetrical (see [33] for detail).

$\rightarrow$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	a	$\top$	a	$\top$
b	b	b	$\top$	$\top$
$\top$	$\bot$	b	a	$\top$

In fact, $a \leq \top \rightarrow b$ though $a \nleq b$, hence the conditions of Theorem 2.1.21 are not satisfied.

Algebra 18. Let $L = \{\bot, a, b, \top\}$ and a and b be incomparable element; the operations $\to$ defined as follows gives to L a structure of cdeo algebra that is not symmetrical (see [33] for detail).

$\to$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	a	$\top$	a	$\top$
b	a	a	$\top$	$\top$
$\top$	a	a	a	$\top$

In fact, $a \leq \top \to b$ though $a \nleq b$, hence the conditions of Theorem 2.1.21 are not satisfied.

Algebra 19. Let $L = \{\bot, a, b, \top\}$ and $\to$ defined by:

$\to$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	a	$\top$	a	$\top$
b	b	b	$\top$	$\top$
$\top$	$\bot$	b	a	$\top$

$(L, \to, \top)$ is a right-distributive ceo algebra and its adjoint product is described in the following table:

$\otimes$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a	a
b	$\bot$	b	$\bot$	b
$\top$	$\bot$	b	a	$\top$

Algebra 20. Let $L = \{\bot, a, b, c, \top\}$ and $\to$ defined by:

$\to$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	$\top$	$\top$	$\top$
b	c	c	$\top$	c	$\top$
c	b	b	b	$\top$	$\top$
$\top$	$\bot$	a	b	c	$\top$

$(L, \to, \top)$ is a right-distributive w-ceo algebra and its adjoint product is described in the following table:

$\otimes$	$\perp$	a	b	c	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	$\perp$	$\perp$	a	a
b	$\perp$	$\perp$	b	$\perp$	b
c	$\perp$	$\perp$	$\perp$	c	c
$\top$	$\perp$	a	b	c	$\top$

Algebra 21. Let $L = \{\perp, b_1, b_2, b_3, c_1, c_2, d, \top\}$ and $\to$ defined by:

$\to$	$\perp$	b_1	b_2	b_3	c_1	c_2	d	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
b_1	b_3	$\top$	b_3	b_3	b_3	$\top$	$\top$	$\top$
b_2	b_2	b_2	$\top$	b_2	$\top$	$\top$	$\top$	$\top$
b_3	b_1	b_1	b_1	$\top$	$\top$	b_1	b_1	$\top$
c_1	$\perp$	$\perp$	b_1	b_2	$\top$	b_1	b_1	$\top$
c_2	$\perp$	b_2	b_3	$\perp$	b_3	$\top$	$\top$	$\top$
d	$\perp$	b_2	b_3	$\perp$	b_3	c_1	$\top$	$\top$
$\top$	$\perp$	$\perp$	$\perp$	$\perp$	b_3	$\perp$	b_1	$\top$

$(L, \to, \top)$ is a right-distributive ceo algebra and its adjoint product is described in the following table:

$\otimes$	$\perp$	b_1	b_2	b_3	c_1	c_2	d	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
b_1	$\perp$	b_1	b_1	$\perp$	b_1	b_1	b_1	b_1
b_2	$\perp$	b_2	$\perp$	b_2	b_2	b_2	b_2	b_2
b_3	$\perp$	$\perp$	b_3	b_3	b_3	b_3	b_3	b_3
c_1	$\perp$	b_2	b_3	c_1	c_1	c_1	c_1	c_1
c_2	$\perp$	c_2	b_1	b_2	c_2	c_2	c_2	c_2
d	$\perp$	d	b_1	b_2	c_2	d	d	d
$\top$	$\perp$	d	$\top$	c_1	$\top$	$\top$	$\top$	$\top$

Algebra 22. Let $L = \{\perp, a, b, \top\}$, $a \leq b$, and $\to$ the binary operation in L described in the table below.

$\to$	$\perp$	a	b	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$
a	a	$\top$	$\top$	$\top$
b	a	b	$\top$	$\top$
$\top$	$\perp$	b	b	$\top$

It is easily seen that $(L, \to, \top)$ is a non-symmetrical cdeo algebra, with $\otimes$ and $\to_\otimes$ described below

$\otimes$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a	a
b	$\bot$	$\bot$	a	b
$\top$	$\bot$	a	a	$\top$

$\to_\otimes$	$\bot$	a	b	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	a	a	$\top$	$\top$
b	a	a	b	$\top$
$\top$	$\bot$	b	b	$\top$

We note that it is easily seen that $\to_\otimes$ is an implicator in the sense of the Definition 2.2.1.

Algebra 23. Let $L = \{\bot, a, b, c, \top\}$, $a \le c, b \le c$ and $\to$ the binary operation in L described in the table below.

$\to$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	b	$\top$	$\top$
b	a	a	$\top$	$\top$	$\top$
c	$\bot$	a	b	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	c	$\top$

It is easily seen that $(L, \to, \top)$ is a non-symmetrical cdeo algebra, with $\otimes$ and $\to_\otimes$ described below

$\otimes$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a	a
b	$\bot$	$\bot$	b	b	b
c	$\bot$	a	b	c	c
$\top$	$\bot$	c	c	c	$\top$

$\to_\otimes$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	b	$\top$	$\top$
b	a	a	$\top$	$\top$	$\top$
c	$\bot$	a	b	$\top$	$\top$
$\top$	$\bot$	c	c	c	$\top$

Also, it can be seen that the condition of Proposition 2.2.6 is satisfied, hence $\to_\otimes$ is a weak-ordered implicator in the sense of the Definition 2.2.1.

Algebra 24. Let $L = \{a, b, c, \top\}$ be the partially ordered set with top element $\top$ and $a \leq b$. Let the binary operation $\rightarrow$ be defined as follows:

$\rightarrow$	a	b	c	$\top$
a	$\top$	$\top$	b	$\top$
b	c	$\top$	c	$\top$
c	b	a	$\top$	$\top$
$\top$	b	b	$\top$	$\top$

$(L, \rightarrow, \top)$ is a w-eo algebra that does not satisfy either (o_5) and (o_5').

Algebra 25. Let $L = \{a, b, c, d, \top\}$ be the partially ordered set with top element $\top$ and $a \leq b$, $c \leq d$. Let the binary operation $\rightarrow$ be defined as follows

$\rightarrow$	a	b	c	d	$\top$
a	$\top$	$\top$	b	a	$\top$
b	b	$\top$	b	a	$\top$
c	b	b	$\top$	$\top$	$\top$
d	b	b	b	$\top$	$\top$
$\top$	b	b	b	a	$\top$

$(L, \rightarrow, \top)$ is a w-eo algebra that satisfies (o_5') but it does not satisfy (o_5).

Algebra 26. Let $L = \{a, b, c, d, \top\}$ be the partially ordered set with top element $\top$ and $a \leq b$, $c \leq d$. Let the binary operation $\rightarrow$ be defined as follows

$\rightarrow$	a	b	c	d	$\top$
a	$\top$	$\top$	a	b	$\top$
b	a	$\top$	c	c	$\top$
c	a	a	$\top$	$\top$	$\top$
d	a	a	a	$\top$	$\top$
$\top$	a	a	c	c	$\top$

$(L, \rightarrow, \top)$ is a w-eo algebra that satisfies (o_5) but it does not satisfy (o_5').

Algebra 27. We describe the MacNeille completion of the left w-eo algebra L described in 25 (something similar could be done by the w-eo algebra described in 26). The elements of the completion are: $\bot_K = [L]$; $\alpha = [a]$; $\beta = [b]$;$\gamma = [c]$; $\delta = [d]$; $\top_K = [\top]$.
The operation $\to_K$ is described in the following table:

$\to_K$	$\bot_K$	α	β	γ	δ	$\top_K$
$\bot_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$	$\top_K$
α	α	$\top_K$	$\top_K$	α	α	$\top_K$
β	α	β	$\top_K$	α	α	$\top_K$
γ	β	β	β	$\top_K$	$\top_K$	$\top_K$
δ	β	β	β	β	$\top_K$	$\top_K$
$\top_K$	α	β	β	α	α	$\top_K$

One can see that $\beta \to_K \gamma = \alpha$, while $b \to c = b$.

Algebra 28. [12] Let $L = \{\bot, a, b, c, \top\}$ be the partially ordered set where $\bot < a < b < c < \top$, equipped with the implication described in the following table:

$\to$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	$\top$	$\top$	$\top$	$\top$
b	$\bot$	c	$\top$	$\top$	$\top$
c	$\bot$	a	b	$\top$	$\top$
$\top$	$\bot$	a	b	c	$\top$

$(L, \to, \top)$ is a (non commutative) symmetrical cdeo algebra and its adjoint product $\otimes$ and its dual implication $\rightsquigarrow$ are described by:

$\otimes$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	a	a
b	$\bot$	a	a	b	b
c	$\bot$	a	a	c	c
$\top$	$\bot$	a	b	c	$\top$

$\rightsquigarrow$	$\bot$	a	b	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	$\top$	$\top$	$\top$	$\top$
b	$\bot$	b	$\top$	$\top$	$\top$
c	$\bot$	b	b	$\top$	$\top$
$\top$	$\bot$	a	b	c	$\top$

We note, moreover, that $(L, \to, \top)$ is not idempotent algebra.

Bibliography

[1] J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Dover Publications (Mineola, New York), 2009.

[2] M. Baczyński, B. Jayaram: *Fuzzy implications*, Studies in Fuzziness and Soft Computing, vol. 231, Springer, Heidelberg, 2008.

[3] W. Bandler, L. Kohout: *Fuzzy powersets and fuzzy implication operators*, Fuzzy Sets Syst., **4**(1) (1980), 13-30.

[4] L. Běhounek, M. Daňková: *Relational compositions in Fuzzy Class Theory*, Fuzzy Sets Syst. **160**(8) (2009), 1005-1036.

[5] R. Bělohlávek: *Fuzzy Galois connections and fuzzy concept lattices: from binary relations to conceptual structures*, In: Novak V., Perfilieva I. (eds.): Discovering the World with Fuzzy Logic. Physica-Verlag (Springer-Verlag Company), Heidelberg, New York, 2000, 462-494.

[6] R. Bělohlávek: *Fuzzy Relational Systems: Foundations and Principles*, IFSR International Series on Systems Science and Engineering, **20**, Kluwer Academic, Plenum Press, Dordrecht, NewYork, 2002.

[7] R. Bělohlávek: *Wath is fuzzy concept lattice?*, in Proceeding of the CLA 2005, 7-9 September 2005, Olomouc, Czech Republic, 34-45.

[8] G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, 1940.

[9] K. Blount, C. Tsinakis: *The structure of residuated lattices*, Internat. J. Algebra Comput., **13**(4) (2003), 437-461.

[10] R. Cignoli, I. M. L. D'Ottaviano, D. Mundici: *Algebraic Foundation of Many-Valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.

[11] M. Ciric, J. Ignjatovic, and S. Bogdanovic, *Uniform fuzzy relations and fuzzy functions*, Fuzzy Sets Syst. **160**(8) (2009), 1054-1081.

[12] L. C. Ciungu: *The radical of a perfect residuated structure*, Information Sciences **179**(15) (2009), 2695-2709.

[13] M. E. Della Stella, C. Guido: *Associativity, commutativity and symmetry in residuated structures*, Order, Published on line 28 February 2012, doi: 10.1007/s11083-012-9250-8.

[14] M. E. Della Stella, C. Guido: *Extended-order algebras and fuzzy implicators*, Soft Comput., **16**(11) (2012), 1883-1892.

[15] M. E. Della Stella, C. Guido: *Functional many-valued relations*, (submitted).

[16] M. E. Della Stella, C. Guido: *L-relations and Galois triangles*, in Algebraic Semantics for Uncertainty and Vagueness, abstract and slides in http://logica.dmi.unisa.it/AlgebraicSemantics2011.

[17] M. E. Della Stella, C. Guido: *The structure of many-valued relations*, in ManyVal'12, abstract and slides in http://logica.dmi.unisa.it/manyval12/index.php?page=home.

[18] M. E. Della Stella, C. Guido: *The structure of many-valued relations I. Galois and Tarski connections*, (work in progress).

[19] M. E. Della Stella, C. Guido: *The structure of many-valued relations II. Relational triangles*, (work in progress).

[20] M. Demirci, *Foundations of fuzzy functions and vague algebra based on many-valued equivalence relations. Part I: fuzzy functions and their applications*, Internat. J. Gen. Systems, **32**(2) (2003), 123-155.

[21] M. Demirci: *Pointed semi-quantales and lattice valued topological spaces*, Fuzzy Sets Syst.,**16**(9) (2010), 1224-1241.

[22] C. De Mitri, C. Guido, *Some remarks on fuzzy powerset operators*, Fuzzy Sets Syst., **126**(2) (2002), 241-251.

[23] J. T. Denniston, S. E. Rodabaugh: *Functorial relationship between lattice-valued topology and topological systems*, Quaest. Math., **32**(2) (2009), 139-186.

[24] J. T. Denniston, A. Melton, S. E. Rodabaugh: *Interweaving algebra and topology: lattice-valued topological systems*, Fuzzy Sets Syst. **192**,(2012), 58-103.

[25] J. T. Denniston, A. Melton, S. E. Rodabaugh: *Formal concept analysis and lattice-valued Chu systems*, Fuzzy Sets Syst., **216** (2013), 52-90.

[26] A. Diego: *Sur Les Algèbres De Hilbert*, Collection de Logique Mathmatique, Sér. A, Fasc, **21**, Paris, Gauthier-Villars, Louvain, E. Nauwelaerts, 1966 (French).

[27] E. W. Dijkstra, *A Discipline of Programming*, Prentice-Hall: Englewood, 1976

[28] R.P. Dilworth: *Non-commutative residuated lattices*, Trans. Amer. Math. Soc., **46**, (1939), 426-444.

[29] R.P. Dilworth, M. Ward: *Residuated lattices*, Trans. Amer. Math. Soc., **45** (1939), 335-354.

[30] F. Durante, E. P. Klement, R. Mesiar, C. Sempi: *Conjunctors and their residual implicators: characterizations and construction methods*, Mediterr. J. Math. **4**(3) (2007), 343-356.

[31] C. Ehresmann: *Catégories et structures*, Dunod, Paris, (1965).

[32] S. Eilenberg, G.M. Kelly: *Closed categories*, in: S. Eilenberg, D.K. Harrison (Eds), Proceedings of the Conference on Categorical Algebra (la Jolla, 1965), Springer, Berlin, 1966, 421-562.

[33] A. Frascella: *Fuzzy Galois connections under weak conditions*, Fuzzy Sets Syst. **172**(1) (2011), 33-50.

[34] A. Frascella, C. Guido: *Structured lattices and topological categories of L-sets*, Fuzzy Sets Syst., **161**(3) (2010), 444-452.

[35] A. Frascella, C. Guido: *Transporting many-valued sets along many-valued relations*, Fuzzy Sets Syst. **159**(1) (2008), 1-22.

[36] H. Furusawa: *Algebraic formalizations of fuzzy relations and their representation theorem*, PhD Thesis, Kyushu University, 1998.

[37] H. Furusawa, Y. Kawahara, M. Winter: *Dedekind Categories with Cutoff Operators*, Fuzzy Sets Syst., **173**(1) (2011), 1-24.

[38] B. Ganter: *Relational Galois connections*, 5th International Conference on Formal Concept Analysis (S. O. Kuznetsov, ed.), Lecture Notes in Computer Science **4390** (2007), 1-17.

[39] B. Ganter, G. Stumme, R. Wille: *Formal concept analysis: foundations and applications*, in: Lecture Notes in Artificial Intelligence, **3626**, Springer Verlag, Berlin, Heidelberg, NewYork, (2005).

[40] B. Ganter, R. Wille: *Formale Begriffsanalyse: Mathematishce Grundlagen*, Springer Verlag, Berlin, Heidelberg, New York, (1999) (German 35 edition 1996/Englishedition).

[41] G. Georgescu, A. Popescu: *Non-commutative fuzzy Galois connections*, Soft Comput. **7**(7) (2003), 458467.

[42] G. Georgescu, A. Popescu: *Non-dual fuzzy connections*, Arch. Math. Logic **43**(8) (2004), 1009-1039.

[43] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott: *A compendium of Continuous Lattices*, Springer-Verlag, Berlin/Heidelberg/New York, 1980.

[44] J. A. Goguen: *L-fuzzy sets*, J. Math. Anal. Appl., **18**(1) (1967), 145-167.

[45] S. Gottwald: *Many-valued logic and fuzzy set theory*, in [55], 5-89.

[46] C. Guido: *Attachment between fuzzy points and fuzzy sets*, U. Bodenhofer, B. De Baets, E. P. Klement and S. Saminger-Platz, eds., Abstract of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009, 52-54.

[47] C. Guido: *Fuzzy points and attachment*, Fuzzy Sets Syst., **161**(16) (2010), 2150-2165.

[48] C. Guido: *Lattice-valued categories*, E. P. Klement, S. E. Rodabaugh and L. N. Stout, Eds., Abstracts of 29th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, 2008, 3537.

[49] C. Guido, L. Ruzittu: *Relational and implicative groupoids*, (preprint).

[50] C. Guido, P. Toto: *Extended-order algebras*, J. Appl. Log., **6**(4) (2008), 609-626.

[51] J. Gutiérrez Garcia, I. Mardones-Prez, M.A. de Prada Vicente, D. Zhang: *Fuzzy Galois connections categorically*, Math. Log. Quart. **56** (2010), 131-147.

[52] P. Hájek: *Metamathematics of Fuzzy Logic*, Trends in Logic-Studia Logica Library, **4**, Kluwer Academic Publishers, Dordrecht, 1998.

[53] H. Herrlich, L. Schröder: *Free Adjunction of Morphisms*, Appl. Cat. Struct., **8**(4) (2000), 595-606.

[54] H. Herrlich, L. Schröder: *Free Factorizations*, Appl. Cat. Struct., **9**(6) (2001), 571-593.

[55] U. Höhle, S. E. Rodabaugh: *Mathematics of Fuzzy Sets: Logic, Topology, And Measure Theory*, The Handbooks of Fuzzy Set Series, **3**, Kluwer Academic Publishers, Boston/Dordrecht/London, 1999.

[56] A. Iorgulescu: *Classes of pseudo-BCK algebras, Part I*, J. Mult. Valued Logic Soft Comput., **12** (2006), 71-130.

[57] A. Iorgulescu: *Classes of pseudo-BCK algebras, Part II*, J. Mult. Valued Logic Soft Comput., **12** (2006), 575-629.

[58] P. Jipsen, C. Tsinakis: *A survey of residuated lattices*, Ordered Algebraic Structures (J. Martinez, Editor), Kluwer Academic Publishers, Dordrecht 2002, 19-56 .

[59] P.T. Johnstone: *Stone Spaces*, Cambridge Univ. Press, Cambridge, 1982.

[60] B. Jónsson and A. Tarski: *Boolean algebras with operators*, I, Amer. J. Math. **73** (1951), 891-939.

[61] B. Jónsson and A. Tarski: *Boolean algebras with operators*, II, Amer. J. Math. **74**, (1952), 127-162.

[62] B. Jónsson, A. Tarski: *Representation problems for relation algebras*, Bull. Amer. Math. Soc. **54** (1948), 79-80.

[63] E. P. Klement, R. Mesiar, E. Pap: *Triangular Norms*, Trends in Logic **8**, Kluwer Academic Publishers, Dordrecht 2000.

[64] E.P. Klement, R. Mesiar, E. Pap: *Triangular norms. Position paper I: basic, analytical and algebraic properties*, Fuzzy Sets Syst., **143**(1) (2004), 5-26.

[65] F. W. Lawvere: *Metric spaces, generalized logic, and closed categories*, Rendiconti del Seminario Matematico e Fisico di Milano **43** (1973), 135-166.

[66] H. M. MacNeille: *Partially ordered sets*, Trans. Amer. Math. Soc., **42** (1937), 416-460.

[67] R. D. Maddux: *Relation Algebras*, Studies in Logic and the Foundations of Mathematics, Elsevier Science, **150** (2006).

[68] C. J. Mulvey: *&*, Rend. Circ. Mat. Palermo (2), **12** (1986), 99-104.

[69] J.P. Olivier, D. Serrato: *Catégories de Dedekind. Morphismes dans les catégories de Schröder*, C. R. Acad. Sci. Paris **290**, (1980), 939-941.

[70] O. Ore: *Galois connexions*, Trans. Amer. Math. Soc. **55** (1944), 493-513.

[71] A. Popescu: *Many-valued Relation Algebras*, Algebra Univers., **53** (2005), 73-108.

[72] A. Popescu: *Some Algeraic Theory for Many-valued Relation Algebras*, Algebra Univers. **56** (2007), 211-235.

[73] V. R. Pratt: *Chu Spaces*, Notes for School on Category Theory and Applications, University of Coimbra, Portugal, 1999.

[74] H. Rasiowa: *An Algebraic Approach to Non-Classical Logics*, Studies in Logics and the Foundations of Mathematics, **78**, North-Holland, Amsterdam, 1974.

[75] K.I. Rosenthal: *Quantales and theirs Applications*, Pitman Research Notes in Mathematics, Series 234, Longman Scientific and Technical, 1990.

[76] L. Schröder: *Monads on Composition Graphs*, Appl. Cat. Struct., **10**(3) (2002), 221-236.

[77] Y. Shi, B. Van Gasse, D. Ruan, E. E. Kerre: *Axioms for fuzzy implications: dependences and independences*, U. Bodenhofer, B. De Baets, E. P. Klement and S. Saminger-Platz, eds., Abstract of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009,, 104-105.

[78] Y. Shi, B. Van Gasse, D. Ruan, E. E. Kerre: *On dependencies and independencies of fuzzy implication axioms*, Fuzzy Sets Syst., **161**(10) (2010), 1388-1405.

[79] S. Solovyov: *Variable-basis topological systems versus variable-basis topological spaces*, Soft Comput. **14**(10) (2010), 1059-1068.

[80] S. A. Solovyov, *Extended-order algebras as a generalization of posets*, Demonstratio Mathematica, Vol. XLIV, No 3 (2011), 589-614.

[81] S. Solovyov: *Categorical foundations of variety-based topology and topological systems*, Fuzzy Sets Syst. **192** (2012), 176-200.

[82] S. Solovyov: *Lattice-valued topological systems as a framework for lattice-valued Formal Concept Analysis*, (submitted to Journal of Mathematics).

[83] A. Tarski: *Sur quelques propriétés caractéristiques des images d'ensembles*, Annales de la Société Polonaise de Mathématique, **6** (1927), 127-128.

[84] S. Vickers: *Topology via Logic*, Cambridge University Press, 1989.

[85] R. Wille: *Formal concept analysis as mathematical theory of concepts and concept hierarchies*, in: B. Ganter, G. Stumme, R. Wille (Eds.), Lecture Notes in Artificial Intelligence, **3626**, Springer Verlag, Berlin, Heidelberg, New York, 2005, 1-33.

[86] M. Winter: *A new algebraic approach to L-fuzzy relations convenient to study crispness*, Information Sciences, **139**(3-4) (2001), 233-252.

[87] M. Winter: *Goguen categories-A categorical approach to L-fuzzy relations*, Trends in Logic, Springer, **25**, 2007.

[88] M. Winter: *Arrow Categories*, Fuzzy Sets Syst., **160**(20) (2009), 2893-2909.

[89] D. Zhang: *Implication structures, fuzzy subsets, and enriched categories*, Fuzzy Sets Syst., **161**(9) (2010), 1205-1223.

[90] L.A. Zadeh: *Fuzzy sets*, Information and Control **8** (1965), 338-353.